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# Estimations sans pertes pour des méthodes asymptotiques et notion de propagation pour des équations dispersives

*par*

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## Titre

Estimations sans pertes pour des méthodes asymptotiques et notion de propagation pour des équations dispersives


## Résumé

Dans cette thèse, nous nous intéressons au comportement d'intégrales oscillantes à une variable d'intégration lorsqu'un paramètre fréquentiel tend vers l'infini. Pour cela, nous considérons tout d'abord la version de la méthode de la phase stationnaire de A. Erdélyi qui couvre le cas d'amplitudes singulières et de phases ayant des points stationnaires d'ordre réel, et qui fournit des estimations explicites de l'erreur. La preuve, seulement esquissée dans le papier original, est entièrement détaillée dans la thèse et la méthode est améliorée. Par ailleurs, nous montrons l'impossibilité de déduire, à partir de cette méthode, des estimations uniformes d'intégrales oscillantes par rapport à la position du point stationnaire dans le cas d'amplitudes singulières. Afin d'obtenir de telles estimations uniformes, nous proposons ensuite une extension du lemme de van der Corput au cas d'amplitudes singulières et de points stationnaires d'ordre réel.

Ces résultats abstraits sont appliqués à des formules de solution de certaines équations dispersives sur la droite réelle, couvrant des équations de type Schrödinger ainsi que des équations hyperboliques. Nous supposons que la transformée de Fourier de la donnée initiale est à support compact et/ou a un point singulier intégrable. Des développements à un terme et des estimations uniformes dans certains cônes de l'espace-temps ainsi que dans l'espace-temps tout entier sont établis, montrant l'influence des hypothèses ci-dessus sur les solutions. En particulier, nous prouvons que les paquets d'ondes tendent à être localisés dans certains cônes lorsque le temps tend vers l'infini, ce qui décrit leurs mouvements asymptotiquement en temps.

Pour finir, nous considérons des solutions approchées de l'équation de Schrödinger avec potentiel sur la droite réelle, telle que la transformée de Fourier du potentiel est supposée avoir un support compact également. En appliquant les méthodes mentionnées ci-dessus, nous prouvons que ces solutions approchées tendent à être concentrées dans certains cônes lorsque le temps tend vers l'infini, ce qui met en évidence des phénomènes de type réflexion et transmission.



## Mots-Clés

Intégrales oscillantes, méthode de la phase stationnaire, lemme de van der Corput, équations dispersives, bande de fréquences, fréquence singulière, décroissance (optimale) $L^\infty$ en temps, cône de l'espace-temps.

## Title

Lossless estimates for asymptotic methods with applications to propagation features for dispersive equations


## Abstract

In this thesis, we study the asymptotic behaviour of oscillatory integrals for one integration variable with respect to a large parameter. We consider first the version of the stationary phase method of A. Erdélyi which covers singular amplitudes and phases with stationary points of real order together with explicit error estimates. The proof, which is only sketched in the original paper, is entirely detailed in the present thesis and the method is improved. Moreover we show the impossibility to derive from this method uniform estimates of oscillatory integrals with respect to the position of the stationary point in the case of singular amplitudes. To obtain such uniform estimates, we propose then an extension of the classical van der Corput lemma to the case of singular amplitudes and stationary points of real order.

These abstract results are applied to solution formulas of certain dispersive equations on the line, covering Schrödinger-type and hyperbolic examples. We suppose that the Fourier transform of the initial condition is compactly supported and/or has a singular point. Expansions to one term and uniform estimates of the solutions in certain space-time cones as well as in the whole space-time are established, exhibiting the influence of the above hypotheses on the solutions. In particular, we show that the waves packets tend to be time-asymptotically localized in space-time cones, describing their motions when the time tends to infinity.

Finally we consider approximate solutions of the Schrödinger equation on the line with potential, where the Fourier transform of the potential is also supposed to have a compact support. Applying the methods mentioned above, we prove that these approximate solutions tend to be time-asymptotically concentrated in certain space-time cones, exhibiting reflection and transmission type phenomena.



## Keywords

Oscillatory integrals, stationary phase method, van der Corput lemma, dispersive equations, frequency band, singular frequency, (optimal) $L^{\infty}$-time decay, space-time cone.

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# Contents





 

# Chapter 0

# Introduction

The asymptotic behaviour of oscillatory integrals with respect to a large parameter is sometimes used to study the time-asymptotic behaviour of solutions of certain evolution equations, like the free Schrödinger equation or the Klein-Gordon equation on the line.
The stationary phase method is a tool which permits to expand oscillatory integrals with respect to the large parameter. Among the various versions of this method, a theorem of A. Erdélyi [16, Section 2.9] treats only the case of one integration variable but has the advantage to cover singular amplitudes and stationary points of real order together with explicit error estimates. The resulting decay rates are optimal and the results are interesting for applications. Unfortunately it turns out that the expansion as well as the remainder blow up when the stationary point of the phase attains the singular point of the amplitude, expressing a change of nature of the integral. This blow-up prevents the derivation of uniform estimates of the oscillatory integral with respect to the position of the stationary point.
However uniform estimates of oscillatory integrals with regular amplitudes and stationary points of integer order can be obtained by employing the classical van der Corput lemma [27, Chapter VIII, Proposition 2]: it furnishes estimates of the modulus of oscillatory integrals exhibiting the decay. Though these estimates are less precise than asymptotic expansions, they are uniform with respect to the position of the stationary point, which can be inside or outside the integration interval.
In this thesis, we provide an improved version of the stationary phase method of Erdélyi with complete proofs, making explicit the above mentioned blow-up. Further an extension of the van der Corput lemma is established, including phases with a stationary point of real order and singular amplitudes. Combining these two methods, we obtain a complete description of the phenomena produced by the stationary point of the phase, by the singular point of the amplitude and by their interaction.

These abstract results are then applied to solution formulas, given by oscillatory integrals depending on space and time, of certain evolution equations. Especially we treat evolution equations defined by Fourier multipliers, covering Schrödinger-type and hyperbolic examples.
In order to exhibit detailed propagation patterns, the initial data are supposed to be in frequency bands, meaning that their Fourier transform is compactly supported. To explain such hypotheses, let us consider the following formal decomposition of the Fourier

transform of an initial condition $u_0$:

$$\mathcal{F}u_0 = \sum_{k \in K} \chi_{I_k}\, \mathcal{F}u_0 \ , \tag{1}$$

where $K$ is a subset of $\mathbb{Z}$, $\{I_k\}_{k\in K}$ is a family of bounded intervals such that

$$\bigcup_{k \in K} I_k = \mathbb{R} \qquad , \qquad \forall\, k \neq l \quad I_k \cap I_l = \emptyset \ ,$$

and $\chi_{I_k}$ is the characteristic function of the interval $I_k$. Applying formally the inverse Fourier transform to equality (1), we obtain

$$u_0 = \sum_{k \in K} \mathcal{F}^{-1} \chi_{I_k}\, \mathcal{F}u_0 =: \sum_{k \in K} u_{0,k} \ ,$$

where $u_{0,k} := \mathcal{F}^{-1}\chi_{I_k}\,\mathcal{F}u_0$. Hence $u_{0,k}$ is a component of $u_0$ which is in the frequency band $I_k$. By supposing $u_0 \in L^2(\mathbb{R})$, this component $u_{0,k}$ can be interpreted in terms of quantum mechanics: it represents the part of the initial state $u_0$ having a momentum localized in the band $I_k$. The idea of using frequency bands, which was used in [7] for the Klein-Gordon equation on $\mathbb{R}$ with potential step, permits to exhibit certain phenomena which disappear when superposing all the bands. Moreover we assume that the Fourier transform of the initial data has a singular point, called singular frequency. Roughly speaking, the presence of a singular frequency shows that there is a concentration of the initial momentum around this singular frequency.

For initial data in frequency bands and having a singular frequency, we rewrite the solution formula as an oscillatory integral with respect to time and apply the stationary phase method to obtain expansions to one term in certain space-time cones, leading to optimal time-decay rates in these regions. This strategy was already employed in [5] to study solutions of the Klein-Gordon equation on a star-shaped network. Due to the blow-up inherited from the abstract setting, these expansions to one term do not cover the entire space-time in a uniform way. This problem is solved by applying our extended version of the van der Corput lemma, which provides uniform estimates of the solution in arbitrary cones as well as in the whole space-time.

The application of the two methods in the case of initial data in frequency bands permits to derive in this thesis not only the optimal time-decay rate for the $L^\infty$-norm of the solutions but also time-asymptotic localizations in cones, describing propagation features of the wave packets. For example, we obtain

- a concentration phenomenon produced by a singular frequency, along a particular space-time direction when the time tends to infinity ;
- the time-asymptotic localization of the solution of the Klein-Gordon equation in the light cone issued by the origin, even for initial data which are not compactly supported; this can be interpreted as an asymptotic version of the notion of causality ;
- a time-asymptotic description of the first interaction of a wave packet with a potential, leading to transmission and reflection type notions.



 

One can mention that this method does not take into account the position of the initial data, which would bring more precision to our results. Indeed one could consider recent rigorous versions of *Ehrenfest theorem* in physics [18], which provides the evolution of the mean position of a wave packet.

Before detailing the contents of the chapters of this thesis, let us remark that the paper [4] contains the essence of Chapters 1 and 2, while the paper [13] contains the essence of Chapters 3 and 4.

***Chapter 1.*** In the first chapter, we start by recalling Erdélyi's result in Theorem 1.1.3 concerning asymptotic expansions with remainder estimates of oscillatory integrals of the type

$$\forall\, \omega > 0 \qquad \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp\ . \tag{2}$$

The amplitude $U : (p_1, p_2) \longrightarrow \mathbb{C}$ may be singular at $p_1$ and $p_2$: we suppose that it is factorized as

$$U(p) = (p - p_1)^{\mu_1 - 1}\, (p_2 - p)^{\mu_2 - 1}\, \tilde{u}(p)\ , \tag{3}$$

where $\mu_1, \mu_2 \in (0, 1)$ and $\tilde{u}$ is supposed to be sufficiently regular. The phase function $\psi$ is allowed to have stationary points of real order at the endpoints as well; more precisely, we suppose the factorization

$$\psi'(p) = (p - p_1)^{\rho_1 - 1}\, (p_2 - p)^{\rho_2 - 1}\, \tilde{\psi}(p)\ , \tag{4}$$

where $\rho_1, \rho_2 \in \mathbb{R}$ are larger than 1 and $\tilde{\psi}$ is supposed to be positive on $[p_1, p_2]$. Since the proof is only sketched in the source, we detail it entirely following the lines of the original demonstration: we start by splitting the integral employing a smooth cut-off function which separates the endpoints of the interval. Then we use explicit substitutions to simplify the phases by exploiting the factorization (4). Afterwards successive integrations by parts create the expansion of the integral and the remainder terms, that we estimate to conclude. The two last steps are carried out using complex analysis in one variable and using the factorization (3). Especially the application of Cauchy's theorem allows to shift the integration path of the integrals defining the primitive functions created by the integrations by parts into a region of controllable oscillations of the integrands. This strategy, coupled with the explicit substitutions, leads finally to a precise estimate of the error.

Then we treat in Theorem 1.1.7 the case of the absence of amplitude singularities, which will be essential for certain applications. The previous error estimate furnishes here only the same decay rate for the highest term of the expansion and the remainder. The remedy proposed by Erdélyi [16, page 55] leads to complicated formulas when written down and does not seem possible in the case of stationary points of non integer order. To refine this analysis, we work on the above mentioned integrals defining the primitive functions created by the integrations by parts and involved in the remainder. Introducing a new parameter, we obtain estimates of these integrals permitting a balance between their singular behaviour with respect to the integration variable of the remainder and their decay with respect to $\omega$. Putting these new estimates into the remainder, we obtain better decay rates of the remainder in the case of regular amplitudes.

To finish the comments on the first chapter, let us present for comparison two classical versions of the stationary phase method coming from the literature. In [17, Chapter 4, Section 5], the author assumes that the amplitude belongs to $\mathcal{C}_c^\infty(\mathbb{R}^d)$ (for $d \geqslant 1$) and supposes that the stationary points of the phase are non-degenerate. Firstly the author employs Morse's lemma to simplify the phase function. Then using Fubini's theorem, he obtains a product of integrals, where each of them is actually the Fourier transform of a tempered distribution. He computes these Fourier transforms explicitly thanks to complex analysis and estimates them, leading to the result. Nevertheless, the use of Morse's lemma implies a loss of precision regarding the estimate of the remainder. Indeed, Morse's lemma is based on the implicit function theorem and so the substitution is not explicit.

In [20, Chapter VII], the author provides a stationary phase method which is different from [17]. It is assumed that the amplitude $U$ and the phase have a certain regularity on $\mathbb{R}^d$ (for $d \geqslant 1$), and that $U$ has a compact support. Asymptotic expansions of the oscillatory integral are given by using Taylor's formula of the phase, where the stationary point is supposed to be non-degenerate. Morse's lemma is not needed in this situation, but stronger hypothesis concerning the phase are required in order to bound uniformly the remainder by a constant. He needs also to use Taylor's expansion of the amplitude, which excludes the case of singular amplitudes. In addition, his expression of the remainder is less explicit than ours.

Finally, we can compare the results of Erdélyi's book [16] with the asymptotic expansions of oscillatory integrals given in the book [15, Chapter IV]. There, the phase is supposed to be equivalent to $x^\alpha$ and the amplitude to be equivalent to $x^\beta$ when $x$ tends to 0, where $0 < \alpha + 1 < \beta$. Hence singular amplitudes are allowed and, in this case, the stationary point of the phase and the singular point of the amplitude coincide. The authors in [15] computes explicitly the first term of the expansions but does not furnish remainder estimates, preventing the application to solution formulas of certain evolution equations for example. The results given in Erdélyi's book are then much more precise than those established in [15].

***Chapter 2.*** In Theorem 2.1.2 of Chapter 2, we modify the stationary phase method of Erdélyi by replacing the smooth cut-off function used in the original proof by a characteristic function. This is motivated by the fact that an inherent blow-up of the expansion of the oscillatory integral and the remainder occurs when a stationary point and a singular point of the amplitude tend to each other, restricting potentially the field of applications. Hence the aim of employing a characteristic function instead of a smooth cut-off function is to make explicit this blow-up in the remainder. For this purpose, we consider a fixed cutting-point $q \in (p_1, p_2)$ and then we follow the lines of the original proof: we carry out explicit substitutions and we integrate by parts. Here the characteristic function has the disadvantage that it produces new terms related to the cutting-point when integrating by parts, and they are not the same due to the different substitutions employed in the two integrals. However by expanding these terms with respect to the parameter $\omega$, we observe that the resulting first terms cancel out but not the remainders. These new remainders related to the cutting-point $q$ are estimated as well as the classical remainders related to

$p_1$ and $p_2$, which permits to conclude.
As previously, we refine the estimate of the remainder related to $p_j$ if this point is a regular point of the amplitude. For this purpose, we follow simply the lines of the method described in the preceding chapter. Moreover, we note that the remainder related to the cutting-point $q$ tends always more quickly to zero than the expansion of the oscillatory integral.

In preparation for applications to the free Schrödinger equation, our improved version of the stationary phase method is applied to oscillatory integrals with amplitudes having a unique singular point at the left endpoint $p_1$ of the integration interval $[p_1, p_2]$, and with phases of the form

$$\psi(p) = -(p - p_0)^2 + c \, ,$$

where $c \in \mathbb{R}$ and where the stationary point $p_0$ is supposed to belong to $[p_1, p_2]$. Thanks to the explicitness of the phase functions and the preciseness given by the previous results, we furnish in Theorem 2.2.2 asymptotic expansions together with remainder estimates depending explicitly on the distance between the stationary point and the singularity. In particular this theorem can be exploited to establish uniform estimates of the above oscillatory integrals in parameter regions defined by,

$$p_1 + \omega^{-\vartheta} \leqslant p_0 < p_2 \, ,$$

where $\vartheta \in \left(0, \frac{1}{2}\right)$. The resulting decay rates are proved to be optimal by expanding the oscillatory integrals on the left boundary of the regions, namely on the curves given by

$$p_0 = p_1 + \omega^{-\vartheta} \, .$$

These abstract results are then exploited to study the time-asymptotic behaviour of the solution of the free Schrödinger equation on the line with initial data satisfying

**Condition ($\mathbf{C1}_{[p_1,p_2],\mu}$).** Let $\mu \in (0,1)$ and $p_1 < p_2$ be two finite real numbers.
A tempered distribution $u_0$ on $\mathbb{R}$ satisfies Condition ($\mathrm{C1}_{[p_1,p_2],\mu}$) if and only if $\mathcal{F}u_0 \equiv 0$ on $\mathbb{R} \backslash [p_1, p_2)$ and

$$\forall \, p \in (p_1, p_2) \qquad \mathcal{F}u_0(p) = (p - p_1)^{\mu - 1} \, \tilde{u}(p) \, ,$$

where $\tilde{u} \in \mathcal{C}^1\big([p_1, p_2], \mathbb{C}\big)$ and $\tilde{u}(p_1) \neq 0$.

The interval $[p_1, p_2]$ is called *frequency band* and the singular point $p_1$ is called *singular frequency*. More precisely, the aim is to study the influence of a restriction to compact frequency bands and of singular frequencies on the dispersion. For this purpose, we rewrite the solution formula as an oscillatory integral of the form (2) in order to apply the preceding abstract results. By employing then Theorem 2.2.2, we establish asymptotic expansions with explicit remainder estimates of the solution with respect to time in certain space-time cones, namely space-time regions given by

$$\mathfrak{C}_S(a, b) = \left\{ (t, x) \in (0, +\infty) \times \mathbb{R} \, \middle| \, 2\, a < \frac{x}{t} < 2\, b \right\} \, , \tag{5}$$

where $a < b$ are two real numbers. Especially the first terms of the expansions exhibit the optimal decay rates in these cones. Let us give the result in the case of weak singular frequencies in the cone $\mathfrak{C}_S\big(p_1 + \varepsilon, p_2\big)$:



 

**0.1 Theorem.** *Suppose that $u_0$ satisfies Condition $(\mathrm{C1}_{[p_1,p_2],\mu})$ with $\mu \in \big(\frac{1}{2},1\big)$, and choose a real number $\varepsilon > 0$ such that*

$$p_1 + \varepsilon < p_2 \, .$$

*For all $(t,x) \in \mathfrak{C}_S\big(p_1+\varepsilon, p_2\big)$, define $H(t,x,u_0) \in \mathbb{C}$ as follows,*

$$H(t,x,u_0) := \frac{1}{2\sqrt{\pi}} \, e^{-i\frac{\pi}{4}} \, e^{i\frac{x^2}{4t}} \, \tilde{u}\Big(\frac{x}{2t}\Big) \left(\frac{x}{2t} - p_1\right)^{\mu-1} \quad ,$$

*Then for all $(t,x) \in \mathfrak{C}_S\big(p_1+\varepsilon, p_2\big)$, we have*

$$\Big| u(t,x) - H(t,x,u_0)\, t^{-\frac{1}{2}} \Big| \leqslant \sum_{k=1}^{9} C_k(u_0,\varepsilon)\, t^{-\sigma_k^{(1)}} \ ,$$

*where $\max\limits_{k \in \{1,\dots,9\}} \Big\{ -\sigma_k^{(1)} \Big\} < -\dfrac{1}{2}$. The exponents $\sigma_k^{(1)}$ and the constants $C_k(u_0,\varepsilon) \geqslant 0$ are given in the proof.*

See Theorem 2.3.2 for a complete statement with proof which includes also the case of strong singular frequencies, and see Theorem 2.3.4 for expansions to one term in cones outside $\mathfrak{C}_S\big(p_1,p_2\big)$. Let us now remark that an initial data satisfying $(\mathrm{C1}_{[p_1,p_2],\mu})$ belongs to $L^2(\mathbb{R})$ if and only if $\mu > \frac{1}{2}$. Hence in the $L^2$-case, we prove that the energy[1] tends to be concentrated in the cone $\mathfrak{C}_S\big(p_1,p_2\big)$ when the time tends to infinity, thanks to the expansion given in Theorem 0.1:

**0.2 Corollary.** *Suppose that the hypotheses of Theorem 0.1 are satisfied and for all $t > 0$, define the interval $I_t$ as follows,*

$$I_t := \Big( 2\,(p_1+\varepsilon)\,t, \; 2\,p_2\,t \Big) \, .$$

*Then we have*

$$\left| \big\| u(t,.) \big\|_{L^2(I_t)} - \frac{1}{\sqrt{2\pi}} \, \big\| \mathcal{F}u_0 \big\|_{L^2(p_1+\varepsilon, p_2)} \right| \leqslant \sum_{k=1}^{9} \tilde{C}_k(u_0,\varepsilon)\, t^{-\sigma_k^{(1)}+\frac{1}{2}} \ ,$$

*where the constants $\tilde{C}_k(u_0,\varepsilon) \geqslant 0$ are given in the proof.*

See Corollary 2.3.5 for the proof. The physical principle of *group velocity* applied to the free Schrödinger equation on the real line says roughly that the energy associated with a frequency $\tilde{p}$ propagates with the speed given by the stationary phase method, in this case $\frac{x}{t} = 2\tilde{p}$. Theorem 0.1 and Corollary 0.2 furnish a precise meaning of this principle in our case.

Now let us remark that it is not possible to derive an asymptotic expansion with uniform error estimate of the solution in the entire cone $\mathfrak{C}_S\big(p_1,p_2\big)$ from Theorem 0.1, due to the inherent blow-up occurring in the abstract setting. In order to study the

[1]For simplicity, we call in Chapter 2 of the thesis *energy* the $L^2$-norm of the solution on subsets of $\mathbb{R}$, in contrast to the usual quantum mechanics interpretation as probability of occurrence of the particle in a subset.



 

time-asymptotic behaviour of the solution in regions containing the space-time direction $\frac{x}{t} = 2p_1$, we start by expanding the solution on this direction, showing that the optimal decay rate is given by $t^{-\frac{\mu}{2}}$. Then we exploit once again the preciseness provided by Theorem 2.2.2 to establish uniform estimates of the solution in regions along the direction associated with the singular frequency $p_1$, which are asymptotically larger than any space-time cone:

**0.3 Theorem.** *Suppose that $u_0$ satisfies Condition $(\mathrm{C1}_{[p_1,p_2],\mu})$ with $\mu \in \left(\frac{1}{2}, 1\right)$, and fix $\vartheta \in \left(0, \frac{1}{2}\right)$. Then for all $(t,x) \in (0,+\infty) \times \mathbb{R}$ satisfying $2\,p_1 + 2\,t^{-\vartheta} \leqslant \frac{x}{t} < 2\,p_2$ and $t > (p_2 - p_1)^{-\frac{1}{\vartheta}}$, we have*

$$\big|u(t,x)\big| \leqslant C_0(u_0)\, t^{-\frac{1}{2}+\vartheta(1-\mu)} + \sum_{k=1}^{9} C_k(u_0)\, t^{-\sigma_k^{(2)}+\vartheta\varsigma_k^{(2)}} \,,$$

*where $\max_{k\in\{1,\ldots,9\}} \left\{-\sigma_k^{(2)} + \vartheta\varsigma_k^{(2)}\right\} < -\dfrac{1}{2} + \vartheta(1-\mu)$ and the decay rate $t^{-\frac{1}{2}+\vartheta(1-\mu)}$ is optimal. The exponents $\sigma_k^{(2)}$, $\varsigma_k^{(2)}$ and the constants $C_k(u_0) \geqslant 0$ are given in the proof.*

The resulting decay rate $t^{-\frac{1}{2}+\vartheta(1-\mu)}$ is optimal because it is attained on the left boundary of the region (see Theorem 2.3.8) and the case $\mu \leqslant \frac{1}{2}$ is also studied (see Theorem 2.3.7 for a complete statement with proof). Theorem 0.3 suggests that the effect of a singular frequency is strong in space-time regions close to the critical direction $\frac{x}{t} = 2p_1$, *i.e.* for $\vartheta$ close to $\frac{1}{2}$.

The method employed in Chapter 2 to study the time-asymptotic behaviour of the Schrödinger equation has been inspired by [5] and [6], where the authors consider the Klein-Gordon equation on a star shaped network with a potential which is constant but different on each semi-infinite branch. The authors are interested in the influence of the coefficients of the potential on the time-asymptotic behaviour of the solution. To do so, they calculate asymptotic expansions to one term with respect to time of the solution with initial data in frequency bands in [5] and they exploit these expansions to describe the time-asymptotic energy flow of wave packets in [6].
They notice that the asymptotic expansion degenerates when the frequency band approaches certain critical values coming from potential steps. These critical values play a similar role as the singularities of the Fourier transform of the initial conditions in the present chapter. Hence our refined version of Erdélyi's expansion theorem could help to improve the comprehension of the problem of the blow-up of the expansion in the setting of [5] and [6].
Moreover the paper [5] shows the way to obtain this type of results for the Schrödinger equation on domains with canonical geometry and canonical potential permitting sufficiently explicit spectral theoretic solution formulas.

We can point out the usefulness of detailed informations on the motion of wave packets in frequency bands by citing [3]. In this paper, the authors study the time-asymptotic behaviour of the solution of the Schrödinger equation on star-shaped networks with a localized potential. They establish a perturbation inequality which shows that the evolution of high frequency signals is close to their evolution without potential.

One can also mention the methods and the results of the papers [11] and [12], in which the time-decay rate of the free Schrödinger equation is considered. In [11], singular initial conditions are constructed to derive the exact $L^p$-time decay rates of the solution, which are slower than the classical results for regular initial conditions. In [12], the authors construct initial conditions in Sobolev spaces (based on the Gaussian function), and they show that the related solutions has no definite $L^p$-time decay rates, nor coefficients, even though upper estimates for the decay rates are established.
The papers [11] and [12] use special formulas for functions and their Fourier transforms, which are themselves based on complex analysis. In our results, we furnish slow decay rates by considering initial conditions with singular Fourier transforms. Here, complex analysis is directly applied to the solution formula of the equation, which permits to obtain results for a whole class of functions. Our method seems therefore to be more flexible.

One-dimensional Schrödinger equations with singular coefficients have been studied in the literature, as for example in [8]. There, dispersion inequalities and Strichartz-type estimates are furnished, exhibiting the influence of the singular coefficients on the dispersion. Under certain hypotheses on the singularities, we observe that the above mentioned estimates remain unchanged as compared with the classical case.

Finally, among all the applications involving oscillatory integrals, let us mention that results on this type of integrals have been employed to establish the first Strichartz-type estimates in [28] for the Schrödinger equation and the Klein-Gordon equation, or to study the time-decay rates for the system of crystal optics in [23].

***Chapter 3.*** In Chapter 3, we consider oscillatory integrals of the type (2). Here the amplitude $U : (p_1, p_2] \longrightarrow \mathbb{C}$ may be singular at $p_1$: it is factorized as

$$\forall\, p \in (p_1, p_2] \qquad U(p) = (p - p_1)^{\mu - 1}\, \tilde{u}(p)\ , \tag{6}$$

where $\mu \in (0, 1]$ and $\tilde{u} : [p_1, p_2] \longrightarrow \mathbb{C}$ has a certain regularity. The phase function $\psi : I \longrightarrow \mathbb{R}$, where $I$ is an open interval containing $[p_1, p_2]$, is allowed to have a stationary point $p_0$ of real order; more precisely, we suppose that

$$\forall\, p \in I \qquad \psi'(p) = |p - p_0|^{\rho - 1}\, \tilde{\psi}(p)\ , \tag{7}$$

where $\rho \in \mathbb{R}$ is larger than 1 and $\tilde{\psi} : I \longrightarrow \mathbb{R}$ satisfies $\big|\tilde{\psi}\big| > 0$ on $[p_1, p_2]$. For example, smooth functions with vanishing first derivatives are included.
The aim of this chapter is to provide uniform estimates of such integrals with respect to the position of the stationary point $p_0$. As explained previously, the stationary phase method can not provide such estimates since an inherent blow-up of the asymptotic expansion occurs when the stationary point attains the singular point, while the oscillatory integral is uniformly bounded with respect to $p_0$ and $\omega$. Hence we give up the idea of expanding the integrals to one term in favour of estimates of their modulus. In the case of regular amplitudes and stationary points of integer order, the classical van der Corput lemma (see [32, Chapter V, Lemma 4.3] or [27, Chapter VIII, Proposition 2]) provides such estimates which are uniform with respect to the position of the stationary point. However the classical version of this lemma is not applicable to the above setting and the constants are not explicitly given, which may prevent certain applications. Hence



 

we propose an extension of this lemma to oscillatory integrals with singular amplitudes and with stationary points of real order. To this end, we shall exploit the factorizations (6) and (7) which are well-suited for the formulation of the results and which have been inspired by [16].

The first part of Chapter 3 is devoted to the case of a phase function having a stationary point $p_0$ which is either inside or outside the interval of integration. In both cases, we furnish an estimate which is uniform with respect to the position of $p_0$ with explicit constant. Moreover we remark that the resulting decay rate, depending on the order of the stationary point and on the strength of the singular point, corresponds to the decay rate given by the stationary phase method of Erdélyi when these two particular points coincide:

**0.4 Theorem.** *Let* $\rho > 1$, $\mu \in (0,1]$ *and choose* $p_0 \in I$. *Suppose that the functions* $\psi : I \longrightarrow \mathbb{R}$ *and* $U : (p_1, p_2] \longrightarrow \mathbb{C}$ *satisfy Assumption* $(\mathrm{P2}_{p_0,\rho})$ *and Assumption* $(\mathrm{A2}_{p_1,\mu})$ *(given in Section 3.1), respectively. Moreover suppose that* $\psi'$ *is monotone on* $\{p \in I \,|\, p < p_0\}$ *and* $\{p \in I \,|\, p > p_0\}$. *Then we have*

$$\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp \right| \leqslant C(U,\psi)\, \omega^{-\frac{\mu}{\rho}} \ ,$$

*for all* $\omega > 0$, *and the constant* $C(U,\psi) > 0$ *is given in the proof.*

See Theorem 3.1.8 for the proof. Assumption $(\mathrm{A2}_{p_1,\mu})$ and Assumption $(\mathrm{P2}_{p_0,\rho})$ are satisfied if and only if the functions $U$ and $\psi$ verify equalities (6) and (7) respectively, with additional hypotheses on the regularity of $\tilde{u}$ and $\tilde{\psi}$.

In the second part of Chapter 3, we suppose the absence of a stationary point inside $[p_1, p_2]$ in preparation for applications to solutions of certain evolution equations in Chapter 4. Thus, if the phase has a stationary point, then it is outside the integration interval; in this case, we furnish a better decay rate than the one obtained in Theorem 0.4, but the estimate is not uniform with respect to the position of the stationary point.
Let us remark that all the decay rates provided in Chapter 3 are proved to be optimal. This optimality is a consequence of an application of the stationary phase method of Erdélyi when the stationary point and the singular point coincide.

In the literature, adaptations of the van der Corput lemma have been developed. For example, on can mention [9] in which the authors study the decay of Fourier transforms on singular surfaces. To do so, they establish a variant of the van der Corput lemma, based on the classical estimate [27], by supposing weaker hypotheses on the phase but stronger hypotheses on the amplitude.

Furthermore, some authors have established versions of the van der Corput lemma for several integration variables, to be applied to solutions of certain evolution equations on $\mathbb{R}^d$ for example. In [26], hypotheses on the radial behaviour of the phase in a neighbourhood of the stationary point permit to reduce the study to oscillatory integrals for one integration variable. By combining the standard calculations in the one-dimensional case and well-chosen assumptions on the phase and the amplitude, the author obtains the desired result. This approach permits to extend the notion of *stationary point of integer order* to several variables and to provide estimates of oscillatory integrals with phases having this type of stationary point, but the resulting constants are not explicit.

On the other hand, the authors in [1] establish a van der Corput-type estimate for several integration variables with explicit constant. To this end, they adapt the proof of the classical lemma to the case of several integration variables, leading to technical computations. Nevertheless their result is restricted to phases whose Hessian is supposed to be invertible, meaning that the order of the stationary can not be larger than one in the one dimensional case.
The amplitudes in both above results are supposed to be smooth and compactly supported, meaning that they do not treat the case of singular amplitudes. An interesting outlook would be to find a suitable extension to several variables of the notion of *singular point* for which van der Corput type estimates can be established.

***Chapter 4.*** In Chapter 4, we consider Fourier solution formulas for the class of initial value problems given by

$$\begin{cases} \Big[i\,\partial_t - f\big(D\big)\Big]u(t) = 0 \\ u(0) = u_0 \end{cases} , \tag{8}$$

for $t \geqslant 0$, where the symbol $f$ of the Fourier multiplier $f(D)$ is supposed to satisfy $f'' > 0$. One can note that the free Schrödinger equation belongs to this class since its symbol $f_S$ is given by $f_S(p) = p^2$. Our aim is to study the phenomena exhibited in Chapter 2 for this class of dispersive equations, not using asymptotic expansions, but by creating a (rougher in a sense but closely related) method based on van der Corput type estimates established in Chapter 3. This approach permits to avoid the blow-up of the expansion occurring in Chapter 2 and hence to give uniform estimates in space-time cones as well as in the whole space-time.

First of all, let us give a new definition of a space-time cone, which is slightly different from (5) but more adapted to the present case:

$$\mathfrak{C}(a,b) = \left\{(t,x) \in (0,+\infty) \times \mathbb{R} \,\middle|\, a < \frac{x}{t} < b\right\} ,$$

where $a < b$ are two finite real numbers. The first result of Chapter 4 furnishes uniform estimates of the solution of (8), for initial data in the frequency band $[p_1, p_2]$ and having a singular frequency at $p_1$, in arbitrary large space-time cones containing $\mathfrak{C}\big(f'(p_1), f'(p_2)\big)$ as well as in their complements:

**0.5 Theorem.** *Suppose that $u_0$ satisfies Condition* $(\mathrm{C2}_{[p_1,p_2],\mu})$ *(given in Section 4.1) and choose two finite real numbers $\tilde{p}_1 < \tilde{p}_2$ such that $[p_1,p_2] \subset (\tilde{p}_1,\tilde{p}_2) =: \tilde{I}$. Then we have*

- $\forall\,(t,x) \in \mathfrak{C}\big(f'(\tilde{p}_1), f'(\tilde{p}_2)\big) \qquad \big|u(t,x)\big| \leqslant c(u_0,f)\,t^{-\frac{\mu}{2}}$ ;
- $\forall\,(t,x) \in \mathfrak{C}\big(f'(\tilde{p}_1), f'(\tilde{p}_2)\big)^c \qquad \big|u(t,x)\big| \leqslant c_{\tilde{I}}(u_0,f)\,t^{-\mu}$ .

*All the constants are given in the proof and the two decay rates are optimal.*



 

See Theorem 4.1.3 for the proof. Condition ($\mathrm{C2}_{[p_1,p_2],\mu}$) is similar to Condition ($\mathrm{C1}_{[p_1,p_2],\mu}$) but it includes also the case $\mu = 1$ and the case $\mathcal{F}u_0(p_2) \neq 0$. Theorem 0.5 highlights the concentration phenomenon produced by the frequency band as explained above in the Schrödinger case. Moreover, this permits to derive an $L^\infty$-norm estimate for the solution of the free Schrödinger equation for initial data satisfying Condition ($\mathrm{C2}_{[p_1,p_2],\mu}$).
To study the influence of a singular frequency $p_1$ of order $\mu - 1$ in regions containing the space-time direction defined by $\frac{x}{t} = f'(p_1)$, we provide estimates of the solution in cones containing this direction as well as in cones which do not contain it:

**0.6 Theorem.** *Suppose that $u_0$ satisfies Condition ($\mathrm{C3}_{p_1,\mu}$) (given in Section 4.1), and choose three finite real numbers $\varepsilon > 0$ and $\tilde{p}_1 < \tilde{p}_2$ such that $p_1 \notin [\tilde{p}_1, \tilde{p}_2]$. Then we have*

- $\forall\, (t,x) \in \mathfrak{C}\big(f'(p_1 - \varepsilon), f'(p_1 + \varepsilon)\big) \qquad \big|u(t,x)\big| \leqslant c^{(1)}(u_0,f)\, t^{-\frac{\mu}{2}} + c^{(2)}_{\varepsilon}(u_0,f)\, t^{-1}$ ;
- $\forall\, (t,x) \in \mathfrak{C}\big(f'(\tilde{p}_1), f'(\tilde{p}_2)\big) \qquad \big|u(t,x)\big| \leqslant c^{(1)}_{\tilde{p}_1,\tilde{p}_2}(u_0,f)\, t^{-\frac{1}{2}} + c^{(2)}_{\tilde{p}_1,\tilde{p}_2}(u_0,f)\, t^{-\mu} + c^{(3)}_{\tilde{p}_1,\tilde{p}_2}(u_0,f)\, t^{-1}$ .

*All the constants are given in the proof.*

See Theorem 4.1.7 and Theorem 4.1.8 for the proofs. This result proves that in cones containing the space-time direction given by the singular frequency, the influences on the decay rate of the order of the singularity and the stationary point are combined. In all other cones, the decay rates given respectively by the order of the singularity and by the stationary point are in concurrence. Here Condition ($\mathrm{C3}_{p_1,\mu}$) is similar to the above Condition ($\mathrm{C1}_{[p_1,p_2],\mu}$) but the initial datum is not supposed to be in a frequency band anymore; for example the support of $\mathcal{F}u_0$ may be equal to an infinite interval but in this case, it must have a sufficiently fast decay at infinity. This permits to study the effect of the singular frequency without the influence of a frequency band.

Then we focus our attention on the fact that the symbol may influence the dispersion of the solution. We suppose now that the symbol $f$ verifies the following condition:

**Condition ($\mathrm{S}_{\beta_+,\beta_-,R}$).** Fix $\beta_- \geqslant \beta_+ > 1$ and $R \geqslant 1$.
A $\mathcal{C}^\infty$-function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Condition ($\mathrm{S}_{\beta_+,\beta_-,R}$) if and only if the second derivative of $f$ is positive on $\mathbb{R}$ and verifies

$$\exists\, c_+ \geqslant c_- > 0 \qquad \forall\, |p| \geqslant R \qquad c_-\, |p|^{-\beta_-} \leqslant f''(p) \leqslant c_+\, |p|^{-\beta_+} \ .$$

We note that the symbol related to the free Schrödinger equation does not satisfy the above condition, unlike the function $f_{KG}$ given by $f_{KG}(p) = \sqrt{c^4 + c^2\, p^2}$, where $c > 0$ is a constant. Under Condition ($\mathrm{S}_{\beta_+,\beta_-,R}$), the first derivative of the symbol is bounded, leading to the existence of a space-time cone in which the decay rate of the solution is slower than outside:

**0.7 Theorem.** *Suppose that the symbol $f$ satisfies Condition ($\mathrm{S}_{\beta_+,\beta_-,R}$) and that $u_0$ satisfies Condition ($\mathrm{C4}_{\mu,\alpha,r}$) (given in Section 4.2), where $\mu \in (0,1]$, $\alpha - \mu > \beta_-$ and $r \leqslant R$. Then we have*

- $\forall\, (t,x) \in \mathfrak{C}(a,b) \qquad \big|u(t,x)\big| \leqslant c^{(1)}(u_0,f)\, t^{-\frac{\mu}{2}} + c^{(2)}(u_0,f)\, t^{-\frac{1}{2}}$
- $\forall\, (t,x) \in \mathfrak{C}(a,b)^c \qquad \big|u(t,x)\big| \leqslant c^{(1)}_c(u_0,f)\, t^{-\mu} + c^{(2)}_c(u_0,f)\, t^{-1}$ .



 

*All the constants are given in the proof. The two finite real numbers $a < b$ are defined by*

$$a := \lim_{p \to -\infty} f'(p) \qquad , \qquad b := \lim_{p \to +\infty} f'(p) \, .$$

See Theorem 4.2.4 for the proof. Roughly speaking, an initial datum satisfies Condition $(C4_{\mu,\alpha,r})$ if and only if it satisfies Condition $(C3_{p_1,\mu})$ with additional hypotheses on the decay of $\mathcal{F}u_0$ at infinity. To illustrate this intrinsic concentration phenomenon, we consider the Klein-Gordon equation on the line. Since the solution formula is in fact a sum of the solutions of evolution equations of the type (8) with symbols $f_{KG}$ and $-f_{KG}$, where $f_{KG}$ is defined above, Theorem 0.7 is applicable to each term. In this case, we observe that the wave packets tend to be concentrated in the light cone issued by the origin, when the time tends to infinity. The appearance of this cone is closely related to the hyperbolic character of the Klein-Gordon equation.

Among all the applications of van der Corput type estimates, one can mention the possibility to derive $L^1$-$L^\infty$ estimates for solutions of the free Schrödinger equation (see [25]) and the Klein-Gordon equation (see [24]) on $\mathbb{R}^d$. Thanks to that, it is possible to establish Strichartz-type estimates and then to study non-linear variants of these equations.

We mention also the paper [2] in which a global (probably not optimal) $L^\infty$-time decay estimate for the Klein-Gordon equation on $\mathbb{R}$ with potential steps has been proved using the van der Corput inequality, spectral theory and the methods of [24].

Moreover $L^1$-$L^\infty$ estimates for more general evolution equations can also be established by using van der Corput type estimates. In [10], the authors consider a family of initial value problems given by (8), where the symbols are of the form $f(p) = |p|^\rho + R(p)$, with $\rho \geqslant 2$ and $R : \mathbb{R} \longrightarrow \mathbb{R}$ is a regular function whose growth at infinity is controlled in a certain sense by $|p|^{\rho-1}$. By establishing estimates of oscillatory integrals adapted to their problem, they derive the following inequality,

$$\big\| u(t,.) \big\|_{L^\infty(\mathbb{R})} \leqslant C \, t^{-\frac{1}{\rho}} \| u_0 \|_{L^1(\mathbb{R})} \, ,$$

for a certain constant $C > 0$. This is then employed to study smoothing effects as well as non-linear variants of the equation.
For comparison with the present chapter, let us remark that our method based on Fourier solution formulas does not furnish a $L^1$-$L^\infty$ estimate but it permits to derive spatial information on the solution. This type of result does not seem possible with the method employed in [10].

Finally let us mention the possibility to study the fractional Schrödinger equation, firstly introduced in [22]. In [21], the authors consider its one-dimensional version and furnish an $L^\infty$-estimate of the free solution for initial data belonging to an appropriate function space by using van der Corput lemma. This is then employed to study a non-linear variant of the equation.

***Chapter 5.*** In the last chapter of this thesis, we propose an approach in order to study time-asymptotic phenomena for approximate solutions of the solution of the Schrödinger equation on the line with potential,

$$\begin{cases} i\,\partial_t u(t) = -\partial_{xx}u(t) + V(x)u(t) \\ u(0) = u_0 \end{cases}, \tag{9}$$

where $t \geqslant 0$. First of all, we prove in Theorem 5.1.2 the well-posedness of this equation in $H^1(\mathbb{R})$ in the case $V \in W^{1,\infty}(\mathbb{R})$, by using the theory of semigroups. Moreover this theory affirms that the solution of the above perturbed equation can be represented as follows,

$$\forall\, t \geqslant 0 \qquad u(t) = \sum_{n \geqslant 0} S_n(t)u_0 \ , \tag{10}$$

where the sequence $\big(S_n(t)\big)_{n\geqslant 0}$ of bounded operators on $H^1(\mathbb{R})$ is defined in a recursive way. The series given in (10) converges in the $H^1$-norm for all $t \geqslant 0$ and is called *Dyson-Phillips series* for the solution of (9).

In order to avoid complicated calculations and to make clear the phenomena, we restrict in the present thesis the study to the two first terms of the above series. Let us note that the first term $S_0(t)u_0$ is the free wave packet while the second term $S_1(t)u_0$ corresponds to the wave packet resulting from the first interaction between the free solution and the potential.

To apply the methods introduced in the preceding chapters, we suppose that the Fourier transform of the potential is compactly supported; more precisely, we suppose that

$$supp\,\mathcal{F}V \subseteq [-b,-a] \cup [a,b] \ , \tag{11}$$

where $a < b$ are two finite real numbers. In particular, this type of hypothesis allows $\mathcal{F}V$ to be even and real-valued, implying in this case that $V$ is also a real-valued function. A family of potentials satisfying (11) is constructed for illustration. For potentials satisfying (11) together with initial data $u_0$ in frequency bands, we prove that the terms $S_0(t)u_0$ and $S_1(t)u_0$ are explicitly represented by oscillatory integrals and that the second term $S_1(t)u_0$ is actually a sum of two wave packets, that we call $S_1^-(t)u_0$ and $S_1^+(t)u_0$. Under some additional hypotheses on the position of the frequency band $[p_1,p_2]$ of $u_0$ with respect to the bands $[-b,-a]$ and $[a,b]$, we expand the terms $S_0(t)u_0$, $S_1^-(t)u_0$ and $S_1^+(t)u_0$ in appropriate space-time cones and we estimate them outside the cones by employing the methods of the preceding chapters. This shows that

- the optimal time-decay rate of the term $S_0(t)u_0$ in the space-time cone $\mathfrak{C}_S(p_1,p_2)$ is $t^{-\frac{1}{2}}$ and this term decays at least like $t^{-1}$ outside the cone ;
- the optimal time-decay rate of the term $S_1^-(t)u_0$ in the cone $\mathfrak{C}_S(p_1-b,p_2-a)$ is $t^{-\frac{1}{2}}$ and this term decays at least like $t^{-1}$ outside the cone ;
- the optimal time-decay rate of the term $S_1^+(t)u_0$ in the cone $\mathfrak{C}_S(p_1+a,p_2+b)$ is $t^{-\frac{1}{2}}$ and this term decays at least like $t^{-1}$ outside the cone.

See Theorems 5.3.4, 5.4.6 and 5.4.7 for complete statements with proof. Especially, we deduce that the terms $S_0(t)u_0$, $S_1^-(t)u_0$ and $S_1^+(t)u_0$ are time-asymptotically localized in the cones $\mathfrak{C}_S(p_1,p_2)$, $\mathfrak{C}_S(p_1-b,p_2-a)$ and $\mathfrak{C}_S(p_1+a,p_2+b)$ respectively.

By studying the inclination of the above space-time cones, we observe that the terms



 

$S_0(t)u_0$ and $S_1^+(t)u_0$ move always in the same direction and the maximal speed of $S_1^+(t)u_0$ is always larger than the maximal speed of $S_0(t)u_0$ when the time tends to infinity. On the other hand, the term $S_1^-(t)u_0$ travels either to the same direction in space as the one of $S_0(t)u_0$ with a minimal speed smaller than the minimal speed of $S_0(t)u_0$ if the absolute value of the initial speed is sufficiently large, or to the opposite direction otherwise. These effects can be interpreted as reflection and transmission type phenomena.

Note that in [7], a phenomenon of retarded reflection for the Klein-Gordon equation on the line with a potential constant but different on the positive and negative parts of the real axis has been detected. For signals in sufficiently narrow frequency bands below the threshold of the tunnel effect, it has been proved that the energy flow of the reflected part is retarded with respect to the incoming signal. The estimate is time-independent and thus carries no time-asymptotic information.

Further one could review other existing results on the Schrödinger equation with localized potential, e.g. [19] and [30]. An interesting issue could be to find optimal decay conditions on the potential, ensuring the decay rate $t^{-\frac{1}{2}}$.

Finally, the article [3] considers $L^\infty$-time decay estimates for the Schrödinger equation with sufficiently localized potential on a star-shaped network. The results contain statements on the Schrödinger equation on the real line as special cases.
For initial conditions with (sufficiently high) lower cut-off frequency, it is proved that the solution tends to the free solution if this cut-off frequency tends to infinity. These results have been obtained by an expansion of the solution in a Neumann-type series similar to the series considered in our Chapter 5.

To finish the introduction, let us mention some ideas for future works:

- It could be interesting to extend the abstract results of this thesis to the multidimensional case. For example, one could extend properly the notion of singular point in higher dimension and establish estimates depending explicitly on the position of the stationary point. This type of result might be applied to solution formulas of certain evolution equations on $\mathbb{R}^d$, where $d > 1$, to derive time-asymptotic information on the solutions.

- As explained previously, our methods give information on the inclination of the cones in space-time but not on their position. An interesting issue would be to find a method in order to localize the origin of a space-time cone, leading to a result comparable with Ehrenfest theorem in physics.

- In Chapter 5 of this thesis, we study only the two first terms of the Dyson-Phillips series representing the solution of the Schrödinger equation with potential. A natural possibility would be to study all the terms of the series in order to derive time-asymptotic information on the solution of the Schrödinger equation with potential.

- Another generalization of the results of Chapter 5 would be to consider potentials which are not in frequency bands. This setting would be more interpretable in terms of physics.

- Finally one could expect some applications to non-homogeneous and non-linear equations.



 

# Chapter 1

# Explicit error estimates for the stationary phase method in one variable

**Abstract**

In this chapter, we consider the version of the stationary phase method of Arthur Erdélyi [16, Section 2.9] for oscillatory integrals in one integration variable with respect to a large parameter. This version allows the amplitude to have integrable singular points and the phase to have stationary points of real order, and it gives explicit error estimates. The first aim of this chapter is to provide a complete proof of Erdélyi's result, since it was only sketched in the source. The second aim is to give a better estimate for the remainder term in the case of regular amplitudes, because the decay rate of the remainder given in the original paper in this case is not sufficiently fast as compared with those of the expansion.

## Contents

## 1.1 Erdélyi's expansion formula: complete proofs and slight improvements

Before formulating the results of this section, let us introduce the assumptions related to the phase and to the amplitude.

Let $p_1, p_2$ be two finite real numbers such that $p_1 < p_2$.

**Assumption $(\mathbf{P1}_{\rho_1,\rho_2,N})$.** Let $\rho_1, \rho_2 \geqslant 1$ and $N \in \mathbb{N}\backslash\{0\}$.
A function $\psi : [p_1, p_2] \longrightarrow \mathbb{R}$ satisfies Assumption $(\mathrm{P1}_{\rho_1,\rho_2,N})$ if and only if $\psi \in \mathcal{C}^1\big([p_1,p_2],\mathbb{R}\big)$ and there exists a function $\tilde{\psi} : [p_1, p_2] \longrightarrow \mathbb{R}$ such that

$$\forall\, p \in [p_1, p_2] \qquad \psi'(p) = (p-p_1)^{\rho_1 - 1}(p_2 - p)^{\rho_2 - 1}\tilde{\psi}(p) \ ,$$

where $\tilde{\psi} \in \mathcal{C}^N\big([p_1,p_2],\mathbb{R}\big)$ is assumed positive.
The points $p_j$ $(j = 1, 2)$ are called *stationary points* of $\psi$ of order $\rho_j - 1$, and $\tilde{\psi}$ the *non-degenerate factor* of $\psi$.

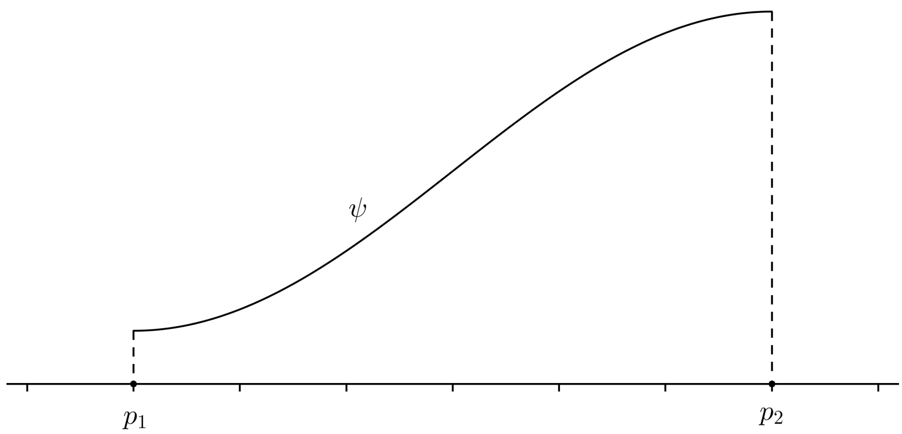


Figure 1.1: Function satisfying Assumption $(\mathrm{P1}_{\rho_1,\rho_2,N})$

**Assumption $(\mathbf{A1}_{\mu_1,\mu_2,N})$.** Let $\mu_1, \mu_2 \in (0, 1]$ and $N \in \mathbb{N}\backslash\{0\}$.
A function $U : (p_1, p_2) \longrightarrow \mathbb{C}$ satisfies Assumption $(\mathrm{A1}_{\mu_1,\mu_2,N})$ if and only if there exists a function $\tilde{u} : [p_1, p_2] \longrightarrow \mathbb{C}$ such that

$$\forall\, p \in (p_1, p_2) \qquad U(p) = (p-p_1)^{\mu_1 - 1}(p_2 - p)^{\mu_2 - 1}\tilde{u}(p) \ ,$$

where $\tilde{u} \in \mathcal{C}^N\big([p_1,p_2],\mathbb{C}\big)$, and $\tilde{u}(p_j) \neq 0$ if $\mu_j \neq 1$ $(j = 1, 2)$.
The points $p_j$ are called *singular points* of $U$, and $\tilde{u}$ the *regular factor* of $U$.

**1.1.1 Remark.** The hypothesis $\tilde{u}(p_j) \neq 0$ if $\mu_j \neq 1$ prevents the function $\tilde{u}$ from affecting the behaviour of the singular point $p_j$.



 

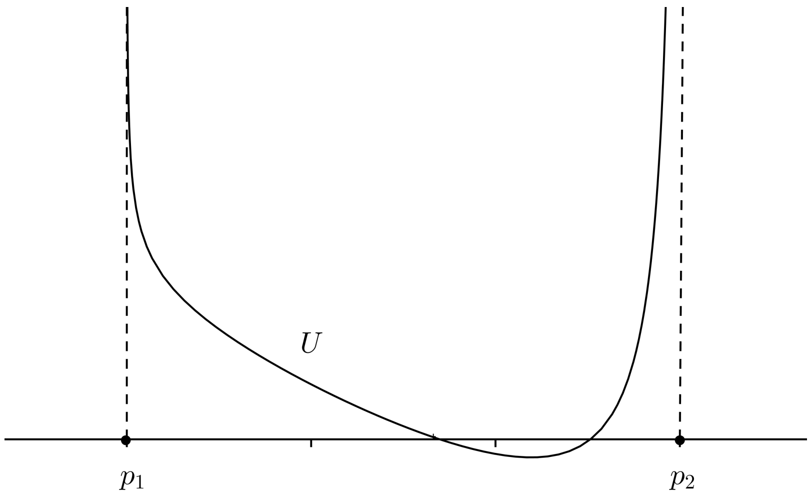


Figure 1.2: Function satisfying Assumption $(\mathrm{A1}_{\mu_1,\mu_2,N})$

## Non-vanishing singularities: Erdélyi's theorem

The aim of this subsection is to state Erdélyi's result [16, Section 2.9] and to provide a complete proof.

Let us define some objects that will be used throughout this chapter.

**1.1.2 Definition.** *Let $\psi : [p_1, p_2] \longrightarrow \mathbb{R}$ and $U : (p_1, p_2) \longrightarrow \mathbb{C}$ be two functions which satisfy the Assumptions ($P1_{\rho_1,\rho_2,N}$) and ($A1_{\mu_1,\mu_2,N}$) respectively.*

*i) Let $\eta \in \big(0, \frac{p_2-p_1}{2}\big)$. For $j = 1,2$, let $\varphi_j : I_j \longrightarrow \mathbb{R}$ be the functions defined by*

$$\varphi_1(p) := \big(\psi(p) - \psi(p_1)\big)^{\frac{1}{\rho_1}} \qquad , \qquad \varphi_2(p) := \big(\psi(p_2) - \psi(p)\big)^{\frac{1}{\rho_2}} ,$$

*with $I_1 := [p_1, p_2 - \eta]$, $I_2 := [p_1 + \eta, p_2]$ and $s_1 := \varphi_1(p_2 - \eta)$, $s_2 := \varphi_2(p_1 + \eta)$.*

*ii) For $j = 1,2$, let $k_j : (0, s_j] \longrightarrow \mathbb{C}$ be the functions defined by*

$$k_j(s) := U\big(\varphi_j^{-1}(s)\big)\, s^{1-\mu_j}\, \big(\varphi_j^{-1}\big)'(s) ,$$

*which can be extended to $[0, s_j]$ (see Proposition 1.2.2).*

*iii) Let $\nu : [p_1, p_2] \longrightarrow \mathbb{R}$ be a smooth function such that*

$$\begin{cases} \nu = 1 & on \quad [p_1, p_1 + \eta] , \\ \nu = 0 & on \quad [p_2 - \eta, p_2] , \\ 0 \leqslant \nu \leqslant 1 , \end{cases}$$

*where $\eta$ is defined above.*
*For $j = 1,2$, let $\nu_j : [0, s_j] \longrightarrow \mathbb{R}$ be the functions defined by*

$$\nu_1(s) := \nu \circ \varphi_1^{-1}(s) \qquad , \qquad \nu_2(s) := (1 - \nu) \circ \varphi_2^{-1}(s) .$$



 

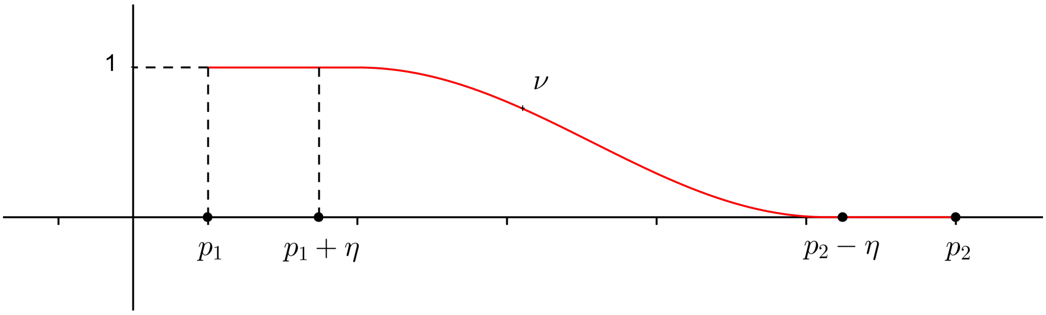


Figure 1.3: Graph of the function $\nu$

*iv) For $s>0$, let $\Lambda^{(j)}(s)$ be the complex curve defined by*

$$\Lambda^{(j)}(s) := \left\{ s + t\, e^{(-1)^{j+1} i \frac{\pi}{2\rho_j}} \;\middle|\; t \geqslant 0 \right\} .$$

**1.1.3 Theorem.** *Let $N \in \mathbb{N}\backslash\{0\}$, $\rho_1, \rho_2 \geqslant 1$ and $\mu_1, \mu_2 \in (0,1)$. Suppose that the functions $\psi : [p_1, p_2] \longrightarrow \mathbb{R}$ and $U : (p_1,p_2) \longrightarrow \mathbb{C}$ satisfy Assumption $(\mathrm{P1}_{\rho_1,\rho_2,N})$ and Assumption $(\mathrm{A1}_{\mu_1,\mu_2,N})$, respectively.*
*Then we have*

$$\begin{cases} \displaystyle\int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp = \sum_{j=1,2} \Big( A_N^{(j)}(\omega) + R_N^{(j)}(\omega) \Big) , \\ \displaystyle\Big| R_N^{(j)}(\omega) \Big| \leqslant \frac{1}{(N-1)!} \frac{1}{\rho_j} \Gamma\Big(\frac{N}{\rho_j}\Big) \int_0^{s_j} s^{\mu_j - 1} \left| \frac{d^N}{ds^N} \big[\nu_j k_j\big](s) \right| ds\, \omega^{-\frac{N}{\rho_j}} , \end{cases}$$

*for all $\omega > 0$, where for $j = 1,2$,*

- $\displaystyle A_N^{(j)}(\omega) := e^{i\omega\psi(p_j)} \sum_{n=0}^{N-1} \Theta_{n+1}^{(j)}(\rho_j, \mu_j) \frac{d^n}{ds^n} \big[k_j\big](0)\, \omega^{-\frac{n+\mu_j}{\rho_j}}$ ,
- $\displaystyle R_N^{(j)}(\omega) := (-1)^{N+1+j} e^{i\omega\psi(p_j)} \int_0^{s_j} \phi_N^{(j)}(s, \omega, \rho_j, \mu_j) \frac{d^N}{ds^N} \big[\nu_j k_j\big](s)\, ds$ ,

*and for $n = 0, ..., N-1$,*

- $\displaystyle \Theta_{n+1}^{(j)}(\rho_j, \mu_j) := \frac{(-1)^{j+1}}{n!\,\rho_j} \Gamma\Big(\frac{n+\mu_j}{\rho_j}\Big) e^{(-1)^{j+1} i \frac{\pi}{2} \frac{n+\mu_j}{\rho_j}}$ ,
- $\displaystyle \phi_{n+1}^{(j)}(s, \omega, \rho_j, \mu_j) := \frac{(-1)^{n+1}}{n!} \int_{\Lambda^{(j)}(s)} (z-s)^n z^{\mu_j - 1} e^{(-1)^{j+1} i\omega z^{\rho_j}} dz$ .

*Proof.* For fixed $\omega > 0$, $\rho_j \geqslant 1$ and $\mu_j \in (0,1)$, we shall note $\phi_n^{(j)}(s,\omega)$ instead of $\phi_n^{(j)}(s,\omega,\rho_j,\mu_j)$ in favour of readability. Now let us divide the proof in five steps.

*First step: Splitting of the integral.* Using the cut-off function $\nu$, we can write the integral as follows,

$$\int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp = \tilde{I}^{(1)}(\omega) + \tilde{I}^{(2)}(\omega) ,$$

where

$$\tilde{I}^{(1)}(\omega) := \int_{p_1}^{p_2-\eta} \nu(p)\, U(p)\, e^{i\omega\psi(p)}\, dp \quad , \quad \tilde{I}^{(2)}(\omega) := \int_{p_1+\eta}^{p_2} \big(1-\nu(p)\big)\, U(p)\, e^{i\omega\psi(p)}\, dp \,.$$

*Second step: Substitution.* Proposition 1.2.1 affirms that $\varphi_j : I_j \longrightarrow [0, s_j]$ is a $\mathcal{C}^{N+1}$-diffeomorphism with $q_1 = p_2 - \eta$ here. Using the substitution $s = \varphi_1(p)$, we obtain

$$\begin{aligned}
\tilde{I}^{(1)}(\omega) &= \int_{p_1}^{p_2-\eta} \nu(p)\, U(p)\, e^{i\omega\psi(p)}\, dp \\
&= e^{i\omega\psi(p_1)} \int_0^{s_1} \nu\big(\varphi_1^{-1}(s)\big)\, U\big(\varphi_1^{-1}(s)\big)\, e^{i\omega s^{\rho_1}} (\varphi_1^{-1})'(s)\, ds \\
&= e^{i\omega\psi(p_1)} \int_0^{s_1} \nu\big(\varphi_1^{-1}(s)\big)\, U\big(\varphi_1^{-1}(s)\big) s^{1-\mu_1} (\varphi_1^{-1})'(s)\, s^{\mu_1-1} e^{i\omega s^{\rho_1}}\, ds \\
&= e^{i\omega\psi(p_1)} \int_0^{s_1} \nu_1(s)\, k_1(s)\, s^{\mu_1-1} e^{i\omega s^{\rho_1}}\, ds \,,
\end{aligned}$$

where $k_1$ and $\nu_1$ are introduced in Definition 1.1.2. In a similar way, we obtain

$$\tilde{I}^{(2)}(\omega) = -e^{i\omega\psi(p_2)} \int_0^{s_2} \nu_2(s)\, k_2(s)\, s^{\mu_2-1} e^{-i\omega s^{\rho_2}}\, ds \,.$$

Note that the minus sign comes from the fact that $\varphi_2$ is a decreasing function.

*Third step: Integrations by parts.* Corollary 1.2.6 provides successive primitives of the function $s \longmapsto s^{\mu_j-1} e^{(-1)^{j+1} i\omega s^{\rho_j}}$. Moreover Proposition 1.2.2 ensures that $k_j \in \mathcal{C}^N\big([0, s_j], \mathbb{C}\big)$. Thus by $N$ integrations by parts, we obtain

$$\begin{aligned}
e^{-i\omega\psi(p_1)}\, \tilde{I}^{(1)}(\omega) &= \int_0^{s_1} \nu_1(s)\, k_1(s)\, s^{\mu_1-1} e^{i\omega s^{\rho_1}}\, ds \\
&= \Big[\phi_1^{(1)}(s,\omega) \big(\nu_1 k_1\big)(s)\Big]_0^{s_1} - \int_0^{s_1} \phi_1^{(1)}(s,\omega)\, \frac{d}{ds}\big[\nu_1 k_1\big](s)\, ds \\
&= \dots \\
&= \sum_{n=0}^{N-1} (-1)^n \left[\phi_{n+1}^{(1)}(s,\omega)\, \frac{d^n}{ds^n}\big[\nu_1 k_1\big](s)\right]_0^{s_1} \\
&\qquad + (-1)^N \int_0^{s_1} \phi_N^{(1)}(s,\omega)\, \frac{d^N}{ds^N}\big[\nu_1 k_1\big](s)\, ds \,.
\end{aligned} \tag{1.1}$$

Let us simplify the sum given in (1.1) by using the properties of the function $\nu_1$: by hypothesis, $\nu(p_1) = 1$, $\nu(p_2 - \eta) = 0$ and $\dfrac{d^n}{dp^n}[\nu](p_1) = \dfrac{d^n}{dp^n}[\nu](p_2 - \eta) = 0$, for $n \geqslant 1$. So the definition of $\nu_1$ implies

$$\nu_1(0) = \nu(p_1) = 1 \qquad , \qquad \nu_1(s_1) = \nu(p_2 - \eta) = 0 \,;$$

and by the product rule applied to $\nu_1 k_1$, it follows

$$\frac{d^n}{ds^n}\big[\nu_1 k_1\big](0) = \frac{d^n}{ds^n}[k_1](0) \qquad , \qquad \frac{d^n}{ds^n}\big[\nu_1 k_1\big](s_1) = 0 \,.$$



 

This leads to

$$\tilde{I}^{(1)}(\omega) = \sum_{n=0}^{N-1} (-1)^{n+1}\, \phi_{n+1}^{(1)}(0,\omega)\, \frac{d^n}{ds^n}\big[k_1\big](0)\, e^{i\omega\psi(p_1)} \\ + (-1)^N\, e^{i\omega\psi(p_1)} \int_0^{s_1} \phi_N^{(1)}(s,\omega)\, \frac{d^N}{ds^N}\big[\nu_1 k_1\big](s)\, ds\ .$$

By similar computations, we obtain

$$\tilde{I}^{(2)}(\omega) = \sum_{n=0}^{N-1} (-1)^{n}\, \phi_{n+1}^{(2)}(0,\omega)\, \frac{d^n}{ds^n}\big[k_2\big](0)\, e^{i\omega\psi(p_2)} \\ + (-1)^{N+1}\, e^{i\omega\psi(p_2)} \int_0^{s_2} \phi_N^{(2)}(s,\omega)\, \frac{d^N}{ds^N}\big[\nu_2 k_2\big](s)\, ds\ .$$

*Fourth step: Calculation of the main terms.* Let us compute the coefficients $\phi_{n+1}^{(j)}(0,\omega)$, for $j=1,2$. From Corollary 1.2.6, we recall that we have

$$\phi_{n+1}^{(j)}(s,\omega) = \frac{(-1)^{n+1}}{n!} \int_{\Lambda^{(j)}(s)} (z-s)^n\, z^{\mu_j - 1}\, e^{(-1)^{j+1} i\omega z^{\rho_j}}\, dz\ ,$$

for all $s \in [0, s_j]$ and $n = 0, ..., N-1$. Choosing $j=1$, putting $s=0$ and parametrizing the curve $\Lambda^{(1)}(0)$ with $z = t\, e^{i\frac{\pi}{2\rho_1}}$ lead to

$$\phi_{n+1}^{(1)}(0,\omega) = \frac{(-1)^{n+1}}{n!}\, e^{i\frac{\pi}{2}\,\frac{n+\mu_1}{\rho_1}} \int_0^{+\infty} t^{n+\mu_1-1}\, e^{-\omega t^{\rho_1}}\, dt\ .$$

Setting $y = \omega\, t^{\rho_1}$ in the previous integral gives

$$\begin{aligned}\phi_{n+1}^{(1)}(0,\omega) &= \frac{(-1)^{n+1}}{n!}\, e^{i\frac{\pi}{2}\,\frac{n+\mu_1}{\rho_1}}\, (\rho_1\,\omega)^{-1} \int_0^{+\infty} \left(\frac{y}{\omega}\right)^{\frac{n+\mu_1}{\rho_1}-1} e^{-y}\, dy \\ &= \frac{(-1)^{n+1}}{n!}\, e^{i\frac{\pi}{2}\,\frac{n+\mu_1}{\rho_1}}\, \frac{1}{\rho_1}\, \Gamma\!\left(\frac{n+\mu_1}{\rho_1}\right) \omega^{-\frac{n+\mu_1}{\rho_1}}\ ,\end{aligned}$$

where $\Gamma$ is the Gamma function defined by

$$\Gamma\ :\ z \in \big\{ z \in \mathbb{C} \,\big|\, \Re(z) > 0 \big\} \longmapsto \int_0^{+\infty} t^{z-1} e^{-t}\, dt \in \mathbb{C}\ .$$

A similar work provides

$$\phi_{n+1}^{(2)}(0,\omega) = \frac{(-1)^{n+1}}{n!}\, e^{-i\frac{\pi}{2}\,\frac{n+\mu_2}{\rho_2}}\, \frac{1}{\rho_2}\, \Gamma\!\left(\frac{n+\mu_2}{\rho_2}\right) \omega^{-\frac{n+\mu_2}{\rho_2}}\ .$$

Then we obtain

$$e^{i\omega\psi(p_j)} \sum_{n=0}^{N-1} (-1)^{n+j}\, \phi_{n+1}^{(j)}(0,\omega)\, \frac{d^n}{ds^n}\big[k_j\big](0) = e^{i\omega\psi(p_j)} \sum_{n=0}^{N-1} \Theta_{n+1}^{(j)}(\rho_j,\mu_j)\, \frac{d^n}{ds^n}\big[k_j\big](0)\, \omega^{-\frac{n+\mu_j}{\rho_j}}\ ,$$



 

where $\Theta_{n+1}^{(j)}(\rho_j, \mu_j) := \dfrac{(-1)^{j+1}}{n!\,\rho_j}\,\Gamma\!\left(\dfrac{n+\mu_j}{\rho_j}\right)\, e^{(-1)^{j+1}\frac{\pi}{2}\,\frac{n+\mu_j}{\rho_j}}$.

*Fifth step: Remainder estimates.* The last step consists in estimating the remainders $R_N^{(j)}(\omega)$. For $j = 1, 2$, we have for all $s \in (0, s_j]$ and for all $t \geqslant 0$,

$$s \leqslant \left| s + te^{(-1)^{j+1} i \frac{\pi}{2\rho_j}} \right| \qquad \Longrightarrow \qquad s^{\mu_j - 1} \geqslant \left| s + te^{(-1)^{j+1} i \frac{\pi}{2\rho_j}} \right|^{\mu_j - 1} , \tag{1.2}$$

since $\mu_j \in (0, 1)$. Now we parametrize the path of integration of the integral defining $\phi_N^{(j)}(s, \omega)$ by

$$z = s + t\, e^{(-1)^{j+1} i \frac{\pi}{2\rho_j}} ,$$

for $t \geqslant 0$, and we employ the previous inequality (1.2) to obtain

$$\begin{aligned}
\left|\phi_N^{(j)}(s,\omega)\right| &\leqslant \frac{1}{(N-1)!} \int_0^{+\infty} t^{N-1} \left| s + te^{(-1)^{j+1} i \frac{\pi}{2\rho_j}} \right|^{\mu_j - 1} \left| e^{(-1)^{j+1} i\omega \left(s + te^{(-1)^{j+1} i \frac{\pi}{2\rho_j}}\right)^{\rho_j}} \right| dt \\
&\leqslant \frac{1}{(N-1)!}\, s^{\mu_j - 1} \int_0^{+\infty} t^{N-1} e^{-\omega t^{\rho_j}}\, dt \\
&= \frac{1}{(N-1)!}\, s^{\mu_j - 1}\, \frac{1}{\rho_j}\, \Gamma\!\left(\frac{N}{\rho_j}\right) \omega^{-\frac{N}{\rho_j}} ,
\end{aligned} \tag{1.3}$$

where the last equality has been obtained by using the substitution $y = \omega\, t^{\rho_j}$. Employing the definition of $R_N^{(j)}(\omega)$ and inequality (1.3) leads to

$$\begin{aligned}
\left| R_N^{(j)}(\omega) \right| &\leqslant \int_0^{s_j} \left| \phi_N^{(j)}(s,\omega) \right| \left| \frac{d^N}{ds^N} \big[\nu_j k_j\big](s) \right| ds \\
&\leqslant \frac{1}{(N-1)!}\, \frac{1}{\rho_j}\, \Gamma\!\left(\frac{N}{\rho_j}\right) \int_0^{s_j} s^{\mu_j - 1} \left| \frac{d^N}{ds^N} \big[\nu_j k_j\big](s) \right| ds\, \omega^{-\frac{N}{\rho_j}} .
\end{aligned}$$

We note that the last integral is well-defined because $\dfrac{d^N}{ds^N}\big[\nu_j k_j\big] : [0, s_j] \longrightarrow \mathbb{R}$ is continuous and $s \longmapsto s^{\mu_j - 1}$ is locally integrable on $[0, s_j]$.

Finally, we remark that the highest term of the expansion $A_N^{(j)}(\omega)$ behaves like $\omega^{-\frac{N-1+\mu_j}{\rho_j}}$ when $\omega$ tends to infinity. Moreover $R_N^{(j)}(\omega)$ is estimated by $\omega^{-\frac{N}{\rho_j}}$. This implies that the decay rate of the remainder with respect to $\omega$ is faster than the one of the highest term of the expansion. This ends the proof. □

## Amplitudes without singularities: refinement of the error estimate

The preceding theorem remains true if we suppose $\mu_j = 1$, that is to say if the amplitude $U$ is regular at the point $p_j$. But in this case, we observe that the decay rates of the highest term of the expansion related to $p_j$ and of the remainder related to $p_j$ are the same, namely $\omega^{-\frac{N}{\rho_j}}$. Hence the aim of this subsection is to refine the estimate of the remainder in this specific case.

For this purpose, we establish the two following lemmas. In the first one, we provide two estimates of the function $s \longmapsto \phi_N^{(j)}(s,\omega,\rho_j,1)$: the first estimate is uniform with respect to $s$ but the decay with respect to $\omega$ is not sufficiently fast; on the other hand, the second one provides a better decay with respect to $\omega$ but is singular with respect to $s$. The first estimate is actually established in the proof of Theorem 1.1.3 and we carry out integrations by parts to establish the second one.

**1.1.4 Lemma.** *Let $j \in \{1,2\}$, $\rho_j \geqslant 1$ and $N \in \mathbb{N}\backslash\{0\}$. Then for all $s, \omega > 0$, we have*

$$\begin{cases} \left|\phi_N^{(j)}(s,\omega,\rho_j,1)\right| \leqslant a_{N,\rho_j}\, \omega^{-\frac{N}{\rho_j}} \;, \\ \left|\phi_N^{(j)}(s,\omega,\rho_j,1)\right| \leqslant b_{N,\rho_j}\, \omega^{-\left(1+\frac{N-1}{\rho_j}\right)}\, s^{1-\rho_j} + c_{N,\rho_j}\, \omega^{-\left(1+\frac{N}{\rho}\right)}\, s^{-\rho_j} \;, \end{cases}$$

*where the constants $a_{N,\rho_j}, b_{N,\rho_j}, c_{N,\rho_j} > 0$ are given in the proof.*

**1.1.5 Remark.** Note that we can extend $\phi_N^{(j)}(.,\omega,\rho_j,1) : [0,s_j] \longrightarrow \mathbb{R}$ to $[0,+\infty)$, according to Remark 1.2.7.

*Proof of Lemma 1.1.4.* Let us fix $s > 0$, $\omega > 0$ and let us choose $j = 1$. We recall the expression of $\phi_N^{(1)}(s,\omega,\rho_1,1)$ with the parametrization of the path $\Lambda^{(1)}(s)$ given in Definition 1.1.2:

$$\phi_N^{(1)}(s,\omega,\rho_1,1) = \frac{(-1)^N}{(N-1)!}\int_0^{+\infty} t^{N-1}\, e^{i\frac{\pi(N-1)}{2\rho_1}}\, e^{i\omega\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}}\, dt\, e^{i\frac{\pi}{2\rho_1}} \;.$$

On the one hand, estimate (1.3) is still valid for $\mu_1 = 1$, namely,

$$\left|\phi_N^{(1)}(s,\omega,\rho_1,1)\right| \leqslant \frac{1}{(N-1)!}\,\frac{1}{\rho_1}\,\Gamma\!\left(\frac{N}{\rho_1}\right)\omega^{-\frac{N}{\rho_1}} =: a_{N,\rho_1}\,\omega^{-\frac{N}{\rho_1}} \;,$$

furnishing the first estimate of the lemma.
On the other hand, we establish the second inequality by using integrations by parts. To do so, we remark that for all $s > 0$ the first derivative of the function

$$t \in (0,+\infty) \longmapsto i\,\omega\left(s + t\, e^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}$$

does not vanish on its domain; therefore we can write

$$e^{i\omega\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}} = (i\omega\rho_1)^{-1}\, e^{-i\frac{\pi}{2\rho_1}}\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{1-\rho_1}\frac{d}{dt}\left[e^{i\omega\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}}\right] \;.$$

Moreover Lemma 1.2.3 implies

$$\forall\, s > 0 \qquad \left|e^{i\omega\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}}\right| \leqslant e^{-\omega t^{\rho_1}} \longrightarrow 0 \quad , \quad t \longrightarrow +\infty \;. \tag{1.4}$$

Now we distinguish the two following cases:



 

- *Case* $N = 1$. Thanks to the two previous observations, we can integrate by parts, providing

$$\begin{aligned}
\phi_1^{(1)}(s,\omega,\rho_1,1) &= -(i\omega\rho_1)^{-1}\int_0^{+\infty}\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{1-\rho_1}\frac{d}{dt}\left[e^{i\omega\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}}\right]dt \\
&= (i\omega\rho_1)^{-1}s^{1-\rho_1}e^{i\omega s^{\rho_1}} \\
&\quad + \frac{1-\rho_1}{i\omega\rho_1}\,e^{i\frac{\pi}{2\rho_1}}\int_0^{+\infty}\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{-\rho_1}e^{i\omega\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}}dt\,,
\end{aligned}$$

where the boundary term at infinity is equal to zero according to (1.4). It follows

$$\begin{aligned}
\left|\phi_1^{(1)}(s,\omega,\rho_1,1)\right| &\leqslant (\omega\rho_1)^{-1}\,s^{1-\rho_1} \\
&\quad + \frac{\rho_1-1}{\omega\rho_1}\int_0^{+\infty}\left|s+te^{i\frac{\pi}{2\rho_1}}\right|^{-\rho_1}\left|e^{i\omega\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}}\right|ds \\
&\leqslant (\omega\rho_1)^{-1}\,s^{1-\rho_1} + \frac{\rho_1-1}{\omega\rho_1}\,s^{-\rho_1}\int_0^{+\infty}e^{-\omega t^{\rho_1}}\,dt \qquad (1.5) \\
&= \frac{1}{\rho_1}\,\omega^{-1}\,s^{1-\rho_1} + \frac{\rho_1-1}{\rho_1^2}\,\Gamma\left(\frac{1}{\rho_1}\right)\omega^{-\left(1+\frac{1}{\rho_1}\right)}\,s^{-\rho_1} \qquad (1.6) \\
&=: b_{1,\rho_1}\,\omega^{-1}\,s^{1-\rho_1} + c_{1,\rho_1}\,\omega^{-\left(1+\frac{1}{\rho_1}\right)}\,s^{-\rho_1}\,.
\end{aligned}$$

Lemma 1.2.3 permits to obtain (1.5) by giving an estimate of the complex exponential and we use the substitution $y = \omega\,t^{\rho_1}$ to get (1.6).

- *Case* $N \geqslant 2$. We proceed as above by using an integration by parts:

$$\begin{aligned}
\phi_N^{(1)}(s,\omega,\rho_1,1) &= \frac{(-1)^N}{(N-1)!}\,e^{i\frac{\pi(N-1)}{2\rho_1}}\,(i\omega\rho_1)^{-1} \\
&\quad\times\int_0^{+\infty}t^{N-1}\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{1-\rho_1}\frac{d}{dt}\left[e^{i\omega\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}}\right]dt \\
&= \frac{(-1)^{N+1}}{(N-1)!}\,e^{i\frac{\pi(N-1)}{2\rho_1}}\,(i\omega\rho_1)^{-1} \\
&\quad\times\int_0^{+\infty}\frac{d}{dt}\left[t^{N-1}\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{1-\rho_1}\right]e^{i\omega\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}}dt \qquad (1.7) \\
&= \frac{(-1)^{N+1}}{(N-1)!}\,e^{i\frac{\pi(N-1)}{2\rho_1}}\,(i\omega\rho_1)^{-1} \\
&\quad\times\Bigg((N-1)\int_0^{+\infty}t^{N-2}\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{1-\rho_1}e^{i\omega\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}}dt \\
&\quad + (1-\rho_1)\,e^{i\frac{\pi}{2\rho_1}}\int_0^{+\infty}t^{N-1}\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{-\rho_1}e^{i\omega\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}}dt\Bigg)\,.
\end{aligned}$$

The boundary terms in (1.7) are equal to zero; indeed we remark that the term at 0 vanishes and we use (1.4) once again to show that the term at infinity is equal to



 

0. Then by similar arguments to those of the preceding case, we obtain

$$\begin{aligned}
\left|\phi_N^{(1)}(s,\omega,\rho_1,1)\right| &\leqslant \frac{\omega^{-1}}{\rho_1(N-2)!}\int_0^{+\infty} t^{N-2}\left|\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{1-\rho_1} e^{i\omega\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}}\right| dt \\
&\quad + \frac{(\rho_1-1)\,\omega^{-1}}{\rho_1(N-1)!}\int_0^{+\infty} t^{N-1}\left|\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{-\rho_1} e^{i\omega\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}}\right| dt \\
&\leqslant \frac{1}{\rho_1(N-2)!}\,\omega^{-1}\,s^{1-\rho_1}\int_0^{+\infty} t^{N-2}\,e^{-\omega t^{\rho_1}}\,dt \\
&\quad + \frac{(\rho_1-1)}{\rho_1(N-1)!}\,\omega^{-1}\,s^{-\rho_1}\int_0^{+\infty} t^{N-1}\,e^{-\omega t^{\rho_1}}\,dt \\
&= \frac{1}{\rho_1^2(N-2)!}\,\Gamma\left(\frac{N-1}{\rho_1}\right)\omega^{-\left(1+\frac{N-1}{\rho_1}\right)}\,s^{1-\rho_1} \\
&\quad + \frac{(\rho_1-1)}{\rho_1^2(N-1)!}\,\Gamma\left(\frac{N}{\rho_1}\right)\omega^{-\left(1+\frac{N}{\rho_1}\right)}\,s^{-\rho_1} \\
&=: b_{N,\rho_1}\,\omega^{-\left(1+\frac{N-1}{\rho_1}\right)}\,s^{1-\rho_1} + c_{N,\rho_1}\,\omega^{-\left(1+\frac{N}{\rho_1}\right)}\,s^{-\rho_1}\ ,
\end{aligned}$$

concluding this point.

A very similar work for $j=2$ provides the conclusion; it is sufficient to replace $\rho_1$ by $\rho_2$ in the expressions of the constants $a_{N,\rho_1}$, $b_{N,\rho_1}$ and $c_{N,\rho_1}$ to obtain $a_{N,\rho_2}$, $b_{N,\rho_2}$ and $c_{N,\rho_2}$. ☐

Given a function satisfying a system of inequalities similar to the one given in Lemma 1.1.4, a new estimate for this function is established by exploiting the balance between blow-up and decay. Note that a technical argument requires $\rho \geqslant 2$.

**1.1.6 Lemma.** *Let $N \in \mathbb{N}\backslash\{0\}$, $\rho \geqslant 2$ and $f : (0,+\infty)\times(0,+\infty) \longrightarrow \mathbb{R}$ be a function which satisfies the following inequalities:*

$$\forall\, s,\omega > 0 \qquad \begin{cases} \left|f(s,\omega)\right| \leqslant a\,\omega^{-\frac{N}{\rho}}\ , \\ \left|f(s,\omega)\right| \leqslant b\,\omega^{-\left(1+\frac{N-1}{\rho}\right)}\,s^{1-\rho} + c\,\omega^{-\left(1+\frac{N}{\rho}\right)}\,s^{-\rho}\ , \end{cases}$$

*where $a,b,c>0$ are constants.*
*Fix $\gamma \in (0,1)$ and define $\delta := \dfrac{\gamma+N}{\rho} \in \left(\dfrac{N}{\rho}, \dfrac{1+N}{\rho}\right)$. Then we have*

$$\forall\, s,\omega > 0 \qquad \left|f(s,\omega)\right| \leqslant L_{\gamma,\rho}\,s^{-\gamma}\,\omega^{-\delta}\ ,$$

*where $L_{\gamma,\rho} := a\,K_\rho^{\gamma} > 0$, with $K_\rho$ the unique positive solution of*

$$aK^\rho - bK - c = 0\ .$$

*Proof.* Let $g_1, g_2 : (0,+\infty)\times(0,+\infty) \longrightarrow \mathbb{R}$ be the functions defined by

$$g_1(s,\omega) := a\,\omega^{-\frac{N}{\rho}} \qquad , \qquad g_2(s,\omega) := b\,\omega^{-\left(1+\frac{N-1}{\rho}\right)}\,s^{1-\rho} + c\,\omega^{-\left(1+\frac{N}{\rho}\right)}\,s^{-\rho}\ .$$



 

Now we fix $\omega > 0$ and we define the function $h_\omega : (0, +\infty) \longrightarrow \mathbb{R}$ by

$$h_\omega(s) := s^\rho \big(g_1(s,\omega) - g_2(s,\omega)\big) = a\,\omega^{-\frac{N}{\rho}}\, s^\rho - b\,\omega^{-\left(1+\frac{N-1}{\rho}\right)}\, s - c\,\omega^{-\left(1+\frac{N}{\rho}\right)} .$$

Then the point $s_\omega := K_\rho\, \omega^{-\frac{1}{\rho}}$, where $K_\rho$ is the unique positive solution of the equation $aK^\rho - bK - c = 0$, is the unique positive solution of the equation $h_\omega(s) = 0$. So $g_1(.,\omega)$ and $g_2(.,\omega)$ intersect each other at the point $s_\omega$ and we have $g_1(.,\omega) \leqslant g_2(.,\omega)$ for $s \in (0, s_\omega]$ and $g_1(.,\omega) \geqslant g_2(.,\omega)$ otherwise.

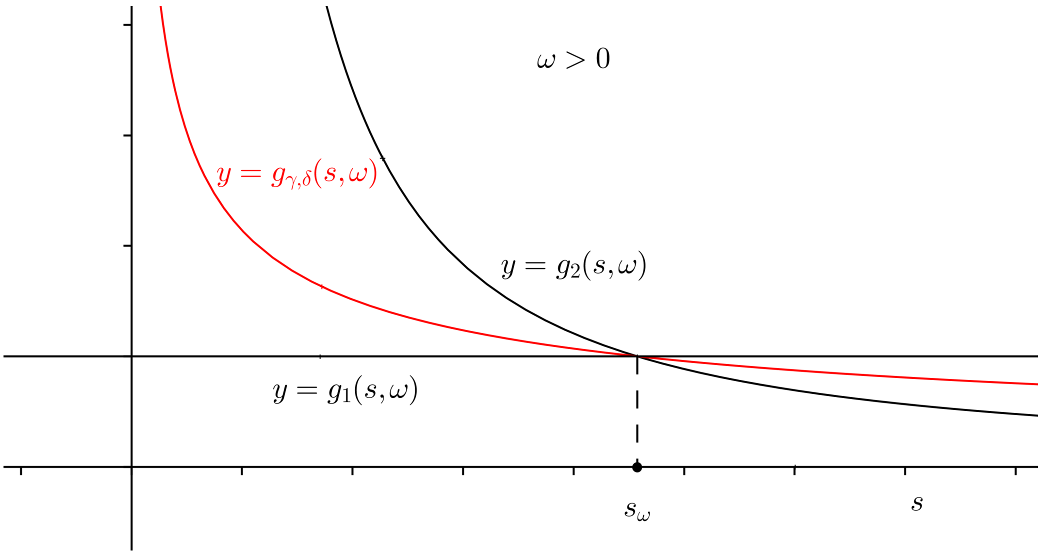


Figure 1.4: The functions $g_1(.,\omega)$, $g_2(.,\omega)$ and $g_{\gamma,\delta}(.,\omega)$

Hence we obtain more precise estimates:

$$\begin{cases} \forall\, s \in (0, s_\omega] & \big|f(s,\omega)\big| \leqslant a\,\omega^{-\frac{N}{\rho}} = g_1(s,\omega)\ , \\ \forall\, s \in [s_\omega, +\infty) & \big|f(s,\omega)\big| \leqslant b\,\omega^{-\left(1+\frac{N-1}{\rho}\right)}\, s^{1-\rho} + c\,\omega^{-\left(1+\frac{N}{\rho}\right)}\, s^{-\rho} = g_2(s,\omega)\ . \end{cases}$$

Now we seek a function $g : (0,+\infty) \times (0,+\infty) \longrightarrow \mathbb{R}$ which is locally integrable with respect to the variable $s$ and which satisfies the following inequalities for any $\omega > 0$:

$$\begin{cases} \forall\, s \in (0, s_\omega] & \big|f(s,\omega)\big| \leqslant g_1(s,\omega) \leqslant g(s,\omega)\ , \\ \forall\, s \in [s_\omega, +\infty) & \big|f(s,\omega)\big| \leqslant g_2(s,\omega) \leqslant g(s,\omega)\ , \end{cases} \tag{1.8}$$

Here we propose $g_{\gamma,\delta}(s,\omega) := L_{\gamma,\rho}\, s^{-\gamma}\omega^{-\delta}$, where $L_{\gamma,\rho}$, $\delta$, $\gamma > 0$ must be clarified. To this end, we require the following condition:

$$\forall\, \omega > 0 \qquad g_{\gamma,\delta}(s_\omega,\omega) = g_1(s_\omega,\omega) = g_2(s_\omega,\omega)\ ,$$

leading to

$$g_{\gamma,\delta}\left(K_\rho\, \omega^{-\frac{1}{\rho}}, \omega\right) = L_{\gamma,\rho}\, K_\rho^{-\gamma}\, \omega^{\frac{\gamma}{\rho}-\delta} = a\,\omega^{-\frac{N}{\rho}}\ .$$

Since this equality holds for all $\omega > 0$, we have

$$\begin{cases} L_{\gamma,\rho} = a\,K_\rho^{\gamma} \\ \dfrac{\gamma}{\rho} - \delta = -\dfrac{N}{\rho} \end{cases} \Longrightarrow \begin{cases} L_{\gamma,\rho} = a\,K_\rho^{\gamma} \\ \delta = \rho^{-1}(\gamma + N) \end{cases} .$$



 

Here we choose $\gamma \in (0,1)$ so that $g_{\gamma,\delta}(.,\omega) : (0,+\infty) \longrightarrow \mathbb{R}$ is locally integrable with respect to $s$; it follows $\delta = \rho^{-1}(\gamma + N) \in \left( \dfrac{N}{\rho}, \dfrac{1+N}{\rho} \right)$. To conclude, we have to check the system of inequalities (1.8):

- *Case* $s \leqslant s_\omega$. We have

$$g_{\gamma,\delta}(s,\omega) = a\, K_\rho^{\,\gamma}\, \omega^{-\delta}\, s^{-\gamma} \geqslant a\, K_\rho^{\,\gamma}\, \omega^{-\delta} \left( K_\rho\, \omega^{-\frac{1}{\rho}} \right)^{-\gamma} = a\, \omega^{\frac{\gamma}{\rho}-\delta} = a\, \omega^{-\frac{N}{\rho}} = g_1(s,\omega) \ ,$$

  since $\dfrac{\gamma}{\rho} - \delta = -\dfrac{N}{\rho}$.

- *Case* $s \geqslant s_\omega$. Here, we want to show that $g_2(s,\omega) \leqslant g_{\gamma,\delta}(s,\omega)$, which is equivalent to

$$s^\rho \big( g_{\gamma,\delta}(s,\omega) - g_2(s,\omega) \big) = a\, K_\rho^{\,\gamma}\, \omega^{-\delta}\, s^{\rho-\gamma} - b\, \omega^{-\left(1+\frac{N-1}{\rho}\right)}\, s - c\, \omega^{-\left(1+\frac{N}{\rho}\right)} \geqslant 0 \ . \quad (1.9)$$

  We define the function $k_\omega : (0,+\infty) \longrightarrow \mathbb{R}$ by $k_\omega(s) := s^\rho \big( g_{\gamma,\delta}(s,\omega) - g_2(s,\omega) \big)$, and we differentiate it,

$$(k_\omega)'(s) = a\, K_\rho^{\,\gamma} (\rho - \gamma)\, \omega^{-\delta}\, s^{\rho-\gamma-1} - b\, \omega^{-\left(1+\frac{N-1}{\rho}\right)} \ .$$

  Since $s > 0$ and $\rho \geqslant 2$, $(k_\omega)'$ is an increasing function, vanishing at the point

$$s'_\omega = \left( \frac{b}{a\, K_\rho^{\,\gamma} (\rho - \gamma)} \right)^{\frac{1}{\rho-\gamma-1}} \omega^{-\frac{1}{\rho}} \ .$$

  Now we want to show the inequality: $s'_\omega \leqslant s_\omega$. Since $\rho \geqslant 2$, we have $\rho - \gamma - 1 > 0$ and so

$$\begin{aligned} & 0 \leqslant b\, K_\rho\, (\rho - \gamma - 1) + (\rho - \gamma)\, c \\ \Longleftrightarrow \quad & \frac{bK_\rho}{\rho - \gamma} \leqslant bK_\rho + c = a\, K_\rho^{\,\rho} \\ \Longleftrightarrow \quad & \frac{b}{a\, K_\rho^{\,\gamma} (\rho - \gamma)} \leqslant K_\rho^{\,\rho-\gamma-1} \ , \end{aligned} \quad (1.10)$$

  where the equality in (1.10) comes from the fact that $K_\rho$ satisfies $a\, K_\rho^{\,\rho} + bK_\rho + c = 0$. It follows that

$$s'_\omega = \left( \frac{b}{a\, K_\rho^{\,\gamma} (\rho - \gamma)} \right)^{\frac{1}{\rho-1-\gamma}} \omega^{-\frac{1}{\rho}} \leqslant K_\rho\, \omega^{-\frac{1}{\rho}} = s_\omega \ .$$

  Hence for all $s \geqslant s_\omega \geqslant s'_\omega$, $k_\omega$ is an increasing function and

$$k_\omega(s) \geqslant k_\omega(s_\omega) = s_\omega^{\,\rho} \big( g_{\gamma,\delta}(s_\omega,\omega) - g_2(s_\omega,\omega) \big) = 0 \ .$$

  Hence inequality (1.9) is satisfied and $g_{\delta,\gamma}(s,\omega) \geqslant g_2(s,\omega)$.

□

Supposing that the amplitude is regular at $p_j$, we derive from the two preceding results a new estimate with faster decay rate of the remainder term related to $p_j$.

**1.1.7 Theorem.** *Let $N \in \mathbb{N}\backslash\{0\}$ and assume $\mu_{j_0} = 1$ and $\rho_{j_0} \geqslant 2$ for a certain $j_0 \in \{1, 2\}$. Suppose that the functions $\psi : [p_1, p_2] \longrightarrow \mathbb{R}$ and $U : (p_1, p_2) \longrightarrow \mathbb{C}$ satisfy Assumption $(\mathrm{P1}_{\rho_1,\rho_2,N})$ and Assumption $(\mathrm{A1}_{\mu_1,\mu_2,N})$, respectively. Then the statement of Theorem 1.1.3 is still true and, for $\gamma \in (0, 1)$ and*

$$\delta := \frac{\gamma + N}{\rho_{j_0}} \in \left( \frac{N}{\rho_{j_0}}, \frac{N+1}{\rho_{j_0}} \right) ,$$

*we have*

$$\left| R_N^{(j_0)}(\omega) \right| \leqslant L_{\gamma,\rho_{j_0},N} \int_0^{s_{j_0}} s^{-\gamma} \left| \frac{d^N}{ds^N} \left[ \nu_{j_0} k_{j_0} \right](s) \right| ds \, \omega^{-\delta} ,$$

*for all $\omega > 0$, where $L_{\gamma,\rho_{j_0},N} > 0$ is the constant given by Lemma 1.1.4 and Lemma 1.1.6.*

*Proof.* We only have to prove the error estimate since the first four steps of the proof of Theorem 1.1.3 remain valid with $\mu_{j_0} = 1$.
Since Lemma 1.1.4 ensures that $\phi_N^{(j_0)}(., \omega, \rho_{j_0}, 1)$ satisfies the assumptions of Lemma 1.1.6, we obtain

$$\forall\, s \in (0, s_{j_0}] \quad \forall\, \omega > 0 \qquad \left| \phi_N^{(j_0)}(s, \omega, \rho_{j_0}, 1) \right| \leqslant L_{\gamma,\rho_{j_0},N} \, s^{-\gamma} \, \omega^{-\delta} ,$$

where $\gamma, \delta > 0$ are defined in the statement of the theorem and $L_{\gamma,\rho_{j_0},N} > 0$ is given in Lemma 1.1.6. Combining the expression of the remainder term from Theorem 1.1.3 with the preceding estimate leads to the conclusion, namely,

$$\begin{aligned} \left| R_N^{(j_0)}(\omega) \right| &\leqslant \int_0^{s_{j_0}} \left| \phi_N^{(j_0)}(s, \omega, \rho_{j_0}, 1) \right| \left| \frac{d^N}{ds^N} [\nu_{j_0} k_{j_0}](s) \right| ds \\ &\leqslant L_{\gamma,\rho_{j_0},N} \int_0^{s_{j_0}} s^{-\gamma} \left| \frac{d^N}{ds^N} [\nu_{j_0} k_{j_0}](s) \right| ds \, \omega^{-\delta} . \end{aligned}$$

And we observe that the decay rate of the remainder term $R_N^{(j_0)}(\omega)$ with respect to $\omega$ is faster than the one of the highest term of the expansion $A_N^{(j_0)}(\omega)$. $\square$

## 1.2 The core of the method: oscillation control by complex analysis

The present section contains the technical but crucial arguments and calculations to fill the considerable gaps left in the original sketch of the proof of Erdélyi. The results will be presented in the order they appear in the proof of Erdélyi's stationary phase method.

Throughout this section, the parameter $\omega > 0$ will be fixed and the integer $j$ will belong to $\{1, 2\}$. We shall prove the propositions in the case $j = 1$ only; the proofs in the case $j = 2$ follow the same lines as in the case $j = 1$ and require only appropriate changes of calculations.

At the beginning of the proof of Erdélyi's theorem, a change of variables is carried out in order to simplify the phase. The aim of the following proposition is to prove that this change of variables is admissible. To this end, we prove that the associated transformation is a diffeomorphism by exploiting substantially the factorization of the zeros of the derivative of the phase.

**1.2.1 Proposition.** *Fix $q_1, q_2 \in (p_1, p_2)$ and let $\psi : [p_1, p_2] \longrightarrow \mathbb{R}$ be a function satisfying Assumption* $(\mathrm{P1}_{\rho_1,\rho_2,N})$. *The real-valued functions $\varphi_1$ and $\varphi_2$ defined by*

$$j = 1, 2 \qquad \varphi_j(p) = \Big( (-1)^{j+1} \big( \psi(p) - \psi(p_j) \big) \Big)^{\frac{1}{\rho_j}} .$$

*are $\mathcal{C}^{N+1}$-diffeomorphism from $[p_1, q_1]$ to $\big[0, \varphi_1(q_1)\big]$ and from $[q_2, p_2]$ to $\big[0, \varphi_2(q_2)\big]$, respectively.*

*Proof.* First of all, we check that $\varphi_1 \in \mathcal{C}^{N+1}\big([p_1, q_1], \mathbb{R}\big)$. We recall that

$$\psi'(p) = (p - p_1)^{\rho_1 - 1} \tilde{\psi}_2(p) ,$$

where we put $\tilde{\psi}_2(p) := (p_2 - p)^{\rho_2 - 1} \tilde{\psi}(p)$, which belongs to $\mathcal{C}^N\big([p_1, q_1], \mathbb{R}\big)$. Applying Taylor's theorem with the integral form of the remainder to $\psi'$, we obtain the following representation of $\varphi_1$:

$$\begin{aligned} \forall\, p \in [p_1, q_1] \qquad \varphi_1(p) &= (p - p_1) \left( \int_0^1 y^{\rho_1 - 1}\, \tilde{\psi}_2\big(y(p - p_1) + p_1\big)\, dy \right)^{\frac{1}{\rho_1}} \\ &=: (p - p_1)\, J_1(p)^{\frac{1}{\rho_1}} . \end{aligned}$$

We fix $k \in \{1, \ldots, N\}$ and we compute formally the $k$th derivative of the above expression by using the product rule:

$$\frac{d^k}{dp^k}\big[\varphi_1\big](p) = (p - p_1) \frac{d^k}{dp^k}\left[ J_1^{\frac{1}{\rho_1}} \right](p) + k\, \frac{d^{k-1}}{dp^{k-1}}\left[ J_1^{\frac{1}{\rho_1}} \right](p) . \tag{1.11}$$

The positivity and the regularity of the function $\tilde{\psi}_2$ allow to differentiate $k$ times under the integral sign the function $J_1$. Hence the $k$ first derivatives of the composite function $J_1^{\frac{1}{\rho_1}}$ exist and are continuous; in particular, the expression (1.11) is well-defined for all $p \in [p_1, q_1]$ and $\frac{d^k}{dp^k}\big[\varphi_1\big]$ is continuous. Concerning the $(N+1)$th derivative, we must refine the analysis because $\tilde{\psi}$ is not supposed to be $\mathcal{C}^{N+1}\big([p_1, p_2], \mathbb{R}\big)$. By applying formally the product rule to $\varphi_1$ once again for $k = N + 1$, we obtain

$$\frac{d^{N+1}}{dp^{N+1}}\big[\varphi_1\big](p) = \underbrace{(p - p_1)\, \frac{d^{N+1}}{dp^{N+1}}\left[ J_1^{\frac{1}{\rho_1}} \right](p)}_{(i)} + (N + 1)\, \underbrace{\frac{d^N}{dp^N}\left[ J_1^{\frac{1}{\rho_1}} \right](p)}_{(ii)} .$$

Note that the term $(ii)$ is well-defined by the previous work. So it remains to study the term $(i)$. Firstly, let us define the function $h_1 : s \longmapsto s^{\frac{1}{\rho_1}}$. Then we obtain by applying Faà di Bruno's Formula to $J_1^{\frac{1}{\rho_1}} = h_1 \circ J_1$,

$$\frac{d^{N+1}}{dp^{N+1}}\left[ J_1^{\frac{1}{\rho_1}} \right](p) = \sum C_N \underbrace{\left( \frac{d^{m_1 + \ldots + m_{N+1}}}{dp^{m_1 + \ldots + m_{N+1}}}\big[h_1\big] \circ J_1 \right)(p)}_{(iii)} \prod_{l=1}^{N+1} \underbrace{\left( \frac{d^l}{dp^l}\big[J_1\big](p) \right)^{m_l}}_{(iv)}$$



 

where the sum is over all the $(N+1)$-tuples $(m_1, ..., m_{N+1})$ satisfying the condition

$$1m_1 + 2m_2 + 3m_3 + \ldots + (N+1)m_{N+1} = N+1 \ .$$

We note that the term $(iii)$ is well-defined by the positivity of $J_1$; moreover by the previous study, the term $(iv)$ is well-defined and continuous for any $l \neq N+1$. So we have to study $\left( \frac{d^{N+1}}{dp^{N+1}} \big[ J_1 \big](p) \right)^{m_{N+1}}$, where $m_{N+1} \leqslant 1$ by the above constraint. Since the case $m_{N+1} = 0$ is clear, we suppose that $m_{N+1} = 1$. We have

$$\begin{aligned}
\frac{d}{dp} \bigg[ \frac{d^N}{dp^N} \big[ J_1 \big] \bigg](p) &= \frac{d}{dp} \bigg[ \int_0^1 y^{N+\rho_1-1} \frac{d^N}{dp^N} \big[ \tilde{\psi}_2 \big] \big( (p-p_1)y + p_1 \big) \, dy \bigg] \\
&= \frac{d}{dp} \bigg[ \frac{1}{(p-p_1)^{\rho_1+N}} \int_{p_1}^{p} (s-p_1)^{\rho_1+N-1} \frac{d^N}{dp^N} \big[ \tilde{\psi}_2 \big](s) \, ds \bigg] \\
&= \frac{-(\rho_1+N)}{(p-p_1)^{\rho_1+N+1}} \int_{p_1}^{p} (s-p_1)^{\rho_1+N-1} \frac{d^N}{dp^N} \big[ \tilde{\psi}_2 \big](s) \, ds \\
&\qquad + \frac{1}{(p-p_1)^{\rho_1+N}} (p-p_1)^{N+\rho_1-1} \frac{d^N}{dp^N} \big[ \tilde{\psi}_2 \big](p) \\
&= \frac{-(\rho_1+N)}{(p-p_1)} \int_0^1 y^{\rho_1+N-1} \frac{d^N}{dp^N} \big[ \tilde{\psi}_2 \big] \big( y(p-p_1) + p_1 \big) \, dy \\
&\qquad + \frac{1}{(p-p_1)} \frac{d^N}{dp^N} \big[ \tilde{\psi}_2 \big](p) \ .
\end{aligned}$$

Multiplying this equality by $(p-p_1)$, we observe that the function

$$p \in [p_1, q_1] \longmapsto (p-p_1) \frac{d^{N+1}}{dp^{N+1}} \bigg[ J_1^{\frac{1}{\rho_1}} \bigg](p)$$

is well-defined and continuous. Then $\frac{d^{N+1}}{dp^{N+1}} \big[ \varphi_1 \big]$ is continuous on $[p_1, q_1]$, proving that $\varphi_1 \in \mathcal{C}^{N+1}\big( [p_1, q_1], \mathbb{R} \big)$.
Furthermore, one remarks that

$$\left\{ \begin{array}{ll} \varphi_1'(p) = \dfrac{1}{\rho_1} \psi'(p) \big( \psi(p) - \psi(p_1) \big)^{\frac{1}{\rho_1}-1} > 0 & \forall \, p \in (p_1, q_1] \ , \\ \varphi_1'(p_1) = \dfrac{1}{\rho_1^{\rho_1}} \tilde{\psi}_2(p_1)^{\frac{1}{\rho_1}} > 0 \ , & \end{array} \right. ,$$

so by the inverse function theorem, $\varphi_1 \, : \, [p_1, q_1] \longrightarrow \big[ 0, \varphi_1(q_1) \big]$ is a $\mathcal{C}^{N+1}$-diffeomorphism.
□

As a result of this change of variables in the proof of the result of Erdélyi, the integrand is factorized into a holomorphic function and a function on a real interval. The aim of the following result is to prove that this second function is regular.

**1.2.2 Proposition.** *Fix $j \in \{1, 2\}$ and let $U : (p_1, p_2) \longrightarrow \mathbb{C}$ be a function satisfying Assumption* $(\mathrm{A1}_{\mu_1, \mu_2, N})$. *Consider the function $k_j : \big( 0, \varphi_j(q_j) \big] \longrightarrow \mathbb{C}$ defined by*

$$k_j(s) = U\big( \varphi_j^{-1}(s) \big) \, s^{1-\mu_j} \, \big( \varphi_j^{-1} \big)'(s) \ ,$$



 

*where the function $\varphi_j$ and the point $q_j$ are defined in Proposition 1.2.1. Then $k_j$ can be extended to $\big[0,\varphi_j(q_j)\big]$ and the extension belongs to $\mathcal{C}^N\left(\big[0,\varphi_j(q_j)\big],\mathbb{C}\right)$.*

*Proof.* We define $\tilde{u}_2(p) := (p_2-p)^{\mu_2-1}\tilde{u}(p)$ for all $p\in[p_1,p_2)$. Then we have by the definition of $k_1$,

$$\begin{aligned}
k_1(s) &= \big(\varphi_1^{-1}(s)-\varphi_1^{-1}(0)\big)^{\mu_1-1}\,\tilde{u}_2\big(\varphi_1^{-1}(s)\big)\,s^{1-\mu_1}\,\big(\varphi_1^{-1}\big)'(s)\\
&= \left(\frac{\varphi_1^{-1}(s)-\varphi_1^{-1}(0)}{s}\right)^{\mu_1-1}\tilde{u}_2\big(\varphi_1^{-1}(s)\big)\big(\varphi_1^{-1}\big)'(s)\\
&= \left(\int_0^1\big(\varphi_1^{-1}\big)'(sy)\,dy\right)^{\mu_1-1}\tilde{u}_2\big(\varphi_1^{-1}(s)\big)\big(\varphi_1^{-1}\big)'(s)\ ,
\end{aligned}\tag{1.12}$$

for all $s\in\big(0,\varphi_1(q_1)\big]$, and $k_1(0):=\tilde{u}_2(p_1)\big(\varphi_1^{-1}\big)'(0)^{\mu_1}$ by taking the limit in (1.12). The conclusion comes from the regularity of $\tilde{u}$ and $\big(\varphi_1^{-1}\big)'$. □

The next step in the proof of Erdélyi consists in creating the expansion of the integral by the classical procedure of integrations by parts. Thanks to the above factorization of the integrand, we differentiate the regular function and we calculate successive primitives under integral forms of the holomorphic factor. The task is to exploit the holomorphy property and Cauchy's theorem to shift the integration path of the primitives in a region where we control the oscillations of the complex exponential, in preparation for precise estimates of the remainder

In the following lemma, we establish an estimate of the complex exponential on a half-line which will be the integration path of the primitives. This result is essential in the proof of Theorem 1.2.4.

**1.2.3 Lemma.** *Fix $j\in\{1,2\}$, let $\rho_j\geqslant 1$ and $s\geqslant 0$. Then we have*

$$\forall\, t\geqslant 0\qquad \left|e^{(-1)^{j+1}i\omega\left(s+te^{(-1)^{j+1}i\frac{\pi}{2\rho_j}}\right)^{\rho_j}}\right|\leqslant e^{-\omega\,t^{\rho_j}}\ .$$

*Proof.* Let us fix $s,t\geqslant 0$. By a simple calculation, we have

$$i\rho_1\omega\int_0^s\left(\xi+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1-1}d\xi = i\omega\left(s+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}+\omega t^{\rho_1}\ .\tag{1.13}$$

Moreover one can see that

$$\forall\,\xi\in[0,s]\qquad 0\leqslant \operatorname{Arg}\left(\xi+te^{i\frac{\pi}{2\rho_1}}\right)\leqslant\frac{\pi}{2\rho_1}\ ;$$

and since $\rho_1\geqslant 1$, it follows

$$0\leqslant\operatorname{Arg}\left(\left(\xi+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1-1}\right)\leqslant\frac{\pi(\rho_1-1)}{2\rho_1}\leqslant\frac{\pi}{2}\ .$$

Hence the imaginary part of the complex number $\big(\xi+te^{i\frac{\pi}{2\rho_1}}\big)^{\rho_1-1}$ is positive and so the real part of the right-hand side in (1.13) is negative. Therefore we have

$$\Re\big(i\omega z^{\rho_1}+\omega t^{\rho_1}\big)\leqslant 0\quad\Longrightarrow\quad\left|e^{i\omega z^{\rho_1}}\right|e^{\omega t^{\rho_1}}=\left|e^{i\omega z^{\rho_1}+\omega t^{\rho_1}}\right|=e^{\Re(i\omega z^{\rho_1}+\omega t^{\rho_1})}\leqslant 1\ ,$$



 

which yields the result in the case $j = 1$. To treat the case $j = 2$, we use the following equality

$$-i\rho_2\,\omega \int_0^s \left(\xi + te^{-i\frac{\pi}{2\rho_2}}\right)^{\rho_2-1} d\xi = -i\omega z^{\rho_2} + \omega t^{\rho_2} \ ,$$

and we carry out a similar work. This ends the proof. □

Now we compute the limit of a sequence of primitives of a certain function related to

$$s \in (0, s_j] \longmapsto s^{\mu_j-1} e^{(-1)^{j+1} i\omega s^{\rho_j}} \in \mathbb{C} \ ,$$

which is the holomorphic factor appearing after Erdélyi's substitution. The sequence is constructed in such a way that each primitive is given by an integral on a finite path, and the sequence of these paths tends to the half-line considered in Lemma 1.2.3, called $\Lambda^{(j)}(s)$. Exploiting the completeness of the space of holomorphic functions $\mathcal{H}(\Omega)$, where $\Omega \subset \mathbb{C}$ is an non-empty open subset of $\mathbb{C}$, and the continuity of the derivative in this space, we show that the resulting limit is also a primitive and its integration path is $\Lambda^{(j)}(s)$. This result will permit to derive the first primitive of the above holomorphic factor, given in Corollary 1.2.6.

**1.2.4 Theorem.** *Fix $j \in \{1, 2\}$ and let $s_j > 0$. Define the parallelogram $D_j \subset \mathbb{C}$ and the domain $U \subset \mathbb{C}$ as follows:*

- $D_j := \left\{ v^* + t_v\, e^{(-1)^{j+1} i\frac{\pi}{2\rho_j}} \in \mathbb{C} \,\middle|\, v^* \in (0, s_j + 1) \ , \ |t_v| < 1 \right\}$
- $U := \mathbb{C} \setminus \left\{ z \in \mathbb{C} \,\middle|\, \Re(z) \leqslant 0 \ , \ \Im(z) = 0 \right\}$

*Fix $\mu_j \in (0, 1]$, $\rho_j \geqslant 1$ and $n \in \mathbb{N}$. Let $F^{(j)}_{n,\omega}(.,.) : U \times \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by*

$$F^{(j)}_{n,\omega}(v, w) := \frac{(-1)^n}{n!} (v - w)^n\, v^{\mu_j-1} e^{(-1)^{j+1} i\omega v^{\rho_j}} \ .$$

*Then for every $w \in D_j$, $F^{(j)}_{n,\omega}(., w)$ has a primitive $H^{(j)}_{n,\omega}(., w)$ on $D_j$ given by*

$$H^{(j)}_{n,\omega}(v, w) := -\int_{\Lambda^{(j)}(v)} F^{(j)}_{n,\omega}(z, w)\, dz = \frac{(-1)^{n+1}}{n!} \int_{\Lambda^{(j)}(v)} (z - w)^n\, z^{\mu_j-1}\, e^{(-1)^{j+1} i\omega z^{\rho_j}} dz \ ,$$

*where*

$$\Lambda^{(j)}(v) := \left\{ v + t\, e^{(-1)^{j+1} i\frac{\pi}{2\rho_j}} \,\middle|\, t \geqslant 0 \right\} \ . \tag{1.14}$$

*Proof.* Let us fix $w \in D_1$ and $n \in \mathbb{N}$. Firstly, we show that the integral defining $H^{(1)}_{n,\omega}(v, w)$ is well-defined for every $v \in D_1$. Since $v \in D_1$, we can write $v = v^* + t_v e^{i\frac{\pi}{2\rho_1}}$ where $0 < v^* < s_1 + 1$ and $-1 < t_v < 1$, and we observe that

$$-H^{(1)}_{n,\omega}(v, w) = \int_{\Lambda^{(1)}(v)} F^{(1)}_{n,\omega}(z, w)\, dz = \int_{\Lambda^{(1)}(v,v^*)} \dots \quad + \int_{\Lambda^{(1)}(v^*)} \dots \ ,$$

where $\Lambda^{(1)}(v, v^*)$ is the segment which starts from the point $v$ and goes to $v^*$, and $\Lambda^{(1)}(v^*)$ is given by (1.14).



 

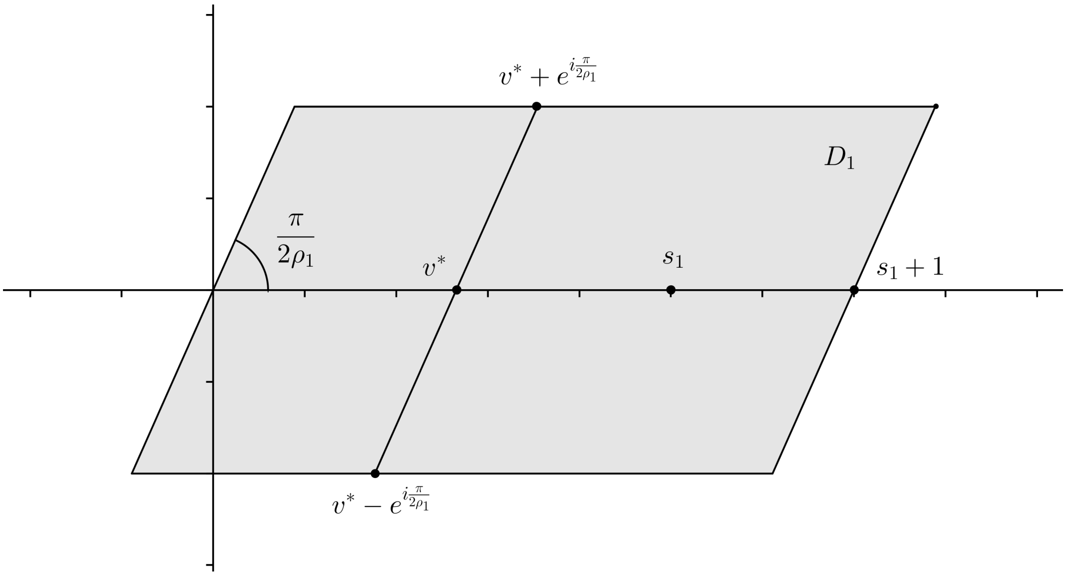


Figure 1.5: The parallelogram $D_1$

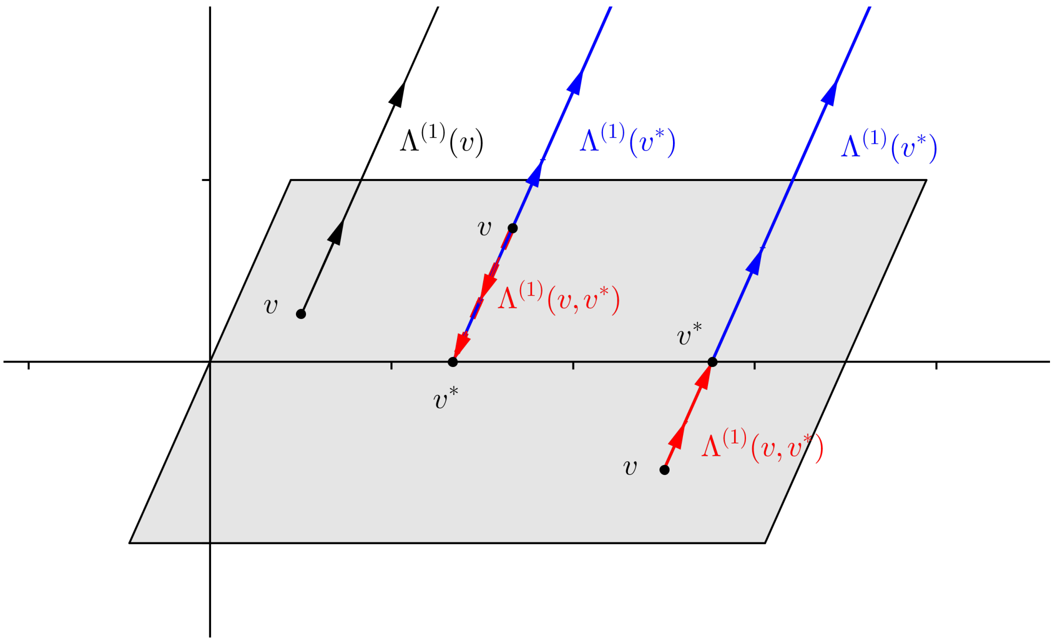


Figure 1.6: The paths $\Lambda^{(1)}(v)$, $\Lambda^{(1)}(v^*)$ and $\Lambda^{(1)}(v, v^*)$ for different positions of $v$

Since $F^{(1)}_{n,\omega}(., w)$ is continuous on the segment $\Lambda^{(1)}(v, v^*)$, then the integral on $\Lambda^{(1)}(v, v^*)$ is well-defined. Concerning the second integral, we give a parametrization of the integration path $\Lambda^{(1)}(v^*)$,

$$\forall\, t \in [0, +\infty) \qquad \lambda^{(1)}_{v^*}(t) := v^* + t\, e^{i\frac{\pi}{2\rho_1}} \in \Lambda^{(1)}(v^*) \; .$$



 

We obtain

$$\left|F^{(1)}_{n,\omega}\Big(\lambda^{(1)}_{v^*}(t),w\Big)\right| \leqslant \frac{1}{n!}\left|v^*+te^{i\frac{\pi}{2\rho_1}}-w\right|^n \left|v^*+te^{i\frac{\pi}{2\rho_1}}\right|^{\mu_1-1}\left|e^{i\omega\left(v^*+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}}\right|$$
$$\leqslant \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}|v^*-w|^{n-k}\,(v^*)^{\mu_1-1}\,t^k\,e^{-\omega t^{\rho_1}}\ , \tag{1.15}$$

where (1.15) comes from the binomial Theorem, Lemma 1.2.3 and the geometric observation:

$$\left|v^*+te^{i\frac{\pi}{2\rho_1}}\right| \geqslant v^*\ .$$

Since the right-hand side of (1.15) defines an integrable function with respect to $t$ on $[0,+\infty)$, and since $\left|(\lambda^{(1)}_{v^*})'(t)\right|=1$, the function $F^{(1)}_{n,\omega}(.,w)$ is integrable on $\Lambda^{(1)}(v^*)$ and hence, $H^{(1)}_{n,\omega}(v,w)$ is well-defined for any $v\in D_1$.
Secondly, we prove that $H^{(1)}_{n,\omega}(.,w)\,:\,D_1\longrightarrow\mathbb{C}$ is a primitive of $F^{(1)}_{n,\omega}(.,w)$ on $D_1$. To this end, we show that $F^{(1)}_{n,\omega}(.,w)$ is a uniform limit on all compact subsets of $D_1$ of a sequence of functions $\big(H^{(1)}_{m,n,\omega}(.,w)\big)_{m\geqslant 1}$ which are primitives of $F^{(1)}_{n,\omega}(.,w)$ on $D_1$. Here we build this sequence of functions as follows: first of all, fix an arbitrary point $v_0\geqslant s_1+1$, for instance $v_0:=s_1+1$, and define the following sequence of complex numbers:

$$\forall\, m\in\mathbb{N}\setminus\{0\}\qquad v_m:=v_0+m\,e^{i\frac{\pi}{2\rho_1}}\ .$$

Let $m\in\mathbb{N}\backslash\{0\}$, let $v=v^*+t_v\,e^{i\frac{\pi}{2\rho_1}}\in D_1$ and let $\Lambda_m(v)$ be the path which is the juxtaposition of the segment that starts from the point $v$ and goes to the point $v^*+me^{i\frac{\pi}{2\rho_1}}$ and of the horizontal segment that joins the points $v^*+me^{i\frac{\pi}{2\rho_1}}$ and $v_m$. We can now define the sequence of functions $\big(H^{(1)}_{m,n,\omega}(.,w):D_1\longrightarrow\mathbb{C}\big)_{m\geqslant 1}$ as follows:

$$H^{(1)}_{m,n,\omega}(v,w):=-\int_{\Lambda_m(v)}F^{(1)}_{n,\omega}(z,w)\,dz\ .$$

It is clear that $F^{(1)}_{n,\omega}(.,w)$ is holomorphic on the domain $U$, which is simply connected, and for any $v\in D_1$, $\Lambda_m(v)$ is included in $U$. The Cauchy integral Theorem affirms that each function $H^{(1)}_{m,n,\omega}(.,w):D_1\longrightarrow\mathbb{C}$ is a primitive of the function $F^{(1)}_{n,\omega}(.,w)$.
Now we prove that this sequence converges to $H^{(1)}_{n,\omega}(.,w)$ uniformly on any compact subset $K$ of $D_1$. Let $K\subset D_1$ be a compact and for every $v\in K$, we have

$$H^{(1)}_{m,n,\omega}(v,w)-H^{(1)}_{n,\omega}(v,w)=\int_{\Lambda^{c,1}_m(v)}F^{(1)}_{n,\omega}(z,w)\,dz+\int_{\Lambda^{c,2}_m(v)}F^{(1)}_{n,\omega}(z,w)\,dz\ ,$$

where $\Lambda^{c,1}_m(v)$ is the horizontal segment which starts from $v_m$ and goes to $v^*+me^{i\frac{\pi}{2\rho_1}}$, and $\Lambda^{c,2}_m(v)$ is the half-line with angle $\frac{\pi}{2\rho_1}$ that starts from $v^*+me^{i\frac{\pi}{2\rho_1}}$ and goes to infinity.
Let $\lambda^{c,1}_m:[0,v_0-v^*]\longrightarrow\mathbb{C}$ and $\lambda^{c,2}_m:[0,+\infty)\longrightarrow\mathbb{C}$ be parametrizations of $\Lambda^{c,1}_m(v)$ and $\Lambda^{c,2}_m(v)$ respectively, and defined by

- $\forall\,t\in\big[0,v_0-v^*\big]\qquad \lambda^{c,1}_m(t):=-t+v_0+me^{i\frac{\pi}{2\rho_1}}\in\Lambda^{c,1}_m(v)\ ,$
- $\forall\,t\in[0,+\infty)\qquad \lambda^{c,2}_m(t):=v^*+(t+m)e^{i\frac{\pi}{2\rho_1}}\in\Lambda^{c,2}_m(v)\ .$



 

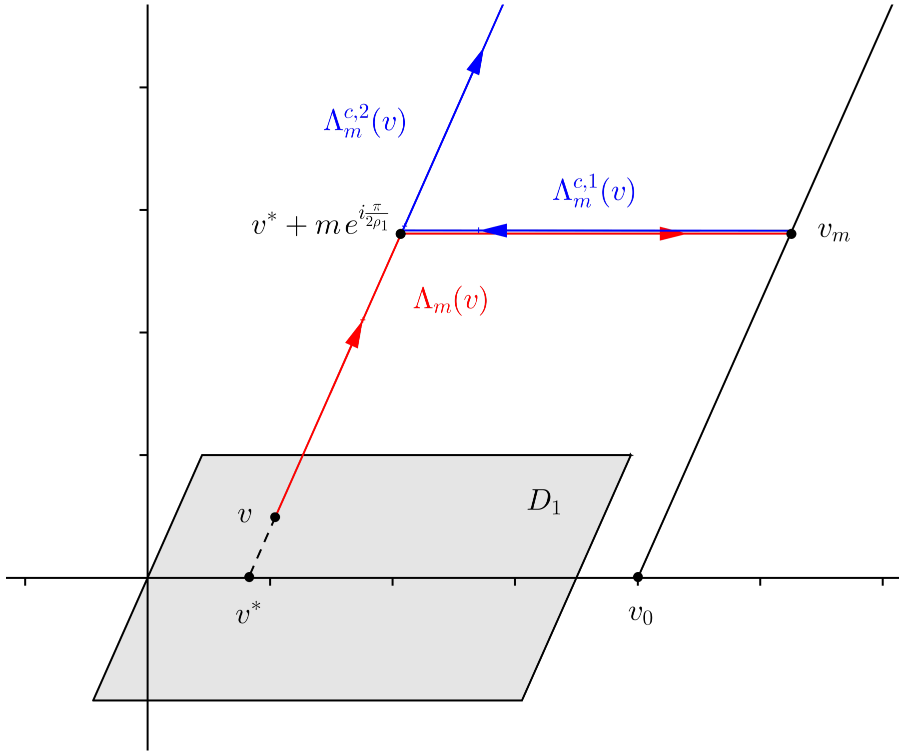


Figure 1.7: The paths $\Lambda_m(v)$, $\Lambda_m^{c,1}(v)$ and $\Lambda_m^{c,2}(v)$

Then we have the following estimates:

$$\left|F_{n,\omega}^{(1)}\big(\lambda_m^{c,1}(t),w\big)\right| \leqslant \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}|v_0-w|^{n-k}\left|-t+me^{i\frac{\pi}{2\rho_1}}\right|^k m^{\mu_1-1}\,e^{-\omega m^{\rho_1}} \tag{1.16}$$

$$\leqslant \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}|v_0-w|^{n-k}\big(C(K)+m\big)^k m^{\mu_1-1}\,e^{-\omega m^{\rho_1}} \tag{1.17}$$

- (1.16): use the binomial Theorem, Lemma 1.2.3 and the fact that $\left|\lambda_m^{c,1}(t)\right| \geqslant m$ ;
- (1.17): use the compactness of $K$ which provides $0 \leqslant t \leqslant v_0 - v^* \leqslant C(K)$, for a certain constant $C(K) > 0$.

Inequality (1.17) permits to estimate uniformly the integral on $\Lambda_m^{c,1}(v)$,

$$\begin{aligned}\left|\int_{\Lambda_m^{c,1}(v)} F_{n,\omega}^{(1)}(z,w)\,dz\right| &\leqslant \int_0^{v_0-v^*}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}|v_0-w|^{n-k}\big(C(K)+m\big)^k m^{\mu_1-1}\,e^{-\omega m^{\rho_1}}\,dt\\ &\leqslant \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}|v_0-w|^{n-k}\big(C(K)+m\big)^k m^{\mu_1-1}\,e^{-\omega m^{\rho_1}}C(K)\\ &\longrightarrow 0 \quad , \quad m \longrightarrow +\infty \,,\end{aligned}$$

where we used the fact that $0 \leqslant v_0 - v^* \leqslant C(K)$ one more time. Here, the convergence is uniform with respect to $v$. Furthermore,

$$\left|F^{(1)}_{n,\omega}\big(\lambda^{c,2}_m(t),w\big)\right| \leqslant \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}|v^*-w|^{n-k}\,|v^*|^{\mu_1-1}\,(t+m)^k\,e^{-\omega(t+m)^{\rho_1}} \tag{1.18}$$

$$\leqslant \frac{C_{n,w}(K)}{n!}\sum_{k=0}^{n}\binom{n}{k}(t+m)^k\,e^{-\omega(t+m)^{\rho_1}} \tag{1.19}$$

$$\leqslant \frac{C_{n,w}(K)}{n!}\sum_{k=0}^{n}\binom{n}{k}m^k\,e^{-\omega m^{\rho_1}}\,(1+t)^k\,e^{-\omega t^{\rho_1}} \tag{1.20}$$

$$\leqslant \frac{C_{n,w}(K)\,M_{k,\omega}}{n!}\sum_{k=0}^{n}\binom{n}{k}(1+t)^k\,e^{-\omega t^{\rho_1}} \tag{1.21}$$

- (1.18): use the binomial Theorem, Lemma 1.2.3 and $\left|\lambda^{c,2}_m(t)\right| \geqslant v^*$ ;
- (1.19): use the compactness of $K$ and the fact that $v \in K$ to bound uniformly $|v^*-w|$ and $|v^*|$ ;
- (1.20): use the inequalities $(m+t)^k \leqslant m^k(1+t)^k$ and $e^{-\omega(t+m)^{\rho_1}} \leqslant e^{-\omega m^{\rho_1}}e^{-\omega t^{\rho_1}}$ ;
- (1.21): use the boundedness of the sequences $\big(m^k e^{-\omega m^{\rho_1}}\big)_{m\geqslant 1}$ for $k=0,\dots,n$.

We remark that (1.20) tends to 0 as $m$ tends to infinity for all $t \geqslant 0$ and (1.21) furnishes an integrable function with respect to $t$ and independent from $m$. So by the dominated convergence Theorem,

$$\left|\int_{\Lambda^{c,2}_m(v)} F^{(1)}_{n,\omega}(z,w)\,dz\right| \leqslant \int_0^{+\infty}\frac{C_{n,w}(K)}{n!}\sum_{k=0}^{n}\binom{n}{k}m^k\,e^{-\omega m^{\rho_1}}\,(1+t)^k\,e^{-\omega t^{\rho_1}}\,dt \tag{1.22}$$
$$\longrightarrow 0 \quad , \quad m \longrightarrow +\infty \ ,$$

and the convergence is uniform with respect to $v$ since the right-hand side in (1.22) is independent from $v$. Hence the hypotheses of a theorem of Weierstrass are satisfied and therefore the function $H^{(1)}_{n,\omega}(.,w) : D_1 \longrightarrow \mathbb{C}$ is holomorphic and its derivative is given by

$$\forall\, v \in D_1 \qquad \frac{\partial}{\partial v}\big[H^{(1)}_{n,\omega}\big](v,w) = \lim_{m\to+\infty}\frac{\partial}{\partial v}\big[H^{(1)}_{m,n,\omega}(v,w)\big] = F^{(1)}_{n,\omega}(v,w) \ ,$$

and the convergence is uniform on every compact subset. □

Since we integrate by parts many times in Erdélyi's proof, we need successive primitives of the holomorphic part of the integrand. For this purpose, we establish this second intermediate but essential result by employing the preceding theorem as well as complex analysis in several variables.
The desired primitives will be deduced from the following result in Corollary 1.2.6.

**1.2.5 Theorem.** *Fix $j \in \{1,2\}$ and let $n \in \mathbb{N}\backslash\{0\}$. Define the function $h : \mathbb{C} \longrightarrow \mathbb{C}\times\mathbb{C}$ by*

$$h(u) := (u,u) \ ,$$



 

*and let* $H^{(j)}_{n,\omega}(.,.) : D_j \times D_j \longrightarrow \mathbb{C}$ *be the function defined in Theorem 1.2.4. Then the composite function* $H^{(j)}_{n,\omega}(.,.) \circ h$ *is holomorphic on* $D_j$ *and its derivative is given by*

$$\forall\, u \in D_j \qquad \frac{d}{du}\big[H^{(j)}_{n,\omega} \circ h\big](u) = \Big(H^{(j)}_{n-1,\omega} \circ h\Big)(u)\ .$$

*Proof.* The aim of the proof is to differentiate the composite function. For this purpose, we must ensure that this function is holomorphic with respect to each variable.
Fix $n \in \mathbb{N}\backslash\{0\}$. We remark that each component of $h$ is holomorphic on $\mathbb{C}$, so is $h$ on $\mathbb{C} \times \mathbb{C}$. Moreover for any fixed $w \in D_1$, $H^{(1)}_{n,\omega}(.,w) : D_1 \longrightarrow \mathbb{C}$ is a primitive of $F^{(1)}_{n,\omega}(.,w)$ on $D_1$ by Theorem 1.2.4, so it is holomorphic. Now let us show that $H^{(1)}_{n,\omega}(v,.) : D_1 \longrightarrow \mathbb{C}$ belongs to $\mathcal{C}^1(D_1)$ and satisfies the Cauchy-Riemann equations for fixed $v \in D_1$. To do so, we employ the holomorphy of $F^{(1)}_{n,\omega}(v,.) : \mathbb{C} \longrightarrow \mathbb{C}$ which provides the following relations:

$$\forall\, w = x+iy \in \mathbb{C} \qquad \frac{\partial}{\partial w}\big[F^{(1)}_{n,\omega}\big](v,w) = \frac{\partial}{\partial x}\big[F^{(1)}_{n,\omega}\big](v,w) = -i\frac{\partial}{\partial y}\big[F^{(1)}_{n,\omega}\big](v,w)\ . \tag{1.23}$$

By a simple calculation, we obtain

$$\frac{\partial}{\partial w}\big[F^{(1)}_{n,\omega}\big](v,w) = \frac{(-1)^{n-1}}{(n-1)!}\,(v-w)^{n-1} v^{\mu_1-1} e^{i\omega v^{\rho_1}} = F^{(1)}_{n-1,\omega}(v,w)\ . \tag{1.24}$$

Furthermore, one can bound $F^{(1)}_{n-1,\omega}(.,w)$ on each path $\Lambda^{(1)}(v,v^*)$ and $\Lambda^{(1)}(v^*)$ by integrable functions independent from $w$. To do so, one can parametrize each path $\Lambda^{(1)}(v,v^*)$ and $\Lambda^{(1)}(v^*)$ and employ similar arguments to those of the proof of Theorem 1.2.4 as well as the boundedness of $D_1$. So we obtain the ability to differentiate under the integral sign which yields the following equalities:

$$\begin{aligned}
-\frac{\partial}{\partial x}\big[H^{(1)}_{n,\omega}\big]\,(v,w) &= \frac{\partial}{\partial x}\bigg[\int_{\Lambda^{(1)}(v,v^*)} F^{(1)}_{n,\omega}(z,w)\,dz\bigg] + \frac{\partial}{\partial x}\bigg[\int_{\Lambda^{(1)}(v^*)} F^{(1)}_{n,\omega}(z,w)\,dz\bigg] \\
&= \int_{\Lambda^{(1)}(v,v^*)} \frac{\partial}{\partial x}\big[F^{(1)}_{n,\omega}\big](z,w)\,dz \;+ \int_{\Lambda^{(1)}(v^*)} \frac{\partial}{\partial x}\big[F^{(1)}_{n,\omega}\big](z,w)\,dz \qquad (1.25) \\
&= \int_{\Lambda^{(1)}(v,v^*)} \frac{\partial}{\partial w}\big[F^{(1)}_{n,\omega}\big](z,w)\,dz \;+ \int_{\Lambda^{(1)}(v^*)} \frac{\partial}{\partial w}\big[F^{(1)}_{n,\omega}\big](z,w)\,dz \qquad (1.26) \\
&= \int_{\Lambda^{(1)}(v,v^*)} F^{(1)}_{n-1,\omega}(z,w)\,dz \;+ \int_{\Lambda^{(1)}(v^*)} F^{(1)}_{n-1,\omega}(z,w)\,dz \qquad (1.27) \\
&= \int_{\Lambda^{(1)}(v)} F^{(1)}_{n-1,\omega}(z,w)\,dz \\
&= -H^{(1)}_{n-1,\omega}(v,w)
\end{aligned}$$

- (1.25): apply the theorem of differentiation under the integral sign ;
- (1.26): use equalities (1.23) coming from the holomorphy of the function $F^{(1)}_{n,\omega}(v,.)$ ;
- (1.27): use relation (1.24) .



 

In a similar way, we obtain

$$-i\frac{\partial}{\partial y}\big[H^{(1)}_{n,\omega}\big](v,w) = H^{(1)}_{n-1,\omega}(v,w)\ .$$

Then the Cauchy-Riemann equations are satisfied and $\dfrac{\partial}{\partial x}\big[H^{(1)}_{n,\omega}\big](v,.)$ and $\dfrac{\partial}{\partial y}\big[H^{(1)}_{n,\omega}\big](v,.)$ are continuous on $D_1$ by the continuity of $F^{(1)}_{n-1,\omega}(z,.) : \mathbb{C} \longrightarrow \mathbb{C}$. So $H^{(1)}_{n,\omega}(v,.) : D_1 \longrightarrow \mathbb{C}$ is holomorphic, with

$$\frac{\partial}{\partial w}\big[H^{(1)}_{n,\omega}\big](v,w) = H^{(1)}_{n-1,\omega}(v,w)\ .$$

Finally the composite function $H^{(1)}_{n,\omega} \circ h$ is holomorphic on $D_1 \times D_1$ and we have the formula

$$\begin{aligned}\frac{d}{du}\big[H^{(1)}_{n,\omega} \circ h\big](u) &= \left(\frac{\partial}{\partial v}\big[H^{(1)}_{n,\omega}\big]\big(h(u)\big) \quad \frac{\partial}{\partial w}\big[H^{(1)}_{n,\omega}\big]\big(h(u)\big)\right)\begin{pmatrix}1\\1\end{pmatrix}\\ &= \frac{\partial}{\partial v}\big[H^{(1)}_{n,\omega}\big](u,u) + \frac{\partial}{\partial w}\big[H^{(1)}_{n,\omega}\big](u,u)\ ;\end{aligned}$$

And a short computation shows that $\dfrac{\partial}{\partial v}\big[H^{(1)}_{n,\omega}\big](u,u) = F^{(1)}_{n,w}(u,u) = 0$, so for all $u \in D_1$,

$$\frac{d}{du}\big[H^{(1)}_{n,\omega} \circ h\big](u) = \Big(H^{(1)}_{n-1,\omega} \circ h\Big)(u) = \frac{(-1)^n}{(n-1)!}\int_{\Lambda^{(1)}(u)} (z-u)^{n-1} z^{\mu_1-1} e^{i\omega z^{\rho_1}}\, dz\ .$$

□

Finally by restricting the domain of definition of the functions introduced in the two preceding theorems to the interval $(0,s_j]$, we derive formulas for the successive primitives of the function $s \in (0,s_j] \longmapsto s^{\mu_j-1}\, e^{(-1)^{j+1} i\omega s^{\rho_j}}$, the holomorphic part of the integrand.

**1.2.6 Corollary.** *Fix $j \in \{1,2\}$, $s_j > 0$, $\rho_j \geqslant 1$ and $\mu_j \in (0,1]$. For any $\omega > 0$, the sequence of functions $\big(\phi^{(j)}_n(.,\omega,\rho_j,\mu_j) : (0,s_j] \longrightarrow \mathbb{C}\big)_{n\geqslant 1}$ defined in Theorem 1.1.3 satisfies the recursive relation:*

$$\forall\, s \in\ (0,s_j] \qquad \left\{\begin{array}{ll}\dfrac{\partial}{\partial s}\Big[\phi^{(j)}_1\Big](s,\omega,\rho_j,\mu_j) = s^{\mu_j-1}\, e^{(-1)^{j+1} i\omega s^{\rho_j}}\ , & \\ \dfrac{\partial}{\partial s}\Big[\phi^{(j)}_{n+1}\Big](s,\omega,\rho_j,\mu_j) = \phi^{(j)}_n(s,\omega,\rho_j,\mu_j) & \forall\, n \geqslant 1\ .\end{array}\right.$$

*Proof.* It suffices to note that $\phi^{(j)}_{n+1}(.,\omega,\rho_j,\mu_j)$ is the restriction to $(0,s_j] \subset D_j$ of the function $H^{(j)}_{n,\omega} \circ h$. Hence Theorem 1.2.4 affirms that $\phi^{(j)}_1(.,\omega,\rho_j,\mu_j) : (0,s_j] \longrightarrow \mathbb{C}$ is a primitive of $s \in (0,s_j] \longmapsto s^{\mu_j-1} e^{(-1)^{j+1} i\omega s^{\rho_j}}$, and use Theorem 1.2.5 to show that a primitive of $\phi^{(j)}_n(.,\omega,\rho_j,\mu_j) : (0,s_j] \longrightarrow \mathbb{C}$ is given by $\phi^{(j)}_{n+1}(.,\omega,\rho_j,\mu_j) : (0,s_j] \longrightarrow \mathbb{C}$, for $n \geqslant 1$. □

**1.2.7 Remark.** The function $\phi^{(j)}_{n+1}(.,\omega,\rho_j,\mu_j) : (0,s_j] \longrightarrow \mathbb{C}$ can be extended to $[0,+\infty)$. Indeed, we recall a parametrization of the curve $\Lambda^{(j)}(s)$:

$$\lambda^{(j)}_s : t \in [0,+\infty) \longmapsto s + t\, e^{(-1)^{j+1} i \frac{\pi}{2\rho_j}} \in \Lambda^{(j)}(s)\ .$$



 

Then we have

$$\forall\, t > 0 \qquad \left| F^{(j)}_{n,\omega}\big(\lambda^{(j)}_s(t), s\big) \right| \leqslant \frac{1}{n!}\, t^{n+\mu_j-1}\, e^{-\omega t^{\rho_j}} \ , \tag{1.28}$$

which was obtained by noting that

$$t \leqslant \left| s + t e^{(-1)^{j+1} i \frac{\pi}{2\rho_j}} \right| = \left| \lambda^{(j)}_s(t) \right| \qquad \Longrightarrow \qquad t^{\mu_j - 1} \geqslant \left| \lambda^{(j)}_s(t) \right|^{\mu_j - 1} \ .$$

We notice that the right-hand side of (1.28) is an integrable function with respect to $t$ on $[0, +\infty)$ and independent from $s > 0$. So $\phi^{(j)}_{n+1}(s, \omega, \rho_j, \mu_j)$ is well-defined for all $s \geqslant 0$. In particular, $\phi^{(j)}_{n+1}(0, \omega, \rho_j, \mu_j)$ is defined as follows,

$$\begin{aligned} \phi^{(j)}_{n+1}(0, \omega, \rho_j, \mu_j) &:= \lim_{s \to 0^+} \phi^{(j)}_{n+1}(s, \omega, \rho_j, \mu_j) \\ &= \frac{(-1)^{n+1}}{n!} \int_{\Lambda^{(j)}(0)} z^{n+\mu_j-1} e^{(-1)^{j+1} i \omega z^{\rho_j}}\, dz \ . \end{aligned}$$



 

# Chapter 2

# Lossless error estimates, optimal parameter domains and applications to the free Schrödinger equation

**Abstract**

This new chapter begins with a modification of the stationary phase method considered in Chapter 1: we replace the smooth cut-off function employed in the original proof by a characteristic function, leading to lossless remainder estimates. For particular oscillatory integrals, we derive optimal parameter domains, depending explicitly on the distance between the stationary point and the singular point, in which the expansion as well as the remainder are uniformly bounded. These abstract refinements are then exploited to study the time-asymptotic behaviour of the solution of the free Schrödinger equation on the line, where the Fourier transform of the initial data is compactly supported and has a singular point. We obtain asymptotic expansions with respect to time in certain space-time cones as well as uniform and optimal estimates in curved regions which are asymptotically larger than any space-time cone. These results show the influence of a restriction to compact frequency bands and of the singularity on the propagation and on the decay of the wave packets.

## Contents

## 2.1 Lossless error estimates

In order to motivate the results of this section, let us consider the following oscillatory integral

$$I(\omega, p_2) = \int_{p_1}^{p_2} (p - p_1)^{-\frac{1}{4}} \, e^{-i\omega(p-p_2)^2} \, dp \ ,$$

with $\omega > 0$ and $p_2 > p_1$; here $p_1$ is a singular point of the amplitude of order $\mu_1 = \frac{3}{4}$ and $p_2$ a stationary point of order $\rho_2 = 2$. In particular, we have

$$\big| I(\omega, p_2) \big| \leqslant \frac{4}{3} \, (p_2 - p_1)^{\frac{3}{4}} \ .$$

Now by applying the results of the preceding chapter, we obtain an asymptotic expansion of the above oscillatory integral, namely,

$$\int_{p_1}^{p_2} (p - p_1)^{-\frac{1}{4}} \, e^{-i\omega(p-p_2)^2} \, dp = \frac{\sqrt{\pi}}{2} \, e^{-i\frac{\pi}{4}} \, (p_2 - p_1)^{-\frac{1}{4}} \, \omega^{-\frac{1}{2}}$$
$$+ \frac{\Gamma\big(\frac{3}{4}\big)}{2^{\frac{3}{4}}} \, e^{i\frac{3\pi}{8}} \, e^{-i\omega(p_2-p_1)^2} \, (p_2 - p_1)^{-\frac{3}{4}} \, \omega^{-\frac{3}{4}} + R_1^{(1)}(\omega, p_2) + R_1^{(2)}(\omega, p_2) \ ,$$

and for $\delta \in \big(\frac{3}{4}, 1\big)$, we have the following estimates of the remainders:

- $$\Big| R_1^{(1)}(\omega, p_2) \Big| \leqslant \int_0^{\frac{8}{9}(p_2-p_1)^2} s^{-\frac{1}{4}} \, \Big| \big(\nu_1 k_1\big)'(s) \Big| \, ds \, \omega^{-1};$$
- $$\Big| R_1^{(2)}(\omega, p_2) \Big| \leqslant L_{\gamma,2,1} \int_0^{\frac{2}{3}(p_2-p_1)} s^{-\gamma} \, \Big| \big(\nu_2 k_2\big)'(s) \Big| \, ds \, \omega^{-\delta},$$

with $\gamma = 2\delta - 1$.

For fixed $\omega > 0$, we note that if $p_2$ tends to $p_1$ then the expansion of the integral blows up due to the presence of the singular factors $(p_2 - p_1)^{-\frac{1}{4}}$ and $(p_2 - p_1)^{-\frac{3}{4}}$. Especially this implies that the expansion does not furnish a good approximation of $I(\omega, p_2)$ for fixed $\omega > 0$ when $p_2$ is too close to $p_1$.

Moreover since the integral is bounded and the expansion blows up, the remainder tends also to infinity when $p_2$ tends to $p_1$. Now let us note that the graphs of the smooth cut-off functions $\nu_j$ compress when $p_2$ tends to $p_1$, implying the fact that the $L^\infty$-norms of $(\nu_j)'$ tend to infinity. This observation leads to the idea that the smooth cut-off function contributes artificially to the blow-up of the remainder.

Hence the aim of this section is to provide lossless error estimate for the stationary phase method by replacing the smooth cut-off function by a characteristic function.

We start by modifying slightly the functions $\varphi_j$ and $k_j$ introduced in Definition 1.1.2 of Chapter 1: we change only their domains of definition. We shall use the notations of Chapter 1 again and these two new definitions will be used throughout the present chapter.

Let $p_1, p_2$ be two finite real numbers such that $p_1 < p_2$, and choose $q \in (p_1, p_2)$.

**2.1.1 Definition.** *Let $\psi : [p_1, p_2] \longrightarrow \mathbb{R}$ and $U : (p_1, p_2) \longrightarrow \mathbb{C}$ be two functions satisfying Assumption $(\mathrm{P1}_{\rho_1,\rho_2,1})$ and Assumption $(\mathrm{A1}_{\mu_1,\mu_2,1})$ (given in Section 1.1), respectively.*



 

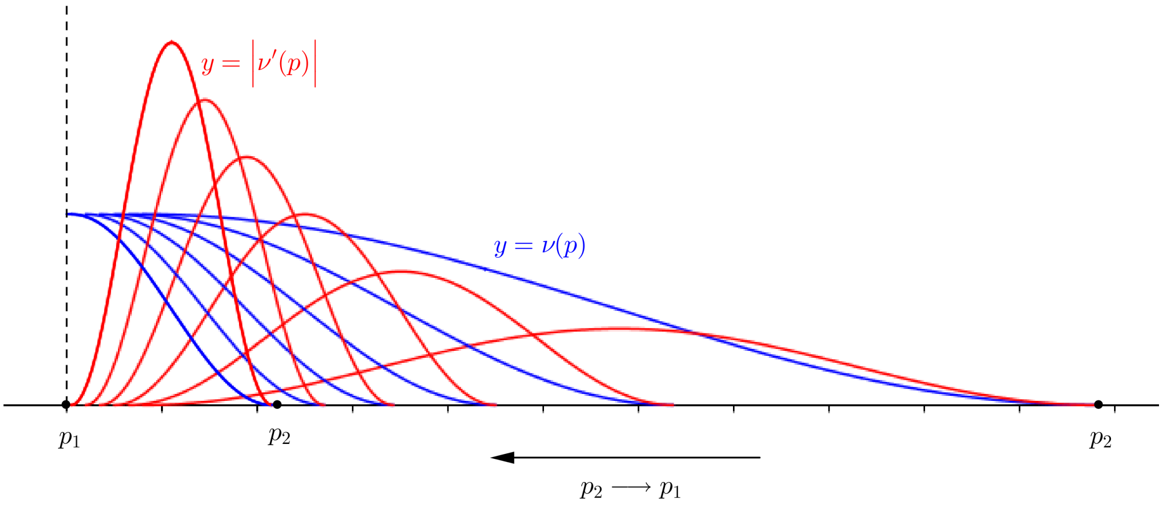


Figure 2.1: Blow-up of the derivative of the cut-off function $\nu$

*i) For $j = 1, 2$, let $\varphi_j : I_j \longrightarrow \mathbb{R}$ be the functions defined by*

$$\varphi_1(p) := \big(\psi(p) - \psi(p_1)\big)^{\frac{1}{\rho_1}} \qquad , \qquad \varphi_2(p) := \big(\psi(p_2) - \psi(p)\big)^{\frac{1}{\rho_2}} ,$$

*with $I_1 := [p_1, q]$, $I_2 := [q, p_2]$ and $s_1 := \varphi_1(q)$, $s_2 := \varphi_2(q)$.*

*ii) For $j = 1, 2$, let $k_j : (0, s_j] \longrightarrow \mathbb{C}$ be the functions defined by*

$$k_j(s) := U\big(\varphi_j^{-1}(s)\big)\, s^{1-\mu_j} \left(\varphi_j^{-1}\right)'(s) ,$$

*which can be extended to $[0, s_j]$ (see Proposition 1.2.2).*

Now we state and prove a refinement of the version of the stationary phase method of Erdélyi which consists in replacing the smooth cut-off function by a characteristic function. The hypotheses on the regularity of the phase and of the amplitude are weakened as compared with Theorem 1.1.3, because we establish an expansion to one term only, which requires a single integration by parts.

**2.1.2 Theorem.** *Let $\rho_1, \rho_2 \geqslant 1$ and $\mu_1, \mu_2 \in (0, 1)$. Suppose that $\psi : [p_1, p_2] \longrightarrow \mathbb{R}$ and $U : (p_1, p_2) \longrightarrow \mathbb{C}$ satisfy Assumption $(\mathrm{P1}_{\rho_1,\rho_2,1})$ and Assumption $(\mathrm{A1}_{\mu_1,\mu_2,1})$, respectively. Then we have*

$$\left\{\begin{array}{l} \displaystyle\int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp = \sum_{j=1,2} \Big(A^{(j)}(\omega) + R_1^{(j)}(\omega, q) + R_2^{(j)}(\omega, q)\Big) , \\ \displaystyle\left|R_1^{(j)}(\omega, q)\right| \leqslant \frac{1}{\rho_j}\, \Gamma\!\left(\frac{1}{\rho_j}\right) \int_0^{s_j} s^{\mu_j - 1} \big|(k_j)'(s)\big|\, ds\; \omega^{-\frac{1}{\rho_j}} , \\ \displaystyle\left|R_2^{(j)}(\omega, q)\right| \leqslant \frac{\rho_j - \mu_j}{\rho_j}\, \Gamma\!\left(\frac{1}{\rho_j}\right) \left|U(q)\, (\varphi_j)'(q)^{-1}\right| \varphi_j(q)^{-\rho_j}\, \omega^{-\left(1 + \frac{1}{\rho_j}\right)} , \end{array}\right.$$



 

*for all $\omega > 0$ and for a fixed $q \in (p_1, p_2)$, where for $j = 1, 2$,*

- $A^{(j)}(\omega) := e^{i\omega\psi(p_j)}\, k_j(0)\, \Theta^{(j)}(\rho_j, \mu_j)\, \omega^{-\frac{\mu_j}{\rho_j}}$ ,
- $R_1^{(j)}(\omega, q) := (-1)^j\, e^{i\omega\psi(p_j)} \int_0^{s_j} \phi^{(j)}(s, \omega, \rho_j, \mu_j)\, (k_j)'(s)\, ds$ ,
- $R_2^{(j)}(\omega, q) := (-1)^j\, i\, \frac{\mu_j - \rho_j}{\rho_j}\, e^{i\omega\psi(p_j)}\, k_j(s_j) \int_{\Lambda^{(j)}(s_j)} z^{\mu_j - \rho_j - 1}\, e^{(-1)^{j+1} i\omega z^{\rho_j}}\, dz\, \omega^{-1}$ ,
- $\Theta^{(j)}(\rho_j, \mu_j) := \frac{(-1)^{j+1}}{\rho_j}\, \Gamma\left(\frac{\mu_j}{\rho_j}\right) e^{(-1)^{j+1} i \frac{\pi}{2} \frac{\mu_j}{\rho_j}}$ ,
- $\phi^{(j)}(s, \omega, \rho_j, \mu_j) := -\int_{\Lambda^{(j)}(s)} z^{\mu_j - 1}\, e^{(-1)^{j+1} i\omega z^{\rho_j}}\, dz$ .

**2.1.3 Remark.** Note that the quantities $\Theta^{(j)}(\rho_j, \mu_j)$ and $\phi^{(j)}(s, \omega, \rho_j, \mu_j)$ correspond respectively to $\Theta_1^{(j)}(\rho_j, \mu_j)$ and $\phi_1^{(j)}(s, \omega, \rho_j, \mu_j)$, which are defined in Theorem 1.1.3. We have removed the subscripts for simplicity.

*Proof of Theorem 2.1.2.* The present proof follows the steps of the proof of Theorem 1.1.3. Hence the steps which are identical will be only sketched and we shall focus on the new arguments coming from the cutting-point $q$.
As previously, we shall note $\phi^{(j)}(s, \omega)$ instead of $\phi^{(j)}(s, \omega, \rho_j, \mu_j)$ in the proof.

*First step: Splitting of the integral.* We fix a point $q \in (p_1, p_2)$ and we split the integral at this point,

$$\int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp = \tilde{I}^{(1)}(\omega, q) + \tilde{I}^{(2)}(\omega, q) ,$$

where

$$\tilde{I}^{(1)}(\omega, q) := \int_{p_1}^{q} U(p)\, e^{i\omega\psi(p)}\, dp \quad , \quad \tilde{I}^{(2)}(\omega, q) := \int_{q}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp .$$

*Second step: Substitution.* Since $\varphi_1 : [p_1, q] \longrightarrow [0, s_1]$ is a $\mathcal{C}^2$-diffeomorphism, we obtain by setting $s = \varphi_1(p)$,

$$\tilde{I}^{(1)}(\omega, q) = e^{i\omega\psi(p_1)} \int_0^{s_1} k_1(s)\, s^{\mu_1 - 1} e^{i\omega s^{\rho_1}}\, ds ,$$

where $k_1$ is given in Definition 2.1.1. In a similar way, we obtain

$$\tilde{I}^{(2)}(\omega, q) = -e^{i\omega\psi(p_2)} \int_0^{s_2} k_2(s)\, s^{\mu_2 - 1} e^{-i\omega s^{\rho_2}}\, ds ,$$

with $k_2$ defined also in Definition 2.1.1.

*Third step: Integration by parts.* An integration by parts leads to

$$\begin{aligned} \tilde{I}^{(1)}(\omega, q) = \phi^{(1)}(s_1, \omega)\, k_1(s_1)\, e^{i\omega\psi(p_1)} - \phi^{(1)}(0, \omega)\, k_1(0)\, e^{i\omega\psi(p_1)} \\ - e^{i\omega\psi(p_1)} \int_0^{s_1} \phi^{(1)}(s, \omega)\, (k_1)'(s)\, ds , \end{aligned}$$

and similarly,

$$\tilde{I}^{(2)}(\omega,q) = \phi^{(2)}(0,\omega)\,k_2(0)\,e^{i\omega\psi(p_2)} - \phi^{(2)}(s_2,\omega)\,k_2(s_2)\,e^{i\omega\psi(p_2)} + e^{i\omega\psi(p_2)}\int_0^{s_2}\phi^{(2)}(s,\omega)(k_2)'(s)\,ds\ . \tag{2.1}$$

*Fourth step: Cancellation.* The aim of this step is to simplify the difference:

$$\phi^{(1)}(s_1,\omega)\,k_1(s_1)\,e^{i\omega\psi(p_1)} - \phi^{(2)}(s_2,\omega)\,k_2(s_2)\,e^{i\omega\psi(p_2)}\ . \tag{2.2}$$

Since the two functions $s \longmapsto \phi^{(j)}(s,\omega)$, for $j=1,2$, are given by oscillatory integrals, we shall expand them with respect to $\omega$ and show that the first terms cancel out.
Since $s_1>0$, we note that the derivative of the function $t \longmapsto \left(s_1 + te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}$ does not vanish for all $t>0$ and thus, one can write

$$e^{i\omega\left(s_1+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}} = e^{-i\frac{\pi}{2\rho_1}}\,(i\omega\rho_1)^{-1}\left(s_1+te^{i\frac{\pi}{2\rho_1}}\right)^{1-\rho_1}\frac{d}{dt}\left[e^{i\omega\left(s_1+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}}\right]\ .$$

Putting this equality in the definition of $\phi^{(1)}(s_1,\omega)$ and carrying out an integration by parts lead to

$$\begin{aligned}\phi^{(1)}(s_1,\omega) &= -(i\omega\rho_1)^{-1}\int_0^{+\infty}\left(s_1+te^{i\frac{\pi}{2\rho_1}}\right)^{\mu_1-\rho_1}\frac{d}{dt}\left[e^{i\omega\left(s_1+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}}\right]dt\\ &= (i\omega\rho_1)^{-1}s_1^{\mu_1-\rho_1}\,e^{i\omega s_1^{\rho_1}}\\ &\quad+\frac{\mu_1-\rho_1}{i\omega\rho_1}\,e^{i\frac{\pi}{2\rho_1}}\int_0^{+\infty}\left(s_1+te^{i\frac{\pi}{2\rho_1}}\right)^{\mu_1-\rho_1-1}e^{i\omega\left(s_1+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}}dt\ .\end{aligned}\tag{2.3}$$

We remark that the boundary term at infinity is 0; indeed, we observe that

$$s_1 \leqslant \left|s_1+te^{i\frac{\pi}{2\rho_1}}\right| \qquad \Longrightarrow \qquad s_1^{\mu_1-\rho_1} \geqslant \left|s_1+te^{i\frac{\pi}{2\rho_1}}\right|^{\mu_1-\rho_1}\ ,$$

because $\mu_1 \leqslant \rho_1$, and by using Lemma 1.2.3, we obtain

$$\forall\, t>0 \qquad \left|\left(s_1+te^{i\frac{\pi}{2\rho_1}}\right)^{\mu_1-\rho_1}e^{i\omega\left(s_1+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}}\right| \leqslant s_1^{\mu_1-\rho_1}e^{-\omega t^{\rho_1}} \longrightarrow 0 \quad , \quad t\to+\infty\ .$$

In a similar way, we have

$$\begin{aligned}\phi^{(2)}(s_2,\omega) &= -(i\omega\rho_2)^{-1}s_2^{\mu_2-\rho_2}\,e^{-i\omega s_2^{\rho_2}}\\ &\quad-\frac{\mu_2-\rho_2}{i\omega\rho_2}\,e^{-i\frac{\pi}{2\rho_2}}\int_0^{+\infty}\left(s_2+te^{-i\frac{\pi}{2\rho_2}}\right)^{\mu_2-\rho_2-1}e^{-i\omega\left(s_2+te^{-i\frac{\pi}{2\rho_2}}\right)^{\rho_2}}dt\ .\end{aligned}$$

Furthermore, by the definitions of $k_j$ and $s_j := \varphi_j(q)$, we obtain

$$k_j(s_j) = U\big(\varphi_j^{-1}(s_j)\big)\,s_j^{1-\mu_j}\,(\varphi_j^{-1})'(s_j) = U(q)\,\varphi_j(q)^{1-\mu_j}\,(\varphi_j)'(q)^{-1}\ .$$

Now, in the case $j = 1$, we multiply the last expression by the expansion of $\phi^{(1)}(s_1, \omega)$ given by (2.3),

$$\phi^{(1)}(s_1,\omega)\,k_1(s_1)\,e^{i\omega\psi(p_1)} = (i\omega\rho_1)^{-1}\,e^{i\omega\left(s_1{}^{\rho_1}+\psi(p_1)\right)}\,U(q)\,\varphi_1(q)^{1-\rho_1}\,(\varphi_1)'(q)^{-1} - i\,\frac{\mu_1-\rho_1}{\rho_1}\,e^{i\omega\psi(p_1)}\,e^{i\frac{\pi}{2\rho_1}}\,k_1(s_1)\int_0^{+\infty}\left(s_1+te^{i\frac{\pi}{2\rho_1}}\right)^{\mu_1-\rho_1-1} e^{i\omega\left(s_1+te^{i\frac{\pi}{2\rho_1}}\right)^{\rho_1}}\,dt\,\omega^{-1}\,. \tag{2.4}$$

The definition of $\varphi_1(q)$ gives $\varphi_1(q)^{\rho_1} = \psi(q) - \psi(p_1)$ and by the regularity of $\varphi_1$, one has

$$\rho_1\,(\varphi_1)'(q)\,\varphi_1(q)^{\rho_1-1} = \frac{d}{dp}\Big[(\varphi_1)^{\rho_1}\Big](q) = \psi'(q)\,,$$

simplifying the first term in (2.4); moreover the integral in (2.4) can be written as an integral on the curve $\Lambda^{(1)}(s_1)$ in the complex plane. These considerations lead to

$$\phi^{(1)}(s_1,\omega)\,k_1(s_1)\,e^{i\omega\psi(p_1)} = -i\omega^{-1}\,e^{i\omega\psi(q)}\,\frac{U(q)}{\psi'(q)} - i\,\frac{\mu_1-\rho_1}{\rho_1}\,e^{i\omega\psi(p_1)}\,k_1(s_1)\int_{\Lambda^{(1)}(s_1)} z^{\mu_1-\rho_1-1}\,e^{i\omega z^{\rho_1}}\,dz\,\omega^{-1}\,.$$

By similar calculations, we obtain

$$\phi^{(2)}(s_2,\omega)\,k_2(s_2)\,e^{i\omega\psi(p_2)} = -i\omega^{-1}\,e^{i\omega\psi(q)}\,\frac{U(q)}{\psi'(q)} - i\,\frac{\mu_2-\rho_2}{\rho_2}\,e^{i\omega\psi(p_2)}\,k_2(s_2)\int_{\Lambda^{(2)}(s_2)} z^{\mu_2-\rho_2-1}\,e^{-i\omega z^{\rho_2}}\,dz\,\omega^{-1}\,.$$

Hence we remark that the difference (2.2) is equal to

$$\sum_{j=1}^{2}(-1)^j\,i\,\frac{\mu_j-\rho_j}{\rho_j}\,e^{i\omega\psi(p_j)}\,k_j(s_j)\int_{\Lambda^{(j)}(s_j)} z^{\mu_j-\rho_j-1}\,e^{(-1)^{j+1}i\omega z^{\rho_j}}\,dz\,\omega^{-1}\,.$$

Consequently, we are able to write the initial integral as follows,

$$\begin{aligned}\int_{p_1}^{p_2} U(p)e^{i\omega\psi(p)}dp &= -\phi^{(1)}(0,\omega)\,k_1(0)\,e^{i\omega\psi(p_1)} - e^{i\omega\psi(p_1)}\int_0^{s_1}\phi^{(1)}(s,\omega)(k_1)'(s)\,ds\\ &\quad - i\,\frac{\mu_1-\rho_1}{\rho_1}\,e^{i\omega\psi(p_1)}\,k_1(s_1)\int_{\Lambda^{(1)}(s_1)} z^{\mu_1-\rho_1-1}e^{i\omega z^{\rho_1}}\,dz\,\omega^{-1}\\ &\quad + \phi^{(2)}(0,\omega)\,k_2(0)\,e^{i\omega\psi(p_2)} + e^{i\omega\psi(p_2)}\int_0^{s_2}\phi^{(2)}(s,\omega)(k_2)'(s)\,ds\\ &\quad + i\,\frac{\mu_2-\rho_2}{\rho_2}\,e^{i\omega\psi(p_2)}\,k_2(s_2)\int_{\Lambda^{(2)}(s_2)} z^{\mu_2-\rho_2-1}e^{-i\omega z^{\rho_2}}\,dz\,\omega^{-1}\\ &=: \sum_{j=1,2}\Big(A^{(j)}(\omega) + R_1^{(j)}(\omega,q) + R_2^{(j)}(\omega,q)\Big)\,.\end{aligned}$$

According to the fourth step of the proof of Theorem 1.1.3, we have

$$\phi^{(j)}(0,\omega) = -\frac{1}{\rho_j}\,\Gamma\!\left(\frac{\mu_j}{\rho_j}\right)\,e^{(-1)^{j+1}i\frac{\pi}{2}\frac{\mu_j}{\rho_j}}\,\omega^{-\frac{\mu_j}{\rho_j}} =: (-1)^j\,\Theta^{(j)}(\rho_j,\mu_j)\,\omega^{-\frac{\mu_j}{\rho_j}}\,.$$



 

leading to the definition of $A^{(j)}(\omega)$,

$$A^{(j)}(\omega) := (-1)^j\,\phi^{(j)}(0,\omega)\,k_j(0)\,e^{i\omega\psi(p_j)} = e^{i\omega\psi(p_j)}\,k_j(0)\,\Theta^{(j)}(\rho_j,\mu_j)\,\omega^{-\frac{\mu_j}{\rho_j}}\ .$$

*Fifth step: Remainder estimates.* Using the last step of the proof of Theorem 1.1.3, we obtain

$$\left|e^{i\omega\psi(p_j)}\int_0^{s_j}\phi^{(j)}(s,\omega,\rho_j,\mu_j)\,(k_j)'(s)\,ds\right| \leqslant \frac{1}{\rho_j}\,\Gamma\!\left(\frac{1}{\rho_j}\right)\int_0^{s_j} s^{\mu_j-1}\left|(k_1)'(s)\right|ds\,\omega^{-\frac{1}{\rho_j}}\ .$$

Now let us estimate $R_2^{(j)}(\omega,q)$. We have

$$\begin{aligned}
&\left| i\,\frac{\mu_j-\rho_j}{\omega\rho_j}\,e^{i\omega\psi(p_j)}\,k_j(s_j)\,e^{(-1)^{j+1}i\frac{\pi}{2\rho_j}} \right. \\
&\qquad\qquad \left. \times\int_0^{+\infty}\left(s_j+te^{(-1)^{j+1}i\frac{\pi}{2\rho_j}}\right)^{\mu_j-\rho_j-1} e^{(-1)^{j+1}i\omega\left(s_j+te^{(-1)^{j+1}i\frac{\pi}{2\rho_j}}\right)^{\rho_j}}\,dt\right| \\
&\qquad \leqslant \frac{\rho_j-\mu_j}{\rho_j}\left|k_j(s_j)\right|\omega^{-1}\,s_j^{\mu_j-\rho_j-1}\int_0^{+\infty}\left|e^{(-1)^{j+1}i\omega\left(s_j+te^{(-1)^{j+1}i\frac{\pi}{2\rho_j}}\right)^{\rho_j}}\right|dt \qquad (2.5)\\
&\qquad \leqslant \frac{\rho_j-\mu_j}{\rho_j}\left|U\big(\varphi_j^{-1}(s_j)\big)\,(\varphi_j^{-1})'(s_j)\right|s_j^{-\rho_j}\,\omega^{-1}\int_0^{+\infty}e^{-\omega t^{\rho_j}}\,dt \qquad (2.6)\\
&\qquad = \frac{\rho_j-\mu_j}{\rho_j}\,\Gamma\!\left(\frac{1}{\rho_j}\right)\left|U(q)\,(\varphi_j)'(q)^{-1}\right|\varphi_j(q)^{-\rho_j}\,\omega^{-\left(1+\frac{1}{\rho_j}\right)}\ ; \qquad (2.7)
\end{aligned}$$

- (2.5): use the fact that $s_j \leqslant \left|s_j+te^{(-1)^{j+1}i\frac{\pi}{2\rho_j}}\right|$ ;
- (2.6): use the definition of the function $k_j$ and Lemma 1.2.3 ;
- (2.7): use the equalities $\displaystyle\int_0^{+\infty}e^{-\omega t^{\rho_j}}dt = \Gamma\!\left(\frac{1}{\rho_j}\right)\omega^{-\frac{1}{\rho_j}}$ and $q=\varphi_j^{-1}(s_j)$.

We remark finally that the decay rates of $A^{(j)}(\omega)$, $R_1^{(j)}(\omega,q)$ and $R_2^{(j)}(\omega,q)$ are $\omega^{-\frac{\mu_j}{\rho_j}}$, $\omega^{-\frac{1}{\rho_j}}$ and $\omega^{-\left(1+\frac{1}{\rho_j}\right)}$, respectively. Thus the decay rates of the remainder $R_1^{(j)}(\omega,q)$ related to $p_j$ and of the remainder $R_2^{(j)}(\omega,q)$ related to $q$ are faster than the one of the first term $A^{(j)}(\omega)$ related to $p_j$. This ends the proof. □

For fixed $q\in(p_1,p_2)$, we observe that $R_2^{(j)}(\omega,q)$ (for $j=1,2$) is always negligible as compared with $A^{(j)}(\omega)$ when $\omega$ tends to infinity, even if $\mu_j=1$. Nevertheless if $\mu_{j_0}=1$ for a certain $j_0\in\{1,2\}$ then the decay rates of $R_1^{(j_0)}(\omega,q)$ and $A^{(j_0)}(\omega)$ with respect to $\omega$ are the same. So we shall use the ideas of the proof of Theorem 1.1.7 to obtain a better decay rate for $R_1^{(j_0)}(\omega,q)$.

**2.1.4 Theorem.** *Assume $\mu_{j_0}=1$ and $\rho_{j_0}\geqslant 2$ for a certain $j_0\in\{1,2\}$. Suppose that the functions $\psi:[p_1,p_2]\longrightarrow\mathbb{R}$ and $U:(p_1,p_2)\longrightarrow\mathbb{C}$ satisfy Assumption* $(\mathrm{P1}_{\rho_1,\rho_2,1})$ *and*



 

*Assumption* $(\mathrm{A1}_{\mu_1,\mu_2,1})$*, respectively. Then the statement of Theorem 2.1.2 is still true and, for* $\gamma \in (0,1)$ *and*

$$\delta := \frac{\gamma+1}{\rho_{j_0}} \in \left( \frac{1}{\rho_{j_0}}, \frac{2}{\rho_{j_0}} \right) ,$$

*we have*

$$\forall\, \omega > 0 \qquad \left| R_1^{(j_0)}(\omega, q) \right| \leqslant L_{\gamma,\rho_{j_0}} \int_0^{s_{j_0}} s^{-\gamma} \left| (k_{j_0})'(s) \right| ds \; \omega^{-\delta} ,$$

*where* $L_{\gamma,\rho_{j_0}} := L_{\gamma,\rho_{j_0},1} > 0$*, with* $L_{\gamma,\rho_{j_0},1}$ *is given in Theorem 1.1.7.*

*Proof.* We obtain the above estimate by following the lines of the proof of Theorem 1.1.7. □

## 2.2 Approaching stationary points and amplitude singularities: the first and the error term between blow-up and decay

In this section, we consider a family of oscillatory integrals with respect to a large parameter $\omega$. In preparation for applications to the solution formula of the free Schrödinger equation, we suppose that the phase function is a polynomial of degree 2 and has its stationary point $p_0$ inside $(p_1, p_2)$, which contains the support of the amplitude. We suppose in addition that the amplitude has a singular point at $p_1$, the left endpoint of the integration interval.
The aim of this section is to furnish remainder estimates with explicit blow-up and to exploit this in order to find curves in the parameter space on which blow-up and decay balance out.

In the first result, we split the integral at the stationary point and then we expand the two resulting integrals by using the results of the previous section.

**2.2.1 Lemma.** *Let* $p_1 < p_2$ *be two finite real numbers. Let* $p_0 \in (p_1, p_2)$ *and* $c \in \mathbb{R}$ *be two parameters, and define* $\psi : [p_1, p_2] \longrightarrow \mathbb{R}$ *by*

$$\psi(p) := -(p-p_0)^2 + c .$$

*Define the following integrals for all* $\omega > 0$*,*

$$I^{(1)}(\omega, p_0) := \int_{p_1}^{p_0} U(p)\, e^{i\omega\psi(p)}\, dp \qquad , \qquad I^{(2)}(\omega, p_0) := \int_{p_0}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp ,$$

*where* $U$ *satisfies Assumption* $(\mathrm{A1}_{\mu,1,1})$ *on* $[p_1, p_2]$ *with* $\mu \in (0,1)$*, and* $U(p_2) = 0$*. Let us define* $\tilde{H}(\omega, \psi, U)$ *and* $\tilde{K}_\mu(\omega, \psi, U)$ *as follows,*

$$\tilde{H}(\omega, \psi, U) := \sqrt{\pi}\, e^{-i\frac{\pi}{4}}\, e^{i\omega c}\, \tilde{u}(p_0) \qquad , \qquad \tilde{K}_\mu(\omega, \psi, U) := \frac{\Gamma(\mu)}{2^\mu}\, e^{i\frac{\pi\mu}{2}}\, e^{i\omega\psi(p_1)}\, \tilde{u}(p_1) .$$

*Then*

- *we have*

$$\Big| I^{(1)}(\omega, p_0) \; - \; \tilde{K}_\mu(\omega, \psi, U)\,(p_0 - p_1)^{-\mu}\,\omega^{-\mu} - \frac{1}{2}\,\tilde{H}(\omega, \psi, U)\,(p_0 - p_1)^{\mu-1}\,\omega^{-\frac{1}{2}} \Big| \leqslant \sum_{k=1}^{6} R_k^{(1)}(U)\,(p_0 - p_1)^{-\alpha_k^{(1)}}\,\omega^{-\beta_k^{(1)}} \;,$$

  *where the constants* $R_k^{(1)}(U) \geqslant 0$ *and the exponents* $\alpha_k^{(1)} \in \mathbb{R}$, $\beta_k^{(1)} > 0$ *are given in the proof ;*

- *we have*

$$\Big| I^{(2)}(\omega, p_0) \; - \; \frac{1}{2}\,\tilde{H}(\omega, \psi, U)\,(p_0 - p_1)^{\mu-1}\,\omega^{-\frac{1}{2}} \Big| \leqslant \sum_{k=1}^{2} R_k^{(2)}(U)\,(p_0 - p_1)^{-\alpha_k^{(2)}}\,\omega^{-\beta_k^{(2)}} \;,$$

  *where the constants* $R_k^{(2)}(U) \geqslant 0$ *and the exponents* $\alpha_k^{(2)} \in \mathbb{R}$, $\beta_k^{(2)} > 0$ *are given in the proof.*

*Proof. Study of* $I^{(1)}(\omega, p_0)$. For all $p \in [p_1, p_0]$, we have

$$\psi'(p) = 2(p_0 - p) \;.$$

By setting $\tilde{\psi} := 2$, we observe that $\psi$ verifies Assumption $(\mathrm{P1}_{1,2,N})$ on $[p_1, p_0]$, for all $N \geqslant 1$, and hence Theorem 2.1.2 is applicable. Here we choose

$$q := q(p_0) = p_1 + \frac{p_0 - p_1}{2} = p_0 - \frac{p_0 - p_1}{2} \;,$$

for simplicity. Then we obtain

$$I^{(1)}(\omega, p_0) = \sum_{j=1,2} \Big( A^{(j)}(\omega, p_0) + R_1^{(j)}(\omega, p_0) + R_2^{(j)}(\omega, p_0) \Big) \;,$$

with

- $A^{(1)}(\omega, p_0) = e^{i\omega\psi(p_1)}\,k_1(0)\,\Theta^{(1)}(1, \mu)\,\omega^{-\mu} = \Gamma(\mu)\,e^{i\frac{\pi\mu}{2}}\,e^{i\omega\psi(p_1)}\,k_1(0)\,\omega^{-\mu}$ ,
- $A^{(2)}(\omega, p_0) = e^{i\omega\psi(p_0)}\,k_2(0)\,\Theta^{(2)}(2, 1)\,\omega^{-\frac{1}{2}} = -\dfrac{\sqrt{\pi}}{2}\,e^{-i\frac{\pi}{4}}\,e^{i\omega\psi(p_0)}\,k_2(0)\,\omega^{-\frac{1}{2}}$ .

To compute the values of $k_1(0)$ and $k_2(0)$, let us study the functions $(\varphi_1^{-1})'$ and $(\varphi_2^{-1})'$. On the one hand, we obtain by the definition of $\varphi_1$,

$$\varphi_1(p) = \psi(p) - \psi(p_1) \qquad \Longrightarrow \qquad (\varphi_1)'(p) = \psi'(p) = 2(p_0 - p) \;, \tag{2.8}$$

for all $p \in [p_1, q]$. On the other hand, by the definition of $\varphi_2$ and the expression of $\psi$, one has

$$\varphi_2(p) = \big(\psi(p_0) - \psi(p)\big)^{\frac{1}{2}} = (p_0 - p) \;,$$

for every $p \in [q, p_0]$. So $(\varphi_2^{-1})'(s) = -1$ and $(\varphi_2^{-1})''(s) = 0$ for all $s \in [0, s_2]$. Then it is possible to compute $k_1(0)$ by using the representation of $k_1$ given in the proof of Proposition 1.2.2,

$$k_1(s) = \left( \int_0^1 (\varphi_1^{-1})'(sy)\, dy \right)^{\mu-1} \tilde{u}\big(\varphi_1^{-1}(s)\big)(\varphi_1^{-1})'(s) \tag{2.9}$$
$$\underset{s \longrightarrow 0^+}{\longrightarrow} (\varphi_1^{-1})'(0)^{\mu}\, \tilde{u}\big(\varphi_1^{-1}(0)\big) = \psi'(p_1)^{-\mu}\, \tilde{u}(p_1) = \big(2(p_0 - p_1)\big)^{-\mu}\, \tilde{u}(p_1) \; .$$

The value $k_2(0)$ can be computed in a direct way since $p_0$ is a regular point of the amplitude:

$$k_2(s) = U\big(\varphi_2^{-1}(s)\big)(\varphi_2^{-1})'(s) = -U\big(\varphi_2^{-1}(s)\big) \underset{s \longrightarrow 0^+}{\longrightarrow} -U(p_0) = -(p_0 - p_1)^{\mu-1}\, \tilde{u}(p_0) \; .$$

Therefore we obtain

- $A^{(1)}(\omega, p_0) = \dfrac{\Gamma(\mu)}{2^{\mu}}\, e^{i\frac{\pi\mu}{2}}\, e^{i\omega\psi(p_1)}\, \tilde{u}(p_1)\, (p_0 - p_1)^{-\mu}\, \omega^{-\mu}$ ,
- $A^{(2)}(\omega, p_0) = \dfrac{\sqrt{\pi}}{2}\, e^{-i\frac{\pi}{4}}\, e^{i\omega c}\, \tilde{u}(p_0)\, (p_0 - p_1)^{\mu-1}\, \omega^{-\frac{1}{2}}$ .

Now let us control precisely the remainder terms. To do so, we have to study the functions $k_j$ and $\varphi_j$. Firstly, the combination of the equalities $\frac{1}{2}(p_0 - p_1) = p_0 - q = q - p_1$ with (2.8) leads to

$$(p_0 - p_1) = 2(p_0 - q) \leqslant (\varphi_1)'(p) \leqslant 2(p_0 - p_1) \; , \tag{2.10}$$

for all $p \in [p_1, q]$. It follows

$$\forall\, s \in [0, s_1] \qquad \big(2(p_0 - p_1)\big)^{-1} \leqslant (\varphi_1^{-1})'(s) \leqslant (p_0 - p_1)^{-1} \; .$$

Moreover by the equality $(\varphi_1^{-1})''(s) = -(\varphi_1)''\big(\varphi_1^{-1}(s)\big)\, (\varphi_1^{-1})'(s)^3$, we have

$$\forall\, s \in [0, s_1] \qquad (\varphi_1^{-1})''(s) = 2\, (\varphi_1^{-1})'(s)^3 \leqslant 2\, (p_0 - p_1)^{-3} \; .$$

Then it is possible compute the value of $s_1$ by using its definition,

$$s_1 = \varphi_1(q) = \psi(q) - \psi(p_0) = \frac{3}{4}\, (p_0 - p_1)^2 \leqslant (p_0 - p_1)^2 \; . \tag{2.11}$$

Concerning $s_2$, we have

$$s_2 = \varphi_2(q) = (p_0 - q) = \frac{1}{2}\, (p_0 - p_1) \leqslant p_0 - p_1 \; . \tag{2.12}$$

Now we study the functions $k_j$. For this purpose, we shall use the expression of $k_1$ given in (2.9). Since $\psi$ satisfies Assumption ($\mathrm{P1}_{1,2,N}$) on $[p_1, p_0]$ for all $N \geqslant 1$, it follows that $\varphi_1$ is a $\mathcal{C}^{N+1}$-diffeomorphism by Proposition 1.2.1. Thus one has the ability to differentiate under the integral sign the function $s \longmapsto \int_0^1 (\varphi_1^{-1})'(sy)dy$. Hence for all $s \in [0, s_1]$, we have

$$\begin{aligned}(k_1)'(s) =&(\mu-1)\left(\int_0^1 y\,(\varphi_1^{-1})''(sy)\,dy\right)\left(\int_0^1 (\varphi_1^{-1})'(sy)\,dy\right)^{\mu-2}\tilde{u}\big(\varphi_1^{-1}(s)\big)\,(\varphi_1^{-1})'(s)\\ &+\left(\int_0^1 (\varphi_1^{-1})'(sy)\,dy\right)^{\mu-1}\tilde{u}'\big(\varphi_1^{-1}(s)\big)\,(\varphi_1^{-1})'(s)^2\\ &+\left(\int_0^1 (\varphi_1^{-1})'(sy)\,dy\right)^{\mu-1}\tilde{u}\big(\varphi_1^{-1}(s)\big)\,(\varphi_1^{-1})''(s)\;.\end{aligned}$$

In consequence, we obtain the following estimate:

$$\begin{aligned}\big\|(k_1)'\big\|_{L^\infty(0,s_1)} &\leqslant \frac{1-\mu}{2}\,2\,(p_0-p_1)^{-3}\,\big(2(p_0-p_1)\big)^{2-\mu}\,\|\tilde{u}\|_{L^\infty(p_1,p_2)}\,(p_0-p_1)^{-1}\\ &\quad+\big(2(p_0-p_1)\big)^{1-\mu}\,\|\tilde{u}'\|_{L^\infty(p_1,p_2)}\,(p_0-p_1)^{-2}\\ &\quad+\big(2(p_0-p_1)\big)^{1-\mu}\,\|\tilde{u}\|_{L^\infty(p_1,p_2)}\,2(p_0-p_1)^{-3}\\ &\leqslant 2^{1-\mu}\,\|\tilde{u}\|_{W^{1,\infty}(p_1,p_2)}\Big(2(2-\mu)(p_0-p_1)^{-(2+\mu)}+(p_0-p_1)^{-(1+\mu)}\Big)\;. \qquad (2.13)\end{aligned}$$

To estimate $(k_2)'$, we start by differentiating the expression of $k_2$ given in Definition 2.1.1 by using the fact that $U(p)=(p-p_1)^{\mu-1}\tilde{u}(p)$. This leads to

$$(k_2)'(s)=\Big((\mu-1)\big(\varphi_2^{-1}(s)-p_1\big)^{\mu-2}\,\tilde{u}\big(\varphi_2^{-1}(s)\big)+\big(\varphi_2^{-1}(s)-p_1\big)^{\mu-1}\,\tilde{u}'\big(\varphi_2^{-1}(s)\big)\Big)(\varphi_2^{-1})'(s)^2\;,$$

for all $s\in[0,s_2]$. We employ then the fact that $\varphi_2^{-1}(s)\in[q,p_2]$ for any $s\in[0,s_2]$ and the equality $(\varphi_2^{-1})'(s)=-1$ to obtain

$$\begin{aligned}\big\|(k_2)'\big\|_{L^\infty(0,s_2)} &\leqslant \Big((1-\mu)\,2^{2-\mu}(p_0-p_1)^{\mu-2}\,\|\tilde{u}\|_{L^\infty(p_1,p_2)}+2^{1-\mu}(p_0-p_1)^{\mu-1}\|\tilde{u}'\|_{L^\infty(p_1,p_2)}\Big)\\ &\leqslant 2^{1-\mu}\,\|\tilde{u}\|_{W^{1,\infty}(p_1,p_2)}\Big(2(1-\mu)(p_0-p_1)^{\mu-2}+(p_0-p_1)^{\mu-1}\Big)\;. \qquad (2.14)\end{aligned}$$

These considerations permit to estimate the four remainders.

- *Estimate of* $R_1^{(1)}(\omega,p_0)$. Theorem 2.1.2 furnishes an estimate of $R^{(1)}(\omega,p_0)$. We combine it with the estimates of $(k_1)'$ given in (2.13) and of $s_1$ given in (2.11):

$$\begin{aligned}\Big|R_1^{(1)}(\omega,p_0)\Big| &\leqslant \int_0^{s_1} s^{\mu-1}\big|(k_1)'(s)\big|\,ds\,\omega^{-1}\\ &\leqslant \frac{1}{\mu}\,s_1^{\mu}\,\big\|(k_1)'\big\|_{L^\infty(0,s_1)}\,\omega^{-1}\\ &\leqslant \frac{2^{1-\mu}}{\mu}\,\|\tilde{u}\|_{W^{1,\infty}(p_1,p_2)}\Big(2(2-\mu)(p_0-p_1)^{\mu-2}+(p_0-p_1)^{\mu-1}\Big)\,\omega^{-1}\\ &=: R_1^{(1)}(U)\,(p_0-p_1)^{-\alpha_1^{(1)}}\,\omega^{-\beta_1^{(1)}}+R_2^{(1)}(U)\,(p_0-p_1)^{-\alpha_2^{(1)}}\,\omega^{-\beta_2^{(1)}}\;, \qquad (2.15)\end{aligned}$$

  where

  - $R_1^{(1)}(U) := \dfrac{2^{2-\mu}}{\mu}\,(2-\mu)\,\|\tilde{u}\|_{W^{1,\infty}(p_1,p_2)}\quad,\quad R_2^{(1)}(U) := \dfrac{2^{1-\mu}}{\mu}\,\|\tilde{u}\|_{W^{1,\infty}(p_1,p_2)}\;,$
  - $\alpha_1^{(1)} := 2-\mu\quad,\quad \alpha_2^{(1)} := 1-\mu\quad,\quad \beta_1^{(1)}=\beta_2^{(1)} := 1\;.$



 

- *Estimate of* $R_2^{(1)}(\omega,p_0)$. The estimate of $R_2^{(1)}(\omega,p_0)$ from Theorem 2.1.2 provides

$$\begin{aligned}\left|R_2^{(1)}(\omega,p_0)\right| &\leqslant (1-\mu)\,\left|U(q)\,(\varphi_1)'(q)^{-1}\right|\,\varphi_1(q)^{-1}\,\omega^{-2}\\ &\leqslant \frac{1-\mu}{2^{\mu-1}}\,\|\tilde{u}\|_{L^\infty(p_1,p_2)}(p_0-p_1)^{\mu-1}\,(p_0-p_1)^{-1}\,\left(\frac{3}{4}\,(p_0-p_1)^2\right)^{-1}\,\omega^{-2}\\ &= (1-\mu)\,\frac{2^{3-\mu}}{3}\,\|\tilde{u}\|_{L^\infty(p_1,p_2)}\,(p_0-p_1)^{\mu-4}\,\omega^{-2}\\ &=: R_3^{(1)}(U)\,(p_0-p_1)^{-\alpha_3^{(1)}}\,\omega^{-\beta_3^{(1)}}\;, \end{aligned}\tag{2.16}$$

where the definition of $U$, inequality (2.10) and the value of $s_1$ given in (2.11) were used. Here we define:

- $R_3^{(1)}(U) := (1-\mu)\,\dfrac{2^{3-\mu}}{3}\,\|\tilde{u}\|_{L^\infty(p_1,p_2)}$ ,
- $\alpha_3^{(1)} := 4-\mu \quad , \quad \beta_3^{(1)} := 2$ .

- *Estimate of* $R_1^{(2)}(\omega,p_0)$. Here $\mu_2 = 1$, so we have to employ the estimate of $R_1^{(2)}(\omega,p_0)$ provided by Theorem 2.1.4,

$$\begin{aligned}\left|R_1^{(2)}(\omega,p_0)\right| &\leqslant L_{\gamma,2}\int_0^{s_2} s^{-\gamma}\left|(k_2)'(s)\right|ds\;\omega^{-\delta}\\ &\leqslant \frac{L_{\gamma,2}}{1-\gamma}\,{s_2}^{1-\gamma}\left\|(k_2)'\right\|_{L^\infty(0,s_2)}\,\omega^{-\delta}\\ &\leqslant \frac{L_{\gamma,2}}{1-\gamma}\,2^{1-\mu}\,\|\tilde{u}\|_{W^{1,\infty}(p_1,p_2)}\Big(2(1-\mu)(p_0-p_1)^{\mu-1-\gamma}+(p_0-p_1)^{\mu-\gamma}\Big)\,\omega^{-\delta}\\ &=: R_4^{(1)}(U)\,(p_0-p_1)^{-\alpha_4^{(1)}}\,\omega^{-\beta_4^{(1)}} + R_5^{(1)}(U)\,(p_0-p_1)^{-\alpha_5^{(1)}}\,\omega^{-\beta_5^{(1)}}\;, \end{aligned}\tag{2.17}$$

where the last inequality was obtained by using (2.12) and (2.14). Here the parameter $\delta$ is arbitrarily chosen in $\left(\frac{1}{2},1\right)$, and the parameter $\gamma\in(0,1)$ is given by $\gamma = 2\delta-1$. Here we define:

- $R_4^{(1)}(U) := \dfrac{L_{\gamma,2}}{1-\gamma}\,2^{2-\mu}\,(1-\mu)\,\|\tilde{u}\|_{W^{1,\infty}(p_1,p_2)}$ , $R_5^{(1)}(U) := \dfrac{L_{\gamma,2}}{1-\gamma}\,2^{1-\mu}\,\|\tilde{u}\|_{W^{1,\infty}(p_1,p_2)}$ ,
- $\alpha_4^{(1)} := -\mu+1+\gamma \quad , \quad \alpha_5^{(1)} := \gamma-\mu \quad , \quad \beta_4^{(1)} = \beta_5^{(1)} := \delta$ .

- *Estimate of* $R_2^{(2)}(\omega,p_0)$. We employ Theorem 2.1.2 once again to control $R_2^{(2)}(\omega,p_0)$. Using the definition of $U$, the relation $(\varphi_2^{-1})' = -1$ and the value of $s_2$ given in (2.12) leads to

$$\begin{aligned}\left|R_2^{(2)}(\omega,p_0)\right| &\leqslant \frac{1}{2}\,\Gamma\!\left(\frac{1}{2}\right)\,\left|U(q)\,(\varphi_2)'(q)^{-1}\right|\,\varphi_2(q)^{-2}\,\omega^{-\frac{3}{2}}\\ &\leqslant \frac{\sqrt{\pi}}{2^\mu}\,\|\tilde{u}\|_{L^\infty(p_1,p_2)}\,(p_0-p_1)^{\mu-1}\,\big(2^{-1}(p_0-p_1)\big)^{-2}\,\omega^{-\frac{3}{2}}\\ &= \frac{\sqrt{\pi}}{2^{\mu-2}}\,\|\tilde{u}\|_{L^\infty(p_1,p_2)}(p_0-p_1)^{\mu-3}\,\omega^{-\frac{3}{2}}\\ &=: R_6^{(1)}(U)\,(p_0-p_1)^{-\alpha_6^{(1)}}\,\omega^{-\beta_6^{(1)}}\;, \end{aligned}\tag{2.18}$$



 

where

- $R_6^{(1)}(U) := \dfrac{\sqrt{\pi}}{2^{\mu-2}} \, \|\tilde{u}\|_{L^\infty(p_1,p_2)}$ ,
- $\alpha_6^{(1)} := 3 - \mu \quad , \quad \beta_6^{(1)} := \dfrac{3}{2}$ .

*Study of* $I^{(2)}(\omega, p_0)$. Firstly we remark that $\psi'$ is negative for all $p \in [p_0, p_2]$. To apply Theorem 2.1.2, we make the change of variables $p \mapsto -p$ in order to have an increasing phase. We obtain

$$I^{(2)}(\omega, p_0) = \int_{p_0}^{p_2} U(p) e^{i\omega\psi(p)} \, dp = \int_{\check{p}_2}^{\check{p}_0} \check{U}(p) e^{i\omega\check{\psi}(p)} \, dp \, ,$$

where we put $\check{U}(p) := U(-p)$, $\check{\psi}(p) := \psi(-p)$, $\check{p}_0 := -p_0$ and $\check{p}_2 := -p_2$.
Thanks to this substitution, $\check{\psi}$ is now an increasing function that satisfies Assumption $(\mathrm{P1}_{1,2,N})$ on $[\check{p}_2, \check{p}_0]$, for all $N \geqslant 1$, and by hypothesis $\check{U}$ verifies $(\mathrm{A1}_{1,1,1})$ on $[\check{p}_2, \check{p}_0]$. Furthermore we remark that $\check{p}_2$ is not a singular point of the amplitude and not a stationary point. This observation indicates the non-necessity of a cutting-point $q$. Hence, in the notations of Theorem 2.1.2, we employ only the expansion of the integral $\tilde{I}^{(2)}(\omega, p_0)$ with $p_2 := \check{p}_0$ and $q := \check{p}_2$. So we obtain from (2.1),

$$\begin{aligned} I^{(2)}(\omega, p_0) = \phi^{(2)}(0, \omega, 2, 1) \, k_2(0) \, e^{i\omega\check{\psi}(\check{p}_0)} - \phi^{(2)}(s_2, \omega, 2, 1) \, k_2(s_2) \, e^{i\omega\check{\psi}(\check{p}_0)} \\ + e^{i\omega\check{\psi}(\check{p}_0)} \int_0^{s_2} \phi^{(2)}(s, \omega, 2, 1) \, (k_2)'(s) \, ds \, . \end{aligned}$$

Let us compute explicitly the first terms by studying the function $\varphi_2$. By the definition of $\varphi_2$ and the expression of $\psi$, we have

$$\varphi_2(p) = \big(\check{\psi}(\check{p}_0) - \check{\psi}(p)\big)^{\frac{1}{2}} = \check{p}_0 - p \, ,$$

for all $p \in [\check{p}_2, \check{p}_0]$. It follows that $(\varphi_2)'(p) = (\varphi_2^{-1})'(s) = -1$, and by the definition of $k_2$, we obtain

$$k_2(s) = \check{U}\big(\varphi_2^{-1}(s)\big) (\varphi_2^{-1})'(s) = -\check{U}\big(\varphi_2^{-1}(s)\big) \, .$$

Since $\varphi_2(\check{p}_0) = 0$ and $\varphi_2(\check{p}_2) = s_2$, we have

$$k_2(0) = -\check{U}(\check{p}_0) = -U(p_0) \qquad , \qquad k_2(s_2) = -U(p_2) = 0 \, ,$$

by the hypothesis on $U$. Combining this with the expression of $\Theta^{(2)}(2,1)$ coming from Theorem 2.1.2, we obtain

$$\begin{aligned} I^{(2)}(\omega, p_0) = \frac{\sqrt{\pi}}{2} \, e^{-i\frac{\pi}{4}} \, e^{i\omega\psi(p_0)} \, \tilde{u}(p_0) \, (p_0 - p_1)^{\mu-1} \, \omega^{-\frac{1}{2}} \\ + e^{i\omega\psi(p_0)} \int_0^{s_2} \phi^{(2)}(s, \omega, 2, 1) \, (k_2)'(s) \, ds \, . \end{aligned}$$

As in the study of $I^{(1)}(\omega, p_0)$, we estimate the remainder term. Firstly, we bound the number $s_2$ as follows:

$$s_2 = \check{p}_0 - \check{p}_2 = p_2 - p_0 \leqslant p_2 - p_1 \, .$$



 

Now we establish an estimate of the first derivative of $k_2$. By the definition of this function and by the fact that $(\varphi_2^{-1})' = -1$, we have

$$(k_2)'(s) = (1-\mu)\left(\check{p}_1 - \varphi_2^{-1}(s)\right)^{\mu-2} \check{\tilde{u}}\left(\varphi_2^{-1}(s)\right) + \left(\check{p}_1 - \varphi_2^{-1}(s)\right)^{\mu-1} \left(\check{\tilde{u}}\right)' \left(\varphi_2^{-1}(s)\right) \ ,$$

where $\check{p}_1 := -p_1$. Since $\varphi_2^{-1}(s) \in [\check{p}_2, \check{p}_0]$, it follows

$$\left\|(k_2)'\right\|_{L^\infty(0,s_2)} \leqslant \left((1-\mu)(p_0-p_1)^{\mu-2} + (p_0-p_1)^{\mu-1}\right) \|\tilde{u}\|_{W^{1,\infty}(p_1,p_2)} \ .$$

We combine this inequality with the estimate of the remainder given in Theorem 2.1.2 to obtain

$$\begin{aligned} \left|R_1^{(2)}(\omega,p_0)\right| &\leqslant \frac{L_{\gamma,2}}{1-\gamma}\,(p_2-p_1)^{1-\gamma}\,\|\tilde{u}\|_{W^{1,\infty}(p_1,p_2)} \\ &\qquad \times \left((1-\mu)(p_0-p_1)^{\mu-2} + (p_0-p_1)^{\mu-1}\right)\omega^{-\delta} \\ &=: R_1^{(2)}(U)\,(p_0-p_1)^{-\alpha_1^{(2)}}\,\omega^{-\beta_1^{(2)}} + R_2^{(2)}(U)\,(p_0-p_1)^{-\alpha_2^{(2)}}\,\omega^{-\beta_2^{(2)}} \ , \end{aligned} \tag{2.19}$$

where $\gamma$ and $\delta$ are defined above. Here we define:

- $R_1^{(2)}(U) := \dfrac{L_{\gamma,2}}{1-\gamma}\,(1-\mu)\,(p_2-p_1)^{1-\gamma}\,\|\tilde{u}\|_{W^{1,\infty}(p_1,p_2)}$ ,

  $R_2^{(2)}(U) := \dfrac{L_{\gamma,2}}{1-\gamma}\,(p_2-p_1)^{1-\gamma}\,\|\tilde{u}\|_{W^{1,\infty}(p_1,p_2)}$ ,
- $\alpha_1^{(2)} := 2-\mu \quad , \quad \alpha_2^{(2)} := 1-\mu \quad , \quad \beta_1^{(2)} = \beta_2^{(2)} := \delta$ .

□

Employing the preceding lemma, we derive asymptotic expansions of the initial oscillatory integral with explicit error estimates. We distinguish three cases depending on the strength of the singularity for readability.

**2.2.2 Theorem.** *Under the assumptions of Lemma 2.2.1, let us define $\tilde{H}(\omega,\psi,U)$ and $\tilde{K}(\omega,\psi,U)$ as follows,*

$$\tilde{H}(\omega,\psi,U) := \sqrt{\pi}\,e^{-i\frac{\pi}{4}}\,e^{i\omega c}\,\tilde{u}(p_0) \qquad , \qquad \tilde{K}_\mu(\omega,\psi,U) := \frac{\Gamma(\mu)}{2^\mu}\,e^{i\frac{\pi\mu}{2}}\,e^{i\omega\psi(p_1)}\,\tilde{u}(p_1) \ .$$

*Then we have*

- *Case $\mu > \frac{1}{2}$ :*

$$\left|\int_{p_1}^{p_2} U(p)\,e^{i\omega\psi(p)}\,dp - \tilde{H}(\omega,\psi,U)\,(p_0-p_1)^{\mu-1}\,\omega^{-\frac{1}{2}}\right| \leqslant \sum_{k=1}^{9} \tilde{R}_k^{(1)}(U)\,(p_0-p_1)^{-\tilde{\alpha}_k^{(1)}}\,\omega^{-\tilde{\beta}_k^{(1)}} \ ,$$

  *where the constants $\tilde{R}_k^{(1)}(U) \geqslant 0$ and the exponents $\tilde{\alpha}_k^{(1)} \in \mathbb{R}$, $\tilde{\beta}_k^{(1)} > \frac{1}{2}$ are given in the proof ;*



 

- *Case* $\mu = \dfrac{1}{2}$ *:*

$$\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp - \Big( \tilde{H}(\omega,\psi,U) + \tilde{K}_\mu(\omega,\psi,U) \Big)\, (p_0 - p_1)^{-\frac{1}{2}}\, \omega^{-\frac{1}{2}} \right| \leqslant \sum_{k=1}^{8} \tilde{R}_k^{(2)}(U)\, (p_0 - p_1)^{-\tilde{\alpha}_k^{(2)}}\, \omega^{-\tilde{\beta}_k^{(2)}} \ ,$$

*where the constants* $\tilde{R}_k^{(2)}(U) \geqslant 0$ *and the exponents* $\tilde{\alpha}_k^{(2)} \in \mathbb{R}$, $\tilde{\beta}_k^{(2)} > \frac{1}{2}$ *are given in the proof ;*

- *Case* $\mu < \dfrac{1}{2}$ *:*

$$\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp - \tilde{K}_\mu(\omega,\psi,U)\, (p_0 - p_1)^{-\mu}\, \omega^{-\mu} \right| \leqslant \sum_{k=1}^{9} \tilde{R}_k^{(3)}(U)\, (p_0 - p_1)^{-\tilde{\alpha}_k^{(3)}}\, \omega^{-\tilde{\beta}_k^{(3)}} \ ,$$

*where the constants* $\tilde{R}_k^{(3)}(U) \geqslant 0$ *and the exponents* $\tilde{\alpha}_k^{(3)} \in \mathbb{R}$, $\tilde{\beta}_k^{(3)} > \mu$ *are given in the proof.*

*Proof.* This result is a consequence of Lemma 2.2.1. We start by splitting the integral as follows,

$$\int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp = \int_{p_1}^{p_0} \dots + \int_{p_0}^{p_2} \dots =: I^{(1)}(\omega, p_0) + I^{(2)}(\omega, p_0) \ .$$

Then we apply Lemma 2.2.1 by distinguishing the three following cases:



 

- *Case* $\mu > \dfrac{1}{2}$ *:*

$$\begin{aligned}
&\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp - \tilde{H}(\omega,\psi,U)\,(p_0-p_1)^{\mu-1}\,\omega^{-\frac{1}{2}} \right| \\
&\quad = \bigg| \int_{p_1}^{p_0} U(p)\, e^{i\omega\psi(p)}\, dp - \frac{1}{2}\,\tilde{H}(\omega,\psi,U)\,(p_0-p_1)^{\mu-1}\,\omega^{-\frac{1}{2}} \\
&\qquad\quad + \int_{p_0}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp - \frac{1}{2}\,\tilde{H}(\omega,\psi,U)\,(p_0-p_1)^{\mu-1}\,\omega^{-\frac{1}{2}} \bigg| \\
&\quad \leqslant \left| \tilde{K}_\mu(\omega,\psi,U)\,(p_0-p_1)^{-\mu}\,\omega^{-\mu} \right| + \sum_{k=1}^{6} R_k^{(1)}(U)\,(p_0-p_1)^{-\alpha_k^{(1)}}\,\omega^{-\beta_k^{(1)}} \\
&\qquad\quad + \sum_{k=1}^{2} R_k^{(2)}(U)\,(p_0-p_1)^{-\alpha_k^{(2)}}\,\omega^{-\beta_k^{(2)}} \\
&\quad \leqslant \frac{\Gamma(\mu)}{2^\mu}\,\|\tilde{u}\|_{L^\infty(p_1,p_2)}\,(p_0-p_1)^{-\mu}\,\omega^{-\mu} + \sum_{k=1}^{6} R_k^{(1)}(U)\,(p_0-p_1)^{-\alpha_k^{(1)}}\,\omega^{-\beta_k^{(1)}} \\
&\qquad\quad + \sum_{k=1}^{2} R_k^{(2)}(U)\,(p_0-p_1)^{-\alpha_k^{(2)}}\,\omega^{-\beta_k^{(2)}} \\
&\quad =: \sum_{k=1}^{9} \tilde{R}_k^{(1)}(U)\,(p_0-p_1)^{-\tilde{\alpha}_k^{(1)}}\,\omega^{-\tilde{\beta}_k^{(1)}}\ ,
\end{aligned}$$

where

- $\tilde{R}_1^{(1)}(U) := \dfrac{\Gamma(\mu)}{2^\mu}\,\|\tilde{u}\|_{L^\infty(p_1,p_2)} \quad , \quad \tilde{\alpha}_1^{(1)} := \mu \quad , \quad \tilde{\beta}_1^{(1)} := \mu\ ;$
- $\tilde{R}_{k+1}^{(1)}(U) := R_k^{(1)}(U) \quad , \quad \tilde{\alpha}_{k+1}^{(1)} := \alpha_k^{(1)} \quad , \quad \tilde{\beta}_{k+1}^{(1)} := \beta_k^{(1)} \qquad k = 1,\dots,6\ ;$
- $\tilde{R}_{k+7}^{(1)}(U) := R_k^{(2)}(U) \quad , \quad \tilde{\alpha}_{k+7}^{(1)} := \alpha_k^{(2)} \quad , \quad \tilde{\beta}_{k+7}^{(1)} := \beta_k^{(2)} \qquad k = 1, 2\ .$

Each $\tilde{\beta}_k^{(1)}$ is strictly larger than $\frac{1}{2}$, ensuring that the decay rate of each remainder term is faster than $\omega^{-\frac{1}{2}}$.



 

- *Case* $\mu = \dfrac{1}{2}$ *:*

$$\begin{aligned}
&\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp - \Big( \tilde{H}(\omega,\psi,U) + \tilde{K}_{\frac{1}{2}}(\omega,\psi,U) \Big)\, (p_0 - p_1)^{\mu-1}\, \omega^{-\frac{1}{2}} \right| \\
&\qquad = \left| \int_{p_1}^{p_0} U(p)\, e^{i\omega\psi(p)}\, dp - \left( \frac{1}{2}\, \tilde{H}(\omega,\psi,U) + \tilde{K}_{\frac{1}{2}}(\omega,\psi,U) \right)\, (p_0 - p_1)^{\mu-1}\, \omega^{-\frac{1}{2}} \right. \\
&\qquad\qquad \left. + \int_{p_0}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp - \frac{1}{2}\, \tilde{H}(\omega,\psi,U)\, (p_0 - p_1)^{\mu-1}\, \omega^{-\frac{1}{2}} \right| \\
&\qquad \leqslant \sum_{k=1}^{6} R_k^{(1)}(U)\, (p_0 - p_1)^{-\alpha_k^{(1)}}\, \omega^{-\beta_k^{(1)}} + \sum_{k=1}^{2} R_k^{(2)}(U)\, (p_0 - p_1)^{-\alpha_k^{(2)}}\, \omega^{-\beta_k^{(2)}} \\
&\qquad =: \sum_{k=1}^{8} \tilde{R}_k^{(2)}(U)\, (p_0 - p_1)^{-\tilde{\alpha}_k^{(2)}}\, \omega^{-\tilde{\beta}_k^{(2)}}\ ,
\end{aligned}$$

  where

  - $\tilde{R}_k^{(2)}(U) := R_k^{(1)}(U) \quad , \quad \tilde{\alpha}_k^{(2)} := \alpha_k^{(1)} \quad , \quad \tilde{\beta}_k^{(2)} := \beta_k^{(1)} \qquad k = 1, \ldots, 6\ ;$
  - $\tilde{R}_{k+6}^{(2)}(U) := R_k^{(2)}(U) \quad , \quad \tilde{\alpha}_{k+6}^{(2)} := \alpha_k^{(2)} \quad , \quad \tilde{\beta}_{k+6}^{(2)} := \beta_k^{(2)} \qquad k = 1, 2\ .$

  As above, we have $\tilde{\beta}_k^{(2)} > \frac{1}{2}$.

- *Case* $\mu < \dfrac{1}{2}$ *:*

$$\begin{aligned}
&\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp - \tilde{K}_\mu(\omega,\psi,U)\, (p_0 - p_1)^{-\mu}\, \omega^{-\mu} \right| \\
&\qquad = \left| \int_{p_1}^{p_0} U(p)\, e^{i\omega\psi(p)}\, dp - \tilde{K}_\mu(\omega,\psi,U)\, (p_0 - p_1)^{-\mu}\, \omega^{-\mu} + \int_{p_0}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp \right| \\
&\qquad \leqslant \left| \frac{1}{2}\, \tilde{H}(\omega,\psi,U)(p_0 - p_1)^{\mu-1}\, \omega^{-\frac{1}{2}} \right| + \sum_{k=1}^{6} R_k^{(1)}(U)\, (p_0 - p_1)^{-\alpha_k^{(1)}}\, \omega^{-\beta_k^{(1)}} \\
&\qquad\qquad + \left| \frac{1}{2}\, \tilde{H}(\omega,\psi,U)\, (p_0 - p_1)^{\mu-1}\, \omega^{-\frac{1}{2}} \right| + \sum_{k=1}^{2} R_k^{(2)}(U)\, (p_0 - p_1)^{-\alpha_k^{(2)}}\, \omega^{-\beta_k^{(2)}} \\
&\qquad \leqslant \sqrt{\pi}\, \|\tilde{u}\|_{L^\infty(p_1,p_2)}\, (p_0 - p_1)^{\mu-1}\, \omega^{-\frac{1}{2}} \\
&\qquad\qquad + \sum_{k=1}^{6} R_k^{(1)}(U)\, (p_0 - p_1)^{-\alpha_k^{(1)}}\, \omega^{-\beta_k^{(1)}} + \sum_{k=1}^{2} R_k^{(2)}(U)\, (p_0 - p_1)^{-\alpha_k^{(2)}}\, \omega^{-\beta_k^{(2)}} \\
&\qquad =: \sum_{k=1}^{9} \tilde{R}_k^{(3)}(U)\, (p_0 - p_1)^{-\tilde{\alpha}_k^{(3)}}\, \omega^{-\tilde{\beta}_k^{(3)}}\ ,
\end{aligned}$$



 

where

- $\tilde{R}_1^{(3)}(U) := \sqrt{\pi}\, \|\tilde{u}\|_{L^\infty(p_1,p_2)}$ , $\tilde{\alpha}_1^{(3)} := 1-\mu$ , $\tilde{\beta}_1^{(3)} := \dfrac{1}{2}$ ;
- $\tilde{R}_{k+1}^{(3)}(U) := R_k^{(1)}(U)$ , $\tilde{\alpha}_{k+1}^{(3)} := \alpha_k^{(1)}$ , $\tilde{\beta}_{k+1}^{(3)} := \beta_k^{(1)}$ $k = 1,\dots,6$ ;
- $\tilde{R}_{k+7}^{(3)}(U) := R_k^{(2)}(U)$ , $\tilde{\alpha}_{k+7}^{(3)} := \alpha_k^{(2)}$ , $\tilde{\beta}_{k+7}^{(3)} := \beta_k^{(2)}$ $k = 1,2$ .

Here we note that $\tilde{\beta}_5^{(3)} = \tilde{\beta}_6^{(3)} = \delta$. So we can choose $\delta \in \left(\frac{1}{2},1\right)$ so that $\delta > \mu$ and thanks to that, each $\tilde{\beta}_k^{(3)}$ is strictly larger than $\mu$.

□

**2.2.3 Remark.** Looking carefully to the value of each $\tilde{\alpha}_k^{(j)}$, we note that only

$$\tilde{\alpha}_6^{(1)} = \tilde{\alpha}_5^{(2)} = \tilde{\alpha}_6^{(3)} = \alpha_5^{(1)} = \gamma - \mu$$

can be negative. To simplify slightly the proof of Theorems 2.3.2 and 2.3.7, and Corollary 2.3.5, we choose $\delta \in \left(\frac{1}{2},1\right)$ so that $\gamma - \mu \geqslant 0$, namely $\delta \geqslant \frac{\mu+1}{2}$.

From the previous theorem, we observe that the blow-up of the asymptotic expansion comes from the terms $(p_0-p_1)^{-\tilde{\alpha}_k^{(j)}}$. This motivates the idea of considering $p_0$ approaching $p_1$ with a certain convergence speed, described by the parameter $\vartheta > 0$, when the large parameter $\omega$ tends to infinity, as for example $p_0 - p_1 = \omega^{-\vartheta}$. This procedure modifies the decay rates of the asymptotic expansions of the integral. We shall exploit the idea that below a certain threshold, the convergence speed to the singular point $p_1$ is sufficiently slow so that the decay with respect to $\omega$ compensates the blow-up. More precisely the proof consists in finding the values of $\vartheta > 0$ for which the decay rate of the remainder is strictly larger than the one of the first term, assuring the optimality of the decay rates. This leads to asymptotic expansions on curves in the space of the parameters.

**2.2.4 Corollary.** *Let $\vartheta \in \left(0, \frac{1}{2}\right)$ and suppose that $p_0 := p_1 + \omega^{-\vartheta}$ for $\omega > (p_2-p_1)^{-\frac{1}{\vartheta}}$. Then under the assumptions of Lemma 2.2.1 and with the notations of Theorem 2.2.2, we have*

- *Case $\mu > \dfrac{1}{2}$ :*

$$\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp - \tilde{H}(\omega,\psi,U)\, \omega^{-\frac{1}{2}+\vartheta(1-\mu)} \right| \leqslant \sum_{k=1}^{9} \tilde{R}_k^{(1)}(U)\, \omega^{-\tilde{\beta}_k^{(1)}+\vartheta\, \tilde{\alpha}_k^{(1)}} ,$$

*where* $\max\limits_{k\in\{1,\dots,9\}} \left\{ -\tilde{\beta}_k^{(1)} + \vartheta\, \tilde{\alpha}_k^{(1)} \right\} < -\dfrac{1}{2} + \vartheta(1-\mu)$ *;*

- *Case $\mu = \dfrac{1}{2}$ :*

$$\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp - \left(\tilde{H}(\omega,\psi,U) + \tilde{K}_\mu(\omega,\psi,U)\right) \omega^{-\frac{1}{2}+\frac{1}{2}\vartheta} \right|$$
$$\leqslant \sum_{k=1}^{8} \tilde{R}_k^{(2)}(U)\, \omega^{-\tilde{\beta}_k^{(2)}+\vartheta\, \tilde{\alpha}_k^{(2)}} ,$$



 

*where* $\max_{k\in\{1,\dots,8\}}\left\{-\tilde{\beta}_k^{(2)}+\vartheta\,\tilde{\alpha}_k^{(2)}\right\}<-\frac{1}{2}+\frac{1}{2}\vartheta$ ;

- *Case* $\mu<\frac{1}{2}$ :

$$\left|\int_{p_1}^{p_2}U(p)\,e^{i\omega\psi(p)}\,dp-\tilde{K}_\mu(\omega,\psi,U)\,\omega^{-\mu+\vartheta\mu}\right|\leqslant\sum_{k=1}^{9}\tilde{R}_k^{(3)}(U)\,\omega^{-\tilde{\beta}_k^{(3)}+\vartheta\,\tilde{\alpha}_k^{(3)}}\ ,$$

*where* $\max_{k\in\{1,\dots,9\}}\left\{-\tilde{\beta}_k^{(3)}+\vartheta\,\tilde{\alpha}_k^{(3)}\right\}<-\mu+\vartheta\mu$ .

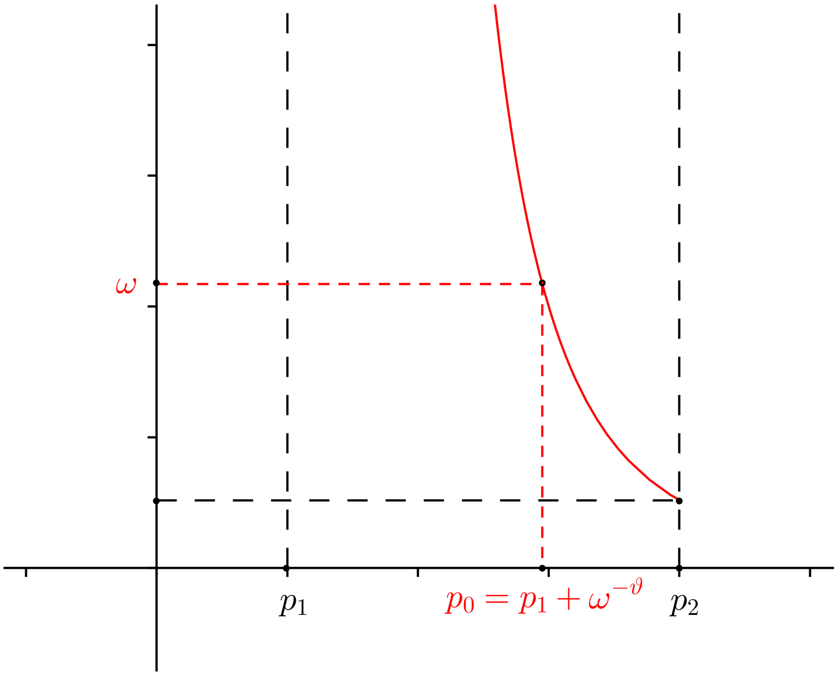


Figure 2.2: The curve $p_0=p_1+\omega^{-\vartheta}$

*Proof.* First of all, note that $I^{(1)}(\omega,p_0)$ and $I^{(2)}(\omega,p_0)$ are well-defined since the hypothesis $\omega>(p_2-p_1)^{-\frac{1}{\vartheta}}$ implies $p_0\in(p_1,p_2)$. Now we replace $p_0-p_1$ by $\omega^{-\vartheta}$ in the estimates of Theorem 2.2.2 and we compare the decay rates of the expansion with those of the remainder. In the following, we choose the parameter $\delta\in\left(\frac{1}{2},1\right)$ in such way that we have $\delta>\frac{1}{2}+\vartheta$ and $\delta>\mu$.

- *Case* $\mu>\frac{1}{2}$*:* here we have

$$\tilde{H}(\omega,\psi,U)\,(p_0-p_1)^{\mu-1}\,\omega^{-\frac{1}{2}}=\tilde{H}(\omega,\psi,U)\,\omega^{-\frac{1}{2}+\vartheta(1-\mu)}\ .$$

We note that the modulus of the coefficient $\tilde{H}(\omega,p_0,U)$ can be bounded from above and below by a non-zero constant when $\omega$ is sufficiently large, due to the hypothesis $\tilde{u}(p_1)\neq0$. It follows that this coefficient does not influence the decay and so the expansion behaves like $\omega^{-\frac{1}{2}+\vartheta(1-\mu)}$ when $\omega$ tends to infinity.



 

To compare the decay rates of the expansion and of the remainder, it is sufficient to compare the exponents of $\omega$. Replacing $p_0 - p_1$ by $\omega^{-\vartheta}$ in (2.15), (2.16), (2.17), (2.18) and (2.19), we obtain new decay rates with respect to $\omega$ for the remainder terms. The resulting exponents have to be less than $-\frac{1}{2} + \vartheta(1-\mu)$, which is equivalent to the fact that the following system of inequalities has to be satisfied:

$$\begin{cases} -\frac{1}{2} + \vartheta(1-\mu) > -\mu + \vartheta\mu & \text{(2.20a)} \\ -\frac{1}{2} + \vartheta(1-\mu) > -1 + \vartheta(2-\mu) & \text{(2.20b)} \\ -\frac{1}{2} + \vartheta(1-\mu) > -1 + \vartheta(1-\mu) & \text{(2.20c)} \\ -\frac{1}{2} + \vartheta(1-\mu) > -2 + \vartheta(4-\mu) & \text{(2.20d)} \\ -\frac{1}{2} + \vartheta(1-\mu) > -\delta + \vartheta(1+\gamma-\mu) & \text{(2.20e)} \\ -\frac{1}{2} + \vartheta(1-\mu) > -\delta + \vartheta(\gamma-\mu) & \text{(2.20f)} \\ -\frac{1}{2} + \vartheta(1-\mu) > -\frac{3}{2} + \vartheta(3-\mu) & \text{(2.20g)} \\ -\frac{1}{2} + \vartheta(1-\mu) > -\delta + \vartheta(1-\mu) & \text{(2.20h)} \\ -\frac{1}{2} + \vartheta(1-\mu) > -\delta + \vartheta(2-\mu) & \text{(2.20i)} \end{cases}$$

- *Inequalities* (2.20a)*,* (2.20b)*,* (2.20d)*,* (2.20e)*,* (2.20g)*.* These inequalities are satisfied because they are equivalent to $\vartheta < \frac{1}{2}$, which is true by hypothesis. Note that we used the relation $\gamma = 2\delta - 1$ to study (2.20e).
- *Inequality* (2.20c)*.* This inequality is equivalent to $\frac{1}{2} < 1$ which is clearly true.
- *Inequality* (2.20f)*.* This inequality is equivalent to $\delta - \frac{1}{2} > \vartheta(\gamma - 1)$. By hypothesis, we have $\delta > \frac{1}{2}$, $\vartheta > 0$ and $\gamma < 1$. So (2.20f) is satisfied.
- *Inequality* (2.20h)*.* This inequality is true because $\delta > \frac{1}{2}$ by hypothesis.
- *Inequality* (2.20i)*.* We have supposed that $\vartheta < \delta - \frac{1}{2}$, which is equivalent to (2.20i). Inequality (2.20i) is then satisfied.

• *Case* $\mu = \frac{1}{2}$*:* in this situation,

$$\Big(\tilde{H}(\omega,\psi,U) + \tilde{K}_{\frac{1}{2}}(\omega,\psi,U)\Big)\,(p_0 - p_1)^{-\frac{1}{2}}\,\omega^{-\frac{1}{2}} = \Big(\tilde{H}(\omega,p_0,U) + \tilde{K}_{\frac{1}{2}}(\omega,p_0,U)\Big)\,\omega^{-\frac{1}{2}+\frac{1}{2}\vartheta}\,.$$

As explained above, the expansion behaves like $\omega^{-\frac{1}{2}+\frac{1}{2}\vartheta}$ when $\omega$ tends to infinity. Here the decay rates of the remainder have to be faster than $\omega^{-\frac{1}{2}+\frac{1}{2}\vartheta}$. So we have to verify if the following system is satisfied:

$$\begin{cases} -\frac{1}{2}+\frac{1}{2}\vartheta > -1+\frac{3}{2}\vartheta & \text{(2.21a)} \\ -\frac{1}{2}+\frac{1}{2}\vartheta > -1+\frac{1}{2}\vartheta & \text{(2.21b)} \\ -\frac{1}{2}+\frac{1}{2}\vartheta > -2+\frac{7}{2}\vartheta & \text{(2.21c)} \\ -\frac{1}{2}+\frac{1}{2}\vartheta > -\delta+\left(\frac{1}{2}+\gamma\right)\vartheta & \text{(2.21d)} \\ -\frac{1}{2}+\frac{1}{2}\vartheta > -\delta+\left(\gamma-\frac{1}{2}\right)\vartheta & \text{(2.21e)} \\ -\frac{1}{2}+\frac{1}{2}\vartheta > -\frac{3}{2}+\frac{5}{2}\vartheta & \text{(2.21f)} \\ -\frac{1}{2}+\frac{1}{2}\vartheta > -\delta+\frac{1}{2}\vartheta & \text{(2.21g)} \\ -\frac{1}{2}+\frac{1}{2}\vartheta > -\delta+\frac{3}{2}\vartheta & \text{(2.21h)} \end{cases}$$

- *Inequalities* (2.21a)*,* (2.21c)*,* (2.21d)*,* (2.21f)*.* These inequalities are satisfied because they are equivalent to $\vartheta < \frac{1}{2}$, which is true by hypothesis. The relation $\gamma = 2\delta - 1$ is used to study (2.21d).
- *Inequality* (2.21b)*.* This inequality is equivalent to $\frac{1}{2} < 1$ which is clearly true.
- *Inequality* (2.21e)*.* This is similar to inequality (2.20f).
- *Inequality* (2.21g)*.* This inequality is true because $\delta > \frac{1}{2}$ by hypothesis.
- *Inequality* (2.21h)*.* This inequality is equivalent to $\vartheta < \delta - \frac{1}{2}$, which is true by hypothesis.

• *Case* $\mu < \frac{1}{2}$*:* replacing $p_0 - p_1$ by $\omega^{-\vartheta}$, we obtain

$$\tilde{K}_\mu(\omega,\psi,U)\,(p_0-p_1)^{-\mu}\,\omega^{-\mu} = K_\mu(\omega,\psi,U)\,\omega^{-\mu+\vartheta\mu} \ ,$$

and in this situation, the system of inequalities to verify is given by

$$\begin{cases} -\mu+\vartheta\mu > -\frac{1}{2}+\vartheta(1-\mu) & \text{(2.22a)} \\ -\mu+\vartheta\mu > -1+\vartheta(2-\mu) & \text{(2.22b)} \\ -\mu+\vartheta\mu > -1+\vartheta(1-\mu) & \text{(2.22c)} \\ -\mu+\vartheta\mu > -2+\vartheta(4-\mu) & \text{(2.22d)} \\ -\mu+\vartheta\mu > -\delta+\vartheta(1+\gamma-\mu) & \text{(2.22e)} \\ -\mu+\vartheta\mu > -\delta+\vartheta(\gamma-\mu) & \text{(2.22f)} \\ -\mu+\vartheta\mu > -\frac{3}{2}+\vartheta(3-\mu) & \text{(2.22g)} \\ -\mu+\vartheta\mu > -\delta+\vartheta(1-\mu) & \text{(2.22h)} \\ -\mu+\vartheta\mu > -\delta+\vartheta(2-\mu) & \text{(2.22i)} \end{cases}$$



 

– *Inequalities* (2.22a), (2.22b), (2.22d), (2.22e), (2.22g). These inequalities are equivalent to $\vartheta < \frac{1}{2}$, which is true by hypothesis. We used once again the relation $\gamma = 2\delta - 1$ and the fact that $\delta > \mu$ to study (2.22e).
– *Inequality* (2.22c). This inequality is equivalent to $\vartheta < \frac{1-\mu}{1-2\mu}$. But we can show that $\frac{1}{2} < \frac{1-\mu}{1-2\mu}$ and we recall that $\vartheta < \frac{1}{2}$, so (2.22c) is verified.
– *Inequality* (2.22f). This inequality is equivalent to $\delta - \mu > \vartheta(\gamma - 2\mu)$. Now we distinguish two cases: if $\gamma - 2\mu \leqslant 0$ then the inequaliy holds since $\delta - \mu$ is positive. In the other case, (2.22f) is equivalent to $\vartheta < \frac{\delta-\mu}{\gamma-2\mu}$. Hence the last inequality is true since $\vartheta < \frac{1}{2}$ and we can show $\frac{1}{2} < \frac{\delta-\mu}{\gamma-2\mu}$.
– *Inequality* (2.22h). It is equivalent to $\vartheta < \frac{\delta-\mu}{1-2\mu}$. Since $\frac{1}{2} < \frac{\delta-\mu}{1-2\mu}$ holds and $\vartheta < \frac{1}{2}$, (2.22h) is satisfied.
– *Inequality* (2.22i). It is equivalent to $\vartheta < \frac{\delta-\mu}{2-2\mu}$. We can show that $\delta - \frac{1}{2} < \frac{\delta-\mu}{2-2\mu}$ and we recall that we have $\vartheta < \delta - \frac{1}{2}$ by hypothesis, which proves that (2.22i) is true.

□

## 2.3 Application to the free Schrödinger equation: propagation of wave packets and anomalous phenomena

In this section, we are interested in the time asymptotic behaviour of the solution of the free Schrödinger equation in one dimension, with initial conditions in a frequency band $[p_1, p_2]$ and having a singular frequency at $p_1$. We establish time asymptotic expansions as well as uniform estimates to explore the influence of the compact frequency band and of the singularity on the dispersion.

We introduce the free Schrödinger equation on the line

$$(S) \qquad \begin{cases} \big[i\partial_t + \partial_{xx}\big]u(t) = 0 \\ u(0) = u_0 \end{cases} ,$$

for $t \geqslant 0$. If $u_0 \in \mathcal{S}'(\mathbb{R})$ then this initial value problem has a unique solution in $\mathcal{C}^1\Big(\mathbb{R}_+, \mathcal{S}'(\mathbb{R})\Big)$, given by the following solution formula,

$$u(t) = \mathcal{F}^{-1}\left(e^{-itp^2}\mathcal{F}u_0\right) ,$$

where $\mathcal{F} : \mathcal{S}'(\mathbb{R}) \longrightarrow \mathcal{S}'(\mathbb{R})$ is the Fourier transform.
Throughout this section, we shall suppose that the initial data satisfy the following condition:

**Condition ($\mathbf{C1}_{[p_1,p_2],\mu}$).** Let $\mu \in (0,1)$ and $p_1 < p_2$ be two finite real numbers.
A tempered distribution $u_0$ on $\mathbb{R}$ satisfies Condition ($\mathrm{C1}_{[p_1,p_2],\mu}$) if and only if $\mathcal{F}u_0 \equiv 0$



 

on $\mathbb{R}\backslash[p_1, p_2]$ and $\mathcal{F}u_0$ verifies Assumption $(\mathrm{A}1_{\mu,1,1})$ (given in Chapter 1, Section 1.1) on $[p_1, p_2]$, with $\mathcal{F}u_0(p_2) = 0$.

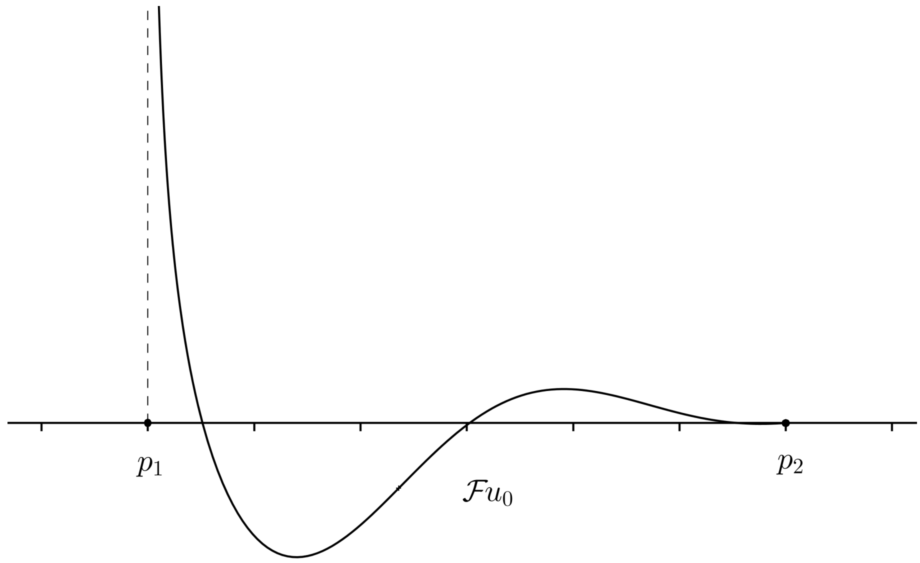


Figure 2.3: Fourier transform of an initial datum satisfying Condition $(\mathrm{C}1_{[p_1,p_2],\mu})$

Under this condition, we note that $\mathcal{F}u_0$ is a function which has a singular point of order $\mu - 1$ at $p_1$ whereas the point $p_2$ is regular. For simplicity, we assume $\mathcal{F}u_0(p_2) = 0$ but a similar work can be carried out in the case $\mathcal{F}u_0(p_2) \neq 0$. Moreover under Condition $(\mathrm{C}1_{[p_1,p_2],\mu})$, the solution formula of the free Schrödinger equation defines a function $u : (0, +\infty) \times \mathbb{R} \longrightarrow \mathbb{C}$ given by the following integral,

$$u(t,x) = \frac{1}{2\pi} \int_{p_1}^{p_2} \mathcal{F}u_0(p)\, e^{-itp^2+ixp}\, dp \; ; \tag{2.23}$$

for $v \in L^1(\mathbb{R})$, the Fourier transform of $v$ is defined by

$$\mathcal{F}v(p) = \int_{\mathbb{R}} v(x)\, e^{-ixp}\, dx \; .$$

Now let us remark that the subset of tempered distributions which satisfy Condition $(\mathrm{C}1_{[p_1,p_2],\mu})$ is non-empty. Indeed if a function $U$ supported on $[p_1, p_2]$ verifies Assumption $(\mathrm{A}1_{\mu,1,1})$ with $U(p_2) = 0$, then $U$ is an integrable function and so it belongs to $\mathcal{S}'(\mathbb{R})$. Since the Fourier transform is a bijection on $\mathcal{S}'(\mathbb{R})$, there exists $u_0 \in \mathcal{S}'(\mathbb{R})$ such that $U = \mathcal{F}u_0$, and hence $u_0$ satisfies Condition $(\mathrm{C}1_{[p_1,p_2],\mu})$.
Finally, we note that $u_0$ is in fact an analytic function on $\mathbb{R}$ due to the fact that $\mathcal{F}u_0$ is compactly supported.

In this section, we shall need the following definitions of certain space-time regions.

**2.3.1 Definition.** *Let $a < b$ be two finite real numbers and let $\vartheta > 0$.*



 

*i) We define the space-time cone $\mathfrak{C}_S(a,b)$ as follows:*

$$\mathfrak{C}_S(a,b) := \left\{ (t,x) \in (0,+\infty) \times \mathbb{R} \;\middle|\; 2\,a < \frac{x}{t} < 2\,b \right\} .$$

*ii) We define the space-time curve $\mathfrak{G}_\vartheta(a)$ as follows:*

$$\mathfrak{G}_\vartheta(a) := \left\{ (t,x) \in (0,+\infty) \times \mathbb{R} \;\middle|\; \frac{x}{t} = 2\,a + 2\,t^{-\vartheta} \right\} .$$

*iii) We define the space-time region $\mathfrak{R}_\vartheta(a,b)$ as follows:*

$$\mathfrak{R}_\vartheta(a,b) = \left\{ (t,x) \in (0,+\infty) \times \mathbb{R} \;\middle|\; 2\,a + 2\,t^{-\vartheta} \leqslant \frac{x}{t} < 2\,b \;,\; t > T_\vartheta(a,b) \right\} ,$$

*where $T_\vartheta(a,b) := (b-a)^{-\frac{1}{\vartheta}}$.*

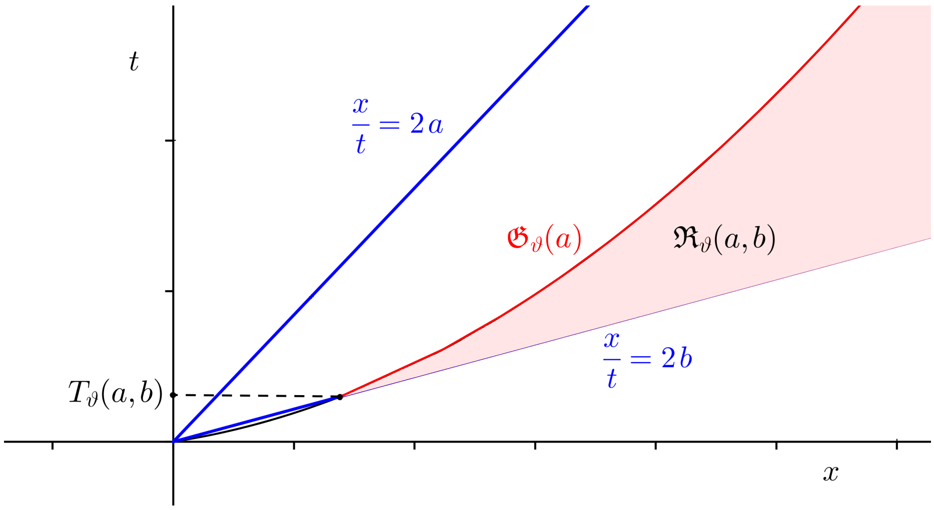


Figure 2.4: Illustration of the region $\mathfrak{R}_\vartheta(a,b)$ and the curve $\mathfrak{G}_\vartheta(a)$

In the first result, we furnish uniform remainder estimates for asymptotic expansions of the solution in the cone $\mathfrak{C}_S\big(p_1+\varepsilon, p_2\big)$. After having rewritten the solution formula as an oscillatory integral, we apply the results of the preceding section and we use the fact that the distance between the stationary point $\frac{x}{2t}$ and the singularity $p_1$ is bounded from below by $\varepsilon$ in order to estimate uniformly the remainder. Let us note that the method employed in the proof furnishes asymptotic expansions of the solution in the entire cone $\mathfrak{C}_S\big(p_1,p_2\big)$ with explicit blow-up when $\frac{x}{2t}$ approaches $p_1$. Especially, we see in the proof that restricting the cone $\mathfrak{C}_S\big(p_1,p_2\big)$ to $\mathfrak{C}_S\big(p_1+\varepsilon,p_2\big)$ is sufficient to obtain uniform estimates.
It is interesting to note that the singular frequency diminishes the time-decay rate in the cone below the rate of quantum mechanic dispersion $t^{-\frac{1}{2}}$, when leaving the $L^2$-setting.

**2.3.2 Theorem.** *Suppose that $u_0$ satisfies Condition $(\mathrm{C1}_{[p_1,p_2],\mu})$ and choose a real number $\varepsilon > 0$ such that*

$$p_1 + \varepsilon < p_2 \,.$$

*For all* $(t,x) \in \mathfrak{C}_S\big(p_1+\varepsilon, p_2\big)$*, define* $H(t,x,u_0) \in \mathbb{C}$ *and* $K_\mu(t,x,u_0) \in \mathbb{C}$ *as follows,*

- $H(t,x,u_0) := \dfrac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, \tilde{u}\Big(\dfrac{x}{2t}\Big)\left(\dfrac{x}{2t}-p_1\right)^{\mu-1}$ ,
- $K_\mu(t,x,u_0) := \dfrac{\Gamma(\mu)}{2^{\mu+1}\pi}\, e^{i\frac{\pi\mu}{2}}\, e^{-itp_1^2+ixp_1}\, \tilde{u}(p_1)\left(\dfrac{x}{2t}-p_1\right)^{-\mu}$ .

*Then for all* $(t,x) \in \mathfrak{C}_S\big(p_1+\varepsilon, p_2\big)$*, we have*

- *Case* $\mu > \dfrac{1}{2}$ *:*

$$\left|u(t,x) - H(t,x,u_0)\, t^{-\frac{1}{2}}\right| \leqslant \sum_{k=1}^{9} C_k^{(1)}(u_0,\varepsilon)\, t^{-\tilde{\beta}_k^{(1)}} ,$$

  *where* $\max\limits_{k\in\{1,\dots,9\}} \left\{-\tilde{\beta}_k^{(1)}\right\} < -\dfrac{1}{2}$*. The exponents* $\tilde{\beta}_k^{(1)}$ *are defined in Theorem 2.2.2 and the constants* $C_k^{(1)}(u_0,\varepsilon) \geqslant 0$ *are given in the proof ;*

- *Case* $\mu = \dfrac{1}{2}$ *:*

$$\left|u(t,x) - \big(H(t,x,u_0) + K_\mu(t,x,u_0)\big)\, t^{-\frac{1}{2}}\right| \leqslant \sum_{k=1}^{8} C_k^{(2)}(u_0,\varepsilon)\, t^{-\tilde{\beta}_k^{(2)}} ,$$

  *where* $\max\limits_{k\in\{1,\dots,8\}} \left\{-\tilde{\beta}_k^{(2)}\right\} < -\dfrac{1}{2}$*. The exponents* $\tilde{\beta}_k^{(2)}$ *are defined in Theorem 2.2.2 and the constants* $C_k^{(2)}(u_0,\varepsilon) \geqslant 0$ *are given in the proof ;*

- *Case* $\mu < \dfrac{1}{2}$ *:*

$$\left|u(t,x) - K_\mu(t,x,u_0)\, t^{-\mu}\right| \leqslant \sum_{k=1}^{9} C_k^{(3)}(u_0,\varepsilon)\, t^{-\tilde{\beta}_k^{(3)}} ,$$

  *where* $\max\limits_{k\in\{1,\dots,9\}} \left\{-\tilde{\beta}_k^{(3)}\right\} < -\mu$*. The exponents* $\tilde{\beta}_k^{(3)}$ *are defined in Theorem 2.2.2 and the constants* $C_k^{(3)}(u_0,\varepsilon) \geqslant 0$*are given in the proof.*

*Proof.* We shall prove the result in the case $\mu > \frac{1}{2}$; the proofs in the other cases are very similar.
In the solution formula (2.23), we factorize the phase function $p \longmapsto -tp^2 + xp$ by $t$, which gives

$$\forall\, (t,x) \in (0,+\infty)\times\mathbb{R} \qquad u(t,x) = \int_{p_1}^{p_2} U(p)\, e^{it\psi(p)}\, dp ,$$

where [2]

$$\begin{cases} \forall\, p \in (p_1,p_2] \qquad U(p) := \dfrac{1}{2\pi}\,\mathcal{F}u_0(p) = \dfrac{1}{2\pi}\,(p-p_1)^{\mu-1}\,\tilde{u}(p) , \\ \forall\, p \in \mathbb{R} \qquad \psi(p) := -p^2 + \dfrac{x}{t}\, p . \end{cases}$$

[2]See Remark 2.3.3



 

By hypothesis, $U$ verifies Assumption $(\mathrm{A1}_{\mu,1,1})$ on $[p_1, p_2]$ and $\psi$ has the form

$$\psi(p) = -\left(p - p_0\right)^2 + c ,$$

where $p_0 := \frac{x}{2t}$ and $c := p_0^2 = \frac{x^2}{4t^2}$. Moreover, we have the following equivalence,

$$(t,x) \in \mathfrak{C}_S\big(p_1+\varepsilon, p_2\big) \qquad \Longleftrightarrow \qquad p_1 + \varepsilon < \frac{x}{2t} = p_0 < p_2 ,$$

implying the fact that the stationary point $p_0$ belongs to $(p_1, p_2)$. Hence Theorem 2.2.2 is applicable and we obtain for all $(t,x) \in \mathfrak{C}_S\big(p_1+\varepsilon, p_2\big)$,

$$\left| u(t,x) - H(t,x,u_0)\, t^{-\frac{1}{2}} \right| \leqslant \sum_{k=1}^{9} \tilde{R}_k^{(1)}\!\left( \frac{1}{2\pi}\, \mathcal{F}u_0 \right) \left( \frac{x}{2t} - p_1 \right)^{-\tilde{\alpha}_k^{(1)}} t^{-\tilde{\beta}_k^{(1)}} ,$$

where the coefficient $H(t,x,u_0)$ is given in the statement of the theorem, the constants $\tilde{R}_k^{(1)}\big(\frac{1}{2\pi}\,\mathcal{F}u_0\big) \geqslant 0$ and the exponents $\tilde{\alpha}_k^{(1)} \geqslant 0$, $\tilde{\beta}_k^{(1)} > \frac{1}{2}$ are provided by Theorem 2.2.2. Note that we can choose $\tilde{\alpha}_k^{(1)} \geqslant 0$ according to Remark 2.2.3 and in this case, if we suppose $(t,x) \in \mathfrak{C}_S\big(p_1+\varepsilon, p_2\big)$ then we have

$$\varepsilon \leqslant \frac{x}{2t} - p_1 \qquad \Longrightarrow \qquad \left( \frac{x}{2t} - p_1 \right)^{-\tilde{\alpha}_k^{(1)}} \leqslant \varepsilon^{-\tilde{\alpha}_k^{(1)}} .$$

By defining

$$\forall\, k \in \{1,...,9\} \qquad C_k^{(1)}(u_0,\varepsilon) := \tilde{R}_k^{(1)}\!\left( \frac{1}{2\pi}\, \mathcal{F}u_0 \right) \varepsilon^{-\tilde{\alpha}_k^{(1)}} ,$$

we obtain the result for the case $\mu > \frac{1}{2}$.
We define in a similar way

- $\forall\, k \in \{1,...,8\} \qquad C_k^{(2)}(u_0,\varepsilon) := \tilde{R}_k^{(2)}\!\left( \frac{1}{2\pi}\, \mathcal{F}u_0 \right) \varepsilon^{-\tilde{\alpha}_k^{(2)}}$ ;
- $\forall\, k \in \{1,...,9\} \qquad C_k^{(3)}(u_0,\varepsilon) := \tilde{R}_k^{(3)}\!\left( \frac{1}{2\pi}\, \mathcal{F}u_0 \right) \varepsilon^{-\tilde{\alpha}_k^{(3)}}$ .

□

**2.3.3 Remark.** At this stage, the authors of [5] introduced the large parameter

$$\omega := \sqrt{t^2 + x^2} ,$$

and replaced $t$ and $x$ by the bounded parameters $\tau := \frac{t}{\omega}$ and $\chi := \frac{x}{\omega}$. This led to a family of phase functions which was globally bounded in $\mathcal{C}^4$ with respect to $\tau$ and $\chi$. This was necessary for the application of the stationary phase method given in [20]. In our context, it is sufficient to control the phase functions in space-time cones. Indeed the explicitness of our remainder estimates shows that their coefficients depend only on the quotient $\frac{x}{t}$, which is bounded in these cones. It is not necessary to have the global boundedness with respect to $t$ and $x$ separately. Therefore we can use $t$ as a large parameter instead of $\sqrt{t^2 + x^2}$, which is conceptually simpler and clearer.



 

In the second result, we study the solution outside the cone $\mathfrak{C}_S\big(p_1, p_2\big)$. In this case, the stationary point is outside the integration interval and so the decay rate is only governed by the singular frequency. As above, we have to restrict slightly the space-time cones in which we expand the solution to bound uniformly the remainder.

**2.3.4 Theorem.** *Suppose that $u_0$ satisfies Condition $(\mathrm{C1}_{[p_1,p_2],\mu})$ and choose $\varepsilon > 0$ such that*

$$-\varepsilon^{-1} < p_1 - \varepsilon \qquad \textit{and} \qquad p_2 + \varepsilon < \varepsilon^{-1} \,.$$

*Define for all $(t,x) \in \mathfrak{C}_S\big(-\varepsilon^{-1}\,,\, p_1 - \varepsilon\big)$,*

$$K_\mu^{(1)}(t,x,u_0) := \frac{\Gamma(\mu)}{2^{\mu+1}\pi}\, e^{-i\frac{\pi\mu}{2}}\, e^{-itp_1^2 + ixp_1}\, \tilde{u}(p_1) \left(p_1 - \frac{x}{2t}\right)^{-\mu} ,$$

*and for all $(t,x) \in \mathfrak{C}_S\big(p_2 + \varepsilon\,,\, \varepsilon^{-1}\big)$,*

$$K_\mu^{(2)}(t,x,u_0) := \frac{\Gamma(\mu)}{2^{\mu+1}\pi}\, e^{i\frac{\pi\mu}{2}}\, e^{-itp_1^2 + ixp_1}\, \tilde{u}(p_1) \left(\frac{x}{2t} - p_1\right)^{-\mu} .$$

*Then*

- *for all $(t,x) \in \mathfrak{C}_S\big(-\varepsilon^{-1}\,,\, p_1 - \varepsilon\big)$, we have*

$$\Big| u(t,x) - K_\mu^{(1)}(t,x,u_0)\, t^{-\mu} \Big| \leqslant C^{(1)}(u_0,\varepsilon)\, t^{-1} \,.$$

  *The constant $C^{(1)}(u_0,\varepsilon) \geqslant 0$ is given in the proof ;*

- *for all $(t,x) \in \mathfrak{C}_S\big(p_2 + \varepsilon\,,\, \varepsilon^{-1}\big)$, we have*

$$\Big| u(t,x) - K_\mu^{(2)}(t,x,u_0)\, t^{-\mu} \Big| \leqslant C^{(2)}(u_0,\varepsilon)\, t^{-1} \,.$$

  *The constant $C^{(2)}(u_0,\varepsilon) \geqslant 0$ is given in the proof.*

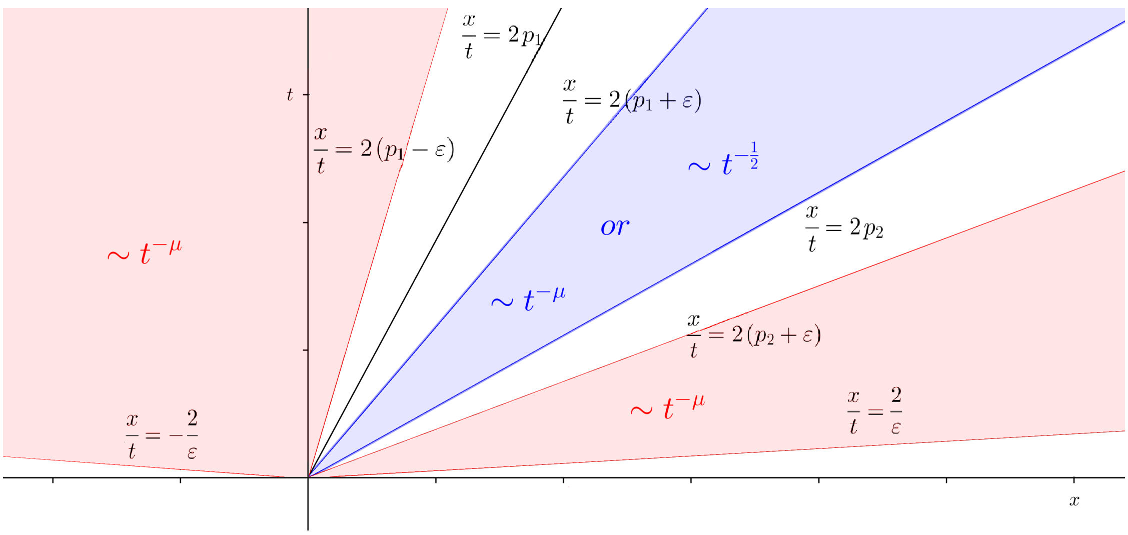


Figure 2.5: Illustration of Theorem 2.3.2 and Theorem 2.3.4 in space-time



 

*Proof.* The first step of the proof consists in rewritting the solution as an oscillatory integral, namely,

$$u(t,x) = \frac{1}{2\pi} \int_{p_1}^{p_2} \mathcal{F}u_0(p)\, e^{-itp^2+ixp}\, dp = \int_{p_1}^{p_2} U(p)\, e^{it\psi(p)}\, dp \;,$$

where the functions $U$ and $\psi$ are defined at the beginning of the proof of Theorem 2.3.2. Here the hypotheses on $(t,x)$ imply

- $(t,x) \in \mathfrak{C}_S\left(-\varepsilon^{-1}\,,\, p_1 - \varepsilon\right) \iff -\varepsilon^{-1} < p_0 < p_1 - \varepsilon \;,$
- $(t,x) \in \mathfrak{C}_S\left(p_2 + \varepsilon\,,\, \varepsilon^{-1}\right) \iff p_2 + \varepsilon < p_0 < \varepsilon^{-1} \;,$

where $p_0 := \frac{x}{2t}$ is the unique stationary point of the phase. Hence the stationary point does not belong to the integration interval $[p_1, p_2]$ in both situations. In the space-time cone $\mathfrak{C}_S\big(p_2+\varepsilon\,,\,\varepsilon^{-1}\big)$, the amplitude satisfies Assumption $(\mathrm{A1}_{\mu,1,1})$ and the phase Assumption $(\mathrm{P1}_{1,1,N})$ (for $N \geqslant 1$) on $[p_1,p_2]$ and so, the hypotheses of Theorem 2.1.2 are verified. Note that it is not necessary to use a cutting-point to split the integral since only $p_1$ is a particular point. This implies that we shall use only the expansion of the integral $\tilde{I}^{(1)}(\omega, q)$ (see the proof of Theorem 2.1.2) with $\omega := t$ and $q := p_2$ in the present proof. In the other cone $\mathfrak{C}_S\big(-\varepsilon^{-1}\,,\, p_1 - \varepsilon\big)$, we note that the phase is a decreasing function. So we employ the substitution $p \longrightarrow -p$ to make it increasing. Finally we use the above inequalities to estimate the remainders, furnishing the following constants:

- $$C^{(1)}(u_0,\varepsilon) := \frac{1}{4\pi\mu}\left(p_2+\varepsilon^{-1}\right)(p_2-p_1)^{\mu}\left\|\tilde{u}\right\|_{W^{1,\infty}(p_1,p_2)}\left(\frac{1-\mu}{2}\left(p_2+\varepsilon^{-1}\right)\varepsilon^{-4} + \varepsilon^{-2}+\varepsilon^{-3}\right),$$
- $$C^{(2)}(u_0,\varepsilon) := \frac{1}{4\pi\mu}\left(\varepsilon^{-1}-p_1\right)(p_2-p_1)^{\mu}\left\|\tilde{u}\right\|_{W^{1,\infty}(p_1,p_2)}\left(\frac{1-\mu}{2}\left(\varepsilon^{-1}-p_1\right)\varepsilon^{-4} + \varepsilon^{-2}+\varepsilon^{-3}\right).$$

□

The following result is a consequence of Theorem 2.3.2. It permits to compute the limit of the $L^2$-norm of the solution on the spatial cross-section of the cone $\mathfrak{C}_S\big(p_1+\varepsilon, p_2\big)$ when the time tends to infinity, assuming $u_0 \in L^2(\mathbb{R})$.

**2.3.5 Corollary.** *Suppose that the hypotheses of Theorem 2.3.2 are satisfied with* $\mu > \frac{1}{2}$ *and define the interval* $I_t$ *as follows for all* $t > 0$,

$$I_t := \Big(2\,(p_1+\varepsilon)\,t\,,\,2\,p_2\,t\Big)\,.$$

*Then we have*

$$\left|\left\|u(t,.)\right\|_{L^2(I_t)} - \frac{1}{\sqrt{2\pi}}\left\|\mathcal{F}u_0\right\|_{L^2(p_1+\varepsilon,p_2)}\right| \leqslant \sum_{k=1}^{9} \tilde{C}_k^{(1)}(u_0,\varepsilon)\, t^{-\tilde{\beta}_k^{(1)}+\frac{1}{2}} \;,$$

*where* $\max_{k\in\{1,\dots 9\}}\left\{-\tilde{\beta}_k^{(1)}\right\} < -\dfrac{1}{2}$*. The exponents* $\tilde{\beta}_k^{(1)}$ *are defined in Theorem 2.3.2 and the constants* $\tilde{C}_k^{(1)}(u_0,\varepsilon) \geqslant 0$ *are given in the proof.*

*Proof.* We start by using the triangle inequality as follows,

$$\begin{aligned}\left| \big\| u(t,.) \big\|_{L^2(I_t)} - \left\| H(t,.,u_0)\, t^{-\frac{1}{2}} \right\|_{L^2(I_t)} \right|^2 &\leqslant \left\| u(t,.) - H(t,x,u_0)\, t^{-\frac{1}{2}} \right\|^2_{L^2(I_t)} \\ &= \int_{I_t} \left| u(t,x) - H(t,x,u_0)\, t^{-\frac{1}{2}} \right|^2 dx \,.\end{aligned}$$

Now we employ the estimate provided by Theorem 2.3.2 in the case $\mu > \frac{1}{2}$:

$$\begin{aligned}\int_{I_t} \left| u(t,x) - H(t,x,u_0)\, t^{-\frac{1}{2}} \right|^2 dx &\leqslant \int_{I_t} \left| \sum_{k=1}^{9} C_k^{(1)}(u_0,\varepsilon)\, t^{-\tilde{\beta}_k^{(1)}} \right|^2 dx \\ &= \sum_{k=1}^{9} C_k^{(1)}(u_0,\varepsilon)^2\, t^{-2\tilde{\beta}_k^{(1)}}\, |I_t| \,,\end{aligned}$$

where $|I_t|$ is the Lebesgue measure of the interval $I_t$; in our context, $|I_t|$ is equal to $2(p_2 - p_1 - \varepsilon)\, t$ . By defining

$$\tilde{C}_k^{(1)}(u_0,\varepsilon) := \sqrt{2(p_2 - p_1 - \varepsilon)}\, C_k^{(1)}(u_0,\varepsilon) \,,$$

we obtain

$$\left| \big\| u(t,.) \big\|_{L^2(I_t)} - \left\| H(t,.,u_0)\, t^{-\frac{1}{2}} \right\|_{L^2(I_t)} \right| \leqslant \sum_{k=1}^{9} \tilde{C}_k^{(1)}(u_0,\varepsilon)\, t^{-\tilde{\beta}_k^{(1)} + \frac{1}{2}} \,.$$

To finish, we compute $\left\| H(t,.,u_0)\, t^{-\frac{1}{2}} \right\|_{L^2(I_t)}$ by using the expression of $H(t,x,u_0)$ given in Theorem 2.3.2,

$$\begin{aligned}\left\| H(t,.,u_0)\, t^{-\frac{1}{2}} \right\|^2_{L^2(I_t)} &= \frac{1}{4\pi} \int_{I_t} \left| \tilde{u}\Big(\frac{x}{2t}\Big) \Big(\frac{x}{2t} - p_1\Big)^{\mu-1} t^{-\frac{1}{2}} \right|^2 dx \\ &= \frac{t}{2\pi} \int_{p_1+\varepsilon}^{p_2} \left| \tilde{u}(y)(y-p_1)^{\mu-1} \right|^2 dy\, t^{-1} \\ &= \frac{1}{2\pi} \big\| \mathcal{F} u_0 \big\|^2_{L^2(p_1+\varepsilon,p_2)} \,.\end{aligned}$$

The proof is now complete. □

The aim of the following result is to show that the time decay rate is $t^{-\frac{\mu}{2}}$ on points moving in space-time with the critical velocity given by the singularity $p_1$. In this case, we do not have to deal with the uniformity of the constants of the remainder since we establish an asymptotic expansion on a line.

**2.3.6 Theorem.** *Suppose that $u_0$ satisfies Condition* $(\mathrm{C1}_{[p_1,p_2],\mu})$*. For all $t > 0$, define $L_\mu(t,u_0)$ as follows:*

$$L_\mu(t,u_0) := \frac{1}{2}\, \Gamma\Big(\frac{\mu}{2}\Big)\, e^{-i\frac{\pi\mu}{4}}\, e^{itp_1^2}\, \tilde{u}(p_1) \,.$$

*Then for all $(t,x) \in (0,+\infty) \times \mathbb{R}$ such that $\frac{x}{t} = 2\, p_1$, we have*

$$\left| u(t,x) - L_\mu(t,u_0)\, t^{-\frac{\mu}{2}} \right| \leqslant C(u_0)\, t^{-\frac{1}{2}} \,.$$

*The constant $C(u_0) \geqslant 0$ is given in the proof.*



 

*Proof.* In this situation, the stationary point $p_0 = \frac{x}{2t}$ and the singular point $p_1$ are equal but the phase is decreasing. So we make the substitution $p \longmapsto -p$ and then we apply Theorem 2.1.2. Since $p_1$ is the unique singular point inside the integration interval, we can employ only the expansion of the integral $\tilde{I}^{(2)}(\omega, q)$ given in the proof of Theorem 2.1.2. Let us give the constant $C(u_0)$ to conclude the proof,

$$C(u_0) := \frac{\sqrt{\pi}}{2\mu} (p_2 - p_1)^{\mu} \left\| \tilde{u}' \right\|_{L^\infty(p_1, p_2)} .$$

□

Uniform estimates of the solution in the curved region $\mathfrak{R}_\vartheta(p_1, p_2)$, which is asymptotically larger than any space-time cone contained in $\mathfrak{C}_S(p_1, p_2)$, are provided in the following theorem. For this purpose, we rewrite the solution as an oscillatory integral that we estimate by using Theorem 2.2.2. The quantities $\left(\frac{x}{2t} - p_1\right)^{-\tilde{\alpha}_k^{(j)}}$, which produce the blow-up, are bounded by $t^{\vartheta\, \tilde{\alpha}_k^{(j)}}$ in $\mathfrak{R}_\vartheta(p_1, p_2)$, furnishing a uniform estimate of the solution with modified decay rates for $\vartheta$ between 0 and $\frac{1}{2}$. Finally we use Corollary 2.2.4 to give the preponderant decay rate.

**2.3.7 Theorem.** *Suppose that* $u_0$ *satisfies Condition* $(\mathrm{C1}_{[p_1,p_2],\mu})$ *and fix* $\vartheta \in \left(0, \frac{1}{2}\right)$. *Then for all* $(t, x) \in \mathfrak{R}_\vartheta(p_1, p_2)$, *we have*

- *Case* $\mu > \dfrac{1}{2}$ :

$$\left| u(t,x) \right| \leqslant C_0^{(1)}(u_0)\, t^{-\frac{1}{2} + \vartheta(1-\mu)} + \sum_{k=1}^{9} C_k^{(1)}(u_0)\, t^{-\tilde{\beta}_k^{(1)} + \vartheta \tilde{\alpha}_k^{(1)}} ,$$

  *where* $\max_{k \in \{1,\dots,9\}} \left\{ -\tilde{\beta}_k^{(1)} + \vartheta \tilde{\alpha}_k^{(1)} \right\} < -\dfrac{1}{2} + \vartheta(1-\mu)$ *and the decay rate* $t^{-\frac{1}{2}+\vartheta(1-\mu)}$ *is optimal. The exponents* $\tilde{\alpha}_k^{(1)}$, $\tilde{\beta}_k^{(1)}$ *are defined in Theorem 2.2.2 and the constants* $C_k^{(1)}(u_0) \geqslant 0$ *are given in the proof ;*

- *Case* $\mu = \dfrac{1}{2}$ :

$$\left| u(t,x) \right| \leqslant C_0^{(2)}(u_0)\, t^{-\frac{1}{2} + \frac{\vartheta}{2}} + \sum_{k=1}^{8} C_k^{(2)}(u_0)\, t^{-\tilde{\beta}_k^{(2)} + \vartheta \tilde{\alpha}_k^{(2)}} ,$$

  *where* $\max_{k \in \{1,\dots,8\}} \left\{ -\tilde{\beta}_k^{(2)} + \vartheta \tilde{\alpha}_k^{(2)} \right\} < -\dfrac{1}{2} + \dfrac{\vartheta}{2}$ *and the decay rate* $t^{-\frac{1}{2}+\frac{\vartheta}{2}}$ *is optimal. The exponents* $\tilde{\alpha}_k^{(2)}$, $\tilde{\beta}_k^{(2)}$ *are defined in Theorem 2.2.2 and the constants* $C_k^{(2)}(u_0) \geqslant 0$ *are given in the proof ;*

- *Case* $\mu < \dfrac{1}{2}$ :

$$\left| u(t,x) \right| \leqslant C_0^{(3)}(u_0)\, t^{-\mu + \vartheta\mu} + \sum_{k=1}^{9} C_k^{(3)}(u_0)\, t^{-\tilde{\beta}_k^{(3)} + \vartheta \tilde{\alpha}_k^{(3)}} ,$$



 

*where* $\max_{k\in\{1,\dots,9\}}\left\{-\tilde{\beta}_k^{(3)}+\vartheta\tilde{\alpha}_k^{(3)}\right\}<-\mu+\vartheta\mu$ *and the decay rate* $t^{-\mu+\vartheta\mu}$ *is optimal. The exponents* $\tilde{\alpha}_k^{(3)}$, $\tilde{\beta}_k^{(3)}$ *are defined in Theorem 2.2.2 and the constants* $C_k^{(3)}(u_0)\geqslant 0$ *are given in the proof.*

*Proof.* First of all, let us establish an estimate for abstract oscillatory integrals that we shall use in the present proof. From Theorem 2.2.2, we derive the following estimate in the case $\mu>\frac{1}{2}$,

$$\begin{aligned}\left|\int_{p_1}^{p_2}U(p)\,e^{i\omega\psi(p)}\,dp\right| &\leqslant \left|\tilde{H}(\omega,p_0,U)\right|(p_0-p_1)^{\mu-1}\,\omega^{-\frac{1}{2}}+\sum_{k=1}^{9}\tilde{R}_k^{(1)}(U)(p_0-p_1)^{-\tilde{\alpha}_k^{(1)}}\,\omega^{-\tilde{\beta}_k^{(1)}}\\ &\leqslant \sqrt{\pi}\left\|\tilde{u}\right\|_{L^\infty(p_1,p_2)}(p_0-p_1)^{\mu-1}\,\omega^{-\frac{1}{2}}+\sum_{k=1}^{9}\tilde{R}_k^{(1)}(U)(p_0-p_1)^{-\tilde{\alpha}_k^{(1)}}\,\omega^{-\tilde{\beta}_k^{(1)}}\ .\end{aligned} \tag{2.24}$$

Now let us prove Theorem 2.3.7 in the case $\mu>\frac{1}{2}$. Firstly, we rewrite the solution formula (2.23) as an oscillatory integral (see the beginning of the proof of Theorem 2.3.2). The hypothesis $(t,x)\in\mathfrak{R}_\vartheta(p_1,p_2)$ implies

$$\frac{x}{2t}-p_1\geqslant t^{-\vartheta}\ , \tag{2.25}$$

and we see that $p_0:=\frac{x}{2t}\in(p_1,p_2)$ in this case. Hence we observe that the hypotheses of Theorem 2.2.2 are satisfied and, in particular, estimate (2.24) is applicable, leading to

$$\left|u(t,x)\right|\leqslant\frac{1}{2\sqrt{\pi}}\left\|\tilde{u}\right\|_{L^\infty(p_1,p_2)}\left(\frac{x}{2t}-p_1\right)^{\mu-1}t^{-\frac{1}{2}}+\sum_{k=1}^{9}\tilde{R}_k^{(1)}\left(\frac{1}{2\pi}\,\mathcal{F}u_0\right)\left(\frac{x}{2t}-p_1\right)^{-\tilde{\alpha}_k^{(1)}}t^{-\tilde{\beta}_k^{(1)}}\ .$$

According to Remark 2.2.3, each $\tilde{\alpha}_k^{(1)}$ is non-negative if the parameter $\delta\in\left(\frac{1}{2},1\right)$ appearing in the proof of Theorem 2.2.2 is such that $\delta\geqslant\frac{\mu+1}{2}$; so let us suppose that $\delta\geqslant\frac{\mu+1}{2}$. Hence we put inequality (2.25) into the last estimate and we obtain

$$\left|u(t,x)\right|\leqslant\frac{1}{2\sqrt{\pi}}\left\|\tilde{u}\right\|_{L^\infty(p_1,p_2)}t^{-\frac{1}{2}+\vartheta(1-\mu)}+\sum_{k=1}^{9}\tilde{R}_k^{(1)}\left(\frac{1}{2\pi}\,\mathcal{F}u_0\right)t^{-\tilde{\beta}_k^{(1)}+\vartheta\tilde{\alpha}_k^{(1)}}\ .$$

Thanks to Corollary 2.2.4, we know that each exponent $-\tilde{\beta}_k^{(1)}+\vartheta\tilde{\alpha}_k^{(1)}$ is strictly smaller than $-\frac{1}{2}+\vartheta(1-\mu)$ if $\delta>\mu$ (which is true since $\delta\geqslant\frac{\mu+1}{2}$) and if $\delta>\frac{1}{2}+\vartheta$; so $\delta$ is chosen in such a way that it is strictly larger than $\frac{1}{2}+\vartheta$. Defining for all $k\in\{1,\dots,9\}$,

$$C_0^{(1)}(u_0):=\frac{1}{2\sqrt{\pi}}\left\|\tilde{u}\right\|_{L^\infty(p_1,p_2)}\quad,\quad C_k^{(1)}(u_0):=\tilde{R}_k^{(1)}\left(\frac{1}{2\pi}\,\mathcal{F}u_0\right)\ ,$$

we obtain the result in the case $\mu>\frac{1}{2}$. The optimality is a direct consequence of Theorem 2.3.8.

We employ the same arguments to establish the estimates in the two other cases, and we define

- $C_0^{(2)}(u_0):=\left(\frac{1}{2\sqrt{\pi}}+\frac{\Gamma(\mu)}{2^{\mu+1}}\right)\left\|\tilde{u}\right\|_{L^\infty(p_1,p_2)}\quad,\quad C_k^{(2)}(u_0):=\tilde{R}_k^{(2)}\left(\frac{1}{2\pi}\,\mathcal{F}u_0\right)$ ;
- $C_0^{(3)}(u_0):=\frac{\Gamma(\mu)}{2^{\mu+1}}\left\|\tilde{u}\right\|_{L^\infty(p_1,p_2)}\quad,\quad C_k^{(3)}(u_0):=\tilde{R}_k^{(3)}\left(\frac{1}{2\pi}\,\mathcal{F}u_0\right)$ ,

for $k \geqslant 1$. □

The last result is devoted to the optimality of the previous uniform estimates. In the region $\mathfrak{R}_\vartheta(p_1, p_2)$, we expect that the decay will be slow in parts which are close to the critical direction given by $p_1$, where the influence of the singular frequency is the strongest. So we use Corollary 2.2.4 to provide asymptotic expansions of the solution on the space-time curve $\mathfrak{G}_\vartheta(p_1)$, the left boundary of the region $\mathfrak{R}_\vartheta(p_1, p_2)$, and we show that the decay rates obtained in the preceding result are attained on this curve, proving the optimality.

**2.3.8 Theorem.** *Suppose that $u_0$ satisfies Condition $(\mathrm{C1}_{[p_1,p_2],\mu})$ and fix $\vartheta \in \big(0, \frac{1}{2}\big)$. For all $t > T_\vartheta(p_1, p_2)$, define $H(t, u_0) \in \mathbb{C}$ and $K_\mu(t, u_0) \in \mathbb{C}$ as follows:*

- $$H(t, u_0) := \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{it(p_1+t^{-\vartheta})^2}\, \tilde{u}\big(p_1 + t^{-\vartheta}\big) ;$$
- $$K_\mu(t, u_0) := \frac{\Gamma(\mu)}{2^\mu}\, e^{i\frac{\pi\mu}{2}}\, e^{-itp_1^2+ixp_1}\, \tilde{u}(p_1) .$$

*Then for all $(t, x) \in \mathfrak{G}_\vartheta(p_1)$ with $t > T_\vartheta(p_1, p_2)$, we have*

- *Case $\mu > \dfrac{1}{2}$ :*
$$\Big| u(t, x) - H(t, u_0)\, t^{-\frac{1}{2}+\vartheta(1-\mu)} \Big| \leqslant \sum_{k=1}^{9} C_k^{(1)}(u_0)\, t^{-\tilde{\beta}_k^{(1)}+\vartheta\tilde{\alpha}_k^{(1)}} ,$$
*where $\max\limits_{k\in\{1,\dots,9\}} \Big\{-\tilde{\beta}_k^{(1)} + \vartheta\tilde{\alpha}_k^{(1)}\Big\} < -\dfrac{1}{2}+\vartheta(1-\mu)$. The exponents $\tilde{\alpha}_k^{(1)}$, $\tilde{\beta}_k^{(1)}$ are defined in Theorem 2.2.2 and the constants $C_k^{(1)}(u_0) \geqslant 0$ are given in Theorem 2.3.7 ;*

- *Case $\mu = \dfrac{1}{2}$ :*
$$\Big| u(t, x) - \big(K_\mu(t, u_0) + H(t, u_0)\big)\, t^{-\frac{1}{2}+\frac{\vartheta}{2}} \Big| \leqslant \sum_{k=1}^{8} C_k^{(2)}(u_0)\, t^{-\tilde{\beta}_k^{(2)}+\vartheta\tilde{\alpha}_k^{(2)}} ,$$
*where $\max\limits_{k\in\{1,\dots,8\}} \Big\{-\tilde{\beta}_k^{(2)} + \vartheta\tilde{\alpha}_k^{(2)}\Big\} < -\dfrac{1}{2} + \dfrac{\vartheta}{2}$. The exponents $\tilde{\alpha}_k^{(2)}$, $\tilde{\beta}_k^{(2)}$ are defined in Theorem 2.2.2 and the constants $C_k^{(2)}(u_0) \geqslant 0$ are given in Theorem 2.3.7 ;*

- *Case $\mu < \dfrac{1}{2}$ :*
$$\Big| u(t, x) - K_\mu(t, u_0)\, t^{-\mu+\vartheta\mu} \Big| \leqslant \sum_{k=1}^{9} C_k^{(3)}(u_0)\, t^{-\tilde{\beta}_k^{(3)}+\vartheta\tilde{\alpha}_k^{(3)}} ,$$
*where $\max\limits_{k\in\{1,\dots,9\}} \Big\{-\tilde{\beta}_k^{(3)} + \vartheta\tilde{\alpha}_k^{(3)}\Big\} < -\mu + \vartheta\mu$. The exponents $\tilde{\alpha}_k^{(3)}$, $\tilde{\beta}_k^{(3)}$ are defined in Theorem 2.2.2 and the constants $C_k^{(3)}(u_0) \geqslant 0$ are given in Theorem 2.3.7.*

*Proof.* The hypothesis $(t, x) \in \mathfrak{G}_\vartheta(p_1)$ is equivalent to $p_0 - p_1 = t^{-\vartheta}$ where $p_0 := \frac{x}{2t}$. Using the solution formula (2.23) and the rewritting given in the proof of Theorem 2.3.2, we can apply Corollary 2.2.4 and we obtain the desired estimates. □



 

# Chapter 3

# Optimal van der Corput estimates describing the interaction between a stationary point of the phase and a singularity of the amplitude

**Abstract**

In the present chapter, we furnish an extension of the van der Corput lemma for oscillatory integrals for one integration variable: the phase is allowed to have a stationary point of real order and the amplitude to have an integrable singular point. This extension permits to obtain uniform estimates with respect to the position of the stationary point, which can be inside or outside the integration interval, for the above class of oscillatory integrals. Moreover in order to localize time-asymptotically solutions of evolution equations in Chapter 4, we consider also the case of the absence of a stationary point inside the integration interval, leading to faster decay rates than those mentioned above. All the resulting decay rates of the present chapter are proved to be optimal by employing the results of Chapter 2.

## Contents





 

## 3.1 Van der Corput type estimates for stationary points of real order and singular amplitudes

We start this section by stating the hypotheses on the phase function. Two examples are then given to illustrate theses assumptions.

Let $p_1, p_2$ be two finite real numbers such that $p_1 < p_2$, and let $I$ be an open interval containing $[p_1, p_2]$.

**Assumption $(\mathbf{P2}_{p_0,\rho})$.** Let $p_0 \in I$ and $\rho > 1$.
A function $\psi : I \longrightarrow \mathbb{R}$ satisfies Assumption $(\mathrm{P2}_{p_0,\rho})$ if and only if $\psi \in \mathcal{C}^1\big(I\big) \cap \mathcal{C}^2\big(I \backslash \{p_0\}\big)$ and there exists a function $\tilde{\psi} : I \longrightarrow \mathbb{R}$ such that

$$\forall\, p \in I \qquad \psi'(p) = |p - p_0|^{\rho - 1}\, \tilde{\psi}(p) \ ,$$

where $\big|\tilde{\psi}\big| : I \longrightarrow \mathbb{R}$ is assumed continuous and does not vanish on $I$.
The point $p_0$ is called *stationary point* of $\psi$ of order $\rho - 1$, and $\tilde{\psi}$ the *non-degenerate factor* of $\psi$.

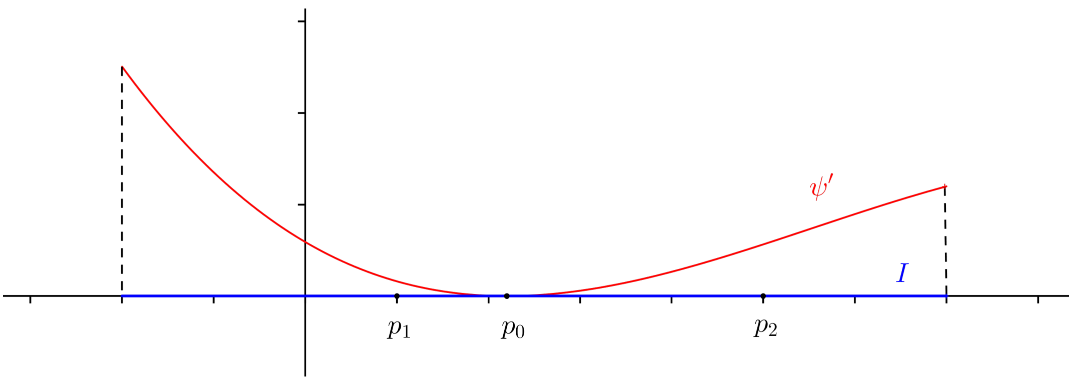


Figure 3.1: First derivative of a function satisfying Assumption $(\mathrm{P2}_{p_0,\rho})$

Let us comment on this choice. Firstly we consider the absolute value of $p - p_0$ because we want to include stationary points of non-integer order in the study. Secondly, the fact that $\tilde{\psi}$ does not vanish prevents this function from affecting the order of the stationary point $p_0$. Finally, the continuity of $\big|\tilde{\psi}\big|$ is sufficient to ensure the fact that $\min\limits_{[p_1,p_2]} |\tilde{\psi}|$ exists and is non-zero; this quantity will be employed several times to establish the results of this chapter. Nevertheless we do not claim that we achieve maximum generality with these hypotheses.
Note that $\tilde{\psi}$ is actually continuously differentiable on $I \backslash \{p_0\}$, because

$$\forall\, p \neq p_0 \qquad \tilde{\psi}(p) = \frac{\psi'(p)}{|p - p_0|^{\rho - 1}} \ .$$

This implies that $\tilde{\psi}$ has a constant sign on $\{p \in I \,|\, p < p_0\}$ and $\{p \in I \,|\, p > p_0\}$; note that the sign of $\tilde{\psi}$ can be different on each interval if $\tilde{\psi}$ has a discontinuity at the point $p_0$.

We illustrate the above Assumption ($\mathrm{P2}_{p_0,\rho}$) in the following two examples. In particular, the first example shows that smooth functions with vanishing first derivatives are included.

**3.1.1 Example.** i) Let $\psi : I \longrightarrow \mathbb{R}$ be a function belonging to $\mathcal{C}^N\big(I\big)$ for a certain $N \geqslant 2$, and let $p_0 \in I$. Suppose that $\psi^{(k)}(p_0) = 0$ for $k = 1, \dots, N-1$. Then by Taylor's formula, we obtain

$$\begin{aligned}\psi'(p) &= \frac{1}{(N-2)!} \int_{p_0}^{p} (p-x)^{N-2}\, \psi^{(N)}(x)\, dx \\ &= \frac{(p-p_0)^{N-1}}{(N-2)!} \int_0^1 (1-y)^{N-2}\, \psi^{(N)}\big(y(p-p_0)+p_0\big)\, dy \ ,\end{aligned}$$

for all $p \in I$. If we define $\tilde{\psi}$ as follows

$$\tilde{\psi}(p) := \begin{cases} \dfrac{1}{(N-2)!} \left( \dfrac{p-p_0}{|p-p_0|} \right)^{N-1} \displaystyle\int_0^1 (1-y)^{N-2}\, \psi^{(N)}\big(y(p-p_0)+p_0\big)\, dy \ , \\ \hfill \text{if } p \neq p_0 \ , \\ \dfrac{1}{(N-1)!}\, \psi^{(N)}(p_0) \ , \quad \text{if } p = p_0 \ , \end{cases}$$

then $\psi'(p) = |p-p_0|^{N-1}\, \tilde{\psi}(p)$. Supposing $\big|\psi^{(N)}\big| > 0$ on $I$ implies that $\psi$ satisfies Assumption ($\mathrm{P2}_{p_0,N}$).

ii) Let $N \in \mathbb{N}$ such that $N \geqslant 2$ and choose $\alpha \in (N-1, N)$. Suppose that $\psi'(p) = |p|^{\alpha-1}$, for all $p \in \mathbb{R}$. In this case, $\psi \in \mathcal{C}^{N-1}\big(\mathbb{R},\mathbb{R}\big)$ but $\psi \notin \mathcal{C}^N\big(\mathbb{R},\mathbb{R}\big)$, and $\tilde{\psi} = 1$. Then Assumption ($\mathrm{P2}_{0,\alpha}$) is satisfied.

Now let us introduce the hypotheses concerning the amplitude function that we shall use throughout this chapter.

**Assumption ($\mathbf{A2}_{p_1,\mu}$).** Let $\mu \in (0,1]$.
A function $U : (p_1, p_2] \longrightarrow \mathbb{C}$ satisfies Assumption ($\mathrm{A2}_{p_1,\mu}$) if and only if there exists a function $\tilde{u} : [p_1, p_2] \longrightarrow \mathbb{C}$ such that

$$\forall\, p \in (p_1, p_2] \qquad U(p) = (p-p_1)^{\mu-1}\, \tilde{u}(p) \ ,$$

where $\tilde{u}$ is assumed continuous on $[p_1, p_2]$, differentiable on $(p_1, p_2)$ with $\tilde{u}' \in L^1(p_1, p_2)$, and $\tilde{u}(p_1) \neq 0$ if $\mu \neq 1$.
The point $p_1$ is called *singular point* of $U$, and $\tilde{u}$ the *regular factor* of $U$.

According to this assumption, the amplitude is singular at the left endpoint of the interval. The results of this chapter remain unchanged if we suppose that the singular point is at the right endpoint of the interval. Moreover in the case of an amplitude function which is singular inside the integration interval, the study can be reduced to the two preceding cases: it suffices to split the integral at the singular point.

Before introducing the first theorem of this section, let us state a basic lemma which will be used several times in this section.



 

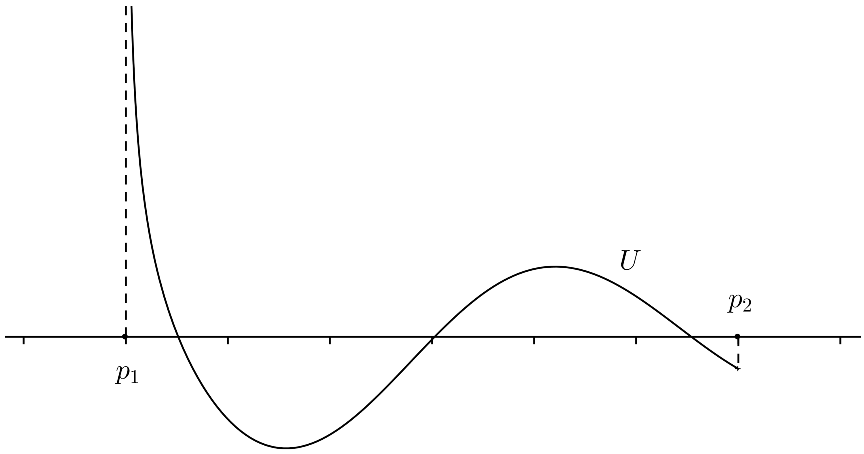


Figure 3.2: Function satisfying Assumption $(\mathrm{A2}_{p_1,\mu})$

**3.1.2 Lemma.** *Let $\alpha \in (0,1]$ and let $x, y \in \mathbb{R}_+$ such that $x \geqslant y$. Then we have*

$$x^\alpha - y^\alpha \leqslant (x-y)^\alpha \ .$$

*Proof.* The case $\alpha = 1$ is trivial so let us assume $\alpha < 1$. If $y = 0$ then the result is clear. Suppose $y \neq 0$, then the above inequality is equivalent to

$$\left(\frac{x}{y}\right)^\alpha - 1 \leqslant \left(\frac{x}{y} - 1\right)^\alpha \ .$$

Define the function $h : [1, +\infty) \longrightarrow \mathbb{R}$ by

$$\forall\, t \in [1, +\infty) \qquad h(t) := (t-1)^\alpha - t^\alpha + 1 \ .$$

Then we note that for all $t > 1$,

$$h'(t) = \alpha \left( (t-1)^{\alpha-1} - t^{\alpha-1} \right) \geqslant 0 \ ,$$

since $\alpha - 1 < 0$. It follows

$$\forall\, t \in [1, +\infty) \qquad h(t) \geqslant h(1) = 0 \ ,$$

which proves the lemma. □

Now let us state a first van der Corput type estimate for integrals of the form

$$\forall\, \omega > 0 \qquad I(\omega) = \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp \ .$$

Here we suppose that the phase function $\psi$ has a stationary point $p_0$ of order $\rho - 1$ which belongs to the integration interval. The resulting estimate is uniform with respect to the position of $p_0$ inside $[p_1, p_2]$; the decay rate is given by $\omega^{-\frac{\mu}{\rho}}$ and an upper bound of the constant is given in terms of the regular factor $\tilde{u}$ of the amplitude and of the non-degenerate factor $\tilde{\psi}$ of the phase function.

The proof of this result is divided with respect to the size of the parameter $\omega$. In the case of small $\omega$, the integral can be estimated by the product of the $L^\infty$-norm of the regular part $\tilde{u}$ and the length of the interval to the power $\mu$, the exponent coming from the singular behaviour of the integrand at the point $p_1$ of order $\mu - 1$. Then we exploit the fact that the interval is smaller than $\omega^{-\frac{1}{\rho}}$ to bound the integral by $\omega^{-\frac{\mu}{\rho}}$. For large $\omega$, we adapt Stein's method [27, Chapter VIII, Proposition 2], which covers the case of stationary points of integer order and which is a generalization of Zygmund's method [32, Chapter V, Lemma 4.3] for simple stationary points. We decompose the integration interval in such way that $p_0$ and $p_1$ are contained in intervals whose length is proportional to $\omega^{-\frac{1}{\rho}}$. The integrals on these intervals are estimated by using the asymptotic smallness of their integration intervals. On the other intervals, we integrate by parts and we employ an upper bound for the amplitude as well as a lower bound for the first derivative of the phase, both bounds depending on $\omega$, to obtain the result. Let us note that we consider also the case of intermediate $\omega$; this situation can be studied by combining the methods used in the case of small and large $\omega$.

**3.1.3 Theorem.** *Let $\rho > 1$, $\mu \in (0,1]$ and choose $p_0 \in [p_1, p_2]$. Suppose that the functions $\psi : I \longrightarrow \mathbb{R}$ and $U : (p_1, p_2] \longrightarrow \mathbb{C}$ satisfy Assumption $(\mathrm{P2}_{p_0,\rho})$ and Assumption $(\mathrm{A2}_{p_1,\mu})$, respectively. Moreover suppose that $\psi'$ is monotone on $I^-_{p_0}$ and $I^+_{p_0}$, where*

$$I^-_{p_0} := \big\{ p \in I \,\big|\, p < p_0 \big\} \qquad , \qquad I^+_{p_0} := \big\{ p \in I \,\big|\, p > p_0 \big\} \ .$$

*Then we have*

$$\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp \right| \leqslant C(U,\psi)\, \omega^{-\frac{\mu}{\rho}} \ ,$$

*for all $\omega > 0$, where the constant $C(U,\psi) > 0$ is given by*

$$C(U,\psi) := \frac{3}{\mu}\, \|\tilde{u}\|_{L^\infty(p_1,p_2)} + \Big( 8\, \|\tilde{u}\|_{L^\infty(p_1,p_2)} + 2\, \|\tilde{u}'\|_{L^1(p_1,p_2)} \Big) \left( \min_{p \in [p_1,p_2]} \Big| \tilde{\psi}(p) \Big| \right)^{-1} \ .$$

Before proving this theorem, let us illustrate the monotonicity hypothesis on $\psi'$ by using the settings given in Example 3.1.1.

**3.1.4 Example.** i) In the setting of Example 3.1.1 i), if $\big|\psi^{(N)}\big| > 0$ on $I$, then $\psi'$ is monotone on both intervals $I^-_{p_0}$ and $I^+_{p_0}$.
Indeed if $N = 2$, then it is clear that the hypothesis $\big|\psi''\big| > 0$ implies the result. Suppose now that $N \geqslant 3$; then applying Taylor's formula to $\psi''$, namely

$$\psi''(p) = \frac{1}{(N-3)!} \int_{p_0}^{p} (p-x)^{N-3}\, \psi^{(N)}(x)\, dx \ ,$$

for all $p \in I$, we observe that $\psi''$ has a constant sign on $I^-_{p_0}$ and $I^+_{p_0}$, which provides the result.

ii) In the setting of Example 3.1.1 ii), we note that $\psi'$ is clearly monotone on $(-\infty, 0)$ and on $(0, +\infty)$.

*Proof of Theorem 3.1.3.* Let $p_0 \in (p_1, p_2)$ and let us suppose $\frac{p_0 - p_1}{2} \geqslant p_2 - p_0$ without loss of generality; the other case can be treated in a similar way. Note that we shall study the cases $p_0 = p_1$ and $p_0 = p_2$ at the end of the proof. Now let us divide the proof.

- *Case* $\omega > (p_2 - p_0)^{-\rho}$. Define $\delta := \omega^{-\frac{1}{\rho}}$ and consider the following splitting of the integral:

$$\int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp = \int_{p_1}^{p_1+\delta} \dots \quad + \int_{p_1+\delta}^{p_0-\delta} \dots \quad + \int_{p_0-\delta}^{p_0+\delta} \dots \quad + \int_{p_0+\delta}^{p_2} \dots$$
$$=: I^{(1)}(\omega) + I^{(2)}(\omega) + I^{(3)}(\omega) + I^{(4)}(\omega)\;.$$

Remark that this splitting is well-defined thanks to the hypothesis $\omega > (p_2 - p_0)^{-\rho}$. Let us estimate each integral.

  – *Study of* $I^{(1)}(\omega)$*.* We bound $I^{(1)}(\omega)$ in a simple way as follows:

$$\left|I^{(1)}(\omega)\right| \leqslant \int_{p_1}^{p_1+\delta} |U(p)|\, dp \leqslant \|\tilde{u}\|_{L^\infty(p_1,p_2)} \int_{p_1}^{p_1+\delta} (p-p_1)^{\mu-1}\, dp = \frac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu}\, \delta^\mu\;.$$

  – *Study of* $I^{(2)}(\omega)$*.* Here we shall suppose that $\tilde{\psi}$ is positive on $I_{p_0}^-$, which implies the positivity of $\psi'$; the case $\tilde{\psi} < 0$ can be studied in the same manner. Since $\psi'$ does not vanish on $[p_1+\delta, p_0-\delta]$, the substitution $s = \psi(p)$ can be employed. Setting $\varphi := \psi^{-1}$, $s_1 := \psi(p_1+\delta)$ and $s_2 := \psi(p_0-\delta)$, we obtain

$$I^{(2)}(\omega) = \int_{s_1}^{s_2} U\big(\varphi(s)\big)\, \varphi'(s)\, e^{i\omega s}\, ds$$
$$= (i\omega)^{-1}\left(\Big[(U\circ\varphi)(s)\, \varphi'(s)\, e^{i\omega s}\Big]_{s_1}^{s_2} - \int_{s_1}^{s_2} \big((U\circ\varphi)\,\varphi'\big)'(s)\, e^{i\omega s}\, ds\right)\;;$$

the last equality was obtained by integrating by parts.
Let us control the boundary terms and the integral. Firstly, we have

$$\big|U(p)\big| \leqslant \delta^{\mu-1}\, \|\tilde{u}\|_{L^\infty(p_1+\delta,p_0-\delta)} \leqslant \delta^{\mu-1}\, \|\tilde{u}\|_{L^\infty(p_1,p_2)}\;, \tag{3.1}$$

for all $p \in [p_1+\delta, p_0-\delta]$, since $U(p) = (p-p_1)^{\mu-1}\tilde{u}(p)$ by hypothesis. Moreover the fact that $\psi'$ satisfies Assumption $(\mathrm{P2}_{p_0,\rho})$ implies

$$\forall\, p \in [p_1+\delta, p_0-\delta] \qquad |\psi'(p)| \geqslant \delta^{\rho-1}\, m\;,$$

where $m := \min_{p\in[p_1,p_2]} \left|\tilde{\psi}(p)\right| > 0$. Combining this with the definition of $\varphi$ leads to

$$\forall\, s \in [s_1, s_2] \qquad |\varphi'(s)| \leqslant \delta^{1-\rho}\, m^{-1}\;. \tag{3.2}$$

Inequalities (3.1) and (3.2) permit to estimate the boundary terms as follows,

$$\left|\Big[(U\circ\varphi)(s)\, \varphi'(s)\, e^{i\omega s}\Big]_{s_1}^{s_2}\right| \leqslant 2\, \|\tilde{u}\|_{L^\infty(p_1,p_2)}\, m^{-1}\, \delta^{\mu-\rho}\;.$$

It remains to control the integral. We have

$$\big((U\circ\varphi)\,\varphi'\big)' = (U'\circ\varphi)\, (\varphi')^2 + (U\circ\varphi)\,\varphi''\;,$$

by the product rule; consequently,

$$\begin{aligned}\left|\int_{s_1}^{s_2}\big((U\circ\varphi)\,\varphi'\big)'(s)\,e^{i\omega s}\,ds\right| &\leqslant \int_{s_1}^{s_2}\Big|(U'\circ\varphi)(s)\,\varphi'(s)^2\Big|\,ds\\ &\quad+\int_{s_1}^{s_2}\Big|(U\circ\varphi)(s)\,\varphi''(s)\Big|\,ds\\ &\leqslant \int_{s_1}^{s_2}\Big|(U'\circ\varphi)(s)\,\varphi'(s)\Big|\,ds\;\delta^{1-\rho}\,m^{-1}\\ &\quad+\|U\|_{L^\infty(p_1+\delta,p_0-\delta)}\int_{s_1}^{s_2}\big|\varphi''(s)\big|\,ds\\ &\leqslant \int_{p_1+\delta}^{p_0-\delta}\big|U'(p)\big|\,dp\;\delta^{1-\rho}\,m^{-1}\\ &\quad+\delta^{\mu-1}\,\|\tilde u\|_{L^\infty(p_1,p_2)}\int_{s_1}^{s_2}\big|\varphi''(s)\big|\,ds\,. \qquad (3.3)\end{aligned}$$

The definition of $U$ implies

$$\begin{aligned}\int_{p_1+\delta}^{p_0-\delta}\big|U'(p)\big|\,dp &\leqslant \int_{p_1+\delta}^{p_0-\delta}\Big|(\mu-1)(p-p_1)^{\mu-2}\,\tilde u(p)\Big|\,dp\\ &\quad+\int_{p_1+\delta}^{p_0-\delta}\Big|(p-p_1)^{\mu-1}\,\tilde u'(p)\Big|\,dp\\ &\leqslant \int_{p_1+\delta}^{p_0-\delta}(1-\mu)(p-p_1)^{\mu-2}\,dp\;\;\|\tilde u\|_{L^\infty(p_1,p_2)}\\ &\quad+\delta^{\mu-1}\int_{p_1+\delta}^{p_0-\delta}\big|\tilde u'(p)\big|\,dp\\ &\leqslant \delta^{\mu-1}\,\|\tilde u\|_{L^\infty(p_1,p_2)}+\delta^{\mu-1}\,\|\tilde u'\|_{L^1(p_1,p_2)}\;; \qquad (3.4)\end{aligned}$$

the last inequality was obtained employing the fact that

$$\int_{p_1+\delta}^{p_0-\delta}(1-\mu)(p-p_1)^{\mu-2}\,dp=\delta^{\mu-1}-(p_0-\delta-p_1)^{\mu-1}\leqslant\delta^{\mu-1}\,.$$

Moreover the relation $\varphi''=\left(\dfrac{-\psi''}{{\psi'}^3}\right)\circ\varphi$ provides the following equalities,

$$\int_{s_1}^{s_2}\big|\varphi''(s)\big|\,ds=\int_{s_1}^{s_2}\left|\frac{-\psi''\big(\varphi(s)\big)}{\psi'\big(\varphi(s)\big)^3}\right|ds=\left|\int_{p_1+\delta}^{p_0-\delta}\frac{-\psi''(p)}{\psi'(p)^2}\,dp\right|\,,$$

the last equality comes from the change of variable $p=\varphi(s)$ and from the constant sign of $\psi''$ on $[p_1+\delta,p_0-\delta]$ thanks to the fact that $\psi'$ is monotonic on $I_{p_0}^-$. Then

$$\int_{s_1}^{s_2}\big|\varphi''(s)\big|\,ds=\left|\int_{p_1+\delta}^{p_0-\delta}\left(\frac{1}{\psi'}\right)'(p)\,dp\right|=\left|\frac{1}{\psi'(p_0-\delta)}-\frac{1}{\psi'(p_1+\delta)}\right|\leqslant\delta^{1-\rho}m^{-1}\,, \qquad (3.5)$$

where we used $|\psi'(p)| \geqslant \delta^{\rho-1}\, m$, for $p \in [p_1+\delta, p_0-\delta]$. Putting (3.4) and (3.5) into (3.3) provides

$$\left| \int_{s_1}^{s_2} \big((U\circ\varphi)\,\varphi'\big)'(s)\, e^{i\omega s}\, ds \right| \leqslant \Big( 2\,\|\tilde{u}\|_{L^\infty(p_1,p_2)} + \|\tilde{u}'\|_{L^1(p_1,p_2)} \Big)\, m^{-1}\, \delta^{\mu-\rho} \;.$$

We are now able to estimate $I^{(2)}(\omega)$:

$$\left| I^{(2)}(\omega) \right| \leqslant \Big( 4\,\|\tilde{u}\|_{L^\infty(p_1,p_2)} + \|\tilde{u}'\|_{L^1(p_1,p_2)} \Big)\, m^{-1}\, \delta^{\mu-\rho}\, \omega^{-1} \;.$$

– *Study of* $I^{(3)}(\omega)$. As for $I^{(1)}(\omega)$, we bound the integral of $|U|$ on $[p_0-\delta, p_0+\delta]$ to provide an estimate of $I^{(3)}(\omega)$:

$$\left| I^{(3)}(\omega) \right| \leqslant \frac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu} \Big( (p_0+\delta-p_1)^\mu - (p_0-\delta-p_1)^\mu \Big) \leqslant 2\, \frac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu}\, \delta^\mu \;;$$

note that we use Lemma 3.1.2 to obtain the last inequality.

– *Study of* $I^{(4)}(\omega)$. On $[p_0+\delta, p_2]$, one can bound from below the absolute value of the first derivative of the phase function as follows,

$$\left|\psi'\right| \geqslant \delta^{\rho-1} \min_{p\in[p_1,p_2]} \left| \tilde{\psi}(p) \right| = \delta^{\rho-1}\, m \;,$$

and we have

$$\forall\, p \in [p_0+\delta, p_2] \qquad (p-p_1)^{\mu-1} \leqslant (p_0+\delta-p_1)^{\mu-1} \leqslant \delta^{\mu-1} \;.$$

Following the lines of the study of $I^{(2)}(\omega)$ and using the two previous estimates, we obtain

$$\left| I^{(4)}(\omega) \right| \leqslant \Big( 4\,\|\tilde{u}\|_{L^\infty(p_1,p_2)} + \|\tilde{u}'\|_{L^1(p_1,p_2)} \Big)\, m^{-1}\, \delta^{\mu-\rho}\, \omega^{-1} \;.$$

To conclude this first case, we replace $\delta$ by $\omega^{-\frac{1}{\rho}}$ leading to the desired estimate:

$$\begin{aligned}
\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp \right| &\leqslant \left|I^{(1)}(\omega)\right| + \left|I^{(2)}(\omega)\right| + \left|I^{(3)}(\omega)\right| + \left|I^{(4)}(\omega)\right| \\
&\leqslant \frac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu}\, \omega^{-\frac{\mu}{\rho}} \; + \; 2\, \frac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu}\, \omega^{-\frac{\mu}{\rho}} \\
&\qquad + \; 2\Big( 4\,\|\tilde{u}\|_{L^\infty(p_1,p_2)} + \|\tilde{u}'\|_{L^1(p_1,p_2)} \Big)\, m^{-1}\, \omega^{-\frac{\mu-\rho}{\rho}}\, \omega^{-1} \\
&=: C(U,\psi)\, \omega^{-\frac{\mu}{\rho}} \;,
\end{aligned}$$

where

$$C(U,\psi) := \frac{3}{\mu}\, \|\tilde{u}\|_{L^\infty(p_1,p_2)} + \Big( 8\,\|\tilde{u}\|_{L^\infty(p_1,p_2)} + 2\,\|\tilde{u}'\|_{L^1(p_1,p_2)} \Big)\, m^{-1} \;.$$



 

- *Case* $\left(\frac{p_0-p_1}{2}\right)^{-\rho} < \omega \leqslant (p_2 - p_0)^{-\rho}$. As above, we define $\delta := \omega^{-\frac{1}{\rho}}$ and we consider the following splitting of the integral:

$$\int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp = \int_{p_1}^{p_1+\delta} \dots + \int_{p_1+\delta}^{p_0-\delta} \dots + \int_{p_0-\delta}^{p_0} \dots + \int_{p_0}^{p_2} \dots\ .$$

The three first integrals can be estimated using the methods of the first case, whereas the last integral can be controlled as follows,

$$\begin{aligned}\left|\int_{p_0}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp\right| &\leqslant \frac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu} \big((p_2-p_1)^\mu - (p_0-p_1)^\mu\big) \\ &\leqslant \frac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu} (p_2-p_0)^\mu \qquad (3.6) \\ &\leqslant \frac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu}\, \delta^\mu\ , \qquad (3.7)\end{aligned}$$

where we used Lemma 3.1.2 to obtain inequality (3.6) and the fact that $p_2 - p_0 \leqslant \delta$ to establish inequality (3.7). These arguments lead to

$$\begin{aligned}\left|\int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp\right| &\leqslant \left|\int_{p_1}^{p_1+\delta} \dots\right| + \left|\int_{p_1+\delta}^{p_0-\delta} \dots\right| + \left|\int_{p_0-\delta}^{p_0} \dots\right| + \left|\int_{p_0}^{p_2} \dots\right| \\ &\leqslant \frac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu}\, \delta^\mu + \Big(4\, \|\tilde{u}\|_{L^\infty(p_1,p_2)} + \|\tilde{u}'\|_{L^1(p_1,p_2)}\Big)\, m^{-1}\, \delta^{\mu-\rho}\, \omega^{-1} \\ &\quad + \frac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu}\, \delta^\mu + \frac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu}\, \delta^\mu\ ; \qquad (3.8)\end{aligned}$$

Replacing $\delta$ by $\omega^{-\frac{1}{\rho}}$ and observing that the constant which appears in (3.8) is smaller than $C(U,\psi)$ provides

$$\left|\int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp\right| \leqslant C(U,\psi)\, \omega^{-\frac{\mu}{\rho}}\ .$$

- *Case* $\omega \leqslant \left(\frac{p_0-p_1}{2}\right)^{-\rho}$. In this last case, we split the integral at the point $p_0$ and using the fact that $\omega \leqslant \left(\frac{p_0-p_1}{2}\right)^{-\rho} \leqslant (p_2-p_0)^{-\rho}$, we obtain

$$\begin{aligned}\left|\int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp\right| &\leqslant \left|\int_{p_1}^{p_0} \dots\right| + \left|\int_{p_0}^{p_2} \dots\right| \\ &\leqslant \frac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu} \big((p_0-p_1)^\mu + (p_2-p_0)^\mu\big) \\ &\leqslant 3\, \frac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu}\, \omega^{-\frac{\mu}{\rho}}\ .\end{aligned}$$

We see that $C(U,\psi)$ is larger than the constant appearing in the right-hand side of the preceding inequality, leading to the result in this case.

Finally the desired estimate holds also for $p_0 = p_1$ and $p_0 = p_2$, since it is sufficient to adapt slightly the different splittings of the integral used in the present proof, and to carry out the same steps. □

**3.1.5 Remark.** i) The choice of the splitting points is optimal in view of the final decay rate. To prove that, we follow the indication given in the proof of Lemma 4.3 of [32]. Let us choose $\delta > 0$ sufficiently small to split the oscillatory integral as follows,

$$\int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp = \int_{p_1}^{p_1+\delta} \dots \quad + \int_{p_1+\delta}^{p_0-\delta} \dots \quad + \int_{p_0-\delta}^{p_0+\delta} \dots \quad + \int_{p_0+\delta}^{p_2} \dots \, .$$

Applying the method employed in the case of large $\omega$ in the preceding proof gives an estimate of the form

$$\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp \right| \leqslant f_\omega(\delta) \, ,$$

where $f_\omega(\delta) := c_1 \delta^\mu + c_2\, \omega^{-1} \delta^{\mu-\rho}$, for certain constants $c_1, c_2 > 0$. We note that $(f_\omega)'$ vanishes at a unique point $\delta_0$ defined by

$$\delta_0 := \left( \frac{\mu}{\rho - \mu} \frac{c_1}{c_2} \right)^{-\frac{1}{\rho}} \omega^{-\frac{1}{\rho}} \, .$$

Since $\lim_{\delta \to 0^+} f_\omega(\delta) = \lim_{\delta \to +\infty} f_\omega(\delta) = +\infty$, $\delta_0$ is then the minimum of $f_\omega$. Therefore the choice $\delta = \omega^{-\frac{1}{\rho}}$ is optimal regarding the decay rate.
In particular, this splitting which depends on the parameter $\omega$ requires a decomposition of the proof with respect to the size of $\omega$. Indeed, the $\omega$-dependent cutting-points may leave the integration interval when $\omega$ is not sufficiently large.
And we note that the constant $C(U, \psi)$ may not be optimal since we do not choose exactly the minimum of $f_\omega$ for simplicity.

ii) Though the constant is surely not optimal, it could be slightly improved in the case of regular amplitudes, namely $\mu = 1$ with $U = \tilde{u}$. Indeed, the study of $I^{(1)}(\omega)$ is not necessary in this situation and inequality (3.4) can be simplified as follows,

$$\int_{p_1}^{p_0-\delta} \left| U'(p) \right| dp \leqslant \| \tilde{u}' \|_{L^1(p_1,p_2)} \, .$$

It follows that we can estimate $I^{(2)}(\omega)$ and $I^{(4)}(\omega)$ more precisely,

$$\left| I^{(j)}(\omega) \right| \leqslant \Big( 3\, \| \tilde{u} \|_{L^\infty(p_1,p_2)} + \| \tilde{u}' \|_{L^1(p_1,p_2)} \Big)\, m^{-1}\, \delta^{1-\rho}\, \omega^{-1} \, , \tag{3.9}$$

with $j = 2, 4$, leading to

$$C(U, \psi) := 2\, \| \tilde{u} \|_{L^\infty(p_1,p_2)} + \Big( 6\, \| \tilde{u} \|_{L^\infty(p_1,p_2)} + 2\, \| \tilde{u}' \|_{L^1(p_1,p_2)} \Big)\, m^{-1} \, .$$

This refined constant will be used several times in Chapter 4, Section 4.1.

In our second result, we assume that the stationary point $p_0$ is outside the interval of integration $[p_1, p_2]$. In this case, the derivative of the phase function does not vanish inside the integration interval but it can be arbitrarily close to 0 if the stationary point is close to this interval. The estimate that we provide does not depend on the position of the stationary point outside $[p_1, p_2]$, which makes that the resulting decay rate is the same as the one obtained in our first theorem. We follow the lines of the proof of Theorem 3.1.3 to prove the following result.

**3.1.6 Theorem.** *Let $\rho > 1$, $\mu \in (0,1]$ and choose $p_0 \in I \backslash [p_1, p_2]$. Suppose that the functions $\psi : I \longrightarrow \mathbb{R}$ and $U : (p_1, p_2] \longrightarrow \mathbb{C}$ satisfy Assumption* $(\mathrm{P2}_{p_0,\rho})$ *and Assumption* $(\mathrm{A2}_{p_1,\mu})$*, respectively. Moreover suppose that $\psi'$ is monotone on $[p_1, p_2]$. Then we have*

$$\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp \right| \leqslant \tilde{C}(U,\psi)\, \omega^{-\frac{\mu}{\rho}} ,$$

*for all $\omega > 0$, where the constant $\tilde{C}(U,\psi) > 0$ is given by*

$$\tilde{C}(U,\psi) := \frac{2}{\mu}\, \|\tilde{u}\|_{L^\infty(p_1,p_2)} + \Big( 4 \, \|\tilde{u}\|_{L^\infty(p_1,p_2)} + \|\tilde{u}'\|_{L^1(p_1,p_2)} \Big) \left( \min_{p \in [p_1,p_2]} \left| \tilde{\psi}(p) \right| \right)^{-1} .$$

*Proof.* We divide the proof with respect to the value of $\omega$.

- *Case* $\omega > \left(\frac{p_2-p_1}{2}\right)^{-\rho}$. We define $\delta := \omega^{-\frac{1}{\rho}}$ and we split the integral,

$$\begin{aligned} \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp &= \int_{p_1}^{p_1+\delta} \ldots \quad + \int_{p_1+\delta}^{p_2-\delta} \ldots \quad + \int_{p_2-\delta}^{p_2} \ldots \\ &=: \tilde{I}^{(1)}(\omega) + \tilde{I}^{(2)}(\omega) + \tilde{I}^{(3)}(\omega) , \end{aligned}$$

  where $\tilde{I}^{(1)}(\omega)$ and $\tilde{I}^{(3)}(\omega)$ are bounded from above by $\dfrac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu}\, \delta^\mu$. To estimate the integral $\tilde{I}^{(2)}(\omega)$, we follow the lines of the method employed to study the integral $I^{(2)}(\omega)$ in the proof of Theorem 3.1.3, which provides

$$\left| \tilde{I}^{(2)}(\omega) \right| \leqslant \Big( 4 \, \|\tilde{u}\|_{L^\infty(p_1,p_2)} + \|\tilde{u}'\|_{L^1(p_1,p_2)} \Big)\, m^{-1}\, \delta^{\mu-\rho}\, \omega^{-1} ;$$

  we used the fact that

$$\forall\, p \in [p_1+\delta, p_2-\delta] \qquad \big| U(p) \big| \leqslant \delta^{\mu-1} \|\tilde{u}\|_{L^\infty(p_1,p_2)} ,$$

  and

$$\forall\, p \in [p_1+\delta, p_2-\delta] \qquad \big| \psi'(p) \big| \geqslant \begin{cases} (p_1+\delta-p_0)^{\rho-1}\, m \geqslant \delta^{\rho-1}\, m , & \text{if } p_0 < p_1 , \\ (p_0-p_2+\delta)^{\rho-1}\, m \geqslant \delta^{\rho-1}\, m , & \text{if } p_0 > p_2 , \end{cases}$$

  with $m := \min\limits_{p\in[p_1,p_2]} \left| \tilde{\psi}(p) \right|$. Finally we replace $\delta$ by $\omega^{-\frac{1}{\rho}}$ to conclude this case.

- *Case* $\omega \leqslant \left(\frac{p_2-p_1}{2}\right)^{-\rho}$. Here we have

$$\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp \right| \leqslant \frac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu} (p_2-p_1)^\mu \leqslant 2\, \frac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu}\, \omega^{-\frac{\mu}{\rho}} \leqslant \tilde{C}(U,\psi)\, \omega^{-\frac{\mu}{\rho}} ,$$

  which ends the proof.

□

**3.1.7 Remark.** In the case of regular amplitudes, one can use the estimate (3.9) of $I^{(2)}(\omega)$ provided in Remark 3.1.5. In this situation, the constant $\tilde{C}(U,\psi)$ becomes

$$\tilde{C}(U,\psi) := 2\,\|\tilde{u}\|_{L^\infty(p_1,p_2)} + \Big(3\,\|\tilde{u}\|_{L^\infty(p_1,p_2)} + \|\tilde{u}'\|_{L^1(p_1,p_2)}\Big)\left(\min_{p\in[p_1,p_2]}\Big|\tilde{\psi}(p)\Big|\right)^{-1} .$$

The following theorem is a consequence of the two previous results. We give an estimate of the oscillatory integral which does not depend on the position of the stationary point, which can be either inside the integration interval (setting of Theorem 3.1.3) or outside (setting of Theorem 3.1.6) .

**3.1.8 Theorem.** *Let $\rho > 1$, $\mu \in (0,1]$ and choose $p_0 \in I$. Suppose that the functions $\psi : I \longrightarrow \mathbb{R}$ and $U : (p_1,p_2] \longrightarrow \mathbb{C}$ satisfy Assumption* $(\mathrm{P2}_{p_0,\rho})$ *and Assumption* $(\mathrm{A2}_{p_1,\mu})$*, respectively. Moreover suppose that $\psi'$ is monotone on $I_{p_0}^-$ and $I_{p_0}^+$, where*

$$I_{p_0}^- := \big\{p \in I \,\big|\, p < p_0\big\} \qquad , \qquad I_{p_0}^+ := \big\{p \in I \,\big|\, p > p_0\big\} .$$

*Then we have*

$$\left|\int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp\right| \leqslant C(U,\psi)\,\omega^{-\frac{\mu}{\rho}} ,$$

*for all $\omega > 0$, where the constant $C(U,\psi) > 0$ is given by*

$$C(U,\psi) := \frac{3}{\mu}\,\|\tilde{u}\|_{L^\infty(p_1,p_2)} + \Big(8\,\|\tilde{u}\|_{L^\infty(p_1,p_2)} + 2\,\|\tilde{u}'\|_{L^1(p_1,p_2)}\Big)\left(\min_{p\in[p_1,p_2]}\Big|\tilde{\psi}(p)\Big|\right)^{-1} .$$

*Proof.* Let us distinguish two cases.

- *Case* $p_0 \in [p_1,p_2]$. This corresponds to the setting of Theorem 3.1.3 and so the integral is bounded by $C(U,\psi)\,\omega^{-\frac{\mu}{\rho}}$, where $C(U,\psi)$ is given in Theorem 3.1.3.
- *Case* $p_0 \notin [p_1,p_2]$. In this case, either $[p_1,p_2] \subset I_{p_0}^-$ or $[p_1,p_2] \subset I_{p_0}^+$. Since $\psi'$ is assumed monotone on both intervals $I_{p_0}^-$ and $I_{p_0}^+$, Theorem 3.1.6 is applicable and then the integral is bounded by $\tilde{C}(U,\psi)\,\omega^{-\frac{\mu}{\rho}}$, where $\tilde{C}(U,\psi)$ is given in Theorem 3.1.6.

Finally we remark that $\tilde{C}(U,\psi) \leqslant C(U,\psi)$, which concludes the proof. □

**3.1.9 Remark.** As previously, we furnish a more precise constant in the case of regular amplitudes:

$$C(U,\psi) := 2\,\|\tilde{u}\|_{L^\infty(p_1,p_2)} + \Big(6\,\|\tilde{u}\|_{L^\infty(p_1,p_2)} + 2\,\|\tilde{u}'\|_{L^1(p_1,p_2)}\Big)\left(\min_{p\in[p_1,p_2]}\Big|\tilde{\psi}(p)\Big|\right)^{-1} .$$

In the following theorem, we prove the optimality of the decay rate given in Theorem 3.1.8 under slightly stronger conditions. We show in fact that this decay rate is attained in the case of $p_0 = p_1$, where one can expect a superposition of the effects of the stationary point $p_0$ and the amplitude singularity $p_1$.
Technically this result is based on an asymptotic expansion of the oscillatory integral to

one term in this case. We use Theorem 1.1.3 and Theorem 1.1.7 of Chapter 1 which are versions of the stationary phase method with explicit error estimates. We recall that Theorem 1.1.3, which covers the case of singular amplitudes, has been stated in [16] with only rough indications of the steps of the proof. In Chapter 1, we have carried out all the details of the proof. Theorem 1.1.7 is an improvement of the expansion result in [16] for the case of regular amplitudes ($\mu = 1$).

**3.1.10 Theorem.** *Suppose that the hypotheses of Theorem 3.1.8 are satisfied. In addition to this, we assume that $\tilde{\psi}$ is right continuously differentiable at $p_0$ and $\tilde{u} \in \mathcal{C}^1\big([p_1, p_2], \mathbb{C}\big)$ with $\tilde{u}(p_1) \neq 0$.*
*Then the decay rate $\omega^{-\frac{\mu}{\rho}}$ given in Theorem 3.1.8 is optimal and it is attained for $p_0 = p_1$.*

*Proof.* First of all, let us suppose that $p_0 = p_1$. Since the phase $\psi$ satisfies Assumption $(\mathrm{P2}_{p_1,\rho})$ in this case, the function $\tilde{\psi}$ has a constant sign on $(p_1, p_2]$ and it belongs to $\mathcal{C}^1\big((p_1, p_2], \mathbb{R}\big)$. Hence the fact that $\tilde{\psi}$ is supposed to be right continuously differentiable at $p_0 = p_1$ implies that $\tilde{\psi}$ has a constant sign on $[p_1, p_2]$ and it belongs to $\mathcal{C}^1\big([p_1, p_2], \mathbb{R}\big)$. Now let us suppose that $\tilde{\psi} > 0$ on $[p_1, p_2]$ without loss of generality. Hence the hypotheses of Theorem 1.1.3 in the case $N = 1$, $\rho_1 = \rho$, $\rho_2 = 1$, $\mu_1 = \mu$ and $\mu_2 = 1$ are satisfied and we obtain the following asymptotic expansion of the oscillatory integral with remainder estimates,

$$\forall\, \omega > 0 \qquad \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp = \sum_{j=1,2} \Big( A_1^{(j)}(\omega) + R_1^{(j)}(\omega) \Big) \ ,$$

where

- $$A_1^{(1)}(\omega) := \frac{\rho^{\frac{\mu}{\rho}}}{\rho}\, \Gamma\Big(\frac{\mu}{\rho}\Big)\, e^{i\frac{\pi}{2}\frac{\mu}{\rho}}\, e^{i\omega\psi(p_1)}\, \frac{\tilde{u}(p_1)}{\tilde{\psi}(p_1)^{\frac{\mu}{\rho}}}\, \omega^{-\frac{\mu}{\rho}} \ ,$$
- $$A_1^{(2)}(\omega) := e^{-i\frac{\pi}{2}}\, e^{i\omega\psi(p_2)}\, \frac{U(p_2)}{\psi'(p_2)}\, \omega^{-1} \ ,$$
- $$\Big| R_1^{(1)}(\omega) \Big| \leqslant C^{(1)}(U, \psi, \nu)\, \omega^{-\frac{1}{\rho}} \ ,$$
- $$\Big| R_1^{(2)}(\omega) \Big| \leqslant C^{(2)}(U, \psi, \nu)\, \omega^{-1} \ .$$

The constants $C^{(1)}(U, \psi, \nu)$ and $C^{(2)}(U, \psi, \nu, )$ are independent from $\omega$ but both depend on a smooth cut-off function $\nu$ which separates the points $p_1$ and $p_2$. The above asymptotic expansion combined with the remainder estimates shows that $\omega^{-\frac{\mu}{\rho}}$ is the optimal decay rate.
Let us remark that if $\mu = 1$, then Theorem 1.1.3 gives the same decay rate for the first term $A_1^{(1)}(\omega)$ and for the remainder term $R_1^{(1)}(\omega)$, namely $\omega^{-\frac{1}{\rho}}$. To avoid this situation, one can employ Theorem 1.1.7 which furnishes an estimate of $R_1^{(1)}(\omega)$ with a better decay rate than $\omega^{-\frac{1}{\rho}}$, and this fact assures that $\omega^{-\frac{1}{\rho}}$ is still the optimal decay rate for the oscillatory integral. □

**3.1.11 Remark.** Theorem 3.1.10 holds also when $\mu = 1$ and $\tilde{u}(p_1) = 0$, and in this case, the optimal decay rate $\omega^{-\frac{1}{\rho}}$ is attained for $p_0 = \tilde{p}$, if $\tilde{p} \in [p_1, p_2]$ satisfies $\tilde{u}(\tilde{p}) \neq 0$. To prove that, one can split the integral at $\tilde{p}$ and apply the stationary phase method to the two resulting integrals as in the preceding proof. We do not state this case in Theorem 3.1.10 in favour of readability.

## 3.2 Non-vanishing first derivative of the phase: improvement of the decay rate

In this section, we suppose the absence of a stationary point inside the integration interval. More precisely we assume that the phase function is twice continuously differentiable and that its first derivative does not vanish on $[p_1, p_2]$. We obtain the decay rate $\omega^{-\mu}$ for singular amplitudes satisfying Assumption $(\mathrm{A2}_{p_1,\mu})$.

When we want to apply the following result in the setting of Theorem 3.1.6 for comparison, we must suppose that the phase function is defined on an open interval $I$ which contains $[p_1, p_2]$ and that the stationary point $p_0$ of order $\rho - 1$ of the phase belongs to $I \backslash [p_1, p_2]$, in other words it is outside the integration interval. In this case, Theorem 3.2.1 furnishes the better decay rate $\omega^{-\mu}$ as compared with the decay rate $\omega^{-\frac{\mu}{\rho}}$ given in Theorem 3.1.6. Nevertheless the constant $C_c(U, \psi)$ of Theorem 3.2.1 tends to infinity when $p_0$ tends to $p_1$ or $p_2$, while the constant $\widetilde{C}(U, \psi)$ provided in Theorem 3.1.6 is uniform with respect to the distance between $p_0$ and $[p_1, p_2]$.

**3.2.1 Theorem.** *Let $\mu \in (0, 1]$. Suppose that the function $U : (p_1, p_2] \longrightarrow \mathbb{C}$ satisfies Assumption $(\mathrm{A2}_{p_1,\mu})$. Moreover suppose that $\psi : I \longrightarrow \mathbb{R}$ belongs to $\mathcal{C}^2\big([p_1, p_2]\big)$, and that $\psi'$ does not vanish and is monotone on $[p_1, p_2]$. Then we have*

$$\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp \right| \leqslant C_c(U, \psi)\, \omega^{-\mu} ,$$

*for all $\omega > 0$, where the constant $C_c(U, \psi) > 0$ is given by*

$$C_c(U, \psi) := \frac{1}{\mu}\, \|\tilde{u}\|_{L^\infty(p_1,p_2)} + \Big( 4\, \|\tilde{u}\|_{L^\infty(p_1,p_2)} + \|\tilde{u}'\|_{L^1(p_1,p_2)} \Big) \Big( \min_{p \in [p_1,p_2]} |\psi'(p)| \Big)^{-1} .$$

*Proof.* We divide the proof with respect to $\omega$ one more time.

- *Case $\omega > (p_2 - p_1)^{-1}$.* We define $\delta := \omega^{-1}$ and we consider the following splitting,

$$\int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp = \int_{p_1}^{p_1+\delta} \dots \quad + \int_{p_1+\delta}^{p_2} \dots$$
$$=: I_c^{(1)}(\omega) + I_c^{(2)}(\omega) .$$

  The integral $I_c^{(1)}(\omega)$ is bounded by $\dfrac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu}\, \delta^\mu$. Then we use the method employed to study the integral $I^{(2)}(\omega)$ in the proof Theorem 3.1.3 in order to bound $I_c^{(2)}(\omega)$, since $\psi'$ does not vanish on $[p_1, p_2]$. But here, we bound $|\psi'|$ from below by $\min\limits_{p \in [p_1,p_2]} |\psi'(p)| =: m > 0$, leading to

$$\left| I_c^{(2)}(\omega) \right| \leqslant \Big( 4\, \|\tilde{u}\|_{L^\infty(p_1,p_2)} + \|\tilde{u}'\|_{L^1(p_1,p_2)} \Big)\, m^{-1}\, \delta^{\mu-1} \omega^{-1} .$$



 

Finally we replace $\delta$ by $\omega^{-1}$ to conclude this case.

- *Case* $\omega \leqslant (p_2 - p_1)^{-1}$. We have

$$\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp \right| \leqslant \frac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu} (p_2 - p_1)^\mu \leqslant \frac{\|\tilde{u}\|_{L^\infty(p_1,p_2)}}{\mu}\, \omega^{-\mu} \ ,$$

and we conclude the proof by noting that the constant which appears in the preceding inequality is smaller than $C_c(U,\psi)$.

□

**3.2.2 Remark.** Let us furnish a refinement of the constant $C_c(U,\psi)$ in the case of regular amplitudes. Here the integral $I_c^{(1)}(\omega)$ is not needed and according to Remark 3.1.5, the estimate of $I_c^{(2)}(\omega)$ is improvable. Then we obtain

$$C_c(U,\psi) := \Big( 3\, \|\tilde{u}\|_{L^\infty(p_1,p_2)} + \|\tilde{u}'\|_{L^1(p_1,p_2)} \Big) \Big( \min_{p \in [p_1,p_2]} \big|\psi'(p)\big| \Big)^{-1} \ .$$

In the last theorem of this chapter, we prove the optimality of the decay rate given in Theorem 3.2.1 by applying Theorem 1.1.3 and Theorem 1.1.7, as we did in Theorem 3.1.10.

**3.2.3 Theorem.** *Suppose that the hypotheses of Theorem 3.2.1 are satisfied. In addition to this, we assume that $\tilde{u} \in \mathcal{C}^1\big([p_1,p_2],\mathbb{C}\big)$ with $\tilde{u}(p_1) \neq 0$.*
*Then the decay rate $\omega^{-\mu}$ given in Theorem 3.2.1 is optimal.*

*Proof.* As in the proof of Theorem 3.1.10, we apply Theorem 1.1.3 whose hypotheses are satisfied in the case $N = 1$, $\rho_1 = \rho_2 = 1$, $\mu_1 = \mu$ and $\mu_2 = 1$. A new asymptotic expansion of the oscillatory integral with remainder estimates is then obtained,

$$\forall\, \omega > 0 \qquad \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp = \sum_{j=1,2} \Big( \tilde{A}_1^{(j)}(\omega) + \tilde{R}_1^{(j)}(\omega) \Big) \ ,$$

where

- $\tilde{A}_1^{(1)}(\omega) := \Gamma(\mu)\, e^{i\frac{\pi}{2}\mu}\, e^{i\omega\psi(p_1)}\, \dfrac{\tilde{u}(p_1)}{\psi'(p_1)^\mu}\, \omega^{-\mu}$ ,
- $\tilde{A}_1^{(2)}(\omega) := e^{-i\frac{\pi}{2}}\, e^{i\omega\psi(p_2)}\, \dfrac{U(p_2)}{\psi'(p_2)}\, \omega^{-1}$ ,
- $\Big| \tilde{R}_1^{(1)}(\omega) \Big| \leqslant \tilde{C}^{(1)}(U,\psi,\nu)\, \omega^{-1}$ ,
- $\Big| \tilde{R}_1^{(2)}(\omega) \Big| \leqslant \tilde{C}^{(2)}(U,\psi,\nu)\, \omega^{-1}$ .

As in the proof of Theorem 3.1.10, $\nu$ is a smooth cut-off function separating the points $p_1$ and $p_2$, and the constants $\tilde{C}^{(1)}(U,\psi,\nu)$ and $\tilde{C}^{(2)}(U,\psi,\nu,)$ are independent from $\omega$. Hence we can conclude that $\omega^{-\mu}$ is the optimal decay rate.
And if $\mu = 1$, then we employ Theorem 1.1.7 to obtain more precise estimates for $\tilde{R}_1^{(1)}(\omega)$ and $\tilde{R}_1^{(2)}(\omega)$, furnishing better decay rates than $\omega^{-1}$ for these remainder terms, and so $\omega^{-1}$ is still the optimal decay rate. □



 

**3.2.4 Remark.** When $\mu = 1$ and $\tilde{u}(p_1) = \tilde{u}(p_2) = 0$, the decay rate may be faster than $\omega^{-1}$. In this case, it depends on the regularity of $\psi$ and $U$, and on the values of the successive derivatives of these functions at the endpoints of the integration interval (see [27, page 331]).



 

# Chapter 4

# Applications to evolution equations given by Fourier multipliers covering Schrödinger-type and hyperbolic examples

**Abstract**

In this chapter, we consider a family of dispersive equations on the line defined by Fourier multipliers, where the Fourier transform of the initial data is compactly supported or has a singular point. By applying the van der Corput type estimates provided in Chapter 3 to the solution formulas, we obtain uniform estimates of the solutions in space-time cones as well as in their complements. This type of results permits to exhibit the effect of a restriction to compact frequency bands and of singular frequencies on the time-decay of the solutions. Furthermore under certain restrictions on the growth of the symbol at infinity, we show that the solutions are time-asymptotically concentrated in a cone which depends on the symbol only. This corresponds to an asymptotic version of the notion of causality for initial data without compact support.

## Contents

## 4.1 Applications to a class of dispersive equations: influence of frequency bands and singular frequencies on decay

In this section, we consider the class of evolution equations on the line defined by Fourier multipliers whose symbols have a positive second derivative. We shall suppose that the support of the Fourier transform of the initial condition is contained in a bounded interval (called *frequency band*) or that this Fourier transform has a singular point of order $\mu - 1$ (called *singular frequency*), where $\mu \in (0,1]$. The influence of the frequency band and the singular frequency on time-decay is studied by establishing estimates of the solution inside certain space-time cones.

Let us describe the setting of this section: let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a $\mathcal{C}^\infty$-function such that all derivatives grow at most as a polynomial at infinity. We can associate with such a *symbol* $f$ an operator $f(D) : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$ defined by

$$\forall\, x \in \mathbb{R} \qquad f(D)u(x) := \frac{1}{2\pi} \int_{\mathbb{R}} f(p)\, \mathcal{F}u(p)\, e^{ixp}\, dp = \mathcal{F}^{-1}\Big(f\, \mathcal{F}u\Big)(x) \;,$$

where $\mathcal{F}u$ is the Fourier transform of $u \in \mathcal{S}(\mathbb{R})$, namely $\mathcal{F}u(p) = \displaystyle\int_{\mathbb{R}} u(x)\, e^{-ixp}\, dx$. Since all the derivatives of the symbol $f$ grow at most as a polynomial at infinity, $f(D)$ can be extended to a map from the tempered distributions $\mathcal{S}'(\mathbb{R})$ to itself. The operator $f(D) : \mathcal{S}'(\mathbb{R}) \longrightarrow \mathcal{S}'(\mathbb{R})$ is called a *Fourier multiplier.*
Then for such an operator, we can introduce the following evolution equation on the line,

$$\left\{ \begin{array}{l} \Big[i\,\partial_t - f\big(D\big)\Big]u(t) = 0 \\ u(0) = u_0 \end{array} \right. \;, \tag{4.1}$$

for $t \geqslant 0$. Supposing $u_0 \in \mathcal{S}'(\mathbb{R})$, this initial value problem has a unique solution in $\mathcal{C}^1\big(\mathbb{R}_+, \mathcal{S}'(\mathbb{R})\big)$, given by the following solution formula,

$$u(t) = \mathcal{F}^{-1}\Big(e^{-itf}\, \mathcal{F}u_0\Big) \;. \tag{4.2}$$

Throughout this section, we shall suppose that $f'' > 0$. In particular, the free Schrödinger equation on the line is included since its symbol $f_S$ is given by $f_S(p) = p^2$. Let us remark that one can also establish similar results to those of the present section when the second derivative of the symbol is supposed to be negative.

Now we give a new definition of a space-time cone $\mathfrak{C}(a,b)$. Let us remark that the definition of the cone $\mathfrak{C}_S(a,b)$, given in Definition 2.3.1 in Chapter 2 and used to study the asymptotic behaviour of the Schrödinger equation, is in fact to the cone $\mathfrak{C}(2\,a, 2\,b)$.

**4.1.1 Definition.** *Let $a < b$ be two real numbers (possibly infinite). We define the space-time cone $\mathfrak{C}(a,b)$ as follows:*

$$\mathfrak{C}(a,b) := \left\{ (t,x) \in (0,+\infty) \times \mathbb{R} \,\middle|\, a < \frac{x}{t} < b \right\} \;.$$

*Let $\mathfrak{C}(a,b)^c$ be the complement of the cone $\mathfrak{C}(a,b)$ in $(0,+\infty) \times \mathbb{R}$ .*



 

In the first result of this section, we consider the solution of the above evolution equation (4.1) for initial data in a compact frequency band $[p_1, p_2]$ where $p_1$ is a singular frequency. We furnish estimates with optimal decay rates of the solution inside a space-time cone, which is related to the frequency band, as well as in its complement. The decay rate is slower inside the cone than outside and in both cases, it is affected by the singular frequency $p_1$. The difference between the two rates shows that the solution tends to be concentrated in the cone when the time tends to infinity. In Theorem 2.3.2 and Theorem 2.3.4 of Chapter 2, different decay rates have already been obtained in the Schrödinger case by expanding the solution in certain space-time cones but without uniformity.
The first step of the proof is to rewrite the solution formula as an abstract oscillatory integral. In particular, the resulting phase function depends explicitly on the parameters $x$ and $t$, and we show that this phase has at most one stationary point which depends on the quotient $\frac{x}{t}$. The following step is to apply the results of the preceding chapter. To do so, we have to divide the proof with respect to the value of $\frac{x}{t}$: if $\frac{x}{t}$ is in a neighborhood of the integration interval, then we apply Theorem 3.1.8 of Chapter 3 leading to a uniform estimate in the above mentioned cone with the slow decay $t^{-\frac{\mu}{2}}$. Otherwise, we obtain the better decay rate $t^{-\mu}$ outside the cone by applying Theorem 3.2.1. The optimality of the rates is a direct consequence of Theorem 3.1.10 and Theorem 3.2.3.

**Condition ($\mathbf{C2}_{[p_1,p_2],\mu}$).** Fix $\mu \in (0,1]$ and let $p_1 < p_2$ be two finite real numbers.
A tempered distribution $u_0$ on $\mathbb{R}$ satisfies Condition $(\mathrm{C2}_{[p_1,p_2],\mu})$ if and only if $\mathcal{F}u_0 \equiv 0$ on $\mathbb{R}\backslash[p_1, p_2]$ and $\mathcal{F}u_0$ verifies Assumption $(\mathrm{A2}_{p_1,\mu})$ (given in Chapter 3, Section 3.1) on $[p_1, p_2]$, where the regular factor $\tilde{u}$ is supposed to belong to $\mathcal{C}^1\big([p_1, p_2], \mathbb{C}\big)$ and $\tilde{u}(p_1) \neq 0$.

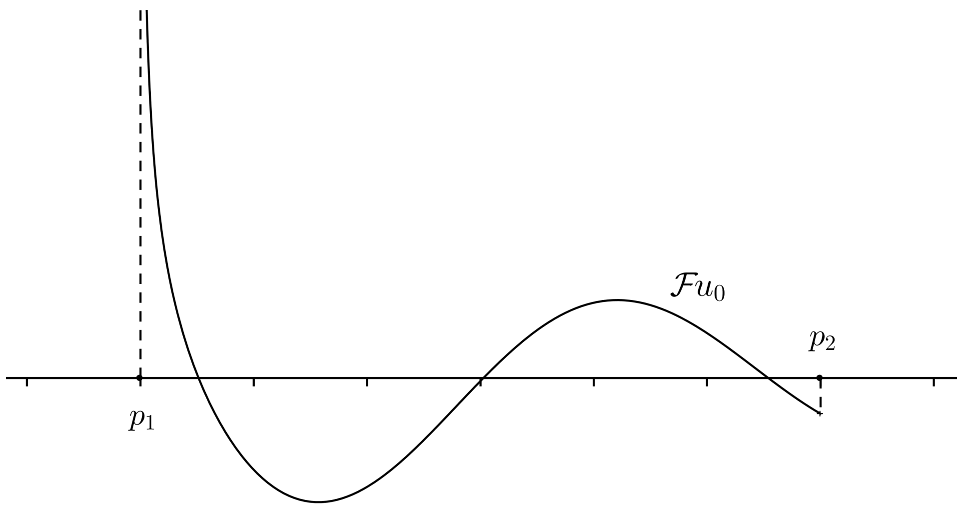


Figure 4.1: Fourier transform of an initial datum satisfying Condition $(\mathrm{C2}_{[p_1,p_2],\mu})$

**4.1.2 Remark.** i) As for Condition $(\mathrm{C1}_{[p_1,p_2],\mu})$ in Chapter 2, let us show that the subset of tempered distributions satisfying Condition $(\mathrm{C2}_{[p_1,p_2],\mu})$ is non-empty. Indeed if a function $U$ verifies Assumption $(\mathrm{A2}_{p_1,\mu})$ with $supp\, U \subseteq [p_1, p_2]$ and with a regular factor belonging to $\mathcal{C}^1\big([p_1, p_2], \mathbb{C}\big)$, then $U$ is an integrable function and so it belongs to $\mathcal{S}'(\mathbb{R})$. Since the Fourier transform is a bijection on $\mathcal{S}'(\mathbb{R})$, there exists $u_0 \in \mathcal{S}'(\mathbb{R})$ such that $U = \mathcal{F}u_0$, and hence $u_0$ satisfies Condition $(\mathrm{C2}_{[p_1,p_2],\mu})$.

ii) Since the support of $\mathcal{F}u_0$ is contained in a compact interval, $u_0$ is in fact an analytic function on $\mathbb{R}$.

iii) Condition ($\mathrm{C2}_{[p_1,p_2],\mu}$) implies that the initial condition has a singular frequency at the left endpoint of its compact frequency band. As explained just after the statement of Assumption ($\mathrm{A2}_{p_1,\mu}$) in Chapter 3, Section 3.1, the result in the case of a singular frequency inside the frequency band is analogous to the result stated in Theorem 4.1.3.

iv) Thanks to the integrability of $\mathcal{F}u_0$, the solution formula given in (4.2) defines a function on $(0,+\infty)\times\mathbb{R}$ as follows,

$$\forall\,(t,x)\in(0,+\infty)\times\mathbb{R} \qquad u(t,x)=\frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{F}u_0(p)\,e^{-itf(p)+ixp}\,dp\,. \tag{4.3}$$

**4.1.3 Theorem.** *Suppose that $u_0$ satisfies Condition ($\mathrm{C2}_{[p_1,p_2],\mu}$) and choose two finite real numbers $\tilde{p}_1<\tilde{p}_2$ such that $[p_1,p_2]\subset(\tilde{p}_1,\tilde{p}_2)=:\tilde{I}$. Then we have*

$$\forall\,(t,x)\in\mathfrak{C}\big(f'(\tilde{p}_1),f'(\tilde{p}_2)\big) \qquad \big|u(t,x)\big|\leqslant c(u_0,f)\,t^{-\frac{\mu}{2}}\,,$$

*where the constant $c(u_0,f)>0$ is given by (4.5). Moreover we have*

$$\forall\,(t,x)\in\mathfrak{C}\big(f'(\tilde{p}_1),f'(\tilde{p}_2)\big)^c \qquad \big|u(t,x)\big|\leqslant c_{\tilde{I}}(u_0,f)\,t^{-\mu}\,,$$

*where the constant $c_{\tilde{I}}(u_0,f)>0$ is given by (4.6). And the two decay rates are optimal.*

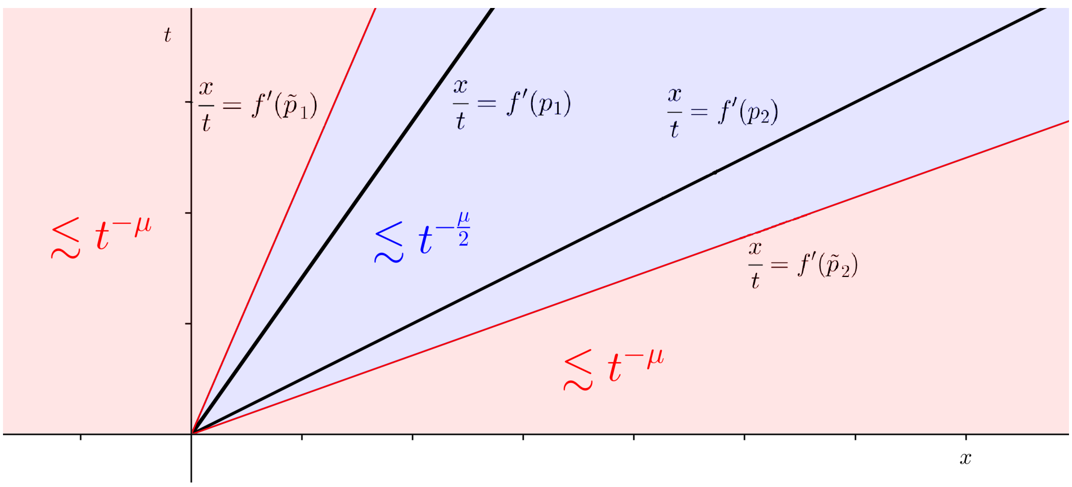


Figure 4.2: Illustration of Theorem 4.1.3 in space-time

*Proof.* We consider the solution formula given by (4.3) and we factorize the phase function $p\longmapsto xp-tf(p)$ by $t$, which gives

$$\forall\,(t,x)\in(0,+\infty)\times\mathbb{R} \qquad u(t,x)=\int_{p_1}^{p_2}U(p)\,e^{it\psi(p)}\,dp\,,$$

where

$$\begin{cases} \forall\, p \in (p_1, p_2] \qquad U(p) := \dfrac{1}{2\pi}\, \mathcal{F}u_0(p) = \dfrac{1}{2\pi}\, (p-p_1)^{\mu-1}\, \tilde{u}(p) \;, \\ \forall\, p \in \mathbb{R} \qquad \psi(p) := \dfrac{x}{t}\, p - f(p) \;. \end{cases}$$

By hypothesis, the function $U$ verifies Assumption $(\mathrm{A2}_{p_1,\mu})$ on $[p_1,p_2]$. Moreover, we recall that $f''$ is supposed to be positive on $\mathbb{R}$, which implies that $f' : \mathbb{R} \longrightarrow f'(\mathbb{R})$ is strictly increasing. It follows that the function $\psi'$ given by

$$\forall\, p \in \mathbb{R} \qquad \psi'(p) = \frac{x}{t} - f'(p) \;,$$

is strictly decreasing on $\mathbb{R}$. In particular, if a stationary point $p_0$ exists then it is unique and it is defined by

$$p_0 = \big(f'\big)^{-1}\Big(\frac{x}{t}\Big) \;.$$

Hence the existence of a stationary point as well as its position with respect to the integration interval depends on the value of $\frac{x}{t}$. This leads us to divide the rest of the proof into two parts.

i) *Case* $\frac{x}{t} \in f'\big(\tilde{I}\big)$. In this case, the stationary point $p_0$ exists and it belongs to the interval $\tilde{I} := (\tilde{p}_1, \tilde{p}_2)$. Moreover the fact that $\psi'' = -f'' < 0$ implies $\psi''(p_0) \neq 0$. Consequently, according to Example 3.1.1 i), the function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Assumption $(\mathrm{P2}_{p_0,2})$ with

$$\tilde{\psi}(p) = \begin{cases} \dfrac{p-p_0}{|p-p_0|} \displaystyle\int_0^1 -f''\big(y(p-p_0)+p_0\big)\, dy \;, \quad \text{if } p \neq p_0 \;, \\ -f''(p_0) \;, \quad \text{if } p = p_0 \;, \end{cases} \tag{4.4}$$

and $\big|\tilde{\psi}(p)\big| \geqslant m > 0$ for all $p \in [p_1,p_2]$, where $m := \min\limits_{p \in [p_1,p_2]} f''(p) > 0$. So we can apply Theorem 3.1.8 with $\rho = 2$, which gives

$$\forall\, (t,x) \in \mathfrak{C}\big(f'(\tilde{p}_1), f'(\tilde{p}_2)\big) \qquad \big|u(t,x)\big| = \left| \int_{p_1}^{p_2} U(p)\, e^{it\psi(p)}\, dp \right| \leqslant c(u_0,f)\, t^{-\frac{\mu}{2}} \;,$$

where

$$c(u_0,f) := \frac{1}{2\pi}\, \frac{3}{\mu}\, \|\tilde{u}\|_{L^\infty(p_1,p_2)} + \frac{1}{\pi} \Big( 4\, \|\tilde{u}\|_{L^\infty(p_1,p_2)} + \|\tilde{u}'\|_{L^1(p_1,p_2)} \Big)\, m^{-1} \;. \tag{4.5}$$

ii) *Case* $\frac{x}{t} \notin f'\big(\tilde{I}\big)$. Firstly, let us suppose $\frac{x}{t} \geqslant f'(\tilde{p}_2)$. Here there is no stationary points inside the integration interval and so it is possible to bound $\psi'$ from below by a non-zero constant,

$$\forall\, p \in [p_1,p_2] \qquad \psi'(p) = \frac{x}{t} - f'(p) \geqslant f'(\tilde{p}_2) - f'(p_2) =: m_{\tilde{p}_2} > 0 \;.$$

Theorem 3.2.1 is then applicable and provides

$$\forall\, t > 0 \qquad \forall\, x \geqslant f'(\tilde{p}_2)\, t \qquad \big|u(t,x)\big| \leqslant c_{x/t \geqslant f'(\tilde{p}_2)}(u_0,f)\, t^{-\mu} \;,$$

with $c_{x/t\geqslant f'(\tilde{p}_2)}(u_0,f) := \dfrac{1}{2\pi}\,\dfrac{1}{\mu}\,\|\tilde{u}\|_{L^\infty(p_1,p_2)} + \dfrac{1}{2\pi}\Big(4\,\|\tilde{u}\|_{L^\infty(p_1,p_2)} + \|\tilde{u}'\|_{L^1(p_1,p_2)}\Big)\, m_{\tilde{p}_2}^{-1}$.
In the other case $\frac{x}{t} \leqslant f'(\tilde{p}_1)$, similar arguments furnish

$$\forall\, t > 0 \qquad \forall\, x \leqslant f'(\tilde{p}_1)\, t \qquad \big|u(t,x)\big| \leqslant c_{x/t\leqslant f'(\tilde{p}_1)}(u_0,f)\, t^{-\mu}\ ,$$

with $c_{x/t\leqslant f'(\tilde{p}_2)}(u_0,f) := \dfrac{1}{2\pi}\,\dfrac{1}{\mu}\,\|\tilde{u}\|_{L^\infty(p_1,p_2)} + \dfrac{1}{2\pi}\Big(4\,\|\tilde{u}\|_{L^\infty(p_1,p_2)} + \|\tilde{u}'\|_{L^1(p_1,p_2)}\Big)\, m_{\tilde{p}_1}^{-1}$,
where we set $m_{\tilde{p}_1} := f'(p_1) - f'(\tilde{p}_1) > 0$.
So we can finally write

$$\forall\, (t,x) \in \mathfrak{C}\big(f'(\tilde{p}_1), f'(\tilde{p}_2)\big)^c \qquad \big|u(t,x)\big| \leqslant c_{\tilde{I}}(u_0,f)\, t^{-\mu}\ ,$$

where

$$c_{\tilde{I}}(u_0,f) := c_{x/t\geqslant f'(\tilde{p}_2)}(u_0,f) + c_{x/t\leqslant f'(\tilde{p}_1)}(u_0,f)\ . \tag{4.6}$$

To prove the optimality of the above rates, we recall that the regular factor of $\mathcal{F}u_0$ is supposed to be continuously differentiable on $[p_1,p_2]$. Hence Theorem 3.2.3 is applicable and it furnishes the optimality of the rate $t^{-\mu}$ in the region $\mathfrak{C}\big(f'(\tilde{p}_1), f'(\tilde{p}_2)\big)^c$. Moreover the definition of the function $\tilde{\psi}$ (see (4.4)) implies that this function is right continuously differentiable at $p_0$, so we can employ Theorem 3.1.10 to prove the optimality of the rate $t^{-\frac{\mu}{2}}$ in $\mathfrak{C}\big(f'(\tilde{p}_1), f'(\tilde{p}_2)\big)$. In particular, the decay rate is attained on the space-time direction defined by $\frac{x}{t} = f'(\tilde{p}_1)$. □

An $L^\infty$-norm estimate for the solution can be easily derived from the preceding result.

**4.1.4 Theorem.** *Suppose that $u_0$ satisfies Condition* $(\mathrm{C2}_{[p_1,p_2],\mu})$. *Then we have*

$$\forall\, t > 0 \qquad \big\|u(t,.)\big\|_{L^\infty(\mathbb{R})} \leqslant c(u_0,f)\, t^{-\frac{\mu}{2}} + c_{\tilde{I}}(u_0,f)\, t^{-\mu}\ ,$$

*where the constants $c(u_0,f) > 0$ and $c_{\tilde{I}}(u_0,f) > 0$ are given by* (4.5) *and* (4.6) *respectively. In particular, we have*

$$\forall\, t \geqslant 1 \qquad \big\|u(t,.)\big\|_{L^\infty(\mathbb{R})} \leqslant \big(c(u_0,f) + c_{\tilde{I}}(u_0,f)\big)\, t^{-\frac{\mu}{2}}\ .$$

*Proof.* Simple consequence of Theorem 4.1.3. □

As an application of the above theorem, we furnish an $L^\infty$-norm estimate of the solution of the free Schrödinger equation on the line for initial data satisfying Condition $(\mathrm{C2}_{[p_1,p_2],\mu})$. The resulting decay rate is given by $t^{-\frac{\mu}{2}}$. Let us remark that this decay rate has been obtained in Theorem 2.3.6 by expanding the solution to one term on the space-time direction given by $\frac{x}{t} = 2\,p_1$.

**4.1.5 Corollary.** *Let $u_S \in \mathcal{C}^1\big(\mathbb{R}_+, \mathcal{S}'(\mathbb{R})\big)$ be the solution of the free Schrödinger equation on $\mathbb{R}$,*

$$\begin{cases} \big[i\,\partial_t + \partial_{xx}\big]u(t) = 0 \\ u(0) = u_0 \end{cases}\ ,$$



 

*for* $t \geqslant 0$, *where* $u_0$ *satisfies Condition* $(\mathrm{C2}_{[p_1,p_2],\mu})$. *Then* $u_S$ *defines a complex-valued function on* $(0,+\infty) \times \mathbb{R}$ *which satisfies*

$$\forall\, t \geqslant 1 \qquad \big\| u_S(t,.) \big\|_{L^\infty(\mathbb{R})} \leqslant c(u_0, f_S)\, t^{-\frac{\mu}{2}} \ ,$$

*where the constant* $c(u_0, f_S) > 0$ *can be computed from Theorem 4.1.4, and the decay rate is optimal.*

*Proof.* Remark 4.1.2 iv) assures that $u_S$ defines a function on $(0,+\infty)\times\mathbb{R}$. Then we apply Theorem 4.1.4 in the case $f_S(p) = p^2$, which gives the differential operator $-\partial_{xx}$. □

In the following result, we establish estimates of the solution in arbitrarily narrow cones containing the space-time direction $\frac{x}{t} = f'(p_1)$. In such regions, the phase function, coming from the rewriting of the solution as an oscillatory integral, has a stationary point which is in a neighbourhood of the singular frequency $p_1$. In this context, these two particular points are expected to interact with each other, producing the slow decay $t^{-\frac{\mu}{2}}$. Here we do not require the initial data to be in frequency bands anymore. This permits to remove the concentration phenomenon produced by the frequency band, which has been exhibited in Theorem 4.1.3, and to focus only on the influence of the singular frequency $p_1$ on the decay rate in the above mentioned cones.

**Condition** $(\mathbf{C3}_{p_1,\mu})$**.** Fix $\mu \in (0,1)$ and choose a finite real number $p_1$.
A tempered distribution $u_0$ on $\mathbb{R}$ satisfies Condition $(\mathrm{C3}_{p_1,\mu})$ if and only if $\mathcal{F}u_0 \in L^1(\mathbb{R})$ and there exists a bounded differentiable function $\tilde{u} : \mathbb{R} \longrightarrow \mathbb{C}$ such that $\tilde{u}(p_1) \neq 0$, $\tilde{u}' \in L^1(\mathbb{R})$ and

$$\forall\, p \in \mathbb{R} \setminus \{p_1\} \qquad \mathcal{F}u_0(p) = |p - p_1|^{\mu-1}\, \tilde{u}(p) \ .$$

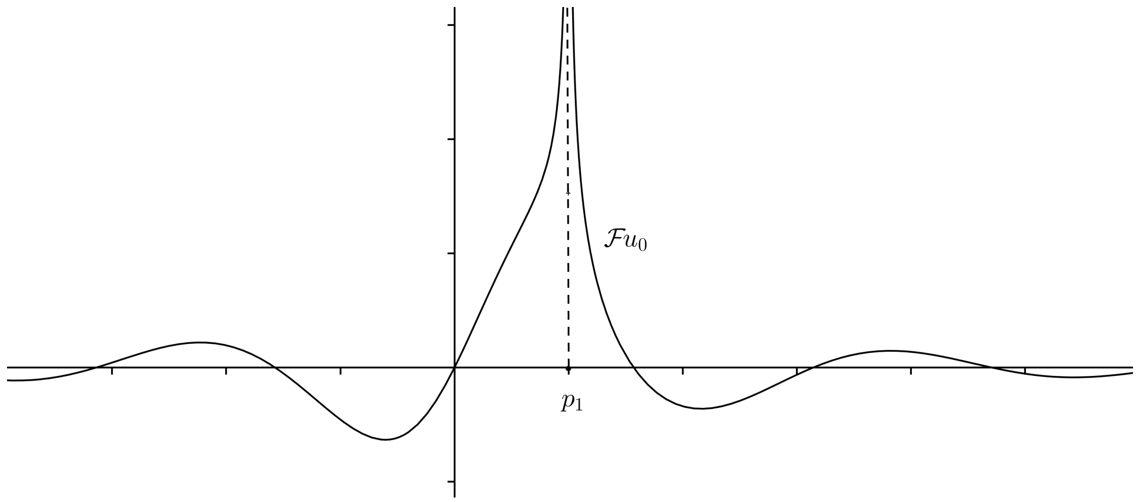


Figure 4.3: Fourier transform of an initial datum satisfying Condition $(\mathrm{C3}_{p_1,\mu})$

**4.1.6 Remark.** i) One can follow the lines of the point i) of Remark 4.1.2 to ensure that the subset of tempered distributions satisfying Condition $(\mathrm{C3}_{p_1,\mu})$ is non-empty.

ii) Here $u_0$ is at least a continuous function on $\mathbb{R}$ but it is not necessarily analytic. Furthermore the solution formula (4.3) is still well-defined for all $t > 0$ and $x \in \mathbb{R}$.

**4.1.7 Theorem.** *Suppose that $u_0$ satisfies Condition* $(\mathrm{C3}_{p_1,\mu})$ *and choose a finite real number $\varepsilon > 0$. Then for all $(t,x) \in \mathfrak{C}\big(f'(p_1-\varepsilon), f'(p_1+\varepsilon)\big)$, we have*

$$\big|u(t,x)\big| \leqslant c^{(1)}(u_0,f)\, t^{-\frac{\mu}{2}} + c_\varepsilon^{(2)}(u_0,f)\, t^{-1} .$$

*The constants $c^{(1)}(u_0,f)$ and $c_\varepsilon^{(2)}(u_0,f)$ are given by* (4.7) *and* (4.9) *respectively.*

*Proof.* We shall employ the rewritting of the solution given in the proof of Theorem 4.1.3, *i.e.*

$$\forall\, (t,x) \in (0,+\infty) \times \mathbb{R} \qquad u(t,x) = \int_{\mathbb{R}} U(p)\, e^{it\psi(p)}\, dp ,$$

where

$$\begin{cases} \forall\, p \in \mathbb{R} \setminus \{p_1\} \qquad U(p) := \dfrac{1}{2\pi}\, \mathcal{F}u_0(p) = \dfrac{1}{2\pi}\, |p-p_1|^{\mu-1}\, \tilde{u}(p) , \\ \forall\, p \in \mathbb{R} \qquad \psi(p) := \dfrac{x}{t}\, p - f(p) . \end{cases}$$

Let $\varepsilon > 0$, choose a finite real number $\eta > 0$ such that $\eta > \varepsilon$ (for example $\eta = \varepsilon + 1$) and split the integral as follows,

$$\begin{aligned} \int_{\mathbb{R}} U(p)\, e^{it\psi(p)}\, dp &= \int_{p_1-\eta}^{p_1+\eta} \dots + \int_{\mathbb{R}\setminus[p_1-\eta,p_1+\eta]} \dots \\ &=: I^{(1)}(t,x,\eta) + I^{(2)}(t,x,\eta) . \end{aligned}$$

Firstly we study $I^{(1)}(t,x,\eta)$. We recall that

$$\psi'(p) = \frac{x}{t} - f'(p) ;$$

since $\dfrac{x}{t}$ is supposed to belong to $\big(f'(p_1-\varepsilon), f'(p_1+\varepsilon)\big)$, then $\psi$ has a stationary point which belongs to $(p_1-\varepsilon, p_1+\varepsilon) \subset [p_1-\eta, p_1+\eta]$. Following the arguments of the point i) of the proof of Theorem 4.1.3, we apply Theorem 3.1.3 on $[p_1-\eta, p_1]$ and on $[p_1, p_1+\eta]$ with $\rho = 2$, leading to

$$\Big|I^{(1)}(t,x,\eta)\Big| \leqslant \left|\int_{p_1-\eta}^{p_1} \dots\right| + \left|\int_{p_1}^{p_1+\eta} \dots\right| \leqslant \Big(c_1^{(1)}(u_0,f) + c_2^{(1)}(u_0,f)\Big)\, t^{-\frac{\mu}{2}} ,$$

where

- $c_1^{(1)}(u_0,f) := \dfrac{1}{2\pi}\, \dfrac{3}{\mu}\, \|\tilde{u}\|_{L^\infty(p_1-\eta,p_1)} + \dfrac{1}{\pi}\Big(4\, \|\tilde{u}\|_{L^\infty(p_1-\eta,p_1)} + \|\tilde{u}'\|_{L^1(p_1-\eta,p_1)}\Big)\, m_{1,\eta}^{-1}$ ,
- $c_2^{(1)}(u_0,f) := \dfrac{1}{2\pi}\, \dfrac{3}{\mu}\, \|\tilde{u}\|_{L^\infty(p_1,p_1+\eta)} + \dfrac{1}{\pi}\Big(4\, \|\tilde{u}\|_{L^\infty(p_1,p_1+\eta)} + \|\tilde{u}'\|_{L^1(p_1,p_1+\eta)}\Big)\, m_{2,\eta}^{-1}$ ,

with $m_{1,\eta} := \min\limits_{p\in[p_1-\eta,p_1]} f''(p) > 0$ and $m_{2,\eta} := \min\limits_{p\in[p_1,p_1+\eta]} f''(p) > 0$. The constant $c^{(1)}(u_0,f)$ is then defined by

$$c^{(1)}(u_0,f) := c_1^{(1)}(u_0,f) + c_2^{(1)}(u_0,f) . \tag{4.7}$$



 

Let us study $I^{(2)}(t,x,\eta)$. Let $k \in \mathbb{N}$ and consider the following sequence,

$$\tilde{I}_k^{(2)}(t,x,\eta) := \int_{p_1+\eta}^{p_1+\eta+k} U(p)\, e^{it\psi(p)}\, dp \; .$$

Since $\dfrac{x}{t} \in \big(f'(p_1-\varepsilon), f'(p_1+\varepsilon)\big)$, we note that the first derivative of the phase function does not vanish on $[p_1+\eta, p_1+\eta+k]$ and more precisely, we have for any $k \in \mathbb{N}$,

$$\forall\, p \in [p_1+\eta, p_1+\eta+k] \quad \big|\psi'(p)\big| = f'(p) - \frac{x}{t} \geqslant f'(p_1+\eta) - f'(p_1+\varepsilon) =: \tilde{m}_{1,\eta,\varepsilon} > 0 \; .$$

Theorem 3.2.1 in the case $\mu = 1$ furnishes for all $(t,x) \in \mathfrak{C}\big(f'(p_1-\varepsilon), f'(p_1+\varepsilon)\big)$,

$$\left|\tilde{I}_k^{(2)}(t,x,\eta)\right| \leqslant \frac{1}{2\pi}\left(3\, \|U\|_{L^\infty(p_1+\eta,p_1+\eta+k)} + \|U'\|_{L^1(p_1+\eta,p_1+\eta+k)}\right) \tilde{m}_{1,\eta,\varepsilon}^{-1}\, t^{-1} \; .$$

Now by using the hypotheses on the initial data, we give estimates for $\|U\|_{L^\infty(p_1+\eta,p_1+\eta+k)}$ and $\|U'\|_{L^1(p_1+\eta,p_1+\eta+k)}$, namely,

- $\forall\, p \in [p_1+\eta, p_1+\eta+k] \qquad \big|U(p)\big| \leqslant \eta^{\mu-1}\, \|\tilde{u}\|_{L^\infty(\mathbb{R})}$ ,
- $\displaystyle\int_{p_1+\eta}^{p_1+\eta+k} \big|U'(p)\big| dp \leqslant \eta^{\mu-1}\left(\|\tilde{u}\|_{L^\infty(\mathbb{R})} + \|\tilde{u}'\|_{L^1(\mathbb{R})}\right)$ .

Consequently, $\tilde{I}_2^{(k)}(t,x,\eta)$ can be estimated as follows,

$$\left|\tilde{I}_k^{(2)}(t,x,\eta)\right| \leqslant \frac{1}{2\pi}\, \eta^{\mu-1}\left(4\, \|\tilde{u}\|_{L^\infty(\mathbb{R})} + \|\tilde{u}'\|_{L^1(\mathbb{R})}\right) \tilde{m}_{1,\eta,\varepsilon}^{-1}\, t^{-1} \; . \tag{4.8}$$

Using the dominated convergence Theorem which claims that

$$\lim_{k \to +\infty} \tilde{I}_k^{(2)}(t,x,\eta) = \int_{p_1+\eta}^{+\infty} U(p)\, e^{it\psi(p)} dp \; ,$$

we can take the limit in (4.8) providing

$$\left|\int_{p_1+\eta}^{+\infty} U(p)\, e^{it\psi(p)} dp\right| \leqslant c_{1,\varepsilon}^{(2)}(u_0,f)\, t^{-1} \; ,$$

with

$$c_{1,\varepsilon}^{(2)}(u_0,f) := \frac{1}{2\pi}\, \eta^{\mu-1}\left(4\, \|\tilde{u}\|_{L^\infty(\mathbb{R})} + \|\tilde{u}'\|_{L^1(\mathbb{R})}\right) \tilde{m}_{1,\eta,\varepsilon}^{-1} \; .$$

Similar arguments furnish the following estimate,

$$\forall\, (t,x) \in \mathfrak{C}\big(f'(p_1-\varepsilon), f'(p_1+\varepsilon)\big) \qquad \left|\int_{-\infty}^{p_1-\eta} U(p)\, e^{it\psi(p)} dp\right| \leqslant c_{2,\varepsilon}^{(2)}(u_0,f)\, t^{-1} \; ,$$

with

$$c_{2,\varepsilon}^{(2)}(u_0,f) := \frac{1}{2\pi}\, \eta^{\mu-1}\left(4\, \|\tilde{u}\|_{L^\infty(\mathbb{R})} + \|\tilde{u}'\|_{L^1(\mathbb{R})}\right) \tilde{m}_{2,\eta,\varepsilon}^{-1} \; ,$$

where $\tilde{m}_{2,\eta,\varepsilon} := f'(p_1-\varepsilon) - f'(p_1-\eta) > 0$.
Finally, by setting

$$c_\varepsilon^{(2)}(u_0,f) := c_{1,\varepsilon}^{(2)}(u_0,f) + c_{2,\varepsilon}^{(2)}(u_0,f) \; , \tag{4.9}$$

we obtain for all $(t,x) \in \mathfrak{C}\big(f'(p_1-\varepsilon), f'(p_1+\varepsilon)\big)$,

$$\left| I^{(2)}(t,x,\eta) \right| \leqslant \left| \int_{-\infty}^{p_1-\eta} \dots \right| + \left| \int_{p_1+\eta}^{+\infty} \dots \right| \leqslant c_\varepsilon^{(2)}(u_0,f)\, t^{-1} \; ,$$

which ends the proof. □

Now we provide estimates of the solution in space-time cones which do not contain the critical direction given by the singular frequency. In this case, the distance between the stationary point and the singular frequency is bounded from below by a positive constant, which removes the superposition of the effects of these particular points. Hence these two points provide two distinct decay rates: $t^{-\frac{1}{2}}$ coming from the stationary point and $t^{-\mu}$ coming from the singular frequency. We note that these two rates are better than $t^{-\frac{\mu}{2}}$. Theorem 4.1.8 combined with Theorem 4.1.7 highlights the fact that the influence of the singular frequency $p_1$ on the decay rate is stronger in space-time regions containing the direction given by $p_1$. These results can be compared with Theorem 2.3.7 of Chapter 2. In the latter, we have furnished estimates of the solution of the free Schrödinger equation in space-time regions along the direction $\frac{x}{t} = 2\,p_1$, and this direction is outside the regions. The estimates show that the decay rate diminishes when the boundary of the region approaches the direction given by $p_1$.

**4.1.8 Theorem.** *Suppose that $u_0$ satisfies Condition $(\mathrm{C3}_{p_1,\mu})$ and choose two finite real numbers $\tilde{p}_1 < \tilde{p}_2$ such that $p_1 \notin [\tilde{p}_1, \tilde{p}_2]$. Then for all $(t,x) \in \mathfrak{C}\big(f'(\tilde{p}_1), f'(\tilde{p}_2)\big)$, we have*

$$\big|u(t,x)\big| \leqslant c_{\tilde{p}_1,\tilde{p}_2}^{(1)}(u_0,f)\, t^{-\frac{1}{2}} + c_{\tilde{p}_1,\tilde{p}_2}^{(2)}(u_0,f)\, t^{-\mu} + c_{\tilde{p}_1,\tilde{p}_2}^{(3)}(u_0,f)\, t^{-1} \; .$$

*The constants $c_{\tilde{p}_1,\tilde{p}_2}^{(1)}(u_0,f)$, $c_{\tilde{p}_1,\tilde{p}_2}^{(2)}(u_0,f)$ and $c_{\tilde{p}_1,\tilde{p}_2}^{(3)}(u_0,f)$ are given by* (4.10), (4.13) *and* (4.14) *respectively.*

*Proof.* The calculations in the present proof are similar to those of the proof of Theorem 4.1.7. Thus we give only the main steps of the proof of Theorem 4.1.8.
Let $\eta \in \big(0, \min\{|\tilde{p}_1 - p_1|, |\tilde{p}_2 - p_1|\}\big)$ and split the integral as follows,

$$\begin{aligned} u(t,x) = \int_{\mathbb{R}} U(p)\, e^{it\psi(p)}\, dp &= \int_{\tilde{p}_1-\eta}^{\tilde{p}_2+\eta} \dots + \int_{\mathbb{R}\setminus[\tilde{p}_1-\eta, \tilde{p}_2+\eta]} \dots \\ &=: I^{(1)}(t,x,\eta) + I^{(2)}(t,x,\eta) \; , \end{aligned}$$

On the interval $[\tilde{p}_1-\eta, \tilde{p}_2+\eta]$, the phase has a unique stationary point and the amplitude has no singular points. Theorem 3.1.3 is applicable with $\rho = 2$ and $\mu = 1$, and it leads to

$$\forall\, (t,x) \in \mathfrak{C}\big(f'(\tilde{p}_1), f'(\tilde{p}_2)\big) \qquad \left| I^{(1)}(t,x,\eta) \right| \leqslant c_{\tilde{p}_1,\tilde{p}_2}^{(1)}(u_0,f)\, t^{-\frac{1}{2}} \; ,$$



 

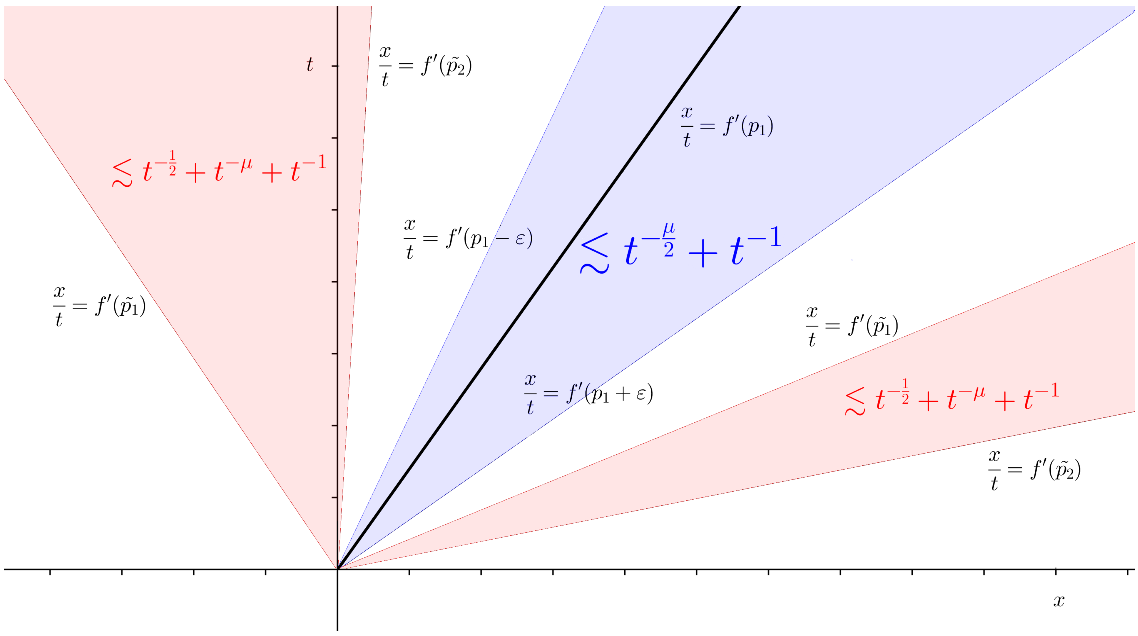


Figure 4.4: Illustration of Theorem 4.1.7 and Theorem 4.1.8 in space-time

where

$$c^{(1)}_{\tilde{p}_1,\tilde{p}_2}(u_0, f) := \begin{cases} \dfrac{(\tilde{p}_1 - \eta - p_1)^{\mu-1}}{\pi} \Big( \|\tilde{u}\|_{L^\infty(\mathbb{R})} \\ \quad + \Big( 4\,\|\tilde{u}\|_{L^\infty(\mathbb{R})} + \|\tilde{u}'\|_{L^1(\mathbb{R})} \Big) m^{-1}_{1,\tilde{p}_1,\tilde{p}_2} \Big) \ , & \text{if } p_1 < \tilde{p}_1 \ , \\ \dfrac{(p_1 - \tilde{p}_2 - \eta)^{\mu-1}}{\pi} \Big( \|\tilde{u}\|_{L^\infty(\mathbb{R})} \\ \quad + \Big( 4\,\|\tilde{u}\|_{L^\infty(\mathbb{R})} + \|\tilde{u}'\|_{L^1(\mathbb{R})} \Big) m^{-1}_{1,\tilde{p}_1,\tilde{p}_2} \Big) \ , & \text{if } p_1 > \tilde{p}_2 \ , \end{cases} \tag{4.10}$$

with $m_{1,\tilde{p}_1,\tilde{p}_2} := \min_{p \in [\tilde{p}_1 - \eta, \tilde{p}_2 + \eta]} f''(p) > 0$.

Now let us study $I^{(2)}(t,x,\eta)$. First of all, we remark that we integrate over two infinite intervals such that one of them contains the singular frequency $p_1$. Consequently we shall suppose that $p_1 < \tilde{p}_1$ without loss of generality; the other case $p_1 > \tilde{p}_2$ can be treated in a similar way.

We consider the following sequence,

$$\forall\, k \in \mathbb{N}^* \qquad \tilde{I}^{(2)}_k(t,x,\eta) := \int_{p_1 - k}^{\tilde{p}_1 - \eta} U(p)\, e^{it\psi(p)}\, dp \ .$$

We note that $[p_1 - k, \tilde{p}_1 - \eta]$ contains the singular frequency $p_1$ and $\psi'$ does not vanish on this interval, and thus Theorem 3.2.1 is applicable on $[p_1 - k, p_1]$ and on $[p_1, \tilde{p}_1 - \eta]$. Then we take the limit when $k$ tends to infinity by using the dominated convergence Theorem and we obtain

$$\forall\, (t,x) \in \mathfrak{C}\big(f'(\tilde{p}_1), f'(\tilde{p}_2)\big) \qquad \left| \int_{-\infty}^{\tilde{p}_1 - \eta} U(p)\, e^{it\psi(p)}\, dp \right| \leqslant c^{(2)}_{\tilde{p}_1,\tilde{p}_2}(u_0, f)\, t^{-\mu} \ , \tag{4.11}$$



 

where

$$c^{(2)}_{\tilde{p}_1,\tilde{p}_2}(u_0,f) := \frac{1}{\pi}\left(\frac{1}{\mu}\left\|\tilde{u}\right\|_{L^\infty(\mathbb{R})} + \Big(4\left\|\tilde{u}\right\|_{L^\infty(\mathbb{R})} + \left\|\tilde{u}'\right\|_{L^1(\mathbb{R})}\Big)\tilde{m}^{-1}_{2,\tilde{p}_1}\right) ,$$

with $\tilde{m}_{2,\tilde{p}_1} := f'(\tilde{p}_1) - f'(\tilde{p}_1 - \eta) > 0$. To study the integral on the other infinite interval, we define $\tilde{I}^{(3)}_k(t,x,\eta)$ as follows,

$$\forall\, k \in \mathbb{N}^* \qquad \tilde{I}^{(3)}_k(t,x,\eta) := \int_{\tilde{p}_2+\eta}^{\tilde{p}_2+\eta+k} U(p)\, e^{it\psi(p)}\, dp \,.$$

Here there is no singular frequency or stationary point, therefore Theorem 3.2.1 in the case $\mu = 1$ is applicable and it furnishes

$$\forall\,(t,x) \in \mathfrak{C}\big(f'(\tilde{p}_1), f'(\tilde{p}_2)\big) \qquad \left|\int_{\tilde{p}_2+\eta}^{+\infty} U(p)\, e^{it\psi(p)}\, dp\right| \leqslant c^{(3)}_{\tilde{p}_1,\tilde{p}_2}(u_0,f)\, t^{-1} \,, \tag{4.12}$$

where

$$c^{(3)}_{\tilde{p}_1,\tilde{p}_2}(u_0,f) := \frac{(\tilde{p}_2+\eta-p_1)^{\mu-1}}{2\pi}\Big(4\left\|\tilde{u}\right\|_{L^\infty(\mathbb{R})} + \left\|\tilde{u}'\right\|_{L^1(\mathbb{R})}\Big)\,\tilde{m}^{-1}_{3,\tilde{p}_2} \,,$$

with $\tilde{m}_{3,\tilde{p}_2} := f'(\tilde{p}_2+\eta) - f'(\tilde{p}_2) > 0$.
Consequently we derive the following estimate for $I^{(2)}(t,x,\eta)$ in the case $p_1 < \tilde{p}_1$ from (4.11) and (4.12),

$$\forall\,(t,x) \in \mathfrak{C}\big(f'(\tilde{p}_1), f'(\tilde{p}_2)\big) \qquad \left|I^{(2)}(t,x,\eta)\right| \leqslant c^{(2)}_{\tilde{p}_1,\tilde{p}_2}(u_0,f)\, t^{-\mu} + c^{(3)}_{\tilde{p}_1,\tilde{p}_2}(u_0,f)\, t^{-1} \,.$$

To conclude, we provide the values of the constants $c^{(2)}_{\tilde{p}_1,\tilde{p}_2}(u_0,f)$ and $c^{(3)}_{\tilde{p}_1,\tilde{p}_2}(u_0,f)$ depending on the position of $p_1$ with respect to the interval $[\tilde{p}_1,\tilde{p}_2]$:

- $$c^{(2)}_{\tilde{p}_1,\tilde{p}_2}(u_0,f) := \begin{cases} \dfrac{1}{\pi}\left(\dfrac{1}{\mu}\left\|\tilde{u}\right\|_{L^\infty(\mathbb{R})} + \Big(4\left\|\tilde{u}\right\|_{L^\infty(\mathbb{R})} + \left\|\tilde{u}'\right\|_{L^1(\mathbb{R})}\Big)\tilde{m}^{-1}_{2,\tilde{p}_1}\right) , & \text{if } p_1 < \tilde{p}_1 \,, \\ \dfrac{1}{\pi}\left(\dfrac{1}{\mu}\left\|\tilde{u}\right\|_{L^\infty(\mathbb{R})} + \Big(4\left\|\tilde{u}\right\|_{L^\infty(\mathbb{R})} + \left\|\tilde{u}'\right\|_{L^1(\mathbb{R})}\Big)\tilde{m}^{-1}_{3,\tilde{p}_2}\right) , & \text{if } p_1 > \tilde{p}_2 \,, \end{cases} \tag{4.13}$$

- $$c^{(3)}_{\tilde{p}_1,\tilde{p}_2}(u_0,f) := \begin{cases} \dfrac{(\tilde{p}_2+\eta-p_1)^{\mu-1}}{2\pi}\Big(4\left\|\tilde{u}\right\|_{L^\infty(\mathbb{R})} + \left\|\tilde{u}'\right\|_{L^1(\mathbb{R})}\Big)\,\tilde{m}^{-1}_{3,\tilde{p}_2} \,, & \text{if } p_1 < \tilde{p}_1 \,, \\ \dfrac{(p_1-\tilde{p}_1+\eta)^{\mu-1}}{2\pi}\Big(4\left\|\tilde{u}\right\|_{L^\infty(\mathbb{R})} + \left\|\tilde{u}'\right\|_{L^1(\mathbb{R})}\Big)\,\tilde{m}^{-1}_{2,\tilde{p}_1} \,, & \text{if } p_1 > \tilde{p}_2 \,. \end{cases} \tag{4.14}$$

□

## 4.2 An intrinsic concentration phenomenon caused by a limited growth of the symbol

In this section, we consider the evolution equation defined in Section 4.1 with the additional hypothesis that the symbols have a growth limitation at infinity. In Theorem 4.2.4, we shall prove the existence of a space-time cone, depending on the symbol only, such that the decay rate of the solution inside this cone is slower than outside. This result highlights a concentration phenomenon of the solution in this cone when the time tends to infinity, which can be viewed as an asymptotic notion of causality for initial data without compact support.

We recall that we study the evolution equation on the line given by

$$\begin{cases} \Big[ i\,\partial_t - f\big(D\big)\Big] u(t) = 0 \\ u(0) = u_0 \end{cases},$$

for $t \geqslant 0$. Throughout this section, the symbol $f$ will satisfy the following condition:

**Condition ($\mathrm{S}_{\beta_+,\beta_-,R}$).** Fix $\beta_- \geqslant \beta_+ > 1$ and $R \geqslant 1$.
A $\mathcal{C}^\infty$-function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Condition ($\mathrm{S}_{\beta_+,\beta_-,R}$) if and only if the second derivative of $f$ is positive on $\mathbb{R}$ and verifies

$$\exists\, c_+ \geqslant c_- > 0 \qquad \forall\, |p| \geqslant R \qquad c_-\, |p|^{-\beta_-} \leqslant f''(p) \leqslant c_+\, |p|^{-\beta_+}\ . \tag{4.15}$$

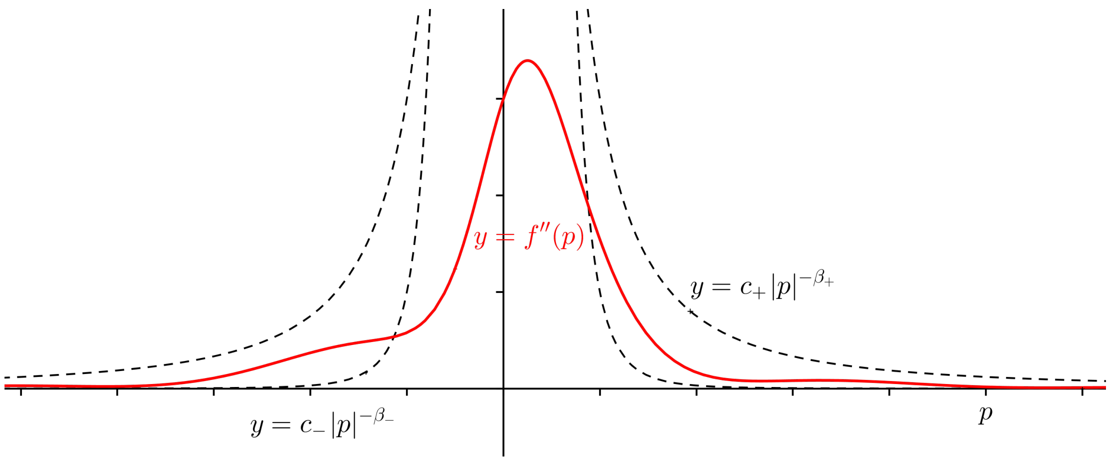


Figure 4.5: Second derivative of a symbol satisfying Condition ($\mathrm{S}_{\beta_+,\beta_-,R}$)

Let us state in the next lemma two important properties for a function $f$ satisfying the above condition. The first property shows that $f'(\mathbb{R}) = (a, b)$ where $a$ and $b$ are the limits of $f'$ at infinity. The existence of the above mentioned cone depending only on the symbol is a consequence of this first property. Indeed by recalling that the stationary point $p_0$ of the phase related to the solution formula (4.3) satisfies $f'(p_0) = \frac{x}{t}$, we see that either $\frac{x}{t}$ belongs to $(a, b)$ leading to the space-time cone $\mathfrak{C}(a, b)$, or it is outside $(a, b)$ providing the complement of $\mathfrak{C}(a, b)$. The second property furnishes estimates from below of the distance between $f'$ and its limits at infinity. This property will be necessary to carry out the proof of Theorem 4.2.4.

**4.2.1 Lemma.** *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function satisfying Condition* ($\mathrm{S}_{\beta_+,\beta_-,R}$). *Then*

*i) we have $f'(\mathbb{R}) = (a, b)$ where*

$$a := \lim_{p \to -\infty} f'(p) \qquad , \qquad b := \lim_{p \to +\infty} f'(p)\ ;$$



 

*ii) we have*

$$\bullet \quad \forall\, p \geqslant R \qquad b - f'(p) \geqslant \frac{c_-}{\beta_- - 1}\, p^{1-\beta_-} \;,$$

$$\bullet \quad \forall\, p \leqslant -R \qquad f'(p) - a \geqslant \frac{c_-}{\beta_- - 1}\, (-p)^{1-\beta_-} \;.$$

*Proof.* i) On the compact interval $[-R, R]$, the function $f'$ is bounded since it is continuous. Now, using the right inequality in (4.15), we have for $p \geqslant R$,

$$f'(p) - f'(R) = \int_R^p f''(x)\, dx \leqslant c_+ \int_R^p x^{-\beta_+} dx \;,$$

which provides

$$f'(p) \leqslant \frac{c_+}{1-\beta_+}\, p^{1-\beta_+} + f'(R) - \frac{c_+}{1-\beta_+}\, R^{1-\beta_+} \leqslant f'(R) - \frac{c_+}{1-\beta_+}\, R^{1-\beta_+} < \infty \;.$$

Consequently $f'$ is bounded from above on $[R, +\infty)$ and similar arguments show that $f'$ is bounded from below on $(-\infty, R]$. Since the function $f'$ is strictly increasing on $\mathbb{R}$, we deduce that $f'$ is bounded on $\mathbb{R}$ and its bounds are given by its limits at $-\infty$ and $+\infty$.

ii) For $p \geqslant R$, we have

$$b - f'(p) = \int_p^{+\infty} f''(x)\, dx \geqslant c_- \int_p^{+\infty} x^{-\beta_-} dx = -\frac{c_-}{1-\beta_-}\, p^{1-\beta_-} \;,$$

where we used the left inequality of (4.15). In the same way, we have for all $p \leqslant -R$,

$$f'(p) - a = \int_{-\infty}^p f''(x)\, dx \geqslant c_- \int_{-\infty}^p (-x)^{-\beta_-} dx = -\frac{c_-}{1-\beta_-}\, (-p)^{1-\beta_-} \;.$$

□

In Theorem 4.2.4, we do not assume that the initial condition is in a frequency band but one frequency can be singular, as in Theorem 4.1.7 and Theorem 4.1.8. But here, if a singular frequency exists, then we put it at 0 in order to avoid a proof with too many technical calculations. Without this assumption on the position of the singular frequency, the result of Theorem 4.2.4 remains unchanged and the proof follows the steps of the proof in the case of the singular frequency put at 0. In addition to this, the Fourier transform of the initial datum is supposed to have a sufficient decay at infinity to carry out the proof:

**Condition ($\mathbf{C4}_{\mu,\alpha,r}$).** Fix $\mu \in (0, 1]$, $\alpha > \mu$ and $r \geqslant 0$.
A tempered distribution $u_0$ on $\mathbb{R}$ satisfies Condition ($C4_{\mu,\alpha,r}$) if and only if there exists a bounded differentiable function $\tilde{u} : \mathbb{R} \longrightarrow \mathbb{C}$ such that $\tilde{u}(0) \neq 0$ if $\mu \neq 1$, with

$$\forall\, p \in \mathbb{R} \setminus \{0\} \qquad \mathcal{F}u_0(p) = |p|^{\mu-1}\, \tilde{u}(p) \;.$$

Moreover we suppose that

$$\exists\, M \geqslant 0 \quad \forall\, p \in \mathbb{R} \qquad \big|\tilde{u}(p)\big| \leqslant M \left(1 + p^2\right)^{-\frac{\alpha}{2}} \;,$$



 

and that $\tilde{u}' \in L^1_{loc}(\mathbb{R})$ with

$$\exists\, M' \geqslant 0 \quad \forall\, n \in \big\{ n \in \mathbb{Z} \,\big|\, |n| \geqslant r \big\} \qquad \|\tilde{u}'\|_{L^1(n,n+1)} \leqslant M' \, |n|^{-\alpha} \; .$$

**4.2.2 Remark and Example.** i) The above condition implies in particular that $\mathcal{F}u_0$ belongs to $L^1(\mathbb{R})$. Indeed $\mathcal{F}u_0 \in L^1_{loc}(\mathbb{R})$ since the function $p \longmapsto |p|^{\mu-1}$ with $\mu \in (0,1]$ belongs to $L^1_{loc}(\mathbb{R})$, and $\tilde{u} \in L^\infty(\mathbb{R})$. Furthermore we have

$$\forall\, p \in \mathbb{R} \setminus \{0\} \qquad \big|\mathcal{F}u_0(p)\big| \leqslant M \left(1+p^2\right)^{-\frac{\alpha}{2}} |p|^{\mu-1} \leqslant M \, |p|^{\mu-1-\alpha} \; ,$$

since $\left(1+p^2\right)^{\frac{1}{2}} \geqslant |p|$. Hence the hypothesis $\alpha > \mu$ leads to the integrability of $\mathcal{F}u_0$ on $\mathbb{R}$.
Thanks to that, one can show that the subset of tempered distributions verifying Condition $(\mathrm{C4}_{\mu,\alpha,r})$ is non-empty by following the lines of Remark 4.1.2 i), and the solution formula (4.3) is well-defined for all $t > 0$ and $x \in \mathbb{R}$.

ii) Let us give an example for the above condition. Choose $u_0 \in \mathcal{S}'(\mathbb{R})$ such that its Fourier transform has the following form:

$$\forall\, p \in \mathbb{R} \setminus \{0\} \qquad \mathcal{F}u_0(p) = |p|^{\mu-1} \, (1+p^2)^{-\frac{\alpha}{2}} \; ,$$

with $\mu \in (0,1]$ and $\alpha > \mu$. Here $\tilde{u} : \mathbb{R} \longrightarrow \mathbb{R}$ is given by $\tilde{u}(p) = (1+p^2)^{-\frac{\alpha}{2}}$ for all $p \in \mathbb{R}$.
In this case, we only have to control $\|\tilde{u}'\|_{L^1(n,n+1)}$ since the other hypotheses are clearly satisfied. One can quickly show that

$$\|\tilde{u}'\|_{L^1(n,n+1)} = \begin{cases} \tilde{u}(n) - \tilde{u}(n+1) \leqslant \tilde{u}(n) \; , & \text{if } n \geqslant 0 \; , \\ \tilde{u}(n+1) - \tilde{u}(n) \leqslant \tilde{u}(n+1) \; , & \text{if } n \leqslant -1 \; . \end{cases}$$

Using the fact that $|n+1|^{-\alpha} \leqslant 2^\alpha \, |n|^{-\alpha}$, if $n \leqslant -2$ according to Lemma 4.2.3 (see below), we obtain

$$\|\tilde{u}'\|_{L^1(n,n+1)} \leqslant \begin{cases} \left(1+n^2\right)^{-\frac{\alpha}{2}} \leqslant n^{-\alpha} \; , & \text{if } n \geqslant 0 \; , \\ \left(1+(n+1)^2\right)^{-\frac{\alpha}{2}} \leqslant |n+1|^{-\alpha} \leqslant 2^\alpha \, |n|^{-\alpha} \; , & \text{if } n \leqslant -2 \; . \end{cases}$$

Hence for all $|n| \geqslant 2$, we have

$$\|\tilde{u}'\|_{L^1(n,n+1)} \leqslant 2^\alpha \, |n|^{-\alpha} \; ,$$

and consequently, $u_0$ satisfies Condition $(\mathrm{C4}_{\mu,\alpha,2})$.

As above, we shall use several times the following basic lemma in the present section.

**4.2.3 Lemma.** *Let $p \in [n, n+1]$, where $n \geqslant 1$ or $n \leqslant -2$. Then we have*

$$\frac{1}{2} \, |n| \leqslant |p| \leqslant 2|n| \; .$$

*Proof.* Firstly let us suppose that $n \geqslant 1$. Then

$$\frac{1}{2}\, n \leqslant n \leqslant p \leqslant n+1 \leqslant 2\, n\ .$$

Now by supposing $n \leqslant -2$, we have

$$\frac{1}{2}\, |n| = -\frac{1}{2}\, n \leqslant -(n+1) \leqslant -p = |p| \leqslant -n = |n| \leqslant 2\, |n|\ .$$

□

To prove Theorem 4.2.4, we start by splitting the infinite integration interval of the integral defining the solution formula (4.3) as follows: the singular frequency 0 is the center of a sufficiently large but bounded interval, and we decompose the two remaining infinite intervals into an infinite union of disjoints bounded intervals. Thanks to that, the solution of the above evolution equation for an initial datum satisfying Condition $(\mathrm{C4}_{\mu,\alpha,r})$ is actually a (infinite) sum of solutions of the same evolution equation but for initial data in frequency bands. Then we follow the lines of the proof of Theorem 4.1.3 to apply the abstract results of Chapter 3, Section 3.1, leading to a uniform estimate of each term of the sum in the cone $\mathfrak{C}(a,b)$ as well as in its complement. Hence the series given by these uniform estimates furnishes a bound for the solution which is studied here. To assure the convergence of this series, we suppose that the decay at infinity of the Fourier transform of the initial datum is sufficiently fast as compared with the decay of the second derivative of the symbol.

**4.2.4 Theorem.** *Suppose that the symbol $f$ satisfies Condition $(\mathrm{S}_{\beta_+,\beta_-,R})$ and that the initial datum $u_0$ satisfies Condition $(\mathrm{C4}_{\mu,\alpha,r})$, where $\mu \in (0,1]$, $\alpha - \mu > \beta_-$ and $r \leqslant R$. Then we have*

$$\forall\, (t,x) \in \mathfrak{C}(a,b) \qquad \big|u(t,x)\big| \leqslant c^{(1)}(u_0,f)\, t^{-\frac{\mu}{2}} + c^{(2)}(u_0,f)\, t^{-\frac{1}{2}}\ ,$$

*where the constants $c^{(1)}(u_0,f)$ and $c^{(2)}(u_0,f)$ are given by (4.16) and (4.18), respectively. Moreover we have*

$$\forall\, (t,x) \in \mathfrak{C}(a,b)^c \qquad \big|u(t,x)\big| \leqslant c_c^{(1)}(u_0,f)\, t^{-\mu} + c_c^{(2)}(u_0,f)\, t^{-1}\ ,$$

*where the constants $c_c^{(1)}(u_0,f)$ and $c_c^{(2)}(u_0,f)$ are given by (4.19) and (4.20), respectively. The two finite real numbers $a < b$ are defined by*

$$a := \lim_{p \to -\infty} f'(p) \qquad , \qquad b := \lim_{p \to +\infty} f'(p)\ .$$

*Proof.* We recall that the solution of the initial value problem (4.1) can be written as follows,

$$\forall\, (t,x) \in (0,+\infty) \times \mathbb{R} \qquad u(t,x) = \int_{\mathbb{R}} U(p)\, e^{it\psi(p)}\, dp\ ,$$

where

$$\begin{cases} \forall\, p \in \mathbb{R} \setminus \{0\} \qquad U(p) := \dfrac{1}{2\pi}\, \mathcal{F}u_0(p) = \dfrac{1}{2\pi}\, |p|^{\mu-1}\, \tilde{u}(p)\ , \\ \forall\, p \in \mathbb{R} \qquad \psi(p) := \dfrac{x}{t}\, p - f(p)\ . \end{cases}$$



 

Let us define $N \in \mathbb{N}$ and $\mathfrak{S}_N \subseteq \mathbb{Z}$ as follows,

$$N := \lceil R \rceil + 1 \qquad , \qquad \mathfrak{S}_N = \mathbb{Z} \setminus \{-N, \dots, N-1\} \, ,$$

where $\lceil . \rceil$ is the ceiling function, and let us split the integral,

$$\begin{aligned}\int_{\mathbb{R}} U(p) e^{it\psi(p)}\, dp &= \int_{\mathbb{R}} \chi_{[-N,N)}(p) U(p)\, e^{it\psi(p)}\, dp + \int_{\mathbb{R}} \sum_{n \in \mathfrak{S}_N} \chi_{[n,n+1)}(p) U(p)\, e^{it\psi(p)}\, dp \\ &= \int_{-N}^{N} U(p)\, e^{it\psi(p)}\, dp + \sum_{n \in \mathfrak{S}_N} \int_n^{n+1} U(p)\, e^{it\psi(p)}\, dp \, ,\end{aligned}$$

where $\chi_{[n,n+1)}$ is the characteristic function of the interval $[n, n+1)$. Now we divide the proof into two parts with respect to the value of $\frac{x}{t}$.

i) *Case* $\frac{x}{t} \in (a,b)$. In this case, $\frac{x}{t}$ belongs to $(a,b)$, that is to say it belongs to $f'(\mathbb{R})$. Therefore the phase $\psi$ has a unique stationary point which belongs to $\mathbb{R}$.
To estimate the integral on $[-N, N]$ in this case, we apply Theorem 3.1.8 for $\rho = 2$ on $[-N, 0]$ and on $[0, N]$ by following the lines of the proof of Theorem 4.1.3 in the case i) which gives

$$\begin{aligned}\forall\, (t,x) \in \mathfrak{C}(a,b) \qquad \left| \int_{-N}^{N} U(p)\, e^{it\psi(p)}\, dp \right| &\leqslant \left| \int_{-N}^{0} \dots \right| + \left| \int_{0}^{N} \dots \right| \\ &\leqslant \left( c_{-N}^{(1)}(u_0, f) + c_{+N}^{(1)}(u_0, f) \right) t^{-\frac{\mu}{2}} \\ &=: c^{(1)}(u_0, f)\, t^{-\frac{\mu}{2}} \, , \qquad (4.16)\end{aligned}$$

with

- $c_{-N}^{(1)}(u_0, f) := \dfrac{1}{2\pi} \dfrac{3}{\mu} \|\tilde{u}\|_{L^\infty(-N,0)} + \dfrac{1}{\pi} \Big( 4 \|\tilde{u}\|_{L^\infty(-N,0)} + \|\tilde{u}'\|_{L^1(-N,0)} \Big) m_{-N}^{-1} \, ,$
- $c_{+N}^{(1)}(u_0, f) := \dfrac{1}{2\pi} \dfrac{3}{\mu} \|\tilde{u}\|_{L^\infty(0,N)} + \dfrac{1}{\pi} \Big( 4 \|\tilde{u}\|_{L^\infty(0,N)} + \|\tilde{u}'\|_{L^1(0,N)} \Big) m_{+N}^{-1} \, ,$

and

$$m_{-N} := \min_{p \in [-N,0]} f''(p) > 0 \qquad , \qquad m_{+N} := \min_{p \in [0,N]} f''(p) > 0 \, .$$

Now let us study each term of the series. By hypothesis, $U$ has no singular points in $[n, n+1]$ for $n \in \mathfrak{S}_N$. As above, we can apply Theorem 3.1.8 for $\rho = 2$ and $\mu = 1$, and we obtain

$$\forall\, (t,x) \in \mathfrak{C}(a,b) \qquad \left| \int_n^{n+1} U(p)\, e^{it\psi(p)}\, dp \right| \leqslant c_n^{(2)}(u_0, f)\, t^{-\frac{1}{2}} \, ;$$

the constant $c_n^{(2)}(u_0, f) > 0$ is given by

$$c_n^{(2)}(u_0, f) := \frac{1}{\pi} \, \|U\|_{L^\infty(n,n+1)} + \frac{1}{\pi} \Big( 3 \|U\|_{L^\infty(n,n+1)} + \|U'\|_{L^1(n,n+1)} \Big) m_n^{-1} \, ,$$



 

with $m_n := \min_{p \in [n,n+1]} f''(p) > 0$.

The following step is to prove the summability of the sequence $\left\{c_n^{(2)}(u_0,f)\right\}_{n \in \mathfrak{S}_N}$. On the one hand, we have by using the hypothesis on $u_0$ and Lemma 4.2.3,

$$\|U\|_{L^\infty(n,n+1)} \leqslant 2^{1-\mu}|n|^{\mu-1}\, M2^\alpha |n|^{-\alpha} = 2^{1-\mu+\alpha} M\, |n|^{\mu-1-\alpha} \ .$$

Moreover

$$\begin{aligned} \|U'\|_{L^1(n,n+1)} &\leqslant \int_n^{n+1} (1-\mu)|p|^{\mu-2}\,|\tilde{u}(p)|\,dp + \int_n^{n+1} |p|^{\mu-1}\,|\tilde{u}'(p)|\,dp \\ &\leqslant \|\tilde{u}\|_{L^\infty(n,n+1)} \int_n^{n+1} (1-\mu)|p|^{\mu-2} dp + 2^{1-\mu}\,|n|^{\mu-1}\,\|\tilde{u}'\|_{L^1(n,n+1)} \\ &\leqslant M2^\alpha|n|^{-\alpha}\, 2^{1-\mu}|n|^{\mu-1} + 2^{1-\mu}|n|^{\mu-1}\, M'|n|^{-\alpha} \qquad (4.17) \\ &= 2^{1-\mu}\left(2^\alpha M + M'\right)|n|^{\mu-1-\alpha} \ , \end{aligned}$$

where the hypothesis $\left\|\tilde{u}'\right\|_{L^1(n,n+1)} \leqslant M'|n|^{-\alpha}$ was used to get (4.17). On the other hand, we have by the hypothesis on the symbol $f$,

$$f''(p) \geqslant c_-\,|p|^{-\beta_-} \geqslant c_-\,2^{-\beta_-}|n|^{-\beta_-} \ .$$

It follows

$$m_n^{-1} \leqslant \frac{2^{\beta_-}}{c_-}\,|n|^{\beta_-} \ .$$

Then we obtain

$$\begin{aligned} c_n^{(2)}(u_0,f) &= \frac{1}{\pi}\,\|U\|_{L^\infty(n,n+1)} + \frac{1}{\pi}\Big(3\,\|U\|_{L^\infty(n,n+1)} + \|U'\|_{L^1(n,n+1)}\Big)\, m_n^{-1} \\ &\leqslant \frac{2^{1-\mu+\alpha}M}{\pi}\,|n|^{\mu-1-\alpha} + 3\,\frac{2^{1-\mu+\alpha+\beta_-}M}{\pi\, c_-}\,|n|^{\mu-1-\alpha+\beta_-} \\ &\quad + \frac{2^{1-\mu+\beta_-}\left(2^\alpha M + M'\right)}{\pi\, c_-}\,|n|^{\mu-1-\alpha+\beta_-} \ . \end{aligned}$$

Since $\alpha - \mu > \beta_-$, the sequence $\left\{c_n^{(2)}(u_0,f)\right\}_{n \in \mathfrak{S}_N}$ is summable. It follows

$$\left|\sum_{n \in \mathfrak{S}_N} \int_n^{n+1} U(p)\, e^{it\psi(p)}\,dp\right| \leqslant \sum_{n \in \mathfrak{S}_N} \left|\int_n^{n+1} U(p)\, e^{it\psi(p)}\,dp\right| \leqslant \left(\sum_{n \in \mathfrak{S}_N} c_n^{(2)}(u_0,f)\right) t^{-\frac{1}{2}} \ .$$

Then it is possible to bound the last series by employing the following estimate of the Riemann Zeta function,

$$\forall\, \sigma > 1 \qquad \sum_{n \in \mathbb{N}^*} n^{-\sigma} \leqslant \frac{\sigma}{\sigma - 1} \ .$$



 

Hence

$$\begin{aligned}\sum_{n\in\mathfrak{S}_N} c_n^{(2)}(u_0,f) &\leqslant \frac{2^{2-\mu+\alpha}M}{\pi}\,\frac{\alpha+1-\mu}{\alpha-\mu} + 3\,\frac{2^{2-\mu+\alpha+\beta_-}M}{\pi\,c_-}\,\frac{\alpha+1-\mu-\beta_-}{\alpha-\mu-\beta_-}\\ &\quad + \frac{2^{2-\mu+\beta_-}\left(2^{\alpha}M+M'\right)}{\pi\,c_-}\,\frac{\alpha+1-\mu-\beta_-}{\alpha-\mu-\beta_-}\\ &= \frac{2^{2-\mu+\alpha}M}{\pi}\,\frac{\alpha+1-\mu}{\alpha-\mu} + \frac{2^{2-\mu+\beta_-}\left(2^{\alpha+2}M+M'\right)}{\pi\,c_-}\,\frac{\alpha+1-\mu-\beta_-}{\alpha-\mu-\beta_-}\\ &=: c^{(2)}(u_0,f)\;. \qquad (4.18)\end{aligned}$$

Hence we obtain finally for all $(t,x)\in\mathfrak{C}(a,b)$,

$$\big|u(t,x)\big| \leqslant c^{(1)}(u_0,f)\,t^{-\frac{\mu}{2}} + c^{(2)}(u_0,f)\,t^{-\frac{1}{2}}\;.$$

ii) *Case* $\frac{x}{t}\notin(a,b)$. We note that $\psi$ has no stationary points on $\mathbb{R}$ and that $(t,x)$ belongs to $\mathfrak{C}(a,b)^c$ in this case.
We start by estimating the integral on $[-N,N]$. To do so, we apply Theorem 3.2.1 on $[-N,0]$ and on $[0,N]$, providing

$$\begin{aligned}\forall\,(t,x)\in\mathfrak{C}(a,b)^c \qquad \left|\int_{-N}^{N} U(p)\,e^{it\psi(p)}\,dp\right| &\leqslant \left|\int_{-N}^{0}\dots\right| + \left|\int_{0}^{N}\dots\right|\\ &\leqslant \left(\tilde{c}_{-N}^{(1)}(u_0,f)+\tilde{c}_{+N}^{(1)}(u_0,f)\right)t^{-\mu}\\ &=: c_c^{(1)}(u_0,f)\,t^{-\mu}\;, \qquad (4.19)\end{aligned}$$

with

- $\tilde{c}_{-N}^{(1)}(u_0,f) := \dfrac{1}{2\pi}\dfrac{1}{\mu}\,\|\tilde{u}\|_{L^\infty(-N,0)} + \dfrac{1}{2\pi}\Big(4\,\|\tilde{u}\|_{L^\infty(-N,0)} + \|\tilde{u}'\|_{L^1(-N,0)}\Big)\,\tilde{m}_{-N}^{-1}\;,$
- $\tilde{c}_{+N}^{(1)}(u_0,f) := \dfrac{1}{2\pi}\dfrac{1}{\mu}\,\|\tilde{u}\|_{L^\infty(0,N)} + \dfrac{1}{2\pi}\Big(4\,\|\tilde{u}\|_{L^\infty(0,N)} + \|\tilde{u}'\|_{L^1(0,N)}\Big)\,\tilde{m}_{+N}^{-1}\;.$

The terms $\tilde{m}_{+N},\tilde{m}_{-N}>0$ are defined as follows,

- $\forall\,p\in[-N,0] \quad |\psi'(p)| = \left|\dfrac{x}{t}-f'(p)\right| \geqslant \min\big\{f'(-N)-a,\,b-f'(0)\big\} =: \tilde{m}_{-N}\;,$
- $\forall\,p\in[0,N] \quad |\psi'(p)| = \left|\dfrac{x}{t}-f'(p)\right| \geqslant \min\big\{f'(0)-a,\,b-f'(N)\big\} =: \tilde{m}_{+N}\;.$

See Figure 4.6 for an illustration of these estimates.

Now we study the terms of the series. The amplitude $U$ has no singular points in $[n,n+1]$ for $n\in\mathfrak{S}_N$ by hypothesis, so Theorem 3.2.1 is applicable once again on the interval $[n,n+1]$ with $\mu=1$ and it furnishes

$$\forall\,(t,x)\in\mathfrak{C}(a,b)^c \qquad \left|\int_{n}^{n+1} U(p)\,e^{it\psi(p)}\,dp\right| \leqslant \tilde{c}_n^{(2)}(u_0,f)\,t^{-1}\;;$$

the constant $\tilde{c}_n^{(2)}(u_0,f)>0$ is defined by

$$\tilde{c}_n^{(2)}(u_0,f) := \frac{1}{2\pi}\Big(3\,\|U\|_{L^\infty(n,n+1)} + \|U'\|_{L^1(n,n+1)}\Big)\,\tilde{m}_n^{-1}\;,$$



 

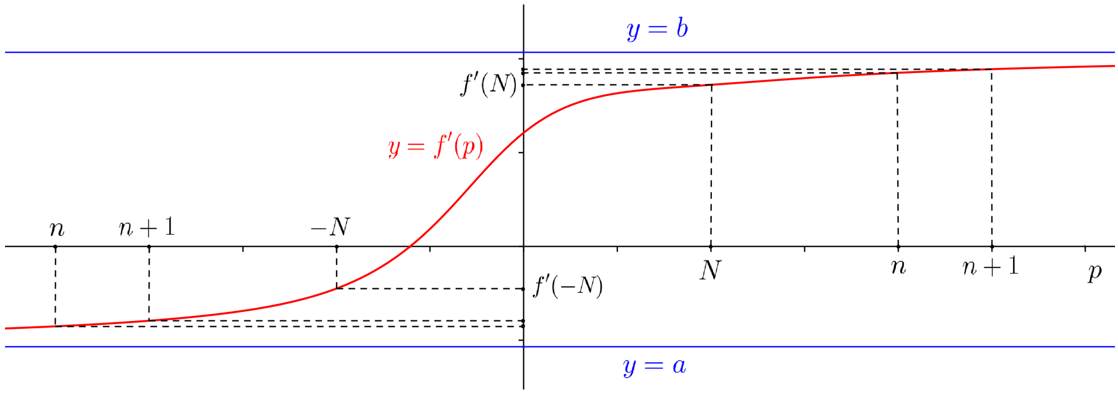


Figure 4.6: First derivative of a symbol satisfying Condition $(\mathrm{S}_{\beta_+,\beta_-,R})$

with $\tilde{m}_n := \min\left\{f'(n) - a, b - f'(n+1)\right\} > 0$. As in the previous case, one can show by using the fact that $u_0$ satisfies Condition $(\mathrm{C4}_{\mu,\alpha,r})$ and by using Lemma 4.2.3 that

$$\|U\|_{L^\infty(n,n+1)} \leqslant 2^{1-\mu+\alpha} M\, |n|^{\mu-1-\alpha} ,$$

and

$$\|U'\|_{L^1(n,n+1)} \leqslant 2^{1-\mu} \left(2^\alpha M + M'\right) |n|^{\mu-1-\alpha} .$$

Furthermore the point ii) of Lemma 4.2.1 implies

$$\tilde{m}_n \geqslant \frac{c_-}{\beta_- - 1} \min\left\{|n|^{1-\beta_-}, |n+1|^{1-\beta_-}\right\} \geqslant \frac{c_-}{\beta_- - 1}\, 2^{1-\beta_-} |n|^{1-\beta_-} ,$$

where we used Lemma 4.2.3 one more time. Then we obtain

$$\begin{aligned}\tilde{c}_n^{(2)}(u_0, f) &= \frac{1}{2\pi}\Big(3\,\|U\|_{L^\infty(n,n+1)} + \|U'\|_{L^1(n,n+1)}\Big)\, \tilde{m}_n^{-1} \\ &\leqslant \frac{\beta_- - 1}{2\pi c_-}\Big(3 \times 2^{-\mu+\alpha+\beta_-} M + 2^{-\mu+\beta_-}\left(2^\alpha M + M'\right)\Big)\, |n|^{\mu-2-\alpha+\beta_-} .\end{aligned}$$

Then the summability of the sequence $\left\{\tilde{c}_n^{(2)}(u_0, f)\right\}_{n\in\mathfrak{S}_N}$ follows from the assumption $\alpha - \mu > \beta_- > \beta_- - 1$, and we have

$$\begin{aligned}\sum_{n\in\mathfrak{S}_N} \tilde{c}_n^{(2)}(u_0, f) &\leqslant \frac{\beta_- - 1}{\pi c_-}\Big(3 \times 2^{-\mu+\alpha+\beta_-} M + 2^{-\mu+\beta_-}(2^\alpha M + M')\Big)\, \frac{\alpha + 2 - \mu - \beta_-}{\alpha + 1 - \mu - \beta_-} \\ &=: c_c^{(2)}(u_0, f) .\end{aligned} \tag{4.20}$$

It follows that

$$\left|\sum_{n\in\mathfrak{S}_N} \int_n^{n+1} U(p)\, e^{it\psi(p)}\, dp\right| \leqslant \left(\sum_{n\in\mathfrak{S}_N} \tilde{c}_n^{(2)}(u_0, f)\right) t^{-1} \leqslant c_c^{(2)}(u_0, f)\, t^{-1} .$$

We obtain finally for all $(t, x) \in \mathfrak{C}(a, b)^c$,

$$\big|u(t, x)\big| \leqslant c_c^{(1)}(u_0, f)\, t^{-\mu} + c_c^{(2)}(u_0, f)\, t^{-1} .$$



 

□

To illustrate the preceding result, let us introduce the Klein-Gordon equation on $\mathbb{R}$,

$$\begin{cases} \left[\partial_{tt} - c^2 \partial_{xx} + c^4\right] u(t) = 0 \\ u(0) = u_0 \quad , \quad \partial_t u(0) = v_0 \end{cases} , \tag{4.21}$$

for $t \geqslant 0$, where $c > 0$ is a constant. In terms of quantum mechanics, the constant $c$ represents the speed of light and the solution is the wave function of a spinless relativistic free particle with mass $m = 1$. By assuming that $u_0, v_0 \in \mathcal{S}'(\mathbb{R})$, one can furnish a solution formula which belongs to $\mathcal{C}^2\big(\mathbb{R}_+, \mathcal{S}'(\mathbb{R})\big)$,

$$u_{KG}(t) = \mathcal{F}^{-1}\Big(e^{-itf_{KG}} a_+(u_0, v_0)\Big) + \mathcal{F}^{-1}\Big(e^{itf_{KG}} a_-(u_0, v_0)\Big) =: u_+(t) + u_-(t) \ , \tag{4.22}$$

where the symbol $f_{KG} : \mathbb{R} \longrightarrow \mathbb{R}$ is given by $f_{KG}(p) = \sqrt{c^4 + c^2\, p^2}$, and the tempered distributions $a_+(u_0, v_0)$ and $a_-(u_0, v_0)$ are defined by

$$a_+(u_0, v_0) := \frac{1}{2}\left(\mathcal{F}u_0 + \frac{i}{f_{KG}}\, \mathcal{F}v_0\right) \quad , \quad a_-(u_0, v_0) := \frac{1}{2}\left(\mathcal{F}u_0 - \frac{i}{f_{KG}}\, \mathcal{F}v_0\right) \ .$$

In this context, we note that $u_+$ and $u_-$ are actually the solutions of

$$\begin{cases} \big[i\, \partial_t - f_{KG}\big(D\big)\big]\, u(t) = 0 \\ u(0) = \mathcal{F}^{-1} a_+(u_0, v_0) \end{cases} , \qquad \begin{cases} \big[i\, \partial_t + f_{KG}\big(D\big)\big]\, u(t) = 0 \\ u(0) = \mathcal{F}^{-1} a_-(u_0, v_0) \end{cases} , \tag{4.23}$$

for $t \geqslant 0$, respectively. In the following result, we furnish estimates of the solution of the Klein-Gordon equation (4.21) coming from estimates of the evolution equations given in (4.23). The proof consists mainly in showing that the symbol $f_{KG}$ satisfies Condition $(\mathrm{S}_{\beta_+, \beta_-, R})$, for certain $\beta_+, \beta_-, R$. Theorem 4.2.4 is then applicable, and the resulting estimates indicates that the solution of the Klein-Gordon equation (4.21) is time-asymptotically localized in the space-time region $\mathfrak{C}(-c, c)$, which is actually the light cone issued by the origin.

**4.2.5 Corollary.** *Let $u_{KG} \in \mathcal{C}^2\big(\mathbb{R}_+, \mathcal{S}'(\mathbb{R})\big)$ be the solution of the Klein-Gordon equation in dimension one where $\mathcal{F}^{-1} a_+(u_0, v_0)$ and $\mathcal{F}^{-1} a_-(u_0, v_0)$ satisfy Condition $(\mathrm{C4}_{\mu, \alpha, r})$, with $\mu \in (0, 1]$, $\alpha - \mu > 3$ and $r \leqslant c$. Then $u_{KG}$ defines a complex-valued function on $(0, +\infty) \times \mathbb{R}$ which satisfies*

$$\forall\, (t, x) \in \mathfrak{C}(-c, c) \qquad \big|u_{KG}(t, x)\big| \leqslant c^{(1)}(u_0, v_0, f_{KG})\, t^{-\frac{\mu}{2}} + c^{(2)}(u_0, v_0, f_{KG})\, t^{-\frac{1}{2}} \ ,$$

*and*

$$\forall\, (t, x) \in \mathfrak{C}(-c, c)^c \qquad \big|u_{KG}(t, x)\big| \leqslant c_c^{(1)}(u_0, v_0, f_{KG})\, t^{-\mu} + c_c^{(2)}(u_0, v_0, f_{KG})\, t^{-1} \ .$$

*All the constants can be computed from Theorem 4.2.4.*



 

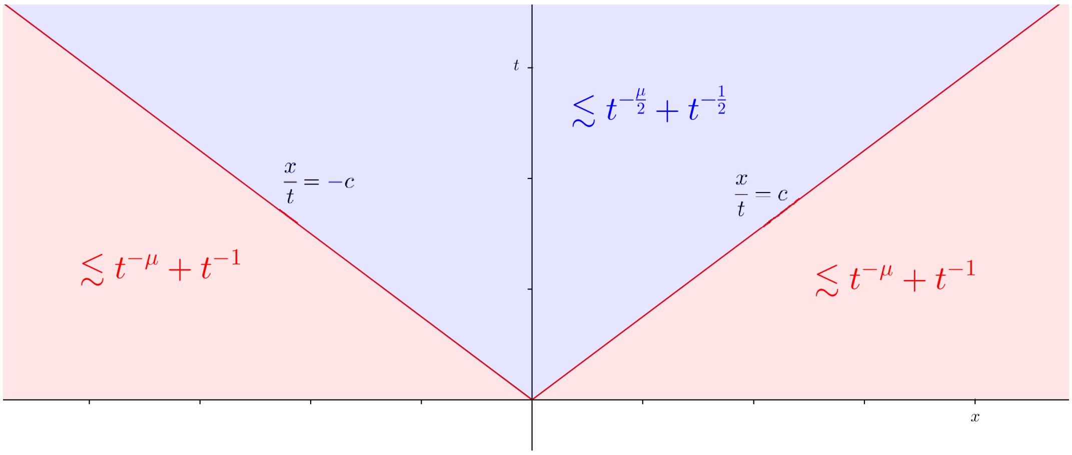


Figure 4.7: Illustration of Corollary 4.2.5 in space-time

*Proof.* Firstly, let us note $a_+(u_0, v_0)$ and $a_-(u_0, v_0)$ define integrable functions when $\mathcal{F}^{-1}a_+(u_0, v_0)$ and $\mathcal{F}^{-1}a_-(u_0, v_0)$ satisfy Condition $(\mathrm{C4}_{\mu,\alpha,r})$. This implies that the solution formula (4.22) defines a complex-valued function on $(0, +\infty) \times \mathbb{R}$.
Secondly, let us remark that one can follow the lines of the proof of Theorem 4.2.4 to establish very similar estimates of the solution of the evolution equation (4.1) when $-f$ satisfies Condition $(\mathrm{S}_{\beta_+,\beta_-,R})$, that is to say when $f'' < 0$. In this case, $a$ is the limit of $f'$ at $+\infty$ and $b$ the limit of $f'$ at $-\infty$. In the present proof, this remark assures that Theorem 4.2.4 is applicable to both equations given in (4.23) if the symbol $f_{KG}$ verifies Condition $(\mathrm{S}_{\beta_+,\beta_-,R})$.
Now we provide the first and the second derivative of $f_{KG}$,

$$\forall\, p \in \mathbb{R} \qquad (f_{KG})'(p) = \frac{c\,p}{\sqrt{c^2 + p^2}} \qquad , \qquad (f_{KG})''(p) = c^3 \left( \frac{c^2}{p^2} + 1 \right)^{-\frac{3}{2}} |p|^{-3} \,.$$

By noting that the following inequalities are true,

$$\forall\, |p| \geqslant c \qquad 2^{-\frac{3}{2}}\, c^3 \leqslant c^3 \left( \frac{c^2}{p^2} + 1 \right)^{-\frac{3}{2}} \leqslant c^3 \,,$$

we deduce that $f_{KG}$ satisfies Condition $(\mathrm{S}_{3,3,c})$. Moreover one can see that the limits of $(f_{KG})'$ at $-\infty$ and $+\infty$ are given by $-c$ and $c$ respectively. It follows that Theorem 4.2.4 is applicable to the solutions of the equations (4.23), furnishing the estimates of the solution $u_{KG} : (0, +\infty) \times \mathbb{R} \longrightarrow \mathbb{C}$ inside the cone $\mathfrak{C}(-c, c)$ and outside. □



 

# Chapter 5

# Time asymptotic behaviour of approximate solutions of Schrödinger equations with both potential and initial condition in frequency bands

**Abstract**

In this final chapter, we consider the Schrödinger equation in one dimension with potential. We start by proving that this equation is well-posed in $H^1(\mathbb{R})$ if the potential belongs to $W^{1,\infty}(\mathbb{R})$. Afterwards we suppose in addition that the Fourier transform of the potential is compactly supported. A family of potentials satisfying this hypothesis is constructed for illustration. We then focus our attention on the two first terms of a series, called Dyson-Phillips series, representing the solution of the equation. The first term is the free wave packet while the second term corresponds to the wave packet resulting from the first interaction between the free solution and the potential. We prove that these terms can be represented by oscillatory integrals. By using the assumptions on the potential and the fact that the initial data are supposed to be in frequency bands, we employ the methods of the previous chapters to describe the asymptotic behaviour of the two above terms. This permits to exhibit dynamic interaction phenomena produced by the potential.

## Contents

## 5.1 Existence and representation of the exact solution

In this section, we consider the Schrödinger equation

$$\begin{cases} i\,\partial_t u(t) = -\partial_{xx} u(t) + V(x)u(t) \\ u(0) = u_0 \end{cases} \tag{5.1}$$

for all $t \geqslant 0$, where the potential $V$ is a function belonging to $W^{1,\infty}(\mathbb{R})$ and not necessarily real-valued. The aim of this section is to assure existence and uniqueness of a solution for this equation in $H^1(\mathbb{R})$ by exploiting the theory of semigroups. Moreover a representation of the solution as a series called Dyson-Phillips series is also provided. Let us note that the general results of the present section will be employed in the next sections where we consider potentials satisfying additional hypotheses.

Before stating the above mentioned result, let us define some objects that will be used throughout the rest of this chapter:

**5.1.1 Definition.** i) Let $\mathcal{F}_{x\to p}$ be the Fourier transform on $L^2(\mathbb{R})$ and $\mathcal{F}^{-1}_{p\to x}$ its inverse. For $f$ in the Schwartz space $\mathcal{S}(\mathbb{R})$, $\mathcal{F}_{x\to p} f$ is a complex-valued function on $\mathbb{R}$ given by

$$\forall\, p \in \mathbb{R} \qquad \big(\mathcal{F}_{x\to p} f\big)(p) = \int_{\mathbb{R}} f(x)\, e^{-ixp}\, dx\ .$$

If there is no risk of confusion, we shall note $\widehat{f} := \mathcal{F}_{x\to p} f$ in favour of readability.

ii) Let $A$ be the operator given by $A := i\frac{d^2}{dx^2}$ with domain $D(A) := H^3(\mathbb{R}) \subset H^1(R)$.

iii) For $V \in W^{1,\infty}(\mathbb{R})$, let $B$ be the operator defined on $H^1(\mathbb{R})$ by

$$\forall\, f \in H^1(\mathbb{R}) \qquad (Bf)(x) := -i\, V(x)\, f(x) \quad a.e\ .$$

In this case, we have $Bf \in H^1(\mathbb{R})$ (see [14, Chapter VI, Section 5, Lemma 5-20]).

**5.1.2 Theorem.** *Suppose that $u_0$ belongs to $H^3(\mathbb{R})$ and that $V$ belongs to $W^{1,\infty}(\mathbb{R})$. Then there exists a unique function $u : \mathbb{R}_+ \longrightarrow H^1(\mathbb{R})$ which is continuously differentiable with respect to the $H^1$-norm, $u(t) \in H^3(\mathbb{R})$ for all $t \geqslant 0$, and $u$ satisfies the Schrödinger equation* (5.1)*.*
*Moreover the function $u : \mathbb{R}_+ \longrightarrow H^1(\mathbb{R})$ can be represented as follows*

$$\forall\, t \geqslant 0 \qquad \lim_{N\to+\infty} \left\| u(t) - \sum_{n=0}^{N} S_n(t) u_0 \right\|_{H^1(\mathbb{R})} = 0\ ,$$

*where*

$$\begin{cases} S_0(t)u_0 := \mathcal{F}^{-1}_{p\to x}\Big(e^{-itp^2}\,\widehat{u_0}\Big) \\ S_{n+1}(t)u_0 := \displaystyle\int_0^t S_n(t-s)\, B\, S_0(s) u_0\, ds \quad , \quad \forall\, n \in \mathbb{N} \end{cases} .$$



 

**5.1.3 Remark.** i) The function $u : \mathbb{R}_+ \longrightarrow H^1(\mathbb{R})$ is called the *classical solution* of the Schrödinger equation (5.1) (see [14, Chapter II, Proposition 6.2]) and the series $\sum_{n\geqslant 0} S_n(t)u_0$ is called the *Dyson-Phillips series* for the solution $u$ (see [14, Chapter III, Theorem 1.10]).

ii) For each $n \in \mathbb{N}$, the term $S_{n+1}(t)u_0$ belongs at least to $H^1(\mathbb{R})$ for all fixed $t \geqslant 0$ if $u_0 \in H^3(\mathbb{R})$, and thus it defines a continuous function on $\mathbb{R}$. To evaluate it at any point $x \in \mathbb{R}$, let us define the operator $E_x : H^1(\mathbb{R}) \longrightarrow \mathbb{C}$ by

$$\forall\, f \in H^1(\mathbb{R}) \qquad E_x f := f(x) \,.$$

Then $E_x$ is a bounded operator from $H^1(\mathbb{R})$ into $\mathbb{C}$ thanks to the continuous embedding of $H^1(\mathbb{R})$ into $\mathcal{C}^0_0(\mathbb{R}) := \left\{ f \in \mathcal{C}^0(\mathbb{R}) \,\middle|\, \lim_{|x|\to 0} f(x) = 0 \right\}$. Hence Proposition 5.6.4 is applicable and provides[1]

$$\begin{aligned} \forall\, x \in \mathbb{R} \qquad \big(S_{n+1}(t)u_0\big)(x) &= E_x\big(S_{n+1}(t)u_0\big) \\ &= E_x\left(\int_0^t S_n(t-s)\, B\, S_0(s)u_0\, ds\right) \\ &= \int_0^t \Big(S_n(t-s)\, B\, S_0(s)u_0\Big)(x)\, ds \,. \end{aligned} \tag{5.2}$$

This will be employed in Section 5.4.

*Proof of Theorem 5.1.2.* In order to apply results from semigroup theory, we start by rewriting the Schrödinger equation (5.1) as an evolution equation of the form

$$\begin{cases} \dot{u}(t) = \big(A + B\big)u(t) \\ u(0) = u_0 \end{cases} ,$$

where $A$ and $B$ are given in Definition 5.1.1.
Now let us recall that the operator $\big(A, D(A)\big)$ is the generator of the strongly continuous semigroup $\big(T(t)\big)_{t\geqslant 0}$ on $H^1(\mathbb{R})$ represented by

$$\forall\, t \geqslant 0 \qquad T(t)f = \mathcal{F}^{-1}_{p\to x}\Big(e^{-itp^2}\mathcal{F}_{x\to p} f\Big) \,, \tag{5.3}$$

for $f \in H^1(\mathbb{R})$. Moreover the operator $B$ belongs to $\mathcal{L}\big(H^1(\mathbb{R})\big)$, the space of bounded operators from $H^1(\mathbb{R})$ into itself. Indeed we have

$$\begin{aligned} \big\|Bf\big\|^2_{H^1(\mathbb{R})} &= \int_{\mathbb{R}} \big|V(x)f(x)\big|^2\, dx + \int_{\mathbb{R}} \big|(Vf)'(x)\big|^2\, dx \\ &= \int_{\mathbb{R}} \big|V(x)f(x)\big|^2\, dx + \int_{\mathbb{R}} \big|V'(x)f(x)\big|^2\, dx + \int_{\mathbb{R}} \big|V(x)f'(x)\big|^2\, dx \\ &\leqslant 2\, \big\|V\big\|^2_{W^{1,\infty}(\mathbb{R})} \big\|f\big\|^2_{H^1(\mathbb{R})} \,. \end{aligned}$$

[1] To apply Proposition 5.6.4, the integral defining $S_{n+1}(t)u_0$ and the integral given in (5.2) are here interpreted as Bochner-integrals. Especially, the integrand of (5.2) is complex-valued and, due to the construction of the Bochner-integral, it is actually an integral of Lebesgue-type.



 

since $V \in W^{1,\infty}(\mathbb{R})$. According to Theorems 5.6.1 and 5.6.2, if $u_0 \in H^3(\mathbb{R}) = D(A)$ then the Schrödinger equation (5.1) has a unique classical solution belonging to $\mathcal{C}^1\big(\mathbb{R}^+, H^1(\mathbb{R})\big)$. More precisely, the solution $u : \mathbb{R}^+ \longrightarrow H^1(\mathbb{R})$ is given by

$$\forall\, t \geqslant 0 \qquad u(t) = S(t)u_0 \;,$$

where $\big(S(t)\big)_{t\geqslant 0}$ is the semigroup generated by the operator $\big(A+B, D(A)\big)$.
Employing Theorem 5.6.3, the solution of the equation (5.1) can be represented as follows,

$$\forall\, t \geqslant 0 \qquad \lim_{N\to+\infty} \left\| u(t) - \sum_{n=0}^{N} S_n(t)u_0 \right\|_{H^1(\mathbb{R})} = 0 \;,$$

where $S_0(t) := T(t)$ and

$$S_{n+1}(t)u_0 := \int_0^t S_n(t-s)\, B\, T(s)u_0\, ds \;.$$

According to equality (5.3), we have $S_0(t)u_0 = T(t)u_0 = \mathcal{F}^{-1}_{p\to x}\Big(e^{-itp^2}\mathcal{F}_{x\to p}u_0\Big)$, which ends the proof. □

## 5.2 A large class of admissible potentials

Throughout the rest of the chapter, we shall use the following hypothesis on the potential.

**Condition ($\mathcal{P}_{a,b}$).** Let $a < b$ be two finite positive real numbers.
An element $V$ of $L^2(\mathbb{R})$ satisfies Condition ($\mathcal{P}_{a,b}$) if and only if $\widehat{V}$ is an even real-valued $\mathcal{C}^1$-function on $\mathbb{R}$ which verifies *supp* $\widehat{V} \subseteq [-b,-a] \cup [a,b]$.

**5.2.1 Remark.** i) If $U$ is an even real-valued $\mathcal{C}^1$-function supported on $[-b,-a]\cup[a,b]$ then it belongs to $L^2(\mathbb{R})$. Hence there exists $V \in L^2(\mathbb{R})$ such that $U = \widehat{V}$ thanks to the fact that $\mathcal{F}_{x\to p} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is bijective. In this case, $V$ satisfies Condition ($\mathcal{P}_{a,b}$) and so the set of functions satisfying Condition ($\mathcal{P}_{a,b}$) is non-empty.

ii) If $V$ satisfies Condition ($\mathcal{P}_{a,b}$) then it is actually a real-valued function on $\mathbb{R}$ which is analytic. Moreover since $\widehat{V}$ is continuous and has a compact support, the function $V$ and its first derivative $V'$ are bounded on $\mathbb{R}$. In particular, the function $V$ belongs to $W^{1,\infty}(\mathbb{R})$ and so the associated operator $B$ defined in Section 5.1 belongs to $\mathcal{L}\big(H^1(\mathbb{R})\big)$.

iii) Let us note that the function $\widehat{V}$ can be written as follows,

$$\widehat{V} = \chi_{[-b,-a]}\widehat{V} + \chi_{[a,b]}\widehat{V} \;,$$

which implies the following decomposition of $V$,

$$V = \mathcal{F}^{-1}_{p\to x}\Big(\chi_{[-b,-a]}\widehat{V}\Big) + \mathcal{F}^{-1}_{p\to x}\Big(\chi_{[a,b]}\widehat{V}\Big) =: V^- + V^+ \;. \tag{5.4}$$

In the following theorem, we construct a family of potentials satisfying Condition $(\mathcal{P}_{a,b})$ and approximately localized in the interval $(-c, c)$ with arbitrary precision if the quantity $(b-a)$ is sufficiently large. To do so, we use essentially Chebyshev's inequality.

**5.2.2 Theorem.** *Let $a, b > 0$ be two finite real numbers and let $v$ be a real-valued $\mathcal{C}^1$-function on $\mathbb{R}$ such that $supp\, v \subseteq [-1,1]$.*
*Let $V_{a,b}$ be the function on $\mathbb{R}$ such that its Fourier transform $\widehat{V_{a,b}}$ is given by*

$$\forall\, p \in \mathbb{R} \qquad \widehat{V_{a,b}}(p) := \sqrt{\frac{2}{b-a}}\, v\bigg(\frac{2p+(a+b)}{a-b}\bigg) + \sqrt{\frac{2}{b-a}}\, v\bigg(\frac{2p-(a+b)}{b-a}\bigg) \ .$$

*Then the function $V_{a,b}$ satisfies Condition $(\mathcal{P}_{a,b})$, and we have for all $c > 0$,*

$$\int_{|x|\geqslant c} \big|V_{a,b}(x)\big|^2 dx \leqslant \frac{16}{c^2}\, \frac{1}{(b-a)^2}\, \big\|v'\big\|^2_{L^2(\mathbb{R})} \ .$$

*Proof.* The function $\widehat{V_{a,b}}$ is clearly an even real-valued $\mathcal{C}^1$-function on $\mathbb{R}$. Moreover combining the hypothesis $supp\, v \subseteq [-1,1]$ and the fact that $p \longmapsto (a-b)^{-1}\big(2p+(a+b)\big)$ and $p \longmapsto (b-a)^{-1}\big(2p-(a+b)\big)$ transform respectively the intervals $[-b,-a]$ and $[a,b]$ into $[-1,1]$, we see that

$$supp\, \widehat{V_{a,b}} \subseteq [-b,-a] \cup [a,b] \ .$$

Hence the function $V_{a,b}$ verifies Condition $(\mathcal{P}_{a,b})$.
Now let us apply Chebyshev's inequality to the function $V_{a,b}$,

$$\forall\, c > 0 \qquad \int_{|x|\geqslant c} \big|V_{a,b}(x)\big|^2 dx \leqslant \frac{1}{c^2} \int_{\mathbb{R}} \big|x\, V_{a,b}(x)\big|^2 dx = \frac{1}{c^2} \int_{\mathbb{R}} \Big|x\, \Big(\mathcal{F}^{-1}_{p\to x} \widehat{V_{a,b}}\Big)(x)\Big|^2 dx \ . \tag{5.5}$$

For convenience, let us define for all $p \in \mathbb{R}$,

$$\widehat{V_{a,b}}^{\,-}(p) := \sqrt{\frac{2}{b-a}}\, v\bigg(\frac{2p+(a+b)}{a-b}\bigg) \quad , \quad \widehat{V_{a,b}}^{\,+}(p) := \sqrt{\frac{2}{b-a}}\, v\bigg(\frac{2p-(a+b)}{b-a}\bigg) \ .$$

Then by simple substitutions, we have for all $x \in \mathbb{R}$,

- $$\begin{aligned}\Big(\mathcal{F}^{-1}_{p\to x} \widehat{V_{a,b}}^{\,-}\Big)(x) &= \frac{1}{2\pi} \sqrt{\frac{2}{b-a}} \int_{\mathbb{R}} v\bigg(\frac{2p+(a+b)}{a-b}\bigg)\, e^{ixp}\, dp \\ &= \frac{1}{2\pi} \sqrt{\frac{b-a}{2}}\, e^{-i\frac{a+b}{2}x}\, \widehat{v}\bigg(-\frac{b-a}{2}\, x\bigg) \ ,\end{aligned} \tag{5.6}$$
- $$\begin{aligned}\Big(\mathcal{F}^{-1}_{p\to x} \widehat{V_{a,b}}^{\,+}\Big)(x) &= \frac{1}{2\pi} \sqrt{\frac{2}{b-a}} \int_{\mathbb{R}} v\bigg(\frac{2p-(a+b)}{b-a}\bigg)\, e^{ixp}\, dp \\ &= \frac{1}{2\pi} \sqrt{\frac{b-a}{2}}\, e^{i\frac{a+b}{2}x}\, \widehat{v}\bigg(\frac{b-a}{2}\, x\bigg) \ .\end{aligned} \tag{5.7}$$



 

Putting this into inequality (5.5) provides

$$\begin{aligned}
\int_{|x|\geqslant c} \left|V_{a,b}(x)\right|^2 dx &\leqslant \frac{1}{c^2} \int_{\mathbb{R}} \left| x \left(\mathcal{F}^{-1}_{p\to x} \widehat{V_{a,b}}^{-}\right)(x) \,+\, x \left(\mathcal{F}^{-1}_{p\to x} \widehat{V_{a,b}}^{+}\right)(x)\right|^2 dx \\
&\leqslant \frac{2}{c^2} \int_{\mathbb{R}} \left| x \left(\mathcal{F}^{-1}_{p\to x} \widehat{V_{a,b}}^{-}\right)(x)\right|^2 dx \,+\, \frac{2}{c^2} \int_{\mathbb{R}} \left| x \left(\mathcal{F}^{-1}_{p\to x} \widehat{V_{a,b}}^{+}\right)(x)\right|^2 dx \quad (5.8)\\
&= \frac{1}{2\pi^2 c^2} \frac{b-a}{2} \int_{\mathbb{R}} \left| x\, \widehat{v}\left(-\frac{b-a}{2}\, x\right)\right|^2 dx \\
&\quad + \frac{1}{2\pi^2 c^2} \frac{b-a}{2} \int_{\mathbb{R}} \left| x\, \widehat{v}\left(\frac{b-a}{2}\, x\right)\right|^2 dx \quad (5.9)\\
&= \frac{1}{2\pi^2 c^2} \left(\frac{2}{b-a}\right)^2 \int_{\mathbb{R}} \left|y\, \widehat{v}(y)\right|^2 dy \,+\, \frac{1}{2\pi^2 c^2} \left(\frac{2}{b-a}\right)^2 \int_{\mathbb{R}} \left|y\, \widehat{v}(y)\right|^2 dy \quad (5.10)\\
&= \frac{16}{c^2} \frac{1}{(b-a)^2} \left\|v'\right\|^2_{L^2(\mathbb{R})} \quad (5.11)
\end{aligned}$$

- (5.8): use the inequality $|z_1+z_2|^2 \leqslant 2\left(|z_1|^2+|z_2|^2\right)$, which holds for all $z_1, z_2 \in \mathbb{C}$ ;
- (5.9): use equalities (5.6) and (5.7) ;
- (5.10): use the substitutions $y = \pm\frac{b-a}{2}\, x$ ;
- (5.11): use the relation $\widehat{(v')}(y) = i\, y\, \widehat{v}(y)$ and Plancherel's theorem.

The proof is now complete. □

## 5.3 Asymptotic behaviour of the free solution

In this section, we focus our attention on the asymptotic behaviour of the term $S_0(t)u_0$ of the Dyson-Phillips series given by Theorem 5.1.2.

First of all, we recall the definition of the space-time cone $\mathfrak{C}_S(a,b)$. Note that we employ Definition 2.3.1 of Chapter 2, which is convenient to describe the asymptotic behaviour of solutions of Schrödinger equations.

**5.3.1 Definition.** *Let $a<b$ be two real numbers (possibly infinite). We define the space-time cone $\mathfrak{C}_S(a,b)$ as follows:*

$$\mathfrak{C}_S(a,b) := \left\{(t,x) \in (0,+\infty)\times\mathbb{R} \,\middle|\, 2\,a < \frac{x}{t} < 2\,b\right\} .$$

*Let $\mathfrak{C}_S(a,b)^c$ be the complement of the cone $\mathfrak{C}_S(a,b)$ in $(0,+\infty)\times\mathbb{R}$ .*

Now let us state the assumptions that the initial data will verify throughout the rest of this chapter. Here we suppose that $u_0$ is in a frequency band (*i.e.* the support of its Fourier transform is contained in a compact interval). For simplicity, we suppose that $\widehat{u_0}$ is a $C^1$-function on $\mathbb{R}$, implying in particular that it vanishes at the endpoints of its support and that singular frequencies are not allowed.

**Condition (C5$_{[p_1,p_2]}$).** Let $p_1 < p_2$ be two finite real numbers.
An element $u_0$ of $H^3(\mathbb{R})$ satisfies Condition (C5$_{[p_1,p_2]}$) if and only if $\widehat{u_0}$ is a $\mathcal{C}^1$-function on $\mathbb{R}$ which verifies *supp* $\widehat{u_0} \subseteq [p_1, p_2]$.

**5.3.2 Remark.** i) One can prove that the set of elements of $H^3(\mathbb{R})$ satisfying Condition (C5$_{[p_1,p_2]}$) is non-empty by using the following argument:
If a $\mathcal{C}^\infty$-function $U$ on $\mathbb{R}$ is supported on $[p_1, p_2]$ then there exists $u_0 \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz space, such that $\widehat{u_0} := U$ since the Fourier transform is a bijection on $\mathcal{S}(\mathbb{R})$. In particular the function $u_0$ belongs to $H^3(\mathbb{R})$ and it satisfies Condition (C5$_{[p_1,p_2]}$).

ii) Since the support of $\widehat{u_0}$ is contained in a compact interval, $u_0$ is analytic on $\mathbb{R}$.

Supposing that the initial datum satisfies Condition (C5$_{[p_1,p_2]}$), the term $S_0(t)u_0$ defines a function on $\mathbb{R}$ for all $t \geqslant 0$ represented by

$$\forall\, x \in \mathbb{R} \qquad \big(S_0(t)u_0\big)(x) = \big(T(t)u_0\big)(x) = \frac{1}{2\pi}\int_{p_1}^{p_2} \widehat{u_0}(p)\, e^{-itp^2+ixp}\, dp\ , \tag{5.12}$$

which is actually the solution of the free Schrödinger equation on the line with initial datum $u_0$. Let us recall that Theorem 2.2.2 has permitted to expand in the space-time cone $\mathfrak{C}_S(p_1+\varepsilon, p_2)$ the solution of the free Schrödinger equation for initial data in frequency bands and having singular frequencies (see Theorem 2.3.2). But in the present setting, Theorem 2.2.2 is not applicable since it is only devoted to amplitudes having a singular point. In the following theorem, we establish a similar result to Theorem 2.2.2 which treats the case of regular amplitudes vanishing at the endpoints of the integration interval in preparation for applications to the integral representing $S_0(t)u_0$. Let us remark that, in this regular case, the remainder estimates are uniformly bounded with respect to the position of the stationary point: no blow-up occurs since the amplitude has no singular point.

**5.3.3 Theorem.** *Let $p_1 < p_2$ be two finite real numbers. Let $p_0 \in (p_1, p_2)$ and $c \in \mathbb{R}$ be two parameters, and define $\psi : [p_1, p_2] \longrightarrow \mathbb{R}$ by*

$$\psi(p) := -(p-p_0)^2 + c\ .$$

*Define the following integral for all $\omega > 0$,*

$$I(\omega, p_0) := \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp\ , \tag{5.13}$$

*where $U$ satisfies Assumption* (A1$_{1,1,1}$) *(see Chapter 1, Section 1.1) on $[p_1, p_2]$ with the additional hypothesis $U(p_1) = U(p_2) = 0$. Let us define $\tilde{H}(\omega, \psi, U)$ as follows,*

$$\tilde{H}(\omega, \psi, U) := \sqrt{\pi}\, e^{-i\frac{\pi}{4}}\, e^{i\omega c}\, U(p_0)\ ,$$

*and let* $\delta \in \left(\frac{1}{2}, 1\right)$. *Then we have*

$$\left| I(\omega, p_0) \;-\; \tilde{H}(\omega, \psi, U)\, \omega^{-\frac{1}{2}} \right| \leqslant C(U)\, \omega^{-\delta} \ ,$$

*where the constant* $C(U) \geqslant 0$ *is given by* (5.17).

*Proof.* Here we split the integral at the stationary point $p_0$,

$$I(\omega, p_0) = \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp = \int_{p_1}^{p_0} \dots + \int_{p_0}^{p_2} \dots \ . \tag{5.14}$$

On the interval $[p_1, p_0]$, the phase $\psi$ is an increasing function that satisfies Assumption ($P1_{1,2,N}$) (see Chapter 1, Section 1.1) for all $N \geqslant 1$, and by hypothesis, $U$ verifies Assumption ($A1_{1,1,1}$) on $[p_1, p_2]$, and so it does on $[p_1, p_0]$ as well. Here the point $p_1$ is neither a stationary point of the phase, nor a singular point of the amplitude. In this context, a cutting-point separating $p_1$ and $p_0$ is not needed to expand the integral on $[p_1, p_0]$ in (5.14), as we have noticed in the proof of Lemma 2.2.1, part *Study of* $I^{(2)}(\omega, p_0)$. Hence, in the notations of the proof of Theorem 2.1.2, we employ only the expansion of the integral $\tilde{I}^{(2)}(\omega, p_0)$ with $q := p_1$ and $p_2 := p_0$ and we obtain

$$\int_{p_1}^{p_0} U(p)\, e^{i\omega\psi(p)}\, dp = \phi^{(2)}(0, \omega, 2, 1)\, k_2(0)\, e^{i\omega\psi(p_0)} - \phi^{(2)}\big(\varphi_2(p_1), \omega, 2, 1\big)\, k_2\big(\varphi_2(p_1)\big)\, e^{i\omega\psi(p_0)} + e^{i\omega\psi(p_0)} \int_0^{\varphi_2(p_1)} \phi^{(2)}(s, \omega, 2, 1)(k_2)'(s)\, ds \ ,$$

where

- $\phi^{(2)}(0, \omega, 2, 1) = -\,\dfrac{\sqrt{\pi}}{2}\, e^{-i\frac{\pi}{4}}\, \omega^{-\frac{1}{2}}$ ,
- $\varphi_2(p) = \big(\psi(p_0) - \psi(p)\big)^{\frac{1}{2}} = p_0 - p \qquad \forall\, p \in [p_1, p_0]$ ,
- $k_2(s) = U\big(\varphi_2^{-1}(s)\big)\big(\varphi_2^{-1}\big)'(s) = -U(p_0 - s) \qquad \forall\, s \in \big[0, \varphi_2(p_1)\big]$ ,
- $k_2(0) = -U(p_0)$ ,
- $k_2\big(\varphi_2(p_1)\big) = -U(p_1) = 0$ .

This leads to the following expansion,

$$\int_{p_1}^{p_0} U(p)\, e^{i\omega\psi(p)}\, dp = \frac{\sqrt{\pi}}{2}\, e^{-i\frac{\pi}{4}}\, e^{i\omega\psi(p_0)}\, U(p_0)\, \omega^{-\frac{1}{2}} + e^{i\omega\psi(p_0)} \int_0^{p_0 - p_1} \phi^{(2)}(s, \omega, 2, 1)\, U'(p_0 - s)\, ds \ . \tag{5.15}$$

By using the remainder estimate provided by Theorem 1.1.7, by estimating $\left|U'(p)\right|$ by $\left\|U'\right\|_{L^\infty(p_1, p_2)}$ and $(p_0 - p_1)$ by $(p_2 - p_1)$, it follows that

$$\left| e^{i\omega\psi(p_0)} \int_0^{p_0 - p_1} \phi^{(2)}(s, \omega, 2, 1)\, U'(p_0 - s)\, ds \right| \leqslant \frac{L_{\gamma,2}}{1-\gamma}\, (p_2 - p_1)^{1-\gamma}\, \left\|U'\right\|_{L^\infty(p_1, p_2)}\, \omega^{-\delta} \ ,$$



 

where $\delta$ is arbitrarily chosen in $\left(\frac{1}{2}, 1\right)$, $\gamma := 2\delta - 1$ and the constant $L_{\gamma,2}$ is the constant $L_{\gamma,2,1}$ given in Theorem 1.1.7.
To study the second integral, we remark firstly that the phase $\psi$ is decreasing on $[p_0, p_2]$. So one can make the substitution $p \longmapsto -p$ to make $\psi$ increasing. Then carrying out the same computations as above leads to

$$\int_{p_0}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp = \frac{\sqrt{\pi}}{2}\, e^{-i\frac{\pi}{4}}\, e^{i\omega\psi(p_0)}\, U(p_0)\, \omega^{-\frac{1}{2}} - e^{i\omega\psi(p_0)} \int_0^{p_2-p_0} \phi^{(2)}(s, \omega, 2, 1)\, U'(p_0 + s)\, ds\ , \tag{5.16}$$

where the remainder is also bounded by $\dfrac{L_{\gamma,2}}{1-\gamma}\,(p_2 - p_1)^{1-\gamma}\, \left\|U'\right\|_{L^\infty(p_1,p_2)}\, \omega^{-\delta}$. Adding up equalities (5.15) and (5.16) and estimating both remainders leads to the result, with

$$C(U) := 2\, \frac{L_{\gamma,2}}{1-\gamma}\, (p_2 - p_1)^{1-\gamma}\, \left\|U'\right\|_{L^\infty(p_1,p_2)}\ . \tag{5.17}$$

□

We are now able to furnish an asymptotic expansion of the integral (5.12) representing the term $S_0(t)u_0$ in the space-time cone $\mathfrak{C}_S\big(p_1, p_2\big)$. As in the preceding chapters, the method consists in rewriting this integral as an oscillatory integral of the form (5.13) with $\omega = t$ and then applying a stationary phase method; here we employ the above Theorem 5.3.3. Remark that the expansion holds in the entire cone $\mathfrak{C}_S\big(p_1, p_2\big)$: since there is no singular frequency in the present setting, the first term as well as the remainder term are uniformly bounded with respect to $x$ in this cone.
Moreover we furnish uniform estimates of the term $S_0(t)u_0$ outside the cone $\mathfrak{C}_S\big(p_1, p_2\big)$ by employing Theorem 3.2.1 of Chapter 3, which is applicable here since it treats also amplitudes without singularities.
Since the resulting decay rates are faster outside $\mathfrak{C}_S\big(p_1, p_2\big)$ than inside, we deduce that the term $S_0(t)u_0$ tends to be time-asymptotically concentrated in this cone, which is in accordance with the results of the preceding chapters.

**5.3.4 Theorem.** *Suppose that $u_0$ satisfies Condition* $(\mathrm{C5}_{[p_1,p_2]})$. *Then we have for all* $(t, x) \in \mathfrak{C}_S\big(p_1, p_2\big)$,

$$\left|\big(S_0(t)u_0\big)(x) - H_0(t, x, u_0)\, t^{-\frac{1}{2}}\right| \leqslant C_0(u_0)\, t^{-\delta}\ ,$$

*where*

$$H_0(t, x, u_0) := \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, \widehat{u_0}\Big(\frac{x}{2t}\Big)\ .$$

*Moreover, if we choose two finite real numbers $\tilde{p}_1 < \tilde{p}_2$ such that $[p_1, p_2] \subset (\tilde{p}_1, \tilde{p}_2) =: \tilde{I}$, then we have for all $(t, x) \in \mathfrak{C}_S\big(\tilde{p}_1, \tilde{p}_2\big)^c$,*

$$\left|\big(S_0(t)u_0\big)(x)\right| \leqslant C_{0,\tilde{I}}(u_0)\, t^{-1}\ .$$

*The constants $C_0(u_0), C_{0,\tilde{I}}(u_0) \geqslant 0$ are given by* (5.18), (5.19) *respectively.*



 

**5.3.5 Remark.** It is straightforward that the $L^\infty$-norm with respect to $x$ of $S_0(t)u_0$ is estimated by $t^{-\frac{1}{2}}$ multiplied by a constant, by applying the results of Chapter 3 as we did in Theorems 4.1.3 and 4.1.4. This is not carried out in the present chapter because our main interest lies in the propagation features.

*Proof of Theorem 5.3.4.* Factorizing the phase function $p \longmapsto -tp^2 + xp$ by $t$ in the formula (5.12) gives

$$\forall\,(t,x) \in (0,+\infty)\times\mathbb{R} \qquad \big(S_0(t)u_0\big)(x) = \int_{p_1}^{p_2} U(p)\, e^{it\psi(p)}\, dp \;,$$

where

$$\begin{cases} \forall\, p \in [p_1,p_2] \qquad U(p) := \dfrac{1}{2\pi}\, \widehat{u_0}(p) \;, \\ \forall\, p \in \mathbb{R} \qquad \psi(p) := -p^2 + \dfrac{x}{t}\, p \;. \end{cases}$$

The function $U$ verifies the assumptions of Theorem 5.3.3 on $[p_1,p_2]$ by hypothesis and $\psi$ has the form

$$\psi(p) = -\,(p-p_0)^2 + c \;,$$

where $p_0 := \frac{x}{2t}$ and $c := p_0^2 = \frac{x^2}{4t^2}$. Moreover the fact that $(t,x) \in \mathfrak{C}_S\big(p_1,p_2\big)$ is equivalent to the fact that the stationary point $p_0$ belongs to $(p_1,p_2)$. Choosing $\delta \in \big(\frac{1}{2},1\big)$, Theorem 5.3.3 is applicable and furnishes

$$\Big|\big(S_0(t)u_0\big)(x) - H_0(t,x,u_0)\, t^{-\frac{1}{2}}\Big| \leqslant C_0(u_0)\, t^{-\delta} \;,$$

where

$$C_0(u_0) := \frac{1}{\pi}\, \frac{L_{\gamma,2}}{2-2\delta}\, (p_2-p_1)^{2-2\delta}\, \big\|(\widehat{u_0})'\big\|_{L^\infty(\mathbb{R})} \;, \tag{5.18}$$

and $L_{\gamma,2}$ is the constant $L_{\gamma,2,1}$ given in Theorem 1.1.7.
Now when $\frac{x}{t} \geqslant 2\,\tilde{p}_2$, we have

$$\forall\, p \in [p_1,p_2] \qquad \psi'(p) = \frac{x}{t} - 2p \geqslant 2(\tilde{p}_2 - p_2) > 0 \;.$$

Moreover the amplitude $U$ is a $\mathcal{C}^1$-function on $\mathbb{R}$ supported on $[p_1,p_2]$. In this context, Theorem 3.2.1 in the case $\mu = 1$ is applicable and it provides for $\frac{x}{t} \geqslant 2\,\tilde{p}_2$,

$$\Big|\big(S_0(t)u_0\big)(x)\Big| \leqslant \frac{1}{4\pi}\, \big(\tilde{p}_2 - p_2\big)^{-1} \Big(3\big\|\widehat{u_0}\big\|_{L^\infty(\mathbb{R})} + \big\|(\widehat{u_0})'\big\|_{L^1(\mathbb{R})}\Big)\, t^{-1} \;.$$

Very similar arguments leads to

$$\Big|\big(S_0(t)u_0\big)(x)\Big| \leqslant \frac{1}{4\pi}\, \big(p_1 - \tilde{p}_1\big)^{-1} \Big(3\big\|\widehat{u_0}\big\|_{L^\infty(\mathbb{R})} + \big\|(\widehat{u_0})'\big\|_{L^1(\mathbb{R})}\Big)\, t^{-1} \;,$$

for $\frac{x}{t} \leqslant 2\,\tilde{p}_1$. Adding up the two previous inequalities furnishes the second estimate of Theorem 5.3.4 with

$$C_{0,\tilde{I}}(u_0) := \frac{1}{4\pi}\, \left(\frac{1}{\tilde{p}_2 - p_2} + \frac{1}{p_1 - \tilde{p}_1}\right) \Big(3\big\|\widehat{u_0}\big\|_{L^\infty(\mathbb{R})} + \big\|(\widehat{u_0})'\big\|_{L^1(\mathbb{R})}\Big) \;. \tag{5.19}$$

□



 

## 5.4 Asymptotic behaviour of the wave packet issued by the first interaction

This section is devoted to the time-asymptotic behaviour of the term $S_1(t)u_0$ of the series expansion of the solution given in Theorem 5.1.2. The aim is a similar result to Theorem 5.3.4 for an integral representation of $S_1(t)u_0$: expanding this integral to one term in a certain space-time cone and to estimate it uniformly outside this cone.

In Proposition 5.4.1 and Proposition 5.4.4, we provide a representation of $S_1(t)u_0$ as a sum of two oscillatory integrals. We assume that $u_0$ and $V$ satisfy Condition ($\mathrm{C5}_{[p_1,p_2]}$) and Condition ($\mathcal{P}_{a,b}$) respectively, which is essential in order to apply the same methods as in the preceding section.
In Proposition 5.4.1, we show that the term $S_1(t)u_0$ can be written as a sum of two functions given by oscillatory integrals with amplitudes depending on time. If we apply Theorem 5.3.3 to these integrals, then the first terms and the remainder estimates may depend intrinsically on time. In particular, deriving the optimal time-decay rates of $S_1(t)u_0$ from these expansions may be not possible.
To prevent that, we furnish in Proposition 5.4.4 an expansion with respect to time of each amplitude given in Proposition 5.4.1: here, the first term does not depend on time, and the remainder is explicit and uniformly bounded by $t^{-1}$. To establish these expansions, we shall make additional hypotheses on the supports of $\widehat{u_0}$ and $\widehat{V}$: roughly speaking the support of $\widehat{u_0}$ does not contain 0 and does not intersect the support of $\widehat{V}$.
Consequently each term of $S_1(t)u_0$ is time-asymptotically equivalent to an oscillatory integral whose amplitude does not depend on time (see Corollary 5.4.5) and this will permit to apply the methods of Section 5.3.

To prove Proposition 5.4.1, we start by decomposing $S_1(t)u_0$ into a sum of two terms by using the splitting of the potential in positive and negative frequencies according to Remark 5.2.1 (iii). Then we employ the Fourier representation of $T(t)u_0$ given in (5.12) as well as Fubini's theorem to show that the terms of the above sum are represented by oscillatory integrals.

**5.4.1 Proposition.** *Suppose that $u_0$ and $V$ satisfy Condition ($\mathrm{C5}_{[p_1,p_2]}$) and Condition ($\mathcal{P}_{a,b}$) respectively. Let $t \geqslant 0$ and let $W^-(t,.), W^+(t,.) : \mathbb{R} \longrightarrow \mathbb{C}$ be the functions defined by*

$$\forall\, p \in \mathbb{R} \qquad W^{\pm}(t,p) := \int_0^t \tilde{W}^{\pm}(s,p)\, e^{isp^2}\, ds \;,$$

*where for all $s \in [0,t]$,*

- $$\tilde{W}^-(s,p) := \Big( \big(\chi_{[-b,-a]} \widehat{V}\big) * \big(e^{-is\,.^2}\, \widehat{u_0}\big) \Big)(p) = \int_{-b}^{-a} \widehat{V}(y)\, \widehat{u_0}(p-y)\, e^{-is(p-y)^2}\, dy$$
- $$\tilde{W}^+(s,p) := \Big( \big(\chi_{[a,b]} \widehat{V}\big) * \big(e^{-is\,.^2}\, \widehat{u_0}\big) \Big)(p) = \int_{a}^{b} \widehat{V}(y)\, \widehat{u_0}(p-y)\, e^{-is(p-y)^2}\, dy \;.$$

*Note that for fixed $t \geqslant 0$, we have*

$$supp\, W^-(t,.) \subseteq [p_1 - b, p_2 - a] \quad , \quad supp\, W^-(t,.) \subseteq [p_1 + a, p_2 + b] \;.$$



 

*Then we have*

$$\forall\, t \geqslant 0 \qquad S_1(t)u_0 = S_1^-(t)u_0 + S_1^+(t)u_0 \ ,$$

*where* $S_1^-(t)u_0, S_1^+(t)u_0 : \mathbb{R} \longrightarrow \mathbb{C}$ *are given by*

- $$\forall\, x \in \mathbb{R} \qquad \big(S_1^-(t)u_0\big)(x) = \frac{1}{2\pi} \int_{p_1-b}^{p_2-a} W^-(t,p)\, e^{-itp^2+ixp}\, dp \ ,$$
- $$\forall\, x \in \mathbb{R} \qquad \big(S_1^+(t)u_0\big)(x) = \frac{1}{2\pi} \int_{p_1+a}^{p_2+b} W^+(t,p)\, e^{-itp^2+ixp}\, dp \ .$$

*Proof.* Let $t \geqslant 0$. According to equality (5.4) from Remark 5.2.1 (iii), we can write the operator $B$ as follows,

$$(Bf)(x) = -i\, V(x) f(x) = -i\, V^-(x) f(x) - i\, V^+(x) f(x) =: (B^- f)(x) + (B^+ f)(x) \quad a.e \ ,$$

for all $f \in H^1(\mathbb{R})$. Putting this into equality (5.2) from Remark 5.1.3 (ii) and using the Fourier representation of $S_0(s) := T(s)$, we obtain for all $x \in \mathbb{R}$,

$$\begin{aligned}
\big(S_1(t)u_0\big)(x) &= \int_0^t \big(T(t-s)\, B^-\, T(s)u_0\big)(x)\, ds + \int_0^t \big(T(t-s)\, B^+\, T(s)u_0\big)(x)\, ds \\
&= -i \int_0^t \mathcal{F}^{-1}_{p\to x}\Big(e^{-i(t-s)p^2} \mathcal{F}_{x\to p}\Big(V^- \mathcal{F}^{-1}_{p\to x}\big(e^{-isp^2}\, \widehat{u_0}\big)\Big)\Big)(x)\, ds \\
&\quad - i \int_0^t \mathcal{F}^{-1}_{p\to x}\Big(e^{-i(t-s)p^2} \mathcal{F}_{x\to p}\Big(V^+ \mathcal{F}^{-1}_{p\to x}\big(e^{-isp^2}\, \widehat{u_0}\big)\Big)\Big)(x)\, ds \\
&= -i \int_0^t \mathcal{F}^{-1}_{p\to x}\Big(e^{-i(t-s)p^2} \Big(\big(\chi_{[-b,-a]}\widehat{V}\big) * \big(e^{-is\,.^2}\, \widehat{u_0}\big)\Big)\Big)(x)\, ds \\
&\quad - i \int_0^t \mathcal{F}^{-1}_{p\to x}\Big(e^{-i(t-s)p^2} \Big(\big(\chi_{[a,b]}\widehat{V}\big) * \big(e^{-is\,.^2}\, \widehat{u_0}\big)\Big)\Big)(x)\, ds \\
&= \int_0^t \mathcal{F}^{-1}_{p\to x}\Big(e^{-i(t-s)p^2}\, \tilde{W}^-(s,p)\Big)(x)\, ds \\
&\quad + \int_0^t \mathcal{F}^{-1}_{p\to x}\Big(e^{-i(t-s)p^2}\, \tilde{W}^+(s,p)\Big)(x)\, ds \ .
\end{aligned} \tag{5.20}$$

By defining the functions $S_1^{\pm}(t)u_0 : \mathbb{R} \longrightarrow \mathbb{C}$ as follows,

$$\forall\, x \in \mathbb{R} \qquad \big(S_1^{\pm}(t)u_0\big)(x) := \int_0^t \mathcal{F}^{-1}_{p\to x}\Big(e^{-i(t-s)p^2}\, \tilde{W}^{\pm}(s,p)\Big)(x)\, ds \ , \tag{5.21}$$

we obtain that: $S_1(t)u_0 = S_1^-(t)u_0 + S_1^+(t)u_0$. Now let us rewrite the term $S_1^+(t)u_0$. Since $\tilde{W}^+(s,.) : \mathbb{R} \longrightarrow \mathbb{C}$ is a convolution of two compactly supported $\mathcal{C}^1$-functions, it is a $\mathcal{C}^1$-function as well. Moreover as a consequence of Condition (C5$_{[p_1,p_2]}$), the support of the function $p \longmapsto e^{-isp^2}\, \widehat{u_0}(p)$ is contained in the interval $[p_1,p_2]$, which implies that

$$supp\, \tilde{W}^+(s,.) = supp\, \Big(\big(\chi_{[a,b]}\widehat{V}\big) * \big(e^{-is\,.^2}\, \widehat{u_0}\big)\Big) \subseteq [a,b] + [p_1,p_2] = [p_1+a, p_2+b] \ ,$$

for any $s \in [0,t]$. It follows that $\tilde{W}^+(s,.)$ is an integrable function and so the term $\mathcal{F}^{-1}_{p\to x}\big(e^{-i(t-s)p^2}\,\tilde{W}^+(s,p)\big)(x)$ can be given by the integral representation of the inverse Fourier transform for integrable functions with respect to the variable $p$:

$$\forall\, x \in \mathbb{R} \qquad \mathcal{F}^{-1}_{p\to x}\Big(e^{-i(t-s)p^2}\,\tilde{W}^+(s,p)\Big)(x) = \frac{1}{2\pi}\int_{p_1+a}^{p_2+b} \tilde{W}^+(s,p)\, e^{-i(t-s)p^2+ixp}\, dp\ .$$

Combining this with equality (5.21), we show that for all $x \in \mathbb{R}$,

$$\big(S_1^+(t)u_0\big)(x) = \int_0^t \left(\frac{1}{2\pi}\int_{p_1+a}^{p_2+b} \tilde{W}^+(s,p)\, e^{-i(t-s)p^2+ixp}\, dp\right) ds\ .$$

And we apply Fubini's theorem to the last equality to obtain the desired equality for $S_1^+(t)u_0$, namely,

$$\begin{aligned}\big(S_1^+(t)u_0\big)(x) &= \frac{1}{2\pi}\int_{p_1+a}^{p_2+b} \left(\int_0^t \tilde{W}^+(s,p)\, e^{isp^2}\, ds\right) e^{-itp^2+ixp}\, dp \\ &=: \frac{1}{2\pi}\int_{p_1+a}^{p_2+b} W^+(t,p)\, e^{-itp^2+ixp}\, dp\ .\end{aligned}$$

Using very similar arguments, we establish the following equality for all $t \geqslant 0$ and $x \in \mathbb{R}$,

$$\big(S_1^-(t)u_0\big)(x) = \frac{1}{2\pi}\int_{p_1-b}^{p_2-a} W^-(t,p)\, e^{-itp^2+ixp}\, dp\ ,$$

with

$$\forall\, p \in [p_1-b, p_2-a] \qquad W^-(t,p) := \int_0^t \tilde{W}^-(s,p)\, e^{isp^2}\, ds\ .$$

□

In Proposition 5.4.4, we shall expand the functions $W^{\pm}(t,.)$ with respect to the parameter $t$. We shall use one of the three following hypotheses:

**Hypothesis ($\mathcal{H}1$).** The initial data $u_0$ and the potential $V$ verify Condition ($\mathrm{C5}_{[p_1,p_2]}$) and Condition ($\mathcal{P}_{a,b}$) respectively, with $a - \frac{b}{2} > 0$ and $\frac{b}{2} - a < p_1 < p_2 < a - \frac{b}{2}$.

**Hypothesis ($\mathcal{H}2$).** The initial data $u_0$ and the potential $V$ verify Condition ($\mathrm{C5}_{[p_1,p_2]}$) and Condition ($\mathcal{P}_{a,b}$) respectively, with $b < p_1$.

**Hypothesis ($\mathcal{H}3$).** The initial data $u_0$ and the potential $V$ verify Condition ($\mathrm{C5}_{[p_1,p_2]}$) and Condition ($\mathcal{P}_{a,b}$) respectively, with $p_2 < -b$.

These three hypotheses are illustrated in Figure 5.1: $u_1$ satisfies Hypothesis ($\mathcal{H}1$) ($\widehat{u_1}$ is in red), $u_2$ satisfies Hypothesis ($\mathcal{H}2$) ($\widehat{u_2}$ is in blue) and $u_3$ satisfies Hypothesis ($\mathcal{H}3$) ($\widehat{u_3}$ is in green).

**5.4.2 Remark.** In terms of quantum mechanics, the three previous hypotheses mean that the momentum of the initial state $u_0$ has to be localized outside the intervals given by the support of $\widehat{V}$.



 

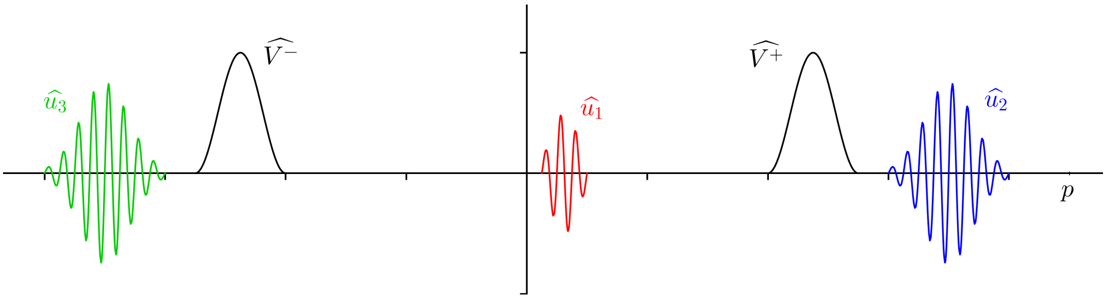


Figure 5.1: Illustration of the hypotheses ($\mathcal{H}1$),($\mathcal{H}2$) and ($\mathcal{H}3$)

If one of the three above hypotheses is satisfied, then the following lemma assures that a certain quadratic function, that will appear in Proposition 5.4.4, does not vanish on a domain related to the supports of $\widehat{u}_0$ and $\widehat{V}$. This result will be employed in the proof of this proposition. Let us not that, actually, the statement of the lemma can be formulated independently from $\widehat{u}_0$ and $\widehat{V}$, only the relations between the real numbers $a, b, p_1, p_2$ are employed.

**5.4.3 Lemma.** *Let $p_1 < p_2$ and $0 < a < b$ be four finite real numbers. Define the domains $D^-, D^+ \subset \mathbb{R}^2$ as follows*

$$D^- := [-b, -a] \times [p_1 - b, p_2 - a] \qquad , \qquad D^+ := [a, b] \times [p_1 + a, p_2 + b] \, .$$

*Let $q : D^- \cup D^+ \longrightarrow \mathbb{R}$ be the function defined by*

$$q(y,p) = (p-y)^2 - p^2 \, .$$

*If one of the three following hypotheses*

($\mathcal{H}1'$) $a - \frac{b}{2} > 0$ *and* $\frac{b}{2} - a < p_1 < p_2 < a - \frac{b}{2}$ ,

($\mathcal{H}2'$) $b < p_1$ ,

($\mathcal{H}3'$) $p_2 < -b$ ,

*is satisfied, then the function $q : D^- \cup D^+ \longrightarrow \mathbb{R}$ does not vanish on its domain.*

*Proof.* Clearly we have

$$q(y,p) = y\,(y - 2p) \, .$$

Since $0 < a \leqslant |y|$ by hypothesis, it suffices to prove that the factor $(y - 2p)$ does not vanish on $D^- \cup D^+$ when one of the three hypothese ($\mathcal{H}1'$), ($\mathcal{H}2'$) or ($\mathcal{H}3'$) is satisfied. To do so, we remark that for $(y,p) \in D^+$,

$$a - 2\,p_2 - 2\,b \;\leqslant\; y - 2p \;\leqslant\; b - 2\,p_1 - 2\,a \, ,$$

and for $(y,p) \in D^-$,

$$-b - 2\,p_2 + 2\,a \;\leqslant\; y - 2p \;\leqslant\; -a - 2\,p_1 + 2\,b \, .$$

Now let us divide the proof.



 

- *Case 1: $(\mathcal{H}1')$ is satisfied.* In this case, we have

$$\bullet \quad \frac{b}{2} - a < p_1 \quad \Longrightarrow \quad b - 2\,p_1 - 2\,a < 0 \;,$$
$$\bullet \quad p_2 < a - \frac{b}{2} \quad \Longrightarrow \quad 0 < -b - 2\,p_2 + 2\,a \;,$$

which proves that the factor $(y - 2p)$ does not vanish on $D^- \cup D^+$ in this case.

- *Case 2: $(\mathcal{H}2')$ is satisfied.* Since $a > 0$, we have

$$b < p_1 \quad \Longrightarrow \quad \left\{ \begin{array}{l} 0 > b - 2\,p_1 > b - 2\,p_1 - 2\,a \\ 0 > 2\,b - 2\,p_1 > -a - 2\,p_1 + 2\,b \end{array} \right. .$$

In this case, the factor $(y - 2p)$ is negative.

- *Case 3: $(\mathcal{H}3')$ is satisfied.* Since $a > 0$, we have

$$p_2 < -b \quad \Longrightarrow \quad \left\{ \begin{array}{l} 0 < -2\,p_2 - 2\,b < a - 2\,p_2 - 2\,b \\ 0 < -b - 2\,p_2 < -b - 2\,p_2 + 2\,a \end{array} \right. .$$

In this case, the factor $(y - 2p)$ is positive.

Lemma 5.4.3 is then proved. □

Thanks to the preceding lemma, we can expand the functions $W^{\pm}(t,.)$ with respect to time:

**5.4.4 Proposition.** *Suppose that $u_0$ and $V$ satisfy Condition $(\mathrm{C5}_{[p_1,p_2]})$ and Condition $(\mathcal{P}_{a,b})$ respectively. Moreover suppose that one of the three hypotheses $(\mathcal{H}1)$, $(\mathcal{H}2)$ or $(\mathcal{H}3)$ is verified and that $0 \notin [p_1,p_2]$. Let $t \geqslant 0$ and let $W_1^{\pm}, W_2^{\pm}(t,.) : \mathbb{R} \longrightarrow \mathbb{C}$ be the functions defined by*

$$\bullet \quad W_1^{\pm}(p) := \mp \int_{\pm a}^{\pm b} \frac{\widehat{V}(y)\, \widehat{u_0}(p-y)}{q(y,p)}\, dy \;,$$
$$\bullet \quad W_2^{\pm}(t,p) := \mp\, i \int_{\pm a}^{\pm b} \partial_y \left[ \frac{\widehat{V}(y)\, \widehat{u_0}(p-y)}{q(y,p)\, \partial_y q(y,p)} \right] e^{-it\, q(y,p)}\, dy \;,$$

*for all $p \in \mathbb{R}$. Note that for fixed $t \geqslant 0$, we have*

$$\bullet \quad supp\, W_1^- = supp\, W_2^-(t,.) \subseteq [p_1 - b, p_2 - a] \;,$$
$$\bullet \quad supp\, W_1^+ = supp\, W_2^+(t,.) \subseteq [p_1 + a, p_2 + b] \;.$$

*Then we have*

$$W^{\pm}(t,p) = W_1^{\pm}(p) + W_2^{\pm}(t,p)\, t^{-1} \;.$$



 

*Proof.* Let $t \geqslant 0$ and let us recall that for all $p \in \mathbb{R}$,

$$W^+(t,p) = \int_0^t \tilde{W}^+(s,p)\, e^{isp^2}\, ds \ ,$$

where

$$\tilde{W}^+(s,p) = \Big( \big( \chi_{[a,b]} \widehat{V} \big) * \big( e^{-is\, .^2}\, \widehat{u_0} \big) \Big)(p) = \int_a^b \widehat{V}(y)\, \widehat{u_0}(p-y)\, e^{-is(p-y)^2}\, dy \ .$$

Since $W^+(t,.)$ is equal to 0 outside $[p_1+a, p_2+b]$, we shall study this function on this interval only. Now let us remark that the function

$$(s,y) \longmapsto \widehat{V}(y)\, \widehat{u_0}(p-y)\, e^{-is(p-y)^2 + isp^2} \ ,$$

is continuous on $\mathbb{R} \times \mathbb{R}$ for any $p \in [p_1+a, p_2+b]$, so it is integrable on the compact domain $[0,t] \times [a,b]$. Using the definition of $\tilde{W}^+(s,p)$ and applying Fubini's theorem gives

$$\begin{aligned} W^+(t,p) &= \int_0^t \tilde{W}^+(s,p)\, e^{isp^2}\, ds \\ &= -i \int_0^t \left( \int_a^b \widehat{V}(y)\, \widehat{u_0}(p-y)\, e^{-is(p-y)^2}\, dy \right) e^{isp^2}\, ds \\ &= -i \int_a^b \widehat{V}(y)\, \widehat{u_0}(p-y) \left( \int_0^t e^{-is\left((p-y)^2 - p^2\right)}\, ds \right) dy \\ &= -i \int_a^b \widehat{V}(y)\, \widehat{u_0}(p-y) \left( \int_0^t e^{-is\, q(y,p)}\, ds \right) dy \ . \end{aligned}$$

The quantity $q(y,p) = (p-y)^2 - p^2$ is never equal to 0 if $y \in [a,b]$ and $p \in [p_1+a, p_2+b]$ according to Lemma 5.4.3, so we can integrate,

$$\int_0^t e^{-is\, q(y,p)}\, ds = \frac{i}{q(y,p)} \int_0^t \frac{\partial}{\partial s} \Big[ e^{-is\, q(y,p)} \Big] ds = \frac{i}{q(y,p)} \left( e^{-it\, q(y,p)} - 1 \right) .$$

Hence we obtain for $p \in [p_1+a, p_2+b]$,

$$\begin{aligned} W^+(t,p) &= \int_a^b \frac{\widehat{V}(y)\, \widehat{u_0}(p-y)}{q(y,p)} \left( e^{-it\, q(y,p)} - 1 \right) dy \\ &= \int_a^b \frac{\widehat{V}(y)\, \widehat{u_0}(p-y)}{q(y,p)}\, e^{-it\, q(y,p)} dy - \int_a^b \frac{\widehat{V}(y)\, \widehat{u_0}(p-y)}{q(y,p)}\, dy \\ &=: \tilde{W}_2^+(t,p) + W_1^+(p) \ . \end{aligned}$$

Now we remark that for fixed $p \in [p_1+a, p_2+b]$, the integrand of the integral defining $\tilde{W}_2^+(t,p)$ is actually equal to zero outside the set $D_p := \big\{ y \in \mathbb{R} \,\big|\, p - y \in [p_1, p_2] \big\}$ since the support of $\widehat{u_0}$ is contained in $[p_1, p_2]$. On the set $D_p$, the function $\partial_y q(y,p) = -2(p-y)$ does not vanish thanks to the hypothesis $0 \notin [p_1, p_2]$. Hence we can integrate by parts, providing

$$\begin{aligned} \tilde{W}_2^+(t,p) &= i \int_a^b \frac{\widehat{V}(y)\, \widehat{u_0}(p-y)}{q(y,p)\, \partial_y q(y,p)}\, \partial_y \Big[ e^{-it\, q(y,p)} \Big] dy\, t^{-1} \\ &= -i \int_a^b \partial_y \left[ \frac{\widehat{V}(y)\, \widehat{u_0}(p-y)}{q(y,p)\, \partial_y q(y,p)} \right] e^{-it\, q(y,p)}\, dy\, t^{-1} \\ &=: W_2^+(t,p)\, t^{-1} \ ; \end{aligned}$$



 

let us remark that the boundary terms in the second equality are equal to 0 because $\widehat{V}(a) = \widehat{V}(b) = 0$.
By changing just the integration interval with respect to $y$, we obtain the desired equality for $W^-(t,p)$. $\square$

From the two preceding propositions, we derive the following corollary:

**5.4.5 Corollary.** *Under the hypotheses and the notations of Proposition 5.4.1 and Proposition 5.4.4, we have for all* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\begin{aligned}\big(S_1^-(t)u_0\big)(x) &= \frac{1}{2\pi}\int_{p_1-b}^{p_2-a} W_1^-(p)\, e^{-itp^2+ixp}\, dp + \frac{1}{2\pi}\int_{p_1-b}^{p_2-a} W_2^-(t,p)\, e^{-itp^2+ixp}\, dp\, t^{-1} \\ &=: \big(S_{1,1}^-(t)u_0\big)(x) \;+\; \big(S_{1,2}^-(t)u_0\big)(x)\, t^{-1}\,,\end{aligned}$$

*and*

$$\begin{aligned}\big(S_1^+(t)u_0\big)(x) &= \frac{1}{2\pi}\int_{p_1+a}^{p_2+b} W_1^+(p)\, e^{-itp^2+ixp}\, dp + \frac{1}{2\pi}\int_{p_1+a}^{p_2+b} W_2^+(t,p)\, e^{-itp^2+ixp}\, dp\, t^{-1} \\ &=: \big(S_{1,1}^+(t)u_0\big)(x) \;+\; \big(S_{1,2}^+(t)u_0\big)(x)\, t^{-1}\,.\end{aligned}$$

*Proof.* It is sufficient to insert the expansions of the function $W^{\pm}(t,p)$ given in Proposition 5.4.4 into the definitions of the functions $S_1^{\pm}(t)u_0$ given in Proposition 5.4.1. $\square$

We are now in position to establish a similar result to Theorem 5.3.4 for the terms $S_1^{\pm}(t)u_0$. We start by furnishing an expansion to one term of $S_1^-(t)u_0$ in the space-time cone $\mathfrak{C}_S\big(p_1-b, p_2-a\big)$ and uniform estimates in its complement. To this end, we apply Theorem 5.3.3 to $\big(S_{1,1}^-(t)u_0\big)(x)$ following the lines of the proof of Theorem 5.3.4: this procedure gives an asymptotic expansion with respect to time such that the first term depends explicitly on $t$ and $x$, and the remainder is uniformly bounded with respect to $x$ in $\mathfrak{C}_S\big(p_1-b, p_2-a\big)$. Outside this cone, we employ Theorem 3.2.1 to provide uniform estimates of $\big(S_{1,1}^-(t)u_0\big)(x)$. Regarding the other term $\big(S_{1,2}^-(t)u_0\big)(x)\, t^{-1}$, we prove that it can be estimated by a constant multiplied by $t^{-1}$ in the whole space-time. Hence this term does not dominate the asymptotic behaviour of $\big(S_1^-(t)u_0\big)$.

**5.4.6 Theorem.** *Suppose that* $u_0$ *and* $V$ *satisfy Condition* $(\mathrm{C5}_{[p_1,p_2]})$ *and Condition* $(\mathcal{P}_{a,b})$ *respectively. Moreover suppose that one of the three assumptions* $(\mathcal{H}1)$, $(\mathcal{H}2)$ *or* $(\mathcal{H}3)$ *is satisfied and that* $0 \notin [p_1,p_2]$. *Then we have for all* $(t,x) \in \mathfrak{C}_S\big(p_1-b, p_2-a\big)$,

$$\Big|\big(S_1^-(t)u_0\big)(x) - H_1^-(t,x,u_0,V)\, t^{-\frac{1}{2}}\Big| \leqslant C_1^-(u_0,V,\delta)\, t^{-\delta} + C_2^-(u_0,V)\, t^{-1}\,,$$

*where* $\delta \in \big(\frac{1}{2}, 1\big)$ *and*

$$H_1^-(t,x,u_0,V) := \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, W_1^-\Big(\frac{x}{2t}\Big)\,.$$

*Moreover, if we choose two finite real numbers* $\tilde{p}_1 < \tilde{p}_2$ *such that* $[p_1,p_2] \subset (\tilde{p}_1,\tilde{p}_2) =: \tilde{I}$, *then we have for all* $(t,x) \in \mathfrak{C}_S\big(\tilde{p}_1-b, \tilde{p}_2-a\big)^c$,

$$\Big|\big(S_1^-(t)u_0\big)(x)\Big| \leqslant C_{1,\tilde{I}}^-(u_0,V)\, t^{-1}\,.$$



 

*The constants* $C_1^-(u_0,V,\delta), C_2^-(u_0,V), C_{1,\tilde{I}}^-(u_0,V) \geqslant 0$ *are given by* (5.22), (5.23), (5.24) *respectively.*

*Proof.* By Corollary 5.4.5, we have for all $t \geqslant 0$ and $x \in \mathbb{R}$,

$$\begin{aligned}\big(S_1^-(t)u_0\big)(x) &= \frac{1}{2\pi}\int_{p_1-b}^{p_2-a} W_1^-(p)\, e^{-itp^2+ixp}\, dp + \frac{1}{2\pi}\int_{p_1-b}^{p_2-a} W_2^-(t,p)\, e^{-itp^2+ixp}\, dp\, t^{-1} \\ &=: \big(S_{1,1}^-(t)u_0\big)(x) \;+\; \big(S_{1,2}^-(t)u_0\big)(x)\, t^{-1} \,.\end{aligned}$$

Following the lines of the proof of Theorem 5.3.4, we prove that for $(t,x)$ belonging to $\mathfrak{C}_S\big(p_1-b, p_2-a\big)$,

$$\Big|\big(S_{1,1}^-(t)u_0\big)(x) - H_1^-(t,x,u_0,V)\, t^{-\frac{1}{2}}\Big| \leqslant C_1^-(u_0,V,\delta)\, t^{-\delta} \,,$$

where $\delta \in \big(\frac{1}{2}, 1\big)$ and

$$C_1^-(u_0,V,\delta) := \frac{1}{\pi}\,\frac{L_{\gamma,2}}{2-2\delta}\,(p_2-p_1)^{2-2\delta}\,\left\| p \longmapsto \int_{-b}^{-a} \partial_p\left[\frac{\widehat{V}(y)\,\widehat{u_0}(p-y)}{q(y,p)}\right] dy \right\|_{L^\infty(\mathbb{R})} . \tag{5.22}$$

Moreover by a rough estimate, we have for all $t > 0$ and $x \in \mathbb{R}$,

$$\Big|\big(S_{1,2}^-(t)u_0\big)(x)\, t^{-1}\Big| \leqslant C_2^-(u_0,V)\, t^{-1} \,,$$

where

$$C_2^-(u_0,V) := \frac{1}{2\pi}\int_{p_1-b}^{p_2-a}\int_{-b}^{-a}\left|\partial_y\left[\frac{\widehat{V}(y)\,\widehat{u_0}(p-y)}{q(y,p)\,\partial_y q(y,p)}\right]\right| dy\, dp \,. \tag{5.23}$$

Therefore for all $(t,x) \in \mathfrak{C}_S\big(p_1-b, p_2-a\big)$, we obtain

$$\begin{aligned}\Big|\big(S_1^-(t)u_0\big)(x) - H_1^-(t,x,u_0,V)\, t^{-\frac{1}{2}}\Big| &\leqslant \Big|\big(S_{1,1}^-(t)u_0\big)(x) - H_1^-(t,x,u_0,V)\, t^{-\frac{1}{2}}\Big| \\ &\qquad + \Big|\big(S_{1,2}^-(t)u_0\big)(x)\, t^{-1}\Big| \\ &\leqslant C_1^-(u_0,V,\delta)\, t^{-\delta} + C_2^-(u_0,V)\, t^{-1} \,.\end{aligned}$$

As in the proof of Theorem 5.3.4, we apply Theorem 3.2.1 to the oscillatory integral giving the term $\big(S_{1,1}^-(t)u_0\big)(x)$ in the space-time regions $\big\{(t,x) \in \mathbb{R}_+^* \times \mathbb{R} \,\big|\, \frac{x}{t} \geqslant \tilde{p}_2 - a\big\}$ and $\big\{(t,x) \in \mathbb{R}_+^* \times \mathbb{R} \,\big|\, \frac{x}{t} \leqslant \tilde{p}_1 - b\big\}$. This furnishes the following estimate,

$$\forall\, (t,x) \in \mathfrak{C}_S\big(\tilde{p}_1-b, \tilde{p}_2-a\big)^c \qquad \Big|\big(S_{1,1}^-(t)u_0\big)(x)\Big| \leqslant \tilde{C}_{1,\tilde{I}}^-(u_0,V)\, t^{-1} \,,$$

where

$$\begin{aligned}\tilde{C}_{1,\tilde{I}}^-(u_0,V) := \frac{1}{4\pi}\left(\frac{1}{\tilde{p}_2-p_2} + \frac{1}{p_1-\tilde{p}_1}\right)\Bigg(3\left\| p \longmapsto \int_{-b}^{-a}\frac{\widehat{V}(y)\,\widehat{u_0}(p-y)}{q(y,p)}\, dy\right\|_{L^\infty(\mathbb{R})} \\ + \int_{p_1-b}^{p_2-a}\int_{-b}^{-a}\left|\partial_p\left[\frac{\widehat{V}(y)\,\widehat{u_0}(p-y)}{q(y,p)\,\partial_y q(y,p)}\right]\right| dy\, dp\Bigg) .\end{aligned}$$

Hence it follows

$$\begin{aligned}\Big|\big(S_1^-(t)u_0\big)(x)\Big| &\leqslant \Big|\big(S_{1,1}^-(t)u_0\big)(x)\Big| + \Big|\big(S_{1,2}^-(t)u_0\big)(x)\Big| \\ &\leqslant \tilde{C}_{1,\tilde{I}}^-(u_0,V)\,t^{-1} + C_2^-(u_0,V)\,t^{-1} \\ &=: C_{1,\tilde{I}}^-(u_0,V)\,t^{-1}\ ,\end{aligned}$$

where

$$\begin{aligned} C_{1,\tilde{I}}^-(u_0,V) &:= \tilde{C}_{1,\tilde{I}}^-(u_0,V) + C_2^-(u_0,V) \\ &= \frac{1}{4\pi}\left(\frac{1}{\tilde{p}_2-p_2}+\frac{1}{p_1-\tilde{p}_1}\right)\left(3\left\|p\longmapsto\int_{-b}^{-a}\frac{\widehat{V}(y)\,\widehat{u_0}(p-y)}{q(y,p)}\,dy\right\|_{L^\infty(\mathbb{R})}\right. \\ &\qquad \left.+\int_{p_1-b}^{p_2-a}\int_{-b}^{-a}\left|\partial_p\left[\frac{\widehat{V}(y)\,\widehat{u_0}(p-y)}{q(y,p)\,\partial_y q(y,p)}\right]\right|dy\,dp\right) \\ &\qquad\qquad + \frac{1}{2\pi}\int_{p_1-b}^{p_2-a}\int_{-b}^{-a}\left|\partial_y\left[\frac{\widehat{V}(y)\,\widehat{u_0}(p-y)}{q(y,p)\,\partial_y q(y,p)}\right]\right|dy\,dp\ . \end{aligned} \tag{5.24}$$

□

In the last result of this section, we provide a very similar result to Theorem 5.4.6 for the term $S_1^+(t)u_0$.

**5.4.7 Theorem.** *Suppose that the hypotheses of Theorem 5.4.6 are satisfied. Then we have for all* $(t,x)\in\mathfrak{C}_S\big(p_1+a,p_2+b\big)$,

$$\Big|\big(S_1^+(t)u_0\big)(x) - H_1^+(t,x,u_0)\,t^{-\frac{1}{2}}\Big| \leqslant C_1^+(u_0,V,\delta)\,t^{-\delta} + C_2^+(u_0,V)\,t^{-1}\ ,$$

*where* $\delta\in\big(\frac{1}{2},1\big)$ *and*

$$H_1^+(t,x,u_0) := \frac{1}{2\sqrt{\pi}}\,e^{-i\frac{\pi}{4}}\,e^{i\frac{x^2}{4t}}\,W_1^+\Big(\frac{x}{2t}\Big)\ .$$

*Moreover, if we choose two finite real numbers* $\tilde{p}_1<\tilde{p}_2$ *such that* $[p_1,p_2]\subset(\tilde{p}_1,\tilde{p}_2)=:\tilde{I}$, *then we have for all* $(t,x)\in\mathfrak{C}_S\big(\tilde{p}_1+a,\tilde{p}_2+b\big)^c$,

$$\Big|\big(S_1^+(t)u_0\big)(x)\Big| \leqslant C_{1,\tilde{I}}^+(u_0,V)\,t^{-1}\ .$$

*The constants* $C_1^+(u_0,V,\delta), C_2^+(u_0,V), C_{1,\tilde{I}}^+(u_0,V)\geqslant 0$ *are given by* (5.25), (5.26), (5.27) *respectively.*

*Proof.* To prove Theorem 5.4.7, it is sufficient to follow the lines of the proof of Theorem 5.4.6 and to change some domains of integration.



 

Let us give the expressions of the resulting constants:

- $$C_1^+(u_0,V,\delta) := \frac{1}{\pi}\,\frac{L_{\gamma,2}}{2-2\delta}\,(p_2-p_1)^{2-2\delta}\,\left\| p \longmapsto \int_a^b \partial_p \left[ \frac{\widehat{V}(y)\,\widehat{u_0}(p-y)}{q(y,p)} \right] dy \right\|_{L^\infty(\mathbb{R})}, \tag{5.25}$$

- $$C_2^+(u_0,V) := \frac{1}{2\pi} \int_{p_1+a}^{p_2+b} \int_a^b \left| \partial_y \left[ \frac{\widehat{V}(y)\,\widehat{u_0}(p-y)}{q(y,p)\,\partial_y q(y,p)} \right] \right| dy\,dp\,, \tag{5.26}$$

- $$\begin{aligned} C_{1,\tilde{I}}^+(u_0,V) := \frac{1}{4\pi}\left(\frac{1}{\tilde{p}_2-p_2}+\frac{1}{p_1-\tilde{p}_1}\right)\Bigg(3\left\| p \longmapsto \int_a^b \frac{\widehat{V}(y)\,\widehat{u_0}(p-y)}{q(y,p)}\,dy \right\|_{L^\infty(\mathbb{R})} \\ + \int_{p_1+a}^{p_2+b}\int_a^b \left| \partial_p \left[ \frac{\widehat{V}(y)\,\widehat{u_0}(p-y)}{q(y,p)\,\partial_y q(y,p)} \right] \right| dy\,dp \Bigg) \\ + \frac{1}{2\pi}\int_{p_1+a}^{p_2+b}\int_a^b \left| \partial_y \left[ \frac{\widehat{V}(y)\,\widehat{u_0}(p-y)}{q(y,p)\,\partial_y q(y,p)} \right] \right| dy\,dp\,. \end{aligned} \tag{5.27}$$

□

## 5.5 Interpretation of the results in terms of transmission and reflection

In this short section, we discuss the results of the two preceding sections by comparing the time-asymptotic behaviours of the terms $S_0(t)u_0$ and $S_1(t)u_0$ under different hypotheses.

Let us recall that the term $S_0(t)u_0$ of the series expansion of the solution given in Theorem 5.1.2 is the solution of the free Schrödinger equation. In Theorem 5.3.4, it is proved that this term tends to be localized in the space-time cone $\mathfrak{C}_S(p_1,p_2)$ when the time tends to infinity. As for the term $S_1(t)u_0$, it is the resulting wave packet from the first interaction of the free solution with the potential. We have proved in Corollary 5.4.5 that this term is actually the sum of the terms $S_1^-(t)u_0$ and $S_1^+(t)u_0$, which tend to be time-asymptotically concentrated in the cones $\mathfrak{C}_S(p_1-b,p_2-a)$ and $\mathfrak{C}_S(p_1+a,p_2+b)$ respectively, according to Theorems 5.4.6 and 5.4.7. Hence these three space-time cones, more precisely their inclinations, permit to describe the propagation of the above wave packets.

We recall also that the results of Section 5.4 have been obtained under one of the three hypotheses $(\mathcal{H}1)$, $(\mathcal{H}2)$ and $(\mathcal{H}3)$.

### Limited and positive initial speed

Assume that Hypothesis $(\mathcal{H}1)$ is satisfied and that $0 < p_1$. In terms of quantum mechanics, this means that the momentum (or speed) of the initial state $u_0$ is localized in a bounded interval containing only positive values and being bounded from above by the quantity $a - \frac{b}{2}$. An illustration of the three cones $\mathfrak{C}_S(p_1,p_2)$, $\mathfrak{C}_S(p_1-b,p_2-a)$ and



 

$\mathfrak{C}_S(p_1 + a, p_2 + b)$ under Hypothesis $(\mathcal{H}1)$ is given in Figure 5.2.

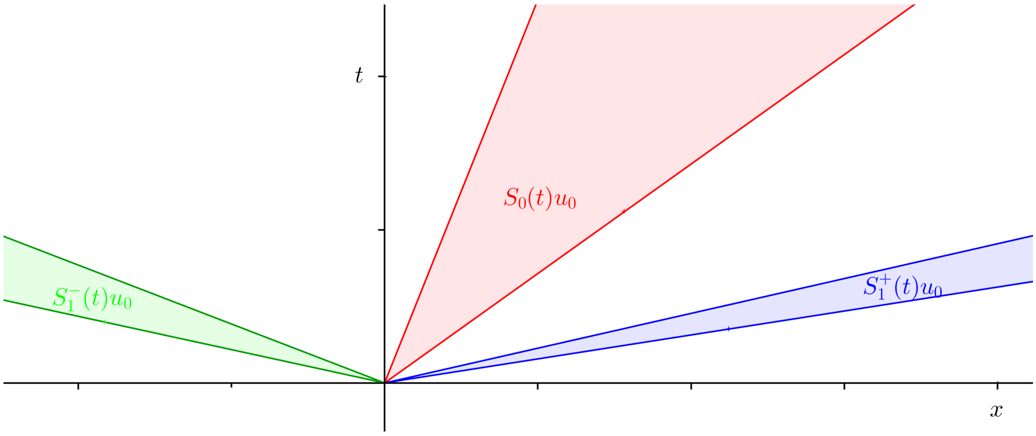


Figure 5.2: The different cones of propagation under Hypothesis $(\mathcal{H}1)$

Hypothesis $0 < p_1$ implies that the two half-lines delimiting the cone $\mathfrak{C}_S(p_1, p_2)$ are inclined to the right in space-time. It follows that the free wave packet $S_0(t)u_0$ tends to move to the right in space when the time tends to infinity. The same conclusion holds for the term $S_1^+(t)u_0$ since the cone $\mathfrak{C}_S(p_1 + a, p_2 + b)$ is also inclined to the right. When the time tends to infinity, we remark that the maximal speed of $S_1^+(t)u_0$, given by $2(p_2 + b)$, is larger than the maximal speed of $S_0(t)u_0$, given by $2p_2$. This means that $S_1^+(t)u_0$ moves more quickly to the right in space than $S_0(t)u_0$. On the other hand, the cone $\mathfrak{C}_S(p_1 - b, p_2 - a)$ is inclined to the left in space-time due to the hypothesis $p_2 < a - \frac{b}{2}$, which implies in particular that $p_2 - a < 0$. Hence the wave packet given by $S_1^-(t)u_0$ tends to move to the left in space when the time tends to infinity. Here the terms $S_1^-(t)u_0$ and $S_1^+(t)u_0$ can be interpreted respectively as the reflected part and the transmitted part of $S_1(t)u_0$.

## High positive initial speed

Assume Hypothesis $(\mathcal{H}2)$. Figure 5.3 provides an illustration of the three cones under this hypothesis.

For the same reasons as previously, the free wave packet $S_0(t)u_0$ and the term $S_1^+(t)u_0$ move time-asymptotically to the right in space. But Hypothesis $(\mathcal{H}2)$ implies that $p_1 - b > 0$ and thus, the cone $\mathfrak{C}_S(p_1 - b, p_2 - a)$ is inclined to the right in space-time too. This means that the wave packet given by $S_1^-(t)u_0$ moves also to the right in space when the time tends to infinity. Hence the term $S_1(t)$ has no time-asymptotic reflected part in the present case. This phenomenon is due to the fact that the speed of the initial wave packet is sufficiently large so that the first interaction of the free wave packet with the potential does not produce a reflection term. This was not the case in the preceding subsection since the initial speed was relatively small.
Now under Hypothesis $(\mathcal{H}2)$, the maximal speed of $S_1^+(t)u_0$ is still larger than the maximal speed of $S_0(t)u_0$. But the minimal speed of $S_1^-(t)u_0$, *i.e.* $2(p_1 - b)$, is smaller than the minimal speed of $S_0(t)u_0$, namely $2p_1$, showing that $S_1^-(t)u_0$ moves more slowly than



 

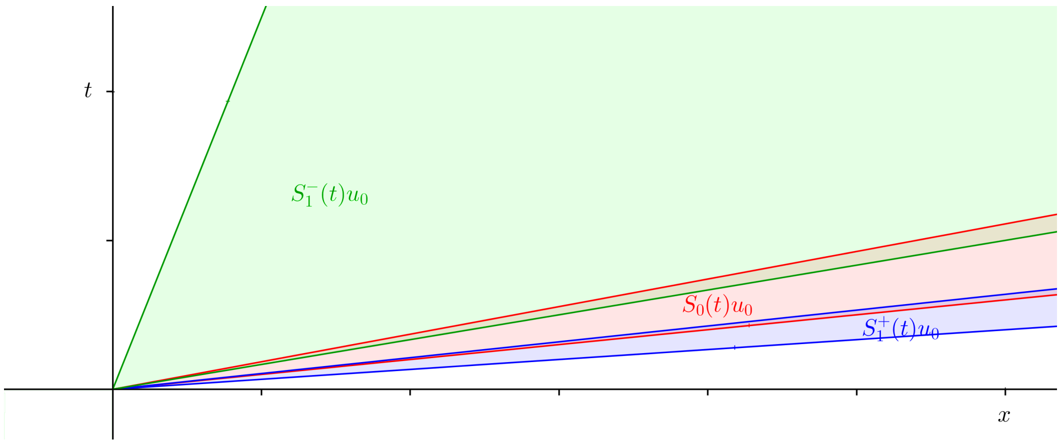


Figure 5.3: The different cones of propagation under Hypothesis ($\mathcal{H}2$)

the free wave packet. In the present case, $S_1^-(t)u_0$ and $S_1^+(t)u_0$ can be interpreted as the retarded transmission and advanced transmission respectively.

Up to now, we do not know if potentials in frequency bands are physically interpretable. Considering potentials which are not in frequency bands is in principle possible but studying the superposition of all the bands seems to be complicated. Further phenomena as retarded reflection have been observed in mathematics as well as in experimental and physical physics; we refer to the paper [7] for the mathematical results and the references provided in this paper for the physical results. However the connection to our calculations is not clear yet.

## 5.6 Some results from functional analysis

In this last section, we start by recalling some results from semigroup theory that we use in Section 5.1. They permit to define the notion of classical solution for the Schrödinger equation (5.1) and to prove existence and uniqueness of this solution. Then a result concerning the possibility of interchanging integration with the application of bounded operators is recalled.

Let us remark that the results of this section are not proved but quoted from the literature containing their proofs. Furthermore, in the present section, the operators $A, B, C$ and the semigroups $\big(T(t)\big)_{t\geqslant 0}$, $\big(S(t)\big)_{t\geqslant 0}$ are general and do not refer to the particular objects which are considered in the preceding sections of this chapter.

We start by recalling the notion of a classical solution of an abstract evolution equation (see [14, Chapter II, Definition 6.1]). If an operator generates a semigroup on a Banach space, then the classical solution of the evolution equation given by this operator exists, is unique and corresponds to the orbit of the initial value under the semigroup (see [14, Chapter II, Proposition 6.2]).



 

**5.6.1 Definition and Proposition.** *Consider the initial value problem*

$$\begin{cases} \dot{u}(t) = A\,u(t) \\ u(0) = v \end{cases}, \tag{5.28}$$

*for $t \geqslant 0$, where $A : D(A) \subset X \longrightarrow X$ is the generator of a semigroup $\big(T(t)\big)_{t\geqslant 0}$ on the Banach space $X$.*
*A function $u : \mathbb{R}_+ \longrightarrow X$ is called a classical solution of (5.28) if $u$ is continuously differentiable with respect to $X$, $u(t) \in D(A)$ for all $t \geqslant 0$, and $u$ satisfies (5.28).*
*If $v \in D(A)$ then the function*

$$u : t \in \mathbb{R}_+ \longmapsto u(t) = T(t)\,v\ ,$$

*is the unique classical solution of (5.28).*

In the following theorem, we recall that the sum of a generator of a semigroup on $X$ and a bounded operator on $X$ generates a semigroup as well (see [14, Chapter III, Bounded Perturbation Theorem 1.3]).

**5.6.2 Theorem.** *Let $\big(A, D(A)\big)$ be the generator of a strongly continuous semigroup on a Banach space $X$. If $B$ is a bounded operator from $X$ into itself, i.e. $B \in \mathcal{L}(X)$, then the operator $\big(C, D(C)\big) := \big(A + B, D(A)\big)$ generates a strongly continuous semigroup on $X$.*

A representation of the semigroup generated by the operator $\big(A+B, D(A)\big)$ as a series is recalled in the following theorem. It is called *Dyson-Phillips series* (see [14, Chapter III, Theorem 1.10]).

**5.6.3 Theorem.** *Let $\big(A, D(A)\big)$ be the generator of a strongly continuous semigroup $\big(T(t)\big)_{t\geqslant 0}$ on a Banach space $X$, and let $B \in \mathcal{L}(X)$. The strongly continuous semigroup $\big(S(t)\big)_{t\geqslant 0}$ generated by $\big(C, D(C)\big) := \big(A + B, D(A)\big)$ can be obtained as*

$$\lim_{N\to+\infty} \left\| S(t) - \sum_{n=0}^{N} S_n(t) \right\|_{\mathcal{L}(X)} = 0\ ,$$

*where $S_0(t) := T(t)$ and*

$$\forall\, v \in X \qquad S_{n+1}(t)v := \int_0^t S_n(t-s)\, B\, T(s)v\, ds\ .$$

In the final result, we recall that Bochner-type integration and the application of bounded operators can be interchanged (see [31, Chapter V, Section 5, Corollary 2]).

**5.6.4 Proposition.** *Let $A$ be a bounded operator acting between two Banach spaces $X$ and $Y$ and let $J \subseteq \mathbb{R}$ be an interval. If $F : J \longrightarrow X$ is a Bochner-integrable function, then $AF : J \longrightarrow Y$ is also a Bochner-integrable function and*

$$A\left(\int_J F(s)\, ds\right) = \int_J A\, F(s)\, ds\ .$$



 

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