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Opérateurs de composition sur les espaces modèles

Thèse de doctorat en cotutelle

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Composition operators on model spaces

Thèse en cotutelle
Doctorat en mathématiques

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Résumé

Cette thèse est consacrée à l'étude des opérateurs de composition sur les espaces modèles. Soit φ une fonction analytique du disque unité dans lui même et soit Θ une fonction intérieure, c'est à dire une fonction holomorphe et bornée par 1 dont les limites radiales sur le cercle sont de module 1 presque partout par rapport à la mesure de Lebesgue. A cette fonction Θ, on associe l'espace modèle K_Θ, défini comme l'ensemble des fonctions $f \in H^2$ qui sont orthogonales au sous-espace ΘH^2. Ici H^2 est l'espace de Hardy du disque unité. Ces sous-espaces sont importants en théorie des opérateurs car ils servent à modéliser une large classe de contractions sur un espace de Hilbert.

Le premier problème auquel nous nous intéressons concerne la compacité d'un opérateur de composition C_φ vu comme opérateur de K_Θ dans H^2. Récemment, Lyubarskii et Malinnikova ont obtenu un joli critère de compacité pour ces opérateurs qui fait intervenir la fonction de comptage de Nevanlinna du symbole φ. Ce critère généralise le critère classique de Shapiro. Dans une première partie de la thèse, nous généralisons ce résultat de Lyubarskii–Malinnikova à une classe plus générale de sous-espaces, à savoir les espaces de de Branges–Rovnyak ou certains de leurs sous-espaces. Les techniques utilisées sont en particulier des inégalités fines de type Bernstein pour ces espaces.

Le deuxième problème auquel nous nous intéressons dans cette thèse concerne l'invariance de K_Θ sous l'action de C_φ. Ce problème nous amène à considérer une structure de groupe sur le disque unité du plan complexe via les automorphismes qui fixent le point 1. A travers cette action de groupe, chaque point du disque produit une classe d'équivalence qui se trouve être une suite de Blaschke. On montre alors que les produits de Blaschke correspondants sont des solutions "minimales" d'une équation fonctionnelle $\psi \circ \varphi = \lambda\psi$, où λ est une constante unimodulaire et φ un automorphisme du disque unité. Ces résultats sont ensuite appliqués au problème d'invariance d'un espace modèle par un opérateur de composition.

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Abstract

This thesis concerns the study of composition operators on model spaces. Let φ be an analytic function on the unit disc into itself and let Θ be an inner function, that is a holomorphic function bounded by 1 such that the radial limits on the unit circle are of modulus 1 almost everywhere with respect to Lebesgue measure. With this function Θ, we associate the model space K_Θ, defined as the set of functions $f \in H^2$, which are orthogonal to the subspace ΘH^2. Here, H^2 is the Hardy space on the unit disc. These subspaces are important in operator theory because they are used to model a large class of contractions on Hilbert space.

The first problem which we are interested in concerns the compactness of the composition operator C_φ as an operator on H^2 into H^2. Recently, Lyubarskii and Malinnikova have obtained a nice criterion for the compactness of these operators which is related to the Nevanlinna counting function. This criterion generalizes the classical criterion of Shapiro. In the first part of the thesis, we generalize this result of Lyubarskii–Malinnikova to a more general class of subspaces, known as de Branges–Rovnyak spaces or some subspaces of them. The techniques that are used are particular Bernstein type inequalities of these spaces.

The second problem in which we are interested in this thesis concerns the invariance of K_Θ under C_φ. We present a group structure on the unit disc via the automorphisms which fix the point 1. Then, through the induced group action, each point of the unit disc produces an equivalence class which turns out to be a Blaschke sequence. Moreover, the corresponding Blaschke products are minimal solutions of the functional equation $\psi \circ \varphi = \lambda \psi$, where λ is a unimodular constant and φ is an automorphism of the unit disc. These results are applied in the invariance problem of the model spaces by the composition operator.

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Contents

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Thèse de Muath Karaki, Lille 1, 2015

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To my parents. . .

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" Go down deep enough into anything and you will find mathematics."

Dean Schlicter

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Chapter 1

Introduction

The study of composition operators lies at the interface of analytic function theory and operator theory. The story begins with fixing an analytic self–map on some domain Ω, and usually denoted by φ. Then the composition mapping is defined as

$$\begin{array}{rccc} C_\varphi : & \mathrm{Hol}(\Omega) & \longrightarrow & \mathrm{Hol}(\Omega) \\ & f & \longmapsto & f \circ \varphi. \end{array}$$

The typical question in this topic is to know under which conditions we have

$$C_\varphi \mathcal{X} \subset \mathcal{Y},$$

where $\mathcal{X}$ and $\mathcal{Y}$ are some Banach spaces of analytic function which reside in $\mathrm{Hol}(\Omega)$ as a subset. In particular, the special case $\mathcal{Y} = \mathcal{X}$ has been extensively studied for various classes of Banach spaces $\mathcal{X}$.

As the Hardy spaces on the unit disc H^p, $0 < p \leq \infty$, are one of the most important function spaces, the composition operator is studied intensively on the Hardy spaces. See for example [8, 29]. The classical *subordination principle* of Littlewood can be rephrased in this language. In terms of composition operators, it says that the mapping $C_\varphi : H^p \longrightarrow H^p$ is a well-defined bounded operator on all Hardy spaces H^p, $0 < p \leq \infty$. See [29, Chapter 1]. This question has also been studied on numerous other function spaces, e.g. on Besov spaces [30], on Bloch spaces [6] or on the Dirichlet space [14, 15, 32].

Studying composition operators on subspaces of the Hardy space is a new topic and still there are several open questions about them. In [20], authors studied the

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compactness and membership in Schatten classes of the mapping $C_\varphi : K_\Theta \longrightarrow H^2$, where K_Θ is the model space corresponding to the inner function Θ. As a continuation, we consider the composition operator on the De Branges–Rovnyak spaces instead of the model spaces.We begin with the boundary behavior of functions on the unit circle, we generalize Hartmann-Ross result which controls the behavior of functions in the model space, [16]. Then, we obtain a weaker sufficient condition for the compactness of the composition operator on the de Branges–Rovnyak spaces, and give a sufficient condition for the compactness of the composition operator on a dense subspace of the de Branges–Rovnyak spaces by generalizing [2, Lemma 5].

In [23], a complete characterization of φ's for which C_φ leaves K_Θ invariant, when Θ is a finite Blaschke product, is given. The paper [24] is devoted to a comprehensive study of C_φ when φ is an inner function. In this situation, authors face with the functional equation

$$\psi\big(\varphi(z)\big) \times \omega(z) = \psi(z), \qquad (z \in \mathbb{D}), \tag{1.1}$$

where all the three functions ψ, φ and ω are inner. With an iteration technique and appealing to the structure of inner function, (1.1) simplifies to

$$\psi\big(\varphi(z)\big) = \lambda\,\psi(z), \qquad (\lambda \in \mathbb{T},\ z \in \mathbb{D}), \tag{1.2}$$

where φ has its fixed (Denjoy–Wolff) point on $\mathbb{T}$. This equation is a special case of the celebrated *Schröder equation* which has a long and rich history. As a matter of fact, its first treatment dates back to 1884 when Königs [18] classified the eigenvalues λ for mappings φ with a fixed points inside $\mathbb{D}$. See [29, Chapter 6] for more detail on Königs' solution. Despite the vast literature on Schröder's equation, not much is known when the Denjoy–Wolff point of φ is on $\mathbb{T}$.

Chapter 3 is devoted to a complete characterization of the Blaschke products ψ which satisfy (1.2), with φ being an automorphism of $\mathbb{D}$. To do so, in Section 3.3 we define a noncommutative group structure on $\mathbb{D}$ which stems from automorphisms of the $\mathbb{D}$ which fix the point 1. Essential properties of this group are discussed in this section. Then in Section 3.4, we introduce a family of abelian subgroups of $\mathbb{D}$. These subgroups will provide the main apparatus to spot all the minimal Blaschke sequences. To achieve this goal, we need an explicit formula for the n-th iterate of an element in the group $\mathbb{D}$; this formula is obtained in Section 3.5. Eventually,

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in Sections 3.6 and 3.7, we study the orbits of the subgroup actions on $\mathbb{D}$ and show that these orbits are two-sided Blaschke sequences. The main outcome is Theorem 3.8, in which we show that the corresponding Blaschke sequence is in fact a minimal solution of the functional equation. Eventually, in Section 3.8, the stage is set to characterize all Blaschke products which satisfy the functional equation. Theorem 3.9 is a complete characterization of Blaschke products which fulfill the functional equation (1.2).

1.1 Hardy spaces

In this section we present the basic theory of the function spaces that we will consider in our work. We begin with the Hardy space and then its closed S^*-invariant subspaces.

Let m be the normalized Lebesgue measure on the unit circle. Let $\mathbb{C}$ denote the complex plane, $\mathbb{D}$ denote the unit disc, and let $\mathbb{T}$ denote the unit circle. For $1 \leq p < \infty$, the Hardy space H^p consists of analytic functions f in $\mathbb{D}$ such that

$$\|f\|_p^p := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

If $p = \infty$, we use H^∞ to denote the space of bounded analytic functions f in the unit disc. Thus

$$\|f\|_\infty := \sup\{|f(z)| : z \in \mathbb{D}\}.$$

For $1 \leq p \leq \infty$, $L^p(\mathbb{T})$ will denote the Lebesgue space of the unit disc induced by the measure m. For functions f in the Hardy space H^p , $1 \leq p \leq \infty$, Fatou's theorem says that the radial limit

$$f(e^{i\theta}) := \lim_{r\to 1} f(re^{i\theta})$$

exists for almost all $e^{i\theta} \in \mathbb{T}$, and f on the boundary belongs to $L^p(\mathbb{T})$, and moreover $\|f\|_{H^p} = \|f\|_{L^p(\mathbb{T})}$. Hence H^p can be regarded as a closed subspace of $L^p(\mathbb{T})$.

For the special case $p = 2$, H^2 is a Hilbert space. The inner product is given by

$$\langle f, g\rangle_2 = \sum_{n=0}^{\infty} a_n \bar{b}_n,$$

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where f, g in H^2 and have the expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

Let P_+ be the orthogonal projection on $L^2(\mathbb{T})$ with range H^2. The operator is given explicitly as a Cauchy integral:

$$(P_+ f)(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1-\bar{\zeta} z} dm(\zeta), \qquad z \in \mathbb{D}, f \in L^2.$$

This operator can be extended to an operator from L^1 into $\mathrm{Hol}(\mathbb{D})$. The kernel function in H^2 for the functional of evaluation at λ in the unit disc will be denoted by k_λ and

$$k_\lambda(z) = \frac{1}{1-\bar{\lambda} z}, \qquad z \in \mathbb{D}.$$

It satisfies

$$f(z) = \langle f, k_z \rangle_2, \qquad z \in \mathbb{D}, f \in H^2.$$

Let b be in the unit ball of H^∞. Then by the canonical factorization theorem b can be factorized as follows:

$$b = \gamma B I_S O,$$

where γ is a constant of modulus one,

$$B(z) = \prod_n \frac{|\lambda_n|}{\lambda_n} \frac{\lambda_n - z}{1-\overline{\lambda_n} z}, \qquad (z \in \mathbb{D}),$$

is the Blaschke product with zeros $\lambda_n \in \mathbb{D}$ satisfying the Blaschke condition $\sum_n (1-|\lambda_n|) < +\infty$,

$$I_S(z) = \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right), \qquad (z \in \mathbb{D}),$$

is the singular inner factor with μ a finite and positive Borel measure on $\mathbb{T}$, singular with respect to the Lebesgue measure,

$$O(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log |b(\zeta)| dm(\zeta) \right), \qquad (z \in \mathbb{D}),$$

is the outer factor.

For $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator is defined on H^2 by

$$T_\varphi f = P_+(\varphi f).$$

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Thèse de Muath Karaki, Lille 1, 2015

It is well known that T_φ is bounded and $\|T_\varphi\| = \|\varphi\|_\infty$. We let S denote the unilateral shift operator on H^2, *i.e* $(Sf)(z) = zf(z)$. Its adjoint, the backward shift, is given by

$$(S^*f)(z) = \frac{f(z) - f(0)}{z}.$$

Note that $S = T_z$ and $S^* = T_{\bar{z}}$. A non-constant analytic bounded function Θ is called inner if its radial limit is of modulus 1 almost everywhere on the unit circle. The model space, corresponding to Θ, $K_\Theta = H^2 \ominus \Theta H^2$, is a proper nontrivial invariant subspace of S^*. Moreover, Beurling's theorem says that these subspaces describe all non proper non trivial invariant subspaces of S^*. The orthogonal projection on $L^2(\mathbb{T})$ with range K_Θ will be denoted by P_Θ. The kernel function in K_Θ for the functional of evaluation at λ will be denoted by k_λ^Θ; it equals $P_\Theta k_\lambda$. Indeed, for any $f \in H^2$ and $\lambda \in \mathbb{D}$ we have

$$\langle f, P_+(\overline{\Theta} k_\lambda)\rangle_2 = \langle \Theta f, k_\lambda\rangle_2 = \langle f, \overline{\Theta(\lambda)} k_\lambda\rangle_2. \tag{1.3}$$

That means $P_+(\overline{\Theta} k_\lambda) = \overline{\Theta(\lambda)} k_\lambda$. Hence,

$$\begin{aligned} k_\lambda^\Theta(z) &= (I - \Theta P_+ \overline{\Theta}) k_\lambda(z) \\ &= k_\lambda(z)(1 - \overline{\Theta(\lambda)}\Theta) \\ &= \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \bar{\lambda} z}, \end{aligned}$$

where $\lambda, z \in \mathbb{D}$. And these kernels satisfy

$$f(z) = \langle f, k_z^\Theta\rangle_2, \qquad z \in \mathbb{D}, \qquad f \in K_\Theta.$$

If B is a finite Blaschke product with zeros $\lambda_1, \lambda_2, \dots, \lambda_N$ with multiplicities $m_1, m_2, \dots m_N$, then K_B is the finite dimensional subspace spanned by

$$\frac{1}{1 - \bar{\lambda}_i z}, \frac{1}{(1 - \bar{\lambda}_i z)^2}, \dots, \frac{1}{(1 - \bar{\lambda}_i z)^{m_i}}, \qquad (1 \le i \le N).$$

The set $\sigma(\Theta)$ denotes the spectrum of Θ. This is the set of all $\zeta \in \overline{\mathbb{D}}$ such that $\liminf_{z \to \zeta} |\Theta(z)| = 0$, that is the smallest closed subset of the closed disc that contains the zeros of Θ and the support of of μ. It is well-known that Θ and all functions in K_Θ admit analytic continuation across any arc lying in $\mathbb{T} \backslash \sigma(\Theta)$, [26].

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1.2 Composition operators

In this section we are going to present some basic facts in the theory of composition operators. For a holomorphic φ from the unit disc to itself, usually called the symbol, one can define the composition operator

$$C_\varphi f = f \circ \varphi.$$

The composition operator can be defined on many different analytic function spaces. The following theorem, called the Littlewood subordination theorem, is essential in the theory of composition operators

Theorem 1.1 (Littlewood subordination theorem). *[8] Let φ be an analytic map of the unit disc into itself such that $\varphi(0) = 0$. If G is a subharmonic function in $\mathbb{D}$, then for $0 < r < 1$*

$$\int_0^{2\pi} G(\varphi(re^{i\theta}))d\theta \leq \int_0^{2\pi} G(re^{i\theta})d\theta. \tag{1.4}$$

It easily implies that C_φ is bounded on H^2, for any $\varphi : \mathbb{D} \to \mathbb{D}$ analytic. For φ an analytic map on the unit disc, let $\omega \neq \varphi(0)$ and let z_j be the points of the disc for which $\varphi(z_j) = \omega$, with multiplicities. The Nevanlinna counting function is

$$N_\varphi(\omega) = \sum_j \log(1/|z_j|).$$

And $N_\varphi(\omega) := 0$ if ω is not in the domain of φ. Since $-\log x \asymp 1 - x$, as $x \to 1$ we have $\sum_j \log(1/|z_j|) \asymp \sum_j (1 - |z_j|)$ which is finite, by the Blaschke condition, so the Nevanlinna counting function is well defined.

The following is a version of Littlewood–Paley identity which gives the H^2 norm of an analytic function in terms of a weighted area integral.

Theorem 1.2 (Littlewood–Paley identity). *[8] If f is analytic in $\mathbb{D}$, then*

$$\|f\|_2^2 = 2\int_{\mathbb{D}} \left|f'(z)\right|^2 \log\frac{1}{|z|^2} dA(z) + |f(0)|^2, \tag{1.5}$$

where dA is the normalized area measure.

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The following is the Stanton formula. It is obtained from Littlewood–Paley identity.

Theorem 1.3. *[8] If f is analytic in the unit disc and φ is a non-constant analytic mapping of $\mathbb{D}$ into itself, then*

$$\|f \circ \varphi\|_2^2 = 2 \int_{\mathbb{D}} \left| f^{'}(z) \right|^2 N_{\varphi}(z) dA(z) + |f(\varphi(0))|^2. \tag{1.6}$$

The Nevanlinna counting function has the following important property, which is the subharmonicity, [28]. That is

$$N_{\varphi}(a) \leq \frac{1}{A(\Delta)} \int_{\Delta} N_{\varphi}(w) dA(w), \tag{1.7}$$

where Δ is an open disc in $\mathbb{D} \backslash \{\varphi(0)\}$ with centre a.

1.2.1 Carleson Measures

Carleson measures are a very useful tool in studying composition operators on Hardy and sub-Hardy spaces. In this subsection we present the definition and well known results about Carleson measures.

For ζ on the unit circle and $h \in (0, 1)$, let

$$S(\zeta, h) = \{z \in \mathbb{D} : |z - \zeta| < h\},$$

that is the intersection of the unit disc with the disc of radius h and centred at ζ. The next result characterizes measures μ on the disc for which H^p is contained in $L^p(\mu)$, and by the closed graph theorem this is equivalent to the continuity of the inclusion map from H^p into $L^p(\mu)$.

Theorem 1.4. *[5, Carleson's theorem] For μ a finite, positive Borel measure on $\mathbb{D}$ and $p \in (0, \infty)$, the following are equivalent:*

1. *There is a constant C_1 such that*

$$\mu(S(\zeta, h)) \leq C_1 h,$$

for all $\zeta \in \mathbb{T}$ and $h \in (0, 1)$.

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2. *There is a constant* C_2 *such that*

$$\int_{\mathbb{D}} |f|^p d\mu \leq C_2 \|f\|_p^p,$$

for all f *in* H^p.

3. *There is a constant* C_3 *such that*

$$\int_{\mathbb{D}} \left|\frac{1}{1-\bar{\omega}z}\right|^2 d\mu(z) \leq C_3 \frac{1}{1-|\omega|^2},$$

for all $\omega \in \mathbb{D}$.

Measures that satisfy these equivalent conditions are called Carleson measures for the Hardy spaces in $\mathbb{D}$ and the set of all these measures will be denoted by $\mathcal{C}$.

1.2.2 Composition operators on Hardy spaces

We begin with the composition operator on Hardy spaces. The operator

$$\begin{aligned} C_\varphi : H^2 &\rightarrow H^2 \\ f &\mapsto f \circ \varphi, \end{aligned}$$

is bounded, for any holomorphic symbol, by the Littlewood's Subordination Theorem. J. Shapiro [28] characterized the compactness of the composition operator on Hardy spaces using the Nevanlinna counting function.

Theorem 1.5. *[8, Theorem 3.20] If* φ *is an analytic map of the unit disc into itself, then the composition operator on* H^2 *is compact if and only if*

$$\lim_{|\omega|\to 1} \frac{N_\varphi(\omega)}{\log|\omega|} = 0. \tag{1.8}$$

Let φ be a self-map on $\mathbb{D}$. We define the pullback measure μ_φ on the closed unit disc $\overline{\mathbb{D}}$ as the image of the Lebesgue measure m of the unit circle under the map φ^* (the boundary limit of φ):

$$\mu_\varphi(E) = m\left(\varphi^{*-1}(E)\right),$$

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for every Borel subset E of $\overline{\mathbb{D}}$. It is clear that the composition operator on the Hardy spaces is bounded if and only if the measure μ_φ is a Carleson measure. Since composition operators on Hardy spaces are always bounded it means that μ_φ is always a Carleson measure.

The composition operator C_φ on H^2 is isometrically equivalent to the embedding of H^2 into $L^2(\mu_\varphi)$. The following theorem characterizes the compactness of the composition operator on H^2 in terms of the measure μ_φ.

Theorem 1.6. *[8, Theorem 3.12(ii)] Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function. Then, the operator*

$$C_\varphi : H^2 \to H^2,$$

is compact if and only if

$$\mu_\varphi(S(\zeta, h)) = o(h), \qquad \text{as } h \to 0,$$

uniformly in $\zeta \in \mathbb{T}$.

1.2.3 Composition operators on finite rank model spaces

In this section, we discuss the invariance of the finite rank model space under C_φ. Let B be a finite Blaschke product. Mashreghi and Shabankhah in [23] give a complete description of bounded composition operator on model subspaces K_B. They show that the collection of composition operators on K_B has a group structure. Let $\mathcal{L}_c(K_B)$ be the collection of all bounded composition operators on K_B into itself. Depending on the zeros of the Blaschke product, the following theorems characterize the collection of the composition operators on the model space. We will use the notation $\varphi^{[n]} = \varphi \circ \ldots \circ \varphi$ n times.

Theorem 1.7. *[23, Theorem 2.1] Let*

$$B(z) = \left(\frac{\lambda - z}{1 - \bar{\lambda} z}\right)^n, \qquad (\lambda \neq 0, n \geq 1).$$

Then

$$\mathcal{L}_c(K_B) = \{C_{(1-\bar{\lambda}a)z+a} : a \in \mathbb{C}, a \neq 1/\bar{\lambda}\}.$$

Theorem 1.8. *[23, Theorem 2.3] Let B be a finite Blaschke product with $B(0) \neq 0$ and with at least two distinct zeros. Then there exists an integer $n \geq 1$ and a linear*

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function $\varphi(z) = az + b$ *such that* $\varphi^{[n]}(z) = z$ *and*

$$\mathcal{L}_c(K_B) = \{C_z, C_\varphi, C_{\varphi^{[2]}}, \ldots, C_{\varphi^{[n-1]}}\}.$$

Theorem 1.9. *[23, Theorem 2.5] Let*

$$B(z) = z\left(\frac{\lambda - z}{1 - \bar{\lambda}z}\right)^n, \qquad (\lambda \neq 0, n \geq 1).$$

Then

$$\mathcal{L}_c(K_B) = \{C_{\frac{az+b}{cz+1}} : a \in \mathbb{C}, b \in \mathbb{C}\backslash\{1/\bar{\lambda}\}, c = (a-1)\bar{\lambda} + b\bar{\lambda}^2\}.$$

Theorem 1.10. *[23, Theorem 2.8] Let B be a finite Blaschke product with $B(0) = 0$, and assume that B has at least two other distinct zeros. Then there is a Mobius function $\varphi(z) = (az+b)/(cz+d)$ such that $\varphi^{[n]}(z) = z$ and*

$$\mathcal{L}_c(K_B) = \{C_z, C_\varphi, C_{\varphi^{[2]}}, \ldots, C_{\varphi^{[n-1]}}\}.$$

Theorem 1.11. *[23, Theorem 2.9] Let*

$$B(z) = z^m\left(\frac{\lambda - z}{1 - \bar{\lambda}z}\right)^n, \qquad (\lambda \neq 0, m \geq 2, n \geq 1).$$

Then

$$\mathcal{L}_c(K_B) = \{C_b : b \in \mathbb{C}\backslash\{1/\bar{\lambda}\}\}.$$

1.2.4 Composition operators on model spaces

Lyubarskii and Malinnikova [20] consider the following operator. Let φ be an analytic map on the unit disc into itself, and let Θ be an inner function. Then we consider the restriction of C_φ to K_Θ,

$$\begin{aligned} C_\varphi : K_\Theta &\to H^2 \\ f &\mapsto f \circ \varphi. \end{aligned}$$

The following theorem characterizes the compactness of the composition operator.

Theorem 1.12. *[20, Theorem 1]Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function, such that $\varphi(0) = 0$. Let Θ be an inner function. Then the following statements are equivalent*

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1. *The operator $C_\varphi : K_\Theta \to H^2$ is compact.*

2. *The Nevanlinna counting function satisfies*

$$N_\varphi(\omega)\frac{1-|\Theta(\omega)|^2}{1-|\omega|^2} \to 0 \ \text{ as } |\omega| \to 1. \tag{1.9}$$

Suppose for a moment that $\Theta \equiv 0$ (of course this is not an inner function but we can think at this case corresponding to $K_\Theta = H^2$). Then (1.9) reduces to

$$\frac{N_\varphi(\omega)}{1-|\omega|^2} \to 0 \ \text{ as } |\omega| \to 1,$$

and since $1-|w|^2 \asymp -\log|w|$, $w \to 0$, (1.9) gives the condition (1.8).

The key point of the proof of Theorem 1.12 is based on a difficult result of Axler, Chang and Sarason. Cohn, in [7] rewrites the inequality in the following form. Let Θ be an inner function. Then for any $f \in K_\Theta$ we have

$$\|f\|_2^2 \geq C_p \int_{\mathbb{D}} |f'(z)|^2 \frac{1-|z|}{(1-|\Theta(z)|)^p} dA(z) + |f(0)|^2, \tag{1.10}$$

which is valid for some $p \in (0,1)$.

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Chapter 2

De Branges–Rovnyak Spaces

In this chapter we introduce the de Branges–Rovnyak spaces $\mathcal{H}(b)$, the definition and some basic properties. In section 2.1 we study the bad boundary behavior of functions in de Branges–Rovnyak spaces. In section 2.2 we give a necessary condition for the compactness of the composition operator from $\mathcal{H}(b)$ into H^2. Then in section 2.3 we obtain a sufficient condition for this property. Then, in section 2.4 we generalize Axler–Chang–Sarason lemma for the de Branges–Rovnyak spaces. Then, in section 2.5 we define a dense subspace, $\mathcal{X}(b)$ in the de Branges–Rovnyak spaces. Finally, in section 2.6 we obtain a sufficient condition for the compactness of the composition operator on this space $\mathcal{X}(b)$.

De Branges–Rovnyak spaces were introduced by de Branges and Rovnyak in [9, 10] as universal model spaces for Hilbert space contractions. Subsequently it was realized that these spaces have numerous connections with other topics in complex analysis and operator theory, [12, 27].

Let b be in the unit ball of H^∞. then the de Branges–Rovnyak space is the range of the operator $(I - T_b T_{\bar{b}})^{1/2}$ on H^2 and denoted by $\mathcal{H}(b)$. The operator T_b is the Toeplitz operator defined on H^2 by $T_b(f) = P_+(bf)$. $\mathcal{H}(b)$ is a Hilbert space with the inner product

$$\langle (I - T_b T_{\bar{b}})^{1/2} f, (I - T_b T_{\bar{b}})^{1/2} g \rangle_b = \langle f, g \rangle_2,$$

where f and g are taken in H^2 such that

$$f, g \perp \ker(I - T_b T_{\bar{b}})^{1/2}.$$

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And its cousin, denoted by $\mathcal{H}(\bar{b})$ is the range of the operator $(I - T_{\bar{b}}T_b)^{1/2}$ on H^2. As a special case, when b is an inner function, the operator $I - T_bT_{\bar{b}}$ is projection and $\mathcal{H}(b)$ becomes a closed subspace of H^2 which coincides with the model space K_b. The reproducing kernel of $\mathcal{H}(b)$ at the point $\lambda \in \mathbb{D}$ is denoted by k_λ^b and given by the following formula

$$k_\lambda^b(z) = \frac{1 - \overline{b(\lambda)}b(z)}{1 - \bar{\lambda}z}, \qquad \lambda, z \in \mathbb{D}.$$

Its norm is

$$\left\|k_\lambda^b\right\|_b^2 = \frac{1 - |b(\lambda)|^2}{1 - |\lambda|^2}, \qquad \lambda \in \mathbb{D}.$$

Hence the normalized reproducing kernel is

$$\widetilde{k_\lambda^b}(z) = \frac{1 - \overline{b(\lambda)}b(z)}{1 - \bar{\lambda}z}\left(\frac{1 - |\lambda|^2}{1 - |b(\lambda)|^2}\right)^{1/2}, \qquad \lambda, z \in \mathbb{D}.$$

In the first section we study the boundary behavior of functions in $\mathcal{H}(b)$. And the rest of the chapter will be devoted for the composition operator on the de Branges–Rovnyak spaces.

The following Lemma is well known and says that $\mathcal{H}(b)$ is contractively contained in H^2.

Lemma 2.1. *Let b be in the unit ball of H^∞. Then $\mathcal{H}(b) \subset H^2$ and for any function in $\mathcal{H}(b)$ we have*

$$\|f\|_2 \leq \|f\|_b. \tag{2.1}$$

Proof. The inclusion $\mathcal{H}(b) \subset H^2$ follows immediately from the definition of $\mathcal{H}(b)$ as

$$\mathcal{H}(b) = (I - T_bT_{\bar{b}})^{1/2} H^2.$$

Now, let $f = (I - T_bT_{\bar{b}})^{1/2} g$, where $g \in H^2$ and $g \perp \ker(I - T_bT_{\bar{b}})^{1/2}$.

On one hand, we have

$$\|f\|_b = \|g\|_2,$$

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and on the other hand, we have

$$\begin{aligned}\|f\|_2^2 &= \left\|(I - T_b T_{\bar{b}})^{1/2}\, g\right\|_2^2 \\ &= \langle (I - T_b T_{\bar{b}})\, g, g\rangle_2 \\ &= \|g\|_2^2 - \|T_{\bar{b}} g\|_2^2 \\ &\leq \|g\|_2^2 = \|f\|_b^2 .\end{aligned}$$

□

The main goal of this chapter is to prove the following implications.

Theorem 2.2. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic such that, $\varphi(0) = 0$. Let b be a function in the unit ball of H^∞. Consider the following statements*

1. *for some $\gamma \in (0, 1/3)$ we have*
$$N_\varphi(z)\left(\frac{(1 - |b(z)|)^\gamma}{1 - |z|^2}\right)^2 \to 0 \ \text{ as } |z| \to 1,$$
2. *the operator $C_\varphi : \mathcal{H}(b) \to H^2$ is compact,*
3. $$N_\varphi(z)\frac{1 - |b(z)|}{1 - |z|^2} \to 0 \ \text{ as } |z| \to 1,$$
4. *the operator $C_\varphi : \mathcal{X}(b) \to H^2$ is compact.*

Then we have the following implications, $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$.

2.1 Boundary behavior of $\mathcal{H}(b)$ spaces

In this section we study the boundary behavior of de Branges–Rovnyak space's functions at some point on the unit circle. We generalize Hartmann–Ross result, [16, Theorem 1.5]. Studying the behavior on the boundary is divided into two types. The first one is the good behavior that is all functions have finite radial limits at a point on the unit circle. While the second type is controlling the growth of the radial limits at a point on the unit circle, and called the bad behavior. The

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following theorem characterizes the good behavior of functions and their derivatives in de Branges–Rovnyak spaces, Ahern and Clark proved the theorem if b is an inner function, [1]. And Fricain and Mashreghi proved the general case.

Theorem 2.3. *[11, Theorem 3.2],[1, Theorem 1.5] Let b be in the unit ball of H^∞ such that*

$$b(z) = \gamma \prod \left(\frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z} \right) \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right) \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log|b(\zeta)| dm(\zeta) \right)$$

is the canonical factorization. Let $\zeta_0 \in \mathbb{T}$ and N be a non-negative integer. Then the following are equivalent.

1. *for every $f \in \mathcal{H}(b)$, $f(z), f'(z), \ldots, f^{(N)}(z)$ have finite limits as z tends radially to ζ_0.*
2. *we have*

$$\sum_n \frac{1 - |a_n|^2}{|\zeta_0 - a_n|^{2N+2}} + \int_0^{2\pi} \frac{d\mu(e^{it})}{|\zeta_0 - e^{it}|^{2N+2}} + \int_0^{2\pi} \frac{|\log|b(e^{it})||}{|\zeta_0 - e^{it}|^{2N+2}} dm(e^{it}) < \infty. \tag{2.2}$$

If (2.2) is no longer satisfied at some point ζ_0 then we lose the good behavior of the functions at that point. But, it is still interesting to see how fast these functions approach infinity. Hartmann and Ross controlled the behavior of functions in model spaces, [16, 17]. They introduced the following auxiliary function, called an admissible function.

Let $\varphi : (0, +\infty) \to \mathbb{R}^+$ be a positive increasing function such that:

1. $x \to \frac{\varphi(x)}{x}$ is bounded,
2. $x \to \frac{\varphi(x)}{x^2}$ is decreasing,
3. $\varphi(x) \asymp \varphi(x + o(x)), \quad x \to 0.$

For example, the functions $f(x) = x^p$, where $1 \leq p < 2$ and $g(x) = -x^p \log x$, where $1 < p < 2$ are admissible functions.

Theorem 2.4. *[16, Theorem 1.5] Let Θ be an inner function with zeros $\{\lambda_n\}$ and associated singular measure μ, φ an admissible function, and $\zeta \in \mathbb{T}$. If*

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$$\sum \frac{1-|\lambda_n|^2}{\varphi(|\zeta-\lambda_n|)} + \int_{\mathbb{T}} \frac{d\mu(z)}{\varphi(|\zeta-z|)} < +\infty, \tag{2.3}$$

then every $f \in K_\Theta$ satisfies

$$|f(r\zeta)| \lesssim \frac{\sqrt{\varphi(1-r)}}{1-r}, \qquad r \to 1^-. \tag{2.4}$$

We noticed that we may have the same behavior for de Branges-Rovnyak spaces with the same technique. The following theorem controls the behavior of $\mathcal{H}(b)$ functions.

Theorem 2.5. *Let b be a function in the unit ball of H^∞ with the canonical factorization*

$$b(z) = \gamma \prod \left(\frac{|a_n|}{a_n} \frac{a_n - z}{1-\bar{a}_n z} \right) \exp\left(-\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} d\mu(\zeta) \right) \exp\left(\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} \log|b(\zeta)| dm(\zeta) \right),$$

φ an admissible function, and $\zeta \in \mathbb{T}$. If

$$\sum \frac{1-|\lambda_n|^2}{\varphi(|\zeta-\lambda_n|)} + \int_{\mathbb{T}} \frac{d\mu(z)}{\varphi(|\zeta-z|)} + \int_{\mathbb{T}} \frac{|\log|b(z)||}{\varphi(|\zeta-z|)} dm(z) < +\infty, \tag{2.5}$$

then every $f \in \mathcal{H}(b)$ satisfies

$$|f(r\zeta)| \lesssim \frac{\sqrt{\varphi(1-r)}}{1-r}, \qquad r \to 1^-. \tag{2.6}$$

It is well known that $\mathcal{H}(b)$ is a reproducing kernel Hilbert space with kernel function

$$k_\lambda^b(z) = \frac{1-\overline{b(\lambda)}b(z)}{1-\bar{\lambda}z},$$

with

$$f(\lambda) = \langle f, k_\lambda^b \rangle_b, \qquad (\lambda \in \mathbb{D}),$$

for every $f \in \mathcal{H}(b)$.

The following lemma [1, Lemma 3] asserts that, under a certain condition, the radial limit of $b(z)$ on $\mathbb{T}$ is of modulus 1.

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Lemma 2.6. *[1] Let b be a function in the unit ball of H^∞ with the canonical factorization*

$$b(z) = \gamma \prod \left(\frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z} \right) \exp \left(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right) \exp \left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log |b(\zeta)| dm(\zeta) \right).$$

Let $\zeta \in \mathbb{T}$. Then the following are equivalent.

1. $$\sum \frac{1 - |\lambda_n|^2}{|\zeta - \lambda_n|} + \int_{\mathbb{T}} \frac{d\mu(z)}{|\zeta - z|} + \int_{\mathbb{T}} \frac{|\log |b(z)||}{|\zeta - z|} dm(z) < +\infty.$$
2. *Every divisor of b has a radial limit of modulus 1 at ζ.*
3. *Every divisor of b has a radial limit at ζ.*

It is enough to prove Theorem 2.5 for $\zeta = 1$. Theorem 2.5 follows from the following theorem and the Cauchy-Schwarz inequality

$$|f(r)| \leq \|f\|_b \left\|k_r^b\right\|_b, \qquad f \in \mathcal{H}(b), \quad r \in (0,1).$$

Theorem 2.7. *Let b be a function in the unit ball of H^∞ with the canonical factorization*

$$b(z) = \gamma \prod \left(\frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z} \right) \exp \left(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right) \exp \left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log |b(\zeta)| dm(\zeta) \right),$$

φ an admissible function, and $\zeta \in \mathbb{T}$. If

$$\sum \frac{1 - |\lambda_n|^2}{\varphi(|\zeta - \lambda_n|)} + \int_{\mathbb{T}} \frac{d\mu(z)}{\varphi(|\zeta - z|)} + \int_{\mathbb{T}} \frac{|\log |b(z)||}{\varphi(|\zeta - z|)} dm(z) < +\infty, \tag{2.7}$$

then

$$\left\|k_r^b\right\|_b \lesssim \frac{\sqrt{\varphi(1-r)}}{(1-r)}, \qquad r \to 1^-. \tag{2.8}$$

Proof. A similar technique to [16, Theorem 2.1] will be used. Since $x \mapsto \varphi(x)/x$ is bounded, (2.7) implies

$$\sum \frac{1 - |\lambda_n|^2}{|\zeta - \lambda_n|} + \int_{\mathbb{T}} \frac{d\mu(z)}{|\zeta - z|} + \int_{\mathbb{T}} \frac{|\log |b(z)||}{|\zeta - z|} dm(z) < +\infty.$$

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Hence, by Lemma 2.6 $|b(r)| \to 1$ as $r \to 1^-$. So

$$\|k_r^b\|_b^2 \asymp \frac{\log|b(r)|^{-2}}{1-r^2}.$$

Let $d\sigma(\zeta) = d\mu(\zeta) + |log|b(\zeta)||dm(\zeta)$. Then

$$\begin{aligned}
\log|b(r)|^{-2} &= -2\log|b(r)| \\
&= -\sum \log\left|\frac{\lambda_n - r}{1-\overline{\lambda_n}r}\right|^2 + 2\int \Re\left(\frac{\zeta+r}{\zeta-r}\right)d\sigma(\zeta) \\
&= -\sum \log\left(1 - \frac{(1-|\lambda_n|^2)^2(1-r^2)}{|1-\overline{\lambda_n}r|^2}\right)^2 + 2\int \Re\left(\frac{\zeta+r}{\zeta-r}\right)d\sigma(\zeta).
\end{aligned}$$

Let $d\Sigma(z) = \sum_n (1-|z|^2)\delta_{\lambda_n}(z) + d\sigma(z)$, where $z \in \overline{\mathbb{D}}$ and δ is the Dirac measure. Hence

$$\begin{aligned}
\frac{\log|b(r)|^{-2}}{1-r^2} &\le \sum \frac{1-|\lambda_n|^2}{|1-\overline{\lambda_n}r|^2} + \int_{\mathbb{T}} \frac{d\sigma(\zeta)}{|1-r\overline{\zeta}|^2} \\
&\asymp \int_{\overline{\mathbb{D}}} \frac{1}{(1-r^2)+\theta^2} d\Sigma(z) \\
&\lesssim \int_{\{\theta:1-r\le\theta\}} \frac{1}{(1-r^2)+\theta^2} d\Sigma(z) \\
&+ \int_{\{\theta:1-r\ge\theta\}} \frac{1}{(1-r^2)+\theta^2} d\Sigma(z) \\
&\lesssim \int_{\{\theta:1-r\le\theta\}} \frac{1}{\theta^2} d\Sigma(z) + \frac{1}{(1-r)^2}\int_{\{\theta:1-r\ge\theta\}} d\Sigma(z).
\end{aligned}$$

For the first integral, note that by Cauchy Schwartz inequality, we have

$$\begin{aligned}
\int_{\{\theta:1-r\le\theta\}} \frac{1}{\theta^2} d\Sigma(z) &= \int_{\{\theta:1-r\le\theta\}} \frac{1}{\sqrt{\varphi(\theta)}\theta^2/\sqrt{\varphi(\theta)}} d\Sigma(z) \\
&\le \left(\int_{\{\theta:1-r\le\theta\}} \frac{d\Sigma(z)}{\varphi(\theta)}\right)^{1/2} \times \qquad (2.9) \\
&\quad \left(\int_{\{\theta:1-r\le\theta\}} \frac{d\Sigma(z)}{\theta^4/\varphi(\theta)}\right)^{1/2}.
\end{aligned}$$

Since φ is an admissible function and $|1-e^{i\theta}| \asymp \theta$ we have $\varphi(\theta) \asymp \varphi(|1-e^{i\theta}|)$, which, according to (2.7) says that

$$\left(\int_{\{\theta:1-r\le\theta\}} \frac{d\Sigma(z)}{\varphi(\theta)}\right)^{1/2} < +\infty.$$

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Hence

$$\begin{aligned}\int_{\{\theta:1-r\leq\theta\}}\frac{1}{\theta^2}d\Sigma(z) &\lesssim \left(\int_{\{\theta:1-r\leq\theta\}}\frac{d\Sigma(z)}{\theta^4/\varphi(\theta)}\right)^{1/2}\\ &\lesssim \left(\int_{\{\theta:1-r\leq\theta\}}\frac{\varphi^2(\theta)}{\theta^4\varphi(\theta)}d\Sigma(z)\right)^{1/2}.\end{aligned}$$

Since $x\mapsto\varphi(x)/x^2$ is decreasing we have

$$\varphi^2(\theta)/\theta^4\leq\frac{\varphi^2(1-r)}{(1-r)^4},\ \text{for } 1-r\leq\theta.$$

Hence

$$\begin{aligned}\int_{\{\theta:1-r\leq\theta\}}\frac{1}{\theta^2}d\Sigma(z) &\lesssim \frac{\varphi(1-r)}{(1-r)^2}\left(\int_{\{\theta:1-r\leq\theta\}}\frac{1}{\varphi(\theta)}d\Sigma(z)\right)^{1/2}\\ &\lesssim \frac{\varphi(1-r)}{(1-r)^2}.\end{aligned}$$

For the second integral, note that

$$\begin{aligned}\int_{\{\theta:1-r\geq\theta\}}d\Sigma(z) &\leq \left(\int_{\{\theta:1-r\geq\theta\}}\varphi(\theta)d\Sigma(z)\right)^{1/2}\cdot\\ &\quad\left(\int_{\{\theta:1-r\geq\theta\}}\frac{1}{\varphi(\theta)}d\Sigma(z)\right)^{1/2}, \qquad (2.10)\end{aligned}$$

and the last integral is finite. Since φ is increasing we get

$$\int_{\{\theta:1-r\geq\theta\}}d\Sigma(z) \lesssim \sqrt{\varphi(1-r)}\left(\int_{\{\theta:1-r\geq\theta\}}d\Sigma(z)\right)^{1/2},$$

therefore,

$$\left(\int_{\{\theta:1-r\geq\theta\}}d\Sigma(z)\right)^{1/2}\lesssim\sqrt{\varphi(1-r)},$$

and finally,

$$\int_{\{\theta:1-r\geq\theta\}}d\Sigma(z)\lesssim\varphi(1-r).$$

□

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2.2 The operator $C_\varphi : \mathcal{H}(b) \to H^2$

As the theory of composition operator on the Hardy spaces is well known, in this section we discuss the composition operator on subspaces in the Hardy space. Since the operator $C_\varphi : H^2 \to H^2$ is bounded then it is easy to get the boundedness of the operator if we restrict the domain on de Branges-Rovnyak spaces.

Proposition 2.8. *Let b be in the unit ball of H^∞ and φ be a self-map on $\mathbb{D}$. Then the operator*

$$\begin{aligned} C_\varphi : \mathcal{H}(b) &\to H^2 \\ f &\mapsto f \circ \varphi \end{aligned}$$

is bounded.

Proof. It is clear that the operator is well defined since $\mathcal{H}(b)$ is a subset of H^2. Let $\widetilde{C}$ denote the composition operator on H^2 into H^2. Then for any $f \in \mathcal{H}(b)$, using Lemma 2.1 we have

$$\begin{aligned} \|C_\varphi f\|_2 &= \left\|\widetilde{C}_\varphi f\right\|_2 \\ &\leq \left\|\widetilde{C}_\varphi\right\| \|f\|_2 \\ &\leq \left\|\widetilde{C}_\varphi\right\| \|f\|_b . \end{aligned}$$

So C_φ acts boundedly as an operator from $\mathcal{H}(b)$ into H^2. □

The composition operator is bounded for all analytic symbols φ, which is not the case for the compactness. Lyubarskii and Malinnikova characterized the compactness of the operator $C_\varphi : K_\Theta \to H^2$, where Θ is any inner function, [20]. In what follows we study the compactness of the operator

$$C_\varphi : \mathcal{H}(b) \to H^2.$$

The Lyubarskii–Malinnikova condition 1.9 is still a necessary condition for the compactness. The following lemmas will be needed. Lemma 2.9 comes from [31, Lemma 7] but we recall the proof for the sake of completeness. Lemma 2.10 was originally proved in [20] for any inner function and it is still true for any function in the unit ball of H^∞.

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Lemma 2.9. *Let* $\lambda, \mu \in \mathbb{D}$*. If*

$$\left|\frac{\lambda-\mu}{1-\bar{\lambda}\mu}\right| \leq \epsilon,$$

where $\epsilon \in (0,1)$*, then*

$$\frac{1-\epsilon}{1+\epsilon} \leq \frac{1-|\lambda|}{1-|\mu|} \leq \frac{1+\epsilon}{1-\epsilon}. \tag{2.11}$$

Proof. First

$$\begin{aligned}
\left|\frac{\lambda-\mu}{1-\bar{\lambda}\mu}\right|^2 &= 1-\frac{|1-\bar{\lambda}\mu|^2-|\lambda-\mu|^2}{|1-\bar{\lambda}\mu|^2} \\
&= 1-\frac{(1-|\lambda|^2)(1-|\mu|^2)}{|1-\bar{\lambda}\mu|^2} \\
&\geq 1-\frac{(1-|\lambda|^2)(1-|\mu|^2)}{(1-|\lambda||\mu|)^2} \\
&\geq \left(\frac{|\lambda|-|\mu|}{1-|\lambda||\mu|}\right)^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\epsilon &\geq \frac{|\lambda|-|\mu|}{1-|\lambda||\mu|} \\
&= \frac{(1-|\mu|)-(1-|\lambda|)}{(1-|\lambda|)+|\lambda|(1-|\mu|)} \\
&\geq \frac{(1-|\mu|)-(1-|\lambda|)}{(1-|\lambda|)+(1-|\mu|)}.
\end{aligned}$$

A straightforward computation leads us to

$$\frac{1-|\mu|}{1-|\lambda|} \leq \frac{1+\epsilon}{1-\epsilon}. \tag{2.12}$$

Since λ and μ are interchangeable we get the other side of the (2.11).

□

Lemma 2.10. *Let* $\{\lambda_n\}$ *be a sequence in the unit disc such that* $|\lambda_n| \to 1$ *and*

$$|b(\lambda_n)| < \delta \tag{2.13}$$

for some $\delta \in (0,1)$*. Then*

1. $\widetilde{k}^b_{\lambda_n} \xrightarrow{w^*} 0$ *as* $n \to \infty$ *(weak* convergence);*

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2. *there is* $\epsilon > 0, c > 0$ *and* n_0 *such that*

$$|(k^b_{\lambda_n})'(\zeta)| > \frac{c}{(1-|\lambda_n|^2)^2}, \qquad \zeta \in D_\epsilon(\lambda_n) \tag{2.14}$$

holds for all $n > n_0$*, where* $D_\epsilon(\lambda) = \{\zeta : |\zeta - \lambda| < \epsilon|1 - \overline{\zeta}\lambda|\}$ *is a hyperbolic disc with center at* λ*.*

Proof. 1. We recall that

$$\widetilde{k^b_{\lambda_n}}(z) = \left(\frac{1-|\lambda_n|^2}{1-|b(\lambda_n)|}\right)^{1/2} \frac{1-\overline{b(\lambda_n)}b(z)}{1-\overline{\lambda_n}z},$$

and for any $f \in \mathcal{H}(b) \cap H^\infty$, we have

$$\begin{aligned} \left|\langle f, \widetilde{k^b_{\lambda_n}}\rangle_b\right| &= \left(\frac{1-|\lambda_n|^2}{1-|b(\lambda_n)|}\right)^{1/2} |f(\lambda_n)| \\ &\le \frac{\|f\|_\infty}{\sqrt{1-\delta}} \left(1-|\lambda_n|^2\right)^{1/2}. \end{aligned}$$

Hence $\langle f, \widetilde{k^b_{\lambda_n}}\rangle_b \to 0$, as $n \to +\infty$, for any function $f \in \mathcal{H}(b) \cap H^\infty$. The result follows immediately using that $\mathcal{H}(b) \cap H^\infty$ is dense in $\mathcal{H}(b)$ (note that the reproducing kernel $k^b_\lambda \in \mathcal{H}(b) \cap H^\infty$ for any $\lambda \in \mathbb{D}$).

2. Since b is a self map on $\mathbb{D}$ we have by the Schwartz–Pick inequality

$$\left|\frac{b(z)-b(\lambda_n)}{1-\overline{b(\lambda_n)}b(z))}\right| \le \left|\frac{z-\lambda_n}{1-\overline{\lambda_n}z}\right|, \quad z \in \mathbb{D},$$

and

$$|b'(z)| \le \frac{1-|b(z)|^2}{1-|z^2|}, \quad z \in \mathbb{D}.$$

Hence,

$$|b(\zeta) - b(\lambda_n)| \le 2\left|\frac{\zeta-\lambda_n}{1-\overline{\lambda_n}\zeta}\right|. \tag{2.15}$$

That is

$$|b(\zeta)| \le |b(\lambda_n)| + 2\left|\frac{\zeta-\lambda_n}{1-\overline{\lambda_n}\zeta}\right|. \tag{2.16}$$

Since $|b(\lambda_n)| < \delta$

$$|b(\zeta)| < c, \qquad \zeta \in \cup_{n\ge1} D_\epsilon(\lambda_n), \tag{2.17}$$

for some $c < 1$ and $\epsilon > 0$.

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We claim now that for sufficiently large n_0

$$|(k_\zeta^b)'(\zeta)| > \frac{C}{(1-|\zeta|^2)^2}, \qquad \zeta \in \cup_{n>n_0} D_\epsilon(\lambda_n).$$

We note that

$$(k_\zeta^b)'(\zeta) = \underbrace{-\frac{b'(\zeta)\overline{b(\zeta)}}{1-|\zeta|^2}}_{A_1} + \underbrace{\bar\zeta\frac{1-|b(\zeta)|^2}{(1-|\zeta|^2)^2}}_{A_2}.$$

Since $|b(\zeta)| < c$ we clearly have

$$|A_2| > \frac{c'}{(1-|\zeta|^2)^2},$$

for some $c' > 0$.

It is enough to show that

$$|A_1| < q|A_2| \quad \text{for some } q \in (0,1), \tag{2.18}$$

when $\zeta \in \cup_{n>n_0} D_\epsilon(\lambda_n)$. Indeed, assuming (2.18), we have

$$\begin{aligned} |(k_\zeta^b)'(\zeta)| &= |A_1 + A_2| \\ &\geq |A_2| - |A_1| \\ &\geq |A_2| - q|A_2| \\ &= (1-q)|A_2| \geq \frac{(1-q)c'}{(1-|\zeta|^2)^2}. \end{aligned}$$

To prove (2.18) note that:

$$|A_1| \leq |b(\zeta)|\frac{1-|b(\zeta)|^2}{(1-|\zeta|^2)^2} < \frac{c}{|\zeta|}|A_2|, \quad \zeta \in \cup_n D_\epsilon(\lambda_n).$$

Since $c < 1$ and $\inf\{|\zeta| : \zeta \in \cup_{n>m} D_\epsilon(\lambda_n)\} \to 1$ as $m \to \infty$, then $q = \sup \frac{c}{|\zeta|} < 1$, and hence the estimate follows for the special case $\zeta = \lambda_n$.

In order to complete the proof consider the function

$$g(\lambda,\zeta) = \overline{(k_\lambda^b)'(\zeta)} = -\frac{\overline{b'(\zeta)}b(\lambda)}{1-\bar\zeta\lambda} + \lambda\frac{1-\overline{b(\zeta)}b(\lambda)}{(1-\bar\zeta\lambda)^2}.$$

We have $|g(\lambda_n,\zeta)| = |(k_{\lambda_n}^b)'(\zeta)|$. On the other hand

$$|g(\lambda_n,\zeta) - g(\zeta,\zeta)| \leq |g'(\tilde w,\zeta)||\zeta - \lambda_n|, \tag{2.19}$$

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for some point $\tilde{w} \in [\zeta, \lambda_n]$, where the derivative is taken with respect to the first variable. By Lemma 2.9 we have

$$|g'(\tilde{w}, \zeta)| < \frac{\text{const}}{(1 - |\lambda_n|)^3}, \;\; \tilde{w}, \zeta \in D_\epsilon(\lambda_n),$$

the constant being independent of n. Now, given any $\eta > 0$ we can choose $\epsilon' < \epsilon$ such that the right-hand side in (2.19) does not exceed $\eta(1 - |\zeta|^2)^{-2}$ when $\zeta \in D_{\epsilon'}(\lambda_n)$. Taking η sufficiently small we obtain the result.

□

In the following theorem we obtain a necessary condition for the compactness of the composition operator on de Branges–Rovnyak spaces.

Theorem 2.11. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic such that $\varphi(0) = 0$. If b is in the unit ball of H^∞, then a necessary condition for the operator $C_\varphi : \mathcal{H}(b) \to H^2$ to be compact is that the Nevanlinna counting function of φ satisfies*

$$N_\varphi(\lambda) \frac{1 - |b(\lambda)|^2}{1 - |\lambda|^2} \to 0 \;\; \textit{as } |\lambda| \to 1. \tag{2.20}$$

Proof. Suppose that C_φ is compact but the condition (2.20) is not satisfied. Then there is a sequence $(\lambda_n)_n$ in $\mathbb{D}$, $|\lambda_n| \to 1$, such that

$$N_\varphi(\lambda_n) \frac{1 - |b(\lambda_n)|^2}{1 - |\lambda_n|^2} \geq c > 0. \tag{2.21}$$

Since $N_\varphi(\omega) \leq \log \frac{1}{|\omega|}$ then there is a constant $a < 1$ such that condition (2.13) is satisfied. Then by Lemma 2.10, $\widetilde{k}^b_{\lambda_n} \xrightarrow{w^*} 0$. Since C_φ is compact we have

$$\left\| C_\varphi \widetilde{k}^b_{\lambda_n} \right\|_2 \to 0. \tag{2.22}$$

On the other hand, we have

$$\begin{aligned}
\left\| C_\varphi \widetilde{k}^b_{\lambda_n} \right\|_2^2 &\geq \int_{\mathbb{D}} |(\widetilde{k}^b_{\lambda_n})'(\zeta)|^2 N_\varphi(\zeta) dA(\zeta) \\
&\geq \int_{\mathbb{D}} |(k^b_{\lambda_n})'(\zeta)|^2 (1 - |\lambda_n|^2) N_\varphi(\zeta) dA(\zeta) \\
&\geq \frac{c}{(1 - |\lambda_n|^2)^3} \int_{D_\epsilon(\lambda_n)} N_\varphi(\zeta) dA(\zeta) \\
&\geq \frac{c_\epsilon N_\varphi(\lambda_n)}{1 - |\lambda_n|^2}.
\end{aligned}$$

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The first inequality follows from Stanton's formula, while the third inequality is an application of Lemma 2.10 and the last one is the subharmonicity of the Nevanlinna counting function (1.7). That contradicts (2.22).

□

2.3 Sufficient condition for compactness

In [20], the authors proved the sufficiency of (2.20) for the compactness of the composition operator on model spaces, and used an inequality derived from [2, Lemma5] by Cohn [7, page 187]. That inequality was essential in the proof. In this section, we obtain a weaker inequality in a slightly different way, using the weighted Bernstein inequality for the de Branges–Rovnyak spaces [3]. Then this inequality is used to obtain a weaker sufficient condition for the compactness of the composition operator on the de Branges–Rovnyak spaces.

We start by recalling the weighted Bernstein inequality. To introduce the inequality we need the following notations. Let $1 < p \leq 2$, and let q be the conjugate exponent of p. Also, let $\rho(e^{i\theta}) = 1 - |b(e^{i\theta})|^2$, $\theta \in [0, 2\pi]$ and

$$\mathfrak{K}_\lambda^\rho(z) = \overline{b(\lambda)}\frac{2\overline{b(\lambda)}b(z) - \overline{b'(\lambda)}b'(z)}{1 - \overline{\lambda}z},$$

where $\lambda, z \in \mathbb{D}$. Finally Let

$$w_p(z) := \min\left\{ \|(k_z^b)^2\|_q^{-p/(p+1)},\ \|\rho^{1/q}\mathfrak{K}_z^\rho\|_q^{-p/(p+1)} \right\},$$

where $z \in \mathbb{D}$.

Then, Baranov, Fricain and Mashreghi obtained the following inequalities.

Lemma 2.12. *[3, Lemma 3.5] For $1 < p \leq 2$, there is a constant $A > 0$ such that*

$$w_p(z) \geq A\frac{1 - |z|}{(1 - |b(z)|)^{\frac{p}{q(p+1)}}}, \qquad z \in \mathbb{D}.$$

Theorem 2.13. *[3, Theorem 4.1] Let μ be a Carleson measure, let $1 < p \leq 2$, and let*

$$(T_p f)(z) = f'(z)w_p(z), \qquad f \in \mathcal{H}(b).$$

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Thèse de Muath Karaki, Lille 1, 2015

If $1 < p < 2$, then T_p is a bounded operator from $\mathcal{H}(b)$ to $L^2(\mu)$, that is, there is a constant $C = C(\mu, p) > 0$ such that

$$\|f' w_p\|_{L^2(\mu)} \leq C\|f\|_b, \qquad f \in \mathcal{H}(b). \tag{2.23}$$

If $p = 2$, then T_2 is of weak type $(2, 2)$ as an operator from $\mathcal{H}(b)$ to $L^2(\mu)$.

The following lemma insures that the subspace of functions that have zero of order n at the origin of any boundedly contained subspace in the Hardy space is closed and its orthogonal complement is finite dimensional space. We recall that a Hilbert space E is said to be boundedly contained into H^2 if $E \subset H^2$ and $\|f\|_2 \lesssim \|f\|_E$, for any $f \in E$.

Lemma 2.14. *Let E be boundedly contained into H^2. Then the subspace*

$$E^{(n)} := \{f \in E; f \text{ has zero of order } n \text{ at the origin}\}$$

is closed in E. Moreover, the subspace $(E^{(n)})^\perp$ is of finite dimension.

Proof. Let us first prove the closedness. Let $(f_k)_k$ be a sequence in $E^{(n)}$ and converges to f in E. Since E is boundedly contained in H^2 we have f_k converges to f in H^2. Hence

$$f_k^{(j)}(0) \to f^{(j)}(0),$$

for all $j \geq 0$. But for $j \leq n-1$ and $k \geq 1$ $f_k^{(j)}(0)$ is zero and hence $f^{(j)}(0)$ is so. That means $f \in E^{(n)}$.

Let us now prove that $\dim(E \ominus E^{(n)}) < +\infty$. Since

$$f \to f^{(j)}(0)$$

is continuous, there exists $k_{j,0} \in E$ such that

$$f^{(j)}(0) = \langle f, k_{j,0}\rangle_E,$$

for all $f \in E$. It is enough to show that

$$E \ominus E^{(n)} = \mathcal{L}\left(k_{0,0}, k_{1,0}, \ldots, k_{n-1,0}\right).$$

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On one hand, for $0 \le j \le n-1$ and $f \in E^{(n)}$ we have,

$$\langle f, k_{j,0} \rangle_E = f^{(j)}(0) = 0.$$

On the other hand, if $f^{(j)}(0) = 0$ for all $j \le n-1$ then $f \in E^{(n)}$. Hence

$$\mathcal{L}\left(k_{0,0}, k_{1,0}, \ldots, k_{n-1,0}\right)^{\perp} \subset E^{(n)},$$

which means

$$(E^{(n)})^{\perp} \subset \mathcal{L}\left(k_{0,0}, k_{1,0}, \ldots, k_{n-1,0}\right).$$

Therefore

$$E \ominus E^{(n)} = \mathcal{L}\left(k_{0,0}, k_{1,0}, \ldots, k_{n-1,0}\right).$$

That is $E \ominus E^{(n)}$ is of finite dimension. □

In the following theorem we obtain a sufficient condition for the compactness of the composition operator on the de Branges–Rovnyak spaces.

Theorem 2.15. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic such that $\varphi(0) = 0$. Let b be in the unit ball of H^∞ and $\gamma \in (0, 1/3)$. If there is $p \in (1, 2)$ such that the Nevanlinna counting function of φ satisfies*

$$N_\varphi(z) w_p^{-2}(z) \to 0, \qquad \textit{as } |z| \to 1, \tag{2.24}$$

then the operator $C_\varphi : \mathcal{H}(b) \to H^2$ is compact.

Proof. Let $\mathcal{H}(b)^{(n)} = \{f \in \mathcal{H}(b);\ f$ has zero of order n at the origin$\}$. Since $\mathcal{H}(b)$ is contractively contained in H^2 then, by Lemma 2.14, we can define

$$\Pi^{(n)} : \mathcal{H}(b) \to \mathcal{H}(b)^{(n)}$$

to be the corresponding orthogonal projection.

Since, by Lemma 2.14, the orthogonal complement of $\mathcal{H}(b)^{(n)}$ is of finite dimension, it is enough to prove that

$$\left\| C_\varphi \Pi^{(n)} \right\|_{\mathcal{H}(b) \to H^2} \to 0, \qquad n \to +\infty.$$

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Indeed, this implies that the operator C_φ can be approximated by the finite rank operator $C_\varphi(I-\Pi^{(n)})$ and thus the operator is compact.

Given a function $f \in \mathcal{H}(b)$ such that $\|f\|_b = 1$, let $g_n = \Pi^{(n)} f$. Then $\|g_n\|_b \leq 1$ and, for each $R<1, \epsilon>0$ we can choose $n(\epsilon, R)$ independent of f such that

$$|g_n(\omega)| < \epsilon,\ |g_n'(\omega)| < \epsilon \text{ for all } n > n(\epsilon, R), \ \text{ and } |\omega| < R.$$

Indeed, since $g_n = \Pi^{(n)} f \in \mathcal{H}(b)^{(n)}$, the Taylor series of g_n at the origin can be written as

$$g_n(w) = \sum_{k=n}^{\infty} \frac{g_n^{(k)}(0)}{k!} w^k.$$

But

$$\begin{aligned} \left| \frac{g_n^{(k)}(0)}{k!} \right| &= \left| \langle g_n, w^k \rangle_2 \right| \\ &\leq \|g_n\|_2 \left\|w^k\right\|_2 \\ &= \|g_n\|_2 \leq \|f\|_b = 1. \end{aligned}$$

Hence,

$$|g_n(w)| \leq \sum |w|^k = \frac{w^n}{1-|w|} \leq \frac{R^n}{1-R}.$$

Since $R^n \to 0$ as $n \to +\infty$, we can find $n(\epsilon, R) \in \mathbb{N}$ such that

$$g_n(w) < \epsilon.$$

For g_n', write

$$g_n'(w) = \sum_{k=n}^{\infty} \frac{g_n^{(k)}(0)}{k!} k w^{k-1}.$$

Hence,

$$\begin{aligned} |g_n'(w)| &\leq \sum_{k=n}^{\infty} k |w|^{k-1} \\ &\leq \sum_{k=n}^{\infty} k R^{k-1} \end{aligned}$$

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So $|g_n'(w)| \to 0$ as $n \to \infty$. Let $\gamma \in (0, 1/3)$ and define

$$p := \frac{1+\gamma}{1-\gamma}.$$

It is easy to check that $1 < p < 2$. Since the area measure is a Carleson measure, Theorem 2.13 implies that

$$\int_{\mathbb{D}} |g'(z)|^2 w_p^2(z) dA(z) \leq C.$$

Therefore, since $\varphi(0) = 0$, Stanton's formula implies that

$$\begin{aligned}
\left\|C_\varphi \Pi^{(n)} f\right\|_2^2 &= 2\int_{\mathbb{D}} |g_n'(z)|^2 N_\varphi(z) dA(z) \\
&= 2\int_{|z|<R} |g_n'(z)|^2 N_\varphi(z) dA(z) \\
&+ 2\int_{R\leq|z|<1} |g_n'(z)|^2 N_\varphi(z) dA(z) \\
&\leq \max_{|z|<R}\{|g_n'(\omega)|^2\} \int_{|z|<R} N_\varphi(z) dA(z) \\
&+ A_{R,\varphi} \int_{R\leq|z|<1} |g_n'(z)|^2 w_p^2(z) dA(z)
\end{aligned}$$

where $A_{R,\varphi} = \max_{R<|z|<1} N_\varphi(z) w_p^{-2}(z)$. Since the last integral is controlled by C, $A_{R,\varphi} \to 0$ as $R \to 1$ and for large n the maximum of $|g_n'|$ in $|z| < R$ is small enough we get the result. □

Corollary 2.16. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic such that $\varphi(0) = 0$. Let b be in the unit ball of H^∞. If there is $\gamma \in (0, 1/3)$ such that the Nevanlinna counting function of φ satisfies*

$$N_\varphi(z) \left(\frac{(1 - |b(z)|)^\gamma}{1 - |z|^2} \right)^2 \to 0 \ \text{as} \ |z| \to 1, \tag{2.25}$$

then the operator $C_\varphi : \mathcal{H}(b) \to H^2$ is compact.

Proof. Take $p = \frac{1+\gamma}{1-\gamma}$. Since $\gamma \in (0, 1/3)$ it easily implies that $p \in (1, 2)$ and if q is the conjugate exponent of p, we have

$$\gamma = \frac{p-1}{p+1} = \frac{p}{q(p+1)}.$$

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Then Lemma 2.12 gives that

$$w_p(z) \geq A\frac{1-|z|}{(1-|b(z)|)^{\gamma}}.$$

And thus (2.25) implies that

$$N_{\varphi}(z)w_p^{-2}(z) \to 0, \tag{2.26}$$

as $|z| \to 1$. Then we can apply Theorem 2.15 which yields that the operator $C_{\varphi} : \mathcal{H}(b) \to H^2$ is compact.

□

2.4 Axler, Chang and Sarasons' lemma

Lemma 5 of [2] is essential for obtaining a sufficient condition for the compactness of the composition operator on model spaces. In this section we generalize this lemma for general function in the unit ball of H^{∞}.

To state the lemma some notations are needed. For $e^{i\theta}$ a point of $\mathbb{T}$, let Γ_{θ} denote the angle with vertex $e^{i\theta}$ and opening $\pi/2$ which bisected by the radius to $e^{i\theta}$. The set of points z in Γ_{θ} satisfying $|e^{i\theta} - z| < \epsilon$ will be denoted by $\Gamma_{\theta,\epsilon}$. For h in L^1 we define

$$A_{\epsilon}h(\theta) = \left[\int_{\Gamma_{\theta,\epsilon}} |\nabla h(z)|^2 dA(z)\right]^{1/2},$$

for $\epsilon \in (0,1]$. Here, ∇h refers to usual gradient of the harmonic extension of h onto $\mathbb{D}$.

The Hardy-Littlewood maximal function of h will be denoted by $h^{\star}$ and is defined as follows

$$h^{\star}(e^{it}) = \sup_{I \ni e^{it}} \frac{1}{|I|}\int_I |h(r^{it})|dt,$$

where I is an arc in the unit circle. For $r > 1$ we let

$$\Lambda_r h = \left[(|h|^r)^{\star}\right]^{1/2}.$$

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For $z \in \mathbb{D}$ let I_z denote the closed subarc of $\mathbb{T}$ with center $z/|z|$ and measure $1-|z|$. Let $1 < p, q < 2$ be fixed such that $r = pq < 2$.

The following theorem generalizes Axler–Chang–Sarasons' lemma, [2, Lemma 5], which was proved for an inner function b.

Lemma 2.17. *If b is a function in the unit ball of H^∞ then for any h in H^2 and any $\delta > 0$,*

$$\left[\int_{\substack{|b|>1-\delta \\ |z|>1/2}} \left|\nabla(I-P)\bar{b}h\right|^2 (1-|z|)dA(z)\right]^{1/2} \leq K\delta^C \|h\|_2 . \tag{2.27}$$

The following lemma, is the key point in the proof of the previous lemma.

Lemma 2.18 (Distribution function inequality)**.** *If b is a function in the unit ball of H^∞, h is in H^2, and z is a point in $\mathbb{D}$ such that $|b(z)| > 1-\delta$ and $|z| = 1-\epsilon > \frac{1}{2}$, then, for $a > 0$ sufficiently large,*

$$|I_z \cap \{A_{2\epsilon}(I-P)\bar{b}h \leq a\delta^C \Lambda_r h\}| \geq K_a |I_z|, \tag{2.28}$$

where $C = 1/p'q$, and where $p' = p/(p-1)$.

First we deduce Lemma 2.17 from Lemma 2.18.

Proof of Lemma 2.17. Let a be fixed so that inequality (2.28) is satisfied. For $e^{i\theta}$ in $\mathbb{T}$ let $\rho(\theta)$ be the maximum of those numbers ϵ for which

$$A_\epsilon(I-P)\bar{b}h(\theta) \leq a\delta^C \Lambda_r h(\theta).$$

Let χ_θ denote the characteristic function of $\Gamma_{\theta,\rho(\theta)}$. Hence,

$$\left(\int_{-\pi}^{\pi} \left(A_{\rho(\theta)}\left(I-P\right)\bar{b}h(\theta)\right)^2 d\theta\right)^{1/2} \leq a\delta^C \left(\int_{-\pi}^{\pi} \left(\Lambda_r h(\theta)\right)^2 d\theta\right)^{1/2}.$$

It is clear that the square of the left side is

$$\int_{\mathbb{D}} \int_{-\pi}^{\pi} \chi_\theta(z) \left|\nabla(I-P)\bar{b}h(z)\right|^2 d\theta dA(z).$$

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Let $E = I_z \cap \{A_{2\epsilon}(I-P)\bar{b}h \leq a\delta^C \Lambda_r h\}$. So, if $|b(z)| > 1-\delta$ and $|z| = 1-\epsilon > 1/2$, then Lemma 2.18 implies that $|E| \geq K|I_z| \geq K'(1-|z|)$. Moreover, if $\theta \in E$ we have

$$A_{2\epsilon}(I-P)\bar{b}h(\theta) \leq a\delta^C \Lambda_r h(\theta).$$

Hence $\rho(\theta) \geq 2\epsilon = 2(1-|z|)$ on E.

Now,

$$\begin{aligned} |e^{i\theta} - z| &\leq \left| e^{i\theta} - \frac{z}{|z|} \right| + \left| \frac{z}{|z|} - z \right| \\ &\leq (1-|z|) + (1-|z|) = 2(1-|z|) \leq \rho(\theta), \end{aligned}$$

because $\theta \in E$. That means $\chi_\theta(z) = 1$ for any such z and θ.

Therefore,

$$\left[\int_{\substack{|b|>1-\delta \\ |z|>1/2}} \left| \nabla (I-P)\bar{b}h \right|^2 (1-|z|) dA(z) \right]^{1/2} \leq K\delta^C \left\| \Lambda_r h \right\|_2 . \tag{2.29}$$

By the Hardy-Littlewood maximal theorem,

$$\left\| \Lambda_r h \right\|_2 = \left\| (|h|^r)^* \right\|_{2/r}^{1/r} \leq K \left\| h \right\|_2 .$$

□

The proof of the distribution function inequality is accomplished in the following lemmas.

Lemma 2.19. *Let $f \in L^2$ and write $f = h + \bar{g}$, where h and g are in H^2. Then*

$$|\nabla (I-P)f| \leq |\nabla f|.$$

Proof. On one hand, we have

$$\begin{aligned} |\nabla (I-P)f|^2 &= |\nabla \bar{g}|^2 \\ &= 2|g'|^2. \end{aligned}$$

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On the other hand

$$\begin{aligned}|\nabla f|^2 &= |h_x + \overline{g_x}|^2 + |h_y + \overline{g_y}|^2 \\ &= 2|h'|^2 + 2|g'|^2 + 2\Re[h_x\overline{g_x} + h_y\overline{g_y}] \\ &= 2|h'|^2 + 2|g'|^2 + 2\Re[h'\overline{g'} - h'\overline{g'}] \\ &= 2|h'|^2 + 2|g'|^2.\end{aligned}$$

□

Lemma 2.20. *Let b be a function in the unit ball on H^∞ and $h \in H^2$. Let z be a point of $\mathbb{D}$ such that $|b(z)| > 1-\delta$. If $h_1 = \chi_{2I_z}(\overline{b} - \overline{b(z)})h$ (where $2I_z$ is the closed subarc of the unit circle with centre $z/|z|$ and measure $2(1-|z|)$), then*

$$\int_0^{2\pi} (A_1(I-P)h_1(\theta))^q \, d\theta \le K\delta^{1/p'}|I_z| \inf_{e^{it}\in I_z} (\Lambda_r h(t))^q$$

Proof. Since, by Lemma 2.19 we have

$$|\nabla(I-P)h_1| \le |\nabla h_1|,$$

it is enough to show the inequality with h_1 in place of $(I-P)h_1$ on the left side.

By the theorem of Marcinkiewicz and Zygmund [21, Theorem 1] we have

$$\begin{aligned}\int_0^{2\pi} (A_1 h_1)^q d\theta &\le K\int_0^{2\pi} |h_1|^q d\theta \\ &= K\int_{2I_z} |b-b(z)|^q |h|^q d\theta \\ &\le K|I_z| \left(\frac{1}{|I_z|}\int_{2I_z} |b-b(z)|^{p'q} d\theta\right)^{1/p'} \left(\frac{1}{|I_z|}\int_{2I_z} |h|^{pq} d\theta\right)^{1/p},\end{aligned}$$

where $1/p + 1/p' = 1$. Note that $p'q - 2 = \frac{pq}{p-1} - 2 = \frac{pq-2p+2}{p-1}$. Using that $p, q \in (1,2)$, we easily check that $p'q > 2$. Let P_z be the Poisson kernel for the point z. It is

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clear that $P_z(e^{i\theta}) \geq K/|I_z|$, for $e^{i\theta} \in 2I_z$. So

$$\begin{aligned}
\frac{1}{|I_z|}\int_{2I_z} |b-b(z)|^{p'q} d\theta &\lesssim \int_0^{2\pi} |b-b(z)|^{p'q} P_z d\theta \\
&\lesssim \int_0^{2\pi} |b-b(z)|^2 P_z d\theta \\
&\lesssim \int_0^{2\pi} |b|^2 P_z d\theta - 2\int_0^{2\pi} \Re\overline{b(z)} b P_z d\theta + |b(z)|^2 \int_0^{2\pi} P_z d\theta \\
&\lesssim 2 - 2\Re\overline{b(z)} b(z) \\
&\lesssim (1-|b(z)|^2) \lesssim \delta.
\end{aligned}$$

Since

$$\frac{1}{|I_z|}\int_{2I_z} |h|^{pq} d\theta \leq (\Lambda_r h(t))^r$$

for all $e^{it} \in 2I_z$, we get the result. □

Lemma 2.21. *Let b be a function in the unit ball of H^∞ and $h \in H^2$. Let z be a point of $\mathbb{D}$ such that $|b(z)| > 1-\delta$ and $\epsilon = 1-|z| < 1/2$. Let*

$$h_2 = (\overline{b} - \overline{b(z)})h - h_1,$$

where h_1 is as in Lemma 2.20. Then, for $e^{i\theta} \in I_z$, we have

$$A_{2\epsilon}(I-P)h_2(t) \leq K\delta^{1/r'} \Lambda_r h(t).$$

Proof. It is enough to show that

$$|\nabla(I-P)h_2(\zeta)| \leq K\delta^{1/r'}(1-|z|)^{-1}\Lambda_r h(t), \quad \zeta \in \Gamma_{t,2\epsilon}, e^{i\theta} \in \mathbb{T}.$$

We recall that $\Gamma_{t,2\epsilon}$ is the set of points $\zeta \in \Gamma_t$ satisfying $|\zeta - e^{it}| < 2\epsilon$. Now,

$$(I-P)h_2(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\overline{\zeta} e^{i\theta} h_2(e^{i\theta})}{1-\overline{\zeta} e^{i\theta}} d\theta.$$

The function $(I-P)h_2$ is an anti-analytic function in $\mathbb{D}$. So the absolute value of its gradient is $2^{1/2}$ times the absolute value of its $\overline{\partial}-$derivative.

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So,

$$\begin{aligned} |\nabla(I-P)h_2(\zeta)| &\leq \frac{2^{1/2}}{2\pi}\int_0^{2\pi}\left|\frac{\partial}{\partial\overline{\zeta}}\frac{\overline{\zeta}e^{i\theta}h_2(e^{i\theta})}{1-\overline{\zeta}e^{i\theta}}\right|d\theta \\ &\leq \frac{2^{1/2}}{2\pi}\int_0^{2\pi}\frac{|h_2(e^{i\theta})|}{|1-\overline{\zeta}e^{i\theta}|^2}d\theta \\ &= \frac{2^{1/2}}{2\pi}\int_{\mathbb{T}\backslash 2I_z}\frac{|b(e^{i\theta})-b(z)||h(e^{i\theta})|}{|e^{i\theta}-\zeta|^2}d\theta. \qquad (2.30) \end{aligned}$$

If $e^{i\theta}\in\mathbb{T}\backslash 2I_z$ and $\zeta\in\Gamma_{t,2\epsilon}$ with $e^{it}\in I_z$, we have

$$\begin{aligned} \left|\frac{e^{i\theta}-z}{e^{i\theta}-\zeta}\right| &\leq 1+\left|\frac{\zeta-z}{e^{i\theta}-\zeta}\right| \\ &\leq 1+\frac{|\zeta-e^{it}|+|e^{it}-\frac{z}{|z|}|+(1-|z|)}{|e^{i\theta}-\zeta|} \\ &\leq 1+\frac{4|I_z|}{|e^{i\theta}-\zeta|} \\ &\leq 1+\frac{4|I_z|}{dist(\mathbb{T}\backslash 2I_z,\Gamma_{t,2\epsilon})}. \end{aligned}$$

An elementary estimate shows that, under the stated condition, the distance in the denominator above is bigger than $K|I_z|$.

Hence

$$\left|\frac{e^{i\theta}-z}{e^{i\theta}-\zeta}\right|\lesssim 1,$$

which enables us to replace $|e^{i\theta}-\zeta|$ by $|e^{i\theta}-z|$ in the integral (2.30).

We thus obtain

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Thèse de Muath Karaki, Lille 1, 2015

$$\begin{aligned}
|\nabla(I-P)h_2(\zeta)| &\lesssim \frac{2^{1/2}}{2\pi}\int_0^{2\pi}\frac{|b-b(z)||h|}{|e^{i\theta}-z|^2}d\theta. \\
&\leq \frac{2^{1/2}}{2\pi(1-|z|^2)}\int_0^{2\pi}|b-b(z)||h|P_z d\theta. \\
&\lesssim \frac{1}{1-|z|^2}\left(\frac{1}{2\pi}\int_0^{2\pi}|b-b(z)|^{r'}P_z d\theta\right)^{1/r'} \\
&\quad \left(\frac{1}{2\pi}\int_0^{2\pi}|h|^r P_z d\theta\right)^{1/r} \\
&\leq \frac{K\delta^{1/r'}}{1-|z|^2}\left(\frac{1}{2\pi}\int_0^{2\pi}|h|^r P_z d\theta\right)^{1/r}.
\end{aligned}$$

The last inequality follows like the argument in the last lemma. Because the non-tangential maximal function is bounded by a constant times the Hardy-Littlewood maximal. the last factor on the right is no larger than $K\Lambda_r h(t)$, and the desired inequality is established.

□

Lemma 2.22. *Let $z \in \mathbb{D}$ such that $|b(z)| > 1-\delta$ and $\epsilon = 1-|z| < 1/2$. For $a > 0$ Let*

$$E(a) := \{e^{i\theta} \in I_z : A_{2\epsilon}(I-P)\bar{b}h(\theta) \leq a\delta^{1/p'q}\Lambda_r h(\theta)\}.$$

Then, $|E(a)| \geq K_a|I_z|$ for a sufficiently large.

Proof. Let h_1, h_2 be as in Lemmas 2.20 and 2.21. We have

$$(I-P)\bar{b}h = (I-P)(\bar{b}-\overline{b(z)})h = (I-P)h_1 + (I-P)h_2.$$

Consequently

$$A_{2\epsilon}(I-P)\bar{b}h \leq A_{2\epsilon}(I-P)h_1 + A_{2\epsilon}(I-P)h_2.$$

Let

$$E_1(a) := \{e^{i\theta} \in I_z : A_{2\epsilon}(I-P)h_1(\theta) \leq a\delta^{1/p'q}\Lambda_r h(\theta)\}.$$

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We have,

$$\begin{aligned}\int_{I_z}(A_{2\epsilon}(I-P)h_1(\theta))^q\,d\theta &\geq \int_{I_z\setminus E_1(a)}(A_{2\epsilon}(I-P)h_1(\theta))^q\,d\theta\\ &\geq \int_{I_z\setminus E_1(a)} a^q\delta^{1/p'}(\Lambda_r h(\theta))^q d\theta\\ &\geq a^q\delta^{1/p'}\inf_{e^{t\theta}\in I_z}(\Lambda_r h(\theta))^q\int_{I_z\setminus E_1(a)}d\theta\\ &\geq a^q\delta^{1/p'}\inf_{e^{t\theta}\in I_z}(\Lambda_r h(\theta))^q(|I_z|-|E_1(a)|).\end{aligned}$$

Therefore, by Lemma 2.20 we have

$$\begin{aligned}a^q\delta^{1/p'}(|I_z|-|E_1(a)|)\inf_{e^{it}\in I_z}(\Lambda_r h(t))^q &\leq \int_{I_z}(A_{2\epsilon}(I-P)h_1(\theta))^q\,d\theta\\ &\leq K\delta^{1/p'}|I_z|\inf_{e^{it}\in I_z}(\Lambda_r h(t))^q\,.\end{aligned}$$

Hence,

$$\frac{|E_1(a)|}{|I_z|}\geq 1-a^{-q}K,$$

and the right side is positive for large a.

Finally, by Lemma 2.21 we have,

$$A_{2\epsilon}(I-P)h_2\leq a\delta^{1/r'}\Lambda_r h\leq a\delta^{1/p'q}\Lambda_r h,$$

everywhere on I_z, which means $E(2a)\supset E_1(a)$. The proof is complete. □

2.5 Conjugation and $\mathcal{X}(b)$ spaces

Let H be a Hilbert space. A map $C: H\to H$ is called a conjugation if C is antilinear, isometric, surjective and $C^2 = Id$. Let Θ be an inner function, and let $K_\Theta = H^2\ominus\Theta H^2$ be the associated model space. Since $K_\Theta = H^2\cap\Theta\overline{H_0^2}$, where $H_0^2 = zH^2$, then it is easy to see that the map Ω_Θ defined on K_Θ by

$$\Omega_\Theta(f)=\bar{z}\bar{f}\Theta,\qquad f\in K_\Theta,$$

is a conjugation on K_Θ.

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Moreover, since $|\Theta| = 1$ *a.e.* on $\mathbb{T}$, we have

$$\begin{aligned}\Omega_\Theta\left(k_\lambda^\Theta\right)(z) &= \bar{z}\Theta(z)\frac{1-\Theta(\lambda)\overline{\Theta(z)}}{1-\lambda\bar{z}} \\ &= \frac{\Theta(z)-\Theta(\lambda)}{z-\lambda},\end{aligned}$$

for almost every $z \in \mathbb{T}$. Denote by

$$\widehat{k}_\lambda^\Theta(z) = \frac{\Theta(z)-\Theta(\lambda)}{z-\lambda}, \qquad \lambda, z \in \mathbb{D}.$$

Then $\widehat{k}_\lambda^\Theta \in K_\Theta$ and the above computations imply that the family $\left(\widehat{k}_\lambda^\Theta\right)_{\lambda\in\mathbb{D}}$ is complete in K_Θ. It appears that a similar construction can be done in $\mathcal{H}(b)$ spaces, where b is extreme. Let $b \in H^\infty$, $\|b\|_\infty \leq 1$. Since the function

$$z \to \frac{1-|b(z)|^2}{|1-b(z)|^2}$$

is positive and harmonic on $\mathbb{D}$, it can be represented as the Poisson integral of a positive measure. In other words, there exists a unique positive Borel measure μ on $\mathbb{T}$ such that

$$\frac{1-|b(z)|^2}{|1-b(z)|^2} = \int_{\mathbb{T}} \frac{1-|z|^2}{|z-\zeta|}d\mu(\zeta), \qquad z \in \mathbb{D}. \tag{2.31}$$

The measure μ is called the Clark measure.

If for $q \in L^2(\mu)$, we put

$$(V_b q)(z) = (1-b(z))\int_{\mathbb{T}} \frac{q(\zeta)}{1-\bar{\zeta}z}d\mu(\zeta), z \in \mathbb{D}. \tag{2.32}$$

Then it is known [27] that V_b is a unitary map from $H^2(\mu)$ onto $\mathcal{H}(b)$, where $H^2(\mu)$ is the closure of the polynomials in the $L^2(\mu)$-norm. We also recall that if b is extreme,[12, 27], then

$$H^2(\mu) = L^2(\mu).$$

We also define the conjugation C on $L^2(\mu)$ by

$$C(f) = \bar{z}\bar{f}.$$

Then we have the following theorem

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Theorem 2.23. *[12] Let b be an extreme point of the closed unit ball of H^∞. Then, the operator*

$$\Omega_b = V_b C V_b^{-1}$$

is a conjugation on $\mathcal{H}(b)$ and we have

$$\Omega_b k_\lambda^b = \widehat{k}_\lambda^b, \qquad \lambda \in \mathbb{D}. \tag{2.33}$$

Proof. Since b is an extreme point of the closed unit ball of H^∞, then $L^2(\mu) = H^2(\mu)$ and thus V_b is a unitary map from $L^2(\mu)$ onto $\mathcal{H}(b)$. Hence, $V_b C V_b^{-1}$ is clearly a conjugation on $\mathcal{H}(b)$. It remains to verify the formula (2.33). We have

$$\begin{aligned} V_b C V_b^{-1} k_\lambda^b &= V_b C \left((1 - \overline{b(\lambda)} k_\lambda) \right) \\ &= (1 - b(\lambda) V_b \left(\frac{e^{-i\theta}}{1 - \lambda e^{-i\theta}} \right) \\ &= (1 - b(\lambda) V_b \left(\frac{1}{e^{i\theta} - \lambda} \right) \end{aligned}$$

The first equality follows from the following known fact, [12, 27]

$$V_b k_\omega = \frac{k_\omega^b}{1 - \overline{b(\omega)}}.$$

Then, using the definition of V_b, we can write

$$\begin{aligned} (V_b C V_b^{-1}) k_\lambda^b(z) &= (1 - b(\lambda))(1 - b(z)) \int_{\mathbb{T}} \frac{1}{1 - e^{-i\theta} z} \frac{1}{e^{i\theta} - \lambda} d\mu(e^{i\theta}) \\ &= (1 - b(\lambda))(1 - b(z)) \mathcal{I}, \end{aligned}$$

where

$$\mathcal{I} = \int_{\mathbb{T}} \frac{e^{i\theta}}{e^{i\theta} - z} \frac{1}{e^{i\theta} - \lambda} d\mu(e^{i\theta}).$$

An easy computation show that

$$\frac{e^{i\theta} + z}{e^{i\theta} - z} - \frac{e^{i\theta} + \lambda}{e^{i\theta} - \lambda} = \frac{2(z - \lambda) e^{i\theta}}{(e^{i\theta} - \lambda)(e^{i\theta} - z)}.$$

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Thus

$$\begin{aligned}\mathcal{I} &= \frac{1}{2(z-\lambda)}\int_{\mathbb{T}}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}-\frac{e^{i\theta}+\lambda}{e^{i\theta}-\lambda}\right]d\mu(e^{i\theta})\\ &= \frac{1}{2(z-\lambda)}\left[\frac{1+b(z)}{1-b(z)}-\frac{1+b(\lambda)}{1-b(\lambda)}\right]\\ &= \frac{b(z)-b(\lambda)}{(z-\lambda)(1-b(z))(1-b(\lambda))}.\end{aligned}$$

Therefore,

$$\begin{aligned}(V_bCV_b^{-1})k_\lambda^b(z) &= (1-b(\lambda))(1-b(z))\frac{b(z)-b(\lambda)}{(z-\lambda)(1-b(z))(1-b(\lambda))}\\ &= \frac{b(z)-b(\lambda)}{z-\lambda}.\end{aligned}$$

□

We will now give another expression for this conjugation. For that purpose, we need to recall a well known result on integral representation for functions in $\mathcal{H}(\bar{b})$.

For b a function in the closed unit ball of H^∞, define

$$\rho(\zeta) = 1-|b(\zeta)|^2, \qquad \zeta\in\mathbb{T}.$$

Since $\rho\in L^\infty(\mathbb{T})$, it is easy to check that the mapping

$$\begin{aligned}K_\rho : L^2(\rho) &\to H^2\\ f &\mapsto P_+(\rho f)\end{aligned}$$

is bounded operator whose norm is at most $\|\rho\|_\infty^{1/2}$.

The following result gives an integral representation for functions in $\mathcal{H}(\bar{b})$.

Theorem 2.24. *[12, 27] The operator K_ρ is a partial isomerty from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$ and* $\ker K_\rho = (H^2(\rho))^\perp$.

In particular, we see that when b is an extreme point, then $H^2(\rho) = L^2(\rho)$ and then K_ρ is a unitary operator from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$.

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Let $\lambda \in \mathbb{D}$. Then, according to Theorem 2.23 we have

$$\begin{aligned}\left(\Omega_b k_\lambda^b\right)(\zeta) &= \frac{b(\zeta)-b(\lambda)}{\zeta-\lambda} \\ &= \frac{\bar{\zeta} b(\zeta)}{1-\bar{\zeta}\lambda} - \bar{\zeta}\frac{b(\lambda)}{1-\bar{\zeta}\lambda}.\end{aligned}$$

Since $T_{\bar{b}} k_\lambda = \overline{b(\lambda)} k_\lambda$ we get

$$\left(\Omega_b k_\lambda^b\right)(\zeta) = \overline{\zeta k_\lambda(\zeta)} b(\zeta) - \bar{\zeta}\overline{\left(T_{\bar{b}} k_\lambda\right)(\zeta)}. \tag{2.34}$$

Now remember that $k_\lambda^b = \left(I - T_b T_{\bar{b}}\right) k_\lambda$, which suggests defining the following operator

$$\mathbb{C}_b\left(\left(I - T_b T_{\bar{b}}\right) f\right) = b\bar{z}\bar{f} - \bar{z}\overline{T_{\bar{b}} f}, \qquad f \in H^2.$$

Theorem 2.25. *Let b be a function in the closed unit ball of H^∞. For $f \in H^2$, define*

$$\mathbb{C}_b\left(\left(I - T_b T_{\bar{b}}\right) f\right) = b\bar{z}\bar{f} - \bar{z}\overline{T_{\bar{b}} f}.$$

Then $\mathbb{C}_b$ extends to an antilinear contraction map from $\mathcal{H}(b)$ into itself, such that for each $\lambda \in \mathbb{D}$, we have

$$\mathbb{C}_b k_\lambda^b = \widehat{k}_\lambda^b. \tag{2.35}$$

Furthermore, if b is extreme, then $\mathbb{C}_b$ is isometric and $\mathbb{C}_b = \Omega_b$.

Proof. Let $f \in H^2$ and $g = \left(I - T_b T_{\bar{b}}\right) f$. Let us first verify that $\mathbb{C}_b g \in H^2$. Recall the conjugation J on $L^2(\mathbb{T})$ which is defined by

$$(Jf)(\zeta) = \bar{\zeta}\overline{f(\zeta)}, \qquad f \in L^2(\mathbb{T}),$$

and which satisfies $JP_+ = P_- J$. Then write

$$\begin{aligned}\mathbb{C}_b g &= \mathbb{C}_b\left(I - T_b T_{\bar{b}}\right) f \\ &= \bar{z}\bar{f} b - \bar{z}\overline{T_{\bar{b}} f} \\ &= \bar{z}\bar{f} b - JP_+(\bar{b} f) \\ &= \bar{z}\bar{f} b - P_- J(\bar{b} f) \\ &= \bar{z}\bar{f} b - P_-(\bar{z} b\bar{f}) \\ &= P_+(\bar{z}\bar{f} b),\end{aligned}$$

and this function $P_+(\bar{z}\bar{f} b)$ clearly belongs to H^2.

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Now, $\mathbb{C}_b g \in \mathcal{H}(b)$ if and only if $T_{\bar{b}}\mathbb{C}_b g$ belongs to $\mathcal{H}(\bar{b})$, [12, 27]. But,

$$\begin{aligned} T_{\bar{b}}\mathbb{C}_b g &= T_{\bar{b}} P_+ \bar{z}\bar{f} b \\ &= P_+ \bar{b} P_+ \bar{z}\bar{f} b \\ &= P_+\left(\bar{b}\bar{z}\bar{f} b\right) \\ &= P_+\left(|b|^2 \bar{z}\bar{f}\right) \\ &= -P_+\left(\rho\bar{z}\bar{f}\right) = -K_\rho(\bar{z}\bar{f}). \end{aligned}$$

Since $\bar{z}\bar{f} \in L^2(\mathbb{T})$, it follows from Theorem 2.24 that $K_\rho(\bar{z}\bar{f}) \in \mathcal{H}(\bar{b})$, whence $\mathbb{C}_b g \in \mathcal{H}(b)$.

Let us verify that $\mathbb{C}_b$ is a contraction. We have

$$\|\mathbb{C}_b g\|_b^2 = \left\|P_+\left(b\bar{z}\bar{f}\right)\right\|_2^2 + \left\|P_{H^2(\rho)}\left(\bar{z}\bar{f}\right)\right\|_{L^2(\rho)}^2,$$

where $P_{H^2(\rho)}$ denotes the orthogonal projection from $L^2(\rho)$ onto $H^2(\rho)$. Hence, we get

$$\begin{aligned} \|\mathbb{C}_b g\|_b^2 &\leq \left\|P_+\left(b\bar{z}\bar{f}\right)\right\|_2^2 + \left\|\rho\bar{z}\bar{f}\right\|_{L^2(\rho)}^2 \\ &= \left\|P_+\left(b\bar{z}\bar{f}\right)\right\|_2^2 + \|f\|_2^2 - \|bf\|_2^2 . \end{aligned} \tag{2.36}$$

Now, using that J is isometric and $JP_+ = P_- J$ we have

$$\begin{aligned} \left\|P_+\left(b\bar{z}\bar{f}\right)\right\|_2^2 &= \left\|JP_+\left(b\bar{z}\bar{f}\right)\right\|_2^2 \\ &= \left\|P_- J\left(b\bar{z}\bar{f}\right)\right\|_2^2 \\ &= \left\|P_-\bar{b}f\right\|_2^2 \\ &= \|bf\|_2^2 - \|T_{\bar{b}}f\|_2^2 \end{aligned}$$

Hence,

$$\begin{aligned} \|\mathbb{C}_b g\|_b^2 &\leq \|f\|_2^2 - \|T_{\bar{b}}f\|_2^2 . \\ &= \left\|\left(I - T_b T_{\bar{b}}\right)^{1/2} f\right\|_2^2 \\ &= \|g\|_b^2 . \end{aligned}$$

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We thus have proved that $\mathbb{C}_b$ is a contraction on the set

$$\{(I - T_b T_{\bar{b}})f : f \in H^2\}$$

which is a dense set of $\mathcal{H}(b)$. Hence, it extends to a contraction map from $\mathcal{H}(b)$ into itself. The formula (2.35) has already been proved.

It remains to note that when b is extreme, then $H^2(\rho) = L(\rho)$. and then K_ρ is an isomerty from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$. In particular we have equality in (2.36) which gives

$$\|\mathbb{C}_b g\|_b = \|g\|_b \,,$$

for any $g \in \mathcal{R}(I - T_b T_{\bar{b}})$. That proves that $\mathbb{C}_b$ extends to an isometric map from $\mathcal{H}(b)$ into itself if b is extreme.

□

Note that when Θ is an inner function, then for any $f \in \mathcal{H}(\Theta) = K_\Theta$, we have $T_{\bar{\Theta}} f = 0$ and that gives

$$\begin{aligned} \mathbb{C}_\Theta f &= \mathbb{C}_\Theta((I - T_\Theta T_{\bar{\Theta}})f) \\ &= \Theta \bar{z} \bar{f}. \end{aligned}$$

Hence, we see that $\mathbb{C}_\Theta$ coincides with the conjugation on K_Θ introduced at the beginning of this section.

We will define a new space. Let

$$\begin{aligned} \mathbb{T}_b : H^2 &\rightarrow H^2 \\ g &\mapsto \mathbb{T}_b g = \bar{z}\bar{g}b - \bar{z}\overline{T_{\bar{b}} g}. \end{aligned}$$

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In other words, $\mathbb{T}_b = \mathbb{C}_b(I - T_bT_{\bar{b}})$. Note that $\mathbb{T}_b$ is antilinear and contraction. Indeed for any $g \in H^2$ we have

$$\begin{aligned}
\|\mathbb{T}_b g\|_2^2 &= \|\mathbb{C}_b(I - T_bT_{\bar{b}})\|_2^2 \\
&\leq \|\mathbb{C}_b(I - T_bT_{\bar{b}})\|_b^2 \\
&\leq \|(I - T_bT_{\bar{b}})g\|_b^2 \\
&= \left\|(I - T_bT_{\bar{b}})^{1/2}g\right\|_2^2 \\
&= \|g\|_2^2 - \|T_{\bar{b}}g\|_2^2 \\
&\leq \|g\|_2^2 .
\end{aligned}$$

Hence it is a contraction.

According to Theorem 2.25, the range of $\mathbb{T}_b$ is a subspace of $\mathcal{H}(b)$. We also note that in the case when b is extreme then $\mathcal{I}m(\mathbb{T}_b)$ is dense in $\mathcal{H}(b)$. Finally, when $b = \Theta$ is inner, then

$$\begin{aligned}
\mathcal{I}m(\mathbb{T}_\Theta) &= \mathbb{C}_\Theta(I - T_\Theta T_\Theta)H^2 \\
&= \mathbb{C}_\Theta K_\Theta \\
&= K_\Theta .
\end{aligned}$$

In fact, we put on the space $\mathcal{I}m(\mathbb{T}_b)$ a new Hilbert structure.

Definition 2.26. Let b be a point in the closed unit ball of H^∞. Define $\mathcal{X}(b) = \mathbb{T}_b H^2$, and for any $f_1, f_2 \in H^2$ such that $f_1, f_2 \perp \ker \mathbb{T}_b$,

$$\langle \mathbb{T}_b f_1, \mathbb{T}_b f_2 \rangle_{\mathcal{X}(b)} := \langle f_2, f_1 \rangle_2 .$$

Theorem 2.27. *Let b be a point in the closed unit ball of H^∞. Then $(\mathcal{X}(b), \|.\|_{\mathcal{X}(b)})$ is a Hilbert space contractively contained into $\mathcal{H}(b)$.*

Proof. The axioms of inner product space follow from straightforward computations. Note that the antilinearity of $\mathbb{T}_b$ gives the linearity of the inner product. To check completeness, let $g_n \in H^2$ and $g_n \perp \ker \mathbb{T}_b$. If $f_n = \mathbb{T}_b g_n$, $n \in \mathcal{N}$, is a Cauchy sequence in $\mathcal{X}(b)$ then $(g_n)_n$ is a Cauchy sequence in H^2. Hence there is $g \in H^2$ such that $g_n \to g$ in H^2. Let $f = \mathbb{T}_b g$, then $g_n \to g$ in $\mathcal{X}(b)$.

By Theorem 2.25 we have

$$\mathcal{X}(b) \subset \mathcal{H}(b),$$

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Thèse de Muath Karaki, Lille 1, 2015

moreover, if $g \in H^2$ such that $g \perp \ker \mathbb{T}_b$ and $f = \mathbb{T}_b g$ then we have

$$\begin{aligned} \|f\|_b^2 &= \|\mathbb{C}_b(I - T_b T_{\bar{b}})g\|_b^2 \\ &\leq \|(I - T_b T_{\bar{b}})g\|_b^2 \\ &= \left\|(I - T_b T_{\bar{b}})^{1/2} g\right\|_2^2 \\ &= \|g\|_2^2 - \|T_{\bar{b}}\|_2^2 \\ &\leq \|g\|_2^2 = \|f\|_{\mathcal{X}(b)}^2 \end{aligned}$$

□

The following theorem says that $\mathcal{X}(b)$ is invariant under S^*.

Lemma 2.28. *For any $f \in H^2$, we have*

$$S^* \mathbb{T}_b f = \mathbb{T}_b S f.$$

In particular, $S^* \mathcal{X}(b) \subset \mathcal{X}(b)$.

Proof. On one hand, we have

$$\begin{aligned} S^* \mathbb{T}_b f &= P_+ \bar{z} \left(\bar{z} \bar{f} b - \bar{z} \overline{T_{\bar{b}} f} \right) \\ &= P_+ \left(\bar{z}^2 \bar{f} b \right) - P_+ \left(\bar{z}^2 \overline{T_{\bar{b}} f} \right) \\ &= P_+ \left(\bar{z}^2 \bar{f} b \right). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \mathbb{T}_b S f &= \mathbb{T}_b(zf) \\ &= \bar{z}\bar{z}\bar{f} b - \bar{z}\overline{T_{\bar{b}} f} \\ &= \bar{z}^2 \bar{f} b - J P_+(z f \bar{b}), \end{aligned}$$

where we recall that $(Jh)(z) = \bar{z}\overline{h(z)}$. But $JP_+ = P_- J$, whence

$$\begin{aligned} \mathbb{T}_b S f &= \bar{z}^2 \bar{f} b - P_- J(z f \bar{b}) \\ &= \bar{z}^2 \bar{f} b - P_-(\bar{z}^2 \bar{f} b) \\ &= P_+(\bar{z}^2 \bar{f} b), \end{aligned}$$

which gives the desired equality. □

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2.6 A sufficient condition for compactness on $\mathcal{X}(b)$

In this section we give a sufficient condition for the compactness of the composition operator on $\mathcal{X}(b)$. Cohn, in [7], wrote Axler–Chang–Sarason's inequality [2] for the model spaces in the following form.

$$\int_{\mathbb{D}} |h'(z)|^2 \frac{1-|z|}{(1-|\Theta(z)|)^p} dA(z) \leq c\, \|h\|_2^2 . \tag{2.37}$$

for all h in K_Θ. We rewrite (2.27) in Cohn's form for functions in $\mathcal{X}(b)$. Hence we will be able to obtain a sufficient condition for the compactness of the composition operator on $\mathcal{X}(b)$.

We have the following lemma.

Lemma 2.29. *Let b be in the unit ball of H^∞. Then, there exists $K > 0$ and $c \in (0,1)$ such that for any $h \in \mathcal{X}(b)$ and $\delta > 0$ we have*

$$\iint_{\substack{|b|>1-\delta \\ |z|>1/2}} |h'(z)|^2 (1-|z|) dA(z) \leq K\delta^C \|h\|^2_{\mathcal{X}(b)} . \tag{2.38}$$

Proof. Let $h \in \mathcal{X}(b)$. Then there is a unique function $g \in H^2$ with $g \perp \ker \mathbb{T}_b$ such that $h = \mathbb{T}_b g$. Then Lemma 2.28 implies $S^\star h = \mathbb{T}_b(zg) = \mathbb{T}_b(g_1) = \overline{\bar{z}(I-P_+)(\bar{b}g_1)}$, which means $h - h(0) = zS^\star h = \overline{(I-P_+)(\bar{b}g_1)}$. Hence

$$\left|\nabla (I-P_+)\bar{b}g_1\right|^2 = 2|h'|^2.$$

Hence, Lemma 2.17 implies that there exists $c' \in (0,1/2)$ and $K > 0$ such that

$$\begin{aligned} \iint_{\substack{|b|>1-\delta \\ |z|>1/2}} |h'(z)|^2 (1-|z|) dA(z) &\leq K\delta^{2c'} \|g_1\|_2^2 \\ &\leq K\delta^C \|g\|_2^2 \\ &\leq K\delta^C \|\mathbb{T}_b g\|^2_{\mathcal{X}(b)} \end{aligned}$$

□

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Instead of dealing with $\mathcal{X}(b)$, we can define the following space $\mathcal{X}_1(b) = \mathcal{M}(T_1)$ where

$$\begin{aligned} T_1 : H^2 &\rightarrow H^2 \\ g &\mapsto \mathbb{T}_b g^*, \end{aligned}$$

where $g^*(\zeta) = \overline{g(\bar{\zeta})}$. In this case T_1 is linear.

Now, we have the Axler-Chang-Sarason Lemma for the $\mathcal{X}(b)$ spaces.

Theorem 2.30. *Let b be in the unit ball of H^∞. Then there is $c > 0$ and $p \in (0,1)$ such that for any $h \in \mathcal{X}(b)$ we have*

$$\int_{\mathbb{D}} |h'(z)|^2 \frac{1-|z|}{(1-|b(z)|)^p} dA(z) \leq c\, \|h\|^2_{\mathcal{X}(b)}. \tag{2.39}$$

Proof. First, let $\delta \in (0,1)$ and $p \in (0,1)$. Then we have

$$\begin{aligned} \int_{\mathbb{D}} |h'(z)|^2 \frac{1-|z|}{(1-|b(z)|)^p} dA(z) &\leq \int_{|z|>1/2} |h'(z)|^2 \frac{1-|z|}{(1-|b(z)|)^p} dA(z) \\ &+ \int_{|z|\leq 1/2} |h'(z)|^2 \frac{1-|z|}{(1-|b(z)|)^p} dA(z). \end{aligned}$$

On the compact set $\{z : |z| \leq 1/2\}$ the function b is continuous and thus bounded and if

$$M = \sup_{|z|\leq 1/2} |b(z)|,$$

then we also have $M < 1$. Hence

$$\int_{|z|\leq 1/2} |h'(z)|^2 \frac{1-|z|}{(1-|b(z)|)^p} dA(z) \leq (1-M)^{-p} \int_{|z|\leq 1/2} |h'(z)|^2 dA(z),$$

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and by Littlewood–Paley identity, the last integral is bounded by $\|h\|_2^2 \leq \|g\|_{\mathcal{X}(b)}^2$. For the first integral, write

$$\int_{|z|>1/2} |h'(z)|^2 \frac{1-|z|}{(1-|b(z)|)^p} dA(z) \leq \underbrace{\int_{\substack{|z|>1/2 \\ |b|>1-\delta}} |h'(z)|^2 \frac{1-|z|}{(1-|b(z)|)^p} dA(z)}_{I_1} + \underbrace{\int_{\substack{|z|>1/2 \\ |b|\leq 1-\delta}} |h'(z)|^2 \frac{1-|z|}{(1-|b(z)|)^p} dA(z)}_{I_2}.$$

For I_2 we argue as before and we get

$$I_2 \leq \frac{1}{\delta^p} \int_{\substack{|z|>1/2 \\ |b|\leq 1-\delta}} |h'(z)|^2 dA(z),$$

and again using Littlewood–Paley identity, that implies that

$$I_2 \leq \frac{1}{\delta^p} \|h\|_2^2 \leq \frac{1}{\delta^p} \|h\|_{\mathcal{X}(b)}^2 .$$

It remains to estimate

$$I_1 = \int_{\substack{|z|>1/2 \\ |b|>1-\delta}} |h'(z)|^2 \frac{1-|z|}{(1-|b(z)|)^p} dA(z).$$

Write

$$I_1 = \int_{\mathbb{D}} |f|^p d\mu,$$

where $d\mu(z) = |h'(z)|^2(1-|z|)dA(z)$, and

$$f(z) = \begin{cases} \frac{1}{1-|b(z)|} & \text{if } |b(z)| > 1-\delta \text{ and } |z| > 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} I_1 &= \int_0^\infty pt^{p-1}\mu(\{z : |f(z)| > t\})dt \\ &= \underbrace{\int_0^2 pt^{p-1}\mu(\{z : |f(z)| > t\})dt}_{I_1'} + \underbrace{\int_2^\infty pt^{p-1}\mu(\{z : |f(z)| > t\})dt}_{I_1''}. \end{aligned}$$

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For I_1', we have

$$\begin{aligned} I_1' &\leq \mu(\mathbb{D})\int_0^2 pt^{p-1}dt \\ &= 2^p\mu(\mathbb{D}) \\ &= 2^p\int_{\mathbb{D}}|h'(z)|^2(1-|z|)dA(z), \end{aligned}$$

and the Littlewood Paley identity implies one more time that

$$I_1' \leq 2^p\,\|h\|_2^2 \leq 2^p\,\|h\|_{\mathcal{X}(b)}^2 .$$

For the last integral, note that

$$\mu(\{z : |f(z)| > t\}) = \int_{\substack{|z|>1/2 \\ |b|>1-\delta}} |h'(z)|^2(1-|z|)dA(z),$$

and Lemma 2.29 implies that there exists $K > 0$ and $c \in (0,1)$ such that

$$\mu(\{z : |f(z)| > t\}) \leq Kt^{-c}\,\|h\|_{\mathcal{X}(b)}^2 .$$

Hence

$$I_1'' \leq Kp\,\|h\|_{\mathcal{X}(b)}^2 \int_2^\infty t^{p-1-c}dt.$$

If $p < c$, we get

$$\begin{aligned} I_1'' &\leq K\frac{p}{p-c}\left[t^{p-c}\right]_2^{+\infty}\|h\|_{\mathcal{X}(b)}^2 \\ &= K\frac{p}{p-c}2^{p-c}\,\|h\|_{\mathcal{X}(b)}^2 . \end{aligned}$$

□

Now we consider the composition operator on $\mathcal{X}(b)$. That is

$$\begin{aligned} C_\varphi : \mathcal{X}(b) &\to H^2 \\ f &\mapsto f\circ\varphi. \end{aligned}$$

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Since $\mathcal{X}(b)$ is contractively contained in H^2 ,it is easy to see that the composition operator on $\mathcal{X}(b)$ is bounded.

Proposition 2.31. *Let b be in the unit ball of H^∞ and φ be a self-map on $\mathbb{D}$. Then, the operator*

$$\begin{aligned} C_\varphi : \mathcal{X}(b) &\to H^2 \\ f &\mapsto f \circ \varphi \end{aligned}$$

is bounded.

Proof. Let $f \in \mathcal{X}(b)$ be such that $f = \mathbb{T}_b g$, where $g \in H^2$ and $g \perp \ker \mathbb{T}_b$. Then it is well known that the operator

$$\begin{aligned} \widetilde{C}_\varphi : H^2 &\to H^2 \\ f &\mapsto f \circ \varphi \end{aligned}$$

is bounded and $C_\varphi = \widetilde{C}_\varphi | \mathcal{X}(b)$. Therefore, for any $f \in \mathcal{X}(b)$ we have

$$\begin{aligned} \|C_\varphi f\|_2 &= \left\| \widetilde{C}_\varphi f \right\| \\ &\leq \left\| \widetilde{C}_\varphi \right\| \|f\|_2 \\ &\leq \left\| \widetilde{C}_\varphi \right\| \|f\|_{\mathcal{X}(b)} . \end{aligned}$$

Hence, C_φ is bounded on $\mathcal{X}(b)$.

□

Now we are ready to give a sufficient condition for the compactness of the composition operator on $\mathcal{X}(b)$.

Theorem 2.32. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic such that $\varphi(0) = 0$. Let b be in the unit ball of H^∞. If the Nevanlinna counting function of φ satisfies*

$$N_\varphi(\omega) \frac{1 - |b(\omega)|^2}{1 - |\omega|^2} \to 0 \ \text{ as } |\omega| \to 1, \tag{2.40}$$

then $C_\varphi : \mathcal{X}(b) \to H^2$ is compact.

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Proof. The proof follows like the proof of Theorem 2.15. Since $N_\varphi(\omega)(1-|b(\omega)|^2)$ and $1-|\omega|^2$ are bounded, the condition (2.40) means that for any $a<1$,

$$\lim_{\substack{|b(\omega)|<a \\ |w|\to 1^-}} N_\varphi(\omega)(1-|\omega|^2)^{-1}=0.$$

In particular, for any $p>0$,

$$N_\varphi(\omega)\frac{(1-|b(\omega)|^2)^p}{1-|\omega|^2}\to 0 \text{ as } |\omega|\to 1. \tag{2.41}$$

Let

$$\mathcal{X}(b)^{(n)}=\{f\in\mathcal{X}(b);\, f \text{ has zero of order } n \text{ at the origin}\},$$

Since $\mathcal{X}(b)$ is contractively contained in H^2 then, by Lemma 2.14, the subspace $\mathcal{X}(b)^{(n)}$ is closed. Let

$$\Pi^{(n)}:\mathcal{X}(b)\to\mathcal{X}(b)^{(n)}$$

be the corresponding orthogonal projection. Again as in the case of K_Θ and $\mathcal{H}(b)$, we will prove that

$$\left\|C_\varphi\Pi^{(n)}\right\|_{\mathcal{X}(b)\to H^2}\to 0, \qquad n\to 0,$$

which is sufficient because the orthogonal complement of $\mathcal{X}(b)^{(n)}$ is of finite dimension, by Lemma 2.14.

Given a function $f\in\mathcal{X}(b)$ such that $\|f\|_{\mathcal{X}(b)}=1$, let $g_n=\Pi^{(n)}f$. Then $\|g_n\|_{\mathcal{X}(b)}\le 1$ and, for each $R<1,\epsilon>0$ we can choose $n(\epsilon,R)$ independent of f such that

$$|g_n(\omega)|<\epsilon,\ |g_n'(\omega)|<\epsilon \text{ for all } n>n(\epsilon,R), \text{ and } |\omega|<R.$$

Now Theorem 2.30 implies that there is a constant C independent of f,n such that

$$\int_{\mathbb{D}}|g_n'(z)|^2\frac{1-|z|}{(1-|b(z)|)^p}dA(z)\le C.$$

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Therefore

$$\begin{aligned}\left\|C_\varphi \Pi^{(n)} f\right\| &= \int_{\mathbb{D}} |g_n'(z)|^2 N_\varphi(z) dA(z) \\ &\leq \int_{|z|<R} |g_n'(z)|^2 N_\varphi(z) dA(z) \\ &+ \int_{R<|z|<1} |g_n'(z)|^2 N_\varphi(z) dA(z) \\ &\leq \max_{|z|<R} \{|g_n'(\omega)|^2\} \int_{|z|<R} N_\varphi(z) dA(z) \\ &+ A_{R,\varphi} \int_{R<|z|<1} |g_n'(z)|^2 \frac{1-|z|}{(1-|b(z)|)^p} dA(z)\end{aligned}$$

where $A_{R,\varphi} = \max_{R<|z|<1} N_\varphi(z) \frac{(1-|b(z)|)^p}{1-|\omega|}$. Since the last integral is controlled by C, $A_{R,\varphi} \to 0$ and for large n the maximum of $|g_n'|$ in $|z| < R$ is small enough we get the result. □

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Chapter 3

A group structure on $\mathbb{D}$ and its application to composition operators

3.1 Introduction

We present a group structure on $\mathbb{D}$ via the automorphisms which fix the point 1. Then, through the induced group action, each point of $\mathbb{D}$ produces an equivalence class which turns out to be a Blaschke sequence. Moreover, the corresponding Blaschke products are minimal solutions of the functional equation $\psi \circ \varphi = \lambda\psi$, where λ is a unimodular contant and φ is an automorphism of the unit disc which fixes the point 1. We also characterize all Blaschke products which satisfy this equation and study its application in the theory of composition operators on model spaces K_θ.

Mashreghi and Shabankhah [24], in studying the inner function φ for which C_φ maps K_Θ into itself, they encountered the functional equation

$$\psi(\varphi(z)) \times \omega(z) = \psi(z), \qquad z \in \mathbb{D}, \tag{3.1}$$

where ψ, ω, φ are inner functions. A variation of (3.1) is known as Schroder equation and has a very long and rich history. In the rest of this section we will present how they, in [24], came up with this equation.

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In [24] they studied the action of C_φ on a given K_θ when the symbol φ is an inner function. In the following theorem they determined the smallest model space that contains the image of C_φ under K_θ.

Theorem 3.1. *[24, Theorem 2.1] Let φ and θ be inner functions, and let*

$$\eta(z) = \begin{cases} (\theta \circ \varphi)(z) & \text{if } \ \theta(0) \neq 0 \text{ and } \varphi(0) = 0, \\ z(\theta \circ \varphi)(z) & \text{if } \ \theta(0) \neq 0 \text{ and } \varphi(0) \neq 0, \\ z\frac{\theta(\varphi(z))}{\varphi(z)} & \text{if } \ \theta(0) = 0. \end{cases}$$

Then the mapping $C_\varphi : K_\theta \to K_\eta$ is well-defined and bounded. Moreover, K_η equals the closed invariant subspace generated by $C_\varphi(K_\Theta)$.

As an immediate result, they have the following corollary.

Corollary 3.2. *[24, Corollary 2.3] For inner functions φ and θ, the composition operator $C_\varphi : K_\theta \to K_{z\theta\circ\varphi}$ is well-defined and bounded.*

A point p is called a Denjoy–Wolff point of φ if $\varphi^{[n]}$ converges to p uniformly on compact subsets of $\mathbb{D}$. Then, it is natural to ask when the inclusion $K_\eta \subset K_\theta$ holds that is the operator C_φ maps K_θ into itself. The following theorem answers the question.

Theorem 3.3. *[24, Theorem 4.1] Let φ and θ be inner functions on $\mathbb{D}$. Then, the operator $C_\varphi : K_\theta \to K_\theta$ is well-defined and bounded if one of the following conditions holds:*

1. *p, the Denjoy-Wolff point of φ, is on $\mathbb{T}$, and θ is of the form $\theta(z) = z\psi(z)$, where ψ fulfills*

$$\psi(\varphi(z)) = \lambda\psi(z), \qquad z \in \mathbb{D},$$

   for some $\lambda \in T$.

2. *p, the Denjoy-Wolff point of φ, is on $\mathbb{T}$, and*

$$\theta(z) = \gamma z\psi(z) \prod_{n=0}^{\infty} \omega\left(\varphi^{[n]}(z)\right),$$

   where ω is a nonconstant inner function such that the product is convergent, and ψ fulfills

$$\psi(\varphi(z)) = \psi(z), \qquad z \in \mathbb{D}.$$

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In fact we have only cited a part of [24, Theorem 4.1]. The complete version of the theorem characterizes all pairs φ, Θ for which C_φ maps K_Θ into itself.

3.2 Automorphisms of $\mathbb{D}$

Let γ be an arbitrary unimodular constant and α be an arbitrary point in $\mathbb{D}$. The Möbius transformation

$$b(z) = \gamma \, \frac{\alpha - z}{1 - \bar{\alpha}\, z}$$

is an automorphism of the open unit disc with a simple zero at α. Conversely, any automorphism of the disc has the above form.

In order to use it in the formation of a Blaschke product, we define the Blaschke factor

$$b_\alpha(z) := \begin{cases} \dfrac{|\alpha|}{\alpha} \, \dfrac{\alpha - z}{1 - \bar{\alpha}\, z} & \text{if} \quad \alpha \neq 0, \\[2ex] z & \text{if} \quad \alpha = 0. \end{cases}$$

But, some other variations of b are needed in our discussion.

Depending on the number of fixed points, apart from the identity, the Möbius transformations divide into two classes: either they have just one fixed point, or they have two distinct fixed points. Since an automorphism of the open unit disc maps bijectively $\mathbb{D}$ into itself, and also $\mathbb{T}$ into itself, there are certain restrictions on the location of these fixed points. See [4, Section 1.2] for more on this topic.

The point 1 is a fixed point of b if and only if

$$1 = \gamma \, \frac{\alpha - 1}{1 - \bar{\alpha}}.$$

Hence, b takes the form

$$\varphi_\alpha(z) := \frac{1 - \bar{\alpha}}{1 - \alpha} \, \frac{z - \alpha}{1 - \bar{\alpha}\, z}, \tag{3.2}$$

where α is a parameter running through $\mathbb{D}$. A simple computation shows that the other fixed point of φ_α is

$$\kappa_\alpha := -\frac{\alpha(1 - \bar{\alpha})}{\bar{\alpha}(1 - \alpha)}. \tag{3.3}$$

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In our calculation, we will also need the quantity

$$A_\alpha := \varphi'_\alpha(1) = \frac{1-|\alpha|^2}{|1-\alpha|^2}. \tag{3.4}$$

As a matter of fact, A_α is the angular derivative (in the sense of Carathéodory) of φ_α at the fixed point 1. Moreover, note that

$$A_\alpha = 1 \quad \Longleftrightarrow \quad \kappa_\alpha = 1.$$

Finally, given $z_0 \in \mathbb{D}$, we define the unimodular constant γ_{α,z_0} by

$$\gamma_{\alpha,z_0} := \begin{cases} \varphi_{z_0}(\kappa_\alpha) & \text{if} \quad A_\alpha < 1, \\ 1 & \text{if} \quad A_\alpha = 1, \\ \overline{\varphi_{z_0}(\kappa_\alpha)} & \text{if} \quad A_\alpha > 1. \end{cases}$$

This constant will appear on several occasions below. For more information on the structure of automorphisms of $\mathbb{D}$ and Blaschke products, see [19, page 65], [25, page 176] or [22, page 155].

3.3 The group $(\mathbb{D}, *)$

In a rather surprising way, the open unit disc $\mathbb{D}$ becomes a group. The law of composition is defined by

$$\alpha * \beta := \frac{\beta(1-\bar{\beta}) + \alpha(1-\beta)}{(1-\bar{\beta}) + \alpha\bar{\beta}(1-\beta)}, \qquad (\alpha, \beta \in \mathbb{D}). \tag{3.5}$$

This algebraic structure is rewarding and has numerous interesting properties. The rather strange law of composition comes from the composition of some judiciously chosen automorphisms of the disc. This is clarified below.

Theorem 3.4. $(\mathbb{D}, *)$ *is a (non-abelian) group. The identity element is* 0*, and the inverse of* α *is* $-\alpha\,\frac{1-\bar{\alpha}}{1-\alpha}$.

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Proof. To reveal the mystery behind the complicated law $*$, consider the collection

$$\mathcal{G} = \{\varphi_\alpha : \alpha \in \mathbb{D}\}.$$

As we observed in Section 3.2, the set $\mathcal{G}$ precisely consists of *all* automorphisms of the open unit disc with a *fixed point* at 1. This fact is essential. Since the automorphisms $\varphi_\alpha \circ \varphi_\beta$ and φ_α^{-1} fix the point 1, we deduce that $\varphi_\alpha \circ \varphi_\beta \in \mathcal{G}$ and $\varphi_\alpha^{-1} \in \mathcal{G}$. Hence, equipped with the law of composition of functions, $\mathcal{G}$ is a group.

Now, we use the parametrization of $\mathcal{G}$ by $\mathbb{D}$ and transfer the algebraic structure of $\mathcal{G}$ to $\mathbb{D}$. Since $\mathcal{G}$ is a group, given $\alpha, \beta \in \mathbb{D}$, there is a *unique* $\gamma \in \mathbb{D}$ such that $\varphi_\alpha \circ \varphi_\beta = \varphi_\gamma$. We define the isomorphism such that $\gamma = \alpha * \beta$, i.e.

$$\varphi_\alpha \circ \varphi_\beta = \varphi_{\alpha*\beta}, \qquad (\alpha, \beta \in \mathbb{D}). \tag{3.6}$$

We proceed to find an explicit formula for γ. In fact, we have

$$\begin{aligned}
\varphi_{\alpha*\beta}(z) &= (\varphi_\alpha \circ \varphi_\beta)(z) \\
&= \frac{1-\bar{\alpha}}{1-\alpha}\,\frac{\frac{1-\bar{\beta}}{1-\beta}\,\frac{z-\beta}{1-\bar{\beta}\,z} - \alpha}{1-\bar{\alpha}\,\frac{1-\bar{\beta}}{1-\beta}\,\frac{z-\beta}{1-\bar{\beta}\,z}} \\
&= \frac{1-\bar{\alpha}}{1-\alpha}\,\frac{(1-\bar{\beta})(z-\beta) - \alpha(1-\beta)(1-\bar{\beta}\,z)}{(1-\beta)(1-\bar{\beta}\,z) - \bar{\alpha}(1-\bar{\beta})(z-\beta)} \\
&= \frac{1-\bar{\alpha}}{1-\alpha}\,\frac{\big((1-\bar{\beta}) + \alpha\bar{\beta}(1-\beta)\big)\,z - \big(\beta(1-\bar{\beta}) + \alpha(1-\beta)\big)}{\big((1-\beta) + \bar{\alpha}\beta(1-\bar{\beta})\big) - \big(\bar{\beta}(1-\beta) + \bar{\alpha}(1-\bar{\beta})\big)\,z}.
\end{aligned}$$

Looking at the zero of the last quotient shows that (3.5) holds.

We constructed the group $(\mathbb{D}, *)$ such that it is an isomorphic copy of $(\mathcal{G}, \circ)$. As the first consequence, since $\varphi_0 = id$, the point 0 is the identity element of $(\mathbb{D}, *)$. Using (3.5), it is also easy to see that

$$\alpha * 0 = 0 * \alpha = \alpha, \qquad (\alpha \in \mathbb{D}).$$

Similarly, the expression

$$\varphi_\alpha^{-1}(z) = \frac{1-\bar{\alpha}}{1-\alpha}\,\frac{z + \alpha\frac{1-\bar{\alpha}}{1-\alpha}}{1 + \bar{\alpha}\frac{1-\alpha}{1-\bar{\alpha}}\,z} = \varphi_{-\alpha\,\frac{1-\bar{\alpha}}{1-\alpha}}(z),$$

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gives the formula for the inverse of α, something that can also be directly verified via (3.5), i.e.

$$\alpha * \left(-\alpha\,\frac{1-\bar{\alpha}}{1-\alpha}\right) = \left(-\alpha\,\frac{1-\bar{\alpha}}{1-\alpha}\right) * \alpha = 0, \qquad (\alpha \in \mathbb{D}).$$

□

Fix $\alpha \in \mathbb{D}$. To avoid the confusion with the law of multiplication in the complex plane, for $n \geq 1$, we write

$$\alpha_n := \alpha * \alpha * \cdots * \alpha, \qquad (n \text{ times}, \ n \geq 1),$$

and, appealing to the formula for the inverse of α in $\mathbb{D}$ given in Theorem 3.4, we define

$$\alpha_{-n} := \left(-\alpha\,\frac{1-\bar{\alpha}}{1-\alpha}\right)_n, \qquad (n \geq 1).$$

Since 0 is the identity element in $\mathbb{D}$, we put $\alpha_0 := 0$. Hence, each $\alpha \in \mathbb{D}$ gives birth to a two-sided sequence $(\alpha_n)_{n\in\mathbb{Z}}$, and with this notation, we have the crucial identity

$$\varphi_\alpha^{[n]} = \varphi_{\alpha_n}, \qquad (n \in \mathbb{Z}). \tag{3.7}$$

The notation $f^{[k]}$ means $f \circ \cdots \circ f$, k times. This observation immediately implies

$$\varphi_{\alpha_m} \circ \varphi_{\alpha_n} = \varphi_{\alpha_{m+n}}, \qquad (m, n \in \mathbb{Z}). \tag{3.8}$$

This identity will be used frequently. Theorem 3.4 and (3.2) also reveal that

$$\varphi_\alpha(0) = -\alpha\frac{1-\bar{\alpha}}{1-\alpha} = \alpha_{-1}, \qquad (\alpha \in \mathbb{D}). \tag{3.9}$$

To obtain another useful formula, note that $\varphi_\beta \circ \varphi_{\alpha_{-1}}$ and $\varphi_{\varphi_\alpha(\beta)}$ both belong to $\mathcal{G}$ and vanish at $\varphi_\alpha(\beta)$. Hence,

$$\varphi_\beta \circ \varphi_{\alpha_{-1}} = \varphi_{\varphi_\alpha(\beta)}, \qquad (\alpha, \beta \in \mathbb{D}).$$

As a special case, we have

$$\varphi_{z_0} \circ \varphi_{\alpha_{-n}} = \varphi_{w_n}, \qquad (\alpha, z_0 \in \mathbb{D}, \ n \in \mathbb{Z}), \tag{3.10}$$

where $w_n = \varphi_{\alpha_n}(z_0)$. The importance of this formula will be revealed below.

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3.4 The subgroup $\mathbb{D}_\kappa$

The fixed points of φ_α are 1 and κ_α. Hence, the study of $\mathbb{D}$ naturally bifurcates into two cases: $\kappa_\alpha = 1$ and $\kappa_\alpha \neq 1$. Note that as α runs through $\mathbb{D}$, the fixed point κ runs through all of $\mathbb{T}$.

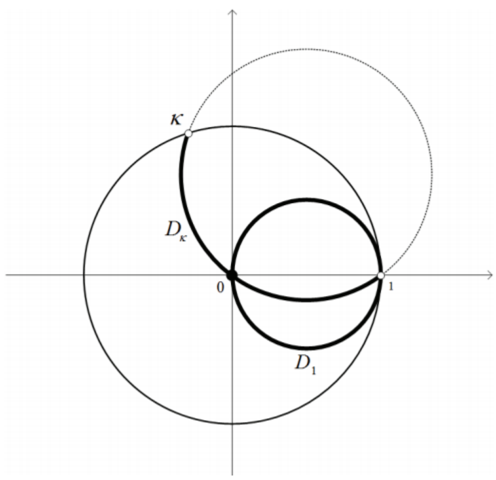
For a fixed $\kappa \in \mathbb{T}$, we also define

$$\mathbb{D}_\kappa := \{\, \alpha \in \mathbb{D} : \kappa_\alpha = \kappa \} = \{\, \alpha \in \mathbb{D} : \alpha + \kappa\bar{\alpha} = (1+\kappa)|\alpha|^2 \}. \tag{3.11}$$

The last expression shows that the points of $\mathbb{D}_\kappa$ are part of the circle passing through the points 1, 0 and κ which is inside $\mathbb{D}$. Also note that for $\kappa = -1$, we have the degenerate case

$$\mathbb{D}_{-1} = \{\, \alpha \in \mathbb{D} : \alpha(1-\bar{\alpha}) = \bar{\alpha}(1-\alpha) \} = (-1, 1).$$

A GROUP STRUCTURE ON $\mathbb{D}$



One spacial case is of special interest. If $\kappa = 1$, then

$$\begin{aligned} \mathbb{D}_1 &= \{\alpha \in \mathbb{D} : \alpha(1-\bar{\alpha}) = -\bar{\alpha}(1-\alpha)\} \\ &= \{\alpha \in \mathbb{D} : \alpha + \bar{\alpha} = 2|\alpha|^2\} \\ &= \{x + iy : (x-1/2)^2 + y^2 = 1/4\} \setminus \{1\}. \end{aligned} \tag{3.12}$$

Hence, $\mathbb{D}_1$ is precisely the circle of radius 1 inside $\mathbb{D}$ which is tangent to point 1, of course without counting the boundary point 1. At such points, φ_α has just one fix point, i.e. the point 1, and this makes the difference in the following. The

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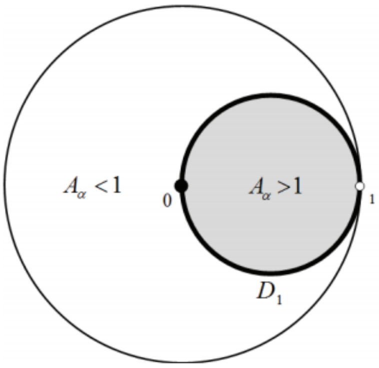
subgroup $\mathbb{D}_1$ is also the border line for the values of A_α. On $\mathbb{D}_1$, we precisely have $A_\alpha = 1$, while inside it $A_\alpha > 1$, and in between $\mathbb{D}$ and $\mathbb{D}_1$ we have $A_\alpha < 1$. This is important when we study the iterates of an element in an equivalence class (Section 3.6).



Theorem 3.5. *Let* $\kappa \in \mathbb{T}$. *Then* $\mathbb{D}_\kappa$ *is an abelian subgroup of* $\mathbb{D}$. *Moreover, on* $\mathbb{D}_\kappa$, *the law of composition simplifies to*

$$\alpha * \beta = \frac{\alpha + \beta - (1+\bar{\kappa})\alpha\beta}{1 - \bar{\kappa}\alpha\beta}, \qquad (\alpha, \beta \in \mathbb{D}_\kappa).$$

Proof. Direct verification of this fact is possible. However, it is easier to just note that if κ is the fixed point of φ_α and φ_β, then it is also stays fixed under $\varphi_\alpha \circ \varphi_\beta^{-1} = \varphi_{\alpha * \beta_{-1}}$. Hence, for each $\alpha, \beta \in \mathbb{D}_\kappa$, we have $\alpha * \beta_{-1} \in \mathbb{D}_\kappa$. Clearly $0 \in \mathbb{D}_\kappa$. Thus, $\mathbb{D}_\kappa$ is a subgroup of $\mathbb{D}$.

To obtain a simpler formula for $*$ in $\mathbb{D}_\kappa$, note that by (3.5) and (3.11), we have

$$\begin{aligned}
\alpha * \beta &= \frac{\beta(1-\bar{\beta}) + \alpha(1-\beta)}{(1-\bar{\beta}) + \alpha\bar{\beta}(1-\beta)} \\
&= \frac{-\kappa\bar{\beta}(1-\beta) + \alpha(1-\beta)}{(1-\bar{\beta}) - \alpha\bar{\kappa}\beta(1-\bar{\beta})} \\
&= \frac{(1-\beta)(-\kappa\bar{\beta} + \alpha)}{(1-\bar{\beta})(1-\bar{\kappa}\alpha\beta)} \\
&= \frac{\alpha + \beta - (1+\bar{\kappa})\alpha\beta}{1-\bar{\kappa}\alpha\beta}, \qquad (\alpha, \beta \in \mathbb{D}_\kappa).
\end{aligned}$$

This formula also reveals that $\mathbb{D}_\kappa$ is abelian. □

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3.5 A formula for the iterates of α

Using an interesting technique of complex analysis, we now obtain an explicit formula for α_n.

Theorem 3.6. *Let $\alpha \in \mathbb{D}$. Then we have*

$$\alpha_n = \begin{cases} \dfrac{\kappa_\alpha(1 - A_\alpha^n)}{1 - \kappa_\alpha A_\alpha^n} & \text{if} \quad \kappa_\alpha \neq 1, \\[2ex] \dfrac{n\alpha}{1 + (n-1)\alpha} & \text{if} \quad \kappa_\alpha = 1, \end{cases} \qquad (n \in \mathbb{Z}).$$

In particular, except for the identity element 0*, no other element of $\mathbb{D}$ is of finite order.*

Proof. Direct verification of the above formula is feasible. But, it is not a pleasant task. We present another more interesting approach. Given $\kappa \in \mathbb{T}$, define

$$\varphi_\kappa(z) := \begin{cases} \dfrac{z-\kappa}{z-1} & \text{if} \quad \kappa \neq 1, \\[2ex] \dfrac{z}{z-1} & \text{if} \quad \kappa = 1. \end{cases}$$

This function satisfies $\varphi_\kappa \circ \varphi_\kappa = id$. Now, we need to consider two cases.

Case I, $\kappa_\alpha \neq 1$: We have

$$(\varphi_{\kappa_\alpha} \circ \varphi_\alpha \circ \varphi_{\kappa_\alpha})(z) = \frac{z}{A_\alpha},$$

and thus we deduce that

$$(\varphi_{\kappa_\alpha} \circ \varphi_\alpha^{[n]} \circ \varphi_{\kappa_\alpha})(z) = \frac{z}{A_\alpha^n}, \qquad (n \in \mathbb{Z}).$$

Therefore,

$$\varphi_\alpha^{[n]}(z) = \varphi_{\kappa_\alpha}\left(\frac{\varphi_{\kappa_\alpha}(z)}{A_\alpha^n}\right), \qquad (n \in \mathbb{Z}), \tag{3.13}$$

which simplifies to

$$\varphi_\alpha^{[n]}(z) = \frac{(1-\kappa_\alpha A_\alpha^n)z - \kappa_\alpha(1-A_\alpha^n)}{(1-A_\alpha^n)z + (A_\alpha^n - \kappa_\alpha)}, \qquad (n \in \mathbb{Z}). \tag{3.14}$$

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Thèse de Muath Karaki, Lille 1, 2015

Now, according to (3.7), $\varphi_\alpha^{[n]} = \varphi_{\alpha_n}$, and by considering the zero of $\varphi_\alpha^{[n]}$ we obtain the required above formula.

Case II, $\kappa_\alpha = 1$: The proof has the same spirit, except that we use φ_1. In this case, we have

$$(\varphi_1 \circ \varphi_\alpha \circ \varphi_1)(z) = z + \frac{\alpha}{1-\alpha},$$

and thus

$$(\varphi_1 \circ \varphi_\alpha^{[n]} \circ \varphi_1)(z) = z + \frac{n\alpha}{1-\alpha}, \qquad (n \in \mathbb{Z}).$$

Therefore,

$$\varphi_\alpha^{[n]}(z) = \varphi_1\left(\varphi_1(z) + \frac{n\alpha}{1-\alpha}\right), \qquad (n \in \mathbb{Z}). \tag{3.15}$$

which simplifies to

$$\varphi_\alpha^{[n]}(z) = \frac{(1-\alpha+n\alpha)z - n\alpha}{n\alpha z + 1 - \alpha - n\alpha}, \qquad (n \in \mathbb{Z}). \tag{3.16}$$

The result now follows. □

With similar techniques, one can also show that

$$\alpha_n = \frac{A_\alpha(1 + A_\alpha + \cdots + A_\alpha^{n-1})B}{1 + A_\alpha(1 + A_\alpha + \cdots + A_\alpha^{n-1})B}, \qquad (n \in \mathbb{Z}),$$

where

$$B = \frac{\alpha - |\alpha|^2}{1 - |\alpha|^2}.$$

But, we do not need this representation in the following.

Note that if $\alpha \in \mathbb{D}_\kappa$, then its iterates form a subgroup in $\mathbb{D}_\kappa$. In particular,

$$(\alpha_n)_{n\in\mathbb{Z}} \subset \mathbb{D}_\kappa.$$

This observation is exploited in the next section.

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3.6 An equivalence relation

The operation

$$\begin{array}{rccc} \diamond : & (\mathbb{D}, *) \times \mathbb{D} & \longrightarrow & \mathbb{D} \\ & (\alpha, z) & \longmapsto & \varphi_\alpha(z) \end{array}$$

defines a group action on the set $\mathbb{D}$. The required condition $\alpha \diamond (\beta \diamond z) = (\alpha * \beta) \diamond z$ is precisely a reformulation of (3.6). Since $(\varphi_{w_{-1}} \circ \varphi_z)(z) = w$ this action is transitive and thus it creates just one orbit on $\mathbb{D}$. Hence, we restrict ourselves to some subgroups of $(\mathbb{D}, *)$ to obtain better equivalence classes.

Fix $\alpha \in \mathbb{D}$. Then the subgroup it generate in $(\mathbb{D}, *)$ is precisely $(\alpha_n)_{n\in\mathbb{Z}}$. The orbits, or equivalent classes, created by this subgroup are as follows. Two points z_1 and z_2 are in the same orbit, and we write $z_1 \sim_\alpha z_2$, if and only if there is an integer $n \in \mathbb{Z}$ such that

$$\varphi_\alpha^{[n]}(z_1) = \varphi_{\alpha_n}(z_1) = z_2.$$

Since φ_α is an automorphism, it maps $\mathbb{D}$ and $\mathbb{T}$ respectively to themselves bijectively. Hence, the equivalence class generated by a $z \in \mathbb{D}$ is entirely in $\mathbb{D}$. A similar statement hold for the points of $\mathbb{T}$. More information on the equivalence classes are gathered below. Since $\alpha = 0$ corresponds to the identity mapping on $\mathbb{D}$, the following result (when properly modified) becomes trivial in this case. Thus, we assume that $\alpha \neq 0$.

Theorem 3.7. *Let $\alpha \in \mathbb{D}$, $\alpha \neq 0$. Then the following assertions hold.*

(i) The equivalence class generated by $z_0 \in \mathbb{D}$ is precisely $\big(\varphi_{\alpha_n}(z_0)\big)_{n\in\mathbb{Z}}$, which consists of distinct points of $\mathbb{D}$. In particular, the equivalence class generated by 0 is the sequence $\big(\alpha_n\big)_{n\in\mathbb{Z}}$.

(ii) We have

$$\lim_{n\to\pm\infty} \varphi_{\alpha_n}(z_0) = 1, \qquad (\textit{if } A_\alpha = 1),$$

and

$$\lim_{n\to+\infty} \varphi_{\alpha_n}(z_0) = \kappa_\alpha \quad \textit{while} \quad \lim_{n\to-\infty} \varphi_{\alpha_n}(z_0) = 1, \qquad (\textit{if } A_\alpha > 1),$$

and

$$\lim_{n\to+\infty} \varphi_{\alpha_n}(z_0) = 1 \quad \textit{while} \quad \lim_{n\to-\infty} \varphi_{\alpha_n}(z_0) = \kappa_\alpha, \qquad (\textit{if } A_\alpha < 1).$$

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Proof. (i): That the equivalence class generated by $z_0 \in \mathbb{D}$ is precisely $\big(\varphi_{\alpha_n}(z_0)\big)_{n\in\mathbb{Z}}$ is rather trivial. This fact says that the equivalence class generated by z_0 consists of the past, present and future of z_0 under the transformation φ_α. See formulas (3.14) and (3.16). For any $\alpha \in \mathbb{D}$, the automorphism φ_α has no fixed point inside $\mathbb{D}$. Hence, the class $\big(\varphi_{\alpha_n}(z_0)\big)_{n\in\mathbb{Z}}$ consists of distinct points. To find the equivalence class of 0, apply (3.2) to get

$$\varphi_{\alpha_n}(0) = -\alpha_n \frac{1-\overline{\alpha_n}}{1-\alpha_n}, \qquad (n \in \mathbb{Z}).$$

But, by Theorem 3.4 and (3.8),

$$-\alpha_n \frac{1-\overline{\alpha_n}}{1-\alpha_n} = \text{ inverse of } \alpha_n \text{ in } (\mathbb{D}, *) = \alpha_{-n}, \qquad (n \in \mathbb{Z}).$$

Thus, by part (i),

$$\big(\varphi_{\alpha_n}(0)\big)_{n\in\mathbb{Z}} = \big(\alpha_{-n}\big)_{n\in\mathbb{Z}} = \big(\alpha_n\big)_{n\in\mathbb{Z}}.$$

(ii): If $A_\alpha = 1$, then we rewrite (3.16) as

$$\varphi_\alpha^{[n]}(z) = \frac{(1-\alpha)z + n\alpha(z-1)}{(1-\alpha) + n\alpha(z-1)}.$$

This representation shows that

$$\lim_{n\to\pm\infty} \varphi_{\alpha_n}(z_0) = 1.$$

Note that $A_\alpha = 1$ happens precisely on $\mathbb{D}_1$. But, if $A_\alpha \neq 1$, then we rewrite (3.14) as

$$\varphi_\alpha^{[n]}(z) = \frac{-A_\alpha^n \kappa_\alpha (z-1) + (z-\kappa_\alpha)}{-A_\alpha^n (z-1) + (z-\kappa_\alpha)}.$$

Now, there are two possibilities. If $A_\alpha > 1$, which corresponds to the points α inside the disc surrounded by $\mathbb{D}_1$, then

$$\lim_{n\to+\infty} \varphi_{\alpha_n}(z_0) = \kappa_\alpha, \quad \text{while} \quad \lim_{n\to-\infty} \varphi_{\alpha_n}(z_0) = 1.$$

But, if $A_\alpha < 1$, which corresponds to the points $\alpha \in \mathbb{D}$, but outside the disc surrounded by $\mathbb{D}_1$, then

$$\lim_{n\to+\infty} \varphi_{\alpha_n}(z_0) = 1, \quad \text{while} \quad \lim_{n\to-\infty} \varphi_{\alpha_n}(z_0) = \kappa_\alpha.$$

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□

We can also provide a geometric interpretation of the equivalence classes. Chapter 3 of [25] contains a comprehensive study of the geometric behavior of Möbius transformation. A very short glimpse of this visual interpretation is provided below.

Theorem 3.7 shows that the points $\left(\varphi_{\alpha_n}(z_0)\right)_{n\in\mathbb{Z}}$ reside on some curves passing through 1, κ_α and z_0, and tend to the frontiers 1 and κ_α as $n \to \pm\infty$.

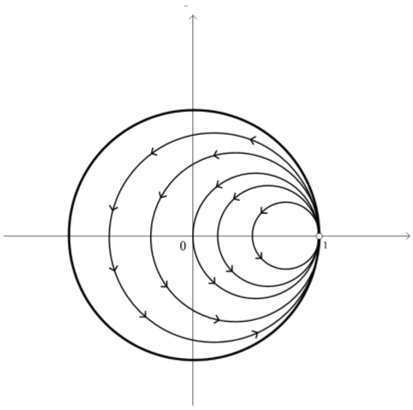
Parabolic case, $\kappa_\alpha = 1$: The relation (3.15) reveals that the equivalence class $\left(\varphi_{\alpha_n}(z_0)\right)_{n\in\mathbb{Z}}$ is on the image of the line

$$t \longmapsto \varphi_1(z_0) + \frac{\alpha}{1-\alpha}\, t, \qquad (t \in \mathbb{R}),$$

under the mapping φ_1. Since $\varphi_1(\infty) = 1$ and $\varphi_1(\varphi_1(z_0)) = z_0$, the image is a circle passing through the points 1 and z_0. Different values of $\varphi_1(z_0)$ corresponds to different parallel lines. Hence, their images are circles which are tangent at 1. One particular circle corresponds to the line passing through $\varphi_1(z_0) = 1/2$. In this case, we have

$$\varphi_1\left(\frac{1}{2} + \frac{\alpha}{1-\alpha}\, t\right) = \frac{t + \frac{1-\alpha}{2\alpha}}{t + \frac{1-\bar{\alpha}}{2\bar{\alpha}}} \in \mathbb{T}.$$

Hence, the image of this last line is the unit circle $\mathbb{T}$. In other words, the iterates of boundary points stay on $\mathbb{T}$ and (except 1) they form a two-sided sequence which converge to 1 from both sides.



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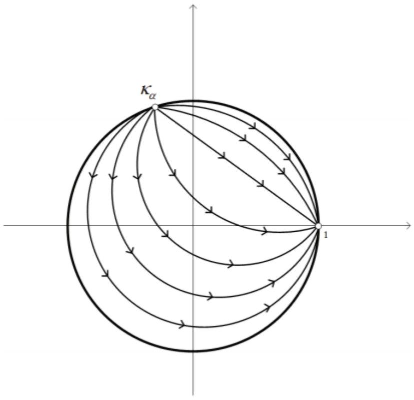
Thèse de Muath Karaki, Lille 1, 2015

Hyperbolic case, $\kappa_\alpha \neq 1$: By (3.13), we see that the equivalence class $\big(\varphi_{\alpha_n}(z_0)\big)_{n\in\mathbb{Z}}$ is on the image of

$$t \longmapsto \frac{\varphi_{\kappa_\alpha}(z_0)}{A_\alpha^t}, \qquad (t \in \mathbb{R}).$$

Since $A_\alpha \in (0,\infty)\setminus\{1\}$, the image is a line passing through 0 and $\varphi_{\kappa_\alpha}(z_0)$. Since $\varphi_{\kappa_\alpha}(\infty) = 1$, $\varphi_{\kappa_\alpha}(0) = \kappa_\alpha$, and $\varphi_{\kappa_\alpha}(\varphi_{\kappa_\alpha}(z_0)) = z_0$ the image is circle passing through the points 1, κ_α and z_0. The following figure shows the pathes when $A_\alpha < 1$.



For $A_\alpha > 1$, we just need to reverse the directions.

3.7 Minimal Blaschke products

In this section, we take the first step in finding the solutions of the equation $\psi\circ\varphi_\alpha = \lambda\psi$ by showing that each equivalence class of $\sim_\alpha$ in $\mathbb{D}$ produces a Blaschke product which is a *minimal solution* of the equation. Therefore, having the freedom to choose $\alpha \in \mathbb{D}$ and any of the equivalence classes it generates, the following result provides a vast variety of solutions of the functional equation. In fact, we can go even further and extract all Blaschke products that satisfy the equation.

Theorem 3.8. *Fix* $\alpha \in \mathbb{D}$, $\alpha \neq 0$. *Let* $z_0 \in \mathbb{D}$, *and let* $(z_n)_{n\in\mathbb{Z}} \subset \mathbb{D}$ *be the corresponding equivalence class generated by* $\sim_\alpha$. *Then* $(z_n)_{n\in\mathbb{Z}}$ *is a two-sided infinite Blaschke sequence and the corresponding Blaschke product*

$$B_{\alpha,z_0} = \prod_{n=-\infty}^{\infty} b_{z_n}$$

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Thèse de Muath Karaki, Lille 1, 2015

satisfies the functional equation

$$B_{\alpha,z_0} \circ \varphi_\alpha = \gamma_{\alpha,z_0} \, B_{\alpha,z_0}.$$

Moreover, no proper divisor ψ *of* B_{α,z_0} *satisfies any functional equations of the form* $\psi \circ \varphi_\alpha = \lambda \, \psi$, $\lambda \in \mathbb{T}$.

Proof. According to Theorem 3.7(i), without loss of generality, we can assume

$$z_n = \varphi_{\alpha_n}(z_0), \qquad (n \in \mathbb{Z}).$$

Hence,

$$\begin{aligned} 1 - |z_n|^2 &= 1 - |\varphi_{\alpha_n}(z_0)|^2 \\ &= \frac{(1-|\alpha_n|^2)\,(1-|z_0|^2)}{|1-\overline{\alpha_n}\;z_0|^2} \\ &\leq \frac{1+|z_0|}{1-|z_0|}\,(1-|\alpha_n|^2). \end{aligned}$$

Therefore, to deal with $(1-|\alpha_n|^2)$, in the light of Theorem 3.6, we consider two cases.

Parabolic case, $\kappa_\alpha = 1$: Using (3.12), we have

$$\begin{aligned} 1 - |z_n|^2 &\leq \frac{1+|z_0|}{1-|z_0|} \left(1 - \left|\frac{n\alpha}{1+(n-1)\alpha}\right|^2\right) \\ &= \frac{1+|z_0|}{1-|z_0|} \, \frac{1+(n-1)(\alpha+\bar{\alpha})-(2n-1)|\alpha|^2}{|1+(n-1)\alpha|^2} \\ &\leq \frac{1+|z_0|}{1-|z_0|} \, \frac{1-|\alpha|^2}{|1+(n-1)\alpha|^2} = O(1/n^2), \qquad (n \longrightarrow \pm\infty). \end{aligned}$$

Hence, $\mathcal{C}$ is a double-sided Blaschke sequence.

Hyperbolic case, $\kappa_\alpha \neq 1$: We have

$$\begin{aligned} 1 - |z_n|^2 &\leq \frac{1+|z_0|}{1-|z_0|} \left(1 - \left|\frac{\kappa_\alpha(1-A_\alpha^n)}{1-\kappa_\alpha A_\alpha^n}\right|^2\right) \\ &= \frac{1+|z_0|}{1-|z_0|} \, \frac{(2-\kappa_\alpha-\bar{\kappa}_\alpha)A_\alpha^n}{|1-\kappa_\alpha A_\alpha^n|^2} = O(q^{|n|}), \qquad (n \longrightarrow \pm\infty), \end{aligned}$$

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where $q := \min\{A_\alpha, 1/A_\alpha\} < 1$. Hence, again $\mathcal{C}$ is a double-sided Blaschke sequence (indeed, with a geometric rate of convergence).

To show that

$$B_{\alpha,z_0} = \prod_{n\in\mathbb{Z}} b_{z_n}$$

satisfies the functional equation $B_{\alpha,z_0} \circ \varphi_\alpha = B_{\alpha,z_0}$, we rewrite B_{α,z_0} in the form

$$B_{\alpha,z_0} = \prod_{n\in\mathbb{Z}} \gamma_n \varphi_{z_n},$$

where γ_n are appropriate constants such that $b_{z_n} = \gamma_n \varphi_{z_n}$, i.e.

$$\gamma_n = -\frac{|z_n|}{z_n} \cdot \frac{1 - z_n}{1 - \bar{w}_n}, \qquad (n \in \mathbb{Z}).$$

Now, by (3.8) and (3.10),

$$\varphi_{z_n} \circ \varphi_\alpha = \varphi_{z_0} \circ \varphi_{\alpha_{-n}} \circ \varphi_\alpha = \varphi_{z_0} \circ \varphi_{\alpha_{-n+1}} = \varphi_{z_{n-1}}. \tag{3.17}$$

Therefore,

$$B_{\alpha,z_0} \circ \varphi_\alpha = \prod_{n\in\mathbb{Z}} \gamma_n \varphi_{z_n} \circ \varphi_\alpha = \prod_{n\in\mathbb{Z}} \gamma_n \varphi_{z_{n-1}} = \left(\prod_{n\in\mathbb{Z}} \frac{\gamma_n}{\gamma_{n-1}} \right) B_{\alpha,z_0}.$$

In the first place, even thought it can be directly verified, the above calculation shows that this last product has to be convergent. Secondly, we have

$$\prod_{n\in\mathbb{Z}} \frac{\gamma_n}{\gamma_{n-1}} = \lim_{N\to+\infty} \prod_{n=-N+1}^{N} \frac{\gamma_n}{\gamma_{n-1}} = \lim_{N\to+\infty} \frac{\gamma_N}{\gamma_{-N}} = \frac{\lim_{N\to+\infty} \gamma_N}{\lim_{N\to-\infty} \gamma_N}.$$

Using Theorem 3.7(*ii*), we can compute both limits. In fact, the formula

$$z_n = \varphi_{\alpha_n}(z_0) = \frac{1 - \bar{\alpha}_n}{1 - \alpha_n} \frac{z_0 - \alpha_n}{1 - \bar{\alpha}_n z_0}, \qquad (n \in \mathbb{Z}),$$

implies

$$\frac{1 - z_n}{1 - \bar{w}_n} = \frac{1 - z_0}{1 - \bar{z}_0} \frac{1 - \bar{\alpha}_n}{1 - \alpha_n} \frac{1 - \alpha_n \bar{z}_0}{1 - \bar{\alpha}_n z_0}, \qquad (n \in \mathbb{Z}).$$

Hence,

$$\gamma_n = \frac{1 - z_0}{1 - \bar{z}_0} \frac{1 - \alpha_n \bar{z}_0}{|1 - \alpha_n \bar{z}_0|} \frac{|\alpha_n - z_0|}{\alpha_n - z_0}, \qquad (n \in \mathbb{Z}),$$

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and thus

$$\alpha_n \to 1 \qquad \Longrightarrow \qquad \gamma_n \to 1$$

while

$$\alpha_n \to \kappa_\alpha \qquad \Longrightarrow \qquad \gamma_n \to \frac{1-z_0}{1-\bar{z}_0}\frac{1-\kappa_\alpha \bar{z}_0}{\kappa_\alpha - z_0}.$$

Therefore, by Theorem 3.7(ii),

$$\prod_{n\in\mathbb{Z}} \frac{\gamma_n}{\gamma_{n-1}} = \begin{cases} \frac{1-z_0}{1-\bar{z}_0}\frac{1-\kappa_\alpha \bar{z}_0}{\kappa_\alpha - z_0} & \text{if} \quad A_\alpha > 1, \\ 1 & \text{if} \quad A_\alpha = 1, \\ \frac{1-\bar{z}_0}{1-z_0}\frac{\kappa_\alpha - z_0}{1-\kappa_\alpha \bar{z}_0} & \text{if} \quad A_\alpha < 1. \end{cases}$$

In fact, the above calculation shows the motivation for the definition of γ_{α,z_0}. It is defined such that $\prod_{n\in\mathbb{Z}} \frac{\gamma_n}{\gamma_{n-1}} = \gamma_{\alpha,z_0}$. Thus, B_{α,z_0} satisfies the functional equation $B_{\alpha,z_0} \circ \varphi_\alpha = \gamma_{\alpha,z_0} B_{\alpha,z_0}$.

Finally, the identity (3.17) reveals that no proper divisor of B_{α,z_0} satisfies a functional equation of the form $\psi \circ \varphi_\alpha = \lambda\psi$. □

By Theorem 3.7(i), the equivalence class generated by 0 is $(\alpha_n)_{n\in\mathbb{Z}}$ and, in this case, $\alpha_{-n} = \bar{\alpha}_n$. Hence, the corresponding minimal Blaschke product is

$$B(z) = z \prod_{n=1}^{+\infty} \frac{(\alpha_n - z)\,(\bar{\alpha}_n - z)}{(1-\alpha_n\, z)\,(1-\bar{\alpha}_n\, z)}.$$

By Theorem 3.8, this is the minimal Blaschke product which satisfies the equation $B \circ \varphi_\alpha = B$ and, moreover, $B(0) = 0$.

3.8 Discussion on the general solution

Let ψ be an inner function satisfying $\psi \circ \varphi_\alpha = \lambda\,\psi$, denote its zero set on $\mathbb{D}$ by $\mathcal{Z}(\psi)$. Then the equation $\psi \circ \varphi_\alpha = \lambda\,\psi$ implies $\psi \circ \varphi_{\alpha_{-1}} = \bar{\lambda}\,\psi$, and by induction we obtain

$$\psi \circ \varphi_{\alpha_n} = \lambda^n\,\psi, \qquad (n \in \mathbb{Z}).$$

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This identity reveals that if z_1 is a zero of ψ, then in fact the whole equivalence class $[z_n]_{n\in\mathbb{Z}}$, generated by $\sim_\alpha$, is among $\mathcal{Z}(\psi)$. Hence, we can write

$$\mathcal{Z}(\psi) = \bigcup_m \mathcal{C}_m,$$

where $(\mathcal{C}_m)_m$ is a (finite or infinite, and repetition allowed) collection of equivalence classes of $\sim_\alpha$ in $\mathbb{D}$. Note that since ψ is a non-constant inner function, we must have

$$\sum_m \sum_{z_{mn}\in\mathcal{C}_m} (1 - |z_{mn}|) < \infty. \tag{3.18}$$

Thus,

$$B_{\alpha,(\mathcal{C}_m)_m} = \prod_m B_{\alpha,\mathcal{C}_m} \tag{3.19}$$

is a well-defined Blaschke product and, by Theorem 3.8, $B_{\alpha,(\mathcal{C}_m)_m}$ satisfies the functional equation

$$B_{\alpha,(\mathcal{C}_m)_m} \circ \varphi_\alpha = \lambda' B_{\alpha,(\mathcal{C}_m)_m},$$

where λ' is an appropriate unimodular constant. These types of Blaschke products form the main building blocks for a description of solutions of the equation $\psi\circ\varphi_\alpha = \lambda\psi$, $\lambda \in \mathbb{T}$.

Again thanks to Theorem 3.8, it is rather trivial that if we have a sequence which can be decomposed as above, then the corresponding Blaschke product is in fact a solution of the functional equation.

Put $S = \psi/B_{\alpha,(\mathcal{C}_m)_m}$. The discussion above shows that S is a zero free inner function (i.e. a singular inner function), which satisfies an equation of the form $S\circ\varphi_\alpha = \lambda'' S$, $\lambda'' \in \mathbb{T}$. The classification of such function is still an open question. However, to conclude we deduce the following result.

Theorem 3.9. *Fix $\alpha \in \mathbb{D}$, $\alpha \neq 0$. If a Blaschke product B satisfies the functional equation $B\circ\varphi_\alpha = \lambda B$ then its zero set is a union of equivalence classes generated by $\sim_\alpha$. Reciprocally, if a sequence $(z_n)_n \subset \mathbb{D}$ is such that:*

- *i) as in (3.19), it can be decomposed as a union of equivalence classes generated by $\sim_\alpha$,*
- *ii) and satisfies (3.18),*

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then the corresponding Blaschke product B *is a solution of the functional equation* $B \circ \varphi_\alpha = \lambda B$*, with some unimodular constant* λ*. In particular, if* $\alpha \in \mathbb{D}_1$*, then* $\lambda = 1$.

3.9 Yet another characterization

If ψ_0 satisfies the equation $\psi \circ \varphi_\alpha = \psi$, and ω is any arbitrary inner function, then we also have

$$(\omega \circ \psi_0) \circ \varphi_\alpha = (\omega \circ \psi_0).$$

Hence, $\psi = \omega \circ \psi_0$ is also a solution of the equation $\psi \circ \varphi_\alpha = \psi$. For example, if B is any of the Blaschke products (3.19) for which $\gamma = 1$, then $\omega \circ B$ is a solution. What is rather surprising is that all solutions are obtained in this manner.

Theorem 3.10. *Let* $\alpha \in \mathbb{D}$, $\alpha \neq 0$*. Then the inner function* ψ *is a solution of the equation* $\psi \circ \varphi_\alpha = \psi$ *if and only if there is an inner function* ω *and a Blaschke product* B *of type* (3.19) *such that*

$$\psi = \omega \circ B.$$

Proof. Without loss of generality, assume that ψ is nonconstant. Then, by a celebrated result of Frostman [13], there is a $\beta \in \mathbb{D}$ such that $\widetilde{\psi} = b_\beta \circ \psi$ is a Blaschke product with simple zeros. As a matter of fact, in a sense (logarithmic capacity), there are many such β's. But, just one choice is enough for us.

Surely, $\widetilde{\psi}$ satisfies $\widetilde{\psi} \circ \varphi_\alpha = \widetilde{\psi}$. By induction, we get

$$\widetilde{\psi} \circ \varphi_{\alpha_n} = \widetilde{\psi}, \qquad (n \in \mathbb{Z}).$$

If z_0 is a zero of $\widetilde{\psi}$, then the above identity shows that $\varphi_{\alpha_n}(z_0)$ is also a zero of $\widetilde{\psi}$. Hence, we can classify the zeros of $\widetilde{\psi}$ as a union of equivalence classes of $\sim_\alpha$, e.g. $(\mathcal{C}_m)_m$. This observation immediately reveals that, up to a unimodular constant, $\widetilde{\psi}$ is precisely a Blaschke product of type (3.19). Since $\psi = b_\beta^{-1} \circ \widetilde{\psi}$, the proof is complete. □

It it important to keep in mind that the representation $\psi = \omega \circ B$, given in Theorem 3.10, is far away from being unique. For example, in the proof of theorem, we

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picked one of the Frostman shifts and then constructed B. Different shifts give different sets of zeros and thus different Blaschke products.

3.10 Application in model spaces

The functional equation (1.1), and its simplified form (1.2), stem from studies on composition operators on model spaces K_Θ. The following question is still wide open:

Open Question: for which symbols φ, does the composition operator C_φ maps K_Θ into itself?

Based on the results obtained above, we can say more about the above question when the symbol φ is inner.

Theorem 3.11. *Let* $\alpha \in \mathbb{D}$, $\alpha \neq 0$. *Let* B *be a Blaschke product whose zeros can be decomposed as a union of equivalence classes generated by* $\sim_\alpha$. *Then* C_{φ_α} *is an isomorphism from* K_{zB} *onto itself.*

Proof. By Theorem 3.9, the Blaschke product satisfies the functional equation $B \circ \varphi_\alpha = \lambda B$, where λ is some unimodular constant. Moreover, the Denjoy–Wolff point of φ_α is either 1 or κ_α. This is because φ_α has just two fixed points on $\overline{\mathbb{D}}$ and one of them has to be the Denjoy–Wolff fixed point. Therefore, by [24, Theorem 4.1(vi)], the operator C_{φ_α} maps K_{zB} into itself. In fact, the main difficulty was to construct an explicit inner function which satisfies the functional equation $\psi \circ \varphi_\alpha = \lambda\psi$, and this is done above.

To show that C_φ is surjective, note that

$$K_{zB} = \mathbb{C} \oplus \mathrm{Span}\{k_{z_j} : B(z_j) = 0\},$$

where k_{z_j} is the Cauchy reproducing kernel

$$k_{z_j}(z) = \frac{1}{1 - \bar{z}_j z}.$$

We have $C_{\varphi_\alpha} 1 = 1$ and, by Theorem 3.4,

$$C_{\varphi_\alpha} k_{z_j}(z) = \frac{1}{1 - \bar{z}_j \varphi_\alpha(z)} = \frac{A + Bz}{1 - \overline{\varphi_{\alpha_{-1}}(z_j)}\, z},$$

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where A and B are some constants. Hence, $k_{\varphi_{\alpha_{-1}}(z_j)}$ belongs to the image of C_{φ_α}. We assumed that the zeros of B can be decomposed as a union of equivalence classes generated by $\sim_\alpha$. Therefore, by Theorem 3.7(i), the image contains all Cauchy kernels k_{z_j}, where z_j runs through the zeros of B. In short, this means that the mapping is surjective. □

Using [24, Theorem 4.1(v)], a similar result can be stated for inner functions of the form

$$\Theta(z) = \gamma z \Theta_1(z) \prod_{n=0}^{\infty} \Theta_2(\, \varphi_{\alpha_n}(z)\,),$$

where the inner function Θ_1 satisfies $\Theta_1 \circ \varphi_\alpha = \Theta_1$, and the inner function Θ_2 fixes 1 and is such that the product is convergent. A sufficient convergence criteria is given in [24, Lemma 3.1]. We leave the formulation of this result to the reader.

We can also interpret Theorem 3.8 in the following way to state some facts about the point spectrum of C_{φ_α}. Writing the functional equation as $C_{\varphi_\alpha} B_{\alpha,z_0} = \gamma_{\alpha,z_0} B_{\alpha,z_0}$, it says that B_{α,z_0} is an eigenvector of C_{φ_α} corresponding to the eigenvalue γ_{α,z_0}. As usual, there are two cases to consider.

If $\alpha \in \mathbb{D}_1$, then for any choice of z_0, we have $\gamma_{\alpha,z_0} = 1$. Hence, there are infinitely many Blaschke products with satisfy $C_{\varphi_\alpha} B_{\alpha,z_0} = B_{\alpha,z_0}$. In the first place, the mere existence of such eigenfunctions was an open question. Secondly, it is still unknown of C_{φ_α} can have other eigenvalues.

If $\alpha \in \mathbb{D} \setminus \mathbb{D}_1$, then $\gamma_{\alpha,z_0} = \varphi_{z_0}(\kappa_\alpha)$ (or its conjugate) and as z_0 ranges over $\mathbb{D}$, the values of $\varphi_{z_0}(\kappa_\alpha)$ cover all of $\mathbb{T} \setminus \{1\}$. Hence, $\sigma_p(C_{\varphi_\alpha}) = \mathbb{T} \setminus \{1\}$ and each eigenvalue has infinitely many Blaschke products as its eigenvectors. (That the eigenvalues of C_{φ_α} must stay on $\mathbb{T}$ is rather elementary to verify.)

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