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Semistable vector bundles on bubble tree surfaces

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Contents

Abstract

Stability, introduced by Mumford in 1963, was used for construction of moduli spaces of vector bundles by methods of GIT. In the boundary of the compactified moduli space appear non locally free sheaves.

The thesis proposes a new stock of more manageable boundary objects, in the case of a smooth algebraic surface $\mathcal{S}$, which are stable bundles on bubble tree surfaces $\mathcal{S}_{\mathsf{T}}$ having $\mathcal{S}$ as root. Motivation comes from gauge theory: from a differential-geometric point of view, degenerations of ASD connections on a 4-manifold can be described by bubbling phenomena. The Kobayashi–Hitchin correspondence suggests that an analogue for vector bundles on algebraic surfaces should exist.

Due to the presence of several irreducible components, semistable sheaves on $\mathcal{S}_{\mathsf{T}}$ are more scarce than in the smooth case. Nevertheless, semistability criteria for bundles on bubble trees are given.

Next, the deformations of tree-like bundles are studied. The main result is that stable bundles on $\mathcal{S}_{\mathsf{T}}$ with trivial restrictions on the intersections are limits of stable bundles on $\mathcal{S}$. This is proven in the case where $\mathcal{S}$ is a K3 surface or its canonical divisor has negative degree.

Finally, a comparison is made between the stock of stable tree-like bundles which are limits of instantons of charge 2 on the projective plane, and the one of Markushevich-Tikhomirov-Trautmann, obtained by a completely different approach.

Introduction

The notion of semistability has a crucial role in constructing moduli spaces of vector bundles. Indeed, restricting the class of objects that one wants to parametrize to semistable ones is a well-known way of obtaining moduli spaces with a scheme structure.

The first definition of stability for bundles on curves by Mumford has known so far several generalizations to varieties of higher dimensions. Stability in the sense of Gieseker comes naturally from Geometric Invariant Theory and provides moduli spaces M^s with a strong algebro-geometric meaning as they corepresent moduli functors. On the other hand, Mumford-Takemoto-stability (μ-stability) is functorially better behaved (with respect to tensor products and restrictions) and the moduli spaces M^μ of μ-stable bundles can be interpreted in purely differential-geometric terms. Both depend on a fixed polarization and coincide on curves. While Gieseker stability is defined on any Notherian scheme, μ-stability needs a slightly different definition on non-integral schemes, that is the $\hat{\mu}$-stability.

Let $\mathcal{S}$ be a smooth complex projective surface with a fixed polarization H. Both stability notions yield quasi-projective moduli spaces parametrizing isomorphism classes of vector bundles on $\mathcal{S}$. More precisely, for any $r \in \mathbb{N}$, $c_1 \in \mathrm{Pic}(\mathcal{S})$, and a class of a 2-cycle $n \in A^2(\mathcal{S})$, there is a coarse moduli space $M^s(r; c_1, n)$ parameterizing the isomorphism classes of rank r stable bundles $\mathcal{F}$ satisfying $\det(\mathcal{F}) = c_1$ and $c_2(\mathcal{F}) = n$. As μ-stability is stronger than the Gieseker one, there exists an open subscheme $M^\mu(r; c_1, n) \subset M^s(r; c_1, n)$. Pursuing the natural desire to work on a complete space, Gieseker and Maruyama investigated the weaker notion of semistability obtaining a projective scheme $M^{ss}(r; c_1, n)$ parametrizing S-equivalence classes of semistable sheaves. This space provides the classical compactification of the moduli space of stable bundles (in the sense of Gieseker). For a compactification of $M^\mu(r; c_1, n)$, one can consider the projective scheme defined by its closure $\overline{M^\mu}(r; c_1, n)$ inside $M^{ss}(r; c_1, n)$.

A different way to read the problem of compactifying $M^\mu(r; c_1, n)$ is via the Kobayashi-Hitchin correspondence, which translates it into a differential-geometric language. Indeed, $M^\mu(r; c_1, n)$ is identified with the moduli space of anti self-dual Yang-Mills connections (Theorem 1 in [7]). In particular, $M^\mu(2; 0, n)$ is the moduli space of instantons of charge n, that are gauge equivalent classes of ASD $SU(2)$-connections. For the latter, Donaldson-Uhlenbeck constructed a compactification $\overline{YM}_n^{DU}$ and Jun Li [20] proved that it can be endowed with a projective scheme structure such that there is a birational morphism $\overline{M^\mu}(2;0,n) \to \overline{YM}_n^{DU}$. In chapter 8 in [16], the construction of the algebraic version of $\overline{YM}_n^{DU}$ is extended to the moduli spaces of bundles with arbitrary fixed first Chern class and rank. This compactification is denoted $M^{\mu ss}(r;c_1,n)$ and called the moduli space of μ-semistable sheaves, though the points in the boundary do not corepresent any funcor of families of μ-semistable sheaves.

Another compactification of the moduli space of instantons has been described by Feehan [8], Taubes [27] and Uhlenbeck by "bubbling off of spheres" phenomena. This space, denoted by $\overline{YM}_n^{FTU}$, encodes the degeneration of an instanton on $\mathcal{S}$ by a connection on a bubble tree surface, obtained from $\mathcal{S}$ by subsequent gluings of four-spheres at a finite set of point. Aware of Li's results on the Donaldson-Uhlenbeck compactification, it is natural to ask if, and how, this construction can be brought back to an algebro-geometric point of view. A partial answer is given in [21] by Markushevich, Tikhomirov and Trautmann in the case of rank 2 bundles. Their compactification $M^g(c_1,n)$ of $M^\mu(2;c_1,n)$ describes limits of stable bundles of rank 2 by bubbling phenomena, where the topological bubbles are replaced by corresponding algebraic ones.

Denote by $\tilde{\mathcal{S}}$ the blowup of $\mathcal{S}$ in a point, an algebraic bubble is a copy of $\mathbb{P}^2$ attached to $\tilde{\mathcal{S}}$ along the exceptional divisor. An algebraic bubble tree surface $\mathcal{S}_\mathsf{T}$ over $\mathcal{S}$ is obtained by iterating this construction in accordance with a rooted tree graph T. Thus, $\mathcal{S}_\mathsf{T}$ is a reducible projective surface with only normal crossing singularities; its irreducible components are indexed by the vertices of T and two of them intersect along a line whenever the corresponding vertices are connected by an edge in T. In particular $\mathcal{S}_\mathsf{T}$ appears as a fiber of the semi-universal family over the compactified configuration spaces of Fulton and MacPherson [9].

Although the points in the boundary of $M^g(c_1,n)$ represent bundles on algebraic bubble tree surfaces, the moduli space does not have the natural properties one might expect from the analogy with $\overline{YM}_n^{FTU}$. For instance, contracting the intersection lines, an algebraic bubble tree surface defines a spherical bubble tree. So it is natural to expect a morphism between $\overline{YM}_n^{FTU}$ and $M^g(0,n)$ setting a correspondence between the bundles on an algebraic bubble tree and the ASD connections on the associated spherical bubble tree. On the contrary, the bundles appearing on the boundary of $M^g(0,n)$ belong to a redundant stock containing objects which are not trivial on the intersection lines. It is not clear if they really occur as limits of stable bundles but if they do, there is no hope of settle the above correspondence by the pushforward via the morphism contracting the intersection lines.

In accordance with Li's result and the above observation, the idea is to replace the ad hoc stock appearing in the boundary of $M^g(c_1, n)$ by a notion of stability on tree surfaces strengthened by a condition of triviality on the intersections lines. This method ensures a certain naturality of the construction and the possibility to extend the results to higher ranks, but a description of semistable bundles on trees of surfaces is not evident.

Notwithstanding that the construction of the moduli space M^{ss} is quite general and does not require particular conditions on the scheme we are working on, the description of its geometry is more complex dealing with non-smooth varieties. So far, several authors have faced this problem. Proving the irreducibility of M^s for rank 2 bundles over a smooth surface, Gieseker and Li [11] studied its specialization into a moduli space of stable sheaves on a reducible surface. For curves, Nagaraj and Seshadri in [25] showed how the geometry of the moduli space of bundles over a reduced curve reflects the geometry of the curve itself. Inalba [17] generalized their construction in the case of reducible surfaces with two components and normal crossing singularities. He described the moduli space of stable sheaves in terms of triples given by torsion free sheaves on the components and a gluing morphism. Although his approach provides a stratification of the moduli space of semistable sheaves, a characterization of stability on reducible surfaces is missing.

Let $\mathcal{S}_{\mathsf{T}}$ be an algebraic tree surface of type T endowed with a polarization H_{T}. Unlike the smooth case, the property of being $\hat{\mu}$-semistable on $\mathcal{S}_{\mathsf{T}}$ is not preserved under twisting by line bundles. Indeed a general line bundle is not even $\hat{\mu}$-stable, for one of its restrictions to components could destabilize it. Hence, a necessary condition for a bundle to be $\hat{\mu}$-stable is that its restriction to subsurfaces of $\mathcal{S}_{\mathsf{T}}$ are not destabilizing quotients. Given a vector bundle $\mathcal{F}$, for a general polarization H_{T} there exists a line bundle $\mathcal{L}$ such that $\mathcal{F} \otimes \mathcal{L}$ satisfies the above property. In particular if the restriction of $\mathcal{L}$ to the root component is trivial outside the intersection lines, then $\mathcal{L}$ is uniquely determined and $\mathcal{F} \otimes \mathcal{L}$ is called the H_{T}-*compatibilization* of $\mathcal{F}$. We prove the following:

Theorem 3.4.1. *Let $\mathcal{F}$ be a rank r vector bundle on $\mathcal{S}_{\mathsf{T}}$ with $\hat{\mu}$-semistable restrictions to components. Then its H_{T}-compatibilization is $\hat{\mu}$-semistable.*

Moreover, if one of the restrictions of $\mathcal{F}$ to components is $\hat{\mu}$-stable, then it is $\hat{\mu}$-stable.

Considering (semi)stability in the sense of Gieseker an analogue of Theorem (3.4.1) holds. The inverse direction of the above result is false in general. Thus, it provides a sufficient condition for a bundle to be $\hat{\mu}$-stable but it does not characterize them.

The impression we get from Theorem (3.4.1) is that in studying semistability on tree surfaces we should consider vector bundles up to a twist by the line bundles appearing in a precise stock. Thus, we focus our attention on (equivalence classes of) admissible bundles, that are bundles on $\mathcal{S}_{\mathsf{T}}$ having a $\hat{\mu}$-stable H_{T}-combatibilization and whose restrictions to the intersection lines are twists of the trivial bundle.

Considering polarizations H_{T} on $\mathcal{S}_{\mathsf{T}}$ in a specific chamber (defined in Theorem (3.6.1)), we get two results describing admissible bundles on a tree surface as limits of $\hat{\mu}$-stable bundles on $\mathcal{S}$.

Theorem 4.3.2. *Let $(\mathcal{S}, H)$ be a polarized surface such that $K_{\mathcal{S}}.H < 0$, and $\mathcal{S}_{\mathsf{T}}$ a tree surface over $\mathcal{S}$. Every admissible bundle of rank r on $\mathcal{S}_{\mathsf{T}}$ is a deformation of stable bundles on $\mathcal{S}$.*

Theorem 4.3.3. *Let $(\mathcal{S}, H)$ be a polarized K3 surface and $\mathcal{S}_{\mathsf{T}}$ a tree surface over $\mathcal{S}$. An admissible bundle $\mathcal{F}$ of rank r on $\mathcal{S}_{\mathsf{T}}$ such that r and $H.c_1(\mathcal{F})$ are coprime, is a deformation of stable bundles on $\mathcal{S}$.*

Structure of the thesis

We start in Chapter 1 recalling the main results on semistable vector bundles on algebraic surfaces needed in the sequel. In Section 1.1 we introduce in a general framework the definitions of $\hat{\mu}$ and Gieseker (semi)stability. Afterward (Section 1.2) we specialize to the case of smooth surfaces. To avoid confusion, these are connected smooth complex projective varieties of dimension 2. In Section 1.3 we consider blown-up surfaces. In particular we focus our attention on two aspects: first, to relate polarizations on the blowup to ample divisors on the blown-down surface; secondly, to describe the pushforward of a vector bundle via the blowdown morphism. To figure out how the stability properties on vector bundles behave under pullback and pushforward is the aim of the subsequent Section 1.4. We conclude the chapter in Section 1.5 by showing two results on $\tilde{\mathbb{P}}^2$, the blowup of $\mathbb{P}^2$ in a point, that are useful for constructing examples in the sequel. We prove a formula to compute the cohomology groups of line bundles on $\tilde{\mathbb{P}}^2$ and we state an existence theorem for rank 2 vector bundles.

Chapter 2 is devoted to the study of sheaves on a reducible surface $\bar{\mathcal{S}}$ with two smooth components meeting transversely along a divisor. The main intent is to relate propeties of the restrictions to components of a sheaf on $\bar{\mathcal{S}}$ to the sheaf itself. In section 2.1 we analyze the case of vector bundles paying particular attention to the Picard group of $\bar{\mathcal{S}}$. We generalize this discussion in Section 2.2, where we characterize sheaves of pure dimension 2 in terms of their restrictions to components. The result is akin to the correspondence with parabolic triples presented in [17]. In Section 2.3, applying a result of Burban and Drodz [6], we provide a precise description of the local structure of reflexive sheaves on $\bar{\mathcal{S}}$. Finally, in Section 2.4 we prove an analogue of Theorem (3.4.1) both for $\hat{\mu}$ and for Gieseker semistability.

Vector bundles on trees of surfaces are presented in Chapter 3. We start introducing the definitions of a rooted tree and a tree surface (Section 3.1). The Picard group of a tree surface $\mathcal{S}_{\mathsf{T}}$, as well as the polarizations on it, are studied in Section 3.2. Approaching the notions of $\hat{\mu}$ and Gieseker (semi)stability on a tree surface, in Section 3.3 appears the concept of H_{T}-compatibilization of a bundle. The feeling is that one should consider semistability for vector bundles on $\mathcal{S}_{\mathsf{T}}$ up to a twist by particular line bundles. With this idea in mind, the theorems proposed in Section 3.4 are the natural generalizations of the results on semistability in the previous chapter. The inverse direction of Theorem (3.4.1) is proven in particular cases imposing some unnatural conditions on the polarization (Section 3.5) and, as shown in some examples, in general they fail. In Section 3.6 we provide the definitions and the first properties of the object we are most interested in: admissible bundles and their equivalence classes. In particular, a representative example clarifies the theory developed so far. In some cases, admissible bundles on tree surfaces can be obtained by lifting those on other surfaces with simpler tree structures. This construction presents an unbounded family of admissible bundle of fixed rank with the same total charge. In Section 3.7 we show how to avoid these situations by imposing conditions on the charges of restrictions. In particular, we have to prohibit the chains with zero charges inside our trees as in the definition of a weighted tree introduced in [21].

In Chapter 4 we discuss the problem of deforming vector bundles. In particular, we are interested in understanding the conditions under which a vector bundle on a tree surface $\mathcal{S}_{\mathsf{T}}$ over $\mathcal{S}$ is a limit of stable bundle on $\mathcal{S}$. The answer is quite immediate in the case of line bundles (Section 4.1). On the contrary, for higher ranks the machinery provided by deformation theory is needed. This is briefly presented in Section 4.2, where a smoothness criterion for the relative moduli space in a stable point is proven. We apply it to the case of admissible bundles in Section 4.3. Choosing the polarization in an appropriate chamber, these bundles are degenerations of bundles on $\mathcal{S}$ whenever $\mathcal{S}$ has canonical class with negative degree. Furthermore, with an extra condition on the first Chern class, an analogous result holds for K3 surfaces.

Chapter 5 is intended to compare admissible bundles with the tree-like bundles appearing in the boundary of the bubble tree compactification presented in [21]. In Section 5.1 we present the description of the compactification of $M^{\mu}(2;0,2)$, the moduli space of rank 2 stable bundles on $\mathbb{P}^2$, as appears in the above mentioned article. Afterwards, in Section 5.2, we provide a complete description of admissible bundles of rank 2 on tree surfaces over $\mathbb{P}^2$ having good charging with total charge 2. There are admissible bundles which do not occur in the boundary of the bubble tree compactification.

Chapter 1

Vector bundles on smooth surfaces

This chapter is intended to introduce the basic definitions of the theory and to describe some application in the well studied case of vector bundles over smooth projective surfaces. In the first section we briefly recall the notions of $\hat{\mu}$ and Gieseker stability in a general setting. Subsequently we will focus on smooth surfaces. In the second sectiont we list some classical results. Details and proofs can be easily found in literature, so we omit them. In the following sections we study vector bundles on blown-up surfaces, paying particular attention to the relation between $\hat{\mu}$-semistability on a surface and on its blowups. The chapter ends by presenting some technical results on the blowup of the projective plane in a point.

1.1 Preliminaries

The framework of this section is highly more general than what we will face in the sequel. On the other hand, it is necessary to provide a notion of semistability on reducible surfaces. The notation and the definitions are as in the first chapter of [16]. We refer to this book for proofs and details.

Let $\mathcal{X}$ be a complex projective scheme of dimension n and $\mathcal{G}$ be a coherent sheaf on it. The *support* of $\mathcal{G}$ is defined as the closed subset of $\mathcal{X}$ where the stalks of $\mathcal{G}$ are non-vanishing. Consequently, by *dimension* of $\mathcal{G}$ we refer to the dimension of $\mathrm{Supp}(\mathcal{G})$.

We denote by $T(\mathcal{G})$ the maximal subsheaf of $\mathcal{G}$ with support of dimension strictly smaller than $\dim(\mathcal{G})$. If $\mathcal{G}$ does not admit subsheaves of this type, i.e. if $T(\mathcal{G})$=0, it is said to be of *equidimensional* (or of pure dimension n, where $n = \dim(\mathcal{G})$). So, we have

$$0 \to T(\mathcal{G}) \to \mathcal{G} \to \overline{\mathcal{G}} \to 0, \tag{1.1}$$

where the upper bar denote the *maximal dimensional quotient* of $\mathcal{G}$.

Assume $\mathcal{X}$ to be equipped with an ample line bundle $\mathcal{L}$. The *Hilbert polynomial* of a coherent sheaf $\mathcal{G}$ is defined to be

$$P_{\mathcal{G}}(t) = \chi(\mathcal{G} \otimes \mathcal{L}^{\otimes t}) \ .$$

It's well known that the degree of $P_{\mathcal{G}}(t)$ coincides with the dimension of $\mathcal{G}$. Hence, if it is of dimension n, the Hilbert polynomial takes the form

$$P_{\mathcal{G}}(t) = \sum_{i=0}^{n} \alpha_i(\mathcal{G}) t^i,$$

where the $\alpha_i(\mathcal{G})$'s are rationals and the leading coefficient $\alpha_n(\mathcal{G})$ is always positive. Dividing $P_{\mathcal{G}}(t)$ by its leading coefficient we obtain the *reduced* Hilbert polinomial of $\mathcal{G}$, denoted by $p_{\mathcal{G}}(t)$.

Definition 1. A coherent sheaf $\mathcal{G}$ is called *(semi)stable* in the sense of Gieseker if it is of pure dimension n and for every proper subsheaf $\mathcal{E} \subset \mathcal{G}$, we have $p_{\mathcal{E}}(t) \underset{(-)}{\leq} p_{\mathcal{G}}(t)$ for t big enough.

It is easy to see that the ordering we use on the polynomials corresponds to the lexicographical ordering on their coefficients. A slightly different notion of semistability can be obtained comparing just the second coefficient of the reduced Hilbert polynomial, and forgetting about the rest.

Definition 2. Let $\mathcal{G}$ be a coherent sheaf of dimension n. The *slope* of $\mathcal{G}$ is defined as

$$\hat{\mu}_{\mathcal{G}} = \frac{\alpha_{n-1}(\mathcal{G})}{\alpha_n(\mathcal{G})}.$$

Moreover, $\mathcal{G}$ is called *$\hat{\mu}$-(semi)stable* if it is of pure dimension n and for every proper subsheaf $\mathcal{E} \subset \mathcal{G}$ such that $\alpha_n(\mathcal{E}) < \alpha_n(\mathcal{G})$, we have $\hat{\mu}_{\mathcal{E}} \underset{(-)}{\leq} \hat{\mu}_{\mathcal{G}}$.

Although the above definitions deal with proper subsheaves, we can easily pass to proper quotients. Let $\mathcal{G}$ be a sheaf of pure dimension n and consider an exact sequence

$$0 \to \mathcal{E} \to \mathcal{G} \to \mathcal{Q} \to 0.$$

The Hilbert polynomial is, as the Euler characteristic does, additive in exact sequences. So, if $\mathcal{Q} \neq 0$ and its dimension is smaller than n, then $\mathcal{G}$ and $\mathcal{E}$ have the same leading term and $p_{\mathcal{E}}(t) < p_{\mathcal{G}}(t)$. The same cannot be said for the slope. Indeed, if the codimension of $\mathrm{Supp}(\mathcal{Q})$ is at least 2, $\mathcal{E}$ and $\mathcal{G}$ have the same slope. The additional condition on the leading coefficient that appears in the definition of $\hat{\mu}$-(semi)stability is meant to avoid this situation.

Suppose now that $\mathcal{Q}$ has the same dimension of $\mathcal{G}$. It can be easily proved that $p_{\mathcal{E}}(t) \underset{(-)}{\leq} p_{\mathcal{G}}(t)$ if and only if $p_{\mathcal{G}}(t) \underset{(-)}{\leq} p_{\mathcal{Q}}(t)$. Furthermore, $\mathcal{Q}$ has a surjection onto its maximal dimensional quotient. This has smaller reduced Hilbert polynomial. Thus, both $\hat{\mu}$ and Gieseker (semi)stability can be verified just by checking exact sequences with quotients of pure dimension d. These are called *saturated sequences* for $\mathcal{G}$.

By checking on saturated sequences, it follows that

$$\hat{\mu}\text{-stable} \implies \text{Gieseker stable} \implies \text{Gieseker semistable} \implies \hat{\mu}\text{-semistable.}$$

Remark. Although both the slope and the reduced Hilbert polynomial are defined for coherent sheaves of dimension n, the definition of $\hat{\mu}$ and Gieseker (semi)stability apply for sheaves of pure dimension only. By a different formulation, Gieseker (semi)stability can be extended to all coherent sheaves, but this does not change the substance. As a result, semistable sheaves turn to be of pure dimension n. On the contrary, allowing non-equidimensional sheaves in the definition of $\hat{\mu}$-semistability, one gets an excessive stock of sheaves, because adding sheaves supported in codimension 2 does not change the slope.

We conclude this section recalling a fundamental property of stable sheaf.

Proposition 1.1.1. *If $\mathcal{G}$ is a stable sheaf, then* $\mathrm{End}(\mathcal{G}) \simeq \mathbb{C}$*, i.e. $\mathcal{G}$ is a simple sheaf.*

1.2 Smooth surfaces

The aim of this section is to study the above definitions in the particular case of a smooth surface $\mathcal{S}$. To avoid confusion, by smooth surface, we mean a connected smooth complex projective variety of dimension 2.

Coherent sheaves on $\mathcal{S}$ are locally defined by finitely generated $\mathcal{O}_{\mathcal{S}}(\mathcal{U})$-modules. Thus, locally, we can consider torsion submodules. By the smoothness hypothesis, these patch together. So, for every coherent sheaf $\mathcal{G}$ on $\mathcal{S}$, we get a subsheaf $T(\mathcal{G})$, called the *torsion part* of $\mathcal{G}$. The cokernel of this inclusion, denoted by $\overline{\mathcal{G}}$, is a torsion free quotient of $\mathcal{G}$.

The notion of torsion is strictly related to the notion of support. Indeed, a local section $s \in \mathcal{G}(\mathcal{U})$ provides torsion if it is annihilated by an element $f \in \mathcal{O}_{\mathcal{S}}(\mathcal{U})$. Thus, s is zero outside the locally closed subvariety defined by $f = 0$. The inverse direction, sections vanishing almost everywhere provide torsion elements, is easily verified. So, torsion sheaves are supported in codimension 1, and the notation above is coherent with the notation in the previous section.

For every coherent sheaf $\mathcal{G}$, the function associating to a point $x \in \mathcal{S}$ the dimension of the fiber $\mathcal{G}(x)$, is upper-semicontinuous (see exercise II.5.8 in [14]). The points where it jumps are called singular points of $\mathcal{G}$ and form a closed subset in $\mathcal{S}$. It is known (chapter 2, [26]) that the singular locus of torsion free sheaves is in codimension 2, while reflexive sheaves have no singularities.

Definition 3. Outside its singular locus, a coherent sheaf $\mathcal{G}$ is a locally free module. Its rank is the *rank* of $\mathcal{G}$.

Denote by $\mathcal{G}^{\check{}} \simeq \mathcal{H}om(\mathcal{G}, \mathcal{O}_{\mathcal{S}})$ the dual sheaf. For every coherent sheaf $\mathcal{G}$ there exists an exact sequence

$$0 \to T(\mathcal{G}) \to \mathcal{G} \to \mathcal{G}^{\check{}\check{}} \to \mathcal{O}_{\mathrm{Sing}(\mathcal{G})} \to 0. \tag{1.2}$$

Comparing the ranks we get $\mathrm{rk}(\mathcal{G}) = \mathrm{rk}(\overline{\mathcal{G}}) = \mathrm{rk}(\mathcal{G}^{\check{}\check{}})$. In particular, a sheaf is torsion if, and only if, it has rank 0.

Let $\mathcal{F}$ be a vector bundle of rank r on $\mathcal{S}$. We use the definition of Chern classes from Appendix A of [14]. Thus, $c_i(\mathcal{F})$ is an algebraic cycle in the i-th Chow group $A^i(\mathcal{S})$, and so $c_1(\mathcal{F}) = \Lambda^r\mathcal{F}$ is an element of $\mathrm{Pic}(\mathcal{S})$.

The splitting principle provides formulas to compute the Chern classes of vector bundles. Among them, recall that for every divisor D in $\mathcal{S}$,

$$c_1(\mathcal{F}(D)) = c_1(\mathcal{F}) + rD\ ,$$
$$c_2(\mathcal{F}(D)) = c_2(\mathcal{F}) + (r-1)c_1(\mathcal{F}).D + \binom{r}{2}D^2\ .$$

Considering free resolutions, we can extend the definition of Chern classes to any coherent sheaf. For example, let $\mathcal{G}$ be a torsion free sheaf. From the exact sequence (1.2) we deduce that, $c_1(\mathcal{G}) = c_1(\mathcal{G}^{\check{}\check{}})$ and $c_2(\mathcal{G}) = c_2(\mathcal{G}^{\check{}\check{}}) + \ell(\mathrm{Sing}(\mathcal{G}))$.

Consider $\mathcal{S}$ to be equipped with an ample polarization H. The Hirzebruch-Riemann-Roch Theorem permits to compute Hilbert polynomial of locally free sheaves on $\mathcal{S}$ from their Chern classes. So, from (1.2), we deduce a formula for the Hilbert polynomial of a torsion free sheaf $\mathcal{G}$:

$$\begin{aligned} P_{\mathcal{G}}(t) &= \deg(\mathrm{ch}(\mathcal{G}^{\check{}\check{}} \otimes \mathcal{O}_{\mathcal{S}}(tH)).\mathrm{td}(\mathcal{T}))_2 - \ell(Sing(\mathcal{G})) \\ &= \frac{r}{2}H^2t^2 + \frac{1}{2}H(2c_1(\mathcal{G}) - rK_{\mathcal{S}})t + \chi(\mathcal{G})\ . \end{aligned} \tag{1.3}$$

where r is the rank of $\mathcal{G}$, and $K_{\mathcal{S}}$ is the canonical divisor of $\mathcal{S}$. Moreover, the Euler characteristic is given by

$$\chi(\mathcal{G}) = \frac{1}{2}(c_1(\mathcal{G})^2 - K_{\mathcal{S}}.c_1(\mathcal{G})) - c_2(\mathcal{G}) + r\chi(\mathcal{O}_X).$$

For general coherent sheaves, we can compute the Hilbert polynomial from the torsion and the torsion free parts via (1.1). As the torsion subsheaf is supported in codimension 1, we recall the formula for the Hilbert polynomial for sheaves on curves. Let $\mathcal{F}_{\mathcal{C}}$ be a vector bundle of rank r on a smooth curve $\mathcal{C}$ in $\mathcal{S}$. Then

$$P_{\mathcal{F}_{\mathcal{C}}}(t) = rH.\mathcal{C}t + deg(\Lambda^r\mathcal{F}_{\mathcal{C}}) + r(1 - g(\mathcal{C})).$$

Clearly, if a sheaf on a curve has torsion, the latter is supported on isolated points and changes the constant term of the Hilbert polynomial by the lenght of its support.

Computing the slope of a torsion free sheaf $\mathcal{G}$ on $\mathcal{S}$, we get

$$\hat{\mu}_{\mathcal{G}} = \frac{2H.c_1(\mathcal{G})}{rH^2} - \frac{H.K_{\mathcal{S}}}{H^2}.$$

Note that there are a term and a factor not depending on $\mathcal{G}$. Define the degree of $\mathcal{G}$ with respect to H to be $H.c_1(\mathcal{G}) = \deg_H(\mathcal{G})$. It follows immediately,

Proposition 1.2.1. *A torsion free sheaf $\mathcal{G}$ on $\mathcal{S}$ is $\hat{\mu}$-(semi)stable if and only if for all proper subsheaves $\mathcal{E}$ of smaller rank,*

$$\frac{\deg_H(\mathcal{E})}{\mathrm{rk}(\mathcal{E})} \underset{(-)}{<} \frac{\deg_H(\mathcal{G})}{\mathrm{rk}(\mathcal{G})}. \tag{1.4}$$

Remark. Historically, the above is called μ-(semi)stability and first appeared as the natural generalization to surfaces of the (semi)stability on curves. For evident computational reasons, when studying sheaves on smooth surfaces we prefer to use this formulation of $\hat{\mu}$-semistability.

It is clear from the above proposition that the equality in (1.4) may hold only if the fraction on the right hand side can be reduced by a common factor.

Lemma 1.2.2. *Let $\mathcal{G}$ be a $\hat{\mu}$-semistable torsion free sheaf on $\mathcal{S}$. If $\deg_H(\mathcal{G})$ and $\mathrm{rk}(\mathcal{G})$ are coprime, then $\mathcal{G}$ is $\hat{\mu}$-stable.*

An important result in the study of $\hat{\mu}$-(semi)stable vector bundles on smooth surfaces is the Bogomolov inequality. It provides a necessary condition for the $\hat{\mu}$-semistability of a torsion free sheaf depending on its Chern classes. Let the *discriminant* of $\mathcal{G}$ be

$$\Delta(\mathcal{G}) = 2rc_2(\mathcal{G}) - (r-1)c_1^2. \tag{1.5}$$

Proposition 1.2.3 (Bogomolov Inequality)**.** *Suppose $\mathcal{G}$ to be a $\hat{\mu}$-semistable torsion free sheaf, then $\Delta(\mathcal{G}) \geq 0$.*

1.3 Vector bundles on monoidal transformations

Let $\mathcal{S}$ be a smooth projective surface and $\sigma : \tilde{\mathcal{S}} \to \mathcal{S}$ the blowup at a finite set of points $P_1, \dots, P_s$. Denote by $E_1, \dots, E_s$ the corresponding exceptional divisors.

It is well known that

$$\mathrm{Pic}(\tilde{\mathcal{S}}) \simeq \mathrm{Pic}(\mathcal{S}) \oplus \langle E_1, \dots, E_s \rangle_{\mathbb{Z}}.$$

Abusing the notation, elements in $\mathrm{Pic}(\tilde{\mathcal{S}})$ arising from $\mathrm{Pic}(\mathcal{S})$ will be written omitting the pullback sign. Moreover, the projection $\mathrm{Pic}(\tilde{\mathcal{S}}) \twoheadrightarrow \mathrm{Pic}(\mathcal{S})$ will be denoted by σ_*. The relation between this group homomorphism and the pushforward functor will be clarified later in this section.

A natural question to ask is how to relate ample divisors on $\mathcal{S}$ and on $\tilde{\mathcal{S}}$. A useful tool to answer to this question is the Nakai-Moishezon criterion (chapter V, [14]). Applying it, it is not hard to prove the following:

Lemma 1.3.1. *Let $\bar{H}$ be an ample divisor on $\tilde{\mathcal{S}}$, then $\sigma_*\bar{H}$ is ample on $\mathcal{S}$.*

Suppose now H to be an ample divisor on $\mathcal{S}$. It is not evident how to construct ample elements in $\mathrm{Pic}(\tilde{\mathcal{S}})$ from H. Necessary conditions for the ampleness of a divisor $H - \sum_{i=1}^{s} a_i E_i$ are immediate:

$$a_i > 0, \quad H^2 < \sum_{i=1}^{s} a_i^2.$$

But, one can easily see that these are not sufficient.

Example 1.3.2. Let $\tilde{\mathcal{S}}$ be the blowup of $\mathbb{P}^2$ in two points and $2H$ the class of conics. The divisor $D = 2H - E_1 - E_2$ has positive intersection with the exceptional divisors and D has positive square. But the intersection with the strict transform of a line passing through the two points is 0. Thus D is not ample.

It turns out that the ampleness of H is not easy to use when lifting it to an ample divisor on $\tilde{\mathcal{S}}$. Surprisingly, things simplify when dealing with very ample divisors.

Lemma 1.3.3. *Let H be a very ample divisor on $\mathcal{S}$, $\mathcal{C}$ a curve in $\mathcal{S}$, and P a point in $\mathcal{C}$. Then, $H.\mathcal{C} \geq mult_P(\mathcal{C})$.*

Proof. As H is very ample, up to taking a different representative of its linear system, we can assume it to intersect $\mathcal{C}$ on a finite set of point including P. So we have

$$H.\mathcal{C} = \sum_{Q \in H \cap \mathcal{C}} (H.\mathcal{C})_Q \geq \sum_{Q \in H \cap \mathcal{C}} \mathrm{mult}_Q(H) \cdot \mathrm{mult}_Q(\mathcal{C}) \geq 0.$$

The result follows immediately. □

Proposition 1.3.4. *Let H be a very ample divisor on $\mathcal{S}$. For every set $\delta_1, \ldots, \delta_s$ of positive rationals such that $0 < \sum_{i=1}^{s} \delta_i < 1$, the $\mathbb{Q}$-divisor $\bar{H} = H - \sum_{i=1}^{s} \delta_i E_i$ lies in the ample cone of $Pic_{\mathbb{Q}}(\tilde{\mathcal{S}})$.*

Proof. The hypotheses easily imply the positivity

$$\bar{H}^2 = H^2 - \sum_{i=1}^{s} \delta_i^2 > 0 \quad \text{and} \quad \bar{H}.E_i = \delta_i > 0.$$

Thus, by the Nakai-Moishezon criterion, to check the ampleness of $\bar{H}$ we have to verify that it has positive intersection with any other irreducible curve, i.e. strict transforms of irreducible curves on $\mathcal{S}$. Let $\mathcal{C}$ be an irreducible curve on $\mathcal{S}$ and $\tilde{\mathcal{C}}$ its strict transform, by the previous lemma,

$$\bar{H}.\tilde{\mathcal{C}} = H.\mathcal{C} - \sum_{i=1}^{s} \delta_i \mathrm{mult}_{P_i}(\mathcal{C}) \geq (1 - \sum_{i=1}^{s} \delta_i) H.\mathcal{C} > 0.$$

□

Example 1.3.5. Clearly, the bounds on δ for the ampleness of $\bar{H}$ provided in the last proposition are sufficient but not necessary. Indeed, consider the blowup of $\mathbb{P}^2$ at three points in general position. The divisor $2H - \frac{1}{2}(E_1 + E_2 + E_3)$, where H is the class of a line, is ample but does not satisfy the above condition. This follows by the existence of a smooth conic passing through three points and by Bézout's theorem.

Remark. Classically, the supremum of the δ_i's such that $H - \sum_{i=1}^{s} \delta_i E_i$ is nef is called s-point Seshadri constant of H and denoted by $\epsilon(H, P_1, \ldots, P_s)$. Except for simple situations Seshadri constants are very difficult to calculate. See chapter 5 in [19] for further details.

As it is well known, for any morphism of schemes the pullback preserves local freeness. In the sequel we will study what can be said about the pushforward by σ. As the blowup is an isomorphism outside the exceptional divisors, we are interested in the behavior of the pushforward of a sheaf at the points $P_i \in \mathcal{S}$. This is a local study so, in order to simplify the notation, in the following we will omit the index distinguishing the points $P_1, \ldots, P_s$ and the respective exceptional divisors.

Let $\mathcal{G}$ a coherent sheaf on $\tilde{\mathcal{S}}$. By the theorem on formal functions (III.11 in [14]),

$$(R^i\sigma_*\mathcal{G})\hat{}_P \simeq \varprojlim H^i(E_{(n)}, \mathcal{G}_n),$$

where for all $n > 0$, $E_{(n)} = \tilde{\mathcal{S}} \times_{\mathcal{S}} \mathrm{Spec}(\mathcal{O}_{\mathcal{S}}/\mathfrak{m}_P^n)$ are the infinitesimal thickenings of the exceptional divisor and $\mathcal{G}_n$ the corresponding restrictions of $\mathcal{G}$.

The structure sheaves of two subsequent infinitesimal thickenings of E are related by the exact sequence

$$0 \to \mathcal{O}_E(n) \to \mathcal{O}_{E_{(n+1)}} \to \mathcal{O}_{E_{(n)}} \to 0, \quad \text{for all } n > 0. \tag{1.6}$$

Indeed, the term on the left is nothing else but the n-th power of the conormal bundle $\mathcal{N}^{\vee}_{E/\tilde{\mathcal{S}}}$, i.e. I^n/I^{n+1}, where I is the ideal sheaf of the exceptional divisor.

Let $\mathcal{F}$ be a locally free sheaf of rank r on $\tilde{\mathcal{S}}$. Tensoring the above sequence by $\mathcal{F}$, we obtain

$$0 \to \mathcal{F}|_E(n) \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to 0, \quad \text{for all } n > 0. \tag{1.7}$$

Recall that $\mathcal{F}_1 \simeq \mathcal{F}|_E$. Thus, the long exact sequences associated to (1.7) provide an inductive way to compute the cohomology groups of $\mathcal{F}_n$. By Grothendieck theorem, $\mathcal{F}|_E$ splits. So, there exist integers a_j such that it is isomorphic to $\bigoplus_{j=1}^r \mathcal{O}_E(a_j)$. The cohomology groups of $\mathcal{F}|_E(n)$ are

$$H^0(E, \mathcal{F}|_E(n)) \simeq \bigoplus_{j=1}^{r} k[x_0, x_1]_{n+a_j},$$

$$H^1(E, \mathcal{F}|_E(n)) \simeq \bigoplus_{j=1}^{r} k[x_0, x_1]_{-n-a_j-2},$$

and the second cohomology group is always vanishing. Note that, from the construction of the blowup, $E \simeq \mathrm{Proj} \bigoplus S^n(\mathfrak{m}_P/\mathfrak{m}_P^2)$. So the above x_0, x_1 are local parameters at P generating $\mathfrak{m}_P$.

A difficulty in applying the induction method comes from the obstruction to lift global sections of $\mathcal{F}_n$ to $\mathcal{F}_{(n+1)}$. That is why in the following we will assume the condition:

(†) For every $n > 0$ the obstruction map

$$\mathfrak{o}_n : H^0(E_{(n)}, \mathcal{F}_n) \to H^1(E, \mathcal{F}|_E(n))$$

in the long exact sequence associated to (1.7) vanishes.

Under this assumption, the inverse systems we want to study are

$$H^0(E_{(n)}, \mathcal{F}_n) \simeq \bigoplus_{j=1}^{r} (\bigoplus_{m=0}^{n-1} k[x_0, x_1]_{m+a_j}),$$

$$H^1(E_{(n)}, \mathcal{F}_n) \simeq \bigoplus_{j=1}^{r} (\bigoplus_{m=0}^{n-1} k[x_0, x_1]_{-m-a_j-2}),$$

Note that the above inverse limits commute with the direct sums. For, the surjectivity of the descending morphisms implies that the Mittag-Leffler condition is satisfied (proposition II.9.1 in [14]).

Focus on the inverse system defined by $(\bigoplus_{m=0}^{n-1} k[x_0, x_1]_{m+a_j})_n$. If $a_j < 0$, the inverse system begins with some zeroes as first groups and then grows up as $(k[x_0, x_1]/(x_0, x_1)^n)_n$. Thus, the limit is the same as for $a_j = 0$, that is $k[[x_0, x_1]]$. On the other hand, if $a_j > 0$ the groups in the inverse system are polynomials of degrees $\geq a_j$. The limit is then $(x_0, x_1)^{a_j} \otimes k[[x_0, x_1]]$. Thus,

$$(\sigma_* \mathcal{F})\hat{}_P \simeq (\bigoplus_{a_j \leq 0} k \oplus \bigoplus_{a_j > 0} (x_0, x_1)^{a_j}) \otimes k[[x_0, x_1]]. \tag{1.8}$$

In a similar way, studying the inverse system defined by $H^1(E_{(n)}, \mathcal{F}_n)$, we get

$$(R^1 \sigma_* \mathcal{F})\hat{}_P \simeq \bigoplus_{a_j < -2} k[x_0, x_1]/(x_0, x_1)^{-a_j-2}.$$

Recall that these results apply only if condition (†) is verified. In general this is not easy to check. But still we can prove it for some particular cases.

Lemma 1.3.6. *Line bundles on $\tilde{\mathcal{S}}$ always satisfy condition* (†).

Proof. Let $\mathcal{L}$ be a line bundle on $\tilde{\mathcal{S}}$. To verify condition (†) is a local property in the points P_i's. So, as above, we refer to a general one and omit the i index. If the obstruction space $H^1(E, \mathcal{L}|_E(n))$ vanishes, so does the obstruction map. By Serre Theorem we know this happens for big n. Suppose now, that there exists $n > 0$ such that the obstruction space is non-vanishing. Then, by Bott formula, the space of global section of $\mathcal{L}|_E(n)$ is trivial. Moreover, this holds for any twist of $\mathcal{L}|_E$ by an m smaller than n. Thus, $H^0(E_{(n)}, \mathcal{L}_n)$ is vanishing, and so does the obstruction map $\mathfrak{o}_n$. □

Therefore, the discussion above applies to line bundles. Patching together the local structures at the points P_i's we get the following result.

Proposition 1.3.7. *Let $\mathcal{L}$ be a line bundle on $\tilde{\mathcal{S}}$. Suppose $\mathcal{L}$ is defined by the divisor $D - \sum_{i=1}^{s} q_i E_i$, where $D \in \mathrm{Pic}(\mathcal{S})$ and $q_i \in \mathbb{Z}$. Then*

$$\sigma_*\mathcal{L} \simeq \mathcal{O}_{\mathcal{S}}(D) \otimes \bigotimes_{q_i>0} \mathcal{I}^{q_i}_{\mathcal{S},P_i}, \text{ and } R^1\sigma_*\mathcal{L} \simeq \bigoplus_{q_i<-2} \mathcal{O}_{\mathcal{S}}/\mathcal{I}^{(-q_i-2)}_{\mathcal{S},P_i}.$$

Remark. The last proposition agrees with the second Riemann extension Theorem. Roughly speaking, it states that a holomorphic function defined on a connected open set (in the classical topology) minus a compact subset of codimension bigger than 2 extends over this compact set in a unique way ([13]). Consider a neighborhood $\mathcal{U}$ of a P_i. A section of $\sigma_*\mathcal{L}$ on $\mathcal{U}$ of P_i is a holomorphic function outside P_i, hence extends to a section of $\mathcal{O}_{\mathcal{S}}(D)(\mathcal{U})$. Therefore we get an injective map

$$0 \to \sigma_*\mathcal{L} \to \mathcal{O}_{\mathcal{S}}(D),$$

whose cokernel is supported in P.

In the following we consider the case of higher rank. Let $\mathcal{F}$ be a rank r vector bundle on $\tilde{\mathcal{S}}$. For every $i = 1, \dots, s$, denote by $(a_{i,1}, \dots, a_{i,r})$ the splitting type of $\mathcal{F}$ on E_i.

Proposition 1.3.8. *If all the coefficient $a_{i,j}$ are in $\{-2, -1, 0\}$, then $\sigma_*\mathcal{F}$ is locally free of rank r. Moreover, if $a_{j,i} \in \{-1, 0\}$ then $\mathcal{F}$ and $\sigma_*\mathcal{F}$ have the same cohomology groups.*

Proof. To prove the local freeness of $\sigma_*\mathcal{F}$, we have to check that its fibers at the P_i's have dimension r as $k(P_i)$-vector spaces. As before, we reduce to a local study at a point P that is one of the P_i's. As the a_j's are bigger or equal to -2, for every $n > 0$ the group $H^1(E, \mathcal{F}|_E(n))$ vanishes and condition (†)

is satisfied. Hence, using the description of the formal completion of $\sigma_*\mathcal{F}$ at P in (1.8),

$$\sigma_*\mathcal{F}(P) \simeq (\sigma_*\mathcal{F})\hat{}_P/\mathfrak{m}_P(\sigma_*\mathcal{F})\hat{}_P \simeq \bigoplus_{a_j\leq 0} k \oplus \bigoplus_{a_j>0} k[x_0,x_1]_{a_j},$$

is a k vector space of dimension bigger than r if $a_j > 0$ for some j.

For the second part of the proposition, the vanishing of $H^1(E, \mathcal{F}|_E)$ implies the vanishing of $H^1(E_{(n)}, \mathcal{F}_n)$, and so that of $R^1\sigma_*\mathcal{F}$. □

Corollary 1.3.9. *If $\mathcal{F}$ is trivial on the exceptional divisors, then $\mathcal{F} \simeq \sigma^*\sigma_*\mathcal{F}$.*

Proof. By the adjoint property of pullback and pushforward, there exists a canonical map

$$\sigma^*\sigma_*\mathcal{F} \to \mathcal{F}.$$

It is an isomorphism outside the exceptional divisors, so the cokernel is supported on the E_i's. On the other hand, the restrictions of both bundles above to an exceptional divisor are trivial. Hence, the map is an isomorphism on the whole surface $\tilde{\mathcal{S}}$. □

Corollary 1.3.10. *If $\mathcal{F}|_{E_i}$ is trivial, then there exists an open subset $\mathcal{V} \subset \tilde{\mathcal{S}}$ such that $E_i \subset \mathcal{V}$ and $\mathcal{F}|_{\mathcal{V}}$ is trivial.*

Proof. Let $P \in \mathcal{U}$ be a trivializing neighborhood for $\sigma_*\mathcal{F}$ and $\mathcal{V}$ its preimage. Then,

$$\mathcal{F}|_{\mathcal{V}} \simeq \sigma^*\sigma_*(\mathcal{F}|_{\mathcal{V}}) \simeq \sigma^*(\sigma_*\mathcal{F})|_{\mathcal{U}} \simeq \sigma^*\mathcal{O}_{\mathcal{S}}|_{\mathcal{U}}^{\oplus r}.$$

□

Corollary 1.3.11. *Let $\mathcal{F}$ be as above, then*

$$H^i(\tilde{\mathcal{S}}, \mathcal{E}nd(\mathcal{F})) \simeq H^i(\mathcal{S}, \mathcal{E}nd(\sigma_*\mathcal{F})).$$

Proof. Consider the vector bundle $\mathcal{E}nd(\sigma_*\mathcal{F})$ on $\mathcal{S}$. By the adjoint property of pullback and pushforward,

$$\mathcal{E}nd(\sigma_*\mathcal{F}) \simeq \sigma_*\mathcal{H}om_{\mathcal{O}_{\tilde{\mathcal{S}}}}(\sigma^*\sigma_*\mathcal{F}, \mathcal{F}) \simeq \sigma_*\mathcal{E}nd(\mathcal{F}).$$

As a consequence of corollary (1.3.10), $\mathcal{E}nd(\mathcal{F})$ has trivial restrictions on the exceptional divisors. So, by proposition (1.3.8),

$$R^i\sigma_*\mathcal{E}nd(\mathcal{F}) = 0, \text{ for } i > 0.$$

The isomorphisms on the cohomology follow. □

The above results hold for bundles with very specific restrictions on the exceptional divisor. Indeed, as we have seen for line bundles, the pushforward of a general vector bundle $\mathcal{F}$ could acquire singularities in the point P_i's. That is why, in these cases, we prefer to study its double dual, denoted by

$$\mathcal{F}_{\mathcal{S}} = (\sigma_* \mathcal{F})^{\check{}\check{}}.$$

Obviously, it coincides with $\sigma_*\mathcal{F}$ when $\mathcal{F}$ satisfies the hypotheses of proposition (1.3.8).

As the singular locus of the pushforward is contained in the set of points P_i's, passing to the double dual does not affect the first Chern class. So,

$$c_1(\mathcal{F}_{\mathcal{S}}) = c_1(\sigma_*\mathcal{F}) = \sigma_* c_1(\mathcal{F}) \in \mathrm{Pic}(\mathcal{S}).$$

We denote the above by $c_{\mathcal{S}}(\mathcal{F})$ and by $e_i(\mathcal{F})$ the degree of the determinant of $\mathcal{F}|_{E_i}$. In this way the first Chern class of $\mathcal{F}$ can be written as

$$c_1(\mathcal{F}) = c_{\mathcal{S}}(\mathcal{F}) - \sum_{i=1}^{s} e_i(\mathcal{F}) E_i.$$

1.4 $\hat{\mu}$-semistability on blown-up surfaces

Suppose now H to be a very ample polarization on $\mathcal{S}$. Lemma (1.3.4) provides a way to get an ample divisor $\bar{H}$ in $\tilde{\mathcal{S}}$. Thus, we have two notions of $\hat{\mu}$-semistability, one on $\mathcal{S}$, one on the blown-up surface. In the following we study how the two notions are related via pullback and pushforward of sheaves.

Proposition 1.4.1. *Let $\mathcal{F}$ be a rank r vector bundle on $\tilde{\mathcal{S}}$ trivial on the exceptional divisor. If $\mathcal{F}$ is $\hat{\mu}$-semistable with respect to $\bar{H}$, then $\sigma_*\mathcal{F}$ is $\hat{\mu}$-semistable with respect to H on $\mathcal{S}$.*

Proof. Suppose we have a saturated sequence on $\mathcal{S}$

$$0 \to \mathcal{G} \to \sigma_*\mathcal{F} \to \mathcal{Q} \to 0,$$

where $\mathcal{G}$ is a rank r' vector bundle. The pull back is a right exact functor, so it defines an exact sequence,

$$\sigma^*\mathcal{G} \to \mathcal{F} \to \sigma^*\mathcal{Q} \to 0.$$

The map on the left is an inclusion, for $\sigma^*\mathcal{G}$ is a vector bundle on $\tilde{\mathcal{S}}$ so it has not subsheaves of rank 0 as $L^1\sigma^*\mathcal{Q}$. Computing the degree, we get

$$rc_1(\mathcal{G}).H = rc_1(\sigma^*\mathcal{G}).\bar{H} \leq r'c_1(\mathcal{F}).\bar{H} = r'c_1(\sigma_*\mathcal{F}).H\,.$$

Thus, the above saturated sequence does not destabilize $\sigma_*\mathcal{F}$. □

Suppose $\sigma_*\mathcal{F}$ is $\hat{\mu}$-semistable with respect to H, and let $\mathcal{G}$ be a subsheaf of $\mathcal{F}$. Though the pushforward defines an inclusion $\sigma_*\mathcal{G} \hookrightarrow \sigma_*\mathcal{F}$, the corresponding inequalities on the slopes do not involve the degrees of $\mathcal{G}$ on the exceptional divisors. One can see that these may contribute positive additions to the slope of $\mathcal{G}$, so we cannot deduce the $\hat{\mu}$-semistability of $\mathcal{F}$. Indeed, as shown in the following example, the inverse direction of the above proposition fails.

Example 1.4.2. Suppose $\mathcal{S}$ is the projective plane, P a point on it, and H the class of a line. Consider $\sigma : \tilde{\mathcal{S}} \to \mathcal{S}$ to be the blowup of $\mathcal{S}$ in P. By lemma (1.3.4), every choice of a rational $\delta \in (0,1)$ defines an ample divisor $\bar{H} = H - \delta E$ in $\tilde{\mathcal{S}}$. Let x be a point on E, then there exists a locally free extension

$$0 \to \mathcal{O}_{\tilde{\mathcal{S}}}(E) \to \mathcal{F} \to \mathcal{I}_{\tilde{\mathcal{S}},x}(-E) \to 0.$$

Using Serre method, one has to check the non-vanishing of the extension group. By Serre duality, it is isomorphic to $H^1(\tilde{\mathcal{S}}, \mathcal{I}_{\tilde{\mathcal{S}},x}(-3H-E))$. Proposition (1.3.7) shows that the higher direct images of the line bundle $\mathcal{O}_{\tilde{\mathcal{S}}}(-3H-E)$ vanish. So,

$$H^i(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}(-3H-E)) \simeq H^i(\mathcal{S}, \mathcal{I}_{\tilde{\mathcal{S}},x}(-3H)).$$

The cohomology groups of the last one can be computed by using Bott formula. In particular, it has no global sections and the first cohomology group is one dimensional. Hence, $\mathrm{Ext}^1(\mathcal{I}_{\tilde{\mathcal{S}},x}(-E), \mathcal{O}_{\tilde{\mathcal{S}}}(E)) \simeq \mathbb{C}^2$ and the above extension can be choosen in order to define a vector bundle $\mathcal{F}$ of rank 2. As it is non-trivial, restricting it to E we get $\mathcal{F}|_E \simeq \mathcal{O}_E^{\oplus 2}$.

Passing to the pushforward, the above exact sequence descends to

$$0 \to \mathcal{O}_{\mathcal{S}} \to \sigma_*\mathcal{F} \to \mathcal{Q} \to 0,$$

where the cokernel $\mathcal{Q}$ is locally isomorphic to the structure sheaf outside P. As the torsion of $\mathcal{Q}$ is supported in codimension 2 and the kernel is a line bundle, this sequence is already saturated. Thus, $\mathcal{Q}$ is a torsion free sheaf on $\mathbb{P}^2$ with $c_1 = 0$ and $c_2(\mathcal{Q}) = c_2(\sigma_*\mathcal{F}) = 2$. That is, it is the ideal of a curvilinear subscheme of lenght 2 supported at P.

Suppose $\mathcal{O}_{\mathcal{S}}(-d)$ is a different subsheaf of $\sigma_*\mathcal{F}$, then $H^0(\mathcal{S}, \mathcal{Q}(d))$ does not vanish. Thus, d is strictly positive. As $c_1(\sigma_*\mathcal{F}) = 0$, this means that $\sigma_*\mathcal{F}$ is strictly $\hat{\mu}$-semistable. On the contrary,

$$\bar{H}.(c_1(\mathcal{F}) - 2E) = -2\delta < 0.$$

So, $\mathcal{F}$ is not $\hat{\mu}$-semistable.

Despite the impression given by the above example, it is in the case of rank 2 bundle that we can say more about the comparison between $\hat{\mu}$-(semi)stability on $\mathcal{S}$ and on $\tilde{\mathcal{S}}$. Tracing Brusse's results ([4]), we prove a sort of inverse of proposition (1.4.1). This does not come for free. Indeed, it requires a strong assumption on the polarization $\bar{H}$ which depends on the particular application we have in mind.

From now till the end of the section, $\mathcal{F}$ is a rank 2 vector bundle on $\tilde{\mathcal{S}}$ with first Chern class $c_1(\mathcal{F}) = c_{\mathcal{S}} - \sum_{i=1}^{s} e_i E_i$. Furthermore, assume the δ_i's defining $\bar{H}$ satisfy

$$\sum_{i=1}^{s} \delta_i < \frac{H^2}{1 + |\Delta(\mathcal{F})| H^2},$$

where $\Delta(\mathcal{F})$ denotes the discriminant of $\mathcal{F}$ as defined in (1.5). We have the following technical lemma.

Lemma 1.4.3. *If there exists an exact sequence*

$$0 \to \mathcal{O}_{\tilde{\mathcal{S}}}(D + \sum_{i=1}^{s} a_i E_i) \to \mathcal{F} \to \mathcal{I}_{\tilde{\mathcal{S}},Z}(c_{\mathcal{S}} - D - \sum_{i=1}^{s}(e_i + a_i)E_i) \to 0, \quad (1.9)$$

where $H.(c_{\mathcal{S}} - 2D) \neq 0$, *then* $|\sum_{i=1}^{s} \delta_i(2a_i + e_i)| < |H.(c_{\mathcal{S}} - 2D)|$.

Proof. From the exact sequence we can compute the second Chern class of $\mathcal{F}$ and its discriminant:

$$\begin{aligned}\Delta &= 4(\ell(Z) + D.(c_{\mathcal{S}} - D) + \sum_{i=1}^{s} a_i(e_i + a_i)) - (c_{\mathcal{S}}^2 + \sum_{i=1}^{s} e_i^2) \\ &= 4\ell(Z) - (c_{\mathcal{S}} - 2D)^2 + \sum_{i=1}^{s}(e_i + 2a_i)^2.\end{aligned}$$

Thus, from the Hodge index formula and Schwartz inequalities,

$$\begin{aligned}(H.(c_{\mathcal{S}} - 2D))^2 &\geq H^2(c_{\mathcal{S}} - 2D)^2 \\ &\geq H^2\left(\sum_{i=1}^{s}(e_i + 2a_i)^2 - \Delta\right) \\ &\geq H^2\left(\frac{1}{\sum_{i=1}^{s}\delta_i^2}\left(\sum_{i=1}^{s}\delta_i(e_i + 2a_i)\right)^2 - \Delta\right).\end{aligned}$$

Rearranging the last inequality, we get

$$\left|\sum_{i=1}^{s}\delta_i(e_i + 2a_i)\right| \leq \sqrt{\sum_{i=1}^{s}\delta_i^2\left[\frac{1}{H^2}(H.(c_{\mathcal{S}} - 2D))^2 + |\Delta|\right]}.$$

The fact that $|H.(c_{\mathcal{S}} - 2D)| \geq 1$ and the assumption on $\sum_{i=1}^{s}\delta_i^2$ yield the desired inequality. □

This lemma allows us to prove some nice results.

Proposition 1.4.4. *If $\mathcal{F}_\mathcal{S}$ is $\hat{\mu}$-stable, then $\mathcal{F}$ is $\hat{\mu}$-stable.*

Proof. Suppose we have a saturated sequence for $\mathcal{F}$ as in (1.9). The push-forward defines an injection $\mathcal{O}_\mathcal{S}(D) \hookrightarrow \mathcal{F}_\mathcal{S}$. The $\hat{\mu}$-stability of $\mathcal{F}_\mathcal{S}$ implies $H.(c_\mathcal{S} - 2D) > 0$. Thus, applying the lemma, we get

$$\sum_{i=1}^{s} \delta_i(2a_i + e_i) < H.(c_\mathcal{S} - 2D).$$

Therefore, $\mathcal{F}$ is $\hat{\mu}$-stable. □

Proposition 1.4.5. *If $\mathcal{F}$ is $\hat{\mu}$-semistable, then $\mathcal{F}_\mathcal{S}$ is $\hat{\mu}$-semistable.*

Proof. Suppose $\mathcal{F}_\mathcal{S}$ is not $\hat{\mu}$-semistable. Then there exists an exact sequence

$$0 \to \mathcal{O}_\mathcal{S}(D) \to \mathcal{F}_\mathcal{S} \to \mathcal{I}_{\mathcal{S},Z}(c_\mathcal{S} - D) \to 0,$$

such that $H.(c_\mathcal{S} - 2D) < 0$. In particular, outside the exceptional divisors, the above defines an injection of $\mathcal{O}_{\tilde{\mathcal{S}}}(D)$ into $\mathcal{F}$. Twisting by a suitable combination of the E_i saturates this injection. Thus, there exists some integers a_i such that

$$\mathcal{O}_{\tilde{\mathcal{S}}}(D + \sum_{i=1}^{s} a_i E_i) \hookrightarrow \mathcal{F}.$$

Again, applying lemma (1.4.3),

$$H.(c_\mathcal{S} - 2D) < - \left| \sum_{i=1}^{s} \delta_i(2a_i + e_i) \right|,$$

and so the above inclusion destabilize $\mathcal{F}$. □

1.5 On the blowup of $\mathbb{P}^2$ in a point

The intention of this section is to summarize some results we will need later on $\tilde{\mathbb{P}}^2$, the blowup of $\mathbb{P}^2$ in a point. In particular we are interested in an existence theorem for rank 2 vector bundles on this surface. To fix the notation, the two generators of $\mathrm{Pic}(\tilde{\mathbb{P}}^2)$ are H, which denotes the pullback of a line in $\mathbb{P}^2$, and E, the exceptional divisor.

First of all, note that $\tilde{\mathbb{P}}^2$ is isomorphic to the rational ruled surface over $\mathbb{P}^1$ defined by $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. The corresponding surjective morphism

$$\pi : \tilde{\mathbb{P}}^2 \twoheadrightarrow \mathbb{P}^1,$$

is the projection on E. Denote by f a fiber and by s the exceptional section of π, then

$$\mathcal{O}_{\tilde{\mathbb{P}}^2}(f) \simeq \pi^*(\mathcal{O}_{\mathbb{P}^1}(1)) \quad \text{and} \quad \pi_*\mathcal{O}_{\tilde{\mathbb{P}}^2}(s) \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Clearly we have $f \sim H - E$ and $s \sim E$. They generate $\mathrm{Pic}(\tilde{\mathbb{P}}^2)$ and

$$s^2 = -1 \quad f^2 = 0 \quad f.s = 1.$$

Lemma 1.5.1. *Let $D = \alpha s + \beta f$ be a divisor on $\tilde{\mathbb{P}}^2$. If $\alpha \geq 0$ then,*

1. *$\pi_* \mathcal{O}_{\tilde{\mathbb{P}}^2}(D)$ is a locally free sheaf of rank $\alpha + 1$.*
2. *$R^i \pi_* \mathcal{O}_{\tilde{\mathbb{P}}^2}(D) = 0$ for $i > 0$ and $H^i(\tilde{\mathbb{P}}^2, \mathcal{O}_{\tilde{\mathbb{P}}^2}(D)) \simeq H^i(\mathbb{P}^1, \pi_* \mathcal{O}_{\tilde{\mathbb{P}}^2}(D))$.*

Otherwise, $\pi_* \mathcal{O}_{\tilde{\mathbb{P}}^2}(D) = 0$.

Proof. Let D be as above. The restriction of $\mathcal{O}_{\tilde{\mathbb{P}}^2}(D)$ to a generic fiber of π is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(\alpha)$. The cohomology groups have constant dimensions along the fibers. Thus the results follow by Grauert base change theorem. □

For further details see [14], section V.2.

Proposition 1.5.2. *Let $D = pH + qE$ be a divisor on $\tilde{\mathbb{P}}^2$, then $h^i(\mathcal{O}_{\tilde{\mathbb{P}}^2}(D))$ are*

	$p+q \geq 0$	$p+q=-1$	$p+q \leq -2$
h^0	$\binom{p+2}{2} - \binom{-q+1}{2}$	0	0
h^1	$\binom{q}{2} - \binom{-p-1}{2}$	0	$\binom{-q+1}{2} - \binom{p+2}{2}$
h^2	0	0	$\binom{-p-1}{2} - \binom{q}{2}$

Remark. The binomial coefficients in the table are defined to be 0 when the upper argument is less than the lower one.

Proof. Rewrite D as $(p+q)s + pf$. By lemma (1.5.1), if $p+q \geq 0$ the direct image of $\mathcal{O}_{\tilde{\mathbb{P}}^2}(D)$ is a vector bundle on $\mathbb{P}^1$ of rank $p+q+1$, namely:

$$\begin{aligned} \pi_* \mathcal{O}_{\tilde{\mathbb{P}}^2}(D) &= \pi_* \mathcal{O}_{\tilde{\mathbb{P}}^2}((p+q)s) \otimes \mathcal{O}_{\mathbb{P}^1}(p) \\ &\simeq \mathrm{S}^{p+q}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(p) \\ &\simeq \mathcal{O}_{\mathbb{P}^1}(p) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-q). \end{aligned}$$

Thus we can compute the cohomology groups of $\mathcal{O}_{\tilde{\mathbb{P}}^2}(D)$ by the Bott formula. For example, the space of global sections is

$$H^0(\mathcal{O}_{\tilde{\mathbb{P}}^2}(D)) = \bigoplus_{j=-q}^{p} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(j)) = \sum_{max\{0,-q\}}^{p} j+1.$$

If $p+q=-1$, then the restriction of $\mathcal{O}_{\tilde{\mathbb{P}}^2}(D)$ to a generic fiber has vanishing cohomology groups. Thus $\pi_* \mathcal{O}_{\tilde{\mathbb{P}}^2}(D) = 0$ and also $\mathcal{O}_{\tilde{\mathbb{P}}^2}(D)$ has vanishing cohomology groups.

If $p+q \leq -2$, using Serre duality,

$$H^i(\mathcal{O}_{\tilde{\mathbb{P}}^2}(pH+qE)) = H^{2-i}(\mathcal{O}_{\tilde{\mathbb{P}}^2}((-p-3)H + (-q+1)E)) \;.$$

As $(-p-3)+(-q+1) \geq 0$ we can compute these groups by the previous case. □

As it is well known in literature (see [1]), a rank 2-bundle $\mathcal{F}$ on $\tilde{\mathbb{P}}^2$ can be classified by a canonical extension defined by two numerical invariants. First restrict $\mathcal{F}$ to a generic fiber, $\mathcal{F}|_f \simeq \mathcal{O}_f(d) \oplus \mathcal{O}_f(d')$ with $d \geq d'$. If $d > d'$ then $\pi_*\mathcal{F}(-d)$ is a line bundle, and we denote by r its degree. Otherwise $\pi_*\mathcal{F}(-d) \simeq \mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1}(s)$ is a 2-bundle and r denotes the maximal degree of its factors. As shown in [5], $\mathcal{F}$ can be expressed as an extension of the form:

$$0 \to \mathcal{O}_{\tilde{\mathbb{P}}^2}(rH + (d-r)E) \to \mathcal{F} \to \mathcal{I}_{\tilde{\mathbb{P}}^2,Z}(r'H + (d'-r')E) \to 0 \ , \qquad (1.10)$$

where Z is a zero-dimensional subscheme.

Clearly d', r' and the length of Z depend only on the Chern classes and the invariants d and r of $\mathcal{F}$:

$$d' = H.c_1 - E.c_1 - d \ , \qquad r' = H.c_1 - r \ ,$$
$$l(Z) = c_2 - E.c_1(d-r) - H.c_1 r - d^2 + 2dr.$$

So, $l(Z) \geq 0$ is a necessary condition for the existence of a 2-bundle with fixed invariants c_1, c_2, d, r, but it is not sufficient. Indeed, in [1] we find the following result:

Theorem 1.5.3. *There exists a rank 2-bundle with invariants c_1, c_2, d, r if and only if $l(Z) \geq 0$ and one of the following is satisfied:*
i) $2d > H.c_1 - E.c_1$
ii) $2d = H.c_1 - E.c_1, \quad H.c_1 - 2r \leq l$

Chapter 2

Sheaves on reducible surfaces

Let $\mathcal{S}_0$ and $\mathcal{S}_1$ be two smooth projective surfaces meeting transversely along a non singular curve $\mathcal{C}$. Denote by $\bar{\mathcal{S}}$ the surface union of $\mathcal{S}_0$ and $\mathcal{S}_1$. The condition on the intersection of $\mathcal{S}_0$ and $\mathcal{S}_1$ translates in a precise local picture. A neighborhood (in the classical topology) of a point on $\mathcal{C}$ is isomorphic to the union of two coordinate planes in $\mathbb{A}^3$, e.g. $\mathrm{Spec}(\mathbb{C}[x, y, z]/xy)$.

Certainly, sheaves supported on smooth surfaces are easier to describe than those on reducible ones. That is why in what follows, we want to relate sheaves on $\bar{\mathcal{S}}$ to their restrictions to components. This turns out to be very natural in the case of vector bundles. Indeed, vector bundles on $\bar{\mathcal{S}}$ are uniquely determined by their restrictions to components and a gluing isomorphism. In section (2.2) we generalize this result to equidimensional sheaves. The correspondence we obtain is similar to the one involving parabolic triple presented in [17]. In the third section we focus on the particular case of reflexive sheaves, and a precise description of their local structure on the intersection curve is provided. The last section of this chapter is devoted to the study of both $\hat{\mu}$ and Gieseker (semi)stability on $\bar{\mathcal{S}}$. In particular, we are interested in understanding how these properties are related to properties of the restrictions to components.

The careful reader may notice that most of the results presented in this chapter are proved by a local study at the intersection. Thus, these properties still hold with a weakened smoothness hypothesis for the components, requiring smoothness only on a neighborhood of the intersection curve.

2.1 Vector bundles

Consider the normalization of $\bar{\mathcal{S}}$ along $\mathcal{C}$,

$$\nu : \mathcal{S}_0 \sqcup \mathcal{S}_1 \to \bar{\mathcal{S}} \ .$$

The pullback ν^* associates a vector bundle $\mathcal{F}$ on $\bar{\mathcal{S}}$ to the pair given by its restrictions. Moreover, the adjoint property of pullback and pushforward provides a natural injection. The corresponding exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}|_{\mathcal{S}_0} \oplus \mathcal{F}|_{\mathcal{S}_1} \to \mathcal{F}|_{\mathcal{C}} \to 0, \tag{2.1}$$

describes the gluing along $\mathcal{C}$ of the restrictions of $\mathcal{F}$ on components. Note that both $\mathcal{F}$ and the middle term have a surjection to $\mathcal{F}|_{\mathcal{S}_i}$. So, by the snake lemma, we get two other exact sequences relating a vector bundle to its restrictions,

$$0 \to \mathcal{F}|_{\mathcal{S}_j}(-\mathcal{C}) \to \mathcal{F} \to \mathcal{F}|_{\mathcal{S}_i} \to 0, \tag{2.2}$$

for $j \neq i \in \{0, 1\}$. Thus, $\mathcal{F}$ is defined by a pair of vector bundles (one on each component) and an isomorphism of their restrictions on $\mathcal{C}$. In particular, ν^* identifies the Picard group of $\bar{\mathcal{S}}$ with the kernel of the map

$$\begin{array}{ccc} \mathrm{Pic}(\mathcal{S}_0) \oplus \mathrm{Pic}(\mathcal{S}_1) & \to & \mathrm{Pic}(\mathcal{C}) \\ (\mathcal{L}_0, \mathcal{L}_1) & \to & \mathcal{L}_0|_{\mathcal{C}} \otimes (\mathcal{L}_1|_{\mathcal{C}})^{\vee} \end{array}$$

Therefore, we can associate to a line bundle a pair of divisors having intersection with $\mathcal{C}$ of the same degree. Obviously, the inverse holds if and only if $\mathcal{C}$ is a rational curve.

Remark. Unlike the smooth case, on a reducible surface Weil divisors and Cartier divisors do not coincide. Indeed, in (2.2) we used the fact that $\mathcal{C}$ is a Cartier divisor on both $\mathcal{S}_i$. On the other hand, on $\bar{\mathcal{S}}$, $\mathcal{C}$ is locally defined by two equations. Thus $\mathcal{C}$ on $\bar{\mathcal{S}}$ is a Weil divisor which is not Cartier.

Lemma 2.1.1. *A line bundle on $\bar{\mathcal{S}}$ is ample if and only if so are both its restrictions.*

Proof. To prove the statement we recall the cohomological criterion for a line bundle to be ample (Chapter III, 5.3 [14]). Suppose $\mathcal{L}$ to be an ample divisor. If $\mathcal{G}_i$ is a coherent sheaf on a component $\mathcal{S}_i$, then for $k = 1, 2$ and n big enough, $H^k(\bar{\mathcal{S}}, \mathcal{G}_i \otimes \mathcal{L}^{\otimes n}) \simeq H^k(\mathcal{S}_i, \mathcal{G}_i \otimes \mathcal{L}|_{\mathcal{S}_i}^{\otimes n}) = 0$. Hence, $\mathcal{L}|_{\mathcal{S}_i}$ is ample.

Suppose now both restrictions to be ample. Let $\mathcal{G}$ a coherent sheaf on $\bar{\mathcal{S}}$. Considering the restriction to a component $\mathcal{S}_i$,

$$0 \to \mathcal{K}_j \to \mathcal{G} \to \mathcal{G}|_{\mathcal{S}_i} \to 0,$$

where the kernel $\mathcal{K}_j$ is supported on $\mathcal{S}_j$ only, for $j \neq i$. Clearly, twisting by $\mathcal{L}$ for a sufficiently big number of times, the higher cohomology groups of both kernel and cokernel vanish. Thus, for $k = 1, 2$ and n big enough, $H^k(\bar{\mathcal{S}}, \mathcal{G} \otimes \mathcal{L}^{\otimes n}) = 0$. □

In the following we will describe a polarization on $\bar{\mathcal{S}}$ by means of the ample divisors defined on the components. In particular, we denote by δ the degree of their intersection with $\mathcal{C}$.

2.2 Sheaves of pure dimension 2

In the spirit of broadening the above results for vector bundles, in this section we would like to relate pure dimension 2 sheaves on $\bar{\mathcal{S}}$ to torsion free sheaves on components. Among all the differences, a first thing one should note about non-locally free sheaves is that their restrictions do not need to coincide on $\mathcal{C}$. Not even the ranks on the components have to be the same. That is why, dealing with non-localy free sheaf, by *rank* we mean the couple of integers defined by the ranks of the restrictions on the $\mathcal{S}_i$'s.

Let $\mathcal{G}$ be a pure dimension 2 sheaf on $\bar{\mathcal{S}}$. Consider the exact sequence (2.1) describing the gluing of the structure sheaves $\mathcal{O}_{\mathcal{S}_i}$. Tensoring it by $\mathcal{G}$, there appears a map from $\mathcal{T}or^1_{\bar{\mathcal{S}}}(\mathcal{G}, \mathcal{O}_{\mathcal{C}})$ on the left. As this sheaf is supported on $\mathcal{C}$ and $\mathcal{G}$ is equidimensional, the map vanishes. Thus, also in this case, there exists an exact sequence relating sheaves of pure dimension 2 to their restrictions. The difference with the locally free case is that in general the property of being of equidimension is not preserved under restriction. Indeed $\mathcal{G}|_{\mathcal{S}_i}$ could present subsheaves supported on $\mathcal{C}$ which are not subsheaves of $\mathcal{G}$. Quotienting out the torsion of the restrictions, we get the following diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & T & = & T & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{G}|_{\mathcal{S}_0}\oplus\mathcal{G}|_{\mathcal{S}_1} & \xrightarrow{\phi} & \mathcal{G}|_{\mathcal{C}} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \overline{\mathcal{G}|_{\mathcal{S}_0}}\oplus\overline{\mathcal{G}|_{\mathcal{S}_1}} & \xrightarrow{\psi} & \mathcal{H}_{\mathcal{G}} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{2.3}$$

where $T \simeq T(\mathcal{G}|_{\mathcal{S}_0}) \oplus T(\mathcal{G}|_{\mathcal{S}_1})$ and $\mathcal{H}_{\mathcal{G}}$ are sheaves with support in $\mathcal{C}$. Note that the central column is the direct sum of two exact sequences. Each of them has a morphism to the right column having as central map the restriction $\phi_i : \mathcal{G}|_{\mathcal{S}_i} \twoheadrightarrow \mathcal{G}|_{\mathcal{C}}$. Denote by ψ_i the corresponding map between the cokernels.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(\mathcal{G}|_{\mathcal{S}_i}) & \longrightarrow & \mathcal{G}|_{\mathcal{S}_i} & \longrightarrow & \overline{\mathcal{G}|_{\mathcal{S}_i}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \psi_i} & & \\ 0 & \longrightarrow & T & \longrightarrow & \mathcal{G}|_{\mathcal{C}} & \longrightarrow & \mathcal{H}_{\mathcal{G}} & \longrightarrow & 0 \end{array}$$

By applying the snake lemma one sees that these maps are surjective. Morever, by construction,

$$\psi = \psi_0 - \psi_1.$$

Thus, a sheaf $\mathcal{G}$ of pure dimension 2 on $\bar{\mathcal{S}}$ defines a sheaf $\mathcal{H}_{\mathcal{G}}$ supported on $\mathcal{C}$ and a pair of surjective maps $\psi_i : \overline{\mathcal{G}|_{\mathcal{S}_i}} \twoheadrightarrow \mathcal{H}_{\mathcal{G}}$ that end in it.

Example 2.2.1. A good picture of what is said above comes from the study of a pure dimension 2 sheaf $\mathcal{G}$ having restrictions of rank 1 on both component. Let $\eta(\mathcal{C})$ be the generic point of $\mathcal{C}$. The stalk $\mathcal{O}_{\bar{\mathcal{S}},\eta(\mathcal{C})}$ is a local ring of dimension 1 and residue field $k(\mathcal{C}) = \mathcal{O}_{\mathcal{C},\eta(\mathcal{C})}$. By the local structure of $\bar{\mathcal{S}}$ at the intersection, it is isomorphic to the local ring at the singular point of two crossing curves over $k(\mathcal{C})$. In particular, this is an A_1 singularity. The classification by Greuel and Knörrer [18] of $\mathcal{O}_{\bar{\mathcal{S}},\eta(\mathcal{C})}$-modules of rank 1 describes two possible situation.

1. $\mathcal{G}_{\eta(\mathcal{C})} \simeq \mathcal{O}_{\bar{\mathcal{S}},\eta(\mathcal{C})}$. Thus $\mathcal{G}|_{\mathcal{C}}$ has a rank 1. That is, its singular locus is in codimension 2. Outside this finite set $\mathcal{G}$, and the quotient map ψ, behave as in the case of invertible sheaves.

2. $\mathcal{G}_{\eta(\mathcal{C})} \simeq \tilde{\mathcal{O}}_{\bar{\mathcal{S}},\eta(\mathcal{C})}$, i.e. the integral closure of $\mathcal{O}_{\mathcal{S},\eta(\mathcal{C})}$ in its field of fractions. In this case $\mathcal{G}$ has rank 2 on $\mathcal{C}$. So, $\mathcal{H}$ is a skyscraper sheaf on finite points in $\mathcal{C}$. Outside these points $\mathcal{G}$ can be described as the direct sum of two torsion free sheaves (one on each component).

From the diagram above, it is clear that the maps (ψ_0, ψ_1) encode how the two restrictions of $\mathcal{G}$ glue together. As shown by the example, already the rank of $\mathcal{H}_{\mathcal{G}}$ on $\mathcal{C}$ provides a lot of information on the geometry of $\mathcal{G}$. Indeed, if it is 0, there is almost no gluing. Outside a finite set of points, $\mathcal{G}$ is a direct sum. If it is equal to the ranks of both restrictions, then outside a finite set of points $\mathcal{G}$ behaves like a vector bundle. As we will see in section (2.4), sheaves of this type gain importance when dealing with semistability.

Definition 4. A purely 2-dimensional sheaf is said to have *constant rank r in codimension* 1 if the restriction to $\mathcal{C}$ has the same rank as both restrictions to components.

The above construction, associating to $\mathcal{G}$ the pair of surjections (ψ_0, ψ_1) from the torsion free part of the restrictions to $\mathcal{H}_{\mathcal{G}}$, can be reversed. Proving this is not straightforward and need some technical results on $\mathcal{T}or$ sheaves.

Lemma 2.2.2. *Let $\mathcal{G}$ be a sheaf of pure dimension 2 on $\bar{\mathcal{S}}$, then the following holds:*

(a) $\mathcal{T}or^k_{\bar{\mathcal{S}}}(\mathcal{O}_{\mathcal{C}}, \mathcal{G}) \simeq \bigoplus_{i\in\{0,1\}} \mathcal{T}or^k_{\bar{\mathcal{S}}}(\mathcal{O}_{\mathcal{S}_i}, \mathcal{G})$ *for all* $k > 0$.

(b) $\mathcal{T}or^{k+1}_{\bar{\mathcal{S}}}(\mathcal{O}_{\mathcal{S}_i}, \mathcal{G}) \simeq \mathcal{T}or^k_{\bar{\mathcal{S}}}(\mathcal{O}_{\mathcal{S}_j}(-\mathcal{C}), \mathcal{G})$ *for all* $k > 0$ *and* $i \neq j \in \{0, 1\}$.

(c) If $\mathcal{G}$ is not supported on $\mathcal{S}_i$, then $\mathcal{T}or^1_{\bar{\mathcal{S}}}(\mathcal{O}_{\mathcal{S}_i}, \mathcal{G}) = 0$.

Proof. To prove (a) recall that the sequence (2.1) remains exact after tensorizing by $\mathcal{G}$. Morever, $\mathcal{T}or$ sheaves with a vector bundle in the argument vanish. So, the long exact sequence provides the isomorphisms

$$\mathcal{T}or^k_{\bar{\mathcal{S}}}\left(\bigoplus_{i\in\{0,1\}} \mathcal{O}_{\mathcal{S}_j}, \mathcal{G}\right) \simeq \mathcal{T}or^k_{\bar{\mathcal{S}}}(\mathcal{O}_{\mathcal{C}}, \mathcal{G}).$$

As $\mathcal{T}or$ sheaves commute with direct sums, the result follows.

Similarly, tensor sequence (2.2) by $\mathcal{G}$. In general we do not get a short exact sequence. Despite that, the long exact sequence yields the isomorphisms in (b).

Now suppose $\mathcal{G}$ to be supported only on $\mathcal{S}_j$. In this case, in accordance with the notation in the following proposition, we denote it by $\mathcal{G}_j$. The exact sequence of $\mathcal{O}_{\bar{\mathcal{S}}}$-modules

$$0 \to \mathcal{O}_{\mathcal{S}_j}(-\mathcal{C}) \to \mathcal{O}_{\mathcal{S}_j} \to \mathcal{O}_{\mathcal{C}} \to 0,$$

remains exact when tensored by $\mathcal{G}_j$. For, $\mathcal{G}_j(-\mathcal{C})$ is torsion free and so does not admit subsheaves supported on a curve. The long exact sequence results in

$$\begin{aligned}\ldots \xrightarrow{\alpha_2} \mathcal{T}or^2_{\bar{\mathcal{S}}}(\mathcal{G}_j, \mathcal{O}_{\mathcal{S}_j}) \to \mathcal{T}or^2_{\bar{\mathcal{S}}}(\mathcal{G}_j, \mathcal{O}_{\mathcal{C}}) \to \mathcal{T}or^2_{\bar{\mathcal{S}}}(\mathcal{G}_j, \mathcal{O}_{\mathcal{S}_i}) \xrightarrow{\alpha_1} \\ \to \mathcal{T}or^1_{\bar{\mathcal{S}}}(\mathcal{G}_j, \mathcal{O}_{\mathcal{S}_j}) \to \mathcal{T}or^1_{\bar{\mathcal{S}}}(\mathcal{G}_j, \mathcal{O}_{\mathcal{C}}) \to 0.\end{aligned}$$

where, by applying (b), we replaced the $\mathcal{T}or$ sheaves concerning $\mathcal{O}_{\mathcal{S}_j}(-\mathcal{C})$. Note that, by (a), all the maps α_i vanish. In particular $\alpha_1 = 0$, and the result follows by applying (a) again. □

Proposition 2.2.3. *Let $\mathcal{G}_0$ and $\mathcal{G}_1$ be torsion free sheaves supported respectively on $\mathcal{S}_0$ and $\mathcal{S}_1$. For any pair of surjective maps $\psi_i : \mathcal{G}_i \to \mathcal{H}$ to a common sheaf supported on $\mathcal{C}$, there exists a sheaf $\mathcal{G}$ of pure dimension 2 such that $\overline{\mathcal{G}|_{\mathcal{S}_i}} \simeq \mathcal{G}_i$ for both $i \in \{0,1\}$ and the gluing is defined by the maps ψ_i's.*

Proof. Define $\mathcal{G}$ to be the kernel of the map obtained by gluing the two surjections,

$$\mathcal{G} := ker\left(\mathcal{G}_0 \oplus \mathcal{G}_1 \xrightarrow{\psi_0 - \psi_1} \mathcal{H}\right).$$

Restricting this map to a component, for example $\mathcal{S}_0$, we get a long exact sequence involving $\mathcal{T}or$ sheaves. By lemma (2.2.2.a), the sheaf $\mathcal{T}or^1_{\bar{\mathcal{S}}}(\mathcal{G}_1, \mathcal{O}_{\mathcal{S}_0})$ vanishes. Thus, the long exact sequence results in

$$\ldots \to \mathcal{T}or^1_{\bar{\mathcal{S}}}(\mathcal{G}_0, \mathcal{O}_{\mathcal{S}_0}) \xrightarrow{\psi_0'} \mathcal{T}or^1_{\bar{\mathcal{S}}}(\mathcal{H}, \mathcal{O}_{\mathcal{S}_0}) \to \mathcal{G}|_{\mathcal{S}_0} \to \mathcal{G}_0 \oplus \mathcal{G}_1|_{\mathcal{C}} \to \mathcal{H} \to 0.$$

The map ψ_0' corresponds to the map in cohomology induced by ψ_0. In particular, it is surjective. For, ψ_0 defines an exact sequence with a torsion free kernel supported on $\mathcal{S}_0$. Hence, it remains exact when tensored by $\mathcal{O}_{\mathcal{S}_0}$.

Recall that, $\mathcal{G}|_{\mathcal{S}_0}$ is a subsheaf of $\mathcal{G}_0 \oplus \mathcal{G}_1|_{\mathcal{C}}$. Its torsion and torsion free part inject respectively in $\mathcal{G}_1|_{\mathcal{C}}$ and $\mathcal{G}_0$. We have the following situation:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(\mathcal{G}|_{\mathcal{S}_0}) & \longrightarrow & \mathcal{G}_1|_{\mathcal{C}} & \xrightarrow{-\psi_1} & \mathcal{H} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{G}|_{\mathcal{S}_0} & \longrightarrow & \mathcal{G}_0 \oplus \mathcal{G}_1|_{\mathcal{C}} & \xrightarrow{\psi_0-\psi_1} & \mathcal{H} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & \overline{\mathcal{G}|_{\mathcal{S}_0}} & \xrightarrow{\sim} & \mathcal{G}_0 & & & & \end{array} \tag{2.4}$$

Thus, $\mathcal{G}$ has restrictions with the desired torsion free parts. □

2.3 Reflexive sheaves

The aim of this section is to provide a better description for the restrictions of reflexive sheaves on $\bar{\mathcal{S}}$. This is possible thanks to the work of Burban and Drozd on Cohen-Macaulay modules on surface singularities.

Lemma 2.3.1 ([6], Theorem 5.3)**.** *Let $A = \mathbb{C}[[x, y, z]]/(xy)$. An indecomposable non-free reflexive A-module is defined by one of the following resolutions:*

1. *$\ldots \xrightarrow{\cdot x} A \xrightarrow{\cdot y} A \xrightarrow{\cdot x} A \to N \to 0$, or the one with permuted multiplication maps. Then N is isomorphic to $A_0 = \mathbb{C}[[x, y, z]]/(x)$, or to $A_1 = \mathbb{C}[[x, y, z]]/(y)$.*

2. *$\ldots \xrightarrow{\varphi_n} A^{\oplus 2} \xrightarrow{\psi_n} A^{\oplus 2} \xrightarrow{\varphi_n} A^{\oplus 2} \to M_n \to 0$, or the one with permuted maps, where*

$$\varphi_n = \begin{pmatrix} x & -z^n \\ 0 & y \end{pmatrix}, \quad \psi_n = \begin{pmatrix} y & z^n \\ 0 & x \end{pmatrix} \quad \textit{and } n > 0.$$

Then M_n is isomorphic to $Ae_1 + Ae_2/\langle xe_1, -z^n e_1 + ye_2\rangle$, or the same with permuted x, y.

Clearly, the first resolution locally describes line bundles supported only on one component. On the other hand, the A-module M_n defined by the second resolution presents a more interesting structure. Indeed, its restrictions are:

$$M_n \otimes A_0 \simeq A_0e_1 + A_0e_2/\langle -z^n e_1 + ye_2\rangle A_0 \simeq (y, z^n)A_0,$$
$$M_n \otimes A_1 \simeq A_1e_1 + A_1e_2/\langle xe_1, -z^n e_1\rangle A_1 \simeq A_1 \oplus T_n,$$

where T_n is a zero dimensional ring of length n isomorphic to $\mathbb{C}[z]/(z^n)$. Furthermore, T_n corresponds to the torsion of the restriction of M_n to the intersection. Hence, comparing the above with the diagram (2.3) and passing to formal completion, the bottom row turns into

$$0 \to M_n \to (y, z^n)A_0 \oplus A_1 \xrightarrow{\psi} \mathbb{C}[[z]].$$

By definition, ψ is surjective.

Proposition 2.3.2. *Let $\mathcal{G}$ be a reflexive sheaf on $\bar{\mathcal{S}}$, then the sheaf $\mathcal{H}_\mathcal{G}$ defined by*

$$0 \to \mathcal{G} \to \overline{\mathcal{G}|_{\mathcal{S}_0}} \oplus \overline{\mathcal{G}|_{\mathcal{S}_1}} \xrightarrow{\psi} \mathcal{H}_\mathcal{G} \to 0.$$

is a vector bundle on $\mathcal{C}$.

Proof. The local structure of $\mathcal{G}$ at a point x on the intersection is given by an A-module of the form

$$\mathcal{G}\hat{_x} \simeq A^{\oplus r} \oplus A_0^{\oplus r_0} \oplus A_1^{\oplus r_1} \oplus (\oplus_{k \in I} M_k).$$

Gluing the torsion free parts of the restrictions, we obtain:

$$0 \to G \to \overline{G \otimes A_0} \oplus \overline{G \otimes A_1} \xrightarrow{\psi} \mathbb{C}[[z]]^{\oplus r} \oplus \mathbb{C}[[z]]^{\oplus \# I} \to 0.$$

The free $\mathbb{C}[[z]]$-module on the right provides the local structure of $\mathcal{H}$. □

In particular, when the restrictions of $\mathcal{G}$ glue almost completely, then we can say more about $\mathcal{H}_\mathcal{G}$.

Proposition 2.3.3. *Let $\mathcal{G}$ be a reflexive sheaf on $\bar{\mathcal{S}}$ of constant rank. Denote by $\mathcal{E}_i$ the double dual of $\mathcal{G}|_{\mathcal{S}_i}$. Then for $i \neq j$, there exists an exact sequence,*

$$0 \to \overline{\mathcal{G}|_{\mathcal{S}_i}} \to \mathcal{E}_i \to T(\mathcal{G}|_{\mathcal{S}_j}) \to 0.$$

Moreover, $\mathcal{H}_\mathcal{G}$ is isomorphic to the kernel of $\mathcal{E}_i|_\mathcal{C} \to T(\mathcal{G}|_{\mathcal{S}_j})$.

Proof. The canonical homomorphism of $\mathcal{G}|_{\mathcal{S}_i}$ into its double dual yields the injection $\overline{\mathcal{G}|_{\mathcal{S}_i}} \hookrightarrow \mathcal{E}_i$. Denote by $\mathcal{Q}$ the cokernel of this map. We check stalkwise that it is isomorphic to the torsion part of $\mathcal{G}|_{\mathcal{S}_j}$.

Suppose x to be a point outside the intersection. Clearly,

$$(\mathcal{G}|_{\mathcal{S}_i})_x \simeq \mathcal{G}_x \simeq (\mathcal{G}^{\check{}\check{}})_x \simeq \mathcal{E}_x.$$

So $\mathcal{Q}$ has support in $\mathcal{C}$. Consider x to be a point on the intersection. As $\mathcal{G}$ has constant rank, its localization at x does not admit factors of type N. Therefore, $\mathcal{G}\hat{_x} \simeq A^{\oplus r} \oplus (\oplus_{k \in I} M_k)$, and its restriction to A^i is

$$(\mathcal{G}|_{\mathcal{S}_i})\hat{_x} \simeq A_i^{\oplus r} \oplus (\oplus_{k \in I_0} (y, z^{n_k}) A_i) \oplus (\oplus_{k \in I_1} A_i \oplus T_{n_k}),$$

where I_0, I_1 is an appropriate partition of I. The stalk of its double dual is then $\mathcal{E}_i\hat{_x} \simeq A_i^{\# I + r}$. Thus, the formal completion of $\mathcal{Q}$ at x corresponds to $\oplus_{k \in I_0} T_{n_k}$, that is the torsion of $(\mathcal{G}|_{\mathcal{S}_j})\hat{_x}$.

To prove the second statement, restrict the exact sequence to $\mathcal{C}$. As the torsion always splits off on curve, we get the isomorphism

$$(\overline{\mathcal{G}|_{\mathcal{S}_i}})|_{\mathcal{C}} \simeq ker(\mathcal{E}_i|_{\mathcal{C}} \to T(\mathcal{G}|_{\mathcal{S}_j})) \oplus T(\mathcal{G}|_{\mathcal{S}_j}).$$

The above has a surjection to the vector bundle $\mathcal{H}_{\mathcal{G}}$, which has the same rank. Thus, the surjection kills the torsion and provides an isomorphism on the torsion free part. □

Example 2.3.4. Let $\mathcal{G}$ be a reflexive sheaf of rank 1 on both components, there are two possible situations. If $\mathcal{G}$ has rank 2 on the intersection, then it is isomorphic to $\mathcal{L}_0 \oplus \mathcal{L}_1$ for some line bundles $\mathcal{L}_i$ on $\mathcal{S}_i$. If it has constant rank, then its restrictions appear as

$$\begin{aligned} \mathcal{G}|_{\mathcal{S}_0} &\simeq \mathcal{L}_0 \otimes \mathcal{I}_{\mathcal{S}_0,Z_0} \oplus \mathcal{O}_{Z_1}, \\ \mathcal{G}|_{\mathcal{S}_1} &\simeq \mathcal{L}_1 \otimes \mathcal{I}_{\mathcal{S}_1,Z_1} \oplus \mathcal{O}_{Z_0}, \end{aligned} \tag{2.5}$$

where Z_0 and Z_1 are 0-dimensional subschemes in $\mathcal{C}$ without intersection. In particular, the sheaf $\mathcal{H}_{\mathcal{G}}$ in lemma (2.3.2) is the line bundle on $\mathcal{C}$ defined by $\mathcal{L}_0|_{\mathcal{C}}(-Z_0) \simeq \mathcal{L}_1|_{\mathcal{C}}(-Z_1)$.

2.4 Gieseker and $\hat{\mu}$-semistability

Once we have fixed a polarization (H_o, H_1) on $\bar{\mathcal{S}}$, we can define the notion of $\hat{\mu}$-stability on it. As we will see, some natural properties of $\hat{\mu}$-(semi)stable bundles do not hold on reducible surfaces.

The Hilbert polynomial (and so the slope) of a vector bundle $\mathcal{F}$ on $\bar{\mathcal{S}}$ can be computed from the Chern classes of its restrictions. Indeed, by the exact sequences (2.1) and (2.2),

$$\begin{aligned} P_{\mathcal{F}}(t) &= P_{\mathcal{F}|_{\mathcal{S}_0}}(t) + P_{\mathcal{F}|_{\mathcal{S}_1}(-\mathcal{C})}(t) \\ &= P_{\mathcal{F}|_{\mathcal{S}_0}}(t) + P_{\mathcal{F}|_{\mathcal{S}_1}}(t) - P_{\mathcal{F}|_{\mathcal{C}}}(t) \end{aligned}$$

Let r be the rank of $\mathcal{F}$, and denote by δ the degree of the intersection, i.e. $\delta = H_i.\mathcal{C}$. Then the Hilbert polynomial of $\mathcal{F}|_{\mathcal{C}}$ is $r\delta t + c_1(\mathcal{F}|_{\mathcal{C}}) + r\chi(\mathcal{O}_{\mathcal{C}})$. Therefore, for the slope of $\mathcal{F}$ we get,

$$(H_0^2 + H_1^2)\hat{\mu}_{\mathcal{F}} = H_0^2 \hat{\mu}_{\mathcal{F}|_{\mathcal{S}_0}} + H_1^2 \hat{\mu}_{\mathcal{F}|_{\mathcal{S}_1}} - 2\delta. \tag{2.6}$$

First thing to note is that every bundle on $\bar{\mathcal{S}}$ (more generally, every sheaf supported on the whole surface) has two natural quotients defined by the restrictions. Thus, a necessary condition for a bundle to be $\hat{\mu}$-(semi)stable is that the restrictions do not destabilize it. Hence, from the exact sequences (2.2), we get two equivalent chains of inequalities

$$\hat{\mu}_{\mathcal{F}|_{\mathcal{S}_i}(-\mathcal{C})} \underset{(-)}{\leq} \hat{\mu}_{\mathcal{F}} \underset{(-)}{\leq} \hat{\mu}_{\mathcal{F}|_{\mathcal{S}_i}}, \tag{2.7}$$

for $i = 0, 1$. Computing the slope of the above bundles on $\mathcal{S}_i$, we see that these confine the slope of $\mathcal{F}$ to stay in an interval of length $2\frac{\delta}{H_i^2}$. So, by (2.6), we can rewrite (2.7) as

$$-2\frac{\delta}{H_1^2}\underset{(-)}{\leq}\hat{\mu}_{\mathcal{F}|_{\mathcal{S}_0}} - \hat{\mu}_{\mathcal{F}|_{\mathcal{S}_1}}\underset{(-)}{\leq}2\frac{\delta}{H_0^2}. \tag{2.8}$$

This highlights that a $\hat{\mu}$-(semi)stable bundle has restrictions with close slopes, where the definition of being close depends only on the polarization.

Remark. On integral surfaces line bundles are $\hat{\mu}$-stable. In general, this is no longer true on $\bar{\mathcal{S}}$. For example, suppose that there exists an effective divisor D on $\mathcal{S}_0$ not intersecting $\mathcal{C}$. Then, for every integer m, $\mathcal{O}_{\mathcal{S}_0}(mD)$ and $\mathcal{O}_{\mathcal{S}_1}$ glue together and form a line bundle $\mathcal{L}_m$ on $\bar{\mathcal{S}}$. The degree of $\mathcal{L}_m|_{\mathcal{S}_0}$ is $mD.H_0$, so it increases linearly in m. On the other hand, the degree of the other restriction is 0. For m sufficiently big, inequalities (2.8) are not satisfied and so $\mathcal{L}_m$ is not $\hat{\mu}$-stable.

As a direct consequence, the property of $\hat{\mu}$-(semi)stability on reducible surfaces is not preserved under twisting by line bundles.

Theorem 2.4.1. *Let $\mathcal{F}$ be a vector bundle of rank r on $\bar{\mathcal{S}}$. If its restrictions to components are $\hat{\mu}$-semistable and satisfy inequalities (2.7), then $\mathcal{F}$ is $\hat{\mu}$-semistable.*

Proof. Let $\mathcal{F} \to \mathcal{Q}$ be a quotient of pure dimension 2. If $\mathcal{Q}$ is supported only on one component, then the quotient map factors through the corresponding restriction of $\mathcal{F}$. Thus, $\hat{\mu}_{\mathcal{F}} \leq \hat{\mu}_{\mathcal{F}|_{\mathcal{S}_i}} \leq \hat{\mu}_{\mathcal{Q}}$. Suppose now $\mathcal{Q}$ to be supported on both components. Let (r_0, r_1) be its rank, then $0 < r_i \leq r$. Assume that $r_0 \geq r_1$.

Consider the exact sequence defined by the restriction to $\mathcal{S}_0$,

$$0 \to \mathcal{K}_1 \to \mathcal{Q} \to \mathcal{Q}|_{\mathcal{S}_0} \to 0. \tag{2.9}$$

Remark that $\mathcal{Q}|_{\mathcal{S}_0}$ could have torsion, but this does not affect the proof. The kernel $\mathcal{K}_1$ is a torsion free sheaf supported on $\mathcal{S}_1$. In particular it corresponds to the torsion free part of $\mathcal{Q}|_{\mathcal{S}_1}(-\mathcal{C})$. Indeed, the above is the truncation of the long exact sequence obtained by tensoring (2.2) by $\mathcal{Q}$. In particular,

$$\begin{array}{ccc} \mathcal{F}|_{\mathcal{S}_1}(-\mathcal{C}) & \twoheadrightarrow & \mathcal{Q}|_{\mathcal{S}_1}(-\mathcal{C}) \\ & \searrow & \downarrow \\ & & \mathcal{K}_1 \end{array}$$

Thus, the $\hat{\mu}$-semistability of the restrictions provides the inequalities, $\hat{\mu}_{\mathcal{F}|_{\mathcal{S}_0}} \leq \hat{\mu}_{\mathcal{Q}|_{\mathcal{S}_0}}$ and $\hat{\mu}_{\mathcal{F}|_{\mathcal{S}_1}(-\mathcal{C})} \leq \hat{\mu}_{\mathcal{K}_1}$. Computing the slope of $\mathcal{Q}$ by the above sequence, we obtain

$$\begin{aligned}(r_0H_0^2 + r_1H_1^2)\hat{\mu}_{\mathcal{Q}} &= r_0H_0^2\hat{\mu}_{\mathcal{Q}|_{\mathcal{S}_0}} + r_1H_1^2\hat{\mu}_{\mathcal{K}_1} \\ &\geq (r_0 - r_1)H_0^2\hat{\mu}_{\mathcal{F}|_{\mathcal{S}_0}} + r_1(H_0^2\hat{\mu}_{\mathcal{F}|_{\mathcal{S}_0}} + H_1^2\hat{\mu}_{\mathcal{F}|_{\mathcal{S}_1}(-\mathcal{C})}) \\ &\geq (r_0H_0^2 + r_1H_1^2)\hat{\mu}_{\mathcal{F}}. \end{aligned} \tag{2.10}$$

Thus, $\mathcal{Q}$ does not destabilize $\mathcal{F}$. □

Corollary 2.4.2. *Suppose $\mathcal{F}$ to have $\hat{\mu}$-semistable restrictions on components, which do not destabilize it (i.e. inequalities (2.7) hold strictly). If $\mathcal{Q}$ is a proper quotient having the same slope as $\mathcal{F}$, then it has constant rank and its restrictions to $\mathcal{S}_i$ have the same slope as $\mathcal{F}|_{\mathcal{S}_i}$.*

Proof. The hypotheses of Theorem (2.4.1) are satisfied, so the chain of inequalities (2.10) in the proof of the theorem provides two equalities. The first one implies that $\hat{\mu}_{\mathcal{F}|_{\mathcal{S}_0}} = \hat{\mu}_{\mathcal{Q}|_{\mathcal{S}_0}}$. In particular, $\mathcal{Q}|_{\mathcal{S}_0}$ cannot have slope bigger than its torsion free part. Thus, $T(\mathcal{Q}|_{\mathcal{S}_0})$ is supported in codimension 2 and so $\mathcal{Q}|_{\mathcal{C}}$ has rank r_0. The second equality (2.10) implies that the rank of the two restrictions of $\mathcal{Q}$ is the same. Therefore $\mathcal{Q}$ has constant rank. Moreover, as $r_0 = r_1$, we can switch the role of the indices in the proof of the theorem. So, $\hat{\mu}_{\mathcal{F}|_{\mathcal{S}_1}} = \hat{\mu}_{\mathcal{Q}|_{\mathcal{S}_1}}$. □

As an immediate consequence we have the following.

Corollary 2.4.3. *Suppose $\mathcal{F}$ to have $\hat{\mu}$-semistable restrictions on components, which do not destabilize it (i.e. inequalities (2.7) hold strictly). If one restriction is $\hat{\mu}$-stable then so is $\mathcal{F}$.*

If we trace the discussion above replacing the slope by the reduced Hilbert polynomial, we notice several similarities. Indeed, by definition, the reduced Hilbert polynomial and the slope of a vector bundle come with the same weights when compared to the restrictions on components.

$$(H_0^2 + H_1^2)p_{\mathcal{F}}(t) = H_0^2 p_{\mathcal{F}|_{\mathcal{S}_0}}(t) + H_1^2 p_{\mathcal{F}|_{\mathcal{S}_1}(-\mathcal{C})}(t).$$

We have a similar equation for equidimensional sheaves, but we should include the ranks on the components into the weights. For example, we get from the exact sequence (2.9):

$$(r_0H_0^2 + r_1H_1^2)p_{\mathcal{Q}}(t) = r_0H_0^2 p_{\mathcal{Q}|_{\mathcal{S}_0}}(t) + r_1H_1^2 p_{\mathcal{K}_1}(t).$$

Mechanically replacing the solpe by the reduce Hilbert polynomial in the proofs of theorem (2.4.1) and its corollaries, we obtain a valid proof of the next proposition.

Proposition 2.4.4. *Let* $\mathcal{F}$ *be a vector bundle of rank* r *on* $\bar{\mathcal{S}}$*. Suppose its restrictions* $\mathcal{F}|_{\mathcal{S}_0}$ *are* $\mathcal{F}|_{\mathcal{S}_1}$ *are semistable and not destabilizing* $\mathcal{F}$*, i.e.*

$$p_{\mathcal{F}|_{\mathcal{S}_i}(-E)}(t) \leq p_{\mathcal{F}}(t) \leq p_{\mathcal{F}|_{\mathcal{S}_i}}(t). \tag{2.11}$$

Then $\mathcal{F}$ *is semistable.*

Furthermore, if the inequalities above are strict and one restriction is stable, then $\mathcal{F}$ *is stable.*

Chapter 3

Trees of surfaces

This chapter is devoted to the study of vector bundles on trees of surfaces. These are deformations of a smooth surface $\mathcal{S}$ to a reducible one where the structure components-intersections corresponds to the structure vertices-edges of a tree graph. In particular, their local structure at an intersection is the same as that of reducible surfaces described in the previous chapter. Thus, most of the results proved in the case of two smooth components apply to tree surfaces. It is worth mentioning that some of the results presented in this chapter are similar to analogous results for trees of curves, see for instance [28], [3].

After giving necessary definitions, we study the Picard group of a tree surface. Particular attention will be given to describing polarizations on tree surfaces and their relation to the ample cone of $\mathcal{S}$. In the third section we approach the notions of $\hat{\mu}$ and Gieseker (semi)stability introducing the definition of a H_{T}-compatibilization of a vector bundle. The feeling is that in studying these notions on tree surfaces, one should consider vector bundles up to a twist by particular invertible sheaves. From this perspective, the fourth section exposes the natural generalizations of the results obtained in section (2.4). Showing some examples, in section 5 we investigate the inverse direction of the above results, i.e. we study the properties descending from a (semi)stable vector bundle to its restrictions. Finally, in the last two sections we present the definition and the properties of admissible bundles, which are the main objects of study in the subsequent chapters.

3.1 Definitions and first properties

First of all, by trees we mean what is commonly called in graph theory a rooted tree. That is, a connected acyclic finite graph, with one fixed special vertex, the root. If not stated otherwise, the root of a tree T will be denoted by 0.

One can see that, by fixing the root, the edges of T acquire a natural orientation pointing to it. Or, equivalently stated, on a tree T there is a partial order on the vertices defined by: $\alpha \leq \beta$ if and only if α lies on the only path connecting β to the root.

From this point of view, we can consider the root as the minimal vertex in T. Due to its peculiarity, the root is often treated as a special case, and we denote by T^* the set of vertices different from 0. It is easy to see that any vertex $\alpha \in \mathsf{T}^*$ has a unique maximal vertex less than α. It is called the predecessor and denoted by α^-.

Let A be a set of vertices in T; we use the following notation. By A^* we denote the subset of vertices in A having predecessor in A. Consequently, we call roots of A all elements in $\mathrm{rt}(A) = A \setminus A^*$. By A^+ we denote the set of direct successors of A, i.e. all the vertices $\beta \in A^c$ having predecessor in A. The vertices in T without successor are called top vertices and form the set T_{top}.

Moreover, we denote by T_α the set given by α and all its successors. Note that T_α has a tree structure with root α. We refer to these subgraphs of T as *complete subtrees*. When considering substructure of a tree T, we are mainly interested in their vertices. So, by a subgraph of T we mean a subset of vertices provided by all the edges in T between them.

Lemma 3.1.1. *Any connected subgraph A of a tree T can be obtained from complete subtrees by unions and complements.*

Proof. As A is connected, it admits a minimal vertex α. Then,

$$A = \left(\mathsf{T}_\alpha^c \cup \bigcup_{\beta \in A^+} \mathsf{T}_\beta \right)^c .$$

□

Thus, complete trees can be considered as a base for the subgraphs of T.

Having defined what a tree is, we can approach the definition of a tree surface. We describe them as special deformations of smooth projective surfaces. These are produced by consecutive blowups of a trivial family, done in accordance with a tree graph. In the following we provide a recursive way to construct them.

Let T be a tree and $\mathcal{S}$ a smooth projective surface. The first step is to blow up the trivial family $\mathcal{S} \times \mathbb{A}^1$ in a set of points $Z_0 = \{x_i \,:\, i \in 0^+\}$ in the central fiber. The result is a 1-parameter family whose central fiber counts a set of exceptional divisors $\mathcal{P}_\alpha$ indexed by the vertices $\alpha \in 0^+$. In particular, every $\mathcal{P}_\alpha$ has a marked line E_α on it, corresponding to the intersection with the strict transform of the previous central fiber. For the recursive step, fix one of these exceptional divisors such that the corresponding α is not in T_{top}. Let Z_α be a set of points in $\mathcal{P}_\alpha$ indexed by α^+ and not lying on E_α. Hence, proceed with a further blowup of the family in Z_α. The outcome is a new family of surfaces. The set of exceptional divisors in its central fiber is indexed by all the previous vertices, apart from α, plus the vertices in α^+. At each step the vertices in T pointing to an exceptional divisor of the central fiber grow with respect to the order of the tree. So, in a finite number of steps these vertices are in T_{top} only, and the process stops.

Definition 5. A *tree surface* of type T, or simply a T-*surface*, with base $\mathcal{S}$ is the central fiber of a family obtained by the above construction.

By the definition, every tree surface is endowed with a morphism to the original projective surface,

$$\sigma : \mathcal{S}_T \to \mathcal{S}.$$

We will refer to this morphism as the *standard contraction*.

Lemma 3.1.2. *A T-surface $\mathcal{S}_\mathsf{T}$ is a connected, reducible surface. Its components $\mathcal{S}_\alpha$ are indexed by the vertices of T:*

- *The root component is obtained by a blowup $\mathcal{S}_0 \xrightarrow{\sigma_0} \mathcal{S}$ in a set of points indexed by 0^+. The corresponding exceptional divisors are denoted by E_α for all $\alpha \in 0^+$.*
- *For any $\alpha \in \mathsf{T}^*$ which is not a top vertex, $\mathcal{S}_\alpha \xrightarrow{\sigma_\alpha} \mathbb{P}^2$ is a blowup in $\#\alpha^+$ points. It comes equipped with a marked line E_α not intersecting the exceptional divisors E_β, for all $\beta \in \alpha^+$.*
- *For any top vertex, $\mathcal{S}_\alpha$ is isomorphic to $\mathbb{P}^2$ and has a marked line E_α.*

The intersections of components corresponds to the edges in T. Thus, for all $\alpha \in \mathsf{T}^*$, $E_\alpha = \mathcal{S}_{\alpha^-} \cap \mathcal{S}_\alpha$ is a normal crossing singularity and there are no triple intersections.

Note that the last assertion on the intersection implies that $\mathcal{S}_\mathsf{T}$ is locally equivalent to two coordinates planes in $\mathbb{A}^3$.

Let A be a subgraph of T. We define

$$\mathcal{S}_A = \bigcup_{\alpha \in A} \mathcal{S}_\alpha.$$

In this way, any subsurface of $\mathcal{S}_\mathsf{T}$ corresponds to a subgraph of T.

3.2 The Picard group

Tracing the argumentation of the previous chapter, we can relate a vector bundle on $\mathcal{S}_\mathsf{T}$ to its restrictions to components. Indeed, consider a vector bundle $\mathcal{F}$ on $\mathcal{S}_T$. Passing to the normalization, we get the exact sequence

$$0 \to \mathcal{F} \to \bigoplus_{\alpha \in \mathsf{T}} \mathcal{F}|_{\mathcal{S}_\alpha} \to \bigoplus_{\gamma \in \mathsf{T}^*} \mathcal{F}|_{E_\gamma} \to 0. \tag{3.1}$$

It is particularly interesting to observe how this applies to the Picard group of $\mathcal{S}_T$. Indeed, as the E_α are lines, line bundles on contiguous components glue together if and only if their degrees on the intersection coincide. On the other hand for any $\alpha \in \mathsf{T}^*$, the Picard group of $\mathcal{S}_\alpha$ is a lattice generated by the intersections with the bordering components. Thus, there is an isomorphism,

$$\mathrm{Pic}(\mathcal{S}) \oplus \mathbb{Z}^{\#\mathsf{T}^*} \xrightarrow{\sim} \mathrm{Pic}(\mathcal{S}_\mathsf{T}).$$

A divisor $D \in \mathrm{Pic}(\mathcal{S})$ and a list of integers $m_\mathsf{T} = \{m_\alpha\}_{\alpha \in \mathsf{T}^*}$ are associated to the line bundle $\mathcal{O}_{\mathcal{S}_\mathsf{T}}(D, m_\mathsf{T})$ defined by the restrictions,

$$\mathcal{O}_{\mathcal{S}_0}(D - \sum_{\alpha \in 0^+} m_\alpha E_\alpha), \text{ and } \mathcal{O}_{\mathcal{S}_\beta}(m_\beta E_\beta - \sum_{\alpha \in \beta^+} m_\alpha E_\alpha) \text{ for } \beta \in \mathsf{T}^*.$$

Note that, dealing with divisors coming from $\mathrm{Pic}(\mathcal{S})$, we omit the pullback sign. Moreover, for every sheaf $\mathcal{G}$ on $\mathcal{S}_\mathsf{T}$, we will write $\mathcal{G}(D, m_\mathsf{T})$ for the tensor product $\mathcal{G} \otimes \mathcal{O}_{\mathcal{S}_\mathsf{T}}(D, m_\mathsf{T})$.

As in the case of blown-up surfaces treated in the first chapter, the standard contraction σ defines two natural morphisms relating the Picard groups of $\mathcal{S}_\mathsf{T}$ and $\mathcal{S}$ via pullback and pushforward. So, it is natural to ask if, and how, polarizations on a tree surface are related to ample divisors on the original surface. Proceeding by induction on the components, lemma (2.1.1) adapts to tree surfaces.

Lemma 3.2.1. *A line bundle on $\mathcal{S}_\mathsf{T}$ is ample if and only if so are its restrictions to components.*

We follow the terminology in which a polarization of $\mathcal{S}_\mathsf{T}$ is a proportionality class of ample line bundles on $\mathcal{S}_\mathsf{T}$, or in other words, a ray of the form $\langle[\mathcal{L}]\rangle_{\mathbb{Q}_+} \subset \mathrm{Pic}_\mathbb{Q}(\mathcal{S}_\mathsf{T})$ for some ample $\mathcal{L}$. Each such ray has a unique representative of the form (H, δ_T), for which $H \in \mathrm{Pic}(\mathcal{S}_\mathsf{T})$ is a primitive integer class and δ_T a set of rationals. We will always normalize representatives of polarizations in this way.

A necessary condition for the ampleness of such a representative follows from lemma (3.2.1) and the Nakai–Moishezon criterion applied to the components $\mathcal{S}_\beta$:

$$H \text{ ample on } \mathcal{S}, \quad H^2 > \sum_{\alpha \in 0^+} \delta_\alpha^2, \quad \text{and } \forall \beta \in \mathsf{T}^*, \begin{cases} \delta_\beta > 0, \\ \delta_\beta^2 > \sum_{\alpha \in \beta^+} \delta_\alpha^2. \end{cases} \tag{3.2}$$

Suppose the projective surface $\mathcal{S}$ is endowed with a very ample divisor H. We would like to have a stock of ample line bundles on $\mathcal{S}_\mathsf{T}$ associated to H. For convenience, we admit rational coefficients, thus working with $\mathrm{Pic}_\mathbb{Q}(\mathcal{S}_\mathsf{T})$, and we say that the elements of the form (H, δ_T) are associated to $H \in \mathrm{Pic}_\mathbb{Q} S$. The conditions in (3.2) are not sufficient; an example of sufficient conditions is given by applying lemma (1.3.4) on each component.

Lemma 3.2.2. *Let $\mathcal{S}$ be endowed with a very ample divisor H and $\delta_{\mathsf{T}} = \{\delta_\alpha\}_{\alpha\in\mathsf{T}^*}$ a set of positive rationals satisfying the following:*

1. $\sum_{\alpha\in 0^+}\delta_\alpha < 1$
2. *For all the others $\beta \in \mathsf{T}\setminus\mathsf{T}_{top}$, $\sum_{\alpha\in\beta^+}\delta_\alpha < \delta_\beta$.*

Then the pair (H, δ_{T}) defines an ample element in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{S}_T)$.

From the perspective of working with families of tree-like surfaces, there might be important to have a stock of polarizations presenting particular symmetries on the coefficients δ_α. These are defined by ample line bundles on the universal family of tree surfaces of type T and base $\mathcal{S}$ invariant under monodromy. The formal definition in terms of subgraphs of T of the set of symmetries we should consider may seem cumbersome. Thus, we restrict ourselves to certain monodromy invariant polarizations that are easier to describe:

Definition 6. A polarization $H_{\mathsf{T}} = (H, \delta_{\mathsf{T}})$ in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{S}_T)$ is said to be *good* if there exists a finite set $\{\bar{\delta}_n\}$ of positive rationals such that, for any connected chain of $n+1$ vertices $0 < \alpha_1 < \cdots < \alpha_n$ in T we have $\delta_{\alpha_n} = \bar{\delta}_{n-1}$.

Remark. Let δ_T define a good polarization and β be a vertex in T^*. For all $\alpha \in \beta^+$, $\delta_\alpha = \bar{\delta}_n$, where n is the number of edges of the unique path connecting β to the root. In particular, suppose to have a connected chain of $n+1$ vertices $0 < \alpha_1 < \cdots < \alpha_n$ in T. Then,

$$\bar{\delta}^2_{n-1} < \frac{H^2}{(\#0^+)(\#\beta_1^+)\dots(\#\beta_n^+)}\,.$$

3.3 H_{T}-compatibility

Fix a polarization $H_{\mathsf{T}} = (H, \delta_{\mathsf{T}})$ on $\mathcal{S}_T$. In order to study semistability for vector bundles on a tree surface, we should understand how their restrictions to subsurfaces behave.

Let $\mathcal{F}$ be a vector bundle of rank r on $\mathcal{S}_{\mathsf{T}}$. The determinant $\Lambda^r\mathcal{F}$ is a line bundle on $\mathcal{S}_{\mathsf{T}}$. Thus, it is defined by a divisor on $\mathcal{S}$ and its degrees on the intersections. We use the following notation:

$$c_1(\mathcal{F}) := (c_{\mathcal{S}}(\mathcal{F}), \{e_\alpha(\mathcal{F})\}_{\alpha\in\mathsf{T}^*})\,,$$

where $e_\alpha(\mathcal{F}) = c_1(\mathcal{F}|_{\mathcal{S}_\alpha}).E_\alpha$ and $c_{\mathcal{S}}(\mathcal{F}) = \sigma_{0*}(c_1(\mathcal{F}|_{\mathcal{S}_0}))$. Moreover, we define the *total charge* of $\mathcal{F}$ to be the sum of the second Chern classes of the restrictions to components, i.e. $n(\mathcal{F}) = \sum_{\alpha\in\mathsf{T}} c_2(\mathcal{F}|_{\mathcal{S}_\alpha})$. We take the liberty to omit $\mathcal{F}$ in the notation whenever it is clear which vector bundle we are referring to.

The exact sequence (3.1) allows us to compute the Hilbert polynomial of $\mathcal{F}$ from its restrictions to components. These can be calculated using Hirzebruch-Riemann-Roch Theorem (1.3). On the root component,

$$P_{\mathcal{F}|_{\mathcal{S}_0}}(t) = \frac{r}{2}H^2t^2 + H.\left(c_{\mathcal{S}} - \frac{r}{2}K_{\mathcal{S}}\right)t + \frac{1}{2}(c_{\mathcal{S}}^2 - K_{\mathcal{S}}.c_{\mathcal{S}}) - n_0 + r\chi\mathcal{O}_{\mathcal{S}} - \sum_{\alpha\in 0^+}\left[\frac{r}{2}\delta_\alpha^2t^2 + \delta_\alpha\left(e_\alpha + \frac{r}{2}\right)t + \frac{1}{2}(e_\alpha^2 + e_\alpha)\right].$$

For $\alpha \in \mathsf{T}^* \setminus \mathsf{T}_{top}$,

$$P_{\mathcal{F}|_{\mathcal{S}_\alpha}}(t) = \frac{r}{2}\delta_\alpha^2t^2 + \delta_\alpha\left(e_\alpha + 3\frac{r}{2}\right)t + \frac{1}{2}(e_\alpha^2 + 3e_\alpha) - n_\alpha + r - \sum_{\beta\in\alpha^+}\left[\frac{r}{2}\delta_\beta^2t^2 + \delta_\beta\left(e_\beta + \frac{r}{2}\right)t + \frac{1}{2}(e_\beta^2 + e_\beta)\right].$$

For a top vertex $\beta \in \mathsf{T}_{top}$,

$$P_{\mathcal{F}|_{\mathcal{S}_\beta}}(t) = \frac{r}{2}\delta_\beta^2t^2 + \delta_\beta\left(e_\beta + 3\frac{r}{2}\right)t + \frac{1}{2}(e_\beta^2 + 3e_\beta) - n_\beta + r.$$

While, for $\alpha \in \mathsf{T}^*$, the Hilbert polynomial of a restriction to E_α is

$$P_{\mathcal{F}|_{E_\alpha}}(t) = r\delta_\alpha t + e_\alpha + r.$$

Summing up the above, we see that all the terms depending on the δ_α's cancel out. The Hilbert polynomial of $\mathcal{F}$ with respect to H has the form,

$$P_{\mathcal{F}}(t) = \frac{r}{2}H^2t^2 + H.\left(c_{\mathcal{S}} - \frac{r}{2}H.K_{\mathcal{S}}\right)t + \frac{1}{2}(c_{\mathcal{S}}^2 - K_{\mathcal{S}}.c_{\mathcal{S}}) - n + r\chi\mathcal{O}_{\mathcal{S}}\ .$$

Note that the behavior of $\mathcal{F}$ outside the root component affects just the constant term of its Hilbert polynomial. Indeed, $P_{\mathcal{F}}(t)$ coincides with the Hilbert polynomial of a rank-2 bundle on $\mathcal{S}$ with first Chern class $c_{\mathcal{S}}$ and charge n.

Once we have computed the Hilbert polynomial of a bundle, we can check its $\hat{\mu}$-(semi)stability by comparing its slope with that of its saturated proper equidimensional quotients. As seen in section (2.4) of the previous chapter, a necessary condition for a bundle to be $\hat{\mu}$-semistable is that none of its restrictions of pure dimension 2 destabilizes it.

Definition 7. A vector bundle on $\mathcal{S}_{\mathsf{T}}$ is said H_{T}-compatible if all its restrictions to subsurfaces of $\mathcal{S}_{\mathsf{T}}$ do not $\hat{\mu}$-destabilize it, i.e.

$$\forall A \subset \mathsf{T}, \quad \hat{\mu}_{\mathcal{F}|_{\mathcal{S}_A}} > \hat{\mu}_{\mathcal{F}}.$$

While in the case of two components this condition translates into the single chain of inequalities (2.7), now we should verify it for all subsurfaces $\mathcal{S}_A$ with $A \subset \mathsf{T}$. The following proposition remarkably reduces the number of cases to check.

Lemma 3.3.1. *A vector bundle is H_{T}-compatible if and only if it is not $\hat{\mu}$-destabilized by restrictions to the subsurfaces arising by complete subtrees or their complements.*

Proof. One direction is trivial. So, suppose $\mathcal{F}$ is a vector bundle on $\mathcal{S}_{\mathsf{T}}$ such that all restrictions defined by complete trees, or their complements, do not $\hat{\mu}$-destabilize it. Let A be a subgraph of T. We will show that $\mathcal{F}|_{\mathcal{S}_A}$ does not destabilize $\mathcal{F}$.

If A is not connected, then $\mathcal{S}_A$ is the union of two surfaces $\mathcal{S}_{A_1}$ and $\mathcal{S}_{A_2}$ without intersection. Thus,

$$\mathcal{F}|_{\mathcal{S}_A} = \mathcal{F}|_{\mathcal{S}_{A_1}} \oplus \mathcal{F}|_{\mathcal{S}_{A_2}}, \quad \text{and} \quad \hat{\mu}_{\mathcal{F}|_{\mathcal{S}_A}} \geq \min_i \{\hat{\mu}_{\mathcal{F}|_{\mathcal{S}_{A_i}}}\}.$$

Hence, we can assume A connected. Lemma (3.1.1) supplies a precise description of the complement of A as a union of disjoint subgraphs A_i. So, $\mathcal{K} = ker(\mathcal{F} \to \mathcal{F}|_{\mathcal{S}_A})$ is supported on the union of the corresponding subsurfaces $\mathcal{S}_{A_i}$. Therefore, $\mathcal{K} \simeq \oplus_i \mathcal{K}|_{\mathcal{S}_{A_i}}$, where each $\mathcal{K}|_{\mathcal{S}_{A_i}}$ is the kernel of the restriction of $\mathcal{F}$ to the subsurface defined by A_i^c. The A_i's are complete subtrees, or their complements. So, by hypothesis, these restrictions do not destabilize $\mathcal{F}$. Summarizing,

$$\hat{\mu}_{\mathcal{K}} \leq \max_i \{\hat{\mu}_{\mathcal{K}|_{\mathcal{S}_{A_i}}}\} \leq \hat{\mu}_{\mathcal{F}}.$$

So $\mathcal{F}|_{\mathcal{S}_A}$ does not destabilize $\mathcal{F}$. □

Remark. The construction of the exact sequence (2.2) can be generalized for any subgraph A of T. Indeed, passing to the partial normalization along the appropriate intersections,

$$0 \to \mathcal{F}|_{\mathcal{S}_A}(-\sum_{\alpha \in \mathrm{rt}(A)} E_\alpha - \sum_{\beta \in A^+} E_\beta) \to \mathcal{F} \to \mathcal{F}|_{\mathcal{S}_{A^c}} \to 0. \tag{3.3}$$

For any α inT^*, the restriction to $\mathcal{S}_{T_\alpha^c}$ destabilizes $\mathcal{F}$ if and only if $\hat{\mu}_{\mathcal{F}|_{\mathcal{S}_{T_\alpha}}(-E_\alpha)} > \hat{\mu}_{\mathcal{F}}$. Computing the contribution of the twist by $-E_\alpha$ on the slope of the restriction, we can reformulate lemma (3.3.1) in the following way.

Proposition 3.3.2. *A vector bundle $\mathcal{F}$ of rank r on $\mathcal{S}_{\mathsf{T}}$ is H_{T}-compatible if and only if, for all $\alpha \in \mathsf{T}^*$,*

$$\hat{\mu}_{\mathcal{F}|_{\mathcal{S}_{T_\alpha}}} - \hat{\mu}_{\mathcal{F}} \in (0, \frac{2}{\delta_\alpha}). \tag{3.4}$$

This interpretation makes it clear that, for a general choice of Chern classes, there are no H_{T}-compatible bundles. Indeed, inequalities (3.4) can be rewritten in terms of $c_1(\mathcal{F})$:

$$2e_\alpha + r < \frac{\delta_\alpha}{H^2} H.(2c_{\mathcal{S}} - rK_{\mathcal{S}}) < 2e_\alpha + 3r,$$

and the following holds:

Proposition 3.3.3. *Let $\mathcal{F}$ be a rank r vector bundle on $\mathcal{S}_{\mathsf{T}}$. For a general polarization H_{T}, there exists a unique set of integers $k_{\mathsf{T}} = \{k_\alpha\}_{\alpha \in \mathsf{T}^*}$ such that $\mathcal{F}(0, k_{\mathsf{T}})$ is H_{T}-compatible.*

Proof. Note that the above twist does not involve pullback of divisors on $\mathcal{S}$. Thus, $c_{\mathcal{S}}(\mathcal{F}(0, k_{\mathsf{T}})) = c_{\mathcal{S}}(\mathcal{F})$, and so $\hat{\mu}_{\mathcal{F}(0,k_{\mathsf{T}})} = \hat{\mu}_{\mathcal{F}}$. On the other hand, the degrees on the intersection do change: $e_\alpha(\mathcal{F}(0, k_{\mathsf{T}})) = e_\alpha(\mathcal{F}) + k_\alpha r$ for all α in T^*. Accordigly, the slope of the restriction to a subsurface $\mathcal{S}_{\mathsf{T}_\alpha}$ is,

$$\hat{\mu}_{\mathcal{F}(0,k_{\mathsf{T}})|_{\mathcal{S}_{\mathsf{T}_\alpha}}} = \hat{\mu}_{\mathcal{F}|_{\mathcal{S}_{\mathsf{T}_\alpha}}} + \frac{2}{\delta_\alpha} k_\alpha.$$

Thus $\mathcal{F}(0, k_{\mathsf{T}})$ satisfies (3.4) if and only if

$$\frac{\delta_\alpha}{2}(\hat{\mu}_{\mathcal{F}|_{\mathcal{S}_{T_\alpha}}} - \hat{\mu}_{\mathcal{F}}) + k_\alpha \in (0, 1).$$

If the term on the left is not an integer, then the unique k_α satisfying the above is

$$k_\alpha = \lceil \frac{1}{2r} \left(\frac{\delta_\alpha}{H^2} H.(2c_{\mathcal{S}} - rK_{\mathcal{S}}) - 2e_\alpha - 3r \right) \rceil.$$

Note that the argument in the ceiling function is linear in δ_α. As $0 < \delta_\alpha < 1$, there is a finite set of bad values for δ_α providing integer solutions for this linear function. Thus, for a general choice of δ_α's the claim holds. ☐

Definition 8. The vector bundle $\mathcal{F}(0, k_{\mathsf{T}})$ is called the H_{T}*-compatibilization* of $\mathcal{F}$.

Remark. Above we described bad values of δ_α's for any choice of Chern classes $(c_{\mathcal{S}}, \{e_\alpha\}_{\alpha \in \mathsf{T}^*})$ and rank r. These define a finite set of walls in the slice of ample divisors associated to H on $\mathcal{S}_{\mathsf{T}}$. For a polarization on a wall, the compatibilization of a bundle is not defined and as the polarization moves across a wall, the twist making the bundle H_{T}-compatible changes.

Example 3.3.4. Let $\mathcal{S} \simeq \mathbb{P}^2$ and consider the tree surface $\mathcal{S}_{\mathsf{T}}$ obtained from $\mathcal{S}$ and the tree with just two vertices $T = \{0, 1\}$. Hence, $\mathcal{S}_0$ is the blowup of $\mathbb{P}^2$ in a point, and $\mathcal{S}_1$ is $\mathbb{P}^2$ itself. The two components intersect along a line E_1 which is the exceptional divisor of $\mathcal{S}_0$. A polarization on $\mathcal{S}_{\mathsf{T}}$ is represented by a pair $H_{\mathsf{T}} = (H, \delta_1)$, where H is the class of a line and $0 < \delta_1 < 1$.

Let $\mathcal{F}$ be a rank r vector bundle on $\mathcal{S}_\mathsf{T}$ with trivial first Chern class, i.e. $c_\mathcal{S} = 0$, and $e_1 = 0$. Its slope and the slopes of its restrictions are:

$$\hat{\mu}_F = 3, \quad \hat{\mu}_{\mathcal{F}|_{\mathcal{S}_0}} = \frac{1}{1-\delta_1^2}(3-\delta_1), \quad \hat{\mu}_{\mathcal{F}|_{\mathcal{S}_1}} = \frac{3}{\delta_1}.$$

In this case, there exists a unique wall and it does not depend on r. Indeed, for $\frac{1}{3} < \delta_1 < 1$, $\mathcal{F}$ is already H_T-compatible. For, $\delta_1 = \frac{1}{3}$ the restriction to $\mathcal{S}_0$ has the same slope as $\mathcal{F}$, whilst, for $0 < \delta_1 < \frac{1}{3}$, the H_T-compatibilization consists in twisting by $\mathcal{O}_{\mathcal{S}_\mathsf{T}}(0,-1)$.

3.4 Semistability

The following result is analogous to that of the previous chapter.

Theorem 3.4.1. *Let $\mathcal{F}$ be a rank r vector bundle on $\mathcal{S}_\mathsf{T}$ with $\hat{\mu}$-semistable restrictions to components. Then its H_T-compatibilization is $\hat{\mu}$-semistable.*

Unfortunately, the method used in the proof of Theorem (2.4.1) does not apply on tree surfaces. Indeed, while working on two components, the above hypotheses guarantee that restrictions to any subsurface are $\hat{\mu}$-semistable. On tree surfaces this claim is too strong. For example, proving lemma (3.3.1), we have seen that, in general, restrictions to disconnected subsurfaces are not $\hat{\mu}$-semistable. So, to prove the above, we use a different strategy.

Proof. First, note that $\hat{\mu}$-semistable sheaves on components are preserved by twist. So, we can assume $\mathcal{F}$ to be H_T-compatible.

Let $\mathcal{F} \twoheadrightarrow \mathcal{Q}$ be a proper quotient of pure dimension 2. We show it is not destabilizing.

The rank of $\mathcal{Q}$ is defined by a set of non-negative integers $\{r_\alpha\}_{\alpha \in T}$. Clearly, $0 \le r_\alpha \le r$, and the set of vertices A where they are non-zero describes the support of $\mathcal{Q}$. By A_i we denote the subset of A defined by the vertices α such that $r_\alpha \ge i$. So, we obtain a nested sequence of subsets

$$A_r \subset \cdots \subset A_1 = A.$$

Similarly to section (2.2) of the previous chapter, tensorizing by $\mathcal{Q}$ the exact sequence (3.1) and quotienting out the torsion, we obtain

$$0 \to \mathcal{Q} \to \bigoplus_{\alpha \in A} \overline{\mathcal{Q}|_{\mathcal{S}_\alpha}} \to \bigoplus_{\alpha \in A^*} \mathcal{H}_{\mathcal{Q},\alpha} \to 0, \tag{3.5}$$

where $\mathcal{H}_{\mathcal{Q},\alpha}$ is a sheaf supported on the intersection E_α only. In particular, the rank of $\mathcal{H}_{\mathcal{Q},\alpha}$ is smaller than the ranks of the restrictions of $\mathcal{Q}$ on the adjacent components. So, if $rk(\mathcal{H}_{\mathcal{Q},\alpha}) \ge i$, then both $r_\alpha, r_{\alpha^-} \ge i$, that is, $\alpha \in A_i^*$.

The exact sequence above allows us to relate the slope of $\mathcal{Q}$ to the slopes of $\overline{\mathcal{Q}|_{\mathcal{S}_\alpha}}$. To simplify the notation, we will denote these sheaves simply by $\mathcal{Q}_\alpha$. By the above observation on the rank of $\mathcal{H}_{\mathcal{Q},\alpha}$ and the $\hat{\mu}$-semistability of $\mathcal{F}|_{\mathcal{S}_\alpha}$, we have:

$$\begin{aligned}(\sum_{\alpha\in A} r_\alpha H_\alpha^2)\hat{\mu}_{\mathcal{Q}} &= \sum_{\alpha\in A} r_\alpha H_\alpha^2 \hat{\mu}_{\mathcal{Q}_\alpha} - 2\sum_{\alpha\in A^*} rk(\mathcal{H}_{\mathcal{Q},\alpha})\delta_\alpha \\ &\geq \sum_{1\leq i\leq r}\left(\sum_{\alpha\in A_i} H_\alpha^2\hat{\mu}_{\mathcal{Q}_\alpha} - 2\sum_{\alpha\in A_i^*}\delta_\alpha\right) \\ &\geq \sum_{1\leq i\leq r}\left(\sum_{\alpha\in A_i} H_\alpha^2\hat{\mu}_{\mathcal{F}|_{\mathcal{S}_\alpha}} - 2\sum_{\alpha\in A_i^*}\delta_\alpha\right).\end{aligned}$$

The exact sequence (3.1) describing the gluing of the restrictions to components for the bundle $\mathcal{F}|_{\mathcal{S}_{A_i}}$, permits to relate the sum inside the parentheses to the slope of $\mathcal{F}|_{\mathcal{S}_{A_i}}$. Therefore, the above can be rewritten as

$$(\sum_{\alpha\in A} r_\alpha H_\alpha^2)\hat{\mu}_{\mathcal{Q}} \geq \sum_{1\leq i\leq r}\left((\sum_{\alpha\in A_i} H_\alpha^2)\hat{\mu}_{\mathcal{F}|_{\mathcal{S}_{A_i}}}\right).$$

The H_{T}-compatibility of $\mathcal{F}$ provides the needed inequalities for the slopes of its restrictions to subsurfaces. Thus,

$$(\sum_{\alpha\in A} r_\alpha H_\alpha^2)\hat{\mu}_{\mathcal{Q}} \geq (\sum_{\alpha\in A} r_\alpha H_\alpha^2)\hat{\mu}_{\mathcal{F}}$$

and so $\mathcal{Q}$ does not destabilize $\mathcal{F}$. □

Observing the chain of inequalities in the above proof, we deduce easily the following.

Corollary 3.4.2. *Let $\mathcal{F}$ be a rank r vector bundle on $\mathcal{S}_{\mathsf{T}}$ with $\hat{\mu}$-semistable restrictions to components. If there exists a quotient $\mathcal{Q}$ of $\mathcal{F}$ with the same slope, then $\mathcal{Q}$ has constant rank $r' < r$ and equality for the slopes holds for the restrictions to all components.*

The results of chapter 2 make us believe that a similar statement still holds for Gieseker semistability. On the other hand, it is not clear how to translate the above proof in terms of Gieseker semistability. Indeed, as we have no information on the Euler characteristic of the sheaves $\mathcal{H}_{\mathcal{Q},\alpha}$, we cannot use exact sequence (3.5) to compare $p_{\mathcal{F}}(t)$ and $p_{\mathcal{Q}}(t)$. The following lemma and the description of reflexive sheaves in section (2.3) help to bypass this problem.

Lemma 3.4.3. *Let* $0 \to \mathcal{G} \xrightarrow{i} \mathcal{F} \to \mathcal{Q} \to 0$ *be a saturated exact sequence on* $\mathcal{S}_{\mathsf{T}}$ *for a vector bundle* $\mathcal{F}$*. Then* $\mathcal{G}$ *is reflexive.*

Proof. Dualizing the inclusion i, we get a map,

$$\mathcal{F}^{\check{}} \xrightarrow{\check{i}} \mathcal{G}^{\check{}} \to \mathcal{E}xt^1(\mathcal{Q}, \mathcal{O}_{\mathcal{S}_{\mathsf{T}}}) \to 0.$$

Note that, as $\mathcal{Q}$ is of pure dimension 2, the cokernel vanishes almost everywhere. Thus dualizing again, its dual vanishes and we get an injection of $\mathcal{G}^{\check{}\check{}}$ into $\mathcal{F}$. As $\mathcal{G}$ is saturated, it follows that $\mathcal{G}^{\check{}\check{}} \simeq \mathcal{G}$. □

Theorem 3.4.4. *Suppose* $\mathcal{F}$ *to be a rank* r *vector bundle on* $\mathcal{S}_{\mathsf{T}}$ *whose restrictions to components are semistable in the sense of Gieseker. Then, so is its* H_{T}*-compatibilization.*

Proof. As above, we can suppose $\mathcal{F}$ to be H_{T}-compatible. The previous theorem implies that $\mathcal{F}$ is $\hat{\mu}$-semistable. So, assume

$$0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{Q} \to 0$$

to be a saturated exact sequence yielding equality on the slopes. To check semistability for $\mathcal{F}$, we have to compare only the constant terms of the Hilbert polynomials.

By corollary (3.4.2) and the above lemma, $\mathcal{G}$ is a reflexive sheaf on $\mathcal{S}_{\mathsf{T}}$ of constant rank $r' < r$.

Restricting the saturated sequence to a component $\mathcal{S}_\alpha$, there appears an injective map from the sheaf $\mathcal{T}or^1_{\mathcal{S}_{\mathsf{T}}}(\mathcal{Q}, \mathcal{O}_{\mathcal{S}_\alpha})$ on the left. As this sheaf has rank 0, it coincides with the torsion part of the $\mathcal{G}|_{\mathcal{S}_\alpha}$. Thus,

$$0 \to \overline{\mathcal{G}|_{\mathcal{S}_\alpha}} \to \mathcal{F}|_{\mathcal{S}_\alpha} \to \mathcal{Q}|_{\mathcal{S}_\alpha} \to 0.$$

Moreover, restrictions to components of pure 2-dimensional sheaves with constant rank have torsion supported at isolated points. Saturating the above sequence, we obtain

$$0 \to \mathcal{E}_\alpha \to \mathcal{F}|_{\mathcal{S}_\alpha} \to \overline{\mathcal{Q}|_{\mathcal{S}_\alpha}} \to 0,$$

where $\mathcal{E}_\alpha$ corresponds to the double dual of $\mathcal{G}|_{\mathcal{S}_\alpha}$. Equality for the slopes still holds, so $r\chi(\mathcal{E}_\alpha) \leq r'\chi(\mathcal{F}|_{\mathcal{S}_\alpha})$.

Recall proposition (2.3.3). The description of reflexive sheaves with constant rank in the case of two smooth components can be easily transcribed for tree surfaces. Indeed, for any $\alpha \in \mathsf{T}^*$ we have an exact sequence

$$0 \to \overline{\mathcal{G}|_{\mathcal{S}_\alpha}} \to \mathcal{E}_\alpha \to T(\mathcal{G}|_{\mathcal{S}_{\alpha^-}})|_{E_\alpha} \oplus_{\beta \in \alpha^+} T(\mathcal{G}|_{\mathcal{S}_\beta})|_{E_\beta} \to 0,$$

and a similar one holds on the root component. Moreover, the sheaves $\mathcal{H}_{\mathcal{G},\alpha}$, defined by the exact sequence (3.5), are isomorphic to the kernel of the quotient $\mathcal{E}_\alpha|_{E_\alpha} \to T(\mathcal{G}|_{\mathcal{S}_{\alpha^-}})|_{E_\alpha}$. Computing the Euler characteristic of $\mathcal{G}$ from these sequences, we get

$$\begin{aligned}\chi(\mathcal{G}) &= \sum_{\alpha\in T} \chi(\overline{\mathcal{G}|_{\mathcal{S}_\alpha}}) - \sum_{\alpha\in T^*} \chi(\mathcal{H}_{\mathcal{G},\alpha}) \\ &= \sum_{\alpha\in T} \Big(\chi(\mathcal{E}_\alpha) - \chi(T(\mathcal{G}|_{\mathcal{S}_{\alpha^-}})|_{E_\alpha} - \sum_{\beta\in\alpha^+} \chi(T(\mathcal{G}|_{\mathcal{S}_\beta})|_{E_\beta})\Big) - \\ &\qquad\qquad - \sum_{\alpha\in T^*} \Big(\chi(\mathcal{E}_\alpha|_{E_\alpha}) - \chi(-T(\mathcal{G}|_{\mathcal{S}_{\alpha^-}})|_{E_\alpha})\Big) \\ &= \sum_{\alpha\in T^*} \chi(\mathcal{E}_\alpha(-E_\alpha)) + \chi(\mathcal{E}_0) - \sum_{\beta\in T^*} \chi(T(\mathcal{G}|_{\mathcal{S}_\beta})|_{E_\beta}).\end{aligned}$$

The Euler characteristic of sheaves supported on finitely many points is non-negative. Thus, by the semistability of the restrictions $\mathcal{F}|_{\mathcal{S}_\alpha}$,

$$r\chi(\mathcal{G}) \le r'\Big(\sum_{\alpha\in T^*} \chi(\mathcal{F}_\alpha(-E_\alpha)) + \chi(\mathcal{F}_0)\Big).$$

Considering exact sequence (3.3), one can show that the sum inside the parentheses corresponds to $\chi(\mathcal{F})$. Hence, $p_{\mathcal{G}}(t) \le p_{\mathcal{F}}(t)$. □

3.5 Restrictions of $\hat{\mu}$-semistable bundles

In what follows we will discuss the opposite direction of Theorem (3.4.1). Let $\mathcal{F}$ be a $\hat{\mu}$-(semi)stable bundle on $\mathcal{S}_{\mathsf{T}}$ and β a vertex in T^*. Suppose

$$0 \to \mathcal{G} \to \mathcal{F}|_{\mathcal{S}_\beta} \to \mathcal{Q} \to 0$$

is a saturated sequence of sheaves on $\mathcal{S}_\beta$. From appropriate exact sequences as in (3.3), one can see that $\mathcal{Q}$ and $\mathcal{G}(-E_\beta - \sum_{\alpha\in\beta^+} E_\alpha)$ are respectively a quotient and a subsheaf of $\mathcal{F}$. The corresponding inequalities on the slopes yield

$$\hat{\mu}_{\mathcal{F}|_{\mathcal{S}_\beta}} - \hat{\mu}_{\mathcal{G}} \underset{(-)}{\geq} \begin{cases} \hat{\mu}_{\mathcal{F}|_{\mathcal{S}_\beta}} - \hat{\mu}_{\mathcal{F}} - \frac{2}{H_\beta^2}(\delta_\beta + \sum_{\alpha\in\beta^+} \delta_\alpha), \\ \frac{r-r'}{r'}(\hat{\mu}_{\mathcal{F}} - \hat{\mu}_{\mathcal{F}|_{\mathcal{S}_\beta}}), \end{cases} \tag{3.6}$$

where r' is the rank of $\mathcal{G}$. Considering the root component, we obtain similar inequalities. The only difference is that δ_0 does not appear.

As both terms on the right side of (3.6) are non-positive, these inequalities do not exclude that $\mathcal{G}$ destabilizes $\mathcal{F}|_{\mathcal{S}_\beta}$. Indeed, both examples (3.5.2) and (3.6.3) show that restrictions to components of a $\hat{\mu}$-(semi)stable bundle on $\mathcal{S}_{\mathsf{T}}$ do not need to be $\hat{\mu}$-semistable. Nevertheless, with some tricks, we can prove this claim to hold under particular assumptions.

Let us restrict ourselves to the rank 2 case. Both the kernel and the quotient of a saturated sequence for $\mathcal{F}|_{\mathcal{S}_\beta}$ have rank 1. Thus, the above inequalities can be reformulated via an absolute value.

The easier cases to consider are restrictions to top components.

Proposition 3.5.1. *Let $\mathcal{F}$ be a rank 2 vector bundle on $\mathcal{S}_\mathsf{T}$. If, for a general polarization H_T, $\mathcal{F}$ is $\hat{\mu}$-semistable, then so are its restrictions to top components.*

Proof. Consider $\mathcal{G}$ to be defined by $\mathcal{O}_{\mathcal{S}_\beta}(q)$. Inequalities (3.6) can be rewritten as follows:

$$\delta_\beta(e_\beta - 2q) \geq \left| \frac{\delta_\beta^2}{H^2} H.(c_\mathcal{S} - K_\mathcal{S}) - \delta_\beta(e_\beta + 2) \right| - \delta_\beta. \tag{3.7}$$

The argument of the absolute value is linear in δ_β. Assuming δ_β to be different from its unique root, leads to a strict inequality $e_\beta - 2q > -1$. Thus $\hat{\mu}_{\mathcal{F}|_{\mathcal{S}_\beta}} \geq \hat{\mu}_\mathcal{G}$. □

Note that the assumption on the polarization is not necessary if e_β is even. Indeed, in this case the strict inequality is not necessary to prove the claim. On the contrary, for odd e_β, the assumption is crucial as shown in the following example.

Example 3.5.2. Consider the tree surface $\mathcal{S}_\mathsf{T}$ with a polarization H_T as in the example (3.3.4). In what follows, we construct a rank 2 vector bundle on $\mathcal{S}_\mathsf{T}$ with odd degree on the intersection. Moreover we show that, for a particular choice of δ_1, it is $\hat{\mu}$-semistable, but its restriction to $\mathcal{S}_1$ is not.

Let x be a point in $\mathcal{S}_1 \simeq \mathbb{P}^2$ not lying on E_1. Computing the extension groups by Serre's duality, $\mathrm{Ext}^1(\mathcal{I}_{\mathcal{S}_1,x}(-1), \mathcal{O}_{\mathcal{S}_1}) \simeq H^1(\mathcal{I}_{\mathcal{S}_1,x}(-4))\check{} \simeq \mathbb{C}$. So, define $\mathcal{F}_1$ by the non-trivial extension

$$0 \to \mathcal{O}_{\mathcal{S}_1} \to \mathcal{F}_1 \to \mathcal{I}_{\mathcal{S}_1,x}(-1) \to 0\ .$$

Serre's method guarantees $\mathcal{F}_1$ to be a rank 2 bundle on $\mathcal{S}_1$ with $c_1(\mathcal{F}_1) = -1$. Furthermore, $\mathcal{F}_1$ is $\hat{\mu}$-unstable, and the above is the unique destabilizing torsion free quotient of $\mathcal{F}_1$. Suppose we have another destabilizing sequence for $\mathcal{F}_1$,

$$\begin{array}{ccccc} 0 \to \mathcal{O}_{\mathcal{S}_1}(s-1) & \xrightarrow{\alpha} & \mathcal{F}_1 & \to \mathcal{I}_{\mathcal{S}_1,Z}(-s) \to 0 \\ & & || & \\ 0 \to \mathcal{O}_{\mathcal{S}_1} & \to & \mathcal{F}_1 & \xrightarrow{\beta} \mathcal{I}_{\mathcal{S}_1,x}(-1) \to 0\ , \end{array}$$

where $s \geq 1$ and Z is 0-dimensional. Composing α and β, we get a map $\mathcal{O}_{\mathcal{S}_1}(s-1) \to \mathcal{I}_{\mathcal{S}_1,x}(-1)$. It has to be zero, because $\mathcal{I}_{\mathcal{S}_1,x}(-s)$ has no global sections. Hence there are vertical maps making the diagram commutative. As all the terms are torsion free, these maps are injective and hence isomorphisms by the snake lemma.

Let $\mathcal{F}_0$ be a non-trivial extension,

$$0 \to \mathcal{O}_{\mathcal{S}_0}(-H + 2E_1) \to \mathcal{F}_0 \to \mathcal{O}_{\mathcal{S}_0}(+H - E_1) \to 0\ .$$

Computing the extension group by proposition (1.5.2), such an extension exists and can be choosen within a 3-dimensional family. Clearly, $\mathcal{F}_0$ is a rank 2 bundle on $\mathcal{S}_0$ with $c_1(\mathcal{F}_0) = E_1$. Moreover, restricting to E_1, the map

$$\mathrm{Ext}^1(\mathcal{O}_{\mathcal{S}_0}(+H - E_1), \mathcal{O}_{\mathcal{S}_0}(-H + 2E_1)) \to \mathrm{Ext}^1(\mathcal{O}_{E_1}(1), \mathcal{O}_{E_1}(-2))$$

is surjective. To see this, one can consider it as a map in the long exact sequence in cohomology given by the restriction to E_1. Hence we can choose $\mathcal{F}_0$ in such a way that

$$\mathcal{F}_0|_{E_1} \simeq \mathcal{O}_{E_1} \oplus \mathcal{O}_{E_1}(-1) \simeq \mathcal{F}_1|_{E_1} \ .$$

Suppose we have a different torsion free quotient of $\mathcal{F}_0$. Then we have

$$\begin{array}{ccc} 0 \to \mathcal{O}_{\mathcal{S}_0}(-pH + (-q+1)E_1) \to & \mathcal{F}_0 & \to \mathcal{I}_{\mathcal{S}_0, Z_0}(pH + qE_1) \to 0 \\ & || & \\ 0 \to \mathcal{O}_{\mathcal{S}_0}(-H + 2E_1) \to & \mathcal{F}_0 & \to \mathcal{O}_{\mathcal{S}_0}(+H - E_1) \to 0 \ . \end{array}$$

As before, $H^0(\mathcal{S}_0, \mathcal{O}_{\mathcal{S}_0}((p+1)H + (q-2)E_1))$ cannot vanish. This condition translates in terms of p and q:

$$\begin{cases} p + q \geq 1 \\ p \geq -1 \end{cases} \tag{3.8}$$

Moreover, restricting the quotient map to E_1, we see that the degree of the cokernel cannot be less than -1. So, $q \leq 1$.

If the above quotient destabilize $\mathcal{F}_0$, then computing the slopes we get

$$\hat{\mu}_{\mathcal{O}_{\mathcal{S}_0}(pH+qE_1)} \underset{(-)}{\leq} \hat{\mu}_{\mathcal{F}_0} \iff \frac{1}{\delta_1} p + q \underset{(-)}{\leq} \frac{1}{2}.$$

Comparing all the above inequalities, the only possible pair (p,q) for a destabilizing torsion free quotient of $\mathcal{F}_0$ is $(-1, 1)$. But this case is excluded as the extension defining $\mathcal{F}_0$ is non-split. Hence, for all $0 < \delta_1 < 1$, $\mathcal{F}_0$ is $\hat{\mu}$-stable.

Define $\mathcal{F}$ to be the vector bundle on $\mathcal{S}$ obtained by gluing $\mathcal{F}_0$ and $\mathcal{F}_1$ along E_1. Its first Chern class is given by $c_{\mathcal{S}} = 0$ and $e_1 = -1$. By proposition (3.3.2), if $0 \leq \delta_1 \leq \frac{2}{3}$, then $\mathcal{F}$ is compatible with respect to the corresponding polarization. On the other hand, as $\mathcal{F}|_{\mathcal{S}_1}$ is $\hat{\mu}$-unstable, lemma (3.5.1) implies that for all $\delta_1 \neq \frac{1}{3}$, $\mathcal{F}$ is not $\hat{\mu}$-semistable. So, fix $\delta_1 = \frac{1}{3}$. Consider a quotient $\mathcal{F} \twoheadrightarrow \mathcal{Q}$ of pure dimension 2. If $\mathcal{Q}|_{\mathcal{S}_1}$ does not destabilize $\mathcal{F}_1$ then we are in the hypothesis of the proof of proposition (3.4.1), and so $\mathcal{Q}$ does not destabilize $\mathcal{F}$.

Assume now $\mathcal{Q}|_{\mathcal{S}_1}$ destabilizes $\mathcal{F}|_{\mathcal{S}_1}$. Then its torsion free part has to be isomorphic to $\mathcal{I}_{\mathcal{S}_1,x}(-1)$. We have three different cases: If $\mathcal{Q}$ has rank $(0, 1)$, it is $\mathcal{I}_{\mathcal{S}_1,x}(-1)$ itself. Its slope is $\hat{\mu}_{\mathcal{Q}} = 3 = \hat{\mu}_{\mathcal{F}}$, so it does not destabilize $\mathcal{F}$. Suppose $\mathcal{Q}$ has rank $(1, 1)$ on $\mathcal{S}$. Let $c_1(\mathcal{Q}|_{\mathcal{S}_0})$ be defined by the divisor

$pH + qE_1$ for some $p, q \in \mathbb{Z}$. The $\hat{\mu}$-stability of $\mathcal{F}_0$ implies that $3p + q \geq 1$. Thus

$$\begin{aligned}\hat{\mu}_{\mathcal{Q}} &= (1-\delta_1^2)\hat{\mu}_{\mathcal{Q}|_{\mathcal{S}_0}} + \delta_1^2 \hat{\mu}_{\mathcal{Q}|_{\mathcal{S}_1}} - 2rk(\mathcal{H}_{\mathcal{Q}})\delta_1 \\ &\geq (2p+3+\frac{1}{3}(2q-1)) + \frac{1}{3} - 2\frac{1}{3} \geq 3 = \hat{\mu}_{\mathcal{F}}.\end{aligned}$$

The last case to check is $\mathcal{Q}$ of rank $(2,1)$. If $\mathcal{Q}$ destabilizes $\mathcal{F}$, then the kernel destabilizes $\mathcal{F}_1(-E_1)$ and so it has to be $\mathcal{O}_{\mathcal{S}_1}(-1)$. But the slope of this line bundle is equal to the slope of $\mathcal{F}$, so $\mathcal{Q}$ does not destabilize $\mathcal{F}$ even in this case.

Finally, for $\delta_1 = \frac{1}{3}$, $\mathcal{F}$ is a strictly $\hat{\mu}$-semistable bundle on $\mathcal{S}_{\mathsf{T}}$ admitting a non-$\hat{\mu}$-semistable restriction to $\mathcal{S}_1$. After a change of polarization, $\mathcal{F}$ becomes $\hat{\mu}$-unstable.

For restrictions to other components, things get more complicated. If $\beta \in \mathsf{T}^* \setminus \mathsf{T}_{top}$, inequalities (3.6) involve a large number of δ_α's which are hard to control. Even considering good polariations, the presence of both δ_β and $\bar{\delta}_\beta$ does not allow us to proceed in the same way as for top components. The case of the root component is quite different, and for a specific choice of $\bar{\delta}_0$, we obtain a similar result.

Proposition 3.5.3. *Let $\mathcal{F}$ be a rank 2 vector bundle on $\mathcal{S}_{\mathsf{T}}$. Suppose $H_{\mathsf{T}} = (H, \delta_{\mathsf{T}})$ is a polarization on $\mathcal{S}_T$ such that, for all $\alpha \in 0^+$, $\delta_\alpha = \bar{\delta}_0 = \frac{m}{n}$, where n, m are coprime positive integers satisfying one of the following:*

i) $n > m \sum_{\alpha \in 0^+}(m\frac{1}{H^2}H.(c_{\mathcal{S}} - K_{\mathcal{S}}) - n(e_\alpha + 1)) > 0$, *or*

ii) $n > m \sum_{\alpha \in 0^+}(n(e_\alpha + 3) - m\frac{1}{H^2}H.(c_{\mathcal{S}} - K_{\mathcal{S}})) > 0$.

If $\mathcal{F}$ is $\hat{\mu}$-semistable with respect to H_{T}, then $\mathcal{F}|_{\mathcal{S}_0}$ is $\hat{\mu}$-semistable with respect to H_0.

Proof. Suppose that $\mathcal{F}$ is $\hat{\mu}$-semistable on $\mathcal{S}_{\mathsf{T}}$ and $\mathcal{G}$ is a destabilizing subsheaf of $\mathcal{F}|_{\mathcal{S}_0}$. Denote the first Chern class of $\mathcal{G}$ by $D + \sum_{\alpha\in 0^+} q_\alpha E_\alpha$. The first inequality (3.6) yields:

$$0 > H.(c_{\mathcal{S}} - 2D) - \bar{\delta}_0 \sum_{\alpha \in 0^+}(e_\alpha - 2q_\alpha) \underset{(-)}{\geq} \sum_{\alpha\in 0^+} -\frac{\bar{\delta}_0^2}{H^2}H.(c_{\mathcal{S}} - K_{\mathcal{S}}) + \bar{\delta}_0(e_\alpha + 1).$$

The term on the right hand side is quadratic in $\bar{\delta}_0$, while the term in the middle is linear. So, multiplying the above chain by n^2, the middle term is a multiple of n. Moreover, it is confined between 0 and

$$-m \sum_{\alpha\in 0^+}(m\frac{1}{H^2}H.(c_{\mathcal{S}} - K_{\mathcal{S}}) - n(e_\alpha + 1)).$$

Thus, if i) holds, there are no elements in $\langle n \rangle_{\mathbb{Z}}$ lying in the above limits.

Similarly, if ii) holds, then the second inequality (3.6) provides the contradiction. □

Observing the proof, one can remark that it still holds without assuming the positivity of i) and ii). But this gives nothing new, for if the positivity fails, then $\mathcal{F}$ will be destabilized by its restrictions.

Though the hypotheses in proposition (3.5.3) seem cryptic, they provide a criterion to construct polarizations with nice properties in specific examples.

Example 3.5.4. Consider the framework of example (3.3.4). Let $\mathcal{F}$ be a rank 2 vector bundle with trivial first Chern class. The conditions on δ_1 provided by proposition (3.5.3) acquire the form:

i) $n > m(3m - n) > 0$, or

ii) $n > 3m(n - m) > 0$.

As $m \geq 1$, it is easy to see that i) and ii) yield respectively $\delta_1 \in (\frac{1}{3}, \frac{2}{3})$ and $\delta_1 \in (\frac{1}{3}, \frac{2}{3})$. Moreover, dividing by n and by the expression between parentheses, we see that

$$\text{i)} \Rightarrow \delta_1 \leq \frac{1}{2}, \quad \text{and ii)} \Rightarrow \delta_1 < \frac{1}{3}.$$

In particular, there are no solutions to ii). On the contrary, i) provides a countable set of solutions in $(\frac{1}{3}, \frac{1}{2}]$. Indeed, for all $m \in \mathbb{N}^*$, $\delta_1 = \frac{m}{3m-1}$ is a solution to i).

Combining the above with proposition (3.5.1) we get the following.

Corollary 3.5.5. *In the setting of example (3.3.4), let H_{T} be a polarization defined by $\delta_1 = \frac{m}{3m-1}$ where $m \in \mathbb{N}^*$. A rank 2 vector bundle $\mathcal{F}$ on $\mathcal{S}_{\mathsf{T}}$ is $\hat{\mu}$-semistable if and only if so are its restrictions to all the components.*

3.6 Admissible bundles

Definition 9. A vector bundle on a tree surface is said to be *admissible* if and only if its H_{T}-compatibilization is $\hat{\mu}$-stable and its restrictions to the intersections are twists of a trivial bundle.

It is clear from the definition that admissible bundles appear in equivalence classes,

$$[\mathcal{F}] := \{\mathcal{F}(0, m_{\mathsf{T}}) | m_{\mathsf{T}} = \{m_\alpha\}_{\alpha \in \mathsf{T}^*}\}.$$

For, if $\mathcal{F}$ is admissible, then for every set of integers $m_{\mathsf{T}} = \{m_\alpha\}_{\alpha \in \mathsf{T}^*}$ the vector bundle $\mathcal{F}(0, m_{\mathsf{T}})$ has the same H_{T}-compatibilization as $\mathcal{F}$. Moreover, the condition on the restrictions to E_α is obviously verified.

In any equivalence class we can distinguish two representatives: the $\hat{\mu}$-stable vector bundle, and the bundle whit trivial restrictions to the E_α. These are both uniquely determined; in most case they do not coincide.

Recall that we defined the standard contraction $\sigma : \mathcal{S}_{\mathsf{T}} \to \mathcal{S}$ and the blowup $\sigma_0 : \mathcal{S}_0 \to \mathcal{S}$ in the first section of this chapter. For any vector bundle $\mathcal{F}$ on $\mathcal{S}_\mathsf{T}$, its pushforward by σ differs from the pushforward by σ_0 of $\mathcal{F}|_{\mathcal{S}_0}$ in a finite set of points. Thus we define

$$\mathcal{F}_\mathcal{S} := (\sigma_*\mathcal{F})^{\check{}\check{}} \simeq (\sigma_{0*}\mathcal{F}|_{\mathcal{S}_0})^{\check{}\check{}}.$$

In particular, all the admissible bundles in the same equivalence class define the same vector bundle $\mathcal{F}_\mathcal{S}$ on $\mathcal{S}$. Moreover, choosing polarization in a specific chamber, we can say more on this bundle.

Theorem 3.6.1. *Let $\mathcal{F}$ be a rank r vector bundle on $\mathcal{S}_\mathsf{T}$ and suppose the polarization H_T on $\mathcal{S}_\mathsf{T}$ satisfies the inequality*

$$\sum_{\alpha\in 0^+} \delta_\alpha < \frac{4}{r^2}. \tag{3.9}$$

If $\mathcal{F}$ is an admissible bundle on $\mathcal{S}_\mathsf{T}$, then $\mathcal{F}_\mathcal{S}$ is a $\hat{\mu}$-semistable bundle on $\mathcal{S}$ with respect to H.

Proof. Clearly we can assume $\mathcal{F}$ to be the $\hat{\mu}$-stable representative of its equivalence class. Suppose $\mathcal{G}$ is a saturated subsheaf of $\mathcal{F}_\mathcal{S}$. So, $\mathcal{G}$ is a vector bundle of rank $r' < r$ and we have the inclusion

$$\mathcal{G}|_\mathcal{U} \hookrightarrow \mathcal{F}|_{\mathcal{S}_0\cap\mathcal{U}},$$

where $\mathcal{U}$ is the complement of the exceptional divisors in $\mathcal{S}_0$. Passing to the map defined on the stalks at the generic point of $\mathcal{S}_0$, one sees that there exists an extension of this inclusion to an inclusion $\mathcal{G}_0 \hookrightarrow \mathcal{F}|_{\mathcal{S}_0}$ with locally free $\mathcal{G}_0$ defined over the whole of $\mathcal{S}_0$. In particular, we can assume $\mathcal{G}_0$ saturated, so that we have a saturated exact triple

$$0 \to \mathcal{G}_0 \to \mathcal{F}|_{\mathcal{S}_0} \to \mathcal{Q}_0 \to 0,$$

in which $\mathcal{Q}_0$ is torsion free of rank $r - r'$. Denoting $D = c_1(\mathcal{G})$, there exist integers $\{a_\alpha\}$ such that $c_1(\mathcal{G}_0) = D + \sum_{\alpha\in 0^+} a_\alpha E_\alpha$. Consequently the first Chern class of $\mathcal{Q}$ is $c_\mathcal{S} - D - \sum_{\alpha\in 0^+}(e_\alpha + a_\alpha)E_\alpha$.

Restricting the above sequence to one of the intersection lines E_α, we get a surjection

$$\mathcal{F}|_{E_\alpha} \simeq \mathcal{O}_{E_\alpha}(e_\alpha/r)^{\oplus r} \twoheadrightarrow \mathcal{Q}_0|_{E_\alpha}.$$

Counting the contribution of the torsion part, the sheaf on the right has first Chern class $c_1(\mathcal{Q}_0).E_\alpha = e_\alpha + a_\alpha$. So, by the $\hat{\mu}$-semistability of the trivial bundle, we get

$$(r - r')e_\alpha \leq r(e_\alpha + a_\alpha) \quad \text{i.e.} \quad ra_\alpha \geq -r'e_\alpha.$$

By the $\hat{\mu}$-stability of $\mathcal{F}$, the injection $\mathcal{G}_0(-\sum_{\alpha\in 0^+} E_\alpha) \hookrightarrow \mathcal{F}$ provides an inequality on the slopes. Computing it, we get

$$r'(H^2 - \sum_{\alpha\in 0^+} \delta_\alpha^2)[H.(2c_\mathcal{S} - rK_\mathcal{S})] \geq rH^2[H.(2D - r'K_\mathcal{S}) + \sum_{\alpha\in 0^+} \delta_\alpha(2a_\alpha + 3r')].$$

Replacing ra_α by $-r'e_\alpha$ and reordering, the above yields

$$2[H.(r'c_{\mathcal{S}} - rD)] \geq r'r \sum_{\alpha \in 0^+} \delta_\alpha^2 [\hat{\mu}_{\mathcal{F}} - \hat{\mu}_{\mathcal{F}|_{\mathcal{S}_{\mathsf{T}_\alpha}}}]. \tag{3.10}$$

This inequality is not sufficient to prove the theorem. Indeed, by proposition (3.3.2), the H_{T}-compatibility of $\mathcal{F}$ confines the term on the right to the interval $(0, -2rr'\delta_\alpha)$. By the way, the surjection $\mathcal{F} \twoheadrightarrow \mathcal{Q}_0$ provides another inequality on the slopes. Proceeding as above, we obtain

$$-2[H.(r'c_{\mathcal{S}} - rD)] \leq r(r - r') \sum_{\alpha \in 0^+} \delta_\alpha^2 \left[\hat{\mu}_{\mathcal{F}} - \hat{\mu}_{\mathcal{F}|_{\mathcal{S}_{\mathsf{T}_\alpha}}} + \frac{2}{\delta_\alpha}\right]. \tag{3.11}$$

Modifying both (3.10) and (3.11) in order to have the same term on the right, we get a chain of inequalities whose opposite terms involve $H.(r'c_{\mathcal{S}} - rD)$ with opposite signs. Comparing them, we obtain

$$2rH.(r'c_{\mathcal{S}} - rD) + 2rr'(r - r') \sum_{\alpha \in 0^+} \delta_\alpha \geq 0.$$

Thus

$$H.(r'c_{\mathcal{S}} - rD) \geq -r'(r - r') \sum_{\alpha \in 0^+} \delta_\alpha \geq -\frac{r^2}{4} \sum_{\alpha \in 0^+} \delta_\alpha > -1,$$

and so $\mathcal{G}$ does not destabilize $\mathcal{F}_{\mathcal{S}}$. □

Although, dealing with bundles of big rank, inequality (3.9) defines small chambers in the space of polarizations on $\mathcal{S}_{\mathsf{T}}$, it applies perfectly to the case we are most interested in. Indeed, comparing Theorem (3.6.1) and lemma (3.2.2), we have the following.

Corollary 3.6.2. *Let H be a very ample divisor on $\mathcal{S}$ and H_{T} a polarization on $\mathcal{S}_{\mathsf{T}}$ constructed from it by lemma (3.2.2). For every admissible bundle $\mathcal{F}$ of rank 2 on $\mathcal{S}_{\mathsf{T}}$, the vector bundle $\mathcal{F}_{\mathcal{S}}$ is $\hat{\mu}$-semistable.*

Naively, one can imagine that Theorem (3.6.1) can be extended in proving the $\hat{\mu}$-semistability of the restrictions to the root component of admissible bundles. This is false in general.

Example 3.6.3. Consider the tree surface $\mathcal{S}_{\mathsf{T}}$ with a polarization H_{T} as in example (3.3.4). Let x be a point in $\mathcal{S}_0$. Consider $\mathcal{F}_0$ to be a vector bundle defined by a non-trivial extension,

$$0 \to \mathcal{O}_{\mathcal{S}_0}(-H + 2E_1) \to \mathcal{F}_0 \to \mathcal{I}_{\mathcal{S}_0,x}(H - 2E_1) \to 0.$$

The space of extensions of the above form is 7-dimensional. By Serre construction, there exists a dense family of vector bundles in it. Thus, restricting to E_1, we can choose $\mathcal{F}_0$ in such a way that $\mathcal{F}_0|_{E_1} \simeq \mathcal{O}_{E_1}^{\oplus 2}$.

Suppose we have a different torsion free quotient $\mathcal{F}_0 \to \mathcal{I}_{\mathcal{S}_0,Z'}(pH+qE_1)$. It defines a non-zero section of $\mathcal{I}_{\mathcal{S}_0,x}((p+1)H+(q-2)E_1)$. So, the space of global sections of $\mathcal{O}_{\mathcal{S}_0}((p+1)H+(q-2)E_1)$ is at least 1-dimensional. By lemma (1.5.2), the pairs (p,q) verifying this condition belong to a region of the lattice $Pic(\mathcal{S}_0)$ defined by

$$\begin{cases} p \geq 1 \\ p+q \geq 0\,. \end{cases}$$

Computing the degree of $\mathcal{I}_{\mathcal{S}_0,Z'}(pH+qE_1)$, we get

$$H_0.(pH+qE_1) = (p+\delta_1 q) \geq (1-\delta_1)p > 0 = \hat{\mu}_{\mathcal{F}},$$

and it does not destabilize $\mathcal{F}_0$. Thus, $\mathcal{F}_0$ is $\hat{\mu}$-(semi)stable if and only if $\mathcal{I}_{\mathcal{S}_0,x}(H-2E_1)$ does not destabilize it. That is, if and only if $\delta_1 \underset{(-)}{\leq} \frac{1}{2}$.

Let $\mathcal{F}_1$ be any $\hat{\mu}$-semistable vector bundle of rank 2 on $\mathcal{S}_1$, trivial on E_1, e.g. $\mathcal{F}_1 \simeq \mathcal{O}_{\mathcal{S}_1}^{\oplus 2}$. Define $\mathcal{F}$ to be the vector bundle on $\mathcal{S}_{\mathsf{T}}$ obtained by gluing together $\mathcal{F}_0$ and $\mathcal{F}_1$.

We observed in example (3.3.4), if we choose a polarization such that $\frac{1}{3} < \delta_1 < 1$, then $\mathcal{F}$ is H_{T}-compatible. Moreover, if $\delta_1 \leq \frac{2}{3}$, then $\mathcal{I}_{\mathcal{S}_0,Z_0}(H-2E_1)$ does not destabilize it. Imitating example (3.5.2), it is not hard to see that there are no quotients of $\mathcal{F}$ that might destabilize it. So, $\mathcal{F}$ is $\hat{\mu}$-(semi)stable on $\mathcal{S}_{\mathsf{T}}$ if and only if $\frac{1}{3} < \delta_1 \underset{(-)}{\leq} \frac{2}{3}$.

For the sake of completeness, we consider also the case of a polarization in the second chamber, that is $\delta_1 \in (0, \frac{1}{3})$. The H_{T}-compatibilization of $\mathcal{F}$ is given by $\mathcal{F}' = \mathcal{F}(0,1)$. Morover, as $\delta_1 < \frac{1}{2}$, both restrictions to components,

$$\mathcal{F}'|_{\mathcal{S}_0} \simeq \mathcal{F}_0(E_1) \quad \text{and} \quad \mathcal{F}'|_{\mathcal{S}_1} \simeq \mathcal{F}_1(-E_1),$$

are $\hat{\mu}$-semistable. Thus, by Theorem (3.4.1), $\mathcal{F}'$ is $\hat{\mu}$-stable.

Summarizing, we described the following situation:

$$\begin{array}{ccc} \mathcal{F} \text{ is } H_{\mathsf{T}}\text{-compatible} & \Leftrightarrow & \delta_1 \in (\frac{1}{3}, 1), \\ \mathcal{F} \text{ is admissible} & \Leftrightarrow & \delta_1 \in (0, \frac{1}{3}) \cup (\frac{1}{3}, \frac{2}{3}), \\ \mathcal{F}|_{\mathcal{S}_0} \text{ is } \hat{\mu}\text{-semistable} & \Leftrightarrow & \delta_1 \in (0, \frac{1}{2}]. \end{array}$$

Clearly, for δ_1 in $(\frac{1}{2}, \frac{2}{3})$, $\mathcal{F}$ is $\hat{\mu}$-stable but its restriction to the root component is not $\hat{\mu}$-semistable. Moreover, note that the interval for δ_1, where both $\mathcal{F}$ and $\mathcal{F}|_{\mathcal{S}_0}$ are $\hat{\mu}$-semistable, coincides with the interval found in example (3.5.4), where the solutions of i) lie.

3.7 Charge of admissible bundles

Besides common $\mathcal{F}_{\mathcal{S}}$, admissible bundles in one equivalence class share other properties. Let $\mathcal{F}$ be an admissible bundle on $\mathcal{S}_{\mathsf{T}}$ and $m_{\mathsf{T}} = \{m_\alpha\}_{\alpha \in \mathsf{T}^*}$ a set of integers. The second Chern class of the restriction of $\mathcal{F}' = \mathcal{F}(0, m_{\mathsf{T}})$ to a component $\mathcal{S}_\alpha$ is

$$c_2(\mathcal{F}'|_{\mathcal{S}_\alpha}) = c_2(\mathcal{F}|_{\mathcal{S}_\alpha}) + (r-1)(m_\alpha e_\alpha - \sum_{\beta \in \alpha^+} m_\beta e_\beta) + m_\alpha^2 - \sum_{\beta \in \alpha^+} m_\beta^2.$$

In the computation of $n(\mathcal{F}')$, all the summands involving the m_α's vanish. Thus the total charge does not depend on the representative of the equivalence class.

Definition 10. Let $\mathcal{F}$ be an admissible vector bundle of rank r. The *charges* of $\mathcal{F}$ are the integers

$$n_\alpha(\mathcal{F}) = c_2(\mathcal{F}(0, \{-\frac{e_\alpha}{r}\})|_{\mathcal{S}_\alpha}), \quad \text{for } \alpha \in \mathsf{T}^*,$$

that is the second Chern classes of the representative of its equivalence class with trivial restrictions to the intersections.

Lemma 3.7.1. *Let H_{T} be a polarization as in Theorem (3.6.1). For every admissible bundle $\mathcal{F}$,*

$$\forall \alpha \in \mathsf{T}^*,\ n_\alpha \geq 0 \quad \textit{and} \quad 4n_0 \geq c_{\mathcal{S}}^2.$$

Proof. Suppose $\mathcal{F}$ is trivial on the intersection lines. So, the restriction of $\mathcal{F}$ to components are vector bundle on blown up surfaces, trivial on the exceptional divisors. By proposition (1.3.9),

$$\forall \alpha \in \mathsf{T}^*, \quad \mathcal{F}|_{\mathcal{S}_\alpha} \simeq \sigma_\alpha^* \sigma_{\alpha *} \mathcal{F}|_{\mathcal{S}_\alpha}.$$

Moreover, the pushforward of $\mathcal{F}|_{\mathcal{S}_\alpha}$ is a vector bundle on $\mathbb{P}^2$, trivial on the line E_α. Thus, it has trivial first Chern class and, as all its subsheaves have negative degree, it is $\hat{\mu}$-semistable. So, by the functoriality of the pullback and Bogomolov inequality, $n_\alpha = c_2(\mathcal{F}|_{\mathcal{S}_\alpha}) = \sigma_\alpha^* c_2(\sigma_{\alpha *}\mathcal{F}|_{\mathcal{S}_\alpha}) \geq 0$. The argument is similar for the restriction to the root component, where the above theorem provides the $\hat{\mu}$-semistability of $\mathcal{F}_{\mathcal{S}}$. □

Lemma 3.7.2. *For every $\beta \in \mathsf{T}^*$, $n_\beta(\mathcal{F}) = 0$ if and only if $\mathcal{F}$ is a twist of the rank r trivial bundle on $\mathcal{S}_\beta$.*

Proof. Let $\mathcal{F}$ be the representative of its equivalence class trivial on the intersections. In particular, denote by $\mathcal{F}_\beta$ the pushforward of $\mathcal{F}|_{\mathcal{S}_\beta}$ via σ_β. By corollary (1.3.9), $\mathcal{F}_\beta$ is a bundle on $\mathbb{P}^2$ trivial on the line E_β and with vanishing c_2. Thus, it is $\hat{\mu}$-semistable and has trivial Chern classes. In the following we will show that there is a unique $\hat{\mu}$-semistable bundle of this type, the trivial one.

Consider the graded object defined by the Jordan-Hölder filtration:

$$\mathcal{F}_\beta \sim_{\mathrm{S}} \bigoplus_{i \in I} \mathcal{E}_i,$$

where $\sim_S$ denotes the S-equivalence and $\mathcal{E}_i$ are stable vector bundles with same Chern classes as $\mathcal{F}_\beta$. Computing the Hilber polynomial of $\mathcal{E}_i$ we get

$$\chi(\mathcal{E}_i) = h^0(\mathcal{E}_i) - h^1(\mathcal{E}_i) + h^2(\mathcal{E}_i) = \mathrm{rk}(\mathcal{E}_i) > 0.$$

By Serre duality, $H^2(\mathbb{P}^2, \mathcal{E}_i) \simeq H^0(\mathbb{P}^2, \check{\mathcal{E}_i}(-3)) = 0$, for a stable bundle with negative degree has no global sections. Thus, the space of global sections of $\mathcal{E}_i$ has at least dimension $\mathrm{rk}(\mathcal{E}_i)$. In particular,

$$\mathcal{O}_{\mathbb{P}^2} \hookrightarrow \mathcal{E}_i$$

is a morphism of stable bundles with same Hilbert polynomial and so an isomorphism. So far we proved $\mathcal{F}_\beta$ to be S-equivalent to the trivial bundle of rank r. This clearly provides an isomorphism for $r = 1$. For higher ranks, the first term of the Jordan-Hölder filtration defines a triple

$$0 \to \mathcal{F}_{\beta,r-1} \to \mathcal{F}_\beta \to \mathcal{O}_{\mathbb{P}^2} \to 0,$$

where $\mathcal{F}_{\beta,r-1}$ is a $\hat{\mu}$-semistable bundle with trivial Chern classes and rank $r - 1$. Thus, the result follows by induction on the rank. □

Before stating the next result, we need to introduce some morphisms on a bubble tree surface $\mathcal{S}_\mathsf{T}$ similar to the standard contraction σ. Reordering the sequence of blowups defining $\mathcal{S}_\mathsf{T}$ in an appropriate way, we can define a morphism which contracts the subsurface associated to the complete subtree T_β for all $\beta \in \mathsf{T}^*$,

$$\theta_\beta : \mathcal{S}_\mathsf{T} \twoheadrightarrow \mathcal{S}'_{\mathsf{T}^c_\beta}.$$

We use a prime in the notation to avoid confusion, for the target of this morphism is a new tree surface, which is not a subsurface of $\mathcal{S}_\mathsf{T}$. Indeed we have a morphism

$$\nu_\beta : \mathcal{S}_{\mathsf{T}^c_\beta} \twoheadrightarrow \mathcal{S}'_{\mathsf{T}^c_\beta},$$

corresponding to the blowup in a point in $\mathcal{S}'_{\beta-}$. By lemma (3.2.1), the polarization H_T defines a polarization on $\mathcal{S}_{\mathsf{T}^c_\beta}$, and hence on $\mathcal{S}'_{\mathsf{T}^c_\beta}$.

Proposition 3.7.3. *Let β be a vertex in T^* and $\mathcal{F}$ an admissible bundle on $\mathcal{S}_\mathsf{T}$ with $n_\alpha(\mathcal{F}) = 0$ for every $\alpha \in \mathsf{T}_\beta$. Then there exists an admissible vector bundle $\mathcal{F}'$ on $\mathcal{S}'_{\mathsf{T}^c_\beta}$ such that $\theta^*_\beta \mathcal{F}' \in [\mathcal{F}]$.*

Proof. Assume $\mathcal{F}$ is the $\hat{\mu}$-stable representative of its equivalence class. Let $\mathcal{F}'$ be the reflexive sheaf defined by

$$\mathcal{F}' := (\nu_{\beta *} \mathcal{F}|_{\mathcal{S}_{\mathsf{T}^c_\beta}})\check{}\check{}.$$

As it is locally free outside a smooth point of $\mathcal{S}'_{\mathsf{T}^c_\beta}$, it is a vector bundle on the whole tree surface. Moreover, its pullback by θ_β is isomorphic to $\mathcal{F}$

outside $\mathcal{S}_{\mathsf{T}_\beta}$ and the trivial bundle on this one. Thus, they belong to the same equivalence class. It remains to prove the admissibility of $\mathcal{F}'$.

Let $\mathcal{G}'$ be a subsheaf of $\mathcal{F}'$ having rank r' on $\mathcal{S}_{\beta-}$. Proceeding as in the proof of Theorem (3.6.1), there exists an integer a_β such that

$$\mathcal{G} := \nu_\beta^* \mathcal{G}' \otimes \mathcal{O}_{\mathcal{S}_{\mathsf{T}_\beta^c}}(-a_\beta E_\beta)$$

is a subsheaf of $\mathcal{F}|_{\mathcal{S}_{\mathsf{T}_\beta^c}}$. Moreover, the injection

$$\mathcal{G}|_{E_\beta} \simeq \mathcal{O}_{E_\beta}(a_\beta)^{\oplus r'} \hookrightarrow \mathcal{F}|_{E_\beta} \simeq \mathcal{O}_{E_\beta}(\frac{e_\beta}{r})^{\oplus r},$$

can be naturally extended to $\mathcal{S}_{\mathsf{T}_\beta}$ where it gives:

$$\mathcal{O}_{\mathcal{S}_{\mathsf{T}_\beta}}(a_\beta E_\beta, \{\frac{e_\alpha}{r}\}_{\alpha\in\mathsf{T}_\beta^*})^{\oplus r'} \hookrightarrow \mathcal{O}_{\mathcal{S}_{\mathsf{T}_\beta}}(\frac{e_\beta}{r} E_\beta, \{\frac{e_\alpha}{r}\}_{\alpha\in\mathsf{T}_\beta^*})^{\oplus r} \simeq \mathcal{F}|_{\mathcal{S}_{\mathsf{T}_\beta}}.$$

Gluing this subsheaf with $\mathcal{G}$, we obtain a subsheaf $\mathcal{G}_\mathsf{T}$ of $\mathcal{F}$.

As $\mathcal{F}$ and $\mathcal{F}'$ have the same $c_\mathcal{S}$ and total charge, the computation done in the beginning of section (3.3) implies that their Hilbert polynomials coincide. The same can be proved for $\mathcal{G}_\mathsf{T}$ and $\mathcal{G}$. Thus, the $\hat{\mu}$-stability of $\mathcal{F}'$ follows from the $\hat{\mu}$-stability of $\mathcal{F}$. □

The above construction allows us to contract top components where an admissible bundle is trivial and provides another admissible bundle. The inverse construction, that is to extend trivially an admissible bundle to a surface with more components, does not always provide admissible bundles.

Example 3.7.4. Recall example (3.6.3). Choosing $\mathcal{F}_1$ to be the trivial bundle of rank 2, $\mathcal{F}$ can be defined as the pullback of $\mathcal{F}_\mathcal{S}$ via σ. Note that in this case the standard contraction acts exactly as the morphism $\theta_1 : \mathcal{S}_\mathsf{T} \to \mathcal{S}$. It is clear from the development of the exercise that $\mathcal{F}_\mathcal{S}$ does not admit subsheaves of positive degree, for their pullback via σ_0 would provide torsion free quotients of $\mathcal{F}_0$ whose first Chern class has negative intersection with H, i.e. $p < 0$. Thus $\mathcal{F}_\mathcal{S}$ is a stable bundle on $\mathcal{S}$. On the other hand, $\mathcal{F} \simeq \theta_1^* \mathcal{F}_\mathcal{S}$ is admissible only for special polarizations on $\mathcal{S}_\mathsf{T}$.

Suppose there exists a vertex $\beta \in \mathsf{T}^*$ with a unique successor. As there is no danger of confusion, we abuse the notation and denote it by β^+. Thus, $\mathcal{S}_\beta \simeq \tilde{\mathbb{P}}^2$, the blowup of $\mathbb{P}^2$ in a point. Recalling the discussion in section (1.5), there exists a surjective morphism $\pi_\beta : \mathcal{S}_\beta \to E_{\beta+}$ defining the ruling on $\tilde{\mathbb{P}}^2$. Moreover, E_β is a section of π_β, so it induces an isomorphism of schemes $E_\beta \simeq E_{\beta+}$. So we can extend π_β to a morphism of tree surfaces contracting the component $\mathcal{S}_\beta$:

$$\Pi_\beta : \mathcal{S}_\mathsf{T} \twoheadrightarrow \mathcal{S}'_{\mathsf{T}(\beta)},$$

where $\mathsf{T}^{(\beta)}$ is the tree obtained from T by removing the vertex β and connecting β^+ and β^- by an edge. Denote by $\mathcal{S}'_1$ and $\mathcal{S}'_0$, the subsurfaces of $\mathcal{S}'_{\mathsf{T}^{(\beta)}}$ associated respectively to the complete subtree with root β^+ and to its complement. The restrictions of the morphism Π_β provide

$$\mathcal{S}_{\mathsf{T}_{\beta+}} \xrightarrow{\sim} \mathcal{S}'_1, \quad \mathcal{S}_{\mathsf{T}^c_\beta} \setminus E_\beta \xrightarrow{\sim} \mathcal{S}'_0 \setminus E_{\beta+}, \quad \text{and } \Pi_\beta|_{\mathcal{S}_\beta} \simeq \pi_\beta : \mathcal{S}_\beta \twoheadrightarrow E_{\beta+}.$$

Note that on $\mathcal{S}'_{\mathsf{T}^{(\beta)}}$ there is a polarization $H_{\mathsf{T}^{(\beta)}}$ obtained in a natural way from H_{T} by omitting the coefficient δ_β.

Proposition 3.7.5. *Let β be a vertex in T as described above and $\mathcal{F}$ be an admissible bundle on $\mathcal{S}_{\mathsf{T}}$ with $n_\beta(\mathcal{F}) = 0$. Then there exists an admissible bundle $\mathcal{F}'$ on the tree surface $\mathcal{S}'_{\mathsf{T}^{(\beta)}}$ such that $\Pi^*\mathcal{F}' \in [\mathcal{F}]$.*

Proof. Assume $\mathcal{F}$ is the $\hat{\mu}$-stable representative of its equivalence class. The condition on the charge n_β implies that

$$\mathcal{F}|_{\mathcal{S}_\beta} \simeq \mathcal{O}_{\mathcal{S}_\beta}(\frac{e_\beta}{r}E_\beta - \frac{e_{\beta+}}{r}E_{\beta+})^{\oplus r}.$$

Suppose $\delta_{\beta+}$ is close enough to δ_β. Then $e_{\beta+} = e_\beta$ by (3.4). By taking account of an annoying twist, the following proof can be rewritten without this assumption.

According to lemma (1.5.1),

$$\Pi_{\beta*}\mathcal{O}_{\mathcal{S}_\beta}(\mathcal{F}|_{\mathcal{S}_\beta}) \simeq \mathcal{O}_{E_{\beta+}}(\frac{e_{\beta+}}{r})^{\oplus r}.$$

In particular, the pushforward of $\mathcal{F}$ by Π_β is a rank r vector bundle $\mathcal{F}'$ whose first Chern class is obtained from $c_1(\mathcal{F})$ by omitting the e_β. Thus $\mathcal{F}'$ is $H_{\mathsf{T}^{(\beta)}}$-compatible.

Let $\mathcal{F}' \twoheadrightarrow \mathcal{Q}'$ a proper quotient of pure dimension 2. If $\mathcal{Q}'$ is not supported on $\mathcal{S}_{\beta-}$, the pullback by Π_β provides an isomorphic quotient of $\mathcal{F}$. The $\hat{\mu}$-stability of the last one implies that $\mathcal{Q}'$ cannot destabilize $\mathcal{F}'$. So, suppose $rk(\mathcal{Q}'|_{\mathcal{S}_{\beta-}}) = r'$ is positive. Recalling diagram (2.3) of the previous chapter, there is an exact sequence

$$0 \to \mathcal{Q}' \to \mathcal{Q}'_0 \oplus \mathcal{Q}'_1 \to \mathcal{H}_{\mathcal{Q}'} \to 0,$$

where $\mathcal{Q}'_i$ is the quotient of $\mathcal{Q}'|_{\mathcal{S}_i}$ by the torsion, and $\mathcal{H}_{\mathcal{Q}'}$ is supported on $E_{\beta+}$ only. Thus we have

$$(\Pi^*_\beta\mathcal{Q}'_1)|_{\mathcal{S}_{\mathsf{T}_{\beta+}}} \simeq \mathcal{Q}'|_{\mathcal{S}_1}, \quad (\Pi^*_\beta\mathcal{Q}'_0)|_{\mathcal{S}_{\mathsf{T}^c_\beta}\setminus E_\beta} \simeq \mathcal{Q}'|_{\mathcal{S}_0\setminus E_{\beta+}},$$
$$\text{and} \quad (\Pi^*_\beta\mathcal{Q}'_0)|_{\mathcal{S}_\beta} \simeq \pi^*_\beta(\mathcal{Q}'_0|_{E_{\beta+}}).$$

From the last one we deduce that $(\Pi^*_\beta\mathcal{Q}'_0)|_{\mathcal{S}_\beta}$ is a rank r' vector bundle on $\mathcal{S}_\beta$ with both restrictions to the intersection lines isomorphic to $\mathcal{Q}'_0|_{E_{\beta+}}$. So, computing the Hilbert polynomial, we get $P_{\Pi^*_\beta \mathcal{Q}'_0}(t) = P_{\mathcal{Q}'_0}(t)$. Denote by $\mathcal{Q}$ the kernel of $\Pi^*_\beta \mathcal{Q}'_0 \oplus \mathcal{Q}'_1 \twoheadrightarrow \mathcal{H}_{\mathcal{Q}'}$. By (2.2.3), it is of pure dimension 2 and we have the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}|_{\mathcal{S}_{\Upsilon^c_{\beta+}}} \oplus \mathcal{F}|_{\mathcal{S}_{\Upsilon_{\beta+}}} & \longrightarrow & \mathcal{F}|_{E_{\beta+}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{Q} & \longrightarrow & \Pi^*_\beta \mathcal{Q}'_0 \oplus \mathcal{Q}'_1 & \longrightarrow & \mathcal{H}_{\mathcal{Q}'} & \longrightarrow & 0 \\
 & & \downarrow\!\!\downarrow & & \downarrow & & \downarrow & & \\
 & & \mathcal{T} & & 0 & & 0 & &
\end{array}$$

By the snake lemma, $\mathcal{T}$ is a sheaf supported on $E_{\beta+}$ only. So, let $\tilde{\mathcal{Q}}$ be the kernel $\mathcal{Q} \to \mathcal{T}$. The $\hat{\mu}$-stability of $\mathcal{F}$ implies

$$\hat{\mu}_{\mathcal{F}'} = \hat{\mu}_{\mathcal{F}} < \hat{\mu}_{\tilde{\mathcal{Q}}} \leq \hat{\mu}_{\mathcal{Q}}.$$

By the above observation on the Hilbert polynomial of $\Pi^*_\beta \mathcal{Q}'_0$, the slope of $\mathcal{Q}$ coincides with the slope of $\mathcal{Q}'$. So $\mathcal{F}'$ is $\hat{\mu}$-stable. □

The last propositions highlight two situations where the study of admissible bundles can be reduced to bundles on simpler tree surfaces. To exclude these situations, we introduce the following definition.

Definition 11. An admissible bundle is said to have *good charges* if the vanishing of some $n_\alpha(\mathcal{F})$ implies that α has at least two successors.

Clearly, the above propositions do not apply for admissible bundles with good charges.

Remark. Here we see the first analogy between our theory and results of [21]. Indeed, the bundles considered in this article have second Chern classes defined by a *weighted tree*. The weights are integers marking the vertices of the tree and they are subject to the same conditions as those imposed on good charges by definition (10) and lemma (3.7.1).

Chapter 4

Deformations of stable bundles

In the previous chapter we studied bundles on tree surfaces. By definition, a tree surface $\mathcal{S}_{\mathsf{T}}$ is the central fiber of a flat family $\mathcal{X}$ over $\mathbb{A}^1$. Thus, it can be seen as a deformation of a smooth surface $\mathcal{S}$. It is natural to ask under which assumptions a vector bundle on a tree surface $\mathcal{S}_{\mathsf{T}}$ is a degeneration of vector bundles on $\mathcal{S}$. In particular, we are interested in describing limits of stable vector bundles. Similar problems have been studied by D. Gieseker and I. Morrison in [10] and [12] to describe degenerations of stable bundles on curves .

We first show the trivial case of line bundles, where a general answer is easily obtained. The second section briefly recalls some tools of deformation theory. In particular, we sketch the proof of a criterion for the smoothness of the relative moduli space at a stable point. In the last section we apply this theory to admissible bundles on tree surfaces.

4.1 Limits of line bundles

Suppose $\mathcal{S}_{\mathsf{T}}$ is a tree surface over the smooth surface $\mathcal{S}$. Denote by $\bar{\sigma} : \mathcal{X} \to \mathcal{S} \times \mathbb{A}^1$ the composite of blowups defining $\mathcal{S}_{\mathsf{T}}$. Recall that $\mathcal{X}$ is a flat family over $\mathbb{A}^1$, with constant fiber $\mathcal{S}$ everywhere except for the central fiber $\mathcal{X}_0$, which is $\mathcal{S}_T$.

Focusing on line bundles, we do not need the machinery of deformation theory developed in the next section. Indeed, global deformations of line bundles are easily constructed by quite elementary means.

Consider the Picard group of $\mathcal{X}$:

$$\mathrm{Pic}(\mathcal{X}) \simeq \mathrm{Pic}(\mathcal{S}) \times \mathbb{A}^1 \oplus \langle \{\mathcal{S}_\alpha\}_{\alpha \in T} \rangle_{\mathbb{Z}},$$

where the intersection theory is determined by the following rules:

$$\mathcal{S}_\alpha.\mathcal{S}_\beta = \begin{cases} -E_\alpha - \sum_{\gamma\in\alpha^+} E_\gamma & \text{if } \alpha=\beta\neq 0 \\ E_\alpha & \text{if } \alpha\in\beta^+ \end{cases} \qquad (\mathcal{S}_0)^2 = -\sum_{\gamma\in 0^+} E_\gamma$$

$$(D\times\mathbb{A}^1).(D'\times\mathbb{A}^1) = (D.D')\times\mathbb{A}^1 \qquad\qquad (D\times\mathbb{A}^1).\mathcal{S}_0 = D$$

Proposition 4.1.1. *Every line bundle on $\mathcal{X}_0$ extends to a line bundle on $\mathcal{X}$.*

Proof. Consider a line bundle $\mathcal{L}$ on $\mathcal{X}_0$. Using the notation introduced in section (3.2), there exists a divisor $D\in\operatorname{Pic}(\mathcal{S})$ and a list of integer $a_{\mathsf{T}} = \{a_\alpha\}_{\alpha\in\mathsf{T}^*}$ such that $\mathcal{L}\simeq\mathcal{O}_{\mathcal{S}_{\mathsf{T}}}(D,a_{\mathsf{T}})$.

Choose $\{b_\alpha\}_{\alpha\in\mathsf{T}}$ such that $a_\alpha = \beta_\alpha - \beta_{\alpha^-}$. By checking on each component $\mathcal{S}_\alpha$, one can see that the line bundle on $\mathcal{X}$ defined by

$$\mathcal{O}_{\mathcal{X}}(D\times\mathbb{A}^1 + \sum_{\alpha\in\mathsf{T}} b_\alpha\mathcal{S}_\alpha),$$

restricts to $\mathcal{L}$ on $\mathcal{X}_0$. □

Remark. Clearly the choice of the extension of $\mathcal{L}$ is not unique and depends on the choice of b_0. Equivalently we can say that it is determined up to a twist by multiples of a fiber, that is by sheaves of the form $\mathcal{O}_{\mathcal{X}}(m\sum_{i\in T}\mathcal{S}_i)$.

4.2 Smooth points of the moduli space

The problem we deal with in this section is to understand when a bundle on a fiber of a family can be deformed to neighboring fibers. Restricting ourselves to stable bundles, we can interpret this question as the one about curves in the relative moduli space.

Denote by (Set) the category of sets and by $(\mathrm{Sch}/\mathbb{A}^1)^o$ the category of schemes over $\mathbb{A}^1$ with reversed arrows. Let P be a polynomial. It is well known that the moduli functor

$$\begin{aligned}\mathcal{M}:\ (\mathrm{Sch}/\mathbb{A}^1)^o &\longrightarrow (\mathrm{Set})\\ T\to &\left\{\begin{array}{c} T\text{-flat families of semistable bundles}\\ \text{on } \mathcal{X}_T \text{ with Hilbert polynomial } P\end{array}\right\}\Big/_{\text{S-equivlaence}}\end{aligned}$$

is universally corepresented by a projective moduli space M. Furthermore the moduli space is a family over $\mathbb{A}^1$, and it is well behaved on fibers. Indeed, for every geometric point t in $\mathbb{A}^1$, the fiber M_t is isomorphic to the absolute moduli spaces of semistable bundles over $\mathcal{X}_t$ with Hilbert polynomial P. Moreover, there exists an open subscheme M^s of the moduli space whose points represent isomorphism classes of stable bundles (for a precise statement see Theorem 4.3.7 in [16]).

Let $\mathcal{F}$ be a stable bundle on the zero fiber $\mathcal{X}_0$ with Hilbert polynomial P. The isomorphism class of $[\mathcal{F}]$ corresponds to a closed point in the zero fiber of $\mathcal{M}^s$. Thus, extensions of $\mathcal{F}$ to bundles on a neighborhood of $\mathcal{X}_{\{0\}}$ represent curves on $\mathcal{M}^s$ passing through the closed point $[\mathcal{F}]$. In this sense, our problem is related to the study of the local structure of the relative moduli space. The following result, first proved by Maruyama ([22], proposition 6.7), provides a smoothness criterion for points in M^s.

Proposition 4.2.1. *Let $\mathcal{F}$ be a stable bundle with Hilbert polynomial P on $\mathcal{X}_{\{0\}}$. Then $\mathrm{Ext}^2_{\mathcal{O}_{\mathcal{X}_{\{0\}}}}(\mathcal{F},\mathcal{F})$ is sufficient for the smoothness of M^s at the point corresponding to $\mathcal{F}$.*

Roughly speaking, the idea behind the proof is that $\mathcal{F}$ represents a smooth point of M^s if and only if all its infinitesimal deformations can be extended to deformations of higher order. This can be formalized in terms of Artin rings and small extensions.

Denote by $\Lambda = \mathcal{O}_{\mathbb{A}^1,\{0\}}$, and by ($\Lambda$-Artin/$\mathbb{C}$) the category of Artin local Λ-algebras with residue field $\mathbb{C}$. Let A be in (Λ-Artin/$\mathbb{C}$). The homomorphism of rings

$$\mathbb{C}[t] \to \Lambda \to A\ ,$$

represents the spectrum of A as an object of $(\mathrm{Sch}/\mathbb{A}^1)$. In particular, it is a zero-dimensional scheme, whose unique closed point is mapped to $\{0\}$. Thus, passing to the spectrum, we can think about (Λ-Artin/$\mathbb{C}$) as a full subcategory of $(\mathrm{Sch}/\mathbb{A}^1)^o$. Define by

$$\mathcal{D}_{[\mathcal{F}]} : (\Lambda\text{-Artin}/\mathbb{C}) \to (\mathrm{Set}),$$

the covariant functor which associates to A the isomorphism classes of A-flat families of sheaves on $\mathcal{X}_A = \mathcal{X} \times_{\mathbb{A}^1} \mathrm{Spec}(A)$ with restriction $\mathcal{F}$ at the closed point of $\mathrm{Spec}(A)$.

Remark. The functor $\mathcal{D}$ is a sort of restriction of the moduli functor $\mathcal{M}^s$ to the subcategory (Λ-Artin/$\mathbb{C}$) with an extra condition on the closed point. Indeed, stability is an open property and the Hilbert polynomial is locally constant in flat families. Thus, an element $\mathcal{F}_A \in \mathcal{D}(A)$ is a family of stable bundles with fixed Hilbert polynomial on the fiber of the morphism $\mathcal{X}_A \to \mathrm{Spec}(A)$. So, $\mathcal{D}(A) \subset \mathcal{M}^s(\mathrm{Spec}(A))$ consists of all the families whose restriction to the zero fiber is $\mathcal{F}$.

Combining Theorem 4.5.1 in [16] and exercise 15.6 (b) in [15], we have the following result on the smoothness at $[\mathcal{F}]$.

Proposition 4.2.2. *The completion of the local ring of $[\mathcal{F}]$ in M^s is regular if and only if, for every surjective map $A' \to A$ in* (Λ-Artin/$\mathbb{C}$)*, the induced map $\mathcal{D}_{[\mathcal{F}]}(A') \to \mathcal{D}_{[\mathcal{F}]}(A)$ is surjective.*

The above property is usually called *unobstructedness* of the functor $\mathcal{D}_{[\mathcal{F}]}$. It easy to see that to prove unobstructedness it is sufficient to check *small extensions*, that are surjections

$$0 \to (\epsilon) \to A' \xrightarrow{\pi} A \to 0, \tag{4.1}$$

where the kernel is generated by a square zero element $\epsilon \in A'$ annihilated by the maximal ideal of A'.

A small extension as in (4.1) defines a closed inclusion $\mathcal{X}_A \hookrightarrow \mathcal{X}_{A'}$. The associated map of structure sheaves has the ideal sheaf of $(\epsilon) \otimes \mathcal{O}_{\mathcal{X}_{A'}}$ as kernel. Thus we obtain the following exact sequence:

$$0 \to \mathcal{O}_{\mathcal{X}_0} \xrightarrow{\epsilon} \mathcal{O}_{\mathcal{X}_{A'}} \to \mathcal{O}_{\mathcal{X}_A} \to 0. \tag{4.2}$$

We see that, when we extend a deformation of $\mathcal{F}$ over a small extension of the base ring, the kernel depends only on $\mathcal{F}$ itself.

From now on we will consider small extensions as in (4.1). Suppose we are given a deformation $\mathcal{F}_A$ of $\mathcal{F}$ on $\mathcal{X}_A$. We look for a bundle $\mathcal{F}_{A'}$ on $\mathcal{X}_{A'}$ isomorphic to $\mathcal{F}_A$ when restricted to $\mathcal{X}_A$.

Lemma 4.2.3. *Suppose an extension $\mathcal{F}_{A'}$ of $\mathcal{F}_A$ exists, then the group of automorphisms of $\mathcal{F}_{A'}$ inducing the identity on $\mathcal{F}_A$ is isomorphic to the group of endomorphisms of $\mathcal{F}$:*

$$(\mathrm{Aut}(\mathcal{F}_{A'}/\mathcal{F}_A), \circ) \simeq (\mathrm{End}(\mathcal{F}), +).$$

Proof. Let Ψ be such an automorphism. The endomorphism $\Psi - \mathrm{Id}_{\mathcal{F}_{A'}}$ vanishes when restricted to $\mathcal{X}_A$. Thus, we construct a 1-to-1 correspondence

$$\mathrm{Aut}(\mathcal{F}_{A'}/\mathcal{F}_A) \xleftrightarrow{1:1} \mathrm{Hom}(\mathcal{F}_{A'}, \mathcal{F}_{A'} \otimes (\epsilon)) \simeq \mathrm{End}(\mathcal{F}).$$

In particular, the composition of two homomorphism in the group in the middle vanishes. Thus, given another element $\Psi' \in \mathrm{Aut}(\mathcal{F}_{A'}/\mathcal{F}_A)$,

$$0 = (\Psi - \mathrm{Id}_{\mathcal{F}_{A'}}) \circ (\Psi' - \mathrm{Id}_{\mathcal{F}_{A'}}) = \Psi \circ \Psi' + \mathrm{Id}_{\mathcal{F}_{A'}} - (\Psi + \Psi').$$

Hence $(\Psi - \mathrm{Id}_{\mathcal{F}_{A'}}) + (\Psi' - \mathrm{Id}_{\mathcal{F}_{A'}}) = (\Psi \circ \Psi' - \mathrm{Id}_{\mathcal{F}_{A'}})$ and the above bijection is an isomorphism of groups. □

Remark. Consider an affine open subset $\mathcal{U} = \mathrm{Spec}(B) \subset \mathcal{X}$. By flatness, tensoring (4.1) by B gives rise to an exact sequence of $\mathbb{C}[t]$-algebras,

$$0 \to B_0 \xrightarrow{\epsilon} B_{A'} \xrightarrow{\pi} B_A \to 0,$$

where $B_0 = B \otimes_{\mathbb{C}[t]} \mathbb{C}$ and $B_A = B \otimes_{\mathbb{C}[t]} A$. As a sequence of complex vector spaces it admits a splitting $\rho : B_A \to B_{A'}$. So, we can decompose an element x in $B_{A'}$ in a unique way as the sum $\rho(x_A) + \epsilon x_0$, where $x_A = \pi(x)$. The same extends to local sections of free modules and their endomorphisms groups. So, we have

$$\mathrm{End}(B_{A'}^{\oplus r}) \stackrel{1:1}{\longleftrightarrow} \mathrm{End}(B_A^{\oplus r}) \oplus \epsilon \mathrm{End}(B_0^{\oplus r}).$$

In this way, ρ defines a lifting of endomorphisms of free modules. In particular, as ρ preserves units, this lifting preserves automorphisms. But ρ is a morphism of complex vector spaces only and does not respect on multiplication. Thus, the lifting does not preserve composition.

Proposition 4.2.4. *There is a constructive way to define an element $\mathfrak{o}(\mathcal{F}_A) \in \mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$ such that its vanishing is equivalent to the existence of an extension $\mathcal{F}_{A'}$ to $\mathcal{X}_{A'}$.*

Proof. By the local freeness of $\mathcal{F}_A$, there exists a covering $\{\mathcal{U}_\alpha\}$ of $\mathcal{X}_0$ by affine open sets, on which the restriction of $\mathcal{F}_A$ are free modules. Hence, we have a family

$$\mathcal{F}_\alpha := \mathcal{O}_{\mathcal{X}_{A'} \cup \mathcal{U}_\alpha}^{\oplus r} \twoheadrightarrow \mathcal{O}_{\mathcal{X}_A \cup \mathcal{U}_\alpha}^{\oplus r} \simeq \mathcal{F}_A|_{\mathcal{U}_\alpha},$$

of local extensions. Below we study the problem of patching them together, and $\mathfrak{o}(\mathcal{F}_A)$ appears as an obstruction to the feasibility of such patching.

We will label multiple intersections of the $\mathcal{U}_\alpha$'s by multi-indices. Two trivializations of $\mathcal{F}_A$ on an intersection $\mathcal{U}_{\alpha\beta}$ provide a transition function $\eta_{\alpha\beta}$, i.e. a $r \times r$ matrix with entries in $\mathcal{O}_{\mathcal{X}_A}(\mathcal{U}_{\alpha\beta})$. As shown in the above remark, for every pair (α, β) we can lift $\eta_{\alpha\beta}$ to an isomorphism

$$\tilde{\eta}_{\alpha\beta} : \mathcal{F}_\beta(\mathcal{U}_{\alpha\beta}) \to \mathcal{F}_\alpha(\mathcal{U}_{\alpha\beta}).$$

By lemma (4.2.3), these extensions are defined up to elements of $\mathrm{End}(\mathcal{F}(\mathcal{U}_{\alpha\beta}))$. Thus, correcting them by the appropriate endomorphisms, we can assume the family $\{\tilde{\eta}_{\alpha\beta}\}$ skew-symmetric, i.e. such that $\tilde{\eta}_{\alpha\beta} = \tilde{\eta}_{\beta\alpha}^{-1}$ and $\tilde{\eta}_{\alpha\alpha} = \mathrm{Id}$. We check the consistency of these maps on a triple intersection. Denote by

$$\Psi_{\alpha\beta\gamma} = \mathrm{Id} - \tilde{\eta}_{\alpha\gamma}\tilde{\eta}_{\gamma\beta}\tilde{\eta}_{\beta\alpha} \in \mathrm{End}(\mathcal{F}_\alpha). \tag{4.3}$$

As it vanishes on $\mathcal{F}_A$, it defines an endomorphism of $\mathcal{F}$ on $\mathcal{U}_{\alpha\beta\gamma}$. In other words, the family $\Psi = \{\Psi_{\alpha\beta\gamma}\}$ is a Čech 2-cochain with values in $\mathcal{E}nd(\mathcal{F})$ for the covering $\{\mathcal{U}_\alpha\}$.

Let us check that is a 2-cocycle. The Čech differential of Ψ is given by

$$\delta(\Psi_{\alpha\beta\gamma})_{\alpha\beta\gamma\delta} = \Psi_{\beta\gamma\delta} - \Psi_{\alpha\gamma\delta} + \Psi_{\alpha\beta\delta} - \Psi_{\alpha\beta\gamma}$$

To apply formula (4.3), we have to note that it gives $\Psi_{\alpha\gamma\delta}$, $\Psi_{\alpha\beta\delta}$, $\Psi_{\alpha\beta\gamma}$ as endomorphisms of $\mathcal{F}_\alpha$, and $\Psi_{\beta\gamma\delta}$ as an endomorphism of $\mathcal{F}_\beta$, so we conjugate the latter by $\tilde{\eta}_{\alpha\beta}$ in order to transform it into an endomorphism of $\mathcal{F}_\alpha$. We thus obtain:

$$\delta(\Psi_{\alpha\beta\gamma})_{\alpha\beta\gamma\delta} = \mathrm{Id} - (\tilde{\eta}_{\alpha\beta}\tilde{\eta}_{\beta\delta}\tilde{\eta}_{\delta\gamma}\tilde{\eta}_{\gamma\beta}\tilde{\eta}_{\beta\alpha}) \circ (\tilde{\eta}_{\alpha\delta}\tilde{\eta}_{\delta\gamma}\tilde{\eta}_{\gamma\alpha})^{-1} \circ$$
$$\circ (\tilde{\eta}_{\alpha\delta}\tilde{\eta}_{\delta\beta}\tilde{\eta}_{\beta\alpha}) \circ (\tilde{\eta}_{\alpha\gamma}\tilde{\eta}_{\gamma\beta}\tilde{\eta}_{\beta\alpha})^{-1}.$$

The elements in parentheses are automorphisms of $\mathcal{F}_{A'}$ extending the identity on $\mathcal{F}_A$, hence they commute with each other, and the differential vanishes. Thus Ψ defines a class $[\Psi] \in H^2(\mathcal{X}_0, \mathcal{E}nd(\mathcal{F}))$. We set Ψ to be $\mathfrak{o}(\mathcal{F}_A)$.

Suppose $[\Psi]$ vanishes. Then $\Psi_{\alpha\beta\gamma} = \delta(\xi_{\alpha\beta})$, where $\xi_{\alpha\beta} \in \mathrm{End}(\mathcal{F}(\mathcal{U}_{\alpha\beta}))$. Let

$$\tilde{\eta}'_{\alpha\beta} = \tilde{\eta}_{\alpha\beta} + \epsilon\xi_{\alpha\beta}.$$

With this adjustment, the functions $\tilde{\eta}'_{\alpha\beta}$ verify the coboundary condition. Patching together the local extensions we get the required bundle on $\mathcal{X}_{n+1}$. □

As proved in the absolute case, first by Mukai in [24], and then by Artamkin in more generality [2], proposition (4.2.1) can be improved.

Let

$$Tr : \mathrm{Ext}^2(\mathcal{F}, \mathcal{F}) \to H^2(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0})$$

be the trace map. It is a surjective morphism and its kernel is denoted by $\mathrm{Ext}^2_0(\mathcal{F}, \mathcal{F})$. The following proposition, relating the trace of $\mathfrak{o}(\mathcal{F}_A)$ and the obstruction to extend $\det(\mathcal{F}_A)$, improves the previous result.

Proposition 4.2.5. *Consider the line bundle* $\det(\mathcal{F}_A)$. *Then the obstructions to extend* $\mathcal{F}_A$ *and* $\det(\mathcal{F}_A)$ *are related by the equality:*

$$\mathfrak{o}(\det(\mathcal{F}_A)) = Tr(\mathfrak{o}(\mathcal{F}_A)).$$

Proof. Using the above notation, the functions $\det(\tilde{\eta}_{\alpha\beta})$ provide a lifting of the transition functions $\det(\eta_{\alpha\beta})$ of $\det(\mathcal{F}_A)$. Hence,

$$\mathfrak{o}(\det(\mathcal{F}_A)) = [\{1 - \det(\tilde{\eta}_{\alpha\gamma})\det(\tilde{\eta}_{\gamma\beta})\det(\tilde{\eta}_{\beta\alpha})\}_{\alpha\beta\gamma}] \in H^2(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}).$$

By definition, $\Psi_{\alpha\beta\gamma}$ are given by matrices with entries in an ideal with square zero. So,

$$\det(\tilde{\eta}_{\alpha\gamma}\tilde{\eta}_{\gamma\beta}\tilde{\eta}_{\beta\alpha}) = \det(\mathrm{Id} - \Psi_{\alpha\beta\gamma}) = 1 - Tr(\Psi_{\alpha\beta\gamma}).$$

This implies for the obstructions:

$$\mathfrak{o}(\det(\mathcal{F}_A)) = [\{Tr(\Psi_{\alpha\beta\gamma})\}] = Tr(\mathfrak{o}(\mathcal{F}_A)).$$

□

We conclude this section by an application to the deformation problem for vector bundles on a tree surface $\mathcal{S}_\mathsf{T}$, viewed as the central fiber of the family $\mathcal{X}$ over $\mathbb{A}^1$ (notation from section 4.1):

Theorem 4.2.6. *Let* $\mathcal{F}$ *be a stable bundle on* $\mathcal{S}_\mathsf{T}$. *Then* $\mathrm{Ext}^2_0(\mathcal{F}, \mathcal{F}) = 0$ *is a sufficient condition for the existence of a family of stable bundles on* $\mathcal{S}$ *with* $c_1 = c_\mathcal{S}(\mathcal{F})$ *and* $c_2 = n(\mathcal{F})$ *degenerating to* $\mathcal{F}$.

Proof. By proposition (4.1.1), deformation of $\det(\mathcal{F})$ are unobstructed, hence the obstructions to extend deformations of $\mathcal{F}$ lie in the traceless part of Ext^2 by proposition (4.2.5).

The claim on the second Chern classes follows by proposition (4.1.1) and by the computation of the Hilbert polynomial of $\mathcal{F}$. □

4.3 Deformations of admissible bundles

In this section we focus our attention on admissible bundles. Providing a direct way to compute their second Ext group, the following lemma provides more evidence of their peculiarity.

Lemma 4.3.1. *Let $\mathcal{F}$ be an admissible bundle on $\mathcal{S}_\mathsf{T}$, then*

$$\mathrm{Ext}^2_{\mathcal{S}_\mathsf{T}}(\mathcal{F}, \mathcal{F}) \simeq \mathrm{Ext}^2_{\mathcal{S}}(\mathcal{F}_\mathcal{S}, \mathcal{F}_\mathcal{S}).$$

Proof. Changing an admissible bundle inside its equivalence class does not change the Ext group we want to compute. Indeed, the Ext groups are invariant under tensoring both arguments by the same line bundle. Thus we assume $\mathcal{F}$ to be the representative of its equivalence class with trivial restrictions to the intersections.

Applying the exact functor $\mathcal{H}om_{\mathcal{O}_{\mathcal{S}_\mathsf{T}}}(\mathcal{F}, -)$ to the exact sequence (3.1), we obtain

$$0 \to \mathcal{E}nd(\mathcal{F}) \to \bigoplus_{\alpha \in \mathsf{T}} \mathcal{E}nd(\mathcal{F}|_{\mathcal{S}_\alpha}) \to \bigoplus_{\gamma \in \mathsf{T}^*} \mathcal{E}nd(\mathcal{F}|_{E_\gamma}) \to 0. \tag{4.4}$$

By the triviality of $\mathcal{F}$ on the E_γ's, the sheaf on the right has vanishing higher cohomology groups. Furthermore, for all $\alpha \in \mathsf{T}^*$, corollary (1.3.11) relates the cohomology groups of the sheaf $\mathcal{E}nd(\mathcal{F}|_{\mathcal{S}_\alpha})$ to those of its pushforward via the blowdown σ_α. By Serre duality,

$$\mathrm{Ext}^2_{\mathcal{S}_\alpha}(\mathcal{F}|_{\mathcal{S}_\alpha}, \mathcal{F}|_{\mathcal{S}_\alpha}) \simeq H^0(\mathbb{P}^2, \mathcal{E}nd(\sigma_{\alpha *}\mathcal{F}|_{\mathcal{S}_\alpha}) \otimes \mathcal{O}_{\mathbb{P}^2}(-3)).$$

Note that, by proposition (1.3.8), $\sigma_{\alpha *}\mathcal{F}|_{\mathcal{S}_\alpha}$ is a vector bundle on $\mathbb{P}^2$. As it is trivial on the line E_α, it is $\hat{\mu}$-semistable, for a destabilizing quotient of $\sigma_{\alpha *}\mathcal{F}|_{\mathcal{S}_\alpha}$ would have negative degree and its restriction to E_α would be a destabilizing quotient of the trivial bundle. In particular, the sheaf $\mathcal{E}nd(\sigma_{\alpha *}\mathcal{F}|_{\mathcal{S}_\alpha})$ is a $\hat{\mu}$-semistable ([16], corollary 3.2.10) vector bundle with trivial first Chern class. As the dualizing sheaf on $\mathbb{P}^2$ has negative degree, the above space of global section vanishes.

The long exact sequence in cohomology associated to (4.4) provides an isomorphism

$$\mathrm{Ext}^2_{\mathcal{S}_\mathsf{T}}(\mathcal{F}, \mathcal{F}) \simeq \mathrm{Ext}^2_{\mathcal{S}_0}(\mathcal{F}|_{\mathcal{S}_0}, \mathcal{F}|_{\mathcal{S}_0}).$$

The result follows by applying again corollary (1.3.11). □

The above lemma allows us, in specific situations, to prove the smoothness of the relative moduli space in points representing isomorphism classes of admissible bundles on a tree surface. In the sequel we present two applications.

Theorem 4.3.2. *Let $(\mathcal{S}, H)$ be a polarized surface such that $K_{\mathcal{S}}.H < 0$, and $\mathcal{S}_{\mathsf{T}}$ a tree surface over $\mathcal{S}$. If the polarization H_{T} on $\mathcal{S}_{\mathsf{T}}$ lies in the chamber defined in Theorem (3.6.1), every admissible bundle of rank r on $\mathcal{S}_{\mathsf{T}}$ is a deformation of stable bundles on $\mathcal{S}$.*

Proof. Let $\mathcal{F}$ be an admissible bundle on a tree surface over $\mathcal{S}$. By Theorem (4.2.6) and lemma (4.3.1), to prove the claim, it is enough to show the vanishing of

$$\mathrm{Ext}^2_{\mathcal{S}}(\mathcal{F}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}}) \simeq H^0(\mathcal{S}, \mathcal{E}nd(\mathcal{F}_{\mathcal{S}}) \otimes \mathcal{O}_{\mathcal{S}}(K_{\mathcal{S}})).$$

The hypothesis of Theorem (3.6.1) is verified, so both $\mathcal{F}_{\mathcal{S}}$ and $\mathcal{E}nd(\mathcal{F}_{\mathcal{S}})$ are $\hat{\mu}$-semistable bundles on $\mathcal{S}$. As shown in the proof of lemma (4.3.1) in the case of $\mathbb{P}^2$, the vanishing of the above group is assured by the positive degree of the anticanonical of $\mathcal{S}$. □

Theorem 4.3.3. *Let $(\mathcal{S}, H)$ be a polarized K3 surface and $(\mathcal{S}_{\mathsf{T}}, H_{\mathsf{T}})$ a polarized tree surface over $\mathcal{S}$. An admissible bundle $\mathcal{F}$ of rank r on $\mathcal{S}_{\mathsf{T}}$ such that*

1. *r and H_{T} satisfy the hypothesis of Theorem (3.6.1),*
2. *r and $\deg_H(c_{\mathcal{S}})$ are coprime,*

is a deformation of stable bundles on $\mathcal{S}$.

Proof. By definition K3 surfaces have trivial canonical bundle. So,

$$\mathrm{Ext}^2_{\mathcal{S}_{\mathsf{T}}}(\mathcal{F}, \mathcal{F}) \simeq \mathrm{End}(\mathcal{F}_{\mathcal{S}}).$$

Recall that $\mathcal{F}_{\mathcal{S}}$ is a $\hat{\mu}$-semistable bundle of rank r on $\mathcal{S}$ with $c_1 = c_{\mathcal{S}}$. By applying lemma (1.2.2), $\mathcal{F}_{\mathcal{S}}$ is $\hat{\mu}$-stable and thus simple. Consider the trace morphism,

$$\mathrm{Ext}^2_{\mathcal{S}_{\mathsf{T}}}(\mathcal{F}, \mathcal{F}) \simeq \mathbb{C} \xrightarrow{Tr} H^2(\mathcal{S}_{\mathsf{T}}, \mathcal{O}_{\mathcal{S}_{\mathsf{T}}}) \simeq H^2(\mathcal{S}, \mathcal{O}_{\mathcal{S}}) \simeq \mathbb{C}.$$

Clearly the kernel $\mathrm{Ext}^2_0(\mathcal{F}, \mathcal{F})$ vanishes and the claim is proven. □

Remark. In the proof of lemma (4.3.1), we observed that admissible bundles in one equivalence class $[\mathcal{F}]$ have isomorphic second Ext groups. Hence, by Theorem (4.2.6), if one representative has a global deformation, so do all the bundles in $[\mathcal{F}]$. This is clear from the following fact: Suppose we have a family of vector bundles on $\mathcal{X}$ with admissible central fiber $\mathcal{F}$. Twisting it by $\mathcal{O}_{\mathcal{X}}(\sum_{\alpha \in \mathsf{T}^*} b_\alpha \mathcal{S}_\alpha)$ affects just the zero fiber and results in a bundle from the same class $[\mathcal{F}]$. In this way, all the admissible bundles in $[\mathcal{F}]$ can be found as limits of the same family of stable bundle on $\mathcal{S} \times \mathbb{A}^1 \setminus \{0\}$.

Chapter 5

Comparison between admissible and limit tree-bundles

In this chapter we deal with the problem that served as motivation for this thesis: Try to describe the boundary of the bubble tree compactification of the moduli space of vector bundles on a surface via a notion of semistability on tree surfaces. The bubble tree compactification was constructed by Markushevich, Tikhomirov and Trautmann in [21]. While in the classical compactification limits of stable vector bundles on a surface $\mathcal{S}$ are semistable torsion free sheaves, the bubble tree compactification describes degenerations via isomorphism classes of *limit tree bundles*. These are pairs $(\mathcal{F}_{\mathsf{T}}, \mathcal{S}_{\mathsf{T}})$ of a tree surface over $\mathcal{S}$ and a vector bundle on it. Both the tree T and the bundle $\mathcal{F}_{\mathsf{T}}$ should satisfy quite an extensive set of conditions. As we do not pretend to give a complete answer to the motivating question, we refer to the above mentioned article for details on the construction of this moduli space.

Focusing our attention on the projective plane, in the first section we describe the bubble tree compactification of the moduli space of rank 2 vector bundles on $\mathbb{P}^2$ with trivial first Chern class and $c_2 = 2$. The rest of the chapter is devoted to comparing the limits found in this specific example to admissible bundles on tree surfaces over $\mathbb{P}^2$ with trivial $c_{\mathcal{S}}$ and good charging of total charge $n = 2$.

5.1 The bubble tree compactification of $M^s_{\mathbb{P}^2}(2;0,2)$

Before presenting the results of Markushevich, Tikhomirov and Trautmann, we recall briefly some properties of $M_{\mathbb{P}^2}(2;0,2)$, the moduli space of Gieseker semistable sheaves of rank 2 on $\mathbb{P}^2$ with $c_1(\mathcal{F}) = 0$ and $c_2(\mathcal{F}) = 2$.

Consider a 3-dimensional complex space V and let $\mathcal{S}$ be $\mathbb{P}^2 \simeq \mathbb{P}(V)$. Let $\mathcal{G}$ be a representative sheaf of an S-equivalence class in $M_{\mathbb{P}^2}(2;0,2)$. It is well known (see [26], [23]), that for a general line ℓ in $\mathbb{P}^2$, $\mathcal{G}|_\ell$ is trivial. The curve defined by

$$\mathcal{C}(\mathcal{G}) := \{\ell \in \mathbb{P}(V\check{}) \mid \mathcal{G}|_\ell \not\simeq \mathcal{O}_\ell^{\oplus 2}\},$$

is called the curve of jumping lines. In our setting, $\mathcal{C}(\mathcal{G})$ is a conic in the dual plane.

Proposition 5.1.1. *Let $\mathcal{G}$ be a Gieseker semistable sheaf of rank 2 on $\mathbb{P}^2$ with $c_1(\mathcal{G}) = 0$ and $c_2(\mathcal{G}) = 2$. The following are equivalent:*

(a) $\mathcal{C}(\mathcal{G})$ is non-singular;

(b) $\mathcal{G}$ is $\hat{\mu}$-stable;

(c) $\mathcal{G}$ is locally free;

(d) There exists a non-trivial extension

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{G} \to \mathcal{I}_Z(1) \to 0,$$

where Z consists of three non-collinear points.

Furthermore, an S-equivalence class of strictly semistable sheaves is represented by a sheaf of the form $\mathcal{I}_{P_0} \oplus \mathcal{I}_{P_1}$, where P_0 and P_1 are two points in $\mathbb{P}^2$. So, it is naturally associated to a pair of linear equation in $\mathbb{P}(V\check{})$ defined by the vanishing at these points.

Therefore, the stable locus $M^s_{\mathbb{P}^2}(2;0,2)$ of the moduli space is isomorphic to the open set of non-singular conics in the dual plane. Its compactification via semistable sheaves is isomorphic to $\mathbb{P}(S^2V)$ and the boundary is defined by the hypersurface of degenerate conics.

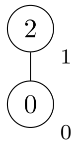
Now we can reproduce the description of M_2 provided in [21], where M_2 stands for the bubble tree compactification of $M^s_{\mathbb{P}^2}(2;0,2)$. The objects parametrized by M_2 are isomorphism classes of pairs $(\mathcal{S}_\mathsf{T}, \mathcal{F})$, where $\mathcal{S}_\mathsf{T}$ is a tree surface with base $\mathbb{P}^2$, and $\mathcal{F}$ is a rank 2 vector bundle on $\mathcal{S}_\mathsf{T}$ of total charge 2 satisfying specific conditions on the restrictions on components (see [21] for the precise definition). Two pairs $(\mathcal{S}_\mathsf{T}, \mathcal{F})$ and $(\mathcal{S}'_\mathsf{T}, \mathcal{F}')$ are isomorphic if there exists an isomorphism $\phi : \mathcal{S}_\mathsf{T} \to \mathcal{S}'_\mathsf{T}$ such that $\mathcal{F} \simeq \phi^*\mathcal{F}'$.

Theorem 5.1.2 (Theorem 8.5, [21])**.** *The moduli space M_2 of limit tree-bundles with base $\mathbb{P}^2$ and total charge $n = 2$ is isomorphic to $\widetilde{\mathbb{P}(S^2V)}$, the blowup of $\mathbb{P}(S^2V)$ along the Veronese surface.*

The different types of trees that occur in the compactification define a stratification of M_2 in locally closed subsets in the following way. Denote by Σ_0 the exceptional divisor of $\widetilde{\mathbb{P}(S^2V)}$ and by Σ_1 the proper transform of the cubic hypersurface of decomposable conics in $\mathbb{P}(S^2V)$. Then:

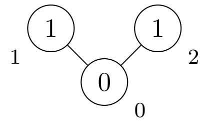
(I) the points of $\widetilde{\mathbb{P}(S^2V)} \setminus (\Sigma_0 \cup \Sigma_1)$ represent stable vector bundles on the original surface $\mathcal{S}$;

(II) *the points of* $\Sigma_0 \setminus \Sigma_1$ *represent vector bundles* $\mathcal{F}$ *on a tree surface of type*



where $\mathcal{F}|_{\mathcal{S}_0} \simeq \mathcal{O}_{\mathcal{S}_0}^{\oplus 2}$ *and* $\mathcal{F}|_{\mathcal{S}_1}$ *is a stable vector bundle on* $\mathbb{P}^2$*.*

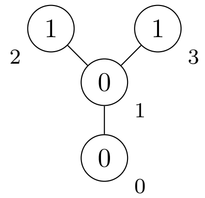
(III) *the points of* $\Sigma_1 \setminus \Sigma_0$ *represent vector bundles* $\mathcal{F}$ *on a tree surface of type*



where $\mathcal{F}|_{\mathcal{S}_0} \simeq \mathcal{O}_{\mathcal{S}_0}^{\oplus 2}$ *and the restrictions to the top components are non-split extensions of the form,*

$$0 \to \mathcal{O}_{\mathcal{S}_i} \to \mathcal{F}|_{\mathcal{S}_i} \to \mathcal{I}_{P_i} \to 0, \text{ where } P_i \in \mathcal{S}_i \setminus E_i;$$

(IV) *the points of* $\Sigma_0 \setminus \Sigma_1$ *represent vector bundles* $\mathcal{F}$ *on a tree surface of type*



where $\mathcal{F}|_{\mathcal{S}_0} \simeq \mathcal{O}_{\mathcal{S}_0}^{\oplus 2}$*,* $\mathcal{F}|_{\mathcal{S}_1} \simeq \mathcal{O}_{\mathcal{S}_1}^{\oplus 2}$*, and the restriction on the top components are non-split extensions of the same form as in the previous case.*

There are two different kinds of labels in the above tree graphs. The index of a vertex is written at the side of each node, while the charge of $\mathcal{F}$ on the corresponding component appears inside a node.

5.2 Tree surfaces over $\mathbb{P}^2$ and admissible bundles with good charge $c_{\mathcal{S}} = 0$ and $n = 2$

In a general situation, the existence of an admissible bundle $\mathcal{F}$ with good charging $\{n_\alpha\}$ on a tree surface imposes some restrictions on the tree T we are considering. From lemma (3.7.1) it follows that:

1. for all $\alpha \in \mathsf{T}^*$, $n_\alpha \geq 0$;

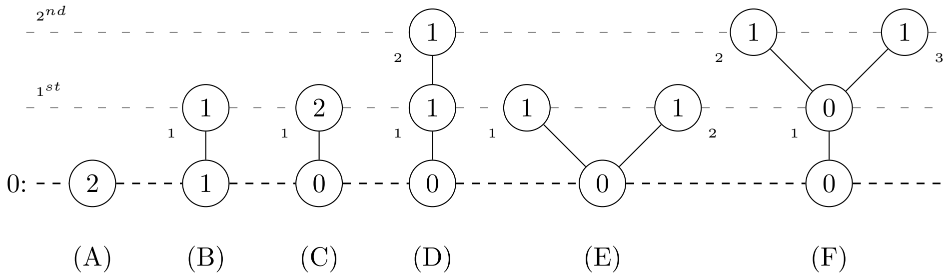
2. If $n_\alpha = 0$, then $\#\alpha^+ \geq 2$;

3. $n_0 \geq \frac{1}{4}c_{\mathcal{S}}^2$.

Clearly, this implies that chains of zeros are forbidden. Moreover, the number of top vertices of T cannot exceed $n - \frac{1}{4}c_{\mathcal{S}}^2$, where n is the total charge of $\mathcal{F}$.

Consider now the case we are interested in. The above observations yield the following:

Lemma 5.2.1. *Let $\mathcal{S} \simeq \mathbb{P}^2$, $c_{\mathcal{S}} = 0$ and $n = 2$. The good charging for these data are represented by the following diagrams:*



(5.1)

Throughout the rest of the section, we will work with these charging only, referring to them by letters (A) through (F)

Remark. At first sight, both chargings (D) and (F) describe situations covered by proposition (3.7.5). However, the zero charge labels the root component, which cannot be contracted. So we shall keep these cases in consideration.

The first step in studying admissibility is to choose a polarization on each tree surface of the above type. The best way to do this is to determine a common set of rational coefficients $\{\delta_\alpha\}$. The polarization for a specific tree surface will be defined by choosing some between these coefficients in accordance with the tree structure.

Let H be the class of a line on $\mathcal{S}$. Recalling the definition of good polarization, it suffices to define a rational $\bar{\delta}_{i-1}$ for each of the two levels appearing in diagram (5.1). By level we mean the set of vertices lying on a dashed line, i.e. having same distance from the root. As the vertices in the above trees have at most two successors, by proposition (3.2.2)

$$0 < \bar{\delta}_0 < \frac{1}{2} \quad \text{and} \quad 0 < \bar{\delta}_1 < \frac{1}{2}\bar{\delta}_0, \tag{5.2}$$

are sufficient and (in this case) necessary conditions for $(H, \{\bar{\delta}_0, \bar{\delta}_1\})$ to define a good polarization on each surface with these tree structures.

We are interested in studying equivalence classes of admissible bundles of rank 2 and $c_{\mathcal{S}} = 0$. To be sure of the existence of such $\hat{\mu}$-stable bundles on tree surfaces of the above type, the hypothesis of proposition (3.3.3) should be verified. So, from now on, we consider polarizations defined by triples $(H, \{\bar{\delta}_0, \bar{\delta}_1\})$ satisfying the inequalities in (5.2) and $\bar{\delta}_0 \neq \frac{1}{3}$.

Remark. Let $\mathcal{S}_\mathsf{T}$ be a tree surface of a type represented in diagram (5.1). Proceeding as in example (3.3.4), we can compute the degrees on the intersection of a $\hat{\mu}$-stable bundle $\mathcal{F}$ on $\mathcal{S}_\mathsf{T}$ of rank 2 and trivial $c_\mathcal{S}$. As $\bar{\delta}_1 < \frac{1}{4}$, $e_\beta(\mathcal{F}) = -2$ for every β in the second level of T. On the contrary, for α in the first level $e_\alpha(\mathcal{F})$ depends on $\bar{\delta}_0$. If it is bigger than $\frac{1}{3}$ than $e_\alpha(\mathcal{F}) = 0$, if it is smaller $e_\alpha(\mathcal{F}) = -2$. In particular, for $\bar{\delta}_0 = \frac{1}{3}$ there are no $\hat{\mu}$-stable bundles with these invariants on all non-trivial tree surface $\mathcal{S}_\mathsf{T}$.

Lemma 5.2.2. *Every rank 2 admissible bundle with $c_\mathcal{S} = 0$ on a tree surface $\mathcal{S}_\mathsf{T}$ of a type represented in diagram (5.1) have $\hat{\mu}$-semistable restrictions both to the root component and to the top components.*

Proof. As the rank is even, the $\hat{\mu}$-semistability of the restrictions on top components follows by propositions (3.5.1) without further restrictions on the polarizations. Focus now on the root component. Aside from the trivial case (A), the other diagrams describe vector bundles with charge $n_0 = 0$ or 1. Let $\mathcal{F}$ be an admissible bundle as above and trivial on the intersections. By corollary (3.6.2), $\mathcal{F}_\mathcal{S} \simeq \sigma_{0*}\mathcal{F}|_{\mathcal{S}_0}$ is $\hat{\mu}$-semistable bundle on $\mathcal{S}$. Moreover, it has the same Chern classes as $\mathcal{F}_0$. If $n_0 = 0$, by applying lemma (3.7.2), $\mathcal{F}|_{\mathcal{S}_0}$ is trivial.

Suppose $\mathcal{F}$ has charging (B). As there are no $\hat{\mu}$-stable bundles on $\mathbb{P}^2$ with trivial first Chern class and $c_2 = 1$, $\mathcal{F}_\mathcal{S}$ appears as an extension

$$0 \to \mathcal{O}_\mathcal{S} \to \mathcal{F}_\mathcal{S} \to \mathcal{I}_{\mathcal{S},x} \to 0,$$

where x is a point in $\mathcal{S}$. If x is different from the blown-up point, then the above exact sequence lifts to a similar one on $\mathcal{S}_0$ and so $\mathcal{F}|_{\mathcal{S}_0}$ is $\hat{\mu}$-semistable. If not $\mathcal{F}|_{\mathcal{S}_0}$ appears in the following exact triple

$$0 \to \mathcal{O}_{\mathcal{S}_0}(+E) \to \mathcal{F}|_{\mathcal{S}_0} \to \mathcal{O}_{\mathcal{S}_0}(-E) \to 0.$$

The above is clearly unstable and so, to conclude the proof, we show that a bundle $\mathcal{F}$ with such restriction is not admissible. Let $\bar{\delta}_0$ in $(\frac{1}{3}, \frac{1}{2})$. Thus $\mathcal{F}$ is H_T-compatible but, from the surjection $\mathcal{F} \twoheadrightarrow \mathcal{O}_{\mathcal{S}_0}(-E)$, we have

$$\hat{\mu}_{\mathcal{O}_{\mathcal{S}_0}(-E)} = 3\frac{1 - \bar{\delta}_0}{1 - \bar{\delta}_0^2} < 3 = \hat{\mu}_\mathcal{F}.$$

Hence $\mathcal{F}$ is not $\hat{\mu}$-stable. On the contrary, if $\bar{\delta}_0$ is less than $\frac{1}{3}$, then the H_T-compatibilization of $\mathcal{F}$ is $\mathcal{F}' := \mathcal{F}(0, -E_1)$. Again, computing the slope in the surjection $\mathcal{F}'|_{\mathcal{S}_0} \twoheadrightarrow \mathcal{O}_{\mathcal{S}_0}$, we get

$$\hat{\mu}_{\mathcal{O}_{\mathcal{S}_0}} = \frac{3 - \bar{\delta}_0}{1 - \bar{\delta}_0^2} < 3 = \hat{\mu}_{\mathcal{F}'}.$$

Thus, even for polarization in this chamber, $\mathcal{F}$ is not admissible. ☐

The following theorem describes, case by case, admissibility on tree surfaces of the above type. As a representative of an equivalence class, we consider the bundle having trivial restrictions to the intersection lines.

Theorem 5.2.3. *Let $\mathcal{S}_\mathsf{T}$ be a tree surface of one of the types appearing in diagram (5.1) and $\mathcal{F}$ a rank 2 vector bundle on it with $c_1(\mathcal{F}) = 0$. Then $\mathcal{F}$ is admissible if and only if,*

(A) $\mathcal{F}$ is a bundle in $M^s_{\mathbb{P}^2}(2;0,2)$;

(B) there exist two extensions

$$0 \to \mathcal{O}_{\mathcal{S}_i} \xrightarrow{s_i} \mathcal{F}|_{\mathcal{S}_i} \to \mathcal{I}_{\mathcal{S}_i,x_i} \to 0, \tag{5.3}$$

where $x_i \in \mathcal{S}_i \setminus E_1$ for both $i = 0,1$, such that $s_0|_{E_1}$ and $s_1|_{E_1}$ are not proportional;

(C) $\mathcal{F}|_{\mathcal{S}_0} \simeq \mathcal{O}^{\oplus 2}_{\mathcal{S}_0}$ and $\mathcal{F}|_{\mathcal{S}_1}$ is a bundle in $M^s_{\mathbb{P}^2}(2;0,2)$ trivial on the intersection line;

(D) $\mathcal{F}|_{\mathcal{S}_0} \simeq \mathcal{O}^{\oplus 2}_{\mathcal{S}_0}$ and $\mathcal{F}|_{\mathcal{S}_{\mathsf{T}^*}}$ is as in (C) and trivial on E_1;

(E) $\mathcal{F}|_{\mathcal{S}_0} \simeq \mathcal{O}^{\oplus 2}_{\mathcal{S}_0}$ and for both $i = 1,2$ there exists an extension as in (5.3) where $x_i \in \mathcal{S}_i \setminus E_i$, such that $s_1|_{E_1}$ and $s_2|_{E_2}$ do not extend to the same global section of $\mathcal{F}|_{\mathcal{S}_0}$;

(F) $\mathcal{F}|_{\mathcal{S}_0} \simeq \mathcal{O}^{\oplus 2}_{\mathcal{S}_0}$ and $\mathcal{F}|_{\mathcal{S}_{\mathsf{T}^*}}$ is as in (E) and it is trivial on E_1.

Proof. The first case is trivial. Indeed, a tree surface of type (A) is $\mathcal{S}$ itself. In this case admissibility coincides with $\hat{\mu}$-stability, and so the equivalence classes of admissible bundles are parametrized by $M^s_{\mathbb{P}^2}(2;0,2)$.

Consider a tree surface $\mathcal{S}_\mathsf{T}$ over $\mathcal{S}$ with just two components. Rank 2 vector bundles on surfaces with this structure were largely studied in the last chapter as they served as reference in almost all the examples.

Suppose $\mathcal{F}$ has charging (B). Both restrictions of $\mathcal{F}$ to components are strictly $\hat{\mu}$-semistable. Thus, there exist two extensions as in (5.3). In particular, as $\mathcal{F}$ is $\hat{\mu}$-stable, the sections defining the inclusions do not glue on the intersection. The if part follows by corollary (3.4.2).

If the charges of $\mathcal{F}$ are as in (C), $\mathcal{F}|_{\mathcal{S}_0}$ is trivial. Suppose $\mathcal{F}|_{\mathcal{S}_1}$ to be strictly $\hat{\mu}$-semistable, i.e.

$$0 \to \mathcal{O}_{\mathcal{S}_1} \xrightarrow{s_1} \mathcal{F}|_{\mathcal{S}_1} \to \mathcal{I}_{\mathcal{S}_1,Z_1} \to 0,$$

where Z_1 is a 0-dimensional subscheme of length 2 in $\mathcal{S}_1 \setminus E_1$. By the triviality on $\mathcal{S}_0$, there exists a global section s_0 of $\mathcal{F}|_{\mathcal{S}_0}$ extending $s_1|_{E_1}$. Gluing s_0 and s_1 together, we obtain

$$\mathcal{O}_{\mathcal{S}_\mathsf{T}} \hookrightarrow \mathcal{F},$$

and so $\mathcal{F}$ is not $\hat{\mu}$-stable. Thus, $\mathcal{F}|_{\mathcal{S}_1}$ is a bundle in $M^s_{\mathbb{P}^2}(2;0,2)$ trivial on E_1. By applying Theorem (3.4.1), we get the inverse direction.

Suppose now $\mathcal{S}_\mathsf{T}$ has the tree structure represented by (D). As $\mathcal{S}_0$ is the blowup of $\mathbb{P}^2$ in a point, proposition (3.7.5) applies to the root surface too. Let

$$\Pi_0 : \mathcal{S}_\mathsf{T} \to \mathcal{S}_{\mathsf{T}^*}$$

be the morphism contracting the root component to E_1. The tree surface $\mathcal{S}_{\mathsf{T}^*}$ has the same structure as in the previous case but not the same polarization. Instead of $(E_1, \{\bar{\delta}_0\})$, we should consider the polarization defined by $(\bar{\delta}_0 E_1, \{\bar{\delta}_1\})$, but this does not affect the proof. Thus, $\mathcal{F}$ can be described by the pullback via Π_0 of an admissible bundle as in (B) trivial on the line corresponding to E_1. The result follows.

Let $\mathcal{S}_\mathsf{T}$ be a surface of type (E). By the vanishing of $n_0(\mathcal{F})$ the restriction of $\mathcal{F}$ to the root component is trivial. On the other hand, both restrictions on the top components are strictly $\hat{\mu}$-semistable and appear as estensions (5.3). Moreover, if there exists a section s_0 of $\mathcal{F}_0$ which restrict to $s_1|_{E_1}$ on E_1 and to $s_2|_{E_2}$ on E_2, then they glue and provide a global section of $\mathcal{F}$. In this case $\mathcal{F}$ would be strictly $\hat{\mu}$-semistable. Clearly, avoiding this situation, the $\hat{\mu}$-semistability of the restrictions implies, by corollary (3.4.2), the $\hat{\mu}$-stability of $\mathcal{F}$.

Consider now the last case. Similarly to (D), (F) can be proven by extending proposition (3.7.5) to the root component. Thus, $\mathcal{F}$ can be described by the pullback via Π_0 of a vector bundle as described in (E). $\square$

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