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Analyse par ondelettes de champs aléatoires stables harmonisables à accroissements stationnaires

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Résumé

L'étude du comportement trajectoriel des champs/processus stochastiques est un sujet de recherche classique en théorie des probabilités et dans des domaines connexes comme la géométrie fractale. Dans cet objectif, plusieurs méthodes ont été développées depuis longtemps afin d'étudier le comportement des trajectoires de champs/processus gaussiens. Ces méthodes reposent souvent sur une structure hilbertienne « sympathique », et peuvent aussi nécessiter la finitude de moments d'ordre élevé. Ainsi, elles sont difficilement transposables dans des cadres de lois à queue lourde. Ces dernières sont importantes en probabilités et en statistiques parce qu'elles constituent une contrepartie naturelle des lois gaussiennes. Dans le cas de certains champs/processus stables linéaires de type moyenne mobile non anticipative, tels que le drap fractionnaire stable linéaire et le mouvement multifractionnaire stable linéaire, des méthodes d'ondelettes, assez nouvelles, se sont déjà avérées fructueuses dans l'étude du comportement trajectoriel. Peut-on adapter cette méthodologie à certains champs/processus stables harmonisables ? Donner une réponse à cette question est un problème assez délicat car, de façon générale, de grandes différences séparent le cadre stable harmonisable de celui de type moyenne mobile. Le principal objectif de la thèse est d'étudier cette question dans le cadre d'un champ stable harmonisable symétrique à accroissement stationnaire de forme générale.

Abstract

Studying sample path behaviour of stochastic fields/processes is a classical research topic in probability theory and related areas such as fractal geometry. To this end, many methods have been developed for a long time in order to study sample path behaviour of Gaussian fields/processes. They often rely on some underlying "nice" Hilbertian structure, and can also require finiteness of moments of high order. Therefore, they can hardly be transposed to frames of heavy-tailed stable probability distributions. Such distributions are very important in probability and statistics because they are a natural counterpart to the Gaussian ones. In the case of some linear non-anticipative moving average stable fields/processes, such as the linear fractional stable sheet and the linear multifractional stable motion, rather new wavelet methods have already proved to be successful in studying sample path behaviour. Can this methodology be adapted to some harmonizable stable fields/processes? Providing an answer to this question is a non trivial problem, since, generally speaking, there are large differences between an harmonizable stable setting and a moving average one. The main goal of the thesis is to study this issue in the case of a stationary increments symmetric stable harmonizable field of a general form.

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Introduction

Studying sample path behaviour of stochastic fields/processes is a classical research topic in probability theory and related areas such as fractal geometry. To this end, many methods have been developed for a long time in order to study sample path behaviour of Gaussian fields/processes (see e.g. [10, 1, 15, 21, 19, 28, 29, 30, 23]). They often rely on some underlying "nice" Hilbertian structure, and can also require finiteness of moments of high order. Therefore, they can hardly be transposed to frames of heavy-tailed stable probability distributions. Such distributions are very important in probability and statistics because they are a natural counterpart to the Gaussian ones. They have been widely examined in the literature; a classical reference on them and related topics, including stable random measures and their associated stochastic integrals, is the book of Samorodnitsky and Taqqu [27].

In the case of some linear non-anticipative moving average stable fields/processes, such as the linear fractional stable sheet and the linear multifractional stable motion, rather new wavelet methods have already proved to be successful in studying sample path behaviour (see [3, 2]). Can this methodology be adapted to some harmonizable stable fields/processes? Providing an answer to this question is a non trivial problem, since, generally speaking, there are large differences between an harmonizable stable setting and a moving average one (see for instance [16, 12, 27]). The main goal of the thesis is to study this issue in the case of a stationary increments real-valued symmetric harmonizable α-stable field $X[f] := \left\{X[f](t),\, t \in \mathbb{R}^d\right\}$. This field has the following general form: for all $t \in \mathbb{R}^d$,

$$X[f](t) := \mathcal{R}e\left\{\int_{\mathbb{R}^d} \left(e^{it\cdot\xi} - 1\right) f(\xi)\, \mathrm{d}\widetilde{M}_\alpha(\xi)\right\}, \tag{1}$$

where, roughly speaking, the two main ingredients of the field $X[f]$ are:

- A symmetric α-stable integral with respect to a complex-valued rotationally invariant stable measure $\widetilde{M}_\alpha$ controlled by Lebesgue measure;
- A measurable complex-valued function f which satisfies some conditions.

Basically, the thesis shows that, despite the difficulties inherent in the frequency domain, the wavelet methodology can be generalized and improved in such way that it works well in the case of this general harmonizable stable field $X[f]$. We mention that when $X[f]$ is a (multi-)operator scaling stable random field satisfying some conditions, interesting results on its Hölder regularity have been obtained in [16, 6, 7]. The methodology employed in these articles relies on a representation of $X[f]$ as a LePage series; it is rather different from the wavelet methodology we use in this thesis.

Not only the study of sample path behaviour of $X[f]$ is interesting in its own right (among other things, for the theoretical reasons given before), but also it may have an impact on future development of new applications related with modelling of anisotropic materials in frames of heavy-tailed stable distribution. It is worthwhile to note that in Gaussian frames such modelling has already proved to be useful, in particular for detecting osteoporosis in human bones through the analysis of their radiographic images (see [20, 9, 8]).

Let us now describe the content of each chapter in this thesis.

The starting point of the first chapter is the well-known Kolmogorov's continuity Theorem and the Kolmogorov-Čentsov Hölder continuity Theorem. Those theorems draw a connection between the pathwise Hölder regularity of a random field and the moments of its increment. They are of a difficult use in the frame of heavy-tailed symmetric stable distributions because sample paths of a symmetric stable stochastic field are not so much connected to the behaviour of the moments of its increments (we mention that some recalls about symmetric stable distributions and symmetrical stable random fields are done in this chapter). However, for a centered Gaussian field, those theorems can be conveniently reformulated in terms of its covariance function. In the second part of this chapter we go further beyond the Kolmogorov's continuity Theorem; we study, through the covariance of a centered Gaussian field, differentiability, at any order, of its sample paths; and more generally their Hölder continuity of an arbitrary non-negative order, which is not necessarily less than 1.

The first part of the second chapter is devoted to some recalls about the symmetric α-stable integral $\int_{\mathbb{R}^d}(\cdot)\,\mathrm{d}\widetilde{M}_\alpha$ with respect to a complex-valued rotationally invariant stable measure $\widetilde{M}_\alpha$ controlled by Lebesgue measure, as well as to the notion of LePage series representation for such an integral. In the second part of the chapter, we precisely define the field $X[f]$ in (1) and provide some basic properties of it. More importantly, we somehow justify the notation $X[f]$ by showing that two fields $X[f]$ and $X[g]$ have the same finite-dimensional distributions *if and only if* one has $|f| = |g|$ almost everywhere. We mention that in this chapter, the function f satisfies very general conditions.

The third chapter is the keystone of the thesis. When the function f in (1) belongs to a wide class of *admissible functions*, we provide a wavelet type random series representation for the field $X[f]$ in which each canonical axis l of $\mathbb{R}^d$ has its own dilatation index j_l; such an additional degree of freedom with respect to the classical wavelet frame allows better analysis of the anisotropy of $X[f]$. Moreover, we express the wavelet type random series representation of the field $X[f]$ as the finite sum $X[f] = \sum_\eta X[f]^\eta$, where the fields $X[f]^\eta$ are called the η-frequency parts, since they extend the usual low-frequency and high-frequency parts. Then, we show that the sample paths of all the X^η's are continuous on $\mathbb{R}^d$, and we connect the existence and continuity of their partial derivative, of an arbitrary order, with the rates of vanishing at infinity of the function f. We mention that this result is valid on a *universal* event Ω_1^* of probability 1 in the sense that it does not depend on the function f associated with the field $X[f]$ through (1).

Let $\omega \in \Omega_1^*$, the *universal* event of probability 1 introduced in Chapter 2, be arbitrary and fixed. The first main goal of Chapter 4 is to derive, in terms of the rates of vanishing at infinity of the function f along the axes of $\mathbb{R}^d$, upper estimates for amplitudes of generalized directional increments and classical (non-directional) iterated increments of the sample path $X[f](\cdot,\omega)$, on an arbitrary compact cube of $\mathbb{R}^d$. The second main goal of this chapter is to connect the behaviour of f in a neighbourhood of 0 to upper estimates for the amplitude of $X[f](t,\omega)$, for large values of $\|t\|$. The third main goal of Chapter 4 is to show that the partial derivative function of $X[f](\cdot,\omega)$, when it exists, is bounded when $\alpha \in (0,1)$, and that it has at most a logarithmic increase at infinity when $\alpha \in [1,2]$.

The main goal of the fifth chapter is to develop a technique that allows to obtain results which, among other things, can be viewed, when $\alpha \in (0,2)$, as counterparts of some results in Chapter 4. This technique relies on the wavelet type random series representation of $X[f]$ obtained in Chapter 3 and on "stability" properties of the family of stationary increments harmonizable stable fields. We mention that the results we obtain in this chapter are valid on an event of probability 1 which a priori depends on the function f in (1).

1 Pathwise regularity of Gaussian fields

Abstract

In this chapter, we provide some important connections between the behaviour of the covariance function of a centered Gaussian field and its sample path behaviour. We go further beyond the classical Kolmogorov's continuity theorem. Indeed, we study, through their covariance function, differentiability, at arbitrary order, of Gaussian sample paths, and more generally their Hölder regularity of an arbitrary non-negative order, which is not necessarily less than 1.

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1.1 Kolmogorov continuity Theorem and Hölder regularity

Let $X := \{X(t), t \in \mathbb{R}^d\}$ be a stochastic field defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. A classical method to ensure the existence of a modification[1] of X which is almost surely continuous on $\mathbb{R}^d$ is to apply the Kolmogorov's continuity Theorem [15, 10, 19] which can be formulated in the following way.

[1]That is a field $\{Y(t), t \in \mathbb{R}^d\}$ such that the equality $\mathbb{P}(X(t) = Y(t)) = 1$ holds for all $t \in \mathbb{R}^d$.

Theorem 1.1.1. *Let $X := \{X(t), t \in \mathbb{R}^d\}$ be a stochastic field which satisfied the following property: for any fixed $T \in (0, +\infty)$, there are three constants $p(T) \in (0, +\infty)$, $\beta(T) \in (0, +\infty)$ and $c(T) \in (0, +\infty)$ such that the inequality*

$$\mathbb{E}\Big[\, |X(t) - X(s)|^{p(T)} \Big] \leq c(T)\, \|t - s\|^{d+\beta(T)} \tag{1.1.1}$$

holds for each $s, t \in [-T, T]^d$. Then, X has a modification $\{Y(t), t \in \mathbb{R}^d\}$ which is almost surely continuous on $\mathbb{R}^d$. That is, for any ω in an event Ω^* of probability 1, the sample path $Y(\cdot, \omega)$ is continuous on $\mathbb{R}^d$.

Remark 1.1.2. *Let X and Y be two almost surely continuous fields on $\mathbb{R}^d$. If Y is a modification of X, then X and Y are indistinguishable. That is,*

$$\mathbb{P}\Big(\forall t \in \mathbb{R}^d, \;\; X(t) = Y(t)\Big) = 1.$$

In fact, the hypothesis of Theorem 1.1.1 allows to obtain a stronger result on the path regularity of Y. In order to provide this stronger version of Theorem 1.1.1, we need to make some recall on the notion of Hölder continuity.
First, we mention that there exist many continuous functions on $\mathbb{R}^d$ which are nowhere differentiable. For instance, when $d = 1$, a famous class of them is formed by 1D-Weierstrass functions [13]. A 1D-Weierstrass function, denoted by $\mathcal{W}$, is defined, for any $t \in \mathbb{R}$, as

$$\mathcal{W}(t) := \sum_{n=0}^{+\infty} a^n \cos(b^n t), \tag{1.1.2}$$

where the parameters $a \in (0, 1)$ and $b \in (1, +\infty)$ satisfies the condition $ab > 1$. Usually, the graph of a continuous nowhere differentiable function seems to be more or less erratic and to have some roughness (see Figure 1.1). The Hölder continuity allows to describe such phenomenon.

Definition 1.1.3. *Let $\gamma \in (0, 1]$. A function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is said to be locally γ-Hölder continuous on $\mathbb{R}^d$ when it satisfies a local Hölder condition of order γ. That is, for each fixed $T \in (0, +\infty)$, there exists a constant $c(T) \in (0, +\infty)$ such that the inequality*

$$|\varphi(t) - \varphi(s)| \leq c(T)\, \|t - s\|^{\gamma} \tag{1.1.3}$$

holds for every $s, t \in [-T, T]^d$.

For instance, the Weierstrass function is a locally Hölder continuous function of order $-\log(a)/\log(b)$. Clearly, when a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is locally Hölder continuous of order $\gamma \in (0, 1]$, then it is locally Hölder continuous of any order $\gamma' \in (0, \gamma]$. We are now in the position to state a stronger version of Theorem 1.1.1: the so-called Kolmogorov-Čentsov Theorem [14].

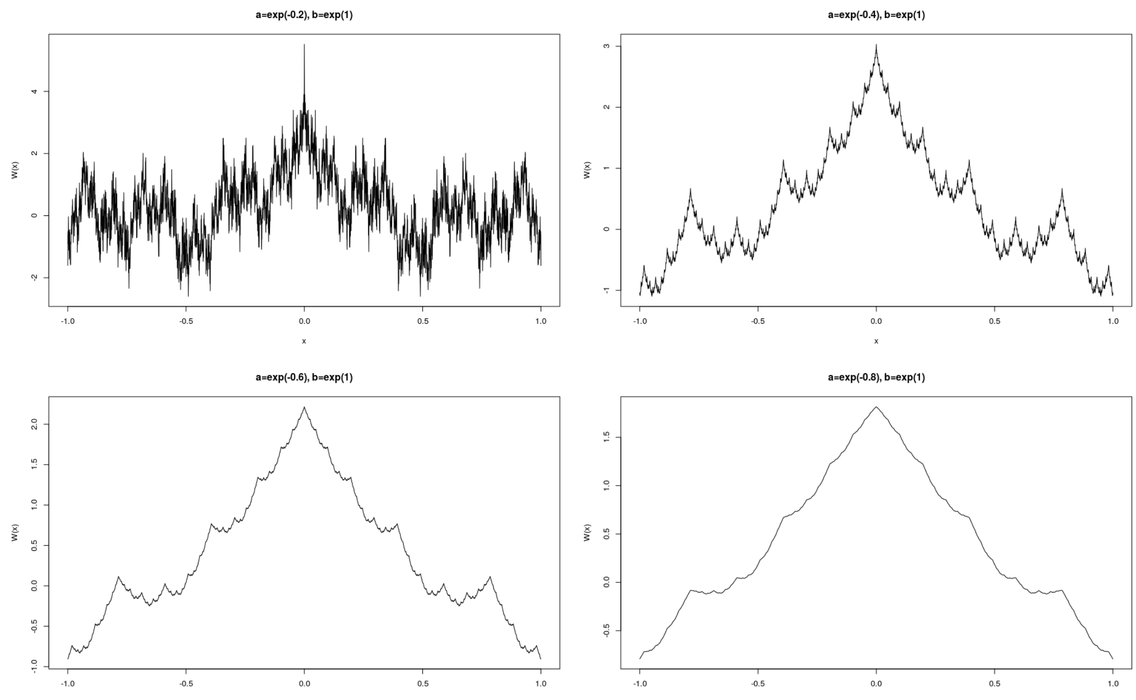

Figure 1.1: Graphs of the Weierstrass function for (1) $a = e^{-0.2}$ and $b = e$, (2) $a = e^{-0.4}$ and $b = e$, (3) $a = e^{-0.6}$ and $b = e$, and (4) $a = e^{-0.8}$ and $b = e$.

Theorem 1.1.4. *Let $X := \{X(t), t \in \mathbb{R}^d\}$ be a stochastic field which satisfied, for some $p \in (0, +\infty)$ and $\beta \in (0, +\infty)$, the following property: for any fixed $T \in (0, +\infty)$, there is a constant $c(T) \in (0, +\infty)$ such that the inequality*

$$\mathbb{E}\Big[\,|X(t) - X(s)|^p\,\Big] \leq c(T)\, \|t - s\|^{d+\beta} \tag{1.1.4}$$

holds for every $s, t \in [-T, T]^d$. Then, X has a modification $\{Y(t), t \in \mathbb{R}^d\}$ which is almost surely locally Hölder continuous of any order $\gamma' \in (0, \beta/p)$ on $\mathbb{R}^d$. That is, for any $\gamma' \in (0, \beta/p)$ and ω in an event Ω^* of probability 1, the sample path $Y(\cdot, \omega)$ is a locally γ'-Hölder continuous function on $\mathbb{R}^d$ in the sense of Definition 1.1.3.

Now, we are going to see that, while Theorem 1.1.4 is of a simple use in the Gaussian setting, it is of a more difficult use in the frame of heavy-tailed stable distributions. A classical reference on such distributions and related fields is the book of Samorodnitsky and Taqqu [27]. We mention that for the sake of simplicity throughout this thesis we restrict to symmetric stable distributions.

Definition 1.1.5. *Let Z be a real-valued random variable and χ_Z its characteristic function defined, for all $\xi \in \mathbb{R}$, as*

$$\chi_Z(\xi) := \mathbb{E}\Big(e^{i\xi Z}\Big). \tag{1.1.5}$$

Then, Z is said to have a symmetric stable distribution of stability parameter $\alpha \in (0, 2]$ and scale parameter $\sigma \in \mathbb{R}_+$, if:

$$\forall\, \xi \in \mathbb{R},\ \chi_Z(\xi) = \exp(-\sigma^\alpha |\xi|^\alpha)\,. \tag{1.1.6}$$

Notice that the law of a real-valued symmetric α-stable random variable is completely determined by its scale parameter σ. That is, two real-valued symmetric α-stable random variables are identically distributed if, and only if, they have the same scale parameter.

When $\alpha = 2$, Z reduces to a real-valued centered Gaussian random variable with variance $2\sigma^2$. The Gaussian distribution presents the advantages of having finite moment at any order. Moreover, when Z is a real-valued centered Gaussian random variable with variance $\sigma^2 \in [0, +\infty)$, for any $k \in \mathbb{Z}_+$, we have that

$$\mathbb{E}\Big[Z^{2k}\Big] = \frac{(2k)!}{k!2^k}\sigma^{2k} \quad \text{and} \quad \mathbb{E}\Big[Z^{2k+1}\Big] = 0, \tag{1.1.7}$$

and, for every $u \in (0 + \infty)$,

$$\mathbb{P}(|Z| > u) \leq \frac{2\sigma e^{-u^2/2\sigma^2}}{\sqrt{2\pi} u}. \tag{1.1.8}$$

The proof of the inequality (1.1.8) is simple; it is obtained by the change of variable $v = t^2$:

$$\begin{aligned}
&\mathbb{P}(|Z| > u) = \frac{2}{\sqrt{2\pi\sigma^2}} \int_u^{+\infty} e^{-t^2/2\sigma^2}\, \mathrm{d}t = \frac{2}{\sqrt{2\pi\sigma^2}} \int_{u^2}^{+\infty} \frac{e^{-v/2\sigma^2}}{2\sqrt{v}}\, \mathrm{d}v \\
&\leq \frac{2}{\sqrt{2\pi\sigma^2} u} \int_{u^2}^{+\infty} \frac{e^{-v/2\sigma^2}}{2}\, \mathrm{d}v = \frac{2\sigma e^{-u^2/2\sigma^2}}{\sqrt{2\pi} u}.
\end{aligned}$$

The situation is very different when $\alpha \in (0,2)$ and $\sigma > 0$. The distribution of Z becomes heavy-tailed: for example the second order moment of Z is infinite. More precisely, it follows from Property 1.2.15 in [27] that

$$\mathbb{P}(|Z| > z) \sim c(\alpha)\sigma^\alpha z^{-\alpha}, \quad \text{when } z \to +\infty, \tag{1.1.9}$$

where the positive and finite constant $c(\alpha)$ is equal to

$$c(\alpha) := \left(\int_0^{+\infty} x^{-\alpha} \sin(x) \,\mathrm{d}x \right)^{-1}.$$

We recall that the symbol "$\sim$" in (1.1.9) means that

$$\lim_{z \to +\infty} z^\alpha \mathbb{P}(|Z| > z) = c(\alpha)\sigma^\alpha.$$

In particular, (1.1.9) implies that:

$$\begin{aligned} \mathbb{E}(|Z|^\gamma) &< +\infty \quad \text{when} \quad \gamma < \alpha, \\ \mathbb{E}(|Z|^\gamma) &= +\infty \quad \text{when} \quad \gamma \geq \alpha. \end{aligned} \tag{1.1.10}$$

In fact, there is a close connection between the moment of order $\gamma \in (0,\alpha)$ of a symmetric α-stable random variable Z and its scale parameter σ. More precisely, there exists a constant $c_\alpha(\gamma)$ such that

$$\mathbb{E}(|Z|^\gamma) = c_\alpha(\gamma)\sigma^\gamma. \tag{1.1.11}$$

We mention that $c_\alpha(\gamma)$ is equal to $\mathbb{E}(|Z_0|^\gamma)$, where Z_0 is a real-valued symmetric α-stable of scale parameter 1.

Symmetric stable stochastic fields are defined as follows.

Definition 1.1.6. *Let $\alpha \in (0,2]$. A real-valued stochastic field $\left\{X(t), t \in \mathbb{R}^d\right\}$ is said to be symmetric α-stable if, for any $N \in \mathbb{N}$, $t^1, \dots, t^N \in \mathbb{R}^d$ and $b_1, \dots b_N \in \mathbb{R}$, the linear combination $\sum_{l=1}^N b_l X(t^l)$ is a real-valued symmetric α-stable random variable.*

Let us now show that in the centered Gaussian setting[2], Theorem 1.1.4 can be expressed in a simple way in terms of covariance function[3]. Indeed, in view of (1.1.7) and the equality

$$\mathbb{E}\Big[|X(t) - X(s)|^2 \Big] = \mathrm{Cov}_X(t,t) - 2\mathrm{Cov}_X(t,s) + \mathrm{Cov}_X(s,s), \tag{1.1.12}$$

we get the following corollary.

[2]That is when $\alpha = 2$.

[3]The covariance function of a Gaussian field $X := \left\{X(t), t \in \mathbb{R}^d\right\}$ is the real-valued function Cov_X defined, for any $s,t \in \mathbb{R}^d$, by

$$\mathrm{Cov}_X(s,t) := \mathbb{E}[X(s)X(t)].$$

Corollary 1.1.7. *Let $X := \{X(t), t \in \mathbb{R}^d\}$ be a centered Gaussian field which satisfies, for some $\gamma \in (0,1]$, the following property: for all fixed $T \in (0,+\infty)$, there exists a constant $c(T) \in (0,+\infty)$ such that the inequality*

$$|\mathrm{Cov}_X(t,t) - 2\mathrm{Cov}_X(t,s) + \mathrm{Cov}_X(s,s)| \leq c(T)\,\|t-s\|^{2\gamma} \tag{1.1.13}$$

folds for any $s,t \in [-T,T]^d$. Then, X has a modification which is almost surely locally Hölder continuous of any order $\gamma' \in (0,\gamma)$ on $\mathbb{R}^d$.

Example 1.1.8. *The fractional Brownian field[4], of Hurst parameter $H \in (0,1)$, is the real-valued centered Gaussian field denoted by $B_H := \left\{B_H(t), t \in \mathbb{R}^d\right\}$ such that, for all $s,t \in \mathbb{R}^d$,*

$$\mathrm{Cov}_{B_H}(t,s) := \mathbb{E}[B_H(t)B_H(s)] = \frac{K_H}{2}\left\{\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H}\right\}, \tag{1.1.14}$$

where $K_H \in (0,+\infty)$ is a constant and $\|\cdot\|$ denotes the Euclidian norm. Observe that, it can be seen that, for all $s,t \in \mathbb{R}^d$,

$$\mathrm{Cov}_{B_H}(t,t) - 2\mathrm{Cov}_{B_H}(t,s) + \mathrm{Cov}_{B_H}(s,s) = K_H\,\|t-s\|^{2H}. \tag{1.1.15}$$

Hence, Corollary 1.1.7 ensures the existence of a modification of B_H which is almost surely locally γ-Hölder continuous on $\mathbb{R}^d$, for any $\gamma \in (0,H)$.

While Theorem 1.1.4 is very efficient in the Gaussian setting, it is less efficient in the frame of heavy-tailed stable distributions. For instance, this theorem does not allow to determine the optimal Hölder regularity of the harmonizable fractional stable motion.

Example 1.1.9. *The harmonizable fractional stable motion of stability parameter $\alpha \in (0,2)$ and Hurst parameter $H \in (0,1)$ is one of the two classical extension of the fractional Brownian motion to the frame of heavy-tailed stable distributions. It is denoted by $X^{hfsm} := \{X^{hfsm}(t), t \in \mathbb{R}\}$ and defined as follows: for all $t \in \mathbb{R}$,*

$$X^{hfsm}(t) = \mathcal{R}e\left\{\int_{\mathbb{R}}\left(e^{it\xi}-1\right)|\xi|^{-H-1/\alpha}\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right\}, \tag{1.1.16}$$

where $\widetilde{M}_\alpha$ is a complex-valued rotationally invariant α-stable random measure on $\mathbb{R}^d$ with Lebesgue control measure[5].

One can show that, for every $s,t \in \mathbb{R}$, the scale parameter $\sigma(X^{hfsm}(t) - X^{hfsm}(s))$ of the symmetric α-stable random variables and $X^{hfsm}(t) - X^{hfsm}(s)$ satisfies

$$\sigma(X^{hfsm}(t) - X^{hfsm}(s)) = c(\alpha,H)\,|t-s|^{H}, \tag{1.1.17}$$

[4] That is, the fractional Brownian motion on $\mathbb{R}^d$.

[5] We present more in details this integral in Chapter 2.

where $c(\alpha, H)$ is a positive and finite constant. Combining (1.1.11) *and* (1.1.17)*, we get, for all $p \in (0, \alpha)$,*

$$\mathbb{E}\Big[\, |X^{hfsm}(t) - X^{hfsm}(s)|^p \,\Big] = c'(\alpha, H, p)\, |t-s|^{Hp}\,, \tag{1.1.18}$$

where $c'(\alpha, H, p)$ is a positive and finite constant. Assume that we have $H > 1/\alpha$. Therefore, the inequality $Hp > 1$ holds for any $p \in (1/H, \alpha)$. Since $p \in (1/H, \alpha)$ is arbitrary, it follows from Theorem 1.1.4 that there exists a modification of X^{hfsm} which is almost surely locally Hölder continuous of any order $\gamma' \in (0, H - 1/\alpha)$.

Although we could apply the Kolmogorov-Čentsov theorem in the case of the harmonizable fractional stable motion, the result we obtained is not optimal. Indeed, one can show that there is a modification of this process which is almost surely locally Hölder continuous of any order $\gamma' \in (0, H)$. This result and a refinement of it can be proved using a LePage series representation of the stable stochastic integral [17, 16]. We state it precisely in Proposition 2.2.1.

The other classical extension of the fractional Brownian motion to the setting of heavy-tailed stable distributions is called linear fractional stable motion. It is defined, for all $t \in \mathbb{R}$, by

$$Y^{\mathrm{lfsm}}(t) := \int_{\mathbb{R}} \left((t-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha} \right) \mathrm{d}M_\alpha(u)\,, \tag{1.1.19}$$

where M_α is a symmetric α-stable real-valued random measure on $\mathbb{R}$[6]. When we have $H > 1/\alpha$, Theorem 1.1.4 allows us to derive the existence a modification of this process which is almost surely locally Hölder continuous of any order $\gamma' \in (0, H - 1/\alpha)$. One can show that this regularity is optimal [3].

Notice that the covariance of a centered Gaussian field X is a useful tool to study the Hölder continuity of its sample paths. It is based on the existence of all the moments of a Gaussian random variable and the equality (1.1.12). In the next section, we will see that the almost sure differentiability of the sample paths of a Gaussian field is also connected to its covariance function.

1.2 Differentiability in quadratic mean and pathwise

In this section, we provide conditions on the covariance of a Gaussian field $X = \Big\{X(t), t \in \mathbb{R}^d\Big\}$ that ensure the existence of a modification of X which is almost surely continuously differentiable on $\mathbb{R}^d$. That is, the existence of a field $\{Y(t), t \in \mathbb{R}^d\}$ and an event Ω^* of probability 1 such that, for any $\omega \in \Omega^*$, the sample path $Y(\cdot, \omega)$ is continuously differentiable on $\mathbb{R}^d$.

[6]We refer to the chapter 3 and section 7.4 in [27] for a detailed study of the stable integral with respect to a symmetric α-stable real-valued random measure on $\mathbb{R}$ and the linear fractional stable motion.

Notation 1.2.1. *Let e_l be the vector of $\mathbb{R}^d$ whose l-th coordinate equals 1 and the others vanish. In the sequel, for all $l \in \{1, \ldots, d\}$ and $\omega \in \Omega^*$, we denote by $\partial^{e_l} Y(\cdot, \omega)$ the partial derivative function of the sample path $Y(\cdot, \omega)$ in the direction l. When $\omega \notin \Omega^*$, we set $\partial^{e_l} Y(\cdot, \omega) := 0$.*

Notice that, in the Gaussian case, the almost sure convergence implies the convergence in $L^2(\Omega)$. More generally, we have the following result.

Proposition 1.2.2. *Let $\{X_n, n \in \mathbb{N}\}$ be a centered Gaussian process and X be a random variable such that the sequence $\{X_n, n \in \mathbb{N}\}$ converges in probability to X. Then, for all $p \in (0, \infty)$, we have*

$$\lim_{n \to +\infty} \mathbb{E}\Big[|X_n - X|^p \Big] = 0. \tag{1.2.1}$$

Proof. First, we prove (1.2.1) when $p = 2$. The fact that the process $\{X_n : n \in \mathbb{N}\}$ is Gaussian with mean zero implies that, for all positive integers m and n, the random variable $X_m - X_n$ has a Gaussian distribution with mean zero. Moreover, for every $n \in \mathbb{N}$, we have

$$\lim_{m \to +\infty} X_n - X_m = X_n - X$$

where the limit holds in probability. Hence, for all positive integers n, the random variable $X_n - X$ also has a centered Gaussian distribution. We denote by σ_n^2 the variance of $X_n - X$. Therefore, its characteristic function is given, for all t in $\mathbb{R}$, by

$$\phi_n(t) = e^{-\sigma_n^2 t/2}. \tag{1.2.2}$$

Moreover, we know that, $\{X_n, n \in \mathbb{N}\}$ converges in probability to X. So, it also converges in distribution. Therefore, for all $t \in \mathbb{R}$,

$$\lim_{n \to +\infty} \phi_n(t) = \mathbb{E}[e^{i0t}] = 1. \tag{1.2.3}$$

The logarithmic function being continuous at 1, putting together (1.2.2) and (1.2.3), we obtain

$$\lim_{n \to +\infty} \sigma_n = 0. \tag{1.2.4}$$

So, (1.2.1) holds when $p = 2$.

Then, when $p \in (0, +\infty)$ is arbitrary, observe that, when $\sigma_n^2 \neq 0$, the random variable $\sigma_n^{-1}(X_n - X)$ has centered Gaussian distribution with variance 1. So,

$$\mathbb{E}[|X_n - X|^p] \;\; = \;\; \sigma_n^p \mathbb{E}\left[\left(\frac{|X_n - X|}{\sigma_n} \right)^p \right] = \sigma_n^p C(p). \tag{1.2.5}$$

where $C(p) := \mathbb{E}\left[|Z|^p\right]$ and Z is a centered Gaussian random variable with variance 1. Notice that $C(p)$ is finite and does not depend on n. On the other hand, if $\sigma_n^2 = 0$, then almost surely, $X_n = X$. Therefore (1.2.5) holds for any $n \in \mathbb{N}$. In view of (1.2.4) and (1.2.5), we get that (1.2.1) holds for any $p \in (0, +\infty)$. □

Let X be a centered Gaussian field which is almost surely continuously differentiable on $\mathbb{R}^d$. In view of Proposition 1.2.2, for all $l \in \{1,\ldots,d\}$ and $t_0 \in \mathbb{R}^d$, the limit of

$$h^{-1}\Big(X(t_0+he_l) - X(t_0)\Big)$$

exists in $L^2(\Omega)$ when the non-vanishing real number h goes to 0.

Definition 1.2.3. *Let $X := \Big\{X(t), t \in \mathbb{R}^d\Big\}$ be a centered Gaussian field. Assume that $l \in \{1,\ldots,d\}$ and $t_0 \in \mathbb{R}^d$ are arbitrary and fixed. X is said to have a partial derivative in quadratic mean at the point t_0 in the direction l if the limit of*

$$h^{-1}\Big(X(t_0+he_l) - X(t_0)\Big) \tag{1.2.6}$$

exists in $L^2(\Omega)$ when the non-vanishing real number h goes to 0. This limit is almost surely unique, and we denote it by $D_l^{\mathrm{qm}}X(t_0)$.

We define as well the differentiability in quadratic mean of X.

Definition 1.2.4. *Let $X := \Big\{X(t), t \in \mathbb{R}^d\Big\}$ be a centered Gaussian field and $t_0 \in \mathbb{R}^d$. X is said to be differentiable in quadratic mean at the point t_0 if there exists a linear mapping $D^{\mathrm{qm}}X(t_0)$ from $\mathbb{R}^d$ to $L^2(\Omega)$ satisfying*

$$\mathbb{E}\Big[\left|X(t_0+h) - X(t_0) - D^{\mathrm{qm}}X(t_0)(h)\right|^2\Big] = \underset{h\to 0}{o}\Big(|h|^2\Big). \tag{1.2.7}$$

If X is differentiable in quadratic mean at any point t_0 in $\mathbb{R}^d$, then X is said to be differentiable in quadratic mean on $\mathbb{R}^d$.

Similarly to the deterministic case, we have the following properties (see Lemma 2.2, Lemma 2.4 and Lemma 2.7 in [26]).

Proposition 1.2.5. *Let $X := \Big\{X(t), t \in \mathbb{R}^d\Big\}$ be a centered Gaussian field and $t_0 \in \mathbb{R}^d$. Assume that X is differentiable in quadratic mean at the point t_0. Then the following properties hold:*

(i) X is continuous in quadratic mean at the point t_0: that is,

$$\lim_{t\to t_0} \mathbb{E}\Big[\left|X(t) - X(t_0)\right|^2\Big] = 0. \tag{1.2.8}$$

(ii) If there exists L another linear mapping from $\mathbb{R}^d$ to $L^2(\Omega)$ satisfying (1.2.7) then

$$\mathbb{P}\Big(\forall h \in \mathbb{R}^d,\ \Big(D^{\mathrm{qm}}X(t_0)\Big)(h) = L(h)\Big) = 1. \tag{1.2.9}$$

(iii) *X has a partial derivative in quadratic mean in t_0 in any direction $l \in \{1, \ldots, d\}$. Moreover, almost surely, for every $h = (h_1, \ldots, h_d) \in \mathbb{R}^d$, we have*

$$\Big(D^{\mathrm{qm}} X(t_0)\Big)(h) = \sum_{l=1}^{d} h_l D_l^{\mathrm{qm}} X(t_0). \tag{1.2.10}$$

Notice that if the partial derivative in quadratic mean at any point t in $\mathbb{R}^d$ in direction l of a centered Gaussian field $X := \Big\{X(t), t \in \mathbb{R}^d\Big\}$ exist, then $D_l^{\mathrm{qm}} X := \{D_l^{\mathrm{qm}} X(t), t \in \mathbb{R}^d\}$ is a centered Gaussian field.

Notation 1.2.6. *For all functions $\varphi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $l \in \{1, \ldots, d\}$ and $s, t \in \mathbb{R}^d$, we denote by*

$$D_1^l \varphi(t, s) := \lim_{\substack{h \to 0 \\ h \neq 0}} h^{-1}\Big(\varphi(t + he_l, s) - \varphi(t, s)\Big), \tag{1.2.11}$$

and

$$D_2^l \varphi(t, s) := \lim_{\substack{h \to 0 \\ h \neq 0}} h^{-1}\Big(\varphi(t, s + he_l) - \varphi(t, s)\Big), \tag{1.2.12}$$

whenever the limits exist.

With this notations, the covariance function of $D_l^{\mathrm{qm}} X$ satisfies the equality

$$\mathrm{Cov}_{D_l^{\mathrm{qm}} X} = D_1^l D_2^l \mathrm{Cov}_X. \tag{1.2.13}$$

We have seen that a centered Gaussian field which is differentiable almost surely is differentiable in quadratic mean. The reciprocal is false in general. We have to assume that the fields $D_l^{\mathrm{qm}} X$, where $l \in \{1, \ldots, d\}$, satisfy some regularity conditions. More precisely, we have the following result which is a consequence of Theorem 3.2 in [26].

Theorem 1.2.7. *Let $X := \Big\{X(t), t \in \mathbb{R}^d\Big\}$ be a centered Gaussian field. Assume that the three following conditions hold:*

(i) *X is differentiable in quadratic mean on $\mathbb{R}^d$.*

(ii) *For any $l \in \{1, \ldots, d\}$, the field $D_l^{\mathrm{qm}} X$ is continuous in quadratic mean on $\mathbb{R}^d$.*

(iii) *For all $l \in \{1, \ldots, d\}$, the field $D_l^{\mathrm{qm}} X$ has a modification which is almost surely continuous on $\mathbb{R}^d$.*

Then, X has a modification which is almost surely continuously differentiable on $\mathbb{R}^d$.

Remark 1.2.8. *Notice that when the conditions (i) and (iii) in Theorem 1.2.7 hold, then the condition (ii) is satisfied as well. In fact, it is convenient for us to add the redundant condition (ii) for the sake of clarity.*

Their exist sufficient conditions on the covariance function of a Gaussian field so that (i), (ii) and (iii) in Theorem 1.2.7 hold. In a first time, we focus on (i) and (ii). One can characterize the differentiability in quadratic mean of a Gaussian field in terms of its covariance function. In order to do so, we need the following notation. For each function $\varphi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $l \in \{1, \dots, d\}$ and $s, t \in \mathbb{R}^d$, the generalized partial second derivative of φ at the point (t, s) in the direction l is defined as

$$D^{l,l}\varphi(t,s) := \lim_{\substack{(h,h')\to(0,0)\\ h\neq 0, h'\neq 0}} \frac{1}{hh'}\Big(\varphi(t+he_l, s+h'e_l)-\varphi(t+he_l, s)-\varphi(t, s+h'e_l)+\varphi(t,s)\Big), \quad (1.2.14)$$

provided that the limit exists. An equivalent condition to the existence of a partial derivative in quadratic mean at a point t_0 in the direction l of a Gaussian field in terms of it covariance function is given by the following result (see Lemma 2.9 in [26]).

Proposition 1.2.9. *Let $X := \left\{X(t), t \in \mathbb{R}^d\right\}$ be a centered Gaussian field and $t_0 \in \mathbb{R}^d$ be fixed. Then, X has a partial derivative in quadratic mean at the point t_0 in the direction l if, and only if,* Cov_X *has a generalized partial second derivative at the point (t_0, t_0) in the direction l.*

Observe that when a function φ is two times continuously differentiable on $\mathbb{R}^d \times \mathbb{R}^d$ then, for any $s, t \in \mathbb{R}^d$, the generalized partial second derivative at the point (t, s) in the direction l of φ exists and satisfies

$$D^{l,l}\varphi(t,s) = D_1^l D_2^l \varphi(t,s) = D_2^l D_1^l \varphi(t,s). \quad (1.2.15)$$

Indeed, we have, for any $s, t \in \mathbb{R}^d$,

$$\begin{aligned}&\frac{1}{hh'}\Big(\varphi(t+he_l, s+h'e_l)-\varphi(t+he_l, s)-\varphi(t, s+h'e_l)+\varphi(t,s)\Big) - D_1^l D_2^l\varphi(t,s)\\ &= \frac{1}{hh'}\int_0^{h'}\int_0^{h}\Big(D_1^l D_2^l\varphi(t+ue_l, s+ve_l) - D_1^l D_2^l\varphi(t,s)\Big)\mathrm{d}u\mathrm{d}v. \quad (1.2.16)\end{aligned}$$

Then, using the continuity of the function $D_1^l D_2^l\varphi$ at the point (t, s), we get that $D^{l,l}\varphi(t,s)$ exists and is equal to $D_1^l D_2^l\varphi(t,s)$. The equality $D_1^l D_2^l\varphi(t,s) = D_2^l D_1^l\varphi(t,s)$ is a consequence of the Schwarz Theorem and the fact that φ is two times continuously differentiable on $\mathbb{R}^d \times \mathbb{R}^d$. Therefore, we get the following corollary (see Corollary 2.11 in [26]).

Corollary 1.2.10. *Let $X := \left\{X(t), t \in \mathbb{R}^d\right\}$ be a centered Gaussian field. If its covariance function* Cov_X *is two times continuously differentiable on $\mathbb{R}^d \times \mathbb{R}^d$, then X is differentiable in quadratic mean in $\mathbb{R}^d$ and the partial derivatives in quadratic mean of X are continuous in quadratic mean at any point $t \in \mathbb{R}^d$ (see* (1.2.8)*).*

In the Gaussian case, thanks to the Kolmogorov-Čenstov Theorem, the point (*iii*) in Theorem 1.2.7 can be satisfied under some simple conditions on the covariance function of the fields $D_l^{\mathrm{qm}} X$, $l \in \{1, \ldots, d\}$. More precisely, combining Corollary 1.2.10, Theorem 1.1.1, (1.2.13), (1.2.15), (1.1.12), and (1.1.7), we have the following result (see Corollary 4.4 in [26]).

Theorem 1.2.11. *Let $X := \{X(t), t \in \mathbb{R}^d\}$ be a centered Gaussian field. Assume that the covariance function of X satisfies the two following properties:*

(i) The covariance function Cov_X of X is two times continuously differentiable on $\mathbb{R}^d \times \mathbb{R}^d$.

(ii) For any $T \in (0, +\infty)$, there are two constants $c(T) \in (0, +\infty)$ and $p(T) \in (0, +\infty)$ such that the inequality

$$\left| D^{ll}\mathrm{Cov}_X(t,t) - 2D^{ll}\mathrm{Cov}_X(t,s) + D^{ll}\mathrm{Cov}_X(s,s) \right| \leq c(T) \|t-s\|^{p(T)} \tag{1.2.17}$$

holds for any $l \in \{1, \ldots, d\}$ and $s, t \in [-T, T]^d$,

Then, X has a modification which is almost surely continuously differentiable on $\mathbb{R}^d$.

Remark 1.2.12. *Let $X := \{X(t), t \in \mathbb{R}^d\}$ be a centered Gaussian field. Assume that Y is a modification of X which is almost surely differentiable. Then, in view of Proposition 1.2.2, the field $\partial^{e_l} Y = \left\{\partial^{e_l} Y(t), t \in \mathbb{R}^d\right\}$ is a modification of $D_l^{\mathrm{qm}} X = \{D_l^{\mathrm{qm}} X(t), t \in \mathbb{R}^d\}$. In particular, it is a centered Gaussian field with covariance function $D_1^l D_2^l \mathrm{Cov}_X$.*

The results presented in this section rely on the Gaussianity of the fields considered. Our approach in this section is based on [26] in which the author considers more general random fields X satisfying, for any $t \in \mathbb{R}^d$, $\mathbb{E}[|X(t)|^2] < +\infty$. Notice that, this approach is not adapted to the frame of heavy-tailed stable distributions, for which the second order moment is infinite. In [10] (Chapter 4), the authors consider random processes $\{X(t), t \in \mathbb{R}\}$ for which the conditions they impose do not suppose the existence of moments of the random variables $X(t)$. However, it would be difficult to verify if those conditions are satisfied in general.

1.3 Generalized Hölder regularity for Gaussian fields

The notion of local Hölder regularity can be extended to the setting of smooth functions for which $\gamma > 1$. To this end, Definition 1.1.3 has to be modified in the following way[7].

Definition 1.3.1. *Let $\gamma \in (1, +\infty)$ be fixed. We set $m(\gamma) := \max\{n \in \mathbb{Z}_+ : n < \gamma\}$. A function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is said to be locally γ-Hölder continuous if it satisfies the two following properties:*

[7] Notice that when $\gamma > 1$ and $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a locally γ-Hölder function on $\mathbb{R}^d$ in the sense of Definition 1.1.3, then φ is a trivial function. Namely, φ is constant on $\mathbb{R}^d$.

(i) For all multi-indices $b := (b_1, \dots, b_d)$ such that $b_1 + \dots + b_d \le m(\gamma)$ the partial derivative function

$$\partial^b \varphi := \frac{\partial^{b_1} \partial^{b_2} \dots \partial^{b_d}}{(\partial t_1)^{b_1} (\partial t_2)^{b_2} \dots (\partial t_d)^{b_d}} \varphi \quad \text{(with the convention that } \partial^0 \varphi := \varphi\text{)}$$

is well-defined and continuous on $\mathbb{R}^d$.

(ii) For all multi-indices $b := (b_1, \dots, b_d)$ such that $b_1 + \dots + b_d = m(\gamma)$ the partial derivative function $\partial^b \varphi$ satisfies a local Hölder condition of order $\gamma - m(\gamma)$. In other words, for every $T \in (0, +\infty)$ there exists a constant $c(T) \in (0, +\infty)$ such that the inequality

$$\left| \partial^b \varphi(t) - \partial^b \varphi(s) \right| \le c(T) \, \|t - s\|^{\gamma - m(\gamma)} \tag{1.3.1}$$

holds for each $s, t \in [-T, T]^d$.

Similarly to the case $\gamma \in (0, 1]$, when a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is locally Hölder continuous of order $\gamma \in (0, +\infty]$, then it locally Hölder continuous of any order $\gamma' \in (0, \gamma]$. The main goal of this section is to prove the following result.

Theorem 1.3.2. *Let $X := \{X(t), t \in \mathbb{R}^d\}$ be a centered Gaussian field. Assume that, for some $\gamma \in (0, +\infty)$, the covariance function of X is a locally Hölder continuous function of order 2γ on $\mathbb{R}^d \times \mathbb{R}^d$. Then, X has a modification which is almost surely locally Hölder continuous of any order $\gamma' \in (0, \gamma)$ on $\mathbb{R}^d$.*

The proof of Theorem 1.3.2 relies on the following two lemmas.

Lemma 1.3.3. *Let $X := \{X(t), t \in \mathbb{R}^d\}$ be a centered Gaussian field. Assume that, for some $\gamma \in (0, 1]$, the covariance function of X is a locally Hölder continuous function of order 2γ on $\mathbb{R}^d \times \mathbb{R}^d$. Then, X has a modification which is almost surely locally Hölder continuous of any order $\gamma' \in (0, \gamma)$ on $\mathbb{R}^d$.*

Lemma 1.3.4. *Let $X := \{X(t), t \in \mathbb{R}^d\}$ be a centered Gaussian field. Assume that, for some $\gamma \in (1, 2]$, the covariance function of X is a locally Hölder continuous function of order 2γ on $\mathbb{R}^d \times \mathbb{R}^d$. Then, X has a modification which is almost surely locally Hölder continuous of any order $\gamma' \in (0, \gamma)$ on $\mathbb{R}^d$. In particular, this modification is almost surely continuously differentiable on $\mathbb{R}^d$.*

Proof of Lemma 1.3.3. We will study two cases: $\gamma \in (0, 1/2]$ and $\gamma \in (1/2, 1]$.
First case: $\gamma \in (0, 1/2]$. Let $T \in (0, +\infty)$ be arbitrary and fixed. The function Cov_X is locally Hölder continuous of order 2γ on $\mathbb{R}^d$. Then, there exists a constant $c(T) \in (0, +\infty)$ such that, for any $s, t \in [-T, T]^d$, we have,

$$|\mathrm{Cov}_X(t, s) - \mathrm{Cov}(s, s)| \le c(T) \, \|t - s\|^{2\gamma}. \tag{1.3.2}$$

Hence, combining (1.1.12) and (1.3.2), we get, for every $s,t \in [-T,T]^d$,

$$|\mathrm{Cov}_X(t,t) - 2\mathrm{Cov}_X(t,s) + \mathrm{Cov}_X(s,s)| \leq 2c(T)\, \|t-s\|^{2\gamma}, \tag{1.3.3}$$

where $\gamma > 0$. Therefore, we get from Corollary 1.1.7 the existence of modification of X which is almost surely locally Hölder continuous of any order $\gamma' \in (0,\gamma)$ on $\mathbb{R}^d$.

Second case: $\gamma \in (1/2,1]$. Let $T \in (0,+\infty)$ be arbitrary and fixed. In this case, we have $2\gamma > 1$, so the function Cov_X is continuously differentiable on $\mathbb{R}^d$. For any $s,t \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\mathrm{Cov}_X(t,t) - \mathrm{Cov}_X(t,s) \\
&= \sum_{j=1}^{d} \int_{s_j}^{t_j} D_2^j \mathrm{Cov}_X(t; t_1, t_2, \ldots, t_{j-1}, u, s_{j+1}, \ldots, s_d)\mathrm{d}u,
\end{aligned} \tag{1.3.4}$$

with the convention that

$$(t_1, t_2, \ldots, t_{j-1}, u, s_{j+1}, \ldots, s_d) = (u, s_2, \ldots, s_d),$$

when $j = 1$, and

$$(t_1, t_2, \ldots, t_{j-1}, u, s_{j+1}, \ldots, s_d) = (t_1, t_2, \ldots, t_{d-1}, u),$$

when $j = d$. On the other hand, the function Cov_X satisfies, for all $s,t \in \mathbb{R}^d$, $\mathrm{Cov}_X(s,t) = \mathrm{Cov}_X(t,s)$. Therefore, we have

$$\begin{aligned}
&\mathrm{Cov}_X(s,s) - \mathrm{Cov}_X(t,s) \\
&= \mathrm{Cov}_X(s,s) - \mathrm{Cov}_X(s,t) \\
&= -\sum_{j=1}^{d} \int_{s_j}^{t_j} D_2^j \mathrm{Cov}_X(s; t_1, t_2, \ldots, t_{j-1}, u, s_{j+1}, \ldots, s_d)\mathrm{d}u.
\end{aligned} \tag{1.3.5}$$

Moreover, for any $j \in \{1,\ldots,d\}$, the function $D_2^j\mathrm{Cov}_X$ is locally Hölder continuous of order $2\gamma - 1$ on $\mathbb{R}^d$. Thus, there exists a constant $c(T,j) \in (0,+\infty)$ such that, for any $s,t,t' \in [-T,T]^d$, we have

$$\left| D_2^j\mathrm{Cov}_X(t,t') - D_2^j\mathrm{Cov}_X(s,t') \right| \leq c(T,j)\, \|t-s\|^{2\gamma-1}. \tag{1.3.6}$$

Therefore, combining (1.3.4) to (1.3.6) we obtain, for any $l \in \{1,\ldots,d\}$ and $s,t \in \mathbb{R}^d$,

$$\begin{aligned}
&|\mathrm{Cov}_X(t,t) - 2\mathrm{Cov}_X(t,s) + \mathrm{Cov}_X(s,s)| \\
&\leq \sum_{j=1}^{d} c(T,j) \left| \int_{s_j}^{t_j} \mathrm{d}u \right| \|t-s\|^{2\gamma-1} \\
&\leq c(T)\, \|t-s\|^{2\gamma}
\end{aligned} \tag{1.3.7}$$

where $c(T) := d\max\left\{c(T,j), j \in \{1,\ldots,d\}\right\}$. So, it follows from Corollary 1.1.7 that their exists a modification of X which is almost surely locally Hölder continuous of any order $\gamma' \in (0,\gamma)$ on $\mathbb{R}^d$. □

Proof of Lemma 1.3.4. We divide this proof into two steps.

Step 1: Using Theorem 1.2.11, we show that there exists a modification $\{Y(t), t \in \mathbb{R}^d\}$ of X which is almost surely continuously differentiable on $\mathbb{R}^d$.
The function Cov_X is locally Hölder continuous of order 2γ on $\mathbb{R}^d \times \mathbb{R}^d$. Hence, it follows from the inequality $2\gamma > 2$ and the Definition 1.3.1 that $D_1^l \mathrm{Cov}_X$, $D_2^{l'} \mathrm{Cov}_X$ and $D_1^l D_2^{l'} \mathrm{Cov}_X$ are continuous functions on $\mathbb{R}^d \times \mathbb{R}^d$. Hence, (i) in Theorem 1.2.11 is satisfied.
Observe that Cov_X is two times continuously differentiable on $\mathbb{R}^d \times \mathbb{R}^d$. So, for any $l \in \{1, \dots, d\}$ and $s, t \in \mathbb{R}^d$, the generalized partial second derivative at the point (t, s) of Cov_X (see (1.2.14)) exists in the direction l . In order to prove that Cov_X satisfies the condition (ii) in Theorem 1.2.11, we will study two cases : $1 < \gamma \leq 3/2$ and $3/2 < \gamma \leq 2$.
First case: $1 < \gamma \leq 3/2$. Let $T \in (0, +\infty)$ be arbitrary and fixed. The function Cov_X satisfies (1.2.15), so, for all $l \in \{1, \dots, d\}$ and $s, t \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\left| D^{ll} \mathrm{Cov}_X(t,t) - 2D^{ll} \mathrm{Cov}_X(t,s) + D^{ll} \mathrm{Cov}_X(s,s) \right| \\
&\quad = \left| D_1^l D_2^l \mathrm{Cov}_X(t,t) - 2 D_1^l D_2^l \mathrm{Cov}_X(t,s) + D_1^l D_2^l \mathrm{Cov}_X(s,s) \right| \\
&\quad \leq \left| D_1^l D_2^l \mathrm{Cov}_X(t,t) - D_1^l D_2^l \mathrm{Cov}_X(t,s) \right| \\
&\qquad + \left| D_1^l D_2^l \mathrm{Cov}_X(t,s) - D_1^l D_2^l \mathrm{Cov}_X(s,s) \right| . \qquad (1.3.8)
\end{aligned}$$

Definition 1.3.1 entails that the function $D_1^l D_2^l \mathrm{Cov}_X$ is locally Hölder continuous of order $2\gamma - 2$ on $\mathbb{R}^d$. Thus, for all $l \in \{1, \dots, d\}$, there exists a constant $c(T, l) \in (0, +\infty)$ such that, for any $s, t \in [-T, T]^d$, we have,

$$\left| D_1^l D_2^l \mathrm{Cov}_X(t,t) - D_1^l D_2^l \mathrm{Cov}_X(t,s) \right| \leq c(T,l) \left\| t - s \right\|^{2(\gamma-1)} . \qquad (1.3.9)$$

Therefore, combining (1.3.8) and (1.3.9), we obtain, for any $l \in \{1, \dots, d\}$ and $s, t \in [-T, T]^d$,

$$\left| D^{ll} \mathrm{Cov}_X(t,t) - 2D^{ll} \mathrm{Cov}_X(t,s) + D^{ll} \mathrm{Cov}_X(s,s) \right| \leq 2c(T) \left\| t - s \right\|^{2(\gamma-1)} , \qquad (1.3.10)$$

where $c(T) := \max \left\{ c(T,l), l \in \{1, \dots, d\} \right\}$. So, (ii) in Theorem 1.2.11 is satisfied.
Second case: $3/2 < \gamma \leq 2$. Let $T \in (0, +\infty)$ be arbitrary and fixed. Definition 1.3.1 entails that the function $D_{1,l} D_{2,l} \mathrm{Cov}_X$ is continuously differentiable on $\mathbb{R}^d$. Then, similarly as in the proof of Lemma 1.3.3, we get, for any $s, t \in \mathbb{R}^d$,

$$\begin{aligned}
&D_1^l D_2^l \mathrm{Cov}_X(t,t) - D_1^l D_2^l \mathrm{Cov}_X(t,s) \\
&\quad = \sum_{j=1}^d \int_{s_j}^{t_j} D_2^j D_1^l D_2^l \mathrm{Cov}_X(t; t_1, t_2, \dots, t_{j-1}, u, s_{j+1}, \dots, s_d) \mathrm{d}u, \qquad (1.3.11)
\end{aligned}$$

and

$$\begin{aligned}
&D_1^l D_2^l \mathrm{Cov}_X(s,s) - D_1^l D_2^l \mathrm{Cov}_X(t,s) \\
&\quad = -\sum_{j=1}^d \int_{s_j}^{t_j} D_2^j D_1^l D_2^l \mathrm{Cov}_X(s; t_1, t_2, \dots, t_{j-1}, u, s_{j+1}, \dots, s_d) \mathrm{d}u. \qquad (1.3.12)
\end{aligned}$$

On the other hand, for any $j,l \in \{1,\dots,d\}$, the function $D_2^j D_1^l D_2^l \mathrm{Cov}_X$ is locally Hölder continuous of order $2\gamma-3$ on $\mathbb{R}^d$. Thus, for all $l \in \{1,\dots,d\}$, there exists a constant $c(T,l,j) \in (0,+\infty)$ such that, for any $s,t,t' \in [-T,T]^d$, we have

$$\left|D_2^j D_1^l D_2^l \mathrm{Cov}_X(t',t) - D_2^j D_1^l D_2^l \mathrm{Cov}_X(t',s)\right| \le c(T,l,j)\, \|t-s\|^{2(\gamma-1)-1}. \tag{1.3.13}$$

Therefore combining (1.2.15) and (1.3.11) to (1.3.13) we obtain, for any $l \in \{1,\dots,d\}$ and $s,t \in \mathbb{R}^d$,

$$\begin{aligned}&\left|D^{ll}\mathrm{Cov}_X(t,t) - 2D^{ll}\mathrm{Cov}_X(t,s) + D^{ll}\mathrm{Cov}_X(s,s)\right|\\ &\le \sum_{j=1}^d c(T,l,j)\left|\int_{s_j}^{t_j} \mathrm{d}u\right| \|t-s\|^{2(\gamma-1)-1}\\ &\le c'(T)\,\|t-s\|^{2(\gamma-1)}\end{aligned}$$

where $2(\gamma-1)>0$ and $c'(T) := d\max\left\{c(T,l,j), l \in \{1,\dots,d\} \text{ and } j \in \{1,\dots,d\}\right\}$. So (ii) in Theorem 1.2.11 is satisfied. Therefore, in all cases, X has a modification which is almost surely continuously differentiable on $\mathbb{R}^d$. We denote it by $Y := \{Y(t), t \in \mathbb{R}^d\}$.

Step 2: We show that, for all $l \in \{1,\dots,d\}$ and $\gamma' \in (0,\gamma-1)$, the field $\partial^{e_l}Y = \left\{\partial^{e_l}Y(t), t \in \mathbb{R}^d\right\}$ is almost surely locally γ'-Hölder continuous on $\mathbb{R}^d$.
Notice that $\partial^{e_l}Y$ is a Gaussian field with covariance function $D_{1,l}D_{2,l}\mathrm{Cov}_X$ (see Remark 1.2.12). Moreover, this latter function is locally Hölder continuous of order $2(\gamma-1)>0$. As $\gamma-1 \in (0,1]$, Lemma 1.3.3 and Remark 1.1.2 entail that, for all $\gamma' \in (0,\gamma-1)$, the field $\partial^{e_l}Y$ is almost surely locally Hölder continuous of any order $\gamma' \in (0,\gamma-1)$ on $\mathbb{R}^d$. As the number of l in $\{1,\dots,d\}$ is finite, we proved that Y is almost surely locally Hölder continuous of any order $\gamma' \in (0,\gamma)$ on $\mathbb{R}^d$. □

Proof of Theorem 1.3.2. For any $\gamma \in (0,+\infty)$, we denote by $m(\gamma) := \max\{n \in \mathbb{Z}_+ : n < \gamma\}$. We prove Theorem 1.3.2 by induction on $n = m(\gamma)$. It follows from Lemma 1.3.3 and Corollary 1.3.4 that Theorem 1.3.2 hold when $n=0$ and $n=1$.
Now, we assume that $n \ge 2$ (that is, $\gamma > 2$). In particular, the covariance function of X is a locally Hölder continuous function of order 4 on $\mathbb{R}^d \times \mathbb{R}^d$. So, Lemma 1.3.4 entails X has a modification which is almost surely continuously differentiable on $\mathbb{R}^d$. We denote it by $Y := \{Y(t), t \in \mathbb{R}^d\}$. Moreover, for any $l \in \{1,\dots,d\}$, the field $\partial^{e_l}Y = \left\{\partial^{e_l}Y(t), t \in \mathbb{R}^d\right\}$ is Gaussian and its covariance function is given by $D_{1,l}D_{2,l}\mathrm{Cov}_X$ (see Remark 1.2.12). Obverse that $D_{1,l}D_{2,l}\mathrm{Cov}_X$ is locally Hölder continuous of order $2(\gamma-1)$ on $\mathbb{R}^d \times \mathbb{R}^d$. The fact that $m(\gamma-1) = m(\gamma)-1 = n-1$ entails, by induction, that $\partial^{e_l}Y$ has an almost surely locally Hölder continuous of any order $\gamma' \in (0,\gamma-1)$ on $\mathbb{R}^d$ modification. In view of Remark 1.1.2 and the fact that the cardinality of $\{1,\dots,d\}$ is finite, we proved that X has a modification which is almost surely locally Hölder continuous of any order $\gamma' \in (0,\gamma)$ on $\mathbb{R}^d$. □

2 Preliminary results related with stationary increments harmonizable stable fields

Abstract

The first part of this chapter consists in some recalls related with sable stochastic fields; we attach particular attention to the notion of stable stochastic integration with respect to a complex-valued rotationally invariant α-stable random measure, as well as the notion of LePage series representation for such integral. In the second part of this chapter we define stationary increments harmonizable stable fields through the stable stochastic integral of a well-chosen kernel function. We also provide some basic properties of them.

Contents

2.1 Stable integrals and their LePage series representations

In chapter 1, we defined real-valued symmetric stable random variable (see Definition 1.1.5). Now we focus on the stable integral $\int_{\mathbb{R}^d}(\,\cdot\,)\,\mathrm{d}\widetilde{M}_\alpha(\xi)$ which has already appeared in (1.1.16). In that way, we define complex-valued rotationally invariant stable random variables and random measures.

Definition 2.1.1. *Let Z be a complex-valued random variable. Its characteristic function χ_Z is defined, for all $\xi = \mathcal{R}e(\xi) + i\mathcal{I}m(\xi) \in \mathbb{C}$, by*

$$\chi_Z(\xi) := \chi_{(\mathcal{R}e(Z),\mathcal{I}m(Z))}(\mathcal{R}e(\xi), \mathcal{I}m(\xi)) = \mathbb{E}\left(e^{i(\mathcal{R}e(Z)\mathcal{R}e(\xi)+\mathcal{I}m(Z)\mathcal{I}m(\xi))}\right). \tag{2.1.1}$$

The random variable Z is said to be rotationally invariant of stability parameter $\alpha \in (0, 2]$ and scale parameter $\sigma \in \mathbb{R}_+$, if:

$$\forall\, \xi \in \mathbb{C}, \quad \chi_Z(\xi) = \exp(-\sigma^\alpha |\xi|^\alpha), \tag{2.1.2}$$

where $|\xi|$ denotes the modulus of ξ.

Remark 2.1.2. *Let Z be a complex-valued rotationally invariant α-stable random variable. The equality* (2.1.2) *entails that $\mathcal{R}e(Z)$ is a real-valued symmetric α-stable random variable with scale parameter σ.*

The term *rotationally invariant* in Definition 2.1.1 comes from the fact that Z satisfies, for any $\theta \in [0, 2\pi)$,

$$e^{i\theta} Z \overset{d}{=} Z, \tag{2.1.3}$$

where $\overset{d}{=}$ means equality in distribution of the two random vectors:

$$\begin{pmatrix} \mathcal{R}e(Z)\cos(\theta) - \mathcal{I}m(Z)\sin(\theta) \\ \mathcal{I}m(Z)\cos(\theta) + \mathcal{R}e(Z)\sin(\theta) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathcal{R}e(Z) \\ \mathcal{I}m(Z) \end{pmatrix}. \tag{2.1.4}$$

Lemma 2.1.3. *In view of* (2.1.4) *when the complex-valued random variable $Z = \mathcal{R}e(Z) + i\mathcal{I}m(Z)$ is rotationally invariant then, for any $(b_1, b_2) \in \mathbb{R}^2$, the real-valued random variables $b_1\mathcal{R}e(Z) + b_2\mathcal{I}m(Z)$ and $\|(b_1, b_2)\|\, \mathcal{R}e(Z)$ have the same distribution.*

Proof. Let $b := (b_1, b_2)$ be an arbitrary vector of $\mathbb{R}^2$. If $\|b\| = 0$, then $b_1 = b_2 = 0$ and $b_1\mathcal{R}e(Z) + b_2\mathcal{I}m(Z)$ and $\|b\|\, \mathcal{R}e(Z)$ have clearly the same distribution. So, from now on, we suppose that $\|b\| \neq 0$.
First, we assume that the norm of b is equal to 1. Then, there exists $\theta_b \in [0, 2\pi)$ such that

$$b_1 = \cos\theta_b \quad \text{and} \quad b_2 = -\sin\theta_b. \tag{2.1.5}$$

Combining (2.1.5), (2.1.4), and the equality $\|b\| = 1$, we have that $b_1\mathcal{R}e(Z) + b_2\mathcal{I}m(Z)$ and $\|b\|\, \mathcal{R}e(Z)$ are identically distributed.
Now, we assume that the norm of b is not necessarily equal to 1. Using the fact that the norm of the vector $\|b\|^{-1} b$ is equal to 1, in view of the above, we get that $\|b\|^{-1} b_1\mathcal{R}e(Z) + \|b\|^{-1} b_2\mathcal{I}m(Z)$ and $\mathcal{R}e(Z)$ are identically distributed. Therefore $b_1\mathcal{R}e(Z) + b_2\mathcal{I}m(Z)$ and $\|b\|\, \mathcal{R}e(Z)$ have the same distribution □

Rotationally invariant α-stable random variables satisfy the convenient following property.

Proposition 2.1.4. *Let Z and Z' be two complex-valued rotationally invariant α-stable random variables. Then the following two statements are equivalent:*

(i) Z and Z' have the same distribution.

(ii) There exists $(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$ such that the real-valued random variables $\xi_1 \mathcal{R}e(Z) + \xi_2 \mathcal{I}m(Z)$ and $\xi_1 \mathcal{R}e(Z') + \xi_2 \mathcal{I}m(Z')$ have the same distribution.

Proof. In view of Lemma 2.1.3, it is enough to show that (i) is equivalent to

$(ii)'$ $\mathcal{R}e(Z)$ and $\mathcal{R}e(Z')$ have the same distribution

It follows from the definition of the characteristic function of a real-valued/complex-valued random variable that (i) implies $(ii)'$. Now, we prove that $(ii)'$ implies (i). It follows from Definition 2.1.1 that the characteristic functions of the two complex-valued rotationally invariant α-stable random variables Z and Z' are respectively given, for all $\xi \in \mathbb{C}$, by

$$\chi_Z(\xi) = e^{-\sigma^\alpha |\xi|^\alpha} \qquad \text{and} \qquad \chi_{Z'}(\xi) = e^{-(\sigma')^\alpha |\xi|^\alpha}, \tag{2.1.6}$$

where σ^α, and $(\sigma')^\alpha$ are two non-negative numbers. It follows from Remark 2.1.2 that σ and σ' are respectively the scale parameters of the two identically distributed random variables $\mathcal{R}e(Z)$ and $\mathcal{R}e(Z')$. Therefore $\sigma = \sigma'$ which implies that (i) holds. $\square$

When the stability parameter α belongs to the interval $(0,2)$ complex-valued rotationally invariant α-stable random variables possess a LePage series representation. This representation will provide a useful series representation of the symmetric α-stable stochastic integral.

Proposition 2.1.5. *We assume that the stability parameter α belongs to the open interval $(0,2)$ and we set*

$$a(\alpha) := \left(\int_0^{+\infty} x^{-\alpha} \sin(x)\,\mathrm{d}x \right)^{-1/\alpha}. \tag{2.1.7}$$

Let $\{\Gamma_m : m \in \mathbb{N}\}$ and $\{Z_m : m \in \mathbb{N}\}$ be two arbitrary mutually independent sequences of random variables, defined on the same probability space $(\Omega, \mathcal{G}, \mathbb{P})$, having the following properties.

- *The Γ_m's, $m \in \mathbb{N}$, are Poisson arrival times with unit rate; that is, for all $m \in \mathbb{N}$, one has*

$$\Gamma_m = \sum_{n=1}^{m} \nu_n, \tag{2.1.8}$$

where $(\nu_n)_{n\in\mathbb{N}}$ denotes a sequence of independent exponential random variables with the same parameter equal to 1.

- *The Z_m's, $m \in \mathbb{N}$, are complex-valued, independent, identically distributed, rotationally invariant (see* (2.1.3)*) and satisfy* $\mathbb{E}(|\mathcal{R}e(Z_m)|^\alpha) < +\infty$.

Then, the random series of complex numbers $\sum_{m=1}^{+\infty} Z_m \Gamma_m^{-1/\alpha}$ is almost surely convergent. It has a rotationally invariant α-stable distribution with scale parameter σ satisfying

$$\sigma := a(\alpha)^{-1}\Big(\mathbb{E}(|\mathcal{R}e(Z_1)|^\alpha)\Big)^{1/\alpha}.$$

The proof of Proposition 2.1.5 can be found in [27, 17, 22]. Now, in order to construct the symmetric stable integrals, we introduce complex-valued rotationally invariant α-stable random measures. In the sequel, we denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel algebra of $\mathbb{R}^d$ and λ the Lebesgue measure on $\mathbb{R}^d$.

Definition 2.1.6. *Let $\alpha \in (0, 2]$. We denote by $\mathcal{E}_0$ the class of sets*

$$\mathcal{E}_0 := \Big\{A \in \mathcal{B}(\mathbb{R}^d), \lambda(A) < +\infty\Big\}. \tag{2.1.9}$$

A complex-valued rotationally invariant α-stable random measure on $\mathcal{E}_0$ with control measure λ is a set function

$$\widetilde{M}_\alpha : \mathcal{E}_0 \to \{\textit{complex-valued random variables } (\Omega, \mathcal{G}, \mathbb{P})\} \tag{2.1.10}$$

satisfying the following properties

(i) *$\widetilde{M}_\alpha$ is independently scattered: if $N \in \mathbb{N}$ and $A_1, \ldots, A_N$ belong to $\mathcal{E}_0$ and are pairwise disjoint sets then the random variables $\widetilde{M}_\alpha(A_1), \widetilde{M}_\alpha(A_2), \ldots, \widetilde{M}_\alpha(A_N)$ are independent.*

(ii) *$\widetilde{M}_\alpha$ is σ-additive: if $A_1, A_2, \ldots$ belong to $\mathcal{E}_0$, are pairwise disjoint sets and the set $\bigcup_{l=1}^{+\infty} A_l$ belongs to $\mathcal{E}_0$, then the equality*

$$\widetilde{M}_\alpha\left(\bigcup_{l=1}^{+\infty} A_l\right) = \sum_{l=1}^{+\infty} \widetilde{M}_\alpha(A_l) \tag{2.1.11}$$

holds almost surely.

(iii) *$\widetilde{M}_\alpha$ is rotationally invariant: for all $\theta \in [0, 2\pi)$*

$$e^{i\theta}\widetilde{M}_\alpha \overset{d}{=} \widetilde{M}_\alpha, \tag{2.1.12}$$

where $\overset{d}{=}$ denotes equality of the finite-dimensional distributions.

(iv) *For every $A \in \mathcal{E}_0$, $\widetilde{M}_\alpha(A)$ is a complex-valued rotationally invariant α-stable random variable with scale parameter $\lambda(A)^{1/\alpha}$.*

Notice that the random variables in (i) are complex-valued, so the independence property of $\widetilde{M}_\alpha(A_1), \dots, \widetilde{M}_\alpha(A_N)$ means the independence of the random vectors

$$\begin{pmatrix} \mathcal{R}e\,\widetilde{M}_\alpha(A_1) \\ \mathcal{I}m\,\widetilde{M}_\alpha(A_1) \end{pmatrix}, \begin{pmatrix} \mathcal{R}e\,\widetilde{M}_\alpha(A_2) \\ \mathcal{I}m\,\widetilde{M}_\alpha(A_2) \end{pmatrix}, \dots, \begin{pmatrix} \mathcal{R}e\,\widetilde{M}_\alpha(A_N) \\ \mathcal{I}m\,\widetilde{M}_\alpha(A_N) \end{pmatrix}.$$

The same remark holds for the condition (iii). The choice of the scale parameter in condition (iv) is motivated by the relation (2.1.2). Indeed, for any $A \in \mathcal{E}_0$, it implies that the characteristic function of $\widetilde{M}_\alpha(A)$ is given, for all $\xi \in \mathbb{C}$, by

$$\chi_{\widetilde{M}_\alpha(A)}(\xi) = e^{-\lambda(A)|\xi|^\alpha}.$$

This equality implies in particular that the real-valued random variable $\mathcal{R}e\,\widetilde{M}_\alpha(A)$ has a symmetric α-stable distribution and its scale parameter is given by $\lambda(A)^{1/\alpha}$.

We are in the position to construct a stochastic stable integral with respect to a complex-valued rotationally invariant α-stable random measure on $\mathbb{R}^d$ with Lebesgue control measure λ. First, we define the integral on simple functions : a function G on $\mathbb{R}^d$ is said to be simple if it can be expressed, for some $N \in \mathbb{N}$, as

$$G = \sum_{l=1}^{N} b_l \mathbb{1}_{A_l},$$

where the b_l's are complex numbers, the A_l's are pairwise disjoint Borel sets, and $\mathbb{1}_{A_l}$ is the indicator function of A_l; that is $\mathbb{1}_{A_l}(x) = 1$, if $x \in A$, and $\mathbb{1}_{A_l}(x) = 0$ else. The integral of a simple function is defined in the following natural way:

$$\int_{\mathbb{R}^d} G(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi) := \sum_{l=1}^{N} b_l \widetilde{M}_\alpha(A_l). \tag{2.1.13}$$

It follows from Definition 2.1.6 that $\int_{\mathbb{R}^d} G(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)$ is a complex-valued rotationally invariant α-stable random variable. Its characteristic function, that is the characteristic function of the random vector $\Big(\mathcal{R}e\Big\{\int_{\mathbb{R}^d} G(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\Big\}, \mathcal{I}m\Big\{\int_{\mathbb{R}^d} G(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\Big\}\Big)$, satisfies for all $\eta \in \mathbb{C}$,

$$\chi_Z(\eta) = \exp\left(-\left(\sum_{l=1}^{N} |b_l|^\alpha \lambda(A_l)\right)|\eta|^\alpha\right). \tag{2.1.14}$$

Using an argument of density, the integral $\int_{\mathbb{R}^d}(\,\cdot\,)\,\mathrm{d}\widetilde{M}_\alpha(\xi)$ can be extended to a general function $G \in L^\alpha\Big(\mathbb{R}^d\Big)$ (see sections 6.2 and 6.3 in [27]). This integral satisfies nice properties. Let us recall some of them.

Proposition 2.1.7. *The stochastic integral* $\int_{\mathbb{R}^d}(\,\cdot\,)\,\mathrm{d}\widetilde{M}_\alpha(\xi)$ *satisfies the following properties:*

(i) For any function $G \in L^\alpha\Big(\mathbb{R}^d\Big)$*, the integral* $\int_{\mathbb{R}^d} G(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)$ *is a complex-valued rotationally invariant* α*-stable random variable, with characteristic function given, for all* $\eta \in \mathbb{C}$*, by*

$$\chi_Z(\eta) = \exp\left(-\left(\int_{\mathbb{R}^d} |G(\xi)|^\alpha \,\mathrm{d}\xi\right)|\eta|^\alpha\right). \tag{2.1.15}$$

(ii) Linearity: for any functions $G_1, G_2 \in L^\alpha\Big(\mathbb{R}^d\Big)$ *and* $b_1, b_2 \in \mathbb{C}$*, the equality*

$$\int_{\mathbb{R}^d} \Big(b_1 G_1(\xi) + b_2 G_2(\xi)\Big)\,\mathrm{d}\widetilde{M}_\alpha(\xi) = b_1 \int_{\mathbb{R}^d} G_1(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi) + b_2 \int_{\mathbb{R}^d} G_2(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi), \tag{2.1.16}$$

holds almost surely.

(iii) $\mathcal{R}e\left\{\int_{\mathbb{R}^d} G(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right\}$ *and* $\mathcal{I}m\left\{\int_{\mathbb{R}^d} G(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right\}$ *are two identically distributed real-valued symmetric* α*-stable random variables of scale parameter*

$$\left(\int_{\mathbb{R}^d} |G(\xi)|^\alpha \,\mathrm{d}\xi\right)^{1/\alpha}.$$

When $\alpha \in (0,2)$, even if the real-part and imaginary-part of the complex valued rotationally invariant α-stable random variable $\int_{\mathbb{R}^d} G(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)$ are identically distributed (see Lemma 2.1.3 with $b_1 = 0$ and $b_2 = 1$), they are not in general independent. Moreover, the scale parameter of the real part $\mathcal{R}e\Big\{\int_{\mathbb{R}^d} G(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\Big\}$ satisfies

$$\sigma\Big(\mathcal{R}e\Big\{\int_{\mathbb{R}^d} G(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\Big\}\Big)^\alpha = \int_{\mathbb{R}^d} \Big|G(\xi)\Big|^\alpha d\xi. \tag{2.1.17}$$

The equality (2.1.17) is reminiscent of the classical isometry property of Wiener integrals. In particular, it allows us to derive the important following proposition which corresponds to Proposition 6.2.3 in [27].

Proposition 2.1.8. *Let* $G_1, G_2, \ldots$ *and* G *be in* $L^\alpha\Big(\mathbb{R}^d\Big)$*. The sequence of random variables* $\Big(\mathcal{R}e\Big\{\int_{\mathbb{R}^d} G_n(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\Big\}\Big)_{n\in\mathbb{N}}$ *converges to* $\mathcal{R}e\Big\{\int_{\mathbb{R}^d} G(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\Big\}$ *in probability if, and only if, the sequence* $(G_n)_{n\in\mathbb{N}}$ *converges to* G *in* $L^\alpha(\mathbb{R}^d)$*.*

When the stability parameter α belongs to $(0,2)$, for any function $G \in L^\alpha\Big(\mathbb{R}^d\Big)$, the complex-valued random variable $\int_{\mathbb{R}^d} G(\xi)\,\mathrm{d}\widetilde{M}(\xi)$ has a rotationally invariant α-stable distribution. Thus, we know from Proposition 2.1.5 that it possesses a LePage series representation. More precisely, the LePage series representation of $\int_{\mathbb{R}^d} G(\xi)\,\mathrm{d}\widetilde{M}(\xi)$ is provided by the following proposition.

Proposition 2.1.9. *We assume that the stability parameter* α *belongs to the open interval* $(0,2)$ *and that* $a(\alpha)$ *is as in (2.1.7). Let* $\{\kappa^m : m \in \mathbb{N}\}$*,* $\{\Gamma_m : m \in \mathbb{N}\}$ *and* $\{g_m : m \in \mathbb{N}\}$ *be three arbitrary mutually independent sequences of random variables, defined on the same probability space* $(\Omega, \mathcal{G}, \mathbb{P})$*, having the following properties.*

(i) *The κ^m's, $m \in \mathbb{N}$, are $\mathbb{R}^d$-valued, independent, identically distributed and absolutely continuous, with a probability density function, denoted by ϕ, such that the measure $\phi(\xi)\mathrm{d}\xi$ is equivalent to the Lebesgue measure $\mathrm{d}\xi$ on $\mathbb{R}^d$; that is, for any measurable set A, one has*

$$\int_A \phi(\xi)\,\mathrm{d}\xi = 0 \quad \Longleftrightarrow \quad \lambda(A) = 0. \tag{2.1.18}$$

Notice that (2.1.18) implies that $\phi(\xi) \neq 0$ for almost all $\xi \in \mathbb{R}^d$.

(ii) *The Γ_m's, $m \in \mathbb{N}$, are Poisson arrival times with unit rate.*

(iii) *The g_m's, $m \in \mathbb{N}$, are complex-valued, independent, identically distributed, rotationally invariant (see (2.1.3)) and satisfy $\mathbb{E}(|\mathcal{R}e(g_m)|^\alpha) = 1$.*

Then, for any function $G \in L^\alpha\Big(\mathbb{R}^d\Big)$, the random series

$$I(G) := a(\alpha) \sum_{m=1}^{+\infty} g_m \Gamma_m^{-1/\alpha} \phi(\kappa^m)^{-1/\alpha} G(\kappa^m) \tag{2.1.19}$$

is almost surely convergent. Moreover, the random variables $\int_{\mathbb{R}^d} G(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)$ and $I(G)$ have the same distribution.

Proof of Proposition 2.1.9. For any $m \in \mathbb{N}$, we define

$$Z_m = g_m \phi(\kappa^m)^{-1/\alpha} G(\kappa^m).$$

Observe that the Z_m's, $m \in \mathbb{N}$, are complex-valued random variables. Moreover, in view the independence property of the g_m's and κ^m's, $m \in \mathbb{N}$, it is clear that the Z_m's, $m \in \mathbb{N}$ are independent. Notice that, for any $m \in \mathbb{N}$, g_m is rotationally invariant and independent of κ^m. Hence, Z_m is also rotationally invariant. In order to apply Proposition 2.1.5 it remains to show that

$$\mathbb{E}\Big(|\mathcal{R}e(Z_m)|^\alpha \Big) < +\infty \tag{2.1.20}$$

for every $n \in \mathbb{N}$. Let $\mathcal{F}_\kappa$ be the sub σ-field of $\mathcal{G}$ generated by the sequence of random variables $\{\kappa_m : m \in \mathbb{N}\}$. We denote by $\mathbb{E}_\kappa[\,\cdot\,]$ the conditional expectation operators with respect to $\mathcal{F}_\kappa$. Applying Lemma 2.1.3 with $Z = g_m$, $b_1 = \mathcal{R}e(G(\kappa^m))$ and $b_2 = -\mathcal{I}m(G(\kappa^m))$, conditionally to $\mathcal{F}_\kappa$, we get that

$$\mathcal{R}e(g_m G(\kappa^m)) = \mathcal{R}e(G(\kappa^m))\mathcal{R}e(g_m) - \mathcal{I}m(G(\kappa^m))\mathcal{I}m(g_m) \overset{d}{=} |G(\kappa^m)|\,\mathcal{R}e(g_m),$$

where $\overset{d}{=}$ denotes the equality in distribution of the random variables $\mathcal{R}e(g_m G(\kappa^m))$ and $|G(\kappa^m)|\,\mathcal{R}e(g_m)$ conditionally to $\mathcal{F}_\kappa$. Combining this equality in distribution to the facts that

κ_m and g_m are independent, ϕ is the probability density function of κ^m, $\mathbb{E}(|\mathcal{R}e(g_m)|^\alpha) = 1$ and $G \in L^\alpha\Big(\mathbb{R}^d\Big)$, we get that

$$\begin{aligned}
\mathbb{E}\Big(\,|\mathcal{R}e(Z_m)|^\alpha\Big) &= \mathbb{E}\Big(\,\Big|\mathcal{R}e(g_m\phi(\kappa^m)^{-1/\alpha}G(\kappa^m))\Big|^\alpha\Big) \\
&= \mathbb{E}\Big(\phi(\kappa^m)^{-1}\,|\mathcal{R}e(g_m G(\kappa^m))|^\alpha\Big) \\
&= \mathbb{E}\Big(\mathbb{E}_\kappa\Big[\phi(\kappa^m)^{-1}\,|\mathcal{R}e(g_m G(\kappa^m))|^\alpha\Big]\Big) \\
&= \mathbb{E}\Big(\mathbb{E}_\kappa\Big[\phi(\kappa^m)^{-1}\,|G(\kappa^m)|^\alpha\,|\mathcal{R}e(g_m)|^\alpha\Big]\Big) \\
&= \mathbb{E}\Big(\phi(\kappa^m)^{-1}\,|G(\kappa^m)|^\alpha\Big)\mathbb{E}\Big(\,|\mathcal{R}e(g_m)|^\alpha\Big) \\
&= \int_{\mathbb{R}^d}|G(\xi)|^\alpha\,\mathrm{d}\xi < +\infty.
\end{aligned}$$

Therefore, thanks to Proposition 2.1.5, the random series

$$\sum_{m=1}^{+\infty} g_m\Gamma_m^{-1/\alpha}\phi(\kappa^m)^{-1/\alpha}G(\kappa^m)$$

is almost surely convergent and has a rotationally invariant α-stable distribution with scale parameter σ satisfying

$$\sigma := a(\alpha)^{-1}\Big(\mathbb{E}(|\mathcal{R}e(Z_1)|^\alpha)\Big)^{1/\alpha} = a(\alpha)^{-1}\left(\int_{\mathbb{R}^d}|G(\xi)|^\alpha\right)^{1/\alpha}$$

So, in view of (2.1.19), $I(G)$ is a complex-valued rotationally invariant α-stable random variable with scale parameter $\sigma\Big(I(G)\Big)$ such that

$$\sigma\Big(I(G)\Big) = \left(\int_{\mathbb{R}^d}|G(\xi)|^\alpha\right)^{1/\alpha}.$$

So, in view of (2.1.2), $\int_{\mathbb{R}^d} G(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)$ and $I(G)$ are identically distributed. □

Observe that $I(\,\cdot\,)$ is a linear function in G; that is, for any $z \in \mathbb{C}$ and $G_1, G_2 \in L^\alpha\Big(\mathbb{R}^d\Big)$, we have

$$I(zG_1 + G_2) = zI(G_1) + I(G_2).$$

Therefore, in view of Proposition 2.1.9 and the linearity of $\int_{\mathbb{R}^d}(\,\cdot\,)\,\mathrm{d}\widetilde{M}(\xi)$ (see (2.1.16)) we get the following result.

Theorem 2.1.10. *We assume that $\alpha \in (0,2)$ and we set $a(\alpha)$ as in (2.1.7). Let $\{\kappa^m : m \in \mathbb{N}\}$, $\{\Gamma_m : m \in \mathbb{N}\}$, and $\{g_m : m \in \mathbb{N}\}$ be three arbitrary mutually independent sequences of random variables defined as in Proposition 2.1.9.*

Then, the stochastic processes

$$\left\{a(\alpha)\sum_{m=1}^{+\infty} g_m\Gamma_m^{-1/\alpha}\phi(\kappa^m)^{-1/\alpha}G(\kappa^m):\ G\in L^\alpha\Big(\mathbb{R}^d\Big)\right\}$$

and

$$\left\{\int_{\mathbb{R}^d} G(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi) : G\in L^\alpha\Big(\mathbb{R}^d\Big)\right\}$$

have the same distribution.

2.2 LePage series representation and study of path behaviour

The main goal of this section is to show that the LePage series representation of the harmonizable fractional stable motion X^{hfsm} allows to derive a stronger result on its almost sure Hölder regularity than the one previously obtained in Example 1.1.9 by making use of the Kolmogorov-Čentsov Theorem.
Before that, we mention that the harmonizable fractional stable motion is a well-defined symmetric stable process in the sense of Definition 1.1.6. Indeed, the fact that the Hurst parameter H is in the open interval $(0,1)$ implies that, for each $t\in\mathbb{R}$, the function $G_t:\xi\mapsto\Big(e^{it\xi}-1\Big)|\xi|^{-H-1/\alpha}$ belongs to $L^\alpha(\mathbb{R})$. Therefore G_t is integrable with respect to $\widetilde{M}_\alpha$.

Proposition 2.2.1. *Let $X^{hfsm}:=\{X^{hfsm}(t),t\in\mathbb{R}\}$ be the harmonizable fractional stable motion of an arbitrary stability parameter $\alpha\in(0,2)$ and Hurst parameter $H\in(0,1)$ defined in* (1.1.16)*. Then, there exists a modification $Y:=\{Y(t),t\in\mathbb{R}\}$ of X^{hfsm} which almost surely satisfies, for any $T\in(0,+\infty)$ and all positive real number δ arbitrarily small,*

$$\sup_{s,t\in[-T,T]}\left\{\frac{|Y(t,\omega)-Y(s,\omega)|}{|t-s|^H\left(1+|\log|t-s||\right)^{1/\alpha+1/2+\delta}}\right\}<+\infty. \tag{2.2.1}$$

This result has already been obtained in [16] by making use of the LePage series representation of X^{hfsm}. Also we mention that more general results than Proposition 2.2.1 can be found in [6]; their proofs rely on LePage series representations as well.

Observe that (2.2.1) implies that there exists a modification of X^{hfsm} which is almost surely locally Hölder continuous of any order $\gamma'\in(0,H)$.

Proof of Proposition 2.2.1. First notice that, in view of Theorem 2.1.10, the processes

$$X^{\text{hfsm}}:=\left\{\mathcal{R}e\left(\int_{\mathbb{R}}\Big(e^{it\xi}-1\Big)|\xi|^{-H-1/\alpha}\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right):\ t\in\mathbb{R}\right\}$$

and

$$\left\{a(\alpha)\mathcal{R}e\left(\sum_{m=1}^{+\infty} g_m\Gamma_m^{-1/\alpha}\phi(\kappa^m)^{-1/\alpha}\Big(e^{it\kappa^m}-1\Big)|\kappa^m|^{-H-1/\alpha}\right):\ t\in\mathbb{R}\right\}$$

have the same distribution. Moreover, we assume that the g_m's, $m\in\mathbb{N}$, are complex-valued centered Gaussian random variables, and that the probability density function ϕ satisfies, for all $\xi\in\mathbb{R}\setminus\{0\}$,

$$\phi(\xi) := \frac{\varepsilon}{4}\,|\xi|^{-1}\Big(1+|\log|\xi||\Big)^{-1-\varepsilon}, \tag{2.2.2}$$

where ε is an arbitrary fixed positive real number. In the sequel, for any $t\in\mathbb{R}$, we set,

$$\widetilde{X}(t) := a(\alpha)\mathcal{R}e\left\{\sum_{m=1}^{+\infty} g_m\Gamma_m^{-1/\alpha}\phi(\kappa^m)^{-1/\alpha}\Big(e^{it\kappa^m}-1\Big)|\kappa^m|^{-H-1/\alpha}\right\}. \tag{2.2.3}$$

Let $\mathcal{F}_\Gamma$ be the sub σ-field of $\mathcal{G}$ generated by the sequence of random variables $\{\Gamma_m : m\in\mathbb{N}\}$. Let also $\mathcal{F}_{\Gamma,\kappa}$ be the sub σ-field of $\mathcal{G}$ generated by the two sequences of random variables $\{\Gamma_m : m\in\mathbb{N}\}$ and $\{\kappa^m : m\in\mathbb{N}\}$. We denote respectively by $\mathbb{E}_\Gamma[\,\cdot\,]$ and $\mathbb{E}_{\Gamma,\kappa}[\,\cdot\,]$ the conditional expectation operators with respect to $\mathcal{F}_\Gamma$ and $\mathcal{F}_{\Gamma,\kappa}$; recall that $\mathbb{E}(\,\cdot\,)$ denotes the classical expectation operator. We know from (2.2.3) that conditionally to $\mathcal{F}_{\Gamma,\kappa}$, for any arbitrary $s,t\in\mathbb{R}$, the random variable $\widetilde{X}(t)-\widetilde{X}(s)$ has a centered Gaussian distribution over $\mathbb{R}$. Moreover, we have almost surely

$$\begin{aligned}
&\mathbb{E}_{\Gamma,\kappa}\left[\left|\widetilde{X}(t)-\widetilde{X}(s)\right|^2\right]\\
&= a(\alpha)^2\mathbb{E}\Big(|g_1|^2\Big)\sum_{m=1}^{+\infty}\Gamma_m^{-2/\alpha}\phi(\kappa^m)^{-2/\alpha}\left|e^{it\kappa^m}-e^{is\kappa^m}\right|^2|\kappa^m|^{-2(H+1/\alpha)}\\
&= a(\alpha)^2\mathbb{E}\Big(|g_1|^2\Big)\sum_{m=1}^{+\infty}\Gamma_m^{-2/\alpha}\phi(\kappa^m)^{-2/\alpha}\left|e^{i(t-s)\kappa^m}-1\right|^2|\kappa^m|^{-2(H+1/\alpha)}.
\end{aligned} \tag{2.2.4}$$

In the sequel, for any $x\in\mathbb{R}$, we set

$$\mathfrak{T}^2(x) := \sum_{m=1}^{+\infty}\Gamma_m^{-2/\alpha}\phi(\kappa^m)^{-2/\alpha}\left|e^{ix\kappa^m}-1\right|^2|\kappa^m|^{-2(H+1/\alpha)}. \tag{2.2.5}$$

Notice that, almost surely, for any $x\in\mathbb{R}$, we have

$$\mathfrak{T}^2(x)\le 4\mathfrak{S}^2(|x|) \tag{2.2.6}$$

where, for any $x\in[0,+\infty)$, the random variable $\mathfrak{S}^2(x)$ is equal to

$$\mathfrak{S}^2(x) := \sum_{m=1}^{+\infty}\Gamma_m^{-2/\alpha}\phi(\kappa^m)^{-2/\alpha}\min\{|x\kappa^m|^2,1\}|\kappa^m|^{-2H-2/\alpha}. \tag{2.2.7}$$

We mention that (2.2.6) easily results from the inequality, for every $y \in \mathbb{R}$,

$$\left|e^{iy} - 1\right| \le \min\{|y|, 2\} \le 2\min\{|y|, 1\}. \tag{2.2.8}$$

For the sake of clarity the rest of the proof is divided into the following 3 steps.
Step 1: We establish that

$$\mathbb{P}\left(\lim_{j\to+\infty} \frac{\mathfrak{S}^2(2^{-j})}{2^{-2jH} j^{2(1+\varepsilon)/\alpha}} = 0\right) = 1. \tag{2.2.9}$$

Step 2: We define, for any arbitrary $T \in [1, +\infty)$ and $n \in \mathbb{N}$ the dyadic set of level n of $[-T, T]$ as

$$D_{n,T} := \{k/2^n, k \in [-2^n T, 2^n T] \cap \mathbb{Z}\}$$

and the set of dyadic numbers of $[-T, T]$ by

$$D_T := \bigcup_{n\in\mathbb{N}} D_{n,T}.$$

Notice that D_T is dense in $[-T, T]$. The main goal of *Step 2* is to show that the probability of the event

$$\Omega_1^*(T) := \bigcup_{J\in\mathbb{N}} \bigcap_{\substack{s,t\in D_T \\ |s-t|\le 2^{-J}}} \left\{|X^{\text{hfsm}}(t) - X^{\text{hfsm}}(s)| \le c_1 |t-s|^H (1 + |\log|t-s||)^{1/\alpha+1/2+\varepsilon/\alpha}\right\}, \tag{2.2.10}$$

is equal to 1, where $c_1 \in (0, +\infty)$ is a deterministic constant which will be defined later.
Step 3: For any $T \in [1, +\infty)$, we construct a modification $Y_T := \{Y_T(t), t \in [-T, T]\}$ of $\{X^{\text{hfsm}}(t), t \in [-T, T]\}$ that satisfies, for any $s, t \in [-T, T]$ and $\omega \in \Omega_1^*(T)$,

$$|Y_T(t, \omega) - Y_T(s, \omega)| \le C_2(\omega, T) |t-s|^H (1 + |\log|t-s||)^{1/\alpha+1/2+\varepsilon/\alpha}, \tag{2.2.11}$$

for some positive and finite constant $C_2(\omega, T)$ which does not depend on s and t.

Proof of Step 1: Combining the independence property of $\{\kappa^m : m \in \mathbb{N}\}$ and $\{\Gamma_m : m \in \mathbb{N}\}$ with (2.2.7) we obtain, for any $x \in (0, +\infty)$, almost surely

$$\mathbb{E}_\Gamma\left[\mathfrak{S}^2(x)\right] = \sum_{m=1}^{+\infty} \Gamma_m^{-2/\alpha} \mathbb{E}\left(\phi(\kappa^m)^{-2/\alpha} \min\{|x\kappa^m|^2, 1\} |\kappa^m|^{-2H-2/\alpha}\right). \tag{2.2.12}$$

We recall that ϕ is the probability density function of the random variables κ^m. Therefore, for any arbitrary $m \in \mathbb{N}$ and $x \in (0, +\infty)$, one has

$$\begin{aligned}
&\mathbb{E}\left(\phi(\kappa^m)^{-2/\alpha} \min\{|x\kappa^m|^2, 1\} |\kappa^m|^{-2(H+1/\alpha)}\right) \\
&= \int_{-\infty}^{+\infty} \phi(\xi)^{1-2/\alpha} \min\left\{|x\xi|^2, 1\right\} |\xi|^{-2(H+1/\alpha)} \,\mathrm{d}\xi \\
&= 2\int_0^{+\infty} \phi(\xi)^{1-2/\alpha} \min\left\{(x\xi)^2, 1\right\} \xi^{-2(H+1/\alpha)} \,\mathrm{d}\xi
\end{aligned} \tag{2.2.13}$$

where the last inequality follows from the fact that ϕ is an even function. Notice that, in view of (2.2.2),

$$\mathbb{E}\left(\phi(\kappa^m)^{-2/\alpha}\min\left\{|x\kappa^m|^2,1\right\}|\kappa^m|^{-2(H+1/\alpha)}\right)$$

can be expressed as

$$2\Big(I_1(x)+I_2(x)\Big)$$

where we have set

$$I_1(x):=(4^{-1}\varepsilon)^{1-2/\alpha}x^2\int_0^{1/x}\xi^{-2H+1}\Big(1+|\log(\xi)|\Big)^{(1+\varepsilon)(2/\alpha-1)}\,\mathrm{d}\xi \tag{2.2.14}$$

and

$$I_2(x):=(4^{-1}\varepsilon)^{1-2/\alpha}\int_{1/x}^{+\infty}\xi^{-2H-1}\Big(1+|\log(\xi)|\Big)^{(1+\varepsilon)(2/\alpha-1)}\,\mathrm{d}\xi. \tag{2.2.15}$$

The change of variable $\eta=x\xi$ entails that

$$\begin{aligned}I_1(x) &= \Big(4^{-1}\varepsilon\Big)^{1-2/\alpha}x^{2H}\int_0^1\eta^{-2H+1}\Big(1+\Big|\log(x^{-1}\eta)\Big|\Big)^{(1+\varepsilon)(2/\alpha-1)}\,\mathrm{d}\eta\\ &\le c_3x^{2H}(1+|\log x|)^{(1+\varepsilon)(2/\alpha-1)},\end{aligned} \tag{2.2.16}$$

where c_3 is the positive finite constant defined as

$$c_3:=(4^{-1}\varepsilon)^{1-2/\alpha}\int_0^1\eta^{-2H+1}\Big(1+|\log\eta|\Big)^{1+\varepsilon}\,\mathrm{d}\eta.$$

Similarly we have

$$\begin{aligned}I_2(x) &= \Big(4^{-1}\varepsilon\Big)^{1-2/\alpha}x^{2H}\int_1^{+\infty}\eta^{-2H-1}\Big(1+\Big|\log(x^{-1}\eta)\Big|\Big)^{(1+\varepsilon)(2/\alpha-1)}\,\mathrm{d}\eta\\ &\le c_4x^{2H}(1+|\log x|)^{(1+\varepsilon)(2/\alpha-1)},\end{aligned} \tag{2.2.17}$$

where c_4 is the positive finite constant defined as

$$c_4:=(4^{-1}\varepsilon)^{1-2/\alpha}\int_1^{+\infty}\eta^{-2H-1}\Big(1+\log\eta\Big)^{(1+\varepsilon)(2/\alpha-1)}\,\mathrm{d}\eta.$$

Notice that c_3 and c_4 are finite since $H\in(0,1)$. Therefore, putting together (2.2.12) to (2.2.17), we obtain, for any $x\in(0,+\infty)$, almost surely

$$\mathbb{E}_\Gamma\left[\mathfrak{S}^2(x)\right]\le 2(c_3+c_4)x^{2H}(1+|\log x|)^{(1+\varepsilon)(2/\alpha-1)}\sum_{m=1}^{+\infty}\Gamma_m^{-2/\alpha}; \tag{2.2.18}$$

which, in particular, implies that, for some well-chosen constant $c_5\in(0,+\infty)$, the event

$$\Omega_2^*:=\bigcap_{j\in\mathbb{N}}\left\{\mathbb{E}_\Gamma\left[\mathfrak{S}^2(2^{-j})\right]\le c_52^{-2Hj}j^{(1+\varepsilon)(2/\alpha-1)}\sum_{m=1}^{+\infty}\Gamma_m^{-2/\alpha}\right\} \tag{2.2.19}$$

has a probability equal to 1. On the other hand, observe that, in view of (2.1.8), it results from the strong law of large number that the probability of the event Ω_3^* defined as

$$\Omega_3^* := \bigcap_{m\in\mathbb{N}} \Big\{ C_6 m \le \Gamma_m \le C_7 m \Big\} \tag{2.2.20}$$

is equal to 1, where C_6 and C_7 are two well-chosen positive finite random variables not depending on m. Moreover, the stability parameter α satisfies $2/\alpha > 1$; so, the inequalities

$$\sum_{m=1}^{+\infty} \Gamma_m(\omega)^{-2/\alpha} \le C_6(\omega)^{-2/\alpha} \sum_{m=1}^{+\infty} m^{-2/\alpha} < +\infty \tag{2.2.21}$$

hold for any $\omega \in \Omega_3^*$. Next, combining (2.2.19) and (2.2.21), for any $\omega \in \Omega_2^* \cap \Omega_3^*$ and $j \in \mathbb{N}$, we obtain

$$\mathbb{E}_\Gamma \left[\sum_{j=1}^{+\infty} \frac{\mathfrak{S}^2(2^{-j})}{2^{-2jH} j^{2(1+\varepsilon)/\alpha}} \right](\omega) \le c_5 C_6(\omega)^{-2/\alpha} \left(\sum_{m=1}^{+\infty} m^{-2/\alpha} \right) \left(\sum_{j=1}^{+\infty} j^{-(1+\varepsilon)} \right) < +\infty. \tag{2.2.22}$$

Therefore, conditionally to $\mathcal{F}_\Gamma$, the random variable $\sum_{j=1}^{+\infty} \frac{\mathfrak{S}^2(2^{-j})}{2^{-2jH} j^{2(1+\varepsilon)/\alpha}}$ is finite almost-surely, which implies that

$$\mathbb{P}\left(\sum_{j=1}^{+\infty} \frac{\mathfrak{S}^2(2^{-j})}{2^{-2jH} j^{2(1+\varepsilon)/\alpha}} < +\infty \right) = 1. \tag{2.2.23}$$

So (2.2.9) is a consequence of (2.2.23)

<u>*Proof of Step 2:*</u> Conditionally to $\mathcal{F}_{\Gamma,\kappa}$, for any arbitrary $s,t \in \mathbb{R}$, the random variable

$$\mathfrak{T}^2(t-s)^{-1/2}\Big(\widetilde{X}(t) - \widetilde{X}(s)\Big)$$

has a centered Gaussian distribution with variance 1. Then, it follows from (1.1.8) that, for any $j \in \mathbb{N}$ and $k \in \mathbb{Z}$,

$$\mathbb{E}_{\Gamma,\kappa}\left[\mathbb{1}_{A_{j,k}}\right] \le \frac{2e^{-(3j\log 2)/2}}{\sqrt{6\pi \log 2}\, j^{1/2}},$$

almost surely, where $A_{j,k} := \Big\{ \Big| \widetilde{X}(k2^{-j}) - \widetilde{X}((k+1)2^{-j}) \Big| > \sqrt{3\log 2}\, j^{1/2} \mathfrak{T}^2(2^{-j})^{1/2} \Big\}$. So, we have, for all $j \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{P}\left(\max_{k\in[-2^jT,2^jT]\cap\mathbb{Z}} \Big| \widetilde{X}(k2^{-j}) - \widetilde{X}((k+1)2^{-j}) \Big| > \sqrt{3\log 2}\, j^{1/2} \mathfrak{T}^2(2^{-j})^{1/2} \right) \\
&\mathbb{P}\left(\bigcup_{k\in[-2^jT,2^jT]\cap\mathbb{Z}} A_{j,k} \right) \\
&\le (2^{j+1}T+1) \frac{2e^{-(3j\log 2)/2}}{\sqrt{6\pi\log 2}\, j^{1/2}} \le \frac{8T}{\sqrt{6\pi \log 2}} 2^{-j/2} j^{-1/2}.
\end{aligned} \tag{2.2.24}$$

The expression in the right hand side of (2.2.24) is the general term of a convergent series. So, the Borel-Cantelli Lemma implies that the event $\Omega_5^*(T)$ defined as

$$\Omega_5^*(T) := \bigcup_{J\in\mathbb{N}} \bigcap_{j\geq J} \left\{ \max_{k\in[-2^jT,2^jT]\cap\mathbb{Z}} \left| \widetilde{X}(k2^{-j}) - \widetilde{X}((k+1)2^{-j}) \right| \leq \sqrt{3\log 2}\, j^{1/2}\mathfrak{T}^2(2^{-j})^{1/2} \right\}, \tag{2.2.25}$$

has a probability equal to 1. We denote by $\Omega_6^*(T)$ the event of probability 1 defined as

$$\Omega_6^*(T) := \Omega_5^*(T) \cap \left\{ \lim_{j\to+\infty} \frac{\mathfrak{S}^2\Big(2^{-j}\Big)}{2^{-2jH}j^{2(1+\varepsilon)/\alpha}} = 0 \right\}.$$

In order to prove that $\mathbb{P}\Big(\Omega_1^*(T)\Big) = 1$, it is enough to show that $\Omega_6^*(T) \subset \Omega_1^*(T)$. Let $\omega \in \Omega_6^*(T)$ be fixed. In view of (2.2.25), (2.2.6) and (2.2.9), there exists $J(\omega) \in \mathbb{N}$ such that, for all $j \geq J(\omega)$ and $k \in [-2^jT, 2^jT] \cap \mathbb{Z}$,

$$\left| \widetilde{X}\Big(k2^{-j},\omega\Big) - \widetilde{X}\Big((k+1)2^{-j},\omega\Big) \right| \leq 2^{-jH} j^{1/\alpha+1/2+\varepsilon/\alpha}. \tag{2.2.26}$$

Let $n \geq J(\omega)$ be fixed. We will show by induction that, for any integer $m > n$, the inequality

$$\left| \widetilde{X}(t,\omega) - \widetilde{X}(s,\omega) \right| \leq 2 \sum_{j=n+1}^{m} 2^{-jH} j^{1/\alpha+1/2+\varepsilon/\alpha} \tag{2.2.27}$$

holds for all $s,t \in D_{m,T}$ satisfying $|t-s| \in (0, 2^{-n})$. With no restriction, we assume that $s < t$. If $m = n+1$, then we can only have $s = k/2^{-m}$ and $t = (k+1)2^{-m}$, for some $k \in [-2^mT, 2^mT] \cap \mathbb{Z}$. So, (2.2.26) implies (2.2.27). Now, suppose that (2.2.27) holds for $m = M-1 > n$. Let $s,t \in D_{M,T}$ with $s < t$. We define $t_1 := \max\{u \in D_{M-1,T}, u \leq t\}$ and $s_1 := \min\{u \in D_{M-1,T}, u \geq s\}$. Notice that we have the inequalities $s \leq s_1 \leq t_1 \leq t$, $s_1 - s \leq 2^{-M}$ and $t - t_1 \leq 2^{-M}$. So, it follows from (2.2.26) that,

$$\left| \widetilde{X}(t,\omega) - \widetilde{X}(t_1,\omega) \right| \leq 2^{-MH} M^{1/\alpha+1/2+\varepsilon/\alpha} \tag{2.2.28}$$

and

$$\left| \widetilde{X}(s,\omega) - \widetilde{X}(s_1,\omega) \right| \leq 2^{-MH} M^{1/\alpha+1/2+\varepsilon/\alpha}. \tag{2.2.29}$$

Moreover, (2.2.27) applied to $s_1, t_1 \in D_{M-1,T}$ entails that

$$\left| \widetilde{X}(t_1,\omega) - \widetilde{X}(s_1,\omega) \right| \leq 2 \sum_{j=n+1}^{M-1} 2^{-jH} j^{1/\alpha+1/2+\varepsilon/\alpha}. \tag{2.2.30}$$

Putting together (2.2.28), (2.2.29) and (2.2.30), we obtain (2.2.27) with $m = M$. Now, assume that s,t belong to D_T and satisfy $0 < t-s < h(\omega) := 2^{-J(\omega)}$. Let $n \geq J(\omega)$ such that

$$2^{-n-1} \leq t - s < 2^{-n}. \tag{2.2.31}$$

So, it follows from (2.2.27) that

$$\begin{aligned}
\left|\widetilde{X}(t,\omega)-\widetilde{X}(s,\omega)\right| &\leq 2\sum_{j=n+1}^{+\infty}2^{-jH}j^{1/\alpha+1/2+\varepsilon/\alpha}\\
&= 2\sum_{j=0}^{+\infty}2^{-(j+n+1)H}(j+n+1)^{1/\alpha+1/2+\varepsilon/\alpha}\\
&= 2^{-(n+1)H+1}(n+1)^{1/\alpha+1/2+\varepsilon/\alpha}\sum_{j=0}^{+\infty}2^{-jH}\left(1+\frac{j}{n+1}\right)^{1/\alpha+1/2+\varepsilon/\alpha}\\
&\leq 2^{-(n+1)H+1}(n+1)^{1/\alpha+1/2+\varepsilon/\alpha}\sum_{j=0}^{+\infty}2^{-jH}(1+j)^{1/\alpha+1/2+\varepsilon/\alpha}\\
&= c_8 2^{-(n+1)H}(n+1)^{1/\alpha+1/2+\varepsilon/\alpha}, \qquad (2.2.32)
\end{aligned}$$

where $c_8 := 2\sum_{j=0}^{+\infty}2^{-jH}(1+j)^{1/\alpha+1/2+\varepsilon/\alpha} < +\infty$ is deterministic. Putting together (2.2.31) and (2.2.32), we obtain

$$\left|\widetilde{X}(t,\omega)-\widetilde{X}(s,\omega)\right| \leq c_9\left|t-s\right|^H\left(1+\left|\log|t-s|\right|\right)^{1/\alpha+1/2+\varepsilon/\alpha}, \qquad (2.2.33)$$

where $c_9 := (\log 2)^{-(1/\alpha+1/2+\varepsilon/\alpha)}c_8$. Since the processes X^{hfsm} and $\widetilde{X}$ have the same distribution, setting $c_1 := c_9$ in (2.2.10), it follows that the probability of $\Omega_1^*(T)$ is equal to 1.

Proof of Step 3: We construct the modification Y_T of $\{X^{\text{hfsm}}(t), t\in[-T,T]\}$ that satisfied (2.2.11) as follows:

(*i*) If $\omega\notin\Omega_1^*(T)$, we set $Y_T(t,\omega)=0$ for all $t\in[-T,T]$.

(*ii*) If $\omega\in\Omega_1^*(T)$ and $t\in D_T$, we set $Y_T(t,\omega)=X^{\text{hfsm}}(t,\omega)$.

(*iii*) Roughly speaking, when $\omega\in\Omega_1^*(T)$ and $t\in[-T,T]\setminus D_T$, we define $Y_T(t,\omega)$ as the limit of the sequence of real-numbers $\{X^{\text{hfsm}}(t_n,\omega), n\in\mathbb{N}\}$ where $\{t_n, n\in\mathbb{N}\}$ is an arbitrary sequence of D_T which converges to t when n tends to $+\infty$.

Let us now precisely present the construction of $Y_T(t,\omega)$ in (*iii*). Since D is dense in $[-T,T]$, there exists a sequence $\{t_n, n\in\mathbb{N}\}$ of D_T which converges to t when n tends to $+\infty$. Then, the fact that $\omega\in\Omega_1^*(T)$ implies that the inequality

$$\left|X^{\text{hfsm}}(t_m,\omega)-X^{\text{hfsm}}(t_n,\omega)\right| \leq c_9\left|t_m-t_n\right|^H\left(1+\left|\log|t_m-t_n|\right|\right)^{1/\alpha+1/2+\varepsilon/\alpha} \qquad (2.2.34)$$

holds for any $m,n\in\mathbb{N}$ big enough. Therefore, $\{X^{\text{hfsm}}(t_n,\omega), n\in\mathbb{N}\}$ is a real-valued Cauchy sequence. The Cauchy criterion implies that this sequence converges to a finite limit. We denote this limit by $Y_T(t,\omega)$. Observe that $Y_T(t,\omega)$ does not depend on the choice of the sequence $\{t_n, n \in \mathbb{N}\}$. Indeed, for any other sequence $\{t'_n, n \in \mathbb{N}\}$ of D_T which converges to t when n tends to $+\infty$, we have

$$\begin{aligned}
|X^{\mathrm{hfsm}}(t'_n,\omega) - Y_T(t,\omega)| &\leq |X^{\mathrm{hfsm}}(t_n,\omega) - X^{\mathrm{hfsm}}(t'_n,\omega)| + |X^{\mathrm{hfsm}}(t_n,\omega) - Y_T(t,\omega)| \\
&\leq c_9 \left|t'_n - t_n\right|^H (1 + |\log|t'_n - t_n||)^{1/\alpha+1/2+\varepsilon/\alpha} \\
&\qquad + \left|X^{\mathrm{hfsm}}(t_n,\omega) - Y_T(t,\omega)\right|. \qquad (2.2.35)
\end{aligned}$$

Observe that, in view of the definition of $Y_T(t,\omega)$ and of the fact that $H \in (0,1)$, the right-hand side of (2.2.35) converges to 0 when n tends to $+\infty$. Therefore, we have that

$$\lim_{n\to+\infty} X^{\mathrm{hfsm}}(t'_n,\omega) = Y_T(t,\omega).$$

So, the process Y_T is well-defined.
Next, we prove that it satisfies (2.2.11). Let $\omega \in \Omega_1^*(T)$ be fixed. By definition of $\Omega_1^*(T)$ (see (2.2.10)) there exists $J(\omega) \in \mathbb{N}$ such that the inequality

$$|X^{\mathrm{hfsm}}(t') - X^{\mathrm{hfsm}}(s')| \leq c_9 \left|t' - s'\right|^H \Big(1 + |\log|t' - s'||\Big)^{1/\alpha+1/2+\varepsilon/\alpha} \qquad (2.2.36)$$

holds for every $s', t' \in D_T$ satisfying $|t' - s'| < 2^{-J(\omega)}$. Moreover, for any $s, t \in [-T, T]$ such that $|t - s| < 2^{-J(\omega)}$ there exist two sequences $\{t_n, n \in \mathbb{N}\}$ and $\{s_n, n \in \mathbb{N}\}$ of D_T such that $\lim_{n\to+\infty} s_n = s$, $\lim_{n\to+\infty} t_n = t$ and $|t_n - s_n| < 2^{-J(\omega)}$, for each $n \in \mathbb{N}$. Then in view of (2.2.36), the inequality

$$|X^{\mathrm{hfsm}}(t_n) - X^{\mathrm{hfsm}}(s_n)| \leq c_9 \left|t_n - s_n\right|^H \Big(1 + |\log|t_n - s_n||\Big)^{1/\alpha+1/2+\varepsilon/\alpha} \qquad (2.2.37)$$

holds for each $n \in \mathbb{N}$. Letting n go to $+\infty$ in (2.2.37), we get (2.2.36) for every $s, t \in [-T, T]$ such that $|t - s| < 2^{-J(\omega)}$. In particular, this implies that the process Y_T is almost surely continuous on $[-T, T]$. So (2.2.11) holds.
It remains to show that Y_T is a modification of $X^{\mathrm{hfsm}} = \{X^{\mathrm{hfsm}}(t), t \in [-T, T]\}$. In view of (ii) and the equality $\mathbb{P}(\Omega_1^*(T)) = 1$, we have that for any $t \in D_T$, $Y_T(t) = X^{\mathrm{hfsm}}(t)$ almost surely. If $t \in [-T, T] \setminus D_T$, we choose $\{t_n, n \in \mathbb{N}\}$ a sequence of D_T such that $\lim_{n\to+\infty} t_n = t$. By definition of the process Y_T (see (iii)), we know that almost surely, $X^{\mathrm{hfsm}}(t_n)$ converges to $Y_T(t)$ when n tends to $+\infty$. Therefore, in order to show that $Y_T(t) = X(t)$ almost surely, it is enough to show that $X^{\mathrm{hfsm}}(t_n)$ converges to $X^{\mathrm{hfsm}}(t)$ in probability when n tends to $+\infty$. In view of Proposition 2.1.8, Theorem 2.1.10 and (1.1.16), this convergence holds as soon as we have that

$$\lim_{n\to+\infty} \Big(e^{it_n\xi} - 1\Big)|\xi|^{-H-1/\alpha} = \Big(e^{it\xi} - 1\Big)|\xi|^{-H-1/\alpha}, \quad \text{in } L^\alpha\Big(\mathbb{R}^d\Big).$$

That is we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}} \left|e^{i(t-t_n)\xi} - 1\right|^\alpha |\xi|^{-H\alpha-1}\,\mathrm{d}\xi = 0. \qquad (2.2.38)$$

We prove (2.2.38) using the Dominated Convergence Theorem. It is clear that

$$\lim_{n\to+\infty}\left|e^{i(t-t_n)\xi}-1\right|^{\alpha}|\xi|^{-H\alpha-1}=0$$

for almost all $\xi\in\mathbb{R}\setminus\{0\}$. Moreover, the sequence $\{t-t_n, n\in\mathbb{N}\}$ is convergent to 0, hence it is bounded by a finite constant c_{10} (notice that c_{10} depends only on t). Thus in view of (2.2.8) the inequality

$$\left|e^{i(t-t_n)\xi}-1\right|^{\alpha}\leq 2^{\alpha}\min\{|c_{10}\xi|^{\alpha},1\} \tag{2.2.39}$$

holds for any $n\in\mathbb{N}$ and $\xi\in\mathbb{R}$. Combining (2.2.39) and the fact that $H\in(0,1)$, we get that the function $\xi\mapsto\left|e^{i(t-t_n)\xi}-1\right|^{\alpha}|\xi|^{-H\alpha-1}$ is bounded, uniformly in $n\in\mathbb{N}$, the measurable function $\xi\mapsto 2^{\alpha}\min\{|c_{10}\xi|^{\alpha},1\}|\xi|^{-H\alpha-1}$ which belongs to $L^1(\mathbb{R})$. So, we can apply the Dominated Convergence Theorem in order to obtain (2.2.38). So, we proved that $Y_T(t)=X^{\mathrm{hfsm}}(t)$ almost surely.

In view of Remark 1.1.2 we can define a modification $\{Y(t), t\in\mathbb{R}\}$ of X^{hfsm} such that, for any ω belonging to the event $\bigcap_{T\in\mathbb{N}}\Omega_1^*(T)$ of probability 1 and $T\in(0,+\infty)$, the inequality (2.2.1) holds. □

Notice that the event of probability 1 where (2.2.1) holds depends on H the Hurst parameter of the harmonizable fractional stable motion. In the next section, we define a general class of real-valued stationary increments harmonizable symmetric stable fields. They depend on a functional parameter f satisfying a very general condition. The harmonizable fractional stable motion is a particular example of them; in its case, one has $f(\xi)=|\xi|^{-H-1/\alpha}$ for almost all $\xi\in\mathbb{R}$. We mention that in Chapter 4 of the thesis we establish that a wide sub-class of those fields satisfies regularity results stronger than the one provided by Proposition 2.2.1. The methodology we use relies on wavelet bases as well as LePage series. Doing so, the advantage of this methodology is that the regularity results are valid on a "*universal*" event of probability 1 *which does not depend on the functional parameter* f (as a consequence, the result on the regularity of the harmonizable fractional stable motion are valid on an event of probability 1 which does not depend on the Hurst parameter H).

2.3 Stationary increments harmonizable stable fields

The symmetric stable fields we focus on are defined through a stochastic stable integral with respect to a complex-valued rotationally invariant α-stable random measure $\widetilde{M}_\alpha$ on $\mathbb{R}^d$ with control measure λ, the Lebesgue measure on $\mathbb{R}^d$. The main ingredient of those fields is a complex-valued function f defined on $\mathbb{R}^d$ satisfying the following condition, denoted by $(\mathcal{H}_0)$.

Definition 2.3.1. *We say that a function f satisfies the condition $(\mathcal{H}_0)$ if it is a complex-valued Lebesgue measurable function on $\mathbb{R}^d$ satisfying the 2 hypotheses:*

$$\int_{\mathbb{R}^d} \min\left(1, \|\xi\|^\alpha\right)\left|f(\xi)\right|^\alpha \mathrm{d}\xi < +\infty \tag{2.3.1}$$

where $\|\cdot\|$ denotes the Euclidian norm on $\mathbb{R}^d$, and, for almost all $\xi \in \mathbb{R}^d$,

$$\overline{f(\xi)} = f(-\xi), \tag{2.3.2}$$

where $\overline{f(\xi)}$ is the complex conjugate of $f(\xi)$.

Let f be a function satisfying $(\mathcal{H}_0)$. Thanks to (2.3.1), for any $t \in \mathbb{R}^d$, the kernel function $\xi \mapsto \left(e^{it\cdot\xi} - 1\right)f(\xi)$ belongs to $L^\alpha(\mathbb{R}^d)$, and thus it is integrable with respect to $\widetilde{M}_\alpha$.

Definition 2.3.2. *Assume that $\widetilde{M}_\alpha$ is a complex-valued rotationally invariant α-stable random measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with control measure λ, the Lebesgue measure on $\mathbb{R}^d$. We define, for any function f satisfying $(\mathcal{H}_0)$, the field $X[f] = \left\{X[f](t), t \in \mathbb{R}^d\right\}$ as follows: for all $t \in \mathbb{R}^d$,*

$$X[f](t) := \mathcal{R}e\left\{\int_{\mathbb{R}^d}\left(e^{it\cdot\xi} - 1\right)f(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right\}, \tag{2.3.3}$$

where $t \cdot \xi$ denotes the usual inner product of t and ξ.

It follows from the definition of $\widetilde{M}_\alpha$ and Proposition 2.1.7 that the real-valued stochastic field $X[f]$ is symmetric α-stable (see Definition 1.1.6). Notice that, by analogy with the Gaussian case (see [9] for instance), the function $|f|^\alpha$ is called *the spectral density of the field* $X[f]$. The following proposition shows that the hypothesis (2.3.2) is not restrictive.

Proposition 2.3.3. *Let f be a complex-valued measurable function on $\mathbb{R}^d$ satisfying the hypothesis* (2.3.1). *We define the real-valued, non-negative, even function function g as follows: for almost all $\xi \in \mathbb{R}^d$,*

$$g(\xi) = 2^{-1/\alpha}\left(\left|f(\xi)\right|^\alpha + \left|f(-\xi)\right|^\alpha\right)^{1/\alpha}. \tag{2.3.4}$$

Then, g satisfies $(\mathcal{H}_0)$ and the field $X[g]$ has the same distribution as the field

$$\left\{\mathcal{R}e\left\{\int_{\mathbb{R}^d}\left(e^{it\cdot\xi} - 1\right)f(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right\}, t \in \mathbb{R}^d\right\}.$$

Before proving Proposition 2.3.3 we recall the following useful result.

Lemma 2.3.4. *Let $X = \{X(t), t \in \mathbb{R}^d\}$ and $Y = \{Y(t), t \in \mathbb{R}^d\}$ be two real-valued random fields. Then the following statements are equivalent:*

(i) The fields X and Y have the same distribution.

(ii) For any $N \in \mathbb{N}$, $t^1, \dots, t^N \in \mathbb{R}^d$ and $b_1, \dots b_N \in \mathbb{R}$, we have

$$\mathbb{E}\left[\exp\left\{i\sum_{l=1}^{N} b_l X\left(t^l\right)\right\}\right] = \mathbb{E}\left[\exp\left\{i\sum_{l=1}^{N} b_l Y\left(t^l\right)\right\}\right]. \tag{2.3.5}$$

(iii) For any $N \in \mathbb{N}$, $t^1, \dots, t^N \in \mathbb{R}^d$ and $b_1, \dots b_N \in \mathbb{R}$, the random variables $\sum_{l=1}^N b_l X\left(t^l\right)$ and $\sum_{l=1}^N b_l Y\left(t^l\right)$ are identically distributed.

Proof of Lemma 2.3.4. By definition, the fields X and Y have the same distribution if, and only if, for any $N \in \mathbb{N}$ and $t^1, \dots, t^N \in \mathbb{R}^d$, the random vectors

$$\left(X\left(t^1\right), \dots, X\left(t^N\right)\right) \qquad \text{and} \qquad \left(Y\left(t^1\right), \dots, Y\left(t^N\right)\right)$$

are identically distributed. That is, if and only if, they have the same characteristic function[1]. Therefore, (i) is equivalent to (ii). In order to prove $(ii) \Rightarrow (iii)$, observe that the random variables $\sum_{l=1}^N b_l X\left(t^l\right)$ and $\sum_{l=1}^N b_l Y\left(t^l\right)$ are real-valued. Therefore, for any $\xi \in \mathbb{R}$, we have

$$\chi_{\sum_{l=1}^N b_l X(t^l)}(\xi) = \mathbb{E}\left[\exp\left\{i\xi\sum_{l=1}^{N} b_l X\left(t^l\right)\right\}\right]$$

and

$$\chi_{\sum_{l=1}^N b_l Y(t^l)}(\xi) = \mathbb{E}\left[\exp\left\{i\xi\sum_{l=1}^{N} b_l Y\left(t^l\right)\right\}\right].$$

So, applying (2.3.5) with b_l replaced by ξb_l we get $(ii) \Rightarrow (iii)$. It is clear that $(iii) \Rightarrow (ii)$ holds. $\square$

Proof of Proposition 2.3.3. In view of (2.3.4) and (2.3.1), it is clear that the function g satisfies $(\mathcal{H}_0)$.

In the sequel, we denote, for any $t \in \mathbb{R}^d$,

$$Y(t) := \mathcal{R}e\left\{\int_{\mathbb{R}^d}\left(e^{it\cdot\xi} - 1\right) f(\xi)\, \mathrm{d}\widetilde{M}_\alpha(\xi)\right\}. \tag{2.3.7}$$

Notice that Proposition 2.1.7 and (2.3.1) entail that $Y := \left\{Y(t),\, t \in \mathbb{R}^d\right\}$ is a well-defined real-valued symmetric α-stable stochastic field (see Definition 1.1.6). Therefore, in view of

[1]Let $m \in \mathbb{N}$ and $(X_1, \dots, X_m)$ be a real-valued random vector. The characteristic function $\chi_{(X_1,\dots,X_m)}$ of $(X_1, \dots, X_m)$ is given, for any $b_1, \dots, b_m \in \mathbb{R}$, by

$$\chi_{(X_1,\dots,X_m)}(b_1, \dots, b_m) := \mathbb{E}\left[\exp\left\{i\sum_{l=1}^{m} b_l X_l\right\}\right]. \tag{2.3.6}$$

Lemma 2.3.4, in order to show that the fields Y and $X[g]$ have the same distribution it is enough to show that, for any $N \in \mathbb{N}$, $t^1, \ldots, t^N \in \mathbb{R}^d$ and real numbers $b_1, \ldots, b_N$, we have that

$$\mathbb{E}\left[\exp\left(i\sum_{l=1}^{N} b_l Y(t^l)\right)\right] = \mathbb{E}\left[\exp\left(i\sum_{l=1}^{N} b_l X[g](t^l)\right)\right]. \tag{2.3.8}$$

Using (2.3.3), the fact that the $b_l's$ are real numbers, and the linearity of the stable integral, we obtain the equalities:

$$\sum_{l=1}^{N} b_l Y(t^l) = \mathcal{R}e\left\{\int_{\mathbb{R}^d}\left(\sum_{l=1}^{N} b_l\Big(e^{it^l\cdot\xi}-1\Big)\right) f(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right\}, \tag{2.3.9}$$

and

$$\sum_{l=1}^{N} b_l X[g](t^l) = \mathcal{R}e\left\{\int_{\mathbb{R}^d}\left(\sum_{l=1}^{N} b_l\Big(e^{it^l\cdot\xi}-1\Big)\right) g(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right\}. \tag{2.3.10}$$

It follows from (iii) in Proposition 2.1.7 that the real-valued random variables

$$\sum_{l=1}^{N} b_l Y(t^l) \quad \text{and} \quad \sum_{l=1}^{N} b_l X[g](t^l)$$

have symmetric α-stable distributions; their scale parameters are given by

$$\sigma\left(\sum_{l=1}^{N} b_l Y(t^l)\right) = \left(\int_{\mathbb{R}^d}\left|\sum_{l=1}^{N} b_l\Big(e^{it^l\cdot\xi}-1\Big)\right|^\alpha |f(\xi)|^\alpha\,\mathrm{d}\xi\right)^{1/\alpha} \tag{2.3.11}$$

and

$$\sigma\left(\sum_{l=1}^{N} b_l X[g](t^l)\right) = \left(\int_{\mathbb{R}^d}\left|\sum_{l=1}^{N} b_l\Big(e^{it^l\cdot\xi}-1\Big)\right|^\alpha |g(\xi)|^\alpha\,\mathrm{d}\xi\right)^{1/\alpha}. \tag{2.3.12}$$

In view of (1.1.6), the equality (2.3.8) holds as soon as

$$\sigma\left(\sum_{l=1}^{N} b_l X[g](t^l)\right) = \sigma\left(\sum_{l=1}^{N} b_l Y(t^l)\right). \tag{2.3.13}$$

Combining (2.3.12), (2.3.4), the change of variable $\eta = -\xi$, the fact that the b_l's are real numbers, and (2.3.11), we obtain

$$\begin{aligned}
&\sigma\left(\sum_{l=1}^{N} b_l X[g](t^l)\right)^{\alpha} \\
&= \frac{1}{2}\int_{\mathbb{R}^d} \left|\sum_{l=1}^{N} b_l\Big(e^{it^l\cdot\xi}-1\Big)\right|^{\alpha} |f(\xi)|^{\alpha}\,\mathrm{d}\xi + \frac{1}{2}\int_{\mathbb{R}^d} \left|\sum_{l=1}^{N} b_l\Big(e^{it^l\cdot\xi}-1\Big)\right|^{\alpha} |f(-\xi)|^{\alpha}\,\mathrm{d}\xi \\
&= \frac{1}{2}\int_{\mathbb{R}^d} \left|\sum_{l=1}^{N} b_l\Big(e^{it^l\cdot\xi}-1\Big)\right|^{\alpha} |f(\xi)|^{\alpha}\,\mathrm{d}\xi + \frac{1}{2}\int_{\mathbb{R}^d} \left|\sum_{l=1}^{N} b_l\Big(e^{-it^l\cdot\eta}-1\Big)\right|^{\alpha} |f(\eta)|^{\alpha}\,\mathrm{d}\eta \\
&= \frac{1}{2}\int_{\mathbb{R}^d} \left|\sum_{l=1}^{N} b_l\Big(e^{it^l\cdot\xi}-1\Big)\right|^{\alpha} |f(\xi)|^{\alpha}\,\mathrm{d}\xi + \frac{1}{2}\int_{\mathbb{R}^d} \left|\overline{\sum_{l=1}^{N} b_l\Big(e^{it^l\cdot\eta}-1\Big)}\right|^{\alpha} |f(\eta)|^{\alpha}\,\mathrm{d}\eta \\
&= \frac{1}{2}\int_{\mathbb{R}^d} \left|\sum_{l=1}^{N} b_l\Big(e^{it^l\cdot\xi}-1\Big)\right|^{\alpha} |f(\xi)|^{\alpha}\,\mathrm{d}\xi + \frac{1}{2}\int_{\mathbb{R}^d} \left|\sum_{l=1}^{N} b_l\Big(e^{it^l\cdot\eta}-1\Big)\right|^{\alpha} |f(\eta)|^{\alpha}\,\mathrm{d}\eta \\
&= \int_{\mathbb{R}^d} \left|\sum_{l=1}^{N} b_l\Big(e^{it^l\cdot\xi}-1\Big)\right|^{\alpha} |f(\xi)|^{\alpha}\,\mathrm{d}\xi \\
&= \sigma\left(\sum_{l=1}^{N} b_l Y(t^l)\right)^{\alpha}.
\end{aligned}$$

Hence, (2.3.13) holds □

In the following proposition, we prove that $X[f]$ has stationary increments.

Proposition 2.3.5. *Let f be an arbitrary function satisfying $(\mathcal{H}_0)$ and $X[f]$ be the field associated with f (see (2.3.3)). The field $X[f]$ has stationary increments: that is (since $X[f](0) = 0$ almost surely), for all $h \in \mathbb{R}^d$, the stochastic fields $\{X[f](t+h) - X[f](h), t \in \mathbb{R}^d\}$ and $X[f]$ share the same distribution.*

Proof of Proposition 2.3.5. Let $h \in \mathbb{R}^d$ be fixed. In view of Lemma 2.3.4, it is enough to show that, for all $N \in \mathbb{N}$, $t^1, \dots, t^N \in \mathbb{R}^d$ and $\theta_1, \dots, \theta_N \in \mathbb{R}$, we have that

$$\mathbb{E}\left[\exp\left\{i\sum_{l=1}^{N}\theta_l\Big(X[f](t^l+h) - X[f](h)\Big)\right\}\right] = \mathbb{E}\left[\exp\left\{i\sum_{l=1}^{N}\theta_l X[f](t^l)\right\}\right]. \tag{2.3.14}$$

Putting together (2.3.3) and the linearity of the stable integral, we obtain that

$$\sum_{l=1}^{N}\theta_l\Big(X[f](t^l+h) - X[f](h)\Big) = \mathcal{R}e\left\{\int_{\mathbb{R}^d} e^{ih\cdot\xi}\left(\sum_{l=1}^{N}\theta_l\Big(e^{it^l\cdot\xi}-1\Big)\right) f(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right\} \tag{2.3.15}$$

and

$$\sum_{l=1}^{N}\theta_l X[f](t^l) = \mathcal{R}e\left\{\int_{\mathbb{R}^d} \left(\sum_{l=1}^{N}\theta_l\Big(e^{it^l\cdot\xi}-1\Big)\right) f(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right\}. \tag{2.3.16}$$

Notice that, for all $x \in \mathbb{R}$, $|e^{ix}| = 1$. Therefore, it follows from (2.1.17) that the scale parameters of the real-valued α-stable random variables in (2.3.15) and (2.3.16) are the same. Hence, in view of (1.1.5), the equality (2.3.14) holds. □

Let f and g be two functions $(\mathcal{H}_0)$ such that $|f(\xi)| = |g(\xi)|$, for almost all $\xi \in \mathbb{R}^d$; using the same arguments as in the proof of Proposition 2.3.5, it can easily be shown that the stochastic fields $X[f]$ and $X[g]$ have the same distribution. The following theorem shows that the converse is also true.

Theorem 2.3.6. *Assume that f and g are two arbitrary functions satisfying $(\mathcal{H}_0)$. If the fields $X[f]$ and $X[g]$ respectively associated with f and g (see Definition 2.3.2) have the same distribution, then, for almost all $\xi \in \mathbb{R}^d$,*

$$\Big|f(\xi)\Big| = \Big|g(\xi)\Big|.$$

In order to prove Theorem 2.3.6 we need the following two lemmas.

Lemma 2.3.7. *Assume that f is a function satisfying $(\mathcal{H}_0)$ and that $X[f]$ is the field associated with f (see Definition 2.3.2). Let $\Big\{Y[f](t), t \in \mathbb{R}^d\Big\}$ be the real-valued symmetric α-stable field defined, for each $t \in \mathbb{R}^d$, as*

$$Y[f](t) := X[f]\Big(t + \overrightarrow{1}\Big) - X[f](t) = \mathcal{R}e\left\{\int_{\mathbb{R}^d} e^{it\cdot\xi}\Big(e^{i\overrightarrow{1}\cdot\xi} - 1\Big) f(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right\}, \qquad (2.3.17)$$

where $\overrightarrow{1}$ is the vector of $\mathbb{R}^d$ whose all coordinates are equal to 1. Next, let $\mu_\theta^{Y[f]}$ be the symmetric α-stable random variable defined as

$$\begin{aligned} \mu_\theta^{Y[f]} &:= \mathcal{R}e\left\{\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} e^{it\cdot\xi}\widehat{\theta}(t)\,\mathrm{d}t\right)\Big(e^{i\xi\cdot\overrightarrow{1}} - 1\Big) f(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right\} \\ &= (2\pi)^d\,\mathcal{R}e\left\{\int_{\mathbb{R}^d} \theta(\xi)\Big(e^{i\xi\cdot\overrightarrow{1}} - 1\Big) f(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right\}, \end{aligned} \qquad (2.3.18)$$

where θ is an arbitrary real-valued even (that is, $\theta(\xi) = \theta(-\xi)$ for every $\xi \in \mathbb{R}^d$) function in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. At last, for every $m \in \mathbb{N}$, let $\mu_{\theta,m}^{Y[f]}$ be the symmetric α-stable random variable defined as

$$\begin{aligned} \mu_{\theta,m}^{Y[f]} &:= \frac{1}{m^d}\sum_{q \in I_d(m)} \widehat{\theta}\Big(m^{-1}q\Big) Y[f]\Big(m^{-1}q\Big) \\ &= \mathcal{R}e\left\{\int_{\mathbb{R}^d}\left(\frac{1}{m^d}\sum_{q \in I_d(m)} \widehat{\theta}\Big(m^{-1}q\Big) e^{im^{-1}q\cdot\xi}\right)\Big(e^{i\xi\cdot\overrightarrow{1}} - 1\Big) f(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right\}, \end{aligned} \qquad (2.3.19)$$

where

$$I_d(m) := \mathbb{Z}^d \cap \Big[- m^{1+\frac{1}{2d}}, m^{1+\frac{1}{2d}}\Big]^d. \qquad (2.3.20)$$

Then, for any $\gamma \in (0, \alpha)$, one has

$$\lim_{m \to +\infty} \mathbb{E}\left[\left| \mu_{\theta,m}^{Y[f]} - \mu_{\theta}^{Y[f]} \right|^{\gamma} \right] = 0. \tag{2.3.21}$$

In Lemma 2.3.7, observe that the Fourier transform $\widehat{\theta}$ is a real-valued function because θ is a real-valued even function. Moreover the second equality in (2.3.17) and the second equality (2.3.19) come from the linearity of the stable integral.

Lemma 2.3.8. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be an arbitrary compactly supported even infinitely differentiable function. Assume that its support satisfies*

$$\operatorname{supp} \varphi \subset \mathcal{B}_d(0,1) := \{\xi \in \mathbb{R}^d, \|\xi\| \leq 1\}. \tag{2.3.22}$$

For each fixed $n \in \mathbb{N}$, let $\varphi_n : \mathbb{R}^d \to \mathbb{R}$ be the function defined, for every $\xi \in \mathbb{R}^d$, by

$$\varphi_n(\xi) := n^d \varphi(n\xi). \tag{2.3.23}$$

Then, for any fixed $\eta \in \mathbb{R}^d$, the equality

$$\frac{1}{2} \int_{\mathbb{R}^d} e^{-it\cdot\xi} \Big(\varphi_n(\xi + \eta) + \varphi_n(\xi - \eta) \Big) \, \mathrm{d}\xi = \cos(t \cdot \eta) \widehat{\varphi}\Big(n^{-1} t\Big) \tag{2.3.24}$$

holds for all $t \in \mathbb{R}^d$.
Moreover, for any $\eta \in \mathbb{R}^d$ with $\|\eta\| > 0$ and for every integer $n > \|\eta\|^{-1}$, one has that

$$\operatorname{supp} \varphi_n(\cdot + \eta) \cap \operatorname{supp} \varphi_n(\cdot - \eta) = \emptyset, \tag{2.3.25}$$

and consequently that, for any $\xi \in \mathbb{R}^d$,

$$\Big| \varphi_n(\xi + \eta) + \varphi_n(\xi - \eta) \Big|^{\alpha} = \Big| \varphi_n(\xi + \eta) \Big|^{\alpha} + \Big| \varphi_n(\xi - \eta) \Big|^{\alpha}. \tag{2.3.26}$$

Proof of Lemma 2.3.7. Definitions 1.1.6 and 2.3.2, Proposition 2.1.7, (2.3.19) and (2.3.18) imply that the random variable $\mu_{\theta,m}^{Y[f]} - \mu_{\theta}^{Y[f]}$ has a symmetric stable distribution. Its scale parameter is given, for any $m \in \mathbb{N}$, by

$$\begin{aligned} &\sigma\Big(\mu_{\theta,m}^{Y[f]} - \mu_{\theta}^{Y[f]} \Big) \\ &:= \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \widehat{\theta}(t) e^{it\cdot\xi} \mathrm{d}t - \frac{1}{m^d} \sum_{q \in I_d(m)} \widehat{\theta}\Big(m^{-1} q\Big) e^{im^{-1} q \cdot \xi} \right|^{\alpha} \left| e^{i\xi\cdot\overrightarrow{1}} - 1 \right|^{\alpha} |f(\xi)|^{\alpha} \, \mathrm{d}\xi \right)^{1/\alpha} \end{aligned} \tag{2.3.27}$$

In view of the equality (1.1.11), the equality (2.3.21) holds as soon as

$$\lim_{m\to+\infty}\sigma\Big(\mu_{\theta,m}^{Y[f]}-\mu_{\theta}^{Y[f]}\Big)=0. \tag{2.3.28}$$

In order to prove (2.3.28), we first show that, for almost all $\xi\in\mathbb{R}^d$,

$$\lim_{m\to+\infty}\frac{1}{m^d}\sum_{q\in I_d(m)}\widehat{\theta}\Big(m^{-1}q\Big)e^{im^{-1}q\cdot\xi}=\int_{\mathbb{R}^d}\widehat{\theta}(t)e^{it\cdot\xi}\mathrm{d}t. \tag{2.3.29}$$

Observe that, for almost all $\xi\in\mathbb{R}^d$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\widehat{\theta}(t)e^{it\cdot\xi}\mathrm{d}t-\frac{1}{m^d}\sum_{q\in I_d(m)}\widehat{\theta}\Big(m^{-1}q\Big)e^{im^{-1}q\cdot\xi}\right|\\
&\leq\int_{\mathbb{R}^d\setminus\left(-m^{\frac{1}{2d}},m^{\frac{1}{2d}}\right)^d}\left|\widehat{\theta}(t)\right|\mathrm{d}t\\
&\quad+\sum_{q\in I_d(m)}\int_{\prod_{l=1}^d\left[\frac{q_l}{m},\frac{q_l+1}{m}\right)}\left|\widehat{\theta}(t)e^{it\cdot\xi}-\widehat{\theta}\Big(m^{-1}q\Big)e^{im^{-1}q\cdot\xi}\right|\mathrm{d}t
\end{aligned} \tag{2.3.30}$$

The function $\widehat{\theta}$ belongs to $\mathcal{S}(\mathbb{R}^d)$, therefore, there exists a constant $c_1\in(0,+\infty)$ such that, for any $t\in\mathbb{R}^d$, we have

$$\left|\widehat{\theta}(t)\right|\leq c_1\prod_{l=1}^d(3+|t_l|)^{-4}\leq c_1\prod_{l=1}^d(1+|t_l|)^{-4}. \tag{2.3.31}$$

Moreover, we have

$$\mathbb{R}^d\setminus\Big(-m^{\frac{1}{2d}},m^{\frac{1}{2d}}\Big)^d\subset\bigcup_{l=1}^d\Big\{\xi\in\mathbb{R}^d,|\xi_l|\geq m^{1/2d}\Big\}. \tag{2.3.32}$$

It follows from (2.3.31) and (2.3.32) that

$$\int_{\mathbb{R}^d\setminus\left(-m^{\frac{1}{2d}},m^{\frac{1}{2d}}\right)^d}\left|\widehat{\theta}(t)\right|\mathrm{d}t\leq\frac{d2^dc_1}{3^d}\Big(1+m^{1/2d}\Big)^{-3} \tag{2.3.33}$$

Hence, (2.3.33) implies that

$$\lim_{m\to+\infty}\int_{\mathbb{R}^d\setminus\left(-m^{\frac{1}{2d}},m^{\frac{1}{2d}}\right)^d}\left|\widehat{\theta}(t)\right|\mathrm{d}t=0. \tag{2.3.34}$$

On the other hand, in view of Mean Value Theorem and (2.2.8), we have, for any $s,t,\xi\in\mathbb{R}^d$, that

$$\begin{aligned}\left|\widehat{\theta}(t)e^{it\cdot\xi}-\widehat{\theta}(s)e^{is\cdot\xi}\right| &\leq \left|\widehat{\theta}(t)\right|\left|e^{it\cdot\xi}-e^{is\cdot\xi}\right|+\left|\widehat{\theta}(t)-\widehat{\theta}(s)\right| \\ &\leq \sup_{u\in\mathbb{R}^d}\left|\widehat{\theta}(u)\right|\|\xi\|\,\|t-s\|+\left(\sum_{l=1}^{d}\sup_{u\in\mathbb{R}^d}\left|\partial^{e_l}\widehat{\theta}(u)\right|\right)\|t-s\|\,,\end{aligned} \tag{2.3.35}$$

where e_l denotes the vector of $\mathbb{R}^d$ whose l-th coordinate equals 1 and the others vanish. So, setting $c_2 := \sqrt{d}\sup_{u\in\mathbb{R}^d}\left|\widehat{\theta}(u)\right|+\sqrt{d}\sum_{l=1}^{d}\sup_{u\in\mathbb{R}^d}\left|\partial^{e_l}\widehat{\theta}(u)\right|\in(0,+\infty)$, the inequality

$$\left|\widehat{\theta}(t)e^{it\cdot\xi}-\widehat{\theta}(s)e^{is\cdot\xi}\right|\leq c_2(1+\|\xi\|)\max_{1\leq l\leq d}|t_l-s_l| \tag{2.3.36}$$

holds for any $\xi,s,t\in\mathbb{R}^d$. It follows from (2.3.36) that, for all $m\in\mathbb{N}$, $q\in I_d(m)$ and almost all $\xi\in\mathbb{R}^d$,

$$\sum_{q\in I_d(m)}\int_{\prod_{l=1}^{d}\left[\frac{q_l}{m},\frac{q_l+1}{m}\right)}\left|\widehat{\theta}(t)e^{it\cdot\xi}-\widehat{\theta}\Big(m^{-1}q\Big)e^{im^{-1}q\cdot\xi}\right|\mathrm{d}t\leq c_2(1+\|\xi\|)m^{-d-1}\mathrm{Card}\Big(I_d(m)\Big). \tag{2.3.37}$$

In view of (2.3.20), for any $m\in\mathbb{N}$, we have that

$$\mathrm{Card}\Big(I_d(m)\Big)\leq\left(2m^{1+\frac{1}{2d}}+1\right)^d\leq 3^d m^{d+1/2}. \tag{2.3.38}$$

Combining (2.3.37) and (2.3.38), we have, for almost all $\xi\in\mathbb{R}^d$,

$$\lim_{m\to+\infty}\sum_{q\in I_d(m)}\int_{\prod_{l=1}^{d}\left[\frac{q_l}{m},\frac{q_l+1}{m}\right)}\left|\widehat{\theta}(t)e^{it\cdot\xi}-\widehat{\theta}\Big(m^{-1}q\Big)e^{im^{-1}q\cdot\xi}\right|\mathrm{d}t=0. \tag{2.3.39}$$

Putting together (2.3.30), (2.3.34) and (2.3.39), we get that (2.3.29) holds.

Now, we show that there exists $c_3\in(0,+\infty)$ satisfying

$$\sup\left\{\left|\frac{1}{m^d}\sum_{q\in I_d(m)}\widehat{\theta}\Big(m^{-1}q\Big)e^{im^{-1}q\cdot\xi}-\int_{\mathbb{R}^d}\widehat{\theta}(t)e^{it\cdot\xi}\mathrm{d}t\right|,(m,\xi)\in\mathbb{N}\times\mathbb{R}^d\right\}\leq c_3. \tag{2.3.40}$$

The function $\widehat{\theta}$ belongs to $\mathcal{S}(\mathbb{R}^d)$ so, the inequality

$$\left|\int_{\mathbb{R}^d}\widehat{\theta}(t)e^{it\cdot\xi}\,\mathrm{d}t\right|\leq\int_{\mathbb{R}^d}\left|\widehat{\theta}(t)\right|\mathrm{d}t<+\infty \tag{2.3.41}$$

holds for all $\xi\in\mathbb{R}^d$. Moreover, notice that, for any $q\in\mathbb{Z}$ and $t\in\Big[m^{-1}q,m^{-1}(q+1)\Big)$, using the triangular inequality, we have that

$$|t|-|q|/m\leq\Big||t|-|q|/m\Big|\leq|t-q/m|\leq 1/m\leq 1. \tag{2.3.42}$$

Combining the first inequality in (2.3.31) and (2.3.42), we get

$$\begin{aligned}
\left|\frac{1}{m^d}\sum_{q\in I_d(m)}\widehat{\theta}\Big(m^{-1}q\Big)e^{im^{-1}q\cdot\xi}\right| &= \left|\int_{\mathbb{R}^d}\sum_{q\in I_d(m)}\widehat{\theta}\Big(m^{-1}q\Big)e^{im^{-1}q\cdot\xi}\mathbb{1}_{\prod_{l=1}^d\left[\frac{q_l}{m},\frac{q_l+1}{m}\right)}(t)\,\mathrm{d}t\right|\\
&\leq \sum_{q\in\mathbb{Z}^d}\int_{\prod_{l=1}^d\left[\frac{q_l}{m},\frac{q_l+1}{m}\right)}\left|\widehat{\theta}\Big(m^{-1}q\Big)\right|\mathrm{d}t\\
&\leq c_1\sum_{q\in\mathbb{Z}^d}\prod_{l=1}^d\int_{\frac{q_l}{m}}^{\frac{q_l+1}{m}}\left(3+\frac{|q_l|}{m}\right)^{-4}\mathrm{d}t\\
&\leq c_1\sum_{q\in\mathbb{Z}^d}\prod_{l=1}^d\int_{\frac{q_l}{m}}^{\frac{q_l+1}{m}}\Big(2+|t_l|\Big)^{-4}\mathrm{d}t\\
&\leq c_1\int_{\mathbb{R}^d}\prod_{l=1}^d\Big(2+|t_l|\Big)^{-4}\mathrm{d}t<+\infty. \qquad (2.3.43)
\end{aligned}$$

Putting together (2.3.41) and (2.3.43) we get (2.3.40). Therefore, for almost all $\xi\in\mathbb{R}^d$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\widehat{\theta}(t)e^{it\cdot\xi}\mathrm{d}t-\frac{1}{m^d}\sum_{q\in I_d(m)}\widehat{\theta}\Big(m^{-1}q\Big)e^{im^{-1}q\cdot\xi}\right|^\alpha\left|e^{i\xi\cdot\overrightarrow{1}}-1\right|^\alpha|f(\xi)|^\alpha\\
&\leq c_3^\alpha\left|e^{i\xi\cdot\overrightarrow{1}}-1\right|^\alpha|f(\xi)|^\alpha. \qquad (2.3.44)
\end{aligned}$$

Since f satisfies (2.3.1), the measurable function $\xi\mapsto\left|e^{i\xi\cdot\overrightarrow{1}}-1\right|^\alpha|f(\xi)|^\alpha\in L^1\Big(\mathbb{R}^d\Big)$. Therefore, in view of (2.3.27) and (2.3.29), applying the Dominated Convergence Theorem, we get (2.3.28). □

Proof of Lemma 2.3.8. First, we prove (2.3.24). Observe that, for any $\eta\in\mathbb{R}^d$, (2.3.23) entails that

$$\begin{aligned}
\int_{\mathbb{R}^d}e^{-it\cdot\xi}\varphi_n(\xi+\eta)\,\mathrm{d}\xi &= \int_{\mathbb{R}^d}e^{-it\cdot(\xi-\eta)}\varphi_n(\xi)\,\mathrm{d}\xi\\
&= n^d\int_{\mathbb{R}^d}e^{-it\cdot(\xi-\eta)}\varphi(n\xi)\,\mathrm{d}\xi\\
&= e^{it\cdot\eta}\int_{\mathbb{R}^d}e^{-in^{-1}t\cdot\xi}\varphi(\xi)\,\mathrm{d}\xi\\
&= e^{it\cdot\eta}\widehat{\varphi}(n^{-1}t). \qquad (2.3.45)
\end{aligned}$$

Hence, for any $\eta\in\mathbb{R}^d$, we have

$$\frac{1}{2}\int_{\mathbb{R}^d}e^{-it\cdot\xi}\Big(\varphi_n(\xi+\eta)+\varphi_n(\xi-\eta)\Big)\,\mathrm{d}\xi=\frac{e^{it\cdot\eta}+e^{-it\cdot\eta}}{2}\widehat{\varphi}(n^{-1}t)=\cos(t\cdot\eta)\widehat{\varphi}\Big(n^{-1}t\Big). \qquad (2.3.46)$$

So, (2.3.24) holds.

Now, we show (2.3.25). In view of (2.3.22) and (2.3.23) we have, for any $\eta \in \mathbb{R}$, that

$$\operatorname{supp} \varphi_n(\cdot - \eta) \subset \left\{\xi \in \mathbb{R}^d, \|\xi - \eta\| \leq n^{-1}\right\}. \tag{2.3.47}$$

Assume that for some $\eta \neq 0$ and $n > \|\eta\|^{-1}$, there exists $x \in \operatorname{supp} \varphi_n(\cdot + \eta) \cap \operatorname{supp} \varphi_n(\cdot - \eta)$. It follows from (2.3.47) that $\|x - \eta\| \leq n^{-1}$ and $\|x + \eta\| \leq n^{-1}$. Hence, we have

$$2n^{-1} < 2\|\eta\| = \|(x+\eta) + (\eta - x)\| \leq \|x + \eta\| + \|x - \eta\| \leq 2n^{-1},$$

which is absurd. Therefore (2.3.25) holds. The equality (2.3.26) is a straightforward consequence of (2.3.25). □

Proof of Theorem 2.3.6. For any $t \in \mathbb{R}^d$, let $Y[f](t)$ be defined as in (2.3.17). Similarly we define $Y[g](t)$. Assume that $\theta : \mathbb{R}^d \to \mathbb{R}$ is an arbitrary real-valued even function in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. For any $m \in \mathbb{N}$, let $\mu_{\theta,m}^{Y[f]}$ be the random variable defined as in (2.3.19). Similarly, we define $\mu_{\theta,m}^{Y[g]}$. The fields $X[f]$ and $X[g]$ have the same distribution, therefore, the fields $\left\{Y[f](t),\, t \in \mathbb{R}^d\right\}$ and $\left\{Y[g](t),\, t \in \mathbb{R}^d\right\}$ also have the same distribution. So, for any $m \in \mathbb{N}$, the random variables $\mu_{\theta,m}^{Y[f]}$ and $\mu_{\theta,m}^{Y[g]}$ are identically distributed. It follows from Lemma 2.3.7 that the random variables $\mu_{\theta}^{Y[f]}$ and $\mu_{\theta}^{Y[g]}$, defined as in (2.3.18), are identically distributed. Hence, we get the equality

$$\int_{\mathbb{R}^d} |\theta(\xi)|^\alpha \left|\sin\left(2^{-1}\overrightarrow{1} \cdot \xi\right)\right|^\alpha |f(\xi)|^\alpha \,\mathrm{d}\xi = \int_{\mathbb{R}^d} |\theta(\xi)|^\alpha \left|\sin\left(2^{-1}\overrightarrow{1} \cdot \xi\right)\right|^\alpha |g(\xi)|^\alpha \,\mathrm{d}\xi. \tag{2.3.48}$$

Let φ and φ_n, $n \in \mathbb{N}$, be the functions defined in Lemma 2.3.8. Assume that we have $\int_{\mathbb{R}^d} \varphi(\xi)\,\mathrm{d}\xi = 1$. For any $n \in \mathbb{N}$ and $\eta \in \mathbb{R}^d \setminus \{0\}$ fixed, we denote by $\theta_{\eta,n}$ the real-valued function defined, for any $\xi \in \mathbb{R}^d$, by

$$\theta_{\eta,n}(\xi) := \frac{\varphi_n(\xi + \eta) + \varphi_n(\xi - \eta)}{2}. \tag{2.3.49}$$

The real-valued function $\theta_{\eta,n}$ is even and belongs to $\mathcal{S}(\mathbb{R}^d)$, so in view of (2.3.48), we have that

$$\int_{\mathbb{R}^d} |\theta_{\eta,n}(\xi)|^\alpha \left|\sin\left(2^{-1}\overrightarrow{1} \cdot \xi\right)\right|^\alpha |f(\xi)|^\alpha \,\mathrm{d}\xi = \int_{\mathbb{R}^d} |\theta_{\eta,n}(\xi)|^\alpha \left|\sin\left(2^{-1}\overrightarrow{1} \cdot \xi\right)\right|^\alpha |g(\xi)|^\alpha \,\mathrm{d}\xi. \tag{2.3.50}$$

On the other hand, it follows from (2.3.49), (2.3.26), (2.3.2), the change of variable $\tilde{\xi} = -\xi$ and the fact that φ is an even function, for every integer $n > \|\eta\|^{-1}$, that

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |\theta_{\eta,n}(\xi)|^\alpha \left|\sin\left(2^{-1}\overrightarrow{1}\cdot\xi\right)\right|^\alpha |f(\xi)|^\alpha \,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^d} \left|\frac{\varphi_n(\xi+\eta)+\varphi_n(\xi-\eta)}{2}\right|^\alpha \left|\sin\left(2^{-1}\overrightarrow{1}\cdot\xi\right)\right|^\alpha |f(\xi)|^\alpha \,\mathrm{d}\xi \\
&= 2^{-\alpha}\int_{\mathbb{R}^d} \left|\varphi_n\Big(-\tilde{\xi}+\eta\Big)\right|^\alpha \left|\sin\left(2^{-1}\overrightarrow{1}\cdot\tilde{\xi}\right)\right|^\alpha \left|f(\tilde{\xi})\right|^\alpha \mathrm{d}\tilde{\xi} \\
&\qquad +2^{-\alpha}\int_{\mathbb{R}^d} |\varphi_n(\xi-\eta)|^\alpha \left|\sin\left(2^{-1}\overrightarrow{1}\cdot\xi\right)\right|^\alpha |f(\xi)|^\alpha \,\mathrm{d}\xi \\
&= 2^{1-\alpha}n^{d\alpha}\int_{\mathbb{R}^d} \left|\varphi\Big(n(\eta-\xi)\Big)\right|^\alpha \left|\sin\left(2^{-1}\overrightarrow{1}\cdot\xi\right)\right|^\alpha |f(\xi)|^\alpha \,\mathrm{d}\xi. \qquad (2.3.51)
\end{aligned}
$$

Similarly to (2.3.51) one has that

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |\theta_{\eta,n}(\xi)|^\alpha \left|\sin\left(2^{-1}\overrightarrow{1}\cdot\xi\right)\right|^\alpha |g(\xi)|^\alpha \,\mathrm{d}\xi \\
&= 2^{1-\alpha}n^{d\alpha}\int_{\mathbb{R}^d} \left|\varphi\Big(n(\eta-\xi)\Big)\right|^\alpha \left|\sin\left(2^{-1}\overrightarrow{1}\cdot\xi\right)\right|^\alpha |g(\xi)|^\alpha \,\mathrm{d}\xi. \qquad (2.3.52)
\end{aligned}
$$

It follows from (2.3.50), (2.3.51) and (2.3.52) that, for all fixed $\eta \in \mathbb{R}^d \setminus \{0\}$ and for any integer n satisfying $n > \|\eta\|^{-1}$ one has

$$
\begin{aligned}
&n^d\int_{\mathbb{R}^d} \left|\varphi\Big(n(\eta-\xi)\Big)\right|^\alpha \left|\sin\left(2^{-1}\overrightarrow{1}\cdot\xi\right)\right|^\alpha |f(\xi)|^\alpha \,\mathrm{d}\xi \\
&= n^d\int_{\mathbb{R}^d} \left|\varphi\Big(n(\eta-\xi)\Big)\right|^\alpha \left|\sin\left(2^{-1}\overrightarrow{1}\cdot\xi\right)\right|^\alpha |g(\xi)|^\alpha \,\mathrm{d}\xi. \qquad (2.3.53)
\end{aligned}
$$

Notice that the function $\xi \mapsto \left|\sin\left(2^{-1}\overrightarrow{1}\cdot\xi\right)\right|^\alpha |f(\xi)|^\alpha$ belongs to $L^1\Big(\mathbb{R}^d\Big)$. Therefore, when n goes to $+\infty$, the convolution function

$$
\eta \mapsto n^d\int_{\mathbb{R}^d} \left|\varphi\Big(n(\eta-\xi)\Big)\right|^\alpha \left|\sin\left(2^{-1}\overrightarrow{1}\cdot\xi\right)\right|^\alpha |f(\xi)|^\alpha \,\mathrm{d}\xi
$$

converges to the function $\eta \mapsto \left|\sin\left(2^{-1}\overrightarrow{1}\cdot\xi\right)\right|^\alpha |f(\xi)|^\alpha$ in $L^1\Big(\mathbb{R}^d\Big)$. So, there exists a subsequence $p \mapsto n_p$ such that, for almost all $\eta \in \mathbb{R}^d$,

$$
\lim_{p\to+\infty} n_p^d\int_{\mathbb{R}^d} \left|\varphi\Big(n_p(\eta-\xi)\Big)\right|^\alpha \left|\sin\left(2^{-1}\overrightarrow{1}\cdot\xi\right)\right|^\alpha |f(\xi)|^\alpha \,\mathrm{d}\xi = \left|\sin\left(2^{-1}\overrightarrow{1}\cdot\xi\right)\right|^\alpha |f(\xi)|^\alpha. \qquad (2.3.54)
$$

Using the same arguments their exists a sub-subsequence $l \mapsto n_{p_l}$ such that, for almost all $\eta \in \mathbb{R}^d$,

$$
\lim_{l\to+\infty} n_{p_l}^d\int_{\mathbb{R}^d} \left|\varphi\Big(n_{p_l}(\eta-\xi)\Big)\right|^\alpha \left|\sin\left(2^{-1}\overrightarrow{1}\cdot\xi\right)\right|^\alpha |g(\xi)|^\alpha \,\mathrm{d}\xi = \left|\sin\left(2^{-1}\overrightarrow{1}\cdot\xi\right)\right|^\alpha |g(\xi)|^\alpha. \qquad (2.3.55)
$$

Hence, It follows from (2.3.50) to (2.3.55) that we have, for almost all $\eta \in \mathbb{R}^d$,

$$\left|\sin\left(2^{-1}\overrightarrow{1}\cdot\eta\right)\right|^{\alpha}|f(\eta)|^{\alpha} = \left|\sin\left(2^{-1}\overrightarrow{1}\cdot\eta\right)\right|^{\alpha}|g(\eta)|^{\alpha}.$$

That is, for almost all $\eta \in \mathbb{R}^d$,

$$|f(\eta)| = |g(\eta)|.$$

□

2.4 Basic properties of these stable fields.

In the previous section, we have seen that the harmonizable stationary increments stable field $X[f]$ defined in (2.3.3) is closely connected to the modulus of the function f. More precisely, in Theorem 2.3.6 we have shown that the finite dimensional distributions of $X[f]$ are completely determined by $|f|$. In the present section, we show that one can derive properties of the field $X[f]$ from properties of $|f|$. We focus on the following properties: global self-similarity, local asymptotic self-similarity and isotropy.

Definition 2.4.1. *Let $\beta \in (0,+\infty)$ be fixed. A stochastic field $\{X(t),\, t \in \mathbb{R}^d\}$ is said to be globally self-similar of order β if, for all real numbers $\lambda \in (0,+\infty)$, the fields*

$$\left\{X(\lambda t),\, t \in \mathbb{R}^d\right\} \quad \text{and} \quad \left\{\lambda^{\beta} X(t),\, t \in \mathbb{R}^d\right\}$$

have the same distribution.

Definition 2.4.2. *Let $\gamma \in \mathbb{R}$ be fixed. A measurable function f is said to be positive homogeneous of order γ if, for any real number $\lambda \in (0,+\infty)$, the equality*

$$f(\lambda\xi) = \lambda^{\gamma} f(\xi)$$

holds for almost all $\xi \in \mathbb{R}^d$.

Proposition 2.4.3. *Let $\beta \in (0,+\infty)$ be fixed. Assume that f is an arbitrary function satisfying $(\mathcal{H}_0)$ and $X[f]$ is the field associated with f (see (2.3.3)). Then, $|f|$ is positive homogeneous of order $-\beta - d/\alpha \in \mathbb{R}$ if, and only if, $X[f]$ is self-similar of order β.*

Proof. In view of Theorem 2.3.6 and the linearity of the stable stochastic integral (see Proposition 2.1.7), it is enough to prove that, for any $\lambda \in (0,+\infty)$, the fields $\left\{X[f](\lambda t),\, t \in \mathbb{R}^d\right\}$ and $\left\{X[f_\lambda](t),\, t \in \mathbb{R}^d\right\}$ are identically distributed, where we have set

$$f_\lambda := \lambda^{-d/\alpha} f(\lambda^{-1}\cdot). \tag{2.4.1}$$

The fact that function f satisfies $(\mathcal{H}_0)$ implies that, for every $\lambda \in (0,+\infty)$, the function f_λ also satisfies $(\mathcal{H}_0)$; therefore, in view of Definition 2.3.2, the field $\left\{X[f_\lambda](t),\, t \in \mathbb{R}^d\right\}$ is well-defined. Assume that $\lambda \in (0,+\infty)$ is fixed. In view of Lemma 2.3.4, in order to show that the fields $\left\{X[f](\lambda t),\, t \in \mathbb{R}^d\right\}$ and $\left\{X[f_\lambda](t),\, t \in \mathbb{R}^d\right\}$ have the same distribution, it is enough to show that for all $N \in \mathbb{N}$, $b_1,\dots,b_N \in \mathbb{R}$ and $t^1,\dots,t^N \in \mathbb{R}^d$, we have

$$\mathbb{E}\left[\exp\left(i\sum_{l=1}^N b_l X[f]\Big(\lambda t^l\Big)\right)\right] = \mathbb{E}\left[\exp\left(i\sum_{l=1}^N b_l X[f_\lambda]\Big(t^l\Big)\right)\right]. \tag{2.4.2}$$

The linearity of the stable stochastic integral and (2.3.3) imply that

$$\sum_{l=1}^N b_l X[f]\Big(\lambda t^l\Big) = \mathcal{R}e\left(\int_{\mathbb{R}^d}\left(\sum_{l=1}^N b_l\Big(e^{i\lambda t^l\cdot\xi}-1\Big)\right) f(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right), \tag{2.4.3}$$

and

$$\sum_{l=1}^N b_l X[f_\lambda](t^l) = \mathcal{R}e\left(\int_{\mathbb{R}^d}\left(\sum_{l=1}^N b_l\Big(e^{it^l\cdot\xi}-1\Big)\right) f_\lambda(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right). \tag{2.4.4}$$

Combining (2.4.3), (2.4.4) and Proposition 2.1.7, the random variables

$$\sum_{l=1}^N b_l X[f]\Big(\lambda t^l\Big) \quad\text{and}\quad \sum_{l=1}^N b_l X[f_\lambda](t^l)$$

have real-valued symmetric α-stable distributions with scale parameters satisfying respectively

$$\sigma_1 := \int_{\mathbb{R}^d}\left|\sum_{l=1}^N b_l(e^{i\lambda t^l\cdot\xi}-1)\right|^\alpha |f(\xi)|^\alpha\,\mathrm{d}\xi \quad\text{and}\quad \sigma_2 := \int_{\mathbb{R}^d}\left|\sum_{l=1}^N b_l(e^{it^l\cdot\xi}-1)\right|^\alpha |f_\lambda(\xi)|^\alpha\,\mathrm{d}\xi. \tag{2.4.5}$$

Therefore, in view of (1.1.6) the equality (2.4.2) holds as soon as $\sigma_1=\sigma_2$. Making the change of variable $\eta = \lambda^{-1}\xi$ in the right-hand side of the second equality in (2.4.5), using (2.4.1) and using the fact that λ is positive, we get that

$$\sigma_2 = \lambda^{-d}\int_{\mathbb{R}^d}\left|\sum_{l=1}^N u_l(e^{it_l\cdot\xi}-1)\right|^\alpha \left|f(\lambda^{-1}\xi)\right|^\alpha\mathrm{d}\xi = \int_{\mathbb{R}^d}\left|\sum_{l=1}^N u_l(e^{i\lambda t_l\cdot\eta}-1)\right|^\alpha |f(\eta)|^\alpha\,\mathrm{d}\eta = \sigma_1. \tag{2.4.6}$$

□

Definition 2.4.4. *Let $\beta \in (0,1)$ be fixed. A stochastic field $\left\{X(t),\, t\in\mathbb{R}^d\right\}$ is said to be locally asymptotically self-similar of order β at a point $t_0 \in \mathbb{R}^d$ if the field*

$$\left\{\frac{X(t_0+\lambda t)-X(t_0)}{\lambda^\beta},\, t\in\mathbb{R}^d\right\}$$

converges in the sense of the finite dimensional distribution to a non trivial field as $\lambda \to 0$. The limit field is called the tangent field at the point t_0.

Definition 2.4.5. *Let $\gamma \in \mathbb{R}$ be fixed. A measurable function g is said to be asymptotically homogeneous of order γ at infinity if there exists a non zero function g_∞ such that, for almost every $\xi \in \mathbb{R}^d$,*

$$\lim_{\lambda \to +\infty} \lambda^\gamma g(\lambda \xi) = g_\infty(\xi). \tag{2.4.7}$$

Proposition 2.4.6. *Let $\beta \in (0,1)$. Assume that f is an arbitrary function satisfying $(\mathcal{H}_0)$ (see Definition 2.3.1) and that $X[f]$ is the field associated with f (see (2.3.3)). Also, assume that there exist two constants $A > 0$ and $c > 0$ such that the inequality*

$$|f(\xi)| \leq c \|\xi\|^{-\beta - d/\alpha}. \tag{2.4.8}$$

holds for almost all $\xi \in \mathbb{R}^d$ satisfying $\|\xi\| > A$. If f is asymptotically homogeneous of order $-\beta - d/\alpha \in \mathbb{R}$ at infinity with limit function f_∞ satisfying $(\mathcal{H}_0)$, then $X[f]$ is, at any point t_0, locally asymptotically self-similar of order β with tangent field $X[f_\infty]$.

Proof. Let $t_0 \in \mathbb{R}^d$ be fixed. We are going to show that the field

$$\left\{ \frac{X[f](t_0 + \lambda t) - X[f](t_0)}{\lambda^\beta}, t \in \mathbb{R}^d \right\}$$

converges in the sense of the finite dimensional distribution to $X[f_\infty]$ as $\lambda \to 0$. That is, for any $N \in \mathbb{N}$, $b_1, \ldots, b_N \in \mathbb{R}$ and $t^1, \ldots, t^N \in \mathbb{R}^d$ we have

$$\lim_{\lambda \to 0} \left(\sum_{l=1}^N b_l \frac{X[f](t_0 + \lambda t^l) - X[f](t_0)}{\lambda^\beta} \right) = \sum_{l=1}^N b_l X[f_\infty](t^l) \tag{2.4.9}$$

in distribution. For any $\lambda > 0$ we set

$$Y_1(\lambda) := \sum_{l=1}^N b_l \frac{X[f](t_0 + \lambda t^l) - X[f](t_0)}{\lambda^\beta} \quad \text{and} \quad Y_2 := \sum_{l=1}^N b_l X[f_\infty](t^l). \tag{2.4.10}$$

Observe that (2.3.3) and Proposition 2.1.7 imply that

$$Y_1(\lambda) \;=\; \mathcal{R}e \left\{ \int_{\mathbb{R}^d} \left(\sum_{l=1}^N b_l \lambda^{-\beta} \Big(e^{i(t_0 + \lambda t^l) \cdot \xi} - e^{i t_0 \cdot \xi} \Big) \right) f(\xi) \mathrm{d}\widetilde{M}_\alpha(\xi) \right\}. \tag{2.4.11}$$

Hence, $Y_1(\lambda)$ is a real-valued symmetric α-stable random variable with scale parameter $\sigma\Big(Y_1(\lambda)\Big)$ satisfying

$$\sigma\Big(Y_1(\lambda)\Big)^\alpha = \int_{\mathbb{R}^d} \left| \sum_{l=1}^N b_l \lambda^{-\beta} \Big(e^{i \lambda t^l \cdot \xi} - 1 \Big) \right|^\alpha |f(\xi)|^\alpha \, \mathrm{d}\xi. \tag{2.4.12}$$

Similarly, Y_2 is a real-valued symmetric α-stable random with scale parameter $\sigma(Y_2)$ satisfying

$$\sigma(Y_2) = \int_{\mathbb{R}^d} \left| \sum_{l=1}^{N} b_l \Big(e^{it^l \cdot \xi} - 1 \Big) \right|^{\alpha} |f_\infty(\xi)|^\alpha \, \mathrm{d}\xi. \tag{2.4.13}$$

In view of (1.1.6) and (2.1.17), we have that (2.4.9) holds as soon as

$$\lim_{\lambda \to 0^+} \sigma\Big(Y_1(\lambda) \Big) = \sigma(Y_2) \tag{2.4.14}$$

Notice that we can express $\sigma\Big(Y_1(\lambda)\Big)^\alpha$ as $I_1(\lambda) + I_2(\lambda)$ where,

$$I_1(\lambda) := \int_{\{\|\xi\| \le A\}} \left| \sum_{l=1}^{N} b_l \lambda^{-\beta} \Big(e^{i\lambda t^l \cdot \xi} - 1 \Big) \right|^{\alpha} |f(\xi)|^\alpha \, \mathrm{d}\xi \tag{2.4.15}$$

and

$$I_2(\lambda) := \int_{\{\|\xi\| > A\}} \left| \sum_{l=1}^{N} b_l \lambda^{-\beta} \Big(e^{i\lambda t^l \cdot \xi} - 1 \Big) \right|^{\alpha} |f(\xi)|^\alpha \, \mathrm{d}\xi. \tag{2.4.16}$$

We will show that

$$\lim_{\lambda \to 0^+} I_1(\lambda) = 0 \quad \text{and} \quad \lim_{\lambda \to 0^+} I_2(\lambda) = \sigma(Y_2). \tag{2.4.17}$$

Let us first establish the first equality in (2.4.17). It follows from the inequality $|e^{ix} - 1| \le |x|$, for all $x \in \mathbb{R}$, that

$$\begin{aligned} I_1(\lambda) &\le \int_{\{\|\xi\| \le A\}} \left| \sum_{l=1}^{N} |b_l| \, |\lambda|^{-\beta} \left| e^{i\lambda t^l \cdot \xi} - 1 \right| \right|^{\alpha} |f(\xi)|^\alpha \, \mathrm{d}\xi \\ &\le c_1(t^1, \ldots, t^N, b_1, \ldots, b_N) \, |\lambda|^{\alpha(1-\beta)}, \end{aligned} \tag{2.4.18}$$

where the positive constant $c_1(t^1, \ldots, t^N, b_1, \ldots, b_N)$, defined as

$$c_1(t^1, \ldots, t^N, b_1, \ldots, b_N) := \left| \sum_{l=1}^{N} |b_l| \, \|t_l\| \right|^{\alpha} \int_{\{\|\xi\| \le A\}} \|\xi\|^\alpha \, |f(\xi)|^\alpha \, \mathrm{d}\xi,$$

is finite because f satisfies (2.3.1). Next using (2.4.18) and the fact that $\beta \in (0,1)$ we obtain the first equality in (2.4.17). Let us now establish the second equality in (2.4.17). Making the change of variable $\eta = \lambda\xi$ in (2.4.16), we get that

$$I_2(\lambda) = \int_{\mathbb{R}^d} \left| \sum_{l=1}^{N} b_l (e^{it^l \cdot \eta} - 1) \right|^{\alpha} \left| \lambda^{-\beta - d/\alpha} f(\lambda^{-1}\eta) \right|^{\alpha} \mathbb{1}_{\{\|\eta\| > \lambda A\}} \mathrm{d}\eta. \tag{2.4.19}$$

As f is asymptotically homogeneous with limit function f_∞ (see (2.4.7)), we have, for almost all $\eta \in \mathbb{R}^d$,

$$\lim_{\lambda \to 0^+} \left| \sum_{l=1}^{N} b_l (e^{it^l \cdot \eta} - 1) \right|^{\alpha} \left| \lambda^{-\beta - d/\alpha} f(\lambda^{-1}\eta) \right|^{\alpha} \mathbb{1}_{\{\|\eta\| > \lambda A\}} = \left| \sum_{l=1}^{N} b_l (e^{it^l \cdot \eta} - 1) \right|^{\alpha} \Big| f_\infty(\eta) \Big|^{\alpha}. \tag{2.4.20}$$

Moreover, (2.4.8) entails that, for almost all $\eta \in \mathbb{R}^d$,

$$\begin{aligned}
&\left|\sum_{l=1}^{N} b_l(e^{it^l\cdot\eta}-1)\right|^{\alpha}\left|\lambda^{-\beta-d/\alpha}f(\lambda^{-1}\eta)\right|^{\alpha}\mathbb{1}_{\{\|\eta\|>\lambda A\}}\\
&\le\left|\sum_{l=1}^{N}2\,|b_l|\min\left\{\left|t^l\cdot\eta\right|,1\right\}\right|^{\alpha}\left|\lambda^{-\beta-d/\alpha}\left\|\lambda^{-1}\eta\right\|^{-\beta-d/\alpha}\right|^{\alpha}\mathbb{1}_{\{\|\eta\|>\lambda A\}}\\
&\le\left|\sum_{l=1}^{N}2\,|b_l|\max_{l=1,\dots,d}\|t_l\|\right|^{\alpha}\min\left\{\|\eta\|,1\right\}^{\alpha}\|\eta\|^{-\beta\alpha-d}. \qquad (2.4.21)
\end{aligned}$$

Using the fact that $\beta \in (0,1)$, (2.4.19), (2.4.20), (2.4.21) and the Dominated Convergence Theorem, we get the second equality in (2.4.17) holds.
Finally, (2.4.15), (2.4.16) and (2.4.17) imply that (2.4.14) holds. □

Definition 2.4.7. *A stochastic field $\left\{X(t),\, t\in\mathbb{R}^d\right\}$ is said to be isotropic if, for each rotation R of $\mathbb{R}^d$ (by a rotation of $\mathbb{R}^d$ we mean a linear map from $\mathbb{R}^d$ into itself such that the matrix M_R associated with this map is an orthogonal matrix, that is $M_R M_R^* = \mathrm{Id}$ where M_R^* is the transpose of M_R and Id is the identity matrix, with determinant equal to 1), the fields*

$$\left\{X(t),\, t\in\mathbb{R}^d\right\} \quad \textit{and} \quad \left\{X(R(t)),\, t\in\mathbb{R}^d\right\}$$

have the same distribution.

Definition 2.4.8. *A measurable function f is said to be rotationally invariant if for any rotation R of $\mathbb{R}^d$, the equality*

$$(f\circ R)(\xi) = f(\xi)$$

holds for almost all $\xi \in \mathbb{R}^d$, where the symbol $f\circ R$ denotes the composition of f with R.

Proposition 2.4.9. *Let f be an arbitrary function satisfying $(\mathcal{H}_0)$ (see Definition 2.3.1) and $X[f]$ be the field associated with f (see (2.3.3)). Then, $X[f]$ is isotropic if, and only if, $|f|$ is rotationally invariant.*

Proof. In view of Theorem 2.3.6, it is enough to prove that, for any rotation R of $\mathbb{R}^d$, the fields

$$\left\{X[f]\big(R(t)\big),\, t\in\mathbb{R}^d\right\} \quad \text{and} \quad \left\{X[f\circ R](t),\, t\in\mathbb{R}^d\right\}$$

are identically distributed. In view of the fact that f satisfies $(\mathcal{H}_0)$ and that R is a rotation, the function $f\circ R$ also satisfies $(\mathcal{H}_0)$; therefore the field $\left\{X[f\circ R](t),\, t\in\mathbb{R}^d\right\}$ is well-defined (see Definition 2.3.2). Assume that R is an arbitrary rotation of $\mathbb{R}^d$. In view of Lemma 2.3.4, in order to show that the fields

$$\left\{X[f]\big(R(t)\big),\, t\in\mathbb{R}^d\right\} \quad \text{and} \quad \left\{X[f\circ R](t),\, t\in\mathbb{R}^d\right\}$$

have the same distribution, it is enough to show that, for all $N \in \mathbb{N}$, $b_1, \dots, b_N \in \mathbb{R}$ and $t^1, \dots, t^N \in \mathbb{R}^d$ we have

$$\mathbb{E}\left[\exp\left(i\sum_{l=1}^{N} b_l X[f]\Big(R\big(t^l\big)\Big)\right)\right] = \mathbb{E}\left[\exp\left(i\sum_{l=1}^{N} b_l X[f]\big(t^l\big)\right)\right]. \tag{2.4.22}$$

The linearity of the stable stochastic integral (see Proposition 2.1.7) and (2.3.3) entail that

$$\sum_{l=1}^{N} b_l X[f]\Big(R\big(t^l\big)\Big) = \mathcal{R}e\left(\int_{\mathbb{R}^d}\left(\sum_{l=1}^{N} b_l(e^{iR(t^l)\cdot\xi} - 1)\right) f(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right), \tag{2.4.23}$$

and

$$\sum_{l=1}^{N} u_l X[f\circ R]\big(t^l\big) = \mathcal{R}e\left(\int_{\mathbb{R}^d}\left(\sum_{l=1}^{N} u_l(e^{it^l\cdot\xi} - 1)\right) f\Big(R(\xi)\Big)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right). \tag{2.4.24}$$

Combining (2.4.23), (2.4.24) and Proposition 2.1.7, the random variables

$$\sum_{l=1}^{N} b_l X[f]\Big(R\big(t^l\big)\Big) \quad \text{and} \quad \sum_{l=1}^{N} b_l X[f\circ R]\big(t^l\big)$$

have a real-valued symmetric α-stable distribution with scale parameter satisfying respectively

$$\sigma_1 := \int_{\mathbb{R}^d}\left|\sum_{l=1}^{N} b_l(e^{iR(t^l)\cdot\xi} - 1)\right|^\alpha |f(\xi)|^\alpha\,\mathrm{d}\xi \tag{2.4.25}$$

and

$$\sigma_2 := \int_{\mathbb{R}^d}\left|\sum_{l=1}^{N} b_l(e^{it^l\cdot\xi} - 1)\right|^\alpha \left|f\Big(R(\xi)\Big)\right|^\alpha\,\mathrm{d}\xi. \tag{2.4.26}$$

Therefore, in view of (1.1.6) the equality (2.4.22) holds as soon as $\sigma_1 = \sigma_2$. Applying the change of variable $\eta = R(\xi)$ in (2.4.26) and using the fact that R is a rotation of $\mathbb{R}^d$, we get that

$$\sigma_2 = \int_{\mathbb{R}^d}\left|\sum_{l=1}^{N} b_l(e^{it^l\cdot R^{-1}(\eta)} - 1)\right|^\alpha |f(\eta)|^\alpha\,\mathrm{d}\eta = \int_{\mathbb{R}^d}\left|\sum_{l=1}^{N} b_l(e^{iR(t^l)\cdot\eta} - 1)\right|^\alpha |f(\eta)|^\alpha\,\mathrm{d}\eta = \sigma_1. \tag{2.4.27}$$

□

3 Wavelet type random series representation

Abstract

In this chapter, we introduce a wavelet type random series representation for the field $X[f]$ in which each canonical axis l of $\mathbb{R}^d$ has its own dilatation index j_l; such an additional degree of freedom with respect to the classical wavelet frame allows better analysis of anisotropy of $X[f]$. Moreover, we express the wavelet type random series representation of $X[f]$ as the finite sum $X[f] = \sum_\eta X[f]^\eta$, where the fields $X[f]^\eta$ are called the η-frequency parts, since they extend the usual low-frequency and high-frequency parts. Then, we show that the sample paths of all the $X[f]^\eta$'s are continuous on $\mathbb{R}^d$, and we connect the existence and continuity of their partial derivative, of an arbitrary order, with the rates of vanishing at infinity of the spectral density along the axes i.e. with the exponents $a_1[f], \dots, a_d[f]$ in (3.1.3)

Contents

3.1 The class of admissible functions

Let f be an arbitrary function satisfying $(\mathcal{H}_0)$ (see Definition 2.3.1) and $X[f]$ be the field associated with f (see (2.3.3)). Typically, $X[f]$ is an anisotropic model when the rate of vanishing at infinity of the corresponding spectral density $|f|^\alpha$ changes from one axis of $\mathbb{R}^d$ to another; therefore, we focus on the class of the so-called admissible functions f, defined in the following way.

Definition 3.1.1. *Let $\lfloor 1/\alpha \rfloor$ be the integer part of $1/\alpha$, the inverse of the stability parameter $\alpha \in (0,2]$. We set*

$$p_* := \max\Big\{2, \lfloor 1/\alpha \rfloor + 1\Big\}. \tag{3.1.1}$$

The function f in (2.3.3) is said to be admissible *when it satisfies $(\mathcal{H}_0)$ (see Definition 2.3.1) and the following three conditions.*

$(\mathcal{H}_1)$ *For all multi-index $p := (p_1, p_2, \dots, p_d) \in \Big\{0,1,2,\dots,p_*\Big\}^d$, the partial derivative function*

$$\partial^p f := \frac{\partial^{p_1}\partial^{p_2}\dots\partial^{p_d}}{(\partial\xi_1)^{p_1}(\partial\xi_2)^{p_2}\dots(\partial\xi_d)^{p_d}} f \qquad \textit{(with the convention that } \partial^0 f := f\textit{)}$$

is well-defined and continuous on the open set $\Big(\mathbb{R}\setminus\{0\}\Big)^d$; that is the Cartesian product of $\mathbb{R}\setminus\{0\}$ with itself d times.

$(\mathcal{H}_2)$ *There are a positive constant c' and an exponent $a'[f] \in [0,1)$ such that, for each $p \in \Big\{0,1,2,\dots,p_*\Big\}^d$, and $\xi \in \Big(\mathbb{R}\setminus\{0\}\Big)^d$,*

$$\|\xi\| \le \frac{8\pi}{3}\sqrt{d} \Longrightarrow \Big|\partial^p f(\xi)\Big| \le c' \,\|\xi\|^{-a'[f]-d/\alpha-\mathrm{l}(p)}, \tag{3.1.2}$$

where $\mathrm{l}(p) := p_1 + p_2 + \dots + p_d$ is the length of the multi-index p.

$(\mathcal{H}_3)$ *There exist a positive constant c and d positive exponents $a_1[f], \dots, a_d[f]$ such that for every $p \in \Big\{0,1,2,\dots,p_*\Big\}^d$, and $\xi \in \Big(\mathbb{R}\setminus\{0\}\Big)^d$,*

$$\|\xi\| \ge \frac{2\pi}{3} \Longrightarrow \Big|\partial^p f(\xi)\Big| \le c\prod_{l=1}^d (1+|\xi_l|)^{-a_l[f]-1/\alpha-p_l}. \tag{3.1.3}$$

Remark 3.1.2. *When f is an admissible function, it is clear that the conditions $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ implies that (2.3.1) holds. Also notice that in (3.1.2) and (3.1.3), the quantities $8\pi\sqrt{d}/3$ and $2\pi/3$ can be replaced by any other fixed positive quantities. More importantly, notice that many functions belong to the admissible class, as, for instance, the function*

$$\xi = (\xi_1,\dots,\xi_d) \longmapsto \left(\sum_{l=1}^d \xi_l^2\right)^{-(u+d/\alpha)/2} \times \prod_{l=1}^d \Big(1+|\xi_l|\Big)^{-v_l},$$

where $u \in (0,1)$ and $v_1,\dots,v_d \in [0,+\infty)$ are arbitrary fixed parameters.

3.2 Wavelet type random series representation

In the general case, where the stability parameter $\alpha \in (0, 2]$ is arbitrary, the strategy, allowing to obtain the wavelet type random series representation of $\{X[f](t), t \in \mathbb{R}^d\}$, that we are looking for, follows, more or less, the main steps as in the Gaussian case, where $\alpha = 2$; yet, the arguments of their proofs have to be significantly modified in order to fit with the general case. First, we intend to present these main steps in a rather heuristic way, by avoiding, as far as possible, to be technical. This is why we restrict, for the time being, our presentation to the Gaussian case which is less difficult to understand than the general one.

We denote by $\left\{\psi_{J,K} : (J, K) \in \mathbb{Z}^d \times \mathbb{Z}^d\right\}$ the orthonormal basis of $L^2(\mathbb{R}^d)$ defined in the following way: for all $(J, K) := (j_1, \ldots, j_d, k_1, \ldots, k_d) \in \mathbb{Z}^d \times \mathbb{Z}^d$ and $x := (x_1, \ldots, x_d) \in \mathbb{R}^d$

$$\psi_{J,K}(x) := \prod_{l=1}^{d} 2^{j_l/2}\psi^1(2^{j_l}x_l - k_l), \tag{3.2.1}$$

where ψ^1 denotes an usual 1D Lemarié-Meyer mother wavelet. We refer to the books of Meyer [24, 25] and to that of Daubechies [11] for a complete description of the wavelet tools used in the present section. It is worthwhile noting that ψ^1 is a real-valued function belonging to the Schwartz class $S(\mathbb{R})$; that is the space of complex-valued C^∞ functions on $\mathbb{R}$ having rapidly decreasing derivatives at any order. Also, we mention that the Fourier transform of ψ^1, denoted by $\widehat{\psi^1}$, is a compactly supported C^∞ function on $\mathbb{R}$, such that

$$\operatorname{supp}\widehat{\psi^1} \subseteq \mathcal{K} := \left\{\lambda \in \mathbb{R} : \frac{2\pi}{3} \leq |\lambda| \leq \frac{8\pi}{3}\right\}. \tag{3.2.2}$$

Observe that it follows from (3.2.1) and elementary properties of the Fourier transform that, for any $\xi \in \mathbb{R}^d$,

$$\widehat{\psi}_{J,K}(\xi) = \prod_{l=1}^{d} 2^{-j_l/2} e^{-i2^{-j_l}k_l\xi_l}\,\widehat{\psi^1}(2^{-j_l}\xi_l). \tag{3.2.3}$$

Therefore combining (3.2.2) and (3.2.3) one gets that

$$\operatorname{supp}\widehat{\psi}_{J,K} \subset \left\{\xi \in \mathbb{R}^{\mathrm{d}} : \text{for all } l = 1, \ldots, d \text{ one has } \frac{2^{j_l+1}\pi}{3} \leq |\xi_l| \leq \frac{2^{j_l+3}\pi}{3}\right\}; \tag{3.2.4}$$

this inclusion will be very useful for us.

Next notice that (2.3.1) and the assumption $\alpha = 2$ imply that, for any fixed $t \in \mathbb{R}^d$, the function $\xi \mapsto \left(e^{it\cdot\xi} - 1\right)f(\xi)$ belongs to $L^2(\mathbb{R}^d)$. Therefore, it can be expressed as

$$\left(e^{it\cdot\xi} - 1\right)f(\xi) = \sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} s_{J,K}[f](t)\overline{\widehat{\psi}_{J,K}(\xi)}, \tag{3.2.5}$$

where

$$s_{J,K}[f](t) := \int_{\mathbb{R}^d} \left(e^{it\cdot\xi} - 1\right) f(\xi)\widehat{\psi}_{J,K}(\xi)\,\mathrm{d}\xi, \tag{3.2.6}$$

and $\overline{\widehat{\psi}_{J,K}(\xi)}$ denotes the complex conjugate of $\widehat{\psi}_{J,K}(\xi)$; observe that, at this stage, the right-hand side in (3.2.5), has to be viewed as a series of functions, of the variable ξ, which converges in the $L^2\Big(\mathbb{R}^d\Big)$ norm. Now, denote by $\Psi_J[f]$ the real-valued function defined, for all $x \in \mathbb{R}^d$, as

$$\Psi_J[f](x) := 2^{(j_1+\cdots+j_d)/2} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f\left(2^J\xi\right)\widehat{\psi}_{0,0}(\xi)\mathrm{d}\xi, \tag{3.2.7}$$

with the convention [1] that $2^J\xi := (2^{j_1}\xi_1, \ldots, 2^{j_d}\xi_d)$. It can easily be derived from (3.2.3), (3.2.6) and (3.2.7) that

$$s_{J,K}[f](t) = \Psi_J[f]\Big(2^J t - K\Big) - \Psi_J[f](-K)\,. \tag{3.2.8}$$

Then, it results from (3.2.5), (3.2.8) and (2.3.3) (with $\alpha = 2$) that

$$X[f](t) = \mathcal{R}e\left\{\int_{\mathbb{R}^d}\left(\sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}\Big(\Psi_J[f]\Big(2^J t - K\Big) - \Psi_J[f](-K)\Big)\overline{\widehat{\psi}_{J,K}(\xi)}\right)\mathrm{d}\widetilde{M}_2(\xi)\right\}. \tag{3.2.9}$$

Finally, in view of (2.1.17), it turns out that, roughly speaking, one can interchange in (3.2.9) the integration and the summation. Thus, we get that

$$X[f](t) = \sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}\Big(\Psi_J[f]\Big(2^J t - K\Big) - \Psi_J[f](-K)\Big)\varepsilon_{J,K}, \tag{3.2.10}$$

where the $\varepsilon_{J,K}$'s are the centered real-valued Gaussian random variables defined as

$$\varepsilon_{J,K} := \mathcal{R}e\left\{\int_{\mathbb{R}^d}\overline{\widehat{\psi}_{J,K}(\xi)}\,\mathrm{d}\widetilde{M}_2(\xi)\right\}.$$

Having presented, in the Gaussian case $\alpha = 2$, the main steps of the strategy allowing to obtain the wavelet type random series representation (3.2.10) of $\{X[f](t), t \in \mathbb{R}^d\}$; from now on we assume that $\alpha \in (0, 2]$ is arbitrary, and that the function f in (2.3.3) is any admissible function in the sense of Definition 3.1.1. Our present goal is to show that the strategy previously employed, in the Gaussian case, for deriving (3.2.10), can be extended to the general case. To this end, the arguments, we have used in the "convenient" framework of the Hilbert space $L^2(\mathbb{R}^d)$, have to be adapted to the "more hostile" framework of the space $L^\alpha\Big(\mathbb{R}^d\Big)$. First we mention that:

Remark 3.2.1. *The space $L^\alpha\Big(\mathbb{R}^d\Big)$ is defined as the space of the Lebesgue measurable complex-valued functions g on $\mathbb{R}^d$, such that*

$$\|g\|_{L^\alpha\left(\mathbb{R}^d\right)} := \left(\int_{\mathbb{R}^d}|g(\xi)|^\alpha\,\mathrm{d}\xi\right)^{1/\alpha} < +\infty. \tag{3.2.11}$$

[1] Notice that such a convention will be extensively used in all the rest of our article, without being recalled.

When $\alpha \in [1,2]$, it is well-known that $\|\cdot\|_{L^\alpha(\mathbb{R}^d)}$ is a norm on $L^\alpha\big(\mathbb{R}^d\big)$ confering to it the structure of a Banach space; the associated distance is

$$\Delta_\alpha(g_1,g_2) := \|g_1-g_2\|_{L^\alpha(\mathbb{R}^d)}. \tag{3.2.12}$$

When $\alpha \in (0,1)$, the definition of the distance Δ_α has to be slightly modified since $\|\cdot\|_{L^\alpha(\mathbb{R}^d)}$ is no longer a norm but only a quasi-norm [2]*. More precisely, Δ_α has to be defined as*

$$\Delta_\alpha(g_1,g_2) := \int_{\mathbb{R}^d} |g_1(\xi)-g_2(\xi)|^\alpha \,\mathrm{d}\xi, \tag{3.2.13}$$

and then $L^\alpha\big(\mathbb{R}^d\big)$ equipped with this distance is a complete metric space. Observe that for any $\alpha \in (0,2]$, Δ_α is invariant under translations, that is for all g_1, g_2, and g_3 in $L^\alpha\big(\mathbb{R}^d\big)$, one has $\Delta_\alpha(g_1+g_3,g_2+g_3)=\Delta_\alpha(g_1,g_2)$.

Let us now come back to our goal. Rather than directly working with the functions $\widehat{\psi}_{J,K}$ (see (3.2.3)), it is more convenient to work with their renormalized versions $\widehat{\psi}_{\alpha,J,K}$ defined, for all $(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d$ and $\xi\in\mathbb{R}^d$, as

$$\widehat{\psi}_{\alpha,J,K}(\xi) := 2^{(j_1+\cdots+j_d)(1/2-1/\alpha)}\,\widehat{\psi}_{J,K}(\xi) = \prod_{l=1}^{d} 2^{-j_l/\alpha} e^{-i2^{-j_l}k_l\xi_l}\,\widehat{\psi^1}(2^{-j_l}\xi_l); \tag{3.2.14}$$

it is clear that, similarly to $\widehat{\psi}_{J,K}$, the function $\widehat{\psi}_{\alpha,J,K}$ is C^∞ on $\mathbb{R}^d$ with a compact support satisfying

$$\operatorname{supp}\widehat{\psi}_{\alpha,\,J,\,K} \subset \left\{\xi\in\mathbb{R}^{\mathrm{d}} : \text{for all } l=1,\ldots,d \text{ one has } \frac{2^{j_l+1}\pi}{3} \le |\xi_l| \le \frac{2^{j_l+3}\pi}{3}\right\}. \tag{3.2.15}$$

The advantage offered by this renormalization is that the (quasi)-norm $\big\|\widehat{\psi}_{\alpha,J,K}\big\|_{L^\alpha(\mathbb{R}^d)}$ does not depend on (J,K), in other words,

$$\big\|\widehat{\psi}_{\alpha,J,K}\big\|_{L^\alpha(\mathbb{R}^d)} = \big\|\widehat{\psi}_{\alpha,0,0}\big\|_{L^\alpha(\mathbb{R}^d)} = \big\|\widehat{\psi^1}\big\|^d_{L^\alpha(\mathbb{R})}. \tag{3.2.16}$$

Therefore, the real-valued symmetric α-stable random variables $\varepsilon_{\alpha,J,K}$ defined, for all $(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d$, as

$$\varepsilon_{\alpha,J,K} := \mathcal{R}e\left\{\int_{\mathbb{R}^d}\overline{\widehat{\psi}_{\alpha,J,K}(\xi)}\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right\}, \tag{3.2.17}$$

have the same distribution.

[2]The difference between a norm and a quasi-norm is that for a quasi-norm the triangle inequality is weakened to $\|g+h\|\le c\big(\|g\|+\|h\|\big)$, where c is a finite constant strictly bigger than 1.

The function $\Psi_{\alpha,J}[f]$ denotes the renormalized version of $\Psi_J[f]$ (see (3.2.7)), such that, for all $x \in \mathbb{R}^d$,

$$\Psi_{\alpha,J}[f](x) = 2^{(j_1+\cdots+j_d)(1/\alpha-1/2)}\Psi_J(x) = 2^{(j_1+\cdots+j_d)/\alpha}\int_{\mathbb{R}^d} e^{ix\cdot\xi} f(2^J\xi)\widehat{\psi}_{0,0}(\xi)\mathrm{d}\xi. \tag{3.2.18}$$

In view of (3.2.14) and (3.2.18), it can easily be seen that, for every $(J,K) \in \mathbb{Z}^d \times \mathbb{Z}^d$ and $(t,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$, one has

$$\Big(\Psi_J[f]\Big(2^J t - K\Big) - \Psi_J[f](-K)\Big)\overline{\widehat{\psi}_{J,K}(\xi)} = \Big(\Psi_{\alpha,J}[f]\Big(2^J t - K\Big) - \Psi_{\alpha,J}[f](-K)\Big)\overline{\widehat{\psi}_{\alpha,J,K}(\xi)}. \tag{3.2.19}$$

The following proposition explains, in a precise way, how the crucial equality (3.2.5) can be extended to the general case where $\alpha \in (0,2]$ is arbitrary.

Proposition 3.2.2. *Assume that f is admissible in the sense of Definition 3.1.1, and denote by F the function defined, for all $(t,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$, as,*

$$F(t,\xi) := \Big(e^{it\cdot\xi} - 1\Big)f(\xi). \tag{3.2.20}$$

Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be an arbitrary increasing (in the sense of the inclusion) sequence of finite subsets of $\mathbb{Z}^d \times \mathbb{Z}^d$ which satisfies $\bigcup_{n\in\mathbb{N}} \mathcal{D}_n = \mathbb{Z}^d \times \mathbb{Z}^d$. Then, for every fixed $t \in \mathbb{R}^d$, one has

$$\lim_{n\to+\infty} \Delta_\alpha\left(\sum_{(J,K)\in\mathcal{D}_n} \Big(\Psi_{\alpha,J}[f](2^J t - K) - \Psi_{\alpha,J}[f](-K)\Big)\overline{\widehat{\psi}_{\alpha,J,K}(\cdot)}, F(t,\cdot)\right) = 0, \tag{3.2.21}$$

where $\Psi_{\alpha,J}[f]$ and $\widehat{\psi}_{\alpha,J,K}$ are as in (3.2.18) and (3.2.14).

The following proposition is a straightforward consequence of Proposition 3.2.2, Remark 3.2.1, (2.1.17), (2.3.3) and (3.2.17). In some sense, it shows that similarly to the Gaussian case (see (3.2.10)), a wavelet type random series representation of the field $\{X[f](t), t \in \mathbb{R}^d\}$ can be obtained in the general case where $\alpha \in (0,2]$ is arbitrary.

Proposition 3.2.3. *Assume that $t \in \mathbb{R}^d$ is arbitrary and fixed. Let $X[f](t)$ be the real-valued symmetric α-stable random variable defined through (2.3.3), where f is supposed to be any admissible function in the sense of Definition 3.1.1. Denote by $(\mathcal{D}_n)_{n\in\mathbb{N}}$ an arbitrary increasing sequence of finite subsets of $\mathbb{Z}^d \times \mathbb{Z}^d$ which satisfies $\bigcup_{n\in\mathbb{N}} \mathcal{D}_n = \mathbb{Z}^d \times \mathbb{Z}^d$. For every fixed $n \in \mathbb{N}$, let $X[f]_n^{\mathcal{D}}(t)$ be the real-valued symmetric α-stable random variable defined as*

$$X[f]_n^{\mathcal{D}}(t) := \sum_{(J,K)\in\mathcal{D}_n} \Big(\Psi_{\alpha,J}[f](2^J t - K) - \Psi_{\alpha,J}[f](-K)\Big)\varepsilon_{\alpha,J,K}, \tag{3.2.22}$$

where $\Psi_{\alpha,J}[f]$ and $\varepsilon_{\alpha,J,K}$ are as in (3.2.18) and (3.2.17). Then, the sequence $(X[f]_n^{\mathcal{D}}(t))_{n\in\mathbb{N}}$ converges in probability to $X[f](t)$.

Proposition 3.2.2 is proved in the section 3.4; we mention that the three main ingredients of its proof are the following two lemmas and Proposition 3.2.6 given below.

Lemma 3.2.4. *Let $\alpha \in (0,2]$ be arbitrary and fixed. Assume that $(g_i)_{i\in\mathbb{Z}^d\times\mathbb{Z}^d}$ is a sequence of functions of $L^\alpha\Big(\mathbb{R}^d\Big)$ which satisfies,*

$$\sum_{i\in\mathbb{Z}^d\times\mathbb{Z}^d} \Delta_\alpha\left(g_i, 0\right) < +\infty. \tag{3.2.23}$$

Then there exists a function $g \in L^\alpha\Big(\mathbb{R}^d\Big)$ such that one has,

$$\lim_{n\to+\infty} \Delta_\alpha\left(\sum_{i\in\mathcal{D}_n} g_i, g\right) = 0, \tag{3.2.24}$$

where $(\mathcal{D}_n)_{n\in\mathbb{N}}$ denotes any arbitrary increasing sequence of finite subsets of $\mathbb{Z}^d\times\mathbb{Z}^d$ satisfying $\bigcup_{n\in\mathbb{N}}\mathcal{D}_n = \mathbb{Z}^d\times\mathbb{Z}^d$; observe that g does not depend on the choice of this sequence of subsets.

The proof of Lemma 3.2.4 is rather classical; it mainly relies on the completeness of $L^\alpha(\mathbb{R}^d)$, the triangle inequality and the fact that the distance Δ_α is invariant under translations. It does not present major difficulties, this is why it has been omitted.

Lemma 3.2.5. *Assume that the real numbers $a' \in [0,1)$, $\alpha \in (0,2]$, and $\delta > 0$ are arbitrary and fixed. Then, for all fixed $r \in \{1,\dots,d\}$, one has*

$$\sum_{J\in\mathbb{Z}_+^d} 2^{-j_r(1-a')}\left(2^{-j_1}+\cdots+2^{-j_d}\right)^{-d/\alpha}\prod_{l=1}^d 2^{-j_l/\alpha}\sqrt{\log\left(3+j_l\right)}(1+j_l)^{1/\alpha+\delta} < +\infty; \tag{3.2.25}$$

which clearly implies that

$$\sum_{J\in\mathbb{Z}_+^d} 2^{-j_r(1-a')}\left(2^{-j_1}+\cdots+2^{-j_d}\right)^{-d/\alpha}\prod_{l=1}^d 2^{-j_l/\alpha}(1+j_l)^{1/\alpha+\delta} < +\infty \tag{3.2.26}$$

and

$$\sum_{J\in\mathbb{Z}_+^d} 2^{-j_r(1-a')}\left(2^{-j_1}+\cdots+2^{-j_d}\right)^{-d/\alpha}\prod_{l=1}^d 2^{-j_l/\alpha}\sqrt{\log\left(3+j_l\right)} < +\infty. \tag{3.2.27}$$

Lemma 3.2.5 is proved in the section 3.4.

For later purposes, we denote by Υ and Υ^* the two sets defined as,

$$\Upsilon := \{0,1\}^d \text{ and } \Upsilon^* := \{0,1\}^d \setminus \{(0,\dots,0)\}. \tag{3.2.28}$$

Also, for any fixed $\eta = (\eta_1,\dots,\eta_d) \in \Upsilon$, we denote by $\mathbb{Z}^d_{(\eta)}$ the subset of $\mathbb{Z}^d$ defined as the Cartesian product

$$\mathbb{Z}^d_{(\eta)} := \prod_{l=1}^d \mathbb{Z}_{\eta_l}, \tag{3.2.29}$$

where

$$\mathbb{Z}_1 := \mathbb{N} = \{1, 2, \dots\} \quad \text{and} \quad \mathbb{Z}_0 := \mathbb{Z}_- = \{\dots, -2, -1, 0\}. \tag{3.2.30}$$

Notice that

$$\mathbb{Z}^d = \bigcup_{\eta \in \Upsilon} \mathbb{Z}^d_{(\eta)}, \quad \text{and} \quad Z^d_{(\eta)} \cap \mathbb{Z}^d_{(\eta')} = \emptyset \quad \text{when } \eta \neq \eta'. \tag{3.2.31}$$

Proposition 3.2.6. *For all $J \in \mathbb{Z}^d$, let $\Psi_{\alpha,J}[f]$ be the function defined through* (3.2.18)*, where f is any admissible function in the sense of Definition 3.1.1. Then $\Psi_{\alpha,J}[f]$ is infinitely differentiable on $\mathbb{R}^d$. Also, its partial derivatives are such that, for all $b \in \mathbb{Z}^d_+$ and $x \in \mathbb{R}^d$,*

$$\partial^b(\Psi_{\alpha,J}[f])(x) = 2^{(j_1+\cdots+j_d)/\alpha}\, i^{\mathrm{l}(b)} \int_{\mathbb{R}^d} e^{ix\cdot\xi}\, \xi^b f(2^J\xi)\widehat{\psi}_{0,0}(\xi)\mathrm{d}\xi, \tag{3.2.32}$$

where $\xi^b := \prod_{l=1}^d \xi_l^{b_l}$ and $\mathrm{l}(b) := \sum_{l=1}^d b_l$ is the length of b. Moreover, the $\partial^b(\Psi_{\alpha,J}[f])$'s, $b \in \mathbb{Z}^d_+$, are well-localized functions, in the sense that they satisfy the following two properties, where p_* is as in (3.1.1).

(i) For each $T > 0$, and $b \in \mathbb{Z}^d_+$, there is a positive constant c, such that for all $J \in \mathbb{Z}^d_+$, and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$,

$$\left|\partial^b(\Psi_{\alpha,-J}[f])(x)\right| \le c\, \frac{\left(2^{-j_1} + \cdots + 2^{-j_d}\right)^{-a'[f]-d/\alpha} \prod_{l=1}^d 2^{-j_l/\alpha}}{\prod_{l=1}^d \left(1 + T + |x_l|\right)^{p_*}}, \tag{3.2.33}$$

where the exponent $a'[f] \in [0, 1)$ and p_* are as in Definition 3.1.1.

(ii) For every $T > 0$, $\eta \in \Upsilon^*$ (see (3.2.28)*), and $b \in \mathbb{Z}^d_+$, there exists a positive constant c, such that for every $J \in \mathbb{Z}^d_{(\eta)}$ (see* (3.2.29) *and* (3.2.30)*), and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$,*

$$\left|\partial^b(\Psi_{\alpha,J}[f])(x)\right| \le c \prod_{l=1}^d \frac{2^{(1-\eta_l)j_l/\alpha}\, 2^{-j_l\eta_l a_l[f]}}{\left(1 + T + |x_l|\right)^{p_*}}, \tag{3.2.34}$$

where the positive exponents $a_1[f], \dots, a_d[f]$, and p_* are as in Definition 3.1.1.

Proposition 3.2.6 is proved in the section 3.3.

Having presented the main ingredients of the proof of the important Proposition 3.2.3 which provides the wavelet type random series representation of $\{X[f](t), t \in \mathbb{R}^d\}$, our present goal is to improve the convergence result concerning this series. First we need to give two useful lemmas. The following one will play a crucial role throughout the rest of the article.

Lemma 3.2.7. *Let $\left\{\varepsilon_{\alpha,J,K} : (J, K) \in \mathbb{Z}^d \times \mathbb{Z}^d\right\}$ be the sequence of the identically distributed real-valued symmetric α-stable random variables defined through* (3.2.17)*. There exists an event Ω_1^* of probability 1 such that the following three results hold.*

1. *Assume that $\alpha \in (0,1)$; then, for all fixed $\delta \in (0,+\infty)$ and $\omega \in \Omega_1^*$, there is a finite constant $C(\omega) > 0$ (depending on α, δ and ω), such that, for every $J = (j_1, \dots, j_d) \in \mathbb{Z}^d$ and $K \in \mathbb{Z}^d$, one has*

$$|\varepsilon_{\alpha,J,K}(\omega)| \leq C(\omega) \prod_{l=1}^{d} (1 + |j_l|)^{1/\alpha+\delta}. \tag{3.2.35}$$

Observe that in this case $|\varepsilon_{\alpha,J,K}(\omega)|$ can be bounded independently of K.

2. *Assume that $\alpha \in [1,2)$; then, for each fixed $\delta \in (0,+\infty)$ and $\omega \in \Omega_1^*$, there exists a finite constant $C(\omega) > 0$ (depending on α, δ and ω), such that for all $(J,K) = (j_1, \dots, j_d, k_1, \dots, k_d) \in \mathbb{Z}^d \times \mathbb{Z}^d$,*

$$|\varepsilon_{\alpha,J,K}(\omega)| \leq C(\omega) \sqrt{\log\left(3 + \sum_{l=1}^{d} \Big(|j_l| + |k_l|\Big)\right)} \prod_{l=1}^{d} (1 + |j_l|)^{1/\alpha+\delta}. \tag{3.2.36}$$

3. *Assume that $\alpha = 2$, then, for every fixed $\omega \in \Omega_1^*$, there is a finite constant $C(\omega) > 0$ (depending on ω), such that for each $(J,K) = (j_1, \dots, j_d, k_1, \dots, k_d) \in \mathbb{Z}^d \times \mathbb{Z}^d$,*

$$|\varepsilon_{\alpha,J,K}(\omega)| \leq C(\omega) \sqrt{\log\left(3 + \sum_{l=1}^{d} \Big(|j_l| + |k_l|\Big)\right)}. \tag{3.2.37}$$

Notice that the event Ω_1^* depends on α; yet, it does not depend on the function f associated with the field $X[f]$ through (2.3.3).

The third result provided by Lemma 3.2.7 (in other words the inequality (3.2.37) which holds in the Gaussian case $\alpha = 2$) is rather classical; its proof can be found in e.g. [4]. The first two results provided by the lemma (in other words the inequalities (3.2.35) and (3.2.36)) are derived in the section 3.5; we mention that their proofs rely on a LePage series representation of the complex-valued α-stable process

$$\left\{ \int_{\mathbb{R}^d} \overline{\widehat{\psi}_{\alpha,J,K}(\xi)} \, \mathrm{d}\widetilde{M}_\alpha(\xi) : (J,K) \in \mathbb{Z}^d \times \mathbb{Z}^d \right\}.$$

On the other hand, it is worth noticing that the elementary inequality

$$\text{for all } u', u'' \in \mathbb{R}_+, \quad \sqrt{\log(3 + u' + u'')} \leq 2\sqrt{\log(3 + u')}\sqrt{\log(3 + u'')}, \tag{3.2.38}$$

will frequently be employed for deriving upper bounds of the logarithmic function in Lemma 3.2.7. In particular it allows to show that:

Remark 3.2.8. *Assume that $\alpha \in (0,2]$ is arbitrary. Let $\left\{\varepsilon_{\alpha,J,K} : (J,K) \in \mathbb{Z}^d \times \mathbb{Z}^d\right\}$ and Ω_1^* be as in Lemma 3.2.7. Then, for each fixed $\delta \in (0,+\infty)$ and $\omega \in \Omega_1^*$, there exists a finite constant $C(\omega) > 0$ (depending on α, δ and ω), such that for all $(J,K) = (j_1,\dots,j_d,k_1,\dots,k_d) \in \mathbb{Z}^d \times \mathbb{Z}^d$,*

$$|\varepsilon_{\alpha,J,K}(\omega)| \leq C(\omega) \prod_{l=1}^{d} \sqrt{\log(3+|j_l|)}(1+|j_l|)^{1/\alpha+\delta}\sqrt{\log(3+|k_l|)}. \tag{3.2.39}$$

The second useful lemma is the following one:

Lemma 3.2.9. *Assume that $\alpha \in (0,2]$ is arbitrary, and let $p_* = p_*(\alpha)$ be as in* (3.1.1)*. Then, there is a positive finite constant c such that, for every $(\theta,v) \in \mathbb{R}_+ \times \mathbb{R}$, the following inequality holds:*

$$\sum_{k\in\mathbb{Z}} \frac{\sqrt{\log(3+\theta+|k|)}}{(2+|v-k|)^{p_*}} \leq c\sqrt{\log(3+\theta+|v|)}. \tag{3.2.40}$$

Lemma 3.2.9 is proved at the end of the section 3.3.

The following proposition is an improvement of Proposition 3.2.3.

Proposition 3.2.10. *We assume that the stability parameter $\alpha \in (0,2]$ is arbitrary, and that Ω_1^* is the event of probability 1 introduced in Lemma 3.2.7. Then, for all $(t,\omega) \in \mathbb{R}^d \times \Omega_1^*$, the series of real numbers,*

$$\sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} \Big(\Psi_{\alpha,J}[f](2^J t - K) - \Psi_{\alpha,J}[f](-K)\Big)\varepsilon_{\alpha,J,K}(\omega), \tag{3.2.41}$$

is absolutely convergent [3]. Thus, in view of Proposition 3.2.3, the sum in (3.2.41) *is equal to $X[f](t,\omega)$ defined through* (2.3.3)*, except when ω belongs to a negligible event [4].*

Before proving Proposition 3.2.10, we introduce a convenient notation. Let T be any fixed positive real number and let g be any real-valued (or complex-valued) function on $\mathbb{R}^d$, then the quantity $\|g\|_{T,\infty}$ is defined as:

$$\|g\|_{T,\infty} := \sup_{s\in[-T,T]^d} |g(s)|; \tag{3.2.42}$$

[3] Therefore, its finite value does not depend on the way the terms of the series are labelled. Moreover, it follows from the Fubini's theorem that:

$$\sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} \big(\Psi_{\alpha,J}[f](2^J t - K) - \Psi_{\alpha,J}[f](-K)\big)\varepsilon_{\alpha,J,K}(\omega)$$
$$= \sum_{J\in\mathbb{Z}^d}\Big(\sum_{K\in\mathbb{Z}^d} \big(\Psi_{\alpha,J}[f](2^J t - K) - \Psi_{\alpha,J}[f](-K)\big)\varepsilon_{\alpha,J,K}(\omega)\Big).$$

[4] Notice that this negligible event does not necessarily coincide with the whole set $\Omega \setminus \Omega_1^*$. On the other hand, this negligible event may depend on t.

observe that $\|\cdot\|_{T,\infty}$ is almost the uniform semi-norm on the cube $[-T,T]^d$; the only difference is that one may have $\|g\|_{T,\infty} = +\infty$, since one does not necessarily impose g to be bounded on $[-T,T]^d$.

Proof of Proposition 3.2.10. We assume that $(t,\omega) \in \mathbb{R}^d \times \Omega_1^*$ is arbitrary and fixed. We have to prove that the series of real numbers in (3.2.41) is absolutely convergent, that is

$$Z[f](t,\omega) < +\infty, \tag{3.2.43}$$

where

$$Z[f](t,\omega) := \sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} \Big|\Psi_{\alpha,J}[f](2^J t - K) - \Psi_{\alpha,J}[f](-K)\Big| \left|\varepsilon_{\alpha,J,K}(\omega)\right|. \tag{3.2.44}$$

Let Υ be as (3.2.28), and, for each fixed $\eta \in \Upsilon$, let $\mathbb{Z}^d_{(\eta)}$ be as in (3.2.29) (see also (3.2.30)). Then, it follows from (3.2.31) and (3.2.44) that $Z(t,\omega)$ can be decomposed as:

$$Z[f](t,\omega) = \sum_{\eta\in\Upsilon} Z[f]^\eta(t,\omega), \tag{3.2.45}$$

where, for all fixed $\eta \in \Upsilon$,

$$Z[f]^\eta(t,\omega) := \sum_{(J,K)\in\mathbb{Z}^d_{(\eta)}\times\mathbb{Z}^d} \Big|\Psi_{\alpha,J}[f](2^J t - K) - \Psi_{\alpha,J}[f](-K)\Big| \left|\varepsilon_{\alpha,J,K}(\omega)\right|. \tag{3.2.46}$$

Next, using (3.2.45) and the fact that Υ is a finite set, it turns out that (3.2.43) is equivalent to:

$$Z[f]^\eta(t,\omega) < +\infty, \quad \text{for all } \eta \in \Upsilon. \tag{3.2.47}$$

In order to prove (3.2.47), we will study two cases: $\eta = 0 := (0,\ldots,0)$ and $\eta \in \Upsilon^* := \Upsilon \setminus \{0\}$.
<u>First case: $\eta = 0$</u>. Notice that, in this case, one has $J \in \mathbb{Z}^d_{(0)} := \mathbb{Z}^d_-$, so it can be rewritten as $J = -J'$, where J' belongs to $\mathbb{Z}^d_+$. In the sequel J' is denoted by J. Using the Mean Value Theorem and the triangle inequality, we get

$$\Big|\Psi_{\alpha,-J}[f](2^{-J}t - K) - \Psi_{\alpha,-J}[f](-K)\Big| \le T\sum_{r=1}^d 2^{-j_r}\left\|\frac{\partial\Psi_{\alpha,-J}[f]}{\partial x_r}\Big(2^{-J}\cdot - K\Big)\right\|_{T,\infty}, \tag{3.2.48}$$

where $T := \max_{1\le l\le d}|t_l|$, the t_l's being the coordinates of t. Moreover, combining (3.2.33) with the inequality,

$$1 + T + \Big|2^{-j_r}s_l - k_l\Big| \ge 1 + |k_l|, \quad \text{for all } l \in \{1,\ldots,d\} \text{ and } s_l \in [-T,T],$$

we obtain, for every $r \in \{1,\ldots,d\}$, that

$$2^{-j_r}\left\|\frac{\partial\Psi_{\alpha,-J}[f]}{\partial x_r}\Big(2^{-J}\cdot - K\Big)\right\|_{T,\infty} \le c_1 \frac{2^{-j_r(1-a'[f])}\left(2^{-j_1}+\cdots+2^{-j_d}\right)^{-d/\alpha}\prod_{l=1}^d 2^{-j_l/\alpha}}{\prod_{l=1}^d\Big(1+|k_l|\Big)^{p_*}}, \tag{3.2.49}$$

where c_1 is a positive finite constant not depending on (J,K). Next, putting together (3.2.46), (3.2.48), (3.2.49), (3.1.1), (3.2.39), and (3.2.25), it follows that (3.2.47) holds when $\eta = 0$.
Second case: $\eta \in \Upsilon^*$. It results from (3.2.46) and the triangle inequality that

$$\begin{aligned} Z[f]^\eta(t,\omega) \quad \le \quad & \sum_{(J,K)\in\mathbb{Z}^d_{(\eta)}\times\mathbb{Z}^d} \left|\Psi_{\alpha,J}[f](2^J t - K)\right| |\varepsilon_{\alpha,J,K}(\omega)| \\ & + \sum_{(J,K)\in\mathbb{Z}^d_{(\eta)}\times\mathbb{Z}^d} |\Psi_{\alpha,J}[f](-K)|\, |\varepsilon_{\alpha,J,K}(\omega)|\,. \end{aligned}$$

Thus, in order to obtain (3.2.47), it is enough to show that,

$$\sum_{(J,K)\in\mathbb{Z}^d_{(\eta)}\times\mathbb{Z}^d} \left|\Psi_{\alpha,J}[f](2^J t - K)\right| |\varepsilon_{\alpha,J,K}(\omega)| < +\infty, \tag{3.2.50}$$

and

$$\sum_{(J,K)\in\mathbb{Z}^d_{(\eta)}\times\mathbb{Z}^d} |\Psi_{\alpha,J}[f](-K)|\, |\varepsilon_{\alpha,J,K}(\omega)| < +\infty. \tag{3.2.51}$$

Notice that (3.2.51) is nothing else than (3.2.50) where $t = 0$. The proof of (3.2.50) can be done in the following way. Using (3.2.39), (3.2.34) (with $T = 1$), (3.2.40) (with $(\theta, v) = (0, 2^{j_l} t_l)$), (3.2.29) and (3.2.30), one gets that,

$$\begin{aligned} & \sum_{(J,K)\in\mathbb{Z}^d_{(\eta)}\times\mathbb{Z}^d} \left|\Psi_{\alpha,J}[f](2^J t - K)\right| |\varepsilon_{\alpha,J,K}(\omega)| \\ & \le C_2(\omega) \sum_{(J,K)\in\mathbb{Z}^d_{(\eta)}\times\mathbb{Z}^d} \prod_{l=1}^d 2^{(1-\eta_l)j_l/2} 2^{-j_l\eta_l a_l[f]} \sqrt{\log\left(3+|j_l|\right)}(1+|j_l|)^{1/\alpha+\delta} \frac{\sqrt{\log\left(3+|k_l|\right)}}{\left(2+|2^{j_l}t_l - k_l|\right)^{p_*}} \\ & \le C_3(\omega) \prod_{l=1}^d \left(\sum_{j_l\in\mathbb{Z}_{\eta_l}} 2^{(1-\eta_l)j_l/2} 2^{-j_l\eta_l a_l[f]} \sqrt{\log\left(3+|j_l|\right)}(1+|j_l|)^{1/\alpha+\delta} \sqrt{\log\left(3+2^{j_l}|t_l|\right)} \right) \\ & < +\infty, \end{aligned}$$

where $C_2(\omega)$ and $C_3(\omega)$ are two positive finite constants. □

Remark 3.2.11. *From now on, for the sake of simplicity, "we forget" the definition of the real-valued symmetric α-stable field $\{X[f](t), t \in \mathbb{R}^d\}$ given by (2.3.3), and we systematically identify this field with its modification provided by Proposition 3.2.10. More precisely, we assume that, for all $(t,\omega) \in \mathbb{R}^d \times \Omega_1^*$, one has*

$$X[f](t,\omega) := \sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} \Big(\Psi_{\alpha,J}[f]\Big(2^J t - K\Big) - \Psi_{\alpha,J}[f](-K)\Big)\varepsilon_{\alpha,J,K}(\omega); \tag{3.2.52}$$

also, we assume that the field $X[f]$ vanishes outside of the event Ω_1^*.

Thanks to (3.2.52), for any $\eta \in \Upsilon$, the η-frequency part $\left\{X[f]^\eta(t), t \in \mathbb{R}^d\right\}$ of the field $\{X[f](t), t \in \mathbb{R}^d\}$ can be precisely defined.

Definition 3.2.12. *For all $\eta \in \Upsilon := \{0,1\}^d$, the η-frequency part of the field $\{X[f](t), t \in \mathbb{R}^d\}$ is the real-valued symmetric α-stable field denoted by $X[f]^\eta := \left\{X[f]^\eta(t), t \in \mathbb{R}^d\right\}$, and defined, for any $(t,\omega) \in \mathbb{R}^d \times \Omega_1^*$, as:*

$$X[f]^\eta(t,\omega) := \sum_{(J,K)\in\mathbb{Z}^d_{(\eta)}\times\mathbb{Z}^d} \Big(\Psi_{\alpha,J}[f]\Big(2^J t - K\Big) - \Psi_{\alpha,J}[f](-K)\Big)\,\varepsilon_{\alpha,J,K}(\omega), \tag{3.2.53}$$

where $\mathbb{Z}^d_{(\eta)}$ is as in (3.2.29) *(see also* (3.2.30)*); moreover, it is assumed that the field $X[f]^\eta$ vanishes outside of the event Ω_1^*. Notice that we know from* (3.2.46) *and* (3.2.47) *that the series of real numbers in* (3.2.53) *is absolutely convergent.*

Remark 3.2.13. *In view of Remark 3.2.11 and Definition 3.2.12, it is clear that the field $X[f]$ can be expressed as the finite sum of all its η-frequency parts: for each $(t,\omega) \in \mathbb{R}^d \times \Omega$ one has*

$$X[f](t,\omega) = \sum_{\eta\in\Upsilon} X[f]^\eta(t,\omega). \tag{3.2.54}$$

In some sense, the two extremes, that is the fields

$$X[f]^0 := X[f]^{(0,\dots,0)} \quad \textit{and} \quad X[f]^1 := X[f]^{(1,\dots,1)},$$

can respectively be viewed as the low-frequency and high-frequency parts. While, for any $\eta \in \{0,1\}^d \setminus \{(0,\dots,0),(1,\dots,1)\}$, the field $X[f]^\eta$ can be viewed as an intermediary part between low-frequency and high-frequency.

Remark 3.2.14. *For the sake of convenience, when $\eta \neq 0$ and $(t,\omega) \in \mathbb{R}^d \times \Omega_1^*$, we sometimes decompose $X[f]^\eta(t,\omega)$ as:*

$$X[f]^\eta(t,\omega) = Y[f]^\eta(t,\omega) - Y[f]^\eta(0,\omega), \tag{3.2.55}$$

where,

$$Y[f]^\eta(t,\omega) := \sum_{(J,K)\in\mathbb{Z}^d_{(\eta)}\times\mathbb{Z}^d} \Psi_{\alpha,J}[f](2^J t - K)\varepsilon_{\alpha,J,K}(\omega). \tag{3.2.56}$$

Notice that we know from (3.2.50) *that the series of real numbers in* (3.2.56) *is absolutely convergent.*

Now, we are going to study some smoothness properties of the sample paths of the η-frequency parts $X[f]^\eta$ of the field $X[f]$. Mainly, we will show that they are always continuous functions, and may even have partial derivatives in some cases; for instance, they are infinitely differentiable in the particular case of the low-frequency part $X[f]^0$. Notice that, in view of (3.2.54), the continuity property of the $X[f]^\eta$'s implies that the sample paths of $X[f]$, itself, are continuous as well.

More precisely, we will show that the following three propositions hold.

Proposition 3.2.15. *For any $\alpha \in (0,2]$, for each $b=(b_1,\ldots,b_d)\in\mathbb{Z}_+^d$, and for all $(T,\omega)\in(0,+\infty)\times\Omega_1^*$, one has*

$$\sum_{(J,K)\in\mathbb{Z}_+^d\times\mathbb{Z}^d}\left\|\partial^b\Big(\Psi_{\alpha,-J}[f](2^{-J}\cdot-K)-\Psi_{\alpha,-J}[f](-K)\Big)\right\|_{T,\infty}|\varepsilon_{\alpha,-J,K}(\omega)|<+\infty. \tag{3.2.57}$$

Thus, when $\eta=0$, the series in (3.2.53) and all its term by term partial derivatives of any order are uniformly convergent in t, on each compact subset of $\mathbb{R}^d$. Therefore, the function

$$X[f]^0(\cdot,\omega): t\mapsto X[f]^0(t,\omega)$$

is infinitely differentiable on $\mathbb{R}^d$, with partial derivatives satisfying, for all $b\in\mathbb{Z}_+^d$ and $t\in\mathbb{R}^d$,

$$\Big(\partial^b X[f]^0\Big)(t,\omega)=\sum_{(J,K)\in\mathbb{Z}_+^d\times\mathbb{Z}^d}\partial^b\left(\Psi_{\alpha,-J}[f](2^{-J}\cdot-K)-\Psi_{\alpha,-J}[f](-K)\right)(t)\,\varepsilon_{\alpha,-J,K}(\omega). \tag{3.2.58}$$

Proposition 3.2.16. *Let $\alpha\in(0,2]$, $\eta=(\eta_1,\ldots,\eta_d)\in\Upsilon^*$, $J=(j_1,\ldots,j_d)\in\mathbb{Z}^d_{(\eta)}$, $b=(b_1,\ldots,b_d)\in\mathbb{Z}_+^d$ and $(T,\omega)\in(0,+\infty)\times\Omega_1^*$ be arbitrary and fixed. One has*

$$\sum_{K\in\mathbb{Z}^d}\left\|\Big(\partial^b(\Psi_{\alpha,J}[f])\Big)(\cdot-K)\right\|_{T,\infty}|\varepsilon_{\alpha,J,K}(\omega)|<+\infty. \tag{3.2.59}$$

Thus, the series

$$\Phi_{\alpha,J}[f](x,\omega):=\sum_{K\in\mathbb{Z}^d}\Psi_{\alpha,J}[f](x-K)\varepsilon_{\alpha,J,K}(\omega), \tag{3.2.60}$$

and all its term by term partial derivatives of any order are uniformly convergent in x, on each compact subset of $\mathbb{R}^d$. Therefore, the real-valued function

$$\Phi_{\alpha,J}[f](\cdot,\omega): x\mapsto\Phi_{\alpha,J}[f](x,\omega)$$

is infinitely differentiable on $\mathbb{R}^d$, with partial derivatives satisfying, for all $b\in\mathbb{Z}_+^d$ and $x\in\mathbb{R}^d$,

$$\Big(\partial^b(\Phi_{\alpha,J}[f])\Big)(x,\omega)=\sum_{K\in\mathbb{Z}^d}(\partial^b(\Psi_{\alpha,J}[f]))(x-K)\varepsilon_{\alpha,J,K}(\omega). \tag{3.2.61}$$

Proposition 3.2.17. *Assume that $\eta=(\eta_1,\ldots,\eta_d)\in\Upsilon^*$ and $b=(b_1,\ldots,b_d)\in\mathbb{Z}_+^d$ satisfy*

$$\eta_l b_l<a_l[f],\quad \text{for all } l\in\{1,\ldots,d\}, \tag{3.2.62}$$

where the positive exponents $a_1[f],\ldots,a_d[f]$ are as in Definition 3.1.1. Let $\alpha\in(0,2]$ and $(T,\omega)\in(0,+\infty)\times\Omega_1^*$ be arbitrary and fixed. Then, one has

$$\sum_{J\in\mathbb{Z}^d_{(\eta)}}\left\|\partial^b\Big(\Phi_{\alpha,J}[f](2^J\cdot,\omega)\Big)\right\|_{T,\infty}<+\infty. \tag{3.2.63}$$

Thus, the series $\sum_{J\in\mathbb{Z}^d_{(\eta)}} \Phi_{\alpha,J}[f](2^J t,\omega)$, *and any of its term by term partial derivatives, of an order* b *satisfying* (3.2.62), *are uniformly convergent in* t *on each compact subset of* $\mathbb{R}^d$. *Therefore, the function*

$$Y[f]^\eta(\cdot,\omega) : t \mapsto Y[f]^\eta(t,\omega),$$

defined on $\mathbb{R}^d$ *through* (3.2.56), *is continuous and has a continuous partial derivative denoted by* $\Big(\partial^b(Y[f]^\eta)\Big)(\cdot,\omega)$ *such that, for all* $t\in\mathbb{R}^d$,

$$\Big(\partial^b(Y[f]^\eta)\Big)(t,\omega) = \sum_{J\in\mathbb{Z}^d_{(\eta)}} \partial^b\Big(\Phi_{\alpha,J}[f](2^J\cdot,\omega)\Big)(t) = \sum_{J\in\mathbb{Z}^d_{(\eta)}} 2^{j_1b_1+\ldots+j_db_d}\Big(\partial^b(\Phi_{\alpha,J}[f])\Big)(2^Jt,\omega). \tag{3.2.64}$$

Notice that, these continuity and differentiability properties are also satisfied by the function $X[f]^\eta(\cdot,\omega)$ *(see Definition 3.2.12) because of the equality* (3.2.55).

Proof of Proposition 3.2.15. We will study two cases: $b=0$ and $b\neq 0$.
First case: $b=0$. Similarly to (3.2.48) and (3.2.49), we can show that, for some finite constant c_1 and for all $(J,K)\in\mathbb{Z}^d_+\times\mathbb{Z}^d$, one has

$$\begin{aligned}
&\Big\|\Psi_{\alpha,-J}[f](2^{-J}\cdot-K)-\Psi_{\alpha,-J}[f](-K)\Big\|_{T,\infty}\\
&\qquad\le T\sum_{r=1}^d 2^{-j_r}\Big\|\frac{\partial\Psi_{\alpha,-J}[f]}{\partial x_r}\Big(2^{-J}\cdot-K\Big)\Big\|_{T,\infty}\\
&\qquad\le c_1\sum_{r=1}^d \frac{2^{-j_r(1-a'[f])}\left(2^{-j_1}+\cdots+2^{-j_d}\right)^{-d/\alpha}\prod_{l=1}^d 2^{-j_l/\alpha}}{\prod_{l=1}^d\Big(1+|k_l|\Big)^{p_*}}.
\end{aligned}\tag{3.2.65}$$

Next putting together (3.2.65), (3.2.39), (3.1.1) and (3.2.25), we get (3.2.57) when $b=0$.
Second case: $b\neq 0$. Notice that in this case the multi-index b has at least one positive coordinate, let us say b_{r_0}. Standard computations and (3.2.33) allow to show that, for some finite constant c_2, and for all $(J,K)\in\mathbb{Z}^d_+\times\mathbb{Z}^d$, one has

$$\begin{aligned}
&\Big\|\partial^b\Big(\Psi_{-J}[f](2^{-J}\cdot-K)-\Psi_{-J}[f](-K)\Big)\Big\|_{T,\infty}\\
&\qquad=\Big(\prod_{l=1}^d 2^{-j_lb_l}\Big)\Big\|\partial^b(\Psi_{-J}[f])(2^{-J}\cdot-K)\Big\|_{T,\infty}\\
&\qquad\le 2^{-j_{r_0}}\Big\|\partial^b(\Psi_{-J}[f])(2^{-J}\cdot-K)\Big\|_{T,\infty}\\
&\qquad\le c_2\frac{2^{-j_{r_0}(1-a'[f])}\left(2^{-j_1}+\cdots+2^{-j_d}\right)^{-d/2}\prod_{l=1}^d 2^{-j_l/2}}{\prod_{l=1}^d(1+|k_l|)^{p_*}}.
\end{aligned}\tag{3.2.66}$$

Next putting together (3.2.66), (3.2.39), (3.1.1) and (3.2.25), we get (3.2.57) when $b\neq 0$. □

Proof of Proposition 3.2.16. It follows from (3.2.39), (3.2.34) and the triangle inequality that, for all $x = (x_1, \ldots, x_d) \in [-T, T]^d$ and $K = (k_1, \ldots, k_d) \in \mathbb{Z}_+^d$, one has

$$\begin{aligned} \left|\left(\partial^b(\Psi_{\alpha,J}[f])\right)(x-K)\right| |\varepsilon_{\alpha,J,K}(\omega)| &\leq C_1(\omega,T,J) \prod_{l=1}^{d} \frac{\sqrt{\log(3+|k_l|)}}{(1+T+|x_l-k_l|)^{p_*}} \\ &\leq C_1(\omega,T,J) \prod_{l=1}^{d} \frac{\sqrt{\log(3+|k_l|)}}{(1+|k_l|)^{p_*}}, \end{aligned} \tag{3.2.67}$$

where $C_1(\omega, T, J)$ is a finite constant depending on T and J, but not on K. In view of (3.1.1), it is clear that (3.2.67) entails that (3.2.59) holds. □

In order to derive Proposition 3.2.17, we need the following lemma.

Lemma 3.2.18. *Assume that $a_1[f], \ldots, a_d[f]$ are the same positive exponents as in Definition 3.1.1. Let $\alpha \in (0, 2]$, $\eta = (\eta_1, \ldots, \eta_d) \in \Upsilon^*$, $J \in \mathbb{Z}^d_{(\eta)}$, $b = (b_1, \ldots, b_d) \in \mathbb{Z}_+^d$ and $(T, \delta, \omega) \in (0, +\infty)^2 \times \Omega_1^*$ be arbitrary and fixed. The following three results are satisfied; notice that $C(\omega)$, in each one of them, is a finite constant not depending on J and T.*

1. *When $\alpha \in (0, 1)$, one has*

$$\left\|\partial^b\left(\Phi_{\alpha,J}[f](2^J\cdot,\omega)\right)\right\|_{T,\infty} \leq C(\omega) \prod_{l=1}^{d} 2^{j_l((1-\eta_l)(1/\alpha+b_l)-\eta_l(a_l[f]-b_l))} (1+|j_l|)^{1/\alpha+\delta}. \tag{3.2.68}$$

2. *When $\alpha \in [1, 2)$, one has*

$$\begin{aligned} &\left\|\partial^b\left(\Phi_{\alpha,J}[f](2^J\cdot,\omega)\right)\right\|_{T,\infty} \\ &\quad \leq C(\omega) \prod_{l=1}^{d} 2^{j_l((1-\eta_l)(1/\alpha+b_l)-\eta_l(a_l[f]-b_l))} (1+|j_l|)^{1/\alpha+\delta} \sqrt{\log(3+|j_l|+2^{j_l}T)}. \end{aligned} \tag{3.2.69}$$

3. *When $\alpha = 2$, one has*

$$\left\|\partial^b\left(\Phi_{\alpha,J}[f](2^J\cdot,\omega)\right)\right\|_{T,\infty} \leq C(\omega) \prod_{l=1}^{d} 2^{j_l((1-\eta_l)(1/\alpha+b_l)-\eta_l(a_l[f]-b_l))} \sqrt{\log(3+|j_l|+2^{j_l}T)}. \tag{3.2.70}$$

Proof of Lemma 3.2.18. We give the proof only in the case where $\alpha \in [1, 2)$; the other two cases, $\alpha \in (0, 1)$ and $\alpha = 2$, can be treated similarly except that one has to use (3.2.35) and

(3.2.37) instead of (3.2.36). It follows from (3.2.61), the triangle inequality, (3.2.34) (with $T=1$), (3.2.36), (3.2.38) and (3.2.40), that, for every $t\in[-T,T]^d$ and $J\in\mathbb{Z}^d_{(\eta)}$, one has

$$\begin{aligned}
&\left|\partial^b\Big(\Phi_{\alpha,J}[f]\Big(2^J\cdot,\omega\Big)\Big)(t)\right|\\
&\le \sum_{K\in\mathbb{Z}^d}\left|\left(\prod_{l=1}^d 2^{j_l b_l}\right)(\partial^b(\Psi_{\alpha,J}[f]))(2^Jt-K)\varepsilon_{\alpha,J,K}(\omega)\right|\\
&\le C_1(\omega)\prod_{l=1}^d 2^{(1-\eta_l)j_l(1/\alpha+b_l)}2^{-\eta_l j_l(a_l[f]-b_l)}\left(1+|j_l|\right)^{1/\alpha+\delta}\sum_{k_l\in\mathbb{Z}}\frac{\sqrt{\log\left(3+|j_l|+|k_l|\right)}}{\left(2+|2^{j_l}t_l-k_l|\right)^{p_*}}\\
&\le C_2(\omega)\prod_{l=1}^d 2^{(1-\eta_l)j_l(1/2+b_l)}2^{-\eta_l j_l(a_l[f]-b_l)}\left(1+|j_l|\right)^{1/\alpha+\delta}\sqrt{\log\left(3+|j_l|+2^{j_l}T\right)},
\end{aligned}$$

where $C_1(\omega)$ and $C_2(\omega)$ are two positive and finite constants not depending on J, t and T. $\square$

We are now ready to prove Proposition 3.2.17.

Proof of Proposition 3.2.17. Using Lemma 3.2.18, (3.2.29), (3.2.30) and standard computations, one can easily obtain (3.2.63). $\square$

Before ending this section let us state the following theorem which easily results from Remark 3.2.13, Proposition 3.2.15 and Proposition 3.2.17.

Theorem 3.2.19. *Assume that f is an admissible function in the sense of Definition 3.1.1, and that the positive exponents $a_1[f],\dots,a_d[f]$ are as in this definition. Then, the field $X[f]$ associated with f (see* (2.3.3) *and Remark 3.2.11) has the following property. For any fixed $\omega\in\Omega_1^*$ (see Lemma 3.2.7), the sample path*

$$X[f](\cdot,\omega): t\mapsto X[f](t,\omega)$$

is continuous on $\mathbb{R}^d$; moreover, when $b=(b_1,\dots,b_d)\in\mathbb{Z}^d_+$ satisfies $b_l<a_l[f]$, for all $l\in\{1,\dots,d\}$, the partial derivative $\Big(\partial^b(X[f])\Big)(\cdot,\omega)$ exists and is continuous on $\mathbb{R}^d$. When $b\neq 0$ and $\omega\in\Omega_1^*$, it is given for all $t\in\mathbb{R}^d$ by

$$\partial^b(X[f])(t,\omega)=\sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}2^{j_1b_1+\dots+j_db_d}\partial^b(\Psi_{\alpha,J}[f])(2^Jt-K)\,\varepsilon_{\alpha,J,K}(\omega) \qquad (3.2.71)$$

Notice that, when $\omega\notin\Omega_1^*$, we have $\Big(\partial^b X[f]\Big)(t,\omega):=0$ for any $t\in\mathbb{R}^d$.

We mention that in view Proposition 3.2.15 and Proposition 3.2.17 we have the following result.

Corollary 3.2.20. *For any* $\alpha \in (0,2]$ *and for all* $(T,\omega) \in (0,+\infty) \times \Omega_1^*$*, one has*

$$\left\| \sum_{(J,K)\in\mathbb{Z}_+^d\times\mathbb{Z}^d} \left|\Psi_{\alpha,-J}[f](2^{-J}\cdot -K) - \Psi_{\alpha,-J}[f](-K)\right| |\varepsilon_{\alpha,-J,K}(\omega)| \right\|_{T,\infty} < +\infty. \tag{3.2.72}$$

Last but not least, we point out that Ω_1^* is an event of probability 1 not depending on f; so, in some sense, Ω_1^* is "universal".

3.3 Proofs of Proposition 3.2.6 and Lemma 3.2.9

Proof of Proposition 3.2.6. Let us first assume that $J \in \mathbb{Z}^d$, and show the infinite differentiability of $\Psi_{\alpha,J}[f]$ and relation (3.2.32). We denote by $\Lambda_{\alpha,J}$ the integrand in (3.2.7), that is, for all $x \in \mathbb{R}^d$, and $\xi \in \mathbb{R}^d$, we set,

$$\Lambda_{\alpha,J}(x,\xi) := 2^{(j_1+\cdots+j_d)/\alpha} e^{ix\cdot\xi} f(2^J\xi)\widehat{\psi}_{0,0}(\xi). \tag{3.3.1}$$

Observe that $\Lambda_{\alpha,J}$ is an infinitely differentiable function on $\mathbb{R}^d$ with respect to the variable x, and that for any $b \in \mathbb{Z}_+^d$,

$$\partial_x^b \Lambda_{\alpha,J}(x,\xi) = 2^{(j_1+\cdots+j_d)/\alpha}\, i^{1(b)} \xi^b e^{ix\cdot\xi} f(2^J\xi)\widehat{\psi}_{0,0}(\xi). \tag{3.3.2}$$

Thus, in view of a classical rule of differentiation under the integral symbol, in order to show that $\Psi_{\alpha,J}[f]$ itself is infinitely differentiable on $\mathbb{R}^d$ and satisfies (3.2.32), it is enough to prove that for any $b \in \mathbb{Z}_+^d$, there exists $G_{\alpha,J}^b \in L^1\left(\mathbb{R}^d\right)$, which does not depend on x, such that the inequality:

$$\left|\partial_x^b \Lambda_{\alpha,J}(x,\xi)\right| \le G_{\alpha,J}^b(\xi), \tag{3.3.3}$$

holds for almost all $\xi \in \mathbb{R}^d$. Recall that $\mathcal{K}$ is the compact subset of $\mathbb{R}$ defined as $\mathcal{K} := \Big\{\lambda \in \mathbb{R} : 2\pi/3 \le |\lambda| \le 8\pi/3\Big\}$; also recall that $\widehat{\psi}_{0,0}$ is a C^∞ function with a compact support included in $\mathcal{K}^d$. Thus the smoothness assumption on the function f (that is $(\mathcal{H}_1)$ in Definition 3.1.1) implies that the supremum $\left\|f(2^j\cdot)\widehat{\psi}_{0,0}(\cdot)\right\|_\infty := \sup_{\xi\in\mathcal{K}^d}\left|f(2^J\xi)\widehat{\psi}_{0,0}(\xi)\right|$ is finite. Then, it turns out that a function $G_{\alpha,J}^b$, belonging to $L^1\left(\mathbb{R}^d\right)$ and satisfying (3.3.3), can simply be obtained by setting, for all $\xi \in \mathbb{R}^d$,

$$G_{\alpha,J}^b(\xi) = 2^{(j_1+\cdots+j_d)/\alpha}\left(\frac{8\pi}{3}\right)^{1(b)} \left\|f(2^j\cdot)\widehat{\psi}_{0,0}(\cdot)\right\|_\infty \mathbb{1}_{\mathcal{K}^d}(\xi).$$

Let us now prove that parts (i) and (ii) of the proposition hold. For the sake of simplicity, we restrict to the case where $x = (x_1,\ldots,x_d) \in \mathbb{R}_+^d$; the other cases can be treated similarly.

It easily follows from (3.2.32), (3.2.3) and (3.2.4) that, for every $T \in (0,+\infty)$, $J \in \mathbb{Z}^d$ and $x \in \mathbb{R}^d_+$,

$$\left|\partial^b(\Psi_{\alpha,J}[f])(x)\right| = 2^{(j_1+\cdots+j_d)/\alpha}\left|\int_{\mathcal{K}^d}\left(\prod_{l=1}^d e^{i(1+T+x_l)\xi_l}\,\widehat{\Phi}_l(\xi_l)\right)f(2^J\xi)\mathrm{d}\xi\right|, \tag{3.3.4}$$

where $\widehat{\Phi}_l(\xi_l) := e^{-i(1+T)\xi_l}\,\xi_l^{b_l}\,\widehat{\psi^1}(\xi_l)$. Next, we set $R_{\alpha,J}(\xi) := f(2^J\xi)\prod_{l=1}^d\widehat{\Phi}_l(\xi_l)$, for all $\xi \in \mathbb{R}^d\setminus\{0\}$. Observe that, similarly to $\widehat{\psi^1}$ (see the beginning of Section 3.2), $\widehat{\Phi}_l$ is a C^∞ function having a compact support included in $\mathcal{K}$. Thus, using the condition $(\mathcal{H}_1)$ in Definition 3.1.1, it turns out that the partial derivative $\partial^{(p_*,\ldots,p_*)}R_{\alpha,J}$ is a well-defined continuous function on $\mathbb{R}^d\setminus\{0\}$ having a compact support included in $\mathcal{K}^d$. Hence, integrating by parts in (3.3.4), we obtain that

$$\begin{aligned}\left|\partial^b(\Psi_{\alpha,J}[f])(x)\right| &= 2^{(j_1+\cdots+j_d)/\alpha}\left|\int_{\mathcal{K}^d}\left(\Big(\partial^{(p_*,\ldots,p_*)}R_{\alpha,J}\Big)(\xi)\prod_{l=1}^d\frac{e^{-i(1+T+x_l)\xi_l}}{(1+T+x_l)^{p_*}}\right)\mathrm{d}\xi\right|\\ &\le c_1\frac{2^{(j_1+\cdots+j_d)/\alpha}}{\prod_{l=1}^d(1+T+x_l)^{p_*}}\sup_{\xi\in\mathcal{K}^d}\left|\Big(\partial^{(p_*,\ldots,p_*)}R_{\alpha,J}\Big)(\xi)\right|,\end{aligned} \tag{3.3.5}$$

where the constant $c_1 > 0$ is the Lebesgue measure of $\mathcal{K}^d$. On the other hand, using the Leibniz formula , we get, for every $\xi \in \mathbb{R}^d\setminus\{0\}$, that

$$\Big(\partial^{(p_*,\ldots,p_*)}R_{\alpha,J}\Big)(\xi) = \sum_{p_1=0}^{p_*}\cdots\sum_{p_d=0}^{p_*}\Big(\partial^{(p_1,\ldots,p_d)}f\Big)(2^J\xi)\prod_{l=1}^d\binom{p_*}{p_l}2^{j_lp_l}\Big(\partial^{p_*-p_l}\widehat{\Phi}_l\Big)(\xi_l). \tag{3.3.6}$$

In view of (3.3.5), it turns out that for deriving (3.2.33), it is enough to show that

$$\sup_{J\in\mathbb{Z}^d_+}\sup_{\xi\in\mathcal{K}^d}\left\{\Big(2^{-j_1}+\cdots+2^{-j_d}\Big)^{a'[f]+d/\alpha}\left|\Big(\partial^{(p_*,\ldots,p_*)}R_{\alpha,-J}\Big)(\xi)\right|\right\} < +\infty, \tag{3.3.7}$$

and for deriving (3.2.34), it is enough to show that, for all $\eta \in \Upsilon^*$,

$$\sup_{J\in\mathbb{Z}^d_{(\eta)}}\sup_{\xi\in\mathcal{K}^d}\left\{\prod_{l=1}^d 2^{j_l/\alpha}2^{-(1-\eta_l)j_l/\alpha}2^{j_l\eta_la_l[f]}\left|\Big(\partial^{(p_*,\ldots,p_*)}R_{\alpha,J}\Big)(\xi)\right|\right\} < +\infty; \tag{3.3.8}$$

recall that sets Υ^* and $\mathbb{Z}^d_{(\eta)}$ are defined respectively in (3.2.28) and (3.2.29).

We now focus on the proof of (3.3.7). In view of (3.3.6) and of the fact that the $\partial^{p_*-p_l}\widehat{\Phi}_l$'s, $l = 1,\ldots,d$ are bounded functions on $\mathcal{K}$, (3.3.7) can be obtained by showing that

$$\sup_{p\in\{0,1,2,\ldots,p_*\}^d}\sup_{J\in\mathbb{Z}^d_+}\sup_{\xi\in\mathcal{K}^d}\left\{\Big(2^{-j_1}+\cdots+2^{-j_d}\Big)^{a'[f]+d/\alpha}2^{-(j_1p_1+\cdots+j_dp_d)}\left|(\partial^pf)(2^{-J}\xi)\right|\right\} < +\infty. \tag{3.3.9}$$

Observe that, for any $\xi \in \mathcal{K}^d$ and $J \in \mathbb{Z}_+^d$, one has $\left\|2^{-J}\xi\right\| \leq 8\pi\sqrt{d}/3$. Thus, assuming that $p \in \{0,1,2,\dots,p_*\}^d$ is arbitrary and using (3.1.2), one gets that

$$\left|\partial^p f(2^{-J}\xi)\right| \leq c_2 \left(2^{-2j_1}\xi_1^2 + \cdots + 2^{-2j_d}\xi_d^2\right)^{-\frac{a'[f]}{2}-\frac{d}{2\alpha}-\frac{\mathrm{l}(p)}{2}}, \tag{3.3.10}$$

where c_2 denotes the constant c' in (3.1.2) which does not depend on b, J, and ξ. On the other hand, the fact that $\xi \in \mathcal{K}^d$ implies that

$$\min_{1\leq l\leq d} |\xi_l| \geq 2\pi/3 \geq 1. \tag{3.3.11}$$

It follows from these inequalities and from the equality $\mathrm{l}(p) = p_1 + \cdots + p_d$ that

$$\begin{aligned}
&\left(2^{-2j_1}\xi_1^2 + \cdots + 2^{-2j_d}\xi_d^2\right)^{-\frac{a'[f]}{2}-\frac{d}{2\alpha}-\frac{\mathrm{l}(p)}{2}} &(3.3.12)\\
&\leq \left(2^{-2j_1} + \cdots + 2^{-2j_d}\right)^{-\frac{a'[f]}{2}-\frac{d}{2\alpha}-\frac{\mathrm{l}(p)}{2}}\\
&= \left(2^{-2j_1} + \cdots + 2^{-2j_d}\right)^{-\frac{a'[f]}{2}-\frac{d}{2\alpha}} \prod_{l=1}^{d}\left(2^{-2j_1} + \cdots + 2^{-2j_d}\right)^{-\frac{p_l}{2}}\\
&\leq c_3 \left(2^{-j_1} + \cdots + 2^{-j_d}\right)^{-a'[f]-\frac{d}{\alpha}} 2^{j_1p_1+\cdots+j_dp_d}, &(3.3.13)
\end{aligned}$$

where $c_3 > 0$ is a constant only depending on d, $a'[f]$ and α. (3.3.9) results from (3.3.10) and (3.3.12).

We now focus on the proof of (3.3.8), where $\eta \in \Upsilon^*$ is arbitrary and fixed. In view of (3.3.6) and of the fact that the $\partial^{p_*-p_l}\widehat{\Phi}_l$'s, $l = 1,\dots,d$, are bounded functions on $\mathcal{K}$, (3.3.8) can be obtained by showing that

$$\sup_{p\in\{0,1,2,\dots,p_*\}^d} \sup_{J\in\mathbb{Z}^d_{(\eta)}} \sup_{\xi\in\mathcal{K}^d} \left\{2^{j_1p_1+\cdots+j_dp_d}\left|(\partial^p f)(2^J\xi)\right| \prod_{l=1}^{d} 2^{j_l/\alpha}2^{-(1-\eta_l)j_l/\alpha}2^{j_l\eta_l a_l[f]}\right\} < +\infty. \tag{3.3.14}$$

Let $p = (p_1,\dots,p_d) \in \{0,1,2,\dots,p_*\}^d$, $J = (j_1,\dots,j_d) \in \mathbb{Z}^d_{(\eta)}$ and $\xi = (\xi_1,\dots,\xi_d) \in \mathcal{K}^d$ be arbitrary. Observe that, we know from the definition of $\mathbb{Z}^d_{(\eta)}$ (see (3.2.29) and (3.2.30)) that J has at least one positive coordinate, let us say j_r. Therefore, using (3.3.11), one gets that $\left\|2^J\xi\right\| \geq |2^{j_r}\xi_r| \geq 2\pi/3$. Then, it follows from (3.1.3) that

$$\left|\partial^p f(2^J\xi)\right| \leq c_4 \prod_{l=1}^{d}\left(1 + 2^{j_l}|\xi_l|\right)^{-a_l[f]-\frac{1}{\alpha}-p_l}, \tag{3.3.15}$$

where c_4 denotes the constant c in (3.1.3) which does not depend on p, J, and ξ. We now provide a convenient upper bound for the right hand side in (3.3.15). To this end, we notice that $\{1,\dots,d\} = \mathbb{L}_+ \cup \mathbb{L}_-$, where disjoint sets $\mathbb{L}_+$ and $\mathbb{L}_-$ are defined by $\mathbb{L}_+ =$

$\{l \in \{1,\dots,d\} : \eta_l = 1\}$ and $\mathbb{L}_- = \{l \in \{1,\dots,d\} : \eta_l = 0\}$. Then, using (3.3.11) and the fact that $-j_l \geq 0$ when $l \in \mathbb{L}_-$, one obtains that

$$\prod_{l\in\mathbb{L}_+} \left(1 + 2^{j_l} |\xi_l|\right)^{-a_l[f]-\frac{1}{\alpha}-p_l} \leq \prod_{l\in\mathbb{L}_+} 2^{-j_l\left(a_l[f]+\frac{1}{\alpha}+p_l\right)} \leq 2^{-(j_1p_1+\cdots+j_dp_d)} \prod_{l=1}^{d} 2^{-j_l\eta_l\left(a_l[f]+\frac{1}{\alpha}\right)}. \tag{3.3.16}$$

On the other hand, one clearly has that

$$\prod_{l\in\mathbb{L}_-} \left(1 + 2^{j_l} |\xi_l|\right)^{-a_l[f]-\frac{1}{\alpha}-p_l} \leq 1, \tag{3.3.17}$$

with the convention that $\prod_{l\in\mathbb{L}_-} \cdots = 1$, when $\mathbb{L}_-$ is the empty set. Next, combining (3.3.16) and (3.3.17), it follows that:

$$\begin{aligned} \prod_{l=1}^{d} \left(1 + 2^{j_l} |\xi_l|\right)^{-a_l[f]-\frac{1}{\alpha}-p_l} &\leq 2^{-(j_1p_1+\cdots+j_dp_d)} \prod_{l=1}^{d} 2^{-\eta_l j_l/\alpha}\, 2^{-j_l\eta_l a_l[f]} \\ &= 2^{-(j_1p_1+\cdots+j_dp_d)} \prod_{l=1}^{d} 2^{-j_l/\alpha}\, 2^{(1-\eta_l)j_l/\alpha}\, 2^{-j_l\eta_l a_l[f]}. \end{aligned} \tag{3.3.18}$$

Finally (3.3.14) results from (3.3.15) and (3.3.18). □

Proof of Lemma 3.2.9. One denotes by $\lfloor v \rfloor$ the integer part of v, and one sets $w(v) = v - \lfloor v \rfloor$. Then, using the triangle inequality and the inequality $|\lfloor v \rfloor| \leq |v| + 1$, one obtains that

$$\sum_{k\in\mathbb{Z}} \frac{\sqrt{\log(3+\theta+|k|)}}{(2+|v-k|)^{p_*}} = \sum_{k\in\mathbb{Z}} \frac{\sqrt{\log(3+\theta+|k+\lfloor v \rfloor|)}}{(2+|v-\lfloor v \rfloor-k|)^{p_*}} \leq \sum_{k\in\mathbb{Z}} \frac{\sqrt{\log(3+\theta+|k|+1+|v|)}}{(2+|w(v)-k|)^{p_*}}. \tag{3.3.19}$$

Next, let c be the constant defined as:

$$c := 2 \sup_{w\in[0,1]} \left\{ \sum_{k\in\mathbb{Z}} \frac{\sqrt{\log\left(4+|k|\right)}}{(2+|w-k|)^{p_*}} \right\}. \tag{3.3.20}$$

Observe that (3.1.1) and the inequality $2 + |w-k| \geq 1 + |k|$, for all $(k,w) \in \mathbb{Z} \times [0,1]$, imply that c is finite. Also, observe that, it follows from (3.2.38), the fact that $w(v) \in [0,1]$, and (3.3.20) that

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \frac{\sqrt{\log(3+\theta+|k|+1+|v|)}}{(2+|w(v)-k|)^{p_*}} &\leq 2\sum_{k\in\mathbb{Z}} \frac{\sqrt{\log(4+|k|)}\sqrt{\log(3+\theta+|v|)}}{(2+|w(v)-k|)^{p_*}} \\ &\leq c\sqrt{\log(3+\theta+|v|)}. \end{aligned} \tag{3.3.21}$$

Finally combining (3.3.19) and (3.3.21), one gets (3.2.40). □

3.4 Proofs of Lemma 3.2.5 and Proposition 3.2.2

Proof of Lemma 3.2.5. Assume that the real numbers $a' \in [0,1)$, $\alpha \in (0,2]$, and $\delta > 0$ are arbitrary and fixed. Also assume that the positive integer d and $r \in \{1,\ldots,d\}$ are arbitrary and fixed. Then, for any fixed $r' \in \{1,\ldots,d\}$, let $\Gamma_{r'}$ be the set defined as

$$\Gamma_{r'} := \Big\{ J = (j_1,\ldots,j_d) \in \mathbb{Z}_+^d : j_{r'} = \min\{j_1,\ldots,j_d\} \Big\},$$

and let $S_{r,r'}$ be the positive quantity defined as

$$S_{r,r'} = \sum_{J\in\Gamma_{r'}} 2^{-j_r(1-a')} \Big(2^{-j_1} + \cdots + 2^{-j_d}\Big)^{-d/\alpha} \prod_{l=1}^{d} 2^{-j_l/\alpha} \sqrt{\log(3+j_l)}(1+j_l)^{1/\alpha+\delta}.$$

The fact that $\mathbb{Z}_+^d = \bigcup_{r'=1}^d \Gamma_{r'}$ implies that

$$\sum_{J\in\mathbb{Z}_+^d} 2^{-j_r(1-a')} \Big(2^{-j_1} + \cdots + 2^{-j_d}\Big)^{-d/\alpha} \prod_{l=1}^{d} 2^{-j_l/\alpha} \sqrt{\log(3+j_l)}(1+j_l)^{1/\alpha+\delta} \le \sum_{r'=1}^{d} S_{r,r'}.$$

On the other hand, standard computations, relying on the definitions of $\Gamma_{r'}$ and $S_{r,r'}$, allow to obtain, for each $r' \in \{1,\ldots,d\}$, that

$$\begin{aligned} S_{r,r'} \quad \le \quad \sum_{n=0}^{+\infty} \Bigg\{ & 2^{-n(1+1/\alpha-a'-d/\alpha)} \sqrt{\log(3+n)}(1+n)^{1/\alpha+\delta} \\ & \qquad \Bigg(\sum_{m=n}^{+\infty} 2^{-m/\alpha} \sqrt{\log(3+m)}(1+m)^{1/\alpha+\delta} \Bigg)^{d-1} \Bigg\}. \end{aligned}$$

Thus, in order to derive (3.2.25), it is enough to show that

$$\begin{aligned} \sum_{n=0}^{+\infty} \Bigg\{ & 2^{-n(1+1/\alpha-a'-d/\alpha)} \sqrt{\log(3+n)}(1+n)^{1/\alpha+\delta} \\ & \qquad \Bigg(\sum_{m=n}^{+\infty} 2^{-m/\alpha} \sqrt{\log(3+m)}(1+m)^{1/\alpha+\delta} \Bigg)^{d-1} \Bigg\} < +\infty. \end{aligned}$$

This can easily be obtained by making use of the inequality

$$\sum_{m=n}^{+\infty} 2^{-m/\alpha} \sqrt{\log(3+m)}(1+m)^{1/\alpha+\delta} \le c 2^{-n/\alpha} \sqrt{\log(3+n)}(1+n)^{1/\alpha+\delta}, \tag{3.4.1}$$

which holds for any non-negative integer n and for some finite constant c only depending on α and δ. Therefore, it remains to prove (3.4.1). The changes of variables $M = m - n$

and (3.2.38) entails that

$$\begin{aligned}
&\sum_{m=n}^{+\infty} 2^{-m/\alpha}\sqrt{\log(3+m)}(1+m)^{1/\alpha+\delta}\\
&= \sum_{M=0}^{+\infty} 2^{-(M+n)/\alpha}\sqrt{\log(3+m+n)}(1+M+n)^{1/\alpha+\delta}\\
&\le 2^{-n/\alpha+1}\sqrt{\log(3+n)}(1+n)^{1/\alpha+\delta}\sum_{M=0}^{+\infty} 2^{-M/\alpha}\sqrt{\log(3+M)}(1+M/(1+n))^{1/\alpha+\delta}\\
&\le 2^{-n/\alpha+1}\sqrt{\log(3+n)}(1+n)^{1/\alpha+\delta}\sum_{M=0}^{+\infty} 2^{-M/\alpha}\sqrt{\log(3+M)}(1+M)^{1/\alpha+\delta}.
\end{aligned}$$

Thus, (3.4.1) holds with $c := 2\sum_{M=0}^{+\infty} 2^{-M/\alpha}\sqrt{\log(3+M)}(1+M)^{1/\alpha+\delta}$. □

The proof of Proposition 3.2.2 is devided into the following two steps which will be obtained separately.

Step 1. We show that, for every fixed $t\in\mathbb{R}^d$, there exists $\widetilde{F}(t,\cdot)$ in $L^\alpha\Big(\mathbb{R}^d\Big)$ such that, for any increasing sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ of finite subsets of $\mathbb{Z}^d\times\mathbb{Z}^d$ which satisfies $\bigcup_{n\in\mathbb{N}}\mathcal{D}_n=\mathbb{Z}^d\times\mathbb{Z}^d$, one has

$$\lim_{n\to+\infty}\Delta_\alpha\left(\sum_{(J,K)\in\mathcal{D}_n}\Big(\Psi_{\alpha,J}[f](2^Jt-K)-\Psi_{\alpha,J}[f](-K)\Big)\overline{\widehat{\psi}_{\alpha,J,K}(\cdot)},\widetilde{F}(t,\cdot)\right)=0. \tag{3.4.2}$$

Step 2. We show that, for all $t\in\mathbb{R}^d$ and almost all $\xi\in\mathbb{R}^d$, $F(t,\xi)=\widetilde{F}(t,\xi)$.

Proof of Proposition 3.2.2 (Step 1). In view of Lemma 3.2.4 and (3.2.31), it is enough to show that, for all fixed $t\in\mathbb{R}^d$ and $\eta\in\Upsilon$, one has

$$\sum_{(J,K)\in\mathbb{Z}^d_{(\eta)}\times\mathbb{Z}^d}\Delta_\alpha\left(\Big(\Psi_{\alpha,J}[f](2^Jt-K)-\Psi_{\alpha,J}[f](-K)\Big)\overline{\widehat{\psi}_{\alpha,J,K}(\cdot)},0\right)<+\infty. \tag{3.4.3}$$

We will study the following 4 cases:

$\alpha\in(0,1)$ and $\eta=0$, $\quad\alpha\in[1,2]$ and $\eta=0$, $\quad\alpha\in(0,1)$ and $\eta\neq0$, $\quad\alpha\in[1,2]$ and $\eta\neq0$.

<u>Case 1: $\alpha\in(0,1)$ and $\eta=0$.</u> Notice that, in this case, one has $J\in\mathbb{Z}^d_{(0)}$, so it can be rewritten as $J=-J'$, where J' belongs to $\mathbb{Z}^d_+$. In the sequel J' is denoted by J. Then (3.2.13), (3.2.14) and the change of variable $\eta=2^{-J}\xi$ imply that, for all $K\in\mathbb{Z}^d$, one has

$$\begin{aligned}
&\Delta_\alpha\left(\Big(\Psi_{\alpha,-J}[f](2^{-J}t-K)-\Psi_{\alpha,-J}[f](-K)\Big)\overline{\widehat{\psi}_{\alpha,-J,K}(\cdot)},0\right)\\
&\qquad= c_1\Big|\Psi_{\alpha,-J}[f](2^{-J}t-K)-\Psi_{\alpha,-J}[f](-K)\Big|^\alpha,
\end{aligned} \tag{3.4.4}$$

where the constant $c_1 := \left(\int_{\mathbb{R}} |\widehat{\psi^1}(\eta)|^\alpha \,\mathrm{d}\xi\right)^d$ is finite. Next, let $T := \max_{1\le l\le d} |t_l|$. Using the mean value Theorem and the triangle inequality, we get that,

$$\left|\Psi_{\alpha,-J}[f](2^{-J}t-K)-\Psi_{\alpha,-J}[f](-K)\right| \le T\sum_{r=1}^{d} 2^{-j_r} \sup_{s\in[-T,T]^d}\left|\frac{\partial\Psi_{\alpha,-J}[f]}{\partial x_r}\Big(2^{-J}s-K\Big)\right|, \quad (3.4.5)$$

Moreover, combining (3.2.33) with the inequality,

$$1+T+\left|2^{-j_l}s_l-k_l\right| \ge 1+|k_l|, \quad \text{for all } l\in\{1,\dots,d\} \text{ and } s_l\in[-T,T],$$

we obtain, for every $r\in\{1,\dots,d\}$, that

$$2^{-j_r}\sup_{s\in[-T,T]^d}\left|\frac{\partial\Psi_{\alpha,-J}[f]}{\partial x_r}\Big(2^{-J}s-K\Big)\right| \le c_2\frac{2^{-j_r(1-a'[f])}\left(2^{-j_1}+\cdots+2^{-j_d}\right)^{-d/\alpha}\prod_{l=1}^d 2^{-j_l/\alpha}}{\prod_{l=1}^d\Big(1+|k_l|\Big)^{p_*}}, \quad (3.4.6)$$

where c_2 is a constant not depending on (J,K). On the other hand (3.1.1) impiles that

$$\sum_{K\in\mathbb{Z}^d}\prod_{l=1}^{d}\Big(1+|k_l|\Big)^{-\alpha p_*} < +\infty. \quad (3.4.7)$$

Finally, using (3.4.4) to (3.4.7), and the same arguments as in the proof of (3.2.25), we get (3.4.3).

Case 2: $\alpha\in[1,2]$ and $\eta=0$. The proof follows the same lines as in the case 1, except that one has to use (3.2.12) instead of (3.2.13).

Case 3: $\alpha\in(0,1)$ and $\eta\neq 0$. It follows from (3.2.13), the triangle inequality, and the sub-additivity on $[0,+\infty)$ of the function $z\mapsto z^\alpha$, that, for all $(J,K)\in\mathbb{Z}^d_{(\eta)}\times\mathbb{Z}^d$, one has

$$\begin{aligned}
&\Delta_\alpha\left(\Big(\Psi_{\alpha,J}[f](2^Jt-K)-\Psi_{\alpha,J}[f](-K)\Big)\overline{\widehat{\psi}_{\alpha,J,K}(\cdot)},0\right)\\
&\quad = c_1\left|\Psi_{\alpha,J}[f](2^Jt-K)-\Psi_{\alpha,J}[f](-K)\right|^\alpha\\
&\quad \le c_1\left|\Psi_{\alpha,J}[f](2^Jt-K)\right|^\alpha+|\Psi_{\alpha,J}[f](-K)|^\alpha\\
&\quad \le c_4\prod_{l=1}^d 2^{(1-\eta_l)j_l}2^{-j_l\eta_l a_l[f]\alpha}\left(\frac{1}{\Big(2+|2^{j_l}t_l-k_l|\Big)^{\alpha p_*}}+\frac{1}{\Big(2+|k_l|\Big)^{\alpha p_*}}\right).
\end{aligned}$$

Notice that c_3 is a constant not depending on (J,K). Also notice that the last inequality is obtained by using (3.2.34) in the case where $T=1$. Next, this inequality, (3.2.29), (3.2.30), and (3.1.1) yield that

$$\begin{aligned}
&\sum_{(J,K)\in\mathbb{Z}^d_{(\eta)}\times\mathbb{Z}^d} \Delta_\alpha\Big(\Big(\Psi_{\alpha,J}[f](2^J t-K)-\Psi_{\alpha,J}[f](-K)\Big)\overline{\widehat{\psi}_{\alpha,J,K}(\cdot)},0\Big)\\
&\le c_3 \sum_{J\in\mathbb{Z}^d_{(\eta)}}\prod_{l=1}^d 2^{(1-\eta_l)j_l}2^{-j_l\eta_l a_l[f]\alpha}\left(\sum_{k_l\in\mathbb{Z}}\frac{1}{\Big(2+|2^{j_l}t_l-k_l|\Big)^{\alpha p_*}}+\sum_{k_l\in\mathbb{Z}}\frac{1}{\Big(2+|k_l|\Big)^{\alpha p_*}}\right)\\
&= c_3 \sum_{J\in\mathbb{Z}^d_{(\eta)}}\prod_{l=1}^d 2^{(1-\eta_l)j_l}2^{-j_l\eta_l a_l[f]\alpha}\left(\sum_{k_l\in\mathbb{Z}}\frac{1}{\Big(2+|2^{j_l}t_l-\lfloor 2^{j_l}t_l\rfloor-k_l|\Big)^{\alpha p_*}}+\sum_{k_l\in\mathbb{Z}}\frac{1}{\Big(2+|k_l|\Big)^{\alpha p_*}}\right)\\
&\le 2^d c_3\prod_{l=1}^d\left\{\left(\sum_{j_l\in\mathbb{Z}_{\eta_l}}2^{(1-\eta_l)j_l}2^{-j_l\eta_l a_l[f]\alpha}\right)\left(\sum_{k_l\in\mathbb{Z}^d}\frac{1}{\Big(1+|k_l|\Big)^{\alpha p_*}}\right)\right\}<+\infty,
\end{aligned}$$

which show that (3.4.3) holds.

Case 4: $\alpha\in[1,2]$ and $\eta\neq 0$. The proof follows the same lines as in the case 3, except that one has to use (3.2.12) instead of (3.2.13). □

Proof of of Proposition 3.2.2 (Step 2). For any fixed $m\in\mathbb{N}$, we denote by Θ_m the closed subset of $\mathbb{R}^d$ defined as

$$\Theta_m := \Big\{\xi=(\xi_1,\dots,\xi_d)\in\mathbb{R}^d : \min\Big\{|\xi_1|,\dots,|\xi_d|\Big\}\ge 2^{-m+1}\pi/3\Big\}. \tag{3.4.8}$$

In view of (3.2.20) and Definition 3.1.1, it can easily be seen that, for any fixed $t\in\mathbb{R}^d$, the function $F(t,\cdot)\mathbb{1}_{\Theta_m}(\cdot):\xi\mapsto F(t,\xi)\mathbb{1}_{\Theta_m}(\xi)$ belongs to the Hilbert space $L^2\Big(\mathbb{R}^d\Big)$. Therefore, using the fact that $\{\widehat{\psi}_{J,K}:(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d\}$ is an orthonormal basis of this space, similarly to (3.2.5), one gets that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^d}\left|F(t,\xi)\mathbb{1}_{\Theta_m}(\xi)-\sum_{(J,K)\in\mathcal{D}_n}w_{J,K}(t)\overline{\widehat{\psi}_{J,K}(\xi)}\right|^2\mathrm{d}\xi=0, \tag{3.4.9}$$

where

$$w_{J,K}(t):=\int_{\mathbb{R}^d}F(t,\xi)\mathbb{1}_{\Theta_m}(\xi)\,\mathrm{d}\xi=\int_{\Theta_m}\Big(e^{it\cdot\xi}-1\Big)f(\xi)\widehat{\psi}_{J,K}(\xi)\,\mathrm{d}\xi, \tag{3.4.10}$$

and $(\mathcal{D}_n)_{n\in\mathbb{N}}$ is an arbitrary increasing sequence of finite subsets of $\mathbb{Z}^d\times\mathbb{Z}^d$ such that $\bigcup_{n\in\mathbb{N}}\mathcal{D}_n=\mathbb{Z}^d\times\mathbb{Z}^d$. Next, we denote $\mathcal{C}_m$ the compact the subset of Θ_m defined as

$$\mathcal{C}_m:=\Big\{\xi=(\xi_1,\dots,\xi_d)\in\mathbb{R}^d : 2^{m+3}\pi/3\ge\max_{l=1,\dots,d}|\xi_l|\ge\min_{l=1,\dots,d}|\xi_l|\ge 2^{-m+3}\pi/3\Big\}. \tag{3.4.11}$$

Let us show that, for all $(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d$ and $\xi\in\mathcal{C}_m$, one has

$$w_{J,K}(t)\overline{\widehat{\psi}_{J,K}(\xi)}=\Big(\Psi_J[f](2^J t-K)-\Psi_J[f](-K)\Big)\overline{\widehat{\psi}_{J,K}(\xi)}, \tag{3.4.12}$$

where the function $\Psi_J[f]$ is as in (3.2.7). To this end, we will study the following two cases: $\min\{j_1,\dots,j_d\} < -m$ and $\min\{j_1,\dots,j_d\} \geq -m$, where the integers $j_1,\dots,j_d$ are the coordinates of J, that is $J=(j_1,\dots,j_d)$. In the first case $\min\{j_1,\dots,j_d\} < -m$, using (3.2.4) and (3.4.11), one gets that $\widehat{\psi}_{J,K}(\xi)=0$, for each $\xi\in\mathcal{C}_m$; therefore (3.4.12) holds. In the second case $\min\{j_1,\dots,j_d\} \geq -m$, it follows from (3.2.4) and (3.4.8) that $\operatorname{supp}\widehat{\psi}_{J,K}\subset\Theta_m$. Thus, (3.4.10), (3.2.3), the change of variable $(\eta_1,\dots,\eta_d)=(2^{-j_1}\xi_1,\dots,2^{-j_d}\xi_d)$, and (3.2.7) imply that

$$w_{J,K}(t)=\Psi_J[f](2^Jt-K)-\Psi_J[f](-K).$$

Therefore (3.4.12) is satisfied.

Next, using (3.4.12), (3.2.19), (3.4.9), and the inclusion $\mathcal{C}_m\subset\Theta_m$ one gets that

$$\lim_{n\to+\infty}\int_{\mathcal{C}_m}\left|F(t,\xi)-\sum_{(J,K)\in\mathcal{D}_n}\Big(\Psi_{\alpha,J}[f]\Big(2^Jt-K\Big)-\Psi_{\alpha,J}[f](-K)\Big)\,\overline{\widehat{\psi}_{\alpha,J,K}(\xi)}\right|^2\mathrm{d}\xi=0.$$

Then the Hölder inequality, combined with the fact that $\mathcal{C}_m$ has a finite Lebesgue measure, implies that

$$\lim_{n\to+\infty}\int_{\mathcal{C}_m}\left|F(t,\xi)-\sum_{(J,K)\in\mathcal{D}_n}\Big(\Psi_{\alpha,J}[f]\Big(2^Jt-K\Big)-\Psi_{\alpha,J}[f](-K)\Big)\,\overline{\widehat{\psi}_{\alpha,J,K}(\xi)}\right|^\alpha\mathrm{d}\xi=0. \quad (3.4.13)$$

On the other hand, (3.4.2) entails that,

$$\lim_{n\to+\infty}\int_{\mathcal{C}_m}\left|\widetilde{F}(t,\xi)-\sum_{(J,K)\in\mathcal{D}_n}\Big(\Psi_{\alpha,J}[f](2^Jt-K)-\Psi_{\alpha,J}[f](-K)\Big)\overline{\widehat{\psi}_{\alpha,J,K}(\xi)}\right|^\alpha\mathrm{d}\xi=0. \quad (3.4.14)$$

Finally, it follows from (3.4.13), and (3.4.14) that, for all $m\in\mathbb{N}$ and for almost all $\xi\in\mathcal{C}_m$, one has $\widetilde{F}(t,\xi)=F(t,\xi)$; this amounts to saying that $\widetilde{F}(t,\xi)=F(t,\xi)$, *for almost all* $\xi\in\mathbb{R}^d$, since $\bigcup_{m\in\mathbb{N}}\mathcal{C}_m=(\mathbb{R}\setminus\{0\})^d$. □

3.5 Proof of Lemma 3.2.7

In order to show that Lemma 3.2.7 holds, we need the following preliminary result

Lemma 3.5.1. *There exists a positive constant c such that for any sequence of complex-valued centered*[5] *Gaussian random variables $\left\{G_{J,K}:(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d\right\}$, defined on $(\Omega,\mathcal{G},\mathbb{P})$, one has*

$$\mathbb{E}\left\{\sup_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}\left(\frac{|G_{J,K}|}{\sqrt{\log\Big(3+\sum_{l=1}^d\Big(|j_l|+|k_l|\Big)\Big)}}\right)\right\}\leq c\sqrt{\sup_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}\mathbb{E}\left[|G_{J,K}|^2\right]}, \quad (3.5.1)$$

where the j_l's and k_l's respectively denote the coordinates of J and K.

[5]That is satisfying $\mathbb{E}(G_{J,K})=0$, for all $(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d$.

Proof. We set,

$$\Sigma(G) := \sqrt{\sup_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} \mathbb{E}\left[|G_{J,K}|^2\right]} \tag{3.5.2}$$

and, for all $(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d$,

$$b_{J,K} := \sqrt{\log\left(3+\sum_{l=1}^{d}\Big(|j_l|+|k_l|\Big)\right)}. \tag{3.5.3}$$

Clearly the lemma holds when $\Sigma(G)=0$, and also when $\Sigma(G)=+\infty$. Thus, in the sequel, we assume that $0<\Sigma(G)<+\infty$. Using the fact that the expectation of an arbitrary non-negative random variable Z can be expressed as $\mathbb{E}[Z]=\int_0^{+\infty}\mathbb{P}(Z>x)\,\mathrm{d}x$, we get that

$$\begin{aligned}
\mathbb{E}\left[\sup_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}\left(\frac{|G_{J,K}|}{\Sigma(G)b_{J,K}}\right)\right] &= \int_0^{+\infty}\mathbb{P}\left(\sup_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}\left(\frac{|G_{J,K}|}{\Sigma(G)b_{J,K}}\right)>x\right)\mathrm{d}x\\
&\le 2^{d+1}+\int_{2^{d+1}}^{+\infty}\mathbb{P}\left(\sup_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}\left(\frac{|G_{J,K}|}{\Sigma(G)b_{J,K}}\right)>x\right)\mathrm{d}x\\
&\le 2^{d+1}+\sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}\int_{2^{d+1}}^{+\infty}\mathbb{P}\left(\frac{|G_{J,K}|}{\Sigma(G)b_{J,K}}>x\right)\mathrm{d}x,
\end{aligned} \tag{3.5.4}$$

where the last inequality follows from the equality

$$\left\{\omega\in\Omega:\sup_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}\left(\frac{|G_{J,K}(\omega)|}{\Sigma(G)b_{J,K}}\right)>x\right\}=\bigcup_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}\left\{\omega\in\Omega:\frac{|G_{J,K}(\omega)|}{\Sigma(G)b_{J,K}}>x\right\}.$$

Next, denoting by $\mathcal{R}e(G_{J,K})$ and $\mathcal{I}m(G_{J,K})$ the real and the imaginary parts of $G_{J,K}$, then, in view of the equality $|G_{J,K}|=\sqrt{|\mathcal{R}e(G_{J,K})|^2+|\mathcal{I}m(G_{J,K})|^2}$, for all $x\ge 2^{d+1}$, one has

$$\mathbb{P}\left(\frac{|G_{J,K}|}{\Sigma(G)b_{J,K}}>x\right)\le\mathbb{P}\left(\frac{|\mathcal{R}e(G_{J,K})|}{\Sigma(G)b_{J,K}}>2^{-1/2}x\right)+\mathbb{P}\left(\frac{|\mathcal{I}m(G_{J,K})|}{\Sigma(G)b_{J,K}}>2^{-1/2}x\right). \tag{3.5.5}$$

Now, we are going to show that

$$\mathbb{P}\left(\frac{|\mathcal{R}e(G_{J,K})|}{\Sigma(G)b_{J,K}}>2^{-1/2}x\right)\le\exp\left(-2^{-2}\,b_{J,K}^2\,x^2\right); \tag{3.5.6}$$

similarly, it can be shown that

$$\mathbb{P}\left(\frac{|\mathcal{I}m(G_{J,K})|}{\Sigma(G)b_{J,K}}>2^{-1/2}x\right)\le\exp\left(-2^{-2}\,b_{J,K}^2\,x^2\right). \tag{3.5.7}$$

We set

$$\sigma(G_{J,K}) := \sqrt{\mathbb{E}\left[|\mathcal{R}e(G_{J,K})|^2\right]};$$

observe that, in view of the first equality in (3.5.2), one has

$$\Sigma(G) \geq \sigma(G_{J,K}). \tag{3.5.8}$$

It is clear that (3.5.6) holds when $\sigma(G_{J,K}) = 0$, since $\mathcal{R}e(G_{J,K})$ is then vanishing almost surely. So, in the sequel we assume that $\sigma(G_{J,K}) > 0$. Hence $\mathcal{R}e(G_{J,K})/\sigma(G_{J,K})$ is a well-defined real-valued standard Gaussian random variable. Therefore, using (3.5.8) and the fact that $2^{-1/2} b_{J,K} x \geq 2^d \sqrt{2\log 3} \geq 1$, we get that

$$\begin{aligned}
\mathbb{P}\left(\frac{|\mathcal{R}e(G_{J,K})|}{\Sigma(G) b_{J,K}} > 2^{-1/2} x\right) &\leq \mathbb{P}\left(\frac{|\mathcal{R}e(G_{J,K})|}{\sigma(G_{J,K}) b_{J,K}} > 2^{-1/2} x\right) \\
&\leq \int_{2^{-1/2} b_{J,K} x}^{+\infty} e^{-y^2/2} \,\mathrm{d}y \\
&\leq \int_{2^{-1/2} b_{J,K} x}^{+\infty} y e^{-y^2/2} \,\mathrm{d}y \\
&= \exp\left(-2^{-2}\, b_{J,K}^2\, x^2\right),
\end{aligned}$$

which shows that (3.5.6) holds.

Next putting together (3.5.5), (3.5.6), (3.5.7) and the inequalities $2^{-2}\, b_{J,K}^2\, x \geq 2^{d-2} \log 3 \geq 1$, we obtain that

$$\int_{2^{d+1}}^{+\infty} \mathbb{P}\left(\frac{|G_{J,K}|}{\Sigma(G) b_{J,K}} > x\right) \mathrm{d}x \leq 2\int_{2^{d+1}}^{+\infty} 2^{-2}\, b_{J,K}^2\, x \exp\left(-2^{-2}\, b_{J,K}^2\, x^2\right) \mathrm{d}x = \exp\left(-2^{2d}\, b_{J,K}^2\right). \tag{3.5.9}$$

Finally, in view of (3.5.2), (3.5.3), (3.5.4) and (3.5.9), it turns out that in order to obtain (3.5.1) it is enough to show that

$$\sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} \left(3 + \sum_{l=1}^d \Big(|j_l| + |k_l|\Big)\right)^{-4^d} < +\infty.$$

This can be shown by noticing that $4^d \geq 4d$ and that

$$\begin{aligned}
\left(3 + \sum_{l=1}^d \Big(|j_l| + |k_l|\Big)\right)^{-4^d} &\leq \left(3 + \sum_{l=1}^d \Big(|j_l| + |k_l|\Big)\right)^{-4d} \\
&= \prod_{m=1}^d \left(3 + \sum_{l=1}^d \Big(|j_l| + |k_l|\Big)\right)^{-4} \\
&\leq \prod_{m=1}^d \left(3 + \Big(|j_m| + |k_m|\Big)\right)^{-4} \leq \prod_{m=1}^d \Big(3 + |j_m|\Big)^{-2} \Big(3 + |k_m|\Big)^{-2}.
\end{aligned}$$

□

We are now in the position to prove Lemma 3.2.7.

Proof of Lemma 3.2.7. First we recall that the third result provided by Lemma 3.2.7 (in other words the inequality (3.2.37) which holds in the Gaussian case $\alpha = 2$) is rather classical. We will skip its proof; it can be found in e.g. [4]. In all the sequel, we assume that $\alpha \in (0, 2)$. Notice that, in view of (3.2.17), for all $(J, K) \in \mathbb{Z}^d \times \mathbb{Z}^d$, one clearly has

$$|\varepsilon_{\alpha,J,K}| \leq \Big| \int_{\mathbb{R}^d} \overline{\widehat{\psi}_{\alpha,J,K}(\xi)} \, \mathrm{d}\widetilde{M}_\alpha(\xi) \Big|. \tag{3.5.10}$$

Thus, in order to get (3.2.35) and (3.2.36), it is enough to show that these two inequalities are satisfied when $\varepsilon_{\alpha,J,K}$ in them is replaced by $\int_{\mathbb{R}^d} \overline{\widehat{\psi}_{\alpha,J,K}(\xi)} \, \mathrm{d}\widetilde{M}_\alpha(\xi)$. The advantage of this strategy is that we know from Proposition 2.1.10, that for each $(J, K) \in \mathbb{Z}^d \times \mathbb{Z}^d$,

$$\int_{\mathbb{R}^d} \overline{\widehat{\psi}_{\alpha,J,K}(\xi)} \, \mathrm{d}\widetilde{M}_\alpha(\xi) = a(\alpha) \sum_{m=1}^{+\infty} g_m \Gamma_m^{-1/\alpha} \phi(\kappa^m)^{-1/\alpha} \overline{\widehat{\psi}_{\alpha,J,K}(\kappa^m)}; \tag{3.5.11}$$

moreover, we can and will assume that the g_m's, $m \in \mathbb{N}$, are complex-valued centred Gaussian random variables, and that the function ϕ is such that for all $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d \setminus \{0\}$, one has,

$$\phi(\xi) := \left(\frac{\epsilon}{4}\right)^d \prod_{l=1}^d |\xi_l|^{-1} \left(1 + |\log|\xi_l||\right)^{-1-\epsilon},$$

where ϵ is an arbitrary fixed positive real number. Therefore, using (3.2.14), and (3.2.15) for every $(J, K) \in \mathbb{Z}^d \times \mathbb{Z}^d$ and $m \in \mathbb{N}^*$, we obtain that

$$\begin{aligned} &\left| \phi(\kappa^m)^{-1/\alpha} \overline{\widehat{\psi}_{\alpha,J,K}(\kappa^m)} \right| \\ &\leq \left(\frac{\epsilon}{4}\right)^{-d/\alpha} \prod_{l=1}^d \left|2^{-j_l} \kappa_l^m\right|^{1/\alpha} \left(1 + |j_l| + \left|\log\left|2^{-j_l}\kappa_l^m\right|\right|\right)^{(1+\epsilon)/\alpha} \left|\widehat{\psi^1}(2^{-j_l}\kappa_l^m)\right| \\ &\leq c_1 \prod_{l=1}^d (1 + |j_l|)^{(1+\epsilon)/\alpha}, \end{aligned} \tag{3.5.12}$$

where c_1 is a deterministic constant not depending on (J, K) and m. On the other hand, in view of the Gaussianity assumption on the g_m's, $m \in \mathbb{N}$, it can be derived from the Borel-Cantelli Lemma that, almost surely, for all $m \in \mathbb{N}$, one has

$$|g_m| \leq C_2 \sqrt{\log(3 + m)}, \tag{3.5.13}$$

where C_2 is a finite random variable not depending on (J, K) and m. Also, observe that, in view of (2.1.8), it results from the strong law of large number, that almost surely, for any $m \in \mathbb{N}$, the Poisson arrival time Γ_m satisfies

$$C_3 m \leq \Gamma_m \leq C_4 m, \tag{3.5.14}$$

where C_3 and C_4 are two positive finite random variables not depending on (J,K) and m. Next, we suppose for a while that $\alpha \in (0,1)$, then the random variable

$$C_5 := a(\alpha) c_1 C_2 C_3 \sum_{m=1}^{+\infty} m^{-1/\alpha} \sqrt{\log(3+m)}$$

is almost surely finite; moreover, it follows from the triangle inequality and from the relations (3.5.11) to (3.5.14) that, almost surely, for all $(J,K) \in \mathbb{Z}^d \times \mathbb{Z}^d$, one has

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \overline{\widehat{\psi}_{\alpha,J,K}(\xi)} \, \mathrm{d}\widetilde{M}_\alpha(\xi) \right| &\leq a(\alpha) \sum_{m=1}^{+\infty} |g_m| \, \Gamma_m^{-1/\alpha} \phi(\kappa^m)^{-1/\alpha} \left| \overline{\widehat{\psi}_{\alpha,J,K}(\kappa^m)} \right| \\ &\leq C_5 \prod_{l=1}^{d} (1+|j_l|)^{(1+\epsilon)/\alpha} . \end{aligned}$$

These inequalities combined with (3.5.10) show that (3.2.35) holds.

From now on, we assume that $\alpha \in [1,2)$ and our goal is to derive (3.2.36); notice that the previous strategy has to be modified since C_5 is no longer finite. Let $\mathcal{F}_{\Gamma,\kappa}$ be the sub $\sigma-$field of $\mathcal{G}$ generated by the two sequences of random variables $\{\Gamma_m : m \in \mathbb{N}\}$ and $\{\kappa^m : m \in \mathbb{N}\}$. We denote by $\mathbb{E}_{\Gamma,\kappa}[\,\cdot\,]$ the conditional expectation operator with respect to $\mathcal{F}_{\Gamma,\kappa}$; recall that $\mathbb{E}(\,\cdot\,)$ denotes the classical expectation operator. We know from (3.5.11) that conditionally to $\mathcal{F}_{\Gamma,\kappa}$, for any arbitrary $(J,K) \in \mathbb{Z}^d \times \mathbb{Z}^d$, the random variable

$$G_{J,K} := \left(\prod_{l=1}^{d} (1+|j_l|)^{-(1+\epsilon)/\alpha} \right) \int_{\mathbb{R}^d} \overline{\widehat{\psi}_{\alpha,J,K}(\xi)} \, \mathrm{d}\widetilde{M}_\alpha(\xi) \tag{3.5.15}$$

has a centred Gaussian distribution over $\mathbb{C}$. Then, assuming that c_6 denotes the constant c in (3.5.1), one can derive from Lemma 3.5.1 that the following inequality holds almost surely:

$$\mathbb{E}_{\Gamma,\kappa}\left[\sup_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} \left(\frac{|G_{J,K}|}{\sqrt{\log\Big(3+\sum_{l=1}^d \Big(|j_l|+|k_l|\Big)\Big)}} \right) \right] \leq c_6 \sqrt{\sup_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} \mathbb{E}_{\Gamma,\kappa}\Big[|G_{J,K}|^2\Big]}. \tag{3.5.16}$$

Next, using the fact that $\mathbb{E}(\,\cdot\,) = \mathbb{E}\Big(\mathbb{E}_{\Gamma,\kappa}[\,\cdot\,]\Big)$, Cauchy-Schwarz inequality, and (3.5.16), one obtains that

$$\begin{aligned}
&\mathbb{E}\left(\sqrt{\sup_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}\left(\frac{|G_{J,K}|}{\sqrt{\log\left(3+\sum_{l=1}^d |j_l|+|k_l|\right)}}\right)}\right)\\
&=\mathbb{E}\left(\mathbb{E}_{\Gamma,\kappa}\left[\sqrt{\sup_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}\left(\frac{|G_{J,K}|}{\sqrt{\log\left(3+\sum_{l=1}^d |j_l|+|k_l|\right)}}\right)}\right]\right)\\
&\leq\mathbb{E}\left(\sqrt{\mathbb{E}_{\Gamma,\kappa}\left[\sup_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}\left(\frac{|G_{J,K}|}{\sqrt{\log\left(3+\sum_{l=1}^d |j_l|+|k_l|\right)}}\right)\right]}\right)\\
&\leq\sqrt{c_6}\,\mathbb{E}\left(\left(\sup_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}\mathbb{E}_{\Gamma,\kappa}\left[|G_{J,K}|^2\right]\right)^{1/4}\right).
\end{aligned} \tag{3.5.17}$$

On the other hand, (3.5.11) and (3.5.15) imply that, one has, almost surely, for any arbitrary $(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d$,

$$\mathbb{E}_{\Gamma,\kappa}\left[|G_{J,K}|^2\right]=c_7\left(\prod_{l=1}^d(1+|j_l|)^{-2(1+\epsilon)/\alpha}\right)\sum_{m=1}^{+\infty}\Gamma_m^{-2/\alpha}\phi(\kappa^m)^{-2/\alpha}\left|\widehat{\psi}_{\alpha,J,K}(\kappa^m)\right|^2,$$

where the deterministic constant $c_7:=a(\alpha)^2\,\mathbb{E}\left(|g_1|^2\right)$ does not depend on (J,K). Then, using (3.5.12), one gets, almost surely, that

$$\sup_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}\mathbb{E}_{\Gamma,\kappa}\left[|G_{J,K}|^2\right]\leq c_8\sum_{m=1}^{+\infty}\Gamma_m^{-2/\alpha}, \tag{3.5.18}$$

where the deterministic constant $c_8:=c_1^2c_7$. Finally, in view of (3.5.10), (3.5.15), (3.5.17) and (3.5.18), it turns out that (3.2.36) can be obtained by showing that

$$\mathbb{E}\left(\left(\sum_{m=1}^{+\infty}\Gamma_m^{-2/\alpha}\right)^{1/4}\right)<+\infty. \tag{3.5.19}$$

We know from Remark 4 on page 29 in [27], that the positive random variable $\sum_{m=1}^{+\infty}\Gamma_m^{-2/\alpha}$ has a stable distribution with a stability parameter equal to $\alpha/2$. Thus combining the fact that $\alpha/2>1/4$ with the Property 1.2.16 on page 18 in [27], one gets (3.5.19). □

4 Upper estimates on path behaviour

Abstract

The first main foal of this chapter is to derive, in terms of the directional rates of vanishing at infinity of f along the axes of $\mathbb{R}^d$, upper estimates for amplitudes of generalized directional increments and classical (non-directional) iterated increments of the sample paths $X[f](\cdot,\omega)$, on an arbitrary compact cube of $\mathbb{R}^d$. The second main goal of this chapter is to obtain, in terms of the exponent $a'[f]$ which governed the behaviour of f in a neighbourhood of 0 (see Definition 3.1.1), upper estimates for the amplitude of $X[f](t,\omega)$, when $\|t\| \geq 1$ (that is, in practive, for large values of $\|t\|$). The third main goal of Chapter 4 is to show that, for any $b \neq 0$, the function $(\partial^b X[f])(\cdot,\omega)$, when it exists, is bounded when $\alpha \in (0,1)$, and that it has at most a logarithmic increase at infinity when $\alpha \in [1,2]$.

Contents

4.1 Generalized directional increments on a compact cube

Let f be an admissible function, $X[f]$ the field associated with f, and $X[f]^\eta$ an arbitrary η-frequency part of $X[f]$, where $\eta = (\eta_1, \ldots, \eta_d) \in \Upsilon := \{0,1\}^d$ (see Definition 3.1.1, (2.3.3), Definition 3.2.12 and Remark 3.2.13). The directional rates of vanishing at infinity of f

along the axes of $\mathbb{R}^d$ are governed by the positive exponents $a_1[f], \dots, a_d[f]$ through the inequality (3.1.3). The main goal of the present section is to draw connections between these exponents and the anisotropic behaviour of the generalized directional increments of $X[f]^\eta$ and $X[f]$, on an arbitrary compact cube of $\mathbb{R}^d$. The methodology we use is based on the wavelet type random series representations (3.2.53) and (3.2.52) of $X[f]^\eta$ and $X[f]$. It is worth mentioning that all the results we obtain are valid on Ω_1^*, the "universal" event of probability 1 which was introduced in Lemma 3.2.7; we recall that "universal" means that Ω_1^* does not depend on f. In order to precisely state our results, first, we need to introduce some notations.

For every fixed $k \in \{1, \dots, d\}$ and $h_k \in \mathbb{R}$, we denote by $\Delta^k_{h_k}$, the operator from the space of the real-valued functions on $\mathbb{R}^d$, into itself; so that, when g is such a function, $\Delta^k_{h_k} g$ is then the function defined, for all $x \in \mathbb{R}^d$, as

$$\Big(\Delta^k_{h_k} g\Big)(x) = g(x + h_k e_k) - g(x), \tag{4.1.1}$$

where e_k denotes the vector of $\mathbb{R}^d$ whose k-th coordinate equals 1 and the others vanish. Clearly $\Delta^k_{h_k} g$ is at least as much regular as g is; in particular, when g belongs to the space $\mathcal{C}^\infty(\mathbb{R}^d)$ of the infinitely differentiable real-valued functions defined on $\mathbb{R}^d$, then $\Delta^k_{h_k} g$ shares the same property. On the other hand, notice that the operators $\Delta^k_{h_k}$ are commutative, in the sense that, for all $(k, k') \in \{1, \dots, d\}^2$ and $(h_k, h'_{k'}) \in \mathbb{R}^2$, one has

$$\Delta^{k'}_{h'_{k'}} \circ \Delta^k_{h_k} = \Delta^k_{h_k} \circ \Delta^{k'}_{h'_{k'}},$$

where the symbol "$\circ$" denotes the usual composition of operators. For every $h = (h_1, \dots, h_d) \in \mathbb{R}^d$ and multi-index $B = (b_1, \dots, b_d) \in \mathbb{Z}^d_+$, we denote by $\Delta^B_{(h)}$, the operator from the space of the real-valued functions on $\mathbb{R}^d$ into itself, defined by

$$\Delta^B_{(h)} := \Delta^{1,b_1}_{h_1} \circ \dots \circ \Delta^{d,b_d}_{h_d}, \tag{4.1.2}$$

where, for all $k \in \{1, \dots, d\}$, $\Delta^{k,b_k}_{h_k}$ is $\Delta^k_{h_k}$ composed with itself b_k times, with the convention that $\Delta^{k,0}_{h_k}$ is the identity.

Definition 4.1.1.

(i) We denote by $\mathcal{L}_2$ the function defined, for each $(a, b) \in \mathbb{R}^2_+$, as

$$\mathcal{L}_2(a, b) := 1/2\, \mathbb{1}_{\{b \geq a\}} + \mathbb{1}_{\{b = a\}}. \tag{4.1.3}$$

More precisely, one has:

$$\mathcal{L}_2(a, b) = 0 \text{ if } a > b, \quad \mathcal{L}_2(a, b) = 3/2 \text{ if } a = b \quad \text{and} \quad \mathcal{L}_2(a, b) = 1/2 \text{ if } a < b.$$

(ii) For any fixed $\alpha \in (0,2)$, we denote by $\mathcal{L}_\alpha$ the function defined, for each $(a,b,\delta) \in \mathbb{R}_+^3$, as

$$\mathcal{L}_\alpha(a,b,\delta) := \Big(1/\alpha + \lfloor \alpha \rfloor/2 + \delta\Big)\mathbb{1}_{\{b \geq a\}} + \mathbb{1}_{\{b=a\}}, \tag{4.1.4}$$

where $\lfloor \alpha \rfloor$ is the integer part of α. More precisely,

- *when $\alpha \in (0,1)$, one has:*

$$\begin{cases} \mathcal{L}_\alpha(a,b,\delta) = 0 & \text{if} \quad a > b \\ \mathcal{L}_\alpha(a,b,\delta) = 1/\alpha + 1 + \delta & \text{if} \quad a = b \\ \mathcal{L}_\alpha(a,b,\delta) = 1/\alpha + \delta & \text{if} \quad a < b; \end{cases}$$

- *when $\alpha \in [1,2)$, one has:*

$$\begin{cases} \mathcal{L}_\alpha(a,b,\delta) = 0 & \text{if} \quad a > b \\ \mathcal{L}_\alpha(a,b,\delta) = 1/\alpha + 3/2 + \delta & \text{if} \quad a = b \\ \mathcal{L}_\alpha(a,b,\delta) = 1/\alpha + 1/2 + \delta & \text{if} \quad a < b; \end{cases}$$

We are now ready to state the first main result of this section.

Theorem 4.1.2. *The positive exponents $a_1[f], \ldots, a_d[f]$ are the same as in Definition 3.1.1. Moreover we assume that $\eta = (\eta_1, \ldots, \eta_d) \in \Upsilon := \{0,1\}^d$, $B = (b_1, \ldots, b_d) \in \mathbb{Z}_+^d$, $T \in (0,+\infty)$ and $\omega \in \Omega_1^*$ are arbitrary and fixed. Then, the following two results hold (with the convention that $0/0 = 0$).*

(i) When $\alpha = 2$, one has

$$\sup_{h \in [-T,T]^d} \left\{ \frac{\Big\| \Delta_{(h)}^B X[f]^\eta(\cdot,\omega) \Big\|_{T,\infty}}{\displaystyle\prod_{l=1}^d |h_l|^{b_l(1-\eta_l)} \, |h_l|^{\min(b_l, a_l[f])\eta_l} \Big(\log\Big(3 + |h_l|^{-1}\Big)\Big)^{\eta_l \mathcal{L}_2(a_l[f], b_l)}} \right\} < +\infty. \tag{4.1.5}$$

(ii) When $\alpha \in (0,2)$, for all arbitrarily small positive real numbers δ, one has

$$\sup_{h \in [-T,T]^d} \left\{ \frac{\Big\| \Delta_{(h)}^B X[f]^\eta(\cdot,\omega) \Big\|_{T,\infty}}{\displaystyle\prod_{l=1}^d |h_l|^{b_l(1-\eta_l)} \, |h_l|^{\min(b_l, a_l[f])\eta_l} \Big(\log\Big(3 + |h_l|^{-1}\Big)\Big)^{\eta_l \mathcal{L}_\alpha(a_l[f], b_l, \delta)}} \right\} < +\infty. \tag{4.1.6}$$

It easily follows from Remark 3.2.13 and Theorem 4.1.2 that:

Corollary 4.1.3. *The positive exponents $a_1[f], \ldots, a_d[f]$ are the same as in Definition 3.1.1. Moreover we assume that $B = (b_1, \ldots, b_d) \in \mathbb{Z}_+^d$, $T \in (0,+\infty)$ and $\omega \in \Omega_1^*$ are arbitrary and fixed. Then, the following two results hold (with the convention that $0/0 = 0$).*

(i) When $\alpha = 2$, one has

$$\sup_{h\in[-T,T]^d}\left\{\frac{\left\|\Delta^B_{(h)}X[f](\cdot,\omega)\right\|_{T,\infty}}{\prod_{l=1}^{d}|h_l|^{\min(b_l,a_l[f])}\left(\log\left(3+|h_l|^{-1}\right)\right)^{\mathcal{L}_2(a_l[f],b_l)}}\right\}<+\infty. \tag{4.1.7}$$

(ii) When $\alpha \in (0,2)$, for all arbitrarily small positive real numbers δ, one has

$$\sup_{h\in[-T,T]^d}\left\{\frac{\left\|\Delta^B_{(h)}X[f](\cdot,\omega)\right\|_{T,\infty}}{\prod_{l=1}^{d}|h_l|^{\min(b_l,a_l[f])}\left(\log\left(3+|h_l|^{-1}\right)\right)^{\mathcal{L}_\alpha(a_l[f],b_l,\delta)}}\right\}<+\infty. \tag{4.1.8}$$

The following proposition is the main ingredient of the proof of Theorem 4.1.2.

Proposition 4.1.4. *The positive exponents $a_1[f],\dots,a_d[f]$ are the same as in Definition 3.1.1. Moreover, we assume that $\eta=(\eta_1,\dots,\eta_d)\in\Upsilon^*:=\{0,1\}^d\setminus\{(0,\dots,0)\}$, $B=(b_1,\dots,b_d)\in\mathbb{Z}^d_+$, $T\in(0,+\infty)$ and $\omega\in\Omega_1^*$ are arbitrary and fixed. Then, the following two results hold (with the convention that $0/0=0$); notice that the notations used in their statements are the same as in* (3.2.29), (4.1.2), (3.2.60), *and Definition 4.1.1.*

(i) When $\alpha = 2$, one has

$$\sup_{h\in[-T,T]^d}\left\{\frac{\sum_{J\in\mathbb{Z}^d_{(\eta)}}\left\|\Delta^B_{(h)}\Big(\Phi_{\alpha,J}[f]\big(2^J\cdot,\omega\big)\Big)\right\|_{T,\infty}}{\prod_{l=1}^{d}|h_l|^{b_l(1-\eta_l)}\,|h_l|^{\min(b_l,a_l[f])\eta_l}\left(\log\left(3+|h_l|^{-1}\right)\right)^{\eta_l\mathcal{L}_2(a_l[f],b_l)}}\right\}<+\infty. \tag{4.1.9}$$

(ii) When $\alpha \in (0,2)$, for all arbitrarily small positive real numbers δ, one has

$$\sup_{h\in[-T,T]^d}\left\{\frac{\sum_{J\in\mathbb{Z}^d_{(\eta)}}\left\|\Delta^B_{(h)}\Big(\Phi_{\alpha,J}[f]\big(2^J\cdot,\omega\big)\Big)\right\|_{T,\infty}}{\prod_{l=1}^{d}|h_l|^{b_l(1-\eta_l)}\,|h_l|^{\min(b_l,a_l[f])\eta_l}\left(\log\left(3+|h_l|^{-1}\right)\right)^{\eta_l\mathcal{L}_\alpha(a_l[f],b_l,\delta)}}\right\}<+\infty. \tag{4.1.10}$$

We now show that Proposition 4.1.4 holds; to this end, we need the three following lemmas.

Lemma 4.1.5. *Denote by* $B = (b_1, \ldots, b_d) \in \mathbb{Z}_+^d$ *an arbitrary multi-index, and by* $\mathrm{l}(B) := b_1 + \ldots + b_d$ *its length. Then for all functions* $g \in \mathcal{C}^\infty(\mathbb{R}^d)$*, for any positive real number* T*, and for each* $h = (h_1, \ldots, h_d) \in [-T, T]^d$*, the following inequality holds:*

$$\left\| \Delta_{(h)}^{B} g \right\|_{T,\infty} \leq 2^{\mathrm{l}(B)} \times \min_{B' \in I(B)} \left\{ \left\| \partial^{B'} g \right\|_{T\,2^{\mathrm{l}(B)},\infty} \times \prod_{l=1}^{d} |h_l|^{b'_l} \right\}, \tag{4.1.11}$$

with the convention that $0^0 = 1$*, and where the set* $I(B)$ *is defined as*

$$I(B) := \left\{ B' = (b'_1, \ldots, b'_d) \in \mathbb{Z}_+^d : \text{ for each } l \in \{1, \ldots, d\},\, b'_l \leq b_l \right\}. \tag{4.1.12}$$

Lemma 4.1.6. *Assume that the real numbers* $T > 0$*,* $\alpha > 0$*,* $\mu \geq 0$ *and* $b \geq 0$ *are arbitrary and fixed. Then, one has*

$$\sup_{z \in [-T,T]} \left\{ \frac{\displaystyle\sum_{j=-\infty}^{0} 2^{j/\alpha} (1 + |j|)^{\mu} \min\left(\left| 2^j z \right|^b, 1 \right)}{\left| z \right|^b} \right\} < +\infty. \tag{4.1.13}$$

with the conventions that $0/0 = 0$ *and* $0^0 = 1$*.*

Lemma 4.1.7. *Assume that the real numbers* $T > 0$*,* $a > 0$*,* $\mu \geq 0$ *and* $b \geq 0$ *are arbitrary and fixed. Then, the following three results hold (with the conventions that* $0/0 = 0$ *and* $0^0 = 1$*).*

1. *When* $b < a$*, one has*

$$\sup_{z \in [-T,T]} \left\{ \frac{\displaystyle\sum_{j=1}^{+\infty} 2^{-ja} (1 + j)^{\mu} \min\left(\left| 2^j z \right|^b, 1 \right)}{\left| z \right|^b} \right\} < +\infty. \tag{4.1.14}$$

2. *When* $b = a$*, one has*

$$\sup_{z \in [-T,T]} \left\{ \frac{\displaystyle\sum_{j=1}^{+\infty} 2^{-ja} (1 + j)^{\mu} \min\left(\left| 2^j z \right|^b, 1 \right)}{|z|^a \left(\log\left(3 + |z|^{-1} \right) \right)^{\mu+1}} \right\} < +\infty. \tag{4.1.15}$$

3. *When* $b > a$*, one has*

$$\sup_{z \in [-T,T]} \left\{ \frac{\displaystyle\sum_{j=1}^{+\infty} 2^{-ja} (1 + j)^{\mu} \min\left(\left| 2^j z \right|^b, 1 \right)}{|z|^a \left(\log\left(3 + |z|^{-1} \right) \right)^{\mu}} \right\} < +\infty. \tag{4.1.16}$$

Proof of Lemma 4.1.5. We intend to proceed by induction on $\mathrm{l}(B)$. More precisely, the proof is structured as follows. *In the Part 1*, we establish the lemma in the particular case where $\mathrm{l}(B) = 0$. *In the Part 2*, we denote by n an arbitrary fixed non-negative integer, and we assume that the lemma holds when $\mathrm{l}(B) = n$ (such a B is denoted by $\widetilde{B}$), then the goal is to derive it in the case where $\mathrm{l}(B) = n+1$.

Part 1: In view of (4.1.12) and of the assumption $\mathrm{l}(B) = 0$, the set $I(B)$ reduces to $\{0\}$. Then, in view of the equalities $\Delta^0_{(h)} g = g$, for all $h \in \mathbb{R}^d$, and $\partial^0 g = g$, it is clear that the lemma is true.

Part 2: Let $B \in \mathbb{Z}^d_+$ be arbitrary and satisfying $\mathrm{l}(B) = n+1$. One has to show that, for all $g \in \mathcal{C}^\infty(\mathbb{R}^d)$, for any positive real number T, and for each $h = (h_1, \dots, h_d) \in [-T,T]^d$, the following inequality holds:

$$\left\|\Delta^B_{(h)} g\right\|_{T,\infty} \le 2^{\mathrm{l}(B)} \times \min_{B' \in I(B)} \left\{ \left\|\partial^{B'} g\right\|_{T\,2^{\mathrm{l}(B)},\infty} \times \prod_{l=1}^d |h_l|^{b'_l} \right\}. \tag{4.1.17}$$

Observe that there exists $\widetilde{B} \in \mathbb{Z}^d_+$ satisfying $\mathrm{l}(\widetilde{B}) = n$, and there exists $k \in \{1, \dots, d\}$, such that B can be expressed as

$$B = \widetilde{B} + e_k, \tag{4.1.18}$$

where $e_k \in \mathbb{Z}^d$ is the multi-index whose k-th coordinate equals 1 and the others vanish. Next, it follows from (4.1.18), (4.1.2) and (4.1.1) that

$$\left\|\Delta^B_{(h)} g\right\|_{T,\infty} = \sup_{x \in [-T,T]^d} \left| \left(\Delta^{\widetilde{B}}_{(h)} g\right)(x + h_k e_k) - \left(\Delta^{\widetilde{B}}_{(h)} g\right)(x) \right|. \tag{4.1.19}$$

Therefore, using the triangle inequality one has that

$$\left\|\Delta^B_{(h)} g\right\|_{T,\infty} \le 2 \left\|\Delta^{\widetilde{B}}_{(h)} g\right\|_{2T,\infty} \le 2^{\mathrm{l}(B)} \times \min_{B' \in I(\widetilde{B})} \left\{ \left\|\partial^{B'} g\right\|_{T\,2^{\mathrm{l}(B)},\infty} \times |h|^{B'}_\pi \right\}, \tag{4.1.20}$$

where the convenient notation $|h|^{B'}_\pi$ is defined by

$$|h|^{B'}_\pi := \prod_{l=1}^d |h_l|^{b'_l}; \tag{4.1.21}$$

notice that the last inequality in (4.1.20) results from the induction hypothesis [1] and the equality $\mathrm{l}(B) = \mathrm{l}(\widetilde{B}) + 1$. On the other hand, one can derive from (4.1.19), the Mean Value Theorem, and the equality $\partial^{e_k}\left(\Delta^{\widetilde{B}}_{(h)} g\right) = \Delta^{\widetilde{B}}_{(h)}\left(\partial^{e_k} g\right)$ that

$$\left\|\Delta^B_{(h)} g\right\|_{T,\infty} \le |h_k| \left\|\Delta^{\widetilde{B}}_{(h)}\left(\partial^{e_k} g\right)\right\|_{2T,\infty}. \tag{4.1.22}$$

[1] In which B is replaced by $\widetilde{B}$ and T by $2T$.

Moreover, applying the induction hypothesis [2] and using (4.1.21), one gets that

$$\left\|\Delta_{(h)}^{\widetilde{B}}\left(\partial^{e_k} g\right)\right\|_{2T,\infty} \leq 2^{\mathrm{l}(\widetilde{B})} \times \min_{B' \in I(\widetilde{B})} \left\{\left\|\partial^{B'+e_k} g\right\|_{T\,2^{\mathrm{l}(B)},\infty} \times |h|_\pi^{B'}\right\}. \tag{4.1.23}$$

Next, putting together (4.1.22), (4.1.23), (4.1.21) and the inequality $\mathrm{l}(\widetilde{B}) < \mathrm{l}(B)$, we obtain that

$$\left\|\Delta_{(h)}^{B} g\right\|_{T,\infty} \leq 2^{\mathrm{l}(B)} \times \min_{B' \in I(\widetilde{B})} \left\{\left\|\partial^{B'+e_k} g\right\|_{T\,2^{\mathrm{l}(B)},\infty} \times |h|_\pi^{B'+e_k}\right\}. \tag{4.1.24}$$

Finally, in view of the fact

$$I(B) = I(\widetilde{B}) \cup \left\{B' + e_k : B' \in I(\widetilde{B})\right\},$$

one can derive from (4.1.21), (4.1.20) and (4.1.24) that (4.1.17) holds. □

Proof of Lemma 4.1.6. Observe that for all $z \in [-T, T]$ and $j \in \mathbb{Z}_-$, one has $\left|2^j T^{-1} z\right|^b \leq 1$. Therefore, one obtains that

$$\begin{aligned}
&\sum_{j=-\infty}^{0} 2^{j/\alpha} (1+|j|)^\mu \min\left(\left|2^j z\right|^b, 1\right) = \sum_{j=-\infty}^{0} 2^{j/\alpha} (1+|j|)^\mu \min\left(T^b \left|2^j T^{-1} z\right|^b, 1\right) \\
&\leq (1+T)^b \sum_{j=-\infty}^{0} 2^{j/\alpha} (1+|j|)^\mu \min\left(\left|2^j T^{-1} z\right|^b, 1\right) = c\,|z|^b,
\end{aligned}$$

where the finite constant c is equal to

$$c := (1+T)^b\, T^{-b} \sum_{j=-\infty}^{0} 2^{j(1/\alpha+b)} (1+|j|)^\mu .$$

□

Proof of Lemma 4.1.7. Let $z \in [-T, T]$ be arbitrary and fixed; there is no restriction to assume that $z \neq 0$. One sets

$$\mathrm{j}_0(z) := \min\left\{j \in \mathbb{N} : \left|2^j z\right| > 1\right\}. \tag{4.1.25}$$

It can easily be shown that there are two constants $0 < c_1 < c_2 < +\infty$, not depending on z, such that

$$c_1 \log\left(3 + |z|^{-1}\right) \leq \mathrm{j}_0(z) \leq c_2 \log\left(3 + |z|^{-1}\right). \tag{4.1.26}$$

Observe that, for any arbitrary fixed real numbers $a > 0$, $\mu \geq 0$ and $b \geq 0$, one has that

$$\sum_{j=\mathrm{j}_0(z)}^{+\infty} 2^{-ja} (1+j)^\mu \min\left(\left|2^j z\right|^b, 1\right) = \sum_{j=\mathrm{j}_0(z)}^{+\infty} 2^{-ja} (1+j)^\mu \tag{4.1.27}$$

[2]In which B is replaced by $\widetilde{B}$, g by $\partial^{e_k} g$, and T by $2T$.

and

$$\sum_{j=1}^{\mathrm{j}_0(z)-1} 2^{-ja}\,(1+j)^{\mu} \min\left(\left|2^j z\right|^b, 1\right) = |z|^b \sum_{j=1}^{\mathrm{j}_0(z)-1} 2^{-j(a-b)}\,(1+j)^{\mu}\,, \tag{4.1.28}$$

with the convention that $\sum_{j=1}^{0} \ldots = 0$. We are going to conveniently bound from above the right-hand side in (4.1.27) and the right-hand side in (4.1.28). First, we show that there exists a finite constant c_3, not depending on z, such that

$$\sum_{j=\mathrm{j}_0(z)}^{+\infty} 2^{-ja}\,(1+j)^{\mu} \le c_3\,|z|^a \left(\log\left(3+|z|^{-1}\right)\right)^{\mu}. \tag{4.1.29}$$

This is indeed the case since one has that

$$\begin{aligned}\sum_{j=\mathrm{j}_0(z)}^{+\infty} 2^{-ja}\,(1+j)^{\mu} &= \sum_{j=0}^{+\infty} 2^{-\mathrm{j}_0(z)a-ja}\,(1+\mathrm{j}_0(z)+j)^{\mu} \\ &= 2^{-\mathrm{j}_0(z)a}\,\mathrm{j}_0(z)^{\mu} \sum_{j=0}^{+\infty} 2^{-ja}\left(1+\frac{1+j}{\mathrm{j}_0(z)}\right)^{\mu} \le c_3\,|z|^a\,\left(\log\left(3+|z|^{-1}\right)\right)^{\mu},\end{aligned}$$

where the last inequality results from (4.1.25) and (4.1.26); notice that the finite constant c_3 is defined as

$$c_3 := c_2^{\mu} \sum_{j=0}^{+\infty} 2^{-ja}(2+j)^{\mu}.$$

Let us now study the right-hand side in (4.1.28). In the case where $b < a$, the constant

$$c_4 := \sum_{j=1}^{+\infty} 2^{-j(a-b)}\,(1+j)^{\mu}$$

is finite, and we have that

$$|z|^b \sum_{j=1}^{\mathrm{j}_0(z)-1} 2^{-j(a-b)}\,(1+j)^{\mu} \le c_4 |z|^b. \tag{4.1.30}$$

In the second case where $b = a$, one has

$$\begin{aligned}|z|^b \sum_{j=1}^{\mathrm{j}_0(z)-1} 2^{-j(a-b)}\,(1+j)^{\mu} &= |z|^a \sum_{j=1}^{\mathrm{j}_0(z)-1} (1+j)^{\mu} \\ &\le |z|^a\,\mathrm{j}_0(z)^{\mu+1} \\ &\le c_2^{\mu+1}\,|z|^a \left(\log\left(3+|z|^{-1}\right)\right)^{\mu+1}, \end{aligned}\tag{4.1.31}$$

where the last inequality results from (4.1.26). In the third and last case where $b > a$, letting c_5 and c_6 be the finite constants defined as $c_5 := 2^{b-a}\big/\left(2^{b-a}-1\right)$ and $c_6 := c_5 c_2^{\mu}$, one has

$$|z|^b \sum_{j=1}^{\mathrm{j}_0(z)-1} 2^{-j(a-b)}\,(1+j)^{\mu} \le c_5 |z|^b\, 2^{(\mathrm{j}_0(z)-1)(b-a)}\,\mathrm{j}_0(z)^{\mu} \le c_6\,|z|^a \left(\log\left(3+|z|^{-1}\right)\right)^{\mu}, \tag{4.1.32}$$

where the last inequality follows from (4.1.25) and (4.1.26).

Finally, putting together (4.1.27) to (4.1.32) one gets the lemma. □

We are now in the position to prove Proposition 4.1.4.

Proof of Proposition 4.1.4. We only give the proof of (4.1.10), since that of (4.1.9) can be done in the same way, except that one has to make use of (3.2.70), instead of (3.2.68) and (3.2.69). So, in the rest of the proof we assume that $\alpha \in (0,2)$.

We know from Proposition 3.2.16 that, for all fixed $J \in \mathbb{Z}^d_{(\eta)}$, the function $\Phi_{\alpha,J}[f]\big(2^J\cdot,\omega\big)$ belongs to the space $\mathcal{C}^\infty(\mathbb{R}^d)$. Thus, it follows from Lemma 4.1.5 that

$$\Big\|\Delta^B_{(h)}\Big(\Phi_{\alpha,J}[f]\big(2^J\cdot,\omega\big)\Big)\Big\|_{T,\infty} \le c_1 \times \min_{B'\in I(B)} \left\{\Big\|\partial^{B'}\Big(\Phi_{\alpha,J}[f]\big(2^J\cdot,\omega\big)\Big)\Big\|_{T_1,\infty} \times \prod_{l=1}^d |h_l|^{b'_l}\right\}, \quad (4.1.33)$$

where $I(B)$ is the same finite set as in (4.1.12), and the finite constants c_1 and T_1 are defined as $c_1 := 2^{\mathrm{l}(B)}$ and $T_1 := T\,2^{\mathrm{l}(B)}$. Moreover, we know from (3.2.68) and (3.2.69) that, for all fixed positive real numbers δ, and for any $B' \in I(B)$, one has

$$\Big\|\partial^{B'}\Big(\Phi_{\alpha,J}[f](2^J\cdot,\omega)\Big)\Big\|_{T_1,\infty} \le C_2(\omega)\prod_{l=1}^d 2^{(1-\eta_l)j_l(1/\alpha+b'_l)}2^{-\eta_l j_l(a_l[f]-b'_l)}\,(1+|j_l|)^{1/\alpha+\lfloor\alpha\rfloor/2+\delta}, \quad (4.1.34)$$

where $\lfloor\alpha\rfloor$ is the integer part of α. Notice that the finite constant $C_2(\omega)$ does not depend on J and h; also, it can be chosen in such a way that it does not depend on B', since $I(B)$ is a finite set. Next setting $C_3(\omega) := c_1 C_2(\omega)$ and using the fact that $\eta_l \in \{0,1\}$, for all $l \in \{1,\dots,d\}$, one can derive from (4.1.33), (4.1.34), and (4.1.12), that

$$\begin{aligned}
&\Big\|\Delta^B_{(h)}\Big(\Phi_{\alpha,J}[f]\big(2^J\cdot,\omega\big)\Big)\Big\|_{T,\infty}\\
&\le C_3(\omega)\prod_{l=1}^d 2^{(1-\eta_l)j_l/\alpha}\,2^{-\eta_l j_l a_l[f]}\,(1+|j_l|)^{1/\alpha+\lfloor\alpha\rfloor/2+\delta}\min_{B'\in I(B)}\left\{\Big|2^{j_l}h_l\Big|^{b'_l}\right\}\\
&\le\ C_3(\omega)\prod_{l=1}^d 2^{(1-\eta_l)j_l/\alpha}\,2^{-\eta_l j_l a_l[f]}\,(1+|j_l|)^{1/\alpha+\lfloor\alpha\rfloor/2+\delta}\min\left\{\Big|2^{j_l}h_l\Big|^{b_l},1\right\}.
\end{aligned}$$

Then, (4.1.10) can be obtained by using (3.2.29), (3.2.30), Lemmas 4.1.6 and 4.1.7, as well as Definition 4.1.1. □

We are now in the position to prove Theorem 4.1.2.

Proof of Theorem 4.1.2. When $\eta = 0 = (0,\dots,0)$ the theorem easily results from Proposition 3.2.15 and Lemma 4.1.5. When $\eta \neq 0$ the theorem can easily be derived from (3.2.53), (3.2.60), the triangle inequality and Proposition 4.1.4. □

In order to state the second main result of this section, we need to introduce some additional notations.

Definition 4.1.8.

(i) We denote by $\widetilde{\mathcal{L}_2}$ *the function defined, for each* $a \in \mathbb{R}_+$, *as*

$$\widetilde{\mathcal{L}_2}(a) := 1/2 + \mathbb{1}_{\{a\in\mathbb{N}\}}. \tag{4.1.35}$$

More precisely, one has:

$$\widetilde{\mathcal{L}_2}(a) = 3/2 \text{ if } a \in \mathbb{N}, \text{ and } \widetilde{\mathcal{L}_2}(a) = 1/2 \text{ if } a \notin \mathbb{N}.$$

(ii) For any fixed $\alpha \in (0,2)$, *we denote by* $\widetilde{\mathcal{L}_\alpha}$ *the function defined, for each* $(a,\delta) \in \mathbb{R}_+^2$, *as*

$$\widetilde{\mathcal{L}_\alpha}(a,\delta) := 1/\alpha + \lfloor \alpha \rfloor/2 + \delta + \mathbb{1}_{\{a\in\mathbb{N}\}}, \tag{4.1.36}$$

where $\lfloor \alpha \rfloor$ *is the integer part of* α. *More precisely,*

- *when* $\alpha \in (0,1)$, *one has:*

$$\widetilde{\mathcal{L}_\alpha}(a,\delta) = 1/\alpha + 1 + \delta \text{ if } a \in \mathbb{N}, \text{ and } \widetilde{\mathcal{L}_\alpha}(a,\delta) = 1/\alpha + \delta \text{ if } a \notin \mathbb{N};$$

- *when* $\alpha \in [1,2)$, *one has:*

$$\widetilde{\mathcal{L}_\alpha}(a,\delta) = 1/\alpha + 3/2 + \delta \text{ if } a \in \mathbb{N}, \text{ and } \widetilde{\mathcal{L}_\alpha}(a,\delta) = 1/\alpha + 1/2 + \delta \text{ if } a \notin \mathbb{N}.$$

For any fixed $h \in \mathbb{R}^d$, we denote by $\mathbf{\Delta}_h$, the operator from the space of the real-valued functions on $\mathbb{R}^d$, into itself; so that, when g is such a function, $\mathbf{\Delta}_h g$ is then the function defined, for all $x \in \mathbb{R}^d$, as

$$\big(\mathbf{\Delta}_h g\big)(x) := g(x+h) - g(x). \tag{4.1.37}$$

Moreover, for each positive integer n, we denote by $\mathbf{\Delta}_h^n$ the operator $\mathbf{\Delta}_h$ composed n times with itself.

We are now ready to state the second main result of this section.

Theorem 4.1.9. *The positive exponents* $a_1[f], \ldots, a_d[f]$ *are the same as in Definition 3.1.1, and we set*

$$n_0 := 1 - d + \sum_{l=1}^{d} \big\lceil a_l[f] \big\rceil,$$

where $\big\lceil a_l[f] \big\rceil := \min\{m \in \mathbb{N} : m \geq a_l[f]\}$, *for any* $l \in \{1,\ldots,d\}$. *Moreover, we assume that* $\eta = (\eta_1,\ldots,\eta_d) \in \Upsilon := \{0,1\}^d$, $T \in (0,+\infty)$ *and* $\omega \in \Omega_1^*$ *are arbitrary and fixed. Let* n *be an arbitrary integer such that* $n \geq n_0$. *Then, the following two results hold (with the convention that* $0/0 = 0$*).*

(i) When $\alpha = 2$, one has

$$\sup_{h\in[-T,T]^d} \left\{ \frac{\left\| \mathbf{\Delta}_h^n X[f]^\eta(\cdot,\omega) \right\|_{T,\infty}}{\sum_{l=1}^d |h_l|^{\eta_l a_l + (1-\eta_l)\lceil a_l[f]\rceil} \left(\log\left(3+|h_l|^{-1}\right)\right)^{\eta_l \widetilde{\mathcal{L}_2}(a_l[f])}} \right\} < +\infty. \tag{4.1.38}$$

(ii) When $\alpha \in (0,2)$, for all arbitrarily small positive real numbers δ, one has

$$\sup_{h\in[-T,T]^d} \left\{ \frac{\left\| \mathbf{\Delta}_h^n X[f]^\eta(\cdot,\omega) \right\|_{T,\infty}}{\sum_{l=1}^d |h_l|^{\eta_l a_l + (1-\eta_l)\lceil a_l[f]\rceil} \left(\log\left(3+|h_l|^{-1}\right)\right)^{\eta_l \widetilde{\mathcal{L}_\alpha}(a_l[f],\delta)}} \right\} < +\infty. \tag{4.1.39}$$

It easily follows from Remark 3.2.13 and Theorem 4.1.9 that:

Corollary 4.1.10. *The positive exponents $a_1[f], \dots, a_d[f]$ are the same as in Definition 3.1.1, and the positive integer $n_0 = n_0(a_1[f], \dots, a_d[f], d)$ is the same as in Theorem 4.1.9. Moreover, we assume that $T \in (0,+\infty)$ and $\omega \in \Omega_1^*$ are arbitrary and fixed. Let n be an arbitrary integer such that $n \geq n_0$. Then, the following two results hold (with the convention that $0/0 = 0$).*

(i) When $\alpha = 2$, one has

$$\sup_{h\in[-T,T]^d} \left\{ \frac{\left\| \mathbf{\Delta}_h^n X[f](\cdot,\omega) \right\|_{T,\infty}}{\sum_{l=1}^d |h_l|^{a_l[f]} \left(\log\left(3+|h_l|^{-1}\right)\right)^{\widetilde{\mathcal{L}_2}(a_l[f])}} \right\} < +\infty. \tag{4.1.40}$$

(ii) When $\alpha \in (0,2)$, for all arbitrarily small positive real numbers δ, one has

$$\sup_{h\in[-T,T]^d} \left\{ \frac{\left\| \mathbf{\Delta}_h^n X[f](\cdot,\omega) \right\|_{T,\infty}}{\sum_{l=1}^d |h_l|^{a_l[f]} \left(\log\left(3+|h_l|^{-1}\right)\right)^{\widetilde{\mathcal{L}_\alpha}(a_l[f],\delta)}} \right\} < +\infty. \tag{4.1.41}$$

Proof of Theorem 4.1.9. We only give the proof of (4.1.39); the strategy of the proof remains the same in the case of (4.1.38), except that (4.1.5) has to be used instead of (4.1.6).

Let $T \in (0,+\infty)$ and $h = (h_1, \dots, h_{k-1}, h_k, \dots, h_d) \in [-T,T]^d$ be arbitrary and fixed. First, we are going to express the operator $\mathbf{\Delta}_h$ (see (4.1.37)) in terms of the operators $\Delta_{h_k}^k$,

$k \in \{1, \ldots, d\}$ (see (4.1.1)), and of some translation operators. To this end, for any fixed $k \in \{1, \ldots, d+1\}$, we denote by $(h)_{k,0}$ the vector of $\mathbb{R}^d$ such that $(h)_{k,0} := (h_1, \ldots, h_{k-1}, 0, \ldots, 0)$, with the convention that $(h)_{1,0}$ is the zero vector and that $(h)_{d+1,0}$ is the vector h itself. Also, for any fixed vector $r \in \mathbb{R}^d$, we denote by Θ_r, the translation operator from the space of the real-valued functions on $\mathbb{R}^d$, into itself; so that, when $\tilde{g}$ is such a function, $\Theta_r \tilde{g}$ is then the function defined, for all $x \in \mathbb{R}^d$, as $\big(\Theta_r \tilde{g}\big)(x) := \tilde{g}(x + r)$. One can easily check that $\Theta_r \circ \Delta^k_{h_k} = \Delta^k_{h_k} \circ \Theta_r$, for every $k \in \{1, \ldots, d\}$, and that

$$\boldsymbol{\Delta}_h = \sum_{k=1}^{d} \Theta_{(h)_{k,0}} \circ \Delta^k_{h_k}. \tag{4.1.42}$$

Now, let n be the same integer as in the statement of Theorem 4.1.9, and let g be an arbitrary real-valued continuous function on $\mathbb{R}^d$. Using (4.1.42), the Multinomial Theorem, the triangle inequality and the inequality $2^n \geq n + 1$, we get that

$$\Big\|\boldsymbol{\Delta}^n_h g\Big\|_{T,\infty} \leq n! \sum_{B \in E_n} \Big\|\Delta^B_{(h)} g\Big\|_{(n+1)T,\infty} \leq n! \sum_{B \in E_n} \Big\|\Delta^B_{(h)} g\Big\|_{2^n T,\infty}, \tag{4.1.43}$$

where the finite set $E_n := \Big\{B = (b_1, \ldots, b_d) \in \mathbb{Z}^d_+ : \mathrm{l}(B) := b_1 + \cdots + b_d = n\Big\}$, and the operators $\Delta^B_{(h)}$ are defined through (4.1.2). Moreover, similarly to (4.1.11), it can be shown, for each $B \in \mathbb{Z}^d_+$, that

$$\Big\|\Delta^B_{(h)} g\Big\|_{2^n T,\infty} \leq 2^{\mathrm{l}(B)} \min_{B' \in I(B)} \Big\|\Delta^{B'}_{(h)} g\Big\|_{2^{\mathrm{l}(B)+n} T,\infty} = 2^n \min_{B' \in I(B)} \Big\|\Delta^{B'}_{(h)} g\Big\|_{2^{2n} T,\infty}, \tag{4.1.44}$$

where the finite set $I(B) := \Big\{B' = (b'_1, \ldots, b'_d) \in \mathbb{Z}^d_+ : \text{ for each } l \in \{1, \ldots, d\},\, b'_l \leq b_l\Big\}$. Next, applying (4.1.43) and (4.1.44) to $g = X[f]^\eta(\cdot, \omega)$, where $\omega \in \Omega^*_1$ is arbitrary and fixed, we obtain that

$$\Big\|\boldsymbol{\Delta}^n_h X[f]^\eta(\cdot, \omega)\Big\|_{T,\infty} \leq 2^n\, n! \sum_{B \in E_n} \min_{B' \in I(B)} \Big\|\Delta^{B'}_{(h)} X[f]^\eta(\cdot, \omega)\Big\|_{2^{2n} T,\infty}. \tag{4.1.45}$$

Let us now provide, for any fixed $B \in E_n$, a suitable upper bound for the quantity

$$\min_{B' \in I(B)} \Big\|\Delta^{B'}_{(h)} X[f]^\eta(\cdot, \omega)\Big\|_{2^{2n} T,\infty}.$$

To this end, we set

$$l_0(B) := \min\Big\{l \in \{1, \ldots, d\} : b_l \geq a_l[f]\Big\}. \tag{4.1.46}$$

Observe that $l_0(B)$ is well-defined since the inequality $n \geq n_0 := 1 - d + \sum_{l=1}^{d} \big\lceil a_l[f]\big\rceil$ implies that there exists at least one $l \in \{1, \ldots, d\}$ satisfying $b_l \geq a_l[f]$. Next, let $B^0 := (b^0_1, \ldots, b^0_d) \in \mathbb{Z}^d_+$ be such that

$$b^0_l = \big\lceil a_l[f]\big\rceil \mathbb{1}_{\{l = l_0(B)\}}, \quad \text{for all } l \in \{1, \ldots, d\}; \tag{4.1.47}$$

that is $b^0_{l_0(B)} = \left\lceil a_{l_0(B)}[f] \right\rceil$, and $b^0_l = 0$ for all $l \neq l_0(B)$. Notice that B^0 belongs to $I(B)$, since (4.1.46) entails that

$$b_{l_0(B)} \geq \left\lceil a_{l_0(B)}[f] \right\rceil = b^0_{l_0(B)}.$$

As a consequence, we have that

$$\min_{B' \in I(B)} \left\| \Delta^{B'}_{(h)} X[f]^\eta(\cdot, \omega) \right\|_{2^{2n}T,\infty} \leq \left\| \Delta^{B^0}_{(h)} X[f]^\eta(\cdot, \omega) \right\|_{2^{2n}T,\infty}.$$

Thus, it follows from (4.1.6), (4.1.47), (4.1.4) and (4.1.36) that, for any fixed $\delta \in (0, +\infty)$, we have

$$\begin{aligned}
&\min_{B' \in I(B)} \left\| \Delta^{B'}_{(h)} X[f]^\eta(\cdot, \omega) \right\|_{2^{2n}T,\infty} \\
&\leq C_2(\omega, B) \prod_{l=1}^{d} |h_l|^{b^0_l(1-\eta_l)} \, |h_l|^{\min(b^0_l, a_l[f])\eta_l} \left(\log\left(3 + |h_l|^{-1}\right)\right)^{\eta_l \mathcal{L}_\alpha(a_l[f], b^0_l, \delta)} \\
&= C_2(\omega, B) \left| h_{l_0(B)} \right|^{(1-\eta_{l_0(B)})\lceil a_{l_0(B)}[f]\rceil + \eta_{l_0(B)} a_{l_0(B)}[f]} \left(\log\left(3 + \left| h_{l_0(B)} \right|^{-1}\right)\right)^{\eta_{l_0(B)} \widetilde{\mathcal{L}_\alpha}(a_{l_0(B)}[f], \delta)},
\end{aligned} \tag{4.1.48}$$

where $C_2(\omega, B)$ is a finite constant not depending on h. Finally, let $C_3(\omega)$ and $C_4(\omega)$ be the two finite constants defined as $C_3(\omega) := (2^n\, n!) \times \max\left\{ C_2(\omega, B) : B \in E_n \right\}$ and $C_4(\omega) := \mathrm{card}(E_n) \times C_3(\omega)$, where $\mathrm{card}(E_n)$ denotes the cardinality of E_n. The inequalities (4.1.45) and (4.1.48), and the fact that, for all $B \in E_n$, the index $l_0(B)$ belongs to $\{1, \ldots, d\}$ imply that

$$\begin{aligned}
&\left\| \boldsymbol{\Delta}^n_h X[f]^\eta(\cdot, \omega) \right\|_{T,\infty} \\
&\leq C_3(\omega) \sum_{B \in E_n} \left| h_{l_0(B)} \right|^{(1-\eta_{l_0(B)})\lceil a_{l_0(B)}[f]\rceil + \eta_{l_0(B)} a_{l_0(B)}[f]} \left(\log\left(3 + \left| h_{l_0(B)} \right|^{-1}\right)\right)^{\eta_{l_0(B)} \widetilde{\mathcal{L}_\alpha}(a_{l_0(B)}[f], \delta)} \\
&\leq C_4(\omega) \sum_{l=1}^{d} |h_l|^{(1-\eta_l)\lceil a_l[f]\rceil + \eta_l a_l} \left(\log\left(3 + |h_l|^{-1}\right)\right)^{\eta_l \widetilde{\mathcal{L}_\alpha}(a_l[f], \delta)},
\end{aligned}$$

which shows that (4.1.39) holds. □

4.2 Behaviour at infinity

Let f be an admissible function, $X[f]$ the field associated with f, and $X[f]^\eta$ an arbitrary η-frequency part of $X[f]$, where $\eta = (\eta_1, \ldots, \eta_d) \in \Upsilon := \{0, 1\}^d$ (see Definition 3.1.1, (2.3.3), Definition 3.2.12 and Remark 3.2.13). The function f may have a singularity at 0; yet, in the neighbourhood of this point, f is governed by the exponent $a'[f] \in [0, 1)$ through the inequality (3.1.2). The main goal of the present section is to draw connections between the exponent $a'[f]$ and the behaviour at infinity of $X[f]^\eta$, that of $X[f]$, and that of their partial derivatives when they exist. The methodology we use is based on the wavelet type random series representations (3.2.53) and (3.2.52) of $X[f]^\eta$ and $X[f]$. It is worth mentioning that all the results we obtain are valid on Ω_1^*, the "universal" event of probability 1 which was introduced in Lemma 3.2.7. Let us first state them.

Theorem 4.2.1. *The exponents $a'[f]\in[0,1)$ and $a_1[f],\dots,a_d[f]\in(0,+\infty)$ are the same as in Definition 3.1.1. Let $\eta=(\eta_1,\dots,\eta_d)\in\Upsilon:=\{0,1\}^d$ and $b=(b_1,\dots,b_d)\in\mathbb{Z}_+^d$ be arbitrary and such that (3.2.62) holds [3]. Then, for each fixed $\delta\in(0,+\infty)$ and $\omega\in\Omega_1^*$, the following three results are satisfied (with the convention that $0/0=0$).*

1. *When $\alpha\in(0,1)$ one has*

$$\sup_{t\in\mathbb{R}^d}\Big\{\Big|\partial^b(X[f]^\eta)(t,\omega)\Big|\Big\}<+\infty \quad \textit{if} \quad \eta\neq 0 \ \textit{or}\ b\neq 0, \tag{4.2.1}$$

and

$$\begin{cases} \sup\limits_{t\in\mathbb{R}^d}\Big\{\|t\|^{-a'[f]}\Big(\log\Big(3+\|t\|\Big)\Big)^{-d/\alpha-\delta}\Big|X[f]^0(t,\omega)\Big|\Big\}<+\infty & \textit{if}\quad a'[f]\in(0,1),\\ \sup\limits_{t\in\mathbb{R}^d}\Big\{\Big(\log\Big(3+\|t\|\Big)\Big)^{-d/\alpha-\delta-1}\Big|X[f]^0(t,\omega)\Big|\Big\}<+\infty & \textit{if}\quad a'[f]=0. \end{cases} \tag{4.2.2}$$

2. *When $\alpha\in[1,2)$ one has*

$$\sup_{t\in\mathbb{R}^d}\Big\{\Big(\log\Big(3+\|t\|\Big)\Big)^{-1/2}\Big|\partial^b(X[f]^\eta)(t,\omega)\Big|\Big\}<+\infty \quad \textit{if} \quad \eta\neq 0 \ \textit{or}\ b\neq 0, \tag{4.2.3}$$

and

$$\begin{cases} \sup\limits_{t\in\mathbb{R}^d}\Big\{\|t\|^{-a'[f]}\Big(\log\Big(3+\|t\|\Big)\Big)^{-d/\alpha-\delta}\Big|X[f]^0(t,\omega)\Big|\Big\}<+\infty & \textit{if}\quad a'[f]\in(0,1),\\ \sup\limits_{t\in\mathbb{R}^d}\Big\{\Big(\log\Big(3+\|t\|\Big)\Big)^{-d/\alpha-\delta-3/2}\Big|X[f]^0(t,\omega)\Big|\Big\}<+\infty & \textit{if}\quad a'[f]=0. \end{cases} \tag{4.2.4}$$

3. *When $\alpha=2$ one has*

$$\sup_{t\in\mathbb{R}^d}\Big\{\Big(\log\Big(3+\|t\|\Big)\Big)^{-1/2}\Big|\partial^b(X[f]^\eta)(t,\omega)\Big|\Big\}<+\infty \quad \textit{if} \quad \eta\neq 0 \ \textit{or}\ b\neq 0, \tag{4.2.5}$$

[3] Notice that when $\eta=0=(0,\dots,0)$, then (3.2.62) holds for any $b=(b_1,\dots,b_d)\in\mathbb{Z}_+^d$.

and

$$\begin{cases} \sup_{t\in\mathbb{R}^d}\Big\{ \|t\|^{-a'[f]}\Big(\log\log\Big(3+\|t\|\Big)\Big)^{-1/2}\Big|X[f]^0(t,\omega)\Big|\Big\}<+\infty & \text{if } \ a'[f]\in(0,1), \\ \sup_{t\in\mathbb{R}^d}\Big\{ \log(3+\|t\|)^{-3/2}\Big(\log\log\Big(3+\|t\|\Big)\Big)^{-1/2}\Big|X[f]^0(t,\omega)\Big|\Big\}<+\infty & \text{if } \ a'[f]=0. \end{cases} \tag{4.2.6}$$

It easily follows from Remark 3.2.13 and Theorem 4.2.1 that:

Corollary 4.2.2. *The exponents $a'[f]\in[0,1)$ and $a_1[f],\dots,a_d[f]\in(0,+\infty)$ are the same as in Definition 3.1.1. Let $b=(b_1,\dots,b_d)\in\mathbb{Z}_+^d$ be arbitrary and such that $b_l<a_l$, for all $l\in\{1,\dots,d\}$. Then, for each fixed $\delta\in(0,+\infty)$ and $\omega\in\Omega_1^*$, the following three results are satisfied (with the convention that $0/0=0$).*

1. *When $\alpha\in(0,1)$ one has*

$$\sup_{t\in\mathbb{R}^d}\Big\{\Big|\partial^b(X[f])(t,\omega)\Big|\Big\}<+\infty \quad \text{if} \quad b\neq 0, \tag{4.2.7}$$

and

$$\begin{cases} \sup_{\|t\|\geq 1}\Big\{ \|t\|^{-a'[f]}\Big(\log\Big(3+\|t\|\Big)\Big)^{-d/\alpha-\delta}|X[f](t,\omega)|\Big\}<+\infty & \text{if } \ a'[f]\in(0,1), \\ \sup_{\|t\|\geq 1}\Big\{ \Big(\log\Big(3+\|t\|\Big)\Big)^{-d/\alpha-\delta-1}|X[f](t,\omega)|\Big\}<+\infty & \text{if } \ a'[f]=0. \end{cases} \tag{4.2.8}$$

2. *When $\alpha\in[1,2)$ one has*

$$\sup_{t\in\mathbb{R}^d}\Big\{\Big(\log\Big(3+\|t\|\Big)\Big)^{-1/2}\Big|\partial^b(X[f])(t,\omega)\Big|\Big\}<+\infty \quad \text{if} \quad b\neq 0, \tag{4.2.9}$$

and

$$\begin{cases} \sup_{\|t\|\geq 1}\Big\{ \|t\|^{-a'[f]}\Big(\log\Big(3+\|t\|\Big)\Big)^{-d/\alpha-\delta}|X[f](t,\omega)|\Big\}<+\infty & \text{if } \ a'[f]\in(0,1), \\ \sup_{\|t\|\geq 1}\Big\{ \Big(\log\Big(3+\|t\|\Big)\Big)^{-d/\alpha-\delta-3/2}|X[f](t,\omega)|\Big\}<+\infty & \text{if } \ a'[f]=0. \end{cases} \tag{4.2.10}$$

3. *When $\alpha=2$ one has*

$$\sup_{t\in\mathbb{R}^d}\Big\{\Big(\log\Big(3+\|t\|\Big)\Big)^{-1/2}\Big|\partial^b(X[f])(t,\omega)\Big|\Big\}<+\infty \quad \text{if} \quad b\neq 0, \tag{4.2.11}$$

and

$$\begin{cases} \sup_{\|t\|\geq 1} \Big\{ \|t\|^{-a'[f]} \Big(\log\log\Big(3+\|t\|\Big)\Big)^{-1/2} |X[f](t,\omega)| \Big\} < +\infty & \textit{if} \quad a'[f]\in(0,1), \\ \sup_{\|t\|\geq 1} \Big\{ \log(3+\|t\|)^{-3/2} \Big(\log\log\Big(3+\|t\|\Big)\Big)^{-1/2} |X[f](t,\omega)| \Big\} < +\infty & \textit{if} \quad a'[f]=0. \end{cases} \tag{4.2.12}$$

We mention that as a by-product of the proof of Theorem 4.2.1, we have the following result.

Corollary 4.2.3. *The exponents $a'[f]\in[0,1)$ is the same as in Definition 3.1.1. Then, for each fixed $\epsilon\in(0,+\infty)$ and $\omega\in\Omega_1^*$, the following inequality is satisfied (with the convention that $0/0=0$):*

$$\sup_{\|t\|\geq 1} \left\{ \|t\|^{-a'[f]-\epsilon} \sum_{J\in\mathbb{Z}^d}\sum_{K\in\mathbb{Z}^d} \Big|\Psi_{\alpha,J}[f]\Big(2^J t-K\Big)-\Psi_{\alpha,J}[f]\Big(-K\Big)\Big|\,|\varepsilon_{\alpha,J,K}(\omega)| \right\} < +\infty. \tag{4.2.13}$$

Moreover, in view of Corollary 3.2.20, one has

$$\sup_{t\in\mathbb{R}^d} \left\{ \Big(1+\|t\|^{-a'[f]-\epsilon}\Big) \sum_{J\in\mathbb{Z}^d}\sum_{K\in\mathbb{Z}^d} \Big|\Psi_{\alpha,J}[f]\Big(2^J t-K\Big)-\Psi_{\alpha,J}[f]\Big(-K\Big)\Big|\,|\varepsilon_{\alpha,J,K}(\omega)| \right\} < +\infty. \tag{4.2.14}$$

Proof of Theorem 4.2.1. The proof is divided into 3 parts. Each part is divided into 3 cases: $\alpha\in(0,1)$, $\alpha\in[1,2)$ and $\alpha=2$.

Part I: we show (4.2.1) when $\eta\neq 0$.

Case 1: $\alpha\in(0,1)$. In view of (3.2.55), it is enough to prove the existence of a positive finite constant $C_1(\omega)$, such that, for all $t\in\mathbb{R}^d$, one has

$$\Big|\partial^b(Y[f]^\eta)(t,\omega)\Big| \leq C_1(\omega). \tag{4.2.15}$$

It follows from (3.2.64), (3.2.61), (3.2.35) and (3.2.34) (with $T=1$) that

$$\Big|\partial^b(Y[f]^\eta)(t,\omega)\Big| \leq C_2(\omega) \sum_{J\in\mathbb{Z}^d_{(\eta)}}\sum_{K\in\mathbb{Z}^d}\prod_{l=1}^d \frac{2^{(1-\eta_l)j_l(b_l+1/\alpha)}2^{-j_l\eta_l(a_l[f]-b_l)}(1+|j_l|)^{1/\alpha+\delta}}{(2+|2^{j_l}t_l-k_l|)^{p_*}},$$

where $C_2(\omega)$ is a positive finite constant not depending on t. Next, following the same line as in the proof of Lemma 3.2.9 and using the fact that $p_*>1$ (see (3.1.1)), one can show that,

$$\sup_{v\in\mathbb{R}}\left\{\sum_{k\in\mathbb{Z}}\frac{1}{(2+|v-k|)^{p_*}}\right\} \leq \sum_{k\in\mathbb{Z}}\frac{1}{(1+|k|)^{p_*}} < +\infty. \tag{4.2.16}$$

Therefore, we have

$$\left|\partial^b (Y[f]^\eta)(t,\omega)\right| \leq C_3(\omega) \sum_{J\in\mathbb{Z}^d_{(\eta)}} \prod_{l=1}^{d} 2^{(1-\eta_l)j_l(b_l+1/\alpha)} 2^{-j_l\eta_l(a_l[f]-b_l)} (1+|j_l|)^{1/\alpha+\delta}, \tag{4.2.17}$$

where $C_3(\omega)$ is a positive finite constant not depending on t. Finally, in view of (4.2.17), (3.2.29), and (3.2.62) that (4.2.15) holds. This implies that (4.2.1) is satisfied when $\eta \neq 0$.

Case 2: $\alpha \in [1,2)$. Similarly to the case 1, it is enough to prove the existence of a positive finite constant $C_1'(\omega)$, such that, for all $t\in\mathbb{R}^d$, one has

$$\left|\partial^b (Y[f]^\eta)(t,\omega)\right| \leq C_1'(\omega)\sqrt{\log\left(3+\|t\|\right)}. \tag{4.2.18}$$

It follows from (3.2.64), (3.2.61), (3.2.36) and (3.2.34) that

$$\begin{aligned}&\left|\partial^b (Y^\eta[f])(t,\omega)\right| \\ &\leq C_2'(\omega) \sum_{J\in\mathbb{Z}^d_{(\eta)}} \sum_{K\in\mathbb{Z}^d} \prod_{l=1}^{d} \sqrt{\log\left(3+\sum_{r=1}^{d}|j_r|+\sum_{r=1}^{d}|k_r|\right)} \frac{2^{(1-\eta_l)j_l(b_l+1/\alpha)} 2^{-j_l\eta_l(a_l[f]-b_l)}(1+|j_l|)^{1/\alpha+\delta}}{(2+|2^{j_l}t_l-k_l|)^{p_*}},\end{aligned}$$

where $C_2'(\omega)$ is a positive finite constant not depending on t. Next, using (3.2.40) and the inequality

$$\|t\| \geq \max_{1\leq l\leq d} |t_l|, \tag{4.2.19}$$

we get that

$$\begin{aligned}&\left|\partial^b (Y[f]^\eta)(t,\omega)\right| \\ &\leq C_3'(\omega) \sum_{J\in\mathbb{Z}^d_{(\eta)}} \sqrt{\log\left(3+\sum_{r=1}^{d}|j_r|+\|t\|\sum_{r=1}^{d}2^{j_r}\right)} \prod_{l=1}^{d} 2^{(1-\eta_l)j_l(b_l+1/2)} 2^{-j_l\eta_l(a_l[f]-b_l)}(1+|j_l|)^{1/\alpha+\delta},\end{aligned} \tag{4.2.20}$$

where $C_3'(\omega)$ is a positive finite constant not depending on t. Finally, in view of (3.2.38) and of the inequalities

$$\|t\|\sum_{l=1}^{d} 2^{j_l} \leq 2\,\|t\|\sum_{l=1}^{d}2^{j_l} \leq \|t\|^2 + \left(\sum_{l=1}^{d}2^{j_l}\right)^2, \tag{4.2.21}$$

one can deduce from (4.2.20), (3.2.29), and (3.2.62) that (4.2.18) holds. This implies that (4.2.3) is satisfied when $\eta \neq 0$.

Case 3: $\alpha = 2$. Similarly to the case 1, it is enough to prove the existence of a positive finite constant $C_1''(\omega)$, such that, for all $t\in\mathbb{R}^d$, one has

$$\left|\partial^b (Y[f]^\eta)(t,\omega)\right| \leq C_1''(\omega)\sqrt{\log\left(3+\|t\|\right)}. \tag{4.2.22}$$

It follows from (3.2.64), (3.2.61), (3.2.37) and (3.2.34) that

$$
\begin{aligned}
&\left|\partial^b(Y^\eta[f])(t,\omega)\right| \\
&\leq C_2''(\omega) \sum_{J\in\mathbb{Z}^d_{(\eta)}} \sum_{K\in\mathbb{Z}^d} \sqrt{\log\left(3+\sum_{l=1}^d |j_l| + \sum_{l=1}^d |k_l|\right)} \prod_{l=1}^d \frac{2^{(1-\eta_l)j_l(b_l+1/2)}2^{-j_l\eta_l(a_l[f]-b_l)}}{(2+|2^{j_l}t_l-k_l|)^{p_*}},
\end{aligned}
$$

where $C_2''(\omega)$ is a positive finite constant not depending on t. Next, using (3.2.40) and (4.2.19), we get that

$$
\begin{aligned}
&\left|\partial^b(Y[f]^\eta)(t,\omega)\right| \\
&\leq C_3''(\omega) \sum_{J\in\mathbb{Z}^d_{(\eta)}} \sqrt{\log\left(3+\sum_{l=1}^d |j_l| + \|t\|\sum_{l=1}^d 2^{j_l}\right)} \prod_{l=1}^d 2^{(1-\eta_l)j_l(b_l+1/2)}2^{-j_l\eta_l(a_l[f]-b_l)},
\end{aligned}
\tag{4.2.23}
$$

where $C_3''(\omega)$ is a positive finite constant not depending on t. Finally, in view of (3.2.38) and (4.2.21), one can deduce from (4.2.23), (3.2.29), and (3.2.62) that (4.2.22) holds. This implies that (4.2.5) is satisfied when $\eta \neq 0$.

Part II: we show (4.2.1) when $\eta = 0$ and $b \neq 0$.

Case 1: $\alpha \in (0,1)$. We know from the assumptions that the multi-index b has at least one non vanishing coordinate; it is denoted by b_s. Thus, using (3.2.58), the triangle inequality, (3.2.35), (3.2.33), and (3.2.26), one gets, for all $t\in\mathbb{R}^d$, that

$$
\begin{aligned}
&\left|\partial^b(X[f]^0)(t,\omega)\right| \\
&\leq \sum_{J\in\mathbb{Z}^d_+}\sum_{K\in\mathbb{Z}^d} \left|\partial^b\left(\Psi_{\alpha,-J}[f]\Big(2^{-J}\cdot-K\Big)-\Psi_{\alpha,-J}[f]\Big(-K\Big)\right)(t)\right| |\varepsilon_{\alpha,-J,K}(\omega)| \\
&= \sum_{J\in\mathbb{Z}^d_+}\sum_{K\in\mathbb{Z}^d} \left|\partial^b\Psi_{\alpha,-J}[f]\Big(2^{-J}t-K\Big)\right| |\varepsilon_{\alpha,-J,K}(\omega)| \prod_{l=1}^d 2^{-j_lb_l} \\
&\leq C_4(\omega)\sum_{J\in\mathbb{Z}^d_+} 2^{-j_s(1-a'[f])}\Big(2^{-j_1}+\cdots+2^{-j_d}\Big)^{-d/\alpha}\prod_{l=1}^d 2^{-j_l/\alpha}(1+|j_l|)^{1/\alpha+\delta} \\
&\leq C_5(\omega),
\end{aligned}
$$

where $C_4(\omega)$ and $C_5(\omega)$ are positive finite constants not depending on t. This shows that (4.2.1) holds when $\eta = 0$ and $b \neq 0$.

Case 2: $\alpha \in [1,2)$. We know from the assumptions that the multi-index b has at least one non vanishing coordinate; it is denoted by b_s. Thus, using (3.2.58), the triangle inequality, (3.2.36), (3.2.33), (3.2.40), (4.2.19), (3.2.38), and (3.2.25), one gets, for all $t \in \mathbb{R}^d$, that

$$\begin{aligned}
&\left|\partial^b(X[f]^0)(t,\omega)\right| \\
&\leq \sum_{J\in\mathbb{Z}_+^d}\sum_{K\in\mathbb{Z}^d}\left|\partial^b\left(\Psi_{-J}[f]\Big(2^{-J}\cdot -K\Big)-\Psi_{-J}[f]\Big(-K\Big)\right)(t)\right|\left|\varepsilon_{\alpha,-J,K}(\omega)\right| \\
&= \sum_{J\in\mathbb{Z}_+^d}\sum_{K\in\mathbb{Z}^d}\left|\partial^b\Psi_{-J}[f]\Big(2^{-J}t-K\Big)\right|\left|\varepsilon_{\alpha,-J,K}(\omega)\right|\prod_{l=1}^d 2^{-j_l b_l} \\
&\leq C_4'(\omega)\sum_{J\in\mathbb{Z}_+^d}2^{-j_s(1-a'[f])}\left(\sum_{r=1}^d 2^{-j_r}\right)^{-d/\alpha}\sqrt{\log\left(3+d\left\|t\right\|+\sum_{r=1}^d j_r\right)}\prod_{l=1}^d 2^{-j_l/\alpha}(1+|j_l|)^{1/\alpha+\delta} \\
&\leq C_5'(\omega)\sum_{J\in\mathbb{Z}_+^d}\sqrt{\log\left(3+\|t\|\right)}2^{-j_s(1-a'[f])}\left(\sum_{r=1}^d 2^{-j_r}\right)^{-d/\alpha}\prod_{l=1}^d 2^{-j_l/\alpha}\sqrt{\log\left(3+j_l\right)}(1+|j_l|)^{1/\alpha+\delta} \\
&\leq C_6'(\omega)\sqrt{\log\left(3+\|t\|\right)},
\end{aligned}$$

where $C_4'(\omega)$, $C_5'(\omega)$ and $C_6'(\omega)$ are positive finite constants not depending on t. This shows that (4.2.3) holds when $\eta = 0$ and $b \neq 0$.

Case 3: $\alpha = 2$. We know from the assumptions that the multi-index b has at least one non vanishing coordinate; it is denoted by b_s. Thus, using (3.2.58), the triangle inequality, (3.2.37), (3.2.33), (3.2.40), (4.2.19), (3.2.38), and (3.2.27), one gets, for all $t \in \mathbb{R}^d$, that

$$\begin{aligned}
&\left|\partial^b(X[f]^0)(t,\omega)\right| \\
&\leq \sum_{J\in\mathbb{Z}_+^d}\sum_{K\in\mathbb{Z}^d}\left|\partial^b\left(\Psi_{-J}[f]\Big(2^{-J}\cdot -K\Big)-\Psi_{-J}[f]\Big(-K\Big)\right)(t)\right|\left|\varepsilon_{-J,K}(\omega)\right| \\
&= \sum_{J\in\mathbb{Z}_+^d}\sum_{K\in\mathbb{Z}^d}\left|\partial^b\Psi_{-J}[f]\Big(2^{-J}t-K\Big)\right|\left|\varepsilon_{-J,K}(\omega)\right|\prod_{l=1}^d 2^{-j_l b_l} \\
&\leq C_4''(\omega)\sum_{J\in\mathbb{Z}_+^d}2^{-j_s(1-a'[f])}\left(2^{-j_1}+\cdots+2^{-j_d}\right)^{-d/2}\sqrt{\log\left(3+d\left\|t\right\|+\sum_{l=1}^d j_l\right)}\prod_{l=1}^d 2^{-j_l/2} \\
&\leq C_5''(\omega)\sum_{J\in\mathbb{Z}_+^d}\sqrt{\log\left(3+\|t\|\right)}2^{-j_s(1-a'[f])}\left(2^{-j_1}+\cdots+2^{-j_d}\right)^{-d/2}\prod_{l=1}^d 2^{-j_l/2}\sqrt{\log\left(3+j_l\right)} \\
&\leq C_6''(\omega)\sqrt{\log\left(3+\|t\|\right)},
\end{aligned}$$

where $C_4''(\omega)$, $C_5''(\omega)$ and $C_6''(\omega)$ are positive finite constants not depending on t. This shows that (4.2.5) holds when $\eta = 0$ and $b \neq 0$.

Part III: we show (4.2.2).

Case 1: $\alpha \in (0,1)$. First notice that, it can easily be derived from the fact that $X[f]^0(\cdot,\omega)$ is an infinitely differentiable function on $\mathbb{R}^d$ vanishing at 0 (see Proposition 3.2.15), that

$$\sup_{\|t\|\le 2}\left\{ \|t\|^{-a'[f]}\Big(\log\Big(3+\|t\|\Big)\Big)^{-d/\alpha-d\delta}\Big|X[f]^0(t,\omega)\Big|\right\} < +\infty \quad \text{if} \quad a'[f]\in(0,1),$$

and

$$\sup_{\|t\|\le 2}\left\{ \Big(\log\Big(3+\|t\|\Big)\Big)^{-d/\alpha-d\delta-1}\Big|X[f]^0(t,\omega)\Big|\right\} < +\infty \qquad \text{if} \quad a'[f]=0. \tag{4.2.24}$$

So, in the sequel, we fix an arbitrary $t\in\mathbb{R}^d$, and we always assume that $\|t\|>2$. Let then $\Gamma_{\inf}(t)$ and $\Gamma_{\sup}(t)$ be the two, non-empty and disjoint, sets of indices $J\in\mathbb{Z}_+^d$ defined as

$$\Gamma_{\sup}(t) := \Big\{ J=(j_1,\dots,j_d)\in\mathbb{Z}_+^d : 2^{\min\{j_1,\dots,j_d\}} > \|t\|\Big\}, \tag{4.2.25}$$

and

$$\Gamma_{\inf}(t) := \Big\{ J=(j_1,\dots,j_d)\in\mathbb{Z}_+^d : 2^{\min\{j_1,\dots,j_d\}} \le \|t\|\Big\}. \tag{4.2.26}$$

Thus, it follows from (3.2.58) (with $b=0$) and from the equality $\mathbb{Z}_+^d=\Gamma_{\sup}(t)\cup\Gamma_{\inf}(t)$ (disjoint union) that

$$X[f]^0(t) = X[f]^0_{\sup}(t) + X[f]^0_{\inf}(t), \tag{4.2.27}$$

where

$$X[f]^0_{\sup}(t,\omega) = \sum_{(J,K)\in\Gamma_{\sup}(t)\times\mathbb{Z}^d}\Big(\Psi_{\alpha,-J}[f](2^{-J}t-K)-\Psi_{\alpha,-J}[f](-K)\Big)\varepsilon_{\alpha,-J,K}(\omega), \tag{4.2.28}$$

and

$$X[f]^0_{\inf}(t,\omega) = \sum_{(J,K)\in\Gamma_{\inf}(t)\times\mathbb{Z}^d}\Big(\Psi_{\alpha,-J}[f](2^{-J}t-K)-\Psi_{\alpha,-J}[f](-K)\Big)\varepsilon_{\alpha,-J,K}(\omega). \tag{4.2.29}$$

From now on, our goal is to derive appropriate upper-bounds for $X[f]^0_{\sup}(t,\omega)$ and $X[f]^0_{\inf}(t,\omega)$.

First, we focus on $X[f]^0_{\sup}(t,\omega)$. In view of (4.2.25), when $J=(j_1,\dots,j_d)\in\Gamma_{\sup}(t)$, then, for any $l\in\{1,\dots,d\}$, one has $|2^{-j_l}t_l|<1$, the t_l's being the coordinates of t. Thus, using the triangle inequality, we get that

$$\prod_{l=1}^d\Big(2+|2^{-j_l}t_l-k_l|\Big) > \prod_{l=1}^d\Big(1+|k_l|\Big), \quad \text{for all } K=(k_1,\dots,k_d)\in\mathbb{Z}^d. \tag{4.2.30}$$

Next applying, as in (3.2.48), the Mean Value Theorem to $\Psi_{\alpha,-J}(2^{-J}t-K)-\Psi_{\alpha,-J}(-K)$, and using (4.2.28), (3.2.33), (4.2.30) and (3.2.35) we obtain that

$$\Big|X[f]^0_{\sup}(t,\omega)\Big| \le C_6(\omega)\,\|t\|\sum_{r=1}^d\sum_{J\in\Gamma_{\sup}(t)}2^{-j_r}\Big(2^{-j_1}+\dots+2^{-j_d}\Big)^{-a'[f]-d/\alpha}\prod_{l=1}^d 2^{-j_l/\alpha}(1+j_l)^{1/\alpha+\delta}, \tag{4.2.31}$$

where $C_6(\omega)$ is a positive finite constant not depending on t. Next, for every fixed $m \in \{1,\dots,d\}$, we let $\Gamma_{\sup}^m(t)$ be the subset of $\Gamma_{\sup}(t)$ defined as

$$\Gamma_{\sup}^m(t) := \Big\{ J=(j_1,\dots,j_d) \in \Gamma_{\sup}(t) : j_m = \min\{j_1,\dots,j_d\} \Big\}. \tag{4.2.32}$$

Observe that, in view of (4.2.25) and (4.2.32), for each fixed $m \in \{1,\dots,d\}$, one has

$$\Gamma_{\sup}^m(t) = \Big\{ J=(j_1,\dots,j_d) \in \mathbb{Z}_+^d : \text{ for all } l \in \{1,\dots,d\},\ j_l \geq j_m \geq N(t)+1 \Big\}, \tag{4.2.33}$$

where

$$N(t) := \Big\lfloor \log(\|t\|)/\log(2) \Big\rfloor \tag{4.2.34}$$

is the integer part of $\log(\|t\|)/\log(2)$. Also, observe that one has $\Gamma_{\sup}(t) = \bigcup_{m=1}^d \Gamma_{\sup}^m(t)$. Combining this equality with (4.2.31) and (4.2.33), we get

$$\begin{aligned}
&\Big| X[f]_{\sup}^0(t,\omega) \Big| \\
&\leq d\, C_6(\omega)\, \|t\| \sum_{m=1}^{d} \sum_{J \in \Gamma_{\sup}^m(t)} 2^{j_m(a'[f]+d/\alpha-1)} \prod_{l=1}^{d} 2^{-j_l/\alpha}(1+j_l)^{1/\alpha+\delta} \\
&= d^2\, C_6(\omega)\, \|t\| \sum_{J \in \Gamma_{\sup}^1(t)} 2^{j_1(a'[f]+d/\alpha-1)} \prod_{l=1}^{d} 2^{-j_l/\alpha}(1+j_l)^{1/\alpha+\delta} \\
&= d^2\, C_6(\omega)\, \|t\| \sum_{j_1=N(t)+1}^{+\infty} 2^{j_1(a'[f]+d/\alpha-1-1/\alpha)}(1+j_1)^{1/\alpha+\delta} \sum_{j_2=j_1}^{+\infty} \dots \sum_{j_d=j_1}^{+\infty} \prod_{l=2}^{d} 2^{-j_l/\alpha}(1+j_l)^{1/\alpha+\delta}.
\end{aligned} \tag{4.2.35}$$

Now, we recall a useful inequality: let ν and μ be two arbitrary fixed positive real numbers, their exists a finite constant c_7, only depending on ν and μ, such that for all $q \in \mathbb{Z}_+$, one has

$$\sum_{j=q}^{+\infty} 2^{-j\nu}(1+j)^{\mu} \leq c_7 2^{-q\nu}(1+q)^{\mu}. \tag{4.2.36}$$

Next, combining (4.2.35) and (4.2.36), we get that,

$$\Big| X[f]_{\sup}^0(t,\omega) \Big| \leq C_8(\omega)\, \|t\| \sum_{j_1=N(t)+1}^{+\infty} 2^{-j_1(1-a'[f])}(1+j_1)^{d/\alpha+d\delta}, \tag{4.2.37}$$

where $C_8(\omega)$ is a positive finite constant not depending on t. Then, (4.2.37), (4.2.36) and (4.2.34) entail that

$$\Big| X[f]_{\sup}^0(t,\omega) \Big| \leq C_9(\omega)\, \|t\|^{a'[f]} \log(3+\|t\|)^{d/\alpha+d\delta}, \tag{4.2.38}$$

for some positive finite constant $C_9(\omega)$ not depending on t.

Now, we focus on $X[f]^0_{\inf}(t,\omega)$. It results from (4.2.29) and the triangle inequality that

$$\left|X[f]^0_{\inf}(t,\omega)\right| \le R[f]^0_{\inf}(t,\omega) + S[f]^0_{\inf}(t,\omega), \tag{4.2.39}$$

where

$$R[f]^0_{\inf}(t,\omega) = \sum_{(J,K)\in\Gamma_{\inf}(t)\times\mathbb{Z}^d} \left|\Psi_{\alpha,-J}[f](2^{-J}t-K)\right| |\varepsilon_{\alpha,-J,K}(\omega)| \tag{4.2.40}$$

and

$$S[f]^0_{\inf}(t,\omega) = \sum_{(J,K)\in\Gamma_{\inf}(t)\times\mathbb{Z}^d} |\Psi_{\alpha,-J}[f](-K)|\, |\varepsilon_{\alpha,-J,K}(\omega)|\,. \tag{4.2.41}$$

Next, for every fixed $m\in\{1,\dots,d\}$, we denote by $\Gamma^m_{\inf}(t)$ the subset of $\Gamma_{\inf}(t)$ defined as

$$\Gamma^m_{\inf}(t) := \Big\{ J=(j_1,\dots,j_d)\in\Gamma_{\inf}(t) : j_m = \min\{j_1,\dots,j_d\}\Big\}. \tag{4.2.42}$$

Observe that, in view of (4.2.26), (4.2.42) and (4.2.34), for each fixed $m\in\{1,\dots,d\}$, one has

$$\Gamma^m_{\inf}(t) = \Big\{ J=(j_1,\dots,j_d)\in\mathbb{Z}^d_+ : j_m \le N(t) \text{ and for all } l\in\{1,\dots,d\},\ j_l\ge j_m\Big\}. \tag{4.2.43}$$

Also, observe that one has $\Gamma_{\inf}(t) = \bigcup_{m=1}^d \Gamma^m_{\inf}(t)$. Combining this equality with (4.2.40), (3.2.33), (3.2.35), (4.2.43), and (4.2.36), we obtain

$$\begin{aligned}
&R[f]^0_{\inf}(t,\omega)\\
&\le C_{10}(\omega)\sum_{J\in\Gamma_{\inf}(t)} \Big(2^{-j_1}+\dots+2^{-j_d}\Big)^{-a'[f]-d/\alpha}\prod_{l=1}^d 2^{-j_l/\alpha}(1+j_l)^{1/\alpha+\delta}\\
&\le C_{10}(\omega)\sum_{m=1}^d\sum_{J\in\Gamma^m_{\inf}(t)} 2^{j_m(a'[f]+d/\alpha)}\prod_{l=1}^d 2^{-j_l/\alpha}(1+j_l)^{1/\alpha+\delta}\\
&= d\,C_{10}(\omega)\sum_{J\in\Gamma^1_{\inf}(t)} 2^{j_1(a'[f]+d/\alpha)}\prod_{l=1}^d 2^{-j_l/\alpha}(1+j_l)^{1/\alpha+\delta}\\
&= d\,C_{10}(\omega)\sum_{j_1=0}^{N(t)} 2^{j_1(a'[f]+d/\alpha-1/\alpha)}(1+j_1)^{1/\alpha+\delta}\sum_{j_2=j_1}^{+\infty}\dots\sum_{j_d=j_1}^{+\infty}\prod_{l=2}^d 2^{-j_l/\alpha}(1+j_l)^{1/\alpha+\delta}\\
&\le C_{11}(\omega)\sum_{j_1=0}^{N(t)} 2^{j_1a'[f]}(1+j_1)^{d/\alpha+d\delta}\\
&\le C_{11}(\omega)(1+N(t))^{d/\alpha+d\delta}\sum_{j_1=0}^{N(t)} 2^{j_1a'[f]},
\end{aligned} \tag{4.2.44}$$

where $C_{10}(\omega)$ and $C_{11}(\omega)$ are two finite constants not depending on t. On the other hand, thanks to standard computations we can show that

$$\sum_{j_1=0}^{N(t)} 2^{j_1 a'[f]} \leq \begin{cases} c_{12} 2^{N(t)a'[f]} & \text{if} \quad a'[f] \in (0,1), \\ \\ N(t)+1 & \text{if} \quad a'[f] = 0, \end{cases} \tag{4.2.45}$$

where c_{12} is a positive and finite constant. Next, combining (4.2.44) and (4.2.45) with (4.2.34), it follows that

$$R[f]^0_{\inf}(t,\omega) \leq \begin{cases} C_{13}(\omega)\, \|t\|^{a'[f]} \log(3+\|t\|)^{d/\alpha+d\delta} & \text{if} \quad a'[f] \in (0,1), \\ \\ C_{14}(\omega)\Big(\log(3+\|t\|)\Big)^{d/\alpha+d\delta+1} & \text{if} \quad a'[f] = 0, \end{cases} \tag{4.2.46}$$

where $C_{13}(\omega)$ and $C_{14}(\omega)$ are finite constants not depending on t. Similarly to (4.2.46), it can be shown that

$$S[f]^0_{\inf}(t,\omega) \leq \begin{cases} C_{13}(\omega)\, \|t\|^{a'[f]} \log(3+\|t\|)^{d/\alpha+d\delta} & \text{if} \quad a'[f] \in (0,1), \\ \\ C_{14}(\omega)\Big(\log(3+\|t\|)\Big)^{d/\alpha+d\delta+1} & \text{if} \quad a'[f] = 0. \end{cases} \tag{4.2.47}$$

Next, combining (4.2.46) and (4.2.47) with (4.2.39), we get that

$$\Big|X[f]^0_{\inf}(t,\omega)\Big| \leq \begin{cases} 2C_{13}(\omega)\, \|t\|^{a'[f]} \log(3+\|t\|)^{d/\alpha+d\delta} & \text{if} \quad a'[f] \in (0,1), \\ \\ 2C_{14}(\omega)\Big(\log(3+\|t\|)\Big)^{d/\alpha+d\delta+1} & \text{if} \quad a'[f] = 0. \end{cases} \tag{4.2.48}$$

Finally (4.2.48), (4.2.38) and (4.2.27) imply that

$$\sup_{\|t\|>2} \left\{ \|t\|^{-a'[f]} \Big(\log\Big(3+\|t\|\Big)\Big)^{-d/\alpha-d\delta} \Big|X[f]^0(t,\omega)\Big| \right\} < +\infty \quad \text{if} \quad a'[f] \in (0,1),$$

and

$$\sup_{\|t\|>2} \left\{ \Big(\log\Big(3+\|t\|\Big)\Big)^{-d/\alpha-d\delta-1} \Big|X[f]^0(t,\omega)\Big| \right\} < +\infty \qquad \text{if} \quad a'[f] = 0. \tag{4.2.49}$$

Then using (4.2.24) and (4.2.49) we obtain (4.2.2).

Case 2: $\alpha \in [1,2)$. Similarly to the case 1, we have that

$$\sup_{\|t\|\leq 2} \left\{ \|t\|^{-a'[f]} \Big(\log\Big(3+\|t\|\Big)\Big)^{-d/\alpha-d\delta} \Big(\log\log(3+\|t\|)\Big)^{-1/2} \Big|X[f]^0(t,\omega)\Big| \right\} < +\infty \quad \text{if} \quad a'[f] \in (0,1),$$

and

$$\sup_{\|t\|\leq 2} \left\{ \Big(\log\Big(3+\|t\|\Big)\Big)^{-d/\alpha-d\delta-3/2} \Big(\log\log(3+\|t\|)\Big)^{-1/2} \Big|X[f]^0(t,\omega)\Big| \right\} < +\infty \quad \text{if} \quad a'[f] = 0. \tag{4.2.50}$$

So, in the sequel, we fix an arbitrary $t \in \mathbb{R}^d$, and we always assume that $\|t\| > 2$. Let then $\Gamma_{\inf}(t)$, $\Gamma_{\sup}(t)$, $X[f]^0_{\sup}(t,\omega)$ and $X[f]^0_{\inf}(t,\omega)$ be defined as in (4.2.25), (4.2.26), (4.2.28) and (4.2.29). From now on, our goal is to derive appropriate upper-bounds for $X[f]^0_{\sup}(t,\omega)$ and $X[f]^0_{\inf}(t,\omega)$.

First, we focus on $X[f]^0_{\sup}(t,\omega)$. Applying, as in (3.2.48), the Mean Value Theorem to $\Psi_{\alpha,-J}(2^{-J}t-K) - \Psi_{\alpha,-J}(-K)$, and using (4.2.28), (3.2.33), (4.2.30), (3.2.36) and (3.2.38), we obtain that

$$\begin{aligned}&\left|X[f]^0_{\sup}(t,\omega)\right| \\ &\le C'_7(\omega)\,\|t\| \sum_{r=1}^{d} \sum_{J\in\Gamma_{\sup}(t)} 2^{-j_r} \left(\sum_{r=1}^{d} 2^{-j_r}\right)^{-a'[f]-d/2} \sqrt{\log\left(3+\sum_{l=1}^{d} j_l\right)} \prod_{l=1}^{d} 2^{-j_l/2}(1+j_l)^{1/\alpha+\delta},\end{aligned} \tag{4.2.51}$$

where $C'_7(\omega)$ is a positive finite constant not depending on t. Next, for every fixed $m \in \{1,\dots,d\}$, we let $\Gamma^m_{\sup}(t)$ and $N(t)$ as in (4.2.32) and (4.2.34). Combining the equality $\Gamma_{\sup}(t) = \bigcup_{m=1}^{d} \Gamma^m_{\sup}(t)$ with (4.2.51) and (4.2.33), we get

$$\begin{aligned}&\left|X[f]^0_{\sup}(t,\omega)\right| \\ &\le d\,C'_7(\omega)\,\|t\| \sum_{m=1}^{d} \sum_{J\in\Gamma^m_{\sup}(t)} 2^{j_m(a'+d/\alpha-1)} \sqrt{\log\left(3+\sum_{l=1}^{d} j_l\right)} \prod_{l=1}^{d} 2^{-j_l/\alpha}(1+j_l)^{1/\alpha+\delta} \\ &= d^2\,C'_7(\omega)\,\|t\| \sum_{J\in\Gamma^1_{\sup}(t)} 2^{j_1(a'+d/\alpha-1)} \sqrt{\log\left(3+\sum_{l=1}^{d} j_l\right)} \prod_{l=1}^{d} 2^{-j_l/\alpha}(1+j_l)^{1/\alpha+\delta} \\ &= d^2\,C'_7(\omega)\,\|t\| \sum_{j_1=N(t)+1}^{+\infty} \Bigg\{ 2^{j_1(a'+d/\alpha-1-1/\alpha)}(1+j_1)^{1/\alpha+\delta} \\ &\qquad \sum_{j_2=j_1}^{+\infty} \dots \sum_{j_d=j_1}^{+\infty} \sqrt{\log\left(3+j_1+\sum_{l=2}^{d} j_l\right)} \prod_{l=2}^{d} 2^{-j_l/\alpha}(1+j_l)^{1/\alpha+\delta} \Bigg\}.\end{aligned} \tag{4.2.52}$$

Now, we recall a useful inequality (which can easily be derived from (3.2.38)): let ν and μ be two arbitrary fixed positive real numbers, there exists a finite constant c'_9, only depending on ν, such that, for all $(q,\theta) \in \mathbb{Z}_+ \times \mathbb{R}_+$, one has

$$\sum_{j=q}^{+\infty} 2^{-j\nu} \sqrt{\log(3+\theta+j)}(1+j_l)^{\mu} \le c'_9 2^{-q\nu} \sqrt{\log(3+\theta+q)}(1+q)^{\mu}. \tag{4.2.53}$$

Therefore, for each $(j_1,\lambda)\in\mathbb{Z}_+\times\mathbb{R}_+$, one has

$$\begin{aligned}&\sum_{j_2=j_1}^{+\infty}\dots\sum_{j_d=j_1}^{+\infty}\sqrt{\log\Big(3+\lambda+\sum_{l=2}^{d}j_l\Big)}\prod_{l=2}^{d}2^{-j_l/\alpha}(1+j_l)^{1/\alpha+\delta}\\&\le c_9'2^{-j_1(d-1)/\alpha}\Big(1+j_1\Big)^{(d-1)(1/\alpha+\delta)}\sqrt{\log\left(3+\lambda+(d-1)j_1\right)},\end{aligned}\tag{4.2.54}$$

Next, combining (4.2.52) and (4.2.54), we get that,

$$\Big|X[f]^0_{\sup}(t,\omega)\Big|\le C'_{10}(\omega)\,\|t\|\sum_{j_1=N(t)+1}^{+\infty}2^{-j_1(1-a'[f])}(1+j_1)^{d/\alpha+d\delta}\sqrt{\log\left(3+dj_1\right)},\tag{4.2.55}$$

where $C'_{10}(\omega)$ is a positive finite constant not depending on t. Then, (4.2.55), (4.2.53) and (4.2.34) entail that

$$\Big|X[f]^0_{\sup}(t,\omega)\Big|\le C'_{11}(\omega)\,\|t\|^{a'[f]}\Big(\log\Big(3+\|t\|\Big)\Big)^{d/\alpha+d\delta}\sqrt{\log\log(3+\|t\|)},\tag{4.2.56}$$

for some constant $C'_{11}(\omega)$ not depending on t.

Now, we focus on $X^0_{\inf}(t,\omega)$. It results from (4.2.29) and the triangle inequality that

$$\Big|X[f]^0_{\inf}(t,\omega)\Big|\le R[f]^0_{\inf}(t,\omega)+S[f]^0_{\inf}(t,\omega),\tag{4.2.57}$$

where $R[f]^0_{\inf}(t,\omega)$ and $S[f]^0_{\inf}(t,\omega)$ are as in (4.2.40) and (4.2.41). Next, for every fixed $m\in\{1,\dots,d\}$, we denote by $\Gamma^m_{\inf}(t)$ the subset of $\Gamma_{\inf}(t)$ defined as in (4.2.42). Combining the equality $\Gamma_{\inf}(t)=\bigcup_{m=1}^{d}\Gamma^m_{\inf}(t)$ with (4.2.40), (3.2.33), (3.2.36), (3.2.40), (4.2.43), and (4.2.54) (where $\lambda=2^{-j_1}\|t\|+j_1$), we obtain

$$\begin{aligned}&R[f]^0_{\inf}(t,\omega)\\&\le C'_{12}(\omega)\sum_{J\in\Gamma_{\inf}(t)}\Big(\sum_{r=1}^{d}2^{-j_r}\Big)^{-a'[f]-d/\alpha}\sqrt{\log\Big(3+\sum_{l=1}^{d}\Big(j_l+2^{-j_l}|t_l|\Big)\Big)}\prod_{l=1}^{d}2^{-j_l/\alpha}(1+j_l)^{1/\alpha+\delta}\\&\le C'_{12}(\omega)\sum_{m=1}^{d}\sum_{J\in\Gamma^m_{\inf}(t)}2^{j_m(a'[f]+d/\alpha)}\sqrt{\log\Big(3+d\,2^{-j_m}\|t\|+\sum_{l=1}^{d}j_l\Big)}\prod_{l=1}^{d}2^{-j_l/\alpha}(1+j_l)^{1/\alpha+\delta}\\&=d\,C'_{12}(\omega)\sum_{J\in\Gamma^1_{\inf}(t)}2^{j_1(a'[f]+d/\alpha)}\sqrt{\log\Big(3+d\,2^{-j_1}\|t\|+\sum_{l=1}^{d}j_l\Big)}\prod_{l=1}^{d}2^{-j_l/\alpha}(1+j_l)^{1/\alpha+\delta}\\&=d\,C'_{12}(\omega)\sum_{j_1=0}^{N(t)}\Bigg\{2^{j_1(a'[f]+d/\alpha-1/\alpha)}\\&\qquad\sum_{j_2=j_1}^{+\infty}\dots\sum_{j_d=j_1}^{+\infty}\sqrt{\log\Big(3+2^{-j_1}\|t\|+j_1+\sum_{l=2}^{d}j_l\Big)}\prod_{l=2}^{d}2^{-j_l/\alpha}(1+j_l)^{1/\alpha+\delta}\Bigg\}\\&\le C'_{13}(\omega)\sum_{j_1=0}^{N(t)}2^{j_1a'[f]}\sqrt{\log\left(3+d\,2^{-j_1}\|t\|+d\,j_1\right)}(1+j_1)^{d/\alpha+d\delta},\end{aligned}\tag{4.2.58}$$

where $C'_{12}(\omega)$ and $C'_{13}(\omega)$ are two finite constants not depending on t. On the other hand, thanks to (4.2.34), we have that

$$\begin{aligned}
&\sum_{j_1=0}^{N(t)} 2^{j_1 a'[f]}(1+j_1)^{d/\alpha+d\delta}\sqrt{\log\left(3+d\,2^{-j_1}\,\|t\|+d\,j_1\right)}\\
&=2^{N(t)a'[f]}\sum_{j_1=0}^{N(t)} 2^{-a'[f](N(t)-j_1)}(1+j_1)^{d/\alpha+d\delta}\sqrt{\log\left(3+d\,2^{-j_1}\,\|t\|+d\,j_1\right)}\\
&=2^{N(t)a'[f]}\sum_{j_1=0}^{N(t)} 2^{-a'[f]j_1}(1+j_1)^{d/\alpha+d\delta}\sqrt{\log\left(3+d\,2^{-N(t)}\,\|t\|\,2^{j_1}+d\,(N(t)-j_1)\right)}\\
&\leq 2^{N(t)a'[f]}(1+N(t))^{d/\alpha+d\delta}\sum_{j_1=0}^{N(t)} 2^{-a'[f]j_1}\sqrt{\log\left(3+d2^{j_1+1}+d\,N(t)\right)}\\
&\leq c_{14}2^{N(t)a'[f]}(1+N(t))^{d/\alpha+d\delta}\sqrt{\log\left(3+d\,N(t)\right)}\sum_{j_1=0}^{N(t)} 2^{-a'[f]j_1}\sqrt{1+j_1},
\end{aligned}$$

where c'_{14} is a positive finite constant not depending on t. Next, combining (4.2.58) and (4.2.59) with (4.2.34), it follows that

$$R[f]^0_{\inf}(t,\omega)\leq\begin{cases} C'_{15}(\omega)\,\|t\|^{a'[f]}\log(3+\|t\|)^{d/\alpha+d\delta}\sqrt{\log\log(3+\|t\|)} & \text{if}\quad a'[f]\in(0,1),\\[2ex] C'_{16}(\omega)\log(3+\|t\|)^{d/\alpha+d\delta+3/2}\sqrt{\log\log(3+\|t\|)} & \text{if}\quad a'[f]=0,\end{cases}\tag{4.2.59}$$

where $C'_{15}(\omega)$ and $C'_{16}(\omega)$ are finite constants not depending on t. Similarly to (4.2.59), it can be shown that

$$S[f]^0_{\inf}(t,\omega)\leq\begin{cases} C'_{15}(\omega)\,\|t\|^{a'[f]}\log(3+\|t\|)^{d/\alpha+d\delta}\sqrt{\log\log(3+\|t\|)} & \text{if}\quad a'[f]\in(0,1),\\[2ex] C'_{16}(\omega)\log(3+\|t\|)^{d/\alpha+d\delta+3/2}\sqrt{\log\log(3+\|t\|)} & \text{if}\quad a'[f]=0.\end{cases}\tag{4.2.60}$$

Next, combining (4.2.59) and (4.2.60) with (4.2.39), we get that

$$\left|X[f]^0_{\inf}(t,\omega)\right|\leq\begin{cases} 2C'_{15}(\omega)\,\|t\|^{a'[f]}\log(3+\|t\|)^{d/\alpha+d\delta}\sqrt{\log\log(3+\|t\|)} & \text{if}\quad a'[f]\in(0,1),\\[2ex] 2C'_{16}(\omega)\log(3+\|t\|)^{d/\alpha+d\delta+3/2}\sqrt{\log\log(3+\|t\|)} & \text{if}\quad a'[f]=0.\end{cases}\tag{4.2.61}$$

Finally (4.2.61), (4.2.56) and (4.2.27) imply that

$$\sup_{\|t\|>2}\left\{\|t\|^{-a'[f]}\Big(\log\big(3+\|t\|\big)\Big)^{-d/\alpha-d\delta}\Big(\log\log(3+\|t\|)\Big)^{-1/2}\Big|X[f]^0(t,\omega)\Big|\right\}<+\infty \quad \text{if} \quad a'[f]\in(0,1),$$

and

$$\sup_{\|t\|>2}\left\{\Big(\log\big(3+\|t\|\big)\Big)^{-d/\alpha-d\delta-3/2}\Big(\log\log(3+\|t\|)\Big)^{-1/2}\Big|X[f]^0(t,\omega)\Big|\right\}<+\infty \quad \text{if} \quad a'[f]=0. \tag{4.2.62}$$

Then using (4.2.50) and (4.2.62) we obtain (4.2.4).

<u>Case 3: $\alpha=2$</u>. Similarly to the case 1, we have that

$$\sup_{\|t\|\le 2}\left\{\|t\|^{-a'[f]}\Big(\log\log\big(3+\|t\|\big)\Big)^{-1/2}\Big|X[f]^0(t,\omega)\Big|\right\}<+\infty, \qquad \text{if} \quad a'[f]\in(0,1),$$

and

$$\sup_{\|t\|\le 2}\left\{\log(3+\|t\|)^{-3/2}\Big(\log\log\big(3+\|t\|\big)\Big)^{-1/2}\Big|X[f]^0(t,\omega)\Big|\right\}<+\infty \quad \text{if} \quad a'[f]=0. \tag{4.2.63}$$

So, in the sequel, we fix an arbitrary $t\in\mathbb{R}^d$, and we always assume that $\|t\|>2$. Let then $\Gamma_{\inf}(t)$, $\Gamma_{\sup}(t)$, $X[f]^0_{\sup}(t)$ and $X[f]^0_{\inf}(t)$ be defined as in (4.2.25), (4.2.26), (4.2.28) and (4.2.29). From now on, our goal is to derive appropriate upper-bounds for $X[f]^0_{\sup}(t)$ and $X[f]^0_{\inf}(t)$.

First, we focus on $X[f]^0_{\sup}(t,\omega)$. Applying, as in (3.2.48), the Mean Value Theorem to $\Psi_{-J}(2^{-J}t-K)-\Psi_{-J}(-K)$, and using (4.2.28), (3.2.33), (4.2.30), (3.2.37) and (3.2.38), we obtain that

$$\Big|X[f]^0_{\sup}(t,\omega)\Big|\le C_7''(\omega)\,\|t\|\sum_{r=1}^{d}\sum_{J\in\Gamma_{\sup}(t)}2^{-j_r}\left(\sum_{u=1}^{d}2^{-j_u}\right)^{-a'[f]-d/2}\sqrt{\log\left(3+\sum_{l=1}^{d}j_l\right)}\prod_{l=1}^{d}2^{-j_l/2}, \tag{4.2.64}$$

where $C_7''(\omega)$ is a positive finite constant not depending on t. Next, for every fixed $m\in\{1,\dots,d\}$, we let $\Gamma^m_{\sup}(t)$ and $N(t)$ as in (4.2.32) and (4.2.34). Combining the equality

$\Gamma_{\sup}(t)=\bigcup_{m=1}^{d}\Gamma_{\sup}^{m}(t)$ with (4.2.64) and (4.2.33), we get

$$\begin{aligned}
&\left|X[f]_{\sup}^{0}(t,\omega)\right| \\
&\le d\,C_7''(\omega)\,\|t\|\sum_{m=1}^{d}\sum_{J\in\Gamma_{\sup}^{m}(t)}2^{j_m(a'+d/2-1)}\sqrt{\log\left(3+\sum_{l=1}^{d}j_l\right)}\prod_{l=1}^{d}2^{-j_l/2} \\
&= d^2\,C_7''(\omega)\,\|t\|\sum_{J\in\Gamma_{\sup}^{1}(t)}2^{j_1(a'+d/2-1)}\sqrt{\log\left(3+\sum_{l=1}^{d}j_l\right)}\prod_{l=1}^{d}2^{-j_l/2} \\
&= d^2\,C_7''(\omega)\,\|t\|\sum_{j_1=N(t)+1}^{+\infty}2^{j_1(a'+d/2-3/2)}\sum_{j_2=j_1}^{+\infty}\dots\sum_{j_d=j_1}^{+\infty}\sqrt{\log\left(3+j_1+\sum_{l=2}^{d}j_l\right)}\prod_{l=2}^{d}2^{-j_l/2}.
\end{aligned} \tag{4.2.65}$$

Now, we recall a useful inequality (which can easily be derived from (3.2.38)): let ν be an arbitrary fixed positive real number, there exists a finite constant c_8'', only depending on ν, such that, for all $(q,\theta)\in\mathbb{Z}_+\times\mathbb{R}_+$, one has

$$\sum_{j=q}^{+\infty}2^{-j\nu}\sqrt{\log\left(3+\theta+j\right)}\le c_8''2^{-q\nu}\sqrt{\log\left(3+\theta+q\right)}. \tag{4.2.66}$$

Therefore, for each $(j_1,\lambda)\in\mathbb{Z}_+\times\mathbb{R}_+$, one has

$$\sum_{j_2=j_1}^{+\infty}\dots\sum_{j_d=j_1}^{+\infty}\sqrt{\log\left(3+\lambda+\sum_{l=2}^{d}j_l\right)}\prod_{l=2}^{d}2^{-j_l/2}\le c_9''2^{-j_1(d-1)/2}\sqrt{\log\left(3+\lambda+(d-1)j_1\right)}, \tag{4.2.67}$$

where c_9'' is a finite constant not depending on (j_1,λ). Next, combining (4.2.65) and (4.2.67) (with $\lambda=j_1$), we get that,

$$\left|X[f]_{\sup}^{0}(t,\omega)\right|\le C_{10}''(\omega)\,\|t\|\sum_{j_1=N(t)+1}^{+\infty}2^{-j_1(1-a'[f])}\sqrt{\log\left(3+d\,j_1\right)}, \tag{4.2.68}$$

where $C_{10}''(\omega)$ is a positive finite constant not depending on t. Then, (4.2.68), (4.2.66) and (4.2.34) entail that

$$\left|X[f]_{\sup}^{0}(t,\omega)\right|\le C_{11}''(\omega)\,\|t\|^{a'[f]}\sqrt{\log\log(3+\|t\|)}, \tag{4.2.69}$$

for some positive finite constant $C_{11}''(\omega)$ not depending on t.

Now, we focus on $X_{\inf}^{0}(t)$. It results from (4.2.29) and the triangle inequality that

$$\left|X[f]_{\inf}^{0}(t,\omega)\right|\le R[f]_{\inf}^{0}(t,\omega)+S[f]_{\inf}^{0}(t,\omega), \tag{4.2.70}$$

where $R[f]^0_{\mathrm{inf}}(t,\omega)$ and $S[f]^0_{\mathrm{inf}}(t,\omega)$ are as in (4.2.40) and (4.2.41). Next, for every fixed $m \in \{1,\ldots,d\}$, we denote by $\Gamma^m_{\mathrm{inf}}(t)$ the subset of $\Gamma_{\mathrm{inf}}(t)$ defined as in (4.2.42). Combining the equality $\Gamma_{\mathrm{inf}}(t) = \bigcup_{m=1}^d \Gamma^m_{\mathrm{inf}}(t)$ with (4.2.40), (3.2.33), (3.2.37), (3.2.40), (4.2.43), and (4.2.67) (where $\lambda = 2^{-j_1}\|t\| + j_1$), we obtain

$$\begin{aligned}
&R[f]^0_{\mathrm{inf}}(t,\omega)\\
&\le C''_{12}(\omega) \sum_{J\in\Gamma_{\mathrm{inf}}(t)} \left(2^{-j_1}+\cdots+2^{-j_d}\right)^{-a'[f]-d/2} \sqrt{\log\left(3+\sum_{l=1}^d \left(j_l + 2^{-j_l}|t_l|\right)\right)} \prod_{l=1}^d 2^{-j_l/2}\\
&\le C''_{12}(\omega) \sum_{m=1}^d \sum_{J\in\Gamma^m_{\mathrm{inf}}(t)} 2^{j_m(a'[f]+d/2)} \sqrt{\log\left(3+d\,2^{-j_m}\|t\| + \sum_{l=1}^d j_l\right)} \prod_{l=1}^d 2^{-j_l/2}\\
&= d\,C''_{12}(\omega) \sum_{J\in\Gamma^1_{\mathrm{inf}}(t)} 2^{j_1(a'[f]+d/2)} \sqrt{\log\left(3+d\,2^{-j_1}\|t\| + \sum_{l=1}^d j_l\right)} \prod_{l=1}^d 2^{-j_l/2}\\
&= d\,C''_{12}(\omega) \sum_{j_1=0}^{N(t)} 2^{j_1(a'[f]+d/2-1/2)} \sum_{j_2=j_1}^{+\infty}\cdots\sum_{j_d=j_1}^{+\infty} \sqrt{\log\left(3+2^{-j_1}\|t\| + j_1 + \sum_{l=2}^d j_l\right)} \prod_{l=2}^d 2^{-j_l/2}\\
&\le C''_{13}(\omega) \sum_{j_1=0}^{N(t)} 2^{j_1 a'[f]} \sqrt{\log\left(3+d\,2^{-j_1}\|t\| + d\,j_1\right)},
\end{aligned} \tag{4.2.71}$$

where $C''_{12}(\omega)$ and $C''_{13}(\omega)$ are two finite constants not depending on t. On the other hand, thanks to (4.2.34), we have that

$$\begin{aligned}
&\sum_{j_1=0}^{N(t)} 2^{j_1 a'[f]} \sqrt{\log\left(3+d\,2^{-j_1}\|t\| + d\,j_1\right)}\\
&= 2^{N(t)a'[f]} \sum_{j_1=0}^{N(t)} 2^{-a'[f](N(t)-j_1)} \sqrt{\log\left(3+d\,2^{-j_1}\|t\| + d\,j_1\right)}\\
&= 2^{N(t)a'[f]} \sum_{j_1=0}^{N(t)} 2^{-a'[f]j_1} \sqrt{\log\left(3+d\,2^{-N(t)}\|t\|\,2^{j_1} + d\,(N(t)-j_1)\right)}\\
&\le 2^{N(t)a'[f]} \sum_{j_1=0}^{N(t)} 2^{-a'[f]j_1} \sqrt{\log\left(3+d 2^{j_1+1} + d\,N(t)\right)}\\
&\le c''_{14} 2^{N(t)a'[f]} \sqrt{\log\left(3+d\,N(t)\right)} \sum_{j_1=0}^{N(t)} 2^{-a'[f]j_1} \sqrt{1+j_1}
\end{aligned} \tag{4.2.72}$$

where c''_{14} is a positive and finite constant. Next, combining (4.2.71) and (4.2.72) with (4.2.34), it follows that

$$R[f]^0_{\inf}(t,\omega) \leq \begin{cases} C''_{15}(\omega)\,\|t\|^{a'[f]}\sqrt{\log\log(3+\|t\|)} & \text{if} \quad a'[f]\in(0,1), \\ \\ C''_{16}(\omega)\sqrt{\log\log(3+\|t\|)}\,\log(3+\|t\|)^{3/2} & \text{if} \quad a'[f]=0. \end{cases} \tag{4.2.73}$$

where $C''_{15}(\omega)$ and $C''_{16}(\omega)$ are two finite constants not depending on t. Similarly to (4.2.73), it can be shown that

$$S[f]^0_{\inf}(t,\omega) \leq \begin{cases} C''_{15}(\omega)\,\|t\|^{a'[f]}\sqrt{\log\log(3+\|t\|)} & \text{if} \quad a'[f]\in(0,1), \\ \\ C''_{16}(\omega)\sqrt{\log\log(3+\|t\|)}\,\log(3+\|t\|)^{3/2} & \text{if} \quad a'[f]=0. \end{cases} \tag{4.2.74}$$

Next, combining (4.2.73) and (4.2.74) with (4.2.70), we get that

$$\left|X[f]^0_{\inf}(t,\omega)\right| \leq \begin{cases} 2C''_{15}(\omega)\,\|t\|^{a'[f]}\sqrt{\log\log(3+\|t\|)} & \text{if} \quad a'[f]\in(0,1), \\ \\ 2C''_{16}(\omega)\sqrt{\log\log(3+\|t\|)}\,\log(3+\|t\|)^{3/2} & \text{if} \quad a'[f]=0. \end{cases} \tag{4.2.75}$$

Finally (4.2.75), (4.2.69) and (4.2.27) imply that

$$\begin{aligned} &\sup_{\|t\|>2}\left\{\|t\|^{-a'[f]}\Big(\log\log\Big(3+\|t\|\Big)\Big)^{-1/2}\left|X[f]^0(t,\omega)\right|\right\}<+\infty, \qquad \text{if} \quad a'[f]\in(0,1), \\ &\text{and} \\ &\sup_{\|t\|>2}\left\{\log(3+\|t\|)^{-3/2}\Big(\log\log\Big(3+\|t\|\Big)\Big)^{-1/2}\left|X[f]^0(t,\omega)\right|\right\}<+\infty \quad \text{if} \quad a'[f]=0. \end{aligned} \tag{4.2.76}$$

Then using (4.2.63) and (4.2.76) we obtain (4.2.6). □

4.3 Monodirectional increments and behaviour at infinity

Let f be an admissible function, $X[f]$ the field associated with f and $X[f]^\eta$ an arbitrary η-frequency part of $X[f]$, where $\eta=(\eta_1,\ldots,\eta_d)\in\Upsilon:=\{0,1\}^d$ (see Definition 3.1.1, (2.3.3), Definition 3.2.12 and Remark 3.2.13). The main goal of this section is to draw a connection between the increments of $X[f]^\eta$ and $X[f]$ in a fixed direction on a compact set whereas the other variables belongs to $\mathbb{R}$. Let $r\in\{1,\ldots,d\}$ be fixed. In order to conveniently state the next result we need some additional notations: assuming that $t=(t_1,\ldots,t_d)$ is an arbitrary vector of $\mathbb{R}^d$, we denote by $\widehat{t_r}$ the vector of $\mathbb{R}^{d-1}$ defined as

$$\widehat{t_r}=(t_1,\ldots,t_{r-1},t_{r+1},\ldots,t_d),$$

with the convention that $\widehat{t_d} = (t_1, \ldots, t_{d-1})$ and $\widehat{t_1} = (t_2, \ldots, t_d)$. Thus the initial vector t is identified with the couple $t = (t_r, \widehat{t}_r)$.

Theorem 4.3.1. *The positive exponents $a_1[f], \ldots, a_d[f]$ are the same as in Definition 3.1.1. Moreover, we assume that $r \in \{1, \ldots, d\}$, $\eta = (\eta_1, \ldots, \eta_d) \in \Upsilon$, $b \in \mathbb{N}$, $T \in (0, +\infty)$ and $\omega \in \Omega_1^*$ are arbitrary and fixed. Then, the following three results hold (with the convention that $0^0 = 1$).*

1. *When $\alpha \in (0,1)$, for all arbitrarily small positive real numbers δ, there exists $C(\omega) \in (0, +\infty)$ such that the inequality*

$$\begin{aligned} &\left|\Delta_{h_r}^{r,b} X[f]^{\eta}(t_r, \widehat{t}_r, \omega)\right| \\ &\leq C(\omega)|h_r|^{b(1-\eta_r)}\, |h_r|^{\min(b, a_r[f])\eta_r} \left(\log\left(3 + |h_r|^{-1}\right)\right)^{\eta_r \mathcal{L}_\alpha(a_r[f], b, \delta)}. \end{aligned} \tag{4.3.1}$$

   holds for any $(h_r, t_r, \widehat{t}_r) \in [-T, T] \times [-T, T] \times \mathbb{R}^{d-1}$.

2. *When $\alpha \in [1,2)$, for all arbitrarily small positive real numbers δ, there exists $C(\omega) \in (0, +\infty)$ such that the inequality*

$$\begin{aligned} &\left|\Delta_{h_r}^{r,b} X[f]^{\eta}(t_r, \widehat{t}_r, \omega)\right| \\ &\leq C(\omega)|h_r|^{b(1-\eta_r)}\, |h_r|^{\min(b, a_r[f])\eta_r} \left(\log\left(3 + |h_r|^{-1}\right)\right)^{\eta_r \mathcal{L}_\alpha(a_r[f], b, \delta)} \sqrt{\log\left(3 + \sum_{\substack{l=1 \\ l \neq r}}^{d} |t_l|\right)} \end{aligned} \tag{4.3.2}$$

   holds for any $(h_r, t_r, \widehat{t}_r) \in [-T, T] \times [-T, T] \times \mathbb{R}^{d-1}$.

3. *When $\alpha = 2$,*

$$\begin{aligned} &\left|\Delta_{h_r}^{r,b} X[f]^{\eta}(t_r, \widehat{t}_r, \omega)\right| \\ &\leq C(\omega)|h_r|^{b(1-\eta_r)}\, |h_r|^{\min(b, a_r[f])\eta_r} \left(\log\left(3 + |h_r|^{-1}\right)\right)^{\eta_r \mathcal{L}_2(a_r[f], b)} \sqrt{\log\left(3 + \sum_{\substack{l=1 \\ l \neq r}}^{d} |t_l|\right)} \end{aligned} \tag{4.3.3}$$

   holds for any $(h_r, t_r, \widehat{t}_r) \in [-T, T] \times [-T, T] \times \mathbb{R}^{d-1}$.

Recall that the functions $\mathcal{L}_\alpha$ and $\mathcal{L}_2$ are defined in Definition 4.1.1.

It easily follows from Remark 3.2.13 and Theorem 4.3.1 that:

Corollary 4.3.2. *The positive exponents $a_1[f], \ldots, a_d[f]$ are the same as in Definition 3.1.1. Moreover, we assume that $r \in \{1, \ldots, d\}$, $b \in \mathbb{N}$, $T \in (0, +\infty)$ and $\omega \in \Omega_1^*$ are arbitrary and fixed. Then, the following three results (with the convention that $0^0 = 1$).*

1. *When $\alpha \in (0,1)$, for all arbitrarily small positive real numbers δ, there exists $C(\omega) \in (0, +\infty)$ such that the inequality*
$$\begin{aligned}&\left|\Delta_{h_r}^{r,b} X[f](t_r, \widehat{t}_r, \omega)\right| \\ &\le C(\omega)|h_r|^{\min(b, a_r[f])} \left(\log\left(3 + |h_r|^{-1}\right)\right)^{\mathcal{L}_\alpha(a_r[f], b, \delta)}. \end{aligned} \tag{4.3.4}$$
holds for any $(h_r, t_r, \widehat{t}_r) \in [-T, T] \times [-T, T] \times \mathbb{R}^{d-1}$.

2. *When $\alpha \in [1,2)$, for all arbitrarily small positive real numbers δ, there exists $C(\omega) \in (0, +\infty)$ such that the inequality*
$$\begin{aligned}&\left|\Delta_{h_r}^{r,b} X[f](t_r, \widehat{t}_r, \omega)\right| \\ &\le C(\omega)|h_r|^{\min(b, a_r[f])} \left(\log\left(3 + |h_r|^{-1}\right)\right)^{\mathcal{L}_\alpha(a_r[f], b, \delta)} \sqrt{\log\left(3 + \sum_{\substack{l=1\\ l \neq r}}^{d} |t_l|\right)} \end{aligned} \tag{4.3.5}$$
holds for any $(h_r, t_r, \widehat{t}_r) \in [-T, T] \times [-T, T] \times \mathbb{R}^{d-1}$.

3. *When $\alpha = 2$,*
$$\begin{aligned}&\left|\Delta_{h_r}^{r,b} X[f](t_r, \widehat{t}_r, \omega)\right| \\ &\le C(\omega)|h_r|^{\min(b, a_r[f])\eta_r} \left(\log\left(3 + |h_r|^{-1}\right)\right)^{\mathcal{L}_2(a_r[f], b)} \sqrt{\log\left(3 + \sum_{\substack{l=1\\ l \neq r}}^{d} |t_l|\right)} \end{aligned} \tag{4.3.6}$$
holds for any $(h_r, t_r, \widehat{t}_r) \in [-T, T] \times [-T, T] \times \mathbb{R}^{d-1}$.

Recall that the functions $\mathcal{L}_\alpha$ and $\mathcal{L}_2$ are defined in Definition 4.1.1.

The proof of Theorem 4.3.1 relies on the following result.

Lemma 4.3.3. *Let $b \in \mathbb{Z}_+$ be an arbitrary integer and $r \in \{1, \ldots, d\}$. Then, for all functions $g \in \mathcal{C}^\infty(\mathbb{R}^d)$, for any positive real number T, for each $h_r \in [-T, T]$ and $\widehat{t}_r \in \mathbb{R}^{d-1}$, the following inequality holds:*
$$\sup_{t_r \in [-T,T]} \left|\Delta_{h_r}^{r,b} g(t_r, \widehat{t}_r)\right| \le 2^b \times \min_{b' \in \{0,1,\ldots,b\}} \left\{ \sup_{t_r \in [-T2^b, T2^b]} \left|\partial^{b' e_r} g(t_r, \widehat{t}_r)\right| \times |h_r|^{b'} \right\}, \tag{4.3.7}$$
with the convention that $0^0 = 1$ and where $e_r \in \mathbb{Z}^d$ is the multi-index whose r-th coordinate equals 1 and the others vanish.

The proof of Lemma 4.3.3 is rather similar to the one of Lemma 4.1.5.

Proof of Lemma 4.3.3. We proceed by induction on b.

Step 1: $b = 0$. In view of the equalities $\Delta_{h_r}^{r,b} g = g$, for all $h_r \in \mathbb{R}$, and $\partial^0 g = g$, it is clear that the lemma is true.

Step 2: Let $b \in \mathbb{Z}_+$ be arbitrary. One has to show that, for all $g \in \mathcal{C}^\infty(\mathbb{R}^d)$, for any positive real number T, and for each $h_r \in [-T, T]$, the following inequality holds:

$$\sup_{t_r \in [-T,T]} \left| \Delta_{h_r}^{r,b+1} g(t_r, \widehat{t_r}) \right| \leq 2^{b+1} \times \min_{b' \in \{0,1,\dots,b+1\}} \left\{ \sup_{t_r \in [-T2^{b+1}, T2^{b+1}]} \left| \partial^{b' e_r} g(t_r, \widehat{t_r}) \right| \times |h_r|^{b'} \right\}. \tag{4.3.8}$$

It follows from (4.1.1) that

$$\sup_{t_r \in [-T,T]} \left| \Delta_{h_r}^{r,b+1} g(t_r, \widehat{t_r}) \right| = \sup_{t_r \in [-T,T]} \left| \Delta_{h_r}^{r,b} g(t_r + h_r, \widehat{t_r}) - \Delta_{h_r}^{r,b} g(t_r, \widehat{t_r}) \right|. \tag{4.3.9}$$

Therefore, using the triangle inequality and the induction hypothesis [4], one has that

$$\begin{aligned} \sup_{t_r \in [-T,T]} \left| \Delta_{h_r}^{r,b+1} g(t_r, \widehat{t_r}) \right| &\leq \sup_{t_r \in [-T,T]} \left| \Delta_{h_r}^{r,b} g(t_r + h_r, \widehat{t_r}) \right| + \sup_{t_r \in [-T,T]} \left| \Delta_{h_r}^{r,b} g(t_r, \widehat{t_r}) \right| \\ &\leq 2 \sup_{t_r \in [-2T,2T]} \left| \Delta_{h_r}^{r,b} g(t_r, \widehat{t_r}) \right| \\ &\leq 2^{b+1} \times \min_{b' \in \{0,1,\dots,b\}} \left\{ \sup_{t_r \in [-T2^{b+1}, T2^{b+1}]} \left| \partial^{b' e_r} g(t_r, \widehat{t_r}) \right| \times |h_r|^{b'} \right\}. \end{aligned} \tag{4.3.10}$$

On the other hand, one can derive from (4.3.9), the Mean Value Theorem, and the equality $\partial^{e_r}\left(\Delta_{h_r}^{r,b} g\right) = \Delta_{h_r}^{r,b}\left(\partial^{e_r} g\right)$ that

$$\sup_{t_r \in [-T,T]} \left| \Delta_{h_r}^{r,b+1} g(t_r, \widehat{t_r}) \right| \leq |h_r| \sup_{t_r \in [-2T,2T]} \left| \Delta_{h_r}^{r,b}\left(\partial^{e_r} g\right)(t_r, \widehat{t_r}) \right|. \tag{4.3.11}$$

Moreover, applying the induction hypothesis [5], one gets that

$$\sup_{t_r \in [-2T,2T]} \left| \Delta_{h_r}^{r,b}\left(\partial^{e_r} g\right)(t_r, \widehat{t_r}) \right| \leq 2^b \times \min_{b' \in \{0,1,\dots,b\}} \left\{ \sup_{t_r \in [-T2^{b+1}, T2^{b+1}]} \left| \partial^{(b'+1) e_r} g(t_r, \widehat{t_r}) \right| \times |h_r|^{b'} \right\}. \tag{4.3.12}$$

Next, putting together (4.3.11) and (4.3.12) we obtain that

$$\sup_{t_r \in [-T,T]} \left| \Delta_{h_r}^{r,b+1} g(t_r, \widehat{t_r}) \right| \leq 2^{b+1} \times \min_{b' \in \{1,2,\dots,b+1\}} \left\{ \sup_{t_r \in [-T2^{b+1}, T2^{b+1}]} \left| \partial^{b' e_r} g(t_r, \widehat{t_r}) \right| \times |h_r|^{b'} \right\}. \tag{4.3.13}$$

Finally, one can derive from (4.3.10) and (4.3.13) that (4.3.8) holds. □

[4] In which T is replaced by $2T$.

[5] In which g is replaced by $\partial^{e_r} g$, and T by $2T$.

Proof of Theorem 4.3.1. We give the proof in the case $\alpha \in [1,2)$ and $r=1$; the other cases can be treated in a similar way.

Let $\omega \in \Omega_1^*$ and $T>0$ be fixed. First, we make a useful remark: for all $T' \in (0,+\infty)$, $(t_1,\ldots,t_d) \in \mathbb{R}^d$ and $J \in \mathbb{Z}^d_{(\eta)}$, we set

$$S_J(t_1,\ldots,t_d) := \sum_{K\in\mathbb{Z}^d} \frac{\sqrt{\log\Big(3+\sum_{l=1}^d \big(|j_l|+|k_l|\big)\Big)}}{\prod_{l=1}^d \Big(2+|2^{j_l}t_l-k_l|\Big)^{p_*}}. \tag{4.3.14}$$

Then, Lemma 3.2.9 and (3.2.38) imply that, for any $t_1 \in [-T',T']$ and $(t_2,\ldots,t_d) \in \mathbb{R}^{d-1}$,

$$\begin{aligned} S_J(t_1,\ldots,t_d) &\le c_1 \sqrt{\log\Big(3+|j_1|+T'2^{j_1}+\sum_{l=2}^d \Big(|j_l|+2^{j_l}|t_l|\Big)\Big)} \\ &\le c_2 \sqrt{\log\Big(3+|j_1|+2^{j_1}+\sum_{l=2}^d \Big(|j_l|+2^{j_l}+|t_l|\Big)\Big)} \\ &\le c_3 \sqrt{\log\Big(3+\sum_{l=2}^d |t_l|\Big)} \prod_{l=1}^d \sqrt{\log\big(3+|j_l|+2^{j_l}\big)}, \end{aligned} \tag{4.3.15}$$

where c_1, c_2 and c_3 are three positive and finite constants which do not depend on $t_1,t_2,\ldots,t_d$. The end of the proof is divided into 2 cases : $\eta=0$ and $\eta\neq 0$.
First case: $\eta=0$. Let $h_1 \in [-T,T]$ and $\widehat{t_1} \in \mathbb{R}^{d-1}$ be arbitrary. In view of Proposition 3.2.15, one can apply Lemma 4.3.3 to the function $X[f]^0(\cdot,\omega)$. Therefore,

$$\sup_{t_1\in[-T,T]} \Big|\Delta^{1,b}_{h_1} X[f]^0(t_1,\widehat{t_1},\omega)\Big| \le 2^b \times \sup_{t_1\in[-T2^b,T2^b]} \Big|\partial^{be_1} X[f]^0(t_1,\widehat{t_1},\omega)\Big| \times |h_1|^b. \tag{4.3.16}$$

Moreover, Proposition 3.2.15 and the fact that $b>0$ entail that, for any $t_1 \in \mathbb{R}$,

$$\partial^{be_1} X[f]^0(t_1,\widehat{t_1},\omega) = \sum_{(J,K)\in\mathbb{Z}^d_{(0)}\times\mathbb{Z}^d} 2^{bj_1} \Big(\partial^{be_1}\Psi_{\alpha,J}[f]\Big)\Big(2^{j_1}t_1-k_1, 2^{\widehat{J_1}}\widehat{t_1}-\widehat{K_1}\Big)\, \varepsilon_{\alpha,J,K}(\omega), \tag{4.3.17}$$

where $2^{\widehat{J_1}}\widehat{t_1}-\widehat{K_1} := (2^{j_2}t_2-k_2,\ldots,2^{j_d}t_d-k_d)$. Therefore, combining (4.3.16), (4.3.17), (3.2.33) (applied with $T = 1$), (3.2.36), (4.3.14) and (4.3.15) (with $T' = 2^b T$), we get that

$$\begin{aligned}
&\sup_{t_1 \in [-T,T]} \left| \Delta_{h_1}^{1,b} X[f]^0(t_1, \widehat{t_1}, \omega) \right| \\
&\leq C_4(\omega) \left| h_1 \right|^b \\
&\qquad \times \sum_{J \in \mathbb{Z}^d_{(0)}} \left(2^{j_1} + \cdots + 2^{j_d} \right)^{-a'[f] - d/\alpha} \sup_{t_1 \in [-T2^b, T2^b]} \left\{ S_J(t_1, \dots, t_d) \right\} \prod_{l=1}^{d} 2^{j_l/\alpha} (1 + |j_l|)^{1/\alpha + \delta} \\
&\leq C_5(\omega) \left| h_1 \right|^b \sqrt{\log \left(3 + \sum_{l=2}^{d} |t_l| \right)} \\
&\qquad \times \sum_{J \in \mathbb{Z}^d_{(0)}} 2^{b j_1} \left(2^{j_1} + \cdots + 2^{j_d} \right)^{-a'[f] - d/\alpha} \prod_{l=1}^{d} 2^{j_l/\alpha} (1 + |j_l|)^{1/\alpha + \delta} \sqrt{\log \left(3 + |j_l| + 2^{j_l} \right)}.
\end{aligned} \tag{4.3.18}$$

Denoting by $C_6(\omega)$ the positive constant defined as

$$C_6(\omega) := C_5(\omega) \sum_{J \in \mathbb{Z}^d_{(0)}} 2^{b j_1} \left(2^{j_1} + \cdots + 2^{j_d} \right)^{-a'[f] - d/\alpha} \prod_{l=1}^{d} 2^{j_l/\alpha} (1 + |j_l|)^{1/\alpha + \delta} \sqrt{\log \left(3 + |j_l| + 2^{j_l} \right)},$$

we obtain (4.3.2) when $\eta = 0$. Notice that, in view of Lemma 3.2.5, the constant $C_6(\omega)$ is finite.

Second case: $\eta \neq 0$. Let $h_1 \in [-T, T]$ and $\widehat{t_1} \in \mathbb{R}^{d-1}$ be arbitrary. We know from Proposition 3.2.16 that, for all $J \in \mathbb{Z}^d_{(\eta)}$, the function $\Phi_{\alpha,J}[f](2^J \cdot, \omega)$ (see (3.2.60)) is infinitely differentiable on $\mathbb{R}^d$. Thus, it follows from lemma 4.3.3 that

$$\begin{aligned}
&\sup_{t_1 \in [-T,T]} \left| \Delta_{h_1}^{1,b} \Big(\Phi_{\alpha,J}[f](2^{j_1} \cdot, 2^{\widehat{J_1}} \cdot, \omega) \Big)(t_1, \widehat{t_1}) \right| \\
&\leq 2^b \times \min_{b' \in \{0,1,\dots,b\}} \left\{ \sup_{t_1 \in [-T2^b, T2^b]} \left| \partial^{b' e_1} \Big(\Phi_{\alpha,J}[f] \Big)(2^{j_1} t_1, 2^{\widehat{J_1}} \widehat{t_1}, \omega) \right| \times \left| 2^{j_1} h_1 \right|^{b'} \right\}.
\end{aligned} \tag{4.3.19}$$

Moreover, Proposition 3.2.16 implies that, for any $t_1 \in \mathbb{R}$,

$$\Big(\partial^{b' e_1} (\Phi_{\alpha,J}[f]) \Big)(2^{j_1} t_1, 2^{\widehat{J_1}} \widehat{t_1}, \omega) = \sum_{K \in \mathbb{Z}^d} (\partial^{b' e_1} (\Psi_{\alpha,J}[f]))(2^{j_1} t_1 - k_1, 2^{\widehat{J_1}} \widehat{t_1} - \widehat{K_1}) \varepsilon_{\alpha,J,K}(\omega). \tag{4.3.20}$$

Therefore, combining (4.3.19), (4.3.20), (3.2.34) (applied with $T = 1$), (3.2.36), (4.3.14) and (4.3.15), we get that

$$\begin{aligned}
&\sup_{t_1\in[-T,T]}\left|\Delta_{h_1}^{1,b}\Big(\Phi_{\alpha,J}[f](2^{j_1}\cdot,2^{\widehat{J}_1}\cdot,\omega)\Big)(t_1,\widehat{t}_1)\right|\\
&\le C_7(\omega)\left(\min_{b'\in\{0,1,\dots,b\}}\left|2^{j_1}h_1\right|^{b'}\right)\left(\sup_{t_1\in[-T2^b,T2^b]}\Big\{S_J(t_1,\dots,t_d)\Big\}\right)\\
&\qquad\times\prod_{l=1}^{d}2^{(1-\eta_l)j_l/\alpha}\,2^{-j_l\eta_l a_l[f]}(1+|j_l|)^{1/\alpha+\delta}\\
&\le C_8(\omega)\min\Big(1,\left|2^{j_1}h_1\right|^{b}\Big)\sqrt{\log\left(3+\sum_{l=2}^{d}|t_l|\right)}\\
&\qquad\times\prod_{l=1}^{d}2^{(1-\eta_l)j_l/\alpha}\,2^{-j_l\eta_l a_l[f]}(1+|j_l|)^{1/\alpha+\delta}\sqrt{\log\left(3+|j_l|+2^{j_l}\right)},
\end{aligned}$$

where $C_7(\omega)$ and $C_8(\omega)$ are two positive and finite constants not depending on $t_1,\dots,t_d$ and h_1. Also, observe that there exists a positive and finite constant c_9 such that, for any $j\in\mathbb{Z}_+$, we have

$$\sqrt{\log\left(3+j+2^{j}\right)}\le c_9(1+j)^{1/2}. \tag{4.3.21}$$

Then, combining (3.2.56), (3.2.59), (4.3.21), (3.2.29), (3.2.30) and (4.3.21), we get that

$$\begin{aligned}
&\sup_{t_1\in[-T,T]}\left|\Delta_{h_1}^{1,b}Y[f]^{\eta}(t_1,\widehat{t}_1,\omega)\right|\\
&\le C_9(\omega)\sqrt{\log\left(3+\sum_{l=2}^{d}|t_l|\right)}\sum_{j_1\in\mathbb{Z}_{\eta_1}}\min\Big(1,\left|2^{j_1}h\right|^{b}\Big)2^{(1-\eta_1)j_1/\alpha}\,2^{-j_1\eta_1 a_1[f]}(1+|j_1|)^{1/\alpha+\delta+1/2},
\end{aligned}\tag{4.3.22}$$

where $C_9(\omega)$ is a positive and finite constant not depending on $t_1,\dots,t_d$ and h_1. Putting together (4.3.22), Lemmas 4.1.6 and 4.1.7, and Definition 4.1.1, we get that

$$\begin{aligned}
&\sup_{t_1\in[-T,T]}\left|\Delta_{h_1}^{1,b}Y[f]^{\eta}(t_1,\widehat{t}_1,\omega)\right|\\
&\le C_{10}(\omega)|h_1|^{b(1-\eta_1)}\,|h_1|^{\min(b,a_1[f])\eta_1}\Big(\log\big(3+|h_1|^{-1}\big)\Big)^{\eta_1\mathcal{L}_\alpha(a_1[f],b,\delta)}\sqrt{\log\left(3+\sum_{l=2}^{d}|t_l|\right)},
\end{aligned}\tag{4.3.23}$$

where $C_{10}(\omega)$ is a positive and finite constant not depending on $t_1,\dots,t_d$ and h_1. In view of the equality (3.2.55), one can conclude that (4.3.23) entails that (4.3.2) holds when $\eta\neq 0$.

□

5 Lower estimates on path behaviour

Abstract
The first main goal of this chapter is to obtain a result which, among other things, can be viewed, when $\alpha \in (0, 2)$, as a counter part to Corollary 4.1.3. The second main goal of this chapter is to derive a result which can be viewed, when $\alpha \in (0, 2)$, as a counterpart to Corollary 4.2.2. This results are proved using wavelet methods which rely on stability properties of the class of stationary increments harmonizable stable fields we introduced in Definition 2.3.3.

Contents

5.1 Stability of the family of stationary increments harmonizable stable fields

First notice that the class of the harmonizable fields $X[f]$ with f admissible is "stable" under the following two elementary operations: the addition and the multiplication by a real number. More precisely, when f and g are two admissible functions and λ is a real number, then the function $\lambda f + g$ is also admissible and we have, for any $(t,\omega) \in \mathbb{R}^d \times \Omega$, the equality

$$\lambda X[f](t,\omega) + X[g](t,\omega) = X[\lambda f + g](t,\omega). \tag{5.1.1}$$

The main goal of the present section is to show that this class of harmonizable fields is "stable" under two more sophisticated operations. First, we will consider the partial derivative operator ∂^b, for some $b \in \mathbb{Z}_+^d$ (see (5.1.2) below), and then, we will consider averages of the sample paths of $X[f]$ (see (5.1.16) below). Let us point out that those "stability" properties will be useful to establish Theorem 5.2.1 and Theorem 5.2.2 in section 5.2.

We begin with the stability by partial derivability. Let $b = (b_1, \dots, b_d) \in \mathbb{Z}_+^d$ satisfying $b_l < a_l[f]$, for all $l \in \{1, \dots, d\}$. Theorem 3.2.19 ensures that almost surely the sample paths of the field $X[f]$ admit continuous partial derivative of order b on $\mathbb{R}^d$. More precisely, for any $\omega \in \Omega_1^*$, the real-valued function $\partial^b\Big(X[f]\Big)(\cdot,\omega)$ exists and is continuous on $\mathbb{R}^d$. So, we are allowed to consider a new stochastic field on $\mathbb{R}^d$ defined as follow: for all $t \in \mathbb{R}^d$, if $\omega \in \Omega_1^*$, we set

$$\mathcal{D}^b\Big(X[f]\Big)(t,\omega) := \partial^b\Big(X[f]\Big)(t,\omega) - \partial^b\Big(X[f]\Big)(0,\omega), \tag{5.1.2}$$

and $\mathcal{D}^b(X[f])(t,\omega) := 0$ else[1]. Notice that, when $b = 0$, in view of (3.2.52), we have, for all $\omega \in \Omega$, $\mathcal{D}^b(X[f])(\cdot,\omega) = X[f](\cdot,\omega)$. Moreover, it follows from (3.2.71) that, for all $(t,\omega) \in \mathbb{R}^d \times \Omega_1^*$,

$$\mathcal{D}^b(X[f])(t,\omega) = \sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} 2^{j_1 b_1 + \cdots + j_d b_d}\Big(\partial^b\Big(\Psi_{\alpha,J}[f]\Big)(2^J t - K) - \partial^b\Big(\Psi_{\alpha,J}[f]\Big)(-K)\Big)\varepsilon_{\alpha,J,K}(\omega). \tag{5.1.3}$$

Observe that the random series (3.2.52) and (5.1.3) are rather similar. So, a natural question is the following one: does the field $\Big\{\mathcal{D}^b(X[f])(t), t \in \mathbb{R}^d\Big\}$ belongs to the frame of the stochastic fields with stationary increments we are interested in? In other word, is there a function g in the admissible class (see Definition 3.1.1), denoted by $\mathcal{A}$, such that, for any $(t,\omega) \in \mathbb{R}^d \times \Omega$, one has $\mathcal{D}^b(X[f])(t,\omega) = X[g](t,\omega)$? In order to answer positively to this question, we need the following preliminary result.

Lemma 5.1.1. *Assume that f is an admissible function in the sense of Definition 3.1.1 and that the positive exponents $a_1[f], \dots, a_d[f]$ are as in this definition. Then, for any $b = (b_1, \dots, b_d) \in \mathbb{Z}_+^d$ satisfying $b_l < a_l[f]$, for all $l \in \{1, \dots, d\}$, the complex-valued function $\mathrm{D}^b f$ defined, for all $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, by*

$$\Big(\mathrm{D}^b f\Big)(\xi) := i^{\mathrm{l}(b)}\xi^b f(\xi) \quad \textit{with } \mathrm{l}(b) = b_1 + \cdots + b_d, \tag{5.1.4}$$

belongs to the class $\mathcal{A}$ of admissible functions; moreover $\mathrm{D}^b f$ satisfies $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ in Definition 3.1.1 with the exponents $a'\Big[\mathrm{D}^b f\Big]$, $a_1\Big[\mathrm{D}^b f\Big], \dots, a_d\Big[\mathrm{D}^b f\Big]$ defined as follows:

$$a'\Big[\mathrm{D}^b f\Big] := 0, \quad \textit{if } b \neq 0 \qquad \textit{and} \qquad a'\Big[\mathrm{D}^b f\Big] := a'[f], \quad \textit{if } b = 0, \tag{5.1.5}$$

[1] The definition of $\mathcal{D}^b(X[f])(t,\omega)$ in this case is rather natural because we assume that the field $X[f]$ vanishes outside Ω_1^* (see Remark 3.2.11)

and, for all $l = 1, \ldots, d$,

$$a_l\Big[\mathrm{D}^b f\Big] := a_l[f] - b_l. \tag{5.1.6}$$

Proof. When the multi-index b is equal to 0, all its coordinates are equal to 0. Therefore, in view of (5.1.4), (5.1.5) and (5.1.6), it is clear that $\mathrm{D}^b f = f$ is admissible. Now, we assume that the multi-index b is non-zero and satisfies $b_l < a_l[f]$, for all $l \in \{1, \ldots, d\}$. So, in view of (5.1.4), $\mathrm{D}^b f$ is a complex-valued Lebesgue measurable function on $\mathbb{R}^d$ satisfying, for almost all $\xi \in \mathbb{R}^d$,

$$\overline{\Big(\mathrm{D}^b f\Big)(\xi)} = \Big(\mathrm{D}^b f\Big)(-\xi). \tag{5.1.7}$$

Next, we show that $\mathrm{D}^b f$ satisfies the conditions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ in Definition 3.1.1. Let $p = (p_1, \ldots, p_d) \in \Big\{0, 1, \ldots, p_*\Big\}^d$ be fixed, where p_* is as in (3.1.1). The function f is admissible, so the hypothesis $(\mathcal{H}_1)$ ensures that it possesses a continuous partial derivative function of order p on $\Big(\mathbb{R} \setminus \{0\}\Big)^d$. Then, in view of (5.1.4), it is clear that the function $\mathrm{D}^b f$ satisfies the condition $(\mathcal{H}_1)$. Moreover, the Leibniz's formula and the triangle inequality imply that, for all $\xi = (\xi_1, \ldots, \xi_d) \in \Big(\mathbb{R} \setminus \{0\}\Big)^d$,

$$\left|\partial^p \Big(\mathrm{D}^b f\Big)(\xi)\right| \leq \sum_{q_1=0}^{p_1} \cdots \sum_{q_d=0}^{p_d} \binom{p_1}{q_1} \cdots \binom{p_d}{q_d} \left|\Big(\partial^{p-q} f\Big)(\xi)\right| \prod_{l=1}^{d} \frac{b_l!}{b_l - q_l} |\xi_l|^{b_l - q_l} \mathbb{1}_{q_l \leq b_l}, \tag{5.1.8}$$

where $(\,\cdot\,)!$ denotes the factorial function, and $\mathbb{1}_{q_l \leq b_l} = 1$ if $q_l \leq b_l$ and $\mathbb{1}_{q_l \leq b_l} = 0$ else. Putting together (5.1.8) and the fact that f satisfies (3.1.2) and (3.1.3), we get the existence of two positive and finite constants c_1 and c_2 satisfying the following property: for all $\xi \in \Big(\mathbb{R} \setminus \{0\}\Big)^d$,

$$\|\xi\| \leq \frac{8\pi}{3}\sqrt{d} \quad \Longrightarrow \quad \left|\partial^p \Big(\mathrm{D}^b f\Big)(\xi)\right| \leq c_1 \|\xi\|^{-a'[\mathrm{D}^b f] - d/\alpha - l(p)} \tag{5.1.9}$$

and

$$\|\xi\| \geq \frac{2\pi}{3} \quad \Longrightarrow \quad \left|\partial^p \Big(\mathrm{D}^b f(\xi)\Big)\right| \leq c_2 \prod_{l=1}^{d} (1 + |\xi_l|)^{-a_l[\mathrm{D}^b f] - 1/\alpha - p_l}, \tag{5.1.10}$$

where the exponents $a'[\mathrm{D}^b f]$ and $a_1[\mathrm{D}^b f], \ldots, a_d[\mathrm{D}^b f]$ are defined through (5.1.5) and (5.1.6). Hence, $\mathrm{D}^b f$ satisfies conditions $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ in Definition 3.1.1. In view of Remark 3.1.2, it satisfies (2.3.1). Therefore $\mathrm{D}^b f$ is admissible. □

We are now in the position to answer the question raised earlier.

Proposition 5.1.2. *Assume that f is an admissible function in the sense of Definition 3.1.1 and that the positive exponents $a_1[f], \ldots, a_d[f]$ are as in this definition. Let $b = (b_1, \ldots, b_d) \in \mathbb{Z}_+^d$ satisfying $b_l < a_l[f]$, for all $l \in \{1, \ldots, d\}$, and let $\mathrm{D}^b f$ be the complex-valued function defined through* (5.1.4)*. Then, for all $\omega \in \Omega_1^*$ and $t \in \mathbb{R}^d$,*

$$\mathcal{D}^b(X[f])(t, \omega) = X[\mathrm{D}^b f](t, \omega), \tag{5.1.11}$$

where $\mathcal{D}^b(X[f])(t,\omega)$ *is defined in* (5.1.2). *Notice that the wavelet series expansions of* $X[\mathrm{D}^b f]$ *and* $X[f]$ *(see Remark 3.2.11) and* (5.1.2) *imply that* $\mathcal{D}^b(X[f])(\cdot,\omega) = X[\mathrm{D}^b f](\cdot,\omega) = 0$ *as soon as* $\omega \notin \Omega_1^*$.

Proof. Notice that when $b = 0$, the proof is clear because $\mathrm{D}^b f = f$ and $\mathcal{D}^b\Big(X[f]\Big) = X[f]$. From now on, we assume that $b = (b_1, \dots, b_d)$ is a non-zero multi-index which satisfies $b_l < a_l[f]$, for all $l \in \{1, \dots, d\}$. Theorem 3.2.19 and (5.1.2) yield that, for all $(t,\omega) \in \mathbb{R}^d \times \Omega_1^*$,

$$\mathcal{D}^b(X[f])(t,\omega) = \sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} 2^{j_1 b_1+\dots+j_d b_d}\bigg(\partial^b\Big(\Psi_{\alpha,J}[f]\Big)(2^J t-K)-\partial^b\Big(\Psi_{\alpha,J}[f]\Big)(-K)\bigg)\varepsilon_{\alpha,J,K}(\omega). \tag{5.1.12}$$

On the other side, Lemma 5.1.1 implies that the function $\mathrm{D}^b f$ is admissible. So, It follows from (3.2.52) that, for all $(t,\omega) \in \mathbb{R}^d \times \Omega_1^*$,

$$X[\mathrm{D}^b f](t,\omega) = \sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} \Big(\Psi_{\alpha,J}[\mathrm{D}^b f](2^J t - K) - \Psi_{\alpha,J}[\mathrm{D}^b f](-K)\Big)\varepsilon_{\alpha,J,K}(\omega). \tag{5.1.13}$$

Moreover, in view of (3.2.18), (3.2.32) and (5.1.4), for all $t \in \mathbb{R}^d$, $J \in \mathbb{Z}^d$ and $K \in \mathbb{Z}^d$, we have that

$$\Psi_{\alpha,J}[\mathrm{D}^b f](2^J t - K) = 2^{j_1 b_1+\dots+j_d b_d}\partial^b(\Psi_{\alpha,J}[f])(2^J t - K). \tag{5.1.14}$$

Hence, one can derive from (5.1.12) and (5.1.13) that (5.1.11) holds. □

From now on, we focus on averages of the sample paths of the field $X[f]$. In order to define those averages, we need some additional notations. Let $q \in \{1, \dots, d\}$ and $1 \le i_1 < i_2 < \dots < i_q \le d$ be arbitrary and fixed. We denote by e_l, for any $l \in \{1, \dots, d\}$, the vector of $\mathbb{R}^d$ whose l-th coordinate equals 1 and the others vanish. Moreover, for any $s = (s_1, \dots, s_q) \in \mathbb{R}^q$, we let $\widetilde{s}$ the vector of $\mathbb{R}^d$, defined as

$$\widetilde{s} := \sum_{l=1}^{q} s_l e_{i_l}. \tag{5.1.15}$$

In other words, we have $\widetilde{s}_l = s_u$ if $l = i_u$ with $u \in \{1, \dots, q\}$, and $\widetilde{s}_l = 0$ else. Let θ be a real-valued Lebesgue measurable function of $\mathbb{R}^q$ such that the quantity

$$P_\theta X[f](t,\omega) := \int_{\mathbb{R}^q} \Big(X[f](t+\widetilde{s},\omega) - X[f](\widetilde{s},\omega)\Big)\theta(s)\,\mathrm{d}s, \tag{5.1.16}$$

is well-defined and finite for any $\omega \in \Omega_1^*$ and $t \in \mathbb{R}^d$. When $\omega \notin \Omega_1^*$, we naturally set $P_\theta X[f](t,\omega) := 0$ for every $t \in \mathbb{R}^d$. So that, in view of Remark 3.2.11, the equality (5.1.16) holds for all $\omega \in \Omega$ and $t \in \mathbb{R}^d$. Hence we have defined a new stochastic field

$$P_\theta X[f] := \{P_\theta X[f](t), t \in \mathbb{R}^d\}.$$

A natural question is the following one: does the field $P_\theta X[f]$ belongs to the frame of the harmonizable stochastic fields with stationary increments we are interested in? In other words, is there a function $g_\theta \in \mathcal{A}$ satisfying, for any $(t,\omega) \in \mathbb{R}^d \times \Omega$,

$$P_\theta X[f](t,\omega) = X[g_\theta](t,\omega)?$$

In order to answer positively to this question, we need the following lemma.

Lemma 5.1.3. *Assume that f is an admissible function in the sense of Definition 3.1.1 and that the positive exponents $a_1[f],\dots,a_d[f]$ are as in this definition. Let $q \in \{1,\dots,d\}$ and $1 \le i_1 < i_2 < \dots < i_q \le d$ be arbitrary and fixed. Let $\phi : \mathbb{R}^q \to \mathbb{C}$ be a complex-valued Lebesgue measurable function such that, for all $\eta \in \mathbb{R}^q$, one has*

$$\overline{\phi(\eta)} = \phi(-\eta). \tag{5.1.17}$$

In addition, we suppose that, for all multi-index $p := (p_1, p_2, \dots, p_q) \in \Big\{0, 1, \dots, p_*\Big\}^q$, the partial derivative function $\partial^p \phi$ is well-defined and continuous on $\mathbb{R}^q$ and satisfies, for all $\eta \in \mathbb{R}^q$,

$$\Big|\partial^p \phi(\eta)\Big| \le c \prod_{l=1}^{q} (1 + |\eta_l|)^{-b_l - p_l}, \tag{5.1.18}$$

where c and $b_1,\dots,b_q$ are non-negative finite constant not depending on η and p.
Then, the complex-valued Lebesgue measurable function g defined, for almost all $(\xi_1,\dots,\xi_d) \in \mathbb{R}^d$, by

$$g(\xi_1,\dots,\xi_d) = f(\xi_1,\dots,\xi_d)\phi(\xi_{i_1},\dots,\xi_{i_q}), \tag{5.1.19}$$

belongs to $\mathcal{A}$, the class of admissible functions; the exponents $a'[g], a_1[g],\dots,a_d[g]$ for which g satisfies $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ in Definition 3.1.1 can be chosen as follows:

$$a'[g] := a'[f] \tag{5.1.20}$$

and, for all $l = 1,\dots,d$,

$$a_l[g] := a_l[f] + b_u, \quad \textit{if} \quad l = i_u \textit{ with } u \in \{1,\dots,q\} \tag{5.1.21}$$

and

$$a_l[g] := a_l[f] \quad \textit{else.} \tag{5.1.22}$$

Proof of Lemma 5.1.3. First observe that it easily follows from (2.3.2), (5.1.17) and (5.1.19) that, for almost all $\xi = (\xi_1,\dots,\xi_d) \in \mathbb{R}^d$,

$$\overline{g(\xi)} = \overline{\phi(\xi_{i_1},\dots,\xi_{i_q})}\,\overline{f(\xi)} = \phi(-\xi_{i_1},\dots,-\xi_{i_q}) f(-\xi) = g(-\xi). \tag{5.1.23}$$

Now we show that g satisfies conditions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ in Definition 3.1.1 (notice that these 3 conditions imply that (2.3.1) is satisfied). Let $p = (p_1,\dots,p_d) \in \Big\{0, 1, \dots, p_*\Big\}^d$ be fixed. Using the fact that f satisfies $(\mathcal{H}_1)$, using the partial differentiability of the function ϕ and using the Leibniz's formula, it follows that the partial derivative $\partial^p g$ is a well-defined continuous function on $\left(\mathbb{R}\setminus\{0\}\right)^d$ satisfying, for all $(\xi_1,\ldots,\xi_d)\in\mathbb{R}^d$,

$$\partial^p g(\xi_1,\ldots,\xi_d)=\sum_{r_{i_1}=0}^{p_{i_1}}\cdots\sum_{r_{i_d}=0}^{p_{i_d}}\binom{p_{i_1}}{r_{i_1}}\ldots\binom{p_{i_q}}{r_{i_q}}\partial^{p-\widetilde{r}}f(\xi_1,\ldots,\xi_d)\,\partial^{(r_{i_1},\ldots,r_{i_q})}\phi(\xi_{i_1},\ldots,\xi_{i_q}),\tag{5.1.24}$$

where $\widetilde{r}$ is defined similarly to $\widetilde{s}$ in (5.1.15). Next putting together (5.1.24), (3.1.2), (3.1.3), (5.1.18) and the definition of $\widetilde{r}$, we get the following property: there are two positive and finite constants c_1 and c_2 such that for all $\xi\in\left(\mathbb{R}\setminus\{0\}\right)^d$,

$$\|\xi\|\leq\frac{8\pi}{3}\sqrt{d}\Longrightarrow\left|\partial^p g(\xi)\right|\leq c_1\left\|\xi\right\|^{-a'[g]-d/\alpha-l(p)},\tag{5.1.25}$$

and,

$$\|\xi\|\geq\frac{2\pi}{3}\Longrightarrow\left|\partial^p g(\xi)\right|\leq c_2\prod_{l=1}^{d}(1+|\xi_l|)^{-a_l[g]-1/\alpha-p_l},\tag{5.1.26}$$

where the exponents $a'[g]$ and $a_1[g],\ldots,a_d[g]$ are defined through (5.1.20), (5.1.21) and (5.1.22). Therefore, in view of Remark 3.1.2, g satisfies (2.3.1). So it satisfies $(\mathcal{H}_0)$, $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$; that is, g is an admissible function. $\square$

We are now ready to answer the question raised earlier.

Proposition 5.1.4. *Assume that f is an admissible function in the sense of Definition 3.1.1 and that the positive exponents $a_1[f],\ldots,a_d[f]$ are as in this definition. Let $q\in\{1,\ldots,d\}$, and $1\leq i_1<i_2<\cdots<i_q\leq d$ be fixed. Let also θ be a real-valued function in the Lebesgue space $L^1(\mathbb{R}^q)$, such that its Fourier transform $\widehat{\theta}$ is a well-defined complex-valued function satisfying, for almost all $\eta\in\mathbb{R}^q$,*

$$\overline{\widehat{\theta}(\eta)}=\widehat{\theta}(-\eta),\tag{5.1.27}$$

and the same hypotheses as ϕ in Lemma 5.1.3. We assume that, for any $t\in\mathbb{R}^d$ and $\omega\in\Omega_1^*$,

$$\int_{\mathbb{R}^q}\sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}\left|\Psi_{\alpha,J}[f](2^J(t+\widetilde{s})-K)-\Psi_{\alpha,J}[f](2^J\widetilde{s}-K)\right||\varepsilon_{\alpha,J,K}(\omega)|\,|\theta(s)|\,\mathrm{d}s<+\infty,\tag{5.1.28}$$

where $\widetilde{s}$ is defined as in (5.1.15). Then, for each $\omega\in\Omega_1^*$ and $t\in\mathbb{R}^d$, the quantity $P_\theta X[f](t,\omega)$ is well-defined; moreover, defining the function g_θ on $\mathbb{R}^d$ by

$$g_\theta(\xi_1,\ldots,\xi_d):=f(\xi_1,\ldots,\xi_d)\overline{\widehat{\theta}(\xi_{i_1},\ldots,\xi_{i_q})},\tag{5.1.29}$$

for almost all $\xi\in\mathbb{R}^d$, we get, for all $\omega\in\Omega_1^*$ and $t\in\mathbb{R}^d$,

$$P_\theta X[f](t,\omega)=X[g_\theta](t,\omega),\tag{5.1.30}$$

where $P_\theta X[f](t,\omega)$ *is defined in* (5.1.16). *Notice that the wavelet series expansions of* $X[g_\theta]$ *and* $X[f]$ *(see Remark 3.2.11) and* (5.1.16) *imply that* $P_\theta X[f](\cdot,\omega) = X[g_\theta](\cdot,\omega) = 0$ *as soon as* $\omega \notin \Omega_1^*$.

Remark 5.1.5. *In view of* (4.2.14)*, the inequality* (5.1.28) *is satisfied as soon as there exists an exponent* $a \in (a'[f]+q, +\infty)$ *such that, for some* $A \in (0,+\infty)$*,*

$$\sup_{\|s\|>A} \Big\{ (1+\|s\|)^a \, |\theta(s)| \Big\} < +\infty. \tag{5.1.31}$$

In particular, when θ *is assumed to be a real-valued function belonging to the Schwartz class* $\mathcal{S}(\mathbb{R}^q)$*,* (5.1.31) *and all the conditions in Proposition 5.1.4 are satisfied.*

Proof. For all $t \in \mathbb{R}^d$ and $\omega \in \Omega_1^*$, combining (5.1.16) and (3.2.52), it follows that

$$\begin{aligned} &P_\theta X[f](t,\omega) \\ &= \int_{\mathbb{R}^q} \left(\sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} \Big(\Psi_{\alpha,J}[f](2^J(t+\widetilde{s})-K) - \Psi_{\alpha,J}[f](2^J\widetilde{s}-K) \Big)\, \varepsilon_{\alpha,J,K}(\omega) \right) \theta(s)\,\mathrm{d}s, \end{aligned} \tag{5.1.32}$$

where $\widetilde{s}$ is defined through (5.1.15). Notice that (5.1.28) ensures that, for any $\omega \in \Omega_1^*$ and $t \in \mathbb{R}^d$, the quantity $P_\theta X[f](t,\omega)$ is well-defined. The function $\widehat{\theta}$ satisfies (5.1.27) and the same hypotheses as ϕ in Lemma 5.1.3; hence, the function g_θ in (5.1.29) is admissible. So, in view of (3.2.52), the wavelet series expansion of the field $X[g_\theta]$ is given, for all $\omega \in \Omega_1^*$ and $t \in \mathbb{R}^d$, by

$$X[g_\theta](t,\omega) := \sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} \Big(\Psi_{\alpha,J}[g_\theta](2^Jt-K) - \Psi_{\alpha,J}[g_\theta](-K) \Big) \varepsilon_{\alpha,J,K}(\omega). \tag{5.1.33}$$

On the other hand, in view of (5.1.28), we can apply the Fubini's Theorem in order to interchange the integration and the summation in (5.1.32). So, for all $\omega \in \Omega_1^*$ and $t \in \mathbb{R}^d$, we get that

$$\begin{aligned} &P_\theta X[f](t,\omega) \\ &= \sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} \left(\int_{\mathbb{R}^q} \Big(\Psi_{\alpha,J}[f](2^J(t+\widetilde{s})-K) - \Psi_{\alpha,J}[f](2^J\widetilde{s}-K) \Big)\, \theta(s)\,\mathrm{d}s \right) \varepsilon_{\alpha,J,K}(\omega). \end{aligned} \tag{5.1.34}$$

Moreover, using (3.2.18) and the Fubini's Theorem, we can show that for all $J \in \mathbb{Z}^d$, $K \in \mathbb{Z}^d$ and $t \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^q} \Psi_{\alpha,J}\,[f](2^J(t+\widetilde{s})-K)\theta(s)\,\mathrm{d}s = \Psi_{\alpha,J}[g_\theta](2^Jt-K). \tag{5.1.35}$$

Thus, combining (5.1.34), (5.1.35) and (5.1.33), we get (5.1.30). $\square$

5.2 Optimality of the anisotropic behaviour

In this section, we focus on the case $\alpha \in (0,2)$. Let f be an admissible function, in the sense of Definition 3.1.1 and $X[f]$ be the stochastic field associated to it. We know from Corollary 4.1.3 that the directional rates of vanishing $a_1[f], \ldots, a_d[f]$ of f along the axes of $\mathbb{R}^d$ provide upper estimates on the anisotropic behaviour of the amplitude of the generalized directional increments of $X[f]$. The main goal of this section is to show the following theorem which can be understood as a counterpart to Corollary 4.1.3. For the sake of simplicity, we state this result in the case of the first direction, yet it remains valid for any other canonical direction of $\mathbb{R}^d$. Also, throughout this section, we mention that we use the notation "$\widehat{t_1}$" introduced at the beginning of Section 4.3.

Theorem 5.2.1. *Assume that f is an admissible function in the sense of Definition 3.1.1. Also assume that there are two constants $A \in (0,+\infty)$ and $c \in (0,+\infty)$ such that we have*

$$\int_{\mathbb{R}^{d-1}} |f(\lambda_1, \xi_2, \ldots, \xi_d)|^\alpha \, \mathrm{d}\xi_2 \ldots \mathrm{d}\xi_d \geq c\, |\lambda_1|^{-\alpha a_1[f]-1} \tag{5.2.1}$$

for all real numbers λ_1 satisfying $|\lambda_1| \geq A$. Then, there exists an event $\Omega_2^*[f] \subset \Omega_1^*$ of probability 1 such that, for all $k \in \mathbb{N} \cap (a_1[f], +\infty)$, $\omega \in \Omega_2^*[f]$, $\rho \in (0,+\infty)$ and $\delta \in (0, 1/\alpha)$, we have that

$$\inf_{(t_1,\widehat{t_1}) \in \mathbb{R}\times\mathbb{R}^{d-1}} \sup_{t_1' \in [t_1-\rho, t_1+\rho]} \sup_{h_1 \in [-\rho,\rho]} \left\{ \frac{\left|\Delta_{h_1}^{1,k} X[f]\left(t_1', \widehat{t_1}, \omega\right)\right|}{|h_1|^{a_1[f]} \left(\log\left(3 + |h_1|^{-1}\right)\right)^{1/\alpha - \delta - \mathbb{1}_{\{a_1[f] \in \mathbb{N}\}}}} \right\} = +\infty, \tag{5.2.2}$$

where the operator $\Delta_{h_1}^{1,k}$ is defined in (4.1.1) *and with the convention that* $0/0 = 1$.

Notice that, for all $k \in \mathbb{Z}_+$ and $h_1 \in \mathbb{R}$, we have

$$\Delta_{h_1}^{1,k} = \boldsymbol{\Delta}_{h_1 e_1}^{k}$$

where the operator $\boldsymbol{\Delta}_{h_1 e_1}^{k}$ is defined in (4.1.37). So, (5.2.2) is equivalent to

$$\inf_{(t_1,\widehat{t_1}) \in \mathbb{R}\times\mathbb{R}^{d-1}} \sup_{t_1' \in [t_1-\rho, t_1+\rho]} \sup_{h_1 \in [-\rho,\rho]} \left\{ \frac{\left|\boldsymbol{\Delta}_{h_1 e_1}^{k} X[f]\left(t_1', \widehat{t_1}, \omega\right)\right|}{|h_1|^{a_1[f]} \left(\log\left(3 + |h_1|^{-1}\right)\right)^{1/\alpha - \delta - \mathbb{1}_{\{a_1[f] \in \mathbb{N}\}}}} \right\} = +\infty. \tag{5.2.3}$$

Before proving Theorem 5.2.1, we will derive the weaker version of it in which we further assume that $a_1[f] \in (0,1]$. That is the following theorem.

Theorem 5.2.2. *Assume that f is an admissible function in the sense of Definition 3.1.1 and that the exponent $a_1[f]$ in this definition belongs to $(0,1]$. Also assume that there are two constants $A \in (0,+\infty)$ and $c \in (0,+\infty)$ such that we have*

$$\int_{\mathbb{R}^{d-1}} |f(\lambda_1, \xi_2, \ldots, \xi_d)|^\alpha \, \mathrm{d}\xi_2 \ldots \mathrm{d}\xi_d \geq c\, |\lambda_1|^{-\alpha a_1[f]-1} \tag{5.2.4}$$

for all real numbers λ_1 *satisfying* $|\lambda_1| \geq A$. *Then, there exists an event* $\Omega_3^*[f] \subset \Omega_1^*$ *of probability 1, which a priori depends on* f*, such that, for all* $\omega \in \Omega_3^*[f]$, $\rho \in (0, +\infty)$ *and* $\delta \in (0, 1/\alpha)$*, one has*

$$\inf_{(t_1,t_2,\ldots,t_d)\in\mathbb{R}^d} \sup_{s_1',s_1''\in[t_1-\rho,t_1+\rho]} \left\{ \frac{|X[f](s_1',t_2,\ldots,t_d,\omega) - X[f](s_1'',t_2,\ldots,t_d,\omega)|}{|s_1'-s_1''|^{a_1[f]} \left(\log\left(3+|s_1'-s_1''|^{-1}\right)\right)^{1/\alpha-\delta}} \right\} = +\infty, \tag{5.2.5}$$

with the convention that $0/0 = 1$.

Remark 5.2.3. *When* $\alpha \in (0,1)$ *and* $a_1[f] \in (0,1)$*, Corollary 4.1.3 implies that, for any* $\omega \in \Omega_1^*$ *and for each positive real number* T *and* δ*, there exists a constant* $C(\omega) \in (0, +\infty)$ *such that the inequality*

$$|X[f](t_1+h,t_2,\ldots,t_d,\omega) - X[f](t_1,t_2,\ldots,t_d,\omega)| \leq C(\omega)\,|h|^{a_1[f]} \log\left(3+|h|^{-1}\right)^{1/\alpha+\delta}$$

holds for every $(t_1,\ldots,t_d) \in [-T,T]^d$ *and* $h \in [-T,T]$. *Thus, when* $\alpha \in (0,1)$ *and* $a_1[f] \in (0,1)$*, Corollary 4.1.3 and Theorem 5.2.2 mean that the exponent* $1/\alpha$ *of the logarithmic factor is optimal.*

The proof of Theorem 5.2.2 relies on five lemmas. Before giving them, we need to fix some other notations. Let θ be a non-zero real-valued function in the Schwartz class $\mathcal{S}(\mathbb{R})$ such that $\widehat{\theta}$ (the Fourier transform of θ) is an even function with a compact support satisfying

$$\operatorname{supp} \widehat{\theta} = \{\zeta \in \mathbb{R} : 1 \leq |\zeta| \leq 2\}. \tag{5.2.6}$$

For instance, we can choose $\widehat{\theta}$ such that, for all $\zeta \in \mathbb{R}$, one has

$$\widehat{\theta}(\zeta) := \begin{cases} \exp\left(-\dfrac{1}{(2-|\zeta|)(|\zeta|-1)}\right) & \text{if} \quad x \in \{x \in \mathbb{R} : 1 < |x| < 2\}, \\ 0 & \text{else.} \end{cases} \tag{5.2.7}$$

Then, θ is defined for every $x \in \mathbb{R}$ as

$$\theta(x) := (2\pi)^{-1} \int_{\mathbb{R}} e^{ix\xi}\widehat{\theta}(\xi)\,\mathrm{d}\xi = (2\pi)^{-1} \int_{\mathbb{R}} \cos(x\xi)\widehat{\theta}(\xi)\,\mathrm{d}\xi;$$

thus θ is a real-valued function. It is worth noticing that, in view of (5.2.6), one has

$$\widehat{\theta}(0) = \int_{\mathbb{R}} \theta(x)\,\mathrm{d}x = 0. \tag{5.2.8}$$

For any $m \in \mathbb{N}$ and $l \in \mathbb{Z}$, we denote by $\theta_{m,l}$ the function defined as

$$\theta_{m,l} := 2^m \theta(2^m \cdot - l), \tag{5.2.9}$$

and we denote by $P_{\theta_{m,l}}X[f] := \left\{P_{\theta_{m,l}}X[f](t),\, t \in \mathbb{R}^d\right\}$ the symmetric α-stable field which vanishes outside the event Ω_1^*, and satisfies, for any $\omega \in \Omega_1^*$ and $t \in \mathbb{R}^d$,

$$P_{\theta_{m,l}}X[f](t,\omega) := 2^m \int_{\mathbb{R}} \Big(X[f]\left(t_1 + s, \widehat{t_1}, \omega\right) - X[f]\left(s, 0, \ldots, 0, \omega\right) \Big) \theta\Big(2^m s - l\Big)\, \mathrm{d}s. \quad (5.2.10)$$

Notice that, in view of Proposition 5.1.4 and of the properties of θ, the field $P_{\theta_{m,l}}X[f]$ is well-defined and satisfies, for all $t \in \mathbb{R}^d$,

$$P_{\theta_{m,l}}X[f](t,\omega) = X[g_{\theta_{m,l}}](t,\omega), \quad (5.2.11)$$

where, for almost all $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$,

$$g_{\theta_{m,l}}(\xi) = f(\xi)\widehat{\theta_{m,l}}(-\xi_1) = f(\xi) e^{2^{-m} l \xi_1} \widehat{\theta}(-2^{-m}\xi_1). \quad (5.2.12)$$

We mention that (5.2.12) comes from (5.2.9).

In the sequel, for every $\omega \in \Omega_1^*$, $m \in \mathbb{N}$, $l \in \mathbb{Z}$ and $t \in \mathbb{R}^d$, we set

$$D_{m,l}(t,\omega) := P_{\theta_{m,l}}X[f](t + 2^{-m}e_1, \omega) - P_{\theta_{m,l}}X[f](t,\omega). \quad (5.2.13)$$

In view of (5.2.11) and Remark 3.2.11, when $\omega \notin \Omega_1^*$, for all $m \in \mathbb{N}$, $l \in \mathbb{Z}$ and $t \in \mathbb{R}^d$, we naturally define $D_{m,l}(t,\omega) := 0$. It follows from (5.2.11) and Proposition 3.2.10 that the field $P_{\theta_{m,l}}X[f]$ is a modification of the symmetric α-stable field

$$\left\{ \mathcal{R}e \left\{ \int_{\mathbb{R}^d} \Big(e^{it\cdot\xi} - 1\Big) g_{\theta_{m,l}}(\xi)\, \mathrm{d}\widetilde{M}_\alpha(\xi) \right\}, t \in \mathbb{R}^d \right\}.$$

Then, (5.2.13) and the linearity of the stochastic stable integral entail that $D_{m,l}(t)$ is a real-valued symmetric α-stable random variable. More precisely, using (5.2.12), for all $t \in \mathbb{R}^d$, we have almost surely

$$D_{m,l}(t) = \mathcal{R}e \left\{ \left(\int_{\mathbb{R}^d} e^{it\cdot\xi} \Big(e^{i2^{-m}\xi_1} - 1 \Big) \widehat{\theta}(-2^{-m}\xi_1) e^{i2^{-m}\xi_1 l} f(\xi)\, \mathrm{d}\widetilde{M}_\alpha(\xi) \right) \right\}. \quad (5.2.14)$$

Therefore, (2.1.17) implies that the scale parameter $\sigma\left(D_{m,l}(t)\right)$ of $D_{m,l}(t)$ satisfies

$$\sigma\left(D_{m,l}(t)\right)^\alpha = \int_{\mathbb{R}^d} \left| e^{i2^{-m}\xi_1} - 1 \right|^\alpha \left| \widehat{\theta}\left(-2^{-m}\xi_1\right) \right|^\alpha \left|f(\xi)\right|^\alpha \mathrm{d}\xi. \quad (5.2.15)$$

Notice that $\sigma\left(D_{m,l}(t)\right)$ does not depend on l, and t. Thus, in the sequel, we denote it by σ_m. Moreover, notice that σ_m is equal to zero, if, and only if, we have, for any $\xi \in \mathbb{R}^d$,

$$\left| e^{i2^{-m}\xi_1} - 1 \right| \left| \widehat{\theta}(-2^{-m}\xi_1) \right| \left|f(\xi)\right| = 0. \quad (5.2.16)$$

The property of the support of $\widehat{\theta}$ (see (5.2.6)) and the fact that

$$e^{i\zeta} \neq 1 \quad \text{for all } \zeta \in \{\zeta \in \mathbb{R} : 1 \leq |\zeta| \leq 2\}$$

imply that the equality (5.2.16) holds if, and only if, for any $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$ satisfying $|\xi_1| \in [2^m, 2^{m+1}]$, we have $f(\xi) = 0$. Therefore, we proved the following Lemma.

Lemma 5.2.4. *For all $m \in \mathbb{N}$, $l \in \mathbb{Z}$ and $t \in \mathbb{R}^d$, the real-valued random variable $D_{m,l}(t)$ (see (5.2.13)) has a symmetric α-stable distribution; its scale parameter satisfies the relation*

$$\sigma_m := \sigma\Big(D_{m,l}(t)\Big) = \left(\int_{\mathbb{R}^d} \left|e^{i2^{-m}\xi_1} - 1\right|^\alpha \left|\widehat{\theta}\left(-2^{-m}\xi_1\right)\right|^\alpha |f(\xi)|^\alpha \,\mathrm{d}\xi\right)^{1/\alpha}. \tag{5.2.17}$$

Moreover, σ_m is equal to zero if, and only if, for almost all $\xi \in \mathbb{R}^d$ satisfying $|\xi_1| \in [2^m, 2^{m+1}]$, we have $f(\xi) = 0$.

Lemma 5.2.5. *Let $M \in \mathbb{N}$ be fixed. Assume that $m_1, \ldots, m_M$ are M positive integers all different. Then, for every $t^1, \ldots, t^M \in \mathbb{R}^d$ and $l_1, \ldots, l_M \in \mathbb{Z}$, the random variables $D_{m_1,l_1}\Big(t^1\Big), \ldots, D_{m_M,l_M}\Big(t^M\Big)$ are independent.*

Proof of Lemma 5.2.5. In order to prove that the random variables

$$D_{m_1,l_1}\Big(t^1\Big), \ldots, D_{m_M,l_M}\Big(t^M\Big)$$

are independent, it is enough to show that, for every $b_1, \ldots, b_M \in \mathbb{R}$, we have

$$\chi_{(D_{m_1,l_1}(t^1),\ldots,D_{m_M,l_M}(t^M))}(b_1, \ldots, b_M) = \prod_{j=1}^M \chi_{D_{m_j,l_j}(t^j)}(b_j), \tag{5.2.18}$$

where $\chi_{(D_{m_1,l_1}(t^1),\ldots,D_{m_M,l_M}(t^M))}$ is the characteristic function of the real-valued random vector $(D_{m_1,l_1}(t^1), \ldots, D_{m_M,l_M}(t^M))$ and $\chi_{D_{m_j,l_j}(t^j)}$ is the characteristic function of the real-valued random variable $D_{m_j,l_j}(t^j)$. In view of (2.3.6) and (1.1.5), the equality (5.2.18) is equivalent to

$$\mathbb{E}\left[\exp\left\{i\sum_{j=1}^M b_j D_{m_j,l_j}\Big(t^j\Big)\right\}\right] = \prod_{j=1}^M \mathbb{E}\left[\exp\left\{ib_j D_{m_j,l_j}\Big(t^j\Big)\right\}\right]. \tag{5.2.19}$$

Moreover, using (5.2.12) and the linearity of the stochastic stable integral $\int_{\mathbb{R}^d}(\,\cdot\,)\,\mathrm{d}\widetilde{M_\alpha}$, we have that, almost surely,

$$\sum_{j=1}^M b_j D_{m_j,l_j}(t_j) = \mathcal{R}e\left\{\int_{\mathbb{R}^d}\sum_{j=1}^M\left[b_j\Big(e^{i2^{-m_j}\xi_1} - 1\Big)e^{it_j\cdot\xi}\widehat{\theta}(-2^{-m_j}\xi_1)e^{i2^{-m_j}\xi_1 l_j}\right]f(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right\}. \tag{5.2.20}$$

Therefore, the real-valued random variable $\sum_{j=1}^M b_j D_{m_j,l_j}(t_j)$ has a symmetric α-stable distribution. Definition 1.1.5 and (2.1.17) imply that its characteristic function satisfies

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(i\mathcal{R}e\left\{\int_{\mathbb{R}^d}\sum_{j=1}^M\left[b_j\Big(e^{i2^{-m_j}\xi_1} - 1\Big)e^{it_j\cdot\xi}\widehat{\theta}(-2^{-m_j}\xi_1)e^{i2^{-m_j}\xi_1 l_j}\right]f(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right\}\right)\right] \\
&= \exp\left(-\int_{\mathbb{R}^d}\left|\sum_{j=1}^M b_j\Big(e^{i2^{-m_j}\xi_1} - 1\Big)e^{it_j\cdot\xi}\widehat{\theta}(-2^{-m_j}\xi_1)e^{i2^{-m_j}\xi_1 l_j}\right|^\alpha |f(\xi)|^\alpha\,\mathrm{d}\xi\right).
\end{aligned} \tag{5.2.21}$$

Notice that the property of the support of $\widehat{\theta}$ (see (5.2.6)) imply that, for any positive integers $m' \neq m''$ and $l', l'' \in \mathbb{Z}$, the set $\mathrm{supp}(\widehat{\theta_{m',l'}}) \cap \mathrm{supp}(\widehat{\theta_{m'',l''}})$ is a negligeable set with respect to the Lebesgue measure. Thus, we have

$$\left|\sum_{j=1}^{M} b_j\Big(e^{i2^{-m_j}\xi_1}-1\Big)e^{it_j\cdot\xi}\widehat{\theta}(-2^{-m_j}\xi_1)e^{i2^{-m_j}\xi_1 l_j}\right|^{\alpha} = \sum_{j=1}^{M}\Big|b_j\Big(e^{i2^{-m_j}\xi_1}-1\Big)e^{it_j\cdot\xi}\widehat{\theta}(-2^{-m_j}\xi_1)e^{i2^{-m_j}\xi_1 l_j}\Big|^{\alpha}. \tag{5.2.22}$$

Hence, combining (5.2.20), (5.2.21), (5.2.22), (5.2.14), and (2.1.17) we get that

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(i\sum_{j=1}^{M} b_j D_{m_j,l_j}(t_j)\right)\right]\\
&= \exp\left(-\sum_{j=1}^{M}\int_{\mathbb{R}^d}\Big|b_j\Big(e^{i2^{-m_j}\xi_1}-1\Big)e^{it_j\cdot\xi}\widehat{\theta}(-2^{-m_j}\xi_1)e^{i2^{-m_j}\xi_1 l_j}\Big|^{\alpha}|f(\xi)|^{\alpha}\,\mathrm{d}\xi\right)\\
&= \prod_{j=1}^{M}\exp\left(-|b_j|^{\alpha}\int_{\mathbb{R}^d}\Big|\Big(e^{i2^{-m_j}\xi_1}-1\Big)e^{it_j\cdot\xi}\widehat{\theta}(-2^{-m_j}\xi_1)e^{i2^{-m_j}\xi_1 l_j}\Big|^{\alpha}|f(\xi)|^{\alpha}\,\mathrm{d}\xi\right)\\
&= \prod_{j=1}^{M}\mathbb{E}\left[\exp\left(ib_j\mathcal{R}e\left\{\int_{\mathbb{R}^d} e^{it_j\cdot\xi}\Big(e^{i2^{-m_j}\xi_1}-1\Big)\widehat{\theta}(-2^{-m_j}\xi_1)e^{i2^{-m_j}\xi_1 l_j}f(\xi)\,\mathrm{d}\widetilde{M}_{\alpha}(\xi)\right\}\right)\right]\\
&= \prod_{j=1}^{M}\mathbb{E}\Big[\exp\Big(ib_j D_{m_j,l_j}(t_j)\Big)\Big].
\end{aligned} \tag{5.2.23}$$

Thereby the equality (5.2.19) is true for all real numbers $b_1,\ldots,b_M$. □

Lemma 5.2.6, 5.2.7 and 5.2.8 are proved in Section 5.3.

Lemma 5.2.6. *Assume that the admissible function f is as in Theorem 5.2.2. Then,*

$$\liminf_{m\to+\infty}\Big\{2^{ma_1[f]}\,|\sigma_m|\Big\} > 0, \tag{5.2.24}$$

where $a_1[f]$ is as in Definition 3.1.1 and σ_m is defined through (5.2.17).

Lemma 5.2.7. *Assume that the admissible function f is as in Theorem 5.2.2. In particular, we have $a_1[f] \in (0,1]$. For any $m \in \mathbb{N}$ and $x \in \mathbb{R}$, we set*

$$l_m(t) := [2^m x], \tag{5.2.25}$$

the integer part of $2^m x$. Let $t_1^0 \in \mathbb{R}$ and $\beta \in (0, 1/\alpha)$ be fixed. Then, there exists an event $\Omega_4^*[f](t_0^1,\beta)$ of probability 1 satisfying the following property: for any $\omega \in \Omega_4^*[f](t_0^1,\beta)$ and $T > 0$,

$$\liminf_{m\to+\infty}\left(\inf_{\widehat{t_1}\in[-T,T]^{d-1}}\max_{n=0,\ldots,m}\Big\{2^{(m+n)a_1[f]}\,m^{-\beta}\,\Big|D_{m+n,l_{m+n}(t_1^0)}(0,\widehat{t}_1,\omega)\Big|\Big\}\right) > 0, \tag{5.2.26}$$

where $D_{m+n,l_{m+n}(t_1^0)}\Big(0,\widehat{t}_1,\omega\Big)$ *is defined by* (5.2.13).

Lemma 5.2.8. *Assume that f is an admissible function in the sense of Definition 3.1.1. Let $\mu\geq 0$ and $\omega\in\Omega_1^*$ (the event of probability 1 introduced in Lemma 3.2.7), be fixed. Assume that there exists $t^0:=(t_1^0,\widehat{t_1^0})\in\mathbb{R}^d$ such that for some $\rho>0$ we have that*

$$\sup_{t_1',t_1''\in[t_1^0-\rho,t_1^0+\rho]}\left\{\frac{\Big|X[f](t_1',\widehat{t_1^0},\omega)-X[f](t_1'',\widehat{t_1^0},\omega)\Big|}{|t_1'-t_1''|^{a_1[f]}\Big(\log\Big(3+|t_1'-t_1''|^{-1}\Big)\Big)^{\mu}}\right\}<+\infty. \tag{5.2.27}$$

Then, the inequality

$$\begin{aligned}\limsup_{m\to+\infty}\Big\{2^{ma_1[f]}m^{-\mu}\sup\Big\{\Big|D_{m,l}(0,\widehat{t_1^0},\omega)\Big|:l\in\mathbb{Z}\text{ such that}\\ \Big|t_1^0-2^{-m}l\Big|\leq\rho/4\text{ and }\Big|t_1^0-2^{-m}(l+1)\Big|\leq\rho/4\Big\}\Big\}<+\infty\end{aligned} \tag{5.2.28}$$

holds, where $D_{m,l}\Big(0,\widehat{t_1^0}\Big)$ *is defined through* (5.2.13).

We are now ready to prove Theorem 5.2.2.

Proof of Theorem 5.2.2. We denote by $\Omega_5^*[f]$ the event defined by

$$\Omega_5^*[f]:=\bigcap_{t_1^0\in\mathbb{Q}}\ \bigcap_{\beta\in(0,1/\alpha)\cap\mathbb{Q}}\Omega_4^*[f](t_1^0,\beta)\cap\Omega_1^*. \tag{5.2.29}$$

Suppose *ad absurdum* that there is $\omega\in\Omega_5^*[f]$ such that (5.2.5) is not satisfied. Then, there exist $(t_1^0,\widehat{t_1^0},\rho,\delta)\in\mathbb{R}\times\mathbb{R}^{d-1}\times(0,+\infty)\times(0,1/\alpha)$, such that (5.2.27) is satisfied with $\mu=1/\alpha-\delta$. Notice that it is possible to find $\widetilde{t_1^0}\in\mathbb{Q}$ and $\widetilde{\rho}>0$ such that $[\widetilde{t_1^0}-\widetilde{\rho},\widetilde{t_1^0}+\widetilde{\rho}]\subset[t_1^0-\rho,t_1^0+\rho]$; so that we have

$$\sup_{t_1',t_1''\in[\widetilde{t_1^0}-\widetilde{\rho},\widetilde{t_1^0}+\widetilde{\rho}]}\left\{\frac{\Big|X[f](t_1',\widehat{t_1^0},\omega)-X[f](t_1'',\widehat{t_1^0},\omega)\Big|}{|t_1'-t_1''|^{a_1[f]}\Big(\log\Big(3+|t_1'-t_1''|^{-1}\Big)\Big)^{1/\alpha-\delta}}\right\}<+\infty. \tag{5.2.30}$$

Hence, we can assume, and we will do it in the sequel, that $t_1^0\in\mathbb{Q}$.

Lemma 5.2.8 implies that there are $M_1\in\mathbb{N}$ and a positive constant $C_1(t^0,\omega)$, such that, for every $m\geq M_1$ and $l\in\mathbb{Z}$ satisfying $|t_1^0-2^{-m}(l+1)|\leq\rho/4$, and $|t_1^0-2^{-m}l|\leq\rho/4$, we have,

$$\Big|D_{m,l}(0,\widehat{t_1^0},\omega)\Big|\leq C_1(t^0,\omega)2^{-ma_1[f]}m^{1/\alpha-\delta}, \tag{5.2.31}$$

where we have set $t^0:=(t_1^0,\widehat{t_1^0})$.

Let $\beta\in(1/\alpha-\delta,1/\alpha)\cap\mathbb{Q}$ be arbitrary. As $\omega\in\Omega_5^*[f]$, Lemma 5.2.7 (applied with this particular $t_1^0\in\mathbb{Q}$ and $\beta\in(1/\alpha-\delta,1/\alpha)\cap\mathbb{Q}$), implies that, for all $T>0$, there exist $M_2\in\mathbb{N}$

and a finite constant $C_2(\beta, T, t_1^0, \omega) > 0$ satisfying the following property: for every $m \geq M_2$, and $\widehat{t_1} \in [-T, T]^{d-1}$, there exists $n \in \{0, 1 \ldots, m\}$ such that,

$$\left| D_{m+n, l_{m+n}(t_1^0)}(0, \widehat{t_1}, \omega) \right| \geq C_2(\beta, T, t_1^0, \omega) 2^{-(m+n)a_1[f]} m^{\beta}. \tag{5.2.32}$$

Observe that, for all $m \geq -\log(\rho/4)/\log(2)$ and $n \in \{0, 1 \ldots, m\}$, the integer $l_{m+n}(t_1^0)$ defined through (5.2.25) satisfies

$$\left| t_1^0 - 2^{-(m+n)}(l_{m+n}(t_1^0) + 1) \right| \leq \rho/4 \quad \text{and} \quad \left| t_1^0 - 2^{-(m+n)} l_{m+n}(t_1^0) \right| \leq \rho/4. \tag{5.2.33}$$

Moreover, choosing $T > \max_{r=2}^d |t_r^0|$, we have $\widehat{t_1^0} \in [-T, T]^{d-1}$. Therefore, putting together (5.2.33), (5.2.31), and (5.2.32), we have for all $m \geq M := \max(M_1, M_2, -\log(\rho/4)/\log(2))$,

$$C_2(\beta, T, t_1^0, \omega) 2^{-(m+n)a_1[f]} m^{\beta} \leq \left| D_{m+n, l_{m+n}(t_1^0)}(0, \widehat{t_1^0}, \omega) \right| \leq C_1(t^0, \omega) 2^{-(m+n)a_1[f]} (m+n)^{1/\alpha - \delta}. \tag{5.2.34}$$

As $n \in \{0, 1, \ldots, m\}$, relation (5.2.34) implies that for some positive and finite constant $C_3(t^0, T, \beta, \omega)$ we have for all $m \geq M$,

$$0 < C_3(t^0, T, \beta, \omega) \leq m^{1/\alpha - \delta - \beta}, \tag{5.2.35}$$

where $1/\alpha - \delta - \beta < 0$. Relation (5.2.35) being valid for any $m \geq M$, it leads to a contradiction. □

In order to prove Theorem 5.2.1, we need the following additional result whose proof is postponed to Section 5.4

Theorem 5.2.9. *Let $t^0 = (t_1^0, \widehat{t_1^0}) \in \mathbb{R}^d$ and $\rho > 0$ be fixed. Assume that $g : \mathbb{R}^d \to \mathbb{R}$ is a real-valued function such that the function $g(\cdot, \widehat{t_1^0})$ is continuous on $\mathbb{R}$. Suppose that there exist $a \in (0, +\infty)$, $\mu \in \mathbb{R}$ and an integer $n \geq a$, such that,*

$$\sup_{t_1 \in [t_1^0 - \rho, t_1^0 + \rho]} \sup_{h_1 \in [-\rho, \rho]} \left\{ \frac{\left| \boldsymbol{\Delta}_{h_1 e_1}^n g(t_1, \widehat{t_1^0}) \right|}{|h_1|^a \log\left(3 + |h_1|^{-1}\right)^{\mu}} \right\} < +\infty. \tag{5.2.36}$$

Then, there is $\widehat{\rho} > 0$ satifying the following properties:

(i) The function $t_1 \mapsto g(t_1, t_1^0)$ has continuous partial derivative functions of any integer order $b < a$ on $[t_1^0 - 2\widehat{\rho}, t_1^0 + 2\widehat{\rho}]$.

(ii) We set $\overline{b} := \max\{p \in \mathbb{Z}_+, p < a\}$. We have the following two properties:

- *if a is not an integer and $\mu \geq 0$ then*

$$\sup_{t_1 \in [t_1^0 - \widehat{\rho}, t_1^0 + \widehat{\rho}]} \sup_{h_1 \in [-\widehat{\rho}, \widehat{\rho}]} \left\{ \frac{\left| \partial_{e_1}^{\overline{b}} g(t_1 + h_1, \widehat{t_1^0}) - \partial_{e_1}^{\overline{b}} g(t_1, \widehat{t_1^0}) \right|}{|h_1|^{a - \overline{b}} \left(\log\left(3 + |h_1|^{-1}\right)\right)^{\mu}} \right\} < +\infty, \tag{5.2.37}$$

- *if a is an integer and $\mu > -1$ then*

$$\sup_{t_1\in[t_1^0-\widehat{\rho},t_1^0+\widehat{\rho}]}\ \sup_{h_1\in[-\widehat{\rho},\widehat{\rho}]}\left\{\frac{\left|\partial_{e_1}^{\overline{b}}g(t_1+h_1,\widehat{t_1^0})-\partial_{e_1}^{\overline{b}}g(t_1,\widehat{t_1^0})\right|}{|h_1|^{a-\overline{b}}\left(\log\left(3+|h_1|^{-1}\right)\right)^{\mu+1}}\right\}<+\infty. \tag{5.2.38}$$

Observe that in (5.2.38)*, we have $a-\overline{b}=1$.*

We are now ready to prove Theorem 5.2.1.

Proof of Theorem 5.2.1. It follows from Theorem 5.2.9 that the case $a_1[f]\in(0,1)$ has already been treated in Theorem 5.2.2. In the sequel, we assume that $a_1[f]\geq 1$. We denote by $\overline{a_1}[f]$ the integer defined as

$$\overline{a_1}[f]:=\max\left\{m\in\mathbb{Z}_+, m<a_1[f]\right\}.$$

In this case, we know from Theorem 3.2.19 that, for all $\omega\in\Omega_1^*$, the function $X[f](\cdot,\omega)$ is $\overline{a_1}[f]$ times continuously differentiable on its first variable. Moreover, Proposition 5.1.2 entails that

$$\mathcal{D}^{\overline{a_1}[f]e_1}X[f](\cdot,\omega)=X\left[\mathrm{D}^{\overline{a_1}[f]e_1}f\right](\cdot,\omega),$$

where $\mathcal{D}^{\overline{a_1}[f]e_1}X[f](\cdot,\omega)$ and $\mathrm{D}^{\overline{a_1}[f]e_1}f$ are respectively defined in (5.1.2) and (5.1.4). Moreover, in view of Lemma 5.1.1, we know that $\mathrm{D}^{\overline{a_1}[f]e_1}f$ is an admissible function satisfying

$$a_1\left[\mathrm{D}^{\overline{a_1}[f]e_1}f\right]=a_1[f]-\overline{a_1}[f]\in(0,1].$$

On the other hand, it follows from (5.2.1) and (5.1.4) that for any $|\lambda_1|\geq A$

$$\int_{\mathbb{R}^{d-1}}\left|\mathrm{D}^{\overline{a_1}[f]e_1}f(\lambda_1,\xi_2,\dots,\xi_d)\right|^{\alpha}\mathrm{d}\xi_2\dots\mathrm{d}\xi_d\geq c\,|\lambda_1|^{-\alpha(a_1[f]-\overline{a_1}[f])-1}, \tag{5.2.39}$$

where c is the constant in (5.2.1). Then, applying Theorem 5.2.2 to the field $X\left[\mathrm{D}^{\overline{a_1}[f]e_1}f\right]$, there is an event $\Omega_3^*[f]\subset\Omega_1^*$ of probability 1 such that, for all $\omega\in\Omega_3^*[f]$, $\rho\in(0+\infty)$ and $\delta\in(0,1/\alpha)$,

$$\inf_{(t_1,\widehat{t_1})\in\mathbb{R}\times\mathbb{R}^d}\ \sup_{t_1',t_1''\in[t_1-\rho,t_1+\rho]}\left\{\frac{\left|X\left[\mathrm{D}^{\overline{a_1}[f]e_1}f\right](t_1',\widehat{t_1},\omega)-X\left[\mathrm{D}^{\overline{a_1}[f]e_1}f\right](t_1'',\widehat{t_1},\omega)\right|}{|t_1'-t_1''|^{a_1[f]-\overline{a_1}[f]}\left(\log\left(3+|t_1'-t_1''|^{-1}\right)\right)^{1/\alpha-\delta}}\right\}=+\infty. \tag{5.2.40}$$

It follows from Proposition 5.1.2 and (5.1.2), that (5.2.40) is equivalent to

$$\inf_{(t_1,\widehat{t_1})\in\mathbb{R}\times\mathbb{R}^d}\ \sup_{t_1',t_1''\in[t_1-\rho,t_1+\rho]}\left\{\frac{\left|\partial_{e_1}^{\overline{a_1}[f]}X[f](t_1',\widehat{t_1},\omega)-\partial_{e_1}^{\overline{a_1}[f]}X[f](t_1'',\widehat{t_1},\omega)\right|}{|t_1'-t_1''|^{a_1[f]-\overline{a_1}[f]}\left(\log\left(3+|t_1'-t_1''|^{-1}\right)\right)^{1/\alpha-\delta}}\right\}=+\infty. \tag{5.2.41}$$

On the other side, suppose *ad absurdum* that there exists ω in the event $\Omega_3^*[f]$ such that (5.2.2) is not satisfied. Then, there exist an integer $n \in \Big[a_1[f], +\infty\Big)$, $(t_1^0, \widehat{t_1^0}) \in \mathbb{R} \times \mathbb{R}^{d-1}$, $\delta \in (0, 1/\alpha)$ and $\rho \in (0, +\infty)$ such that,

$$\sup_{t_1' \in [t_1^0 - \rho, t_1^0 + \rho]} \sup_{h_1 \in [-\rho, \rho]} \left\{ \frac{\left| \mathbf{\Delta}_{h_1 e_1}^n X[f]\Big(t_1', \widehat{t_1^0}, \omega\Big) \right|}{|h_1|^{a_1[f]} \Big(\log\Big(3 + |h_1|^{-1}\Big)\Big)^{1/\alpha - \delta - \mathbb{1}_{\{a_1[f] \in \mathbb{N}\}}}} \right\} < +\infty. \tag{5.2.42}$$

Theorem 5.2.9 applied with $\mu = 1/\alpha - \delta - \mathbb{1}_{\{a_1[f] \in \mathbb{N}\}}$ implies that for some $\widehat{\rho} > 0$, we have

$$\sup_{t_1 \in [t_1^0 - \widehat{\rho}, t_1^0 + \widehat{\rho}]} \sup_{h_1 \in [-\widehat{\rho}, \widehat{\rho}]} \left\{ \frac{\left| \partial_{e_1}^{\overline{a_1}[f]} X[f](t_1 + h_1, t_1^0) - \partial_{e_1}^{\overline{a_1}[f]} X[f](t_1, t_1^0) \right|}{|h_1|^{a_1[f] - \overline{a_1}[f]} \Big(\log\Big(3 + |h_1|^{-1}\Big)\Big)^{1/\alpha - \delta}} \right\} < +\infty, \tag{5.2.43}$$

which is in contradiction with (5.2.41). □

5.3 Proof of the Lemmas 5.2.6, 5.2.7 and 5.2.8

Proof of Lemma 5.2.6. We show that there is a positive constant c_1 such that for any integer m big enough, we have

$$\sigma_m \geq c_1 2^{-m a_1[f]}. \tag{5.3.1}$$

It follows from (5.2.15), the change of variables $\lambda_1 = 2^{-m}\xi_1$ and the Fubini-Tonelli's Theorem that, for any integer $m \in \mathbb{N}$,

$$\begin{aligned}
\sigma_m^\alpha &= \int_{\mathbb{R}^d} \left| e^{i2^{-m}\xi_1} - 1 \right|^\alpha \left| \widehat{\theta}(-2^{-m}\xi_1) \right|^\alpha |f(\xi)|^\alpha \, \mathrm{d}\xi \\
&\geq \int_{[2^m, 2^{m+1}] \times \mathbb{R}^{d-1}} \left| e^{i2^{-m}\xi_1} - 1 \right|^\alpha \left| \widehat{\theta}(-2^{-m}\xi_1) \right|^\alpha |f(\xi)|^\alpha \, \mathrm{d}\xi \\
&= 2^m \int_{[1,2] \times \mathbb{R}^{d-1}} \left| e^{i\lambda_1} - 1 \right|^\alpha \left| \widehat{\theta}(-\lambda_1) \right|^\alpha |f(2^m\lambda_1, \xi_2, \ldots, \xi_d)|^\alpha \, \mathrm{d}\lambda_1 \mathrm{d}\xi_2 \ldots \mathrm{d}\xi_d \\
&\geq c_2^\alpha 2^m \int_1^2 \left| \widehat{\theta}(-\lambda_1) \right|^\alpha \left(\int_{\mathbb{R}^{d-1}} |f(2^m\lambda_1, \xi_2, \ldots, \xi_d)|^\alpha \, \mathrm{d}\xi_2 \ldots \mathrm{d}\xi_d \right) \mathrm{d}\lambda_1,
\end{aligned} \tag{5.3.2}$$

where $c_2 := \min_{1 \leq x \leq 2} |e^{ix} - 1|$. We mention that c_2 is non-zero because, for any $x \in [1, 2]$, we have that

$$\left| e^{ix} - 1 \right| = 2 \left| \sin(x/2) \right| \neq 0.$$

Moreover, if m is greater than $\log(A)/\log(2)$, then, for any $\lambda_1 \in [1, 2]$, we have

$$|2^m \lambda_1| \geq 2^m \geq A.$$

Thus, combining (5.3.2) and (5.2.4), we have, for every such m,

$$\begin{aligned}
\sigma_m^\alpha &\geq c c_2^\alpha 2^m \int_1^2 \left| \widehat{\theta}(-\lambda_1) \right|^\alpha |2^m \lambda_1|^{-a_1[f]\alpha - 1} \, \mathrm{d}\lambda_1 \\
&= c_3 2^{-m a_1[f] \alpha},
\end{aligned}$$

where c is the constant in (5.2.4) and c_3 is the constant equal to

$$c_3 := cc_2^\alpha \int_1^2 \left|\widehat{\theta}(-x)\right|^\alpha |x|^{-a_1[f]\alpha-1} \, \mathrm{d}x.$$

Notice that the constant c_3 is positive and finite because the non-zero function $\widehat{\theta}$ belongs to the Schwartz class $S(\mathbb{R})$. Therefore (5.3.1) holds. $\square$

The proof of Lemma 5.2.7 relies on Corollary 4.3.2 and the following result.

Lemma 5.3.1. *Assume that the function f belongs to the class $\mathcal{A}$ of admissible functions. Let $t_1^0 \in \mathbb{R}$, $\beta \in (0, 1/\alpha)$ and $R > 1$ be fixed. Assume that $\gamma \in (0, 1 - \alpha\beta]$ is arbitrary and fixed. There exists an event $\Omega_5^*[f](t_1^0, \beta, \gamma, R)$ of probability 1 and a positive constant $c(\alpha)$ such that, for any $\omega \in \Omega_5^*[f](t_1^0, \beta, \gamma, R)$ and $T > 0$, we have*

$$\liminf_{m\to+\infty} \left(\inf_{\widehat{l_1}\in\left[-TR^{m^\gamma}, TR^{m^\gamma}\right]^{d-1}\cap\mathbb{Z}^{d-1}} \max_{n=0,\dots,m} \left\{ \left(m^\beta \sigma_{m+n}\right)^{-1} \left| D_{m+n,l_{m+n}(t_1^0)}(0, R^{-m^\gamma}\widehat{l_1}, \omega)\right| \right\} \right) > c(\alpha), \tag{5.3.3}$$

where $l_{m+n}(t_1^0)$, $D_{m+n,l_{m+n}(t_1^0)}(0, R^{-m}\widehat{l_1})$ and σ_{m+n} are defined respectively through (5.2.25), (5.2.13) *and* (5.2.17).

Proof of lemma 5.3.1. Let $t_1^0 \in \mathbb{R}$, $T > 0$, $R > 1$, $\beta \in (0, 1/\alpha)$ and $\gamma \in (0, 1 - \alpha\beta]$ be arbitrary and fixed. With no restriction we can suppose that $T \in \mathbb{N}$. Assume that $M_0 \in \mathbb{N}$ is the integer part of $\log(A)/\log(2)$ (where A is as in Theorem 5.2.2); hence Lemma 5.2.4, and (5.2.1) imply that $\sigma_m \neq 0$, as soon as $m \geq M_0 + 1$. Then, we set, for all integers $m \geq M_0 + 1$ and $\widehat{l_1} \in [-TR^{m^\gamma}, TR^{m^\gamma}]^{d-1} \cap \mathbb{Z}^{d-1}$,

$$D_m^*(R^{-m^\gamma}\widehat{l_1}) := \max_{n\in\{0,\dots,m\}} \left| \frac{D_{m+n,l_{m+n}(t_1^0)}(0, R^{-m^\gamma}\widehat{l_1})}{\sigma_{m+n}} \right|, \tag{5.3.4}$$

where $l_{m+n}(t_1^0)$ is defined through (5.2.25). In view of the Borel-Cantelli Lemma, in order to prove Lemma 5.3.1, it is enough to show that, for some finite constant $c_0 > 0$, the series of general term

$$\mathbb{P}\left(\bigcup_{\widehat{l_1}\in[-TR^{m^\gamma}, TR^{m^\gamma}]^{d-1}\cap\mathbb{Z}^{d-1}} \left\{ D_m^*\left(R^{-m^\gamma}\widehat{l_1}\right) \leq c_0 m^\beta \right\} \right) \tag{5.3.5}$$

converges. Indeed, this implies that the probability of the event

$$\Omega_4^*[f](t_1^0, \beta, \gamma, R) := \bigcup_{M=1}^{+\infty} \bigcap_{m=M}^{+\infty} \bigcap_{\widehat{l_1}\in[-TR^{m^\gamma}, TR^{m^\gamma}]^{d-1}\cap\mathbb{Z}^{d-1}} \left\{ D_m^*\left(R^{-m^\gamma}\widehat{l_1}\right) > c_0 m^\beta \right\} \tag{5.3.6}$$

is equal to 1. That is, using the definition of $D_m^*\Big(R^{-m^\gamma}\widehat{l_1}\Big)$ (see (5.3.4)), for all $\omega \in \Omega_6^*[f](t_1^0,\beta,\gamma,R)$, there is $M \geq 1$, such that for every $m \geq M$ and $\widehat{l_1} \in [-TR^{m^\gamma}, TR^{m^\gamma}]^{d-1} \cap \mathbb{Z}^{d-1}$, there exists $n \in \{0,1,\dots,m\}$ satisfying,

$$\Big|D_{m+n,l_{m+n}(t_1^0)}(0,R^{-m^\gamma}\widehat{l_1},\omega)\Big| > c_0 m^\beta \sigma_{m+n}, \tag{5.3.7}$$

which completes the proof. It remains to show the convergence of the series of general term (5.3.5). To this end, we need the following classical result on symmetric α-stable distributions, with stable parameter $\alpha \in (0,2)$ (see (1.1.9)): if U is a symmetric α-stable random variable with scale parameter 1, then for all real number $u \geq 1$, one has

$$\mathbb{P}(|U| > u) \geq c_1(\alpha) u^{-\alpha}, \tag{5.3.8}$$

where $c_1(\alpha)$ is a strictly positive finite constant, only depending on α. Let $c_2 \in \Big((d-1)\log(R), +\infty\Big)$ be a fixed positive constant. We denote by M_1 the integer part of

$$\max\Big\{M_0, c_2^{1/(\alpha\beta)}, \Big(c_2 c_1(\alpha)^{-1}\Big)^{1/(\alpha\beta)}\Big\}. \tag{5.3.9}$$

Therefore, for any $m \geq M_1+1$, we have $c_2^{-1/\alpha} c_1(\alpha)^{1/\alpha} m^\beta \geq 1$. Moreover, Lemmas 5.2.4 and 5.2.5 entail that the normalized symmetric stable random variables

$$\Big\{D_{m+n,l_{m+n}(t_1^0)}(0,R^{-m}\widehat{l_1})/\sigma_{m+n},\, n \in \{0,1,\dots,m\}\Big\}$$

are independent and identically distributed. They all have the same distribution as U. So, combining the latter property to (5.3.4) and (5.3.8) (applied with $u = c_2^{-1/\alpha} c_1(\alpha)^{1/\alpha} m^\beta$), we obtain that, for any $m \geq M_1+1$,

$$\begin{aligned}
&\mathbb{P}\left(\bigcup_{\widehat{l_1}\in[-TR^{m^\gamma},TR^{m^\gamma}]^{d-1}} \left\{D_m^*\left(R^{-m^\gamma}\widehat{l_1}\right) \leq c_2^{-1/\alpha} c_1(\alpha)^{1/\alpha} m^\beta\right\}\right)\\
&\leq \sum_{\widehat{l_1}\in[-TR^{m^\gamma},TR^{m^\gamma}]^{d-1}} \mathbb{P}\left(D_m^*\left(R^{-m^\gamma}\widehat{l_1}\right) \leq c_2^{-1/\alpha} c_1(\alpha)^{1/\alpha} m^\beta\right)\\
&= \sum_{\widehat{l_1}\in[-TR^{m^\gamma},TR^{m^\gamma}]^{d-1}} \mathbb{P}\left(\bigcap_{n=0}^{m}\left\{\left|\frac{D_{m+n,l_{m+n}(t_1^0)}\Big(0,R^{-m^\gamma}\widehat{l_1}\Big)}{\sigma_{m+n}}\right| \leq c_2^{-1/\alpha} c_1(\alpha)^{1/\alpha} m^\beta\right\}\right)\\
&= \sum_{\widehat{l_1}\in[-TR^{m^\gamma},TR^{m^\gamma}]^{d-1}} \mathbb{P}\left(|U| \leq c_2^{-1/\alpha} c_1(\alpha)^{1/\alpha} m^\beta\right)^{m+1}\\
&\leq c_3 R^{(d-1)m^\gamma}\left(1 - c_2 m^{-\alpha\beta}\right)^m,
\end{aligned} \tag{5.3.10}$$

where c_3 is a constant only depending on T and d. Then, in view of the definition of M_1 (see (5.3.9)), the inequality

$$\log(1-x) \leq -x, \tag{5.3.11}$$

which holds for all $x \in [0,1)$, implies that, for all $m > M_1$,

$$\begin{aligned} R^{(d-1)m^\gamma}\left(1-c_2 m^{-\alpha\beta}\right)^m &= \exp\left\{m^\gamma (d-1)\log(R)\right\}\exp\left\{m\log\left(1-c_2 m^{-\alpha\beta}\right)\right\} \\ &\le \exp\left\{m^\gamma (d-1)\log(R)\right\}\exp\left\{-c_2 m^{1-\alpha\beta}\right\} \\ &= \exp\left\{-m^{1-\alpha\beta}\left[-m^{-((1-\alpha\beta)-\gamma)}(d-1)\log(R)+c_2\right]\right\}. \end{aligned} \tag{5.3.12}$$

Recall that $\beta \in (0,1/\alpha)$, $\gamma \in (0, 1-\alpha\beta]$ and $c_2 > (d-1)\log(R)$. Hence, it follows from relations (5.3.10) and (5.3.12) that

$$\mathbb{P}\left(\bigcup_{\widehat{l_1}\in[-TR^{m^\gamma},TR^{m^\gamma}]^{d-1}}\left\{D_m^*\left(R^{-m^\gamma}\widehat{l_1}\right)\le c_1(\alpha)^{1/\alpha}m^\beta\right\}\right) = \underset{m\to+\infty}{\mathcal{O}}\left(\frac{1}{m^2}\right).$$

Therefore the series of general term given in (5.3.5) converges. □

We are now ready to prove Lemma 5.2.7.

Proof of Lemma 5.2.7. Let $\beta \in (0,1/\alpha)$ and $t_1^0 \in \mathbb{R}$. Lemma 5.3.1 applied with $\gamma := 1-\alpha\beta$ and $R := 2$ entails that there exists $\Omega_6^*[f](t_1^0,\beta) := \Omega_5^*[f](t_1^0,\beta,\gamma,R)$ an event of probability 1 satisfying the following property: for all $\omega \in \Omega_6^*[f](t_1^0,\beta)$ and $T>0$, there are $c_1>0$ and $M_1 \in \mathbb{N}$ such that the inequality

$$\max_{n\in\{0,1,\dots,m\}}\left\{\left|\left(m^\beta\sigma_{m+n}\right)^{-1}D_{m+n,l_{m+n}(t_1^0)}\left(0,R^{-m^\gamma}\widehat{l_1},\omega\right)\right|\right\} > c_1, \tag{5.3.13}$$

holds for every $m \ge M_1$ and $\widehat{l_1} \in [-TR^{m^\gamma}, TR^{m^\gamma}]^{d-1} \cap \mathbb{Z}^{d-1}$.

Therefore, in order to derive (5.2.26) it is enough to show that for every $\omega \in \Omega_1^* \cap \Omega_6^*[f](t_1^0,\beta)$, $\varepsilon \in (0,+\infty)$ arbitrarily small and $m \in \mathbb{N}$ big enough, there exists a positive and finite constant $C_2(T,t_1^0,\omega)$ satisfying the following property: for every $\widehat{t_1} = (t_2,\dots,t_d) \in [-T,T]^{d-1}$ and $n \in \{0,1,\dots,m\}$, we have that

$$\begin{aligned} &\left|D_{m+n,l_{m+n}(t_1^0)}(0,\widehat{t_1},\omega) - D_{m+n,l_{m+n}(t_1^0)}(0,R^{-m}\widehat{l_1},\omega)\right| \\ &\qquad \le C_2(T,t_1^0,\omega)2^{-(m+n)a_1[f]}m^{\mathcal{L}_\alpha(a_1[f],1,\delta)}\sum_{r=2}^{d}R^{-m^\gamma(\min(1,a_r[f])-\varepsilon)} \end{aligned} \tag{5.3.14}$$

where $\widehat{l_1} := (l_2,\dots,l_d)$ is such that it belongs to $[-TR^{m^\gamma},TR^{m^\gamma}]^{d-1}\cap\mathbb{Z}^{d-1}$ and it satisfies the inequality

$$\left|t_r - R^{-m^\gamma}l_r\right| \le R^{-m^\gamma}, \tag{5.3.15}$$

for any $r = 2, \ldots, d$. Indeed, the triangle inequality, (5.3.13), (5.3.14) and Lemma 5.2.6 imply that there is $M_2 \in \mathbb{N}$ such that for all $m \geq M_2$, $\widehat{t_1} \in [-T,T]^{d-1}$, there exists $n \in \{0,1,\ldots,m\}$ satisfying

$$\begin{aligned}
&\left|D_{m+n,l_{m+n}(t_1^0)}(0,\widehat{t_1},\omega)\right| \\
&\geq \left|D_{m+n,l_{m+n}(t_1^0)}(0,R^{-m}\widehat{l_1},\omega)\right| \\
&\quad - \left|D_{m+n,l_{m+n}(t_1^0)}(0,\widehat{t_1},\omega) - D_{m+n,l_{m+n}(t_1^0)}(0,R^{-m}\widehat{l_1},\omega)\right| \\
&\geq c_1 m^{\beta}\sigma_{m+n} - C_2(T,t_1^0,\omega)2^{-(m+n)a_1[f]}m^{\mathcal{L}_\alpha(a_1[f],1,\delta)}\sum_{r=2}^{d} R^{-m^{\gamma}(\min(1,a_r[f])-\varepsilon)} \\
&\geq c_3 m^{\beta}2^{-(m+n)a_1[f]} - C_2(T,t_1^0,\omega)2^{-(m+n)a_1[f]}m^{\mathcal{L}_\alpha(a_1[f],1,\delta)}\sum_{r=2}^{d} R^{-m^{\gamma}(\min(1,a_r[f])-\varepsilon)} \\
&= c_3 m^{\beta}2^{-(m+n)a_1[f]}\left(1 - C_4(T,t_1^0,\omega)\sum_{r=2}^{d} m^{-\beta}m^{\mathcal{L}_\alpha(a_1[f],1,\delta)}R^{-m^{\gamma}(\min(1,a_r[f])-\varepsilon)}\right).
\end{aligned} \tag{5.3.16}$$

Moreover, the positive real number ε is arbitrarily small; therefore, one can assume that, for any $r = 2, \ldots, d$, we have $\varepsilon \in \Big(0, \min(1,a_r[f])\Big)$. Doing so, for any $r \in \{2,\ldots,d\}$, the sequence of positive real numbers

$$\left(m^{-\beta}m^{\mathcal{L}_\alpha(a_1[f],1,\delta)}R^{-m^{\gamma}(\min(1,a_r[f])-\varepsilon)}\right)_{m\in\mathbb{N}}$$

converges to 0 when m goes to infinity. Hence, there is $M_3 \in \mathbb{N}$ such that, for every $m \geq M_3$ and $\widehat{t_1} \in [-T,T]^{d-1}$, there exists $n \in \{0,1,\ldots,m\}$ satisfying the inequality

$$\left|D_{m+n,l_{m+n}(t_1^0)}(0,\widehat{t_1},\omega)\right| \geq C_5(t_1^0,T,\omega)m^{\beta}2^{-(m+n)a_1[f]}, \tag{5.3.17}$$

where $C_5(t_1^0,T,\omega)$ is a positive and finite constant. That is, (5.2.26) holds.

It remains to prove (5.3.14). Let $m \in \mathbb{N}$, $n \in \{0,1,\ldots,m\}$ and $l_1 \in \mathbb{Z}$ be fixed. We define $T_1 := \max(1+|t_1^0|, 1+T)$. Using (5.2.13), (5.2.10), the change of variable $u = 2^{m+n}s - l_1$ and (5.2.8), for all $\widehat{t_1} \in [-T,T]^{d-1}$ and $\widehat{l_1} \in \mathbb{Z}^{d-1}$, we have that

$$\begin{aligned}
&D_{m+n,l_1}(0,\widehat{t_1},\omega) - D_{m+n,l_1}(0,R^{-m^{\gamma}}\widehat{l_1},\omega) \\
&= 2^{m+n}\int_{\mathbb{R}}\Big(X[f](s+2^{-m-n},\widehat{t_1},\omega) - X[f](s+2^{-m-n},R^{-m^{\gamma}}\widehat{l_1},\omega)\Big)\theta(2^{m+n}s-l_1)\,\mathrm{d}s \\
&\quad -2^{m+n}\int_{\mathbb{R}}\Big(X[f](s,\widehat{t_1},\omega) - X[f](s,R^{-m^{\gamma}}\widehat{l_1},\omega)\Big)\theta(2^{m+n}s-l_1)\,\mathrm{d}s \\
&= \int_{\mathbb{R}}\Big(X[f](2^{-m-n}(u+l_1+1),\widehat{t_1},\omega) - X[f](2^{-m-n}(u+l_1+1),R^{-m^{\gamma}}\widehat{l_1},\omega)\Big)\theta(u)\,\mathrm{d}u \\
&\quad - \int_{\mathbb{R}}\Big(X[f](2^{-m-n}(u+l_1),\widehat{t_1},\omega) - X[f](2^{-m-n}(u+l_1),R^{-m^{\gamma}}\widehat{l_1},\omega)\Big)\theta(u)\,\mathrm{d}u \\
&= I^1_{m,n,l_1+1}(\widehat{t_1},\omega) + I^2_{m,n,l_1+1}(\widehat{t_1},\omega) + I^1_{m,n,l_1}(\widehat{t_1},\omega) + I^2_{m,n,l_1}(\widehat{t_1},\omega),
\end{aligned} \tag{5.3.18}$$

where we have defined, for any $\kappa \in \{0,1\}$,

$$\begin{aligned} I^1_{m,n,l_1+\kappa}(\widehat{t_1},\omega) &:= \int_{\{|s|\le 2^{m+n}T_1\}} \Big(\Delta^1_{2^{-m-n}s} X[f](2^{-m-n}(l_1+\kappa),\widehat{t_1},\omega) \\ &\quad - \Big(\Delta^1_{2^{-m-n}s} X[f](2^{-m-n}(l_1+\kappa), R^{-m^\gamma}\widehat{l_1},\omega)\Big)\Big)\theta(s)\,\mathrm{d}s, \quad (5.3.19) \end{aligned}$$

and

$$\begin{aligned} I^2_{m,n,l_1+\kappa}(\widehat{t_1},\omega) &:= \int_{\{|s|> 2^{m+n}T_1\}} \Big(\Delta^1_{2^{-m-n}s} X[f](2^{-m-n}(l_1+\kappa),\widehat{t_1},\omega) \\ &\quad - \Big(\Delta^1_{2^{-m-n}s} X[f](2^{-m-n}(l_1+\kappa), R^{-m^\gamma}\widehat{l_1},\omega)\Big)\Big)\theta(s)\,\mathrm{d}s. \quad (5.3.20) \end{aligned}$$

Recall that, for any $k \in \{1,\dots,d\}$ and $h_k \in \mathbb{R}$, the operator $\Delta^k_{h_k}$ is defined though (4.1.1). First, we provide an upper-bound of $I^1_{m,n,l_1+\kappa}(\widehat{t_1},\omega)$. Let $\widehat{t_1} \in [-T,T]$ be arbitrary and $\widehat{l_1} := (l_2,\dots,l_d)$ be such that it belongs to $[-TR^{m^\gamma}, TR^{m^\gamma}]^{d-1} \cap \mathbb{Z}^{d-1}$ and satisfies (5.3.15), for all $r = 2,\dots,d$. Therefore, for all $m \in \mathbb{N}$ and $n \in \{0,1\dots,m\}$,

$$\left| I^1_{m,n,l_{m+n}(t_1^0)+\kappa}(\widehat{t_1},\omega)\right| \le \int_{\{|s|\le 2^{m+n}T_1\}} |\theta(s)| \sum_{r=2}^d \left\| \Delta^{e_1}_{2^{-m-n}s} \Delta^{e_r}_{t_r - R^{-m^\gamma} l_r} X[f](\cdot,\omega)\right\|_{T_1,\infty} \mathrm{d}s, \quad (5.3.21)$$

where $T_1 = \max(1+|t_1^0|, 1+T)$ and $\|\cdot\|_{T_1,\infty}$ is defined through (3.2.42). Observe that, in view of (5.3.15), for any $s \in [-2^{m+n}T_1, 2^{m+n}T_1]$, we get that

$$(2^{-m-n}s, t_1 - R^{-m^\gamma} l_1, \dots, t_d - R^{-m^\gamma} l_d) \in [-T_1,T_1]^d.$$

Therefore, setting

$$\widetilde{a_r}[f] := \min(a_r[f],1), \quad (5.3.22)$$

for all $r \in \{2,\dots,d\}$, Corollary 4.1.3, (3.2.38), (5.3.15) and the fact that $\theta \in S(\mathbb{R})$ imply that, for any $\varepsilon > 0$ and $\delta > 0$ arbitrarily small,

$$\begin{aligned} &\left| I^1_{m+n,l_{m+n}(t_1^0)+\kappa}(\widehat{t_1},\omega)\right| \\ &\le C_6(T,t_1^0,\omega) \int_{\{|s|\le 2^{m+n}T_1\}} \Bigg\{ |\theta(s)| \\ &\qquad \times \sum_{r=2}^d \left|2^{-m-n}s\right|^{a_1[f]} \left(\log(3+2^{m+n}|s|^{-1})\right)^{\mathcal{L}_\alpha(a_1[f],1,\delta)} \left|t_r - R^{-m^\gamma} l_r\right|^{\widetilde{a_r}[f]-\varepsilon} \Bigg\} \mathrm{d}s \\ &\le C_7(T,t_1^0,\omega) 2^{-(m+n)a_1[f]} \sum_{r=2}^d R^{-m^\gamma(\widetilde{a_r}[f]-\varepsilon)} (m+n)^{\mathcal{L}_\alpha(a_1[f],1,\delta)} \\ &\qquad \times \int_{\mathbb{R}} |s|^{a_1[f]} \left(\log(3+|s|^{-1})\right)^{\mathcal{L}_\alpha(a_1[f],1,\delta)} |\theta(s)|\,\mathrm{d}s \\ &\le C_8(T,t_1^0,\omega) 2^{-(m+n)a_1[f]} m^{\mathcal{L}_\alpha(a_1[f],1,\delta)} \sum_{r=2}^d R^{-m^\gamma(\widetilde{a_r}[f]-\varepsilon)}, \quad (5.3.23) \end{aligned}$$

where $C_6(T,t_1^0,\omega)$, $C_7(T,t_1^0,\omega)$ and $C_8(T,t_1^0,\omega)$ are positive and finite constants which do not depend on m, n, t_1, $\widehat{t_1}$ and $\widehat{l_1}$. We mention that in the last inequality in (5.3.23) we used that $n \in \{0,\dots,m\}$.
Next, we focus on $I^2_{m,n,l_1+\kappa}(\widehat{t_1},\omega)$. The triangle inequality, Corollary 4.3.2, (3.2.38) and (5.3.15) yield that, for all $\omega \in \Omega_1^*$, $v \in \mathbb{R}$ and $\varepsilon > 0$ arbitrarily small, we have that

$$\left|X[f](v,\widehat{t_1},\omega) - X[f](v,R^{-m^\gamma}\widehat{l_1},\omega)\right| \le C_9(\omega)\sqrt{\log\left(3+|v|\right)}^{\lfloor\alpha\rfloor}\sum_{r=2}^{d} R^{-m^\gamma(\widetilde{a_r}[f]-\varepsilon_2)}, \quad (5.3.24)$$

where $\lfloor\alpha\rfloor$ is the integer part of $\alpha\in(0,2)$ and where $C_9(\omega)$ is a positive and finite constant which does not depend on m, t_1, $\widehat{t_1}$ and $\widehat{l_1}$. Moreover, the function θ belongs to $\mathcal{S}(\mathbb{R})$; therefore, we have that

$$\sup_{|s|\ge 1}\left\{|s|^2\,|\theta(s)|\right\} < +\infty. \quad (5.3.25)$$

So, it follows from (5.3.20), the triangle inequality, (5.3.24), (3.2.38), the change of variables $u = 2^{-m-n}s$ and (5.3.25) that, for all $m\in M$ and $\widehat{t_1}\in[-T,T]^d$,

$$\begin{aligned}
&\left|I^2_{m,n,l_{m+n}(t_1^0)+\kappa}(\widehat{t_1},\omega)\right| \\
&\le C_{10}(T,t_1^0,\omega)\sum_{r=2}^{d} R^{-m(\widetilde{a_r}[f]-\varepsilon_2)}\int_{\{|s|>2^{m+n}T_1\}}\Bigg(\sqrt{\log\left(3+2^{-m-n}\left|l_{m+n}(t_1^0)+\kappa\right|\right)}^{\lfloor\alpha\rfloor} \\
&\qquad +\sqrt{\log\left(3+2^{-m-n}\left|l_{m+n}(t_1^0)+\kappa\right|+\left|2^{-m-n}s\right|\right)}^{\lfloor\alpha\rfloor}\Bigg)|\theta(s)|\,\mathrm{d}s \\
&\le C_{11}(T,t_1^0,\omega)\sum_{r=2}^{d} R^{-m^\gamma(\widetilde{a_r}[f]-\varepsilon)}\int_{\{|s|>2^{m+n}T_1\}}\sqrt{\log\left(3+2^{-m-n}|s|\right)}^{\lfloor\alpha\rfloor}|\theta(s)|\,\mathrm{d}s \\
&\le C_{12}(T,t_1^0,\omega)\sum_{r=2}^{d} R^{-m^\gamma(\widetilde{a_r}[f]-\varepsilon)}\int_{\{|u|>T_1\}}\sqrt{\log\left(3+|u|\right)}^{\lfloor\alpha\rfloor}\left|\theta(2^{m+n}u)\right|\,\mathrm{d}u \\
&\le C_{13}(T,t_1^0,\omega)2^{-2(m+n)}\sum_{r=2}^{d} R^{-m^\gamma(\widetilde{a_r}[f]-\varepsilon)},
\end{aligned} \quad (5.3.26)$$

where the constants $C_{10}(T,t_1^0,\omega)$ to $C_{13}(T,t_1^0,\omega)$ are positive, finite and do not depend on m, n, t_1, $\widehat{t_1}$ and $\widehat{l_1}$. In view of (5.3.22), combining (5.3.18), (5.3.23) and (5.3.26), we get (5.3.14). □

Proof of Lemma 5.2.8. Observe that, relations (5.2.10), (5.2.13) and the change of variables $u = s + 2^{-m}$ yield that, for every $m\in\mathbb{N}$, $l\in\mathbb{Z}$, $t\in\mathbb{R}^d$, and $\omega\in\Omega_1^*$, we have

$$\begin{aligned}
&D_{m,l}(t,\omega) \\
&= 2^m\int_{\mathbb{R}} X[f](t_1+u,t_2,\dots,t_d,\omega)\theta(2^m u-l-1)\,\mathrm{d}u \\
&\qquad -2^m\int_{\mathbb{R}} X[f](t_1+s,t_2,\dots,t_d,\omega)\theta(2^m s-l)\,\mathrm{d}s \\
&= \delta_{m,l+1}(t,\omega)-\delta_{m,l}(t,\omega),
\end{aligned} \quad (5.3.27)$$

where for any $m \in \mathbb{N}$, $l \in \mathbb{Z}$, $t \in \mathbb{R}^d$ and $\omega \in \Omega_1^*$, we have set

$$\delta_{m,l}(t,\omega) := 2^m \int_{\mathbb{R}} X[f](t_1+s,t_2,\dots,t_d,\omega)\theta\Big(2^m s - l\Big)\, \mathrm{d}s. \tag{5.3.28}$$

Let $\omega \in \Omega_1^*$, $t^0 := (t_1^0, \widehat{t_1^0}) \in \mathbb{R}^d$ and $\kappa \in \{0,1\}$ be fixed. Using (5.2.8), for any $m \in \mathbb{N}$ and $l \in \mathbb{Z}$, we have that

$$\Big|\delta_{m,l+\kappa}(0,\widehat{t_1^0},\omega)\Big| \le 2^m \int_{\mathbb{R}} \Big|X(s,\widehat{t_1^0},\omega) - X(2^{-m}(l+\kappa),\widehat{t_1^0},\omega)\Big|\,|\theta(2^m s - l - \kappa)|\, \mathrm{d}s. \tag{5.3.29}$$

Then, in view of (5.3.27), in order to prove (5.2.28), for every $\kappa \in \{0,1\}$, it is enough to show that

$$\limsup_{m\to+\infty} \Big\{2^{m(a_1[f]+1)} m^{-\mu} \sup\Big\{\Big|A_{m,l+\kappa}(t^0,\omega)\Big| : l \in \mathbb{Z} \text{ such that } \Big|t_1^0 - 2^{-m}(l+\kappa)\Big| \le \rho/4\Big\}\Big\} < +\infty, \tag{5.3.30}$$

and,

$$\limsup_{m\to+\infty} \Big\{2^{3m} \sup\Big\{\Big|B_{m,l+\kappa}(t^0,\omega)\Big| : l \in \mathbb{Z} \text{ such that } \Big|t_1^0 - 2^{-m}(l+\kappa)\Big| \le \rho/4\Big\}\Big\} < +\infty, \tag{5.3.31}$$

where, we have defined

$$A_{m,l+\kappa}(t^0,\omega) := \int_{\big\{s\in\mathbb{R}: \left|s-t_1^0\right| \le \rho/2\big\}} \Big|X[f](s,\widehat{t_1^0},\omega) - X[f](2^{-m}(l+\kappa),\widehat{t_1^0},\omega)\Big|\,|\theta(2^m s - l - \kappa)|\, \mathrm{d}s, \tag{5.3.32}$$

$$B_{m,l+\kappa}(t^0,\omega) := \int_{\big\{s\in\mathbb{R}: \left|s-t_1^0\right| > \rho/2\big\}} \Big|X[f](s,\widehat{t_1^0},\omega) - X[f](2^{-m}(l+\kappa),\widehat{t_1^0},\omega)\Big|\,|\theta(2^m s - l - \kappa)|\, \mathrm{d}s. \tag{5.3.33}$$

Let us first show (5.3.30). Notice that it is possible to find $M \in \mathbb{N}$ such that for all integers $m \ge M$, the set

$$\Big\{l \in \mathbb{Z} \text{ such that } \qquad \Big|t_1^0 - 2^{-m}(l+\kappa)\Big| \quad \text{and} \quad \Big|t_1^0 - 2^{-m}(l+\kappa)\Big|\Big\}$$

is not empty. Then, for such $m \ge M$ and l in this non-empty set, we get from (5.2.27), the change of variables $u = 2^m s - l - \kappa$ and (3.2.38) that there exists a finite constant $C_1(t^0,\omega) > 0$ such that, for all $m \in \mathbb{N}$,

$$\begin{aligned}
&A_{m,l+\kappa}(t^0,\omega) \\
&\le C_1(t^0,\omega) \int_{\big\{s\in\mathbb{R}: \left|s-t_1^0\right| \le \rho/2\big\}} \Big|s - 2^{-m}(l+\kappa)\Big|^{a_1[f]} \left(\log\left(3 + \Big|s - 2^{-m}(l+\kappa)\Big|^{-1}\right)\right)^{\mu} |\theta(2^m s - l - \kappa)|\, \mathrm{d}s \\
&\le C_1(t^0,\omega) \int_{\mathbb{R}} \Big|s - 2^{-m}(l+\kappa)\Big|^{a_1[f]} \left(\log\left(3 + \Big|s - 2^{-m}(l+\kappa)\Big|^{-1}\right)\right)^{\mu} |\theta(2^m s - l - \kappa)|\, \mathrm{d}s \\
&= C_1(t^0,\omega) 2^{-m} \int_{\mathbb{R}} \Big|2^{-m}u\Big|^{a_1[f]} \left(\log\left(3 + \Big|2^{-m}u\Big|^{-1}\right)\right)^{\mu} |\theta(u)|\, \mathrm{d}u \\
&\le C_2(t^0,\omega) 2^{-m(1+a_1[f])} m^{\mu} \int_{\mathbb{R}} |u|^{a_1[f]} \left(\log\left(3 + |u|^{-1}\right)\right)^{\mu} |\theta(u)|\, \mathrm{d}u,
\end{aligned} \tag{5.3.34}$$

where $C_2(t^0,\omega) := (6\log 2)^{\mu} C_1(t^0,\omega)$. Notice that the integral in (5.3.34) is finite because $\theta \in S(\mathbb{R})$, $a_1[f] \in (0,1]$ and $\mu \geq 0$. Therefore, (5.3.30) holds.

Now, we focus on (5.3.31). Let $m \geq M$ and $l \in \mathbb{Z}$ such that $|t_1^0 - 2^{-m}l| \leq \rho/4$ and $|t_1^0 - 2^{-m}(l+1)| \leq \rho/4$. Using the fact that θ belongs to $S(\mathbb{R})$, for some constant $c_3 \in (0,+\infty)$, the inequality

$$|\theta(x)| \leq c_3(1+|x|)^{-3} \tag{5.3.35}$$

holds for any $x \in \mathbb{R}$. In view of (4.2.14) and (3.2.52), for any $\varepsilon > 0$ arbitrarily small, there exists $C_4(t^0,\omega)$ satisfying, for every $s \in \mathbb{R}$,

$$\left|X[f](s,t_1^0,\omega)\right| \leq C_4(t^0,\omega)\left(1 + \left\|(s,\widehat{t_1^0})\right\|^{a'[f]+\varepsilon}\right). \tag{5.3.36}$$

Moreover, for every $s \in \mathbb{R}$ satisfying $|s - t_1^0| > \rho/2$, the inequality $|t_1^0 - 2^{-m}(l+\kappa)| \leq \rho/4$ implies that

$$\begin{aligned} |2^m s - l - \kappa| &\geq 2^m \left|s - t_1^0\right| - 2^m \left|t_1^0 - 2^{-m}(l+\kappa)\right| \\ &\geq 2^m \left|s - t_1^0\right| - 2^m \rho/4 \\ &\geq 2^m \left|s - t_1^0\right| - 2^m \left|s - t_1^0\right|/2 \\ &\geq 2^m \left|s - t_1^0\right|/2 \end{aligned} \tag{5.3.37}$$

Putting together (5.3.33), (5.3.36), (5.3.35) and (5.3.37), one can derive from the Triangle inequality that

$$\begin{aligned} B_{m,l+\kappa}(t^0,\omega) &\leq c_3 C_4(t^0,\omega)\Bigg(\int_{\left\{s\in\mathbb{R}:\left|s-t_1^0\right|>\rho/2\right\}} \frac{1+\left\|(s,\widehat{t_1^0})\right\|^{a'[f]+\varepsilon}}{(1+|2^m s - l - \kappa|)^3}\,\mathrm{d}s \\ &\qquad + \int_{\left\{s\in\mathbb{R}:\left|s-t_1^0\right|>\rho/2\right\}} \frac{1+\left\|(2^{-m}(l+\kappa),\widehat{t_1^0})\right\|^{a'[f]+\varepsilon}}{(1+|2^m s - l - \kappa|)^3}\,\mathrm{d}s\Bigg) \\ &\leq c_3 C_4(t^0,\omega)\Bigg(\int_{\left\{s\in\mathbb{R}:\left|s-t_1^0\right|>\rho/2\right\}} \frac{1+(|s|+\|t^0\|)^{a'[f]+\varepsilon}}{(1+2^{m-1}\left|s-t_1^0\right|)^3}\,\mathrm{d}s \\ &\qquad + \int_{\left\{s\in\mathbb{R}:\left|s-t_1^0\right|>\rho/2\right\}} \frac{1+\left\|(\rho/4+\left|t_1^0\right|,\widehat{t_1^0})\right\|^{a'[f]+\varepsilon}}{(1+2^{m-1}\left|s-t_1^0\right|)^3}\,\mathrm{d}s\Bigg) \\ &\leq C_5(t^0,\omega)2^{-3m}\Bigg(\int_{\left\{s\in\mathbb{R}:\left|s-t_1^0\right|>\rho/2\right\}} \frac{1+(|s|+\|t^0\|)^{a'[f]+\varepsilon}}{\left|s-t_1^0\right|^3}\,\mathrm{d}s \\ &\qquad + \int_{\left\{s\in\mathbb{R}:\left|s-t_1^0\right|>\rho/2\right\}} \frac{1+\left\|(\rho/4+\left|t_1^0\right|,\widehat{t_1^0})\right\|^{a'[f]+\varepsilon}}{\left|s-t_1^0\right|^3}\,\mathrm{d}s\Bigg) \end{aligned} \tag{5.3.38}$$

As $a'[f] \in (0,1]$ and $\varepsilon > 0$ is arbitrary small, the integrals in (5.3.38) are finite, which finishes the proof of (5.3.31). □

5.4 Proof of Theorem 5.2.9

Throughout this section, we use the notation " $\widehat{t_1}$ " introduced at the beginning of Section 4.3. The main goal of this section is to prove Theorem 5.2.9. Before that, we need to introduce the following definition.

Definition 5.4.1. *For any $h \in \mathbb{R}^d$, we denote by $\widetilde{\mathbf{\Delta}}_h$ the operator from the space of real-valued functions on $\mathbb{R}^d$ into itself, so that when g is such a function, $\widetilde{\mathbf{\Delta}}_h g$ is then the function defined, for all $x \in \mathbb{R}^d$, as*

$$\Big(\widetilde{\mathbf{\Delta}}_h g\Big)(x) := g(x+h) - g(x-h). \tag{5.4.1}$$

Moreover, for any integer n, we denote by $\widetilde{\mathbf{\Delta}}_h^n$ the operator $\widetilde{\mathbf{\Delta}}_h$ composed with itself n times, with the convention that $\widetilde{\mathbf{\Delta}}_h^0$ is the identity. Notice that, for each integer n, the equality

$$\Big(\widetilde{\mathbf{\Delta}}_h^n g\Big)(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} g\Big(x + (n-2k)h\Big) \tag{5.4.2}$$

holds for all x and h in $\mathbb{R}^d$.

Remark 5.4.2. *Let $x \in \mathbb{R}^d$ and $h \in \mathbb{R}^d$ be arbitrary. In view of (4.1.37), for all integer n, we have that*

$$\Big(\mathbf{\Delta}_h^n g\Big)(x) = \sum_{k=0}^{n} (-1)^{k+n} \binom{n}{k} g(x+kh). \tag{5.4.3}$$

Hence, we get the following two equalities:

$$\Big(\widetilde{\mathbf{\Delta}}_h^n g\Big)(x) = (-1)^n \Big(\mathbf{\Delta}_{-2h}^n g\Big)(x+nh), \tag{5.4.4}$$

and

$$\Big(\mathbf{\Delta}_h^n g\Big)(x) = (-1)^n \Big(\widetilde{\mathbf{\Delta}}_{-2^{-1}h}^n g\Big)(x+2^{-1}nh). \tag{5.4.5}$$

The following lemma is very useful. It shows that we can work either with $\mathbf{\Delta}_h^n$ or $\widetilde{\mathbf{\Delta}}_h^n$.

Lemma 5.4.3. *Assume that $n \in \mathbb{N}$. Let $a > 0$ and $\mu \in \mathbb{R}$ be two real numbers, then for all real-valued function g on $\mathbb{R}^d$ the following properties hold:*

(i) If there exist $\rho > 0$ and $(t_1^0, \widehat{t_1^0}) \in \mathbb{R} \times \mathbb{R}^{d-1}$ such that

$$\sup_{t_1 \in [t_1^0-\rho, t_1^0+\rho]} \sup_{h_1 \in [-\rho,\rho]} \left\{ \frac{\left|\widetilde{\mathbf{\Delta}}_{h_1 e_1}^n g(t_1, \widehat{t_1^0})\right|}{|h_1|^a \log\left(3 + |h_1|^{-1}\right)^{\mu}} \right\} < +\infty, \tag{5.4.6}$$

then

$$\sup_{t_1\in[t_1^0-\rho/2,t_1^0+\rho/2]}\ \sup_{h_1\in[-\rho/n,\rho/n]}\left\{\frac{\left|\mathbf{\Delta}^n_{h_1e_1}g(t_1,\widehat{t_1^0})\right|}{|h_1|^a\log\left(3+|h_1|^{-1}\right)^{\mu}}\right\}<+\infty. \tag{5.4.7}$$

(ii) If there exists $\rho>0$ and $(t_1^0,\widehat{t_1^0})\in\mathbb{R}\times\mathbb{R}^{d-1}$ such that

$$\sup_{t_1\in[t_1^0-\rho,t_1^0+\rho]}\ \sup_{h_1\in[-\rho,\rho]}\left\{\frac{\left|\mathbf{\Delta}^n_{h_1e_1}g(t_1,\widehat{t_1^0})\right|}{|h_1|^a\log\left(3+|h_1|^{-1}\right)^{\mu}}\right\}<+\infty, \tag{5.4.8}$$

then

$$\sup_{t_1\in[t_1^0-\rho/2,t_1^0+\rho/2]}\ \sup_{h_1\in[-\rho/(2n),\rho/(2n)]}\left\{\frac{\left|\widetilde{\mathbf{\Delta}}^n_{h_1e_1}g(t_1,\widehat{t_1^0})\right|}{|h_1|^a\log\left(3+|h_1|^{-1}\right)^{\mu}}\right\}<+\infty. \tag{5.4.9}$$

Proof of Lemma 5.4.3. The proof of Lemma 5.4.3 is a straighforward application of Remark 5.4.2. □

The proof of Theorem 5.2.9 is based on the one of Theorem 6.1 in [18] (page 214). It mainly relies on the following three lemmas.

Lemma 5.4.4. *Let $t^0=(t_1^0,\widehat{t_1^0})\in\mathbb{R}^d$ and $\rho>0$ be fixed. Assume that $g:\mathbb{R}^d\to\mathbb{R}$ is a real-valued function such that the function $g(\cdot,\widehat{t_1^0})$ is continuous on $\mathbb{R}$. Suppose that there exist $a\in(0,+\infty)$, $\mu\in\mathbb{R}$ and an integer $n\geq a$, such that,*

$$\sup_{t_1\in[t_1^0-\rho,t_1^0+\rho]}\ \sup_{h_1\in[-\rho,\rho]}\left\{\frac{\left|\widetilde{\mathbf{\Delta}}^n_{h_1e_1}g(t_1,\widehat{t_1^0})\right|}{|h_1|^a\log\left(3+|h_1|^{-1}\right)^{\mu}}\right\}<+\infty. \tag{5.4.10}$$

Then, for any integer $n'>n$, there exists $\widetilde{\rho}>0$ such that

$$\sup_{t_1\in[t_1^0-\widetilde{\rho},t_1^0+\widetilde{\rho}]}\ \sup_{h_1\in[-\widetilde{\rho},\widetilde{\rho}]}\left\{\frac{\left|\widetilde{\mathbf{\Delta}}^{n'}_{h_1e_1}g(t_1,\widehat{t_1^0})\right|}{|h_1|^a\log\left(3+|h_1|^{-1}\right)^{\mu}}\right\}<+\infty. \tag{5.4.11}$$

Lemma 5.4.5. *Let $t^0=(t_1^0,\widehat{t_1^0})\in\mathbb{R}^d$ and $\rho>0$ be fixed. Assume that $g:\mathbb{R}^d\to\mathbb{R}$ is a real-valued function such that the function $g(\cdot,\widehat{t_1^0})$ is continuous on $\mathbb{R}$. Suppose that there exist $a\in(0,+\infty)$, $\mu\in\mathbb{R}$ and an integer $n\in[a,\infty)$ such that,*

$$\sup_{t_1\in[t_1^0-\rho,t_1^0+\rho]}\ \sup_{h_1\in[-\rho,\rho]}\left\{\frac{\left|\widetilde{\mathbf{\Delta}}^n_{h_1e_1}g(t_1,\widehat{t_1^0})\right|}{|h_1|^a\log\left(3+|h_1|^{-1}\right)^{\mu}}\right\}<+\infty. \tag{5.4.12}$$

Then, there exist $c\in(0,+\infty)$, $\widetilde{\rho}\in(0,\rho)$ and ϕ an infinitely differentiable, compactly supported, function from $\mathbb{R}$ to $\mathbb{R}$ satisfying the following properties:

(i) The support of ϕ is such that

$$\operatorname{supp}(\phi) \subset [-\widetilde{\rho}/2, \widetilde{\rho}/2]. \tag{5.4.13}$$

(ii) For all $\delta \in (0,1]$,

$$\sup_{t_1 \in [t_1^0 - \widetilde{\rho}, t_1^0 + \widetilde{\rho}]} \left\{ (g * \phi_\delta)(t_1, \widehat{t_1^0}) - g(t_1, \widehat{t_1^0}) \right\} \leq c\, \delta^a \log(3 + \delta^{-1})^\mu, \tag{5.4.14}$$

where the function ϕ_δ is defined by

$$\phi_\delta := \delta^{-1} \phi(\delta^{-1} \cdot), \tag{5.4.15}$$

and, the "partial convolution product" $(g * \phi_\delta)(\cdot, \widehat{t_1^0})$ is defined, for all $t_1 \in \mathbb{R}$, as

$$(g * \phi_\delta)(t_1, \widehat{t_1^0}) := \int_{\mathbb{R}} g(t_1 - x, \widehat{t_1^0}) \phi_\delta(x)\, \mathrm{d}x. \tag{5.4.16}$$

Notice that the integral in (5.4.16) is well-defined because $g(\cdot, \widehat{t_1^0})$ is continuous on $\mathbb{R}$ and ϕ is infinitely differentiable and compactly supported.

Lemma 5.4.6. *Let $t^0 = (t_1^0, \widehat{t_1^0}) \in \mathbb{R}^d$, $\rho > 0$, $a \in (0, +\infty)$ and $\mu \in \mathbb{R}$ be fixed. Assume that $g : \mathbb{R}^d \to \mathbb{R}$ is a real-valued function such that the function $g(\cdot, \widehat{t_1^0})$ is continuous on $\mathbb{R}$. Assume also that ϕ is an infinitely differentiable, compactly supported, function of $\mathbb{R}$ such that*

$$\operatorname{supp}(\phi) \subset [-\rho/2, \rho/2]. \tag{5.4.17}$$

If there exists $c > 0$ such that, for all $\delta \in (0,1]$,

$$\sup_{t_1 \in [t_1^0 - \rho, t_1^0 + \rho]} \left\{ \left| (g * \phi_\delta)(t_1, \widehat{t_1^0}) - g(t_1, \widehat{t_1^0}) \right| \right\} \leq c\, \delta^a \log(3 + \delta^{-1})^\mu, \tag{5.4.18}$$

where $\phi_\delta := \delta^{-1}\phi(\delta^{-1}\cdot)$. Then, for any integer $n \in \mathbb{Z}_+$, there exists $\widetilde{c} > 0$ such that, for all $\delta \in (0,1]$, we have that

$$\sup_{t_1 \in [t_1^0 - \rho/2, t_1^0 + \rho/2]} \left\{ \left| \partial_{e_1}^n \left((g * \phi_\delta)(t_1, \widehat{t_1^0}) - (g * \phi_{2\delta})(t_1, \widehat{t_1^0}) \right) \right| \right\} \leq \widetilde{c}\, \delta^{a-n} \log(3 + \delta^{-1})^\mu. \tag{5.4.19}$$

Before proving those lemmas, we show that Theorem 5.2.9 holds.

Proof of Theorem 5.2.9. It follows from (5.2.36) and *(ii)* in Lemma 5.4.3 that g satisfies

$$\sup_{t_1 \in [t_1^0 - \rho', t_1^0 + \rho']} \sup_{h_1 \in [-\rho', \rho']} \left\{ \frac{\left| \widetilde{\mathbf{\Delta}}_{h_1 e_1}^n g(t_1, \widehat{t_1^0}) \right|}{|h_1|^a \log\left(3 + |h_1|^{-1}\right)^\mu} \right\} < +\infty, \tag{5.4.20}$$

for some $\rho' \in (0, \rho)$. Hence, Lemma 5.4.5 implies that there exist $c_1 > 0$, $\widetilde{\rho} \in (0, \rho')$ and ϕ an infinitely differentiable, compactly supported, function from $\mathbb{R}$ to $\mathbb{R}$ satisfying the following properties:

(*i*) The support of ϕ is such that

$$\mathrm{supp}(\phi) \subset [-\widetilde{\rho}/2, \widetilde{\rho}/2]. \tag{5.4.21}$$

(*ii*) For all $\delta \in (0,1]$,

$$\sup_{t_1 \in [t_1^0 - \widetilde{\rho}, t_1^0 + \widetilde{\rho}]} \left\{ \left| (g * \phi_\delta)(t_1, \widehat{t_1^0}) - g(t_1, \widehat{t_1^0}) \right| \right\} \leq c_1\, \delta^a \log(3 + \delta^{-1})^\mu, \tag{5.4.22}$$

where the function ϕ_δ has been defined in (5.4.15).

In the sequel $\widetilde{\rho}$ is denoted by ρ. For all integer $m \geq 1$, we define the function Λ_m as follows:

$$\begin{aligned} &\Lambda_m := g * \phi_{2^{-m}}, && \text{if } m = 1, \\ &\Lambda_m := g * \phi_{2^{-m}} - g * \phi_{2^{-m+1}}, && \text{if } m > 1. \end{aligned} \tag{5.4.23}$$

Notice that, for any integer $m \geq 2$, we have that

$$m \log(2) \leq \log(3 + 2^m) = m\log(2) + \log(32^{-m} + 1) \leq m \log 2 + 2\log 2 \leq 2m \log 2$$

Therefore, Lemma 5.4.6 (with $\delta = 2^{-m}$) implies that, for all integer $b \in \mathbb{Z}_+$, there is $c_{2,b} \in (0, +\infty)$ such that, for all integer $m > 1$,

$$\sup_{t_1 \in [t_1^0 - \rho/2, t_1^0 + \rho/2]} \left\{ \left| \partial_{e_1}^b \Lambda_m(t_1, \widehat{t_1^0}) \right| \right\} \leq c_{2,b}\, 2^{-m(a-b)} m^\mu. \tag{5.4.24}$$

Notice that the function Λ_1 is infinitely differentiable on $\mathbb{R}$. Then, replacing $c_{2,b}$ in (5.4.24) by the constant $c_{3,b}$ defined as

$$c_{3,b} := \max\left(c_{2,b}, \sup_{t_1 \in [t_1^0 - \rho/2, t_1^0 + \rho/2]} \left\{ \left| \partial_{e_1}^b \Lambda_1(t_1, \widehat{t_1^0}) \right| \right\} \right),$$

the inequality in (5.4.24) holds for any $m \geq 1$. Moreover, it follows from (5.4.23), (5.4.22) and the inequality $a > 0$ that,

$$\begin{aligned} &\lim_{M \to +\infty} \left\{ \sup_{t_1 \in [t_1^0 - \rho/2, t_1^0 + \rho/2]} \left| g(t_1, \widehat{t_1^0}) - \sum_{m=1}^{M} \Lambda_m(t_1, \widehat{t_1^0}) \right| \right\} \\ &= \lim_{M \to +\infty} \left\{ \sup_{t_1 \in [t_1^0 - \rho/2, t_1^0 + \rho/2]} \left| g(t_1, \widehat{t_1^0}) - g * \phi_{2^{-M}}(t_1, \widehat{t_1^0}) \right| \right\} \\ &\leq \lim_{M \to +\infty} \left\{ c_1 2^{-Ma} \log\left(3 + 2^M\right)^\mu \right\} = 0. \end{aligned}$$

So, the series of infinitely differentiable functions $\sum_{m \geq 1} \Lambda_m(\cdot, \widehat{t_1^0})$ converges to $g(\cdot, \widehat{t_1^0})$ uniformly on the compact set $[t_1^0 - \rho/2, t_1^0 + \rho/2]$. On the other side, (5.4.24) entails that for all integer $b < a$, the series of functions $\sum_{m\ge 1} \partial_{e_1}^{b} \Lambda_m(\cdot, \widehat{t_1^0})$ converges uniformly on the compact set $[t_1^0 - \rho/2, t_1^0 + \rho/2]$. So, for any integer $b < a$, the partial derivative function $\partial_{e_1}^{b} g(\cdot, \widehat{t_1^0})$ exists and is continuous on the compact set $[t_1^0 - 2\widehat{\rho}, t_1^0 + 2\widehat{\rho}]$ for any arbitrary $\widehat{\rho} \in (0, \rho/4)$.

From now on, it remains to prove (5.2.37) and (5.2.38). Assume that $\widehat{\rho} \in (0, \rho/4)$ is arbitrary. Let $t_1 \in [t_1^0 - \widehat{\rho}, t_1^0 + \widehat{\rho}]$ and $h_1 \in [-\widehat{\rho}, \widehat{\rho}]$ be fixed. Recall that $\overline{b} := \max\{p \in \mathbb{Z}_+, p < a\}$. With no restriction, we can assume that $\widehat{\rho} < 1$, so that, there exists $m_0(h_1) \in \mathbb{N}$ satisfying

$$2^{-m_0(h_1)} \le |h_1| < 2^{-m_0(h_1)+1}. \tag{5.4.25}$$

So, using the fact that the series of infinitely differentiable functions $\sum_{m\ge 1} \partial_{e_1}^{\overline{b}} \Lambda_m(\cdot, \widehat{t_1^0})$ converges to $\partial_{e_1}^{\overline{b}} g(\cdot, \widehat{t_1^0})$ uniformly on the compact set $[t_1^0 - 2\widehat{\rho}, t_1^0 + 2\widehat{\rho}]$, the Mean Value Theorem, (5.4.24) and (5.4.25), we have

$$\begin{aligned}
\left|\partial_{e_1}^{\overline{b}} g(t_1+h_1, \widehat{t_1^0}) - \partial_{e_1}^{\overline{b}} g(t_1, \widehat{t_1^0})\right| &\le \sum_{m=1}^{m_0(h_1)} \left|\partial_{e_1}^{\overline{b}} \Lambda_m(t_1+h_1, \widehat{t_1^0}) - \partial_{e_1}^{\overline{b}} \Lambda_m(t_1, \widehat{t_1^0})\right| \\
&\quad + \sum_{m=m_0(h_1)+1}^{+\infty} \left|\partial_{e_1}^{\overline{b}} \Lambda_m(t_1+h_1, \widehat{t_1^0}) - \partial_{e_1}^{\overline{b}} \Lambda_m(t_1, \widehat{t_1^0})\right| \\
&\le \sum_{m=1}^{m_0(h_1)} |h_1| \sup_{t\in[t_1^0-2\widehat{\rho}, t_1^0+2\widehat{\rho}]} \left|\partial_{e_1}^{\overline{b}+1} \Lambda_m(t, \widehat{t_1^0})\right| \\
&\quad +2 \sum_{m=m_0(h_1)+1}^{+\infty} \sup_{t\in[t_1^0-2\widehat{\rho}, t_1^0+2\widehat{\rho}]} \left|\partial_{e_1}^{\overline{b}} \Lambda_m(t_1, \widehat{t_1^0})\right| \\
&\le c_{3,\overline{b}+1} |h_1| \sum_{m=1}^{m_0(h_1)} 2^{-m(a-\overline{b}-1)} m^{\mu} \\
&\quad +2 c_{3,\overline{b}} \sum_{m=m_0(h_1)+1}^{+\infty} 2^{-m(a-\overline{b})} m^{\mu}.
\end{aligned} \tag{5.4.26}$$

From now on, we focus on (5.2.37): that is, we assume that a is not an integer and $\mu \ge 0$. It follows from (5.4.26) and (5.4.25) that

$$\begin{aligned}
\left|\partial_{e_1}^{\overline{b}} g(t_1+h_1, \widehat{t_1^0}) - \partial_{e_1}^{\overline{b}} g(t_1, \widehat{t_1^0})\right| &\le c_{3,\overline{b}+1} |h_1|\, m_0(h_1)^{\mu} \sum_{m=1}^{m_0(h_1)} 2^{-m(a-\overline{b}-1)} \\
&\quad +2 c_{3,\overline{b}} 2^{-m_0(h_1)(a-\overline{b})} m_0(h_1)^{\mu} \sum_{m=1}^{+\infty} 2^{-m(a-\overline{b})} (1+m)^{\mu} \\
&\le c_{4,\overline{b}} \Big(\log\big(3 + |h_1|^{-1}\big)\Big)^{\mu} \left(|h_1| \sum_{m=1}^{m_0(h_1)} 2^{-m(a-\overline{b}-1)} + |h_1|^{a-\overline{b}} \right),
\end{aligned} \tag{5.4.27}$$

where $c_{4,\overline{b}}$ is a positive and finite constant which does not depend on t_1 and h_1. The exponent a is not an integer, so we have that $a - \overline{b} - 1 < 0$. Therefore, (5.4.25) implies that

$$\sum_{m=1}^{m_0(h_1)} 2^{-m(a-\overline{b}-1)} = \frac{2^{m_0(h_1)(\overline{b}+1-a)} - 1}{2^{\overline{b}+1-a} - 1} \leq c_5 \left|h_1\right|^{a-\overline{b}-1}, \tag{5.4.28}$$

where $c_5 := \left(2^{\overline{b}+1-a} - 1\right)^{-1} \in (0, +\infty)$. Putting together (5.4.26), (5.4.27) and (5.4.28) we get

$$\left|\partial_{e_1}^{\overline{b}} g(t_1 + h_1, \widehat{t_1^0}) - \partial_{e_1}^{\overline{b}} g(t_1, \widehat{t_1^0})\right| \leq c_{6,\overline{b}} \left|h_1\right|^{a-\overline{b}} \left(\log\left(3 + \left|h_1\right|^{-1}\right)\right)^{\mu}, \tag{5.4.29}$$

where $c_{6,\overline{b}}$ is a positive and finite constant which does not depend on t_1 and h_1. So, (5.2.37) holds.

Now we focus on (5.2.38). In this case a is an integer and $\mu > -1$. Notice that we have $a - \overline{b} = 1$, hence

$$\sum_{m=1}^{m_0(h_1)} 2^{-m(a-\overline{b}-1)} m^{\mu} = \sum_{m=1}^{m_0(h_1)} m^{\mu} = \sum_{m=1}^{m_0(h_1)} \int_{m-1}^{m} m^{\mu} \, \mathrm{d}x \leq \int_{0}^{m_0(h)} x^{\mu} \, \mathrm{d}x = (\mu+1)^{-1} m_0(h_1)^{\mu+1}. \tag{5.4.30}$$

Moreover, using the fact that $\mu + 1 > 0$, we get

$$\begin{aligned} \sum_{m=m_0(h_1)+1}^{+\infty} 2^{-m(a-\overline{b})} m^{\mu} &\leq \sum_{m=m_0(h_1)+1}^{+\infty} 2^{-m(a-\overline{b})} m^{\mu+1} \\ &= \sum_{m=1}^{+\infty} 2^{-(m+m_0(h_1))} (m + m_0(h_1))^{\mu+1} \\ &\leq m_0(h_1)^{\mu+1} 2^{-m_0(h_1)} \sum_{m=1}^{+\infty} 2^{-m} (1+m)^{\mu+1}. \end{aligned} \tag{5.4.31}$$

Putting together (5.4.26), (5.4.30), (5.4.31) and (5.4.25), we get

$$\left|\partial_{e_1}^{\overline{b}} g(t_1 + h_1, \widehat{t_1^0}) - \partial_{e_1}^{\overline{b}} g(t_1, \widehat{t_1^0})\right| \leq c_{7,\overline{b}} \left|h_1\right| \left(\log\left(3 + \left|h_1\right|^{-1}\right)\right)^{\mu+1}, \tag{5.4.32}$$

where $c_{7,\overline{b}}$ is a positive and finite constant which does not depend on t_1 and h_1. Finally, (5.2.38) holds. □

Proof of Lemma 5.4.4. Let $n' > n$ be fixed. We define

$$\widetilde{\rho} := \frac{\rho}{1 + n' - n} \in (0, \rho]. \tag{5.4.33}$$

Let $t_1 \in [t_1^0 - \widetilde{\rho}, t_1^0 + \widetilde{\rho}]$ and $h_1 \in [-\widetilde{\rho}, \widetilde{\rho}]$ be arbitrary. For any $x_1 \in [t_1^0 - \widetilde{\rho}, t_1^0 + \widetilde{\rho}]$ and $h_1 \in [-\widetilde{\rho}, \widetilde{\rho}]$, it follows from (5.4.10) that

$$\left|\widetilde{\boldsymbol{\Delta}}_{h_1 e_1}^{n} g(x_1, \widehat{t_1^0})\right| \leq c_1(t^0) \left|h_1\right|^{a} \left(\log\left(3 + \left|h_1\right|^{-1}\right)\right)^{\mu}, \tag{5.4.34}$$

where the positive and finite constant $c_1(t^0)$ does not depend on x_1, t_1 and h_1. Moreover, it follows from (5.4.2)

$$\begin{aligned}\left|\widetilde{\mathbf{\Delta}}^{n'}_{h_1e_1} g(t_1,\widehat{t^0_1})\right| &= \left|\left(\widetilde{\mathbf{\Delta}}^{n'-n}_{h_1e_1}\left(\widetilde{\mathbf{\Delta}}^{n}_{h_1e_1} g\right)\right)(t_1,\widehat{t^0_1})\right| \\ &\leq \sum_{k=0}^{n'-n}\binom{n'-n}{k}\left|\widetilde{\mathbf{\Delta}}^{n}_{h_1e_1} g\Big(t_1+(n'-n-2k)h_1,\widehat{t^0_1}\Big)\right|. \end{aligned} \tag{5.4.35}$$

Observe that, for any $k\in\{0,1,\dots,n'-n\}$, we have $|n'-n-2k|\leq n'-n$. Then it follows from the triangle inequality and (5.4.33) that

$$\left|t_1+(n'-n-2k)h_1-t^0_1\right|\leq\widetilde{\rho}+\widetilde{\rho}(n'-n)=\widetilde{\rho}(1+n'-n)=\rho. \tag{5.4.36}$$

Therefore combining (5.4.35), (5.4.36), (5.4.33) and (5.4.34) we get that

$$\left|\widetilde{\mathbf{\Delta}}^{n'}_{h_1e_1} g(t_1,\widetilde{t^0_1})\right|\leq c_1(t^0)\sum_{k=0}^{n'-n}\binom{n'-n}{k}|h_1|^a\log\left(3+|h_1|^{-1}\right)^{\mu}=c_2(t^0)\,|h_1|^a\log\left(3+|h_1|^{-1}\right)^{\mu}, \tag{5.4.37}$$

where the positive and finite constant $c_2(t^0)$ is equal to $2^{n'-n}c_1(t^0)$ and does not depend on t_1 and h. Therefore (5.4.11) holds. □

Proof of Lemma 5.4.5. Assume that n' is an integer satisfying $n'\geq n$ and $n'=4p+2$ for some $p\in\mathbb{N}$. It follows from (5.4.12) and Lemma 5.4.4, that for some $\widetilde{\rho}\leq\rho$ we have

$$\sup_{t_1\in[t^0_1-\widetilde{\rho},t^0_1+\widetilde{\rho}]}\ \sup_{h_1\in[-\widetilde{\rho},\widetilde{\rho}]}\left\{\frac{\left|\widetilde{\mathbf{\Delta}}^{n'}_{h_1e_1} g(t_1,\widehat{t^0_1})\right|}{|h_1|^a\log\left(3+|h_1|^{-1}\right)^{\mu}}\right\}<+\infty. \tag{5.4.38}$$

In the sequel, we denote respectively n' and $\widetilde{\rho}$ by n and ρ. Let $\varphi:\mathbb{R}\to\mathbb{R}$ be an infinitely differentiable, compactly supported, even function satisfying the following properties:

$$\operatorname{supp}(\varphi)\subset[-\rho/(2n),\rho/(2n)]\quad\text{and}\quad\int_{\mathbb{R}}\varphi(t)\,\mathrm{d}t=1. \tag{5.4.39}$$

Then, we denote by $\widetilde{\phi}$ the real-valued function defined, for any $x\in\mathbb{R}$, by

$$\widetilde{\phi}(x):=\sum_{k=0}^{\frac{n}{2}-1}(-1)^k\binom{n}{k}\frac{1}{n-2k}\varphi\left(\frac{x}{n-2k}\right). \tag{5.4.40}$$

Then, we have

$$c_n:=\int_{\mathbb{R}}\widetilde{\phi}(x)\mathrm{d}x=\sum_{k=0}^{\frac{n}{2}-1}(-1)^k\binom{n}{k}=\frac{1}{2}\binom{n}{\frac{n}{2}}>0. \tag{5.4.41}$$

The latter equality in (5.4.41) is proved at the end of the proof. Finally, the real-valued function ϕ defined, for all $x \in \mathbb{R}^d$, by

$$\phi(x) = \frac{\widetilde{\phi}(x)}{c_n}, \tag{5.4.42}$$

is infinitely differentiable, compactly supported, and satisfies the following properties:

$$\operatorname{supp}(\phi) \subset [-\rho/2, \rho/2] \quad \text{and} \quad \int_{\mathbb{R}} \phi(x)\,\mathrm{d}x = 1. \tag{5.4.43}$$

It follows from (5.4.16), (5.4.15), the change of variables $y = \delta^{-1}x$, (5.4.42), (5.4.40) and the change of variables $z = y/(n-2k)$ that, for any $t_1 \in [t_1^0 - \rho, t_1^0 + \rho]$ and $\delta \in (0,1]$, we have

$$\begin{aligned}
&g * \phi_\delta\Big(t_1, \widehat{t_1^0}\Big) - g\big(t_1, \widehat{t_1^0}\big)\\
&= \int_{\mathbb{R}} g\Big(t_1 - x, \widehat{t_1^0}\Big)\phi_\delta(x)\,\mathrm{d}x - g\Big(t_1, \widehat{t_1^0}\Big)\\
&= \int_{\mathbb{R}} g\Big(t_1 - \delta y, \widehat{t_1^0}\Big)\phi(y)\,\mathrm{d}y - g\Big(t_1, \widehat{t_1^0}\Big)\\
&= \frac{1}{c_n}\left\{\sum_{k=0}^{\frac{n}{2}-1}(-1)^k\binom{n}{k}\frac{1}{n-2k}\int_{\mathbb{R}} g\Big(t_1 - \delta y, \widehat{t_1^0}\Big)\varphi\left(\frac{y}{n-2k}\right)\,\mathrm{d}y - c_n g\Big(t_1, \widehat{t_1^0}\Big)\right\}\\
&= \frac{1}{c_n}\left\{\sum_{k=0}^{\frac{n}{2}-1}(-1)^k\binom{n}{k}\int_{\mathbb{R}} g\Big(t_1 - (n-2k)\delta z, \widehat{t_1^0}\Big)\varphi(z)\,\mathrm{d}z - c_n g\Big(t_1, \widehat{t_1^0}\Big)\right\},
\end{aligned} \tag{5.4.44}$$

where c_n has been defined in (5.4.41). Using the change of indices $l = n - k$, the facts that n is twice an odd integer and that φ is an even function, we obtain that

$$\begin{aligned}
&\sum_{k=0}^{\frac{n}{2}-1}(-1)^k\binom{n}{k}\int_{\mathbb{R}} g\Big(t_1 - (n-2k)\delta x, \widehat{t_1^0}\Big)\varphi(x)\,\mathrm{d}x\\
&= \sum_{l=\frac{n}{2}+1}^{n}(-1)^{n-l}\binom{n}{n-l}\int_{\mathbb{R}} g\Big(t_1 - (n-2(n-l))\delta x, \widehat{t_1^0}\Big)\varphi(x)\,\mathrm{d}x\\
&= \sum_{l=\frac{n}{2}+1}^{n}(-1)^{l}\binom{n}{l}\int_{\mathbb{R}} g\Big(t_1 - (n-2l)\delta x, \widehat{t_1^0}\Big)\varphi(x)\,\mathrm{d}x.
\end{aligned} \tag{5.4.45}$$

it follows from (5.4.44), (5.4.45), (5.4.39), (5.4.41) and the fact that $n/2$ is an odd number that

$$
\begin{aligned}
& g * \phi_\delta\Big(t_1, \widehat{t_1^0}\Big) - g\big(t_1, \widehat{t_1^0}\big) \\
&= \frac{1}{2c_n}\left\{\sum_{k=0}^{\frac{n}{2}-1}(-1)^k\binom{n}{k}\int_{\mathbb{R}} g\Big(t_1-(n-2k)\delta x, \widehat{t_1^0}\Big)\varphi(x)\ \mathrm{d}x\right. \\
&\quad \left. + \sum_{k=\frac{n}{2}+1}^{n}(-1)^k\binom{n}{k}\int_{\mathbb{R}} g\Big(t_1-(n-2k)\delta x, \widehat{t_1^0}\Big)\varphi(x)\ \mathrm{d}x - 2c_n g\Big(t_1, \widehat{t_1^0}\Big)\right\}, \\
&= \frac{1}{2c_n}\left\{\int_{\mathbb{R}}\sum_{k=0, k\neq\frac{n}{2}}^{n}\left((-1)^k\binom{n}{k} g\Big(t_1-(n-2k)\delta x, \widehat{t_1^0}\Big)\varphi(x)\right) + (-1)^{n/2}\binom{n}{\frac{n}{2}} g\Big(t_1, \widehat{t_1^0}\Big)\,\mathrm{d}x\right\} \\
&= \frac{1}{2c_n}\left\{\int_{\mathbb{R}}\sum_{k=0}^{n}(-1)^k\binom{n}{k} g\Big(t_1-(n-2k)\delta x, \widehat{t_1^0}\Big)\varphi(x)\ \mathrm{d}x\right\} \\
&= \frac{1}{2c_n}\left\{\int_{\mathbb{R}}\varphi(x)\widetilde{\boldsymbol{\Delta}}^n_{-\delta x e_1} g(t_1, \widehat{t_1^0})\,\mathrm{d}t\right\}.
\end{aligned} \tag{5.4.46}
$$

Therefore, it follows from (5.4.46), (5.4.39) and (5.4.12) that, for all $t_1 \in [t_1^0-\rho, t_1^0+\rho]$ and $\delta \in (0,1]$,

$$\Big|g * \phi_\delta\Big(t_1, \widehat{t_1^0}\Big) - g\big(t_1, \widehat{t_1^0}\big)\Big| \ \leq\ c_1\int_{\mathbb{R}} |\delta x|^a \log\left(3+|\delta x|^{-1}\right)^{\mu}\varphi(x)\,\mathrm{d}x, \tag{5.4.47}$$

where c_1 is a positive finite constant which does not depend on t_1 and δ. Moreover, when $\mu \geq 0$, for all $\delta \in (0,1]$ and $x \in \mathbb{R}\setminus\{0\}$, we have that

$$\log\left(3+|\delta x|^{-1}\right)^{\mu} \leq 2^{\mu}\log\left(3+\delta^{-1}\right)^{\mu}\log\left(3+|x|^{-1}\right)^{\mu}. \tag{5.4.48}$$

On the other hand, when $\mu < 0$, with no restriction, we can assume that $\rho \leq 1$. Therefore, for all $\delta \in (0,1]$ and $x \in [-\rho, \rho]$, one can derive from $|\delta x| \leq \delta$, the inequality

$$\log\left(3+|\delta x|^{-1}\right)^{\mu} \leq \log\left(3+\delta^{-1}\right)^{\mu}. \tag{5.4.49}$$

Putting together (5.4.47), (5.4.48) and (5.4.49) we have

$$\Big|g * \phi_\delta\Big(t_1, \widehat{t_1^0}\Big) - g\big(t_1, \widehat{t_1^0}\big)\Big| \leq c_2\left(\int_{\mathbb{R}}|x|^a\log\left(3+|x|^{-1}\right)^{\mu\mathbb{1}_{\{\mu\geq 0\}}}\varphi(t)\,\mathrm{d}t\right)\delta^a\log\left(3+\delta^{-1}\right)^{\mu}, \tag{5.4.50}$$

where the latter integral is finite because a is a positive real number and the function φ is compactly supported and infinitely differentiable on $\mathbb{R}$. That is (5.4.14) holds

Now, it remains to show (5.4.41). The Binomial Theorem and the fact that $n/2$ is an odd number entail that

$$0 = (1-1)^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k} = \sum_{k=0}^{\frac{n}{2}-1}(-1)^k\binom{n}{k} - \binom{n}{\frac{n}{2}} + \sum_{k=\frac{n}{2}+1}^{n}(-1)^k\binom{n}{k}. \tag{5.4.51}$$

Moreover, the change of indices $l = n - k$ and the fact that n is an even integer entail that

$$\sum_{k=\frac{n}{2}+1}^{n} (-1)^k \binom{n}{k} = \sum_{l=0}^{\frac{n}{2}-1} (-1)^{n-l} \binom{n}{n-l} = \sum_{l=0}^{\frac{n}{2}-1} (-1)^l \binom{n}{l}. \tag{5.4.52}$$

Putting together (5.4.51) and (5.4.52), we get (5.4.41) □

Proof of Lemma 5.4.6. Let $n \in \mathbb{Z}_+$ be fixed and $G(\cdot, \widehat{t}_1^0)$ be an arbitrary continuous function on $\mathbb{R}$. The function ϕ is infinitely differentiable and compactly supported. Hence, in view of (5.4.16) and (5.4.15), for any $\delta \in (0,1]$, the function $(G*\phi_\delta)(\cdot, \widehat{t}_1^0)$ is infinitely differentiable on $\mathbb{R}$. Moreover, we have that

$$\Big(\partial_{e_1}^n (G * \phi_\delta)\Big)(\cdot, \widehat{t}_1^0) = \Big(G * \phi_\delta^{(n)}\Big)(\cdot, \widehat{t}_1^0). \tag{5.4.53}$$

where $\phi_\delta^{(n)}$ is the derivative function of order n of ϕ_δ. So, applying (5.4.53) to $G(\cdot, \widehat{t}_1^0) = g(\cdot, \widehat{t}_1^0)$, we have, for any $\delta \in (0, 1/2]$ and $t_1 \in [t_1^0 - \rho, t_1^0 + \rho]$, that

$$\begin{aligned} \Big(\partial_{e_1}^n (g * \phi_\delta - g * \phi_{2\delta})\Big)(t_1, \widehat{t}_1^0) &= \partial_{e_1}^n (g * \phi_\delta)(t_1, \widehat{t}_1^0) - \partial_{e_1}^n (g * \phi_{2\delta})(t_1, \widehat{t}_1^0) \\ &= \Big(g * \phi_\delta^{(n)}\Big)(t_1, \widehat{t}_1^0) - \Big(g * \phi_{2\delta}^{(n)}\Big)(t_1, \widehat{t}_1^0). \end{aligned} \tag{5.4.54}$$

We define $I_{\delta,1}(t_1, \widehat{t}_1^0)$, $I_{\delta,2}(t_1, \widehat{t}_1^0)$ and $I_{\delta,3}(t_1, \widehat{t}_1^0)$ as follows

$$I_{\delta,1}(t_1, \widehat{t}_1^0) := \Big(\phi_\delta^{(n)} * (g * \phi_\delta - g * \phi_{2\delta})\Big)(t_1, \widehat{t}_1^0), \tag{5.4.55}$$

$$I_{\delta,2}(t_1, \widehat{t}_1^0) := \Big(\phi_\delta^{(n)} * (g - g * \phi_\delta)\Big)(t_1, \widehat{t}_1^0), \tag{5.4.56}$$

$$I_{\delta,3}(t_1, \widehat{t}_1^0) := -\Big(\phi_{2\delta}^{(n)} * (g - g * \phi_\delta)\Big)(t_1, \widehat{t}_1^0). \tag{5.4.57}$$

Observe that (5.4.53) and the Fubini's Theorem imply that

$$I_{\delta,1}(t_1, \widehat{t}_1^0) = \Big(\phi_\delta * \Big(g * \phi_\delta^{(n)}\Big)\Big)(t_1, \widehat{t}_1^0) - \Big(\phi_\delta * \Big(g * \phi_{2\delta}^{(n)}\Big)\Big)(t_1, \widehat{t}_1^0) \tag{5.4.58}$$

$$I_{\delta,2}(t_1, \widehat{t}_1^0) = \Big(g * \phi_\delta^{(n)}\Big)(t_1, \widehat{t}_1^0) - \Big(\phi_\delta * \Big(g * \phi_\delta^{(n)}\Big)\Big)(t_1, \widehat{t}_1^0) \tag{5.4.59}$$

$$I_{\delta,3}(t_1, \widehat{t}_1^0) = \Big(\phi_\delta * \Big(g * \phi_{2\delta}^{(n)}\Big)\Big)(t_1, \widehat{t}_1^0) - \Big(g * \phi_{2\delta}^{(n)}\Big)(t_1, \widehat{t}_1^0). \tag{5.4.60}$$

In view of (5.4.54) to (5.4.60), we have

$$\Big(\partial_{e_1}^n (g * \phi_\delta - g * \phi_{2\delta})\Big)(t_1, \widehat{t}_1^0) = I_{\delta,1}(t_1, \widehat{t}_1^0) + I_{\delta,2}(t_1, \widehat{t}_1^0) + I_{\delta,3}(t_1, \widehat{t}_1^0). \tag{5.4.61}$$

Therefore, in order to prove Lemma 5.4.6, it is enough to show that there exists $c_1 > 0$ such that for $l \in \{1,2,3\}$ satisfying for any $\delta \in (0, 1/2]$,

$$\sup_{t_1 \in [t_1^0 - \rho/2, t_1^0 + \rho/2]} \Big| I_{\delta,l}(t_1, \widehat{t}_1^0) \Big| \le c_1 \delta^{a-n} \log\Big(3 + \delta^{-1}\Big)^{\mu}. \tag{5.4.62}$$

We only do the proof of (5.4.62) when $l = 1$ (the proof in the other cases is similar). It follows from (5.4.16), (5.4.17), (5.4.15) and the triangle inequality that, for every $\delta \in (0,1]$ and $t_1 \in [t_1^0 - \rho/2, t_1^0 + \rho/2]$,

$$\begin{aligned} \left|I_{\delta,1}(t_1, \widehat{t_1^0})\right| &\leq \int_{-\delta\rho/2}^{\delta\rho/2} \left|\phi_\delta^{(n)}(x)\right| \left|g * \phi_\delta(t_1 - x, \widehat{t_1^0}) - g(t_1 - x, \widehat{t_1^0})\right| \mathrm{d}x \\ &\quad + \int_{-\delta\rho/2}^{\delta\rho/2} \left|\phi_\delta^{(n)}(x)\right| \left|g * \phi_{2\delta}(t_1 - x, \widehat{t_1^0}) - g(t_1 - t, \widehat{t_1^0})\right| \mathrm{d}x. \qquad (5.4.63) \end{aligned}$$

Observe that when $\delta \in (0,1]$, $x \in [-\delta\rho/2, \delta\rho/2]$ and $t_1 \in [t_1^0 - \rho/2, t_1^0 + \rho/2]$, we have $t_1 - x \in [t_1^0 - \rho, t_1^0 + \rho]$. Hence, for all $\delta \in (0, 1/2]$ and $t_1 \in [t_1^0 - \rho/2, t_1^0 + \rho/2]$, it follows from (5.4.63), (5.4.18) (applied with δ and 2δ), (5.4.15) and the change of variables $y = \delta^{-1}x$ that

$$\left|I_{\delta,1}(t_1, \widehat{t_1^0})\right| \leq c_2 \delta^a \log\left(3 + \delta^{-1}\right)^\mu \int_{\mathbb{R}} \left|\phi_\delta^{(n)}(x)\right| \mathrm{d}x \leq c_4 \left(\int_{\mathbb{R}} \left|\phi^{(n)}(x)\right| \, \mathrm{d}x\right) \delta^{a-n} \log\left(3 + \delta^{-1}\right)^\mu, \qquad (5.4.64)$$

where c_2 does not depend on δ and t_1. Notice that the integral in the right-hand side of (5.4.64) is finite since ϕ is infinitely differentiable and compactly supported. Hence, (5.4.62) holds for every $\delta \in (0, 1/2]$. Therefore, (5.4.19) holds for all $\delta \in (0, 1/2]$. When $\delta \in [1/2, 1]$, it is enough to show that the function

$$\mathtt{G} : (\delta, t_1) \mapsto \delta^{-(a-n)} \log\left(3 + \delta^{-1}\right)^{-\mu} \left(\left(g * \phi_\delta^{(n)}\right)(t_1, \widehat{t_1^0}) - \left(g * \phi_{2\delta}^{(n)}\right)(t_1, \widehat{t_1^0})\right) \qquad (5.4.65)$$

is continuous on the compact set $[1/2, 1] \times [t_1^0 - \rho/2, t_1^0 + \rho/2]$. Observe that, for every $(\delta, t_1) \in [1/2, 1] \times [t_1^0 - \rho/2, t_1^0 + \rho/2]$, (5.4.16), the changes of variables $y = \delta^{-1}x$ and $y = (2\delta)^{-1}x$, and (5.4.17) entail that

$$\left(g * \phi_\delta^{(n)}\right)(t_1, \widehat{t_1^0}) - \left(g * \phi_{2\delta}^{(n)}\right)(t_1, \widehat{t_1^0}) = \delta^{-n} \int_{-\rho/2}^{\rho/2} \left(g(t_1 - \delta x, \widehat{t_1^0}) - 2^{-n} g(t_1 - 2\delta x, \widehat{t_1^0})\right) \phi(x) \, \mathrm{d}x. \qquad (5.4.66)$$

Next, recall that $g(\cdot, \widehat{t_1^0})$ is continuous on $\mathbb{R}$ and ϕ is infinitely differentiable on $\mathbb{R}$. Therefore, the integrand in the latter integral is continuous on $[1/2, 1] \times [t_1^0 - \rho/2, t_1^0 + \rho/2]$ with respect to (t_1, δ). Moreover, the inequality

$$\left|\left(g(t_1 - \delta x, \widehat{t_1^0}) - g(t_1 - 2\delta x, \widehat{t_1^0})\right) \phi(x)\right| \leq 2^{n+1} \sup_{y \in [t_1^0 - 3\rho/2, t_1^0 + 3\rho/2]} \left|g(y, \widehat{t_1^0})\right| |\phi(x)| \qquad (5.4.67)$$

holds for all $(\delta, t_1) \in [1/2, 1] \times [t_1^0 - \rho/2, t_1^0 + \rho/2]$. Notice that ϕ is in particular integrable on $\mathbb{R}$. Therefore, combining (5.4.65) and (5.4.66), the Dominated Convergence Theorem entails that $\mathtt{G}$ is continuous on $[1/2, 1] \times [t_1^0 - \rho/2, t_1^0 + \rho/2]$. Hence is it bounded on this compact set, which implies that (5.4.19) holds for all $\delta \in [1/2, 1]$. □

5.5 Optimality of the behaviour at infinity

In this section, we focus on the case $\alpha \in (0,2)$. Let f be an admissible function, in the sense of Definition 3.1.1 and $X[f]$ be the stochastic field associated to it (see Remark 3.2.11). We know from Corollary 4.2.2 that the exponent $a'[f]$, which controls the behaviour of f in a neighbourhood of 0, provides upper estimates on the behaviour of the amplitude $X[f]$. The main goal of this section is to show the following theorem which can be understood as a counterpart to Corollary 4.2.2.

Theorem 5.5.1. *Assume that f is an admissible function in the sense of Definition 3.1.1 and that the exponent $a'[f]$ in this definition belongs to $(0,1)$. Let also $A \in (0,+\infty)$ and $c \in (0,+\infty)$ be two finite constants such that for all $\xi \in (\mathbb{R}\setminus\{0\})^d$,*

$$\|\xi\| \leq A \quad \Longrightarrow \quad |f(\xi)| \geq c\,\|\xi\|^{-a'[f]-d/\alpha} \tag{5.5.1}$$

where c is a positive and finite constant. Then, there exists an event $\Omega_7^*[f] \subset \Omega_1^*$ of probability 1, which a priori depends on f, such that, for all $\omega \in \Omega_7^*[f]$ and $\delta \in (0,1/\alpha)$, we have that

$$\sup_{\|t\|\geq 1}\left\{\frac{|X[f](t,\omega)|}{\|t\|^{a'[f]}\left(\log\left(3+|t|\right)\right)^{1/\alpha-\delta}}\right\} = +\infty. \tag{5.5.2}$$

Before proving Theorem 5.5.1, we introduce some notations. Let $\widehat{\Theta}$ be the even function defined, for any $\xi \in \mathbb{R}^d$, as

$$\widehat{\Theta}(\xi) := \widehat{\theta}\Big(\|\xi\|\Big), \tag{5.5.3}$$

where θ is the inverse Fourier transform of the function in (5.2.7). We recall that θ is a real-valued non-zero function in the Schwartz class $\mathcal{S}(\mathbb{R})$ such that $\widehat{\theta}$ is real-valued, even, compactly supported and satisfies (5.2.6). Moreover, the function $\|\cdot\|$ is infinitely differentiable on $\mathbb{R}^d\setminus\{0\}$. Therefore, the function $\widehat{\Theta}$ is infinitely differentiable on $\mathbb{R}^d$ and its support satisfies

$$\operatorname{supp}\widehat{\Theta} = \Big\{\xi \in \mathbb{R}^d : 1 \leq \|\xi\| \leq 2\Big\}. \tag{5.5.4}$$

Hence, in particular $\widehat{\Theta}$ belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^d)$. Therefore, its inverse Fourier transform, denoted by Θ[2], also belongs to $\mathcal{S}(\mathbb{R}^d)$. Next, for any $n \in \mathbb{N}$, we let δ_n be the random variable defined as

$$\delta_n := \int_{\mathbb{R}^d}\Big(X[f]\Big(2^n\overrightarrow{1}+s\Big) - X[f](s)\Big)\Theta_n(s)\mathrm{d}s, \tag{5.5.5}$$

[2]Namely, the function given, for any $x \in \mathbb{R}^d$, by

$$\Theta(x) := (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\widehat{\Theta}(\xi)\,\mathrm{d}\xi.$$

where $\overrightarrow{1}$ is the vector of $\mathbb{R}^d$ whose all coordinates equal 1. Notice that

$$\delta_n = P_{\Theta_n} X[f]\Big(2^n \overrightarrow{1}\Big), \tag{5.5.6}$$

Applying Proposition 5.1.4 to $P_{\Theta_n} X[f]$ and using the equality $\widehat{\Theta_n} = \widehat{\Theta}(2^n \cdot)$ we obtain that

$$\delta_n = X\Big[f\widehat{\Theta}(2^n \cdot)\Big]\Big(2^n \overrightarrow{1}\Big). \tag{5.5.7}$$

So, in view of Remark 3.2.11, δ_n is a real-valued symmetric α-stable random variable with a scale parameter $\mathfrak{s}_n$ satisfying the following equality:

$$\mathfrak{s}_n^\alpha = \int_{\mathbb{R}^d} \left| e^{i2^n\left(\overrightarrow{1}\cdot\xi\right)} - 1\right|^\alpha \left|\widehat{\Theta}(2^n\xi)\right|^\alpha |f(\xi)|^\alpha \,\mathrm{d}\xi. \tag{5.5.8}$$

We are now in the position to prove Theorem 5.5.1

Proof of Theorem 5.5.1. We divide the proof into three steps.

Step 1: We show that the random variables $\{\delta_n, n \in \mathbb{N}\}$ are independent.
Let $M \in \mathbb{N}$ and let $m_1, \dots, m_M \in \mathbb{N}$ be such that $m_k \neq m_l$ when $k \neq l$. Similarly to the proof of Lemma 5.2.5, in order to show that the random variables

$$\delta_{m_1}, \dots, \delta_{m_M}$$

are independent, it is enough to show that, for every $b_1, \dots, b_M \in \mathbb{R}$, we have

$$\mathbb{E}\left[\exp\left\{i\sum_{j=1}^M b_j\delta_{m_j}\right\}\right] = \prod_{j=1}^M \mathbb{E}\left[\exp\left\{ib_j\delta_{m_j}\right\}\right]. \tag{5.5.9}$$

Notice that, in view of (5.2.12) and of the linearity of the stochastic stable integral $\int_{\mathbb{R}^d}(\,\cdot\,)\,\mathrm{d}\widetilde{M_\alpha}$, we have, almost surely, that

$$\sum_{j=1}^M b_j\delta_{m_j} = \mathcal{R}e\left\{\int_{\mathbb{R}^d}\sum_{j=1}^M b_j\left(e^{i2^{m_j}\left(\overrightarrow{1}\cdot\xi\right)} - 1\right)\widehat{\Theta}(2^{m_j}\xi)f(\xi)\,\mathrm{d}\widetilde{M_\alpha}(\xi)\right\}. \tag{5.5.10}$$

Therefore, the real-valued random variable $\sum_{j=1}^M b_j\delta_{m_j}$ has a symmetric α-stable distribution. Definition 1.1.5 and (2.1.17) imply that its characteristic function satisfies

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(i\mathcal{R}e\left\{\int_{\mathbb{R}^d}\sum_{j=1}^M b_j\left(e^{i2^{m_j}\left(\overrightarrow{1}\cdot\xi\right)} - 1\right)\widehat{\Theta}(2^{m_j}\xi)f(\xi)\,\mathrm{d}\widetilde{M_\alpha}(\xi)\right\}\right)\right]\\
&= \exp\left(-\int_{\mathbb{R}^d}\left|\sum_{j=1}^M b_j\left(e^{i2^{m_j}\left(\overrightarrow{1}\cdot\xi\right)} - 1\right)\widehat{\Theta}(2^{m_j}\xi)\right|^\alpha |f(\xi)|^\alpha\,\mathrm{d}\xi\right).
\end{aligned} \tag{5.5.11}$$

Observe that for any positive integer $m' \neq m''$, the Lebesgue measure of the set

$$\operatorname{supp}\widehat{\Theta}(2^{m'}\cdot)\cap\operatorname{supp}\widehat{\Theta}(2^{m''}\cdot)$$

is equal to 0. Therefore, putting together (5.5.10) and (5.5.11) we have that

$$\begin{aligned}\mathbb{E}\Bigg[\exp\Big\{i\sum_{j=1}^{M} b_j\delta_{m_j}\Big\}\Bigg] &= \exp\left(-\int_{\mathbb{R}^d}\sum_{j=1}^{M}\left|b_j\left(e^{i2^{m_j}\left(\overrightarrow{1}\cdot\xi\right)}-1\right)\widehat{\Theta}(2^{m_j}\xi)\right|^{\alpha}|f(\xi)|^{\alpha}\,\mathrm{d}\xi\right)\\ &= \prod_{j=1}^{M}\exp\left(-\int_{\mathbb{R}^d}\left|b_j\left(e^{i2^{m_j}\left(\overrightarrow{1}\cdot\xi\right)}-1\right)\widehat{\Theta}(2^{m_j}\xi)\right|^{\alpha}|f(\xi)|^{\alpha}\,\mathrm{d}\xi\right).\end{aligned} \tag{5.5.12}$$

Hence, combining (5.5.12), (5.5.7), (2.1.17) and Definition 1.1.5, we obtain (5.5.9).

Step 2: Let $\varepsilon \in (0, 1/\alpha)$ be arbitrarily. We show that for some constant $c_0 \in (0, +\infty)$, on an event $\Omega_8^*[f](\varepsilon)$ of probability 1, we have

$$\liminf_{m\to+\infty}\max_{n\in\{0,1\dots,m\}}\left\{\left(m^{1/\alpha-\varepsilon}2^{(n+m)a'[f]}\right)^{-1}|\delta_{n+m}|\right\} > c_0. \tag{5.5.13}$$

Using (5.5.8), the change of variable $\eta = 2^m\xi$, (5.5.4) and (5.5.1), we obtain, for any positive integer $m > \log(2/A)/\log(2)$,

$$\begin{aligned}\mathfrak{s}_m^{\alpha} &= 2^{-md}\int_{1\le\|\eta\|\le 2}\left|e^{i\left(\overrightarrow{1}\cdot\eta\right)}-1\right|^{\alpha}\left|\widehat{\Theta}(\eta)\right|^{\alpha}\left|f(2^{-m}\eta)\right|^{\alpha}\,\mathrm{d}\eta\\ &\ge c^{\alpha}2^{-md}\int_{1\le\|\eta\|\le 2}\left|e^{i\left(\overrightarrow{1}\cdot\eta\right)}-1\right|^{\alpha}\left|\widehat{\Theta}(\eta)\right|^{\alpha}\left\|2^{-m}\eta\right\|^{-a'[f]\alpha-d}\,\mathrm{d}\eta\\ &= c_2 2^{ma'[f]\alpha},\end{aligned} \tag{5.5.14}$$

where $c_2 := c^{\alpha}\left(\int_{1\le\|\eta\|\le 2}\left|e^{i\left(\overrightarrow{1}\cdot\eta\right)}-1\right|^{\alpha}\left|\widehat{\Theta}(\eta)\right|^{\alpha}\|\eta\|^{-a'[f]\alpha-d}\,\mathrm{d}\eta\right)$. The function $\widehat{\Theta}$ belongs to $\mathcal{S}(\mathbb{R}^d)$; so c_2 is finite. More importantly, one has that $c_2 \neq 0$ since $\widehat{\Theta}$ does not vanish everywhere.

Next, in view of *Step 1*, (5.5.7) and (5.5.8), the random variables $\delta_n/\mathfrak{s}_n$, $n \in \mathbb{N}$, are independent symmetric α-stable random variables of scale parameter equal to 1. Therefore, for all positive integer $m > c_1(\alpha)^{-1/(1-\varepsilon\alpha)}$, using (5.3.8) and (5.3.11), we have that

$$\begin{aligned}\mathbb{P}\left(\max_{n=0,\dots,m}\frac{|\delta_{n+m}|}{\mathfrak{s}_{n+m}}\le c_1(\alpha)^{1/\alpha}m^{1/\alpha-\varepsilon}\right) &= \mathbb{P}\left(|U|\le c_1(\alpha)^{1/\alpha}m^{1/\alpha-\varepsilon}\right)^{m}\\ &\le \left(1-m^{-1+\alpha\varepsilon}\right)^{m}\\ &= \exp\left(m\log\left(1-m^{-1+\alpha\varepsilon}\right)\right)\\ &\le \exp\left(-m^{\varepsilon\alpha}\right)\\ &= \mathop{\mathcal{O}}_{m\to+\infty}\left(\frac{1}{m^2}\right),\end{aligned} \tag{5.5.15}$$

where U denotes an arbitrary real-valued symmetric α-stable random variable of scale parameter 1. Hence, the series of general term

$$\mathbb{P}\left(\max_{n=0,\dots,m}\frac{|\delta_{n+m}|}{\mathfrak{s}_{n+m}}\leq c_1(\alpha)^{1/\alpha}m^{1/\alpha-\varepsilon}\right)$$

converges. Therefore, the Borel-Cantelli Lemma entails that the probability of the event

$$\bigcup_{M\in\mathbb{N}}\bigcap_{m\geq M}\left\{\max_{n=0,\dots,m}\frac{|\delta_{n+m}|}{\mathfrak{s}_{n+m}}>c_1(\alpha)^{1/\alpha}m^{1/\alpha-\varepsilon}\right\} \tag{5.5.16}$$

is equal to 1. Thus, one can derive from (5.5.14), that (5.5.13) holds on this event where $c_0=c_2c_1(\alpha)^{1/\alpha}$.

Step 3: We define the event $\Omega_9^*[f]$ of probability 1 by

$$\Omega_9^*[f]:=\bigcap_{\varepsilon\in(0,1/\alpha)\cap\mathbb{Q}}\Omega_8^*[f](\varepsilon)\cap\Omega_1^*$$

Assume *ad absurdum* that there exists $\omega\in\Omega_9^*[f]$ and $\delta\in(0,1/\alpha)$ such that

$$\sup_{\|t\|\geq 1}\left\{\frac{|X[f](t,\omega)|}{\|t\|^{a'[f]}\left(\log\left(3+\|t\|\right)\right)^{1/\alpha-\delta}}\right\}<+\infty. \tag{5.5.17}$$

Therefore, in view of Theorem 3.2.19, for some finite constant $C_2(\omega)$, the inequality

$$|X[f](t,\omega)|\leq C_2(\omega)\Big(1+\|t\|^{a'[f]}\Big)\left(\log\left(3+\|t\|\right)\right)^{1/\alpha-\delta} \tag{5.5.18}$$

holds for every $t\in\mathbb{R}^d$. Hence, using (5.5.5), the change of variables $t=2^{-n}s$, (5.5.18) and (3.2.38), we get that

$$\begin{aligned}|\delta_n(\omega)| &\leq \int_{\mathbb{R}^d}\Big(\Big|X[f]\Big(2^n(\overrightarrow{1}+t)\Big)\Big|+|X[f](2^nt)|\Big)|\Theta(t)|\,\mathrm{d}t\\ &\leq C_3(\omega)2^{na'[f]}n^{1/\alpha-\delta}\int_{\mathbb{R}^d}\Big(1+\|t\|^{a'[f]}\Big)\left(\log\left(3+\|t\|\right)\right)^{1/\alpha-\delta}|\Theta(t)|\,\mathrm{d}t. \end{aligned}\tag{5.5.19}$$

The non-zero function Θ belongs to $\mathcal{S}(\mathbb{R}^d)$, therefore the positive constant

$$C_4(\omega):=C_3(\omega)\int_{\mathbb{R}^d}\Big(1+\|t\|^{a'[f]}\Big)\left(\log\left(3+\|t\|\right)\right)^{1/\alpha-\delta}|\Theta(t)|\,\mathrm{d}t$$

is finite. Next we assume that $\varepsilon\in(0,\delta)\cap\mathbb{Q}$ is arbitrary and fixed. The inequality (5.5.13) entails that there exists $M(\omega)\in\mathbb{N}$ such that for any $m\geq M(\omega)$, there is $n_m\in\{0,\dots,m\}$ such that

$$\delta_{n+m}(\omega)>c_0m^{1/\alpha-\varepsilon}2^{(n_m+m)a'[f]}. \tag{5.5.20}$$

Putting together (5.5.19) and (5.5.20), we get, for every $m \geq M(\omega)$

$$c_0 m^{1/\alpha-\varepsilon} \leq C_4(\omega)(n_m + m)^{1/\alpha-\delta} \leq 2^{1/\alpha-\delta} C_4(\omega) m^{1/\alpha-\delta}. \tag{5.5.21}$$

This implies that, for any integer $m \geq M(\omega)$, one has

$$C_5(\omega) \leq m^{-(\delta-\varepsilon)}, \tag{5.5.22}$$

where $C_5(\omega)$ is a positive and finite constant which does not depend on m. Since $\varepsilon < \delta$, (5.5.22) leads to a contradiction. $\square$

Projection of the harmonizable fractional stable field

It is natural to believe that the "projection" of a fractal set is "more regular" than the set itself. In the case of the graph of the Gaussian fractional Brownian field $B_H := \left\{B_H(t),\, t \in \mathbb{R}^d\right\}$, with $d \geq 2$, this turns out to be true when one considers a "φ-weighted projection" [20, 9]. In order to be more precise one needs the following two definitions.

Definition 6.1. *Assume that φ is a continuous compactly supported function from $\mathbb{R}^{d-1}$ to $\mathbb{R}$ such that*

$$\int_{\mathbb{R}^{d-1}} \varphi(s)\,\mathrm{d}s = 1. \tag{6.1.1}$$

Also, assume that $X = \left\{X(t),\, t \in \mathbb{R}^d\right\}$ is a stochastic field with almost surely continuous sample paths. The φ-weighted projection of X is the process $\left\{p(X,\varphi)(x),\, x \in \mathbb{R}\right\}$ defined, for any $x \in \mathbb{R}$, by

$$p(X,\varphi)(x) := \int_{\mathbb{R}^{d-1}} X(x,s)\varphi(s)\,\mathrm{d}s. \tag{6.1.2}$$

Notice that the notion of φ-weighted projection has been introduced in [20, 9].

Definition 6.2. *The critical local Hölder exponent β_X of a stochastic field $X := \{X(t), t \in \mathbb{R}^d\}$ is the non-negative real-number defined as*

$$\begin{aligned} \beta_X \;:=\; \sup\Big\{\beta \in (0,+\infty) : & \text{ the field } X \text{ has a modification which is} \\ & \text{almost surely locally Hölder continuous of order } \beta\Big\} \end{aligned} \tag{6.1.3}$$

We recall that the notion of locally Hölder continuous function of order β has been introduced in Definition 1.3.1.

The following result shows that the graph of $p(B_H,\varphi)$ is more regular than that of B_H in the sense of local Hölder exponent.

Theorem 6.3. *Let β_{B_H} be the critical local Hölder exponent of the fractional Brownian field B_H. Let also $\beta_{p(B_H,\varphi)}$ be the critical local Hölder exponent of the projection $p(B_H,\varphi)$ of B_H. Then one has*

$$\beta_{p(B_H,\varphi)} = \beta_{B_H} + (d-1)/2. \tag{6.1.4}$$

We mention that Theorem 6.3 was first obtained in [20] for the fractional Brownian field over $\mathbb{R}^2$ (that is $d = 2$). This theorem was extended to more general classes of Gaussian fields in [9]; but no extension of it is known, so far, in the frame of stable fields with heavy-tailed distributions. An important consequence of the results of the thesis is that Theorem 6.3 can be extended to this frame. More precisely, we recall that the harmonizable fractional stable field $X^{\text{hfsf}} := \left\{X^{\text{hfsf}}(t),\, t \in \mathbb{R}^d\right\}$ of Hurst parameter $H \in (0,1)$ is defined, for all $t \in \mathbb{R}^d$, as

$$X^{\text{hfsf}}(t) := \int_{\mathbb{R}^d} \left(e^{it\cdot\xi} - 1\right) \|\xi\|^{-H-d/\alpha}\, \mathrm{d}\widetilde{M}_\alpha(\xi). \tag{6.1.5}$$

Observe that, in view of Remark 3.1.2 (with $u = M$ and $v_1 = \cdots = v_d = 0$), the real-valued function $\xi \mapsto \|\xi\|^{-H-d/\alpha}$ on $\mathbb{R}^d$ is admissible in the sense of Definition 3.1.1. Notice that the fractional Brownian field B_H is nothing else than the harmonizable fractional stable field X^{hfsf} in the particular Gaussian case (that is $\alpha = 2$). Also, we recall that the critical pathwise Hölder regularity of X^{hfsf} is equal to the Hurst parameter H [5]; that is

$$\beta_{X^{\text{hfsf}}} = H. \tag{6.1.6}$$

Let us emphasize that Corollary 4.1.3 and Theorem 5.2.1 in the thesis allows to extend Theorem 6.3 to the harmonizable fractional stable field X^{hfsf} with heavy-tailed; more precisely:

Theorem 6.4. *Let φ be a real-valued, compactly supported, continuous function defined on $\mathbb{R}$ satisfying* (6.1.1)*. Assume that there exist a positive exponent M and a positive constant B satisfying the following property: for each $p \in \left\{0,1,2,\dots,p_*\right\}^{d-1}$ (see* (3.1.1)*) the inequality*

$$|\partial^p \widehat{\varphi}(\eta)|^\alpha \le B\, |\eta|^{-\alpha M-(d-1)-\alpha \mathrm{l}(p)}, \tag{6.1.7}$$

holds for all $\eta \in \mathbb{R}^{d-1}$, where $\partial^p\widehat{\varphi}$ is the partial derivative function of order p of the Fourier transform $\widehat{\varphi}$ of φ. Then, Theorem 6.3 remains true in the general case where the stability parameter $\alpha \in (0,2]$ is arbitrary and the fractional Brownian field B_H is replaced by the harmonizable fractional stable field X^{hfsf} introduced in (6.1.5)*. Yet, $(d-1)/2$ has to be replaced by $(d-1)/\alpha$. More precisely,*

$$\begin{pmatrix} \textit{Critical local Hölder regularity of} \\ \textit{the } \varphi\textit{-weighted harmonizable} \\ \textit{projection } p(X^{hfsf},\varphi) \end{pmatrix} = \begin{pmatrix} \textit{Critical local Hölder regularity of} \\ \textit{the harmonizable fractionnal} \\ \textit{stable field } X^{hfsf} \end{pmatrix} + \frac{d-1}{\alpha}. \tag{6.1.8}$$

Proof. In view of (5.1.16) and 5.1.15 (with $q = d-1$, $i_1 = 2, \dots, i_{d-1} = d$) notice that, for every $x \in \mathbb{R}$,

$$p(X^{\text{hfsf}},\varphi)(x) = P_\varphi X^{\text{hfsf}}(x,0,\dots,0) + P_\varphi X^{\text{hfsf}}(0,0,\dots,0). \tag{6.1.9}$$

It is worth pointing out that $p(X^{\text{hfsf}},\varphi)$ is a process, whereas $P_\varphi X^{\text{hfsf}}$ is a stochastic field over $\mathbb{R}^d$. Using (6.1.7) and the fact that φ is a real-valued, compactly supported, continuous function defined on $\mathbb{R}^{d-1}$ one can show that $\widehat{\varphi}$ satisfies the same hypothesis as ϕ in Lemma 5.1.3 (with $b_1=\cdots=b_{d-1}=M/(d-1)+1/\alpha$). Moreover, in view of Corollary 4.2.14 and the fact that the function φ is continuous and compactly supported, we get that (5.1.28) holds. Thus, Proposition 5.1.4 can be applied to the field X^{hfsf} and to the function φ. That is, for any $\omega \in \Omega_1^*$ and $t\in\mathbb{R}^d$, we have that

$$P_\varphi X^{\text{hfsf}}(t,\omega) = X[g_\varphi](t,\omega), \tag{6.1.10}$$

where, the admissible function g_φ (see Definition 3.1.1 and Lemma 5.1.3) is defined, for every $\xi=(\xi_1,\ldots,\xi_d)\in\mathbb{R}^d\setminus\{0\}$, as

$$g_\varphi := \|\xi\|^{-H-d/\alpha}\,\overline{\widehat{\varphi}(\xi_2,\ldots,\xi_d)}. \tag{6.1.11}$$

Moreover, in view of (6.1.11) and (6.1.7), the exponents $a'[g_\varphi], a_1[g_\varphi],\ldots,a_d[g_\varphi]$ in $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ in Definition 3.1.1 can be chosen as follows:

$$a'[g_\varphi]=H \quad\text{and}\quad a_1[g_\varphi]=H+(d-1)/\alpha \tag{6.1.12}$$

and

$$a_l[g_\varphi]=M/(d-1), \quad \text{for all } l=2,\ldots,d. \tag{6.1.13}$$

At this point, we are in the position to apply Theorem 3.2.19 and Corollary 4.1.3 to the field $X[g_\varphi]$: in view of (6.1.10), for any $\omega\in\Omega_1^*$ and non-negative integers $p<H+(d-1)/\alpha$, the partial derivative function $\partial^{pe_1}\Big(P_\varphi X^{\text{hfsf}}\Big)(\cdot,\omega)$ exists and is continuous on $\mathbb{R}^d$, where e_1 denotes the vector of $\mathbb{R}^d$ whose 1-th coordinate equals 1 and the others vanish. Moreover, for all integer $n\in[H+(d-1)/\alpha,+\infty)$, $\omega\in\Omega_1^*$, $T\in(0,+\infty)$ and $\delta\in(0,1/\alpha)$,

$$\sup_{h_1\in[-T,T]}\left\{\frac{\left\|\Delta_{h_1}^{1,n}\Big(P_\varphi X^{\text{hfsf}}\Big)(\cdot,\omega)\right\|_{T,\infty}}{|h_1|^{H+(d-1)/\alpha}\Big(\log\Big(3+|h_1|^{-1}\Big)\Big)^{1/\alpha+\lfloor\alpha\rfloor/2+\delta+\mathbb{1}_{\{n=H+(d-1)/\alpha\}}}}\right\}<+\infty, \tag{6.1.14}$$

where the operator $\Delta_{h_1}^{1,n}$ is defined in (4.1.1) and $\lfloor\alpha\rfloor$ is the integer part of α. In particular, in view of (6.1.9) and Theorem 5.2.9, the sample paths of the process $p(X^{\text{hfsf}},\varphi)$ are almost surely locally Hölder continuous of any order $\gamma\in\Big(0,H+(d-1)/\alpha\Big)$ in the sense of Definition 1.3.1. That is, in view of Definition 6.2, we have that

$$\beta_{p(X^{\text{hfsf}},\varphi)} \geq H+(d-1)/\alpha. \tag{6.1.15}$$

On the other hand, in order to apply Theorem 5.2.2, we will show that, for some positive and finite constant c, the inequality

$$\int_{\mathbb{R}}\Big|\|(\lambda_1,\xi_2,\ldots,\xi_d)\|^{-H-d/\alpha}\,\widehat{\varphi}(\xi_2,\ldots,\xi_d)\Big|^\alpha \mathrm{d}\xi_2\ldots\mathrm{d}\xi_d \geq c\,|\lambda_1|^{-(H\alpha+d-1)-1} \tag{6.1.16}$$

holds for every $|\lambda_1| \geq 1$. Notice that, for all $|\lambda_1| \geq 1$, we have that

$$\begin{aligned}\|(\lambda_1, \xi_2, \ldots, \xi_d)\|^{-H-d/\alpha} &= |\lambda_1|^{-H-d/(2\alpha)} \left(1 + \frac{\xi_2^2}{\lambda_1^2} + \cdots + \frac{\xi_d^2}{\lambda_1^2}\right)^{-H/2-d/(2\alpha)} \\ &\geq |\lambda_1|^{-(H+(d-1)/\alpha)-1/\alpha} \left(1 + \xi_2^2 + \cdots + \xi_d^2\right)^{-H/2-d/(2\alpha)}. \quad (6.1.17)\end{aligned}$$

Therefore, (6.1.16) holds where the constant c is equal to

$$c := \int_{\mathbb{R}} \left|1 + \xi_2^2 + \cdots + \xi_d^2\right|^{-H\alpha/2-d/2} |\widehat{\varphi}(\xi_2, \ldots, \xi_d)|^{\alpha} \, \mathrm{d}\xi_2 \ldots \mathrm{d}\xi_d. \quad (6.1.18)$$

Notice that the non-zero function $\widehat{\varphi}$ satisfies (6.1.7) with $p = 0$, therefore the constant c is finite and positive. Finally, in view of (6.1.10) and Theorem 5.2.1, we get that for every integer $n \in (H + (d-1)/\alpha, +\infty)$, $\omega \in \Omega_2^*[g_\varphi]$, $\rho \in (0, +\infty)$ and $\delta \in (0, 1/\alpha)$, one has

$$\inf_{(t_1,t_2,\ldots,t_d) \in \mathbb{R}\times\mathbb{R}^{d-1}} \sup_{t_1' \in [t_1-\rho, t_1+\rho]} \sup_{h_1 \in [-\rho,\rho]} \left\{ \frac{\left|\Delta_{h_1}^{1,n}\left(P_\varphi X^{\mathrm{hfsf}}\right)\left(t_1', t_2, \ldots, t_d, \omega\right)\right|}{|h_1|^{H+(d-1)/\alpha} \left(\log\left(3 + |h_1|^{-1}\right)\right)^{1/\alpha-\delta-\mathbb{1}_{\{H+(d-1)/\alpha \in \mathbb{N}\}}}} \right\} = +\infty. \quad (6.1.19)$$

Hence, in view of (6.1.9) and the Mean Value Theorem, we have that

$$\beta_{p(X^{\mathrm{hfsf}},\varphi)} \leq H + (d-1)/\alpha. \quad (6.1.20)$$

Therefore, in view of (6.1.15), (6.1.20) and (6.1.6) we get that

$$\beta_{p(X^{\mathrm{hfsf}},\varphi)} = \beta_{X^{\mathrm{hfsf}}} + \frac{d-1}{\alpha}.$$

In other words, we have that (6.1.8) holds. □

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