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Pre-Lie Algebras and Operads in Positive Characteristic

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Contents

Introduction

The main purpose of this thesis is to study the categories of algebras over operads in the context of a category of modules defined over a field of positive characteristic. Several well known theorems of algebraic operads which are valid over the fields of characteristic 0 fail to be true in this more general setting.

Let $\mathbb{K}$ be the ground ring of our category of modules. Briefly recall that an operad P consists of a collection $\{P(n)\}_{n\in\mathbb{N}}$ where $P(n)$ is a $\mathbb{K}$ module equipped with an action of the symmetric group on n-letters $\mathbb{S}_n$, together with composition products which model the composition schemes of abstract operations. The standard category of algebras associated to an operad P is governed by a monad on the category of $\mathbb{K}$-modules denoted by $S(P,-)$. This monad $S(P,-)$ is given by a generalization of the classical construction of the symmetric algebra. We explicitly have :

$$S(P,V) = \bigoplus_{n\in\mathbb{N}} P(n) \otimes_{\mathbb{S}_n} V^{\otimes n},$$

for every $\mathbb{K}$-module V, where $P(n)\otimes_{\mathbb{S}_n} V^{\otimes n}$ denotes the coinvariant quotient of the tensor product of the component $P(n)$ of our operad P and the tensor power $V^{\otimes n}$ under the diagonal action of the symmetric group $\mathbb{S}_n$. The composition products of an operad actually reflect the composition product associated to a monad of this shape. The classical categories of algebras, like notably the category of commutative algebras, and the category of Lie algebras, are associated to operad.

In [Fre00] B.Fresse observe that we can associate another monad $\Gamma(P,-)$ to any operad P by replacing the coinvariants in the definition of $S(P,-)$ by invariants. We explicitly have :

$$\Gamma(P,V) = \bigoplus_{n\in\mathbb{N}} P(n) \otimes^{\mathbb{S}_n} V^{\otimes n},$$

for every $\mathbb{K}$-module V, where we use the notation $\otimes^{\mathbb{S}_n}$ for this invariant construction.

Two important examples of algebraic structures come from this construction. The category of divided power algebras is governed by $\Gamma(Com,-)$, where Com is the operad of commutative algebras. The category of p-restricted Lie algebras is governed by $\Gamma(Lie,-)$, where Lie is the operad of Lie algebras. The monads $S(P,-)$ and $\Gamma(P,-)$ are related by a natural transformation of monads $trace : S(P,-) \to \Gamma(P,-)$. The epi-mono factorization of this trace map defines a third interesting monad denoted by $\Lambda(P,-)$. These three monads coincide when the commutative ring $\mathbb{K}$ contains $\mathbb{Q}$. But in general they are different.

For a given operad P, we have no general method to obtain a description of the structure of an algebra over the monads $\Lambda(P,-)$ and $\Gamma(P,-)$ in terms of generating operations and relations. The first goal of my thesis is to find such presentations for a significant example of operad, $PreLie$, which is associated to a category of algebras called pre-Lie algebras. Pre-Lie algebras were introduced by M. Gerstenhaber [Ger63] in the deformation theory of associative algebras. A pre-Lie algebra explicitly consists of a $\mathbb{K}$-module V equipped with a bilinear product $\{-,-\}$ such that we have the relation :

$$\{\{x,y\},z\} - \{x,\{y,z\}\} = \{\{x,z\},y\} - \{x,\{z,y\}\},$$

for every $x,y,z \in V$. A pre-Lie algebra inherits a Lie bracket which is given by the commutator of the pre-Lie product :

$$[x,y] = \{x,y\} - \{y,x\},$$

for every $x,y \in V$. In [CL01] F. Chapoton and M. Livernet prove that the category of pre-Lie algebras is associated to an operad that has an explicit description in terms of rooted trees.

Examples of pre-Lie algebras notably appear in deformation theory of algebraic structures (see [DSV15]), in operad theory (see Section 5.4.6 [LV12]), and in renormalization theory for quantum field theories (see [CK99]).

The main results of this thesis on pre-Lie algebras are explained in Chapter 1. First we show that over a field of characteristic $p > 0$ the category of $\Lambda(PreLie, -)$ algebras is isomorphic to the category of p-restricted pre-Lie algebras introduced by A. Dzhumadil'daev in [Dzh01]. Explicitly :

Theorem A (Chapter 1, Theorem 1.4.16). *We assume that the ground ring $\mathbb{K}$ is a field of characteristic p. A $\Lambda PreLie$-algebra is equivalent to a $\mathbb{K}$-module V equipped with an operation $\{-,-\} : V \otimes V \longrightarrow V$ satisfying the $PreLie$-relation and the following p-restricted $PreLie$-algebra relation :*

$$\{\{\ldots\{\{x,\underbrace{y\},y\}\ldots\}y}_{p}\} = \{x,\{\ldots\{\underbrace{\{y,y\}\ldots\}y}_{p}\}\},$$

for every $x, y \in V$.

Then we give a presentation by generating operations and relations of the structure of an algebra over the monad $\Gamma(PreLie, -)$ which is valid over any commutative ring :

Theorem B (Chapter 1, Theorem 1.5.19). *If V is a free module over the ground ring $\mathbb{K}$, then providing the module V with a $\Gamma PreLie$-algebra structure is equivalent to providing V with a collection of polynomial maps*

$$\{-;\underbrace{-,\ldots,-}_{n}\}_{r_1,\ldots,r_n} : V \times \underbrace{V \times \ldots \times V}_{n} \longrightarrow V,$$

for all $n \in \mathbb{N}$, where $r_1, \ldots, r_n \in \mathbb{N}$ and which are linear in the first variable and such that the following relations hold :

$$\{x; y_1, \ldots, y_n\}_{r_{\sigma(1)},\ldots,r_{\sigma(n)}} = \{x; y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(n)}\}_{r_1,\ldots,r_n}, \tag{1}$$

for any $\sigma \in \mathbb{S}_n$;

$$\{x; y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_n\}_{r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_n} = \{x; y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n\}_{r_1,\ldots,r_{i-1},r_{i+1},\ldots,r_n}, \tag{2}$$

$$\{x; y_1, \ldots, \lambda y_i, \ldots, y_n\}_{r_1,\ldots,r_i,\ldots,r_n} = \lambda^{r_i}\{x; y_1, \ldots, y_i, \ldots, y_n\}_{r_1,\ldots,r_i,\ldots,r_n}, \tag{3}$$

for any λ in $\mathbb{K}$;

if $y_i = y_{i+1}$

$$\{x; y_1, \ldots, y_i, y_{i+1}, \ldots, y_n\}_{r_1,\ldots,r_i,r_{i+1},\ldots,r_n} = \binom{r_i + r_{i+1}}{r_i}\{x; y_1, \ldots, y_i, y_{i+2}, \ldots, y_n\}_{r_1,\ldots,r_i+r_{i+1},r_{i+2},\ldots,r_n}. \tag{4}$$

$$\{x; y_1, \ldots, y_{i-1}, a+b, y_{i+1}, \ldots, y_n\}_{r_1,\ldots,r_i,\ldots,r_n} = \sum_{s=0}^{r_i}\{x; y_1, \ldots, a, b, \ldots, y_n\}_{r_1,\ldots,s,r_i-s,\ldots,r_n}, \tag{5}$$

$$\{-;\} = id, \tag{6}$$

$$\begin{aligned}\{\{x;y_1,\ldots,y_n\}_{r_1,\ldots,r_n};z_1,\ldots,z_m\}_{s_1,\ldots,s_n} = \\ \sum_{s_i=\beta_i+\sum\alpha_i^{\cdot,\cdot}} \frac{1}{\prod(r_j!)}\{x;\{y_1;z_1,\ldots,z_m\}_{\alpha_1^{1,1},\ldots,\alpha_m^{1,1}},\ldots,\{y_1;z_1,\ldots,z_m\}_{\alpha_1^{1,r_1},\ldots,\alpha_m^{1,r_1}}, \\ \ldots,\{y_n;z_1,\ldots,z_m\}_{\alpha_1^{n,1},\ldots,\alpha_m^{n,1}},\ldots,\{y_n;z_1,\ldots,z_m\}_{\alpha_1^{n,r_n},\ldots,\alpha_m^{n,r_n}}, \\ z_1,\ldots,z_m\}_{1,\ldots,1,\beta_1,\ldots,\beta_m},\end{aligned} \tag{7}$$

where, to give a sense to the latter formula, we use that the denominators $r_j!$ divide the coefficient of the terms of the reduced expression which we get by applying relations (1) and (4) to simplify terms with repeated inputs on the right hand side (see Example 1.5.11).

It turns out that some important examples of pre-Lie algebra have such a $\Gamma(PreLie,-)$ algebra structure. For example the $\mathbb{K}$-module $\bigoplus_n P(n)$ associated to an operad P is a $\Gamma(PreLie,-)$ algebra.

Let P be an operad. Let V be a $\mathbb{K}$-module. By a classical statement of the theory of operads, providing V with the structure of an $S(P,-)$-algebra amounts to giving an operad morphism $\phi : P \to End_V$, where End_V is a universal operad associated to V (the endomorphism operad of V). But we do not have an analogue of this universal operad for the study of $\Gamma(P,-)$-algebra structures, at least if we only consider operads in the classical sense. In Chapter 2, we explain how to define a suitable generalisation of the notion of an operad in order to work out this problem.

The functors $S_n(P,V) = P(n) \otimes_{\mathbb{S}_n} V^{\otimes n}$ and $\Gamma_n(P,V) = P(n) \otimes^{\mathbb{S}_n} V^{\otimes n}$ which define the summands of the monads $S(P,-)$ and $\Gamma(P,-)$ associated to an operad P are examples of (strict) polynomial functors of degree n in the sense of Friedlander-Suslin.

In a first step we explain the definition of a category of (cohomological) Mackey functors, which generalize the $\mathbb{S}_n$-modules considered in the definition of an operad, to get a combinatorial "model" of the category of strict polynomial funtors of degree n. To give an idea of our definition, we consider the collection Par_n formed by the subgroups of the symmetric group $\mathbb{S}_n$ which are conjugate to a group of the form $\mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_r} \leq \mathbb{S}_n$ where $i_1 + \cdots + i_r = n$. The cohomological Mackey functors which we consider can be defined by giving a collection of $\mathbb{K}$-modules $M(\pi)$, where $\pi \in Par_n$, together with induction morphisms $\mathrm{Ind}_{\pi_1}^{\pi_2} : M(\pi_1) \to M(\pi_2)$ and restriction morphisms $\mathrm{Res}_{\pi_1}^{\pi_2} : M(\pi_2) \to M(\pi_1)$ for each pair of subgroup $\pi_1, \pi_2 \in Par_n$ such that $\pi_1 \leq \pi_2$, and conjugation operations $c_\sigma : M(\pi) \to M(\pi^\sigma)$ for each $\sigma \in \mathbb{S}_n$, where π^σ denotes the conjugate subgroup of π in $\mathbb{S}_n$ under the action of σ. We suppose that these operations satisfy natural relations. We notably assume $\mathrm{Ind}_{\pi_1}^{\pi_2} \mathrm{Res}_{\pi_1}^{\pi_2} = [\pi_2 : \pi_1] \mathrm{Id}_{\pi_2}$ in our category of cohomological Mackey functors.

We denote the category of strict polynomial functors of degree n by $PolFun_n$, and the category of cohomological Mackey functors on Par_n by $\mathrm{Mac}^{coh}(\mathcal{H}Par_n)$. We associate a strict polynomial functor $ev(M)_n$ of degree n to every object $M \in \mathrm{Mac}^{coh}(\mathcal{H}Par_n)$ and we prove that :

Theorem C (Chapter 2, Theorem 2.2.18)**.** *Our mapping $ev_n\colon \mathrm{Mac}^{coh}(\mathcal{H}Par_n) \to PolFun_n$ defines an equivalence of categories from $\mathrm{Mac}^{coh}(\mathcal{H}Par_n)$ the category of cohomological Mackey functors on Par_n to the category $PolFun_n$ of strict polynomial functors of degree n.*

We then consider a category of analytic functors, denoted by AnFun, whose objects are direct sums $F = \bigoplus_{n\in\mathbb{N}} F_n$ where F_n is a strict polynomial functor of degree n on the category of $\mathbb{K}$-modules. We check that the composition of functors lifts to the category AnFun, so that the triple $(\mathrm{AnFun}, \circ, \mathrm{Id})$, where $\circ$ is this composition operation and Id is the identity functor, forms a monoidal category. We consider on the other hand a category of $\mathbb{M}$-modules $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$ whose objects are collections $M = \{M_n\}_{n\in\mathbb{N}}$ such that $M_n \in \mathrm{Mac}^{coh}(\mathcal{H}Par_n)$, for each n. We consider the obvious functor $ev : \mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}} \to \mathrm{AnFun}$ such that $ev(M) = \bigoplus_{n\in\mathbb{N}} ev_n(M_n)$, for every $M \in \mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$. Theorem C implies that this functor defines an equivalence of categories.

We make explicit a composition product $\square$ and a unit object $\mathbb{I}$ in the category of $\mathbb{M}$-modules $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$ such that $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \square, \mathbb{I})$ forms a monoidal category and we establish the following results :

Theorem D (Chapter 2, Theorem 2.4.28)**.** *The mapping $ev : M \to ev(M)$, defines a (strongly) monoidal functor from the category of $\mathbb{M}$-modules equipped with the composition product $\square$ to the category of analytic functors* AnFun *equipped with the composition product $\circ$.*

We then define an $\mathbb{M}$-operad as a monoid object in the monoidal category of $\mathbb{M}$-modules. We denote the category of $\mathbb{M}$-operads by $\mathbb{M}$-Op. Theorem C and Theorem D have the following corollary :

Corollary . *The mapping $P \mapsto ev(P)$ induces an equivalence of categories between the category of $\mathbb{M}$-operads and the category of analytic monads*

Let us mention that our notion of an $\mathbb{M}$-operad is equivalent to the notion of a Schur operad defined in the Ph.D. thesis of Q. Xantcha [Xan10]. The main novelty of our approach is the definition of our objects in terms of monoidal structures whereas Xantcha define his notion of a Schur operad by using an abstract notion of polynomial operation. Xantcha's approach is a reminiscence of Lazard's definition of an analizeur [Laz55].

We already mentioned that the summands $S_n(P,-)$ and $\Gamma_n(P,-)$ of the monads $S(P,-)$ and $\Gamma(P,-)$ associated to an operad P are examples of strict polynomial functors. These monads $S(P,-)$ and $\Gamma(P,-)$ form examples of analytic monads ; and so does the other third monad which we associate to an operad $\Lambda(P,-)$. We make explicit the $\mathbb{M}$-operads $S_-(P)$, $\Gamma_-(P)$ and $\Lambda_-(P)$ which correspond to these analytic monads. We prove that the category of p-restricted Poisson algebras introduced by R. Bezrukavnikov and D. Kaled [BK08] in the study of deformation theory of manifolds in positive characteristics is also associated to an $\mathbb{M}$-operad p-$Pois$ which is not of this form.

To any $\mathbb{K}$-module V is associated an $\mathbb{M}$-operad denoted by $Poly_V$. If P is an $\mathbb{M}$-operad then the set of P-algebras structures on V is recovered from the set $\mathrm{Hom}_{\mathbb{M}\text{-Op}}(P, Poly_V)$

Theorem E (Chapter 2, Theorem 2.5.8)**.** *Let P be an $\mathbb{M}$-operad and V be a $\mathbb{K}$-module. The set of P-algebras structures on V is in bijection with* $\mathrm{Hom}_{\mathbb{M}\text{-Op}}(P, Poly_V)$.

We also have a notion of an $\mathbb{M}$-PROP which generalizes MacLane's concept of a PROP, and which can be used to govern categories of bialgebras. We define for instance an $\mathbb{M}$-PROP $\Gamma BiAlg_{Com}$ which governs the category of commutative-coassociative bialgebras with divided powers (This category of bialgebras is equivalent to the André's category of divided power Hopf algebras [And71] when we work in a category of connected graded $\mathbb{K}$-modules).

Outlook

The work-in-progress [Ces] is devoted to the study of applications of divided symmetries pre-Lie algebras on the theory of combinatorial Hopf algebras. J.-M. Oudom, D. Guin [OG08], and T. Schedler [Sch13] showed that for a Lie algebra coming from a pre-Lie algebra a strongest version of Poincaré-Birkhoff-Witt's Theorem holds, namely the quantum PBW theorem. The aim of this work [Ces] is to study the p-restricted case and a generalization to divided power algebras of this result.

Plan of the thesis

The thesis is divided in three chapters.

Chapter 1 is devoted to the study of pre-Lie algebras. We examine the definition of $\Lambda(PreLie,-)$-algebras and $\Gamma(PreLie,-)$-algebras in terms of generating operations and relations, and we establish the results of Theorem A and Theorem B. We also give a bunch of examples of $\Lambda(PreLie,-)$-algebra and $\Gamma(Pre, Lie,-)$-algebra structures in the concluding section of Chapter 1.

Chapter 2 is devoted to our study of the generalized operads which model the structure of analytic monads. We explain the definition of our categories of cohomological Mackey functors associated to our subset Par_n of the set of subgroups of the symmetric group $\mathbb{S}_n$. We define the equivalence between these categories of cohomological Mackey functors and the category of strict polynomial functors asserted in Theorem C. Then we explain the definition of our monoidal structure and of our notion of operad in the category of $\mathbb{M}$-modules, which fit in the results of Theorem D and its corollary. We eventually establish the result of Theorem E and we give examples of $\mathbb{M}$-operads and of our more general notion of an $\mathbb{M}$-PROP which naturally occur in the field of algebra.

These Chapters 1 and 2 are independent articles of the author and each of these chapter includes a self contained introduction and its own reminders.

In Appendix A, we summarize the basic results of the theory of operads, of the theory of polynomial functors and of the theory of Mackey functors, which we use in this thesis.

Chapter 1

On *PreLie* Algebras with Divided Symmetries

abstract

We study an analogue of the notion of p-restricted *Lie*-algebra and of the notion of divided power algebra for *PreLie*-algebras. We deduce our definitions from the general theory of operads. We consider two variants $\Lambda(P,-)$ and $\Gamma(P,-)$ of the monad $S(P,-)$ which governs the category of algebras classically associated to an operad P. For the operad *PreLie* corresponding to *PreLie*-algebras, we prove that the category of algebras over the monad $\Lambda(PreLie,-)$ is identified with the category of p-restricted *PreLie*-algebras introduced by A. Dzhumadil'daev. We give an explicit description of the structure of an algebra over the monad $\Gamma(PreLie,-)$ in terms of brace-type operations and we compute the relations between these generating operations. We prove that classical examples of *PreLie*-algebras occurring in deformation theory actually form $\Gamma(PreLie,-)$-algebras.

Introduction

In this chapter, we study an analogue of the notion of p-restricted *Lie*-algebra and of the notion of divided power algebra for *PreLie*-algebras.

PreLie-algebras were introduced by Gerstenhaber in [Ger63] to encode structures related to the deformation complex of algebras. In recent years, applications of *PreLie*-algebras appear in many other topics. Notably it has been discovered that they play a fundamental role in Connes-Kreimer's renormalization methods.

The category of *PreLie*-algebras is associated to an operad denoted by *PreLie*. To define our notion of *PreLie*-algebras with divided symmetries, we use the general theory of B. Fresse [Fre00], who showed how to associate a monad $\Gamma(P,-)$ to any operad P in order to encode this notion of algebra with divided symmetries.

Recall that the usual category of algebras associated to an operad P is governed by a monad $S(P,-)$ given by a generalized symmetric algebra functor with coefficients in the components of the operad P. To define $\Gamma(P,-)$ we merely replace the modules of coinvariant tensors, which occur in the generalized symmetric algebra construction, by modules of invariants. We denote by $\Lambda(P,-)$ the monad given by the image of the trace map between $S(P,-)$ and $\Gamma(P,-)$. For short, we call ΓP-algebras the category of algebras governed by the monad $\Gamma(P,-)$, and we similarly call ΛP-algebras the category of algebras governed by the monad $\Lambda(P,-)$. It turns out that many variants of algebra categories associated to these monads are governed by these monads. For instance, for the operad $P = Lie$ a ΛLie-algebra is equivalent to a Lie algebra equipped with an alterned Lie bracket $[x,x]=0$, while the ordinary category of algebras over the operad *Lie* only depicts Lie algebras equipped with an antisymmetric Lie bracket $[x,y]=-[y,x]$ (which differs from the latter when the ground field has characteristic two). The category of ΓLie-algebras, on the other hand, turns out to be equivalent to the classical notion of a p-restricted Lie algebra, where p is the characteristic of the ground field (see [Fre00] and [Fre04])

We aim to give a description in terms of generating operations of the structure of an algebra over the monads $\Lambda(PreLie,-)$ and $\Gamma(PreLie,-)$. Our main motivations come from the applications of *PreLie*-algebras in deformation theory. We will see that significant examples of *PreLie*-algebras occurring in deformation theory are actually $\Gamma PreLie$-algebras.

To be explicit, recall that a *PreLie*-algebra is a module V equipped with an operation $\{-,-\}: V \otimes V \longrightarrow V$ such that :

$$\{\{x,y\},z\} - \{x,\{y,z\}\} = \{\{x,z\},y\} - \{x,\{z,y\}\},$$

for all x, y, and z in V.

First, we study the algebras over $\Lambda(PreLie,-)$. We prove that these algebras are identified with the notion of p-restricted *PreLie*-algebras in the sense of [Dzh01]. A p-restricted *PreLie*-algebra is a *PreLie*-algebra where the following relation is satisfied

$$\{\ldots\{x,\underbrace{y\},\ldots\}y\}}_{p} = \{x,\{\ldots\underbrace{\{y,y\}\ldots\}y\}}_{p}.$$

Our result explicitly reads :

Theorem A (Theorem 1.4.16)**.** *We assume that the ground ring $\mathbb{K}$ is a field of characteristic p. A $\Lambda PreLie$-algebra is equivalent to a $\mathbb{K}$-module V equipped with an operation $\{-,-\}: V\otimes V \longrightarrow V$ satisfying the PreLie-relation and the p-restricted PreLie-algebra relation.*

Let V be a free $\mathbb{K}$-module with a fixed basis. We prove the existence of an isomorphism of graded free $\mathbb{K}$-modules between $S(PreLie,V)$ and $\Gamma(PreLie,V)$. Using this isomorphism we express the composition morphism of the free algebra $\Gamma(PreLie,V)$ and find a normal form for its elements. We combine these results to give a presentation of $\Gamma(PreLie,-)$: this monad is determined by $n+1$-fold polynomial "corollas" operations $\{-;\underbrace{-,\ldots,-}_{n}\}_{r_1,\ldots,r_n}$ of degree $(1,r_1,\ldots,r_n)$ and which satisfy some relations. We obtain the following theorem :

Theorem B (Theorem 1.5.19)**.** *If V is a free module over the ground ring $\mathbb{K}$, then providing the module V with a $\Gamma PreLie$-algebra structure is equivalent to providing V with a collection of polynomial maps*

$$\{-;\underbrace{-,\ldots,-}_{n}\}_{r_1,\ldots,r_n} : V \times \underbrace{V\times\ldots\times V}_{n} \longrightarrow V,$$

where $r_1,\ldots,r_n \in \mathbb{N}$ and which are linear in the first variable and the following relations hold :

$$\{x;y_1,\ldots,y_n\}_{r_{\sigma(1)},\ldots,r_{\sigma(n)}} = \{x;y_{\sigma^{-1}(1)},\ldots,y_{\sigma^{-1}(n)}\}_{r_1,\ldots,r_n}, \tag{1}$$

for any $\sigma \in \mathbb{S}_n$;

$$\{x;y_1,\ldots,y_{i-1},y_i,y_{i+1},\ldots,y_n\}_{r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_n} = \{x;y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_n\}_{r_1,\ldots,r_{i-1},r_{i+1},\ldots,r_n}, \tag{2}$$

$$\{x;y_1,\ldots,\lambda y_i,\ldots,y_n\}_{r_1,\ldots,r_i,\ldots,r_n} = \lambda^{r_i}\{x;y_1,\ldots,y_i,\ldots,y_n\}_{r_1,\ldots,r_i,\ldots,r_n}, \tag{3}$$

for any λ in $\mathbb{K}$;

if $y_i = y_{i+1}$

$$\{x;y_1,\ldots,y_i,y_{i+1},\ldots,y_n\}_{r_1,\ldots,r_i,r_{i+1},\ldots,r_n} = \binom{r_i+r_{i+1}}{r_i}\{x;y_1,\ldots,y_i,y_{i+2},\ldots,y_n\}_{r_1,\ldots,r_i+r_{i+1},r_{i+2},\ldots,r_n}. \tag{4}$$

$$\{x;y_1,\ldots,y_{i-1},a+b,y_{i+1},\ldots,y_n\}_{r_1,\ldots,r_i,\ldots,r_n} = \sum_{s=0}^{r_i}\{x;y_1,\ldots,a,b,\ldots,y_n\}_{r_1,\ldots,s,r_i-s,\ldots,r_n}, \tag{5}$$

$$\{-;\} = id, \tag{6}$$

$$\{\{x; y_1, \ldots, y_n\}_{r_1,\ldots,r_n}; z_1, \ldots, z_m\}_{s_1,\ldots,s_m} =$$
$$\sum_{s_i=\beta_i+\sum \alpha_i^{\cdot\cdot}} \frac{1}{\prod(r_j!)} \{x; \{y_1; z_1, \ldots, z_m\}_{\alpha_1^{1,1},\ldots,\alpha_m^{1,1}}, \ldots, \{y_1; z_1, \ldots, z_m\}_{\alpha_1^{1,r_1},\ldots,\alpha_m^{1,r_1}},$$
$$\ldots, \{y_n; z_1, \ldots, z_m\}_{\alpha_1^{n,1},\ldots,\alpha_m^{n,1}}, \ldots, \{y_n; z_1, \ldots, z_m\}_{\alpha_1^{n,r_n},\ldots,\alpha_m^{n,r_n}},$$
$$z_1, \ldots, z_m\}_{1,\ldots,1,\beta_1,\ldots,\beta_m}, \tag{7}$$

where, to give a sense to the latter formula, we use that the denominators $r_j!$ divide the coefficient of the terms of the reduced expression which we get by applying relations (1) and (4) to simplify terms with repeated inputs on the right hand side (see Example 1.5.11).

We give explicit examples of $\Gamma PreLie$-algebras in the last section. Notably we explain that $\Gamma PreLie$-algebras naturally occur in the study of $Brace$-algebras in characteristic different from 0. The already alluded to applications of $\Gamma PreLie$-algebras in deformation theory actually arise from this relationship.

Contents

We devote sections $1-2$ to general recollections on the operadic background of our constructions and to the definition of the operad $PreLie$.

In section 3 we construct a normal form for $S(PreLie, V)$ and a basis for $\Gamma(PreLie, V)$. We establish the equivalence between $\Lambda PreLie$-algebras and p-restricted $PreLie$-algebras in section 4. We give the construction of a presentation of $\Gamma(PreLie, -)$ in section 5. We conclude with examples of $\Gamma PreLie$-algebras in section 6.

1.1 Operads and their monads

In this section, we briefly survey the general definitions of operad theory which we use in the chapter. This section does not contain any original result. We follow the presentation of [Fre00] for the definition of the monads $\Lambda(P,-)$ and $\Gamma(P,-)$ associated to an operad. We refer to the books [MSS02], [LV12], and [Fre09] for a comprehensive account of the theory of operads. We work in a category of modules, $\mathrm{Mod}_{\mathbb{K}}$, over a fixed commutative ground ring $\mathbb{K}$. For simplicity, we assume that $\mathbb{K}$ is a field in the statement of the general results and in the account of the general constructions of this section. We only consider the general case of a ring in concluding remarks at the end of each subsection (1.1.1-1.1.4). We explain in these remarks the extra assumptions which we need to make our constructions work when we work over a ring.

1.1.1 $\mathbb{S}$-Modules

We recall the notion of $\mathbb{S}$-module, which underlies the notion of operad, and the definition of three monoidal structures on the category of $\mathbb{S}$-modules.

Definition 1.1.1. *We denote by $\mathbb{S}_n$ the symmetric group on a finite set of n elements. An $\mathbb{S}$-module M is a collection $\{M(n)\}_{n\in\mathbb{N}}$ where $M(n)$ is an $\mathbb{S}_n$-$\mathbb{K}$-module for each $n \in \mathbb{N}$.*

A morphism between $\mathbb{S}$-modules $f : M \longrightarrow N$ is a collection $\{f_n : M(n) \longrightarrow N(n)\}_{n\in\mathbb{N}}$ where f_n is a morphism of $\mathbb{S}_n$-$\mathbb{K}$-modules for each $n \in \mathbb{N}$.

We denote by $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}$ the category of $\mathbb{S}$-modules.

We recall the definition of generalized symmetric (respectively, divided symmetries) tensors associated to an $\mathbb{S}$-module. These functors determine the monads that define algebras over an operad and divided symmetries algebras over an operad.

Definition 1.1.2. *We denote by $End(\mathrm{Mod}_{\mathbb{K}})$ the category of endofunctors of $\mathrm{Mod}_{\mathbb{K}}$. Let V be an $\mathbb{K}$-module and M be an $\mathbb{S}$-module. On $V^{\otimes n}$ the monoidal structure of the tensor product induces a natural $\mathbb{S}_n$-action. The $\mathbb{K}$-module $M(n) \otimes V^{\otimes n}$ is equipped with the diagonal $\mathbb{S}_n$ action. The Schur functor $S(M,-) : \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ is defined as :*

$$S(M,V) := \bigoplus_n M(n) \otimes_{\mathbb{S}_n} V^{\otimes n},$$

where $\otimes_{\mathbb{S}_n}$ means the $\mathbb{K}$-module of co-invariants of the tensor product $M(n) \otimes V^{\otimes n}$ under the diagonal action; and the coSchur functor $\Gamma(M,-) : \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ is defined as :

$$\Gamma(M,V) := \bigoplus_n M(n) \otimes^{\mathbb{S}_n} V^{\otimes n},$$

where $\otimes^{\mathbb{S}_n}$ means the $\mathbb{K}$-module of invariants.

We then have two functors $S : \mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}} \longrightarrow End(\mathrm{Mod}_{\mathbb{K}})$ defined as $S(M) \mapsto S(M,-)$, and $\Gamma : \mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}} \longrightarrow End(\mathrm{Mod}_{\mathbb{K}})$ defined as $\Gamma(M) \mapsto \Gamma(M,-)$.

Remark 1.1.3. *The functors $S(M,-)$ and $\Gamma(M,-)$ are full and faithful when the ground ring is an infinite field.*

Between the coinvariant space and the invariant space there is a map called the trace (or norm) map.

Definition 1.1.4. *Let M be an $\mathbb{S}$-module. The trace map is the natural transformation $Tr : S(M,-) \longrightarrow \Gamma(M,-)$ such that :*

$$Tr(m \otimes v_1 \otimes \ldots \otimes v_n) = \sum_{\sigma \in \mathbb{S}_n} \sigma^*(m \otimes v_1 \otimes \ldots \otimes v_n),$$

for each $m \in M$, $v_1, \ldots v_n \in V$, and where we take the diagonal action of $\sigma \in \mathbb{S}_n$ on the tensor $m \otimes v_1 \otimes \ldots \otimes v_n \in M(n) \otimes V^{\otimes n}$.

Remark 1.1.5. *The natural transformation Tr is an isomorphism in characteristic 0, but this is no longer the case in positive characteristic.*

Definition 1.1.6. *We consider three monoidal structures on $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}$, let M, N be two $\mathbb{S}$-modules :*

1. *the tensor product $- \boxtimes - : \mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}} \times \mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}$ of $\mathbb{S}$-modules, defined by*

$$(M \boxtimes N)(n) := \bigoplus_{i+j=n} \mathrm{Ind}_{\mathbb{S}_i \times \mathbb{S}_j}^{\mathbb{S}_n} M(i) \otimes N(j),$$

and whose unit is the $\mathbb{S}$-module such that $\mathbb{K}(n) = \begin{cases} \mathbb{K} & if\ n = 0 \\ 0 & if\ n \neq 0, \end{cases}$

2. *the coinvariant composition product $-\underset{\sim}{\square}- : \mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}} \times \mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}$ of $\mathbb{S}$-modules, defined by*

$$M \underset{\sim}{\square} N := \bigoplus_r M(r) \otimes_{\mathbb{S}_r} N^{\boxtimes r},$$

and whose unit is the $\mathbb{S}$-module such that $I(n) = \begin{cases} \mathbb{K} & if\ n = 1 \\ 0 & if\ n \neq 1, \end{cases}$.

3. *and the invariant composition product $-\tilde{\square}- : \mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}} \times \mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}$ of $\mathbb{S}$-modules, defined by*

$$M \tilde{\square} N := \bigoplus_r M(r) \otimes^{\mathbb{S}_r} N^{\boxtimes r},$$

with the same unit object as the coinvariant tensor product.

The tensor product $- \boxtimes -$ is symmetric, while the composition products $-\underset{\sim}{\square}-$ and $-\tilde{\square}-$ are not.

The two functors S and Γ are monoidal, more precisely :

Proposition 1.1.7. *The functors $S : \mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}} \longrightarrow End(\mathrm{Mod}_{\mathbb{K}})$ and $\Gamma : \mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}} \longrightarrow End(\mathrm{Mod}_{\mathbb{K}})$ define :*

1. *strongly symmetric monoidal functors*

$$(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \boxtimes, \mathbb{K}) \xrightarrow{S} (End(\mathrm{Mod}_{\mathbb{K}}), \otimes, \mathbb{K}) \quad and \quad (\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \boxtimes, \mathbb{K}) \xrightarrow{\Gamma} (End(\mathrm{Mod}_{\mathbb{K}}), \otimes, \mathbb{K}),$$

where $\otimes$ is the pointwise tensor product, inherited from the tensor product of $\mathbb{K}$-modules, on the category of functors ;

2. *strongly monoidal functors*
$$(\mathrm{Mod}^{\mathbb{S}}_{\mathbb{K}}, \underset{\sim}{\square}, \mathbb{K}) \xrightarrow{S} (End(\mathrm{Mod}_{\mathbb{K}}), \circ, Id) \quad and \quad (\mathrm{Mod}^{\mathbb{S}}_{\mathbb{K}}, \tilde{\square}, \mathbb{K}) \xrightarrow{\Gamma} (End(\mathrm{Mod}_{\mathbb{K}}), \circ, Id),$$
where $\circ$ is the composition of functors.

Proof: These assertions are classical for S (see for instance [LV12, Ch. 5]) and the analogue of these relations for Γ is established in [Fre00]. □

Remark 1.1.8. *The statements of Proposition 1.1.7 remain valid without any change when we work with a commutative ground ring $\mathbb{K}$ in the case of the functor $S : M \mapsto S(M,-)$.*

For the functor $\Gamma(P,-)$ the statement of Proposition 1.1.7 is still valid if $\mathbb{K}$ is an hereditary ring, we restrict ourself to $\mathbb{S}$-modules M whose components $M(r)$ are projective as $\mathbb{K}$-modules for all $r \in \mathbb{N}$, and we consider the restriction of our functor $\Gamma(M,-)$ to the category of projective $\mathbb{K}$-modules.

In short the tensor product $M(r) \otimes V^{\otimes r}$ form a projective $\mathbb{K}$-module as soon as $M(r)$ and V do so. We just use the assumption that the ring $\mathbb{K}$ is hereditary to ensure that $M(r) \otimes^{\mathbb{S}_r} V^{\otimes r} \subseteq M(r) \otimes V^{\otimes r}$ is still projective as a $\mathbb{K}$-module. We accordingly get that the map $\Gamma(M,-) : V \mapsto \Gamma(M,V)$ defines an endofunctor of the category of projective $\mathbb{K}$-modules in this case. We then use that the tensor product with a projective module preserves kernels (and hence invariants) to check the validity of the claims of our proposition, after observing that the invariant composition of $\mathbb{S}$-modules also consists of projective $\mathbb{K}$-modules in this setting.

1.1.2 Operads and P-algebras

We now recall the definition of an operad and the definition of the monads associated to an operad which we use in this chapter. To be specific, when we use the name operad, we mean symmetric operad, and we define this structure by using the coinvariant composition product recalled in the previous subsection.

Definition 1.1.9. *We define an operad to be a triple (P, μ, η) where P is an $\mathbb{S}$-module, $\mu : P \underset{\sim}{\square} P \longrightarrow P$, is a multiplication morphism, and $\eta : I \longrightarrow P$ a unit morphism such that P forms a monoid in $(\mathrm{Mod}^{\mathbb{S}}_{\mathbb{K}}, \underset{\sim}{\square}, I)$.*

If P is an operad, then $S(P,-)$ is a monad by Proposition 1.1.7.

In what follows, we also use that the composition structure of an operad is determined by composition operations $\circ_i : P(m) \otimes P(n) \longrightarrow P(m+n-1)$ defined for any $m, n \in \mathbb{N}$ and $1 \leq i \leq m$, and which satisfies natural equivariance and associative relations. The unit morphism can then be given by a unit element $1 \in P(1)$ which satisfies natural unit relations with respect to these composition products. We refer to [Fre00] for instance for more details on this correspondence.

Since a general theory of free operads and their ideals can be set up (see [Fre09]) we can present operads by generating operations and relations.

Definition 1.1.10. *Let P be an operad, we define a P-algebra to be an algebra over the monad $S(P,-)$.*

We have the following classical statement.

Proposition 1.1.11. *Let V be a $\mathbb{K}$-module and (P, μ, η) be an operad. The $\mathbb{K}$-module $S(P,V)$ equipped with the morphisms induced by μ and η is itself a P-algebra.*

Proof: See [LV12, Sec. 5.2.5]. □

Remark 1.1.12. *The statement of 1.1.11 remains valid without any extra assumption on our objects nor change when we work over a general ring $\mathbb{K}$.*

1.1.3 $\Gamma(P,-)$ and $\Lambda(P,-)$ monads

Under a connectivity condition any operad structure on an $\mathbb{S}$-module P induces a monad structure on $\Gamma(P,-)$. We define ΓP-algebras as the algebras for the monad $\Gamma(P,-)$. The trace map is a natural transformation of monads. The concept of ΓP-algebra was introduced by B. Fresse in [Fre00]. We recall the definition of these concepts in this section.

Definition 1.1.13. *An $\mathbb{S}$-module N is connected if $N(0) = 0$.*

We rely on the following observation :

Proposition 1.1.14. *Let M and N be two $\mathbb{S}$-modules. If N is connected, then we have an isomorphism $Tr_{M,N} : M\underset{\sim}{\square}N \longrightarrow M\tilde{\square}N$.*

Proof: See [Fre00]. □

This proposition has the following consequence :

Proposition 1.1.15. *Let (P,μ,η) be a connected operad. There exists a product $\tilde{\mu} : P\tilde{\square}P \longrightarrow P$ given by :*

$$P\tilde{\square}P \xleftarrow{\cong} P\underset{\sim}{\square}P \xrightarrow{\mu} P$$

and making $(P,\tilde{\mu},\eta)$ into a monoid in the monoidal category $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \tilde{\square}, I)$. □

Corollary 1.1.16. *Let (P,μ,η) be a connected operad ; then $(\Gamma(P,-),\tilde{\mu},\eta)$ is a monad.* □

Definition 1.1.17. *Let (P,μ,η) be a connected operad. A ΓP-algebra is an algebra over the monad $\Gamma(P,-)$.*

From now on we only consider connected operads.

Proposition 1.1.18. *Let P a connected operad. The natural transformation $Tr : S(P,-) \to \Gamma(P,-)$ is a morphism of monads.*

Proof: See [Fre00]. □

We introduce a third kind of algebras called ΛP-algebras.

Definition 1.1.19. *We denote by $\Lambda(P,-) : \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ the functor defined by the epi-mono factorization of the trace map.*

Proposition 1.1.20. *Let P a connected operad. The functor $\Lambda(P,-)$ forms a submonad of $\Gamma(P,-)$ and the factorization*

$$S(P,-) \to \Lambda(P,-) \to \Gamma(P,-)$$

forms a monad morphism.

Proof: We use that Tr is a morphism of monads and that the functor $S(P,-)$ preserves the epimorphisms to obtain that we have a commutative diagram of the form :

$$\begin{tikzcd}
S(P,S(P,-)) \arrow[r, two heads] \arrow[d] \arrow[rr, bend left, "Tr\circ S(P,Tr)=\Gamma(P,Tr)\circ Tr"] & \Lambda(P,\Lambda(P,-)) \arrow[r] \arrow[d, dashed, "\exists"] & \Gamma(P,\Gamma(P,-)) \arrow[d] \\
S(P,-) \arrow[r, two heads] \arrow[rr, bend right, "Tr"'] & \Lambda(P,-) \arrow[r, hook] & \Gamma(P,-).
\end{tikzcd}$$

We deduce from this diagram that the composition product of the monad $\Gamma(P,-)$ and factor through $\Lambda(P,-)$. The unit of $\Gamma(P,-)$ similarly factors through $\Lambda(P,-)$. The conclusion of the proposition follows. □

Definition 1.1.21. *Let P be a connected operad, a ΛP-algebra is an algebra for the monad $\Lambda(P,-)$.*

Remark 1.1.22. *Any ΛP-algebra V is a P-algebra. Any ΓP-algebra W is a ΛP-algebra by the following commutative diagram :*

$$\begin{tikzcd} \Lambda(P,V) \arrow[r] \arrow[d] & V \\ \Gamma(P,V). \arrow[ur] & \end{tikzcd}$$

Remark 1.1.23. *The statements of this subsection have a generalization when we work over a hereditary ring. We then assume that the components of our operads $P(r)$ form projective $\mathbb{K}$-modules, for all $r \in \mathbb{N}$, and we use that the map $\Gamma(P,-) : V \mapsto \Gamma(P,V)$ defines an endofunctor of the category of projective $\mathbb{K}$-modules, according to the observation of Remark 1.1.8. We get that this functor $\Gamma(P,-)$ forms a monad in this case, and that $\Lambda(P,-)$ is a submonad of this monad over the category of projective $\mathbb{K}$-modules.*

We can actually forget the assumption that $\mathbb{K}$ is hereditary in the case of the $PreLie$ operad which we study in the following section. We will actually see that $\Gamma(PreLie,-) : V \mapsto \Gamma(PreLie,V)$ induces an endofunctor of the category of free $\mathbb{K}$-module without any further assumption on the ground ring $\mathbb{K}$.

1.1.4 Non-symmetric operads and TP-algebras

We mostly use symmetric operads in this chapter. But we also consider a monad $T(P,-)$ which is naturally associated to any non-symmetric operad. We explain this auxiliary construction in this subsection.

Notations 1.1.24. *We denote by $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{N}}$ the category of $\mathbb{K}$-modules graded on $\mathbb{N}$.*

Definition 1.1.25. *Let A be in $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{N}}$. There is a functor $T(A,-) : \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ defined as follows :*

$$T(A,V) = \bigoplus_n A(n) \otimes V^{\otimes n}.$$

Forgetting the action of the symmetric groups we get a functor $U : \mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}^{\mathbb{N}}$. Composing U with $T(-,-)$ we have a functor $T : \mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}} \longrightarrow End(\mathrm{Mod}_{\mathbb{K}})$.

Definition 1.1.26. *Let M, N be two graded modules. We define the graded module $M \square N$ by :*

$$M \square N(n) = \bigoplus_r M(r) \otimes (\bigoplus_{n_1 + \ldots n_r = n} N(n_1) \otimes \ldots \otimes N(n_r)),$$

This operation gives a monoidal structure on $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{N}}$.

We have the following proposition :

Proposition 1.1.27. *The functor $T : (\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \square, I) \longrightarrow (End(\mathrm{Mod}_{\mathbb{K}}), \circ, Id)$ is strongly monoidal.*

Proof: We easily adapt the proof of the counterpart of this statement for S and Γ. □

Definition 1.1.28. *Let P a non-symmetric operad, a TP-algebra is an algebra over the monad $T(P,-)$.*

Let P be an operad and V a $\mathbb{K}$-module; then $T(P,V)$ is a TP-algebra with a structure map given by the map μ on P and juxtaposition of words formed by elements of V.

Definition 1.1.29. *There is a natural transformation given by the quotient*

$$pr : T(P,-) \longrightarrow S(P,-).$$

Proposition 1.1.30. *Let P be a connected operad. The two natural transformations in and pr are monad morphisms.*

Proof: Let V be a $\mathbb{K}$-module. This statement follows from the commutativity of the following diagrams :

$$\begin{tikzcd} T(P,T(P,V)) \arrow[r] \arrow[d, "\cong"'] & S(P,S(P,V)) \arrow[d, "\cong"] \\ T(P \square P,V) \arrow[d] & S(P\tilde{\square}P,V) \\ T(P\tilde{\square}P,V). \arrow[ur, "pr"'] & \end{tikzcd}$$

The verification of this commutative property is immediate. ☐

Remark 1.1.31. *The results of this subsection remain valid without change when we work over a commutative ring* $\mathbb{K}$.

1.2 On $PreLie$ and rooted trees operads

We recall the definition of $PreLie$-algebras. These algebras have a binary product and a relation, sometimes called right associativity.

The $PreLie$-algebras were introduced in [Ger63] by Gerstenhaber. We refer to [CL01] for the definition of the operad which governs this category of algebras. We also refer to [Man11] for a survey on the theory and to [Dok13] for some applications of PreLie-algebras in positive characteristic.

Definition 1.2.1. *A* $\mathbb{K}$*-module* V *is a* $PreLie$*-algebra if it is endowed with a bilinear product :*

$$\{-,-\} : V \otimes V \longrightarrow V,$$

such that

$$\{\{x,y\},z\} - \{x,\{y,z\}\} = \{\{x,z\},y\} - \{\{x,y\},z\}.$$

The $PreLie$ bracket defines a Lie bracket by : $[a,b] = \{a,b\} - \{b,a\}$.

This structure appears naturally in different contexts. We recall some examples which we revisit in the context of $\Gamma PreLie$-algebras.

Example 1.2.2.
1. *Let* P *be an operad; we can define a* $PreLie$*-algebra structure on the following* $\mathbb{K}$*-module* $\bigoplus_n P(n)$*. Explicitly the* $PreLie$*-product is given by the following formula :*

$$\{p,q\} = \sum_{i\in\{1,\dots,n\}} p \circ_i q$$

where $p \in P(n)$ *and* $q \in P(m)$*. We go back to this example in Section 1.6 where we study the relation between* $PreLie$*-systems and* $\Gamma PreLie$*-algebras.*

2. *The Hochschild complex of an associative algebra* A *defined as* $C^r(A,A) = \mathrm{Hom}(A^{\otimes r},A)$ *has a dg-*$PreLie$*-algebra structure. For* $f \in C^m(A,A)$ *and* $g \in C^n(A,A)$*, we explicitly have :*

$$\{f,g\}(x_1,\dots,x_{n+m-1}) = \\ \sum_{i=1}^{m} (-1)^{(n-1)(i-1)} f(x_1,\dots,x_{i-1},g(x_i,\dots,x_{n+i-1}),x_{n-i}\dots,x_{n+m-1}),$$

This structure was introduced by Gerstenhaber in [Ger63] and can actually be defined on the deformation complex of any algebra over an operad (see [LV12, Ch. 12]). This $PreLie$*-algebra structure on the Hochschild complex of an algebra is also a special case of the previous example, where we take* $P = \Lambda End_A$*, the operadic suspension* Λ *of the endomorphism operad* End_A *of* A.

We have a new type of $PreLie$-algebras, called p-restricted $PreLie$-algebras, which occur when the ground ring is a field of characteristic $p > 0$. As for p-restricted Lie-algebras, introduced by N. Jacobson in [Jac79], p-restricted $PreLie$-algebras appear naturally in the study of $PreLie$ structures in positive characteristic p. This kind of algebras was introduced by A. Dzhumadil'daev in [Dzh01].

Definition 1.2.3. *Fixed a field $\mathbb{K}$ of characteristic p. Let $(L, \{,\})$ be a $PreLie$-algebra. It is a p-restricted $PreLie$, or $p - PreLie$-algebra if the following equation holds :*

$$\{\{\ldots\{\underbrace{\{x,y\},y\}\ldots\}y}_{p}\} = \{x,\{\ldots\{\underbrace{\{y,y\}\ldots\}y}_{p}\}\}.$$

Remark 1.2.4. *In [Dok13] I. Dokas introduces a more general notion of p-restricted $PreLie$-algebra. A "generalized" p-restricted $PreLie$-algebra is a $PreLie$-algebra V endowed with a Frobenius map $\phi : V \to V$ satisfying some relations. If we assume $\phi = \{\{\cdots\underbrace{\{y,y\},\cdots\},y}_{p}\}$ we retrieve the definition of A. Dzhumadil'daev (Definition 1.2.3).*

Example 1.2.5. ***Simple Lie algebra*** $sl(2,\mathbb{K})$. *In characteristic* 0 *a semisimple Lie algebra does not admit a $PreLie$ structure. But this is no longer the case in positive characteristic. In [Dok13] it is shown that $sl(2,\mathbb{K})$ admits a $PreLie$ structure if and only if $char(\mathbb{K}) = 3$. In this case the $PreLie$ structure is* 3*-restricted. For details and proof see [Dok13].*

Rota-Baxter algebras. *In [Dok13] it is shown that the Rota-Baxter algebras, introduced by Gian-Carlo Rota in [Rot95], admit a p-restricted $PreLie$ structure.*

$PreLie$-algebras in the sense of 1.2.1 are identified with a category of algebras over an operad defined by generators and relations. We recall another description of this operad in terms of trees.

1.2.1 Non labelled trees

In this section we introduce the definition of non labelled tree.

Definition 1.2.6. *We use the name non labelled tree to refer to a non-empty, finite, connected graph, without loops, with one special vertex called the root. The edges of such a tree admit a canonical orientation with the root as ultimate outgoing vertex, we have a pre-order corresponding to this orientation on the set of vertices of the tree, with the root as least element. Two non labelled trees are isomorphic if they are isomorphic as graphs by an isomorphism which preserves the root.*

If necessary, we speak about a non labelled n-tree to specify the number n of vertices.

Definition 1.2.7. *Let τ be a non labelled tree, a sub-tree is a connected sub-graph with root its minimum vertex by the pre-order defined by τ.*

Definition 1.2.8. *Let τ be a non labelled rooted tree, a branch B of τ is a maximal subtree of τ that does not contain the root, where maximal has to be understood as a maximal element in the poset, defined by inclusion, of non labelled sub-trees of τ.*

Definition 1.2.9. *Let τ be a non labelled tree and B be a branch of τ, the set $iso(B)$ is the set of all branches of τ isomorphic, as non labelled trees, to B.*

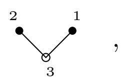
1.2.2 Labelled trees

We define the concept of labelled tree.

Definition 1.2.10. *We call labelled tree, or just tree, a non labelled tree with a fixed bijection, called labelling, between its vertices and the set $\{1,\ldots,n\}$, where n is the number of vertices. We denote by $\mathcal{RT}(n)$ the set of labelled trees with n vertices. The group $\mathbb{S}_n$ acts on this set by permuting the labelling.*

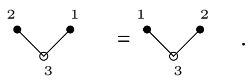
If necessary, we use the expression of n-tree to specify the number of vertices of a tree.

Example 1.2.11. *The following is a 3-tree :*



with root the vertex labelled by 3.

Notice that our trees are not planar. For example, we have :

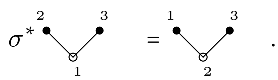


Definition 1.2.12. *The $\mathbb{S}$-module RT of rooted trees is*

$$RT(n) := \mathbb{K}[\mathcal{RT}(n)],$$

where $\mathbb{K}[X]$ is the $\mathbb{K}$-module freely generated by the base set X.

Example 1.2.13. *Let σ be the permutation of $\mathbb{S}_3$ that permutes* 1 *with* 2 *and fixes* 3 *:*



1.2.3 The rooted trees operad

The $\mathbb{S}$-module RT can be endowed with a structure of operad. This new operad is isomorphic to $PreLie$. We review this result in this section. The proof of the isomorphism is given in [CL01].

Definition 1.2.14. *We define the following partial compositions :*

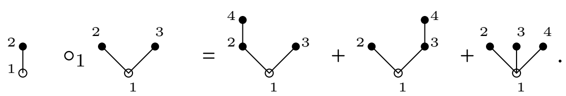
$$- \circ_i - : RT(m) \times RT(n) \longrightarrow RT(n+m-1),$$

with $1 \leq i \leq m$ as follows, let $In(\tau, i)$ be the set of incoming edges of the vertex of τ labelled i :

$$\tau \circ_i \upsilon := \sum_{f : In(\tau,i) \longrightarrow \{1,\ldots,n\}} \tau \circ_i^f \upsilon,$$

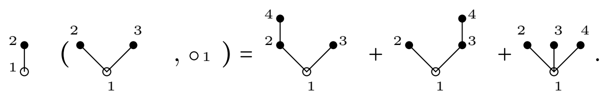
where $\tau \circ_i^f \upsilon$ is the $n+m-1$-tree obtained by substituting the tree υ to the ith vertex of the tree τ, by attaching the outgoing edge of this vertex in τ, if it exists, to the root of υ, and the ingoing edges to vertices of υ following the attaching map f and then labelling following the labelling of τ and the labelling of υ after obvious the shift. The sum runs over all these attachment maps $f : In(\tau, i) \longrightarrow \{1, \ldots n\}$.

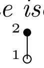
Example 1.2.15.



Lemma 1.2.16. *These partial compositions define a total composition $\gamma : RT \circ RT \longrightarrow RT$ that is an operad structure on the $\mathbb{S}$-module RT.*

Example 1.2.17.



Theorem 1.2.18 (Chapoton, Livernet)**.** *The $PreLie$ operad is isomorphic to the RT operad. The isomorphism $\varphi : PreLie \longrightarrow RT$ is realized by sending the generating operations of $PreLie$ to*  .

Proof: See [CL01]. □

From now on we do not make any difference between RT and $PreLie$ if it is not strictly necessary and therefore we will talk about trees as elements of $PreLie$.

1.3 A basis of $\Gamma(PreLie, V)$

The aim of this section is to make explicit a basis of the module $\Gamma(PreLie, V)$ when V is a $\mathbb{K}$-module equipped with a fixed basis $\mathcal{V}$.

Definition 1.3.1. *Let $x_1, \ldots, x_n$ be elements of V, and τ be an n-tree. We denote the element $\tau \otimes x_1 \otimes \ldots \otimes x_n$ in $T(RT, V)$ by $\tau\langle x_1, \ldots, x_n\rangle$ and the class $[\tau \otimes x_1 \otimes \ldots \otimes x_n]$ in $S(RT, V)$ by $\tau(x_1, \ldots, x_n)$. If we fix a basis $\mathcal{V}$ of V, then we call :*
— *canonical basis of $T(RT, V)$ the set $\mathcal{T}(\mathcal{RT}, \mathcal{V}) = \{\tau\langle x_1, \ldots, x_n\rangle | \tau \in \mathcal{RT}(n), x_i \in \mathcal{V}\}$,*
— *and canonical basis of $S(RT, V)$ the set $\mathcal{S}(\mathcal{RT}, \mathcal{V}) = \{\tau(x_1, \ldots, x_n) | \tau \in \mathcal{RT}(n), x_i \in \mathcal{V}\}$.*
The epimorphism $pr : T(RT, V) \longrightarrow S(RT, V)$ restricts to a surjective function

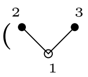
$$pr : \mathcal{T}(\mathcal{RT}, \mathcal{V}) \longrightarrow \mathcal{S}(\mathcal{RT}, \mathcal{V}).$$

Definition 1.3.2. *Let V be a free $\mathbb{K}$-module with a fixed basis $\mathcal{V}$. Let $\mathfrak{t} = \tau\langle x_1, \ldots x_n\rangle$ be an element of $\mathcal{T}(\mathcal{RT}, \mathcal{V})$. The stabilizer of $\mathfrak{t}$, denoted by $\mathrm{Stab}(\mathfrak{t})$, is the subgroup of $\mathbb{S}_n$ defined by :*

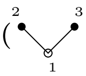
$$\mathrm{Stab}(\mathfrak{t}) := \{\sigma \in \mathbb{S}_n \mid \sigma^* \, \mathfrak{t} = \mathfrak{t}\},$$

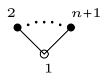
where we consider the diagonal action of permutations $\sigma \in \mathbb{S}_n$ on the tensor $\tau \otimes x_1 \otimes \ldots \otimes x_n$ which represents our element $\mathfrak{t} = \tau\langle x_1, \ldots, x_n\rangle$.

Example 1.3.3. *Let V be a free $\mathbb{K}$-module with a fixed basis $\mathcal{V}$, and x, y, z be elements of $\mathcal{V}$. We have the following formulas :*

Stab( $\{x, y, z\}) = \{id\}$,

and

Stab( $\{y, x, x\}) = \{id, (2, 3)\}$.

Definition 1.3.4. *We define F_n to be the following labelled $n+1$-tree*  .

Proposition 1.3.5. *Let V be a free $\mathbb{K}$-module with a fixed basis $\mathcal{V}$ and $x_0, \ldots, x_r$ be elements of $\mathcal{V}$ such that $x_p \neq x_q$ if $p \neq q$ and $p, q \neq 0$. Consider the element $F_n\langle x_0, \underbrace{x_1, \ldots, x_1}_{i_1}, \ldots, \underbrace{x_r, \ldots, x_r}_{i_r}\rangle$ in $\mathcal{T}(\mathcal{RT}, \mathcal{V})$; then $\mathrm{Stab}(F_n\langle x_0, \underbrace{x_1, \ldots, x_1}_{i_1}, \ldots, \underbrace{x_r, \ldots, x_r}_{i_r}\rangle)$ is isomorphic to $\mathbb{S}_{i_1} \times \ldots \times \mathbb{S}_{i_r}$.*

Proof: An element σ of $\mathbb{S}_{n+1}$ is in $\mathrm{Stab}(F_n\langle x_0, \underbrace{x_1, \ldots, x_1}_{i_1}, \ldots, \underbrace{x_r, \ldots, x_r}_{i_r}\rangle)$ if its action fixes both F_n and $x_0, \underbrace{x_1, \ldots, x_1}_{i_1}, \ldots, \underbrace{x_r, \ldots, x_r}_{i_r}$. Since F_n should be fixed we have that x_0 is fixed and then σ has to be in the stabilizer of $\underbrace{x_1, \ldots, x_1}_{i_1}, \ldots, \underbrace{x_r, \ldots, x_r}_{i_r}$ that is isomorphic to $\mathbb{S}_{i_1} \times \ldots \times \mathbb{S}_{i_r}$. $\square$

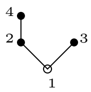
Definition 1.3.6. *Let V be a free $\mathbb{K}$-module with a fixed basis $\mathcal{V}$. Let $\mathtt{t}$ be an element of $\mathcal{S}(\mathcal{RT}, \mathcal{V})$. We define $\mathrm{Dec}(\mathtt{t})$ to be the element of $\mathcal{S}(\mathcal{RT}, \mathcal{S}(\mathcal{RT}, \mathcal{V}))$:*

$$F_r(x_0, B_1, \ldots, B_r),$$

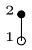
where F_r is isomorphic as non labelled rooted tree to the full sub-corolla with root x_0 the element of $\mathcal{V}$ corresponding to the root of $\mathtt{t}$, and B_j are elements of $\mathcal{S}(\mathcal{RT}, \mathcal{V})$ corresponding to the branches of $\mathtt{t}$.

In the literature the elements $F_r(x, T_1, \dots, T_r)$ are sometimes denoted $B(x, T_1, \dots, T_r)$, see [CK98].

Example 1.3.7. *Let* $\mathtt{t}$ *be the element :*

 (x_0, x_1, x_2, x_3).

We have :

$$\mathrm{Dec}(\mathtt{t}) = F_2(x_0,$$  $$(x_1, x_3), \circ_1 (x_2)).$$

Remark 1.3.8. *Let* V *be a free* $\mathbb{K}$*-module with a fixed basis* $\mathcal{V}$*, and* $\mathtt{t}$ *be an element of* $\mathcal{S}(\mathcal{RT}, \mathcal{V})$*. If* $\mu : S(RT, S(RT, V)) \longrightarrow S(RT, V)$ *is the composition product for the operad* RT *then* $\mu(\mathrm{Dec}(\mathtt{t})) = \mathtt{t}$.

Definition 1.3.9. *Let* V *be a free* $\mathbb{K}$*-module with a fixed basis* $\mathcal{V}$*. By iterating the process of Definition 1.3.6, we can decompose any element* $\mathtt{t} \in S(\mathcal{RT}, \mathcal{V})$ *into a composition of corollas whose roots are labelled by elements of the basis* $\mathcal{V}$*. We refer to this decomposition as the normal form of* $\mathtt{t}$*. It is unique up to the permutations of the non root entry of corollas.*

Definition 1.3.10. *Let* V *be a free* $\mathbb{K}$*-module with a fixed basis* $\mathcal{V}$*. Let* $\mathfrak{t}$ *be an element of* $\mathcal{T}(\mathcal{RT}, \mathcal{V})$*, and* σ *be an element in* $\mathbb{S}_n$*. Then* $\mathrm{Stab}(\mathfrak{t})$ *is isomorphic to* $\mathrm{Stab}(\sigma^* \mathfrak{t})$*. Therefore we can define the group* $\mathrm{Stab}(\mathtt{t})$ *where* $\mathtt{t}$ *is an element of* $\mathcal{S}(\mathcal{RT}, \mathcal{V})$ *as* $\mathrm{Stab}(\mathfrak{t})$ *where* $\mathfrak{t}$ *is in the pre-image of* $\mathtt{t}$ *under* $pr : \mathcal{T}(\mathcal{RT}, \mathcal{V}) \longrightarrow \mathcal{S}(\mathcal{RT}, \mathcal{V})$*.*

The group $\mathrm{Stab}(\mathfrak{t})$ can be computed by induction.

Proposition 1.3.11. *Let* V *be a free* $\mathbb{K}$*-module with a fixed basis* $\mathcal{V}$*, and* $\mathfrak{t}$ *be an element of* $\mathcal{T}(\mathcal{RT}, \mathcal{V})$*. Then* $\mathrm{Stab}(\mathfrak{t})$ *is isomorphic to* $\mathrm{Stab}(\mathrm{Dec}(\mathfrak{t})) \ltimes (\mathrm{Stab}(B_1) \times \dots \times \mathrm{Stab}(B_r))$*, where the semi-direct product is defined by the action of* $\mathrm{Stab}(\mathrm{Dec}(\mathfrak{t}))$ *which permutes isomorphic branches.*

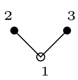
Proof: There is an obvious inclusion of $\mathrm{Stab}(\mathrm{Dec}(\mathfrak{t})) \ltimes (\mathrm{Stab}(B_1) \times \dots \times \mathrm{Stab}(B_r))$ into $\mathrm{Stab}(\mathfrak{t})$. Since any element of $\mathrm{Stab}(\mathfrak{t})$ can be written in a unique way as a product of an element in $\mathrm{Stab}(\mathrm{Dec}(\mathfrak{t}))$ and an element in $(\mathrm{Stab}(B_1) \times \dots \times \mathrm{Stab}(B_r))$ the inclusion is actually an isomorphic. □

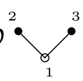
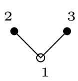
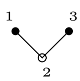
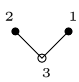
Definition 1.3.12. *Let* V *be a free* $\mathbb{K}$*-module with a fixed basis* $\mathcal{V}$*. Let* $\mathfrak{t}$ *be an element of* $\mathcal{T}(\mathcal{RT}, \mathcal{V})$*. We set :*

$$\mathcal{O}\,\mathfrak{t} := \sum_{\sigma \in \mathbb{S}_n / \mathrm{Stab}(\mathfrak{t})} \sigma^* \mathfrak{t},$$

Remark 1.3.13. *Let* V *be a free* $\mathbb{K}$*-module with a fixed basis* $\mathcal{V}$*. Let* $\mathfrak{t}$ *be an element of* $\mathcal{T}(\mathcal{RT}, \mathcal{V})$*. We clearly have* $\mathcal{O}\,\mathfrak{t} = \mathcal{O}\,\sigma^* \mathfrak{t}$ *for any permutation* σ*. Hence, this map passes to the quotient over coinvariants and induces a map of* $\mathbb{K}$*-modules* $\mathcal{O} : S(RT, V) \longrightarrow \Gamma(RT, V)$ *by linearity. Let* x *be an element of* V*. It is easy to show that* $\mathcal{O} \circ_1 (x)$ *is equal to* $\circ_1 \otimes x$ *(where* $\circ_1$ *is the unique* 1*-tree). If there is no risk of confusion we will denote this element just by* x*.*

Notice that, in general, $Tr(\mathfrak{t})$ differs from $\mathcal{O}\,\mathfrak{t}$.

Example 1.3.14. *Let* V *be a free* $\mathbb{K}$*-module with a fixed basis* $\mathcal{V}$*, and* x, y *be elements of* $\mathcal{V}$*, we compute* $\mathcal{O}$  $\{x, y, y\}$ *:*

$\mathcal{O}$  $(x, y, y) =$  $\otimes x \otimes y \otimes y +$  $\otimes y \otimes x \otimes y +$  $\otimes y \otimes y \otimes x.$

Given a free $\mathbb{K}$-module V, we want to compare the $\mathbb{K}$-modules $\Gamma(RT,V)$ and $S(RT,V)$. We show that they are isomorphic and that this isomorphism is realized by the map $\mathcal{O} : S(PreLie,V) \longrightarrow \Gamma(PreLie,V)$.

We use the following elementary result :

Lemma 1.3.15. *Let G be a group and X be a G-set. There exists a isomorphism between $\mathbb{K}[X]_G$ and $\mathbb{K}[X]^G$, where $\mathbb{K}[X]$ is the free $\mathbb{K}$-module over the set X.* □

Proposition 1.3.16. *Let V be a free $\mathbb{K}$-module with a fix basis $\mathcal{V}$. We define the set $\mathcal{O}\mathcal{S}(\mathcal{RT},\mathcal{V}) = \{\mathcal{O}\mathtt{t} \,|\, \mathtt{t} \in \mathcal{S}(\mathcal{RT},\mathcal{V})\}$. The set $\mathcal{O}\mathcal{S}(\mathcal{RT},\mathcal{V})$ forms a basis for the $\mathbb{K}$-module $\Gamma(RT,V)$.*

Proof: By definition the map $\mathcal{O} : S(RT,V) \longrightarrow \Gamma(RT,V)$ gives a set-map $\mathcal{O} : \mathcal{S}(\mathcal{RT},\mathcal{V}) \longrightarrow \mathcal{O}\mathcal{S}(\mathcal{RT},\mathcal{V})$ defined by the bijection of Lemma (1.3.15). □

1.4 The equivalence between $\Lambda PreLie$-algebras and p–$PreLie$-algebras

In this section we assume that $\mathbb{K}$ is a field of positive characteristic p . We show that the categories of $\Lambda PreLie$-algebras and $p-PreLie$-algebras are isomorphic. Let us observe that this implies that the category of $p-PreLie$-algebras is a monadic subcategory of $PreLie$-algebras.

In [Dok13] I. Dokas proves that $\Gamma PreLie$-algebra are p-restricted $PreLie$-algebras. Here we improve this result by showing that the restricted $PreLie$ structure is given by the $\Lambda PreLie$ action on $\Gamma PreLie$.

Remark 1.4.1. *In [Dok13] I. Dokas introduces a more general notion of p-restricted PreLie-algebras, here we consider the less general definition given by in A. Dzhumadil'daev in [Dzh01].*

Recall that $\Lambda(PreLie,V)$ is the target of the epimorphism given by the epi-mono decomposition of the trace map.

$$\begin{tikzcd}
Ker(Tr_V) \arrow[dr, hook] & & \\
& S(PreLie,V) \arrow[rr, "Tr_V"] \arrow[dr] & & \Gamma(PreLie,V) \\
& & \Lambda(PreLie,V) \arrow[ur, hook] &
\end{tikzcd}$$

We compute the kernel of the trace map.

Proposition 1.4.2. *Let V be a $\mathbb{K}$-module with a fixed basis $\mathcal{V}$. Let $\mathtt{t}$ be an element of $\mathcal{S}(\mathcal{RT},\mathcal{V})$. We have $Tr(\mathtt{t}) = |\mathrm{Stab}(\mathtt{t})|\mathcal{O}\mathtt{t}$.*

Proof: Let $\mathtt{t}$ be equal to $\tau(x_1,\ldots,x_n)$ for some tree τ and $x_1,\ldots,x_n$ elements of $\mathcal{V}$. Then the following equation holds :

$$Tr(\tau(x_1,\ldots,x_n)) = \sum_{\sigma\in\mathbb{S}_n} \sigma^* \tau\langle x_1,\ldots,x_n\rangle =$$

$$\sum_{\mathrm{Stab}(\tau(x_1,\ldots,x_n))} \sum_{\sigma\in\frac{\mathbb{S}_n}{\mathrm{Stab}(\tau(x_1,\ldots,x_n))}} \sigma^* \tau\langle x_1,\ldots,x_n\rangle = |\mathrm{Stab}(\mathtt{t})|\mathcal{O}\mathtt{t}.$$

□

Corollary 1.4.3. *Let V be a $\mathbb{K}$-module with a fixed basis $\mathcal{V}$. The kernel of the trace map is linearly generated by the elements $\mathtt{t}$ of $\mathcal{S}(\mathcal{RT},\mathcal{V})$ such that $|\mathrm{Stab}(\mathtt{t})|$ is a multiple of p.*

Proof: The proof follows from Proposition 1.4.2 and from the observation that the map $\mathtt{t} \mapsto \mathcal{O}\mathtt{t}$ defines a one-to-one correspondence from a basis of $S(RT,V)$ to a basis of $\Gamma(RT,V)$. □

Lemma 1.4.4. *Let V be a $\mathbb{K}$-module with a fixed basis $\mathcal{V}$. Let $\mathtt{t}$ be an element of $\mathcal{S}(\mathcal{RT},\mathcal{V})$. Then $\mathtt{t}$ has trace zero if and only if the expression $F_{n+p}(x,\underbrace{B,\dots,B}_{p},B_1,\dots,B_n)$ with B and $B_i \in \mathcal{S}(\mathcal{RT},\mathcal{V})$ appears in its normal form.*

Proof: The proof follows from Proposition 1.3.11, since $\mathrm{Stab}(\mathtt{t})$ is an iterated product of semi-direct products of symmetric groups representing the stabilizers of the corollas which compose the normal form of $\mathtt{t}$. □

We improve this result and find a smaller collection of generators. First we fix the notation for multinomial coefficients.

Notations 1.4.5. *Let $k_0,\dots,k_r$ be natural numbers and $n=\sum_{i=0}^r k_i$. We define the multinomial coefficient $(k_0,\dots,k_r)$ to be $\dfrac{n!}{k_0!\dots k_r!}$.*

Lemma 1.4.6. *Let V be a $\mathbb{K}$-module, $x\in V$ and $B,B_i\in S(PreLie,V)$. The following equation holds :*

$$F_p(F_{n-p}(x,B_1,\dots,B_{n-p}),\underbrace{B,\dots,B}_{p}) = F_n(x,\underbrace{B,\dots,B}_{p},B_1,\dots,B_{n-p})$$
$$+\sum_{\substack{i_0+\dots+i_{v-p}=p\\ i_0<p}}(i_0,\dots,i_{n-p})F_{n-p+i_0}(x,\underbrace{B,\dots,B}_{i_0},F_{i_1}(B_1,\underbrace{B,\dots,B}_{i_1}),\dots,F_{i_{n-p}}(B_{n-p},\underbrace{B,\dots,B}_{i_{n-p}}))$$

in $S(PreLie,S(PreLie(S(PreLie,V)))$.

Proof: Immediate consequence of the definition of composition of trees. □

For $g_i\in S(\mathcal{RT},(S(\mathcal{RT},\mathcal{V}))$ and f in $\mathcal{RT}(n)$ we denote by $f(g_1,\dots,g_n)$ the element in $S(\mathcal{RT},S(\mathcal{RT},S(\mathcal{RT},\mathcal{V})))$ representing their composite.

Definition 1.4.7. *Let V be a $\mathbb{K}$-module with a fixed basis $\mathcal{V}$. A subset G of $\mathcal{S}(\mathcal{RT},\mathcal{S}(\mathcal{RT},\mathcal{V}))$ is said to generate $Ker(Tr)$ if any element $\mathtt{t}$ of $\mathcal{S}(\mathcal{RT},\mathcal{V})$ is in $Ker(Tr)$ if and only if it is the image by $\mu\circ\mu : S(RT,S(RT,S(RT,V)))\longrightarrow S(RT,V)$ of a linear combination of elements of the form $f(g_1,\dots,g_n)$, where at least one g_i is in G.*

Lemma 1.4.8. *Let V be a $\mathbb{K}$-module with a fixed basis $\mathcal{V}$. The set $K:=\{F_p(A,\underbrace{B,\dots,B}_{p})|A,B\in \mathcal{S}(\mathcal{RT},\mathcal{V})\}$ generates $Ker(Tr)$.*

Proof: We compute $F_p(F_v(x,B_1,\dots,B_v),\underbrace{B,\dots,B}_{p})$. By Lemma (1.4.6) it is equal to :

$$\sum_{i_0+\dots+i_v=p}(i_0,\dots,i_v)F_{v+i_0}(x,\underbrace{B,\dots,B}_{i_0},F_{i_1}(B_1,\underbrace{B,\dots,B}_{i_1}),\dots,F_{i_v}(B_v,\underbrace{B,\dots,B}_{i_v})),$$

We have that $(i_0,\dots,i_v)=\binom{p}{i_h}k_{i_h}\,\forall 0\le h\le n$ for an integer k_{i_h} and so this coefficient differs from 0 modulo p if and only if $i_k=0$ for all but one index i_k. Then we get a multinomial coefficient $(0,\dots,p,\dots,0)=1$. Therefore we have :

$$F_p(F_v(x,B_1,\dots,B_v),\underbrace{B,\dots,B}_{p}) = F_{v+p}(x,\underbrace{B,\dots,B}_{p},B_1,\dots,B_v)$$
$$+\sum_{i\in\{1,\dots,v\}}F_v(x,B_1,\dots,B_{i-1},F_p(B_i,\underbrace{B,\dots,B}_{p}),B_{i+1},\dots,B_v).$$

Let $\mathtt{t}$ be an element of $\mathcal{S}(\mathcal{RT},\mathcal{V})$. By Lemma 1.4.4, $\mathtt{t}\in Ker(Tr)$ if and only if its normal form contains a corolla of the form $g_i=F_{v+p}(x,\underbrace{B,\dots,B}_{p},B_1\dots,B_v)$. We use the above computation and the multi-linearity of a tree component to express this factor $F_{v+p}(x,\underbrace{B,\ldots,B}_{p},B_1,\ldots,B_v)$ as difference of terms of the form $f(g_1,\ldots,g_n)$ where at least one g_i is in K. This proves the "only if" part of our claim.

To check the "if" part of our statement we use the above formula to express a factor $F_p(A,\underbrace{B,\ldots,B}_{p})$ with $\mathrm{Dec}(A)=F_v(x,B_1,\ldots,B_v)$ as sum of terms with either a factor

$$F_{v+p}(x,\underbrace{B,\ldots,B}_{p},B_1,\ldots,B_v)\in Ker(Tr)$$

or a factor

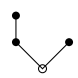
$$F_p(B_i,\underbrace{B,\ldots,B}_{p})$$

where B_i has strictly less vertices than A. Repeating the computation inductively of the equation and using the multi-linearity of the tree components we obtain, on the right side of the equation, a sum of elements in $Ker(Tr)$. Since Tr is a morphism of monads any $f(g_1,\ldots,g_n)\in S(RT,S(RT,S(RT,V)))$ such that at least one g_i is in K is in $Ker(Tr)$. □

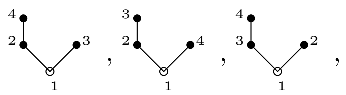
The following definition appears in the literature with the name of heap order trees, see [CK98].

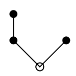
Definition 1.4.9. *Let τ be a non labelled tree. A labelling of vertices of τ is said to be an increasing labelling if it defines a total order refinement of the partial order on vertices induced by the tree, with the root as least element. We denote the number of possible increasing labellings by $\lambda(\tau)$.*

Example 1.4.10. *Consider the following non labelled tree :*

 .

Then the following are the only three possible increasing labellings of τ :



and $\lambda($  $)=3$.

Definition 1.4.11. *We denote by $n-ILTrees$ the set of n-trees with an increasing labelling on vertices.*

The following lemma is already treated in the literature using a different notation, see for example the operator N in [CK98], the growth operator in [Hof03]. It is used in the Butcher series for example in [Bro00] and in [Liv06].

Lemma 1.4.12. *In $PreLie$ the following equation holds :*

$$\varphi(\{\{\ldots\{\underbrace{\{-,-\},-\}\ldots\}-}_{n}\})=\sum_{\tau\in n-ILTrees}\tau(\underbrace{-,\ldots,-}_{n}),$$

where φ is the natural isomorphism between $PreLie$ and RT.

Proof: We proceed by induction on n. If n is equal to 1 then the result is obviously true. Suppose the statement true at rank $n-1$. We then have :

$$\begin{gathered}\varphi_V(\{\{\ldots\{\{y_1,y_2\},y_3\}\ldots\}y_n\}) = \\ \mu\Big(\begin{smallmatrix}2\bullet\\ \vert\\ 1\circ\end{smallmatrix}\Big(\sum_{\tau\in(n-1)-ILTrees}\tau(y_1,\ldots,y_{n-1})\Big),y_n\Big)) = \\ \sum_{\tau\in(n-1)-ILTrees}\mu\Big(\begin{smallmatrix}2\bullet\\ \vert\\ 1\circ\end{smallmatrix}(\tau(y_1,\ldots,y_{n-1})),y_n\Big).\end{gathered}$$

If τ is in $(n-1)-ILTree$, then $\begin{smallmatrix}2\bullet\\ \vert\\ 1\circ\end{smallmatrix}(\tau(y_1,\ldots,y_{n-1})),y_n)$ is the sum of $n-1$ distinct increasing labelling n-trees. To any increasing labelling of an n-tree is associated a labelling $(n-1)$-tree obtained by dropping the leave labelled with n. We readily conclude that all the increasing labellings n-tree appear once in the sum. $\square$

Proposition 1.4.13. *Let V be a $\mathbb{K}$-module with a fixed basis $\mathcal{V}$ and x,y be elements of $\mathcal{V}$. In $S(PreLie,V)$, the following equation holds :*

$$\varphi_V(\{\{\ldots\{\{x,\underbrace{y\},y\}\ldots\}y}_{n}\}) = \sum_{\tau\in(n+1)-Trees}\lambda(\tau)\,\tau(x,y,\ldots,y),$$

where $\lambda(\tau)$ is the number of increasing labellings of τ, and φ_V the natural isomorphism induced by the isomorphism of operads between $PreLie$ and RT.

Proof: This identity follows from Lemma 1.4.12. $\square$

Lemma 1.4.14. *Let V be a $\mathbb{K}$-module. Let x,y be elements of V. The equation :*

$$\{\{\ldots\{\{x,\underbrace{y\},y\}\ldots\}y}_{p}\} = \{x,\{\ldots\{\underbrace{\{y,y\}\ldots\}y}_{p}\}\}$$

holds in a $PreLie$-algebra (V,γ) if and only if $\gamma(F_p(x,\underbrace{y,\ldots,y}_{p})) = 0$.

Proof: By Proposition 1.4.13 the left side of the equation can be expressed as a sum of trees with coefficients the number of possible increasing labellings. Let τ be a tree and B be a branch of τ. That is :

$$\tau = F_{m_B+r}(\circ_1, \underbrace{B,\ldots,B}_{m_B}, B_1,\ldots,B_r),$$

where $B_i \ncong B$ for all $i\in\{1,\ldots,r\}$. We denote by the symbol n_B the number of vertices of B and by S the tree

$$F_r(\circ_1, B_1,\ldots,B_r).$$

It is easy to check that the coefficient $\lambda(\tau)$ is equal to $\binom{p}{n_B m_B}(\underbrace{n_B,\ldots,n_B}_{m_B})\frac{1}{m_B!}\lambda(B)^{m_B}\lambda(S)$, where the symbol $(\underbrace{n_B,\ldots,n_B}_{m_B})$ refers to the Notation 1.4.5. Since p is a prime number $p\nmid\lambda(\tau)$ just in two cases :

1. $n_B = p$, and $m_B = 1$;
2. $n_B = 1$, and $m_B = p$.

In the first case we obtain the following sum

$$\sum_{\tau\in p-Trees}\lambda(\tau)(x,\tau(y,\ldots,y)).$$

But $\lambda(\tau)$ is equal to $\lambda(x,\tau(y,\ldots,y))$, and therefore applying φ^{-1} we get $\{x,\{\ldots\{\underbrace{\{y,y\}\ldots\}y}_{p}\}\}$.

In the second case we obtain $F_p(x,y,\ldots,y)$. This completes the proof. $\square$

Proposition 1.4.15. *If (V,γ) is a $\Lambda PreLie$-algebra, then $\{-,-\}: V\otimes V \longrightarrow V$ deduced from the $PreLie$-algebra structure of V is a $p-PreLie$-algebra.*

Proof: By Lemma 1.4.14 (V,γ) satisfies the relation of p-restricted $PreLie$-algebra. □

Theorem 1.4.16. *The construction of Proposition 1.4.15 gives an isomorphism between the categories of $\Lambda PreLie$-algebras and of $p-PreLie$-algebras.*

Proof: The category of $\Lambda PreLie$-algebras is isomorphic to the subcategory of $PreLie$-algebras (V,γ) such that the following diagram admits a factorization :

$$\begin{tikzcd} S(PreLie,V) \arrow[r, "\gamma"] \arrow[d, two heads] & V \\ \Lambda(PreLie,V) \arrow[ur, dashed] & \end{tikzcd} .$$

This diagram admits an extension if and only if the composition

$$Ker(Tr) \longrightarrow S(PreLie,V) \longrightarrow V$$

is zero. By Lemma 1.4.8 this is equivalent to say that (V,γ) is a $p-PreLie$-algebra. □

Proposition 1.4.17. *The morphism $\Gamma(Lie,-) \longrightarrow \Gamma(PreLie,-)$ factors through $\Lambda(PreLie,-)$.*

Proof: To determine the image of the map it is enough to compute the image of a general bracket $[-,-]$ and Frobenius power $-^{[p]}$. They are respectively

$$\{-,-\} - (1,2)^*\{-,-\}$$

and

$$\underbrace{\{\ldots\{\{-,-\},-\}\ldots,-\}}_{p}.$$

These operations are contained in the sub-monad $\Lambda(PreLie,-)$. □

1.5 The $\Gamma(PreLie,-)$ monad

We go back to the case where $\mathbb{K}$ is a commutative ring. We express the formula to compute the composition morphism of the monad $\Gamma(PreLie,-)$. We use this formula to recover a normal form for the elements of $\Gamma(PreLie,V)$.

1.5.1 A formula for the $\Gamma(PreLie,-)$ composition

Let V be a free $\mathbb{K}$-module, we show an explicit formula for the composition in $\Gamma(PreLie,V)$.

By Proposition 1.3.16 we have an explicit basis of $\Gamma(PreLie,V)$. So we compute the composition on it, and then extend by linearity.

Theorem 1.5.1. *Let V be a free $\mathbb{K}$-module with a fixed basis $\mathcal{V}$. Let υ be an element of $\mathcal{RT}(n)$ and $\mathtt{t}_1,\ldots,\mathtt{t}_n$ be elements of $\mathcal{S}(\mathcal{RT},\mathcal{V})$. We assume that the composite of $\upsilon\ \mathtt{t}_1,\ldots,\mathtt{t}_n$ in $S(PreLie,\mathbb{Z}[\mathcal{V}])$, where $\mathbb{Z}[\mathcal{V}]$ denotes the free $\mathbb{Z}$-module generated by $\mathcal{V}$, has the expansion :*

$$\mu(\upsilon(\mathtt{t}_1,\ldots,\mathtt{t}_n)) = \sum_{\mathtt{t}\in\mathcal{S}(\mathcal{RT},\mathcal{V})} \chi(\mathtt{t})\,\mathtt{t},$$

We then have the identity :

$$\tilde{\mu}(\mathcal{O}\,\upsilon(\mathcal{O}\,\mathtt{t}_1,\ldots,\mathcal{O}\,\mathtt{t}_n)) = \sum \frac{\chi(\mathtt{t})|\mathrm{Stab}(\mathtt{t})|}{|\mathrm{Stab}(\upsilon(\mathtt{t}_1,\ldots,\mathtt{t}_n))|\prod_i|\mathrm{Stab}(\mathtt{t}_i)|}\,\mathcal{O}\,\mathtt{t},$$

in $\Gamma(RT,V)$ where we consider the map $\tilde{\mu}:\Gamma(RT,\Gamma(RT,V))\to\Gamma(RT,V)$ and $\mathrm{Stab}(\upsilon(\mathtt{t}_1,\ldots,\mathtt{t}_n))$ is the stabilizer of $\upsilon(\mathtt{t}_1,\ldots,\mathtt{t}_n)\in S(\mathcal{RT},S(\mathcal{RT},\mathcal{V}))$. (In the latter case we apply the definition of the stabilizer to the set $S(\mathcal{RT},\mathcal{W})$, where we take $\mathcal{W}=S(\mathcal{RT},\mathcal{V})$.)

Proof: We start with a preliminary step. We use the notation $PreLie_{\mathbb{K}}$ to distinguish the coefficient in which is defined the operad $PreLie$. We consider the following morphism

$$\Gamma(PreLie_{\mathbb{Z}}, \mathbb{Z}[\mathcal{V}]) \to \Gamma(PreLie_{\mathbb{K}}, \mathbb{K}[\mathcal{V}]),$$

induced by the canonical morphism $i_{PreLie} : PreLie_{\mathbb{Z}} \to PreLie_{\mathbb{K}}$ and $i_V : \mathbb{Z}[\mathcal{V}] \to \mathbb{K}[\mathcal{V}]$. We have the following commutative diagram :

$$\begin{tikzcd}
\Gamma(PreLie_{\mathbb{Z}}, \Gamma(PreLie_{\mathbb{Z}}, \mathbb{Z}[\mathcal{V}])) \arrow[r] \arrow[d, "\cong"'] & \Gamma(PreLie_{\mathbb{K}}, \Gamma(PreLie_{\mathbb{K}}, \mathbb{K}[\mathcal{V}])) \arrow[d, "\cong"] \\
\Gamma(PreLie_{\mathbb{Z}} \underset{\sim}{\square} PreLie_{\mathbb{Z}}, \mathbb{Z}[\mathcal{V}])) \arrow[r] \arrow[dd, "\tilde{\mu}_{\mathbb{Z}}"'] & \Gamma(PreLie_{\mathbb{K}} \underset{\sim}{\square} PreLie_{\mathbb{K}}, \mathbb{K}[\mathcal{V}])) \arrow[d, "\cong"] \\
 & \Gamma((PreLie \underset{\sim}{\square} PreLie)_{\mathbb{K}}, \mathbb{K}[\mathcal{V}])) \arrow[d, "\tilde{\mu}_{\mathbb{K}}"] \\
\Gamma(PreLie_{\mathbb{Z}}, \mathbb{Z}[\mathcal{V}]) \arrow[r] & \Gamma(PreLie_{\mathbb{K}}, \mathbb{K}[\mathcal{V}])).
\end{tikzcd}$$

We assume $\mathbb{K} = \mathbb{Q}$ first and we check the relation in this case. We use the fact that $\mathcal{O}\,\mathtt{t} = \dfrac{Tr(\mathtt{t})}{|Stab(\mathtt{t})|}$ and that $Tr : S(PreLie_{\mathbb{Q}}, -) \to \Gamma(PreLie_{\mathbb{Q}}, -)$ is an isomorphism of monads to get the identity :

$$\begin{aligned}
\mu(Tr(\upsilon(Tr(\mathtt{t}_1), \ldots, Tr(\mathtt{t}_n)))) &= Tr(\mu(\upsilon(\mathtt{t}_1, \ldots, \mathtt{t}_n))) \\
&= \sum_{\mathtt{t} \in \mathcal{S}(\mathcal{RT}, \mathcal{V})} \chi(\mathtt{t}) Tr(\mathtt{t}) \\
&= \sum_{\mathtt{t} \in \mathcal{S}(\mathcal{RT}, \mathcal{V})} \chi(\mathtt{t}) |Stab(\mathtt{t})| \mathcal{O}\,\mathtt{t},
\end{aligned}$$

and

$$\mu(Tr(\upsilon(Tr(\mathtt{t}_1), \ldots, Tr(\mathtt{t}_n)))) = |Stab(\upsilon(\mathtt{t}_1, \ldots, \mathtt{t}_n))| \prod_i |\mathrm{Stab}(\mathtt{t}_i)| \mu(\mathcal{O}\,\upsilon(\mathcal{O}\,\mathtt{t}_1, \ldots, \mathcal{O}\,\mathtt{t}_n))).$$

We now consider the case $\mathbb{K} = \mathbb{Z}$. We have a monomorphism $\Gamma(PreLie_{\mathbb{Z}}, \mathbb{Z}[\mathcal{V}]) \hookrightarrow \Gamma(PreLie_{\mathbb{Q}}, \mathbb{Q}[\mathcal{V}])$ which respects the composition product. Thus the coefficients computed for the basis of $\Gamma(PreLie_{\mathbb{Q}}, \mathbb{Q}[\mathcal{V}])$ correspond to the coefficients for the basis of $\Gamma(PreLie_{\mathbb{Z}}, \mathbb{Z}[\mathcal{V}])$.

We consider the general case. The canonical morphism $\Gamma(PreLie_{\mathbb{Z}}, \mathbb{Z}[\mathcal{V}]) \to \Gamma(PreLie_{\mathbb{K}}, \mathbb{K}[\mathcal{V}])$ carries the relation, which is verified over $\mathbb{Z}$ for our basis elements, to the same relation over $\mathbb{K}$. □

1.5.2 Decompositions in corollas and normal form

Let V be a free $\mathbb{K}$-module with a fixed basis $\mathcal{V}$. Let t be an element of $\Gamma(PreLie, V)$; recall that by Proposition 1.3.16, t is a linear combination of elements in $\mathcal{O}\,\mathcal{S}(\mathcal{RT}, \mathcal{V})$.

We present how to construct the elements of $\Gamma(PreLie, V)$ from corollas.

Lemma 1.5.2. *Let V be a free $\mathbb{K}$-module with a fixed basis $\mathcal{V}$. Let x be an element of $\mathcal{V}$, and $\mathtt{t}_1, \ldots, \mathtt{t}_r$ be elements of $\mathcal{S}(\mathcal{RT}, \mathcal{V})$. Then*

$$\tilde{\mu}(\mathcal{O}\,F_r\{x, \mathcal{O}\,\mathtt{t}_1, \ldots, \mathcal{O}\,\mathtt{t}_r\}) = \mathcal{O}(\mu(F_r(x, \mathtt{t}_1, \ldots, \mathtt{t}_r))).$$

Proof: The only thing to check is that the coefficient which appears in the left terms is one. □

Lemma 1.5.3. *Let V be a free $\mathbb{K}$-module with a fixed basis $\mathcal{V}$. If $\mathtt{t}$ is an element of $\mathcal{S}(\mathcal{RT}, \mathcal{V})$, then $\mathcal{O}\,\mathtt{t}$ is equal to $\tilde{\mu}(\mathcal{O}\,F_r(x_a, \mathcal{O}\,B_1, \ldots, \mathcal{O}\,B_r))$ where $F_r[x_a, B_1, \ldots, B_r]$ is $Dec(\mathtt{t})$.*

Proof: We apply Lemma 1.5.2 to $Dec(\mathtt{t})$. □

Definition 1.5.4. *Let V be a free $\mathbb{K}$-module with a fixed basis $\mathcal{V}$, and $\mathtt{t}$ be an element of $\mathcal{S}(\mathcal{RT},\mathcal{V})$. We call normal form of $\mathcal{O}\mathtt{t}$ its expression in iterated composition of elements of the form $\mathcal{O}(F_r(x,-,\dots,-))$ with x an element of $\mathcal{V}$ deduced from the normal form of $\mathtt{t}$.*

Proposition 1.5.5. *Let V be a free $\mathbb{K}$-module with a fixed basis $\mathcal{V}$. If $\mathtt{t}$ is an element of $\mathcal{S}(\mathcal{RT},\mathcal{V})$ then $\mathcal{O}\mathtt{t}$ admits a unique normal form.*

Proof: We apply Lemma 1.5.3 recursively to get a bijection between the normal form in $S(RT,V)$ and the normal form of $\Gamma(RT,V)$. □

Proposition 1.5.6. *The set of monomials in normal form gives a basis of the $\mathbb{K}$-module $\Gamma(RT,V)$.*

Proof: It is easy to prove that the set of monomials in normal forms is in bijection with the set $\mathcal{S}(\mathcal{RT},\mathcal{V})$. By Proposition 1.3.16 the set $\mathcal{O}\mathcal{S}(\mathcal{RT},\mathcal{V})$ forms a basis for $\Gamma(RT,V)$ and it is in bijection with $\mathcal{S}(\mathcal{RT},\mathcal{V})$. □

1.5.3 A presentation for $\Gamma(PreLie,-)$

To describe the structure of $\Gamma PreLie$-algebras we first show how to construct some polynomial abstract operations from a tree. We define a new type of algebras, Cor-algebras, using just the abstract operations defined by corollas. We conclude the section by proving that Cor-algebras coincide with $\Gamma PreLie$-algebras.

Let (V,γ) be a $\Gamma PreLie$-algebra and E_n be the free $\mathbb{K}$-module generated by a set of variables $\mathcal{E}_n = \{e_i\}_{i\in\{1,\dots,n\}}$. We consider an element of $\Gamma(RT,E_n)$. It can be written as a linear combination of elements of the form :

$$\mathcal{O}(\rho(\underbrace{e_1\otimes\dots\otimes e_1}_{r_1}\otimes\dots\otimes\underbrace{e_n\otimes\dots\otimes e_n}_{r_n})),$$

for some $\rho\in\mathcal{RT}$.

Let $v_1,\dots,v_n$ be elements in V, we define the morphism $\psi_{v_1,\dots,v_n}: E_n\longrightarrow V$ by linear extension of $\psi_{v_1,\dots,v_n}(e_i)=v_i$. By functoriality it induces a morphism $\psi_{v_1,\dots,v_n}:\Gamma(RT,E_n)\longrightarrow\Gamma(RT,V)$.

Definition 1.5.7. *Any element α of $\Gamma(\mathcal{RT},\mathcal{E}_n)$ is of the form*

$$\alpha=\mathcal{O}(\rho(\underbrace{e_1\otimes\dots\otimes e_1}_{r_1}\otimes\dots\otimes\underbrace{e_n\otimes\dots\otimes e_n}_{r_n}))$$

and induces a function

$$\varphi_{\underbrace{e_1\otimes\dots\otimes e_1}_{r_1}\otimes\dots\otimes\underbrace{e_n\otimes\dots\otimes e_n}_{r_n}}:V^{\times n}\longrightarrow V$$

defined by :

$$\varphi_{\underbrace{e_1\otimes\dots\otimes e_1}_{r_1}\otimes\dots\otimes\underbrace{e_n\otimes\dots\otimes e_n}_{r_n}}(v_1,\dots,v_n)=\gamma(\psi_{v_1,\dots,v_n}(\alpha)).$$

The elements e_i have the role of abstract variables. We denote the set of these functions by $AbsOp_n$ and we set $AbsOp=\coprod_{n\in\mathbb{N}}AbsOp_n$.

Definition 1.5.8. *The group $\mathbb{S}_n$ acts on the set $AbsOp_n$ by permutation of the indices $\{1,\dots,n\}$. Let σ be an element of $\mathbb{S}_n$. Let $\varphi\in AbsOp$ be the element associated to*

$$\alpha=\mathcal{O}(\rho(\underbrace{e_1\otimes\dots\otimes e_1}_{r_1}\otimes\dots\otimes\underbrace{e_n\otimes\dots\otimes e_n}_{r_n}))$$

we define :

$$(\sigma_*\varphi)_{\underbrace{e_1\otimes\ldots\otimes e_1}_{r_1}\otimes\ldots\otimes\underbrace{e_n\otimes\ldots\otimes e_n}_{r_n}}$$

to be the element associated to

$$\mathcal{O}(\rho(\underbrace{e_{\sigma(1)}\otimes\ldots\otimes e_{\sigma(1)}}_{r_1}\otimes\ldots\otimes\underbrace{e_{\sigma(n)}\otimes\ldots\otimes e_{\sigma(n)}}_{r_n}))$$

From now on, we denote $\varphi_{\underbrace{e_1\otimes\ldots\otimes e_1}_{r_1}\otimes\ldots\otimes\underbrace{e_n\otimes\ldots\otimes e_n}_{r_n}}$ *by* $\varphi_{r_1,\ldots,r_n}$ *or* φ.

Proposition 1.5.9. *The following equations hold :*

$$\sigma_*\varphi(v_1,\ldots,v_n) = \varphi(v_{\sigma(1)},\ldots,v_{\sigma(n)}), \tag{1.1}$$

where $\sigma\in\mathbb{S}_n$,

$$\varphi_{r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_n}(v_1,\ldots,v_n) = \varphi_{r_1,\ldots,r_{i-1},r_{i+1},\ldots,r_n}(v_1,\ldots,v_{i-1},v_{i+1}\ldots,v_n). \tag{1.2}$$

$$\varphi_{r_1,\ldots,r_i\ldots,r_n}(v_1,\ldots,\lambda v_i,\ldots,v_n) = \lambda^{r_i}(\varphi_{r_1,\ldots,r_i,\ldots,r_n}(v_1,\ldots,v_i,\ldots,v_n)). \tag{1.3}$$

If the function

$$\varphi_{r_1,\ldots,r_i,r_{i+1},\ldots,r_n}$$

is commutative in the variables i *and* $i+1$ *i.e.* $(i,i+1)^*\varphi=\varphi$, *and* $v_i = v_{i+1}$, *then*

$$\begin{aligned}\varphi_{r_1,\ldots,r_i,r_{i+1},\ldots,r_n}&(v_1,\ldots,v_i,v_{i+1},\ldots,v_n) =\\ &\binom{r_i+r_{i+1}}{r_i}\varphi_{r_1,\ldots,r_{i-1},r_i+r_{i+1},r_{i+2},\ldots,r_n}(v_1,\ldots,v_{i-1},v_i,v_{i+2},\ldots,v_n).\end{aligned} \tag{1.4}$$

We have

$$\varphi_{r_1,\ldots,r_i\ldots,r_n}(v_1,\ldots,a\underset{i}{+}b,\ldots,v_n) = \sum_{s=0}^{r_i}\varphi_{r_1,\ldots,s,r_i-s,\ldots,r_n}(v_1,\ldots,a,b,\ldots,v_n). \tag{1.5}$$

where $v_i = a+b$

Proof: These identities are immediate consequences of the multi-linearity of the operadic composition. ☐

Proposition 1.5.10. *Let* $\{-;\underbrace{-,\ldots,-}_{n}\}_{r_1,\ldots,r_n}$ *be the function defined by the corolla :*

$$\mathcal{O}\,F_{(\sum r_\bullet)}(e_1,\underbrace{e_2,\ldots,e_2}_{r_1},\ldots,\underbrace{e_{n+1},\ldots,e_{n+1}}_{r_n}).$$

We have the unit relation :

$$\{-;\} = id, \tag{1.6}$$

and a distribution relation, which we formally write :

$$\begin{aligned}\{\{x;y_1,\ldots,y_n\}_{r_1,\ldots,r_n};z_1,\ldots,z_m\}_{s_1,\ldots,s_m} &=\\ \sum_{s_i=\beta_i+\sum\alpha_i^{\bullet,\bullet}}\frac{1}{\prod(r_j!)}\{x;\{y_1;z_1,\ldots,z_m\}_{\alpha_1^{1,1},\ldots,\alpha_m^{1,1}},&\ldots,\{y_1;z_1,\ldots,z_m\}_{\alpha_1^{1,r_1},\ldots,\alpha_m^{1,r_1}},\\ \ldots,\{y_n;z_1,\ldots,z_m\}_{\alpha_1^{n,1},\ldots,\alpha_m^{n,1}},&\ldots,\{y_n;z_1,\ldots,z_m\}_{\alpha_1^{n,r_n},\ldots,\alpha_m^{n,r_n}},\\ &z_1,\ldots,z_m\}_{1,\ldots,1,\beta_1,\ldots,\beta_m},\end{aligned} \tag{1.7}$$

where, to give a sense to the latter formula, we use that the denominators $r_j!$ *divide the coefficient of the terms of the reduced expression which we get by applying relations (1.1) and (1.4) to simplify terms with repeated inputs on the right hand side.*

Proof: Let $\mathcal{V}$ be a basis of V and $x, y_1, \ldots, y_n, z_1, \ldots, z_m \in \mathcal{V}$ with possible repetition, the general case follows from relation (1.3) and relation (1.5). The Proposition is an immediate consequence of the formula of Theorem 1.5.1, where we take $\upsilon = F_{s_1+\ldots+s_m}$,

$$\mathtt{t}_0 = F_{r_1+\ldots+r_n}(x, \underbrace{y_1, \ldots, y_1}_{r_1}, \ldots, \underbrace{y_n, \ldots, y_n}_{r_n})$$

which we plug into the root of υ and $\mathtt{t}_1 = \circ_1\ (z_1), \ldots, \mathtt{t}_{s_1} = \circ_1\ (z_1), \ldots, \mathtt{t}_{s_1+\ldots+s_{m-1}+1} = \circ_1\ (z_m), \ldots, \mathtt{t}_{s_1+\ldots+s_m} = \circ_1\ (z_m)$ which we plug into the leaves of υ. More precisely, the expansion of the composite is a linear combination of elements of the form $\mathcal{O}\,\mathtt{t}$ where

$$\begin{aligned}
\mathtt{t} = F_{\sum b^{\bullet}_{\bullet} + \sum \beta_\bullet}(x,\\
&\underbrace{F_{\alpha_1^{1,1}+\ldots+\alpha_m^{1,1}}(y_1, \underbrace{z_1, \ldots, z_1}_{\alpha_1^{1,1}}, \ldots, \underbrace{z_m, \ldots, z_m}_{\alpha_m^{1,1}}), \ldots, F_{\alpha_1^{1,1}+\ldots+\alpha_m^{1,1}}(y_1, \underbrace{z_1, \ldots, z_1}_{\alpha_1^{1,1}}, \ldots, \underbrace{z_m, \ldots, z_m}_{\alpha_m^{1,1}})}_{b_1^1}, \ldots\\
&\underbrace{F_{\alpha_1^{1,\gamma_1}+\ldots+\alpha_m^{1,\gamma_1}}(y_1, \underbrace{z_1, \ldots, z_1}_{\alpha_1^{1,\gamma_1}}, \ldots, \underbrace{z_m, \ldots, z_m}_{\alpha_m^{1,\gamma_1}}), \ldots, F_{\alpha_1^{1,\gamma_1}+\ldots+\alpha_m^{1,\gamma_1}}(y_1, \underbrace{z_1, \ldots, z_1}_{\alpha_1^{1,\gamma_1}}, \ldots, \underbrace{z_m, \ldots, z_m}_{\alpha_m^{1,\gamma_1}})}_{b_{\gamma_1}^1} \ldots\\
&\underbrace{F_{\alpha_1^{n,1}+\ldots+\alpha_m^{n,1}}(y_n, \underbrace{z_1, \ldots, z_1}_{\alpha_1^{n,1}}, \ldots, \underbrace{z_m, \ldots, z_m}_{\alpha_m^{n,1}}), \ldots, F_{\alpha_1^{n,1}+\ldots+\alpha_m^{1,1}}(y_n, \underbrace{z_1, \ldots, z_1}_{\alpha_1^{n,1}}, \ldots, \underbrace{z_m, \ldots, z_m}_{\alpha_m^{n,1}})}_{b_1^n}, \ldots\\
&\underbrace{F_{\alpha_1^{n,\gamma_1}+\ldots+\alpha_m^{1,1}}(y_n, \underbrace{z_1, \ldots, z_1}_{\alpha_1^{n,\gamma_n}}, \ldots, \underbrace{z_m, \ldots, z_m}_{\alpha_m^{n,\gamma_n}}), \ldots, F_{\alpha_1^{n,\gamma_n}+\ldots+\alpha_m^{n,\gamma_n}}(y_1, \underbrace{z_1, \ldots, z_1}_{\alpha_1^{n,\gamma_n}}, \ldots, \underbrace{z_m, \ldots, z_m}_{\alpha_m^{n,\gamma_n}})}_{b_{\gamma_n}^n} \ldots\\
&\underbrace{y_1, \ldots, y_1}_{b_{\gamma_1+1}^1}, \ldots, \underbrace{y_n, \ldots, y_n}_{b_{\gamma_n+1}^n}, \underbrace{z_1, \ldots, z_1}_{\beta_1}, \ldots, \underbrace{z_m, \ldots, z_m}_{\beta_m}),
\end{aligned}$$

where $1 \leq \gamma_i \leq r_i$ for all $1 \leq r \leq n$

We first compute the coefficient in front of $\mathcal{O}\,\mathtt{t}$ by the formula of Theorem 1.5.1. We get

$$\chi(\mathtt{t}) = \frac{\prod r_\bullet!}{\prod b^{\bullet}_{\bullet}!} \frac{\prod s_\bullet!}{\prod \alpha^{\bullet,\bullet}_{\bullet}! \prod \beta_\bullet!},$$

$$|Stab(\upsilon(\mathtt{t}_0, \ldots, \mathtt{t}_{s_1+\ldots\,\mathtt{t}\,m}))| = \prod s_\bullet!,$$

and

$$|\prod Stab(\mathtt{t}_i)| = \prod r_\bullet!.$$

If $y_i = z_i$ for all $i < t$, we have

$$|Stab(\mathtt{t})| = \prod \alpha^{\bullet,\bullet}_{\bullet}! \prod b^{\bullet}_{\bullet}! \prod \beta_\bullet! \prod_{i=1}^{t} \binom{b^i_{\gamma_i+1} + \beta_i}{\beta_i}.$$

Thus the coefficient in front of $\mathcal{O}\,\mathtt{t}$ is equal to $\prod_{i=1}^t \binom{b^i_{\gamma_i+1}+\beta_i}{\beta_i}$.

On the other hand in the relation (1.7) we first use relation (1.1) to sum all the terms associated to $\mathtt{t}$. We find the coefficients

$$\frac{1}{\prod r_\bullet!} \frac{\prod r_\bullet!}{\prod b^{\bullet}_{\bullet}!}.$$

Then we apply relation (1.4) merging the common variables, hence we multiply the coefficient by

$$\prod b^{\bullet}_{\bullet}! \prod_{i=1}^{p} \binom{b^i_{\gamma_i+1} + \beta_i}{\beta_i}.$$

We obtain the same coefficient as before. □

Example 1.5.11. *We have :*

$$\begin{aligned}
\{\{x;y\}_2;y,z\}_{2,1} = \frac{1}{2}(&\{x;y,y,y,z\}_{1,1,2,1} + \{x;\{y;y\}_1,\{y;y\}_1,z\}_{1,1,1}\\
&+\{x;\{y;y\}_1,y,y,z\}_{1,1,1,1} + \{x;y,\{y;y\}_1,y,z\}_{1,1,1,1}\\
&+\{x;\{y;z\},y,y\}_{1,1,1} + \{x;y,\{y;z\},y\}_{1,1,1}\\
&+\{x;\{y;y\},\{y;z\},y\}_{1,1,1} + \{x;\{y;z\},\{y;y\},y\}_{1,1,1}\\
&+\{x;\{y;y,z\}_{1,1},y,y\}_{1,1,1} + \{x;y,\{y;y,z\}_{1,1},y\}_{1,1,1}\\
&+\{x;\{y;y\}_2,y,z\}_{1,1,1} + \{x;y,\{y;y\}_2,z\}_{1,1,1}\\
&+\{x;\{y;y\}_2,\{y;z\}_1\}_{1,1} + \{x;\{y;z\}_1,\{y;y\}_2\}_{1,1}\\
&+\{x;\{y,y\}_1,\{y;y,z\}_{1,1}\}_{1,1} + \{x;\{y;y,z\}_{1,1},\{y,y\}_1\}_{1,1}\\
&+\{x;\{y;y,z\}_{2,1},y\}_{1,1} + \{x;y,\{y;y,z\}_{2,1}\}_{1,1})\\
\overset{(1.1)}{=} \frac{1}{2}(&\{x;y,y,y,z\}_{1,1,2,1} + \{x;\{y;y\}_1,\{y;y\}_1,z\}_{1,1,1}\\
&+2\{x;\{y;y\}_1,y,y,z\}_{1,1,1,1} + 2\{x;\{y;z\},y,y\}_{1,1,1}\\
&+2\{x;\{y;y\},\{y;z\},y\}_{1,1,1} + 2\{x;\{y;y,z\}_{1,1},y,y\}_{1,1,1}\\
&+2\{x;\{y;y\}_2,y,z\}_{1,1,1} + 2\{x;\{y;y\}_2,\{y;z\}_1\}_{1,1}\\
&+2\{x;\{y,y\}_1,\{y;y,z\}_{1,1}\}_{1,1} + 2\{x;\{y;y,z\}_{2,1},y\}_{1,1})\\
\overset{(1.4)}{=} \frac{1}{2}(&12\{x;y,z\}_{4,1} + 2\{x;\{y;y\}_1,z\}_{2,1}\\
&+4\{x;\{y;y\}_1,y,z\}_{1,2,1} + 4\{x;\{y;z\},y\}_{1,2}\\
&+2\{x;\{y;y\},\{y;z\},y\}_{1,1,1} + 4\{x;\{y;y,z\}_{1,1},y\}_{1,2}\\
&+2\{x;\{y;y\}_2,y,z\}_{1,1,1} + 2\{x;\{y;y\}_2,\{y;z\}_1\}_{1,1}\\
&+2\{x;\{y,y\}_1,\{y;y,z\}_{1,1}\}_{1,1} + 2\{x;\{y;y,z\}_{2,1},y\}_{1,1})\\
= 6\{x;y,z\}_{4,1} &+ \{x;\{y;y\}_1,z\}_{2,1}\\
&+2\{x;\{y;y\}_1,y,z\}_{1,2,1} + 2\{x;\{y;z\},y\}_{1,2}\\
&+\{x;\{y;y\},\{y;z\},y\}_{1,1,1} + 2\{x;\{y;y,z\}_{1,1},y\}_{1,2}\\
&+\{x;\{y;y\}_2,y,z\}_{1,1,1} + \{x;\{y;y\}_2,\{y;z\}_1\}_{1,1}\\
&+\{x;\{y,y\}_1,\{y;y,z\}_{1,1}\}_{1,1} + \{x;\{y;y,z\}_{2,1},y\}_{1,1},
\end{aligned}$$

where we apply relation 1.1 to get our second identity and relation 1.4 to get our third identity.

Definition 1.5.12. *A Cor-algebra is a $\mathbb{K}$-module V with a family of functions*

$$\{-;\underbrace{-,\dots,-}_{n}\}_{r_1,\dots,r_n} : V^{n+1} \longrightarrow V$$

satisfying relations (1.2-1.7) as axioms. A morphism of Cor-algebra is a linear map commuting with the operations $\{-;-,\dots,-\}_{r_1,\dots,r_n}$ i.e.

$$f(\{-;-,\dots,-\}_{r_1,\dots,r_n}) = \{f(-);f(-),\dots,f(-)\}_{r_1,\dots,r_n}.$$

Proposition 1.5.13. *Let V be a $\mathbb{K}$-module. A $\Gamma PreLie$-algebra structure $\gamma : \Gamma(PreLie, V) \longrightarrow V$ on V induces a natural Cor-algebra structure on V.*

Proof: We set $\{v;w_1,\dots,w_n\}_{r_1,\dots,r_n} = \gamma(\mathcal{O}F_{r_1+\dots+r_n})(v,\underbrace{w_1,\dots,w_1}_{r_1},\dots,\underbrace{w_n,\dots,w_n}_{r_n}))$. The statements of Propositions 1.5.9 and 1.5.10 show that it defines a Cor-algebra. □

Our aim is to show that when we restrict to free $\mathbb{K}$-modules the structures of $\Gamma PreLie$-algebra and Cor-algebra are equivalent. From now on, let V be a free $\mathbb{K}$-module with a basis $\mathcal{V}$ endowed with a Cor-algebra structure. We aim to define a $\Gamma PreLie$-algebra structure on V i.e. we define a morphism $\gamma : \Gamma(RT, V) \longrightarrow V$ compatible with the action of $\Gamma(PreLie, -)$ on $\Gamma(PreLie, V)$.

Construction 1.5.14. *We set*

$$\gamma(\mathcal{O}(F_{(\sum r_\bullet)}(x,\underbrace{y_1,\dots,y_1}_{r_1},\dots,\underbrace{y_n,\dots,y_n}_{r_n}))) = \{x;y_1,\dots,y_n\}_{r_1,\dots,r_n}$$

where $x,y_1,\dots,y_n \in V$. By the normal form any element of $\Gamma(\mathcal{RT},\mathcal{V})$ can be decomposed in the iterated composition of corollas, the morphism γ is defined on the basis by composition of the function associated to corollas and then computed iteratively.

Lemma 1.5.15. *Let V be a free $\mathbb{K}$-module with a basis $\mathcal{V}$. If V is a Cor-algebra then the assignment of construction 1.5.14 is well defined and does not depend on the choice of the basis $\mathcal{V}$ of the $\mathbb{K}$-module V.*

Proof: This follows from the relations (1.1), (1.2), (1.3), (1.5). More precisely given two basis $\mathcal{V}$, $\mathcal{W}$ of V, we check that the two maps $\gamma_{\mathcal{V}},\gamma_{\mathcal{W}} : \Gamma(PreLie,V) \longrightarrow V$ are equal. Let $\mathfrak{t}$ be a general element of $\Gamma(\mathcal{RT},\mathcal{V})$ such that $\mathrm{Dec}(\mathfrak{t}) = \mathcal{O}F_m(v;\underbrace{\mathfrak{q}_1,\dots,\mathfrak{q}_1}_{t_1},\dots,\underbrace{\mathfrak{q}_r,\dots,\mathfrak{q}_r}_{t_r})$, for some $\mathfrak{q}_j \in \Gamma(\mathcal{RT},\mathcal{V})$, $v \in \mathcal{V}$. Let $\sum_{j_s\in J_s} \lambda_{j_s}\mathfrak{p}_s^{j_s}$ be the linear decomposition of $\mathfrak{q}_s$ in the basis $\Gamma(\mathcal{RT},\mathcal{W})$ and $v = \sum_{w_i\in\mathcal{W}} \xi_i w_i$. We proceed by induction on n, the number of corollas appearing in the normal form of $\mathfrak{t}$. If n is equal to 0 then $\mathfrak{t}$ is the identity. Suppose the statement true at rank $n-1$. We then have by definition :

$$\begin{aligned}
\mathrm{Dec}(\mathfrak{t}) &= \mathcal{O}F_m(v;\underbrace{\mathfrak{q}_1,\dots,\mathfrak{q}_1}_{t_1},\dots,\underbrace{\mathfrak{q}_r,\dots,\mathfrak{q}_r}_{t_r}) = \\
&\mathcal{O}F_m(v;\underbrace{\sum_{j_1\in J_1}\lambda_{j_1}\mathfrak{p}_1^{j_1},\dots\sum_{j_1\in J_1}\lambda_{j_1}\mathfrak{p}_1^{j_1}}_{t_1},\dots,\underbrace{\sum_{j_r\in J_r}\lambda_{j_r}\mathfrak{p}_r^{j_r},\dots\sum_{j_r\in J_r}\lambda_{j_r}\mathfrak{p}_r^{j_r}}_{t_r}) = \\
&\sum_{\substack{j_k^h\in J_k\\ \sum_{u=1}^{a_k} s_{j_k^u}=t_k}} \lambda_{j_1^1}^{s_{j_1^1}}\dots\lambda_{j_r^{a_r}}^{s_{j_r^{a_r}}} |\mathrm{Stab}(F):\mathbb{S}_{s_{j_1^1}}\times\dots\times\mathbb{S}_{s_{j_r^{a_r}}}|\,\mathcal{O}F_m(v;\underbrace{\mathfrak{p}_1^{j_1^1},\dots,\mathfrak{p}_1^{j_1^1}}_{s_{j_1^1}}, \\
&\qquad\dots,\underbrace{\mathfrak{p}_1^{j_1^{a_1}},\dots,\mathfrak{p}_1^{j_1^{a_1}}}_{s_{j_1^{a_1}}},\dots,\underbrace{\mathfrak{p}_r^{j_r^1},\dots,\mathfrak{p}_r^{j_r^1}}_{s_{j_r^1}},\dots,\underbrace{\mathfrak{p}_r^{j_r^{a_r}},\dots,\mathfrak{p}_r^{j_r^{a_r}}}_{s_{j_r^{a_r}}}) = \\
&\sum_{w_i\in\mathcal{W}}\xi_i \sum_{\substack{j_k^h\in J_k\\ \sum_{u=1}^{a_k} s_{j_k^u}=t_k}} \lambda_{j_1^1}^{s_{j_1^1}}\dots\lambda_{j_r^{a_r}}^{s_{j_r^{a_r}}} |\mathrm{Stab}(C_{s_{j_1^1},\dots,s_{j_1^{a_1}}}):\mathbb{S}_{s_{j_1^1}}\times\dots\times\mathbb{S}_{s_{j_r^{a_r}}}|\,\mathcal{O}F_m(w_i;\underbrace{\mathfrak{p}_1^{j_1^1},\dots,\mathfrak{p}_1^{j_1^1}}_{s_{j_1^1}}, \\
&\qquad\dots,\underbrace{\mathfrak{p}_1^{j_1^{a_1}},\dots,\mathfrak{p}_1^{j_1^{a_1}}}_{s_{j_1^{a_1}}},\dots,\underbrace{\mathfrak{p}_r^{j_r^1},\dots,\mathfrak{p}_r^{j_r^1}}_{s_{j_r^1}},\dots,\underbrace{\mathfrak{p}_r^{j_r^{a_r}},\dots,\mathfrak{p}_r^{j_r^{a_r}}}_{s_{j_r^{a_r}}})
\end{aligned}$$

where $\mathrm{Stab}(C_{s_{j_1^1},\dots,s_{j_1^{a_1}}})$ is the group

$$\mathrm{Stab}(\mathcal{O}F_m(v;\underbrace{\mathfrak{p}_1^{j_1^1},\dots,\mathfrak{p}_1^{j_1^1}}_{s_{j_1^1}},\dots,\underbrace{\mathfrak{p}_1^{j_1^{a_1}},\dots,\mathfrak{p}_1^{j_1^{a_1}}}_{s_{j_1^{a_1}}})).$$

We have :

$$\gamma_{\mathcal{W}}(\mathtt{t}) =$$

$$\sum_{w_i \in \mathcal{W}} \xi_i \sum_{\substack{j_k^h \in J_k \\ \sum_{u=1}^{a_k} s_{j_k^u} = t_k}} \lambda_{j_1^1}^{s_{j_1^1}} \dots \lambda_{j_r^{a_r}}^{s_{j_r^{a_r}}} |\mathrm{Stab}(F) : \mathbb{S}_{s_{j_1^1}} \times \dots \times \mathbb{S}_{s_{j_r^{a_r}}}| \{w_i; \gamma_{\mathcal{W}}(\mathfrak{p}_1^{j_1^1}), \dots, \gamma_{\mathcal{W}}(\mathfrak{p}_1^{j_1^{a_1}}),$$

$$\dots, \gamma_{\mathcal{W}}(\mathfrak{p}_r^{j_r^1}), \dots, \gamma_{\mathcal{W}}(\mathfrak{p}_r^{j_r^{a_r}})\}_{s_{j_1^1}, \dots, s_{j_r^{a_r}}} =$$

$$\sum_{\substack{j_k^h \in J_k \\ \sum_{u=1}^{a_k} s_{j_k^u} = t_k}} \lambda_{j_1^1}^{s_{j_1^1}} \dots \lambda_{j_r^{a_r}}^{s_{j_r^{a_r}}} |\mathrm{Stab}(F) : \mathbb{S}_{s_{j_1^1}} \times \dots \times \mathbb{S}_{s_{j_r^{a_r}}}| \{v; \gamma_{\mathcal{W}}(\mathfrak{p}_1^{j_1^1}), \dots, \gamma_{\mathcal{W}}(\mathfrak{p}_1^{j_1^{a_1}}),$$

$$\dots, \gamma_{\mathcal{W}}(\mathfrak{p}_r^{j_r^1}), \dots, \gamma_{\mathcal{W}}(\mathfrak{p}_r^{j_r^{a_r}})\}_{s_{j_1^1}, \dots, s_{j_r^{a_r}}}.$$

Applying the Cor-algebra relations (1.4)-(1.6) we have :

$$\gamma_{\mathcal{W}}(\mathtt{t}) = \{v; \gamma_{\mathcal{W}}(\mathfrak{p}_1), \dots, \gamma_{\mathcal{W}}(\mathfrak{p}_r)\}_{t_1, \dots, t_r}$$

by induction hypothesis

$$\gamma_{\mathcal{W}}(\mathtt{t}) = \{v; \gamma_{\mathcal{W}}(\mathfrak{p}_1), \dots, \gamma_{\mathcal{W}}(\mathfrak{p}_r)\}_{t_1, \dots, t_r} = \{v; \gamma_{\mathcal{V}}(\mathfrak{p}_1), \dots, \gamma_{\mathcal{V}}(\mathfrak{p}_r)\}_{t_1, \dots, t_r} = \gamma_{\mathcal{V}}(\mathtt{t}).$$

□

Definition 1.5.16. *Let V be a free $\mathbb{K}$-module with a basis $\mathcal{V}$. Let $\mathtt{t}$ be an element of $\mathcal{RT}$ and $w_1, \dots, w_m \in \mathcal{V}$. We say that an element of $\Gamma(\mathcal{RT}, \Gamma(\mathcal{RT}, \mathcal{V}))$ is simple if it is of the form $\mathcal{O}(\mathtt{t}(\circ_1(w_1), \dots, \circ_1(w_m)))$.*

Lemma 1.5.17. *Let V be a free $\mathbb{K}$-module with a basis $\mathcal{V}$. Recall that $\circ_1$ is the unique 1-tree. If $\mathcal{O}(\mathtt{t}(\circ_1(w_1), \dots, \circ_1(w_m)))$ is a simple element then*

$$\tilde{\mu}(\mathcal{O}(\mathtt{t}(\circ_1(w_1), \dots, \circ_1(w_m)))) = \mathcal{O}(\mathtt{t}(w_1, \dots, w_m)).$$

Lemma 1.5.18. *The Construction 1.5.14 is compatible with unit and composition in $\Gamma(PreLie, V)$.*

Proof: This follows from the relations (1.4), (1.6), (1.7). More precisely, let $v_1^1, \dots, v_{m_1}^1, \dots, v_1^n, \dots, v_{m_n}^n$ be elements of $\mathcal{V}$, let $\mathtt{s}$ be an element of $\mathcal{RT}(n)$ and for any $i \in \{1, \dots, n\}$ let $\mathtt{t}_i$ be an element of $\mathcal{RT}(m_i)$ such that

$$\mathrm{Dec}(\mathtt{t}_i(v_1^i, \dots, v_{m_i}^i)) = F_{r_i}(v_1^i; \mathfrak{p}_1^i, \dots, \mathfrak{p}_{r_i}^i),$$

for some $\mathfrak{p}_j^i \in \Gamma(\mathcal{RT}, \mathcal{V})$.

We consider an element of $\Gamma(\mathcal{RT}, \Gamma(\mathcal{RT}, \mathcal{V}))$

$$T = \mathcal{O}(\mathtt{s}(\mathcal{O}(\mathtt{t}_1(v_1^1, \dots, v_{m_1}^1)), \dots, \mathcal{O}(\mathtt{t}_n(v_1^n, \dots, v_{m_n}^n)))).$$

We want to compute the image of T under the map

$$\tilde{\mu} : \Gamma(PreLie, \Gamma(PreLie, V)) \longrightarrow \Gamma(PreLie, V).$$

Our strategy is to find a linear combination of simple elements with the same image of T under the map $\tilde{\mu}$. We suppose $m_i = max\{m_j | j = 1, \dots, n\}$ i.e. the tree $\mathtt{t}_i$ has the highest number of vertices. The normal form of T in $\Gamma(\mathcal{RT}, \Gamma(\mathcal{RT}, \mathcal{V}))$ is a composition of corollas of the form :

$$\mathcal{O}(F_s(\mathcal{O}\,\mathtt{t}_i(v_1^i, \dots, v_{m_i}^i); \mathfrak{q}_1, \dots, \mathfrak{q}_t)),$$

that is the image of

$$S = \mathcal{O}(F_s(\mathcal{O}\,F_{r_i}(v_1^i; \mathfrak{p}_1^i, \dots, \mathfrak{p}_{r_i}^i); \mathfrak{q}_1, \dots, \mathfrak{q}_t))$$

under the map

$$id \circ \tilde{\mu} : \Gamma(PreLie, \Gamma(PreLie, \Gamma(PreLie, V))) \longrightarrow \Gamma(PreLie, \Gamma(PreLie, V)).$$

Since $\Gamma(PreLie, V)$ is a $\Gamma PreLie$-algebra, the following diagram commutes :

$$\begin{tikzcd}
\Gamma(PreLie, \Gamma(PreLie, \Gamma(PreLie, V))) \arrow[r, "\tilde{\mu} \circ id"] \arrow[d, "id \circ \tilde{\mu}"'] & \Gamma(PreLie, \Gamma(PreLie, V)) \arrow[d, "\tilde{\mu}"] \\
\Gamma(PreLie, \Gamma(PreLie, V))) \arrow[r, "\tilde{\mu}"'] & \Gamma(PreLie, V).
\end{tikzcd}$$

To compute the image of T we apply first $\tilde{\mu} \circ id$ on S as composition of corollas. The result is a linear combination of elements of $\Gamma(PreLie, \Gamma(PreLie, V))$ whose normal forms are compositions of corollas which have as roots

$$\mathcal{O}(\mathtt{t}_1(v_1^1, \ldots, v_{m_1}^1)), \ldots, \circ_1 (v_1^i), \mathfrak{p}_1^i, \ldots, \mathfrak{p}_{r_i}^i, \ldots \mathcal{O}(\mathtt{t}_n(v_1^n, \ldots, v_{m_n}^n)).$$

Since the number of vertices of $\mathfrak{p}_j^i$ is strictly smaller than m_i, repeating the same computation inductively we obtain, in a finite number of passages, a sum of simple elements of $\Gamma(PreLie, \Gamma(PreLie, V))$. This procedure of computing T use just the compositions of corollas and it is performed the same way by using Cor-algebra relations for corollas. □

This verification completes the proof of the following statement.

Theorem 1.5.19. *The construction of Proposition 1.5.13 induces an isomorphism between the subcategories of $\Gamma(PreLie, -)$-algebras and of Cor-algebras formed by objects with a free $\mathbb{K}$-modules structure.* □

Remark 1.5.20. *The previous discussion shows that the functor $\Gamma(PreLie, -)$ corresponds to an analyseur de Lazard (see [Laz55]) with non-commutative variables.*

1.6 Examples

In this last section we give some particular examples of $\Gamma PreLie$-algebras.

1.6.1 Brace algebras are $\Gamma PreLie$-algebras

We recall the definition of the operad $Brace$ and prove that any $Brace$-algebra is a $\Gamma PreLie$-algebra.

Definition 1.6.1. *Let V be a $\mathbb{K}$-module. It is a brace algebra if it is endowed with a sequence of operations $\langle -; \underbrace{-, \ldots, -}_{n-1} \rangle : V^{\otimes n} \longrightarrow V$, subject to the following relations :*

1. $\langle x; \rangle = x$,
2. $$\langle \langle x; y_1, \ldots, y_n \rangle; z_1 \ldots, z_r \rangle = \sum \langle x; Z_1, \langle y_1; Z_2 \rangle, Z_3, \ldots, Z_{2n-1}, \langle y_n; Z_{2n} \rangle, Z_{2n+1} \rangle,$$
   where the sum runs over the partitions of the ordered set $\{z_1 \ldots, z_r\}$ into (possibly empty) consecutive ordered intervals $Z_1 \sqcup \ldots \sqcup Z_{2n+1}$.

The operad corresponding to brace-algebras is denoted by $Brace$.

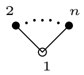
The $Brace$ algebras naturally appears in the study of Hochschild complex (see for example [LM05]).

We embed the operad $PreLie$ into the operad $Brace$:

$$\psi : PreLie \hookrightarrow Brace$$

by sending $\{-, -\}$ into $\langle -, - \rangle$. This inclusion induces a monomorphism from the $\Gamma PreLie$ free algebra into the $\Gamma Brace$ free algebra which is isomorphic to the $Brace$ free algebra, since the symmetric action on the operad $Brace$ is free. We accordingly have an inclusion from the $\Gamma PreLie$ free algebra into the $Brace$ free algebra. For more details see [Cha02].

This monomorphism is given by the following correspondence :



$$(x, \underbrace{y_1, \ldots, y_1}_{i_1}, \ldots, \underbrace{y_r, \ldots, y_r}_{i_r}) := \sum_{\sigma \in \mathrm{Sh}(i_1, \ldots, i_r)} \sigma * \langle x; \underbrace{y_1, \ldots, y_1}_{i_1}, \ldots, \underbrace{y_r, \ldots, y_r}_{i_r} \rangle,$$

where $\mathrm{Sh}(i_1, \ldots, i_r)$ is the set of the $(i_1, \ldots, i_r)$-shuffles. More precisely :

Definition 1.6.2. *We call n-planar-tree an n-tree with an order on the set $In(\tau, i)$ for any vertex i of the n-tree τ. Let $\{PRT(n)\}$ be the $\mathbb{S}$-module with $PRT(n)$ generated by the n-planar labelled rooted trees. We define partial compositions*

$$- \circ_i - : PRT(m) \otimes PRT(n) \longrightarrow PRT(n + m - 1),$$

with $1 \leq i \leq m$ as follows :

$$(\tau, ord(\tau)) \circ_i (\upsilon, ord(\upsilon)) := \sum_{f : In(\tau, i) \longrightarrow (1, \ldots, n)} \sum_{j_1, \ldots, j_n}^{|s(1)+1|, \ldots, |s(n)+1|} (\tau \circ_i^f \upsilon, ord(j_1, \ldots, j_n)),$$

where $\tau \circ_i^f \upsilon$ is the $n+m-1$-tree obtained by substituting the tree υ to the ith vertex of the tree τ, by attaching the outgoing edges of this vertex in τ to the root of υ, and the ingoing edges accordantly with the attaching map f. The sum runs over all these attachment maps $f : In(\tau, i) \longrightarrow (1, \ldots n)$ preserving $ord(\tau)$ and $ord(\upsilon)$.

Lemma 1.6.3. *The operad Brace is isomorphic to the operad PRT.*

Proof: See [Foi10]. □

Proposition 1.6.4. *The action of symmetric groups on the operad Brace is free. The brace algebras therefore coincide with $\Gamma Brace$-algebras for any field and any brace-algebra inherits a $\Gamma PreLie$-algebra structure. More precisely we have a morphism from $\Gamma(PreLie, V)$ into $S(Brace, V)$, and we can make it explicit :*

$$\{x; y_1, \ldots, y_n\}_{r_1, \ldots, r_n} \mapsto \sum_{\sigma \in \mathrm{Sh}(r_1, \ldots, r_n)} \langle x; \overline{y_{\sigma(1)}}, \ldots, \overline{y_{\sigma(r_1 + \ldots + r_n)}} \rangle,$$

where the ordered set $(\overline{y_1}, \ldots, \overline{y_{r_1 + \ldots + r_n}})$ is $(\underbrace{y_1, \ldots, y_1}_{r_1}, \ldots, \underbrace{y_n, \ldots, y_n}_{r_n})$. □

Notations 1.6.5. *Let (P, μ, η) be a connected operad. Let $\phi : \{1, \ldots r\} \to \{1, \ldots, n\}$ be an injective function. Let $p \in P(n)$ and $q_i \in P(m_i)$ where $i \in \{1, \ldots, r\}$. We denote by $p \circ_\phi (q_1, \ldots, q_r)$ the following element of P :*

$$\mu(p \otimes x_1 \otimes \ldots \otimes x_r)$$

where

$$x_j = \begin{cases} q_i & \text{if } j = \phi(i) \text{ for some } i, \\ 1 & \text{the operadic unit, otherwise.} \end{cases}$$

Example 1.6.6. *Let (P, μ, η) be a connected operad. It is a well known fact that the $\mathbb{K}$-module $\bigoplus_i P(i)$ is a PreLie-algebra. This structure is induced by a Brace-algebra structure. Therefore the PreLie-algebra structure extends to a $\Gamma PreLie$-algebra structure. More explicitly :*

$$\{p; q_1, \ldots, q_m\}_{r_1, \ldots r_m} = \sum_{\phi \in Sh_n(r_1, \ldots, r_m)} p \circ_\phi (\underbrace{q_1, \ldots, q_1}_{r_1}, \ldots, \underbrace{q_m, \ldots, q_m}_{r_m})$$

where $Sh_n(r_1, \ldots, r_m)$ is the set of injective functions from $\{1, \ldots, r_1 + \ldots + r_m\}$ to $\{1, \ldots, n\}$ such that they are $(r_1, \ldots, r_m)$-shuffle when we identify their image with $\{1, \ldots, r_1 + \ldots + r_m\}$.

Significant examples of *PreLie*-algebras are associated to *PreLie*-systems (see [Ger63] and [GV95]). We revisit the definition of this notion and we check that any *PreLie*-system gives rise to a $\Gamma PreLie$-algebra.

Definition 1.6.7. *Let $\mathfrak{S}^\bullet$ be a $\mathbb{N}$-graded free $\mathbb{K}$-module. A $PreLie$-system on $\mathfrak{S}^\bullet$ is a family of maps :*

$$\circ_k : \mathfrak{S}^n \otimes \mathfrak{S}^m \longrightarrow \mathfrak{S}^{n+m-1}$$

for any $1 \leq k \leq n$, such that for any $f \in \mathfrak{S}^n$, $g \in \mathfrak{S}^m$, and $h \in \mathfrak{S}^l$ we have :

$$f \circ_u (g \circ_v h = (f \circ_u g) \circ_{v+u-1} h$$

for any $1 \leq u \leq n$ and $1 \leq v \leq m$, and

$$(f \circ_u g) \circ_{v+m-1} h = (f \circ_v h) \circ_u g$$

for any $1 \leq u < v \leq n$.

Proposition 1.6.8. *Let f be an element of $\mathfrak{S}^m$ and $g_1, \ldots, g_n$ be elements of $\mathfrak{S}^\bullet$ with $n \leq m$. We define :*

$$\langle f; g_1, \ldots, g_n \rangle = \sum_{1 \leq i_1 < \ldots < i_n \leq m} (\ldots((f \circ_{i_n} g_n) \ldots) \circ_{i_1} g_1);$$

Then $\mathfrak{S}^\bullet$ endowed with these operations is a Brace-algebra, and hence inherits a $\Gamma PreLie$-algebra structure. □

Example 1.6.9. *Let P a connected operad. The $PreLie$-algebra structure on the module $\bigoplus_n P(n)$ of the Example 1.2.2 (1) is clearly induced by a $PreLie$-system therefore it extends to a $\Gamma PreLie$-algebra structure.*

1.6.2 Dendriform algebras are $\Gamma PreLie$-algebras

I. Dokas proved in [Dok13] that dendriform algebras in positive characteristic admits a p-restricted $PreLie$-algebra structure (and hence a $\Lambda PreLie$-algebra structure by Theorem 1.4.16). We prove that any dendriform algebra is a $\Gamma PreLie$-algebra.

Definition 1.6.10. *A dendriform algebra, denoted by Dend-algebra, is a free $\mathbb{K}$-module A endowed with two binary operations $<, >: A \otimes A \longrightarrow A$, such that :*

$$\begin{aligned} (x < y) < z &= x < (y * z), \\ (x > y) < z &= x > (y < z), \\ (x * y) > z &= x > (y > z), \end{aligned}$$

where $x * y = x > y + y < x$. It is easy to show that $*$ is associative.

The category of dendriform algebras is governed by an operad denoted Dend.

Dendriform algebras were introduced by J.L. Loday in [Lod01] as Koszul dual of diassociative algebras in the study of K-Theory periodicity. They appear naturally in other fields such as combinatorial algebra, physics and algebraic topology.

Definition 1.6.11. *Let $(A, <, >)$ be a Dend-algebra. We define the following binary operation $\{x, y\} = x > y - y < x$.*

Proposition 1.6.12. *Let $(A, <, >)$ be a Dend-algebra. Then $(A, \{-, -\})$ is a p-restricted $PreLie$-algebra.*

Proof: See [Dok13]. □

We deduce from Theorem 1.4.16 and the previous proposition that a $Dend$-algebra is a $\Lambda PreLie$-algebra.

Proposition 1.6.13. *Let V be a free $\mathbb{K}$-module then the $PreLie$-algebra structure defined in $S(Dend, V)$ extends to a $\Gamma PreLie$-algebra structure.*

Proof: Let V be a free $\mathbb{K}$-module then the $\Gamma PreLie$-algebra structure defined on $S(Dend, V)$ is given by the inclusions $PreLie \longrightarrow Brace \longrightarrow Dend$, and the construction of Proposition 1.6.4. $\square$

By the same kind of argument we prove that any Zinbiel algebra is a $\Gamma(PreLie, -)$-algebra. Zinbiel algebras are encoded by the operad $Zinb$ which was introduced by J.L. Loday in [Lod95], it is the Koszul dual of the operad $Leib$ which encodes the Leibniz algebras. Then the cohomology of a $Leib$-algebra inherits a $Zinb$-algebra structure.

Definition 1.6.14. *Let A be a free $\mathbb{K}$-module, then it is a Zinbiel algebra if it is endowed with a bilinear product $\circ$ such that :*

$$(a \circ b) \circ c = a \circ (b \circ c + c \circ b).$$

Proposition 1.6.15. *Let V be a free $\mathbb{K}$-module then $S(Zinb, V)$ is a $\Gamma PreLie$-algebra.*

Proof: This proposition follows from the inclusion of $S(Dend, V)$ into $S(Zinb, V)$. $\square$

Chapter 2

Mackey Functors, Generalized Operads and Analytic Monads

abstract

Let $\mathbb{K}$ be a field. We denote by $\mathrm{Mod}_{\mathbb{K}}$ the category of $\mathbb{K}$-modules. We study a generalization of cohomological Mackey functors defined on $\mathcal{H}Par_n$, a subcategory of the Hecke category of the symmetric group $\mathbb{S}_n$. We denote the category of cohomological Mackey functors defined on $\mathcal{H}Par_n$ by $\mathrm{Mac}^{coh}(\mathcal{H}Par_n)$ and the category of strict polynomial functors of degree n by $PolFun_n$. We show that $\mathrm{Mac}^{coh}(\mathcal{H}Par_n)$ is equivalent to $PolFun_n$. An $\mathbb{M}$-module is a collection of objects in $\mathrm{Mac}^{coh}(\mathcal{H}Par_n)$ parametrized by $n \in \mathbb{N}$. We denote the category of $\mathbb{M}$-modules by $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$. We introduce two monoidal structures on $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$: the tensor product $\boxtimes$ and the composition $\square$ product. A strict analytic functor is a collection of objects in $PolFun_n$ parametrized by $n \in \mathbb{N}$. We denote the category of strict analytic functors by AnFun. We show that the monoidal structures of tensor product and the composition of endofunctors of $\mathrm{Mod}_{\mathbb{K}}$ induce two monoidal structures on the category of strict analytic functors. We call these structures tensor product and composition of strict analytic functors. We show that the equivalence between $\mathrm{Mac}^{coh}(\mathcal{H}Par_n)$ and $PolFun_n$ induces an equivalence of symmetric monoidal categories between $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \boxtimes)$ and $(\mathrm{AnFun}, \otimes)$ as well an equivalence of monoidal categories $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \square)$ and $(\mathrm{AnFun}, \circ)$. Based on this new constructions we define the concept of an $\mathbb{M}$-Operad, of an $\mathbb{M}$-PROP, and of their categories of algebras. We give examples of categories of algebras governed by $\mathbb{M}$-operads and $\mathbb{M}$-PROPs.

Introduction

We fix a field $\mathbb{K}$ and a non-negative integer n. We denote by $\mathrm{Mod}_{\mathbb{K}}$ the category of $\mathbb{K}$-modules.

Polynomial functors were introduced by Eilenberg and MacLane in [EML54] in the study of homology of Eilenberg-MacLane spaces $K(\pi, n)$. Strict polynomial functors of degree n are particular polynomial functors endowed with an additional structure. They were introduced by Friedlander and Suslin in [FS97a] in the study of the cohomology of finite group schemes. We denote the category of strict polynomial functors of degree n by $PolFun_n$.

We define the category $\mathcal{H}Par_n$, a generalization of the Hecke category associated to the symmetric group $\mathbb{S}_n$. A Cohomological $\mathcal{H}Par_n$-Mackey functor is an additive functor from $\mathcal{H}Par_n$ to $\mathrm{Mod}_{\mathbb{K}}$. We denote the category of Cohomological $\mathcal{H}Par_n$-Mackey functors by $\mathrm{Mac}^{coh}(\mathcal{H}Par_n)$. We show that $\mathrm{Mac}^{coh}(\mathcal{H}Par_n)$ is equivalent to the category of strict polynomial functors of degree n. Our result explicitly reads :

Theorem A (Theorem 2.2.18). *There exists an equivalence of categories*

$$ev_n : \mathrm{Mac}^{coh}(\mathcal{H}Par_n) \to PolFun_n.$$

A strict analytic functor F is a collection $\{F_n\}_{n \in \mathbb{N}}$ such that F_n is a strict polynomial functor of degree n for each $n \in \mathbb{N}$. We denote the category of strict analytic functors by AnFun. There exists a forgetful functor $\mathcal{U} : \mathrm{AnFun} \to \mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}})$ from the category of strict analytic functors to the category of endofunctor of $\mathrm{Mod}_{\mathbb{K}}$. The tensor product and the composition in $\mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}})$ extend along $\mathcal{U}$ and define two monoidal structures on the category of strict analytic functors which we denote $(\mathrm{AnFun}, \otimes, \mathbb{K})$ and $(\mathrm{AnFun}, \circ, \mathrm{Id})$.

An $\mathbb{M}$-module is a collection $\{M_n\}_{n\in\mathbb{N}}$ such that $M_n \in \mathrm{Mac}^{coh}(\mathcal{H}Par_n)$ for each $n \in \mathbb{N}$. We denote the category of $\mathbb{M}$-modules by $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$. We endow $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$ with two monoidal structures, the tensor product $\boxtimes$ of $\mathbb{M}$-modules with unit $\mathbb{K}$ and the composition $\square$ of $\mathbb{M}$-modules with unit $\mathbb{I}$. We show the following result :

Theorem B (Theorem 2.4.28)**.** *The equivalence of Theorem 2.2.18 extends to an equivalence of symmetric monoidal categories* $ev : (\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \boxtimes, \mathbb{K}) \to (\mathrm{AnFun}, \otimes, \mathbb{K})$ *as well as to an equivalence of monoidal categories* $ev : (\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \square, \mathbb{I}) \to (\mathrm{AnFun}, \circ, \mathrm{Id})$.

We introduce the category of $\mathbb{M}$-operads, denoted by $\mathbb{M}$-Op. An $\mathbb{M}$-operad is defined as a monoid in the category of $\mathbb{M}$-modules with the monoidal product $\square$. To any $\mathbb{M}$-operad we associate a monad and a category of algebras. An $\mathbb{M}$-operad encodes an algebraic structure with polynomial operations. Any operad P defines an $\mathbb{M}$-operad $S_-(P)$ such that the category of P-algebras is isomorphic to the category of $S_-(P)$-algebras. Moreover, if the operad P is connected then we associate to it two additional $\mathbb{M}$-operads : $\Lambda_-(P)$ and $\Gamma_-(P)$. The corresponding monads are isomorphic, respectively to $\Lambda(P,-)$ and $\Gamma(P,-)$ (see Appendix A).

Let V be a $\mathbb{K}$-module. We define the $\mathbb{M}$-operad $Poly_V$, it replaces the operad End_V in the following sense :

Theorem C (Theorem 2.5.8)**.** *Let P be an $\mathbb{M}$-operad and V be a $\mathbb{K}$-module. The set of P-algebra structures on V is in bijection with* $\mathrm{Hom}_{\mathbb{M}\text{-Op}}(P, Poly_V)$.

We generalize the construction of $\mathbb{M}$-modules and we define the category of $\mathbb{M}$-PROPs. To any $\mathbb{M}$-PROP we associate a category of algebras. An $\mathbb{M}$-PROP is an object which encodes algebraic structures with polynomial operations with possible multiple inputs and outputs. The category of $\mathbb{M}$-PROPs generalizes the category of PROPs (see Appendix A).

We give examples of categories of algebras governed by $\mathbb{M}$-operads and $\mathbb{M}$-PROPs which are not governed by operads nor by PROPs. More precisely we show that the category of p-restricted Poisson algebras, that appears in the theory of quantization of manifolds in positive characteristic (see [BK08]), is governed by an $\mathbb{M}$-operad. The categories of divided power bi-algebras, related to the category of divided powers Hopf algebras (see [And71]), and p-restricted Lie bi-algebras are governed by $\mathbb{M}$-PROPs.

Contents

In Section 2.1 we introduce the concept of a cohomological Mackey functor from an admissible collection of subgroups. In Section 2.2 we recall the definition of a strict polynomial functor and we prove the equivalence of categories between $\mathrm{Mac}^{coh}(\mathcal{H}Par_n)$ and $PolFun_n$. In Section 2.3 we introduce the category $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$ and the monoidal structures $\boxtimes$, and $\square$. In Section 2.4 we recall the definition of a strict analytic functor and we prove the equivalence of monoidal categories between $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$ and AnFun. We conclude with applications to operads and PROPs, in Sections 2.5 and 2.6.

2.1 Admissible cohomological Mackey functors on partition subgroups of the symmetric group

We introduce the definition of a cohomological Mackey functor on an admissible collection of subgroups of a finite group. We apply this general definition to a collection of partition subgroups of the symmetric group $\mathbb{S}_n$.

2.1.1 Admissible cohomological Mackey functors

We fix a finite group G. We introduce the concept of an admissible collection of subgroups of G. Any admissible collection of subgroups $\mathcal{D}$ defines a category denoted by $\mathcal{HD}$ and a category of cohomological Mackey $\mathcal{HD}$-functors.

Definition 2.1.1 (The Hecke category $\mathcal{H}G$)**.** *We denote by $\mathcal{H}G$ the full subcategory of $\mathbb{K}[G]$-modules whose objects are permutation modules over $\mathbb{K}[G]$, i.e. it is the category defined as follow :*

1. *the objects are direct sums of $\mathbb{K}[G]$-modules of the form $\mathbb{K}[{}^{G}/_{H}]$ where H is a subgroup of G,*

2. *if* $\mathbb{K}[{}^{G}/_{H_1}]$ *and* $\mathbb{K}[{}^{G}/_{H_2}]$ *are two objects of* $\mathcal{H}G$ *then*

$$\mathrm{Hom}_{\mathcal{H}G}(\mathbb{K}[{}^{G}/_{H_1}], \mathbb{K}[{}^{G}/_{H_2}]) = \mathbb{K}[{}_{H_1}\backslash {}^{G}/_{H_2}].$$

From this definition, we see that the category $\mathcal{H}G$ *is self dual, with an isomorphism* $\mathcal{H}G^{op} \to \mathcal{H}G$ *which is the identity map on objects, and which is induced by the inversion of* G *on morphisms.*

Definition 2.1.2 (Admissible collection)**.** *A collection* $\mathcal{D}$ *of subgroups of* G *is admissible if it is closed under intersection and conjugation by elements of* G*.*

Notations 2.1.3. *Let* G *be a finite group,* $K \leq H$ *be subgroups of* G *and* $g \in G$*. We use the following notation*
— $\pi_K^H : {}^{G}/_{K} \to {}^{G}/_{H}$ *is the projection of cosets,*
— ${}^{g}H = \{ghg^{-1} | h \in H\}$*, and*
— $H^g = \{g^{-1}hg | h \in H\}$*.*

We associate a category to any admissible collection.

Definition 2.1.4 (The category $\mathcal{HD}$)**.** *Let* $\mathcal{D}$ *be an admissible collection of subgroups of* G*. We define the category* $\mathcal{HD}$ *to be the full subcategory of* $\mathcal{H}G$ *with objects* $\bigoplus_{i=1}^{n} \mathbb{K}[{}^{G}/_{H_i}]$ *where* H_i *is in* $\mathcal{D}$*.*

Let us mention that $\mathcal{HD}$ *is self dual (like the Hecke category* $\mathcal{H}G$*).*

For any admissible collection we define a category of cohomological Mackey functors.

Definition 2.1.5 (The category $\mathrm{Mac}^{coh}(\mathcal{HD})$)**.** *Let* $\mathcal{D}$ *be an admissible collection of subgroups of* G*. The category of cohomological* $\mathcal{HD}$*-Mackey functors is the category of* $\mathbb{K}$*-linear functors from* $\mathcal{HD}$ *to* $\mathrm{Mod}_{\mathbb{K}}$ *with natural transformations. We denote this category by* $\mathrm{Mac}^{coh}(\mathcal{HD})$*.*

We present an equivalent definition of cohomological $\mathcal{HD}$-Mackey functors.

Proposition 2.1.6. *Let* $\mathcal{D}$ *be an admissible collection of subgroups of* G*. A cohomological* $\mathcal{HD}$*-Mackey functor is equivalent to the following data assignment : a function* $A : \mathcal{D} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$*; for any inclusion between elements of* $\mathcal{D}$*,* $H_1 \longrightarrow H_2$*, a pair of morphisms* $\mathrm{Ind}_{H_1}^{H_2} : A(H_1) \longrightarrow A(H_2)$ *and* $\mathrm{Res}_{H_1}^{H_2} : A(H_2) \longrightarrow A(H_1)$ *and for any element* $g \in G$ *and* H *in* $\mathcal{D}$ *an isomorphism* $\mathrm{c}_g : A(H) \longrightarrow A({}^{g}H)$ *such that the following relations are satisfied :*

1. $\mathrm{Ind}_{H_2}^{H_3} \mathrm{Ind}_{H_1}^{H_2} = \mathrm{Ind}_{H_1}^{H_3}$,
2. $\mathrm{Res}_{H_1}^{H_2} \mathrm{Res}_{H_2}^{H_3} = \mathrm{Res}_{H_1}^{H_3}$,
3. $\mathrm{c}_g \mathrm{c}_h = \mathrm{c}_{gh}$,
4. $\mathrm{c}_g \mathrm{Ind}_{H_1}^{H_2} = \mathrm{Ind}_{{}^{g}H_1}^{{}^{g}H_2} \mathrm{c}_g$,
5. $\mathrm{c}_g \mathrm{Res}_{H_1}^{H_2} = \mathrm{Res}_{{}^{g}H_1}^{{}^{g}H_2} \mathrm{c}_g$,
6. $\mathrm{Res}_J^H \mathrm{Ind}_K^H = \sum\limits_{x \in J\backslash H/K} \mathrm{Ind}_{J \cap {}^{x}K}^{J} \mathrm{c}_x \mathrm{Res}_{J^x \cap K}^{K}$,
7. $\mathrm{Ind}_{H_1}^{H_2} \mathrm{Res}_{H_1}^{H_2} = [H_2 : H_1] \mathrm{Id}_{H_2}$,

for all $H_1, H_2, H_3, H, J, K \in \mathcal{D}$ *such that* $H_1 \leq H_2 \leq H_3$*, and* $J, K \leq H$*.*

Proof: Suppose we have an assignment A of this type. It defines a cohomological $\mathcal{HD}$-Mackey functor M as follows :

1. let $\mathbb{K}[{}^{G}/_{H}]$ be an object of $\mathcal{HD}$, we set $M(\mathbb{K}[{}^{G}/_{H}]) = A(H)$,
2. let $\mathbb{K}[{}^{G}/_{H_1}]$ and $\mathbb{K}[{}^{G}/_{H_2}]$ be two objects of $\mathcal{HD}$ and $[g]$ an element of

$$\mathrm{Hom}_{\mathcal{HD}}(\mathbb{K}[{}^{G}/_{H_1}], \mathbb{K}[{}^{G}/_{H_2}]),$$

we set

$$M([g])(x) = \mathrm{Ind}_{H_1^g \cap H_2}^{H_2} \mathrm{Res}_{H_1^g \cap H_2}^{H_1^g} \mathrm{c}_g(x).$$

The statement then follows from the Theorem of Yoshida ; see [Yos83, Thm. 4.3]. □

From now on we will define cohomological $\mathcal{HD}$-Mackey functors giving their values on the subgroups in $\mathcal{D}$ and the morphisms $\mathrm{Ind}_{H_1}^{H_2}$, $\mathrm{Res}_{H_1}^{H_2}$ and c_g for all $g \in G$, and $H_1, H_2 \in \mathcal{D}$ such that $H_1 \leq H_2$.

Proposition 2.1.7. *Let $\mathcal{D}$ be an admissible collection of subgroups of G and $K, H \in \mathcal{D}$. We have that $\mathrm{Hom}_{\mathcal{HD}}({}^{G}/_{K}, {}^{G}/_{K})$ is isomorphic to the $\mathbb{K}$-free module generated by the diagram of the form :*

$$\begin{tikzcd} & {}^{G}/_{K^g \cap H} \arrow[dl, "\pi^K_{L^g} \circ c_g"'] \arrow[dr, "\pi^H_L"] & \\ {}^{G}/_{K} & & {}^{G}/_{H} \end{tikzcd}$$

where $g \in K \backslash G / H$ and $L = K^g \cap H$.

Moreover, let M be a cohomological $\mathcal{HD}$-Mackey functors and suppose $H \leq K$. We have

$$\mathrm{Res}^K_H = M({}^{G}/_{K} \overset{\pi^K_H}{\leftarrow} {}^{G}/_{H} \overset{\mathrm{Id}}{\rightarrow} {}^{G}/_{H}),$$

$$\mathrm{Ind}^K_H = M({}^{G}/_{H} \overset{\mathrm{Id}}{\leftarrow} {}^{G}/_{H} \overset{\pi^K_H}{\rightarrow} {}^{G}/_{K}),$$

and

$$\mathrm{c}_{g,H} = M({}^{G}/_{H} \overset{\mathrm{Id} \circ c_g}{\leftarrow} {}^{G}/_{H^g} \overset{\mathrm{Id}}{\rightarrow} {}^{G}/_{H}).$$

Proof: It follows directly by Proposition A.5.9. □

2.1.2 The collection Par_n

Let n be a non-negative integer, we denote by $\mathbb{S}_n$ the symmetric group of n letters set. In this paper we are interested in cohomological Mackey functors for a particular admissible collection of subgroups of $\mathbb{S}_n$ denoted by Par_n.

Definition 2.1.8 (The collection Par_n)**.** *We define Par_n to be the collection of $\mathbb{S}_n$-subgroups conjugated to*

$$\mathbb{S}_{r_1} \times \ldots \times \mathbb{S}_{r_t} \hookrightarrow \mathbb{S}_n$$

for some non-negative integers $r_1, \ldots, r_t$ such that $r_1 + \ldots + r_t = n$ where the inclusion is induced by the ordering preserving bijection $\coprod_{i \in \{1,\ldots,t\}} \{1, \ldots, r_i\} \to \{1, \ldots n\}$.

These subgroups of $\mathbb{S}_n$ appear in the literature under the name "Young subgroups".

Notations 2.1.9. *The elements of Par_n are in bijection with the partitions of the set $\mathbf{n} := \{1, \ldots, n\}$. From now on we identify the subgroups $\pi \in Par_n$ with the partitions of $\mathbf{n}$.*

We denote a partition of $\mathbf{n}$ by $(p_1), \ldots, (p_r)$, where p_i is a subset of $\mathbf{n}$ and $\coprod_{i=1}^r p_i = \mathbf{n}$. We denote by δ_n the discrete partition ; i.e. the partition associated to the trivial subgroup $\underbrace{\mathbb{S}_1 \times \ldots \times \mathbb{S}_1}_{n}$.

Proposition 2.1.10. *The set Par_n is an admissible collection of subgroups of $\mathbb{S}_n$.*

Proof: It is easy to check that the collection Par_n is closed by conjugations and intersections. □

In what follows we consider the Hecke category $\mathcal{H}Par_n$ associated to the admissible collection Par_n.

Example 2.1.11. *Let V be a vector space endowed with an action of $\mathbb{S}_n$. Since the functors $H_k(-, V)$ and $H^k(-, V)$ are cohomological Mackey functors (See [Yos83], Example 2.1) by restriction they are cohomological $\mathcal{H}Par_n$-Mackey functors.*

2.2 The equivalence between strict polynomial functors and cohomological $\mathcal{H}Par_n$-Mackey functors

In this section we recall the general theory of strict polynomial functors and we show that their category is equivalent to the category of cohomological $\mathcal{H}Par_n$-Mackey functors.

2.2.1 Strict polynomial functors

We fix a non-negative integer n. We recall the definition of the category of strict polynomial functors of degree n. This category was introduced by Friedlander and Suslin in [FS97b] for the study of group schemes.

Definition 2.2.1 (The functor $\Gamma_n(-)$). *The functor $\Gamma_n(-) : \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ is defined as follows :*

$$\Gamma_n(V) = (\underbrace{V \otimes \ldots \otimes V}_{n})^{\mathbb{S}_n},$$

where $\underbrace{V \otimes \ldots \otimes V}_{n}$ is endowed with the natural $\mathbb{S}_n$-action induced by permutations.

We set :

$$\Gamma_\pi(V) = (\underbrace{V \otimes \ldots \otimes V}_{n})^{\pi},$$

for any $\pi \in Par_n$.

In what follows we use that these functor preserves filtered colimits. This claim follows from the observation that the tensor powers preserve filtered colimits (see for instance [Fre09, Proposition 1.2.3]) and that finite limits commute with filtered colimits in module categories (see [Bor94, Theorem 2.13.4] for the counterpart of this statement in the category of sets).

Notations 2.2.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We denote by $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$.*

Definition 2.2.3 (The category $\Gamma_n \mathrm{Mod}_{\mathbb{K}}$). *We denote by $\Gamma_n \mathrm{Mod}_{\mathbb{K}}$ the category defined by :*

1. *the objects are $\mathbb{K}$-modules,*
2. *if V and W are $\mathbb{K}$-modules then*

$$\mathrm{Hom}_{\Gamma_n \mathrm{Mod}_{\mathbb{K}}}(V,W) = \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V,W)),$$

3. *composition is the following :*

$$\Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(W,U)) \otimes \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V,W)) \longrightarrow \\ \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(W,U) \otimes \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V,W)) \longrightarrow \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V,U)).$$

where the first morphism is given by the natural transformation $\zeta_{A,B} : \Gamma_n(A) \otimes \Gamma_n(B) \to \Gamma_n(A \otimes B)$, and the second is given by the composition in $\mathrm{Mod}_{\mathbb{K}}$.

We have a functor $\gamma_n : \mathrm{Mod}_{\mathbb{K}} \to \Gamma_n \mathrm{Mod}_{\mathbb{K}}$ defined as the identity on the objects and for a morphism $f : X \to Y$ in $\mathrm{Mod}_{\mathbb{K}}$ we have $f \mapsto \gamma_n(f) = \underbrace{f \otimes \cdots \otimes f}_{n} \in \Gamma_n(\mathrm{Hom}_{ModK}(X,Y))$.

Definition 2.2.4 (Strict polynomial functors). *A strict polynomial functor of degree n is a $\mathbb{K}$-linear functor $F : \Gamma_n \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ such that the functor $\mathcal{U}_n(F) = F \circ \gamma_n : \mathrm{Mod}_{\mathbb{K}} \to \mathrm{Mod}_{\mathbb{K}}$ preserves filtered colimits. We denote the category of strict polynomial functors of degree n by $PolFun_n$. The map $\mathcal{U}_n : F \mapsto F \circ \gamma_n$ induces a functor $\mathcal{U}_n : PolFun_n \to \mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}})$. As a consequence to any strict polynomial functor we associate an endofunctor of the category $\mathrm{Mod}_{\mathbb{K}}$.*

Example 2.2.5. *The following functors have a natural strict polynomial structure of degree n :*

1. *the n-symmetric powers : S_n,*
2. *the n-divided powers : Γ_n,*
3. *the n-external powers : Λ_n.*

Proposition 2.2.6. *Let $F : \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ be a functor. Providing F with the structure of a strict polynomial functor of degree n amounts to giving a natural transformation*

$$\zeta = \zeta_{X,Y} : \Gamma_n(X) \otimes F(Y) \to F(X \otimes Y),$$

for $X, Y \in \mathrm{Mod}_{\mathbb{K}}$ such that the following diagrams commute :

$$\begin{tikzcd}
\Gamma_n(X) \otimes \Gamma_n(Y) \otimes F(Z) \arrow[rr, "id\otimes\zeta_{Y,Z}"] \arrow[d, "\zeta_{X,Y}\otimes \mathrm{Id}_{F(Z)}"] & & \Gamma_n(X) \otimes F(Y \otimes Z) \arrow[d, "\zeta_{X,Y\otimes Z}"] \\
\Gamma_n(X \otimes Y) \otimes F(Z) \arrow[r, "\zeta_{X\otimes Y,Z}"'] & F((X \otimes Y) \otimes Z) \arrow[r, "\cong"'] & F(X \otimes (Y \otimes Z)),
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Gamma_n(\mathbb{K}) \otimes F(X) \arrow[r, "\zeta_{\mathbb{K},X}"] \arrow[dr, "\cong"'] & F(\mathbb{K} \otimes X) \arrow[d, "\cong"] \\
 & F(X).
\end{tikzcd}$$

Proof: Suppose we have such natural transformation ζ. We have :

$$\begin{tikzcd}
\Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(X,Y)) \otimes F(X) \arrow[r] \arrow[dr, "\exists\alpha"'] & F(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(X,Y) \otimes X) \arrow[d] \\
 & F(Y).
\end{tikzcd}$$

and we take the adjoint $\alpha_\sharp : \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(X,Y)) \to \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(F(X), F(Y))$. In the converse direction, we assume F is a strict polynomial functor of degree n. We have $\mathrm{Id}_\sharp : X \to \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(Y, X \otimes Y)$ defined by $\mathrm{Id}_\sharp(x) : y \mapsto x \otimes y$ the adjoint of $\mathrm{Id} : X \otimes Y \to X \otimes Y$. We take

$$\begin{tikzcd}
\Gamma_n(X) \otimes F(Y) \arrow[r] \arrow[dr, "\exists\zeta"'] & \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(Y, X \otimes Y)) \otimes F(X) \arrow[d] \\
 & F(X \otimes Y).
\end{tikzcd}$$

□

We recall some properties of the category of strict polynomial functors.

Proposition 2.2.7. *Let $\pi \in Par_n$. The functor $\Gamma_\pi(-) : V \mapsto \Gamma_\pi(V)$ is canonically a strict polynomial functor of degree n. The action is given by the following composition*

$$\Gamma_n(X) \otimes \Gamma_\pi(Y) \to \Gamma_\pi(X) \otimes \Gamma_\pi(Y) \to \Gamma_\pi(X \otimes Y),$$

where the first morphism is the restriction $\Gamma_n(X) \hookrightarrow \Gamma_\pi(X)$. □

Proposition 2.2.8 (Krause [Kra13]). *The set $\{\Gamma_\pi(-)\}_{\pi\in Par_n}$ is a set of small projective generators for the category $PolFun_n$.* □

We recall a result on the Hom-sets between the projective generators $\Gamma_\pi(-)$ in the category of strict polynomial functors of degree n.

Lemma 2.2.9. *Let $\pi_1 = (p_1)\dots(p_c)$ and $\pi_2 = (q_1),\dots,(q_l)$ be in Par_n. The set B of $l \times c$ $\mathbb{N}$-matrix such that $\sum_{j\in\{1,\dots,c\}} \alpha_{i,j} = |q_i|$ and $\sum_{i\in\{1,\dots,l\}} \alpha_{i,j} = |p_j|$ is in bijection with the set ${\pi_1}\backslash \mathbb{S}_n/{\pi_2}$.*

Proof: Let $g \in \mathbb{S}_n$ we define the $l \times c$ Set-matrix $m(g)$ by $m(g)_{i,j} = p_i^g \cap q_j$. We have a function $\mathbb{S}_n \to B$ defined by $g \mapsto M(g)_{i,j} = \{|m(i,j)|\}_{i,j}$. Let g_1 and g_2 in $\mathbb{S}_n$. We have $|M_{g_1}| = |M_{g_2}|$ if and only if there exist $h_1 \in \pi_1$ and $h_2 \in \pi_2$ such that $h_1 g_1 h_2 = g_2$. Thus the map pass to the quotient defining an injective function ${\pi_1}\backslash \mathbb{S}_n/{\pi_2} \to B$.

For the surjectivity suppose that the elements inside (q_j) are ordered by the usual order for every j. Let $b = b_{i,j} \in B$. We take $(q_i)_b = (q_{i,1}), \dots, (q_{i,c})$ a partition of q_i such that $|q_{i,j}| = b_{i,j}$ and we consider the associated matrix $q_{i,j}$. We consider a permutation σ which map the element of p_j in the elements of $\bigsqcup_i q_{i,j}$. We have that $M(\sigma) = b$. □

Example 2.2.10. *Let $\pi_1 = (1,3),(2,4)$ and $\pi_2 = (1),(2),(3,4) \in Par_4$. We consider $g = (1,2) \in \mathbb{S}_4$ we have $\pi_1^g = (2,3)(1,4)$ and $m(g)$ is* $\begin{array}{|c|c|}\hline \varnothing & \{1\} \\ \hline \{2\} & \varnothing \\ \hline \{3\} & \{4\} \\ \hline \end{array}$.

Definition 2.2.11. *Let $\pi_1 = (p_1)\dots(p_c)$ and $\pi_2 = (q_1),\dots,(q_l)$ be in Par_n. Let $A = \{\alpha_{i,j}\}$ be a $l \times c$ $\mathbb{N}$-matrix such that $\sum_{j\in\{1,\dots,c\}} \alpha_{i,j} = |q_i|$ and $\sum_{i\in\{1,\dots,l\}} \alpha_{i,j} = |p_j|$. Using the permutation of Lemma 2.2.9 it defines a morphism :*

$$\gamma_A : \Gamma_{\pi_1}(-) \cong \bigotimes_j \Gamma_{p_j}(-) \longrightarrow \bigotimes_j (\bigotimes_i \Gamma_{\alpha_{i,j}}(-)) \cong \bigotimes_i (\bigotimes_j \Gamma_{\alpha_{i,j}}(-)) \longrightarrow \bigotimes_j \Gamma_{q_j}(-) \cong \Gamma_{\pi_2}(-).$$

We call the morphisms defined in this way "standard morphisms".

Lemma 2.2.12 (Totaro [Tot97], Krause [Kra14])**.** *Let π_1 and π_2 be in Par_n. The set of standard morphisms of Definition 2.2.11 forms a basis for the $\mathbb{K}$-module*

$$\mathrm{Hom}_{PolFun_n}(\Gamma_{\pi_1}(-), \Gamma_{\pi_2}(-)).$$

2.2.2 Cohomological $\mathcal{H}Par_n$-Mackey functors and strict polynomial functors

In what follows we prove the equivalence between $\mathrm{Mac}^{coh}(\mathcal{H}Par_n)$ and $PolFun_n$.
We recall the notion of coend.

Definition 2.2.13. *Let $\mathfrak{C}$ be a small category enriched over $\mathrm{Mod}_\mathbb{K}$ (see [Kel05]). Let $F : \mathfrak{C} \times \mathfrak{C}^{op} \longrightarrow \mathrm{Mod}_\mathbb{K}$ be a $\mathrm{Mod}_\mathbb{K}$-enriched functor (a $\mathbb{K}$-linear functor in the terminology used in the previous sections). A extranatural transformation $g : F \longrightarrow x$ with $x \in \mathrm{Mod}_\mathbb{K}$, is a collection $\{g_c : F(c,c) \longrightarrow x\}_{c\in\mathfrak{C}}$ of morphisms in $\mathrm{Mod}_\mathbb{K}$, such that the following diagram commutes :*

$$\begin{tikzcd} F(c,d) \otimes \mathrm{Hom}_\mathfrak{C}(c,d) \arrow[r] \arrow[d] & F(c,c) \arrow[d] \\ F(d,d) \arrow[r] & x. \end{tikzcd}$$

A coend of F is an object $\int^{c\in\mathfrak{C}} F(c,c)$ in $\mathrm{Mod}_\mathbb{K}$ with a extranatural transformation $f : F \longrightarrow \int^{c\in\mathfrak{C}} F(c,c)$ such that any extranatural transformation $g : F \longrightarrow x$ factorizes uniquely through f.

A coend of F is equivalent to a coequalizer of the form :

$$\bigoplus_{c,d\in\mathfrak{C}} F(c,d) \otimes \mathrm{Hom}_\mathfrak{C}(c,d) \rightrightarrows \bigoplus_c F(c,c) \to \int^{c\in\mathfrak{C}} F(c,c),$$

see [Kel05] for more details on this definition.

Definition 2.2.14 (The functor ev_n)**.** *Let M_n be a cohomological $\mathcal{H}Par_n$-Mackey functor. It defines a functor :*

$$M_n(-) : \mathrm{Mod}_\mathbb{K} \longrightarrow \mathrm{Mod}_\mathbb{K}$$
$$V \mapsto \int^{\pi\in Par_n} M_n(\pi) \otimes \Gamma_\pi(V),$$

where we use that the mapping $\pi \mapsto \Gamma_\pi(V)$ gives a covariant functor $\Gamma_-(V) : \mathcal{H}Par_n \to \mathrm{Mod}_\mathbb{K}$ and we compose this functor with the anti-isomorphism $\mathcal{H}Par_n^{op} \to \mathcal{H}Par_n$ of Definition 2.1.4 to form the contravariant functor $\Gamma_-(V) : \mathcal{H}Par_n^{op} \to \mathrm{Mod}_\mathbb{K}$ of this coend formula.

This mapping is functorial in M_n, we then have :

$$ev_n : \mathrm{Mac}^{coh}(\mathcal{H}Par_n) \longrightarrow \mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}})$$

$$M_n \mapsto ev_n(M_n)(-).$$

Proposition 2.2.15. *Let M_n be a cohomological $\mathcal{H}Par_n$-Mackey functor. We have that $ev_n(M_n)$ extends canonically to a strict polynomial functor of degree n.*

Proof: We have $ev_n(M_n) = \int^{\pi\in\mathcal{H}Par_n} M_n(\pi)\otimes\Gamma_\pi(-)$. If V and W are two objects in $\Gamma_n\,\mathrm{Mod}_{\mathbb{K}}$ then the morphism $\Gamma_\pi(V)\otimes\mathrm{Hom}_{\Gamma_n\,\mathrm{Mod}_{\mathbb{K}}}(V,W) \longrightarrow \Gamma_\pi(W)$ induces a morphism :

$$\Big(\int^{\pi\in\mathcal{H}Par_n} M_n(\pi)\otimes\Gamma_\pi(V)\Big)\otimes\mathrm{Hom}_{\Gamma_n\,\mathrm{Mod}_{\mathbb{K}}}(V,W) \longrightarrow \int^{\pi} M_n(\pi)\otimes\Gamma_\pi(W).$$

□

Corollary 2.2.16. *The functor $ev_n : \mathrm{Mac}^{coh}(\mathcal{H}Par_n) \longrightarrow \mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}})$ extends to a functor*

$$\begin{tikzcd} & PolFun_n \arrow[d, "\mathcal{U}_n"] \\ \mathrm{Mac}^{coh}(\mathcal{H}Par_n) \arrow[ur, dashed, "ev_n"] \arrow[r, "ev_n"'] & \mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}}). \end{tikzcd}$$

Proposition 2.2.17. *Let π_1 and π_2 be partitions of n. We have a natural isomorphism :*

$$\mathrm{Hom}_{PolFun_n}(\Gamma_{\pi_1}(-),\Gamma_{\pi_2}(-)) \cong \mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1,\pi_2).$$

Proof: We have to check that 2.2.9 is compatible with composition. This follows by Proposition 2.1.7 and the observation that a "standard morphism" is the composition of a permutation with $g \in {}_{\pi_1}\backslash\mathbb{S}_n/_{\pi_2}$, a restriction to $\pi_1^g\cap\pi_2$ and an induction to π_2. □

As a direct consequence we have the following theorem.

Theorem 2.2.18. *The functor $ev_n : \mathrm{Mac}^{coh}(\mathcal{H}Par_n) \to PolFun_n$ induces an equivalence between the category of cohomological $\mathcal{H}Par_n$-Mackey functors, and the category of strict polynomial functors of degree n.*

Proof: The theorem follows applying Yoneda's Lemma, Proposition 2.2.8 and Lemma 2.2.17.

We define explicitly an inverse of ev by using Yoneda's Lemma. Let P be a strict polynomial functor of degree n. We define the cohomological $\mathcal{H}Par_n$-Mackey functor :

$$P(\pi) = \mathrm{Hom}_{PolFun_n}(\Gamma_\pi(-),P).$$

Let π_1, and π_2 be in Par_n such that $\pi_1\le\pi_2$, and $\sigma\in Par_n$. We recall that by Lemma 2.2.17 we have a natural isomorphism $\mathrm{Hom}_{PolFun_n}(\Gamma_{\pi_1}(-),\Gamma_{\pi_2}(-)) \cong \mathbb{K}[{}_{\pi_1}\backslash\mathbb{S}_n/_{\pi_2}]$. We define the morphisms $P(\mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1,\pi_2))$ by precomposition with $\mathrm{Hom}_{PolFun_n}(\Gamma_{\pi_1}(-),\Gamma_{\pi_2}(-))$. Using the isomorphism $\mathcal{H}Par_n^{op}\to\mathcal{H}Par_n$ we deduce that the relations of cohomological $\mathcal{H}Par_n$-Mackey functors are satisfied. □

2.3 The category $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$

The aim of this section is to define the category of $\mathbb{M}$-modules, denoted by $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$, and to introduce the two monoidal structures $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}},\boxtimes,\mathbb{K})$ and $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}},\square,\mathbb{I})$.

2.3.1 $\mathbb{M}$-modules

We introduce the concept of $\mathbb{M}$-module. It generalizes the definition of $\mathbb{S}$-module (see Appendix A).

Definition 2.3.1 ($\mathbb{M}$-module)**.** *An $\mathbb{M}$-module M is a sequence $\{M_n\}_{n\in\mathbb{N}}$ of cohomological $\mathcal{H}Par_n$-Mackey functors. A morphism between two $\mathbb{M}$-modules $\{M_n\}_{n\in\mathbb{N}}$ and $\{N_n\}_{n\in\mathbb{N}}$ is a sequence of natural transformations $\{f_n : M_n \longrightarrow N_n\}_{n\in\mathbb{N}}$. Their category is denoted by $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$.*

We introduce some special classes of $\mathbb{M}$-modules.

Definition 2.3.2 (The $\Gamma(M)$ and $S(M)$ $\mathbb{M}$-modules)**.** *Let M be a $\mathbb{S}$-module (see Appendix A). We set $\Gamma_n(M)(-) = H^0(-, M(n))$ and we consider the $\mathbb{M}$-module $\Gamma(M)$ defined by the collection of these cohomological Mackey functors. We also set $S_n(M)(-) = H_0(-, M(n))$ and consider the $\mathbb{M}$-module $S(M)$ defined by the collection of these cohomological Mackey functors. Remark that $H^k(-, M(n))$ and $H_k(-, M(n))$ are $\mathbb{M}$-module for all k.*

Definition 2.3.3 (The trace map)**.** *Let M be a $\mathbb{S}$-module (see Appendix A). There exists a natural morphism of $\mathbb{M}$-modules $\mathrm{tr}_M : S(M) \longrightarrow \Gamma(M)$ called trace map defined by : for any $n \in \mathbb{N}$ and any $\pi \in Par_n$ we set $\mathrm{tr}_M(\pi) : S_n(M)(\pi) \to \Gamma_n(M)(\pi)$ as $[x] \mapsto \sum_{\sigma\in\pi} \sigma^* x$.*

2.3.2 The monoidal structures $\boxtimes$ and $\square$

We introduce the two monoidal structures $(\boxtimes, \mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \mathbb{K})$ and $(\square, \mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \mathbb{I})$.
We recall some properties of coends.

Lemma 2.3.4 (Fubini Theorem for coends)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be small categories and $F : (\mathcal{A} \times \mathcal{B})^{op} \times (\mathcal{A} \times \mathcal{B}) \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ be a functor. We have, if the coend exists :*

$$\int^{(A,B)\in\mathcal{A}\times\mathcal{B}} F(A,B,A,B) \cong \int^{A\in\mathcal{A}}\int^{B\in\mathcal{B}} F(A,B,A,B) \cong \int^{B\in\mathcal{B}}\int^{A\in\mathcal{A}} F(A,B,A,B).$$

Lemma 2.3.5 (coYoneda Lemma for coends)**.** *Let $\mathcal{A}$ be a small category enriched over $\mathrm{Mod}_{\mathbb{K}}$ and $F : \mathcal{A} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ be a functor. We have :*

$$F(-) \cong \int^{A\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(A,-) \otimes F(A).$$

Proof: For more details and proofs see [Kel05, Sec. 3.10]. □

We introduce the two monoidal structures on $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$. They correspond to tensor product and composition.

Definition 2.3.6 (The product $\boxtimes$)**.** *Let M and N be two $\mathbb{M}$-modules. We set :*

$$(M \boxtimes N)_n(\pi) = \bigoplus_{i+j=n} \int^{\pi_1\times\pi_2\in\mathcal{H}Par_i\times\mathcal{H}Par_j} (M(\pi_1) \otimes N(\pi_2)) \otimes \mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1 \times \pi_2, \pi).$$

for each $\pi \in Par_n$ and for all $n \in \mathbb{N}$.

The action of $\mathcal{H}Par_n$ is given by the action on $\mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1 \times \pi_2, \pi)$ inside the coend.

Proposition 2.3.7. *Let $\mathbb{K}$ be the following $\mathbb{M}$-module :*

$$\mathbb{K}_i := \begin{cases} \mathbb{K} & i = 0; \\ 0 & i \neq 0. \end{cases}$$

The triple $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \boxtimes, \mathbb{K})$ forms a symmetric monoidal category.

Proof: Let A, B, C be $\mathbb{M}$-modules. We consider the following isomorphism :

$$\begin{aligned}(A \boxtimes (B \boxtimes C))(\pi) &= \bigoplus_{i+j=n} \int^{\pi_1 \times \pi_2 \in \mathcal{H}Par_i \times \mathcal{H}Par_j} A(\pi_1) \otimes (B \boxtimes C)(\pi_2) \otimes \mathrm{Hom}_{Par_n}(\pi_1 \times \pi_2, \pi) \\ &= \bigoplus_{i+j=n} \int^{\pi_1 \times \pi_2} A(\pi_1) \otimes (\bigoplus_{s+t=j} \int^{\rho_1 \times \rho_2 \in \mathcal{H}Par_s \times \mathcal{H}Par_t} B(\rho_1) \otimes C(\rho_2) \otimes \\ &\quad \mathrm{Hom}_{\mathcal{H}Par_j}(\rho_1 \times \rho_2, \pi_2)) \otimes \mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1 \times \pi_2, \pi) \\ &\cong \bigoplus_{i+s+t=n} \int^{\pi_1 \times \pi_2 \times \rho_1 \times \rho_2} A(\pi_1) \otimes B(\rho_1) \otimes C(\rho_2) \otimes \mathrm{Hom}_{\mathcal{H}Par_j}(\rho_1 \times \rho_2, \pi_2) \otimes \\ &\quad \mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1 \times \pi_2, \pi) \\ &\cong \bigoplus_{i+s+t=n} \int^{\pi_1 \times \rho_1 \times \rho_2} A(\pi_1) \otimes B(\rho_1) \otimes C(\rho_2) \otimes \mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1 \times \rho_1 \times \rho_2, \pi),\end{aligned}$$

where we first expand the tensor product and then we use the isomorphisms given by Lemma 2.3.4 and by Lemma 2.3.5.

We get the same formula for $((A \boxtimes B) \boxtimes C)(\pi)$ hence we have $A \boxtimes (B \boxtimes C) \cong (A \boxtimes B) \boxtimes C$. For the unit $\eta_A : A \boxtimes \mathbb{K} \to A$ morphism we consider the following isomorphism :

$$\begin{aligned}A \boxtimes \mathbb{K} &= \bigoplus_{i+j} \int^{\pi_1 \times \pi_2 \in \mathcal{H}Par_i \times \mathcal{H}Par_j} A(\pi_1) \otimes \mathbb{K}(\pi_2) \otimes \mathrm{Hom}_{Par_n}(\pi_1 \times \pi_2, \pi) \\ &= \int^{\pi_1 \in \mathcal{H}Par_n} A(\pi_1) \otimes \mathbb{K} \otimes \mathrm{Hom}_{Par_n}(\pi_1, \pi) \cong A(\pi),\end{aligned}$$

where we use the isomorphism of Lemma 2.3.5.

For the symmetry isomorphism $\beta_{A,B} : A \boxtimes B \to B \boxtimes A$ we consider the following isomorphism :

$$\begin{aligned}(A \boxtimes B)(\pi) &= \bigoplus_{i+j=n} \int^{\pi_1 \times \pi_2 \in \mathcal{H}Par_i \times \mathcal{H}Par_j} A(\pi_1) \otimes B(\pi_2) \otimes \mathrm{Hom}_{Par_n}(\pi_1 \times \pi_2, \pi) \\ &\cong \bigoplus_{i+j=n} \int^{\pi_2 \times \pi_1 \in \mathcal{H}Par_j \times \mathcal{H}Par_i} B(\pi_2) \otimes A(\pi_1) \otimes \mathrm{Hom}_{Par_n}(\pi_2 \times \pi_1, \pi) = (A \boxtimes B)(\pi).\end{aligned}$$

□

Definition 2.3.8 (The product $\square$)**.** *Let M and N be two $\mathbb{M}$-modules we set :*

$$(M \square N)_n(\pi) = \bigoplus_{r \in \mathbb{N}} (\int^{\rho \in \mathcal{H}Par_r} M(\rho) \otimes (N^{\boxtimes r}(\pi))^{\rho}),$$

for all $\pi \in Par_n$, where we use that $N^{\boxtimes r}(\pi)$ forms a $\mathbb{K}[\mathbb{S}_r]$-module by the symmetry of the tensor product $\boxtimes$ and again we consider the contravariant functor $(N^{\boxtimes r}(\pi))^{-})$ induced by the duality isomorphism $\mathcal{H}Par_n^{op} \to \mathcal{H}Par_n$.

Let $\mathbb{I}$ be the following $\mathbb{M}$-module :

$$\mathbb{I}_i = \begin{cases} \mathbb{K} & i = 1, \\ 0 & i \neq 1. \end{cases}$$

The proof that the triple $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \square, \mathbb{I})$ forms a monoidal category is postponed to Theorem 2.4.28.

2.4 The equivalence between strict analytic functors and $\mathbb{M}$-modules

In this section we recall the definition of AnFun, the category of strict analytic functors. We prove that the equivalence of Theorem 2.2.18 extends to a monoidal equivalence between $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$ and AnFun.

2.4.1 Strict analytic functors

We recall the definition of strict analytic functors and we introduce two monoidal structures.

Definition 2.4.1 (Strict analytic functor). *A strict analytic functor is a collection $\{F_n\}_{n\in\mathbb{N}}$ where F_n is a strict polynomial functor of degree n. Let $\{F_n\}_{n\in\mathbb{N}}$ and $\{G_n\}_{n\in\mathbb{N}}$ be strict analytic functors. A morphism of strict analytic functors is a collection $\{f_n\}:\{F_n\}_{n\in\mathbb{N}}\to\{G_n\}_{n\in\mathbb{N}}$ where f_n is a morphism of strict polynomial functors. We denote the category of strict analytic functors by* AnFun*. We accordingly have* $\mathrm{AnFun}=\prod_{n\in\mathbb{N}} PolFun_n$.

Definition 2.4.2 (The functor $\mathcal{U}$). *We define the functor $\mathcal{U}:\mathrm{AnFun}\to\mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}},\mathrm{Mod}_{\mathbb{K}})$. Let $F=\{F_n\}_{n\in\mathbb{N}}$ be a strict analytic functor we set $\mathcal{U}F=\bigoplus_{n\in\mathbb{N}}\mathcal{U}F_n$. This functor $\mathcal{U}:\mathrm{AnFun}\to\mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}},\mathrm{Mod}_{\mathbb{K}})$ is faithful, because this is clearly the case for each functor $\mathcal{U}_n:PolFun_n\to\mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}},\mathrm{Mod}_{\mathbb{K}})$ in Definition 2.2.4.*

The category AnFun is equipped with two monoidal structures $(\mathrm{AnFun},\otimes,\mathbb{K})$ and $(\mathrm{AnFun},\circ,\mathrm{Id})$.

Definition 2.4.3 (The product $\otimes$). *Let $F=\{F_n\}_{n\in\mathbb{N}}$ and $\{G_n\}_{n\in\mathbb{N}}$ be strict analytic functors we set :*

$$(F\otimes G)_n(-)=\bigoplus_{i+j=n}F_i(-)\otimes G_j(-).$$

Let $F=\{F_n\}_{n\in\mathbb{N}},\{G_n\}_{n\in\mathbb{N}},A=\{A_n\}_{n\in\mathbb{N}}$, and $B=\{B_n\}_{n\in\mathbb{N}}$ be strict analytic functors and $\{f_n\}_{n\in\mathbb{N}}:F\to A$, $\{g_n\}_{n\in\mathbb{N}}:G\to B$ be strict analytic functor morphisms we set :

$$\{f\otimes g\}_n=\sum_{i+j=n}f_i\otimes g_j.$$

Definition 2.4.4 (The strict analytic functor $\mathbb{K}$). *We define the strict analytic functor $\mathbb{K}=\{\mathbb{K}_n:\Gamma_n\mathrm{Mod}_{\mathbb{K}}\to\mathrm{Mod}_{\mathbb{K}}\}_{n\in\mathbb{N}}$ such that $\mathbb{K}_0:\Gamma_0\mathrm{Mod}_{\mathbb{K}}\to\mathrm{Mod}_{\mathbb{K}}$ is the constant functor $V\mapsto\mathbb{K}$, and $\mathbb{K}_n:\Gamma_n\mathrm{Mod}_{\mathbb{K}}\to\mathrm{Mod}_{\mathbb{K}}$ is the constant functor $V\mapsto 0$ when $n\neq 0$.*

Proposition 2.4.5. *The triple $(\mathrm{AnFun},\otimes,\mathbb{K})$ forms a symmetric monoidal category. In particular, for F and G strict analytic functors the collection $F\otimes G=\{(F\otimes G)_n\}_{n\in\mathbb{N}}$ is canonically a strict analytic functor. We moreover have a natural isomorphism $\mathcal{U}(F\otimes G)\cong\mathcal{U}(F)\otimes\mathcal{U}(G)$.*

Proof: We show that $(F\otimes G)_n(-)$ is a strict polynomial functor of degree n using the characterization of Proposition 2.2.6. We have :

$$\Gamma_n(X)\otimes(F\otimes G)_n(Y)=\bigoplus_{i+j=n}\Gamma_n(X)\otimes F_i(Y)\otimes G_j(Y)\overset{(*)}{\to}\bigoplus_{i+j=n}\Gamma_{\mathbb{S}_i\times\mathbb{S}_j}(X)\otimes F_i(Y)\otimes G_j(Y)\cong$$

$$\bigoplus_{i+j=n}\Gamma_i(X)\otimes F_i(Y)\otimes\Gamma_j(X)\otimes G_j(Y)\to\bigoplus_{i+j=n}F_i(X\otimes Y)\otimes G_j(X\otimes Y),$$

where the morphism $(*)$ is given by the restriction map $\Gamma_n(X)=\Gamma_{\mathbb{S}_n}(X)\to\Gamma_{\mathbb{S}_i\times\mathbb{S}_j}(X)=\Gamma_{\mathbb{S}_i}(X)\otimes\Gamma_{\mathbb{S}_j}(X)$. The unit and the associativity property of this action of $\Gamma_n(X)$ on $(F\otimes G)_n$ follows from the commutativity of the following diagrams :

$$\begin{tikzcd}
\Gamma_n(\mathbb{K}) \arrow[r] \arrow[d] & \bigoplus_{i+j=n}\Gamma_i(\mathbb{K})\otimes\Gamma_j(\mathbb{K}) \arrow[d] \\
\mathbb{K} \arrow[r, "="'] & \mathbb{K},
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Gamma_n(X)\otimes\Gamma_n(Y) \arrow[r] \arrow[d] & \bigoplus_{i+j=n}\Gamma_i(X)\otimes\Gamma_j(X)\otimes\Gamma_i(Y)\otimes\Gamma_j(Y) \arrow[d] \\
\Gamma_n(X\otimes Y) \arrow[r, "="'] & \bigoplus_{i+j=n}\Gamma_i(X\otimes Y)\otimes\Gamma_j(X\otimes Y).
\end{tikzcd}$$

The relation $\mathcal{U}(F\otimes G)\cong\mathcal{U}(F)\otimes\mathcal{U}(G)$ follows from the distributivity of tensor product with respect to direct sums.

There are evident isomorphisms :

$$(\mathbb{K}\otimes F)_n(-)\cong F_n(-)\cong(F\otimes\mathbb{K})_n(-),$$

and

$$((A\otimes B)\otimes C)_n(-)\cong\bigoplus_{i+j+k=n}A_i(-)\otimes B_j(-)\otimes C_k(-)\cong(A\otimes(B\otimes C))_n(-),$$

the compatibility of these isomorphisms with polynomial structures follows from the unit, associativity and symmetry of the restriction maps used in our definition. □

We recall some relations between polynomial functors, in the sense of Eilenberg-MacLane (see [EML54]), and strict polynomial functors.

Definition 2.4.6 (Cross-effect)**.** *Let $F:\mathrm{Mod}_{\mathbb{K}}\longrightarrow\mathrm{Mod}_{\mathbb{K}}$ be a functor. We set*

$$\Delta_0(F)=F(0).$$

Let n be a non-negative integer. We define the nth cross-effect $\Delta_n(F):\mathrm{Mod}_{\mathbb{K}}{}^{\times n}\longrightarrow\mathrm{Mod}_{\mathbb{K}}$ by :

$$\Delta_n(F)(V_1,\dots,V_n)=Ker(F(V_1\oplus\ldots\oplus V_n)\longrightarrow\bigoplus_{i=1}^{n}F(V_1\oplus\ldots\oplus\overset{i}{0}\oplus\ldots\oplus V_n)).$$

Proposition 2.4.7. *Let F be an endofunctor of the category $\mathrm{Mod}_{\mathbb{K}}$. We have the following canonical decomposition :*

$$F(V_1\oplus\ldots\oplus V_n)=\bigoplus_{r=1}^{n}\bigoplus_{1\le i_1\le\ldots\le i_r\le n}\Delta_r(F)(V_{i_1},\dots,V_{i_r}).$$

Proof: We refer to [EML54] for a proof of the statement. □

Definition 2.4.8 (Homogeneous cross-effect)**.** *We assume that $\pi_i:V_1\oplus\cdots\oplus V_s\longrightarrow V_1\oplus\cdots\oplus V_s$ is the endomorphism of $V_1\oplus\cdots\oplus V_s$ induced by the projection on the summand V_i. For $\alpha_1+\ldots+\alpha_s=n$ we consider the following elements of $\Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V_1\oplus\cdots\oplus V_s,V_1\oplus\cdots\oplus V_s))$:*

$$\gamma_{\alpha_1}(\pi_1)\ldots\gamma_{\alpha_s}(\pi_s)=\sum_{\sigma\in\frac{\mathbb{S}_n}{\mathbb{S}_{\alpha_1}\times\cdots\times\mathbb{S}_{\alpha_s}}}\sigma^*(\pi_1^{\otimes\alpha_1}\otimes\cdots\otimes\pi_s^{\otimes\alpha_s}),$$

where the notation γ_α refers to the fact that $\Gamma(-)$ represents the free divided power algebra. In this expression, we use the action of a set of representative of the class $\sigma\in\frac{\mathbb{S}_n}{\mathbb{S}_{\alpha_1}\times\cdots\times\mathbb{S}_{\alpha_s}}$ in the group of permutation $\mathbb{S}_n$ to shuffle the factors $\pi_i^{\otimes\alpha_i}$ in the tensor product $(\pi_1^{\otimes\alpha_1},\dots,\pi_s^{\otimes\alpha_s})$. We equivalently have :

$$\gamma_{\alpha_1}(\pi_1)\ldots\gamma_{\alpha_s}(\pi_s)=\sum_{|\{i_k=i\}|=\alpha_i}\pi_{i_1}\otimes\cdots\otimes\pi_{i_n},$$

where the sum runs over the set of n-tuples $(i_1,\dots,i_n)$ with α_i terms such that $i_k=i$ for each i.

The addition formula for divided powers (see Definition A.1.40) implies that we have the identity :

$$\gamma_n(\mathrm{Id})=\gamma_n(\pi_1+\cdots+\pi_s)=\sum_{\alpha_1+\cdots+\alpha_s=n}\gamma_{\alpha_1}(\pi_1)\cdots\gamma_{\alpha_s}(\pi_s),$$

in $\Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V_1 \oplus \cdots \oplus V_s, V_1 \oplus \cdots \oplus V_s))$. *From the relation*

$$(\pi_{i_1} \otimes \cdots \otimes \pi_{i_n}) \circ (\pi_{j_1} \otimes \cdots \otimes \pi_{j_n}) \\ = (\pi_{i_1}\pi_{j_1} \otimes \cdots \otimes \pi_{i_n}\pi_{j_n}) = \begin{cases} \pi_{i_1} \otimes \cdots \otimes \pi_{i_n}, & \textit{if } (i_1, \dots, i_n) = (j_1, \dots, j_n), \\ 0, & \textit{otherwise}, \end{cases}$$

in $Hom(V_1 \oplus \cdots \oplus V_s, V_1 \oplus \cdots \oplus V_s)^{\otimes n}$, *we also deduce that :*

$$(\gamma_{\alpha_1}(\pi_1)\dots\gamma_{\alpha_s}(\pi_s)) \circ (\gamma_{\beta_1}(\pi_1)\dots\gamma_{\beta_s}(\pi_s)) \\ = \begin{cases} \gamma_{\alpha_1}(\pi_1)\dots\gamma_{\alpha_s}(\pi_s), & \textit{if } (\alpha_1, \dots, \alpha_n) = (\beta_1, \dots, \beta_n), \\ 0, & \textit{otherwise}, \end{cases}$$

we also deduce that these elements $(\gamma_{\alpha_1}(\pi_1)\dots\gamma_{\alpha_s}(\pi_s))$ *forms a complete set of orthogonal idempotents in* $\Gamma_n(Hom(V_1 \oplus \cdots \oplus V_s, V_1 \oplus \cdots \oplus V_s))$. *We refer to [Bou67] for this result.*

Let F be a strict polynomial functor of degree n. We define the homogeneous cross-effect of degrees $(\alpha_1, \dots, \alpha_s)$ of F as follows :

$$F^{(\alpha_1,\dots,\alpha_s)}(V_1, \dots, V_s) = Im(F(\gamma_{\alpha_1}(\pi_1)\dots\gamma_{\alpha_s}(\pi_s))).$$

Proposition 2.4.9. *Let F be a strict polynomial functor of degree n. We have the following canonical decomposition of the nth cross-effect :*

$$\Delta_s(\mathcal{U}(F))(V_1, \dots, V_s) = \bigoplus_{\substack{\alpha_1+\dots+\alpha_s=n \\ \alpha_i>0}} F^{(\alpha_1,\dots,\alpha_s)}(V_1, \dots, V_s).$$

Proof: We refer to [Bou67] for this statement. □

Remark 2.4.10. *Let $F : \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ be a functor. We say that F is polynomial, in the sense of Eilenberg-MacLane [EML54], of degree lower or equal to n if $\Delta_{n+1}(F) = 0$. We say that F is of degree n if it is of degree lower or equal to n and $\Delta_n \neq 0$.*

Let F be a strict polynomial functors of degree n. The functor $\mathcal{U}(F) : \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ is a polynomial functor, in the sense of Eilenberg-MacLane [EML54], of degree lower or equal to n. The statement is an obvious consequence of the formula of Proposition 2.4.9 when $n > s$.

On the other hand the functor $\mathcal{U} : PolFun_n \longrightarrow \mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}})$ does not preserve the polynomial degree. In general if F is a strict polynomial functor of degree n then $\mathcal{U}(F)$ is a polynomial functor of degree m where $m \leq n$.

In what follows, we mainly use the following variation on the results of Proposition 2.4.7 and Proposition 2.4.9 :

Proposition 2.4.11 (Bousfield [Bou67])**.** *Let F be a strict polynomial functor of degree n. We have the isomorphism :*

$$F(V_1 \oplus \cdots \oplus V_s) = \bigoplus_{\substack{\alpha_1+\cdots+\alpha_s=n \\ \alpha_i\geq 0}} F^{(\alpha_1,\dots,\alpha_s)}(V_1, \dots, V_s),$$

where the sum runs over all s-tuples of non-negative integers $\alpha_i \in \mathbb{N}$ such that $\alpha_1 + \cdots + \alpha_s = n$.

Proof: The proof follows directly from the decomposition of $\gamma_n(\mathrm{Id})$ in orthogonal idempotents as in Definition 2.4.8. □

Proposition 2.4.12. *Let F be a strict polynomial functor of degree n.*

1. *If $\alpha_i = 0$ for some i, then we have*

$$F^{(\alpha_1,\dots,\alpha_i,\dots,\alpha_s)}(V_1, \dots, V_i, \dots, V_s) = F^{(\alpha_1,\dots,\widehat{\alpha_i},\dots,\alpha_s)}(V_1, \dots, \widehat{V_i}, \dots, V_s),$$

2. if we assume $V_i = \bigoplus_{j=1}^{k_i} V_i^j$ for each i, then we have

$$F^{(\alpha_1,\dots,\alpha_i,\dots,\alpha_s)}(V_1,\dots,V_i,\dots,V_s) = \bigoplus_{\substack{(\beta_i^j)\\ \sum_j \beta_i^j=\alpha_i}} F^{(\beta_1^1,\dots,\beta_1^{k_1},\dots,\beta_s^1,\dots,\beta_s^{k_s})}(V_1^1,\dots,V_1^{k_1},\dots,V_s^1,\dots,V_s^{k_s}).$$

3. $\Gamma_n^{(\alpha_1,\dots,\alpha_n)}(X_1,\dots,X_r) = \Gamma_{\alpha_1}(X_1)\otimes\cdots\otimes\Gamma_{\alpha_n}(X_n)$.

Proof: The first relation is trivial. The second relation follows from decomposition rules for divided power operations :

$$\gamma_{\alpha_i}(\pi_i) = \gamma_{\alpha_i}(\pi_i^1+\cdots+\pi_i^{k_i}) = \sum_{\beta_i^1+\cdots+\beta_i^{k_i}=\alpha_i} \gamma_{\beta_i^1}(\pi_i^1)\dots\gamma_{\beta_i^{k_i}}(\pi_i^{k_i}),$$

with the obvious notation for the projectors associated to the direct sum $V_i = \bigoplus_{j=1}^{k_i} V_i^j$. To get the third relation, we use the isomorphism :

$$(X_1\oplus\cdots\oplus X_r)^{\otimes n} \cong \bigoplus_{(i_1,\dots,i_n)} X_{i_1}\otimes\cdots\otimes X_{i_n}.$$

The action of a permutation $\sigma\in\mathbb{S}_n$ on the tensor power maps the term $X_{i_1}\otimes\cdots\otimes X_{i_n}$ associated to $(i_1,\dots,i_n)$ to the term $X_{i_{\sigma(1)}}\otimes\cdots\otimes X_{i_{\sigma(n)}}$ in this sum. We then have the relation :

$$im(\Gamma_n(\gamma_{\alpha_1}(\pi_1)\cdots\gamma_{\alpha_s}(\pi_s))) = (\bigoplus_{|\{i_k=i\}|=\alpha_i} X_{i_1}\otimes\cdots\otimes X_{i_n})^{\mathbb{S}_n},$$

from which the requested identity follows. □

Lemma 2.4.13. *Let F be a strict polynomial functor. We have a natural morphism :*

$$\Gamma_{\alpha_1}(X_1)\otimes\dots\otimes\Gamma_{\alpha_r}(X_r)\otimes F^{(\alpha_1,\dots,\alpha_r)}(Y_1,\dots,Y_r) \longrightarrow F^{(\alpha_1,\dots,\alpha_r)}(X_1\otimes Y_1,\dots,X_r\otimes Y_r).$$

This pairing verifies an evident generalization of unit relation of 2.2.6 when we suppose $X_i=\mathbb{K}$ for some i as well as an evident generalization of associativity relation of Proposition 2.2.6 when we compose our pairing to get an operation of the form :

$$(\Gamma_{\alpha_1}(X_1)\otimes\cdots\otimes\Gamma_{\alpha_r}(X_r))\otimes(\Gamma_{\alpha_1}(Y_1)\otimes\cdots\otimes\Gamma_{\alpha_r}(Y_r))\otimes F^{(\alpha_1,\dots,\alpha_r)}(Z_1,\dots,Z_r) \to F^{(\alpha_1,\dots,\alpha_r)}(X_1\otimes Y_1\otimes Z_1,\dots,X_r\otimes Y_r\otimes Z_r)$$

Proof: The morphism is deduced from the following commutative diagram :

$$\begin{tikzcd}
\Gamma_n(X_1\oplus\dots\oplus X_r)\otimes F(Y_1\oplus\dots\oplus Y_r) \arrow[r, "(*)"] \arrow[d, "\Gamma_n(\gamma_{\alpha_1}(\pi_1)\dots\gamma_{\alpha_r}(\pi_r))\otimes F(\gamma_{\alpha_1}(\pi_1)\dots\gamma_{\alpha_r}(\pi_r))"'] & F(X_1\otimes Y_1\oplus\dots X_r\otimes Y_r) \arrow[d, "F(\gamma_{\alpha_1}(\pi_1)\dots\gamma_{\alpha_r}(\pi_r))"] \\
\Gamma_{\alpha_1}(X_1)\otimes\dots\otimes\Gamma_{\alpha_r}(X_r)\otimes F^{(\alpha_1,\dots,\alpha_r)}(Y_1,\dots,Y_r) \arrow[r, dashed] & F^{(\alpha_1,\dots,\alpha_r)}(X_1\otimes Y_1,\dots,X_r\otimes Y_r),
\end{tikzcd}$$

where $(*)$ is yielded by the morphism of Proposition 2.2.6 and the projection morphism

$$(X_1\oplus\cdots\oplus X_r)\otimes(Y_1\oplus\cdots\oplus Y_r)\to X_1\otimes Y_1\oplus\cdots\oplus X_r\otimes Y_r.$$

We apply the idempotent construction of Definition 2.4.8 to $F(X_1\oplus\cdots\oplus X_r)$, $F(Y_1\oplus\cdots\oplus Y_r)$, and $F(X_1\otimes Y_1\oplus\cdots\oplus X_r\otimes Y_r)$ to get the vertical morphisms of this diagram. We actually consider the corestriction of these idempotent morphisms to their image in our diagram. We check that these idempotents commute with the horizontal morphism $(*)$ to establish the existence of the dotted map of our diagram. We deduce this statement from the associativity of Proposition 2.2.6. To be more precise if we set $X' = X = X_1 \oplus \cdots \oplus X_r$ and $Y' = Y = Y_1 \oplus \cdots \oplus Y_r$, then this associativity property implies that we have a commutative diagram :

$$\begin{tikzcd}
\Gamma_n(Hom(X,X')) \otimes \Gamma_n(X) \otimes \Gamma_n(Hom(Y,Y')) \otimes F(Y) \arrow[r] \arrow[d] & \Gamma_n(X') \otimes F(Y') \arrow[dd] \\
\Gamma_n(Hom(X,X') \otimes Hom(Y,Y')) \otimes F(X \otimes Y) \arrow[d] & \\
\Gamma_n(Hom(X \otimes Y, X' \otimes Y')) \otimes F(X \otimes Y) \arrow[r] & F(X' \otimes Y')
\end{tikzcd}$$

We take the morphisms induced by the projection of $X \otimes Y = (\bigoplus_i X_i) \otimes (\bigoplus_j Y_j)$ onto $\bigoplus_i X_i \otimes Y_i$ to prolong the vertical morphism of this diagram. We then get a commutative diagram

$$\begin{tikzcd}
\Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(X \otimes Y, X' \otimes Y')) \otimes F(X \otimes Y) \arrow[r] \arrow[d] & F(X' \otimes Y') \arrow[d] \\
\Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\bigoplus_i X_i \otimes Y_i, \bigoplus_i X'_i \otimes Y'_i)) \otimes F(\bigoplus_i X_i \otimes Y_i) \arrow[r] & F(\bigoplus_i X'_i \otimes Y'_i)
\end{tikzcd}$$

We just take $\gamma_{\alpha_1}(\pi_1)\cdots\gamma_{\alpha_r}(\pi_r) \in \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(X,X))$ and $\gamma_{\alpha_1}(\pi_1)\cdots\gamma_{\alpha_r}(\pi_r) \in \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(Y,Y))$ to check our assertion.

The associativity of the pairing for $F^{(\alpha_1,\dots,\alpha_r)}$ comes from the associativity of the pairing for F with the direct sum inside. □

We use the observation of the previous proposition to give a sense to homogeneous cross-effects over a countable sequence of variables :

Definition 2.4.14. *Let F be a strict polynomial functor of degree n. Let $\underline{X} = (X_0,\dots,X_i,\dots)$ be a collection of modules $X_i \in Mod_K$. Let $\underline{\alpha} = (\alpha_0,\dots,\alpha_n,\dots)$ denote a sequence of non-negative integers $\alpha_i \in N$ such that $\alpha_i = 0$ for all but a finite number of indices i and $\sum_i \alpha_i = n$. Let $i_1 < \cdots < i_r$ be the collection of these indices $i = i_k$ such that $\alpha_i > 0$. We set :*

$$F^{\underline{\alpha}}(\underline{X}) = F^{(\alpha_{i_1},\dots,\alpha_{i_r})}(X_{i_1},\dots,X_{i_r}).$$

We then have the following generalization of the result of Proposition 2.4.11 :

Proposition 2.4.15. *Let F be a strict polynomial functor of degree n. Let $\underline{X} = (X_0,\dots,X_i,\dots)$ be a collection of modules $X_i \in Mod_K$. We have the isomorphism :*

$$F(X_0 \oplus \cdots \oplus X_i \oplus \dots) = \bigoplus_{\underline{\alpha}} F^{\underline{\alpha}}(\underline{X}),$$

where the sum runs over all the sequences of non-negative integers $\underline{\alpha} = (\alpha_0,\dots,\alpha_i,\dots)$ which satisfy the constraints of the previous definition.

Proof: The statement follows from the fact that F commutes with the filtered colimits (see Definition 2.2.4) □

Definition 2.4.16. *Let $(a_1,\dots,a_s)$ be any collection of non-negative integers $a_i \geq 0$. Let $n = a_1 + \cdots + a_s$. For an analytic functor $F = (F_n)_{n\in N}$, we set $F^{(a_1,\dots,a_s)} = F_n^{(a_1,\dots,a_s)}$, where we consider the homogeneous cross effect of the component of F of degree $n = a_1 + \cdots + a_s$. Let $\underline{\alpha} = (\alpha_0,\dots,\alpha_i,\dots)$ be any sequence of non-negative integers such that $\alpha_i = 0$ for all but a finite number of indices $i \geq 0$. Let $n = \sum_i \alpha_i$. We also set $F^{\underline{\alpha}} = F_n^{\underline{\alpha}}$, where we use the construction of Definition 2.4.14 for the component of F of degree $n = \sum_i \alpha_i$. The formulas of Proposition 2.4.11 and of Proposition 2.4.15 have an obvious generalization for analytic functors (we just forget about the constraints $\sum_i \alpha_i = n$ in this case).*

Proposition 2.4.17. *Let $F = \{F_n\}_{n\in\mathbb{N}}$ and $G = \{G_n\}_{n\in\mathbb{N}}$ be two strict analytic functors. The composition functor $\mathcal{U}F \circ \mathcal{U}G : \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ has a natural structure of strict analytic functor such that :*

$$(F \circ G)_n = \bigoplus_{\substack{s \\ 1\le t\le s}} \bigoplus_{\substack{0\le i_1<\ldots<i_t \\ \alpha_{i_1}+\ldots+\alpha_{i_t}=s \\ i_1\alpha_{i_1}+\ldots+i_t\alpha_{i_t}=n}} F_s^{(\alpha_{i_1},\ldots,\alpha_{i_t})}(G_{i_1},\ldots,G_{i_t}).$$

Proof: Proposition 2.4.15 implies that the functor $\mathcal{U}(F)\circ\mathcal{U}(G)(X)$ is given by the sum of the expression of the statement. The structure is given by the composition of the following morphisms :

$$\Gamma_n(X)\otimes F_s^{(\alpha_{i_1},\ldots,\alpha_{i_t})}(G_{i_1}(Y),\ldots,G_{i_t}(Y))$$

$$(1)\downarrow$$

$$\Gamma_{\alpha_{i_1}}(\Gamma_{i_1}(X))\otimes\ldots\otimes\Gamma_{\alpha_{i_t}}(\Gamma_{i_t}(X))\otimes F_s^{(\alpha_{i_1},\ldots,\alpha_{i_t})}(G_{i_1}(Y),\ldots,G_{i_t}(Y))$$

$$(2)\downarrow$$

$$F_s^{(\alpha_{i_1},\ldots,\alpha_{i_t})}(\Gamma_{i_1}(X)\otimes G_{i_1}(Y),\ldots,\Gamma_{i_t}(X)\otimes G_{i_t}(Y))$$

$$(3)\downarrow$$

$$F_s^{(\alpha_{i_1},\ldots,\alpha_{i_t})}(G_{i_1}(X\otimes Y),\ldots,G_{i_t}(X\otimes Y)),$$

To define our map (1), we use that any composite $\Gamma_k(\Gamma_l(X))$ is identified with the submodule of $X^{\otimes kl}$ spanned by the tensors which are invariant under a certain subgroup of S_{kl}, denoted by $S_k \wr S_l$, and which is classically called the wreath product in the literature. We then have $\Gamma_{\alpha_1}(\Gamma_{i_1}(X))\otimes\cdots\otimes\Gamma_{\alpha_r}(\Gamma_{i_r}(X)) = (X^{\otimes n})^{S_{\alpha_1}\wr S_{i_1}\times\cdots\times S_{\alpha_r}\wr S_{i_r}}$, and morphism (1) is given by the obvious embedding $\Gamma_n(X) = (X^{\otimes n})^{S_n} \hookrightarrow (X^{\otimes n})^{S_{\alpha_1}\wr S_{i_1}\times\cdots\times S_{\alpha_r}\wr S_{i_r}}$. The morphism (2) is the morphism of Lemma 2.4.13, and the morphism (3) is induced by the morphism of Proposition 2.2.6. □

Definition 2.4.18 (The product $\circ$)**.** *We define the product $\circ$ on* AnFun *by the construction of Proposition 2.4.17. It is compatible with the usual composition of functors in the sense that the following diagram commutes :*

$$\begin{tikzcd}
\mathrm{AnFun}\times\mathrm{AnFun} \arrow[r, "\circ"] \arrow[d, "\mathcal{U}\times\mathcal{U}"'] & \mathrm{AnFun} \arrow[d, "\mathcal{U}"] \\
\mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}},\mathrm{Mod}_{\mathbb{K}})\times\mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}},\mathrm{Mod}_{\mathbb{K}}) \arrow[r, "\circ"'] & \mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}},\mathrm{Mod}_{\mathbb{K}}).
\end{tikzcd}$$

Lemma 2.4.19. *Let F, G be analytic functors. We use the short notation $\underline{X} = (X_1,\ldots,X_r)$ for any r-tuple of K-modules X_i. We also use the short notation $\underline{b}$ for any collection $\underline{b} = (b_1,\ldots,b_r) \in \mathbb{N}^r$ and we set $\Gamma_{\underline{b}}(\underline{X}) = \bigotimes_{i=1}^r \Gamma_{b_i}(X_i)$ for short. We equip the set of collections $\mathbb{N}^r$ with a total ordering and we fix $\underline{c} = (c_1,\ldots,c_r)\in\mathbb{N}^r$.*

1. *We have :*

$$(F\circ G)^{(c_1,\ldots,c_r)}(\underline{X}) = \bigoplus_{\substack{\underline{b}^1<\cdots<\underline{b}^l \\ a_i>0 \\ \sum_i a_i b_j^i = c_j\,(\forall j)}} F^{(a_1,\ldots,a_l)}(G^{\underline{b}^1}(\underline{X}),\ldots,G^{\underline{b}^l}(\underline{X})),$$

where the sum runs over all sequences $(a_1,\ldots,a_l)$, $l\ge 0$, of positive integers $a_i > 0$, and over all ordered sequences of collections $\underline{b}^1<\cdots<\underline{b}^l$ such that we have $\sum_i a_i b_j^i = c_j$, for all $j\in\{1,\ldots,r\}$.

2. *For this object $(F \circ G)^{\underline{c}}(-)$, the pairing of Lemma 2.4.13 is given by a composite of the form :*

$$\begin{aligned}
\Gamma_{\underline{c}}(\underline{X}) \otimes (F \circ G)^{\underline{c}}(\underline{Y}) \to & \bigoplus_{\substack{\underline{b}^1 < \cdots < \underline{b}^l \\ a_i > 0 \\ \sum_i a_i b^i_j = c_j\, (\forall j)}} \Gamma_{a_1}(\Gamma_{\underline{b}^1}(\underline{X})) \otimes \cdots \otimes \Gamma_{a_l}(\Gamma_{\underline{b}^l}(\underline{X})) \otimes F^{(a_1,\dots,a_l)}(G^{\underline{b}^1}(\underline{Y}), \dots, G^{\underline{b}^l}(\underline{Y})) \\
\to & \bigoplus_{\substack{\underline{b}^1 < \cdots < \underline{b}^l \\ a_i > 0 \\ \sum_i a_i b^i_j = c_j\, (\forall j)}} F^{(a_1,\dots,a_l)}(\Gamma_{\underline{b}^1}(\underline{X}) \otimes G^{\underline{b}^1}(\underline{Y}), \dots, \Gamma_{\underline{b}^l}(\underline{X}) \otimes G^{\underline{b}^l}(\underline{Y})) \\
\to & \bigoplus_{\substack{\underline{b}^1 < \cdots < \underline{b}^l \\ a_i > 0 \\ \sum_i a_i b^i_j = c_j\, (\forall j)}} F^{(a_1,\dots,a_l)}(G^{\underline{b}^1}(\underline{X} \otimes \underline{Y}), \dots, G^{\underline{b}^l}(\underline{X} \otimes \underline{Y})),
\end{aligned}$$

where we use the notation $\underline{Y} = (Y_1, \dots, Y_r)$ for another r-tuple of variables, and we set $\underline{X} \otimes \underline{Y} = (X_1 \otimes Y_1, \dots, X_r \otimes Y_r)$. In this composite, the first morphism is given term-wise by a canonical inclusion $\Gamma_{\underline{c}}(\underline{X}) \hookrightarrow \Gamma_{a_1}(\Gamma_{\underline{b}^1}(\underline{X})) \otimes \cdots \otimes \Gamma_{a_l}(\Gamma_{\underline{b}^l}(\underline{X}))$, and the next morphisms are given by the pairing of Lemma 2.4.13 for the functors F and G.

Proof: We have by definition :

$$(F \circ G)_n(X_1 \oplus \cdots \oplus X_r) = \bigoplus_{\substack{n_1 < \cdots < n_l \\ \alpha_i > 0\, (\forall i) \\ \sum n_i \alpha_i = n}} F^{(\alpha_1,\dots,\alpha_l)}(G_{n_1}(X_1 + \cdots + X_r), \dots, G_{n_l}(X_1 + \cdots + X_r)).$$

Let $\pi_i : X_1 \oplus \cdots \oplus X_r \to X_1 \oplus \cdots \oplus X_r$ be the morphism given by the projection onto the summand X_i in the sum $X = X_1 \oplus \cdots \oplus X_r$. For any collection $\underline{b} = (b_1, \dots, b_r)$, we set $\gamma_{\underline{b}}(\underline{\pi}) = \prod_{i=1}^r \gamma_{b_i}(\pi_i)$ for short.

We use the expansion

$$G_{n_i}(X_1 + \cdots + X_r) = \bigoplus_{\beta_1 + \cdots + \beta_r = n_i} G^{(\beta_1,\dots,\beta_r)}(\underline{X})$$

of Proposition 2.4.13. We adopt the short notation $\Pi^{\underline{\beta}} = \gamma_{\underline{\beta}}(\underline{\pi})$ for the morphism which induces the projection onto the summand $G^{\underline{\beta}}(\underline{X})$ in this sum, where we still write $\underline{\beta} = (\beta_1, \dots, \beta_r)$ for short. We also use the notation $|\underline{\beta}| = \beta_1 + \cdots + \beta_r$ for any collection $\underline{\beta} = (\beta_1, \dots, \beta_r)$ in what follows.

We aim to determine the image of the element $\gamma_{\underline{c}}(\underline{\pi}) \in \Gamma_n(X)$ under the morphism $\Delta : \Gamma_n(X) \to \Gamma_{\alpha_1}(\Gamma_{n_1}(X)) \otimes \cdots \otimes \Gamma_{\alpha_r}(\Gamma_{n_r}(X))$ which we use in the construction of Proposition 2.4.17. We explicitly get :

$$\Delta(\gamma_{\underline{c}}(\underline{\pi})) = \sum_{\substack{\underline{b}^{i,1} < \cdots < \underline{b}^{i,k_i} \\ s.t.\, |\underline{b}^{i,j}| = n_i\, (\forall i,j) \\ a_i^1, \dots, a_i^{k_i} > 0 \\ s.t.\, a_i^1 + \cdots + a_i^{k_i} = \alpha_i\, (\forall i) \\ \sum_{ij} a_i^j b_s^{i,j} = c_s\, (\forall s)}} \underbrace{\gamma_{a_1^1}(\Pi^{\underline{b}^{1,1}}) \dots \gamma_{a_1^{k_1}}(\Pi^{\underline{b}^{1,k_1}})}_{\in \Gamma_{\alpha_1}(\Gamma_{n_1}(X))} \otimes \cdots \otimes \underbrace{\gamma_{a_l^1}(\Pi^{\underline{b}^{r,1}}) \dots \gamma_{a_l^{k_l}}(\Pi^{\underline{b}^{r,k_l}})}_{\in \Gamma_{\alpha_l}(\Gamma_{n_l}(X))}, \quad (*)$$

where the sum runs over collections of positive integers $a_i^1, \dots, a_i^{k_i} > 0$, $k_i \geq 1$, $i = 1, \dots, r$, and over sequences $\underline{b}^{i,1} < \cdots < \underline{b}^{i,k_i}$ of collections $\underline{b}^{i,j} = (b_1^{i,j}, \dots, b_r^{i,j})$ which satisfy the constraints given in our expression. We put off the verification of this identity until the end of this proof.

We deduce from this result that we have an identity :

$$(F\circ G)^{(c_1,\dots,c_r)}(\underline{X}) = \bigoplus_{\substack{n_1<\dots<n_l\\ \alpha_i>0\ (\forall i)\\ \sum n_i\alpha_i=n}} \Bigg(\bigoplus_{\substack{\underline{b}^{i,1}<\dots<\underline{b}^{i,k_i}\\ s.t.\ |\underline{b}^{i,j}|=n_i\ (\forall i,j)\\ a_i^1,\dots,a_i^{k_i}>0\\ s.t.\ a_i^1+\dots+a_i^{k_i}=\alpha_i\ (\forall i)\\ \sum_{ij} a_i^j b_s^{i,j}=c_s\ (\forall s)}} F^{a_1^1,\dots,a_1^{k_1},\dots,a_l^1,\dots,a_l^{k_l}}(G^{\underline{b}^{1,1}}(\underline{X}),\dots,G^{\underline{b}^{1,k_1}}(\underline{X}),\dots,G^{\underline{b}^{l,1}}(\underline{X}),\dots,G^{\underline{b}^{l,k_l}}(\underline{X}))\Bigg),$$

and we use a straightforward re-indexing of the direct sum which we get in this formula to get the decomposition of the lemma.

The second assertion of the lemma follows from a straightforward expansion of the definition of our pairing in Proposition 2.4.12 for objects of the form $F^{(\alpha_1,\dots,\alpha_l)}(G_{n_1}(X_1+\dots+X_r),\dots,G_{n_l}(X_1+\dots+X_r))$ and from the expansion of our pairing for the objects $G^{\underline{b}}(\underline{X})$ in Lemma 2.4.13. We also use that these constructions are compatible with the isomorphisms of Proposition 2.4.12 which we use to get the expansion of the first assertion of this lemma.

We now explain the proof of Formula $(*)$. We argue as follows. We use a scalar extension $\mathbb{K}[t_1,\dots,t_r]\otimes_{\mathbb{K}} -$, where $(t_1,\dots,t_r)$ denote formal variables and we work in $\mathbb{K}[t_1,\dots,t_r]\otimes_{\mathbb{K}} \Gamma_n(Hom_{\mathrm{Mod}_{\mathbb{K}}}(X,X)) = \Gamma_n(\mathbb{K}[t_1,\dots,t_r]\otimes_{\mathbb{K}} Hom_{\mathrm{Mod}_{\mathbb{K}}}(X,X))$. We have the formula $\gamma_n(t_1\pi_1+\dots+t_r\pi_r) = \sum_{m_1+\dots+m_r=n}\gamma_{m_1}(\pi_1)\dots\gamma_{m_r}(\pi_r)t_1^{m_1}\dots t_r^{m_r}$ by properties of divided powers (see Definition A.1.40). We can accordingly identify $\gamma_{\underline{c}}(\underline{\pi})$ with the coefficient of $t^{\underline{c}} = t_1^{c_1}\dots t_r^{c_r}$ in the expansion of $\gamma_n(t_1\pi_1+\dots+t_r\pi_r)$. We use that for an element of this form $\gamma_n(\phi)$, where $\phi = t_1\pi_1+\dots+t_r\pi_r$, we have the formula $\Delta(\gamma_n(\phi)) = \gamma_{\alpha_1}(\gamma_{n_1}(\phi))\otimes\dots\otimes\gamma_{\alpha_r}(\gamma_{n_r}(\phi))$ in $\Gamma_{\alpha_1}(\Gamma_{n_1}(X))\otimes\dots\otimes\Gamma_{\alpha_r}(\Gamma_{n_r}(X))$. The terms of Formula $(*)$ correspond to the coefficients of the monomial $t_1^{c_1}\dots t_r^{c_r}$ when we use the properties of the divided powers to expand the factors $\gamma_{\alpha_i}(\gamma_{n_i}(\phi)) = \gamma_{\alpha_i}(\gamma_{n_i}(t_1\pi_1+\dots+t_r\pi_r))$ in this tensor product. □

Lemma 2.4.20. *Let F be an analytic functor. We have an isomorphism $Id\circ F\simeq F\simeq F\circ Id$ in the category of analytic functors which realizes the obvious identity* $\mathrm{Id}\circ\mathcal{U}(F)=\mathcal{U}(F)=\mathcal{U}(F)\circ\mathrm{Id}$ *in the category of ordinary functors.*

Let A,B,C be analytic functors. We have an isomorphism $(A\circ B)\circ C\simeq A\circ(B\circ C)$ in the category of analytic functors which realizes the obvious identity $(\mathcal{U}(A)\circ\mathcal{U}(B))\circ\mathcal{U}(C) = \mathcal{U}(A)\circ(\mathcal{U}(B)\circ\mathcal{U}(C))$ in the category of ordinary functors.

Proof: The verification of the unit relation is easy and we focus on the proof of the associativity relation.

We use the following conventions in this proof. We set $\underline{F}(X) = (F_0(X),\dots,F_n(X),\dots)$ for the sequence of modules $F_n(X)$ which we obtain by taking the image of a module X under the components of an analytic functor $F_n\in AnPol_n$. For a sequence $\underline{\alpha} = (\alpha_0,\dots,\alpha_n,\dots)$ such that $\alpha_i = 0$ for all but a finite number of indices i, we also set $w(\alpha) = \sum_i i\alpha_i$.

We have a straightforward generalization of the result of the previous lemma in the case where $\underline{X}$ is a countable sequence of modules $\underline{X} = (X_0,\dots,X_n,\dots)$. We then assume that the set of sequences $\underline{b} = (b_0,\dots,b_i,\dots)$ such that $b_i = 0$ for all but a finite number of indices i is equipped with a total ordering such that $\underline{b}^1<\underline{b}^2$ if we have $\sum_i b_i^1 < \sum_i b_i^2$. We get :

$$(A\circ B)^{\underline{c}}(\underline{X}) = \bigoplus_{\substack{\underline{b}^1<\dots<\underline{b}^l\\ a_i>0\\ \sum_i a_i b_j^i=c_j\ (\forall j)}} A^{(a_1,\dots,a_l)}(B^{\underline{b}^1}(\underline{X}),\dots,B^{\underline{b}^l}(\underline{X})),$$

where the sum runs over all sequences $(a_1,\dots,a_l)$, $l\geq 0$, of positive integers $a_i>0$, and over all ordered sequences of collections $\underline{b}^1<\dots<\underline{b}^l$ such that we have $\sum_i a_i b_j^i = c_j$, for all $j\in\{1,\dots,r\}$.

We use this identity to determine the expansion of $(A \circ B) \circ C$. We explicitly have :

$$\begin{aligned}((A \circ B) \circ C)_n(X) &= \bigoplus_{\underline{c}\, s.t.\, w(\underline{c})=n} (A \circ B)^{\underline{c}}(\underline{C}(X)) \\ &= \bigoplus_{\substack{\underline{b}^1 < \cdots < \underline{b}^l \\ a_i > 0 \\ \sum_i a_i b^i_j j = n}} A^{(a_1,\ldots,a_l)}(B^{\underline{b}^1}(\underline{C}(X)),\ldots,B^{\underline{b}^l}(\underline{C}(X))).\end{aligned}$$

We also get that the pairing $\Gamma_n(X) \otimes ((A \circ B) \circ C)_n(Y) \to ((A \circ B) \circ C)_n(X \otimes Y)$ which we associate to the composite functor $((A \circ B) \circ C)$ is carried to the direct sum of the morphisms

$$\Gamma_n(X) \otimes A^{(a_1,\ldots,a_l)}(B^{\underline{b}^1}(\underline{C}(Y)),\ldots,B^{\underline{b}^l}(\underline{C}(Y))) \to A^{(a_1,\ldots,a_l)}(B^{\underline{b}^1}(\underline{C}(X \otimes Y)),\ldots,B^{\underline{b}^l}(\underline{C}(X \otimes Y)))$$

which we obtain by using the operation of the previous lemma for the composite $A \circ B$, and by using the pairing $\zeta_{X,Y} : \Gamma_i(X) \otimes C_i(Y) \to C_i(X \otimes Y)$ associated to each functor $C_i(-)$ inside the functors $B^{\underline{b}^j}$.

We have on the other hand :

$$(A \circ (B \circ C))_n(X) = \bigoplus_{\substack{n_1 < \cdots < n_r \\ \alpha_1,\ldots,\alpha_r > 0 \\ \alpha_1 n_1 + \cdots + \alpha_r n_r = n \\ r \geq 0}} A^{(\alpha_1,\ldots,\alpha_r)}((B \circ C)_{n_1}(X),\ldots,(B \circ C)_{n_r}(X)).$$

We then use the expression of each $(B \circ C)_{n_i}(X)$ as a direct sum of cross-effects in Proposition 2.4.17, and the result of Proposition 2.4.12 to get the identity :

$$\begin{aligned}(A \circ (B \circ C))_n(X) = \bigoplus_{\substack{n_1 < \cdots < n_r \\ \alpha_1,\ldots,\alpha_r > 0 \\ \alpha_1 n_1 + \cdots + \alpha_r n_r = n \\ r \geq 0}} \Bigg(\bigoplus_{\substack{\underline{b}_1^1 < \cdots < \underline{b}_1^{l_1} < \cdots < \underline{b}_1^1 < \cdots < \underline{b}_1^{l_r} \\ w(\underline{b}_i^j) = n_i\ (\forall j) \\ a_i^j > 0, \sum_j a_i^j = \alpha_i}} \\ A^{(a_1^1,\ldots,a_1^{l_1},\ldots,a_r^1,\ldots,a_r^{l_r})}(B^{\underline{b}_1^1}(\underline{C}(\underline{X})),\ldots,B^{\underline{b}_1^{l_1}}(\underline{C}(\underline{X})),\ldots,B^{\underline{b}_r^1}(\underline{C}(\underline{X})),\ldots,B^{\underline{b}_r^{l_r}}(\underline{C}(\underline{X}))) \Bigg).\end{aligned}$$

We use a straightforward re-indexing operation in this sum to retrieve the expression of $((A \circ B) \circ C)_n(X)$. We can also check by using the correspondence of Lemma 2.4.13 and of Proposition 2.4.17 inside each input of the functor $A^{(\alpha_1,\ldots,\alpha_r)}(-,\ldots,-)$ that the pairing $\Gamma_n(X) \otimes (A \circ (B \circ C))_n(Y) \to (A \circ (B \circ C))_n(X \otimes Y)$ which we obtain for this expression of the composite $(A \circ (B \circ C))_n(X)$ agrees with the pairing which we obtain for the composite $((A \circ B) \circ C)_n(X)$.

We conclude that we have an isomorphism of strict polynomial functor $((A \circ B) \circ C)_n \simeq (A \circ (B \circ C))_n$, for each $n \in N$. ☐

Proposition 2.4.21. *The triple* $(\mathrm{AnFun}, \circ, \mathrm{Id})$ *forms a monoidal category.*

Proof: This statement follows from the result of the previous lemma. Let us simply mention that our structure isomorphisms fulfil the coherence constraints of monoidal categories since we observe that these isomorphisms correspond to the obvious unit and associativity identities of the composition in the category of functors and because the functor $\mathcal{U} : AnFun \to Fun(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}})$ is faithful. ☐

2.4.2 The functor ev

We introduce the equivalence of categories $ev : \mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}} \longrightarrow \mathrm{AnFun}$ which extends the functor $ev_n : \mathrm{Mac}^{coh}(\mathcal{H}Par_n) \to PolFun_n$ of Definition 2.2.14. We prove that ev is strongly monoidal; i.e. it reflects the two monoidal structures on $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$ into the tensor product and the composition of functors.

Definition 2.4.22 (The functor ev)**.** *Let M be an $\mathbb{M}$-module, it defines a strict analytic functor*

$$\{ev_n(M_n)(V)\}_{n\in\mathbb{N}} = \{\int^{\pi\in\mathcal{H}Par_n} M_n(\pi)\otimes\Gamma_\pi(V)\}_{n\in\mathbb{N}}.$$

The mapping ev is functorial in M, so it defines a functor :

$$ev : \mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}} \longrightarrow \mathrm{AnFun},$$

$$M \mapsto \{ev_n(M)(-)\}_{n\in\mathbb{N}}.$$

Since ev_n is an equivalence of categories for any $n\in\mathbb{N}$, we have that ev is an equivalence of categories as well.

We devote the rest of this section to the study of the image of monoidal structures under the functor ev. We establish a series of intermediate lemmas before formulating our main theorem.

Lemma 2.4.23. *We have a natural isomorphism*

$$ev(M\boxtimes N)\to ev(M)\otimes ev(N)$$

for any pair $M,N\in\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$, where we consider the functor $ev(M\boxtimes N)\in\mathrm{AnFun}$ associated to $M\boxtimes N$ on the left hand side, the pointwise tensor product of the analytic functors $ev(M), ev(N)\in\mathrm{AnFun}$ such as in Definition 2.4.3 on the right hand side.

Proof: We prove that there exists a natural isomorphism $ev(M\boxtimes N)\longrightarrow ev(M)\otimes ev(N)$. It follows from a sequence of natural isomorphims given by $\Gamma_{\pi_1\times\pi_2}(V)\cong\Gamma_{\pi_1}(V)\otimes\Gamma_{\pi_2}(V)$, Lemma 2.3.4, and Lemma 2.3.5. More precisely :

$$\begin{aligned}
&ev(M\boxtimes N)(V)\\
&\quad=\bigoplus_n\bigoplus_{i+j=n}\int^{\pi\in\mathcal{H}Par_n}\int^{\pi_1\times\pi_2\in\mathcal{H}Par_i\times\mathcal{H}Par_j}(M(\pi_1)\otimes N(\pi_2))\otimes\mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1\times\pi_2,\pi)\otimes\Gamma_\pi(V)\\
&\quad\cong\bigoplus_n\bigoplus_{i+j=n}\int^{\pi_1\times\pi_2}(M(\pi_1)\otimes N(\pi_2))\otimes\int^{\pi}\mathrm{Hom}_{Par_n}(\pi_1\times\pi_2,\pi)\otimes\Gamma_\pi(V)\\
&\quad\cong\bigoplus_{i,j}\int^{\pi_1\times\pi_2}(M(\pi_1)\otimes N(\pi_2))\otimes\Gamma_{\pi_1}(V)\otimes\Gamma_{\pi_2}(V)\\
&\quad\cong(\bigoplus_i\int^{\pi_1}M(\pi_1)\otimes\Gamma_{\pi_1}(V))\otimes(\bigoplus_j\int^{\pi_2}N(\pi_2)\otimes\Gamma_{\pi_2}(V)),
\end{aligned}$$

where we use the isomorphisms given by Lemma 2.3.4 and by Lemma 2.3.5.

The isomorphism commute with the action of $\Gamma_n(X)$ on $ev_n(M\boxtimes N)(Y)$. This claim follows from the commutativity of the following diagram :

$$\begin{tikzcd}
\Gamma_n(X)\otimes\int^{\pi\in\mathcal{H}Par_n}\mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1\times\pi_2,\pi)\otimes\Gamma_\pi(Y) \arrow[rr] & & \int^{\pi}\mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1\times\pi_2,\pi)\otimes\Gamma_\pi(X\otimes Y)\\
\Gamma_n(X)\otimes\Gamma_{\pi_1}(Y)\otimes\Gamma_{\pi_2}(Y) \arrow[u, "\cong"] \arrow[d] & & \\
\Gamma_i(X)\otimes\Gamma_j(X)\otimes\Gamma_{\pi_1}(Y)\otimes\Gamma_{\pi_2}(Y) \arrow[rr] & & \Gamma_{\pi_1}(X\otimes Y)\otimes\Gamma_{\pi_2}(X\otimes Y), \arrow[uu, "\cong"']
\end{tikzcd}$$

where $i+j=n$, $\pi_1\in Par_i$, $\pi_2\in Par_j$ and the morphism $\Gamma_n(X)\to\Gamma_i(X)\otimes\Gamma_j(X)$ is given by the restriction from $\mathbb{S}_n$ to $\mathbb{S}_i\times\mathbb{S}_j$. (We then use that the Fubini isomorphism of Lemma 2.3.4 is given by the canonical morphism from the object $\Gamma_{\pi_1\times\pi_2}(V)=\mathrm{Id}_{\pi_1\times\pi_2}\otimes\Gamma_{\pi_1\times\pi_2}(V)\subset\mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1\times\pi_2,\pi_1\times\pi_2)\otimes\Gamma_n(V)$ into the coend and that $\Gamma_n(X)$ acts on $\Gamma_{\pi_1\times\pi_2}(-)=\Gamma_{\pi_1}(-)\otimes\Gamma_{\pi_2}(-)$ through the diagonal morphism $\Gamma_n(X)\to\Gamma_i(X)\otimes\Gamma_j(X)$.) $\square$

Lemma 2.4.24. *The isomorphisms of Lemma 2.4.23 make the unit, associativity, and symmetry isomorphisms of the symmetric monoidal category of $\mathbb{M}$-modules, such as defined in Proposition 2.3.7, correspond to the unit, associativity and symmetry isomorphisms of the symmetric monoidal category of analytic functors such as defined in Proposition 2.4.5.*

Proof: The proof of this lemma follows from straightforward verifications. □
We show a similar result for $\square$.

Lemma 2.4.25. *Let M be an $\mathbb{M}$-module. We have interchange formula :*

$$ev_n((N^{\boxtimes r})^{\rho}) = (ev_n(N^{\boxtimes r}))^{\rho}$$

for every ρ subgroup of $\mathbb{S}_r$.

Proof: Since the functor ev_n is an equivalence of category it is an exact functor and hence preserves invariants. □

Lemma 2.4.26. *We have a natural isomorphism*

$$ev(M \square N) \cong ev(M) \circ ev(N),$$

for every $M, N \in \mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$, where we consider the functor $ev(M \square N) \in \mathrm{AnFun}$ associated to $M \square N$ on the left hand side, the composition product of the analytic functors $ev(M), ev(N) \in \mathrm{AnFun}$ such as in Definition 2.4.18 on the right hand side.

Proof: We have :

$$\begin{aligned}
ev(M \square N)(V) &\cong \bigoplus_n \int^{\pi \in \mathcal{H}Par_n} \int^{\rho \in \mathcal{H}Par_r} (M(\rho) \otimes (N^{\boxtimes r}(\pi))^{\rho}) \otimes \Gamma_{\pi}(V) \\
&\cong \bigoplus_{r,n} \int^{\rho} \int^{\pi} (M(\rho) \otimes (N^{\boxtimes r}(\pi))^{\rho}) \otimes \Gamma_{\pi}(V) \\
&\cong \bigoplus_{r,n} \int^{\rho} M(\rho) \otimes \int^{\pi} (N^{\boxtimes r}(\pi))^{\rho} \otimes \Gamma_{\pi}(V) \\
&= \bigoplus_{r,n} \int^{\rho} M(\rho) \otimes ev_n((N^{\boxtimes r})^{\rho})(V) \\
&\overset{(1)}{\cong} \bigoplus_{r} \int^{\rho} M(\rho) \otimes (ev(N^{\boxtimes r})(V))^{\rho} \\
&\cong \bigoplus_{r} \int^{\rho} M(\rho) \otimes (ev(N)(V)^{\otimes r})^{\rho} \\
&\cong \bigoplus_{r} \int^{\rho} M(\rho) \otimes \Gamma_{\rho}(ev(N)(V)) = ev(M)(ev(N)(V)),
\end{aligned}$$

where we use the isomorphisms given by Lemma 2.3.4 and by Lemma 2.3.5, and the isomorphism (1) is given by Lemma 2.4.25.

To check that this isomorphism commutes with the action of $\Gamma_n(X)$ we use that the isomorphisms inside the coends preserve the natural action of $\Gamma_n(X)$ on our objects. In the final step, we get an action of $\Gamma_n(X)$ on $\Gamma_{\rho}(ev(N)(-)) \subset ev(N)(-)^{\otimes r}$ which coincides with the action defined in Proposition 2.4.17 for this composite functor, and the conclusion readily follows. □

Lemma 2.4.27. *The composition product $\square$ inherits unit and associativity isomorphisms which correspond to the unit and associativity isomorphisms of the composition of analytic functors, such as defined in Definition 2.4.18. These unit and associativity isomorphisms satisfy the coherence constraints of a monoidal category in $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$. Thus the triple $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \square, \mathbb{I})$, where $\mathbb{I}$ denotes the obvious $\mathbb{M}$-module which corresponds to the identity functor, forms a monoidal category.*

Proof: This statement follows from the result of Lemma 2.4.26 and from the observation that ev is an equivalence of categories. □

Theorem 2.4.28. *The mapping $ev : M \mapsto ev(M)$ defines an equivalence of symmetric monoidal categories $ev : (\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \boxtimes, \mathbb{K}) \to (\mathrm{AnFun}, \otimes, \mathbb{K})$ as well as an equivalence of monoidal categories $ev : (\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \square, \mathbb{I}) \to (\mathrm{AnFun}, \circ, \mathrm{Id})$.*

Proof: The proof follows from Theorem 2.2.18, Lemma 2.4.24 and Lemma 2.4.27. □

Remark 2.4.29. *Let A,B and C be three $\mathbb{M}$-modules. We have an isomorphism :*

$$(A \square C) \boxtimes (B \square C) \cong (A \boxtimes B) \square C$$

which reflects the formula :

$$(ev(A) \circ ev(C)) \otimes (ev(B) \circ ev(C)) \cong (ev(A) \otimes ev(B)) \circ ev(C).$$

Recall that by Definition 2.3.2, to a $\mathbb{S}$-module M we can associate the $\mathbb{M}$-modules $\Gamma(M)$ and $S(M)$. By definition 2.4.22 we have the strict analytic functors $ev(\Gamma(M))$ and $ev(S(M))$. In the following proposition we identify these strict analytic functors.

Proposition 2.4.30. *Let $M = \{M_n\}_{n\in\mathbb{N}}$ be a $\mathbb{S}$-module (see Appendix A). If V is a free $\mathbb{K}$-module, then we have $ev(S(M))(V) \cong \{S_n(M,V)\}_{n\in\mathbb{N}}$ and $ev(\Gamma(M))(V) \cong \{\Gamma_n(M,V)\}_{n\in\mathbb{N}}$, where $S_n(M,V) = M(n) \otimes_{\mathbb{S}_n} V^n$ and $\Gamma_n(M,V) = M(n) \otimes^{\mathbb{S}_n} V^n$ (see Appendix A).*

Proof: We first consider the cohomological $\mathcal{H}Par_n$-Mackey functor $T(M) = T_n(M)$ where $T_n(M) = M_n \otimes \mathbb{I}^{\boxtimes n}$. We have that $ev_n(T_n(M))(V) = ev_n(M_n \otimes \mathbb{I}^{\boxtimes n})(V) \cong M_n \otimes V^{\otimes n}$. The unit object $\mathbb{I}$ is given by $\mathbb{I}_1 = \mathbb{K}$ (the constant functor on the category $\mathcal{H}Par_1$ with object set pt and Hom-object $\mathbb{K}$) and $\mathbb{I}_i = 0$ for $i \neq 1$. Let $\pi \in Par_n$. We accordingly have

$$\begin{aligned}
\mathbb{I}^{\boxtimes r}(\pi) &= \bigoplus_{i_1+\cdots+i_r=n} \int^{\pi_1\times\cdots\times\pi_r\in\mathcal{H}Par_{i_1}\times\cdots\times\mathcal{H}Par_{i_r}} \mathbb{I}(\pi_1)\otimes\cdots\otimes\mathbb{I}(\pi_r)\otimes \mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1\times\cdots\times\pi_r,\pi)\\
&= \int^{\pi_1\times\cdots\times\pi_r\in\mathcal{H}Par_1^{\times r}} \mathbb{I}(\pi_1)\otimes\cdots\otimes\mathbb{I}(\pi_1)\otimes \mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1\times\cdots\times\pi_r,\pi)\\
&= \mathbb{K}[\mathbb{S}_n/\pi]
\end{aligned}$$

by the associativity of $\boxtimes$ and the definition of $\mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1,\pi_2)$. We therefore have $T_n(M)(\pi) = M_n \otimes \mathbb{K}[\mathbb{S}_n/\pi]$.

The mapping $S_n(M)(\pi) \longrightarrow M_n \otimes_{\mathbb{S}_n} \mathbb{K}[\mathbb{S}_n/\pi]$ defined by $[m] \mapsto [m\otimes e]$ where e is the unit of $\mathbb{S}_n$ induces an isomorphism $S_n(M) \cong M_n \otimes_{\mathbb{S}_n} \mathbb{K}[\mathbb{S}_n/\pi] \cong (T_n(M))_{\mathbb{S}_n}$. The mapping $\Gamma_n(M)(\pi) \longrightarrow M_n \otimes^{\mathbb{S}_n} \mathbb{K}[\mathbb{S}_n/\pi]$ defined by $m \mapsto \sum_{\alpha\in\mathbb{S}_n/\pi} \alpha^*(m)\otimes\alpha$ induces an isomorphism $\Gamma_n(M) \cong M_n \otimes^{\mathbb{S}_n} \mathbb{K}[\mathbb{S}_n/\pi] \cong (T_n(M))^{\mathbb{S}_n}$.

Since ev_n is an equivalence of categories it preserves invariants and coinvariants. The conclusion follows. □

Corollary 2.4.31. *Let M and N be two $\mathbb{S}$-modules (see Appendix A). We have*
- $S(M) \boxtimes S(N) \cong S(M\boxtimes N)$*, and*
- $\Gamma(M) \boxtimes \Gamma(N) \cong \Gamma(M \boxtimes N)$*.*

Proof: The statement is a direct consequence of Proposition 2.4.30, of Proposition A.1.23 and of Theorem 2.4.28. □

Corollary 2.4.32. *Let M and N be two $\mathbb{S}$-modules (see Appendix A). We have*
- $S(M) \square S(N) \cong S(M \square_{\mathbb{S}} N)$*, and*
- $\Gamma(M) \square \Gamma(N) \cong \Gamma(M \square^{\mathbb{S}} N)$*.*

Proof: The statement is a direct consequence of Proposition 2.4.30, of Proposition A.1.23 and of Theorem 2.4.28. □

2.5 $\mathbb{M}$-Operads and their algebras

In this section we introduce the definition of $\mathbb{M}$-operad. Roughly speaking an $\mathbb{M}$-operad is an object governing the category of "type of algebras" with polynomial operations with multiple inputs and one output. Our definition of $\mathbb{M}$-operad is equivalent to the definition of Schur operads introduced by Ekedahl and Salomonsson in [ES04], [Sal03] and studied by Xantcha in [Xan10].

2.5.1 $\mathbb{M}$-Operads

We introduce the definition of $\mathbb{M}$-operads. They are a generalization of operads (see Appendix A).

Definition 2.5.1 ($\mathbb{M}$-operad)**.** *An $\mathbb{M}$-operad is an $\mathbb{M}$-module P together with two $\mathbb{M}$-module morphisms $\mu : P \square P \longrightarrow P$ and $\eta : \mathbb{I} \longrightarrow P$ such that the following diagrams commute :*

$$\begin{tikzcd} P \square P \square P \arrow[r, "\mathrm{Id}_P \square \mu"] \arrow[d, "\mu \square \mathrm{Id}_P"'] & P \square P \arrow[d, "\mu"] \\ P \square P \arrow[r, "\mu"'] & P, \end{tikzcd} \qquad \text{(associativity)}$$

$$\begin{tikzcd} P \square \mathbb{I} \arrow[r, "\mathrm{Id}_P \square \eta"] \arrow[dr, "\pi_1"'] & P \square P \arrow[d, "\mu"] & \mathbb{I} \square P \arrow[l, "\eta \square \mathrm{Id}_P"'] \arrow[dl, "\pi_2"] \\ & P, & \end{tikzcd} \qquad \text{(unity)}$$

i.e. (P, μ, η) *is a monoid in the monoidal category* $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \square, \mathbb{I})$.

A morphism of $\mathbb{M}$-operad is a morphism of monoid in the monoidal category $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \square, \mathbb{I})$*. We denote the category of $\mathbb{M}$-operad by* $\mathbb{M}$-Op.

Proposition 2.5.2. *Let P be a connected operad (see Appendix A). The $\mathbb{M}$-modules $S(P)$, $\Gamma(P)$ and $\Lambda(P)$ are $\mathbb{M}$-operads.*

Proof: Let $\mu : P \square_{\mathbb{S}} P \longrightarrow P$ and $\eta : \mathbb{I} \longrightarrow P$ be the structure maps of the operad P. We have two induced morphisms $S(\mu) : S(P \square_{\mathbb{S}} P) \longrightarrow S(P)$ and $\eta : \mathbb{I} \longrightarrow S(P)$. From Proposition 2.4.32 we have isomorphisms $S(P \square_{\mathbb{S}} P) \cong S(P) \square S(P)$ and $\Gamma(P \square^{\mathbb{S}} P) \cong \Gamma(P) \square \Gamma(P)$. $\square$

Proposition 2.5.3. *Let (P, μ, η) be an $\mathbb{M}$-operad. The endofunctor $ev(P)$ endowed with the morphisms $ev(\mu)$ and $ev(\eta)$ is a monad.*

Proof: It is a consequence of Theorem 2.4.27. $\square$

Definition 2.5.4 (P-algebra)**.** *Let (P, μ, η) be an $\mathbb{M}$-operad. The category of P-algebras is the category of algebras governed by the monad $ev(P)$. More explicitly, a P-algebra is a pair (V, γ), where V is an object of $\mathrm{Mod}_{\mathbb{K}}$ and $\gamma : ev(P)(V) \longrightarrow V$ is a morphism in $\mathrm{Mod}_{\mathbb{K}}$ such that the following diagrams commute :*

$$\begin{tikzcd} ev(P)(ev(P)(V)) \arrow[r, "\gamma"] \arrow[d, "\mu"'] & ev(P)(V) \arrow[d, "\gamma"] \\ ev(P)(V) \arrow[r, "\gamma"'] & V, \end{tikzcd} \qquad \text{(associativity)}$$

$$\begin{tikzcd} V \arrow[r, "\eta"] \arrow[dr, "\mathrm{Id}_A"'] & ev(P)(V) \arrow[d, "\gamma"] \\ & V. \end{tikzcd} \qquad \text{(unity)}$$

2.5.2 The $\mathbb{M}$-operad $Poly_V$

Let (P, μ, η) be an $\mathbb{M}$-operad and V be a $\mathbb{K}$-module. The set of P-algebra structures over V is governed by the set of morphisms of $\mathbb{M}$-operads between P and an $\mathbb{M}$-operad denoted by $Poly_V$.

Lemma 2.5.5. *Let M be an $\mathbb{M}$-module and V be a $\mathbb{K}$-module. We denote by $\bar{M} : \mathcal{H}Par_n^{op} \to \mathrm{Mod}_{\mathbb{K}}$ the functor obtained by the composition of M with the isomorphism $\mathcal{H}Par_n^{op} \to \mathcal{H}Par_n$. The V-dual of M is the $\mathbb{M}$-module defined by $\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\bar{M}(-), V) : \mathcal{H}Par_n \to \mathrm{Mod}_{\mathbb{K}}$.*

Proof: It follows from the linearity of $\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(-, V)$. □

Definition 2.5.6 (The $\mathbb{M}$-module $Poly_V$)**.** *Let V be a $\mathbb{K}$-module. We define the $\mathbb{M}$-module $Poly_V$ to be the V-dual of $\Gamma_-(V)$, explicitly :*

1. *let π be an object of $\mathcal{H}Par_n$, we set $Poly_V(\pi) := \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_\pi(V), V)$,*
2. *let π_1 and π_2 be objects in $\mathcal{H}Par_n$ such that π_1 is a subgroup of π_2, we set $\mathrm{Ind}_{\pi_1}^{\pi_2} := (\mathrm{Res}_{\pi_1}^{\pi_2})^*$ and $\mathrm{Res}_{\pi_1}^{\pi_2} := (\mathrm{Ind}_{\pi_1}^{\pi_2})^*$.*

Proposition 2.5.7. *Let V be a $\mathbb{K}$-module. The $\mathbb{M}$-module $Poly_V$ inherits the structure of an $\mathbb{M}$-operad.*

Proof: We aim to define

$$(Poly_V \,\square\, Poly_V)(\pi) \to Poly_V(\pi),$$

which is equivalent to give a morphism as follows :

$$(Poly_V \,\square\, Poly_V)(\pi) \otimes \Gamma_\pi(V) \to V.$$

We have :

$$\begin{aligned}
&(Poly_V \,\square\, Poly_V)(\pi) \otimes \Gamma_\pi(V)\\
&\quad = (\bigoplus_r \int^{\rho \in \mathcal{H}Par_r} Poly_V(\rho) \otimes (\bigoplus_{n_1+\cdots+n_r=n} \int^{\pi_1 \times \ldots \times \pi_r \in \mathcal{H}Par_{n_1} \times \cdots \times \mathcal{H}Par_{n_r}} Poly_V(\pi_1) \otimes \cdots \otimes Poly_V(\pi_r) \otimes \\
&\qquad \mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1 \times \ldots \pi_r, \pi))^\rho) \otimes \Gamma_\pi(V)\\
&\quad \cong \bigoplus_r \int^{\rho} Poly_V(\rho) \otimes (\bigoplus_{n_1+\cdots+n_r=n} \int^{\pi_1 \times \ldots \times \pi_r} Poly_V(\pi_1) \otimes \cdots \otimes Poly_V(\pi_r) \otimes \\
&\qquad \mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1 \times \ldots \pi_r, \pi) \otimes \Gamma_\pi(V))^\rho)\\
&\quad \overset{(1)}{\cong} \bigoplus_r \int^{\rho} Poly_V(\rho) \otimes (\bigoplus_{n_1+\cdots+n_r=n} \int^{\pi_1 \times \ldots \times \pi_r} Poly_V(\pi_1) \otimes \cdots \otimes Poly_V(\pi_r) \otimes \\
&\qquad \Gamma_{\pi_1}(V) \otimes \cdots \Gamma_{\pi_r}(V))^\rho)\\
&\quad \overset{(2)}{\to} \bigoplus_r \int^{\rho} Poly_V(\rho) \otimes (V^{\otimes r})^\rho) \cong \bigoplus_r \int^{\rho \in \mathcal{H}Par_r} Poly_V(\rho) \otimes \Gamma_\rho(V)) \overset{(3)}{\to} V
\end{aligned}$$

where we first expand the composite, the isomorphism (1) is given by $\Gamma_{\pi_1 \times \cdots \times \pi_r}(V) \cong \Gamma_{\pi_1}(V) \otimes \cdots \otimes \Gamma_{\pi_r}(V)$, and the morphisms (2) and (3) by the maps $Poly_V(\pi) \otimes \Gamma_\pi(V) = \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_\pi(V), V) \otimes \Gamma_\pi(V) \to V$.

Unit and associativity follow from straightforward verifications. □

Theorem 2.5.8. *Let P be an $\mathbb{M}$-operad and V be in $\mathrm{Mod}_{\mathbb{K}}$, the set of P-algebra structures over V is in bijection with $\mathrm{Hom}_{\mathbb{M}\text{-}\mathrm{Op}}(P, Poly_V)$.*

Proof: We define a function between the set of monoids morphisms between P and $Poly_V$ and the set of P-algebra structures of V :

$$\phi : \mathrm{Hom}_{\mathbb{M}\text{-}\mathrm{Op}}(P, Poly_V) \longrightarrow \{\gamma : ev(P)(V) \longrightarrow V | \gamma \text{ P-Algebra structure}\}.$$

Let $f : P(-) \to \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_{-}(V), V)$ be an $\mathbb{M}$-operad morphism between P and $Poly_V$. We denote by $f^* : \bigoplus_\pi P(\pi) \otimes \Gamma_\pi(V) \to V$ the morphism defined by the adjoint of f. We set $\phi(f) : P(V) = \int^{\pi \in \mathcal{H}Par_n} P(\pi) \otimes \Gamma_\pi(V) \longrightarrow V$ by the universal property of the coend :

$$\begin{tikzcd}
\bigoplus\limits_{\alpha\to\beta} P(\alpha)\otimes\Gamma_\beta(V) \arrow[r] \arrow[d] & \bigoplus\limits_{\pi} P(\pi)\otimes\Gamma_\pi(V) \arrow[d] \arrow[ddr, "f^*"] & \\
\bigoplus\limits_{\pi} P(\pi)\otimes\Gamma_\pi(V) \arrow[r] \arrow[drr, "f^*"'] & P(V) \arrow[dr] & \\
 & & V
\end{tikzcd}$$

The Theorem follows from the following sequences of isomorphisms :

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}\Big(\bigoplus_n \int^{\pi\in\mathcal{H}Par_n} P(\pi)\otimes\Gamma_\pi(V), V\Big) &\cong \bigoplus_n \int^{\pi} \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(P(\pi)\otimes\Gamma_\pi(V), V) \\
&\cong \bigoplus_n \int^{\pi} \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(P(\pi), \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_\pi(V), V)).
\end{aligned}$$

More precisely an $\mathbb{M}$-operad morphism between P and $Poly_V$ is a morphism of $\mathbb{M}$-modules $g : P \longrightarrow Poly_V$ such that the following diagram commutes :

$$\begin{tikzcd}
\bigoplus\limits_r \int^{\rho\in\mathcal{H}Par_r} P(\rho)\otimes(P^{\boxtimes r}(\pi))^\rho \arrow[r, "\mathrm{Id}_P \,\square\, g"] \arrow[dddd, "\mu"] & \bigoplus\limits_r \int^{\rho} P(\rho)\otimes(Poly_V^{\boxtimes r}(\pi))^\rho \arrow[d] \\
 & \bigoplus\limits_r \int^{\rho} P(\rho)\otimes \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_\pi(V), \Gamma_\rho(V)) \arrow[d, "f\,\square\,\mathrm{Id}_P"] \\
 & \bigoplus\limits_r \int^{\rho} \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_\rho(V), V)\otimes \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_\pi(V), \Gamma_\rho(V)) \arrow[dd] \\
 & \\
P(\pi) \arrow[r, "g"'] & Poly_V(\pi) = \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_\pi(V), V).
\end{tikzcd}$$

Applying the isomorphism we get the following commutative diagram :

$$\begin{tikzcd}
\bigoplus\limits_n \int^{\pi\in\mathcal{H}Par_n} \Big(\bigoplus\limits_r \int^{\rho\in\mathcal{H}Par_r} P(\rho)\otimes(P^{\boxtimes r}(\pi))^\rho\Big)\otimes\Gamma_\pi(V) \arrow[r] \arrow[d] & \bigoplus\limits_n \int^{\pi} \bigoplus\limits_r \int^{\rho} P(\rho)\otimes \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_\pi(V), \Gamma_\rho(V))\otimes\Gamma_\pi(V) \arrow[d] \\
\bigoplus\limits_n \int^{\pi} P(\pi)\otimes\Gamma_\pi(V) \arrow[r, "="] & \bigoplus\limits_r \int^{\rho} P(\rho)\otimes\Gamma_\rho(V).
\end{tikzcd}$$

□

2.5.3 Examples

We present some examples of categories of algebras governed by $\mathbb{M}$-operads.

We only aim to give an idea of future applications of our constructions in this example section. We therefore posit the existence of free objects in the category of $\mathbb{M}$-operads, which generalize the ordinary free operads, without giving further details on the construction of such objects.

Proposition 2.5.9. *Let P be a connected operad. We have that the category of $S(P)$-algebras is isomorphic to the category of $S(P,-)$-algebras, and the category of $\Gamma(P)$-algebras is isomorphic to the category of $\Gamma(P,-)$-algebras (see Appendix A).*

Proof: By Proposition 2.4.30 $S(P)$ is an $\mathbb{M}$-operad such that $ev(S(P)) \cong S(P,-)$ and the structure maps are induced by the structure maps of P. The same argument works for $\Gamma(P)$. $\square$

Example 2.5.10. *A $\Gamma(Com)$-algebra structure corresponds to a divided power algebra. That is a triple $(V, \mu, \{\gamma_i\}_{i\in\mathbb{N}})$ such that (V,μ) is a commutative algebra and $\gamma_i : V \longrightarrow V$ are set-theoretical functions such that :*

$$\gamma_n(x+y) = \sum_{i=0}^{n} \gamma_{n-i}(x)\gamma_i(y),$$
$$\gamma_i(\lambda x) = \lambda^i \gamma_i(x),$$
$$\gamma_1(x) = x,$$
$$\gamma_m(x)\gamma_n(x) = \binom{m+n}{n}\gamma_{m+n}(x),$$
$$\gamma_m(\gamma_n(x)) = \frac{mn!}{(n!)^m m!}\gamma_{mn}(x).$$

Let $\mathbb{K}$ be a field of positive characteristic p. A $\Gamma(Lie)$-algebra structure corresponds to a p-restricted Lie algebra (see [Fre00]) . That is a triple $(V, [-,-], -^{[p]})$ such that $(V,[-,-])$ is a Lie algebra, and $-^{[p]} : V \longrightarrow V$ is a set-theoretical function such that :

$$(\lambda x)^{[p]} = \lambda^p (x)^{[p]},$$
$$(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} \frac{s_i(x,y)}{i},$$
$$ad(x^{[p]}) = (ad(x))^{[p]}.$$

There are explicit descriptions for $\Gamma(Pois)$-algebras (see [Fre00]) and for $\Lambda(PreLie)$ and $\Gamma(PreLie)$-algebras (see Chapter 1), where $Pois$ is the operad governing Poisson algebras, and $PreLie$ is the operad governing the category of pre-Lie algebras.

Definition 2.5.11 (2-restricted Poisson algebra)**.** *Let $\mathbb{K}$ be a field of characteristic 2. A 2-restricted Poisson algebra is a triple*

$$(A, [-,-] : A \otimes A \longrightarrow A, (-)^2 : A \longrightarrow A)$$

where A is a commutative algebra and $(A, [-,-] : A \otimes A \longrightarrow A, (-)^{[2]} : A \longrightarrow A)$ is a 2-restricted Lie algebra structure, such that :

1. *$[x, yz] = y[x,z] + [x,y]z$, and*
2. *$(xy)^{[2]} = x^2(y)^{[2]} + x[x,y]y + (x)^{[2]}y^2$.*

Proposition 2.5.12. *Let $\mathbb{K}$ a field of characteristic 2. The $\mathbb{M}$-module $S(Com) \square \Gamma(Lie)$ is an $\mathbb{M}$-operad, denoted by 2-Pois, which encodes the category of 2-restricted Poisson algebras.*

Sketch: For the partition $(1)(2)$ of the set $\mathbf{2} = \{1,2\}$ let $\mu \in Com((1)(2))$ and $[-,-] \in Lie((1)(2))$ be respectively the generators of the operads Com and Lie. Consider the $\mathbb{M}$-module $S(Com) \square \Gamma(Lie)$. We show that the relation 1 of Definition 2.5.11 defines a distributive law of monads in the sense of Beck [Bec69]. We define the morphism of $\mathbb{M}$-modules $\rho_- : \Gamma(Lie) \square S(Com) \longrightarrow S(Com) \square \Gamma(Lie)$ using this relation. $\square$

Remark 2.5.13. *Let $\mathbb{K}$ be a field of positive characteristic $p > 2$. The $\mathbb{M}$-module $S(Com) \square \Gamma(Lie)$ still forms an $\mathbb{M}$-operad by using the distributive law of monads induced by relation 1 of Definition 2.5.11. In this case the relation 2 of Definition 2.5.11 is replaced by the more complicated :*

$$(xy)^{[p]} = x^p y^{[p]} + x^{[p]} y^p + P(x,y)$$

where $P(x,y)$ is a Poisson polynomial that can be made explicit. This structure was first introduced by Bezrukavnikov and Kaledin in [BK08] in the study of quantization of algebraic manifolds in positive characteristic.

2.6 $\mathbb{M}$-PROPs and their algebras

In this section we introduce the definition of $\mathbb{M}$-PROPs. A $\mathbb{M}$-PROP is an algebraic object which governs algebraic structures with (polynomial) operations with multiple inputs and multiple outputs.

2.6.1 The category $\mathrm{Mod}_{\mathbb{K}}^{\mathrm{Bi}\mathbb{M}}$

Definition 2.6.1 (Cohomological $(\mathcal{H}Par_n, \mathcal{H}Par_m)$-Mackey bifunctor). *Let n and m be two non-negative integers. A cohomological $(\mathcal{H}Par_n, \mathcal{H}Par_m)$-Mackey bifunctor M is a biadditive bifunctor :*

$$M : \mathcal{H}Par_n \times \mathcal{H}Par_m \longrightarrow \mathrm{Mod}_{\mathbb{K}}\,.$$

Definition 2.6.2 (Bi$\mathbb{M}$-module). *A* Bi$\mathbb{M}$*-module $M_{\bullet,\bullet}$ is a collection $\{M_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ of cohomological $(\mathcal{H}Par_n, \mathcal{H}Par_m)$-Mackey bifunctors. A morphism between two* Bi$\mathbb{M}$*-modules is a collection of natural transformations. Their category is denoted by $\mathrm{Mod}_{\mathbb{K}}^{\mathrm{Bi}\mathbb{M}}$.*

We define two monoidal structures $(\mathrm{Mod}_{\mathbb{K}}^{\mathrm{Bi}\mathbb{M}}, \boxdot, \mathbb{K})$ and $(\mathrm{Mod}_{\mathbb{K}}^{\mathrm{Bi}\mathbb{M}}, \boxminus, \mathbb{I})$, respectively the horizontal and the vertical composition.

Definition 2.6.3 (The product $\boxdot$). *For any M and N* Bi$\mathbb{M}$*-modules we set :*

$$(M \boxdot N)(\pi,\rho) = \bigoplus_{\substack{i_1+j_1=n_1\\ i_2+j_2=n_2}} \int^{\substack{\pi_1\times\pi_2\in\mathcal{H}Par_{i_1}\times\mathcal{H}Par_{i_2}\\ \rho_1\times\rho_2\in\mathcal{H}Par_{j_1}\times\mathcal{H}Par_{j_2}}} M(\pi_1,\pi_2)\otimes N(\rho_1,\rho_2)\otimes \mathrm{Hom}_{\mathcal{H}Par_{n_1}\times\mathcal{H}Par_{n_2}}((\pi_1\times\rho_1, \pi_2\times\rho_2),(\pi,\rho)).$$

Proposition 2.6.4. *The product $\boxdot$ forms a symmetric monoidal structure together with the* Bi$\mathbb{M}$*-module $\mathbb{K}$:*

$$\mathbb{K}_{i_1,i_2} := \begin{cases} \mathbb{K} & (i_1,i_2) = (0,0), \\ 0 & (i_1,i_2) \neq (0,0) \end{cases}$$

as unit.

Proof: A prove similar to the one for Definition 2.3.6 works. □

Definition 2.6.5 (The product $\boxminus$). *Let M and N be two* Bi$\mathbb{M}$*-modules we define*

$$(M \boxminus N)(\pi,\rho) = \bigoplus_{w} \int^{\upsilon\in\mathcal{H}Par_w} M(\pi,\upsilon)\otimes N(\upsilon,\rho).$$

Proposition 2.6.6. *The product $\boxminus$ forms a monoidal structure together with the* Bi$\mathbb{M}$*-module $\mathbb{I}$:*

$$\mathbb{I}_{i_1,i_2} := \begin{cases} \mathbb{K} & i_1 = i_2, \\ 0 & i_1 \neq i_2) \end{cases}$$

as unit.

Proof: It follows directly from the monoidal structure of the tensor product of $\mathbb{K}$-modules. □

2.6.2 $\mathbb{M}$-PROPs

We introduce the concept of an $\mathbb{M}$-PROP which generalizes the concept of an $\mathbb{M}$-operad.

Definition 2.6.7 ($\mathbb{M}$-PROP)**.** *An $\mathbb{M}$-PROP is a* Bi$\mathbb{M}$*-module P endowed with two associative multiplication maps $\mu_h : P \boxbar P \longrightarrow P$, $\mu_v : P \boxminus P \longrightarrow P$ and a unit $\eta : \mathbb{I} \to P$ for μ_v such that :*

— *the restriction of $\eta : \mathbb{I} \to P$ to $\mathbb{K} \hookrightarrow \mathbb{I}$ is a unit for μ_h,*

— *for any $f_1 \in P(\pi_1, \upsilon_1), f_2 \in P(\pi_2, \upsilon_2)$ we have*

$$\mu_h(f_2), f_1) = c_{\sigma,\tau}(\mu_h(f_1, f_2)))$$

where σ (resp. τ) is the permutation in $\mathbb{S}_{n_1+n_2}$ (resp. $\mathbb{S}_{m_1+m_2}$) which permutes the blocks $\{1, \dots, n_1\}$ and $\{n_1+1, \dots, n_1+n_2\}$ (resp. $\{1, \dots, m_1\}$ and $\{m_1+1, \dots, m_1+m_2\}$) and fix the orders inside the blocks.

— *for any $f_1 \in P(\pi_1, \upsilon_1), f_2 \in P(\pi_2, \upsilon_2), g_1 \in P(\upsilon_1, \rho_1), g_2 \in P(\upsilon_2, \rho_2)$ we have :*

$$\mu_h(\mu_v(f_1, g_1), \mu_v(f_2, g_2)) = \mu_v(\mu_h(f_1, f_2), \mu_h(g_1, g_2)).$$

A morphism of $\mathbb{M}$-PROPs is a natural transformation compatible with this structure.

Example 2.6.8. *Let P be an $\mathbb{M}$-operad then it is, in particular, an $\mathbb{M}$-PROP.*

Proposition 2.6.9. *Let P be a PROP (see Appendix A). It defines different $\mathbb{M}$-PROPs as follows :*

— *$S(P)$ is defined by :*

$$S_{n,m}(P)(\pi, \rho) = {}_{\pi}(P(n,m))_{\rho}$$

where $\pi \in \mathcal{H}Par_n$ and $\rho \in \mathcal{H}Par_m$, and

— *if P is biconnected, $\Gamma(P)$ is defined by :*

$$\Gamma_{n,m}(P)(\pi, \rho) = {}^{\pi}(P(n,m))^{\rho}$$

where $\pi \in \mathcal{H}Par_n$ and $\rho \in \mathcal{H}Par_m$.

Proof: We show how the composition on P induces a composition on $\Gamma(P)$.

$$\int^{\pi \in \mathcal{H}Par_n} \Gamma_{n,m}(P)(\pi, \rho) \otimes \Gamma_{s,n}(P)(\sigma, \pi) \cong$$

$$\Gamma_{s,m}(\int^{\pi} {}^{\pi}P(n,m) \otimes P(s,n)^{\pi})(\sigma, \rho) \to \Gamma_{s,m}(\int^{\pi} (P(n,m) \otimes P(s,n))^{\pi})(\sigma, \rho) \cong$$

$$\Gamma_{s,m}(P(n,m) \otimes^{\mathbb{S}_n} P(s,n))(\sigma, \rho) \to \Gamma_{s,m}(P(s,m))(\sigma, \rho) =$$

$$\Gamma_{s,m}(P)(\sigma, \rho)$$

a similar proof works for $S(P)$. □

2.6.3 Algebras over an $\mathbb{M}$-PROP

Fix a $\mathbb{K}$-module V. We define an $\mathbb{M}$-PROP denoted by $BiPoly_V$ which we use to define the category of algebras over an $\mathbb{M}$-PROP.

Definition 2.6.10 (The $\mathbb{M}$-PROP $BiPoly_V$)**.** *We define the $\mathbb{M}$-PROP $BiPoly_V$ by :*

$$BiPoly_V(\pi, \rho) = \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_{\pi}(V), \Gamma_{\rho}(V)).$$

The horizontal composition is induced by the tensor product of morphism in $\mathrm{Mod}_{\mathbb{K}}$ and the vertical composition by the composition of morphisms in $\mathrm{Mod}_{\mathbb{K}}$.

Definition 2.6.11 ($\mathbb{M}$-PROP algebras)**.** *Let P be a $\mathbb{M}$-PROP. A P-algebra over the $\mathbb{K}$-module V is a morphism of $\mathbb{M}$-PROPs $\gamma : P \longrightarrow BiPoly_V$.*

2.6.4 Examples

Let P be a PROP (see Appendix A). We prove that the category of $S(P)$-algebras is equivalent to the category of P-algebras. We prove that the category of p-restricted Lie bialgebras and the category of divided power bialgebras are governed by two $\mathbb{M}$-PROPs. These two categories are not governed by any PROPs.

We again only aim to give an idea of future applications of our constructions in this example section. We still posit the existence of free objects in the category of $\mathbb{M}$-PROPs, which generalize the ordinary free PROPs, without giving further details on the construction of such objects.

Proposition 2.6.12. *Let P be an $\mathbb{M}$-operad. It defines an $\mathbb{M}$-PROP where*

$$P(\pi,\rho) \coloneqq (P^{\boxtimes r}(\pi))^{\rho},$$

for all $\pi \in Par_n$ and $\rho \in Par_r$.

Proposition 2.6.13. *Let P be a PROP (see Appendix A). The category of algebras associated to the $\mathbb{M}$-PROP $S(P)$ is equivalent to the category of P-algebras.*

Proof: Let V be a $\mathbb{K}$-module. Let $\phi : S(P) \longrightarrow BiPoly_V$ be a $S(P)$-algebra then if restricted to the discrete partitions it defines a P-algebra structure. Vice-versa since inductions are epimorphisms in $S(P)$ any P-algebra structure can be extended to a unique $S(P)$-algebra structure. $\square$

Definition 2.6.14 (2-restricted Lie bialgebra)**.** *Let $\mathbb{K}$ be a field of characteristic 2. We say that $(A, [-,-], -^{[2]}, \delta)$, where*

- $A \in \mathrm{Mod}_{\mathbb{K}}$,
- $[-,-] : A \otimes A \longrightarrow A$,
- $-^{[2]} : A \longrightarrow A$,
- $\delta : A \longrightarrow A \otimes A$,

is a 2-restricted Lie bialgebra if $(A, [-,-], (-)^{[2]})$ is a 2-restricted Lie algebra, $(A, [-,-], \delta)$ is a Lie bialgebra and

$$\delta(-^{[2]}) = 0.$$

Proposition 2.6.15. *Let $\mathbb{K}$ be a field of characteristic 2. There exists an $\mathbb{M}$-PROP, denoted by $\Gamma BiLie$, which encodes the category of 2-restricted Lie bialgebras.*

Sketch: Let $BiLie$ be the PROP which governs the category of Lie bialgebras. We consider the $\mathbb{M}$-PROP $\Gamma(BiLie)$.

We prove that the $\Gamma(BiLie)$-algebras correspond to the 2-restricted Lie bialgebras.

Let $\phi : \Gamma(BiLie) \longrightarrow Poly_V$ be a $\Gamma(BiLie)$-algebra. There exists a monomorphism from the $\mathbb{M}$-PROP defined by the $\mathbb{M}$-operad $\Gamma(Lie)$ and $\Gamma(BiLie)$ that we denote i. From this inclusion ϕ defines a 2-restricted Lie algebra $(V, [-,-], -^{[2]})$. The restriction of ϕ to the discrete partitions is equivalent to a $BiLie$-algebra $(V, [-,-], \delta)$ where $[-,-] = \phi(m)$ and $\delta = \phi(c)$. For $i = 2$ we have that :

$$c(m) = e \otimes m(c \otimes e) + m \otimes e(e \otimes c) + m \otimes e((2,3)c \otimes e) + e \otimes m((1,2)e \otimes c).$$

By applying this relation to the image by ϕ of $\Gamma_{2,2}(BiLie)((1,2),(1)(2))$ we obtain :

$$\delta(-^{[2]}) = 0.$$

Let $(V, [-,-], -^{[2]}, \delta)$ be a 2-restricted Lie bialgebra. In particular (V, μ, δ) is a bialgebra, that is equivalent to a morphism $\psi : BiLie \longrightarrow \mathrm{BiEnd}_V$. We identify indexes of these two PROPs with the discrete partitions and partially extend the morphism ψ by the inductions morphisms. The 2-restricted Lie bialgebra $(V, [-,-], -^{[2]})$ is in particular a $\Gamma(Lie)$-algebra. Extending ϕ by the inclusion of the $\mathbb{M}$-PROP defined by $\Gamma(Lie)$ into $\Gamma(BiLie)$ we obtain a $\Gamma(BiLie)$-algebra structure. $\square$

Remark 2.6.16. *Let $\mathbb{K}$ be a field of positive characteristic $p > 2$. It is possible to define p-restricted Lie bialgebra.*

Definition 2.6.17 (Divided power bialgebra)**.** *We say that $(A, \mu, \{\gamma_i\}_{i\in\mathbb{N}}, \Delta)$, where :*
- $A \in \mathrm{Mod}_{\mathbb{K}}$,
- $\mu : A \otimes A \longrightarrow A$,
- $\gamma_i : A \longrightarrow A$
- $\gamma_i : A \longrightarrow A$,
- $\Delta : A \longrightarrow A \otimes A$,

is a divided power bialgebra if $(A, \mu, \{\gamma_i\}_{i\in\mathbb{N}})$ is a divided powers algebra, Δ is co-associative and a map of divided power algebras. In particular (A, μ, Δ) is a commutative bialgebra.

The notion of divided power bialgebras have been studied by André in [And71], Bulliksen and Levin in [GL69], and Block in [Blo85] for its relations with the enveloping algebra of a Lie algebra over a field of positive characteristic and the Hopf algebra associated.

Proposition 2.6.18. *There exists an $\mathbb{M}$-PROP, denoted by $\Gamma BiAlg_{Com}$, which encodes the category of divided power bialgebras.*

Sketch: Let $BiAlg_{Com}$ be the PROP which governs the category of commutative bialgebras. We denote $m \in BiAlg_{Com}(2,1)$ and $c \in BiAlg_{Com}(1,2)$ the generating elements. We consider the $\mathbb{M}$-PROP $\Gamma(BiAlg_{Com})$.

We prove that $\Gamma(BiAlg_{Com})$-algebras correspond to divided power bialgebras.

Let $\phi : \Gamma(BiAlg_{Com}) \longrightarrow BiPoly_V$ be a $\Gamma(BiAlg_{Com})$-algebra. There exists a monomorphism from the $\mathbb{M}$-PROP defined by the $\mathbb{M}$-operad $\Gamma(Com)$ and $\Gamma(BiAlg_{Com})$ that we denote i. By this inclusion ϕ defines a divided power algebra $(V, \mu, \{\gamma_i\}_{i\in\mathbb{N}})$. The restriction of ϕ to the discrete partitions is equivalent to a $BiAlg_{Com}$-algebra (V, μ, Δ) where $\mu = \phi(m)$ and $\Delta = \phi(c)$. For $i = 2$ we have that :

$$c(m) = m \otimes m((2,3)c \otimes c).$$

Applying this relation to the image by ϕ of $\Gamma_{2,2}(BiAlg_{Com})((1,2),(1)(2))$ we obtain :

$$\Delta(\gamma_2) = \gamma_2 \otimes \gamma_2((2,3)\Delta \otimes \Delta).$$

This is equivalent to say that Δ is compatible with γ_2. Similar computations work for the general γ_i.

Let $(V, \mu, \{\gamma_i\}_{i\in\mathbb{N}}, \Delta)$ be a divided power bialgebra. In particular (V, μ, Δ) is a bialgebra. This is equivalent to a morphism $\psi : BiAlg_{Com} \longrightarrow \mathrm{BiEnd}_V$. We identify indexes of these two PROPs with the discrete partitions and partially extend the morphism ψ by the inductions morphisms. The divided power bialgebra $(V, \mu, \{\gamma_i\}_{i\in\mathbb{N}})$ is in particular a $\Gamma(Com)$-algebra. Extending ϕ by the inclusion of the $\mathbb{M}$-PROP defined by $\Gamma(Com)$ into $\Gamma(BiAlg_{Com})$ we obtain a $\Gamma(BiAlg_{Com})$-algebra structure. $\square$

Appendix A

Background

The aim of this chapter is to recall the basic definitions and notions of the theory of operads, of the theory of polynomial functors and of the theory of Mackey functors, which we use in this thesis.

A.1 Operads and PROPs

We fix a commutative ring $\mathbb{K}$. We denote the category of $\mathbb{K}$-modules by $\mathrm{Mod}_{\mathbb{K}}$. In this section we recall the definitions and properties of symmetric modules, of (symmetric) operads and of (symmetric) PROPs in the category $\mathrm{Mod}_{\mathbb{K}}$.

A.1.1 Symmetric modules

We recall the definition of the notion of a symmetric module.

Definition A.1.1 (Symmetric modules)**.** *A symmetric module A is a collection $\{A_n\}_{n\in\mathbb{N}}$ of $\mathbb{K}$-modules with an action of the symmetric group $\mathbb{S}_n$ on A_n for all $n \in \mathbb{N}$.*

A morphism of symmetric modules $f : A \to B$ is a collection of $\mathbb{K}$-morphisms $f_n : A_n \to B_n$ commuting with the symmetric group actions.

We denote the category of symmetric modules by $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}$.

A symmetric module $A = \{A_n\}_{n\in\mathbb{N}}$ is said to be connected if $A_0 = 0$.

Remark A.1.2. *In Chapter 1 we use the notation $\{A(n)\}_{n\in\mathbb{N}}$ instead of $\{A_n\}_{n\in\mathbb{N}}$.*

The category $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}$ has three important monoidal structures, namely $\boxtimes, \square_{\mathbb{S}}$ and $\square^{\mathbb{S}}$. The first two correspond to the classical tensor product and to the composition of symmetric modules. They are used to define the notions of operads and algebras over an operad. The product $\square^{\mathbb{S}}$ was introduced by Fresse in [Fre00] and it is used to define the categories of ΓP-algebras or algebras with divided symmetries for any connected operad P.

We recall the definition of unit objects which we associate to these monoidal structures in the paragraph. We explain the definition of the operations $\boxtimes$, $\square_{\mathbb{S}}$, and $\square^{\mathbb{S}}$ afterwards.

Definition A.1.3. *1. The tensor unit symmetric module $\mathbb{K}$ is the symmetric module*

$$\mathbb{K}_n = \begin{cases} \mathbb{K} & n = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

2. The composition unit symmetric module $\mathbb{I}$ is the symmetric module

$$\mathbb{I}_n = \begin{cases} \mathbb{K} & n = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Definition A.1.4 (The product $\boxtimes$)**.** *Let $A = \{A_n\}_{n\in\mathbb{N}}$ and $B = \{B_n\}_{n\in\mathbb{N}}$ be symmetric modules. We define the symmetric module $A \boxtimes B$ as follows :*

$$A \boxtimes B = \bigoplus_{i+j=n} \mathrm{Ind}_{\mathbb{S}_i\times\mathbb{S}_j}^{\mathbb{S}_n} A_i \otimes B_j,$$

where $\mathrm{Ind}_{\mathbb{S}_i\times\mathbb{S}_j}^{\mathbb{S}_n} A_i \otimes B_j$ *stands for the* $\mathbb{K}[\mathbb{S}_n]$*-module induced by the* $\mathbb{K}[\mathbb{S}_i \times \mathbb{S}_j]$*-module* $A_i \otimes B_j$*.*

The product $\boxtimes$ *forms a bifunctor. To be explicit, let* $f : A \to B$ *and* $g : A' \to B'$ *be symmetric module morphisms. We define* $f \boxtimes g$ *to be the collection*

$$(f \boxtimes g)_n = \bigoplus_{i+j=n} \mathrm{Ind}_{\mathbb{S}_i\times\mathbb{S}_j}^{\mathbb{S}_n} f_i \otimes g_j .$$

Proposition A.1.5. *The triple* $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \boxtimes, \mathbb{K})$ *forms a symmetric monoidal category.*

Proof: See [Fre00, Proposition 1.1.6]. □

Definition A.1.6 (The product $\otimes_{\mathbb{S}_n}$)**.** *Let* R *and* S *be* $\mathbb{K}[\mathbb{S}_n]$*-modules. We denote by* $R \otimes_{\mathbb{S}_n} S$ *the* $\mathbb{K}$*-module of coinvariants of the* $\mathbb{K}[\mathbb{S}_n]$*-module* $R \otimes S$ *endowed with the diagonal action of* $\mathbb{S}_n$*. In what follows we use the notation* $[x \otimes y]$ *for the class of a tensor* $x \otimes y \in R \otimes S$ *in this quotient.*

Definition A.1.7 (The product $\square_{\mathbb{S}}$)**.** *Let* $A = \{A_n\}_{n\in\mathbb{N}}$ *and* $B = \{B_n\}_{n\in\mathbb{N}}$ *be symmetric modules. We define the symmetric module* $A \square_{\mathbb{S}} B$ *by :*

$$(A \square_{\mathbb{S}} B)_n = \bigoplus_{r\in\mathbb{N}} A_r \otimes_{\mathbb{S}_r} (B^{\boxtimes r})_n .$$

The product $\square_{\mathbb{S}}$ *forms a bifunctor. To be explicit, let* $f : A \to B$ *and* $g : A' \to B'$ *be symmetric module morphisms. We define* $f \square_{\mathbb{S}} g$ *to be the collection*

$$(f \square_{\mathbb{S}} g)_n = \sum_{r\in\mathbb{N}} f_r \otimes_{\mathbb{S}_r} (\sum_{t_1+\dots+t_r=n} g_{t_1} \otimes \dots \otimes g_{t_r}).$$

Remark A.1.8. *In Chapter 1 we use the notation* $\underset{\sim}{\square}$ *instead of* $\square_{\mathbb{S}}$*.*

Proposition A.1.9. *The triple* $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \square_{\mathbb{S}}, \mathbb{I})$ *forms a monoidal category.*

Proof: We refer to [Fre00, Proposition 1.1.9]. □

Definition A.1.10 (The product $\otimes^{\mathbb{S}_n}$)**.** *Let* A *and* B *be* $\mathbb{K}[\mathbb{S}_n]$*-modules. We denote by* $A \otimes^{\mathbb{S}_n} B$ *the* $\mathbb{K}$*-module of invariants of the* $\mathbb{K}[\mathbb{S}_n]$*-module* $A \otimes B$ *endowed with the diagonal action of* $\mathbb{S}_n$*.*

Definition A.1.11 (The product $\square^{\mathbb{S}}$)**.** *Let* $A = \{A_n\}_{n\in\mathbb{N}}$ *and* $B = \{B_n\}_{n\in\mathbb{N}}$ *be symmetric modules. We define the symmetric module* $A \square^{\mathbb{S}} B$ *by :*

$$(A \square^{\mathbb{S}} B)_n = \bigoplus_{r\in\mathbb{N}} A_r \otimes^{\mathbb{S}_r} (B^{\boxtimes r})_n .$$

The product $\square^{\mathbb{S}}$ *forms a bifunctor. To be explicit, let* $f : A \to B$ *and* $g : A' \to B'$ *be symmetric module morphisms. We define* $f \square^{\mathbb{S}} g$ *as the collection*

$$(f \square^{\mathbb{S}} g)_n = \sum_{r\in\mathbb{N}} f_r \otimes^{\mathbb{S}_r} (\sum_{t_1+\dots+t_r=n} g_{t_1} \otimes \dots \otimes g_{t_r}).$$

Remark A.1.12. *In Chapter 1 we use the notation* $\tilde{\square}$ *instead of* $\square^{\mathbb{S}}$*.*

Proposition A.1.13. *The triple* $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \square^{\mathbb{S}}, \mathbb{I})$ *forms a monoidal category.*

Proof: We refer to [Fre00, Proposition 1.1.9]. □

Let G be a finite group and X be a $\mathbb{K}[G]$-module. We consider the $\mathbb{K}$-module of coinvariant X_G and the $\mathbb{K}$-module of invariant X^G. There is a natural map, called trace or norm map, $tr : X_G \to X^G$ defined by $[x] \mapsto \sum_{g\in G} g^* x$, for any $x \in X$. We apply this observation to our composition product :

Definition A.1.14 (The natural transformation tr)**.** *Let $A = \{A_n\}_{n\in\mathbb{N}}$ and $B = \{B_n\}_{n\in\mathbb{N}}$ be symmetric modules. We define the morphism of symmetric modules*

$$tr : A \,\square_{\mathbb{S}}\, B \to A \,\square^{\mathbb{S}}\, B,$$

by

$$tr([a \otimes b_1 \otimes \cdots \otimes b_r]) = \sum_{\sigma\in\mathbb{S}_n} \sigma^*(a \otimes b_1 \otimes \cdots \otimes b_r),$$

for any $[a \otimes b_1 \otimes \cdots \otimes b_r] \in A_r \otimes_{\mathbb{S}_r} (B^{\boxtimes r})_n$.

We use the epi-mono factorization of tr to define a third product $\square_{\mathrm{tr}}$ intermediate between $\square_{\mathbb{S}}$ and $\square^{\mathbb{S}}$:

Proposition A.1.15. *The natural transformation tr is monoidal, i.e. it preserves unit and associativity isomorphisms.*

Proof: See [Fre00, Lemma 1.1.19]. □

Definition A.1.16 (The product $\square_{\mathrm{tr}}$)**.** *Let $A = \{A_n\}_{n\in\mathbb{N}}$ and $B = \{B_n\}_{n\in\mathbb{N}}$ be symmetric modules. We define the symmetric module $A \,\square_{\mathrm{tr}}\, B$ by :*

$$(A \,\square_{\mathrm{tr}}\, B)_n = Im(tr : (A \,\square_{\mathbb{S}}\, B)_n \to (A \,\square^{\mathbb{S}}\, B)_n),$$

for each $n \in \mathbb{N}$.

The product $\square_{\mathrm{tr}}$ forms a bifunctor. To be explicit, let $f : A \to B$ and $g : A' \to B'$ be symmetric module morphisms. We define $f \,\square_{\mathrm{tr}}\, g$ as the collection

$$(f \,\square_{\mathrm{tr}}\, g)_n = (f \,\square^{\mathbb{S}}\, g)_n\big|_{(A \,\square_{\mathrm{tr}}\, B)_n},$$

the restrictions of $(f \,\square^{\mathbb{S}}\, g)_n$ to $(A \,\square_{\mathrm{tr}}\, B)_n$.

Proposition A.1.17. *Let $\mathbb{K}$ be a field. The triple $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \square_{\mathrm{tr}}, \mathbb{I})$ forms a monoidal category.*

Proof: We use that tr is monoidal and the observations that $- \,\square_{\mathbb{S}}\, -$ preserves the epimorphisms and $- \,\square^{\mathbb{S}}\, -$ preserves the monomorphisms to obtain a diagram of the form :

$$\begin{tikzcd}
(A \,\square_{\mathbb{S}}\, B) \,\square_{\mathbb{S}}\, C \arrow[r, two heads] \arrow[d, "\cong"'] & (A \,\square_{\mathrm{tr}}\, B) \,\square_{\mathrm{tr}}\, C \arrow[r, hook] \arrow[d, dashed, "\exists"] & (A \,\square^{\mathbb{S}}\, B) \,\square^{\mathbb{S}}\, C \arrow[d, "\cong"] \\
A \,\square_{\mathbb{S}}(B \,\square_{\mathbb{S}}\, C) \arrow[r, two heads] & A \,\square_{\mathrm{tr}}(B \,\square_{\mathrm{tr}}\, C) \arrow[r, hook] & A \,\square^{\mathbb{S}}(B \,\square^{\mathbb{S}}\, C).
\end{tikzcd}$$

We deduce the associativity diagram for $\square_{\mathrm{tr}}$, the unit follows easily. □

Let G be a group of cardinality n and X be a $\mathbb{K}[G]$-module. If $\mathbb{K}$ is a field of characteristic 0 then the natural map $tr^{-1} : X^G \to X_G$ defined as follows $x \mapsto \frac{1}{n}[x]$ is the inverse of the trace map. Thus, the natural transformation tr is an isomorphism of bifunctors.

Proposition A.1.18. *If $\mathbb{K}$ is a field of characteristic 0 then the trace induces an isomorphism of monoidal categories*

$$(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \square_{\mathbb{S}}, \mathbb{I}) \cong (\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \square_{\mathrm{tr}}, \mathbb{I}) \cong (\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \square^{\mathbb{S}}, \mathbb{I}).$$

□

If $\mathbb{K}$ does not contain $\mathbb{Q}$ we still have :

Proposition A.1.19 (Fresse [Fre00], Proposition 1.1.15)**.** *Let $A = \{A_n\}_{n\in\mathbb{N}}$ and $B = \{B_n\}_{n\in\mathbb{N}}$ be symmetric modules. If B is connected then $tr : A \,\square_{\mathbb{S}}\, B \to A \,\square^{\mathbb{S}}\, B$ is an isomorphism of symmetric modules.* □

We are interested in symmetric modules because they are combinatorial models of a special kind of endofunctors of the category $\mathrm{Mod}_{\mathbb{K}}$. We explain this correspondence in the following definition.

Definition A.1.20 (The functors $S(A,-)$, $\Gamma(A,-)$ and $\Lambda(A,-)$)**.** *Let $A=\{A_n\}_{n\in\mathbb{N}}$ be a symmetric module. We have an obvious inclusion $in : \mathrm{Mod}_{\mathbb{K}} \hookrightarrow \mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}$ such that :*

$$in(V)_n = \begin{cases} V & n=0, \\ 0 & \textit{otherwise.} \end{cases}$$

We then consider the functors $S(A,-)$, $\Lambda(A,-)$, and $\Gamma(A,-) : \mathrm{Mod}_{\mathbb{K}} \to \mathrm{Mod}_{\mathbb{K}}$ such that :

$$S(A,V) = A \,\square_{\mathbb{S}}\, in(V),$$

$$\Lambda(A,V) = A \,\square_{\mathrm{tr}}\, in(V),$$

$$\Gamma(A,V) = A \,\square^{\mathbb{S}}\, in(V).$$

We have natural transformations :

$$S(A,-) \to \Lambda(A,-) \to \Gamma(A,-)$$

given by the epi-mono factorization of the trace map on these composition products.

The functor $S(A,-)$ is the standard functor of the theory of operads and is usually called the Schur functor associated to A.

Let A be a symmetric module. In general the functors $S(A,-)$, $\Lambda(A,-)$ and $\Gamma(A,-)$ are not isomorphic. But we have the following statement :

Proposition A.1.21 (Fresse [Fre00], Proposition 1.1.2)**.** *Let $A=\{A_n\}_{n\in\mathbb{N}}$ be a symmetric module. If A is projective as a symmetric sequence then*

$$tr : S(A,-) \to \Gamma(A,-)$$

is an isomorphism. □

Corollary A.1.22. *We have that $S(As,-)$ is isomorphic to $\Gamma(As,-)$.* □

The functors $S(-,-)$, $\Lambda(-,-)$, and $\Gamma(-,-)$ are compatible with the monoidal structures $\boxtimes$, $\square_{\mathbb{S}}$, $\square_{\mathrm{tr}}$, and $\square^{\mathbb{S}}$:

Proposition A.1.23 (Fresse [Fre00], Propositions 1.1.6 and 1.1.9)**.** *The bifunctors $S(-,-)$, $\Gamma(-,-)$ and $\Lambda(-,-) : \mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}} \to Fun(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}})$ are :*

- *(strongly) symmetric monoidal functors with respect to the two symmetric monoidal structures $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \boxtimes, \mathbb{K})$ and $(\mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}}), \otimes, \mathbb{K})$, hence we have :*

  $$S(A\boxtimes B,-) \cong S(A,-)\otimes S(B,-), \Gamma(A\boxtimes B,-) \cong \Gamma(A,-)\otimes \Gamma(B,-),$$

  if $\mathbb{K}$ is a field

  $$\Lambda(A\boxtimes B,-) \cong \Lambda(A,-)\otimes \Lambda(B,-),$$

- *(strongly) monoidal functors with respect to the two monoidal structures $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \square, \mathbb{I})$ and $(\mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}}), \circ, \mathrm{Id}_{\mathrm{Mod}_{\mathbb{K}}})$, hence we have :*

  $$S(A\,\square_{\mathbb{S}}\, B,-) \cong S(A,-)\circ S(B,-), \Gamma(A\,\square^{\mathbb{S}}\, B,-) \cong \Gamma(A,-)\circ \Gamma(B,-),$$

  if $\mathbb{K}$ is a field

  $$\Lambda(A\,\square_{\mathrm{tr}}\, B,-) \cong \Lambda(A,-)\circ \Lambda(B,-).$$

□

A.1.2 Operads and their associated monads

We recall the definitions and the properties of operads and of the categories of algebras associated to operads.

Operads and algebras over an operad

Since $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \square_{\mathbb{S}}, \mathbb{I})$ is a monoidal category we can define the category of monoids with respect to this structure.

Definition A.1.24 (Operads)**.** *Let $P = \{P_n\}_{n\in\mathbb{N}}$ be a symmetric module. Let $\mu : P\square_{\mathbb{S}} P \to P$ and $\eta : \mathbb{I} \to P$ be morphisms of symmetric modules. The triple (P,μ,η) is an operad if it is a monoid in the monoidal category $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \square_{\mathbb{S}}, \mathbb{I})$. More explicitly the triple (P,μ,η) is an operad if the following diagrams commute :*

$$\begin{tikzcd} P\square_{\mathbb{S}} P\square_{\mathbb{S}} P \arrow[r, "\mathrm{Id}_P\square_{\mathbb{S}}\mu"] \arrow[d, "\mu\square_{\mathbb{S}}\mathrm{Id}_P"'] & P\square_{\mathbb{S}} P \arrow[d, "\mu"] \\ P\square_{\mathbb{S}} P \arrow[r, "\mu"'] & P, \end{tikzcd} \qquad \text{(Associativity)}$$

and

$$\begin{tikzcd} \mathbb{I}\square_{\mathbb{S}} P \arrow[r, "\eta\square_{\mathbb{S}}\mathrm{Id}_P"] \arrow[dr] & P\square_{\mathbb{S}} P \arrow[d, "\mu"] & P\square_{\mathbb{S}}\mathbb{I} \arrow[l, "\mathrm{Id}_P\square_{\mathbb{S}}\eta"'] \arrow[dl] \\ & P. & \end{tikzcd} \qquad \text{(Unit)}$$

Let $(P = \{P_n\}_{n\in\mathbb{N}},\mu,\eta)$ and $(P' = \{P'_n\}_{n\in\mathbb{N}},\mu',\eta')$ be operads. A morphism of operads is a morphism of symmetric modules $\phi : P \to P'$ such that the following diagrams commute :

$$\begin{tikzcd} P\square_{\mathbb{S}} P \arrow[r, "\phi\square_{\mathbb{S}}\phi"] \arrow[d, "\mu"'] & P'\square_{\mathbb{S}} P' \arrow[d, "\mu'"] \\ P \arrow[r, "\phi"'] & P', \end{tikzcd}$$

and

$$\begin{tikzcd} & P \arrow[dd, "\phi"] \\ \mathbb{K} \arrow[ur, "\eta"] \arrow[dr, "\eta'"'] & \\ & P'. \end{tikzcd}$$

We denote the category of operads by Op.

We use that $P\square_{\mathbb{S}} P$ is spanned by tensors of the form $[p\otimes q_1\otimes\cdots\otimes q_n]$ with $p\in P_n$ and $q_1,\dots,q_n\in P$ to give an explicit definition of μ.

Remark A.1.25. *The general theory of operads allows us to define the free operad generated by a symmetric module, and the ideals of an operad. We can present any operad by generators and relations. Since this theory goes beyond the purpose of this section we do not give more details. For the interested reader we refer to the books of Fresse [Fre09, Section 3.1], Loday and Vallette [LV12, Section 5.5], and Markl, Schnider and Stasheff [MSS02].*

Let $(P = \{P_n\}_{n\in\mathbb{N}},\mu,\eta)$ be an operad, the elements $p\in P_n$ can be interpreted as n-ary operations and μ as the rule for composing them. The morphism η represents the identity operation. We can present operads by generating operations and relations.

We introduce a different and useful definition of operad structure on a symmetric module.

Definition A.1.26 (System of partial compositions)**.** *Let $P = \{P_n\}_{n\in\mathbb{N}}$ be a symmetric module. A system of partial compositions $(\{\circ_i\}_{i\in\mathbb{N}^*},\eta)$ is a collection of $\mathbb{K}$-modules morphisms $-\circ_i - : P_n\otimes P_m \to P_{n+m-1}$ and a morphism of $\mathbb{K}$-module $\eta : \mathbb{K}\to P_1$ such that :*

1. *$-\circ_i - : P_n\otimes P_m \to P_{n+m-1}$ is the zero map if $i > n$,*
2. *$-\circ_i(-\circ_j -) = (-\circ_i -)\circ_{i+j-1} -$, and*

3. *$p \circ_i (\eta(1)) = (\eta(1)) \circ_1 p = p$ for any $p \in P_n$ and $i \leq n$,*

and which respect the symmetric action. That is :

$$x \circ_i \sigma^*(y) = \overline{\sigma}^*(x \circ_i y)$$

for all $\sigma \in \mathbb{S}_n$ where $\overline{\sigma}$ is the $\mathbb{S}_{n+m-1}$ permutation that act as the identity on the set $\{1, \dots, i-1, i+n, \dots, n+m\}$ and as σ on the set $\{i, \dots, i+n-1\}$, and

$$\rho^*(x) \circ_i y = \underline{\rho}^*(x \circ_i y)$$

for all $\rho \in \mathbb{S}_m$ where $\underline{\sigma}$ is the $\mathbb{S}_{n+m-1}$ permutation that act as ρ on the blocks $\{(1), \dots, (i-1), (i, \dots, i+n-1), (i+n), \dots, (n+m)\}$ and identity inside the block $(i, \dots, i+n-1)$.

Proposition A.1.27. *Let $P = \{P_n\}_{n \in \mathbb{N}}$ be a symmetric module. An operad structure (P, μ, η) is equivalent to a system of partial compositions $(P, \{\circ_i\}^*_{i \in \mathbb{N}}, \eta)$.*

Proof: For more details see [LV12, Section 5.3.7] □

The compatibility of $S(-,-)$ with the composition products $\square_{\mathbb{S}}$ and $\circ$ has an important consequence. Any monoid with respect to $\square_{\mathbb{S}}$ defines a monoid in the category of endofunctor of $\mathrm{Mod}_{\mathbb{K}}$ with respect to the composition of functors $\circ$, a monad is the usual terminology of category theory :

Proposition A.1.28. *Let $(P = \{P_n\}_{n \in \mathbb{N}}, \mu, \eta)$ be an operad. The triple $(S(P,-), S(\mu,-), S(\eta,-))$ is a monad.*

Proof: The statement is a direct consequence of Proposition A.1.23. □

To any monad we associate a category of algebras. Thus, to any operad we associate a category of algebras.

Definition A.1.29 (P-algebra)**.** *Let $(P = \{P_n\}_{n \in \mathbb{N}}, \mu, \eta)$ be an operad. The category of P-algebras is the category of algebras over the monad $(S(P,-), S(\mu,-), S(\eta,-))$. It is denoted by Alg_P. More explicitly an object of Alg_P is a couple (V, γ) such that the following diagrams commute :*

$$\begin{tikzcd}
S(P \square_{\mathbb{S}} P, V) \arrow[r, "\cong"] \arrow[d, "\mu \circ \mathrm{Id}_V"'] & S(P, S(P,V)) \arrow[r, "\mathrm{Id}_P \square_{\mathbb{S}} \gamma"] & S(P,V) \arrow[d, "\gamma"] \\
S(P,V) \arrow[rr, "\gamma"'] & & V,
\end{tikzcd}$$

and

$$\begin{tikzcd}
S(\mathbb{I}, V) \arrow[r, "\eta \circ \mathrm{Id}_V"] \arrow[dr] & S(P,V) \arrow[d, "\gamma"] \\
& V.
\end{tikzcd}$$

Example A.1.30. 1. *The symmetric module As defined by $As_0 = 0$ and $As_n = \mathbb{K}[\mathbb{S}_n]$ with multiplicative actions for all $n > 0$ is an operad with the composition product such that :*

$$\mu([\rho \otimes \tau_1 \otimes \dots \otimes \tau_r]) = \tau_{\rho(1)} \oplus \dots \oplus \tau_{\rho(r)}$$

for $\rho \in \mathbb{S}_r$ and $\tau_i \in \mathbb{S}_{n_i}$ and $\eta = \mathrm{Id}_{\mathbb{K}}$. Alternatively the operad As can be defined as the free operad generated by a binary operation m quotient by the ideal generated by the relation $m(m(-,-),-) = m(-,m(-,-))$.

The category of As-algebras is isomorphic to the category of non unital associative algebras.

2. *The symmetric module Com is defined by $Com_0 = 0$ and $Com_n = \mathbb{K}$ with trivial action for all $n > 0$, is an operad if endowed with the morphisms $\mu = \mathrm{Id}_{\mathbb{K}}$ and $\eta = \mathrm{Id}_{\mathbb{K}}$. Alternatively the operad Com can be defined as the free operad generated by a commutative binary operation c quotient by the ideal generated by the relation $c(c(-,-),-) = c(-,c(-,-))$.*

The category of Com-algebras is isomorphic to the category of associative commutative algebras,

3. *The symmetric module Lie is defined by $Lie_0 = 0$ and $Lie_n = \mathrm{Ind}_{\mathbb{Z}/n\mathbb{Z}}^{\mathbb{S}_n}(\rho)$, where ρ is the one dimensional representation of the n-cyclic group given by an irreducible nth-root for all $n > 0$. We can define an operad structure on the symmetric module Lie as the free operad generated by an anti-symmetric binary operation $[-,-]$ quotient by the ideal generated by the relation $[[1,2],3] + [[2,3],1] + [[3,1],2] = 0$.*

   The category of Lie-algebras is isomorphic to the category of Lie algebras.

Not every monad is in the image of $S(-,-)$. Operads correspond, in some sense, to the category of monads presented by multilinear operations and multilinear relations between them. The advantage of working with the category of operads instead of the whole category of monads is their combinatorial nature that allows us to make explicit computations.

Definition A.1.31 (The operad End_V)**.** *Let V be a $\mathbb{K}$-module. We define the symmetric module End_V by :*

$$\mathrm{End}_{V,n} = \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V^{\otimes n}, V),$$

with the symmetric group action induced by the permutation action on $V^{\otimes n}$ for any $n \in \mathbb{N}$. The composition of morphisms in the category $\mathrm{Mod}_{\mathbb{K}}$ and the identity of V gives an operad structure on End_V. We explicit set :

$$\mu([f(\underbrace{-,\ldots,-}_{r}) \otimes g_1(\underbrace{-,\ldots,-}_{n_1}) \otimes \cdots \otimes g_r(\underbrace{-,\ldots,-}_{n_r})]) = f(g_1,\ldots g_r)(\underbrace{-,\ldots,-}_{n_1+\ldots+n_r}),$$

for $f \in \mathrm{End}_{V,r}$, and $g_i \in \mathrm{End}_{V,n_i}$ and

$$\eta(1) = \mathrm{Id}_V .$$

Remark A.1.32. *The construction of the symmetric module End_V is not functorial on V.*

Proposition A.1.33. *Let $(P = \{P_n\}_{n\in\mathbb{N}}, \mu, \eta)$ be an operad and V be a $\mathbb{K}$-module. We have the following bijection :*

$$\{\gamma | (V,\gamma) \in \mathrm{Alg}_P\} \cong \mathrm{Hom}_{\mathrm{Op}}(P, \mathrm{End}_V).$$

Let (V,γ) and (V',γ') be P-algebras and $f : V \to V'$ be a $\mathbb{K}$-morphism. It is a morphism of P-algebras if and only if

$$\gamma'(p)(\underbrace{f \otimes \cdots \otimes f}_{n}(\underbrace{-,\ldots,-}_{n})) = f(\gamma(p)(\underbrace{-,\ldots,-}_{n})),$$

for all $p \in P_n$ and $n \in \mathbb{N}$.

Proof: We refer to Fresse [Fre09, Proposition 3.4.2] and Loday and Vallette [LV12, Proposition 5.2.13]. □

ΛP and ΓP-algebras

Let $\mathbb{K}$ be a field. o Since the map $P \square_{\mathbb{S}} P \to P \square^{\mathbb{S}} P$ induced by the trace is an isomorphism for connected symmetric modules. Hence the category of connected operads coincides with the category of connected monoids with respect to $\square_{\mathrm{tr}}$ and $\square^{\mathbb{S}}$. Let P be a connected operad. We use the compatibility of the functors $\Lambda(-,-)$ and $\Gamma(-,-)$ with the composition to define other two monads associated to P.

Proposition A.1.34. *Let $(P = \{P_n\}_{n\in\mathbb{N}}, \mu, \eta)$ be a connected operad. The triples*

$$\Lambda(P,-), \Lambda(\mu,-), \Lambda(\eta,-)), \qquad (\Gamma(P,-), \Gamma(\mu,-), \Gamma(\eta,-))$$

are monads such that the morphisms given by the epi-mono factorization of tr :

$$S(P,-) \to \Lambda(P,-) \to \Gamma(P,-)$$

are monad morphisms. □

Definition A.1.35 (ΛP-algebras)**.** *Let $(P = \{P_n\}_{n\in\mathbb{N}}, \mu, \eta)$ be a connected operad. We define the category of ΛP-algebras as the category of algebras over the monad $(\Lambda(P,-), \Lambda(\mu,-), \Lambda(\eta,-))$.*

Proposition A.1.36. *Let $(P = \{P_n\}_{n\in\mathbb{N}}, \mu, \eta)$ be a connected operad and A be a ΛP-algebra. The monad $S(P,-)$ acts on A through the morphism $S(P,-) \to \Lambda(P,-)$ so that A inherits a natural P-algebra structure.* □

Let P be a connected operad. Since the functor $\Lambda(P,-)$ is, in general, different from the functor $S(P,-)$ the category of ΛP-algebras is, in general, not equivalent to the category of P-algebras. The category of ΛP-algebras can be interpreted as the subcategory of P-algebras satisfying some additional non-linear relations.

Example A.1.37. *We have :*

1. *let $\mathbb{K}$ be a field of positive characteristic p, a Com-algebra C is a ΛCom-algebra if $c^p = 0$ for any $c \in C$,*
2. *let $\mathbb{K}$ be a field of characteristic 2, a Lie-algebra L is a ΛLie-algebra if $[l,l] = 0$ for any $l \in L$,*

see [Fre04, Proposition 1.2.15-1.2.16].

Definition A.1.38 (ΓP-algebras)**.** *Let $(P = \{P_n\}_{n\in\mathbb{N}}, \mu, \eta)$ be a connected operad. We define the category of ΓP-algebras as the category of algebras over the monad $(\Gamma(P,-), \Gamma P(\mu,-), \Gamma P(\eta,-))$.*

Proposition A.1.39. *Let $(P = \{P_n\}_{n\in\mathbb{N}}, \mu, \eta)$ be a connected operad and A be a ΓP-algebra. The monad $\Lambda(P,-)$ acts on A through the morphism $\Lambda(P,-) \to \Gamma(P,-)$ so that A inherits a natural ΛP-algebra structure.* □

As for ΛP-algebras, since $\Gamma(P,-)$ is, in general, different from $S(P,-)$ the category of ΓP-algebras is, in general, not equivalent to the categories of P-algebras. The category of ΓP-algebras can be interpreted as the category of ΛP-algebras with an additional structure.

Definition A.1.40 (Divided power algebras)**.** *A divided power algebra is a commutative algebra C endowed with a collection of operations $\gamma_i : C \longrightarrow C$ such that :*

$$\gamma_n(x+y) = \sum_{i=0}^{n} \gamma_{n-i}(x)\gamma_i(y),$$
$$\gamma_i(\lambda x) = \lambda^i \gamma_i(x),$$
$$\gamma_1(x) = x,$$
$$\gamma_m(x)\gamma_n(x) = \binom{m+n}{n}\gamma_{m+n}(x),$$
$$\gamma_m(\gamma_n(x)) = \frac{mn!}{(n!)^m m!}\gamma_{mn}(x).$$

Let C and D be divided power algebras. A commutative algebra morphism $\phi : C \to D$ is a morphism of divided power algebras if

$$\phi(\gamma_i(-)) = \gamma_i(\phi(-))$$

for any $i \in \mathbb{N}$.

Definition A.1.41 (p-restricted Lie algebras)**.** *Let $\mathbb{K}$ be a field of positive characteristic p. A p-restricted Lie algebra is a Lie algebra L equipped with an operation $-^{[p]} : L \longrightarrow L$ such that :*

$$(\lambda x)^{[p]} = \lambda^p (x)^{[p]},$$
$$(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} \frac{s_i(x,y)}{i},$$
$$ad(x^{[p]}) = (ad(x))^{[p]},$$

where $s_i(x,y)$ is the coefficient of t^{i-1} on the expression of $ad_{(tx+y)^{p-1}}(x)$.

Let L and G be p-restricted Lie algebras. A Lie algebra morphism $\phi : L \to G$ is a p-restricted Lie algebra morphism if

$$\phi((-)^{[p]}) = (\phi(-))^{[p]}.$$

Example A.1.42. *We have :*

1. *the category of* ΓCom*-algebras is isomorphic to the category of divided power algebras (see Fresse [Fre00, Proposition 1.2.3]),*
2. *let* $\mathbb{K}$ *be a field of positive characteristic* p *; the category of* ΓLie*-algebras is isomorphic to the category of* p*-restricted Lie algebras (see Fresse [Fre00, Theorem 1.2.5]).*

A.1.3 PROPs and their algebras

We recall the notion of a PROP and of the category of algebras associated to a PROP. These notions were first introduced by MacLane. We first introduce the concept of symmetric bimodule.

Definition A.1.43 ((G,H)-modules)**.** *Let* G *and* H *be groups. A* (G,H)*-module is a* $\mathbb{K}$*-module* V *endowed with a left* G*-action and a right* H*-action such that the two actions commute with each other. A morphism of* (G,H)*-modules is a morphism of left* $\mathbb{K}[G]$*-modules and right* $\mathbb{K}[H]$*-modules.*

Definition A.1.44 (Symmetric bimodule)**.** *A symmetric bimodule* $A = \{A_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ *is a collection of* $(\mathbb{S}_n,\mathbb{S}_m)$*-modules.*

Let A *and* B *be symmetric bimodules. A morphism of symmetric bimodules* $f : A \to B$ *is a collection* $\{f_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ *of* $(\mathbb{S}_n,\mathbb{S}_m)$*-module morphisms.*

We denote their category by $\mathrm{BiMod}_{\mathbb{K}}^{\mathbb{S}}$*.*

We define two monoidal structures, namely $\boxplus$ and $\boxminus$. They correspond to tensor and composition products. We recall the definition of unit objects for these monoidal structures.

Definition A.1.45.

1. *the horizontal tensor unit* $\mathbb{K}$ *is the symmetric bimodule defined as follows :*
$$\mathbb{K}_{n,m} = \begin{cases} \mathbb{K} & n = 0 \text{ and } m = 0, \\ 0 & \text{otherwise,} \end{cases}$$
2. *the vertical tensor unit* $\mathbb{K}$ *is the symmetric bimodule defined as follows :*
$$\mathbb{I}_{n,m} = \begin{cases} \mathbb{K} & n = m, \\ 0 & \text{otherwise,} \end{cases}$$
where we take the trivial action of symmetric groups on $\mathbb{K}$*.*

Definition A.1.46 (The product $\boxplus$)**.** *Let* $A = \{A_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ *and* $B = \{B_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ *be symmetric bimodules. We define the symmetric bimodule* $A \boxplus B$ *by :*

$$(A \boxplus B)_{n,m} = \bigoplus_{\substack{n_1+n_2=n \\ m_1+m_2=m}} \mathrm{Ind}_{\mathbb{S}_{n_1}\times\mathbb{S}_{n_2},\mathbb{S}_{m_1}\times\mathbb{S}_{m_2}}^{\mathbb{S}_n,\mathbb{S}_m} A_{n_1,m_1} \otimes A_{n_2,m_2},$$

where

$$\mathrm{Ind}_{\mathbb{S}_{n_1}\times\mathbb{S}_{n_2},\mathbb{S}_{m_1}\times\mathbb{S}_{m_2}}^{\mathbb{S}_n,\mathbb{S}_m}(-) = \mathrm{Ind}_{\mathbb{S}_{n_1}\times\mathbb{S}_{n_2}}^{\mathbb{S}_n}(\mathrm{Ind}_{\mathbb{S}_{m_1}\times\mathbb{S}_{m_2}}^{\mathbb{S}_m}(-)) = \mathrm{Ind}_{\mathbb{S}_{m_1}\times\mathbb{S}_{m_2}}^{\mathbb{S}_m}(\mathrm{Ind}_{\mathbb{S}_{n_1}\times\mathbb{S}_{n_2}}^{\mathbb{S}_n}(-)).$$

The product $\boxplus$ *is a bifunctor. To be explicit let* $f : A \to B$ *and* $g : A' \to B'$ *be symmetric bimodule morphisms. We define* $f \boxplus g : A \boxplus A' \to B \boxplus B'$ *by :*

$$(f \boxplus g)_{n,m} = \sum_{\substack{n_1+n_2=n \\ m_1+m_2=m}} \mathrm{Ind}_{\mathbb{S}_{n_1}\times\mathbb{S}_{n_2},\mathbb{S}_{m_1}\times\mathbb{S}_{m_2}}^{\mathbb{S}_n,\mathbb{S}_m} f_{n_1,m_1} \otimes g_{n_2,m_2}.$$

Proposition A.1.47. *The triple* $(\mathrm{BiMod}_{\mathbb{K}}^{\mathbb{S}}, \boxplus, \mathbb{K})$ *forms a symmetric monoidal category.*

Proof: It easily follow by adapting the proof [Fre00, Proposition 1.1.6]. $\square$

Definition A.1.48 (The product $\boxminus$)**.** *Let $A = \{A_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ and $B = \{B_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ be symmetric bimodules. We define the symmetric bimodule $A \boxminus B$ by :*

$$(A \boxminus B)_{n,m} = \bigoplus_{r\in\mathbb{N}} A_{n,r} \otimes_{\mathbb{S}_r} B_{r,m}.$$

The product $\boxminus$ is a bifunctor. To be explicit let $f : A \to B$ and $g : A' \to B'$ be symmetric bimodule morphisms. We define $f \boxminus g : A \boxminus A' \to B \boxminus B'$ to be the morphism defined such that :

$$(f \boxminus g)_{n,m} = \sum_{r\in\mathbb{N}} f_{n,r} \otimes_{\mathbb{S}_r} g_{r,m}.$$

Proposition A.1.49. *The triple* $(\mathrm{BiMod}^{\mathbb{S}}_{\mathbb{K}}, \boxminus, \mathbb{I})$ *forms a monoidal category.*

Proof:[Sketch] The unit is given by

$$(A \boxminus \mathbb{I})_{n,m} = \bigoplus_{r\in\mathbb{N}} A_{n,r} \otimes_{\mathbb{S}_r} \mathbb{I}_{r,m} = A_{n,n} \otimes_{\mathbb{S}_n} \mathbb{K} = A_{n,m},$$

the associativity morphism is given by :

$$\begin{aligned}(A \boxminus B) \boxminus C)_{n,m} &= \bigoplus_{r} (A \boxminus B)_{n,r} \otimes_{\mathbb{S}_r} C_{r,m} \\ &= \bigoplus_{r} (\bigoplus_{s} A_{n,s} \otimes_{\mathbb{S}_s} B_{s,r}) \otimes_{\mathbb{S}_r} C_{r,m} \\ &= \bigoplus_{r} \bigoplus_{s} A_{n,s} \otimes_{\mathbb{S}_s} (B_{s,r} \otimes_{\mathbb{S}_r} C_{r,m}) \\ &= (A \boxminus (B \boxminus C))_{n,m}\end{aligned}$$

□

Let $A = \{A_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ be a symmetric bimodule. As for operads, we want to identify the elements of $A_{n,m}$ with some abstract operations with n inputs and m outputs. A PROP is a symmetric bimodule endowed with a structure that encodes the composition of these abstract operations.

Definition A.1.50 (PROP)**.** *Let $P = \{P_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ be a symmetric bimodule, $\mu_h : P \boxbar P \to P$, $\mu_v : P \boxminus P \to P$, and $\eta : \mathbb{I} \to P$ be symmetric bimodule morphisms. The set of data (P, μ_h, μ_v, η) is a PROP if the following diagrams commute :*

$$\begin{tikzcd} P \boxbar P \boxbar P \arrow[r, "\mathrm{Id}_P \square_{\mathbb{S}} \mu_h"] \arrow[d, "\mu_h \square_{\mathbb{S}} \mathrm{Id}_P"'] & P \boxbar P \arrow[d, "\mu_h"] \\ P \boxbar P \arrow[r, "\mu_h"'] & P, \end{tikzcd}$$ (Horizontal associativity)

$$\begin{tikzcd} \mathbb{I}\,\mathbb{K} \boxbar P \arrow[r, "\eta|_{\mathbb{K}} \boxbar \mathrm{Id}_P"] \arrow[dr] & P \boxbar P \arrow[d, "\mu_v"] & P \boxbar \mathbb{K} \arrow[l, "\mathrm{Id}_P \boxbar \eta|_{\mathbb{K}}"'] \arrow[dl] \\ & P, & \end{tikzcd}$$ (Horizontal unit)

$$\begin{tikzcd} P \boxminus P \boxminus P \arrow[r, "\mathrm{Id}_P \square_{\mathbb{S}} \mu_v"] \arrow[d, "\mu_v \square_{\mathbb{S}} \mathrm{Id}_P"'] & P \boxminus P \arrow[d, "\mu_v"] \\ P \boxminus P \arrow[r, "\mu_v"'] & P, \end{tikzcd}$$ (Vertical associativity)

$$\begin{tikzcd} \mathbb{I} \boxminus P \arrow[r, "\eta \boxminus \mathrm{Id}_P"] \arrow[dr] & P \boxminus P \arrow[d, "\mu_v"] & P \boxminus \mathbb{I} \arrow[l, "\mathrm{Id}_P \boxminus \eta"'] \arrow[dl] \\ & P, & \end{tikzcd}$$ (Vertical unit)

and the following equations holds :

$$\mu_h(f \otimes g) = \sigma^*(\mu_h(g \otimes f))\tau_*, \qquad \text{(Horizontal commutativity)}$$

for all $f \in P_{n_1,m_1}$ *and* $g \in P_{n_2,m_2}$ *where* σ *(resp.* τ*) is the permutation in* $\mathbb{S}_{n_1+n_2}$ *(resp.* $\mathbb{S}_{m_1+m_2}$*) which permutes the blocks* $\{1,\dots,n_1\}$ *and* $\{n_1+1,\dots,n_1+n_2\}$ *(resp.* $\{1,\dots,m_1\}$ *and* $\{m_1+1,\dots,m_1+m_2\}$*) and fix the orders inside the blocks.*

$$\mu_h(\mu_v(f_1,g_1),\mu_v(f_2,g_2)) = \mu_v(\mu_h(f_1,f_2),\mu_h(g_1,g_2)). \qquad \text{(Distributivity)}$$

A morphism of symmetric bimodules $f : P \to Q$ *is a morphism of PROPs if it commutes with all* μ_v, μ_h *and* η.

We denote the category of PROPs by $PROP$.

Proposition A.1.51. *The category of PROPs is equivalent to the category of symmetric monoidal categories* $(P, \odot, S, e)$ *enriched over* $\mathrm{Mod}_{\mathbb{K}}$ *such that :*

1. *the class of objects is identified with the set of natural numbers* $\mathbb{N}$,
2. *the product on objects is defined by* $m \odot n = m + n$ *for any* $m, n \in \mathbb{N}$.

Proof: We refer to Markl [Mar08, Section 8] for more details. □

Remark A.1.52. *As for operads, the general theory of PROPs allows us to define the free PROP generated by a set of operations and ideals generated by relations. Any PROP can be presented by generators and relations.*

Example A.1.53. *We can define the following PROPs :*

1. *the PROP* $BiAlg$ *is the PROP generated by a product* $m \in BiAlg_{2,1}$ *and a coproduct* $\Delta \in BiAlg_{1,2}$*, quotiented by the ideal generated by the following relations :*

$$m(m(-,-),-) = m(-,m(-,-)), \qquad (\Delta \otimes \mathrm{Id})\Delta(-) = (\mathrm{Id} \otimes \Delta)(\Delta(-)),$$
$$\Delta(m(1,2)) = (m \otimes m)((2,3)^*(\Delta(1),\Delta(2)))$$

2. *the PROP* $Frob$ *is the PROP generated by a product* $m \in Frob_{2,1}$*, a unit* $e \in Frob_{0,1}$*, a coproduct* $\Delta \in Frob_{1,2}$ *and a counit* $c \in Frob_{1,0}$*, quotiented by the ideal generated by the following relations :*

$$m(m(-,-),-) = m(-,m(-,-)), \qquad m(-,e) = m(e,-) = \mathrm{Id}(-),$$
$$(\Delta \otimes \mathrm{Id})\Delta(-) = (\mathrm{Id} \otimes \Delta)(\Delta(-)), \qquad (\mathrm{Id} \otimes c)(\Delta(-)) = (c \otimes \mathrm{Id})(\Delta(-)) = \mathrm{Id}(-),$$

and the Frobenius relation :

$$(\mathrm{Id} \otimes m)(\Delta \otimes \mathrm{Id})(-,-) = (m \otimes \mathrm{Id})(\mathrm{Id} \otimes \Delta)(-,-) = \Delta(m(-,-)),$$

3. *if* $\mathbb{K}$ *has characteristic different from* 2*, the PROP* $BiLie$ *is the PROP generated by an antisymmetric product* $[-,-] \in BiLie_{2,1}$ *and an antisymmetric product* $\delta \in BiLie_{1,2}$*, quotiented by the ideal generated by the following relations :*

$$[[1,2],3] + [[2,3],1] + [[3,1],2] = 0,$$

$$(1,2,3)(\delta \otimes \mathrm{Id})(\delta(-)) + (2,3,1)(\delta \otimes \mathrm{Id})(\delta(-)) + (3,1,2)(\delta \otimes \mathrm{Id})(\delta(-)) = 0,$$

and

$$(1,2)\delta([1,2]) - (1,2)([-,-] \otimes \mathrm{Id})(\mathrm{Id} \otimes \delta)(1,2) - (2,1)([-,-] \otimes \mathrm{Id})(\mathrm{Id} \otimes \delta)(1,2)$$
$$- (2,1)([-,-] \otimes \mathrm{Id})(\mathrm{Id} \otimes \delta)(2,1) - (1,2)([-,-] \otimes \mathrm{Id})(\mathrm{Id} \otimes \delta)(2,1) = 0.$$

Definition A.1.54 (The PROP End_V). *Let V be a $\mathbb{K}$-module. The PROP End_V is the strict symmetric monoidal category $(\mathrm{End}_V, \odot, S, e)$ such that :*

$$\mathrm{End}_{V;n,m} = \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V^{\otimes n}, V^{\otimes m}).$$

The PROP structure is given by the permutation action on $V^{\otimes n}$ and $V^{\otimes m}$, the tensor product and the composition of morphisms in $\mathrm{Mod}_{\mathbb{K}}$.

As for operads, PROPs are combinatorial objects that govern categories of algebras which are described by multilinear operations and multilinear relations. The major difference between operads and PROPs is that PROPs allow operations with more than one outputs. Another important difference between operads and PROPs is that, in general, a PROP is not associated to any monad. To define the category of algebras we are forced to use the PROP End_V.

Definition A.1.55 (P-algebras). *Let $(P = \{P_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}, \mu_h, \mu_v, \eta)$ be a PROP. A P-algebra is a pair $(V, \phi : P \to \mathrm{End}_V)$ where V is a $\mathbb{K}$-modules and ϕ a morphism of PROPs.*

Let (V, ϕ) and (V', ϕ') be P-algebras. A morphisms of $\mathbb{K}$-module $f : V \to V'$ is a P-algebra morphism if

$$\underbrace{f \otimes \cdots \otimes f}_{m}(\phi(p)(\underbrace{-, \ldots, -}_{n})) = \phi'(p)(\underbrace{f \otimes \cdots \otimes f}_{n})(\underbrace{-, \ldots, -}_{n}),$$

for any $p \in P_{n,m}$.

We denote the category of algebras over the PROP P by Alg_P.

Example A.1.56. *We have :*

1. *the category of $BiAlg$-algebras is equivalent to the category of associative, coassociative bialgebras,*
2. *the category of $Frob$-algebras is equivalent to the category of Frobenius algebras,*
3. *the category of $BiLie$-algebras is equivalent to the category of Lie bialgebras.*

A.2 Pre-Lie algebras

We introduce the definition of a pre-Lie algebra. The monad governing this category of algebras comes from an operad. Pre-Lie algebras were introduced by Gerstenhaber [Ger63] in the study of deformations of associative algebras. They appear in several contexts, notably in operad theory (see Loday and Vallette [LV12, Section 5.4.6], Chapoton and Livernet [CL01]), in deformation theory (see Dotsenko, Shadrin and Vallette preprint [DSV15]), and in quantum field theory (see Connes and Kreimer [CK99], Foïssy [Foi14]).

Definition A.2.1 (Pre-Lie algebras). *A pre-Lie algebra is a pair $(V, \{-,-\})$ where V is a $\mathbb{K}$-module and $\{-,-\} : V \otimes V \to V$ is a morphism of $\mathbb{K}$-modules such that :*

$$\{\{x,y\},z\} - \{x,\{y,z\}\} = \{\{x,z\},y\} - \{x,\{z,y\}\},$$

for all $x, y, z \in V$.

The name pre-Lie, which refers to Lie algebras, was chosen for the following reason.

Proposition A.2.2. *If $(V, \{-,-\})$ is a pre-Lie algebra then $(V, [-,-])$, where we set $[x,y] = \{x,y\} - \{y,x\}$ for all $x, y \in V$, is a Lie algebra.*

Proof: Immediate from a direct inspection. □

Example A.2.3. *The following structures are pre-Lie algebras :*

1. *let $(P = \{P_n\}_{n\in\mathbb{N}}, \mu = \{\circ_i\}_{i\in\mathbb{N}}, \eta)$ be an operad, the $\mathbb{K}$-module $\bigoplus_{n\in\mathbb{N}} P_n$ endowed with the product :*

   $$\{f, g\} = \sum_{i=1}^{n} f \circ_i g$$

   for all $f \in P_n$ and $g \in P$,

2. the set of vector fields of a n dimensional affine $\mathbb{R}$-space endowed with the product :

$$\{f(x)\frac{\partial}{\partial x}, g(x)\frac{\partial}{\partial x}\} = f'(x)g(x)\frac{\partial}{\partial x}.$$

Since a pre-Lie product is a multilinear operation satisfying a multilinear relation, the category of pre-Lie algebras is governed by an operad.

Definition A.2.4 (The operad $PreLie$)**.** *We define the operad $PreLie$ as the quotient of the free operad generated by a binary operation m by the ideal generated by the following relation :*

$$m(m(1,2),3) - m(1,m(2,3)) = m(m(1,3),2) - m(1,m(3,2)).$$

Dzhumadil'daev [Dzh01] introduces a special class of pre-Lie algebras, called p-restricted pre-Lie algebras, in the context where the ground ring is a field of positive characteristic p.

Definition A.2.5 (p-restricted pre-Lie algebra)**.** *A p-restricted pre-Lie algebra is a pre-Lie algebra $(V,\{-,-\})$ such that the following equation is satisfied :*

$$\{\cdots\{x,\underbrace{y\},\cdots\};y}_{p}\} = \{x,\{\cdots\underbrace{\{y,y\},\cdots\},y}_{p}\}\},$$

for any $x,y \in V$.

The notion of a p-restricted pre-Lie algebra is the first approximation of the notion of a $\Gamma PreLie$-algebra.

Proposition A.2.6 (Dokas [Dok13])**.** *There is a forgetful functor from $\Gamma PreLie$-algebras to p-restricted pre-Lie algebras.* □

Remark A.2.7. *In [Dok13] I. Dokas introduces a more general notion of p-restricted $PreLie$-algebra. A "generalized" p-restricted $PreLie$-algebra is a $PreLie$-algebra V endowed with a Frobenius map $\phi : V \to V$ satisfying some relations. If we assume $\phi = \{\{\cdots\underbrace{\{y,y\},\cdots\},y}_{p}\}$ we retrieve the definition of A. Dzhumadil'daev (Definition 1.2.3).*

Pre-Lie algebras play an important role in operad theory over a field of characteristic zero. In the context of operad theory over a field of positive characteristic p pre-Lie algebras should be replaced by $\Lambda PreLie$ or $\Gamma PreLie$-algebras. The aim of Chapter 1 is to give presentations by generators and relations of the monads which govern these categories we also give examples of $\Lambda PreLie$ and $\Gamma PreLie$-algebras.

A.3 Strict polynomial functors

In this section we briefly recollect the notions of polynomial functors and strict polynomial functors over $\mathrm{Mod}_{\mathbb{K}}$.

A.4 Polynomial Functors

Polynomial functors were introduced by Eilenberg and MacLane [EML54] in the study of the cohomology of Eilenberg-MacLane spaces. Strict polynomial functors were introduced by Friedlander and Suslin [FS97a] in the study of cohomology of group schemes. The definition of polynomial functors à la Eilenberg-MacLane of degree lower or equal to n is given by induction from the notion of additive functor.

Definition A.4.1 (Cross-effect)**.** *Let $F : \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ be a functor. The cross-effect of F is the bifunctor*

$$\Delta_2(F) : \mathrm{Mod}_{\mathbb{K}} \times \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$$

such that :

$$\Delta_2(F)(V_1,V_2) \coloneqq Ker(F(V_1 \oplus V_2) \longrightarrow F(V_1) \oplus F(V_2)).$$

We define the nth cross-effect $\Delta_n(F) : \mathrm{Mod}_{\mathbb{K}}{}^{\times n} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ *inductively by :*

$$\Delta_n(F)(V_1, \ldots, V_n) = \Delta_2(\Delta_{n-1}(F)(V_1, \ldots, V_{n-2}, -))(V_{n-1}, V_n).$$

We also have :

$$\Delta_n(F)(V_1, \ldots, V_n) := Ker(F(V_1 \oplus \ldots \oplus V_n) \longrightarrow \bigoplus_{i=1}^{n} F(V_1 \oplus \ldots \oplus \overset{i}{0} \oplus \ldots \oplus V_n)).$$

Roughly speaking the cross-effect measures the additivity defect of a functor.

Proposition A.4.2 ([EML54])**.** *Let* $F : \mathrm{Mod}_{\mathbb{K}} \to \mathrm{Mod}_{\mathbb{K}}$ *be a functor. We have the following canonical decomposition :*

$$F(V_1 \oplus \ldots \oplus V_n) = \bigoplus_{1 \leq i_1 < \ldots < i_r \leq n} \Delta_r F(V_{i_1}, \ldots, V_{i_r}).$$

Definition A.4.3 (Polynomial functor)**.** *Let* $F : \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ *be a functor. We say that* F *is polynomial of degree lower or equal to* n *if* $\Delta_{n+1}(F) = 0$*. We say that* F *is of degree* n *if it is of degree lower or equal to* n *and* $\Delta_n(F) \neq 0$*.*

We recall the notion of a strict polynomial functor on the underlying category $\mathrm{Mod}_{\mathbb{K}}$.

Definition A.4.4 (The functor $\Gamma_n(-)$)**.** *We define the functor* $\Gamma_n : \mathrm{Mod}_{\mathbb{K}} \to \mathrm{Mod}_{\mathbb{K}}$ *as follows :*

$$\Gamma_n(V) = \bigoplus_{n \in \mathbb{N}^*} (V^{\otimes n})^{\mathbb{S}_n},$$

for any $V \in \mathrm{Mod}_{\mathbb{K}}$ *and where* $(V^{\otimes n})^{\mathbb{S}_n}$ *stands for the* $\mathbb{K}$*-module of invariants of the* $\mathbb{K}[\mathbb{S}_n]$*-module* $V^{\otimes n}$*.*

Definition A.4.5 (The category $\Gamma_n \mathrm{Mod}_{\mathbb{K}}$)**.** *Let* n *be a non-negative integer. We denote by* $\Gamma_n \mathrm{Mod}_{\mathbb{K}}$ *the category objects are* $\mathbb{K}$*-modules, and morphisms :*

$$\mathrm{Hom}_{\Gamma_n \mathrm{Mod}_{\mathbb{K}}}(V, W) = \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V, W)).$$

The composition is defined by the composite :

$$\Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(W, U)) \otimes \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V, W)) \longrightarrow \\ \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(W, U) \otimes \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V, W)) \longrightarrow \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V, U)).$$

where the first morphism is the natural transformation $\Gamma_n(A) \otimes \Gamma_n(B) \to \Gamma_n(A \otimes B)$*, and the second morphism is induced by the composition in* $\mathrm{Mod}_{\mathbb{K}}$*.*

Definition A.4.6 (Strict polynomial functors)**.** *A strict polynomial functor of degree* n *is a linear functor* $F : \Gamma_n \mathrm{Mod}_{\mathbb{K}} \to \mathrm{Mod}_{\mathbb{K}}$*. Strict polynomial functors and natural transformations form a category denoted by* $PolFun_n$*.*

Definition A.4.7 (The functor $\mathcal{U}_n$)**.** *We have an embedding of categories* $in : \mathrm{Mod}_{\mathbb{K}} \to \Gamma_n \mathrm{Mod}_{\mathbb{K}}$ *defined as follows*

1. *on objects the functor is the identity map,*
2. *on morphisms the mapping* $f : X \to Y$ *is induced by the map* $V \to \Gamma_n(V)$ *such that*

$$v \mapsto \gamma_n(v) = \underbrace{v \otimes \cdots \otimes v}_{n}.$$

By precomposition, we get a functor $\mathcal{U}_n : PolFun_n \to \mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}})$ *from strict polynomial functors of degree* n *to endofunctors of the category* $\mathrm{Mod}_{\mathbb{K}}$*.*

The functor $\mathcal{U}_n$ allows us to compare the notions of a polynomial functor and the notion of a strict polynomial functor.

Proposition A.4.8 ([Bou67]). *Let F be a strict polynomial functor of degree n. The functor $\mathcal{U}_n(F)$ is polynomial of degree lower or equal to n.*

Remark A.4.9.

1. *There exists a strict polynomial functors F of degree n such that $\mathcal{U}_n(F)$ is a polynomial functor of degree strictly lower than n,*
2. *there exist non isomorphic strict polynomial functors F and G such that $\mathcal{U}_n(F) \cong \mathcal{U}_n(G)$,*
3. *there exists a polynomial functors F such that F is not in the image of $\mathcal{U}$.*

The functors associated to a symmetric module M which are considered in this thesis, $S(M,-)$, $\Lambda(M,-)$ and $\Gamma(M,-)$ are strict analytic functors, i.e. direct sums of strict polynomial functors. The aim of part 2 is to replace the category of symmetric modules with the category of $\mathbb{M}$-modules. The category of $\mathbb{M}$-modules is isomorphic to the category of strict analytic functors and allows us to introduce the concept of $\mathbb{M}$-operad and of category of algebra governed by an $\mathbb{M}$-operad. Some important categories of algebras are governed by $\mathbb{M}$-operads.

In this context we introduce a generalization of PROPs namely $\mathbb{M}$-PROPs and the categories of their algebras. These objects govern categories of bialgebras with possible divided symmetry operations.

A.5 Mackey functor and cohomological Mackey functors

In Chapter 2 we introduce the notion of a cohomological Mackey functors over the admissible category Par_n. This structure is used in the definition of $\mathbb{M}$-modules.

In this section we recall the definitions of the notions of a Mackey functor and of a cohomological Mackey functor.

A.5.1 Mackey functors

The notion of Mackey functors was introduced by Dress in [Dre71], and Green in [Gre71] in the study of the representations of finite groups. We refer to [TW95] for an exhaustive treatment of the subject.

We introduce the category $\Omega_{\mathbb{K}}(G)$ to give a functorial definition of Mackey functors for a finite group G.

Definition A.5.1 (The category $\Omega_{\mathbb{K}}(G)$). *Let G be a finite group. We denote by $\omega(G)$ the category such that :*

1. *the objects are finite G-sets,*
2. *if X and Y are finite G-sets then the set of morphisms $\mathrm{Hom}_{\omega(G)}(X,Y)$ is the set of equivalence classes of diagrams of the form*

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X & & Y, \end{tikzcd}$$

in the category of finite G-sets. Two diagrams $X \longleftarrow A \longrightarrow Y$ and $X \longleftarrow B \longrightarrow Y$ are equivalent if there is an isomorphism of finite G-sets $\sigma : A \longrightarrow B$ such that the following diagram commutes :

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] \arrow[dd, "\sigma"] & \\ X & & Y \\ & B \arrow[ul] \arrow[ur] & \end{tikzcd},$$

3. *the composition of two morphisms $X \longleftarrow A \longrightarrow Y$ and $Y \longleftarrow B \longrightarrow Z$ is defined by the*

following pull-back diagram of finite G-sets :

$$\begin{tikzcd}[column sep=small] & & C \arrow[dl] \arrow[dr] & & \\ & A \arrow[dl] \arrow[dr] & & B \arrow[dl] \arrow[dr] & \\ X & & Y & & Z. \end{tikzcd}$$

The disjoint union of G-sets gives to $\mathrm{Hom}_{\omega(G)}(X,Y)$ *a commutative monoid structure. More explicitly let* $f = X \longleftarrow A \longrightarrow Y$ *and* $g = X \longleftarrow B \longrightarrow Y$ *be two elements of* $\mathrm{Hom}_{\omega(G)}(X,Y)$*. We set*

$$f + g := \begin{tikzcd}[column sep=small] & A \sqcup B \arrow[dl] \arrow[dr] & \\ X & & Y. \end{tikzcd}$$

We define the abelian group $\mathbb{Z}\,\mathrm{Hom}_{\omega(G)}(X,Y)$ *by the usual Grothendieck group construction on* $\mathrm{Hom}_{\omega(G)}(X,Y)$*. The category* $\Omega(G)$ *is defined as follows :*

1. *the objects are finite G-sets,*
2. *if* X *and* Y *are finite G-sets then* $\mathrm{Hom}_{\Omega(G)}(X,Y) = \mathbb{Z}\,\mathrm{Hom}_{\omega(G)}(X,Y)$*.*

The category $\Omega_{\mathbb{K}}(G)$ *is the category defined to have the same objects as* $\Omega(G)$ *and*

$$\mathrm{Hom}_{\Omega_{\mathbb{K}}(G)}(X,Y) := \mathbb{K} \otimes \mathrm{Hom}_{\Omega(G)}(X,Y).$$

The Hom-sets of the category $\omega(G)$ admit a useful description as free abelian monoids over some equivalent classes of diagrams.

Notations A.5.2. *Let* G *be a finite group,* $H \leq K$ *be subgroups of* G *and* $g \in G$*. We use the following notation :*
— $\pi_H^K : {}^G/_H \to {}^G/_K$ *the coset projection,*
— ${}^gH = \{ghg^{-1} | h \in H\}$, *and*
— $H^g = \{g^{-1}hg | h \in H\}$.

Definition A.5.3 (Basic morphisms)**.** *Let* G *be a finite group and* K*,* H *be two subgroups of* G*. In the category* $\omega(G)$*, a morphism between* ${}^G/_K$ *and* ${}^G/_H$ *is a basic morphism if it is represented by a diagram of the following type :*

$$\begin{tikzcd}[column sep=small] & {}^G/_L \arrow[dl, "\pi^K_{{}^gL} \circ c_g"'] \arrow[dr, "\pi^H_L"] & \\ {}^G/_K & & {}^G/_H \end{tikzcd}$$

where we consider a class $g \in K \backslash G / H$ *and* L *is a subgroup of* $K^g \cap H$*.*

Proposition A.5.4. *Let* G *be a finite group and* K*,* H *be two subgroups of* G*. The set of morphisms* $\mathrm{Hom}_{\omega(G)}({}^G/_K, {}^G/_H)$ *is the free abelian monoid generated by the set of basic morphisms.*

In particular the set of morphisms $\mathrm{Hom}_{\Omega(G)}({}^G/_K, {}^G/_H)$ *is the free abelian group generated by the set of basic morphisms.*

Proof: See [TW95, Chp. 2]. □

We recall the definition of Mackey functor for a finite group G and a useful characterization.

Definition A.5.5 (Mackey functor)**.** *Let* G *be a finite group. A Mackey functor for* G *is an additive functor* $M : \Omega_{\mathbb{K}}(G) \longrightarrow \mathrm{Mod}_{\mathbb{K}}$*. Mackey functors and natural transformations form a category which we denote by* $\mathrm{Mac}(G)$*.*

There are some important instances of Mackey functors in different contexts.

Example A.5.6. *Fix a finite group G :*

1. *The group cohomology and group homology functors $H^n(G,V)$ and $H_n(G,V)$ with coefficients in the $\mathbb{K}[G]$-module V, are Mackey functors,*
2. *the functor $K_n(\mathbb{Z}[G])$, the K-theory of $\mathbb{Z}[G]$, is a Mackey functor,*
3. *the functor $B(G)$, the Burnside ring of G, is a Mackey functor,*

for details see [Bou10].

In the literature Mackey functors appear with different equivalent definitions. We recall one of them :

Proposition A.5.7. *Let G be a finite group, the definition of a Mackey functor is equivalent to the following data assignment : a function from the set $\{G\text{-sub-groups}\}$ to $\mathrm{Mod}_{\mathbb{K}}$ for any inclusion of G-sub-groups $H_1 \longrightarrow H_2$ a pair of $\mathbb{K}$-linear morphisms $\mathrm{Ind}_{H_1}^{H_2} : M(H_1) \longrightarrow M(H_2)$ and $\mathrm{Res}_{H_1}^{H_2} : M(H_2) \longrightarrow M(H_1)$, and for any element $g \in G$ and any G-subgroup H a $\mathbb{K}$-linear isomorphism $\mathrm{c}_g : M(H) \longrightarrow M({}^gH)$ such that the following relations are satisfied :*

1. $\mathrm{Ind}_{H_2}^{H_3} \mathrm{Ind}_{H_1}^{H_2} = \mathrm{Ind}_{H_1}^{H_3}$,
2. $\mathrm{Res}_{H_1}^{H_2} \mathrm{Res}_{H_2}^{H_3} = \mathrm{Res}_{H_1}^{H_3}$,
3. $\mathrm{c}_g \,\mathrm{c}_h = \mathrm{c}_{gh}$,
4. $\mathrm{c}_g \mathrm{Ind}_{H_1}^{H_2} = \mathrm{Ind}_{{}^gH_1}^{{}^gH_2} \mathrm{c}_g$,
5. $\mathrm{c}_g \mathrm{Res}_{H_1}^{H_2} = \mathrm{Res}_{{}^gH_1}^{{}^gH_2} \mathrm{c}_g$,
6. $\mathrm{Res}_J^H \mathrm{Ind}_K^H = \sum\limits_{x \in J\backslash H/K} \mathrm{Ind}_{J\cap {}^xK}^J \mathrm{c}_x \mathrm{Res}_{J^x \cap K}^K$,

the last relation is called the Mackey formula.

Proof: See [TW95, Chp. 2]. □

A.5.2 Cohomological Mackey functors

We recall the definition of a cohomological Mackey functor. We follow the characterization given by Yoshida, in [Yos83], using the Hecke category $\mathcal{H}G$.

Definition A.5.8 (The Hecke category $\mathcal{H}G$). *We denote by $\mathcal{H}G$ the full sub-category of $\mathbb{K}[G]$-modules whose objects are permutation modules over $\mathbb{K}[G]$, i.e. it is the category defined as follows :*

1. *the objects are direct sums of $\mathbb{K}[G]$-modules of the form $\mathbb{K}[{}^G/_H]$, where H is a subgroup of G,*
2. *if $\mathbb{K}[{}^G/_{H_1}]$ and $\mathbb{K}[{}^G/_{H_2}]$ are two objects of $\mathcal{H}G$, then*

$$\mathrm{Hom}_{\mathcal{H}G}(\mathbb{K}[{}^G/_{H_1}], \mathbb{K}[{}^G/_{H_2}]) \coloneqq \mathbb{K}[H_1 \backslash {}^G/_{H_2}].$$

Proposition A.5.9. *Let G be a finite group. The set of morphisms $\mathrm{Hom}_{\mathcal{H}G}(\mathbb{K}[{}^G/_K], \mathbb{K}[{}^G/_H])$ is the free abelian group generated by the set of equivalent diagrams of the following type :*

$$\begin{tikzcd}
 & {}^G/_{K^g \cap H} \arrow[dl, "\pi^K_{L^g}\circ c_g"'] \arrow[dr, "\pi^H_L"] & \\
{}^G/_K & & {}^G/_H ,
\end{tikzcd}$$

where $g \in K\backslash G/H$ and $L = K^g \cap H$.

Proof: The statement easily follows from Proposition A.5.4. □

Definition A.5.10 (Cohomological Mackey functors)**.** *A cohomological Mackey functor is an additive functor* $M : \mathcal{H}G \longrightarrow \mathrm{Mod}_{\mathbb{K}}$*. We denote the category of cohomological Mackey functors by* $\mathrm{Mac}^{coh}(G)$*.*

Cohomological Mackey functors are particular Mackey functors.

Proposition A.5.11. *Let G be a finite group. There exists a full and faithful functor from* $\mathrm{Mac}^{coh}(G)$*, the category of cohomological Mackey functors, to* $\mathrm{Mac}(G)$*, the category of Mackey functors.*

Proof: We define a functor $in : \Omega_{\mathbb{K}}(G) \longrightarrow \mathcal{H}G$. We proceed as follows.

1. Let X be a G-set. It decomposes uniquely as the disjoint union of transitive G-sets $X = \coprod_{i\in I} {}^{G}\!/_{H_i}$, with H_i a subgroup of G. We set :

$$in(X) := \bigoplus_{i\in I} \mathbb{K}[{}^{G}\!/_{H_i}].$$

2. Let f be a basic morphism in $\Omega_{\mathbb{K}}(G)$ of the form :

$$\begin{tikzcd} & G/L \arrow[dl, "\pi^K_{{}^gL}\circ c_g"'] \arrow[dr, "\pi^H_L"] & \\ G/K & & G/H. \end{tikzcd}$$

We set

$$in\left(\begin{tikzcd} & G/L \arrow[dl, "\pi^K_{{}^gL}\circ c_g"'] \arrow[dr, "\pi^H_L"] & \\ G/K & & G/H \end{tikzcd}\right) := |K \cap {}^gH : L| \begin{tikzcd} & G/K^g \cap H \arrow[dl, "\pi^K_{{}^gL}\circ c_g"'] \arrow[dr, "\pi^H_L"] & \\ G/K & & G/H. \end{tikzcd}$$

A cohomological Mackey functor defines a Mackey functor by pre-composition with in. $\square$

As for Mackey functors there is a description of cohomological Mackey functors in terms of operations of conjugation, induction and restriction.

Theorem A.5.12 (Yoshida)**.** *Let G be a finite group. The definition of a cohomological Mackey functors* $\mathrm{Mac}^{coh}(G)$ *is equivalent to the following data assignment : a function from the set* $\{G\text{-sub-groups}\}$ *to the class* $\mathrm{Mod}_{\mathbb{K}}$ *of $\mathbb{K}$-modules; for any inclusion of G-sub-groups $H_1 \longrightarrow H_2$, a pair of $\mathbb{K}$-linear morphisms* $\mathrm{Ind}_{H_1}^{H_2} : M(H_1) \longrightarrow M(H_2)$ *and* $\mathrm{Res}_{H_1}^{H_2} : M(H_2) \longrightarrow M(H_1)$*, and for any element $g \in G$ and any G-subgroup H, a $\mathbb{K}$-linear isomorphism* $\mathrm{c}_g : M(H) \longrightarrow M({}^gH)$ *such that the relations $(1-6)$ of Proposition A.5.7 are satisfied and the following additional relation*

$$\mathrm{Ind}_{H_1}^{H_2} \mathrm{Res}_{H_1}^{H_2} = |H_2 : H_1| \,\mathrm{Id}_{H_2}$$

holds.

Proof: See [Yos83, Thm. 4.3]. $\square$

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