HOMOTOPIE RATIONNELLE DES ESPACES D'INTERSECTION
RATIONAL HOMOTOPY THEORY OF INTERSECTION SPACES

MATHIEU KLIMCZAK

Thèse de Doctorat
Mathématiques
Ecole Doctorale Sciences Pour l'Ingénieur
Université des Sciences et Technologies, Lille
N° Ordre : 42070

Thèse présentée et soutenue publiquement le 29 Juin 2016 pour l'obtention du

DOCTORAT DE L'UNIVERSITÉ LILLE 1, SPÉCIALITÉ MATHÉMATIQUES PURES

en présence de :

DIRECTEUR DE THÈSE :

David Chataur — Université de Picardie

CODIRECTEUR DE THÈSE :

Patrick Popescu-Pampu — Université des Sciences et Technologies

RAPPORTEUR :

Markus Banagl — Universität Heidelberg

EXAMINATEURS :

Benoit Fresse — Université des Sciences et Technologies
Martin Saralegui — Université d'Artois
Yves Félix — Université Catholique de Louvain-la-Neuve
Ivan Marin — Université de Picardie

RAPPORTEUR ABSENT LORS DE LA SOUTENANCE :

Francisco Guillén — Universitat de Barcelona

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Mathieu Klimczak: *Rational Homotopy Theory of Intersection Spaces,* Homotopie Rationnelle des Espaces d'Intersection, ©

DIRECTEUR DE THÈSE:
David Chataur — Université de Picardie
CODIRECTEUR DE THÈSE:
Patrick Popescu-Pampu — Université des Sciences et Technologies
EXAMINATEURS:
Benoit Fresse — Université des Sciences et Technologies
Martin Saralegui — Université d'Artois
Yves Félix — Université Catholique de Louvain-la-Neuve
Ivan Marin — Université de Picardie
RAPPORTEURS:
Markus Banagl — Universität Heidelberg
Francisco Guillén — Universitat de Barcelona

LOCATION:
Lille

© 2016 Tous droits réservés.

Le fait est que les chameaux sont plus intelligents que les dauphins[...]. Ce chameau ci s'appelait Sale-Bête. Il était, de fait, le plus grand mathématicien du monde. Sale-Bête réfléchissait : on dirait qu'on a affaire à une instabilité dimensionnelle croissante, oscillant à première vue de zéro à quarante-cinq degrés. Soit ν égal à 3. Soit τ égal à $\chi/4$. Soit κ/y un ordre de tenseur différentiel à quatre coefficients de spin imaginaires...

Terry Pratchett - *Pyramides*

Dédié à mes parents, qui m'ont toujours soutenu.

© 2016 Tous droits réservés.

lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

© 2016 Tous droits réservés.

lilliad.univ-lille.fr

REMERCIEMENTS

Tout d'abord, je souhaiterais remercier mon directeur de thèse, David Chataur, qui m'a guidé et aidé tout au long de ces années de thèse et avant lors de mon mémoire de Master 2.

Merci aux rapporteurs, Francisco Guillen et Markus Banagl, personne à l'origine du concept d'espaces d'intersection. J'espère qu'il trouve cette thèse intéressante.

Un grand merci aux membres du jury qui sont présents aujourd'hui : Yves Félix, Benoit Fresse, Ivan Marin et Martin Saralegui.

Ces trois dernières années auraient été beaucoup moins intéressantes, et drôles, sans tous les membres de l'équipe de topologie de l'université de Lille et en particuliers les autres doctorants. Un grand merci à eux pour avoir fait en sorte que la thèse ne se résume pas simplement à la lecture d'articles en anglais derrière un bureau.

Je souhaiterais aussi remercier Patrick Popescu-Pampu, qui a eu la gentillesse d'accepter d'être mon codirecteur de thèse pour cette dernière année.

Pour finir, je souhaiterais évidemment remercier mes parents qui m'ont toujours soutenu et encouragé.

© 2016 Tous droits réservés.

lilliad.univ-lille.fr

© 2016 Tous droits réservés.

lilliad.univ-lille.fr

ABSTRACT

This thesis is concerned with the rational homotopy theory of intersection spaces. It is composed of three parts, each of them being more or less independent.

Unless stated otherwise, all homology and cohomology groups involved in this thesis have to been understood with rational coefficients. These coefficients won't be denoted unless if there is ambiguity, for instance if we consider homology or cohomology with coefficients in **Z** or **C**.

The first part concerns the notion of Poincaré duality associated to the intersection spaces $I^{\overline{p}}X$. When X is a compact connected oriented pseudo-manifold of dimension $n = 4s$ with only isolated singularities, we then have a well defined middle perversities intersection spaces

$$I^{\overline{m}}X = I^{\overline{n}}X$$

with a non degenerate symmetric intersection form

$$b_{HI}\colon\ HI^{2s}_{\overline{m}}(X) \times HI^{2s}_{\overline{m}}(X) \longrightarrow \mathbf{Q}.$$

This intersection form comes from a generalized Poincaré duality defined on intersection spaces, but is not defined as the evaluation of a cup product against a fundamental class. We construct rational Poincaré duality spaces $\mathcal{DP}(X)$ such that when $\dim X = 4s$ the Witt class of the intersection form $b_{\mathcal{DP}(X)}$ associated to $\mathcal{DP}(X)$ is the same that b_{HI} in the Witt group $W(\mathbf{Q})$. We also show how to construct Poincaré duality spaces $\mathcal{DP}(X)$ when $n = 2s+1$.

The second part develop the notion of Lagrangian intersection spaces introduced in the first part to construct $\mathcal{DP}(X)$ when $\dim X = 2s+1$. We show that the rational homology of these spaces lies between the two middle perversities intersection homology $IH^{\overline{m}}_{*}(X)$ and $IH^{\overline{n}}_{*}(X)$ in a sense that we call a $(s+1, s)$-bireflective diagram. In a second section we extend the notion of homology truncation to nilpotent rational spaces of finite type.

The last part is devoted to the interaction between Hodge theory and the rational cohomology of intersection spaces when X is a complex projective algebraic varieties with only isolated singularities. We show that theses spaces carry a natural mixed Hodge structure at the algebraic models level. We then use these mixed Hodge structures to derive results about the formality of intersection spaces.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

© 2016 Tous droits réservés.
lilliad.univ-lille.fr

RÉSUMÉ

Cette thèse se concentre sur l'homotopie rationnelle des espaces d'intersection, espaces définis et développés par M. Banagl dans [6]. Le présent manuscrit se décompose en trois chapitres, chacun étant plus ou moins indépendants des autres.

Le premier chapitre traite de la dualité de Poincaré associée aux espaces d'intersection. Étant donnée X une pseudovariété compacte, connexe et orientée de dimension $n = 4s$ à singularités isolées, les espaces d'intersections associés aux deux perversités milieux coïncident

$$I^{\overline{m}}X = I^{\overline{n}}X.$$

Cela nous permet de définir une forme d'intersection bilinéaire symétrique non dégénérée

$$b_{HI}\colon\ HI^{2s}_{\overline{m}}(X) \times HI^{2s}_{\overline{m}}(X) \longrightarrow \mathbf{Q}$$

provenant d'une dualité de Poincaré généralisée définie sur l'homologie rationnelle des espaces d'intersection. Cette dualité ne provient pas de l'évaluation d'un cup produit contre une classe fondamentale. En utilisant le formalisme des espaces d'intersection nous montrons, dans le cas de la dimension paire, qu'il est possible de construire un espace à dualité de Poincaré rationnelle $\mathcal{DP}(X)$. Lorsque $\dim X = 4s$ la classe de Witt associée à la forme d'intersection $b_{\mathcal{DP}(X)}$, définie via dualité de Poincaré, est la même que la classe Witt de b_{HI} dans le groupe $W(\mathbf{Q})$. Nous montrons aussi comment construire de tels espaces $\mathcal{DP}(X)$ de le cas d'une dimension impaire.

Le second chapitre développe la notion d'espace d'intersection lagrangien, notion introduite dans le premier chapitre pour construire $\mathcal{DP}(X)$ lorsque $\dim X = 2s+1$. Nous montrons que l'homologie rationnelle de ces espaces interagit avec les homologies d'intersection milieu $IH^{\overline{m}}_{*}(X)$ et $IH^{\overline{n}}_{*}(X)$ au travers d'un diagramme commutatif que nous appelons un diagramme $(s+1, s)$-biréflexif. Dans une seconde partie, nous étendons la notion de troncation homologique utilisée pour définir les espaces d'intersection au cas des espaces rationnels nilpotents de type fini.

Pour finir, le troisième chapitre étudie l'interaction entre la théorie de Hodge mixte et la cohomologie rationnelle des espaces d'intersection pour X une variété algébrique projective complexe à singularités isolées. Nous montrons que la cohomologie de ces espaces d'intersection possède de façon naturelle une structure de Hodge mixte définie au niveau des modèles rationnels. Ces structures de Hodge mixte nous permettent alors de déduire des résultats sur la formalité des espaces d'intersection.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

© 2016 Tous droits réservés.
lilliad.univ-lille.fr

INTRODUCTION

GENERAL SETTING : THE PROBLEM OF SINGULAR SPACES AND ANSWERS TO THIS PROBLEM

"Pour une variété fermée, les nombres de Betti également distants des extrêmes sont égaux."

With that foundational theorem, that Poincaré stated in [38] and corrected in [39], began algebraic topology.

Nowadays it is stated in terms of homology and cohomology. Homology and cohomology assign in a natural way to each manifold M of dimension n graded abelian groups

$$\begin{cases} M \longmapsto H_*(M;\mathbf{Z}), \\ M \longmapsto H^*(M;\mathbf{Z}). \end{cases}$$

Then, when M is an oriented closed manifold, this theorem states that we have and isomorphism between the integral homology and cohomology

$$H^*(M;\mathbf{Z}) \cong H_{n-*}(M;\mathbf{Z}).$$

This isomorphism, given a choice of orientation of M, is defined by the cap product with the fundamental class of M

$$-\cap [M] \colon H^*(M;\mathbf{Z}) \xrightarrow{\cong} H_{n-*}(M;\mathbf{Z}).$$

When passing to rational coefficients, the universal coefficients theorem states that

$$H^i(M;\mathbf{Q}) \cong \mathrm{Hom}(H_i(M;\mathbf{Q}),\mathbf{Q}).$$

When M is of dimension $n = 4s$, Poincaré duality defines a non degenerate symmetric bilinear form

$$H_{2s}(M) \times H_{2s}(M) \longrightarrow \mathbf{Q},$$

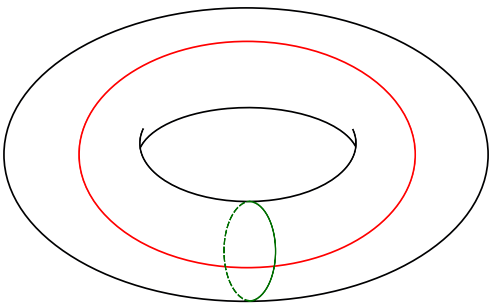


Figure 1: Poincaré duality on the torus. The red and green cycles are dual to each other.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

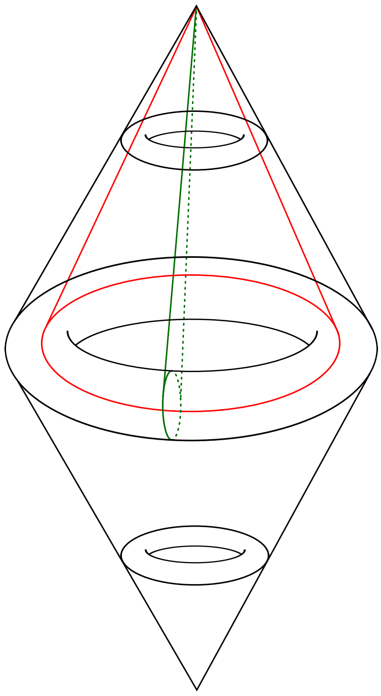


Figure 2: Failure of Poincaré duality in the suspended torus. The red and green cycles become boundaries.

for which we can computes its signature. It is a theorem of Thom, [46], that this signature is

- a cobordism invariant,
- multiplicative under cartesian product,
- additive under disjoint union.

As cohomology and homology are defined for any topological spaces, one can ask whether the Poincaré duality holds for general spaces. Poincaré already answered this question in [38, p. 232] by giving a counterexample to his isomorphism. The historical and prototypical example that Poincaré gave was the suspension of the torus, denoted by ΣT. That topological space is a manifold at every points unless for the two suspension points and this is already enough to break Poincaré duality. A classical Mayer-Vietoris long exact sequence gives the following rational homology groups

$$H_*(\Sigma T) \cong \begin{cases} \mathbf{Q} & * = 0, \\ 0 & * = 1, \\ \mathbf{Q} \oplus \mathbf{Q} & * = 2, \\ \mathbf{Q} & * = 3, \end{cases}$$

this computation show that ΣT does not satisfies Poincaré duality.

Characteristic numbers of manifolds are intimately related to Poincaré duality. In the 1960's and the 1970's some characteristic classes were ex-

© 2016 Tous droits réservés.

lilliad.univ-lille.fr

tended to the realm of singular spaces. Motivated by applications to surgery theory, D. Sullivan asked whether it is possible to define a signature for a new homology theory for singular spaces. This new homology theory was to restore Poincaré duality, and one would be able to determine families of spaces together with a signature that is a bordism invariant.

Actually, there are two strategies to restore Poincaré duality for singular spaces:

1. The "algebraic" answer : we change the definition of our homology. This is the solution followed by Mark Goresky and Robert MacPherson when defining intersection homology groups.
2. The "topological" answer : we try to modify our class of spaces or the spaces themselves. This is the solution followed by Markus Banagl when defining intersection spaces.

Intersection homology theory was defined by Mark Goresky and Robert MacPherson in [26] and [27], first with the use of simplicial techniques and then using sheaf theory. Intersection homology with rational coefficients assign to each singular space (they use stratified pseudomanifold) a family of graded vector spaces

$$X \rightsquigarrow IH_*^{\overline{p}}(X).$$

These vector spaces depend of a multi index $\overline{p}$ which is called a perversity and these perversities form a finite poset $\mathcal{P}_n$ with a unique maximal element $\bar{t}$. Then, when X is a compact, connected, oriented pseudomanifold and $\overline{p}$ and $\overline{q}$ two complementary perversities, meaning $\overline{p}+\overline{q}$ is defined in $\mathcal{P}_n$ and equal to $\bar{t}$, we have a generalized Poincaré duality isomorphism

$$IH_i^{\overline{p}}(X) \cong IH^{n-i}_{\overline{q}}(X)$$

with $IH^{n-i}_{\overline{q}}(X) = IH^{\overline{q}}_{n-i}(X)^\vee = \hom(IH^{\overline{q}}_{n-i}(X), \mathbf{Q})$. In particular, when X is of dimension $n = 4s$ and stratified by only even dimensional strata, the middle perversity $\overline{m}$ is self-complimentary, $\overline{m}+\overline{m} = \bar{t}$. This implies that we have a well defined non degenerate symmetric bilinear form

$$IH^{\overline{m}}_{2s}(X) \times IH^{\overline{m}}_{2s}(X) \longrightarrow \mathbf{Q}.$$

The signature of this bilinear form is also a cobordism invariant. This solved the problem Sullivan asked in the 1970's.

Intersection spaces were defined by Markus Banagl in [6] as a way to implement Poincaré duality at the level of topological spaces.

Suppose given a stratified pseudomanifold X of dimension n with only isolated singularities. We also suppose the links of the singularities are simply connected. We assign to such a space a family of topological spaces

$$X \rightsquigarrow I^{\overline{p}}X,$$

called its intersection spaces, also indexed by perversities $\overline{p} \in \mathcal{P}_n$. Let us mention here that in the isolated singularity case a perversity is just an integer $0 \leqslant \overline{p} \leqslant n-2$.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

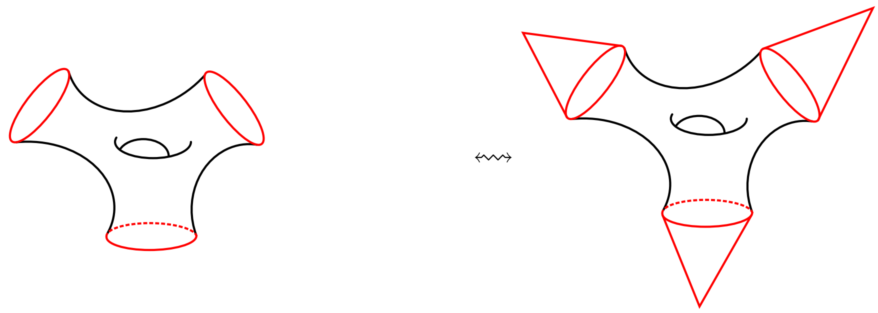


Figure 3: Construction on a pseudomanifold with isolated singularities by coning off the boundaries of the left manifold.

Before explaining the construction of the intersection spaces, let us recall how to construct stratified spaces with only isolated singularities.

Consider X a stratified pseudomanifold with isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$. Removing an open neighbourhood of each singularities leaves us with a manifold with boundary $(X_{reg}, \partial X_{reg})$ where the boundary

$$\partial X_{reg} = \bigsqcup_{\sigma_i \in \Sigma} L_i$$

is the disjoint union of the links L_i for each singularities $\sigma_i \in \Sigma$.

Conversely, assume that is given a manifold with boundary $(M, \partial M)$ such that the boundary ∂M is the union of m connected components

$$\partial M = \bigsqcup_{i=1}^{m} \partial_i M.$$

Let $(I(1), \dots, I(\ell))$ be a partition of m and consider

$$\partial_{I(k)} M := \bigsqcup_{\sum_{r=1}^{k-1} I(r) < j \leqslant \sum_{r=1}^{k} I(r)} \partial_j M$$

with $k \geqslant 1$ and $I(0) = 0$. The space

$$X := M \bigcup_{\partial M} \left(\bigsqcup_{k=1}^{\ell} c(\partial_{I(k)} M) \right)$$

formed by glueing the cones $c(\partial_{I(k)} M)$ onto the boundary is then a stratified pseudomanifold with ℓ isolated singularities. The manifold M becomes then the regular part X_{reg} of X. An example is shown in Figure 3.

The process to construct intersection spaces is similar in the sense that the $I^{\overline{p}}X$ are constructed by conning off only a part of the boundary.

Suppose that is given X a stratified pseudomanifold of dimension n with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ and a perversity $0 \leqslant \overline{p} \leqslant n-2$. Suppose also the links L_i are simply connected.

Consider the associated manifold with boundary $(X_{reg}, \partial X_{reg})$, we have $\partial X_{reg} = \sqcup_{\sigma_i} L_i$. Spatial homology truncation theory (originally defined in

© 2016 Tous droits réservés. lilliad.univ-lille.fr

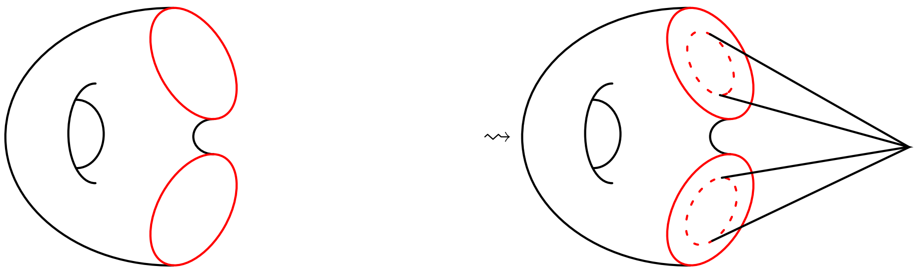


Figure 4: Contruction of an intersection space by coning off only a part of the boundaries.

[6]) provides, for each L_i and for an integer $k(\overline{p}) := n-1-\overline{p}$, a CW-complex $t_{k(\overline{p})}L_i$ together with a comparison map

$$f_i\colon t_{k(\overline{p})}L_i \longrightarrow L_i$$

such that the induced map in homology $H_r(f;\mathbf{Z})$ is an isomorphism for $r < k(\overline{p})$ and the zero map otherwise. The intersection space $I^{\overline{p}}X$ is then defined as the mapping cone of the compostion

$$\bigsqcup_{\sigma_i} t_{k(\overline{p})}L_i \xrightarrow{\sqcup_{\sigma_i} f_i} \bigsqcup_{\sigma_i} L_i \longrightarrow X_{reg}.$$

The ordinary homology of these spaces satisfies generalized Poincaré duality when X is a compact, connected, oriented pseudomanifold of dimension n with only isolated singularities and simply connected links. By analogy with intersection homology we denote $\widetilde{HI}^{\overline{p}}_*(X) := \widetilde{H}_*(I^{\overline{p}}X)$ and $\widetilde{HI}^*_{\overline{p}}(X) := \widetilde{H}^*(I^{\overline{p}}X)$. We have the following isomorphism

$$\widetilde{HI}^{\overline{p}}_i(X) \cong \widetilde{HI}^{n-i}_{\overline{q}}(X).$$

where $\overline{p}$ and $\overline{q}$ are complementary perversities.

The homology of the intersection spaces $I^{\overline{p}}X$ is not isomorphic to the intersection homology of X. Intersection spaces can in fact be understood as an enrichment of intersection homology theory and we can recover information on intersection homology by studying the intersection spaces. In particular, when X is a stratified pseudomanifold of dimension $n = 4s$ with isolated singularities, the cohomology of the middle perversity intersection space $I^{\overline{m}}X$ defines a non degenerate symmetric bilinear form

$$\widetilde{HI}^{2s}_{\overline{m}}(X) \times \widetilde{HI}^{2s}_{\overline{m}}(X) \longrightarrow \mathbf{Q}$$

and both this bilinear form and the form induced by intersection cohomology share the same Witt element in the Witt group $W(\mathbf{Q})$.

Of the problem of restoring Poincaré duality on singular spaces, this thesis focuses on the second solution : the intersection spaces, from the point of view of rational homotopy theory. Let us first briefly recall the notion of rational homotopy theory. The rational homotopy type of a topological space X is given by the commutative differential graded algebra $A_{PL}(X)$ in

© 2016 Tous droits réservés. lilliad.univ-lille.fr

the homotopy category $\mathsf{Ho}(\mathsf{CDGA}_{\mathbf{Q}})$ defined by formally inverting quasi-isomorphisms and where $A_{PL}(-)\colon \mathsf{Top} \to \mathsf{CDGA}_{\mathbf{Q}}$ is the polynomial De Rham functor defined by Sullivan.

Rational homotopy theory provides a powerful tool since it allows us to work in a completely algebraic way. The most important result being the fact that one can completely classify connected nilpotent rational topological space of finite type. In fact to any rational homotopy type belongs a unique isomorphism class of cdga's. To be more precise, define by $\mathsf{CW}^0_{\mathrm{rat,ft}}$ the full subcategory of connected rational nilpotent CW-complexes of finite type and by $\mathsf{SulAlg}^0_{\mathrm{ft}}$ the full subcategory of connected Sullivan algebras of finite type. Passing to homotopy categories we have an equivalence of categories [45]

$$\mathsf{Ho}(\mathsf{CW}^0_{\mathrm{rat,ft}}) \rightleftarrows \mathsf{Ho}(\mathsf{SulAlg}^0_{\mathrm{ft}})^{op}.$$

Poincaré duality and intersection spaces

Intersection spaces theory provide, when $\dim X = n = 2s$, a topological space $I^{\overline{m}}X$ such that its homology satisfies generalized Poincaré duality.

$$\widetilde{H}I^k_{\overline{m}}(X) \cong \widetilde{H}I^{\overline{m}}_{2s-k}(X).$$

The natural question which comes to mind is : "Is it a manifold ? If not, how close it is from a manifold ?"

Note we gave the generalized Poincaré duality of intersection spaces with *reduced* homology. This is because the intersection spaces $I^{\overline{p}}X$ do not carry a fundamental class for any perversity $\overline{p}$. Thus for the middle perversity $\overline{m}$ the Poincaré duality isomorphism

$$\widetilde{H}I^{\overline{m}}_i(X) \cong \widetilde{H}I^{n-i}_{\overline{m}}(X).$$

cannot be described as a cap product with a fundamental class. Therefore the $I^{\overline{p}}X$ are not manifolds.

In this thesis we modify the preceding construction in order to get a Poincaré duality space. That is a topological space Y, which is not necessarily a manifold, but whose rational homology satisfies Poincaré duality. Moreover, this duality isomorphism is given by the cap product with an orientation class [Y]

$$-\cap [Y] : H^r(Y;\mathbf{Q}) \longrightarrow H_{n-r}(Y;\mathbf{Q}).$$

Let X be a compact, connected and oriented stratified pseudomanifold X of dimension n with only isolated singularities and simply connected links. We can associate to X a rational Poincaré duality space $\mathcal{DP}(X)$ such that $\mathcal{DP}(X)$ lies between the regular part X_{reg} of X and its normalisation $\overline{X}$. We call $\mathcal{DP}(X)$ a Poincaré duality approximation space of X, see the definition 1.1.0.1. Moreover, when $\dim X \equiv 0 \bmod 4$ the Witt element of the intersection form of $\mathcal{DP}(X)$ is the same as the one from $I^{\overline{m}}X$. Being more precise, we have the following theorem.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Theorem 0.0.1 (Multiple isolated singularities case)**.** *Let X be a compact, connected oriented pseudomanifold of dimension n with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu; \nu > 1\}$ of links L_i simply connected. Then,*

1. *If $n = 2s$, there exists a rational Poincaré approximation $\mathcal{DP}(X)$ of X. Moreover, if $\dim X \equiv 0 \mod 4$, then the Witt class associated to the intersection form $b_{\mathcal{DP}(X)}$ is the same as the Witt class associated to the middle intersection cohomology of X.*
2. *If $n = 2s + 1$ and X is either a Witt space or an L-space then there exists a rational Poincaré approximation $\mathcal{DP}(X)$ of X.*

What this theorem means is that, for instance in the case where $n = 2s$, the intersection space $I^{\overline{m}}X$ can be "completed" into a rational Poincaré duality space by glueing a single $2s$-dimensional cell e^{2s}. Moreover, when X is simply connected, the rational homotopy type of this new space $\mathcal{DP}(X) := I^{\overline{m}}X \cup e^{2s}$ is determined by $I^{\overline{m}}X$.

MIXED HODGE STRUCTURES

One of the other points investigated here is the study of the rational cohomology algebra $HI^*_{\overline{p}}(X)$ of the $I^{\overline{p}}X$ when X is a complex projective algebraic variety with only isolated singularities. The rational cohomology of X already bears interesting information since we know it can be endowed with a mixed Hodge structure and that this mixed Hodge structure is functorial with respect to algebraic morphisms, [17]. In fact the mixed Hodge structure is defined at the algebraic models level, meaning that the cdga $A_{PL}(X)$ can be endowed with two filtrations $(A_{PL}(X), W, F)$ such that the cohomology of the triple $(A_{PL}(X), W, F)$ induces a mixed Hodge structure on $H^*(X)$.

This notion of mixed Hodge structure can then be used to derive results about the rational homotopy type of X, like its formality. A space X is formal if there is a string of quasi-isomorphisms from the cdga $A_{PL}(X)$ to its cohomology with rational coefficients $H^*(A_{PL}(X)) \cong H^*(X)$ seen as a cdga with trivial differential. In particular if X is simply connected and formal then its rational homotopy type is a formal consequence of its cohomology algebra, its higher order Massey products vanish and the rational homotopy groups of X, $\pi_*(X) \otimes \mathbf{Q}$ can be computed in a purely algebraic way directly from $H^*(X)$. One of the landmark result combining the two notions of rational homotopy theory and Hodge structures is the result of Deligne, Griffiths, Morgan and Sullivan [18] that compact Kähler manifolds, in particular smooth projective varieties, are formal topological spaces.

The notion of Hodge structure dates back to the result of William Hodge [31] on compact Kähler manifolds. A complex form α has type (p, q) if for any local system of holomorphic coordinates $(z_1, \dots, z_n)$, the form α is a linear combination of forms $g \cdot dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$ with g

© 2016 Tous droits réservés. lilliad.univ-lille.fr

a differentiable function. Suppose given X a compact Kähler manifold of complex dimension n. For all m, we have the following decomposition

$$H^m(X;\mathbf{C}) = \bigoplus_{p+q=m} H^{p,q}(X)$$

where $H^{p,q}(X)$ is the subspace of cohomology classes whose harmonic representative is of type (p,q). The space $H^{p,q}(X)$ is the complex conjugate of $H^{q,p}(X)$. This modelization by a direct sum decomposition leads to the more general definition of a Hodge structure of weight k with the use of a decreasing filtration. Let H be a $\mathbf{Z}$-module of finite rank and a decreasing filtration

$$H \otimes \mathbf{C} \supset \cdots \supset F^p \supset F^{p+1} \supset \cdots$$

such that $F^p \cap \overline{F^q} = 0$ whenever $p+q = k+1$, then H has a (pure) Hodge structure of weight k if one has

$$H \otimes \mathbf{C} = \bigoplus_{p+q=k} F^p \cap F^q.$$

The existence of a pure Hodge structure on a cohomology group puts restrictions on its rank. For instance, if X a compact Kähler manifold we have

$$H^1(X;\mathbf{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$$

and $H^1(X;\mathbf{C})$ is then of even rank. For a non-compact or singular complex algebraic variety the above decomposition doesn't work, for instance $H^1(X)$ can have odd rank. This led Deligne to generalize the Hodge structure notion into the notion of mixed Hodge structure [16], [17]. The idea is that on each cohomology groups $H^k(X)$ there is a increasing filtration W_*, the weight filtration, such that the m-th graded quotient has a pure Hodge structure of weight m.

Morgan [36] endowed homotopy groups of smooth algebraic varieties with mixed Hodge structures through the notion of mixed Hodge diagram. He not only put a mixed Hodge structure on the higher homotopy groups of a complex algebraic manifold X, he showed that the minimal model of the Sullivan algebra $A_{PL}(X)$ also has a mixed Hodge structure. Navarro Aznar then extended the results of Morgan [4] to possibly singular complex algebraic varieties. Independently, Hain also extended the results of Morgan with the use of Chen's iterated integrals [28] [29].

Given a cdga (A,d) with an increasing bounded filtration $\{W_*A\}$ one can consider the associated spectral sequence $E(A,W)$ with

$$E_1^{r,s}(A,W) := H^{r+s}(\mathrm{gr}_r^W(A^{r+s})).$$

In the case of the weight filtration on the rational models of a complex projective variety X that spectral sequence is called the weight spectral sequence and is denoted by $E_1(X,W)$. Cirici and Guillen proved [15] that complex algebraic varieties are E_1-formal. This means that the rational

© 2016 Tous droits réservés.

lilliad.univ-lille.fr

homotopy type of complex algebraic varieties is determined by the E_1-term of the weight spectral sequence. In other words, there is a string of quasi-isomorphisms

$$A_{PL}(X) \longrightarrow * \longleftarrow E_1(X, W).$$

Let us consider the family $\{I^{\overline{p}}X\}_{\overline{p}\in\mathcal{P}_n}$ of intersection spaces associated to the space X. Each one of them admits a rational model which we denote by $AI_{\overline{p}}(X)$, in fact this family can be considered as a functor

$$AI_{\overline{\bullet}}(X)\colon \mathcal{P}_n^{op} \longrightarrow \mathsf{CDGA}_{\mathbf{Q}}$$

which we call a coperverse cdga, see definition 3.2.2.1. These coperverse cdga's enjoy an extended product

$$AI_{\overline{p}}(X) \otimes AI_{\overline{q}}(X) \longrightarrow AI_{\overline{q}}(X) \quad \overline{p} \leqslant \overline{q} \text{ in } \mathcal{P}_n^{op}.$$

When $\overline{p} = \overline{q}$, this is the product that each one of the cdga $AI_{\overline{p}}(X)$ naturally has as rational model of a topological space. Moreover when $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$, that is, when X is a complex projective variety with only isolated singularities and simply connected links, the coperverse cdga $AI_{\overline{\bullet}}(X)$ can be endowed with two filtrations. One increasing filtration $W^{\overline{\bullet}}$ on the rational algebra $AI_{\overline{\bullet}}(X)$, its weight filtration, and one decreasing filtration $F_{\overline{\bullet}}$ on its complexification $AI_{\overline{\bullet}}(X) \otimes \mathbf{C}$, its Hodge filtration, in a way such that the triple

$$(AI_{\overline{\bullet}}(X), W^{\overline{\bullet}}, F_{\overline{\bullet}})$$

is a coperverse mixed Hodge cdga. This means that for every perversity $\overline{p}$, we have a bifiltered cdga

$$(AI_{\overline{p}}(X), W^{\overline{p}}, F_{\overline{p}})$$

such that the two filtrations define a mixed Hodge structure on $AI_{\overline{p}}(X)$ compatible with the poset maps $AI_{\overline{p}}(X) \to AI_{\overline{q}}(X)$.

The work of Chataur and Cirici [12] made for studying the interactions between mixed hodge structures and the intersection cohomology of a complex projective variety with only isolated singularities can be modified to suit the study of mixed hodge structures on $HI^*_{\overline{\bullet}}(X)$. This allow us to get the following theorem, which was stated in [12, theorem 3.10] for intersection cohomology. In a more precise way we have the following theorem.

Theorem 0.0.2. *Let $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$ of complex dimension n. There exists a coperverse mixed Hodge cdga $MI_{\overline{\bullet}}(X)$ together with a string of quasi-isomorphisms*

$$MI_{\overline{\bullet}}(X) \leftarrow * \rightarrow AI_{\overline{\bullet}}(X)$$

such that this mixed Hodge structure is compatible with the one on X_{reg} and the one on X.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

A more precise statement is given is the theorem 3.4.1.

It is interesting to remark here that the intersection spaces $I^{\bar{\bullet}}X$ aren't complex algebraic varieties. The fact that their cohomology carries a mixed Hodge structure isn't clear at all at first glance.

Next we study the weight spectral sequence associated to $(MI_{\bar{\bullet}}(X), W)$ which we denote by $EI_{\bar{\bullet}}(X)$. We then use this spectral sequence to derive results about the formality of the intersection spaces $I^{\bar{p}}X$. The key ingredient to obtain results about formality is the following theorem about what we call the $EI_{1,\bar{\bullet}}$-formality of complex projective algebraic varieties with isolated singularities.

Theorem 0.0.3. *Let $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$ with only isolated singularities. There is a string of quasi-isomorphisms of coperverse cdga's from $MI_{\bar{\bullet}}(X) \otimes \mathbf{C}$ to $EI_{1,\bar{\bullet}}(X) \otimes \mathbf{C}$. In particular, there is an isomorphism in $\mathsf{Ho}(\widehat{\mathcal{P}_n}^{op}\mathsf{CDGA}_{\mathbf{C}})$ from $AI_{\bar{\bullet}}(X) \otimes \mathbf{C}$ to $EI_{1,\bar{\bullet}}(X) \otimes \mathbf{C}$.*

One remarkable fact is that a similar theorem is also true in the case of intersection cohomology [12, theorem 3.12]. This shows the strong resemblance between intersection cohomology and the cohomology of intersection spaces when we consider complex projective varieties. This result allows us to state the theorem 3.5.4 of the type "purity implies formality" and the theorem 3.6.1 which in particular says that the intersection spaces $I^{\bar{p}}X$ of any nodal hypersurfaces X in $\mathbf{C}P^4$ are formal.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

CONTENTS

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Part I

POINCARÉ DUALITY AND SINGULAR SPACES

In this part we assign, under reasonable hypothesis, to each pseudomanifold with isolated singularities a rational Poincaré duality space. These spaces are constructed with the formalism of intersection spaces defined by Markus Banagl and are indeed related to them in the even dimensional case.

© 2016 Tous droits réservés.

© 2016 Tous droits réservés.
lilliad.univ-lille.fr

1
POINCARÉ DUALITY FOR SPACES WITH ISOLATED SINGULARITIES

© 2016 Tous droits réservés.
lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

1.1 INTRODUCTION

We are concerned with rational Poincaré duality for singular spaces. There is at least two ways to restore it in this context :

- As a self-dual sheaf or as a self-dual cohomology. This is for instance the case with rational intersection homology.
- As a spatialization. That is, given a singular space X, trying to associate to it a new topological space X_{DP} that satisfies Poincaré duality. This strategy is at the origin of the concept of intersection spaces.

Let us briefly recall this two approaches.

While seeking for a theory of characteristic numbers for complex analytic varieties and other singular spaces, Mark Goresky and Robert MacPherson discovered (and then defined in [26] for PL pseudomanifolds and in [27] for topological pseudomanifolds) a family of groups $IH_*^{\overline{p}}(X)$ called intersection homology groups of X. These groups depend on a multi-index $\overline{p}$ called a perversity. A n-perversity $\overline{p}$ is a function

$$\overline{p}\colon \{2,3,\ldots,n\} \longrightarrow \{0,1,\ldots,n-2\}$$

such that $\overline{p}(2) = 0$ and $\overline{p}(k) \leqslant \overline{p}(k+1) \leqslant \overline{p}(k)+1$, where n is the dimension of X. The n-perversities form a finite poset $\mathcal{P}_n$ endowed with a partial addition $+$ and a unique maximal element called the top perversity $\overline{t}$ defined by $\overline{t}(k) = k-2$ for all k. Two n-perversities $\overline{p}$ and $\overline{q}$ are said to be complementary if $\overline{p}+\overline{q}$ exists in $\mathcal{P}_n$ and is equal to $\overline{t}$. Intersection homology is able to restore Poincaré duality on topological stratified pseudomanifolds.

If X is a compact oriented pseudomanifold of dimension n and $\overline{p}, \overline{q}$ are two complementary perversities, over $\mathbf{Q}$ we have an isomorphism

$$IH_i^{\overline{p}}(X) \cong IH_{\overline{q}}^{n-i}(X),$$

With $IH_{\overline{q}}^{n-i}(X) := \hom(IH_{n-i}^{\overline{q}}(X), \mathbf{Q})$.

In particular, suppose given an oriented pseudomanifold X of dimension $n = 4s$. Moreover, suppose that X has only even dimensional strata. Then for the lower middle perversity $\overline{m}(k) := \lfloor \frac{k}{2} \rfloor - 1$ we have a well defined non degenerate symmetric bilinear pairing

$$IH_{2s}^{\overline{m}}(X) \times IH_{2s}^{\overline{m}}(X) \longrightarrow \mathbf{Q}.$$

We can define the Witt class of its associated quadratic form in $W(\mathbf{Q})$. The signature of this bilinear form is a bordism invariant. This pairing can be generalized to Witt spaces. A Witt space is a topological stratified pseudomanifold such that for each stratum S of odd codimension $2r+1$ the lower middle perversity intersection homology of the link L_S satisfies

$$IH_r^{\overline{m}}(L_S) = 0.$$

This duality has been extended to non Witt spaces by Markus Banagl [5]. He introduces a category $\mathcal{SD}(X)$ of complexes of sheaves on X. Those

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

sheaves are required to be Verdier self-dual and satisfy stalk axioms, which make them lie between the lower and upper-middle perversity intersection chain sheaves on X. When the category $\mathcal{SD}(X)$ is not empty we can choose a self dual chain complex of sheaves F, this allows the definition of a signature. The signature does not depend of F in $\mathcal{SD}(X)$.

Intersection spaces were defined by Markus Banagl in [6] as an attempt to spatialize Poincaré duality for singular spaces. Given a stratified pseudomanifold X on dimension n with only isolated singularities and simply connected links we have a family of topological spaces $I^{\overline{p}}X$ indexed by perversities $\overline{p}$. By analogy with intersection homology, denote by $\widetilde{HI}^{\overline{p}}_*(X) := \widetilde{H}_*(I^{\overline{p}}X)$ and $\widetilde{HI}^*_{\overline{p}}(X) := \widetilde{H}^*(I^{\overline{p}}X)$. Over $\mathbf{Q}$ and for complementary perversities $\overline{p}$, $\overline{q}$, we have an isomorphism

$$\widetilde{HI}^{\overline{p}}_i(X) \cong \widetilde{HI}^{n-i}_{\overline{q}}(X).$$

One may regard the theory of intersection spaces as an enrichment of intersection homology and we can recover informations about intersection homology thanks to those intersection spaces. In particular they have the same information when it comes to the signature of the intersection form as shown in [6, theorem 2.28]. Suppose X is a compact oriented pseudomanifold of dimension $n = 4s$ with only isolated singularities and simply connected links. Considering the middle perversity intersection space $I^{\overline{m}}X$ gives us the isomorphism

$$\widetilde{HI}^{\overline{m}}_i(X) \cong \widetilde{HI}^{4s-i}_{\overline{m}}(X).$$

Then, by the way of some nice algebraic tools it is shown that the two bilinear pairings over $\mathbf{Q}$

$$b_{HI} : \widetilde{HI}^{\overline{m}}_{2s}(X) \times \widetilde{HI}^{\overline{m}}_{2s}(X) \longrightarrow \mathbf{Q}$$

and

$$b_{IH} : IH^{\overline{m}}_{2s}(X) \times IH^{\overline{m}}_{2s}(X) \longrightarrow \mathbf{Q}$$

have the same Witt element in $W(\mathbf{Q})$.

It must be noticed here that the pairing $\widetilde{HI}^{\overline{m}}_{2s}(X) \times \widetilde{HI}^{\overline{m}}_{2s}(X) \to \mathbf{Q}$ is not realized as the quadratic form associated to the generalized Poincaré duality of the space $I^{\overline{m}}X$. In fact, the space $I^{\overline{m}}X$ does not have a fundamental class. That is we can't express $b_{HI}(x, y)$ as $\langle [I^{\overline{m}}X], x \cup y\rangle$ where $[I^{\overline{m}}X]$ would be a fundamental class of the space $I^{\overline{m}}X$.

What we show is the existence of a rational Poincaré duality space $\mathcal{DP}(X)$. In particular the pairing defined above for the middle perversity intersection space can be realized as a pairing induced by a classical Poincaré duality and we have

$$b_{\mathcal{DP}(X)}(x, y) = \langle [\mathcal{DP}(X)], x \cup y\rangle.$$

Basically, we can separate the construction of the space $\mathcal{DP}(X)$ in two cases, whether the pseudomanifold X has only one isolated singularity or more than one.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Suppose given a compact, connected oriented pseudomanifold X of dimension n with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ and denote by L_i the link of the singularity σ_i. We also suppose that the L_i are all simply connected.

The case where X has only one singularity is very particular in the sense that the link of the singularity is always a boundary. If we denote by X_{reg} the complementary in X of an open neighbourhood of the singularity, then X_{reg} is a manifold with boundary and $\partial X_{reg} = L$. By a classical result of Thom, see [30], the Witt class $[b_L] \in W(\mathbf{Q})$ of the intersection form associated to L is zero. This result allows us to perform what we call a Lagrangian truncation, which we explain in 1.2.4, and then construct a rational Poincaré duality space when the dimension of X is odd. When the dimension of X is even, we do not need this assumption and the construction of $\mathcal{DP}(X)$ is easier. We have the following definition and theorems

Definition 1.1.0.1. *Let X be a compact, connected pseudomanifold with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$. Denote by $\overline{X}$ the normalization of X. A **good** rational Poincaré approximation of X is a topological space $\mathcal{DP}(X)$ such that*

1. *$\mathcal{DP}(X)$ is a rational Poincaré duality space,*
2. *there is a rational factorization of the inclusion $i\colon X_{reg} \to \overline{X}$ in two maps*

$$X_{reg} \xrightarrow{\phi} \mathcal{DP}(X) \xrightarrow{\psi} \overline{X}.$$

That is $\psi_r \circ \phi_r = i_r\colon H_r(X_{reg}) \to H_r(\overline{X})$ such that

- *If $\dim X = 2s$, then*
  - *a) $\phi_r\colon H_r(X_{reg}) \longrightarrow H_r(\mathcal{DP}(X))$ is an isomorphism for $2s-1 > r > s$ and an injection for $r = s$,*
  - *b) $\psi_r\colon H_r(\mathcal{DP}(X)) \longrightarrow H_r(\overline{X})$ is an isomorphism for $r < s$ and $r = 2s$.*
- *If $\dim X = 2s+1$, then*
  - *a) $\phi_r\colon H_r(X_{reg}) \longrightarrow H_r(\mathcal{DP}(X))$ is an isomorphism for $2s > r > s+1$ and an injection for $r = s+1$,*
  - *b) $\psi_r\colon H_r(\mathcal{DP}(X)) \longrightarrow H_r(\overline{X})$ is an isomorphism for $r < s$ and $r = 2s+1$ and a surjection for $r = s$.*

*We say that $\mathcal{DP}(X)$ is a **very good** rational Poincaré approximation of X if*

- *when $\dim X = 2s$, then ϕ_s is also an isomorphism.*
- *when $\dim X = 2s+1$, then ϕ_{s+1} and ψ_s are also isomorphisms.*

Theorem 1.1.1 (Unique isolated singularity case)**.** *Let X be a compact, connected oriented pseudomanifold of dimension n with one isolated singularity of link L simply connected. There exists a good rational Poincaré approximation $\mathcal{DP}(X)$ of X. Moreover if $\dim X \equiv 0 \mod 4$, then the Witt class associated to the intersection form $b_{\mathcal{DP}(X)}$ is the same that the Witt class associated to the middle intersection cohomology of X.*

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Depending of the dimension of X we use two different types of truncations on the link of the singularity L. When $\dim X = 2s$ this is the classical homology truncation defined by Markus Banagl in [6]. The odd dimensional case also use the Lagrangian truncation which will be defined in section 1.2.4.

The case where we have more than one singularity is a bit more delicate in the odd dimensional case because we do not have anymore the result of Thom we used above. Consider for example $S\mathbf{C}P^2$, the suspension on the 2 dimensional complex projective space. As a pseudomanifold, it has two isolated singularities which are the two suspension points and each of these points has a copy of $\mathbf{C}P^2$ as link. But $\mathbf{C}P^2$ is not the boundary of the regular part which is homeomorphic to $\mathbf{C}P^2 \times [0,1]$, and $S\mathbf{C}P^2$ is neither a Witt space nor an L-space as stated in definitions 1.2.7.3 and 1.2.11.1.

The odd dimensional case then breaks down in two subcases corresponding to the type of space we are dealing with. If the space is a Witt space we can construct a rational Poincaré duality space, and if the space is an L-space we can perform a Lagrangian truncation to also get a rational Poincaré duality space. In the even dimensional case we can always construct a rational Poincaré duality space. We then have the following theorem.

Theorem 1.1.2 (Multiple isolated singularities case)**.** *Let X be a compact, connected oriented pseudomanifold of dimension n with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu; \nu > 1\}$ of links L_i simply connected. Then,*

1. *If $n = 2s$, there exists a good rational Poincaré approximation $\mathcal{DP}(X)$ of X. Moreover, if $\dim X \equiv 0 \mod 4$, then the Witt class associated to the intersection form $b_{\mathcal{DP}(X)}$ is the same as the Witt class associated to the middle intersection cohomology of X.*

2. *If $n = 2s+1$ and is either a Witt space or an L-space then there exists a good rational Poincaré approximation $\mathcal{DP}(X)$ of X. Moreover when X is a Witt space $\mathcal{DP}(X)$ is a very good rational Poincaré approximation of X.*

The first section of this paper contains the various and already known definitions and results we will use. We first recall the definitions of pseudomanifolds, perversities and we give a brief account of rational homotopy theory. The third part is devoted to the theory of homological truncation theory and intersection spaces defined by Markus Banagl in [6], we also give a rational model of the intersection spaces in 1.2.7.3. We extend the homological truncation to a Lagrangian truncation in the fourth part 1.2.4.

The second section is devoted to the construction of the spaces $\mathcal{DP}(X)$. We first recall the notions we use about Poincaré duality. We then completely develop the method of construction, first with a unique isolated singularity and then explain how to modify the results to get the general theorem in the context of multiple isolated singularities.

We finish with a section of examples, the real algebraic varieties, the nodal hypersurfaces and the Thom spaces.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

1.2 BACKGROUND, TRUNCATIONS AND INTERSECTION SPACES

1.2.1 *Pseudomanifold and Goresky MacPherson perversity*

We first recall the definitions of stratified pseudomanifolds, links and perversities.

Definition 1.2.0.1. *A (topologically) stratified pseudomanifold is defined inductively on dimension.*

- *A 0-dimensional stratified pseudomanifold X is a countable set of points with the discrete topology.*
- *An n-dimensional stratified pseudomanifold is a paracompact Hausdorff topological space X equipped with a filtration*

$$X = X_n \supset X_{n-1} = X_{n-2} \supset X_{n-3} \supset \cdots \supset X_1 \supset X_0 \supset X_{-1} = \emptyset$$

such that

1. *Every non-empty $X_{n-k} - X_{n-k-1}$ is a topological manifold of dimension $n-k$, called an (open) stratum of X.*
2. *$X - X_{n-2}$, the top stratum, is dense in X.*
3. *For each point $x \in X_{n-k} - X_{n-k-1}$, there exists an open neighbourhood U of $x \in X$, a compact topological stratified pseudomanifold L of dimension $k-1$ with stratification*

   $$L = L_{k-1} \supset L_{k-3} \supset \cdots \supset L_0 \supset L_{-1} = \emptyset$$

   and a homeomorphism

   $$\phi : U \xrightarrow{\simeq} \mathbf{R}^{n-k} \times \mathring{c}L$$

   where $\mathring{c}L := L \times [0,1)/((t,0) \sim (t',0))$ is the open cone on L, ϕ restricts to homeomorphisms

   $$\phi_| : U \cap X_{n-l} \xrightarrow{\simeq} \mathbf{R}^{n-k} \times \mathring{c}L_{k-l-1}.$$

The pseudomanifold L is called a link of the point x in X.

In our case all the topological spaces are supposed compact. So a 0-dimensional stratified pseudomanifold X is a finite set of points with the discrete topology.

In the rest of this paper we will work with pseudomanifold with isolated singularities of dimension n. That is a compact Hausdorff topological space X with stratification

$$\emptyset \subset \Sigma \subset X_n = X$$

© 2016 Tous droits réservés.

Thèse de Mathieu Klimczak, Lille 1, 2016

where $\Sigma = \{\sigma_1, \ldots, \sigma_\nu\}$ is a finite set of points, the isolated singularities. We will denote by $L_i := L(\sigma_i, X)$ the link in X of the singularity σ_i and by

$$L(\Sigma, X) := \bigsqcup_{\sigma_i \in \Sigma} L_i$$

the disjoint union of the links. The space $L(\Sigma, X)$ is then a disjoint union of topological manifolds of dimension $n-1$.

Such a pseudomanifold X can be obtained by coning off a manifold with boundary. Let $(M, \partial M)$ be manifold with boundary such that ∂M is the disjoint union of m connected components,

$$\partial M = \bigsqcup_{j=1}^{m} \partial_j M.$$

Let $I = (I(1), \ldots, I(\nu))$ be a partition of m such that $I(k) \neq 0$ for all $k \in \{1, \ldots, \nu\}$. This partition of m induces the following partition on ∂M. For $k \geqslant 1$, denote by

$$\partial_{I(k)} M := \bigsqcup_{\sum_{r=1}^{k-1} I(r) < j \leqslant \sum_{r=1}^{k} I(r)} \partial_j M$$

with $I(0) = 0$, then $\partial M = \sqcup_{k=1}^{\nu} \partial_{I(k)} M$. The homotopy pushout diagram

$$\begin{tikzcd}
\sqcup_{k=1}^{\nu} \partial_{I(k)} M \arrow[r] \arrow[d] & \sqcup_{k=1}^{\nu} c(\partial_{I(k)} M) \arrow[d] \\
M \arrow[r] & X
\end{tikzcd}$$

defines X as a stratified pseudomanifold with ν isolated singularities $\Sigma = \{\sigma_1, \ldots, \sigma_\nu\}$. The link L_i of σ_i is then

$$L_i = \partial_{I(i)} M.$$

Conversely, let X be a stratified pseudomanifold with only isolated singularities $\Sigma = \{\sigma_1, \ldots, \sigma_\nu\}$. Removing a small open neighbourhood of each singularities σ_i gives us a manifold with boundary $(X_{reg}, \partial X_{reg})$ where the number of connected components of ∂X_{reg} is the number of connected components of $L(\Sigma, X)$. The manifold X_{reg} is called the regular part of X.

Definition 1.2.0.2. *Let X is a compact, connected oriented pseudomanifold of dimension n with only isolated singularities $\Sigma = \{\sigma_1, \ldots, \sigma_\nu\}$. We say that X is a normal pseudomanifold if the link L_i of each singularities σ_i is connected.*

If X is constructed by coning off the boundary ∂M of $(M, \partial M)$, saying that X is normal is then equivalent to say the partition used was

$$(I(1), I(2), \ldots, I(m)) = (1, 1, \ldots, 1).$$

The integer m being the number of connected components of ∂M as before.

© 2016 Tous droits réservés.
lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

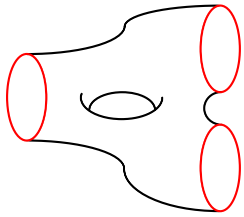


Figure 5: Manifold with boundary with $m = 3$ connected components.

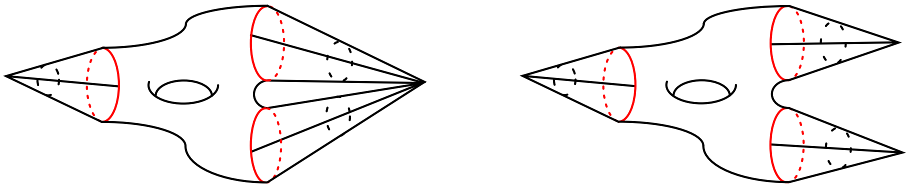


Figure 6: Pseudomanifolds with isolated singularities obtained by coning off the boundary with the partition $(1, 2)$ and $(1, 1, 1)$.

Given X is a compact, connected oriented pseudomanifold of dimension n with only isolated singularities, one can construct its normalization $\overline{X}$ by considering its regular part X_{reg} and using the partition $(1, 1, \ldots, 1)$ for coning off the boundary. The space $\overline{X}$ is then a normal pseudomanifold with only isolated singularities. We have a map $\overline{X} \longrightarrow X$.

Since the definition of a Poincaré duality approximation space of X 1.1.0.1 involves a map from $\mathcal{DP}(X)$ to its normalization $\overline{X}$, we will always assume for the rest of this part that the pseudomanifolds considered here are normal pseudomanifolds.

Since we only work with isolated singularities, a perversity is just a number $\overline{p} \in \{0, 1, \ldots, \dim X - 2\}$. We denote by $\overline{m}$ and $\overline{n}$ the following perversities

$$\begin{cases} \overline{m} & := \lfloor \frac{\dim X}{2} \rfloor - 1, \\ \overline{n} & := \lceil \frac{\dim X}{2} \rceil - 1. \end{cases}$$

Example 1.2.1. 1. ***Thom spaces***. *Let B be a compact, connected, oriented manifold of dimension n and E a fiber bundle over B of rank n',*

$$\mathbf{R}^{n'} \longrightarrow E \longrightarrow B.$$

The Thom space $Th(E)$ of the fiber bundle E is defined as the homotopy cofiber of the map

$$S_E \longrightarrow D_E$$

where S_E and D_E are respectively the sphere bundle and disk bundle associated to E.

$Th(E)$ is then a pseudomanifold of dimension $n + n'$, the singularity is the compactification point, its link is the sphere bundle S_E and the regular part of $Th(E)$ is the disk bundle D_E.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

2. ***Real algebraic varieties***. *Using Whitney stratifications, one can show that any real algebraic set X has a stratification that makes it a stratified pseudomanifold in the sense of the definition above unless for the condition about the strata of codimension 1. For example the figure 8 can be defined as a real algebraic variety, its singular point is then a codimension 1 stratum. See for instance the section 3 of the chapter 4 in [2].*

1.2.2 *Rational homotopy theory*

We recall here the definitions involving minimal and rational models we will need later. The results of this section are true for any fields $\mathbf{k}$ of characteristic zero, we focus on $\mathbf{Q}$. We refer to [24] and [25] for the following definitions and results.

Denote by $\mathsf{DGA}_{\mathbf{Q}}$ the category of differential graded algebras over $\mathbf{Q}$, dga for short, and by $\mathsf{CDGA}_{\mathbf{Q}}$ its full subcategory of commutative differential graded algebras over $\mathbf{Q}$, cdga for short, and by Top the category of topological spaces.

We have the contravariant functor $C^*(-;\mathbf{Q})\colon \mathsf{Top} \to \mathsf{DGA}_{\mathbf{Q}}$ that to each topological space K assigns its cochain algebra of normalized singular cochains on K. This algebra is almost never commutative but homotopy commutative. In fact, the only case where this algebra is commutative is when X is a disjoint union of points.

Nevertheless over the rationals we can restrict our attention to commutative dga.

Theorem 1.2.1 ([45]). *There exists a contravariant functor $A_{PL}(-)\colon \mathsf{Top} \to \mathsf{CDGA}_{\mathbf{Q}}$ such that, for all topological spaces X, there is a natural cochain algebra quasi-isomorphism of dgas $C^*(K;\mathbf{Q}) \xrightarrow{\simeq} \cdot \xleftarrow{\simeq} A_{PL}(K)$.*

Given a graded vector space V, $\wedge V$ denotes the free commutative graded algebra generated by V and not just its exterior algebra. We denote by $\wedge^n V$ the vector space generated by the elements of the form $x_1 \cdots x_n$ with the $x_i \in V$. We also have by $\wedge^+ V = \oplus_{n\geqslant 1} \wedge^n V$ and $\wedge^{\geqslant q} V = \oplus_{n\geqslant q} \wedge^n V$.

Definition 1.2.1.1. *A Sullivan algebra is a cdga $(\wedge V, d)$, where*

1. $V = \{V^p\}_{p\geqslant 1}$ *is a graded vector space over* $\mathbf{Q}$*,*
2. $V = \cup_{k=0}^{\infty} V(k)$*, where* $V(0) \subset V(1) \subset \cdots$ *is an increasing sequence of graded subspaces such that*

$$d = 0 \text{ in } V(0) \text{ and } d : V(k) \to \wedge V(k-1),\ k \geqslant 1.$$

Definition 1.2.1.2. 1. *A Sullivan model for a cdga (A, d) is a quasi isomorphism*

$$m\colon (\wedge V, d) \longrightarrow (A, d)$$

from a Sullivan algebra $(\wedge V, d)$.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

2. *If* K *is a path connected topological space then a Sullivan model for* $A_{PL}(K)$

$$m\colon (\wedge V, d) \longrightarrow A_{PL}(K)$$

is called a Sullivan model for K.

3. *A Sullivan algebra,* $(\wedge V, d)$ *is called minimal if*

$$\operatorname{im} d \subset \wedge^+ V \cdot \wedge^+ V$$

Proposition 1.2.1.1. *Any cdga* (A, d) *satisfying* $H^0(A) = \mathbf{Q}$ *and any path connected topological space* K *have a unique minimal Sullivan model up to isomorphism.*

Definition 1.2.1.3 (Rational model)**.** *A rational model for a space* K *is a cdga* $(A^*(K), d)$ *with a quasi-isomorphism*

$$m\colon (M(K), d) \to (A^*(K), d),$$

where $(M(K), d)$ *is the minimal model of* K.

When needed, we will denote by $(M(K), d)$ the minimal model of the topological space K and by $(A^*(K), d)$ a rational model of K, not necessarily minimal.

We will also need the notion of relative minimal models.

Definition 1.2.1.4. *A relative minimal cdga is a morphism of cdga's of the form*

$$i\colon (A, d_A) \longrightarrow (A \otimes \wedge V, d),$$

where $i(a) = a$, $d_{|A} = d_A$, $d(V) \subset (A^+ \otimes \wedge V) \oplus \wedge^{\geqslant 2} V$ *and such that* V *admits a basis* x_α *indexed by a well ordered set such that* $d(x_\alpha) \in A \otimes (\wedge(x_\beta))_{\beta<\alpha}$.

Theorem 1.2.2. *Let* $f\colon (A, d) \longrightarrow (B, d)$ *be a morphism of cdga's. We then have a commutative diagram*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & B \\ & (A \otimes \wedge V, d) \arrow[ur, "g"'] & \end{tikzcd}$$

where i *is a relative minimal cdga and* g *a quasi-isomorphism. This property characterizes* $(A \otimes \wedge V, d)$ *up to isomorphism.*

Example 1.2.2.

1. *For* n *odd, the minimal model of the sphere* S^n *is the cdga* $(\wedge(e), d = 0)$ *with* $|e| = n$.

2. *For* n *even, the minimal model of the sphere* S^n *is the cdga* $(\wedge(e, e'), d)$ *with* $|e| = n, |e'| = 2n - 1$ *and* $de' = e^2$.

3. *The minimal model of* $S^3 \times S^5$ *is the cdga* $(\wedge(s_3, s_5), d = 0)$ *with* $|e_i| = i$.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

4. *The minimal model of* $S^3 \vee S^5$ *is the cdga* $(\wedge(s_3, s_5, d_7), d)$ *with* $d(d_7) = s_3 s_5$, $|s_i| = i$ *and* $|d_7| = 7$.

5. *The minimal model of the torus* T^n *of dimension* n *is the cdga*

$$(\wedge(x_1, \dots, x_n), 0)$$

where all the x_i *have degree 1.*

6. *The minimal model of the complex projective spaces* $\mathbf{C}P^n$ *are the cdgas* $(\wedge(x, y), d)$ *with* $dy = x^{n+1}$, $|x| = 2$ *and* $|y| = 2n + 1$.

Definition 1.2.2.1. *Let* $f\colon (\wedge V, d) \to (\wedge W, d)$ *be a morphism between Sullivan algebras. The linear map*

$$Q(f)\colon V \to W$$

defined on the graded vector spaces V *and* W *by*

$$f(v) - Q(f)(v) \in \wedge^{\geqslant 2} W,\; v \in V$$

is called the linear part of the morphism f.

Let K be a path connected topological space and suppose $\alpha \in \pi_k K$ is represented by $a\colon (S^k, *) \to (K, *)$. Then $Q(a)\colon V_K^k \to \mathbf{Q} \cdot e$ depends only on α and the choice of the morphism $m_K\colon (\wedge V_K, d) \to A_{PL}(K)$. We define the pairing

$$\langle -, - \rangle\colon V \times \pi_* K \longrightarrow \mathbf{Q}$$

by

$$\langle v; \alpha \rangle = \begin{cases} Q(a)(v) & \text{if } v \in V_K^k \\ 0 & \text{if } \deg v \neq \deg \alpha. \end{cases}$$

That pairing is bilinear and when K is simply connected and $H^*(K)$ has finite type it induces the following theorem.

Theorem 1.2.3 ([24]). *Suppose* K *is simply connected and* $H^*(K)$ *has finite type, let* $(\wedge V, d)$ *be its minimal Sullivan model. Then we have an isomorphism :*

$$V^k \xrightarrow{\cong} \hom_{\mathbf{Z}}(\pi_k(K), \mathbf{Q}).$$

One of the advantage of rational homotopy theory is that it also extends the range of the Hurewicz theorem

Theorem 1.2.4 (Rational Hurewicz theorem,[23],[34]). *Let* K *be a simply connected topological space with* $\pi_i(K) \otimes \mathbf{Q} = 0$ *for* $1 < i < r$. *Then the Hurewicz map induces an isomorphism*

$$\mathrm{Hur}_i : \pi_i(K) \otimes \mathbf{Q} \longrightarrow H_i(K)$$

for $1 \leqslant i < 2r - 1$ *and a surjection for* $i = 2r - 1$.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Given the family of rational homotopy groups of K, $\{\pi_*(X)\otimes\mathbf{Q}\}$, we can endow them with a structure of graded Lie algebra by considering the pair $(\{\pi_*(K)\otimes\mathbf{Q}\},[-,-])$ where $[-,-]$ is the Whitehead product. We first recall the definition of the Whitehead product and then see it has a nice property when considering rational homotopy theory.

For $k\geqslant 1$, recall the homeomorphisms $I^k/\partial I^k\to S^k$ and $\partial I^{k+1}\to S^k$. Regard the first one as a continuous map $a_k:(I^k,\partial I^k)\to(S^k,*)$. Thus

$$a_k\times a_n:(I^{k+n},\partial I^{k+n},y_0)\longrightarrow(S^k\times S^n,S^k\vee S^n,*),y_0=(1,1,\dots,1).$$

Use the second homeomorphism to identify $(a_k\times a_n)_{|\partial I^{k+n}}$ as a continuous map

$$a_{k,n}:(S^{k+n-1},x_0)\longrightarrow(S^k\vee S^n,*),x_0=(\frac{1}{\sqrt{k+n}},\dots,\frac{1}{\sqrt{k+n}}).$$

Definition 1.2.4.1. *Let K be a path connected topological space. The Whitehead product of $\gamma_0\in\pi_k(K)\otimes\mathbf{Q}$ and $\gamma_1\in\pi_n(K)\otimes\mathbf{Q}$ is the homotopy class $[\gamma_0,\gamma_1]\in\pi_{k+n-1}(K)\otimes\mathbf{Q}$ represented by the map*

$$[c_0,c_1]:S^{k+n-1}\xrightarrow{a_{k,n}}S^k\vee S^n\xrightarrow{(c_0,c_1)}K,$$

where $c_0:S^k\to K$ represents γ_0 and $c_1:S^n\to K$ represents γ_1.

Let's now relate it to rational homotopy.

Consider a minimal Sullivan algebra $(\wedge V,d)$. The restriction of d to V decomposes as the sum of linear maps

$$\alpha_i:V\longrightarrow\wedge^{i+1}V,\ i\geqslant 1.$$

Each α_i extends uniquely to a derivation d_i of $\wedge V$ increasing wordlength by i. Moreover, d decomposes as the sum

$$d=d_1+d_2+\cdots$$

of the derivations d_i. The square d_1^2 raises wordlength by 2 and $d^2-d_1^2$ by at least 3. Since $d^2=0$ we must have $d_1^2=0$.

Definition 1.2.4.2. *The differential d_1 is called the quadratic part of the differential d.*

We now define a trilinear map

$$\langle -;-,-\rangle:\wedge^2V_K\times\pi_*(K)\otimes\mathbf{Q}\times\pi_*(K)\otimes\mathbf{Q}\longrightarrow\mathbf{Q}$$

by

$$\langle vw;\gamma_0,\gamma_1\rangle=\langle v;\gamma_1\rangle\langle w;\gamma_0\rangle+(-1)^{|w||\gamma_0|}\langle v;\gamma_0\rangle\langle w;\gamma_1\rangle$$

We then have the

Proposition 1.2.4.1. *The Whitehead product in π_*K is dual to the quadratic part of the differential of $(\wedge V_K,d)$. That is*

$$\langle d_1v;\gamma_0,\gamma_1\rangle=(-1)^{k+n-1}\langle v;[\gamma_0,\gamma_1]\rangle$$

for $v\in V_K$, $\gamma_0\in\pi_kK$, $\gamma_1\in\pi_nK$.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

The key property of the functor $A_{PL}(-)$ and the minimal models is the following theorem.

Theorem 1.2.5. *There is a bijection*

$$\{\textit{rational homotopy types}\} \xrightarrow{\cong} \{\textit{isomorphism classes of minimal models over } \mathbf{Q}\}$$

where on the left side we restrict to simply connected spaces with rational homology of finite type and on the right side to Sullivan algebras $(\wedge V, d)$ with $V^1 = 0$ and each V^k finite dimensional.

We recall the following theorem of minimal cellular models.

Theorem 1.2.6 ([24, theorem 9.11 p.111])**.** *Every simply connected space K is rationally modelled by a CW-complex $\widetilde{K}$ for which the differential in the integral cellular chain complex is identically zero.*

Sketch of proof. Let us give here a sketch of the proof on how to inductively construct such a CW-complex $\widetilde{K}$ since we will need this argument later.

By cellular approximation, for example see [24, theorem 1.4], we restrict ourselves to the case where K is a CW-complex with $K^0 = K^1 = pt$, and all cells are attached by based maps $(S^k, *) \to (K^k, pt)$.

Let $(C_*(K), \partial_*)$ denote the cellular chain complex for K, we also use the same symbol to denote an k-cell of K and the corresponding basis element of $C_k(K)$.

Choose k-cells a_i^k and b_j^k so that in the rational chain complex $(C_*(K) \otimes \mathbf{Q}, \partial_*)$

$$C_k(K) \otimes \mathbf{Q} = \ker \partial_k \oplus \bigoplus_i \mathbf{Q} a_i^k = \operatorname{im} \partial_{k+1} \oplus \bigoplus_j \mathbf{Q} b_j^k \oplus \bigoplus_i \mathbf{Q} a_i^k.$$

Define subcomplexes $W(k) \subset Z(k) \subset K^k$ by

$$W(k) := K^{k-1} \cup (\bigcup_i a_i^k),$$

$$Z(k) := W(k) \cup (\bigcup_j b_j^k).$$

Since $\partial\colon \bigoplus_i \mathbf{Q} a_i^k \xrightarrow{\cong} \operatorname{im} \partial_k$ the Cellular chain models theorem [24, theorem 4.18] asserts that $H_*(W(k), Z(k-1)) = 0$. Thus the inclusion $\lambda\colon Z(k-1) \to W(k)$ is an isomorphism in rational homology, thus $\pi_*(\lambda) \otimes \mathbf{Q}$ is an isomorphism. In particular, since the cells b_j^k are attached by maps $f_j\colon (S^{k-1}, *) \to K^{k-1} \subset W(k)$, there are maps $g_j\colon : (S^{k-1}, *) \to (Z(k-1), *)$ and non-zero integers r_j so that

$$\pi_{n-1}(\lambda)[g_j] = r_j [f_j].$$

The construction of $\varphi\colon \widetilde{K} \to K$ is made inductively so that φ restrict to rational homotopy equivalences $\varphi_k\colon \widetilde{K}^k \to Z(k) \subset K^k$. We explain the induction for $\widetilde{K}$. Begin with $\widetilde{K}^0 = \widetilde{K}^1 = pt$. Suppose $\varphi_{k-1}\colon \widetilde{K}^{k-1} \to Z(k-$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

1) is constructed. Then there are maps $h_j\colon (S^{k-1},*) \to (K^{k-1},pt)$ and non-zero integers s_j such that

$$\pi_{k-1}(\varphi_{k-1})[h_j] = s_j[g_j].$$

Set $h = \{h_j\}\colon \bigvee_j S_j^{k-1} \to \widetilde{K}^{k-1}$ and set

$$\widetilde{K}^k := \widetilde{K}^{k-1} \cup_h (\bigvee_j S_j^k).$$

□

Remark 1.2.7. *1. Let K_1 and K_2 be two simply connected CW-complexes such that $K_1^s = K_2^s$ for all $s \leqslant k$. Then $\widetilde{K_1}^s = \widetilde{K_2}^s$ for all $s \leqslant k$ and the map $\varphi_s^{K_1}$ and $\varphi_s^{K_2}$ are equal for all $s \leqslant k$.*

2. If K is a CW-complex of dimension n such that for the cellular chain complex $(C_*(K), \partial_*)$ we have $\ker \partial_n = 0$. Then $\widetilde{K}$ is a CW-complex of dimension $n-1$.

1.2.3 *Homological truncation and intersection spaces*

In [6] Markus Banagl constructed, for a given perversity $\overline{p}$, a space called the perversity $\overline{p}$ intersection space of X denoted by $I^{\overline{p}}X$. We briefly recall the construction.

Definition 1.2.7.1. *Given an integer $k \geqslant 3$, a (homological) k-truncation structure is a quadruple $(K, K/k, h, t_kK)$, where*

- *K is a simply connected CW-complex,*
- *K/k is an k-dimensional CW-complex with $(K/k)^{k-1} = K^{k-1}$ and such that the group of k-cycles of K/k has a basis of cells,*
- *$h\colon K/k \to K^k$ is the identity on K^{k-1} and a cellular homotopy equivalence rel K^{k-1}, and*
- *$t_kK \subset K/k$ is a subcomplex such that*

$$H_r(t_kK;\mathbf{Z}) \cong \begin{cases} H_r(K;\mathbf{Z}) & r < k, \\ 0 & r \geqslant k, \end{cases} \tag{1}$$

and such that $(t_kK)^{k-1} = K^{k-1}$.

Proposition 1.2.7.1 ([6])**.** *Given any integer $k \geqslant 3$, every simply connected CW-complex K can be completed to an k-truncation structure $(K, K/k, h, t_kK)$.*

We won't rewrite the whole proof but we can at least give more precisions about the construction of K/k. Let K be a simply connected CW-complex and for $k \geqslant 3$ we consider the short exact sequence

$$0 \longrightarrow \ker \partial_k \longrightarrow C_k(K) \longrightarrow \operatorname{im} \partial_k \longrightarrow 0$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

where $C_k(K)$ is the abelian group of the cellular k-chains of K. All the groups are freely generated so we have the existence of a section

$$s\colon \operatorname{im}\partial_k \longrightarrow C_k(K),$$

call $Y := \operatorname{im} s$ and consider the direct sum

$$C_k(K) = Z_k(K) \oplus Y.$$

where $Z_k(K)$ is the abelian group of the cellular n-cycles of K. Suppose chosen a basis $\{\xi_\beta\}$ of $Z_k(K)$ and a basis $\{\eta_\alpha\}$ of Y. For $k \geqslant 3$, the simple connectivity of K implies the simple connectivity K^{k-1} and the Hurewicz map gives us an isomorphism

$$\text{HUR}_k\colon C_k(K) \longrightarrow \pi_k(K^k, K^{k-1}).$$

For a k-cell e^k, the connecting homomorphism

$$d\colon \pi_k(K^k, K^{k-1}) \to \pi_{k-1}(K^{k-1})$$

send the class of its characteristic map $\chi(e^k)$ to the class of its attaching map. Let

$$a_\alpha\colon S^{k-1} \to K^{k-1}$$

be choices of representatives for the homotopy classes of $d\eta_\alpha$, and let

$$b_\beta\colon S^{k-1} \to K^{k-1}$$

be choices of representatives for the homotopy classes of $d\xi_\beta$. For $\{y_\alpha\}$ and $\{z_\beta\}$ families of k-cells, we define K/k by taking the new cells $\{y_\alpha\}$ and $\{z_\beta\}$ and attaching them to K^{k-1} by the mean of the maps a_α for y_α and the b_β for z_β

$$K/k := K^{k-1} \cup \bigcup_{a_\alpha} y_\alpha \cup \bigcup_{b_\beta} z_\beta.$$

We then define $t_k K$ to be

$$t_k K := K^{k-1} \cup \bigcup_{a_\alpha} y_\alpha.$$

For the construction of the cellular homotopy equivalence rel K^{n-1}, $h : K/n \simeq K^n$, we send the reader to [6].

Given any k-truncation structure $(K, K/k, h, t_k K)$, we have a homotopy class of maps $f : t_k K \to K$ given by the composition of the following maps

$$t_k K \hookrightarrow K/k \xrightarrow{h} K^k \hookrightarrow K$$

where the maps at the extremities are cellular inclusions.

Let now X be a compact, connected oriented pseudomanifold of dimension n with isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ of simply connected

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

links $L_i = L(\sigma_i, X)$, the L_i are then $(n-1)$-dimensional compact manifolds.

Given a Goresky MacPherson perversity $\overline{p}$, put $k(\overline{p}) := n - 1 - \overline{p}(n)$, we apply the $k(\overline{p})$-truncation on each links L_i to get a family of CW-complexes $t_{k(\overline{p})}L_i$ together with homotopy classes of maps

$$f_i \colon t_{k(\overline{p})}L_i \to L_i.$$

We denote by

$$t_{k(\overline{p})}L(\Sigma, X) \xrightarrow{f} L(\Sigma, X)$$

the disjoint union of these maps, with $f := \sqcup_{\sigma_i} f_i$.

We define the two following homotopy cofibers.

If the link L_i of σ_i, has more than one connected component, that is $L_i = \sqcup_{j \in J} L_{i,j}$ with J a finite set, denote by $f_{i,j} \colon t_{k(\overline{p})}L_{i,j} \to L_{i,j}$ the corresponding map.

First, consider the homotopy cofiber of the map $f_{i,j}$, which we denote by $t^{k(\overline{p})}L_{i,j}$ and call it the $k(\overline{p})$-cotruncation of $L_{i,j}$. We have maps

$$f^i = \sqcup_j f^{i,j} \colon L_i = \sqcup_j L_{i,j} \longrightarrow t^{k(\overline{p})}L_i = \sqcup_{j \in J} t^{k(\overline{p})}L_i$$

and

$$H_r(t^{k(\overline{p})}L_i; \mathbf{Z}) \cong \begin{cases} \mathbf{Z} & r = 0, \\ 0 & 1 \leqslant r < k(\overline{p}), \\ H_r(L_i; \mathbf{Z}) & r \geqslant k(\overline{p}). \end{cases}$$

We then have a family of maps

$$\partial X_{reg} = L(\Sigma, X) = \bigsqcup_i L_i \longrightarrow \bigsqcup_i t^{k(\overline{p})}L_i.$$

Then, we define by $t^{k(\overline{p})}L(\Sigma, X)$ to be the homotopy cofiber of the map f.

Definition 1.2.7.2. *1. The intersection space $I^{\overline{p}}X$ of the space X is the homotopy pushout of the solid arrows diagram.*

$$\begin{tikzcd} L(\Sigma, X) \arrow[r] \arrow[d] & X_{reg} \arrow[d, dashed] \\ t^{k(\overline{p})}L(\Sigma, X) \arrow[r, dashed] & I^{\overline{p}}X \end{tikzcd}$$

2. The normal intersection space $\mathcal{I}^{\overline{p}}X$ of the space X is the homotopy pushout of the solid arrows diagram.

$$\begin{tikzcd} L(\Sigma, X) \arrow[r] \arrow[d] & X_{reg} \arrow[d, dashed] \\ \bigsqcup_i t^{k(\overline{p})}L_i \arrow[r, dashed] & \mathcal{I}^{\overline{p}}X \end{tikzcd}$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

When X is normal with only one isolated singularity, there is no difference between the two definitions. Differences may arise only for the first homology group. In the first case, which is the original definition of [6], we have

$$H_1(I^{\overline{p}}X) = H_1(X_{reg}) \oplus \mathbf{Q}^{\beta_0(\partial X_{reg})-1}$$

where $\beta_0(\partial X_{reg})$ is the number of connected components of ∂X_{reg}. For the normal intersection space $\mathcal{I}^{\overline{p}}X$ we have

$$H_1(\mathcal{I}^{\overline{p}}X) = H_1(X_{reg}).$$

We now determine rational models of the truncation, cotruncation and the intersection space of X. Let L be a simply connected CW-complex of finite dimension and $(M(L), d)$ be its unique minimal Sullivan model.

Let us make some changes which will be useful when working rational models. Recall that L admits a cellular model $\widetilde{L}$ for which the differential in the integral cellular chain complex is identically zero by theorem 1.2.6. Denote by $\varphi\colon \widetilde{L} \to L$ the rational homotopy equivalence given by this theorem.

Since $(t_kL)^{k-1} = L^{k-1}$, the first point of the remark 1.2.7 implies that

$$\widetilde{t_kL}^{k-1} = \widetilde{L}^{k-1}.$$

By definition t_kL is a CW-complex of dimension k such that for its cellular chain complex $(C_*(t_kL), \partial_*)$ we have $\ker \partial_k = 0$. The CW-complex $\widetilde{t_kL}$ is then of dimension $k-1$ and is equal to $\widetilde{L}^{k-1}$ by the second point of the remark 1.2.7.

Consider then the following diagram

$$\begin{tikzcd}
\widetilde{L}^{k-1} \arrow[rrr, "\mathrm{incl}"] \arrow[d, "\varphi_|"'] & & & \widetilde{L} \arrow[d, "\varphi"] \\
t_kL \arrow[r, "i"'] & L/k \arrow[r, "h"'] & L^k \arrow[r, "j"'] & L
\end{tikzcd}$$

where $\varphi_|$ is the restriction of φ to the $(k-1)$-cellular skeleton of $\widetilde{L}$, which is then a rational homotopy equivalence. Since h is a homotopy equivalence relative to L^{k-1} this diagram is commutative. The fact that φ is a rational homotopy equivalences imply that the minimal models of L and $\widetilde{L}$ are isomorphic. The same is true for the minimal models of t_kL and $\widetilde{L}^{k-1}$.

Proposition 1.2.7.2. *For $k \geqslant 0$. Let $m : (M(L), d) \to (A^*(L), d)$ be a rational model of L. Denote by $C_{\overline{k-1}}$ a supplement of*

$$\ker(d^{k-1} : A^{k-1}(L) \to A^k(L))$$

and by I_{k-1} be the differential ideal of $A^*(L)$ generated by $C_{\overline{k-1}} \oplus A^{\geqslant k}(L)$.

A rational model of t_kL is given by $A^*(L)/I_{k-1}$ and a Sullivan representative of $f : t_kL \longrightarrow L$ is given by the projection to the equivalence class

$$A^*(L) \twoheadrightarrow A^*(L)/I_{k-1}.$$

© 2016 Tous droits réservés.

Thèse de Mathieu Klimczak, Lille 1, 2016

Proof. By the discussion above, we have the isomorphism of minimal models

$$M(\varphi)\colon (M(L), d) \xrightarrow{\cong} (M(\widetilde{L}), d).$$

Composing its inverse with the quasi-isomorphism m gives us the following rational model of L

$$M := m \circ M(\varphi)^{-1}\colon (M(\widetilde{L}), d) \longrightarrow (A^*(L), d).$$

Consider now the map $M(\mathrm{incl})\colon M(\widetilde{L}) \longrightarrow M(\widetilde{L}^{k-1})$. By theorem 1.2.2 there is a relative minimal model

$$\begin{tikzcd}
M(\widetilde{L}) \arrow[r, "M(\mathrm{incl})"] \arrow[d, "="'] & M(\widetilde{L}^{k-1}) \\
M(\widetilde{L}) \arrow[r, "i"'] & M(\widetilde{L}) \otimes \wedge V \arrow[u, "g"']
\end{tikzcd}$$

where g is a quasi-isomorphism and i the canonical inclusion. The fact that $H^*(M(\widetilde{L})) = H^*(M(\widetilde{L}^{k-1})) = H^*(L)$ for $* \leqslant k-1$ implies that the elements of V are either

- of degree greater than or equal to k,
- or of degree $k-1$ and not in $\ker d^{k-1}$.

Let then I_{k-1} the differential ideal defined in the proposition and consider the following diagram.

$$\begin{tikzcd}
M(\widetilde{L}) \arrow[r, "i"] \arrow[d, "M"'] & M(\widetilde{L}) \otimes \wedge V \arrow[d, dashed, "\overline{M}"] \\
A^*(L) \arrow[r, "p"'] & A^*(L)/I_{k-1}
\end{tikzcd}$$

Where p is the projection map. We define the map $\overline{M}$ by

$$\begin{cases} \overline{M}(a) = [M(a)] & a \in M(\widetilde{L}), \\ \overline{M}(\wedge V) = 0. \end{cases}$$

The image of a product is defined by

$$\begin{cases} \overline{M}(ab) = [M(a)M(b)] & a, b \in M(\widetilde{L}), \\ \overline{M}(vw) = 0 = \overline{M}(v)\overline{M}(w), & v, w \in V, \end{cases}$$

and for all $v \in V$ and all $a \in M(\widetilde{L})$, the degree of av is greater than or equal to k, so we define

$$\overline{M}(av) := 0 = \overline{M}(a)\overline{M}(v).$$

This diagram then commutes and $\overline{M}$ defines a quasi-isomorphism. $\square$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

We now determine a rational model for the intersection spaces and the normal intersection spaces. The cotruncation being a homotopy cofiber, the next lemma follows from [24, Proposition 13.6]

Lemma 1.2.7.1. *The* k*-cotruncation of* L*,* $t^k L$*, being defined as the homotopy cofiber of the map* $t_k L \to L$*, a rational model is given by*

$$\mathbf{Q} \oplus \ker p = \mathbf{Q} \oplus I_{k-1}$$

where p *is the map* $p\colon A^*(L) \twoheadrightarrow A^*(L)/I_{k-1}$.

Proposition 1.2.7.3. *Let* X *be a compact, connected oriented pseudomanifold of dimension* n *with only isolated singularities* $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ *of simply connected links* $L_i = L(\sigma_i, X)$*. Let* $\overline{p}$ *be a Goresky MacPherson perversity and*

$$\phi\colon (A^*(X_{reg}), d) \to (A^*(L(\Sigma, X)), d)$$

a surjective model of the inclusion $i\colon L(\Sigma, X) = \partial X_{reg} \to X_{reg}$*. A rational model of* $I^{\overline{p}}X$ *is given by*

$$AI_{\overline{p}}(X) := (A^*(X_{reg}), d) \oplus_{A^*(L)} (\mathbf{Q} \oplus I_{k(\overline{p})}, d)$$

where $(A^*(X_{reg}), d)$ *is a rational model of the regular part of the pseudomanifold and* $(\mathbf{Q} \oplus I_{k(\overline{p})}, d)$ *a rational model of* $t^{k(\overline{p})}L(\Sigma, X)$*.*

Proof. The intersection space $I^{\overline{p}}X$ of the space X is the homotopy pushout of the diagram.

$$\begin{tikzcd} L(\Sigma, X) \arrow[r, "i"] \arrow[d] & X_{reg} \arrow[d] \\ t^{k(\overline{p})}L(\Sigma, X) \arrow[r] & I^{\overline{p}}X \end{tikzcd}$$

Then applying $A_{PL}(-)$ we have a diagram of pullback,

$$\begin{tikzcd} A_{PL}(I^{\overline{p}}X) \arrow[r] \arrow[d] & A_{PL}(t^{k(\overline{p})}L(\Sigma, X)) \arrow[d] \\ A_{PL}(X_{reg}) \arrow[r, "A_{PL}(i)"'] & A_{PL}(L(\Sigma, X)) \end{tikzcd}$$

and then the quasi isomorphism.

$$A_{PL}(I^{\overline{p}}X) \simeq A_{PL}(X_{reg}) \oplus_{A_{PL}(L(\Sigma,X))} A_{PL}(t^{k(\overline{p})}L(\Sigma, X)).$$

Given the rational models of X_{reg}, $L(\Sigma, X)$ and $t^{k(\overline{p})}L(\Sigma, X)$ thanks to the lemma 1.2.7.1, we get a map

$$(A^*(X_{reg}), d) \oplus_{A^*(L(\Sigma,X))} (\mathbf{Q} \oplus I_{k(\overline{p})}, d) \to A_{PL}(X_{reg}) \oplus_{A_{PL}(L(\Sigma,X))} A_{PL}(t^{k(\overline{p})}L(\Sigma, X)).$$

With the surjective model $\phi : (A^*(X_{reg}), d) \to (A^*(L(\Sigma, X)), d)$, we get the following morphism of short exact sequences. The result follows from an application of the five lemma to the associated long exact sequences.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

$$\begin{tikzcd}
\ker \phi \arrow[r] \arrow[d] & (A^*(X_{reg}), d_{X_{reg}}) \oplus_{A^*(L(\Sigma,X))} (\mathbf{Q} \oplus I_{k(\overline{p})}, d) \arrow[r] \arrow[d] & (\mathbf{Q} \oplus I_{k(\overline{p})}, d) \arrow[d] \\
\ker A_{PL}(i) \arrow[r] & A_{PL}(X_{reg}) \oplus_{A_{PL}(L(\Sigma,X))} A_{PL}(t^{k(\overline{p})}L(\Sigma,X)) \arrow[r] & A_{PL}(t^{k(\overline{p})}L(\Sigma,X))
\end{tikzcd}$$

□

Proposition 1.2.7.4. *Let X be a compact, connected oriented pseudomanifold of dimension n with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ of simply connected links $L_i = L(\sigma_i, X)$. Let $\overline{p}$ be a Goresky MacPherson perversity and*

$$\phi\colon (A^*(X_{reg}), d) \to (A^*(L(\Sigma, X)), d)$$

a surjective model of the inclusion $i\colon L(\Sigma, X) = \partial X_{reg} \to X_{reg}$. A rational model of the normal intersection space $\mathcal{I}^{\overline{p}}X$ is given by

$$A\mathcal{I}_{\overline{p}}(X) := (A^*(X_{reg}), d) \oplus_{A^*(L)} (\bigoplus_i \mathbf{Q} \oplus I_{k(\overline{p},i)}, d)$$

where $(A^*(X_{reg}), d)$ is a rational model of the regular part of the pseudomanifold and $(\mathbf{Q} \oplus I_{k(\overline{p},i)}, d)$ a rational model of $t^{k(\overline{p})}L_i$.

Proof. The proof is exactly the same as the previous proposition unless the normal intersection space $\mathcal{I}^{\overline{p}}X$ is the homotopy pushout of the following diagram.

$$\begin{tikzcd}
L(\Sigma, X) \arrow[r, "i"] \arrow[d] & X_{reg} \arrow[d] \\
\bigsqcup_i t^{k(\overline{p})}L_i \arrow[r] & \mathcal{I}^{\overline{p}}X
\end{tikzcd}$$

□

In the odd dimensional case, there is a class of pseudomanifolds X for which the truncations $t_{k(\overline{m})}$ and $t_{k(\overline{n})}$ will coincide.

Definition 1.2.7.3. *Let X be a compact, connected oriented pseudomanifold of dimension $n = 2s + 1$ with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ of links L_i simply connected. X is a Witt space if $H^s(L_i) = 0$ for all $\sigma_i \in \Sigma$.*

Example 1.2.3.
1. *The suspension of the complex projective space $S\mathbf{C}P^3$ is a Witt space since $H^3(\mathbf{C}P^3) = 0$.*
2. *The suspension of the complex projective plane $S\mathbf{C}P^2$ is not a Witt-space since $H^2(\mathbf{C}P^2) = \mathbf{Q}$.*

Remark 1.2.8. *Being a Witt space is a condition on singularities of odd codimension, there is no condition on singularities of even codimension.*

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

1.2.4 *Lagrangian truncation and Lagrangian intersection spaces*

First, we recall some facts about quadratic spaces that we will need later. The only field we'll work with is $\mathbf{Q}$, but the results are true over any field $\mathbf{k}$ such that $\mathrm{char}(\mathbf{k}) \neq 2$.

Definition 1.2.8.1. *A regular quadratic space* (E, b) *is a vector space of finite dimension* E *together with a non degenerate bilinear form*

$$b : E \times E \to \mathbf{k},$$

b *being either be a symmetric form or an skew-symmetric one.*

When needed, we'll denote by q the quadratic form associated to b, that is $q(x) := b(x, x)$. If the context is clear, we will use without any distinctions both notations.

Definition 1.2.8.2. *An isotropic subspace* V *of* (E, b) *is a subspace of* E *such that for all* $x \in V$, $q(x) = b(x, x) = 0$*. If* $2 \dim V = \dim E$, V *is then called a Lagrangian subspace.*

Theorem 1.2.9 ([40]). *Let* (E, b) *be a regular quadratic space of dimension* p *and suppose that* E *posses an isotropic subspace* V *of dimension* m*. Then there exists a subspace* H *of* E *such*

- $V \subset H$,
- $\dim H = 2 \dim V$,
- *there exists a basis*

$$(a_1, \dots, a_m, a_1^*, \dots, a_m^*, b_1, \dots, b_{p-2m})$$

of E *such that* $(a_1, \dots, a_m, a_1^*, \dots, a_m^*)$ *is an hyperbolic basis of* H*, that is, for all* $i, j \in \{1, \dots, m\}^2$,

$$\begin{cases} b(a_i, a_j) &= 0, \\ b(a_i^*, a_j^*) &= 0, \\ b(a_i, a_j^*) &= \delta_{ij}. \end{cases}$$

In particular, any hyperbolic basis of H *is a basis in the usual sense of* H.

When the bilinear form is non degenerate skew-symmetric, that is the form is symplectic, the theorem above simplifies because the classification of skew-symmetric bilinear forms is determined by the dimension of E and the rank of the form.

Theorem 1.2.10 ([40]). *Let* (E, b) *be a regular quadratic space such that the bilinear form* b *is skew-symmetric, then there exists a basis*

$$(a_1, \dots, a_m, a_1^*, \dots, a_m^*)$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

of E *such that*

$$\begin{cases} b(a_i, a_j) & = 0, \\ b(a_i^*, a_j^*) & = 0, \\ b(a_i, a_j^*) & = \delta_{ij}. \end{cases}$$

In particular $(a_1, \dots, a_m, a_1^*, \dots, a_m^*)$ *is a basis in the usual sense of* E *and* $\dim E = 2m$*. We also call this base an hyperbolic basis.*

The spaces generated respectively by $(a_1, \dots, a_m)$ and $(a_1^*, \dots, a_m^*)$ are then Lagrangian subspaces.

We now gather the various hypothesis we need to define the Lagrangian truncation

Consider K as a simply connected n-dimensional CW-complex satisfying Poincaré duality with $n = 2s$. We denote by b the non degenerate bilinear form induced by the Poincaré duality with $\mathbf{Q}$ coefficients, consider $\dim H^s(K) = 2m$ and

$$b : H^s(K) \times H^s(K) \longrightarrow \mathbf{Q}$$

where $b(x, y) := \langle x \cup y, [K] \rangle$ with $[K] \in H_{2s}(K)$ the fundamental class and $\langle -, - \rangle$ the evaluation form.

If b is symmetric suppose that $H^s(K)$ posses a Lagrangian subspace V of dimension m, let then $(a_1, \dots, a_m)$ be a basis of V and thanks to the theorem 1.2.9, complete $(a_1, \dots, a_m)$ into a hyperbolic basis

$$(a_, \dots, a_m, a_1^*, \dots, a_m^*)$$

of $H^s(K)$.

If b is skew-symmetric then thanks to the theorem 1.2.10, there exists a hyperbolic basis

$$(a_, \dots, a_m, a_1^*, \dots, a_m^*)$$

of $H^s(K)$.

Either way, denote by V and V^* the subspaces respectively generated by $(a_, \dots, a_m)$ and $(a_1^*, \dots, a_m^*)$, we have

$$H^s(K) = V \oplus V^*.$$

Remark that since $b(a_i, a_i^*) = 1$, a_i and a_i^* are Poincaré duals to each other. Denote by $\overline{V}$ and $\overline{V^*}$ the Poincaré duals in $H_s(K)$ of respectively V and V^* and by $(\overline{a_1}, \dots, \overline{a_m})$ the basis of $\overline{V}$ and by $(\overline{a_1^*}, \dots, \overline{a_m^*})$ the basis of $\overline{V^*}$. We have the direct sum

$$H_s(K) = \overline{V} \oplus \overline{V^*}.$$

Applying theorem 1.2.6 to K we then have a rational cellular homotopy equivalence $\varphi \colon \widetilde{K} \longrightarrow K$ where $(C_*(\widetilde{K}), 0)$ is the integral cellular chain complex. We then perform and define the Lagrangian truncation on $\widetilde{K}$.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Since the differential of $C_*(\widetilde{K})$ is zero we have

$$C_s(\widetilde{K}) \otimes \mathbf{Q} = H_s(\widetilde{K}) \cong H_n(K) = \overline{V} \oplus \overline{V^*}$$

By simply connectivity and the Hurewicz theorem we have the isomorphism

$$\mathrm{Hur}_s : \overline{V} \oplus \overline{V^*} \xrightarrow{\cong} \pi_s(\widetilde{K}^s, \widetilde{K}^{s-1}).$$

Let

$$\lambda_i, \lambda_i^* \colon S^{s-1} \to \widetilde{K}^{s-1}$$

be choices of representatives for the homotopy classes of $d\overline{a_i}$, $d\overline{a_i^*}$. Then for s-cells $\{t_i\}, \{t_i^*\}$ and using λ_i, λ_i^* as attaching maps we define

$$\widetilde{K}/\mathcal{L} := \widetilde{K}^{s-1} \cup \bigcup_{\lambda_i} t_i \cup \bigcup_{\lambda_i^*} t_i^*.$$

We get the cellular homotopy equivalence $h \colon \widetilde{K}/\mathcal{L} \to \widetilde{K}^s$ rel $\widetilde{K}^{s-1}$ the same way as the classical spatial homology truncation in [6, proposition 1.6].

Definition 1.2.10.1. *The Lagrangian truncation of the CW-complex* K *is defined by*

$$t_{\overline{\mathcal{L}}} K := \widetilde{K}^{s-1} \cup \bigcup_{\lambda_i} t_i^*.$$

Moreover, we have $h_s \colon H_s(\widetilde{K}/\mathcal{L}) \cong H_s(\widetilde{K}) \cong H_s(\widetilde{K}^s, \widetilde{K}^{s-1})$. Denote by $(C_*(\widetilde{K}/\mathcal{L}), \partial)$ the integral cellular chain complex of $\widetilde{K}/\mathcal{L}$ then this implies that

$$(C_{\leqslant s}(\widetilde{K}/\mathcal{L}), \partial) \otimes \mathbf{Q} \longrightarrow (C_s(\widetilde{K}), 0) \otimes \mathbf{Q}$$

yields an isomorphism of homology in degree n. This implies that

$$\partial \colon C_s(\widetilde{K}/\mathcal{L}) \to C_{s-1}(\widetilde{K}/\mathcal{L})$$

is zero.

The comparison map $t_{\overline{\mathcal{L}}} K \to K$ is then defined as the composition of the maps

$$t_{\overline{\mathcal{L}}} K \hookrightarrow \widetilde{K}/\mathcal{L} \xrightarrow{h} \widetilde{K}^s \hookrightarrow \widetilde{K} \xrightarrow{\varphi} K$$

where the arrows $\hookrightarrow$ denote cellular inclusions, h is a cellular homotopy equivalence rel $\widetilde{K}^{s-1}$ and φ a cellular rational homotopy equivalence. We have

$$H_r(t_{\overline{\mathcal{L}}} K) \cong \begin{cases} H_r(K) & r \leqslant s-1 \\ \overline{V^*} & r = s \\ 0 & r > s. \end{cases} \tag{2}$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

We now define, with the help of Lagrangian truncation, the space called the Lagrangian intersection space associated to X

Let X a compact, connected oriented pseudomanifold of dimension n with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ of links L_i simply connected. This time we fix the dimension of X to be odd, $n = 2s + 1$.

Denote by

$$b_i \colon H^s(L_i) \times H^s(L_i) \longrightarrow \mathbf{Q}$$

the non degenerate bilinear form induced by Poincaré duality on the links L_i. Suppose that for all i, $H^s(L_i)$ admit a Lagrangian subspace V_i with respect to the bilinear form b_i. To each L_i we then apply the Lagrangian truncation process to get maps

$$f_i \colon t_{\overline{\mathcal{L}}} L_i \longrightarrow L_i.$$

Denote by $t^{\overline{\mathcal{L}}} L_i$ the homotopy cofiber of the map f_i and call it the Lagrangian cotruncation of L_i, we then have a map

$$f^i \colon L_i \longrightarrow t^{\overline{\mathcal{L}}} L_i.$$

And

$$H_r(t^{\overline{\mathcal{L}}} L_i) \cong \begin{cases} \mathbf{Q} & r = 0, \\ 0 & 1 \leqslant r < s, \\ \overline{V_i} & r = s, \\ H_r(K) & s+1 \leqslant r \leqslant 2s. \end{cases}$$

Definition 1.2.10.2. *The normal Lagrangian intersection space $\mathcal{I}^{\overline{\mathcal{L}}} X$ of the space X is the homotopy pushout of the solid arrows diagram.*

$$\begin{tikzcd} L(\Sigma, X) \arrow[r] \arrow[d] & X_{reg} \arrow[d, dashed] \\ \bigsqcup_i t^{\overline{\mathcal{L}}} L_i \arrow[r, dashed] & \mathcal{I}^{\overline{\mathcal{L}}} X \end{tikzcd}$$

Remark 1.2.11. *We use the adjective normal to differentiate it from the Lagrangian intersection space we will used in the second part.*

We want to know when we can perform Lagrangian truncation to get Lagrangian intersection spaces. Lets us first define the class of spaces for which it is possible.

Definition 1.2.11.1. *Let X be a compact, connected oriented pseudomanifold of dimension $2s + 1$ with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ of links L_i simply connected. X is an L-space if $H^s(L_i)$ has a Lagrangian subspace with respect to the non degenerate bilinear form $b_i \colon H^s(L_i) \times H^s(L_i) \to \mathbf{Q}$ for all $\sigma_i \in \Sigma$.*

© 2016 Tous droits réservés.

lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Example 1.2.4. *1. The suspension of the torus* ST^2 *is an L-space since*

$$H^1(L) = H^1(T^2) = \mathbf{Q} \oplus \mathbf{Q}.$$

2. The suspension of the complex projective plane $S\mathbf{C}P^2$ *is not an L-space since* $H^2(\mathbf{C}P^2) = \mathbf{Q}$.

Remark 1.2.12. *Being an L-space implies that* $\dim H^n(L_i)$ *is an even number for all* i.

We would like to have a simple criterion saying when X is an L-space and thus when we can perform a Lagrangian truncation. We in fact have two cases to consider, following the parity of s.

Suppose first that $\dim L_i \equiv 2 \mod 4$, that is $\dim X = n = 4s+3$. In that case the non-degenerate bilinear form induced by Poincaré duality

$$b_i \colon H^{2s+1}(L_i) \times H^{2s+1}(L_i) \longrightarrow \mathbf{Q}$$

is skew-symmetric due to the graded commutativity. The form b_i is then called a non-degenerate symplectic form for all $\sigma_i \in \Sigma$, we always have a Lagrangian subspace in that case.

Now, if $\dim L_i \equiv 0 \mod 4$, that is $\dim X = 4s+1$. We can't always apply the Lagrangian truncation.

Let L_i be a link of a singularity of X, L_i is then a connected compact manifold of dimension $4s$. Consider its non-degenerate bilinear form induced by Poincaré duality

$$b_i \colon H^{2s}(L_i) \times H^{2s}(L_i) \longrightarrow \mathbf{Q}$$

and let $\sigma(b_i)$ be its reduced signature, $\sigma(b_i)$ is related to the Pontryagin numbers by the Hirzebruch signature formula.

The existence of a Lagrangian subspace for $H^{2s}(L_i)$ is then given by the theorem of Sullivan and Barge, see [9] and [45], about rational classification of simply connected manifolds. We recall here the part of the Sullivan-Barge theorem we need.

Theorem 1.2.13 ([45]). *Let* $(\wedge V, d)$ *be a Sullivan model whose cohomology satisfies Poincaré duality with a fundamental class in dimension* $n = 4s$ *and* $V^1 = 0$. *We also choose cohomology classes* $p = \{p_j\} \in H^{4j}(\wedge V, d)$.

If the signature is zero, there is a compact simply connected manifold that realizes the pair $((\wedge V, d), p)$, *if and only if the quadratic form on* H^{2s} *is equivalent over* $\mathbf{Q}$ *to a quadratic form* $\sum \pm x_k^2$.

Lemma 1.2.13.1. *Let* L *be a simply connected compact manifold of dimension* $n = 4s$ *and such that* $\sigma(b_L) = 0$. *Then* $H^{2s}(L)$ *has a Lagrangian subspace.*

Proof. Let $(\wedge V, d)$ be a Sullivan model of L and $p = \{p_i\}$ the Pontryagin numbers of L related to $\sigma(b_L) = 0$. Obviously L realizes the pair $((\wedge V, d), p)$, this implies and then the quadratic form on $H^{2s}(L)$ is equivalent over $\mathbf{Q}$ to a quadratic form

$$\sum_{k=1}^{m} x_k^2 - \sum_{k'=1}^{m} x_{k'}^2.$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

This quadratic form is then hyperbolic and posses a Lagrangian subspace. □

Proposition 1.2.13.1. *Let* X *a compact, connected oriented pseudomanifold of dimension* n *with only isolated singularities* $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ *of links* L_i *simply connected. Then*

1. *If* $n = 4s + 3$, X *is an L-space.*
2. *If* $n = 4s + 1$, $\sigma(b_i) = 0$ *for all* i *if and only if* X *is an L-space.*

1.3 THE CONSTRUCTION OF POINCARÉ DUALITY SPACES

1.3.1 *Poincaré duality*

We recall here the definitions and results about Poincaré duality needed in the rest of the paper.

Definition 1.3.0.1. *Let* (X, Y) *be a pair of CW-complex, we say that* (X, Y) *is a rational Poincaré duality pair of dimension* n *if :*

1. $\dim_{\mathbf{Q}} H_r(X; \mathbf{Q})$ *is finite for all* r,
2. Y *is a sub-CW-complex of* X *with the same property,*
3. *there exists a class* $[x] \in H_n(X, Y; \mathbf{Q})$ *such that*

$$- \cap [x] : H^r(X; \mathbf{Q}) \longrightarrow H_{n-r}(X, Y; \mathbf{Q})$$

is an isomorphism. We call $[x]$ *an orientation class of* (X, Y)

Remark 1.3.1. *Let* (X, Y) *be a Poincaré duality pair of dimension* n*, then* $Y = (Y, \emptyset)$ *is a Poincaré duality pair without boundary of dimension* $n-1$*. Indeed, if* $[x] \in H_n(X, Y)$ *is an orientation class, then* $[y] = \partial[x] \in H_{n-1}(Y)$ *is an orientation class of* Y*. We then say that* Y *is a Poincaré complex of dimension* $n-1$*. We also say that* $(Y, [y])$ *is the oriented boundary of* $(X, Y, [x])$.

Let $(X_1, Y, [x_1])$ and $(X_2, Y, [x_2])$ be two oriented Poincaré duality pairs of dimension n with the same oriented boundary $(Y, [y])$. Let $\hat{X} := X_1 \cup_Y X_2$ the CW-complex obtained by glueing X_1 and X_2 along their common boundary Y. We have the two classical Mayer-Vietoris sequences.

$$H_{r+1}(\hat{X}) \xrightarrow{\partial_0} H_r(Y) \longrightarrow H_r(X_1) \oplus H_q(X_2) \longrightarrow H_r(\hat{X})$$

$$H_r(Y) \longrightarrow H_r(\hat{X}) \xrightarrow{j_1 - j_2} H_r(X_1, Y) \oplus H_q(X_2, Y) \xrightarrow{\partial_1 + \partial_2} H_{r-1}(Y)$$

Together with the commutative diagram

$$\begin{tikzcd}
H_{r+1}(\hat{X}) \arrow[r, "\partial_0"] \arrow[d, "j_i"'] & H_r(Y) \\
H_{r+1}(X_i, Y) \arrow[ur, "\partial_i"'] &
\end{tikzcd}$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Theorem 1.3.2 ([11], Glueing of 2 oriented Poincaré duality pairs)**.** *Consider the following diagram*

$$\begin{tikzcd} & 0 \arrow[d] \\ & H_n(\hat{X}) \arrow[d, "i"] \\ H_n(X_1,Y) \oplus H_n(X_2,Y) \arrow[r, "i_1 \oplus i_2"] & H_n(\hat{X},Y) \arrow[d, "\partial"] \\ & H_{n-1}(Y) \end{tikzcd}$$

and let $[\hat{x}] = i^{-1}(i_1 \oplus i_2([x_1],[-x_2])) \in H_n(\hat{X})$. *Then any two of the following conditions imply the third.*

1. $(\hat{X},[\hat{x}])$ *is an oriented Poincaré complex of dimension* n *without boundary,*
2. $(Y,[y])$ *is a Poincaré complex of dimension* $n-1$ *with orientation class* $[y] = \partial_1[x_1] = \partial_2[x_2] = \partial_0[\hat{x}] \in H_{n-1}(Y)$,
3. $(X_i, Y, [x_i])$ *are Poincaré duality pairs of dimension* n *with orientation classes* $[x_i] \in H_n(X_i,Y)$.

Proof. This is a direct consequence of the commutativity (up to sign) of the following ladder and the 5-lemma.

$$\begin{tikzcd} \cdots \arrow[r] & H^{r-1}(Y) \arrow[r, "\delta_0"] \arrow[d, "-\cap \partial_0[\hat{x}]"] & H^r(\hat{X}) \arrow[r] \arrow[d, "-\cap[\hat{x}]"] & H^r(X_1) \oplus H^r(X_2) \arrow[r] \arrow[d, "(-\cap[x_1]) + (-\cap[x_2])"] & \cdots \\ \cdots \arrow[r] & H_{n-r}(Y) \arrow[r] & H_{n-r}(\hat{X}) \arrow[r] & H_{n-r}(X_1,Y) \oplus H_{n-r}(X_2,Y) \arrow[r] & \cdots \end{tikzcd}$$

□

Let $(X,Y,[x])$ be an oriented Poincaré duality pair of dimension $n = 4s$, the following diagram

$$\begin{tikzcd} & H^{2s}(X,Y) \arrow[r] \arrow[d, "-\cap[x]"', "\cong"] & H^{2s}(X) \arrow[d, "\cong"] \\ H_{2s}(Y) \arrow[r, "i_*"] & H_{2s}(X) \arrow[r] & \hom(H_{2s}(X),\mathbf{Q}) \end{tikzcd}$$

gives the vector space $H_{2s}(X)$ a symmetric bilinear form for which the kernel is $i_*(H_{2n}(Y))$. We denote it by b_X.

Lemma 1.3.2.1 (Novikov)**.** *Suppose given* $(X_1,Y,[x_1])$ *and* $(X_2,Y,[x_2])$ *2 oriented Poincaré duality pairs of dimension* $n = 4s$ *with the same oriented boundary* $(Y,[y])$. *If* $(\hat{X},[\hat{x}])$ *is the space obtained by glueing as in theorem 1.3.2, then*

$$[b_{\hat{X}}] = [b_{X_1}] - [b_{X_2}] \text{ in } W(\mathbf{Q}).$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

The rational homotopy type of rational Poincaré duality spaces does not depend of the fundamental class. As stated in the following theorem.

Theorem 1.3.3 ([42]). *Let H be a Poincaré duality algebra of top dimension n and $H^1 = 0$. Let X be a simply connected rational space with $H(X) \cong H$ except $H^n(X) = 0$. If $Y = X \cup e^n$ with $H(Y) \cong H$, then the rational homotopy type of Y is determined by X. Moreover, the cell e^n is attached by ordinary Whitehead products (not iterated) with respect to some basis of $\pi_*(X) \otimes \mathbf{Q}$.*

Example 1.3.1. *Consider the two spaces $Y = S^3 \times S^5$ and $X = S^3 \vee S^5$. The space Y, as a manifold of dimension 8, satisfies Poincaré duality. The theorem then says that, rationally*

$$S^3 \times S^5 \simeq_{\mathbf{Q}} (S^3 \vee S^5) \cup_{[s_3^\sharp, s_5^\sharp]} e^8.$$

Indeed, denote by $s_3^\sharp$ and $s_5^\sharp$ the elements of respectively $\pi_3 X \otimes \mathbf{Q}$ and $\pi_5 X \otimes \mathbf{Q}$ obtained by theorem 1.2.3, then $[s_3^\sharp, s_5^\sharp] \in \pi_7(S^3 \vee S^5)$ and the rational model of $(S^3 \vee S^5) \cup_{[s_3^\sharp, s_5^\sharp]} e^8$ is the rational model of a cell attachment to the space X, see [24, 13.d p.173], given by

$$((\wedge V_{S^3 \vee S^5}) \oplus \mathbf{Q} e_8, D)$$

where $\wedge(V_{S^3 \vee S^5})$ is given in item 4 of the example 1.2.2 and

1. $\deg e_8 = 8$,
2. $\wedge(V_{S^3 \vee S^5})$ *is a subalgebra and* $e_8 \cdot \wedge^+(V_{S^3 \vee S^5}) = 0 = e_8^2$,
3. $De_8 = Ds_3 = Ds_5 = 0$ *and* $D(d_7) = d(d_7) + \langle d_7; [s_3^\sharp, s_5^\sharp] \rangle e_8$.

Using proposition 1.2.4.1 and the fact that $d(d_7) = d_1(d_7)$ we have

$$\langle d_7; [s_3^\sharp, s_5^\sharp] \rangle e_8 = -\langle d(d_7); s_3^\sharp, s_5^\sharp \rangle e_8 = -\langle s_3 s_5; s_3^\sharp, s_5^\sharp \rangle e_8 = -e_8$$

then

$$D(d_7) = d(d_7) - e_8 = s_3 s_5 - e_8.$$

The Whitehead bracket in fact encodes the Poincaré duality of the space $S^3 \times S^5$.

1.3.2 *The unique isolated singularity case*

In this part we prove the following theorem

Theorem 1.3.4 (Unique isolated singularity case). *Let X be a compact, connected oriented pseudomanifold of dimension n with one isolated singularity of link L simply connected. Then, there exists a good rational Poincaré approximation $\mathcal{DP}(X)$ of X. Moreover if $\dim X \equiv 0 \mod 4$, then the Witt class associated to the intersection form $b_{\mathcal{DP}(X)}$ is the same that the Witt class associated to the middle intersection cohomology of X.*

The strategy adopted here is to transform the pair (tL, L), where tL is a given cotruncation of the link L, into a Poincaré duality pair and to glue it to the already existing Poincaré duality pair $(X_{reg}, \partial X_{reg})$ via theorem 1.3.2 to obtain our spaces $\mathcal{DP}(X)$.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

1.3.2.1 *The even dimensional case*

Consider now X a compact, connected oriented pseudomanifold of dimension $n = 2s$ with one isolated singularity σ of simply connected link L. Since $\overline{m} = \overline{n}$ we have a well defined intersection space $IX := I^{\overline{m}}X = I^{\overline{n}}X$.

Let $\phi : S^{2s-1} \to tL$ be an arbitrary continuous map with $tL := t^{k(\overline{m})}L$ the middle cotruncation of the link. We denote by $t^{\phi}L$ the space obtained as the result of the following homotopy pushout :

$$\begin{tikzcd} S^{2s-1} \arrow[r, hook] \arrow[d, "\phi"'] \arrow[dr, phantom, "\mathrm{HPO}"] & D^{2s} \arrow[d, dashed] \\ tL \arrow[r, dashed] & t^{\phi}L \end{tikzcd}$$

Lemma 1.3.4.1. $(t^{\phi}L, L)$ *is a Poincaré duality pair if and only if* ϕ_{2s-1} *is an isomorphism. Where* ϕ_{2s-1} *is the connecting homomorphism*

$$\phi_{2s-1} \colon H_{2s}(t^{\phi}L, L) \longrightarrow H_{2s-1}(L)$$

in the long exact sequence of the pair $(t^{\phi}L, L)$ *induced by the attaching map* ϕ.

Proof. Suppose $(t^{\phi}L, L)$ is a Poincaré duality pair and denote by $[e_{\phi}]$ a choice of orientation class for the pair. By definition we have $\partial_{2s}[e_{\phi}] = [L]$, but $\partial_{2s} = \phi_{2s-1}$.

On the other hand, if $\phi_{2s-1} : H_{2s}(t^{\phi}L, L) \cong H_{2s-1}(L)$ is an isomorphism let us denote by $[e_{\phi}] := \phi_{2s-1}^{-1}([L])$. We then have to check the commutativity of the following square for the different values of i

$$\begin{tikzcd} H^r(t^{\phi}L) \arrow[r, "- \cap [e_{\phi}]"] \arrow[d, "\mathrm{incl}^*"'] & H_{2s-r}(t^{\phi}L, L) \arrow[d, "\partial_{2s-i}"] \\ H^r(L) \arrow[r, "- \cap [L]", "\cong"'] & H_{2s-1-r}(L) \end{tikzcd}$$

and the fact that this induces an isomorphism on the upper row. Which are straightforward calculations. □

We now look at a condition on ϕ to be an isomorphism, condition that is given by the rational Hurewicz theorem 1.2.4.

Lemma 1.3.4.2. ϕ_{2s-1} *is an isomorphism if and only if*

$$\mathrm{H{\small UR}}_{2s-1}([\phi]) \neq 0$$

in $H_{2s-1}(tL) = H_{2s-1}(L) = \mathbf{Q}$.

Since the pair $(X_{reg}, \partial X_{reg})$, with $\partial X_{reg} = L$, satisfies Poincaré-Lefschetz duality this is a Poincaré duality pair with the same boundary that the pair $(t^{\phi}L, L)$. The space

$$\mathcal{DP}(X) := X_{reg} \cup_L t^{\phi}L$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

is then a Poincaré complex of dimension $2s$ whitout boundary and of orientation class given by

$$[\mathcal{DP}(X)] = i^{-1}(i_1 \oplus i_2([X_{reg}, L], [-e_\phi]))$$

We now show the relation between IX and $\mathcal{DP}(X)$. We recall the two following results (see for exemple [44])

Proposition 1.3.4.1. *If* $i\colon A \hookrightarrow B$ *is a cofibration in the diagram* F *given by*

$$C \xleftarrow{f} A \overset{i}{\hookrightarrow} B$$

then the comparison morphism

$$\xi\colon \mathrm{hocolim} F \to \mathrm{colim} F$$

is a homotopy equivalence.

Proposition 1.3.4.2 (2 out 3 for homotopy pushouts)**.** *Consider the following commutative diagram*

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "(I)"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "(II)"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

and denote the outside square by (T)*.*

1. *If* (I) *and* (II) *are homotopy pushout squares, then* (T) *is also a homotopy pushout square.*

2. *If* (I) *and* (T) *are homotopy pushout squares, then* (II) *is also a homotopy pushout square.*

Proposition 1.3.4.3. *Let* X *be a compact, connected oriented pseudomanifold of dimension* $n = 2s$ *with one isolated singularity of link* L *simply connected. The space* $\mathcal{DP}(X)$ *is then rationally homotopy equivalent to* $IX \cup e^{2s}$*. If moreover* $H^1(X_{reg}) = 0$*, then* e^{2s} *is attached by ordinary Whitehead products (not iterated) with respect to some basis of* $\pi_*(IX) \otimes \mathbf{Q}$ *and the rational homotopy type of* $\mathcal{DP}(X)$ *is determined by* IX*.*

Proof. Consider the following diagram.

$$\begin{tikzcd}
 & L = \partial X_{reg} \arrow[r, hook, "i_1"] \arrow[d] \arrow[dr, phantom, "\text{(I) HPO}"] & X_{reg} \arrow[d] \\
S^{2s-1} \arrow[r, "\phi"] \arrow[d, "i_2"'] \arrow[dr, phantom, "\text{(III) HPO}"] & tL \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(II)}"] & IX \arrow[d] \\
D^{2s} \arrow[r] & t^{\phi}L \arrow[r] & \mathcal{DP}(X)
\end{tikzcd}$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

The square (I) is a homotopy pushout by definition of the construction of the space IX, since i_1 is a cofibration $\mathcal{DP}(X)$ is then the homotopy pushout of the diagram $tL_\phi \leftarrow tL \leftarrow L \hookrightarrow M$. By the proposition 1.3.4.2 the square (II) is a homotopy pushout. The square (III) is also a homotopy pushout by definition of $t^\Phi L$, again by the proposition 1.3.4.2 this time applied to the squares (II) and (III) the corresponding outside square is a homotopy pushout. We have the commutative square

$$\begin{tikzcd} S^{2s-1} \arrow[r] \arrow[d] & IX \arrow[d] \\ D^{2s} \arrow[r] & \mathcal{DP}(X) \end{tikzcd}$$

which is then a homotopy pushout.

So we have a rational homotopy equivalence between $\mathcal{DP}(X)$ and $IX \cup e^{2s}$.

Suppose now that we also have $H^1(X_{reg}) = 0$. The space $\mathcal{DP}(X)$ is then simply connected and the theorem 1.3.3 then tells us how e^{2s} is attached to IX. □

In the case of a pseudomanifold of dimension $n = 4s$ with isolated singularities, Markus Banagl showed in [6, theorem 2.28] that the intersection form

$$b_{HI}\colon \widetilde{HI}^{\overline{m}}_{2s}(X) \otimes \widetilde{HI}^{\overline{m}}_{2s}(X) \longrightarrow \mathbf{Q}$$

has the same Witt element that the Goresky-MacPherson intersection form

$$b_{IH}\colon IH^{\overline{m}}_{2s}(X) \otimes IH^{\overline{m}}_{2s}(X) \longrightarrow \mathbf{Q}.$$

that is $[b_{HI}] = [b_{IH}] \in W(\mathbf{Q})$, where $W(\mathbf{Q})$ is the Witt group of the rationals.

Applying the lemma 1.3.2.1 to the Poincaré duality pair $(t^\Phi L, L)$ constructed above shows that this pair is endowed with a symmetric bilinear form of kernel $i_*(H_{2s}(L)) = H_{2s}(t^\Phi L)$, that is the form is the zero form. We then see that the Witt class of the intersection form $[b_{\mathcal{DP}(X)}]$ of $\mathcal{DP}(X)$ is completely determined by the intersection form of the regular part $(X_{reg}, \partial X_{reg})$. Which is also the case of IX as showed in [6, theorem 2.28]. Therefore we have the following corollary.

Corollary 1.3.4.1. *$\mathcal{DP}(X)$ is a Poincaré duality rational space whose Witt class associated to the intersection form $b_{\mathcal{DP}(X)}$ is the same that the Witt class associated to the middle intersection cohomology of X.*

1.3.2.2 *The odd dimensional case*

Consider now X a compact, oriented pseudomanifold of dimension $n = 2s+1$ with one isolated singularity σ of simply connected link L. If we want to extend the construction we just made in the even dimensional case we will have to consider more hypothesis on the link to get our second

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Poincaré duality pair. The pair $(X_{reg}, \partial X_{reg})$ being a Poincaré duality pair whatever the dimension of M is.

We then consider two case :

1. Either we have $t^{k(\overline{m})}L = t^{k(\overline{n})}L$, $k(\overline{m}) = 2n - \overline{m}$ and $k(\overline{n}) = 2n - \overline{n}$ which gives us the case of Witt spaces.
2. Or $t^{k(\overline{m})}L \neq t^{k(\overline{n})}L$ and we must put more algebraic structures on the link to get our Poincaré duality pair. This is the case of L-spaces.

First let us consider that X is a Witt space, that is $H_s(L) = 0$. We then have the following proposition.

Proposition 1.3.4.4. *Let X be a compact, oriented pseudomanifold of dimension $n = 2s + 1$ with one isolated singularity σ of simply connected link L. Suppose moreover that X is a Witt space. The constructions of the even dimensional case extend to this case and there exists a rational Poincaré approximation $\mathcal{DP}(X)$ of X.*

Proof. We have to show that there exists a cotruncation tL of the link of the singularity and a map $\phi \in \pi_{2s}(tL) \otimes \mathbf{Q}$ such that the pair $(t^{\phi}L, L)$ is a Poincaré duality pair. If so, the theorem 1.3.2 and the proposition 1.3.4.3 can be applied.

Consider the cotruncation $t^{k(\overline{n})}L$ given by the upper middle perversity $\overline{n}$. By definition of Witt spaces and of the cotruncation we have

$$H_s(t^{k(\overline{n})}L) = H_s(L) = 0.$$

So in fact $t^{k(\overline{n})}L$ is s-connected and we have

$$t^{k(\overline{m})}L = t^{k(\overline{n})}L := tL.$$

By the rational Hurewicz theorem we have the isomorphism

$$\pi_{2s}(tL) \otimes \mathbf{Q} \xrightarrow{\cong} H_{2s}(tL).$$

We still denote by ϕ the map obtained by this isomorphism, ϕ_{2s} as in the lemma 1.3.4.1 is then a isomorphism and the pair $(t^{\phi}L, L)$ is a Poincaré duality pair. □

When we only have only one isolated singularity, every pseudomanifold X of dimension $n = 2s + 1$ which is not a Witt space is in fact an L-space due to the following result of Thom.

Lemma 1.3.4.3 (Thom, [30]). *Let $(X, [x])$ be a rational Poincaré complex of dimension $n = 4s$ such that $(X, [x])$ is the boundary of a rational Poincaré duality pair $(Y, [y])$. Then $[b_X] = 0 \in W(\mathbf{Q})$.*

The link L of the singularity of X is a manifold of dimension $2s$ and we have the non degenerate bilinear form induced by Poincaré duality.

$$b_L : H^s(L) \times H^s(L) \longrightarrow \mathbf{Q}$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Suppose that X is not a Witt space. By this result of Thom, the Witt class $[b_L] \in W(\mathbf{Q})$ of the intersection form associated to L is zero. This implies that b_L is hyperbolic and we have the existence of a Lagrangian subspace. The pseudomanifold X is then an L-space. This can be resumed in the following corollary to the Thom's lemma.

Corollary 1.3.4.2. *Any compact, connected oriented pseudomanifold of dimension* $4s+i$, $i=1,3$, *with one isolated singularity* σ *of simply connected link* L *is an L-space.*

Let us then fix X a compact oriented L-space of dimension $4s+i$, $i=1,3$, with one isolated singularity σ of simply connected link L. The link L of the singularity is of dimension $4s+i-1$ and thanks to the lemma 1.3.4.3 we have

$$H^{2s+\frac{i-1}{2}}(L) = V \oplus V^*$$

where V is a Lagrangian subspace of dimension $\frac{1}{2}\dim H^{2s+1}(L) := m$.

Suppose $(a_1,\dots,a_m)$ is a basis of V and complete it into a hyperbolic basis using theorem 1.2.9. Apply then the Lagrangian truncation to L and denote by $t^{\overline{\mathcal{L}}}L$ the homotopy cofiber of the map

$$t_{\overline{\mathcal{L}}}L \longrightarrow L.$$

Recall that

$$H^r(t^{\overline{\mathcal{L}}}L) = \begin{cases} \mathbf{Q} & \text{if } r=0, \\ 0 & \text{if } 1 \leqslant r \leqslant 2s, \\ V & \text{if } r = 2s+\frac{i-1}{2}, \\ H^*(L) & \text{if } r > 2s+\frac{i-1}{2}. \end{cases}$$

Let $(M(t^{\overline{\mathcal{L}}}L), d)$ be the Sullivan minimal model of this Lagrangian cotruncation and let ϖ be the element of degree $4s+i-1$ in $(M(t^{\overline{\mathcal{L}}}L), d)$ representing the fundamental class $[L]^* \in H^{4s+i-1}(L)$.

Proposition 1.3.4.5. *Suppose* ϖ *is an indecomposable element of* $(M(t^{\overline{\mathcal{L}}}L), d)$, *then there exists a map* $\phi \in \pi_{4s+i-1}(t^{\overline{\mathcal{L}}}L) \otimes \mathbf{Q}$ *such that* $\mathrm{Hur}_{4s+i-1}([\phi]) = 1$.

Proof. Suppose ϖ is an indecomposable element, that is $\varpi \in W^{4s+i-1}$ where $W = \oplus_{k\geqslant 0} W^k$ is the graded vector space generating $M(t^{\overline{\mathcal{L}}}L)$. So using theorem 1.2.3, we have the natural isomorphism

$$W^{4s+i-1} \xrightarrow{\cong} \hom(\pi_{4s+i-1}(t^{\overline{\mathcal{L}}}L), \mathbf{Q}).$$

That natural isomorphism is the same as saying that the bilinear pairing

$$\langle -,- \rangle : W \times \pi_*(t^{\overline{\mathcal{L}}}L) \longrightarrow \mathbf{Q}$$

defined by

$$\langle v;\alpha\rangle = \begin{cases} Q(a)(v) & \text{if } v \in W^k \\ 0 & \text{if } \deg v \neq \deg \alpha. \end{cases}$$

© 2016 Tous droits réservés.

Thèse de Mathieu Klimczak, Lille 1, 2016

is non degenerate, where $\alpha \in \pi_*(t^{\overline{\mathcal{L}}}L)$ is represented by a. So there is a map $\varphi \in \pi_{4s+i-1}(t^{\overline{\mathcal{L}}}L)$ such that $\langle \varpi, [\varphi] \rangle \neq 0$.

Recall that

$$m_{t^{\overline{\mathcal{L}}}L} : (\wedge W, d) = (M(t^{\overline{\mathcal{L}}}L), d) \longrightarrow A_{PL}(t^{\overline{\mathcal{L}}}L)$$

denotes the minimal Sullivan model of $t^{\overline{\mathcal{L}}}L$ and denote by

$$\textsc{Hur}_k \colon \pi_k(t^{\overline{\mathcal{L}}}L) \otimes \mathbf{Q} \longrightarrow H_k(t^{\overline{\mathcal{L}}}L)$$

the Hurewicz map. Since $\operatorname{im} d \subset \wedge^{\geqslant 2} W$, division by $\operatorname{im} d$ defines a linear map $\xi \colon H^+(\wedge W) \to W$, since ϖ represents the fundamental class $[L]^* \in H^{4s+i-1}(L) = H^{4s+i-1}(t^{\overline{\mathcal{L}}}L)$, we clearly have a element $[\varpi] \in H^+(\wedge W)^{4s+i-1}$ such that $\xi([\varpi]) = \varpi$.

Denote by $\{-,-\}$ the bilinear pairing between cohomology and homology defined by $\{[f],[c]\} := f(c)$, since we work on $\mathbf{Q}$ that pairing is also non degenerate. By the definition of the pairing $\langle -,- \rangle$ we have

$$\langle \xi([\varpi]), [\varphi] \rangle = \{H(m_{t^{\overline{\mathcal{L}}}L})[\varpi], \textsc{Hur}_{4s+i-1}(\varphi)\} \neq 0.$$

Since $H(m_{t^{\overline{\mathcal{L}}}L})[\varpi] = [L]^*$, we have $\textsc{Hur}_{4s+i-1}(\varphi) = q[L]$ with $q \in \mathbf{Q} - \{0\}$. Then

$$\phi := \frac{1}{q}\varphi \in \pi_{4s+i-1}(t^{\overline{\mathcal{L}}}L) \otimes \mathbf{Q}$$

is the map we wanted. □

Lemma 1.3.4.4. *ϖ is an indecomposable element of $(M(t^{\overline{\mathcal{L}}}L), d)$, that is $\varpi \in W^{4s+i-1}$ where $W = \oplus_{k \geqslant 0} W^k$ is the graded vector space generating $M(t^{\overline{\mathcal{L}}}L)$.*

Proof. The elements of degree $2s + \frac{i-1}{2}$ of $(M(t^{\overline{\mathcal{L}}}L), d)$ come from the Lagrangian V so for all elements $x, y \in M(t^{\overline{\mathcal{L}}}L)^{2n+\frac{i-1}{2}}$ there exists an element $z \in M(t^{\overline{\mathcal{L}}}L)^{4n+i-2}$ such that $dz = x \cdot y$, in particular, none of these products are equal to ϖ. For degree reasons these were the only elements we had to care about. □

Denote by ϕ the element of $\pi_{4s+i-1}(t^{\overline{\mathcal{L}}}L) \otimes \mathbf{Q}$ obtained by this process, like in the general case, and consider the homotopy pushout.

$$\begin{tikzcd}
S^{4s+i-1} \arrow[r, hook] \arrow[d, "\phi"'] \arrow[dr, phantom, "\text{HPO}"] & D^{4s+i} \arrow[d] \\
t^{\overline{\mathcal{L}}}L \arrow[r] & t^{\overline{\mathcal{L}},\phi}L
\end{tikzcd}$$

Proposition 1.3.4.6. *$(t^{\overline{\mathcal{L}},\phi}L, L)$ is a Poincaré duality pair if and only if ϕ_{4s+i-1} is an isomorphism.*

We denote by $\mathcal{DP}(X)$ the space obtained by the glueing of the two Poincaré duality pairs $(X_{reg}, \partial X_{reg}) = (X_{reg}, L)$ and $(t^{\overline{\mathcal{L}}}L_\phi, L)$ following the theorem 1.3.2. We then have the last part of the theorem 1.1.1.

© 2016 Tous droits réservés.
lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Proposition 1.3.4.7. *Let X be a compact, oriented pseudomanifold of dimension $n = 4s+1$ or $n = 4s+3$ with one isolated singularity σ of simply connected link L. Suppose moreover that X is an L-space. Then there exists a rational Poincaré approximation $\mathcal{DP}(X)$ of X.*

Just like in the even dimensional case with the proposition 1.3.4.3. We can relate the spaces $\mathcal{DP}(X)$ to the intersection and Lagrangian intersections spaces, and get more precision on how to attach the top cell in the simply connected case. When X is a Witt space, we denote by $IX = I^{\overline{m}}X = I^{\overline{n}}X$.

Proposition 1.3.4.8. *Let X be a compact, connected oriented pseudomanifold of dimension $n = 2s+1$ with one isolated singularity of link L simply connected.*

1. *Suppose X is a Witt space. The space $\mathcal{DP}(X)$ is then rationally homotopy equivalent to $IX \cup e^{2s+1}$. If moreover $H^1(X_{reg}) = 0$, then e^{2s+1} is attached by ordinary Whitehead products (not iterated) with respect to some basis of $\pi_*(IX) \otimes \mathbf{Q}$ and the rational homotopy type of $\mathcal{DP}(X)$ is determined by IX.*

2. *Suppose X is an L-space. The space $\mathcal{DP}(X)$ is then rationally homotopy equivalent to $\mathcal{I}^{\overline{\mathcal{L}}}X \cup e^{2s+1}$. If moreover $H^1(X_{reg}) = 0$, then e^{2s+1} is attached by ordinary Whitehead products (not iterated) with respect to some basis of $\pi_*(\mathcal{I}^{\overline{\mathcal{L}}}X) \otimes \mathbf{Q}$ and the rational homotopy type of $\mathcal{DP}(X)$ is determined by $\mathcal{I}^{\overline{\mathcal{L}}}X$.*

Proof. The proof is the same as for the proposition 1.3.4.3 unless we consider the following diagram when X is a Witt space.

$$\begin{tikzcd}
& L = \partial X_{reg} \arrow[r, hook, "i_1"] \arrow[d] \arrow[dr, phantom, "\text{(I) HPO}"] & X_{reg} \arrow[d] \\
S^{2s} \arrow[r, "\phi"] \arrow[d, "i_2"'] \arrow[dr, phantom, "\text{(III) HPO}"] & tL \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(II)}"] & IX \arrow[d] \\
D^{2s+1} \arrow[r] & t^{\phi}L \arrow[r] & \mathcal{DP}(X)
\end{tikzcd}$$

and the following diagram when X is an L-space.

$$\begin{tikzcd}
& L = \partial X_{reg} \arrow[r, hook, "i_1"] \arrow[d] \arrow[dr, phantom, "\text{(I) HPO}"] & X_{reg} \arrow[d] \\
S^{2s} \arrow[r, "\phi"] \arrow[d, "i_2"'] \arrow[dr, phantom, "\text{(III) HPO}"] & t^{\overline{\mathcal{L}}}L \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(II)}"] & \mathcal{I}^{\overline{\mathcal{L}}}X \arrow[d] \\
D^{2s+1} \arrow[r] & t^{\overline{\mathcal{L}},\phi}L \arrow[r] & \mathcal{DP}(X)
\end{tikzcd}$$

□

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

The considerations whether the approximations are good or very good come from the study of the rational homology of these diagrams of homotopy pushouts

$$\begin{tikzcd} L \arrow[r, hook] \arrow[d] & X_{reg} \arrow[d, "\phi"'] \arrow[dd, bend left, "i"] \\ t^{\Phi}L \arrow[r] \arrow[d] & \mathcal{DP}(X) \arrow[d, "\psi"'] \\ * \arrow[r] & X \end{tikzcd}$$

for the even dimensional case and the Witt space case, or

$$\begin{tikzcd} L \arrow[r, hook] \arrow[d] & X_{reg} \arrow[d, "\phi"'] \arrow[dd, bend left, "i"] \\ t^{\overline{\mathcal{L}},\Phi}L \arrow[r] \arrow[d] & \mathcal{DP}(X) \arrow[d, "\psi"'] \\ * \arrow[r] & X \end{tikzcd}$$

for the L-space case.

1.3.3 *The multiple isolated singularities case*

The theorem 1.3.2 did not make any assumptions on the connectivity of the pairs $(X_j, Y_j, [x_j])$, so in fact we can apply everything that was above to the case of a pseudomanifold with more than one isolated singularity.

Theorem 1.3.5 (Multiple isolated singularities case)**.** *Let* X *be a compact, connected oriented pseudomanifold of dimension* n *with only isolated singularities* $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$, $\nu > 1$, *of links* L_i *simply connected. Then,*

1. *If* $n = 2s$, *there exists a good rational Poincaré approximation* $\mathcal{DP}(X)$ *of* X. *Moreover, if* $\dim X \equiv 0 \mod 4$, *the Witt class associated to the intersection form* $b_{\mathcal{DP}(X)}$ *is the same as the Witt class associated to the middle intersection cohomology of* X *in* $W(\mathbf{Q})$.

2. *If* $n = 2s + 1$ *and is either a Witt space or an L-space there exists a good rational Poincaré approximation* $\mathcal{DP}(X)$ *of* X. *Moreover is* X *is Witt space* $\mathcal{DP}(X)$ *is a very good rational Poincaré approximation of* X

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Just like before, the considerations whether the approximations are good or very good come from the study of the rational homology of these diagrams of homotopy pushouts

$$\begin{tikzcd}
\bigsqcup_{\sigma_i} L_i \arrow[r, hook] \arrow[d] & X_{reg} \arrow[d, "\phi"'] \arrow[dd, bend left, "i"] \\
\bigsqcup_{\sigma_i} t^{\phi_i} L_i \arrow[r] \arrow[d] & \mathcal{DP}(X) \arrow[d, "\psi"'] \\
\bigsqcup_{\sigma_i} * \arrow[r] & X
\end{tikzcd}$$

for the even dimensional case and the Witt space case, or

$$\begin{tikzcd}
\bigsqcup_{\sigma_i} L_i \arrow[r, hook] \arrow[d] & X_{reg} \arrow[d, "\phi"'] \arrow[dd, bend left, "i"] \\
\bigsqcup_{\sigma_i} t^{\mathcal{L},\phi_i} L \arrow[r] \arrow[d] & \mathcal{DP}(X) \arrow[d, "\psi"'] \\
\bigsqcup_{\sigma} * \arrow[r] & X
\end{tikzcd}$$

for the L-space case.

1.3.3.1 *The even dimensional case*

Let X be a compact oriented pseudomanifold of dimension $n = 2s$ with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu; \nu > 1\}$ of simply connected links L_i.

The rational Hurewicz theorem gives us maps ϕ_i such that the pairs $(t^{\phi_i}L_i, L_i)$ are Poincaré duality pairs for all i. Denote by $[e_\phi]_i$ the induced orientation class in $H_{2s}(t^{\phi_i}L_i, L_i)$.

The pair $(X_{reg}, \partial X_{reg})$ with $\partial X_{reg} = \sqcup_{\sigma_i} L_i$ is still a manifold with boundary, thus satisfies Poincaré-Lefschetz duality and is a Poincaré duality pair. The theorem 1.3.2 then applies and, with the same notation, $\mathcal{DP}(X)$ is an oriented Poincaré complex of dimension $2s$ without boundary and of orientation class given by

$$[\mathcal{DP}(X)] = i^{-1}(i_1 \oplus i_2([X_{reg}, \partial X_{reg}], [-e_\phi]_1, \dots, [-e_\phi]_r)),$$

and all the results obtained before remain true except for the proposition 1.3.4.3 which has to be modified. We have to take the normal intersection space $\mathcal{I}X = \mathcal{I}^{\overline{m}}X = \mathcal{I}^{\overline{n}}X$ to modify the proposition 1.3.4.3. Which then becomes

Proposition 1.3.5.1. *Let X be a compact, connected oriented pseudomanifold of dimension $n = 2s$ with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu; \nu >$*

© 2016 Tous droits réservés.
lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

1\} of links L_i simply connected. Suppose moreover that $H^1(X_{reg}) = 0$. Then $\mathcal{DP}(X)$ is rationally homotopy equivalent to $t_{2s-1}\mathcal{I}X \cup e^{2s}$ where e^{2s} is attached by ordinary Whitehead products (not iterated) with respect to some basis of $\pi_*(t_{2s-1}\mathcal{I}X) \otimes \mathbf{Q}$ and the rational homotopy type of $\mathcal{DP}(X)$ is determined by $t_{2s-1}\mathcal{I}X$.

Proof. Consider the following diagram, obtained by the construction of $\mathcal{DP}(X)$

$$\begin{tikzcd}
 & \bigsqcup_{\sigma_i} L_i = \partial X_{reg} \arrow[r, hook, "i_1"] \arrow[d] \arrow[dr, phantom, "\text{(I) HPO}"] & X_{reg} \arrow[d] \\
\bigsqcup_{\sigma_i} S_i^{2s-1} \arrow[r, "\bigsqcup_{\sigma_i} \phi_i"] \arrow[d, "i_2"'] \arrow[dr, phantom, "\text{(III) HPO}"] & \bigsqcup_{\sigma_i} tL_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(II)}"] & \mathcal{I}X \arrow[d] \\
\bigsqcup_{\sigma_i} D_i^{2s} \arrow[r] & \bigsqcup_{\sigma_i} t^{\phi_i} L_i \arrow[r] & \mathcal{DP}(X)
\end{tikzcd}$$

With the same arguments than for the unique isolated singularity case, $\mathcal{DP}(X)$ is rationally homotopy equivalent to

$$\mathcal{DP}(X) \simeq \mathcal{I}X \cup (\bigcup_{\phi_i} e_i^{2s}).$$

Now, $H^1(X_{reg}) = 0$ so $\mathcal{DP}(X)$ is simply connected and the theorem 1.2.6 gives a rational homotopy equivalence

$$\varphi \colon \widetilde{\mathcal{DP}(X)} \longrightarrow \mathcal{DP}(X)$$

such that the differential in the integral cellular chain complex of $\widetilde{\mathcal{DP}(X)}$ is identically zero. This implies that there is only one top dimensional cell on $\widetilde{\mathcal{DP}(X)}$, we have a attaching map θ and a cell e^{2s} such that

$$\widetilde{\mathcal{DP}(X)} = X_0 \cup_\theta e^{2s}.$$

Let us now determine X_0. By the theorem 1.2.6 X_0 is a CW-complex of dimension $2s-1$ and by the Poincaré duality of $\mathcal{DP}(X)$ we have

$$H_{2s-1}(X_0) = H_{2s-1}(\mathcal{DP}(X)) \cong H^1(\mathcal{DP}(X)) = H^1(X_{reg}) = 0.$$

For any other $r \leqslant 2s-2$ we have by construction $H_r(X_0) = H_r(\mathcal{I}X)$.

$$\begin{tikzcd}
X_0 \arrow[rrrr, "\text{incl}"] \arrow[d, "\varphi_|"'] & & & & X_0 \cup_\theta e^{2s} \arrow[d, "\varphi"] \\
t_{2s-1}\mathcal{I}X \arrow[r, "i"'] & \mathcal{I}X/2s-1 \arrow[r, "h"'] & \mathcal{I}X^{2s-1} \arrow[r, "j"'] & \mathcal{I}X \arrow[r] & \mathcal{DP}(X)
\end{tikzcd}$$

© 2016 Tous droits réservés.
lilliad.univ-lille.fr

The above diagram commutes and by the same argument that the one given before the proposition 1.2.7.2, φ restricts to a rational homotopy equivalence

$$\varphi_{|}\colon X_0 \longrightarrow t_{2s-1}\mathcal{I}X.$$

Then, up to rational homotopy equivalence, we have

$$\mathcal{DP}(X) = t_{2s-1}\mathcal{I}X \cup_\theta e^{2s}.$$

The theorem 1.3.3 then tells us that e^{2s} is attached by ordinary Whitehead products (not iterated) with respect to some basis of $\pi_*(t_{2s-1}\mathcal{I}X)\otimes\mathbf{Q}$ and the rational homotopy type of $\mathcal{DP}(X)$ is determined by $t_{2s-1}\mathcal{I}X$. □

The others results remain true, in particular $\mathcal{DP}(X)$ is a rational Poincaré duality space and we have the first part of the theorem 1.1.2.

Proposition 1.3.5.2. *If $\dim X = 2s$, then $\mathcal{DP}(X)$ is a good rational Poincaré approximation of X. Moreover, if $\dim X \equiv 0 \mod 4$, then the Witt class associated to the intersection form $b_{\mathcal{DP}(X)}$ is the same that the Witt class associated to the middle intersection cohomology of X in $W(\mathbf{Q})$.*

1.3.3.2 *The odd dimensional case*

Let X be a compact, connected oriented pseudomanifold of dimension $n = 2s+1$ with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu; \nu > 1\}$ of links L_i simply connected.

Suppose that X is a Witt space. Just as in the case of a unique isolated singularity we have

$$H_s(t^{k(\overline{m})}L_i) = H_s(L_i) = 0.$$

So in fact $t^{k(\overline{m})}L_i$ are s-connected and we have

$$t^{k(\overline{m})}L_i = t^{k(\overline{n})}L_i := tL_i.$$

The rational Hurewicz theorem gives us maps ϕ_i such that the pairs $(t^{\phi_i}L_i, L_i)$ are Poincaré duality pairs for all i. Applying the theorem 1.3.2 in the case of multiple isolated singularities gives us then a rational Poincaré duality space $\mathcal{DP}(X)$ and the following part of the theorem 1.1.2.

Proposition 1.3.5.3. *Let X be a compact, connected oriented pseudomanifold of dimension $n = 2s+1$ with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu; \nu > 1\}$ of links L_i simply connected. Suppose moreover X is a Witt space, then $\mathcal{DP}(X)$ is a very good rational Poincaré duality space.*

Suppose now X is an L-space. For each link L_i of the pseudomanifold X^{4s+i}, $i = 1, 3$ we have

$$H^{2s+j}(L_i) = V_i \oplus V_i^* \quad j = 0, 2$$

where V_i and V_i^* are Lagrangian subspaces of dimensions

$$\frac{1}{2}\dim H^{2n+j}(L_i) := m_i.$$

© 2016 Tous droits réservés.

Thèse de Mathieu Klimczak, Lille 1, 2016

Applying the Lagrangian truncation to each of the links, the lemma 1.3.4.4 then give us maps ϕ_i such that the pairs $(t^{\overline{\mathcal{L}},\phi_i}L_i, L_i)$ are Poincaré duality pairs for all i. The theorem 1.3.2 in the case of multiples isolated singularities gives us then a rational Poincaré duality space, which concludes the theorem 1.1.2.

Proposition 1.3.5.4. *Let X be a compact, oriented pseudomanifold of dimension $n = 2s+1$ with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu; \nu > 1\}$ of simply connected links L_i. Suppose moreover X is an L-space, then $\mathcal{DP}(X)$ is a good rational Poincaré approximation of X.*

1.4 EXAMPLES AND APPLICATIONS

1.4.1 *Real Algebraic varieties*

If V is a real algebraic variety of even dimension with isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ and an oriented regular part V_{reg}. We can apply the homological truncation and then by the use of the precedents results construct a rational Poincaré approximation $\mathcal{DP}(V)$.

The odd dimensional is more interesting. Suppose that V is a real algebraic variety of with multiple isolated singularities of odd dimension. If the regular part V_{reg} of V is oriented then V is automatically an L-space due to the following result of Selman Akbulut and Henry King :

Theorem 1.4.1 ([1]). *Let V be a compact topological space. Then the following are equivalent :*

1. *V is homeomorphic to a real algebraic set with isolated singularities.*
2. *V is homeomorphic to the quotient obtained by taking a smooth closed manifold M and collapsing each L_i to point a point where L_i, $i = 1, \dots, \nu$ is a collection of disjoint smooth subpolyhedra of M.*
3. *$V = M \cup \bigcup_{i=1}^{\nu} cL_i$ where M and L_i are smooth compact manifolds, ∂M is the disjoint union of the L_i's, each L_i bounds a smooth compact manifolds and $L_i \times 1 \subset cL_i$ is identified with $L_i \subset M$.*

Consider then V a oriented real algebraic variety of dimension $n = 4s+1$ with ν isolated singularities. Then by the third equivalence we have

$$V = M \cup \bigcup_{i=1}^{\nu} cL_i$$

and each link L_i is a smooth compact manifold of dimension $4s$ and is the boundary of a $4s+1$ smooth compact manifold. Then by the lemma 1.3.4.3 we have that

$$[b_i] = 0 \in W(\mathbf{Q}) \,\forall i.$$

We then can perform a Lagrangian truncation and we have our rational Poincaré approximation $\mathcal{DP}(V)$.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Note that if V is of dimension $4n+3$ then we don't need this result because the bilinear form b_L would be skew-symmetric.

We then have the following result

Proposition 1.4.1.1. *Every oriented real algebraic variety V with only isolated singularities and simply connected links admits at least a good rational Poincaré approximation $\mathcal{DP}(V)$.*

1.4.2 *Hypersurfaces with nodal singularities*

Let V be a complex projective hypersurface with one nodal singularity such that $\dim_{\mathbf{C}} V = 3$. The link of this singularity is then $L = S^2 \times S^3$, applying the method of theorem 1.3.3 and example 1.3.1 to this case the link of the singularity is then rationally homotopy equivalent to

$$L \simeq (S^2 \vee S^3) \bigcup_{[s_2^\sharp, s_3^\sharp]} e^5.$$

Since $\overline{m}(6) = \overline{n}(6) = 2$, the homological truncation of the link is

$$t_2 L = S^2$$

and the cotruncation is rationally homotopy equivalent to

$$t^2 L = S^3 \vee S^5.$$

To see this, just compute the cohomology algebra of the cotruncation. The rational Hurewics theorem 1.2.4 then says that we have the isomorphism

$$\pi_5(t^2 L) \otimes \mathbf{Q} \xrightarrow{\cong} H_5(t^2 L) \cong H_5(S^5).$$

The cell attachment ϕ obtained by this isomorphism then kill the 5-sphere of the cotruncation. That is we have

$$t^2 L_\phi = (S^3 \vee S^5) \cup_{s_5^\sharp} e^6 \simeq S^3.$$

But $\tau L_\phi = S^3 \simeq D^3 \times S^3$ ans since $\partial(D^3 \times S^3) = S^2 \times S^3$, the pair $(\tau L_\phi, L) = (D^3 \times S^3, S^2 \times S^3)$ is a Poincaré duality pair. The space $\mathcal{DP}(V)$ is then a good rational Poincaré approximation of X.

This construction extends to multiple isolated singularities and higher dimension complex hypersurfaces with nodal singularities.

1.4.3 *Thom Spaces*

Definition 1.4.1.1. *Let B be a compact, connected, oriented manifold of dimension m and E a fiber bundle over B of rank m',*

$$\mathbf{R}^{m'} \longrightarrow E \longrightarrow B.$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

The Thom space $\mathrm{Th}(E)$ *of the fiber bundle* E *is defined as the homotopy cofiber of the map*

$$S_E \longrightarrow D_E$$

where S_E *and* D_E *are respectively the sphere bundle and disk bundle associated to* E*.*

$\mathrm{Th}(E)$ is then a pseudomanifold of dimension $m+m'$, the singularity is the compactification point, its link is the sphere bundle S_E and the regular part of $\mathrm{Th}(E)$ is the disk bundle D_E.

We show that in the case of an odd dimensional Thom space $\mathrm{Th}(E)$ is either an L-space or a Witt space whether the rank of the vector bundle is lesser than the dimension of the base space or not.

Theorem 1.4.2. *Suppose* $m' > 0$*.*

1. *Let* $\mathbf{R}^{2m'} \longrightarrow E \longrightarrow B^{2m+1}$ *with* B *be a manifold of dimension* $2m+1$ *and* E *a fiber bundle over* B *of rank* $2m'$*. Then,*
   - *if* $m' \leqslant m+1$, $\mathrm{Th}(E)$ *is an L-space,*
   - *if* $m' > m+1$, $\mathrm{Th}(E)$ *is a Witt space if and only if* $H^{m+m'}(B) = 0$*.*
2. *Let* $\mathbf{R}^{2m'+1} \longrightarrow E \longrightarrow B^{2m}$ *with* B *be a manifold of dimension* $2m$ *and* E *a fiber bundle over* B *of rank* $2m'+1$*. Then,*
   - *if* $m' \leqslant m$, $\mathrm{Th}(E)$ *is an L-space,*
   - *if* $m' > m$, $\mathrm{Th}(E)$ *is a Witt space if and only if* $H^{m+m'}(B) = 0$*.*

Proof. Consider $\mathbf{R}^{2m'} \longrightarrow E \longrightarrow B^{2m+1}$.

In order to know if $\mathrm{Th}(E)$ is an L-space or a Witt space we have to look at $H^{m+m'}(S_E)$ where S_E is the sphere bundle associated to the vector bundle E (see definitions 1.2.7.3 and 1.2.11.1). To compute $H^{m+m'}(S_E)$ we use the cohomological Leray-Serre spectral sequence associated to the fiber bundle

$$S^{2m'-1} \longrightarrow S_E \longrightarrow B^{2m+1}.$$

We have $E_2^{p,q} = H^p(B; H^q(S^{2m'-1})) = 0$ if $q \neq 0, 2m'-1$ and

$$d_{2m'} : E_{2m'}^{p,2m'-1} \longrightarrow E_{2m'}^{p+2m',0}$$

is the only non-zero differential which is defined by

$$d_{2m'} : E_{2m'}^{0,2m'-1} \longrightarrow E_{2m'}^{2m',0}$$

with $d_{2m'}(a) = eu(E) \in H^{2m'}(B)$ with a the generator of $H^{2m'-1}(S^{2m'-1})$ and $eu(E) \in H^{2m'}(B)$ the Euler class of the sphere bundle. Since this is the only non-zero differential we have

$$H^{m+m'}(S_E) = E_{2m'+1}^{m+m',0} \oplus E_{2m'+1}^{m-m'+1,2m'-1}.$$

Suppose that $m' \leqslant m+1$.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

The summand $E_{2m'+1}^{m-m'+1,2m'-1}$ is well defined and using the structure product of the spectral sequence, we see that the product of two elements belonging to the same summand of $H^{m+m'}(S_E)$ is zero. Then by Poincaré duality the symmetric bilinear form

$$E_{2m'+1}^{m+m',0} \times E_{2m'+1}^{m-m'+1,2m'-1} \longrightarrow E_{2m'+1}^{2m+1,2m'-1} \cong \mathbf{Q}\omega a$$

induced by the product and where $\omega \in H^{2m+1}(B)$ is the fundamental class of the manifold B is non degenerate. Thus, provided than one of the two summand is non zero, the symmetric bilinear form is then hyperbolic and S_E is an L-space.

Suppose that $m' > m+1$.

Then $E_{2m'+1}^{m-m'+1,2m'-1} = 0$ and $E_{2m'+1}^{m+m',0} = H^{m+m'}(B)$ and S_E is a Witt space if and only is $H^{m+m'}(B) = 0$.

We now consider $\mathbf{R}^{2m'+1} \longrightarrow E \longrightarrow B^{2m}$. By the same arguments we have

$$H^{m+m'}(S_E) = E_{2m'+2}^{m+m',0} \oplus E_{2m'+2}^{m-m',2m'}.$$

If $m' \leqslant m$ then $E_{2m'+2}^{m-m',2m'}$ is well defined, the same arguments about the product structure and Poincaré duality imply that S_E is an L-space.

If $m' > m$ then $E_{2m'+2}^{m-m',2m'} = 0$, $E_{2m'+2}^{m+m',0} = H^{m+m'}(B)$ and $Th(E)$ is a Witt space if and only if $H^{m+m'}(B) = 0$. $\square$

Corollary 1.4.2.1. *For any complex line bundle $\mathbf{C} \longrightarrow E \longrightarrow B^k$ with $k \geqslant 1$, the Thom space $Th(E)$ is an L-space.*

Remark 1.4.3. *The cases where the fiber and the base space have the same parity, that is they both are of odd dimension or they both are of even dimension has already been taken care of by the definition 1.2.7.3. The Thom space is then an even dimensional space and the singularity is of even codimension. Therefore $Th(E)$ is automatically a Witt space by definition.*

Let $B^9 := (S^3 \times S^6)\sharp(S^4 \times S^5)$, where $\sharp$ denotes the connected sum, and let $f : B^9 \to S^4$ the composition of the following contraction map q and projection map p :

$$B^9 \xrightarrow{q} S^4 \times S^5 \xrightarrow{p} S^4.$$

Let E be the fiber bundle over B^9 that is the pullback along f of the tangent space over S^4,

$$\begin{tikzcd}
\mathbf{R}^4 \arrow[d] & \mathbf{R}^4 \arrow[d] \\
E := f^*(TS^4) \arrow[r] \arrow[d] & TS^4 \arrow[d] \\
B^9 \arrow[r, "f"] & S^4
\end{tikzcd}$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

The Thom space $\mathrm{Th}(E)$ associated to this bundle is a pseudomanifold of dimension 13 and is an L-space, indeed we show using the Leray Serre spectral sequence that $H^6(S_E) = \mathbf{Q} \oplus \mathbf{Q}$ and that both of these factors are Lagrangian subspaces. The sphere bundle associated to E is the bundle

$$S^3 \longrightarrow S_E \longrightarrow B^9,$$

applying the cohomological Leray Serre spectral sequence to this bundle, we have the following E_2 page

$$E_2^{p,q} = H^p(B^9; H^q(S^3))$$

$$\begin{tikzcd}
3 & \mathbf{Q}a \arrow[rrrrd, "d_4"] & 0 & 0 \arrow[rrrrd] & \mathbf{Q}s_3a \arrow[rrrrd] & \mathbf{Q}s_4a & \mathbf{Q}s_5a & \mathbf{Q}s_6a & 0 & 0 & \mathbf{Q}\omega a \\
0 & \mathbf{Q}1 & 0 & 0 & \mathbf{Q}s_3 & \mathbf{Q}s_4 & \mathbf{Q}s_5 & \mathbf{Q}s_6 & 0 & 0 & \mathbf{Q}\omega \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{tikzcd}$$

In this page, s_i represents the generator of the sphere S^i in B^9, ω the fundamental class of B^9 with $\omega = s_4 s_5 = s_3 s_6$ and a the generator of the fiber S^3 of the sphere bundle.

The only non zero differential is d_4 so $E_2 = E_3 = E_4$ and $E_\infty = E_5$, d_4 is completely determined by the Euler class $eu(E)$ of the sphere bundle because $d_4 a = eu(E)$. We know that $eu(TS^4) = 2[S^4]$ so by naturality of the Euler class we have

$$eu(E) = eu(f^*(TS^4)) = f^*(eu(TS^4)) = 2s_4.$$

We then have

$$H^6(S_E) = \mathbf{Q}s_6 \oplus \mathbf{Q}s_3 a.$$

Using the product structures of the spheres S^6 and S^3, we have $s_6 s_6 = 0$ and $(s_3 a)(s_3 a) = 0$, but since $\omega = s_3 s_6$ the matrix of the intersection form

$$H^6(S_E) \times H^6(S_E) \longrightarrow \mathbf{Q}$$

in the base $(s_6, s_3 a)$ is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

The intersection form is then hyperbolic and both factors $\mathbf{Q}s_6$ and $\mathbf{Q}s_3a$ are Lagrangian subspaces.

We now construct a rational model of $\mathcal{DP}(\mathrm{Th}(E))$. For that we'll need a surjective model of $S_E \hookrightarrow D_E$ and a model of the Lagrangian truncation $t^{\overline{\mathcal{L}}}S_E$. A surjective model of $S_E \hookrightarrow D_E$ is given by

$$A(D_E) \overset{\varphi}{\twoheadrightarrow} A(S_E)$$

with

$$\begin{cases} A(S_E) & = (A(B) \otimes \wedge a, d) \text{ with } da = s_4 \\ A(D_E) & = (A(B) \otimes \wedge(a,b), D) \text{ with } Da = s_4 - b \\ \varphi_{|A(B)\otimes\wedge a} & = \mathrm{id} \\ \varphi(b) & = 0 \end{cases}$$

where $A(B)$ is a rational model of the base space $(S^3 \times S^6)\sharp(S^4 \times S^5)$ which is given by

$$A(B) = (\wedge(s_3, s_4, s_5, s_6, \beta_6, \beta_{7,1}, \beta_{7,2}, \beta_8, \dots), d)$$

with $|s_i| = |\beta_i| = i$ and

$$\begin{cases} d\beta_6 & = s_3 s_4 \\ d\beta_{7,1} & = s_4^2 \\ d\beta_{7,2} & = s_5 s_3 \\ d\beta_8 & = s_3 s_6 - s_4 s_5. \end{cases}$$

Since $\dim B = 9$ we only gave elements of the model of B up to degree 9, the rest of the model being an acyclic part. That is for every element α_k of degree $k \geqslant 10$ such that $d\alpha_k = 0$, there is an element β_{k-1} such that $d\beta_{k-1} = \alpha_k$. In fact we can take take a better model for $A(S_E)$ and $A(D_E)$ because the base space B is a formal space, that is we have a quasi isomorphism

$$\psi : (A(B), d) \longrightarrow (H(B), 0)$$

given by

$$\begin{cases} \psi(s_i) & = s_i \\ \psi(\beta_i) & = 0 \\ \psi(A^{\geqslant 10}(B)) & = 0. \end{cases}$$

The models we use are then

$$\begin{cases} A(S_E) & = (H(B) \otimes \wedge a, d) \text{ with } da = s_4 \\ A(D_E) & = (H(B) \otimes \wedge(a,b), D) \text{ with } Da = s_4 - b \\ \varphi_{|H(B)\otimes\wedge a} & = \mathrm{id} \\ \varphi(b) & = 0 \end{cases}$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

By adapting the proposition 1.2.7.2 and then using the lemma 1.2.7.1 we can show that a model of the Lagrangian cotruncation is given by

$$(A(t^{\overline{\mathcal{L}}}S_E), d) = (\mathbf{Q} \oplus I_{\mathcal{L}}, d)$$

where $I_{\mathcal{L}}$ is the differential ideal given in this case by a choice of generator for one of the Lagrangian subspaces, a complementary of $\ker(d : A^6(S_E) \to A^7(S_E))$ and all the cochains of degree greater or equal to 7 of $A(S_E)$. A choice of $A(t^{\overline{\mathcal{L}}}S_E))$ is given by

$$\mathbf{Q} \oplus (\mathbf{Q}s_6 \otimes (H(B) \otimes \wedge a)^{\geqslant 7})$$

and the model of $I^{\overline{\mathcal{L}}}Th(E)$ is

$$A^*(I^{\overline{\mathcal{L}}}Th(E)) = (H(B) \otimes \wedge(a, b), D) \oplus_{H(B)\otimes\wedge a} (\mathbf{Q} \oplus I_{\mathcal{L}}, d).$$

We attach the top cell inducing Poincaré duality by Whitehead products with respect to some basis of $\pi_*(I^{\overline{\mathcal{L}}}Th(E)) \otimes \mathbf{Q}$ and denote the resulting space by $\mathcal{DP}(Th(E))$, its cohomology algebra is then

$$H^*(\mathcal{DP}(Th(E))) = \frac{\mathbf{Q}[e_4, e_6] \otimes \wedge(e_7, e_9, e_{13})}{(e_4^2, e_6^2, e_4e_6, e_4e_7, e_6e_9, e_7e_9, e_6e_7 = e_4e_9 = e_{13})} \text{ with } |e_i| = i.$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Part II

INTERSECTION SPACES AND RATIONAL HOMOTOPY THEORY

In this part, we first develop the notion of Lagrangian intersection spaces we introduced in the first chapter. We compute their rational homology and show that they are related to the middle perversities intersection homology. We then talk about the rational homology truncation.

© 2016 Tous droits réservés.

lilliad.univ-lille.fr

© 2016 Tous droits réservés. lilliad.univ-lille.fr

2

BIREFLECTIVE ALGEBRAS, LAGRANGIAN INTERSECTION SPACES AND RATIONAL TRUNCATION

© 2016 Tous droits réservés.

lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

2.1 BIREFLECTIVE ALGEBRA

Definition 2.1.0.1. *Let* (H'_*, H_*, H''_*, B_*) *be* $\mathbf{Z}$*-graded* R*-modules and let*

$$(A, C^1_-, C^1_+, C^2_-, C^2_+)$$

be R*-modules. A* $(k+1, k)$*-bireflective diagram is a commutative diagram of the form*

$$\cdots \longrightarrow B_{k+2} \longrightarrow H_{k+2} \longrightarrow H'_{k+2} \longrightarrow B_{k+1} \longrightarrow \cdots$$

$$\begin{tikzcd}
 & & H'_{k+1} \arrow[dr, "r^1_{k+1}"] & & & & H''_k \arrow[dr, "r^1_k"] & & \\
B_{k+1} \arrow[r] & C^1_- \arrow[ur, "d^1_{k+1}"] \arrow[dr, "d^2_{k+1}"'] \arrow[rr, "\alpha_{k+1}"] & & C^1_+ \arrow[r, "l_1"] & A \arrow[r, "l_2"] & C^2_- \arrow[ur, "d^1_k"] \arrow[dr, "d^2_k"'] \arrow[rr, "\alpha_k"] & & C^2_+ \arrow[r] & B_{k-1} \\
 & & H_{k+1} \arrow[ur, "r^2_{k+1}"'] & & & & H_k \arrow[ur, "r^2_k"'] & &
\end{tikzcd}$$

$$B_{k-1} \longrightarrow H''_{k-1} \longrightarrow H_{k-1} \longrightarrow B_{k-2} \longrightarrow \cdots$$

Such that we have the following exact sequences

1. $\cdots \to B_{k+2} \to H_{k+2} \to H'_{k+2} \to B_{k+1} \to C^1_- \xrightarrow{d^1_{k+1}} H'_{k+1} \to 0,$
2. $\cdots \to B_{k+1} \to C^1_- \to C^1_+ \to A \to C^2_- \to H_k \to C^2_+ \to 0,$
3. $0 \to H''_k \xrightarrow{r^1_k} C^2_+ \to B_{k-1} \to H''_{k-1} \to H_{k-1} \to B_{k-2} \to \cdots,$
4. $0 \to C^1_- \to H_{k+1} \to C^1_+ \to A \to C^2_- \to C^2_+ \to B_{k-1} \to \cdots,$
5. $\cdots \to B_{k+1} \to C^1_- \to C^1_+ \to A \to C^2_- \to C^2_+ \to B_{k-1} \to \cdots.$

Such a diagram will be denoted by $\Delta_H(k+1, k)$.

Definition 2.1.0.2. *Let* (H'_*, H_*, H''_*) *be* $\mathbf{Z}$*-graded* R*-modules. We say that the triple* (H'_*, H_*, H''_*) *is* $(k+1, k)$*-bireflective along the* $\mathbf{Z}$*-graded* R*-module* B_* *if there exist* R*-modules* $(A, C^1_-, C^1_+, C^2_-, C^2_+)$ *such that they all fit into* $(k+1, k)$*-reflective as defined above.*

Let $\Delta_H(k+1, k)$ be a $(k+1, k)$-reflective diagram. Suppose there exist R-modules A_1, A_2, D_1, D_2 such that $A = A_1 \oplus A_2$, $D_i \subset A_i$, and maps $\gamma_1, \gamma_2, \gamma_3$ and γ_4 creating two new commutative squares in the following diagram

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

$$\begin{tikzcd}
0 \arrow[dr] & & H'_{k+1} \arrow[dr, hook, "r^1_{k+1}"] & & & & H''_k \arrow[dr, hook, "r^1_k"] & & 0 \\
B_{k+1} \arrow[r, "h_1"] & C^1_- \arrow[ur, two heads, "d^1_{k+1}"] \arrow[dr, hook, "d^2_{k+1}"'] \arrow[rr, "\alpha_{k+1}"] & & C^1_+ \arrow[r, "l_1"] & A_1 \oplus A_2 \arrow[r, "l_2"] \arrow[dr, two heads, "\gamma_3"] & C^2_- \arrow[ur, two heads, "d^1_k"] \arrow[dr, "d^2_k"'] \arrow[rr, "\alpha_k"] & & C^2_+ \arrow[ur] \arrow[r, "h_2"] & B_{k-1} \\
& & H_{k+1} \arrow[ur, "r^2_{k+1}"] \arrow[r, two heads, "\gamma_1"'] & D_1 \arrow[ur, hook, "\gamma_2"'] & & D_2 \arrow[r, hook, "\gamma_4"'] & H_k \arrow[ur, "r^2_k"'] & &
\end{tikzcd}$$

Definition 2.1.0.3. *Such a* $(k+1,k)$*-bireflective diagram is called split if we have the two following short exact sequences*

$$0 \longrightarrow C^1_- \xrightarrow{d^2_{k+1}} H_{k+1} \xrightarrow{\gamma_1} D_1 \longrightarrow 0$$

and

$$0 \longrightarrow D_2 \xrightarrow{\gamma_4} H_k \xrightarrow{r^2_k} C^2_+ \longrightarrow 0.$$

Let $\Delta_H(k+1,k)$ be a split $(k+1,k)$-reflective diagram, we have the two T-shaped diagrams of short exact sequences

$$\begin{tikzcd}
& 0 & & & \\
& H'_{k+1} \arrow[u] & & & \\
0 \arrow[r] & C^1_- \arrow[u, two heads] \arrow[r, hook] & H_{k+1} \arrow[r, two heads] & D_1 \arrow[r] & 0 \\
& \operatorname{im} h_1 \arrow[u, hook] & & & \\
& 0 \arrow[u] & & &
\end{tikzcd}$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

and

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & H_k'' \arrow[d, hook] & \\
0 \arrow[r] & D_2 \arrow[r, hook] & H_k \arrow[r, two heads] & C_+^2 \arrow[r] \arrow[d, two heads] & 0 \\
 & & & \operatorname{im} h_2 \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

if we are on a field $\mathbf{k}$ we have the isomorphisms

$$H_{k+1} \cong \operatorname{im} h_1 \oplus H''_{k+1} \oplus D_1$$

and

$$H_k \cong \operatorname{im} h_2 \oplus H'_k \oplus D_2.$$

2.2 LAGRANGIAN TRUNCATION AND ASSOCIATED BIREFLECTIVE ALGEBRA

We first briefly recall the notion of Lagrangian truncation defined in the section 1.2.4 of the first chapter.

Let K be a simply connected CW-complex of dimension $n = 2s$ satisfying Poincaré duality. We denote by b the non degenerate bilinear form induced by the Poincaré duality with $\mathbf{Q}$ coefficients, consider $\dim H^s(K) = 2m$ and

$$b : H^s(K) \times H^s(K) \longrightarrow \mathbf{Q}$$

where $b(x, y) := \langle x \cup y, [K] \rangle$ with $[K] \in H_{2s}(K; \mathbf{Q})$ the fundamental class and $\langle -, - \rangle$ the evaluation form.

If b is symmetric suppose that $H^{2k}(K; \mathbf{Q})$ posses a Lagrangian subspace V of dimension m, if b in skew-symmetric then one has automatically a Lagrangian subspace. By hyperbolic completion we have in both cases a hyperbolic basis

$$(a_, \dots, a_m, a_1^*, \dots, a_m^*)$$

of $H_s(K)$ such that, if we denote by $\overline{V}$ and $\overline{V^*}$ the subspaces respectively generated by $(a_, \dots, a_m)$ and $(a_1^*, \dots, a_m^*)$, we have

$$H_s(K) = \overline{V} \oplus \overline{V^*}.$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

We are then able to construct a topological space $t_{\overline{\mathcal{L}}}K$

$$t_{\overline{\mathcal{L}}}K \longrightarrow K$$

such that

$$H_r(t_{\overline{\mathcal{L}}}K;\mathbf{Q}) \cong \begin{cases} H_r(K) & \text{if } r \leqslant s-1 \\ \overline{V^*} & \text{if } r = s \\ 0 & \text{if } r > s. \end{cases} \tag{3}$$

We recall the definition 1.2.10.1 from the first chapter.

Definition 2.2.0.1. *The space $t_{\overline{\mathcal{L}}}K$ is called the Lagrangian truncation of the CW-complex K.*

Let now X be an L-space. That is a compact, connected oriented pseudomanifold of dimension $n = 2s+1$ with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ such that, for each singularities $\sigma_i \in \Sigma$ $H^s(L_i)$ has a Lagrangian subspace with respect to the non degenerate bilinear form

$$b_i \colon H^s(L_i) \times H^s(L_i) \to \mathbf{Q}.$$

Denote by $L(\Sigma, X) := \sqcup_{\sigma_i} L_i$ and let

$$j \colon L(\Sigma, X) \longrightarrow X_{reg}$$

be the inclusion of the links into the regular part of X.

Applying the Lagrangian truncation to $L(\Sigma, X)$ one then has the map

$$t_{\overline{\mathcal{L}}}L(\Sigma, X) \xrightarrow{f} L(\Sigma, X) \xrightarrow{j} X_{reg}.$$

Definition 2.2.0.2. *Let X be an L-space with simply connected links. The Lagrangian intersection space of the L-space X, denoted by $I^{\overline{\mathcal{L}}}X$, is defined by the homotopy cofiber of the maps $t_{\overline{\mathcal{L}}}L(\Sigma, X) \longrightarrow L(\Sigma, X) \longrightarrow X_{reg}$.*

$$I^{\overline{\mathcal{L}}}X := \mathrm{hocof}(t_{\overline{\mathcal{L}}}L(\Sigma, X) \xrightarrow{f} L(\Sigma, X) \xrightarrow{j} X_{reg})$$

We note by $HI_*^{\overline{\mathcal{L}}}(X) := H_*(I^{\overline{\mathcal{L}}}X)$ and by $HI^*_{\overline{\mathcal{L}}}(X)$ the corresponding cohomology.

Remark 2.2.1. *In the first chapter, the definition 1.2.10.2 of $I^{\overline{\mathcal{L}}}X$ we used was the normal Lagrangian intersection space it is easier for the construction of Poincaré approximation spaces. The definition 2.2.0.2 used here is as a homotopy cofiber with Lagrangian truncation because of the availability of braid diagrams for homotopy cofiber sequence. The only difference between these two definitions is at the first homology group level where loops appear in $HI_1^{\overline{\mathcal{L}}}(X)$ when it is defined as a homotopy cofiber.*

Theorem 2.2.2. 1. *Let X be an L-space of dimension $2s+1$ with only isolated singularities. The triple*

$$(IH_*^{\overline{m}}(X), HI_*^{\overline{\mathcal{L}}}(X), IH_*^{\overline{n}}(X))$$

is a split $(s+1, s)$-bireflective diagram along $H_*(L)$.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

2. *Let* X *be a Witt space of dimension* $2s+1$ *with only isolated singularities. The triple*

$$(IH_*^{\overline{m}}(X), HI_*^{\overline{m}}(X), IH_*^{\overline{n}}(X))$$

is a split $(s+1,s)$*-reflective diagram along* $H_*(L)$.

Proof. We denote by $g\colon t_{\overline{\mathcal{L}}}L \to X_{reg}$ the composition $j \circ f$, the triple

$$t_{\overline{\mathcal{L}}}L \xrightarrow{f} L(\Sigma, X) \xrightarrow{j} X_{reg}$$

induces the following commutative braid

$$\begin{tikzcd}[column sep=tiny]
H_{r+2}(j) \arrow[dr] \arrow[rr, bend left, dashed] & & H_{r+1}(f) \arrow[dr, dashed] \arrow[rr, bend left] & & H_r(t_{\overline{\mathcal{L}}}L) \arrow[dr] \arrow[rr, bend left] & & H_r(X_{reg}) \arrow[dr] \arrow[rr, bend left] & & H_r(j) \arrow[dr] \arrow[rr, bend left, dashed] & & H_{r-1}(f) \arrow[dr, dashed] \arrow[rr, bend left] & & H_{r-2}(t_{\overline{\mathcal{L}}}L) \\
& H_{r+1}(L) \arrow[ur] \arrow[dr] & & H_{r+1}(g) \arrow[ur] \arrow[dr, dashed] & & H_r(L) \arrow[ur] \arrow[dr] & & H_r(g) \arrow[ur, dashed] \arrow[dr] & & H_{r-1}(L) \arrow[ur] \arrow[dr] & & H_{r-1}(g) \arrow[ur] \arrow[dr, dashed] & \\
H_{r+1}(t_{\overline{\mathcal{L}}}L) \arrow[ur] \arrow[rr, bend right] & & H_{r+1}(X_{reg}) \arrow[ur] \arrow[rr, bend right] & & H_{r+1}(j) \arrow[ur] \arrow[rr, bend right, dashed] & & H_r(f) \arrow[ur, dashed] \arrow[rr, bend right] & & H_{r-1}(t_{\overline{\mathcal{L}}}L) \arrow[ur] \arrow[rr, bend right] & & H_{r-1}(X_{reg}) \arrow[ur] \arrow[rr, bend right] & & H_{r-1}(j)
\end{tikzcd}$$

where the paths of color red, blue and green are long exact sequences, by a result of Wall [10, lemma 6.16 p.189], the dashed arrow is also an long exact sequence. We have

$$H_r(g) = HI_r^{\overline{\mathcal{L}}}(X)$$

and

$$HI_r^{\overline{\mathcal{L}}}(X) = \begin{cases} H_r(X) = H_r(j) & r < s \\ H_r(X_{reg}) & r > s+1. \end{cases}$$

By composing with the indicated isomorphisms and their inverses, we may replace $H_r(f)$ by $H_r(t^{\overline{\mathcal{L}}}L)$ and $H_r(L_i)$ by $\overline{V}_i \oplus \overline{V^*}_i$. The braid diagram around $r = s$ then becomes

$$\begin{tikzcd}[column sep=tiny]
H_{s+2}(X) \arrow[dr] \arrow[rr, bend left, dashed] & & H_{s+1}(L) \arrow[dr, dashed] \arrow[rr, bend left, "0"] & & \bigoplus_{i=1}^{\nu} \overline{V}_i \arrow[dr, "i"] \arrow[rr, bend left, "\delta_1"] & & H_s(X_{reg}) \arrow[dr, "d_s^2"] \arrow[rr, bend left, "\alpha_s"] & & H_s(X) \arrow[dr, "h_2"] \arrow[rr, bend left, dashed] & & 0 \\
& H_{s+1}(L) \arrow[ur, "="] \arrow[dr, "h_1"] & & HI_{s+1}^{\overline{\mathcal{L}}}(X) \arrow[ur, "\beta_1"] \arrow[dr, dashed, "r_{s+1}^2"] & & \bigoplus_{i=1}^{\nu} \overline{V}_i \oplus \overline{V^*}_i \arrow[ur, "l_2"] \arrow[dr, "p"] & & HI_s^{\overline{\mathcal{L}}}(X) \arrow[ur, dashed, "r_s^2"] \arrow[dr] & & H_{s-1}(L) \arrow[ur] \arrow[dr] & \\
0 \arrow[ur, "0"] \arrow[rr, bend right] & & H_{s+1}(X_{reg}) \arrow[ur, "d_{s+1}^2"] \arrow[rr, bend right, "\alpha_{s+1}"'] & & H_{s+1}(X) \arrow[ur, "l_1"] \arrow[rr, bend right, dashed, "\delta_2"'] & & \bigoplus_{i=1}^{\nu} \overline{V}_i^* \arrow[ur, dashed, "\beta_2"] \arrow[rr, bend right, "0"'] & & H_{s-1}(L) \arrow[ur, "="] \arrow[rr, bend right] & & H_{s-1}(X_{reg})
\end{tikzcd}$$

where the notations for the maps are the one of the previous section. For

© 2016 Tous droits réservés.

Thèse de Mathieu Klimczak, Lille 1, 2016

the middle perversities, we have the following intersection cohomologies for X

$$IH_r^{\overline{n}}(X) = \begin{cases} H_r(X_{reg}) & r < s \\ \operatorname{im} \alpha_s \colon H_s(X_{reg}) \to H_s(X) & r = s \\ H_r(X) & r > s \end{cases}$$

and

$$IH_r^{\overline{m}}(X) = \begin{cases} H_r(X_{reg}) & r < s+1 \\ \operatorname{im} \alpha_{s+1} \colon H_{s+1}(X_{reg}) \to H_{s+1}(X) & r = s+1 \\ H_r(X) & r > s+1 \end{cases}$$

This implies the following maps for the braid diagram.

$$\begin{tikzcd}
 & & & & & & & IH_s^{\overline{n}}(X) \arrow[dr, hook, "r_s^1"] & & & \\
H_{s+2}(X) \arrow[dr] \arrow[rr, dashed, bend left] & & H_{s+1}(L) \arrow[rr, bend left, "0"] \arrow[dr, dashed] & & \bigoplus_{i=1}^{\nu} \overline{V}_i \arrow[rr, bend left, "\delta_1"] \arrow[dr, hook, "i"] & & H_s(X_{reg}) \arrow[ur, "d_s^1"] \arrow[rr, bend left, "\alpha_s"] \arrow[dr, "d_s^2"] & & H_s(X) \arrow[rr, dashed, bend left] \arrow[dr, "h_2"] & & 0 \\
 & H_{s+1}(L) \arrow[ur, "="] \arrow[dr, "h_1"] & & HI_{s+1}^{\mathcal{L}}(X) \arrow[ur, "\beta_1"] \arrow[dr, dashed, "r_{s+1}^2"] & & \bigoplus_{i=1}^{\nu} \overline{V}_i \oplus \overline{V^*}_i \arrow[ur, "l_2"] \arrow[dr, "p"] & & HI_s^{\mathcal{L}}(X) \arrow[ur, dashed, "r_s^2"] \arrow[dr] & & H_{s-1}(L) \arrow[ur] \arrow[dr] & \\
0 \arrow[ur, "0"] \arrow[rr, bend right] & & H_{s+1}(X_{reg}) \arrow[ur, "d_{s+1}^2"] \arrow[rr, bend right, "\alpha_{s+1}"] \arrow[dr, "d_{s+1}^1"] & & H_{s+1}(X) \arrow[ur, "l_1"] \arrow[rr, dashed, bend right, "\delta_2"] & & \bigoplus_{i=1}^{\nu} \overline{V}_i^* \arrow[ur, dashed, "\beta_2"] \arrow[rr, bend right, "0"] & & H_{s-1}(L) \arrow[ur, "="] \arrow[rr, bend right] & & H_{s-1}(X_{reg}) \\
 & & & IH_{s+1}^{\overline{m}}(X) \arrow[ur, hook, "r_{n+1}^1"] & & & & & & &
\end{tikzcd}$$

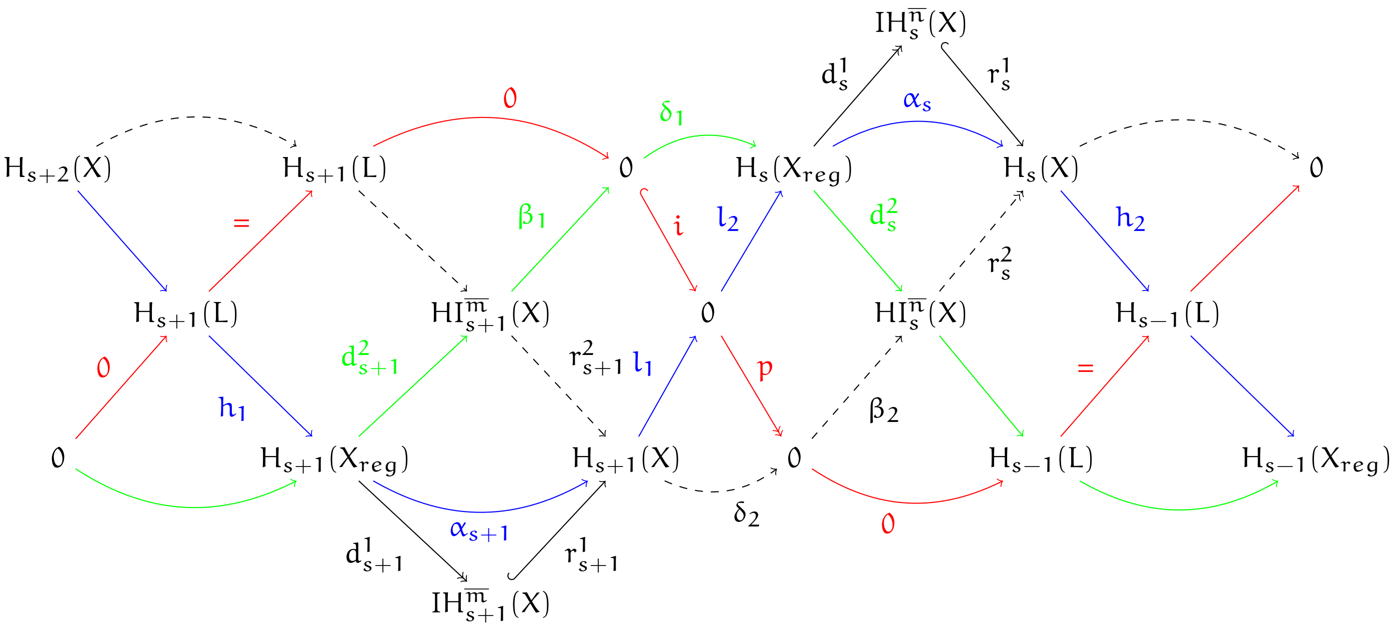
The braid then contains the desired bireflective diagram and all the required exact sequences. The split is given by the following diagram.

$$\begin{tikzcd}
0 \arrow[dr] & & IH_{s+1}^{\overline{m}}(X) \arrow[dr, hook, "r_{s+1}^1"] & & & & IH_s^{\overline{n}}(X) \arrow[dr, hook, "r_s^1"] & & 0 \\
H_{s+1}(L) \arrow[r, "h_1"] & H_{s+1}(X_{reg}) \arrow[ur, "d_{s+1}^1"] \arrow[rr, "\alpha_{s+1}"] \arrow[dr, hook, "d_{s+1}^2"'] & & H_{s+1}(X) \arrow[r, "l_1"] & \bigoplus_{i=1}^{\nu} \overline{V}_i \oplus \overline{V^*}_i \arrow[r, "l_2"] \arrow[dr, "p"] & H_s(X_{reg}) \arrow[ur, "d_s^1"] \arrow[rr, "\alpha_s"] \arrow[dr, "d_s^2"] & & H_s(X) \arrow[r, "h_2"] \arrow[ur] & H_{s-1}(L) \\
 & & HI_{s+1}^{\mathcal{L}}(X) \arrow[ur, "r_{s+1}^2"] \arrow[r, two heads, "\beta_1"'] & \operatorname{im} \beta_1 \arrow[ur, hook, "i"] & & \frac{\bigoplus_{i=1}^{\nu} \overline{V}_i^*}{\ker \beta_2} \arrow[r, hook, "\beta_2"'] & HI_s^{\mathcal{L}}(X) \arrow[ur, "r_s^2"] & &
\end{tikzcd}$$

The case of a Witt space is easier. Consider X to be a Witt space, then $H_s(L_i) = 0$ for all i and we have $HI_*^{\overline{m}}(X) = HI_*^{\overline{n}}(X)$ and $IH_*^{\overline{m}}(X) = IH_*^{\overline{n}}(X)$ the braid diagram becomes

© 2016 Tous droits réservés.

Thèse de Mathieu Klimczak, Lille 1, 2016



As before the braid then contains the desired bireflective diagram and all the required exact sequences. □

Now that we have our braid diagrams, we are able to give the T-shaped that are verified when X is an L-space or a Witt space. For X an L-space we have

$$\begin{tikzcd}
& 0 \\
& IH^{\overline{m}}_{s+1}(X) \arrow[u] \\
0 \arrow[r] & H_{s+1}(X_{reg}) \arrow[u, two heads] \arrow[r, hook] & HI^{\mathcal{L}}_{s+1}(X) \arrow[r, two heads] & \bigoplus_{i=1}^{\nu}(\ker l_2)\cap \overline{V}_i \arrow[r] & 0 \\
& \operatorname{im} h_1 \arrow[u, hook] \\
& 0 \arrow[u]
\end{tikzcd}$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

and

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & IH_s^{\overline{n}}(X) \arrow[d, hook] & \\
0 \arrow[r] & \bigoplus_{i=1}^{\nu} \frac{\overline{V}_i^*}{(\operatorname{im} l_1)\cap \overline{V}_i^*} \arrow[r, hook] & HI_s^{\overline{\mathcal{L}}}(X) \arrow[r, two heads] & H_s(X) \arrow[r] \arrow[d, two heads] & 0 \\
 & & & \operatorname{im} h_2 \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

For a Witt space it is easier since $H_s(L) = 0$, we then have $HI_s^{\overline{n}}(X) = H_s(X)$ and $HI_{s+1}^{\overline{m}}(X) = H_{s+1}(X_{reg})$, with $I^{\overline{m}}X = I^{\overline{n}}X$. The intersection homology groups $IH_s^{\overline{n}}(X)$ and $IH_{s+1}^{\overline{m}}(X)$ are respectively subvector spaces of $H_s(X)$ and $H_{s+1}(X_{reg})$.

Proposition 2.2.2.1. *The rational homology of* $I^{\overline{\mathcal{L}}}X$ *is given by*

$$HI_r^{\overline{\mathcal{L}}}(X) \cong \begin{cases} \mathbf{Q} & r = 0 \\ H_1(X) \oplus \mathbf{Q}^{\beta_0(L)-1} & r = 1 \\ H_r(X) & 1 < r < s \\ H_s(X) \oplus \left(\bigoplus_{i=1}^{\nu} \frac{\overline{V}_i^*}{(\operatorname{im} l_1)\cap \overline{V}_i^*}\right) & r = s \\ H_{s+1}(X_{reg}) \oplus \left(\bigoplus_{i=1}^{\nu} (\ker l_2) \cap \overline{V}_i\right) & r = s+1 \\ H_r(X_{reg}) & r > s+1 \end{cases}$$

where $\beta_0(L)$ *is the number of connected components of* $L(\Sigma, X)$*, that is the number of isolated singularities.*

For X *a Witt space we have*

$$HI_r^{\overline{m}}(X) = HI_r^{\overline{n}}(X) \cong \begin{cases} \mathbf{Q} & r = 0, \\ H_1(X) \oplus \mathbf{Q}^{\beta_0(L)-1} & r = 1, \\ H_r(X) & 1 < r \leqslant s, \\ H_r(X_{reg}) & r \geqslant s+1. \end{cases}$$

For example, consider the suspension of $S^{2s+1} \times S^{2s+1}$, denoted by $\Sigma(S^{2s+1} \times S^{2s+1})$. This is an L-space of dimension $4s+3$ with two isolated singularities which are the suspension points $\Sigma = \{\sigma_1, \sigma_2\}$. We have

$$H_{2s+1}(L_1) \cong H_{2s+1}(S_1^{2s+1}) \oplus H_{2s+1}(S_1^{2s+1}) := \overline{V}_1 \oplus \overline{V}_1^*,$$

© 2016 Tous droits réservés.

Thèse de Mathieu Klimczak, Lille 1, 2016

$$H_{2s+1}(L_2) \cong H_{2s+1}(S_2^{2s+1}) \oplus H_{2s+1}(S_2^{2s+1}) := \overline{V}_2 \oplus \overline{V}_2^*.$$

The regular part is $X_{reg} = S^{2s+1} \times S^{2s+1} \times [0,1]$.

Then

$$HI_r^{\overline{\mathcal{L}}}(\Sigma(S^{2s+1} \times S^{2s+1})) \cong \begin{cases} \mathbf{Q} & r = 0, \\ \mathbf{Q} & r = 1, \\ H_r(\Sigma(S^{2s+1} \times S^{2s+1})) & 1 < r \leqslant 2s, \\ H_{2s+1}(S_2^{2s+1}) & r = 2s+1, \\ H_{2s+1}(S_1^{2s+1}) & r = 2s+2, \\ H_r(S^{2s+1} \times S^{2s+1}) & r \geqslant 2s+3. \end{cases}$$

Which gives

$$HI_r^{\overline{\mathcal{L}}}(\Sigma(S^{2s+1} \times S^{2s+1})) \cong \begin{cases} \mathbf{Q} & r = 0, \\ \mathbf{Q} & r = 1, \\ 0 & 1 < r \leqslant 2s, \\ \mathbf{Q} & r = 2s+1, \\ \mathbf{Q} & r = 2s+2, \\ 0 & 2s+3 \leqslant r < 4s+2, \\ \mathbf{Q} & r = 4s+2. \end{cases}$$

2.3 EXTENSION OF THE RATIONAL HOMOLOGICAL TRUNCATION

For the rest of this section, we will consider the following categories, with models categories of the left and full subcategories on the right.

$$\begin{tikzcd}
\mathsf{Top}_{\mathrm{Serre}} \arrow[d, shift left, "S_*(-)"] & \mathsf{CW}^0_{\mathrm{rat,ft}} \arrow[l, hook'] \\
\mathsf{sSet}_{\mathrm{Quillen}} \arrow[u, shift left, "|-|"] \arrow[d, shift left, "A_{PL}(-)"] & \mathsf{sSet}^0_{\mathrm{rat,ft}} \arrow[l, hook'] \\
(\mathsf{CDGA}_{\mathbf{Q},\mathrm{proj}})^{\mathrm{op}} \arrow[u, shift left, "\langle - \rangle"] & \mathsf{SulAlg}^0_{\mathrm{ft}} \arrow[l, hook']
\end{tikzcd}$$

The different categories and the relations between them in the diagram are

1. The category of topological spaces $\mathsf{Top}_{\mathrm{Serre}}$ endowed with the model structure where
   - The fibrations are the Serre fibrations.

© 2016 Tous droits réservés.
lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

- The weak equivalences are the weak homotopy equivalences of topological spaces, that is $f\colon X \to Y$ is a weak homotopy equivalence if

$$\pi_k(f)\colon \pi_k X \longrightarrow \pi_k Y$$

 is a bijection for all k.

2. The category of simplicial sets $\mathsf{sSet}_{\mathrm{Quillen}}$ endowed with the Quillen model structure where
   - The cofibrations $f\colon X \to Y$ are the levelwise injection $f_n\colon X_n \to Y_n$ of simplicial sets.
   - The weak equivalences are the weak homotopy equivalences, that is the morphisms whose Milnor realization is a weak homotopy equivalence of topological spaces.
3. The category of commutative differential graded algebras over $\mathbf{Q}$ $\mathsf{CDGA}_{\mathbf{Q},\mathrm{proj}}$ endowed with the projective model structure where
   - The fibrations are the degreewise surjection.
   - The weak equivalences are the quasi-isomorphisms.

Between theses categories there are the following Quillen adjunctions

1. The Milnor realization and the singular set functors $|-| \dashv S_*(-)$, this pair also sets a Quillen equivalence.
2. The polynomial De Rham functor and the Sullivan simplicial realization functors $A_{PL}(-) \dashv \langle - \rangle$.

Denote by $\mathsf{CW}^0_{\mathrm{rat,ft}}$ the full subcategory of connected nilpotent rational CW-complexes of finite type and by $\mathsf{sSet}^0_{\mathrm{rat,ft}}$ the category of simplicial sets such that their realization is in $\mathsf{CW}^0_{\mathrm{rat,ft}}$. The objects of $\mathsf{CW}^0_{\mathrm{rat,ft}}$ are CW-complexes X such that

- The CW-complex X is a connected topological space.
- The fundamental group $\pi_1 X$ is a nilpotent group and $\pi_n X$ is a nilpotent $\pi_1 X$-module.
- The reduced homology of X, $\widetilde{H}_k(X, \mathbf{Z})$ is a $\mathbf{Q}$-vector space of finite dimension.

The category $\mathsf{SulAlg}^0_{\mathrm{ft}}$ denotes the category of connected Sullivan algebras of finite type. By connected we mean we have a morphism of cdga's

$$\varepsilon\colon \mathbf{Q} \longrightarrow A$$

where $\mathbf{Q}$ is seen as a cdga $(\mathbf{Q}, 0)$ with $\mathbf{Q}$ of degree zero and such that we have the isomorphism $H^0(\varepsilon)\colon H^0(\mathbf{Q}) = \mathbf{Q} \xrightarrow{\cong} H^0(A, d)$. Finite type means that their homology is finite dimensional is every degree.

When have the following theorem

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Theorem 2.3.1 ([45]). *When passing to homotopy categories, the adjunctions induces the following equivalences of categories.*

$$\mathsf{Ho}(\mathsf{CW}^0_{\mathrm{rat,ft}}) \underset{S_*(-)}{\overset{|-|}{\rightleftarrows}} \mathsf{Ho}(\mathsf{sSet}^0_{\mathrm{rat,ft}}) \underset{Q(A_{PL}(-))}{\overset{\langle - \rangle}{\rightleftarrows}} \mathsf{Ho}(\mathsf{SulAlg}^0_{\mathrm{ft}})^{\mathrm{op}}$$

We recall some of the properties we will need. The adjunction

$$\mathsf{Top}(|X|, Y) \cong \mathsf{sSet}(X, S_*(Y))$$

induces the following unit and counit

$$\begin{cases} \theta_X \colon X \longrightarrow S_*(|X|) \\ \nu_Y \colon |S_*(Y)| \longrightarrow Y \end{cases}$$

When we are considering CW-complex we have the following proposition.

Proposition 2.3.1.1. *[24, Proposition 17.3]*

1. *If Y is a simply connected CW-complex, then ν_Y is a homotopy equivalence.*
2. *If X is a simplicial set such that $|X|$ is simply connected then $|\theta_X|$ is a homotopy equivalence, that is θ_X is a weak homotopy equivalence.*

Let (A, d) be a cdga, combining Sullivan's functor with Milnor realization gives the following definition.

Definition 2.3.1.1. *The spatial realization of a cdga (A, d) is the CW-complex $\|A, d\| := |\langle A, d\rangle|$. The spatial realization of a morphism $\varphi\colon (A, d) \to (B, d)$ is the continuous map $\|\varphi\| := |\langle \varphi \rangle|$.*

This defines a contravariant functor

$$\| - \| \colon \mathsf{CDGA}_{\mathbb{Q}} \longrightarrow \mathsf{CW}.$$

Definition 2.3.1.2. *Let $X \in \mathsf{Top}$, we define the cdga $A_{PL}(X)$ to be the cdga associated to the simplicial set $S_*(X)$. That is*

$$A_{PL}(X) := A_{PL}(S_*(X)).$$

Definition 2.3.1.3. *A Sullivan algebra for a space X is the functorial cofibrant replacement of $A_{PL}(S_*(X))$.*

A more algebraic definition was already given in the definition 1.2.1.1. Note that we do not consider taking the minimal model of a cdga (A, d) as a cofibrant replacement even if by definition the minimal model of (A, d) is a Sullivan algebra. This is because taking the minimal model defines a functor only up to homotopy, unlike taking a cofibrant replacement. We denote by

$$Q(-)\colon \mathsf{CDGA}_{\mathbb{Q}} \longrightarrow \mathsf{SulAlg}$$

the cofibrant replacement (Sullivan algebra) functor.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Proposition 2.3.1.2. *Let $X \in \mathsf{CW}^0_{\mathrm{rat,ft}}$, then the map $X \to \|\mathsf{Q}(\mathsf{A}_{\mathsf{PL}}(X))\|$ coming from adjunctions is a homotopy equivalence.*

Definition 2.3.1.4. *Let (A, d) be a cdga and $p \in \mathbf{N}$. Denote by $\mathsf{C}_{\overline{p}}$ a supplement of*

$$\ker(d^p : A^p(L) \to A^{p+1}(L))$$

and by I_p be the differential ideal of $A^*(L)$ generated by $\mathsf{C}_{\overline{p}} \oplus A^{\geqslant p+1}(L)$.

The p-truncation of the cdga (A, d) is defined by the following cdga $(\xi_{\overline{p}}A, d)$ where $\xi_{\overline{p}}A := A/I_p$.

We then have a surjection $\pi_{\overline{p}}(A)\colon\ A \twoheadrightarrow \xi_{\overline{p}}A$.

This defines an assignment

$$\xi_{\overline{p}}\colon\ \mathsf{CDGA}_{\mathbf{Q}} \longrightarrow \mathsf{CDGA}_{\mathbf{Q}}$$

with a comparison map $\pi_{\overline{p}}(-)\colon\ \mathrm{Id} \twoheadrightarrow \xi_{\overline{p}}$.

Note that up to quasi-isomorphism the p-truncation of (A, d) is given by

$$\xi_{\overline{p}}A^i \simeq \begin{cases} A^i & i < p \\ \ker d^p & i = p \\ 0 & i > p \end{cases}$$

but since this definition doesn't define a cdga, we are forced to take quotient by I_p.

Recall that $\mathsf{CDGA}_{\mathbf{Q}}$ has a model category structure where the fibrations are the degreewise surjections and where the weak equivalences are the quasi-isomorphisms. We then have the obvious following property.

Proposition 2.3.1.3. *The p-truncation assignment*

$$\xi_{\overline{p}}\colon\ \mathsf{CDGA}_{\mathbf{Q}} \longrightarrow \mathsf{CDGA}_{\mathbf{Q}}$$

preserves fibrations and weak equivalences.

Definition 2.3.1.5. *Let $(A, d) \in \mathsf{CDGA}_{\mathbf{Q}}$ and $p \in \mathbf{N}$. We define the p-cotruncation of the cdga (A, d), denoted by $\xi^{\overline{p}}_{+}A$, by the following pullback of cdga's.*

$$\begin{tikzcd} \xi^{\overline{p}}_{+}A \arrow[r] \arrow[d, hook, "\lambda^{\overline{p}}(A)"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbf{Q} \arrow[d, "\varepsilon"] \\ A \arrow[r, two heads, "\pi_{\overline{p}}(A)"'] & \xi_{\overline{p}}A \end{tikzcd}$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

By universal property of pullback diagrams, we have the following commutative diagram

$$\begin{tikzcd}
\mathbf{Q} \oplus \ker \pi_{\overline{p}}(A) \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\
& \xi_+^{\overline{p}} A \arrow[r] \arrow[d, hook, "\lambda^{\overline{p}}(A)"] & \mathbf{Q} \arrow[d] \\
& A \arrow[r, two heads, "\pi_{\overline{p}}(A)"'] & \xi_{\overline{p}} A
\end{tikzcd}$$

and the map $m\colon \mathbf{Q} \oplus \ker \pi_{\overline{p}}(A) \to \xi_+^{\overline{p}} A$ is an isomorphism.

This defines an assignment

$$\xi_+^{\overline{p}}\colon \mathsf{CDGA}_{\mathbf{Q}} \longrightarrow \mathsf{CDGA}_{\mathbf{Q}}$$

with a comparison map $\lambda^{\overline{p}}(-)\colon \xi_+^{\overline{p}} \hookrightarrow \mathrm{Id}$. The following property also holds thanks to the isomorphism m.

Proposition 2.3.1.4. *The p-cotruncation assignment*

$$\xi_+^{\overline{p}}\colon \mathsf{CDGA}_{\mathbf{Q}} \longrightarrow \mathsf{CDGA}_{\mathbf{Q}}$$

preserves fibrations and weak equivalences.

We now define the truncation and cotruncation of a nilpotent rational space of finite type.

By the proposition 1.2.7.2 of the first chapter, if (A, d) is a rational model of a simply connected space X, then $\xi_{\overline{p}} A$ is a rational model of the homological $(p+1)$-truncation of X, we recall that

$$H^r(t_{p+1}X) \cong \begin{cases} H^r(X) & r < p+1, \\ 0 & r \geqslant p+1. \end{cases}$$

Being a rational model implies we have a quasi-isomorphism

$$A_{PL}(t_{p+1}X) \xrightarrow{\sim} \xi_{\overline{p}} A.$$

Then if X is simply connected of finite type, applying cofibrant replacement and geometric realization of cdga's we have the following homotopy equivalence by proposition 2.3.1.2.

$$(t_{p+1}X)_{\mathbf{Q}} \simeq \|Q(A_{PL}(t_{p+1}X))\| \xrightarrow{\simeq} \|Q(\xi_{\overline{p}} A)\|.$$

The homological truncation defined in the first chapter is done in a topological way by constructing a new CW-complex and only concerns simply connected space. What we want to do here, with the help of algebraic tools, is to extend this construction to the case of nilpotent rational topological spaces of finite type. Suppose we have such a truncation and denote it by $t_{\leqslant p}^{\mathbf{Q}}(-)$. Such a truncation should verify the following properties.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

1. It should defines a covariant assignment

$$t^{\mathbf{Q}}_{\leqslant p}(-)\colon \mathsf{CW}^0_{\mathrm{rat,ft}} \longrightarrow \mathsf{CW}^0_{\mathrm{rat,ft}},$$

   and not necessarily a functor. The homological truncation on simply connected CW-complex defined in [6] is not a functor unless for very specific spaces and maps, which from what is called a rigid subcategory of the CW-complexes.

2. It should come with a comparison map $\mathrm{emb}^{\mathbf{Q}}_p(X)\colon t^{\mathbf{Q}}_{\leqslant p}(X) \to X$ such that $H_r(\mathrm{emb}^{\mathbf{Q}}_p(X))$ is an isomorphism for $r \leqslant p$ and the zero map otherwise. Again $\mathrm{emb}^{\mathbf{Q}}_p(X)$ shouldn't necessarily be a natural transformation.

3. It should preserve homotopy equivalences, meaning that if X and Y have the same homotopy type, then $t^{\mathbf{Q}}_{\leqslant p}(X)$ and $t^{\mathbf{Q}}_{\leqslant p}(Y)$ should at least have the same rational homotopy type.

4. If $X \in \mathsf{CW}^1_{\mathrm{rat,ft}}$, that is X is a simply connected rational CW-complex of finite type, then $t^{\mathbf{Q}}_{\leqslant p}(X)$ should be homotopy equivalent to $t_{p+1}X$.

The following definition verifies these 4 properties. We denote by $\mathcal{P}_n$ the poset $\{0, 1, \ldots, n-2\}$.

Definition 2.3.1.6. *Let $X \in \mathsf{CW}^0_{\mathrm{rat,ft}}$, $n = \dim X$ and $p \in \mathcal{P}_n$, the rational cohomological p-truncation of X is defined by*

$$t^{\mathbf{Q}}_{\leqslant p}(X) := \|\mathsf{Q}(\xi_{\overline{p}}\mathsf{A}_{\mathsf{PL}}(X))\|.$$

We denote by $\mathrm{emb}^{\mathbf{Q}}_p(X)\colon t^{\mathbf{Q}}_{\leqslant p}(X) \to \|\mathsf{Q}(\mathsf{A}_{\mathsf{PL}}(X))\|$ the continuous map induced by the spatial realization of the comparison map $\pi_{\overline{p}}(\mathsf{A}_{\mathsf{PL}}(X))$,

$$\mathrm{emb}^{\mathbf{Q}}_p(X) := \|\mathsf{Q}(\pi_{\overline{p}}(\mathsf{A}_{\mathsf{PL}}(X)))\|.$$

Here we have

$$\begin{aligned} H^r(t^{\mathbf{Q}}_{\leqslant p}(X)) &= H^r(\|\mathsf{Q}(\xi_{\overline{p}}\mathsf{A}_{\mathsf{PL}}(X))\|) \\ &\cong H^r(\xi_{\overline{p}}\mathsf{A}_{\mathsf{PL}}(X)) \\ &\cong \begin{cases} H^r(\mathsf{A}_{\mathsf{PL}}(X)) \cong H^r(X) & r \leqslant p \\ 0 & r > p \end{cases} \end{aligned}$$

and the map $\mathrm{emb}^{\mathbf{Q}}_p(X)$ induces

$$H^r(\mathrm{emb}^{\mathbf{Q}}_p(X))\colon H^r(\|\mathsf{Q}(\mathsf{A}_{\mathsf{PL}}(X))\|) \longrightarrow H^r(t^{\mathbf{Q}}_{\leqslant p}(X))$$

which is an isomorphism for $r \leqslant p$ and the map

$$H^r(\mathrm{emb}^{\mathbf{Q}}_p(X))\colon H^r(\|\mathsf{Q}(\mathsf{A}_{\mathsf{PL}}(X))\|) \to 0$$

for $r > p$. Note that by the universal coefficients theorem $t^{\mathbf{Q}}_{\leqslant p}(X)$ also truncates the rational homology of X in the same way. We define the cotruncation in the same way.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Definition 2.3.1.7. *Let* $X \in \mathsf{CW}^0_{\mathrm{rat,ft}}$ *and* $p \in \mathcal{P}_n$*, the rational cohomological* p*-cotruncation of* X *is defined by*

$$t^{\mathbf{Q}}_{>p}(X) := \|\mathsf{Q}(\xi^{\overline{p}}_{+}\mathsf{A}_{\mathsf{PL}}(X))\|.$$

We denote by $\mathrm{pro}^{\mathbf{Q}}_{p}(X)\colon \|\mathsf{Q}(\mathsf{A}_{\mathsf{PL}}(X))\| \to t^{\mathbf{Q}}_{>p}(X)$ *the continuous map induced by the spatial realization of the comparison map* $\lambda^{\overline{p}}(\mathsf{A}_{\mathsf{PL}}(X))$*,*

$$\mathrm{pro}^{\mathbf{Q}}_{p}(X) := \|\mathsf{Q}(\lambda^{\overline{p}}(\mathsf{A}_{\mathsf{PL}}(X)))\|.$$

We have

$$\begin{aligned} H^r(t^{\mathbf{Q}}_{>p}(X)) &= H^r(\|\xi^{\overline{p}}_{+}\mathsf{A}_{\mathsf{PL}}(X)\|) \\ &\cong H^r(\xi^{\overline{p}}_{+}\mathsf{A}_{\mathsf{PL}}(X)) \\ &\cong \begin{cases} H^0(\mathsf{A}_{\mathsf{PL}}(X)) \cong H^0(X) = \mathbf{Q} & r = 0 \\ 0 & 1 \leqslant r \leqslant p \\ H^r(\mathsf{A}_{\mathsf{PL}}(X)) \cong H^r(X) & r > p \end{cases} \end{aligned}$$

With the definitions of the truncation and cotruncation assignment $\xi_{\overline{p}}$, $\xi^{\overline{p}}_{+}$ and the different Quillen pairs involved, it is then clear that $t^{\mathbf{Q}}_{\leqslant p}(-)$ and $t^{\mathbf{Q}}_{>p}(-)$ preserve homotopy equivalences.

Definition 2.3.1.8. *Let* $F\colon \mathsf{A} \longrightarrow \mathsf{B}$ *be a covariant assignment between the categories* A *and* B*.*

- *The assignment* F *is said to be augmented if it comes with a comparison map*

$$\alpha\colon \mathrm{Id} \longrightarrow F.$$

- *The assignment* F *is said to be coaugmented if it comes with a comparison map*

$$\alpha\colon F \longrightarrow \mathrm{Id}.$$

Since $\|\mathsf{Q}(\mathsf{A}_{\mathsf{PL}}(X))\| \simeq X$ for $X \in \mathsf{CW}^0_{\mathrm{rat,ft}}$, $\mathrm{emb}^{\mathbf{Q}}_{p}(-)$ defines a coaugmentation $\mathrm{emb}^{\mathbf{Q}}_{p}(X)\colon t^{\mathbf{Q}}_{\leqslant p}(X) \to X$ in $\mathsf{HoCW}^0_{\mathrm{rat,ft}}$ and $\mathrm{pro}^{\mathbf{Q}}_{p}(-)$ defines an augmentation $\mathrm{pro}^{\mathbf{Q}}_{p}(X)\colon X \to t^{\mathbf{Q}}_{>p}(X)$ in $\mathsf{HoCW}^0_{\mathrm{rat,ft}}$. We then have

Theorem 2.3.2.
1. $(t^{\mathbf{Q}}_{\leqslant p}(-), \mathrm{emb}^{\mathbf{Q}}_{p}(-))\colon \mathsf{HoCW}^0_{\mathrm{rat,ft}} \longrightarrow \mathsf{HoCW}^0_{\mathrm{rat,ft}}$ *is a coaugmented assignment.*
2. $(t^{\mathbf{Q}}_{>p}(-), \mathrm{pro}^{\mathbf{Q}}_{p}(-))\colon \mathsf{HoCW}^0_{\mathrm{rat,ft}} \longrightarrow \mathsf{HoCW}^0_{\mathrm{rat,ft}}$ *is an augmented assignment.*

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Part III

MIXED HODGE STRUCTURES

Let X be a complex projective variety of complex dimension n with only isolated singularities of simply connected links. We show that we can endow the rational cohomology of the family of the $\overline{p}$-perverse intersection spaces $\{I^{\overline{p}}X\}_{(\overline{p})}$ with compatible mixed Hodge structures.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

© 2016 Tous droits réservés. lilliad.univ-lille.fr

3

MIXED HODGE STRUCTURES ON THE RATIONAL HOMOTOPY TYPE OF INTERSECTION SPACES

© 2016 Tous droits réservés.

lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

3.1 INTRODUCTION

This part deals with the notion of mixed Hodge structure associated to the intersection spaces of a complex projective variety X of complex dimension n with only isolated singularities and simply connected links.

Intersection spaces were defined by Markus Banagl in [6] as a way to spatialize Poincaré duality for singular spaces. Suppose given a compact, connected pseudomanifold of dimension n with only isolated singularities and simply connected links. We assign to this space a family of topological spaces $I^{\overline{p}}X$, its intersection spaces, where $\overline{p}$ is an element called a perversity varying in a poset $\mathcal{P}_n$ called the poset of perversities. We then have for complementary perversities a generalized Poincaré duality isomorphism

$$\widetilde{H}I^k_{\overline{p}}(X) \cong \widetilde{H}I^{\overline{q}}_{n-k}(X)^{\vee}.$$

with $\widetilde{H}I^{\overline{q}}_{n-k}(X)^{\vee} = \hom(\widetilde{H}I^{\overline{q}}_{n-k}(X), \mathbf{Q})$

The theory of intersection spaces can be seen as an enrichment of intersection homology since they both gives complementary informations about X.

The aim of this part is twofold. First we want to get a better understanding of the family of cohomology algebras $\{HI^*_{\bullet}(X)\}_{\overline{p}\in\mathcal{P}_n}$ when we take all the spaces into consideration. We then want to put mixed Hodge structures on these algebras in order to get results about the formality of intersections spaces.

Formality is a notion tied to the rational homotopy theory of topological spaces. The rational homotopy type of a topological space X is given by the commutative differential graded algebra $\mathsf{A_{PL}}(X)$ in the homotopy category $\mathsf{Ho}(\mathsf{CDGA}_{\mathbf{Q}})$ defined by formally inverting quasi-isomorphisms and where $\mathsf{A_{PL}}(-)\colon \mathsf{Top} \to \mathsf{CDGA}_{\mathbf{Q}}$ is the polynomial De Rham functor defined by Sullivan. The space X is then formal if there is a string of quasi-isomorphisms from the cdga $\mathsf{A_{PL}}(X)$ to its cohomology with rational coefficients $H^*(\mathsf{A_{PL}}(X)) \cong H^*(X, \mathbf{Q})$ seen as a cdga with trivial differential. In particular is X is formal then its rational homotopy type is a formal consequence of its cohomology ring and its higher order Massey products vanish.

The combination of rational homotopy theory and Hodge theory has already been showed to be fruitful. Using Hodge theory, Deligne, Griffiths, Morgan and Sullivan proved in [18] that compact Kähler manifolds, in particular smooth projective varieties, are formal. It was also shown by Simpson in [41] that every finitely presented group G is the fundamental group of a singular projective variety X and then Kapovich and Kollár showed in [33] that this X could be chosen to be complex projective with only simple normal crossing singularities. More recently, Chataur and Cirici proved in [13] that every complex projective variety of dimension n with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ such that the link L_i of each singularities σ_i is $(n-2)$-connected is then a formal topological space.

The intersection spaces $I^{\overline{p}}X$ of X are not complex nor algebraic varieties, even if X is. Thus at first glance there should be no reasons the cohomology of these spaces carry a mixed Hodge structure. On second thought,

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

when X is a complex projective variety of complex dimension n with only isolated singularities and that we look at the rational cohomology of their normal intersection spaces

$$HI^k_{\overline{p}}(X) = \begin{cases} \mathbf{Q} & k = 0 \\ H^k(X) & 1 \leqslant k \leqslant p \\ H^k(X) \oplus \operatorname{im} H^k(X_{reg}) \to H^k(L) & k = p+1 \\ H^k(X_{reg}) & k > p+1 \end{cases}$$

it becomes a bit more natural to think that there is a mixed Hodge structure since each part of their rational cohomology can be endowed with a natural mixed Hodge structure coming from X. We show here that in fact all these structures naturally come from a mixed Hodge structure at the algebraic models level and that this structure is compatible with the different operations defined on intersection spaces.

It must be pointed out here that the question of a Hodge structure on the intersection spaces as already been looked at in the work of Banagl and Hunsicker [7] where they use L^2-cohomology to provide a Hodge theoretic structure. We do not follow this path here and rather modify the rational homotopy theory tools developed in [12] for the mixed Hodge structures in intersection cohomology.

We explain the contents of this paper.

The section 3.2 is devoted to collect the different definitions needed. We recall what we call a perversity, the definition of the intersection spaces and the convention we use to construct them. We also introduce the notion of a coperverse cdga which is the main tool for the rational algebraic models of the intersection spaces. We then define a model category structure on the category of coperverse cdga's 3.2.5.

The section 3.3 is a direct application of the previous section. We define the notion of a coperverse cdga associated to a morphism of cdga's. As a result we show that the whole family of algebraic model $AI_{\overline{\bullet}}(X)$ computing the rational cohomology of intersection spaces carry a structure of coperverse algebra and that we have a external product on that family, extending the cup product that each $I^{\overline{p}}X$ naturally has as a topological space.

The section 3.4 is the main section of this chapter, we extend our notion of coperverse cdga to the notion of coperverse mixed Hodge cdga. These coperverse mixed Hodge algebras carry a mixed Hodge structure which is compatible the differential, product and poset maps of the underlying coperverse cdga. After developing their algebraic definitions we show in theorem 3.4.1 that given a complex projective variety X of complex dimension n with only isolated singularities and simply connected links, there is a coperverse mixed Hodge cdga $MI_{\overline{\bullet}}(X)$ quasi-isomorphic to the coperverse cdga $AI_{\overline{\bullet}}(X)$. As a result the whole family $HI^*_{\overline{\bullet}}(X)$ carry a well defined mixed Hodge structure defined at the algebraic models level.

The section 3.5 is devoted to the computation of the associated weight spectral sequence. If X is a complex projective algebraic variety with only

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

isolated singularities and such that X admits a resolution of singularities where the exceptional divisor is smooth, we are able to compute the weight spectral sequence associated to the mixed Hodge structure. We then use this spectral sequence to show a result of "purity implies formality" in theorem 3.5.4.

The section 3.6 is completely devoted to the proof of the theorem 3.6.1 : suppose X to be a complex projective algebraic threefold with isolated singularities such that there exists a resolution of singularities with a smooth exceptional divisor, then if the links are simply connected the intersection spaces $I^{\overline{p}}X$ are formal topological spaces for any perversity $\overline{p}$. The proof being rather long and intricate, we made the choice of giving it its own section. This result goes well with the result of [14, Theorem E p.76] stating that any nodal hypersurface in $\mathbf{C}P^4$ is intersection-formal.

The last section 3.7 deals with computations, with for instance the computations for the Calabi-Yau generic quintic 3-fold 3.7.3 and the Calabi-Yau quintic 3-fold 3.7.4 where we are able to retrieve the cohomology of the associated smooth deformation as stated in [8].

3.2 BACKGROUND, INTERSECTION SPACES AND COPERVERSE ALGEBRAS

3.2.1 *Perversities and intersection spaces*

Since we are concerned about complex algebraic varieties of complex dimension n with only isolated singularities we use the following definition of a perversity.

Definition 3.2.0.1. *A perversity $\overline{p}$ is determined by a integer $0 \leqslant p \leqslant 2n-2$.*

We denote by $\mathcal{P}_n^{op}$ the poset $\{0,\dots,2n-2;\leqslant\}$ with the reverse order, we set $\widehat{\mathcal{P}_n}^{op} := \mathcal{P}_n^{op} \cup \{\infty\}$.

The posets $\mathcal{P}_n^{op}$ and $\widehat{\mathcal{P}_n}^{op}$ are then totally ordered and look like

$$2n-2 \to 2n-3 \to \cdots \to 2 \to 1 \to 0.$$

$$\overline{\infty} \to 2n-2 \to 2n-3 \to \cdots \to 2 \to 1 \to 0.$$

The maximal element is the zero perversity $\overline{0} = 0$, the minimal element is the top perversity $\overline{t} = 2n-2$ for $\mathcal{P}_n^{op}$ and ∞ for $\widehat{\mathcal{P}_n}^{op}$.

The posets $\mathcal{P}_n^{op}$ and $\widehat{\mathcal{P}_n}^{op}$ are endowed with a partial addition $\oplus$ defined by $\overline{p} \oplus \overline{q} := \overline{p+q}$ if $p+q \leqslant 2n-2$ for $\mathcal{P}_n^{op}$ and $\widehat{\mathcal{P}_n}^{op}$. The complementary perversity $\overline{q}$ of $\overline{p}$ is then $\overline{q} = \overline{t} - \overline{p} = \overline{t-p}$.

If we do not consider complex varieties but just pseudomanifold of dimension n with only isolated singularities, we will still use a linear poset

$$\overline{\infty} \to n-2 \to n-3 \to \cdots \to 2 \to 1 \to 0.$$

Throughout this paper, every equation involving perversities will be considered in $\widehat{\mathcal{P}_n}^{op}$. For example $\max(\overline{p},\overline{0}) = \overline{0}$, $\min(\overline{\infty},\overline{p}) = \overline{\infty}$ for all $\overline{p}$ and if $\overline{p} = 2$ and $\overline{q} = 1$, then $\overline{p} < \overline{q}$.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Intersection spaces were defined by Markus Banagl in [6] in an attempt to spatialize Poincaré duality for singular spaces. The construction of these spaces rely on the notion of *spatial homology truncation* also introduced in [6].

Definition 3.2.0.2. *Given a simply connected CW-complex K of dimension n and an integer $k \leqslant n$. A spatial homology truncation of cut-off degree k of K is a CW-complex $t_k K$ together with a comparison map*

$$f\colon t_k K \longrightarrow K$$

such that the induced map $H_r(f)$ gives the following isomorphisms

$$H_r(t_k K) \cong \begin{cases} H_r(K) & r < k, \\ 0 & r \geqslant k. \end{cases} \tag{4}$$

The integer k is called the cut-off degree of the truncation.

Remark 3.2.1. *Such a truncation always exists provided that K is simply connected and this truncation is in fact defined on $\mathbf{Z}$ and not just on $\mathbf{Q}$, see [6].*

Definition 3.2.1.1. *Let X be a compact, connected, oriented pseudomanifold of dimension n and denote by $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ the singular locus of X. The pseudomanifold X is called supernormal if the link L_i of each singularity $\sigma_i \in \Sigma$ is simply connected.*

We denote by $\mathsf{Super}\mathcal{V}_{\mathbf{C}}$ the class of supernormal complex projective varieties with only isolated singularities.

For the rest of this chapter, we assume that the definition of a supernormal pseudomanifold X includes the fact that X is a connected pseudomanifold of dimension n (the compacity and orientability assumptions being automatic since we work in projective spaces $\mathbf{C}P^n$).

Before recalling the definition of intersection spaces given in 1.2.7.2, let us make changes on how we define the cut off degree $k(\overline{p})$ for the spatial homological truncation. This new definition will be more suited for our notion of coperverse cdga we will introduce in definition 3.2.2.1.

Let K be a simply connected CW-complex of dimension n and suppose given a perversity $\overline{p}$. Since

$$k(\overline{p}) := n - 1 - \overline{p}(n) = \overline{t}(n) - \overline{p}(n) + 1 = \overline{q}(n) + 1$$

with $\overline{q}$ the complementary perversity of $\overline{p}$, we now set that the cut-off degree is directly given by the perversity $\overline{p}$ and we denote it by $t_{\overline{p}} K$. That is

$$H_r(t_{\overline{p}} K) \cong \begin{cases} H_r(K) & \text{if } r \leqslant p \\ 0 & \text{if } r > p. \end{cases} \tag{5}$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Note that we also swap the strict and large inequalities in the definition. We will use this convention for the rest of this chapter.

By convention we also define $t_{\overline{\infty}}K = K$.

Suppose given a supernormal pseudomanifold X with isolated singularities,

$$L(\Sigma, X) := \sqcup_{\sigma_i} L_i$$

is then the disjoint union of simply connected topological manifold of dimension $n-1$. Denote by $X_{reg} := X - \Sigma$ the regular part of X. We denote by $t^{\overline{p}}L_i$ the homotopy cofiber of the map

$$f_i \colon t_{\overline{p}}L_i \to L_i.$$

We have maps

$$f^i \colon L_i \longrightarrow t^{\overline{p}}L_i.$$

If the link L_i of the singularity σ_i has more than one connected component, we apply the above homotopy cofiber separately on each connected components.

Definition 3.2.1.2. *The intersection space $I^{\overline{p}}X$ of the space X is defined by the following homotopy pushout diagram*

$$\begin{tikzcd} L(\Sigma, X) \arrow[r] \arrow[d] & X_{reg} \arrow[d, dashed] \\ \bigsqcup_i t^{\overline{p}}L_i \arrow[r, dashed] & I^{\overline{p}}X \end{tikzcd}$$

We shall use this definition of intersection spaces for the rest of the chapter. Note that with this definition we have $I^{\overline{\infty}}X = \overline{X}$ which is the normalization of X. We will denote by $HI^*_{\overline{p}}(X) := H^*(I^{\overline{p}}X)$ and by $\widetilde{HI}^*_{\overline{p}}(X)$ the reduced cohomology. We then have

$$HI^r_{\overline{p}}(X) = \begin{cases} \mathbf{Q} & r = 0 \\ H^r(X) & 1 \leqslant r \leqslant p \\ H^r(X) \oplus \operatorname{im} H^r(X_{reg}) \to H^r(L) & r = p+1 \\ H^r(X_{reg}) & r > p+1 \end{cases}$$

In particular, we have $HI^*_{\overline{0}}(X) = H^*(X_{reg})$ and $HI^*_{\overline{\infty}}(X) = H^*(\overline{X})$.

Remark 3.2.2. 1. *With this definition of the cut-off degree, our intersection spaces $I^{\overline{p}}X$ are the normal intersection spaces $\mathcal{I}^{\overline{q}}X$ originally defined in 1.2.7.2. Since we will only work with normal intersection spaces here, we drop the adjective normal and just call them intersection spaces.*

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

2. *This convention also has to be compared at the level of algebraic models with [14], where a $\overline{p}$-perverse rational model of a cone* cL *on a topological space* L *of dimension* n *is given by a truncation in degree* $\overline{p}(n)$ *of the rational model of* L. *In our case, a rational model of the intersection space* $I^{\overline{p}}cL$ *is then given by a unitary cotruncation in degree* $\overline{p}(n)$ *of the rational model of* L.

Let's compute the bounds of the different weight filtrations involved in $HI^r_{\overline{p}}(X)$ for a general perversity $\overline{p}$. Denote by $R^r(X_{reg}, L) := \operatorname{im} H^r(X_{reg}) \to H^r(L)$.

Lemma 3.2.2.1. *For* $r < n$, $R^r(X_{reg}, L)$ *is pure of weight* r. *For* $r \geqslant n$, *we have*

$$0 = W_r \subset W_{r+1} \subset \cdots \subset W_{2r} = R^r(X_{reg}, L).$$

Proof. This follows from the semi purity of the link, see [43]. Since

$$\dim(\Sigma) = 0,$$

the weight filtration on the cohomology of the link is semi-pure, this means that :

- the weights on $H^r(L)$ are less than or equal to r for $r < n$,
- the weights on $H^r(L)$ are greater or equal to $r + 1$ for $r \geqslant n$.

Combined with the two following facts

- The filtration $0 \subset W_r \subset \cdots \subset W_{2r} = H^r(X_{reg})$.
- $H^r(X_{reg}) \to H^r(L)$ is a morphism of mixed Hodge structures.

□

We have three cases

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

First case : $\overline{p} < \overline{m} = n-1$

	$1 \leqslant r \leqslant \overline{p}$	$r = p+1$			$\overline{p}+1 < r < n$	$n \leqslant r$
-1	0	0				
0	W_0	W_0				
1	W_1	W_1				
$\vdots$	$\vdots$	$\vdots$				
$r-1$	W_{r-1}	W_{r-1}		$W_{r-1} = 0$	$W_{r-1} = 0$	$W_{r-1} = 0$
r	W_r	W_r	$\oplus$	W_r	W_r	W_r
$r+1$						W_{r+1}
$\vdots$						$\vdots$
$2r-1$						W_{2r-1}
$2r$						W_{2r}
	$H^r(X)$	$H^r(X)$	$\oplus$	$R^r(X_{reg}, L)$	$H^r(X_{reg})$	$H^r(X_{reg})$

Second case : $\overline{p} = \overline{m} = n-1$

	$1 \leqslant r \leqslant n-1$	$r = p+1 = n$			$n \leqslant r$
-1	0	0			
0	W_0	W_0			
1	W_1	W_1			
$\vdots$	$\vdots$	$\vdots$			
$r-1$	W_{r-1}	W_{r-1}			$W_{r-1} = 0$
r	W_r	W_r	$\oplus$	$W_r = 0$	W_r
$r+1$				W_{r+1}	W_{r+1}
$\vdots$				$\vdots$	$\vdots$
$2r-1$				W_{2r-1}	W_{2r-1}
$2r$				W_{2r}	W_{2r}
	$H^r(X)$	$H^r(X)$	$\oplus$	$R^r(X_{reg}, L)$	$H^r(X_{reg})$

© 2016 Tous droits réservés.
lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Third case : $\overline{p} > \overline{m} = n-1$						
	$1 \leqslant r \leqslant n$	$n < r \leqslant p$	$r = p+1$			$p+1 < r$
-1	0					
0	W_0					
1	W_1					
$\vdots$	$\vdots$					
$r-1$	W_{r-1}	$W_{r-1} = 0$	$W_{r-1} = 0$			$W_{r-1} = 0$
r	W_r	W_r	W_r	$\oplus$	$W_r = 0$	W_r
$r+1$					W_{r+1}	W_{r+1}
$\vdots$					$\vdots$	$\vdots$
$2r-1$					W_{2r-1}	W_{2r-1}
$2r$					W_{2r}	W_{2r}
	$H^r(X)$	$H^r(X)$	$H^r(X)$	$\oplus$	$R^r(X_{reg}, L)$	$H^r(X_{reg})$

3.2.2 *Coperverse algebras and their homotopy theory*

3.2.2.1 *Coperverse algebras*

Let $\mathbf{k}$ be a fixed field of characteristic zero.

Definition 3.2.2.1. *A n-coperverse commutative differential graded algebra over $\mathbf{k}$, coperverse cdga for short, is a functor*

$$A_{\bar{\bullet}} \colon \widehat{\mathcal{P}_n}^{op} \longrightarrow \mathsf{CDGA}_{\mathbf{k}}.$$

That is for all perversities $\overline{p} \in \widehat{\mathcal{P}_n}^{op}$, $A_{\overline{p}}$ is a bigraded $\mathbf{k}$-algebra $(A^k_{\overline{p}})_{k \in \mathbf{N}}$, together with a linear differential $d \colon A^k_{\overline{p}} \to A^{k+1}_{\overline{p}}$ and an associative product $\mu \colon A^i_{\overline{p}} \times A^j_{\overline{p}} \to A^{i+j}_{\overline{p}}$.

We assume that products and differentials satisfy graded commutativity, Leibniz rules, and are compatible with poset maps. That is for every $\overline{p} \leqslant \overline{q}$ in $\widehat{\mathcal{P}_n}^{op}$ we have the following commutative diagrams.

$$\begin{tikzcd} A_{\overline{p}} \times A_{\overline{p}} \arrow[r, "\mu"] \arrow[d, "(\varphi_{\overline{p},\overline{q}}, \varphi_{\overline{p},\overline{q}})"'] & A_{\overline{p}} \arrow[d, "\varphi_{\overline{p},\overline{q}}"] \\ A_{\overline{q}} \times A_{\overline{q}} \arrow[r, "\mu"'] & A_{\overline{q}} \end{tikzcd} \qquad \begin{tikzcd} A_{\overline{p}} \arrow[r, "d"] \arrow[d, "\varphi_{\overline{p},\overline{q}}"'] & A_{\overline{p}} \arrow[d, "\varphi_{\overline{p},\overline{q}}"] \\ A_{\overline{q}} \arrow[r, "d"'] & A_{\overline{q}} \end{tikzcd}$$

© 2016 Tous droits réservés.
lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

We denote by $H_{\overline{\bullet}}(A, \mathbf{k}) := H(A_{\overline{\bullet}}, d)$.

Alternatively, a coperverse cdga is a diagram

$$A_{\overline{\infty}} \overset{\varphi_{\overline{\infty},\overline{2n-2}}}{\longrightarrow} A_{\overline{2n-2}} \overset{\varphi_{\overline{2n-2},\overline{2n-3}}}{\longrightarrow} \dots \overset{\varphi_{\overline{2},\overline{1}}}{\longrightarrow} A_{\overline{1}} \overset{\varphi_{\overline{1},\overline{0}}}{\longrightarrow} A_{\overline{0}}$$

where the vertices are cdga's and the edges are morphisms of cdga's. The assumption about the compatibility of the poset maps with the differentials and the products implies that passing to cohomology gives a coperverse cdga with the zero differential.

$$H^*_{\overline{\infty}}(A) \overset{H(\varphi_{\overline{\infty},\overline{2n-2}})}{\longrightarrow} H^*_{\overline{2n-2}}(A) \overset{H(\varphi_{\overline{2n-2},\overline{2n-3}})}{\longrightarrow} \dots \overset{H(\varphi_{\overline{2},\overline{1}})}{\longrightarrow} H^*_{\overline{1}}(A) \overset{H(\varphi_{\overline{1},\overline{0}})}{\longrightarrow} H^*_{\overline{0}}(A)$$

Morphisms of coperverse cdga's $f_{\overline{\bullet}}\colon A_{\overline{\bullet}} \to B_{\overline{\bullet}}$ are then morphisms of cdga's $\{f_{\overline{\infty}}, \dots, f_{\overline{0}}\}$ such that the following ladder commutes.

$$\begin{tikzcd}
\dots \arrow[r, "\varphi_{\overline{p+2},\overline{p+1}}"] & A_{\overline{p+1}} \arrow[r, "\varphi_{\overline{p+1},\overline{p}}"] \arrow[d, "f_{\overline{p+1}}"] & A_{\overline{p}} \arrow[r, "\varphi_{\overline{p},\overline{p-1}}"] \arrow[d, "f_{\overline{p}}"] & A_{\overline{p-1}} \arrow[r, "\varphi_{\overline{p-1},\overline{p-2}}"] \arrow[d, "f_{\overline{p-1}}"] & \dots \\
\dots \arrow[r, "\varphi'_{\overline{p+2},\overline{p+1}}"'] & B_{\overline{p+1}} \arrow[r, "\varphi'_{\overline{p+1},\overline{p}}"'] & B_{\overline{p}} \arrow[r, "\varphi'_{\overline{p},\overline{p-1}}"'] & B_{\overline{p-1}} \arrow[r, "\varphi'_{\overline{p-1},\overline{p-2}}"'] & \dots
\end{tikzcd}$$

Composition is given by the compostion of the vertical arrows.

We denote by $\widehat{\mathcal{P}_n}^{op}\mathsf{CDGA}_{\mathbf{k}}$ the category of coperverse cdga's over $\mathbf{k}$.

Note that with this definition, we have an extended product over the whole family $(A_{\overline{p}})_{\overline{p}\in\widehat{\mathcal{P}_n}^{op}}$. Indeed, for every $\overline{p} \leqslant \overline{q}$ in $\widehat{\mathcal{P}_n}^{op}$, denote by $\mu_{\overline{p},\overline{q}}$ the following composition

$$\mu_{\overline{p},\overline{q}}\colon A_{\overline{p}} \times A_{\overline{q}} \overset{(\varphi_{\overline{p},\overline{q}}, id)}{\longrightarrow} A_{\overline{q}} \times A_{\overline{q}} \overset{\mu}{\longrightarrow} A_{\overline{q}}.$$

Definition 3.2.2.2. *The map* $\mu_{\overline{\bullet},\overline{\bullet}}$ *defined for all* $\overline{p} \leqslant \overline{q}$ *in* $\mathcal{P}^{op}$ *by the above composition is called the extended product over the family* $(A_{\overline{p}})_{\overline{p}\in\mathcal{P}^{op}}$.

Remark 3.2.3. 1. *The following diagram, where* T *is the twist isomorphism* $T(a, b) := (-1)^{|a|\cdot|b|}(b, a)$, *commutes. Because of that and for the sake of simplicity, we will then adopt the following convention. Each time a product* $A_{\overline{p}} \times \dots \times A_{\overline{q}}$ *will appear, we will consider that the perversities are put in order, that is* $\overline{p} \leqslant \dots \leqslant \overline{q}$ *in* $\widehat{\mathcal{P}_n}^{op}$.

$$\begin{tikzcd}
A_{\overline{p}} \times A_{\overline{q}} \arrow[r, "(\varphi_{\overline{p},\overline{q}}, id)"] \arrow[d, "T"'] & A_{\overline{q}} \times A_{\overline{q}} \arrow[r, "\mu"] \arrow[d, "T"] & A_{\overline{q}} \\
A_{\overline{q}} \times A_{\overline{p}} \arrow[r, "(id, \varphi_{\overline{p},\overline{q}})"] & A_{\overline{q}} \times A_{\overline{q}} \arrow[ur, "\mu"'] &
\end{tikzcd}$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

2. *The extended product* $\mu_{\overline{\bullet},\overline{\bullet}}$ *verifies Leibniz rule, is associative and compatible with poset maps and morphisms of coperverse algebras. That is all* $\overline{p} \leqslant \overline{q} \leqslant \overline{r}$ *in* $\widehat{\mathcal{P}_n}^{op}$ *we have the commutative diagram,*

$$\begin{tikzcd}
A_{\overline{p}} \times A_{\overline{q}} \times A_{\overline{r}} \arrow[r, "{(\mathrm{id}, \mu_{\overline{q},\overline{r}})}"] \arrow[d, "{(\mu_{\overline{p},\overline{q}}, \mathrm{id})}"'] & A_{\overline{p}} \times A_{\overline{r}} \arrow[d, "\mu_{\overline{p},\overline{r}}"] \\
A_{\overline{q}} \times A_{\overline{r}} \arrow[r, "\mu_{\overline{q},\overline{r}}"'] & A_{\overline{r}}
\end{tikzcd}$$

and for all $\overline{p_1} \leqslant \overline{p_2} \leqslant \overline{q_1} \leqslant \overline{q_2}$ *in* $\widehat{\mathcal{P}_n}^{op}$ *we have the commutative diagram.*

$$\begin{tikzcd}
A_{\overline{p_1}} \times A_{\overline{q_1}} \arrow[r, "\mu_{\overline{p_1},\overline{q_1}}"] \arrow[d, "\varphi_{\overline{p_1},\overline{p_2}} \times \varphi_{\overline{q_1},\overline{q_2}}"'] & A_{\overline{q_1}} \arrow[d, "\varphi_{\overline{q_1},\overline{q_2}}"] \\
A_{\overline{p_2}} \times A_{\overline{q_2}} \arrow[r, "\mu_{\overline{p_2},\overline{q_2}}"'] & A_{\overline{q_2}}
\end{tikzcd}$$

Since $\mu_{\overline{p},\overline{p}} = \mu$ for all $\overline{p}$ we will always consider the family $(A_{\overline{p}})_{\overline{p}\in\widehat{\mathcal{P}_n}^{op}}$ endowed with the extended product. We then denote a coperverse cdga by $(A_{\overline{\bullet}}, \mu_{\overline{\bullet},\overline{\bullet}})$.

3.2.2.2 *Homotopy theory of coperverse algebras*

We now define a model structure on the category of coperverse cdga's by using the formalism of Reedy categories. The definitions and results involving Reedy categories can be found in [32].

First, recall the model structure of $\mathsf{CDGA}_{\mathbf{k}}$. The projective model structure on $\mathsf{CDGA}_{\mathbf{k}}$ is given by the following

- the weak equivalences are the quasi-isomorphims,
- the fibrations are the degreewise surjections,
- the cofibrations are the retracts of relative Sullivan algebras.

For $n \in \mathbf{N}$, consider the semifree dga's

$$S(n) := (\wedge \mathbf{k}[n], d = 0)$$

where $\mathbf{k}[n]$ denotes the graded vector space which is $\mathbf{k}$ in degree n and 0 otherwise. For $n \geqslant 1$, consider the semifree dga's

$$D(n) := \begin{cases} 0 & n = 0, \\ (\wedge(\mathbf{k}[n+1] \oplus \mathbf{k}[n]), d = 0) & n > 0 \end{cases}$$

and write

$$i_n \colon S(n) \to D(n)$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

for the morphism that send the generator of degree n to the generator of degree n. If $n = 0$ then this is the unique morphism $0 \to 0$, and for $n > 0$

$$j_n \colon 0 \to D(n).$$

Proposition 3.2.3.1. *The sets* $I := \{i_n\}_n \cup \{S(0) \to 0\}$, *and* $J := \{j_n\}_{n>0}$ *are the sets of generating cofibrations and acyclic cofibrations, respectively, of* $\mathsf{CDGA}_{\mathbf{k}}$. *The category* $\mathsf{CDGA}_{\mathbf{k}}$ *is then cofibrantly generated.*

Before talking about Reedy categories, note that we have an exact evaluation functor

$$Ev_{\overline{p}} \colon \widehat{\mathcal{P}_n}^{op}\mathsf{CDGA}_{\mathbf{k}} \longrightarrow \mathsf{CDGA}_{\mathbf{k}}$$

that send $A_{\overline{\bullet}}$ to $A_{\overline{p}}$, this functor admits an exact left adjoint $F_{\overline{p}}$ defined by $F_{\overline{p}}(A)_{\overline{q}} = A$ if $\overline{p} \leqslant \overline{q}$ and zero otherwise.

Definition 3.2.3.1. *Let* $\mathcal{C}$ *be a small category and* $\mathcal{C}' \subset \mathcal{C}$ *a subcategory. The subcategory* $\mathcal{C}'$ *is said to be a lluf subcategory if the objects of* $\mathcal{C}'$ *and* $\mathcal{C}$ *are the same.*

This implies that for all $x, y \in \mathcal{C}$, the sets $\mathcal{C}(x,y)$ and $\mathcal{C}'(x,y)$ might be different.

Definition 3.2.3.2 (Reedy category)**.** *Let* $\mathcal{C}$ *be a small category together with a degree function* $\deg \colon \mathcal{C} \longrightarrow \mathbf{N}$ *defined on the objects and suppose that we have two lluf subcategories* $\overrightarrow{\mathcal{C}}$ *and* $\overleftarrow{\mathcal{C}}$. *We say that* $(\mathcal{C}, \overrightarrow{\mathcal{C}}, \overleftarrow{\mathcal{C}})$ *is a Reedy category if the two following conditions are satisfied.*

1. *If* $\alpha \colon c \to c'$ *is a non-identity map in* $\overrightarrow{\mathcal{C}}$ *(resp. in* $\overleftarrow{\mathcal{C}}$*) then* $\deg(c) < \deg(c')$ *(resp.* $\deg(c) > \deg(c')$*).*
2. *Every map* α *in* $\mathcal{C}$ *has a unique factorization*

$$\begin{cases} \alpha & = \overrightarrow{\alpha} \circ \overleftarrow{\alpha}, \\ \overrightarrow{\alpha} & \in \overrightarrow{\mathcal{C}}, \\ \overleftarrow{\alpha} & \in \overleftarrow{\mathcal{C}}. \end{cases}$$

Example 3.2.1. 1. *A discrete category* $\mathcal{C}$, *that is a category where* $\mathcal{C}(x,y) = \{\mathrm{id}_x\}$ *if and only if* $x = y$ *and the empty set otherwise, is a Reedy category where all the objects are of degree* 0.

2. *Let* $\mathcal{P}$ *be a finite poset. We define every minimal element to be of degree 0 and we define the degree of an element* $p \in \mathcal{P}$ *to be the length of the longest path of non-identity maps from an element of degree zero to* p. *If we have* $p \to p'$ *with* $p \neq p'$ *then necessarily we have* $\deg p < \deg p'$. *The poset* $\mathcal{P}$ *is then endowed with a structure of Reedy category with*

$$\begin{cases} \overrightarrow{\mathcal{P}} & = \mathcal{P}, \\ \overleftarrow{\mathcal{P}} & = \mathrm{Disc}(\mathcal{P}). \end{cases}$$

where $\mathrm{Disc}(\mathcal{P})$ *is the discrete category underlying the poset* $\mathcal{P}$, *every elements of* $\mathrm{Disc}(\mathcal{P})$ *are of degree* 0.

© 2016 Tous droits réservés.

lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

For every Reedy category $\mathcal{C}$ there exists subcategories $\mathcal{C}_{<n}$ of objects of degree strictly inferior to n. Let then $F\colon \mathcal{C} \to \mathcal{M}$ a functor which we suppose covariant, consider $c \in \mathcal{C}$ with $\deg c = n$, we have the two objects and maps

$$L^c F \xrightarrow{\ell_c} F(c) \xrightarrow{m_c} M^c X,$$

where

$$L^c F := \operatorname{colim}(\partial(\overrightarrow{\mathcal{C}}_{<k}/c) \xrightarrow{U_c} \mathcal{C} \xrightarrow{F} \mathcal{M}),$$

$$M^c F := \lim(\partial(c/\overleftarrow{\mathcal{C}}_{<k}) \xrightarrow{U_c} \mathcal{C} \xrightarrow{F} \mathcal{M}).$$

with $\partial(\overrightarrow{\mathcal{C}}_{<k}/c)$ and $\partial(c/\overleftarrow{\mathcal{C}}_{<k})$ are the two full subcategories of respectively $\overrightarrow{\mathcal{C}}_{<k}/c$ and $c/\overleftarrow{\mathcal{C}}_{<k}$ where we have removed the identity object $c \to c$.

Definition 3.2.3.3. *The objects $L^c F$ and $M^c F$ are respectively called the c-th latching and c-th matching objects. The maps ℓ_c and m_c are then the c-th latching and c-th matching maps.*

Given a map $F \to G$ in $\mathsf{Fun}(\mathcal{C}, \mathcal{M})$, we define the c-th relative latching map by the following diagram of pushout

$$\begin{tikzcd}
L^c F \arrow[r] \arrow[d] & F(c) \arrow[d] \arrow[ddr] & \\
L^c G \arrow[r] \arrow[drr] \arrow[ur, phantom, "\lrcorner"] & \cdot \arrow[dr, dashed] & \\
& & G(c)
\end{tikzcd}$$

and the c-th relative matching map by the following diagram of pullback

$$\begin{tikzcd}
F(c) \arrow[drr] \arrow[dr, dashed] \arrow[ddr] & & \\
& \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & M^c F \arrow[d] \\
& G(c) \arrow[r] & M^c G
\end{tikzcd}$$

Theorem 3.2.4 ([32], 5.2.5)**.** *Let $\mathcal{M}$ be a model category et let $\mathcal{C}$ be a Reedy category. Then there is a model category on $\mathsf{Fun}(\mathcal{C}, \mathcal{M})$ such that :*

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

1. *the weak equivalences are defined pointwise,*
2. *the cofibrations are the maps* $F \to G$ *such that each relative latching map*

$$L^c G \coprod_{L^c F} F(c) \longrightarrow G(c)$$

is a cofibration in $\mathcal{M}$*,*

3. *the fibrations are the maps* $F \to G$ *such that each relative matching map*

$$F(c) \longrightarrow G(c) \times_{M^c G} M^c F$$

is a fibration in $\mathcal{M}$*.*

We now apply this result to our context. We endow $\widehat{\mathcal{P}_n}^{op}$ with the structure of a Reedy category defined in the item 2 of the last example.

Let $A_{\overline{\bullet}} \colon \widehat{\mathcal{P}_n}^{op} \to \mathsf{CDGA}_{\mathbf{k}}$ be a coperverse cdga and $\overline{p} \in \widehat{\mathcal{P}_n}^{op}$ such that $\deg \overline{p} = k$. We have

$$L^{\overline{p}} A_{\overline{\bullet}} := \operatorname{colim}(\partial(\mathcal{P}^{op}_{<k}/\overline{p}) \xrightarrow{U_{\overline{p}}} \mathcal{P}^{op} \xrightarrow{A_{\overline{\bullet}}} \mathcal{M}) = \operatorname{colim}_{\overline{p}<\overline{q}} A_{\overline{q}}$$

and

$$M^{\overline{p}} A_{\overline{\bullet}} := \lim(\partial(\overline{p}/\mathrm{Disc}(\mathcal{P})) \xrightarrow{U_{\overline{p}}} \mathcal{P}^{op} \xrightarrow{A_{\overline{\bullet}}} \mathcal{M}) = 0.$$

Computing the relative latching and matching map we get the following result

Theorem 3.2.5. *The category* $\widehat{\mathcal{P}_n}^{op}\mathsf{CDGA}_{\mathbf{k}}$ *has a structure of a cofibrantly generated model category which we call the projective model structure. In this model category, the weak equivalences are the quasi-isomorphisms and the fibrations are the surjections.*

Proof. The computations of weak equivalences and fibrations are clear.

The fact that $\widehat{\mathcal{P}_n}^{op}\mathsf{CDGA}_{\mathbf{k}}$ is cofibrantly generated comes from [32, Remark 5.1.8], the generating cofibrations are the $\{F_{\overline{p}}(i)\}_{i\in I, \overline{p}\in\widehat{\mathcal{P}_n}^{op}}$ and the generating acyclic cofibrations are the $\{F_{\overline{p}}(j)\}_{i\in J, \overline{p}\in\widehat{\mathcal{P}_n}^{op}}$ where I and J are the sets defined in the proposition 3.2.3.1. □

For clarity, we give the following definition as a result of the previous theorem.

Definition 3.2.5.1. *Let* $f_{\overline{\bullet}} \colon A_{\overline{\bullet}} \to B_{\overline{\bullet}}$ *be a morphism of coperverse cdga's. The morphism* $f_{\overline{\bullet}}$ *is*

1. *A quasi-isomorphism if, for every perversity* $\overline{p} \in \widehat{\mathcal{P}_n}^{op}$*, the induced map* $H^*_{\overline{p}}(A) \to H^*_{\overline{p}}(B)$ *is an isomorphism.*
2. *A fibration if, for every perversity* $\overline{p} \in \widehat{\mathcal{P}_n}^{op}$*, the induced map* $f_{\overline{p}} \colon A_{\overline{p}} \to B_{\overline{p}}$ *is a degreewise surjection.*

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

We denote by $\mathsf{Ho}(\widehat{\mathcal{P}_n}^{op}\mathsf{CDGA}_{\mathbf{k}})$ the homotopy category associated to the model category structure on $\widehat{\mathcal{P}_n}^{op}\mathsf{CDGA}_{\mathbf{k}}$. That is the category defined by formally inverting quasi-isomorphisms.

Remark 3.2.6. *There are many ways to put a model structures on $\widehat{\mathcal{P}_n}^{op}\mathsf{CDGA}_{\mathbf{k}}$. Indeed the category $\mathsf{CDGA}_{\mathbf{k}}$ also has an injective model structure where the weak equivalences are the quasi-isomorphisms and the cofibrations are the injections and we could have choose this model structure to do the computations.*

On the other hand we could have chose the projective or injective model structure on $\widehat{\mathcal{P}_n}^{op}\mathsf{CDGA}_{\mathbf{k}}$ coming from $\mathsf{CDGA}_{\mathbf{k}}$ rather than doing computations using Reedy categories. But since $\mathsf{CDGA}_{\mathbf{k}}$ is a combinatorial model category all the ways mentioned above are guaranteed to be Quillen equivalent to the projective model structure on $\widehat{\mathcal{P}_n}^{op}\mathsf{CDGA}_{\mathbf{k}}$.

By the way, all these model structures share the same weak equivalences.

3.3 COPERVERSE RATIONAL MODELS

3.3.1 *Coperverse cdga's associated with a morphism of cdga's*

The tools in this chapter are modified versions of the one appearing the work of Chataur and Cirici [12] on the interactions between intersection cohomology and mixed Hodge structures.

Let $(A, d) \in \mathsf{CDGA}_{\mathbf{k}}$. We denote by $\mathbf{k}(t, dt) := \wedge(t, dt)$ the free cdga generated by t and dt with $\deg t = 0$, $\deg dt = 1$ and $d(t) = dt$.

Definition 3.3.0.1. *We denote by $A(t, dt) := A \otimes_{\mathbf{k}} \mathbf{k}(t, dt)$. For $\lambda \in \mathbf{k}$ we also define the evaluation map*

$$\delta_\lambda \colon A(t, dt) \longrightarrow A$$

by $\delta_\lambda(t) = \lambda$ and $\delta_\lambda(dt) = 0$.

For all $r \geqslant 0$, we have the following short exact sequence

$$0 \longrightarrow \ker d^r \longrightarrow A^r \longrightarrow \operatorname{Coim} d^r \longrightarrow 0$$

where $\operatorname{Coim} d^r := A^r / \ker d^r$. Denote by $s_r \colon \operatorname{Coim} d^r \to A^r$ a choice of section. For all $r \geqslant 0$, we denote by $C_{\overline{r}} := \operatorname{im} s_r$, the differential d^r induces the isomorphism $C_{\overline{r}} \to \operatorname{im} d^r$.

Definition 3.3.0.2. *Let $\overline{p} \in \widehat{\mathcal{P}_n}^{op}$, the unitary $\overline{p}$-cotruncation of $A(t, dt)$ is defined by*

$$\xi_+^{\overline{p}} A(t, dt) := A^0 \oplus \xi^{\overline{p}} A(t, dt).$$

where $\xi^{\overline{p}} A(t, dt)$ is defined by

$$\xi^{\overline{p}} A(t, dt)^r := \begin{cases} A^r \otimes \mathbf{k}[t]t \oplus A^{r-1} \otimes \mathbf{k}[t]dt & r < p \\ A^{p-1} \otimes \mathbf{k}[t]dt \oplus A^p \otimes \mathbf{k}[t]t \oplus C_{\overline{p}} & r = p \\ A^{r-1} \otimes \mathbf{k}[t]dt \oplus A^r \otimes \mathbf{k}[t] & r > p \end{cases}$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Lemma 3.3.0.1. *$\xi_+^{\overline{\bullet}}A(t, dt)$ is a coperverse cdga.*

Proof. Consider first $\xi^{\overline{p}}A(t, dt)$.

The compatibility of $\xi^{\overline{\bullet}}A(t, dt)$ with the differential $d(\xi^{\overline{p}}A(t, dt)) \subset \xi^{\overline{p}}A(t, dt)$ and product $\xi^{\overline{p}}A(t, dt) \times \xi^{\overline{p}}A(t, dt) \to \xi^{\overline{p}}A(t, dt)$ is clear by construction. We detail the compatibility with the poset maps. By unicity of the maps $\varphi_{\overline{p},\overline{q}}$, every $\varphi_{\overline{p},\overline{q}}$ is a composition of poset maps $\varphi_{\overline{k+1},\overline{k}}$ so we only detail these ones. We have

$$\xi^{\overline{k+1}}A(t, dt)^r := \begin{cases} A^r \otimes \mathbf{k}[t]t \oplus A^{r-1} \otimes \mathbf{k}[t]dt & r < k+1 \\ A^k \otimes \mathbf{k}[t]dt \oplus A^{k+1} \otimes \mathbf{k}[t]t \oplus C_{\overline{k+1}} & r = k+1 \\ A^{r-1} \otimes \mathbf{k}[t]dt \oplus A^r \otimes \mathbf{k}[t] & r > k+1 \end{cases}$$

and

$$\xi^{\overline{k}}A(t, dt)^r := \begin{cases} A^r \otimes \mathbf{k}[t]t \oplus A^{r-1} \otimes \mathbf{k}[t]dt & r < k \\ A^{k-1} \otimes \mathbf{k}[t]dt \oplus A^{k} \otimes \mathbf{k}[t]t \oplus C_{\overline{k}} & r = k \\ A^{r-1} \otimes \mathbf{k}[t]dt \oplus A^r \otimes \mathbf{k}[t] & r > k. \end{cases}$$

For $r \leqslant k$ or $r > k+1$, $\varphi_{\overline{k+1},\overline{k}}$ is the identity map. For $r = k+1$, since $A^{k+1} \otimes \mathbf{k}[t] = A^{k+1} \oplus A^{k+1} \otimes \mathbf{k}[t]t$, $\varphi_{\overline{k+1},\overline{k}}$ is an injection.

Now for $\xi_+^{\overline{p}}A(t, dt) := A^0 \oplus \xi^{\overline{p}}A(t, dt)$ the compatibility with the differential and the poset maps is clear by the same arguments than above. The product $\xi_+^{\overline{p}}A(t, dt) \times \xi_+^{\overline{p}}A(t, dt) \to \xi_+^{\overline{p}}A(t, dt)$ is also clear by construction. $\square$

Let now $f \colon A \longrightarrow B$ be a morphism of cdga's. Given a perversity $\overline{p} \in \widehat{\mathcal{P}_n}^{op}$, we consider the following pull-back diagram in the category $\mathsf{CDGA}_{\mathbf{k}}$.

$$\begin{tikzcd} \mathcal{J}_{\overline{p}}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \xi_+^{\overline{p}}B(t, dt) \arrow[d, "\delta_1"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

The product and the differential are defined component-wise, the pull-back $\mathcal{J}_{\overline{\bullet}}(f)$ is compatible with poset maps. We then have

Proposition 3.3.0.1. *The pull-back $\mathcal{J}_{\overline{\bullet}}(f)$ is a coperverse cdga.*

Definition 3.3.0.3. *$\mathcal{J}_{\overline{\bullet}}(f)$ is the coperverse cdga associated to the morphism of cdga's $f \colon A \longrightarrow B$.*

Definition 3.3.0.4. *Let $(A_{\overline{\bullet}}, \mu_{\overline{\bullet},\overline{\bullet}})$ be a coperverse cdga and $r \in \mathbf{Z}$. We say that $(A_{\overline{\bullet}}, \mu_{\overline{\bullet},\overline{\bullet}})$ is a r-sharp coperverse cdga if the product satisfies the two following conditions*

© 2016 Tous droits réservés.
lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

1. ***Unity*** *For* $A^i_{\overline{p}} \times A^j_{\overline{0}} \to A^{i+j}_{\overline{0}}$ *the product lifts to*

$$\begin{tikzcd} & A^{i+j}_{\overline{p}} \arrow[d, "\varphi_{\overline{p},\overline{0}}"] \\ A^i_{\overline{p}} \times A^j_{\overline{0}} \arrow[ur, dashed] \arrow[r, "\mu_{\overline{p},\overline{0}}"] & A^{i+j}_{\overline{0}} \end{tikzcd}$$

2. ***Factorization*** *For* $\overline{p}, \overline{q} \neq \overline{0}$ *and* $i, j \neq 0$ *the product lifts to*

$$\begin{tikzcd} & A^{i+j}_{\overline{p+q+r}} \arrow[d, "\varphi_{\overline{p+q+r},\overline{q}}"] \\ A^i_{\overline{p}} \times A^j_{\overline{q}} \arrow[ur, dashed] \arrow[r, "\mu_{\overline{p},\overline{q}}"] & A^{i+j}_{\overline{q}} \end{tikzcd}$$

We assume that this lift satisfies all the properties of the product μ. *That is Leibniz rule with respect to the differential, graded commutativity and compatibility with poset maps and morphisms of cdga's.*

Lemma 3.3.0.2. $\xi^{\overline{\bullet}}_{+}A(t, dt)$ *is a* (-1)*-sharp coperverse cdga.*

Corollary 3.3.0.1. *Let* $f\colon A \longrightarrow B$ *be a morphism of cdga's, then* $\mathcal{J}_{\overline{\bullet}}(f)$ *is a* (-1)*-sharp coperverse cdga*

Remark 3.3.1.
1. *The first condition means that the final cdga* $A_{\overline{0}}$, *since* $\overline{0}$ *is the maximal element of* $\widehat{\mathcal{P}_n}^{op}$, *plays the role of the unit for the family* $(A_{\overline{p}})_{\overline{p} \in \widehat{\mathcal{P}_n}^{op}}$ *and in particular for the unit* $\eta_{\overline{0}}\colon \mathbf{k} \to A^0_{\overline{0}}$ *we have* $A^i_{\overline{p}} \times A^0_{\overline{0}} \to A^i_{\overline{p}}$ *for every* $\overline{p}$ *and every* $i \geqslant 0$.
2. *coperverse cdga's are meant to model the rational cohomology of intersection spaces* $HI^k_{\overline{p}}(X)$. *Since the* $I^{\overline{p}}X$ *are topological spaces their cohomology bear an inner cup-product which is reflected in the definition of the coperverse cdga's. The lift is here to show the interactions between the different* $HI^k_{\overline{p}}(X)$.

3.3.2 *Coperverse rational model of intersection spaces*

Let $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$ of complex dimension n, we denote by Σ the singular locus of X.

Let T be a closed algebraic neighbourhood of the singular locus in X such that the inclusion $\Sigma \subset T$ is a homotopy equivalence. Such a neighbourhood exists and is constructed with "rug functions", see [37, p.144] or [22].

The link $L := L(\Sigma, X)$ of Σ in X is defined by $L := \partial T \simeq T^* := T - \Sigma$. The inclusion $i\colon L \hookrightarrow X_{reg}$ of the link into the regular part of X induces a morphism of cdga's over $\mathbf{Q}$

$$i^*\colon A_{PL}(X_{reg}) \longrightarrow A_{PL}(L).$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Let $\overline{p} \in \widehat{\mathcal{P}_n}^{op}$ be a perversity, the rational model of the intersection space $I^{\overline{p}}X$ is given by $AI_{\overline{p}}(X) := \mathcal{I}_{\overline{p}}(i^*)$, which is the following pull-back diagram by proposition 1.2.7.4.

$$\begin{tikzcd}
\mathcal{I}_{\overline{p}}(i^*) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \xi_+^{\overline{p}} A_{PL}(L)(t, dt) \arrow[d, "\delta_1"] \\
A_{PL}(X_{reg}) \arrow[r, "i^*"] & A_{PL}(L)
\end{tikzcd}$$

Definition 3.3.1.1. *The coperverse cdga* $AI_{\overline{\bullet}}(X)$ *is called the coperverse rational model of the intersection spaces* $I^{\overline{\bullet}}X$.

If $A_{\overline{\bullet}}$ is a coperverse cdga, its cohomology is also a coperverse cdga with trivial differential. We then have the following proposition.

Proposition 3.3.1.1. $HI^*_{\overline{\bullet}}(X)$ *is a coperverse cdga.*

We have an isomorphism of coperverse cdga $H^*(AI_{\overline{\bullet}}(X)) \cong HI^*_{\overline{\bullet}}(X)$. This defines a map

$$AI_{\overline{\bullet}} \colon \mathsf{Super}\mathcal{V}_{\mathbf{C}} \longrightarrow \mathsf{Ho}(\widehat{\mathcal{P}_n}^{op}\mathsf{CDGA}_{\mathbf{k}}).$$

If we only consider the coperverse rational model of $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$, we then have that $AI_{\overline{\bullet}}(X)$ is a (-1)-sharp coperverse cdga by corollary 3.3.0.1. But if we only want to consider the cohomology coperverse algebra $HI^*_{\overline{\bullet}}(X)$, we can have an even sharper result.

Proposition 3.3.1.2. *Let* $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$. *Then* $(HI^*_{\overline{\bullet}}(X), 0)$ *is a 1-sharp coperverse cdga. That is we have*

$$\begin{cases} HI^i_{\overline{0}}(X) \otimes \widetilde{HI}^j_{\overline{p}}(X) \longrightarrow \widetilde{HI}^{i+j}_{\overline{p}}(X) \\ \widetilde{HI}^i_{\overline{p}}(X) \otimes \widetilde{HI}^j_{\overline{q}}(X) \longrightarrow \widetilde{HI}^{i+j}_{\overline{p+q+1}}(X) \quad p+q+1 \leqslant 2n-2. \end{cases}$$

Remark 3.3.2. *It is important to make a difference between the extended product* $\mu_{\overline{\bullet},\overline{\bullet}}$ *and the property of sharpness. The existence of the extended product is a consequence of the definition 3.2.2.1 and as such every coperverse cdga defined in the same way naturally has an extended product.*

The property of sharpness of our coperverse algebras defined in 3.3.0.3 is a consequence of our methods of construction. There might be coperverse algebras which do not have any property of sharpness, but still have an extended product.

3.4 HODGE THEORY

3.4.1 *Coperverse mixed Hodge algebras*

We now want to put finer structures on our coperverse cdga's. This is done with the help of filtered cdga's and mixed Hodge cdga's, see for example [15].

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Definition 3.4.0.1. *A filtered cdga (A, W) is a cdga (A, d) together with a filtration $\{W_m A\}_{m \in \mathbf{Z}}$ such that*

1. $W_{m-1}A \subset W_m A$ *and* $d(W_m A) \subset W_m A$*, for all* $m \in \mathbf{Z}$*,*
2. $W_m A.W_n A \subset W_{m+n} A$*,*
3. *The filtration W is exhaustive and biregular : for all $n \geqslant 0$ there exist integers m and l such that $W_m A^n = 0$ and $W_l A^n = A^n$.*

Definition 3.4.0.2. *A mixed Hodge cdga over $\mathbf{Q}$ is a filtered cdga (A, W) with a filtration F on $A \otimes \mathbf{C}$ such that for all $n \geqslant 0$,*

1. *the triple $(A^n, \mathrm{Dec}(W), F)$ is a mixed Hodge structure,*
2. *the differential $d\colon A^k \to A^{k+1}$ and the product $\mu\colon A^i \times A^j \to A^{i+j}$ are morphisms of mixed Hodge structures.*

The filtration W is called the weight filtration and the filtration F is called the Hodge filtration.

We extend these definitions to make them compatible with our notion of coperverse cdga.

Definition 3.4.0.3. *A coperverse filtered cdga $(A_{\overline{\bullet}}, W)$ is a coperverse cdga $A_{\overline{\bullet}}$ together with a filtration $\{W_m A_{\overline{\bullet}}\}_{m \in \mathbf{Z}}$ such that*

1. $W_{m-1}A_{\overline{p}} \subset W_m A_{\overline{p}}$ *and* $d(W_m A_{\overline{p}}) \subset W_m A_{\overline{p}}$*, for all $m \in \mathbf{Z}$ and all* $\overline{p} \in \widehat{\mathcal{P}_n}^{op}$*,*
2. $W_m A_{\overline{p}}.W_n A_{\overline{p}} \subset W_{m+n} A_{\overline{p}}$*,*
3. $\varphi_{\overline{p},\overline{q}}(W_m A_{\overline{p}}) \subset W_m A_{\overline{q}}$ *for all $\overline{p} \leqslant \overline{q}$ in* $\widehat{\mathcal{P}_n}^{op}$*,*
4. *The filtration W is exhaustive and biregular : for all $n \geqslant 0$ and all $\overline{p} \in \mathcal{P}^{op}$ there exist integers m and l such that $W_m A^n_{\overline{p}} = 0$ and $W_l A^n_{\overline{p}} = A^n_{\overline{p}}$.*

Definition 3.4.0.4. *A coperverse mixed Hodge cdga over $\mathbf{Q}$ is a coperverse filtered cdga $(A_{\overline{\bullet}}, W)$ with a filtration F on $A_{\overline{\bullet}} \otimes \mathbf{C}$ such that for all $n \geqslant 0$ and all $\overline{p} \in \widehat{\mathcal{P}_n}^{op}$,*

1. *the triple $(A^n_{\overline{p}}, \mathrm{Dec}(W), F)$ is a mixed Hodge structure,*
2. *the differential $d\colon A^k_{\overline{p}} \to A^{k+1}_{\overline{p}}$, the product $\mu\colon A^i_{\overline{p}} \times A^j_{\overline{p}} \to A^{i+j}_{\overline{p}}$ and the poset maps $\varphi_{\overline{p},\overline{q}}\colon A^k_{\overline{p}} \to A^k_{\overline{q}}$ are morphisms of mixed Hodge structures.*

The filtration W is called the weight filtration and the filtration F is called the Hodge filtration.

We will denote, by an abuse of notations, such a coperverse mixed Hodge cdga by the triple $(A_{\overline{\bullet}}, W, F)$ with in mind the fact that F is not defined on $A_{\overline{\bullet}}$ but on its complexification $A_{\overline{\bullet}} \otimes \mathbf{C}$. The filtration $\mathrm{Dec}(W)$ is the Deligne's décalage of the weight filtration defined in [16, p. 15] which is given by

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

$$\mathrm{Dec}(W_p)A^n_{\bar\bullet} := W_{p-n}A^n_{\bar\bullet} \cap d^{-1}(W_{p-n-1}A^{n+1}_{\bar\bullet}).$$

We denote by $\widehat{\mathcal{P}_n}^{op}\mathbf{MH}\mathsf{CDGA}_{\mathbf{Q}}$ the category of coperverse mixed Hodge cdga's over $\mathbf{Q}$.

Lemma 3.4.0.1. *Let $(A_{\bar\bullet}, W, F)$ be a coperverse mixed Hodge cdga, then the extended product $\mu_{\bar\bullet,\bar\bullet}$ is a morphism of mixed Hodge structure.*

Definition 3.4.0.5. *A coperverse filtered cdga $(A_{\bar\bullet}, W)$ is said to be r-sharp if $A_{\bar\bullet}$ is a filtered coperverse cdga such that the lift is compatible with the filtration $\{W_m A_{\bar\bullet}\}_{m\in\mathbf{Z}}$. That is we have the two following conditions*

1. ***Filtered unity*** *For $W_m A^i_{\bar p} \times W_n A^j_{\bar 0} \to W_{m+n}A^{i+j}_{\bar 0}$ the product lifts to*

$$\begin{tikzcd}
 & W_{m+n}A^{i+j}_{\bar p} \arrow[d, "\varphi_{\bar p,\bar 0}"] \\
W_m A^i_{\bar p} \times W_n A^j_{\bar 0} \arrow[ur, dashed] \arrow[r, "\mu_{\bar p,\bar 0}"] & W_{m+n}A^{i+j}_{\bar 0}
\end{tikzcd}$$

2. ***Filtered factorization*** *For $\bar p, \bar q \neq \bar 0$ and $i, j \neq 0$ the product lifts to*

$$\begin{tikzcd}
 & W_{m+n}A^{i+j}_{\overline{p+q+r}} \arrow[d, "\varphi_{\overline{p+q+r},\bar q}"] \\
W_m A^i_{\bar p} \times W_n A^j_{\bar q} \arrow[ur, dashed] \arrow[r, "\mu_{\bar p,\bar q}"] & W_{m+n}A^{i+j}_{\bar q}
\end{tikzcd}$$

Definition 3.4.0.6. *A r-sharp coperverse mixed Hodge cdga over $\mathbf{Q}$ is a coperverse mixed Hodge cdga $(A_{\bar\bullet}, W, F)$ such that the lift is a morphism of mixed Hodge structure.*

Consider $\mathbf{Q}(t, dt)$ together with the *bête* filtration σ, that is the multiplicative filtration with t of weight 0 and dt of weight -1. We endow $\mathbf{C}(t, dt) := \mathbf{Q}(t, dt) \otimes \mathbf{C}$ with the bête filtration σ and the trivial filtration t, that is decreasing filtration given by

$$0 = t^1\mathbf{C}(t, dt) \subset t^0\mathbf{C}(t, dt) = \mathbf{C}(t, dt).$$

Since $\mathrm{Dec}(\sigma) = t$ the triple $(\mathbf{Q}(t, dt), \sigma, t)$ is a mixed Hodge cdga.

Given another mixed Hogde cdga (A, W, F), since the category of mixed Hodge structure is abelian the triple

$$(A(t, dt), W * \sigma, F * t)$$

is again a mixed Hodge cdga where the filtrations are defined by convolution. That is we have

$$(W * \sigma)_m A(t, dt)^n := W_m A^n \otimes \mathbf{Q}[t] \oplus W_{m+1}A^{n-1} \otimes \mathbf{Q}[t]dt$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

and

$$(F * t)^k A(t, dt) := F^k A \otimes \mathbf{C}(t, dt).$$

The evaluation map δ_1 is strictly compatible with filtrations.

Lemma 3.4.0.2. *Let* (A, W, F) *be a mixed Hodge cdga. Then* $\xi_+^{\bullet} A(t, dt)$ *is a* (-1)*-sharp coperverse mixed Hodge cdga.*

Proof. The triple $(A(t, dt), W * \sigma, F * t)$ is a mixed Hodge cdga, for all $\overline{p} \in \mathcal{P}^{op}$, $\xi_+^{\overline{p}} A(t, dt)$ is a sub-algebra with the filtrations induced by restriction.

The differential is a morphism of mixed Hodge structure since the differential on $(A(t, dt), W * \sigma, F * t)$ is and $d(\xi_+^{\overline{p}} A(t, dt)) \subset \xi_+^{\overline{p}} A(t, dt)$.

The poset maps $\varphi_{\overline{k+1}, \overline{k}}$, $k \geqslant 0$, are the identity everywhere but at the cut-off degree $k+1$ where they are canonical inclusions, $\varphi_{\overline{k+1}, \overline{k}}$ in then compatible with both filtrations and by composition so are the $\varphi_{\overline{p}, \overline{q}}$.

The extended product $\xi_+^{\overline{p}} A(t, dt)^i \times \xi_+^{\overline{q}} A(t, dt)^j \to \xi_+^{\overline{q}} A(t, dt)^{i+j}$ being defined as the composition of μ with poset maps $\varphi_{\overline{p}, \overline{q}}$, it is a morphism of mixed Hodge structure.

The sharpness comes from the fact that $\xi_+^{\overline{p}} A(t, dt)$ is (-1)-sharp and that the product is a morphism of mixed Hodge structure. □

Let then $f\colon (A, W, F) \to (B, W, F)$ be a morphism of mixed Hodge cdga. Since the category of mixed Hodge structures is abelian, see [16, Theorem 2.3.5], we have the following proposition.

Proposition 3.4.0.1. *The coperverse cdga* $\mathcal{I}_{\overline{\bullet}}(f)$ *is a coperverse mixed Hodge cdga.*

3.4.2 *Mixed Hodge structure on the coperverse rational model of the intersection spaces* $I^{\overline{\bullet}} X$

Definition 3.4.0.7 ([15]). *A mixed Hodge diagram of cdga's over* $\mathbf{Q}$ *consists of a filtered cdga* $(A_{\mathbf{Q}}, W)$ *over* $\mathbf{Q}$*, a bifiltered cdga* $(A_{\mathbf{C}}, W, F)$ *over* $\mathbf{C}$*, together with a string of filtered* E_1*-quasi-isomorphisms from* $(A_{\mathbf{Q}}, W) \otimes \mathbf{C}$ *to* $(A_{\mathbf{C}}, W)$*. in addition, the following axioms must hold :*

- *The weight filtrations* W *are regular and exhaustive. The Hodge filtration* F *is biregular. The cohomology* $H(A_{\mathbf{Q}})$ *has finite type.*
- *For all* $p \in \mathbf{Z}$*, the differential of* $\mathrm{gr}_p^W(A_{\mathbf{C}})$ *is strictly compatible with* F*.*
- *For all* $n \geqslant 0$ *and all* $p \in \mathbf{Z}$*, the filtration* F *induced on* $H^n(\mathrm{gr}_p^W(A_{\mathbf{C}}))$ *defines a pure Hodge structure of weight* $p+n$ *on* $H^n(\mathrm{gr}_p^W(A_{\mathbf{Q}}))$*.*

Morphisms of mixed Hodge diagrams are defined by level-wise morphisms of bifiltered cdga's such that the associated diagram is strictly commutative. Forgetting the multiplicative structure gives back the notion of mixed Hodge complex defined by Deligne in [17, section 8.1].

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Definition 3.4.0.8. *Let X be a topological space. A mixed Hodge diagram for X is a mixed Hodge diagram $M(X)$ such that $M(X)_{\mathbf{Q}} \simeq A_{PL}(X)$, that is its rational component is quasi-isomorphic to the rational algebra of piecewise linear forms on X.*

The following theorem is a modified version of a theorem appearing in [12] stating that the intersection homotopy type of a complex variety X with only isolated singularities carries well-defined mixed Hodge structures.

Theorem 3.4.1. *Let $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$ of complex dimension n. There exists a coperverse mixed Hodge cdga $MI_{\bar{\bullet}}(X)$ together with a string of quasi-isomorphisms*

$$MI_{\bar{\bullet}}(X) \leftarrow * \rightarrow AI_{\bar{\bullet}}(X)$$

such that :

1. $MI_{\bar{\bullet}}(X) = \mathcal{J}_{\bar{\bullet}}(\tilde{\iota})$ *where* $\tilde{\iota}\colon M(X_{reg}) \to M(L)$ *is a model of mixed Hodge cdga's for the rational homotopy type of the inclusion* $i\colon L \hookrightarrow X_{reg}$.
2. *there is an isomorphism of coperverse mixed Hodge cdga's*

$$H^*(MI_{\bar{\bullet}}(X)) \cong HI^*_{\bar{\bullet}}(X).$$

3. *The mixed Hodge cdga's* $MI_{\bar{0}}(X)$ *and* $MI_{\overline{\infty}}(X)$ *defines respectively the mixed Hodge structure on the rational homotopy type of the regular part* X_{reg} *of* X *and on the normalisation* $\overline{X}$ *of* X.
4. *The differential of* $MI_{\bar{\bullet}}(X)$ *satisfies* $d(W_p MI_{\bar{\bullet}}(X)) \subset W_{p-1} MI_{\bar{\bullet}}(X)$.

This defines a map

$$MI_{\bar{\bullet}}\colon \mathsf{Super}\mathcal{V}_{\mathbf{C}} \longrightarrow \mathsf{Ho}(\widehat{\mathcal{P}_n}^{op}\mathbf{MH}\mathsf{CDGA}_{\mathbf{Q}}).$$

Proof. By [20, theorem 3.2.1], there is a morphism of mixed Hodge diagrams $M(X_{reg}) \to M(L)$ induced by the inclusion $i\colon L \hookrightarrow X_{reg}$. The rational component of this morphism is the morphism $i^*\colon A_{PL}(X_{reg}) \to A_{PL}(L)$ of rational piecewise linear forms induced by the inclusion $i\colon L \hookrightarrow X_{reg}$. By [15, theorem 3.19], there is a commutative diagram of mixed Hodge diagrams

$$\begin{tikzcd}
A_{PL}(X_{reg}) \arrow[r, "i^*"] & A_{PL}(L) \\
* \arrow[u] \arrow[r] \arrow[d] & * \arrow[u] \arrow[d] \\
M(X_{reg}) \arrow[r, "\tilde{\iota}"] & M(L)
\end{tikzcd}$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

where the vertical maps are quasi-isomorphisms and $\tilde{\iota}$ is a map of mixed Hodge cdga's whose differential satisfies $d(W_p) \subset W_{p-1}$. We then let $MI_{\overline{\bullet}}(X) := \mathcal{J}_{\overline{\bullet}}(\tilde{\iota})$. The above commutative diagram defines a string of quasi-isomorphisms from $MI_{\overline{\bullet}}(X)$ to $AI_{\overline{\bullet}}(X)$ by proposition 1.2.7.4.

Let now show that $MI_{\overline{\bullet}}(X)$ is a coperverse mixed Hodge cdga. Consider the mixed Hodge cdga $M(L)(t, dt)$ defined as in definition 3.3.0.1. Then $\xi_+^{\overline{p}} M(L)(t, dt)$ is a complex of mixed Hodge structure for every perversities $\overline{p} \in \widehat{\mathcal{P}_n}^{op}$. The product

$$\xi_+^{\overline{p}} M(L)(t, dt) \times \xi_+^{\overline{q}} M(L)(t, dt) \longrightarrow \xi_+^{\overline{q}} M(L)(t, dt)$$

and the poset maps

$$\xi_+^{\overline{p}} M(L)(t, dt) \longrightarrow \xi_+^{\overline{q}} M(L)(t, dt)$$

for $\overline{p} \leqslant \overline{q} \in \widehat{\mathcal{P}_n}^{op}$ are strictly compatible with filtrations. Since the category of mixed Hodge structures is abelian, for each $n \geqslant 0$ and each $\overline{p} \in \widehat{\mathcal{P}_n}^{op}$, the vector space $MI_{\overline{p}}(X)^n$ carries a mixed Hodge structure. The compatibility with product and poset maps is a matter of verifications. This proves the first three properties.

The differential on $MI_{\overline{p}}(X)$ being defined via the pull-back of cdga's whose differential satisfies $d(W_p) \subset W_{p-1}$, this also holds for $MI_{\overline{p}}(X)$. $\square$

From this result we can deduce the two following product structure.

Corollary 3.4.1.1. *Let* $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$, $MI_{\overline{\bullet}}(X)$ *is then a* (-1)*-sharp coperverse mixed Hodge cdga.*

Corollary 3.4.1.2. *Let* $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$*, then the family of algebras*

$$\{HI^*_{\overline{0}}(X), \widetilde{H}I^*_{\overline{1}}(X), \dots, \widetilde{H}I^*_{\overline{2n-2}}(X)\}$$

is endowed with a product

$$\begin{cases} HI^i_{\overline{0}}(X) \otimes \widetilde{H}I^j_{\overline{p}}(X) \longrightarrow \widetilde{H}I^{i+j}_{\overline{p}}(X) \\ \widetilde{H}I^i_{\overline{p}}(X) \otimes \widetilde{H}I^j_{\overline{q}}(X) \longrightarrow \widetilde{H}I^{i+j}_{\overline{p+q+1}}(X) \quad p+q+1 \leqslant 2n-2. \end{cases}$$

This product is a morphism of mixed Hodge structure.

Due to the method of construction of the coperverse mixed Hodge cdga $MI_{\overline{\bullet}}(X)$, we have the following commutative diagram of mixed Hodge cdga's.

$$\begin{tikzcd}
M(X_{reg}) \arrow[r, "\tilde{\iota}"] \arrow[d] & M(L) \arrow[d] & \xi_+^{\overline{k+1}} M(L)(t, dt) \arrow[l, "\delta_1"'] \arrow[d, hook, "\varphi_{\overline{k+1},\overline{k}}"] \\
M(X_{reg}) \arrow[r, "\tilde{\iota}"] \arrow[d] & M(L) \arrow[d] & \xi_+^{\overline{k}} M(L)(t, dt) \arrow[l, "\delta_1"'] \arrow[d, two heads] \\
0 \arrow[r] & 0 & M(L, k) \arrow[l]
\end{tikzcd}$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Where each elements of the last row is the quotient of the previous elements in the same column. That is $M(L,k)$ is the mixed Hodge cdga quotient such that $H^i(M(L,k)) = H^k(L)$ for $i = k$ and zero otherwise. Taking the pullback on each rows we then have a short exact sequence of mixed Hodge structure

$$0 \longrightarrow MI_{\overline{k+1}}(X) \longrightarrow MI_{\overline{k}}(X) \longrightarrow M(L,k) \longrightarrow 0.$$

This short exact sequence induces a long exact sequence of mixed Hodge structure and extends to arbitrary perversities. That is we have

Corollary 3.4.1.3. *Suppose given* $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$ *and two perversities* $\overline{p} \leqslant \overline{q} \in \widehat{\mathcal{P}_n}^{op}$. *We have a long exact sequence of mixed Hodge structures*

$$\cdots \to HI^i_{\overline{p}}(X) \to HI^i_{\overline{q}}(X) \to H^i(M(L,q,p)) \to HI^{i+1}_{\overline{p}}(X) \to \cdots$$

where

$$H^i(M(L,q,p)) = \begin{cases} H^i(L) & q \leqslant i < p, \\ 0 & \textit{otherwise.} \end{cases}$$

3.5 WEIGHT SPECTRAL SEQUENCE

Let (B, W, F) a mixed Hodge cdga, then $(B(t,dt), W * \sigma, F * t)$ is again a mixed Hodge cdga where the filtrations are given by

$$(W * \sigma)_m B(t,dt)^n := W_m B^n \otimes \mathbf{Q}[t] \oplus W_{m+1} B^{n-1} \otimes \mathbf{Q}[t]dt$$

and

$$(F * t)^k B(t,dt) := F^k B \otimes \mathbf{C}(t,dt).$$

The graded subspace associated to the the weight filtration is then given by

$$gr^{W*\sigma}_m (B(t,dt)^n) = gr^W_m (B^n) \otimes \mathbf{Q}[t] \oplus gr^W_{m+1}(B^{n-1}) \otimes \mathbf{Q}[t]dt.$$

Given a mixed Hodge cdga (B, W, F), we then have a cohomological weight spectral sequence $E(B, W)$ whose E_1 term is defined by

$$E_1^{r,s}(B,W) := H^{r+s}(gr^W_{-r}(B^{r+s})).$$

The spectral sequence associated to a coperverse filtered cdga $(A_{\overline{\bullet}}, W)$ is compatible with the multiplicative structure. Thus, for all $r \geqslant 0$, The term $E_r(A_{\overline{\bullet}}, W)$ is a coperverse bigraded algebra with differential d_r of degree $(r, 1-r)$.

Lemma 3.5.0.1. *Let* (B, W, F) *a mixed Hodge cdga, we have a canonical isomorphism of differential bigraded algebras*

$$E_1(B(t,dt), W * \sigma) \cong E_1(B, W)(t, dt)$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Lemma 3.5.0.2. *Let* $f\colon (A, W, F) \to (B, W, F)$ *be a morphism of mixed Hodge cdga's. There is a quasi-isomorphism of coperverse differential bigraded algebras*

$$E_1(\mathcal{J}_{\overline{\bullet}}(f), W) \xrightarrow{\sim} \mathcal{J}_{\overline{\bullet}}(E_1(f, W)).$$

Proof. The evaluation map δ_1 is strictly compatible with filtrations. Since f is a morphism of mixed Hodge structures, the morphism of complexes $f - \delta_1\colon A \times \xi_+^{\overline{p}} B(t, dt) \to B$ is strictly compatible with filtrations. Therefore we have $E_1(\ker(f - \delta_1)) = \ker E_1(f - \delta_1)$. The lemma 3.5.0.1 and the observations that we have a quasi-isomorphism $E_1(\xi_+^{\overline{\bullet}} B(t, dt), W * \sigma) \xrightarrow{\sim} \xi_+^{\overline{\bullet}} E_1(B, W)(t, dt)$ finish the proof. □

Lemma 3.5.0.3. *Let* $(A_{\overline{\bullet}}, W, F)$ *be a coperverse mixed Hodge cdga such that*

$$d(W_p A_{\overline{\bullet}}) \subset W_{p-1} A_{\overline{\bullet}}.$$

There is an isomorphism of complex coperverse cdga's

$$A_{\overline{\bullet}} \otimes \mathbf{C} \cong E_1(A_{\overline{\bullet}} \otimes \mathbf{C}, W).$$

Proof. The proof is is the same as the proof of [12, lemma 3.4] for perverse mixed Hodge cdga's. □

Remark 3.5.1. *Let* (A, W) *be a filtered cdga of finite type over a field* $\mathbf{k}$ *and* $\mathbf{k} \subset \mathbf{K}$ *a field extension. By [15, theorem 2.26] we have that* $A \cong E_r(A, W)$ *if and only if* $A \otimes_{\mathbf{k}} \mathbf{K} \cong E_r(A \otimes_{\mathbf{k}} \mathbf{K}, W)$. *For a coperverse cdga of finite type the same proof is valid. This implies the isomorphism of lemma 3.5.0.3 descends to an isomorphism over* $\mathbf{Q}$.

Let $X \in \mathrm{Super}\mathcal{V}_{\mathbf{C}}$ of complex dimension n. The inclusion $i\colon L \hookrightarrow X_{reg}$ of the link into the regular part induces a morphism of multiplicative weight spectral sequence $E_1(i^*)\colon E_1(X_{reg}) \to E_1(L)$. We define

$$EI_{1,\overline{\bullet}}(X) := \mathcal{J}_{\overline{\bullet}}(E_1(i^*)).$$

This is a coperverse differential bigraded algebra whose cohomology satifies

$$EI_{2,\overline{p}}^{r,s}(X) := H^{r,s}(EI_{1,\overline{p}}(X)) \cong gr_s^W(HI_{\overline{p}}^{r+s}(X))$$

Definition 3.5.1.1. *Let* $X \in \mathrm{Super}\mathcal{V}_{\mathbf{C}}$ *of complex dimension* n. *The spectral sequence* $EI_{1,\overline{\bullet}}(X)$ *defined by*

$$EI_{1,\overline{\bullet}}(X) := \mathcal{J}_{\overline{\bullet}}(E_1(i^*))$$

is called the coperverse weight spectral sequence associated to $I^{\overline{\bullet}}X$.

In [12, theorem 3.12], Chataur and Cirici prove the existence of a quasi-isomorphism between the rational perverse model $IA_{\overline{\bullet}}(X)$ of a complex projective variety with only isolated singularities and the first term of its perverse weight spectral sequence $IE_{1,\overline{\bullet}}(X)$. This theorem can be modified to get the following one.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Theorem 3.5.2. *Let $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$ with only isolated singularities. There is a string of quasi-isomorphisms of coperverse cdga's from $MI_{\bar{\bullet}}(X) \otimes \mathbf{C}$ to $EI_{1,\bar{\bullet}}(X) \otimes \mathbf{C}$. In particular, there is an isomorphism in $\mathrm{Ho}(\widehat{\mathcal{P}_n}^{op}\mathsf{CDGA}_{\mathbf{C}})$ from $AI_{\bar{\bullet}}(X) \otimes \mathbf{C}$ to $EI_{1,\bar{\bullet}}(X) \otimes \mathbf{C}$.*

Proof. Let $(MI_{\bar{\bullet}}(X), W, F)$ be the coperverse mixed Hodge cdga given by the theorem 3.4.1. Since the differential satisfies

$$d(W_p MI_{\bar{\bullet}}(X)) \subset W_{p-1} MI_{\bar{\bullet}}(X)$$

by the lemma 3.5.0.3 we have an isomorphism of complex coperverse cdga's $MI_{\bar{\bullet}}(X) \otimes \mathbf{C} \cong E_1(MI_{\bar{\bullet}}(X) \otimes \mathbf{C}, W)$.

By construction, we have $MI_{\bar{\bullet}}(X) := \mathcal{J}_{\bar{\bullet}}(\tilde{\iota})$, where

$$\tilde{\iota} \colon (M(X_{reg}), W, F) \to (M(L), W, F)$$

is a morphism of mixed Hodge cdga's which computes the rational homotopy type of $\iota \colon L \to X_{reg}$. Thus by lemma 3.5.0.2 we have a quasi-isomorphism of coperverse cdga's $E_1(MI_{\bar{\bullet}}(X), W) \longrightarrow \mathcal{J}_{\bar{\bullet}}(E_1(\tilde{\iota}, W))$. It remains to note that we have a string of quasi-isomorphisms from $\mathcal{J}_{\bar{\bullet}}(E_1(\tilde{\iota}))$ to $EI_{1,\bar{\bullet}}(X) := \mathcal{J}_{\bar{\bullet}}(E_1(\iota^*))$ □

Remark 3.5.3. *Suppose we have a topological space X such that its rational model is endowed with an increasing filtration W, then one can consider the associated spectral sequence $E_1(X, W)$. The existence of a string of quasi-isomorphisms between the rational model of X and the first page $E_1(X, W)$ is called the E_1-formality and is a property of complex algebraic varieties, see [15] and [13]. It is an interesting result that the intersection spaces of complex projective varieties have this property although they are not algebraic varieties.*

Definition 3.5.3.1. *Let X be a compact, connected oriented pseudomanifold of dimension n with only isolated singularities. We say that X is a $EI_{r,\bar{\bullet}}$-formal topological space if its coperverse rational model $AI_{\bar{\bullet}}(X)$ can be endowed with an increasing filtration W such that there exists a string of quasi-isomorphisms between $AI_{\bar{\bullet}}(X)$ and the r-th term of its associated spectral sequence $EI_{r,\bar{\bullet}}(X, W)$.*

With this definition, the theorem 3.5.2 can be rephrased in the following corollary.

Corollary 3.5.3.1. *Let $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$. The space X is $EI_{1,\bar{\bullet}}$-formal with respect to the weight filtration.*

Corollary 3.5.3.2. *The complex homotopy type of $I^{\bar{p}}X$ is completely determined by the first term of its weight spectral sequence $EI_{1,\bar{p}}(X)$.*

3.5.1 *The case of a smooth exceptional divisor*

3.5.1.1 *Notations*

Let X be a complex projective variety of complex dimension n with only normal isolated singularities. We denote by $\Sigma = \{\sigma_1, \ldots, \sigma_\nu\}$ the singular

© 2016 Tous droits réservés.
lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

locus of X and by $X_{reg} := X - \Sigma$ its regular part. We also denote by $L := L(\Sigma, X)$ the link of Σ in X and by $i\colon L \hookrightarrow X_{reg}$ the natural inclusion of the link into the regular part.

Since Σ is discrete, we can write L as a disjoint union $L = \sqcup_{\sigma_i} L_i$ where $L_i := L(\sigma_i, X)$ is the link of $\sigma_i \in \Sigma$ in X. The assumption that X is normal implies that L_i is connected for all $\sigma_i \in \Sigma$.

From now on, we will always assume X admits a resolution of singularities

$$\begin{tikzcd} D \arrow[r, hook, "j"] \arrow[d] & \widetilde{X} \arrow[d, "f"] \\ \Sigma \arrow[r, hook] & X \end{tikzcd}$$

such that the exceptional divisor $D := f^{-1}(\Sigma)$ is smooth.

We denote by

$$j^k\colon H^k(\widetilde{X}) \longrightarrow H^k(D) \text{ and } \gamma^k\colon H^{k-2}(D) \longrightarrow H^k(\widetilde{X})$$

the restriction maps and the Gysin maps induced by the inclusion j.

For all $k \geqslant 2$ we also denote by

$$j^k_\sharp\colon H^{k-2}(D) \xrightarrow{\gamma^k} H^k(\widetilde{X}) \xrightarrow{j^k} H^k(D)$$

the composition of the two maps.

The morphism $E_1(i^*)\colon E_1^{*,*}(X_{reg}) \to E_1^{*,*}(L)$ of weight spectral sequence induced by the inclusion $i\colon L \hookrightarrow X_{reg}$ is defined by

$$\begin{tikzcd} E_1^{-1,s}(X_{reg}) \arrow[r, "d"] \arrow[d, "E_1^{-1,s}(i^*)"'] & E_1^{0,s}(X_{reg}) \arrow[d, "E_1^{0,s}(i^*)"] \\ E_1^{-1,s}(L) \arrow[r, "d"'] & E_1^{0,s}(L) \end{tikzcd} = \begin{tikzcd} H^{s-2}(D) \arrow[r, "\gamma^s"] \arrow[d, "\mathrm{id}"'] & H^s(\widetilde{X}) \arrow[d, "j^s"] \\ H^{s-2}(D) \arrow[r, "j^s_\sharp"'] & H^s(D) \end{tikzcd}$$

The algebra structure on $E_1^{*,*}(X_{reg})$ is given by the cup product of $H^*(\widetilde{X})$, together with the map

$$\begin{array}{ccc} H^s(\widetilde{X}) \times H^{s'}(D) & \longrightarrow & H^{s+s'}(D) \\ (x, a) & \longmapsto & j^s(x) \cdot a. \end{array}$$

This algebra structure is compatible with the differential γ because

$$\gamma(j^s(x) \cdot a) = x \cdot \gamma(a).$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

In other words, the following diagram commutes.

$$\begin{tikzcd} H^s(\widetilde{X}) \times H^{s'}(D) \arrow[r] \arrow[d, "{(\mathrm{id}, \gamma^{s'+2})}"'] & H^{s+s'}(D) \arrow[d, "\gamma^{s+s'+2}"] \\ H^s(\widetilde{X}) \times H^{s'+2}(D) \arrow[r] & H^{s+s'+2}(\widetilde{X}) \end{tikzcd}$$

The non-trivial products on $E_1^{*,*}(L)$ are the maps

$$E_1^{0,s}(L) \times E_1^{r,s'}(L) \longrightarrow E_1^{r,s+s'}(L) \quad r \in \{0,1\}, s, s' \geqslant 0$$

induced by the cup-product on $H^*(D)$.

The coperverse weight spectral sequence $EI_{1,\overline{\bullet}}(X) := \mathcal{J}_{\overline{\bullet}}(E_1(i^*))$ for X is then given by

$s > p+1$	$H^{s-2}(D) \otimes \mathbf{Q}[t]$	$\to$	$\mathcal{I}_0^s \oplus H^{s-2}(D) \otimes \mathbf{Q}[t]dt$	$\to$	$H^s(D) \otimes \mathbf{Q}[t]dt$
$s = p+1$	$C_{\overline{p}} \oplus H^{s-2}(D) \otimes \mathbf{Q}[t]t$	$\to$	$\mathcal{I}_0^s \oplus H^{s-2}(D) \otimes \mathbf{Q}[t]dt$	$\to$	$H^s(D) \otimes \mathbf{Q}[t]dt$
$1 \leqslant s < p+1$	$H^{s-2}(D) \otimes \mathbf{Q}[t]t$	$\to$	$\mathcal{I}_1^s \oplus H^{s-2}(D) \otimes \mathbf{Q}[t]dt$	$\to$	$H^s(D) \otimes \mathbf{Q}[t]dt$
$s = 0$	0		$\mathcal{I}_0^0$	$\to$	$H^0(D) \otimes \mathbf{Q}[t]dt$
$EI_{1,\overline{p}}^{r,s}(X)$	$r = -1$		$r = 0$		$r = 1$

Where

1. $C_{\overline{p}}$ is the image of the section of $d_1^{-1,s}\colon E_1^{-1,s}(L) \to E_1^{0,s}(L)$, ie a section of $j_\sharp^s\colon H^{s-2}(D) \to H^s(D)$. Note that $C_{\overline{p}}$ is just a computational tool and does not impact the value of the EI_2 term since it has been shown in [6, theorem 2.18] that the values of $HI_{\overline{p}}^k(X)$ for rational coefficients are independent of the choices made during the construction.

2. $\mathcal{I}_k^s$, $k \in \{0,1\}$, is the vector space given by the following pullback square.

$$\begin{tikzcd} \mathcal{I}_k^s \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & H^s(D) \otimes \mathbf{Q}[t]t^k \arrow[d, "\delta_1"] \\ H^s(\widetilde{X}) \arrow[r, "j^s"] & H^s(D) \end{tikzcd}$$

3. The differential $d_{\overline{p}}^{-1,s}\colon EI_{1,\overline{p}}^{-1,s}(X) \to EI_{1,\overline{p}}^{0,s}(X)$ is defined by

$$\sum a_i t^i \mapsto \Big((\sum \gamma^s(a_i), \sum j_\sharp^s(a_i)t^i), \sum i a_i t^{i-1} dt \Big) \quad a_i \in H^{s-2}(D).$$

4. The differential $d_{\overline{p}}^{0,s}\colon EI_{1,\overline{p}}^{0,s}(X) \to EI_{1,\overline{p}}^{1,s}(X)$ is defined by

$$\Big((x, \sum a_i t^i), \sum b_i t^i dt \Big) \mapsto \sum i a_i t^{i-1} dt + \sum j_\sharp^s(b_i) t^i dt$$

Thèse de Mathieu Klimczak, Lille 1, 2016

with

$$\begin{cases} a_i \in H^s(D), b_i \in H^{s-2}(D), \\ x \in H^s(\widetilde{X}), j^s(x) = \sum a_i. \end{cases}$$

We describe the internal algebra structure of the coperverse weight spectral sequence $EI_{1,\overline{p}}^{r,s}(X)$. Due to the method of construction, this algebra structure is similar to the external one on the perverse weight spectral sequence for intersection cohomology in [12].

The algebra structure is described by the following maps. We set $x, x' \in H^*(\widetilde{X})$ and $a, a', b, b' \in H^*(D) \otimes \mathbf{Q}[t]$.

$$\begin{array}{ccc} EI_{1,\overline{p}}^{0,s}(X) \times EI_{1,\overline{p}}^{0,s'}(X) & \longrightarrow & EI_{1,\overline{p}}^{0,s+s'}(X) \\ ((x, a + b \cdot dt), (x', a' + b' \cdot dt)) & \longmapsto & (xx', aa' + (a'b + b'a)dt) \end{array}$$

$$\begin{array}{ccc} EI_{1,\overline{p}}^{0,s}(X) \times EI_{1,\overline{p}}^{1,s'}(X) & \longrightarrow & EI_{1,\overline{p}}^{1,s+s'}(X) \\ ((x, a + b \cdot dt), (a' \cdot dt)) & \longmapsto & aa' \cdot dt \end{array}$$

$$\begin{array}{ccc} EI_{1,\overline{p}}^{-1,s}(X) \times EI_{1,\overline{p}}^{1,s'}(X) & \longrightarrow & EI_{1,\overline{p}}^{0,s+s'}(X) \\ (a, a' \cdot dt) & \longmapsto & aa' \cdot dt \end{array}$$

$$\begin{array}{ccc} EI_{1,\overline{p}}^{-1,s}(X) \times EI_{1,\overline{p}}^{0,s'}(X) & \longrightarrow & EI_{1,\overline{p}}^{-1,s+s'}(X) \\ (a, (x, a' + b' \cdot dt)) & \longmapsto & aa' \end{array}$$

Note that since $C_{\overline{p}} \subset H^{s-2}(D)$ and $\mathcal{I}_1^s \subset \mathcal{I}_0^s$, $\varphi_{\overline{k+1},\overline{k}}$ induces a morphism of spectral sequences of bidegree $(0,0)$

$$EI_1(\varphi_{\overline{k+1},\overline{k}}) \colon EI_{1,\overline{k+1}}(X) \to EI_{1,\overline{k}}(X)$$

and we get a diagram of spectral sequences

$$EI_{1,\overline{\infty}}(X) \to EI_{1,\overline{2n-2}}(X) \to \cdots \to EI_{1,\overline{1}}(X) \to EI_{1,\overline{0}}(X).$$

The internal algebra structure extends into an external one, we have an extended product

$$EI_{1,\overline{p}}^{r,s}(X) \times EI_{1,\overline{q}}^{r',s'}(X) \longrightarrow EI_{1,\overline{q}}^{r+r',s+s'}(X)$$

defined with the same map as before for the internal structure and following the same rules for r, r', s, s'.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

By computing the cohomology of $EI_{1,\overline{p}}(X)$ we have

$s > p+1$	$\ker \gamma^s$	$\operatorname{coker} \gamma^s$	0
$s = p+1$	0	$\operatorname{coker} \gamma^s_{\vert C_{\overline{p}}}$	0
$1 \leqslant s < p+1$	0	$\ker j^s$	$\operatorname{coker} j^s$
$s = 0$	0	$H^0(\widetilde{X})$	0
$EI^{r,s}_{2,\overline{p}}(X)$	$r = -1$	$r = 0$	$r = 1$

Where $\gamma^s_{\vert C_{\overline{p}}}$ is the restriction of γ^s to

$$C_{\overline{p}} \to H^s(\widetilde{X}).$$

We then have the following isomorphisms

$$HI^k_{\overline{p}}(X) = \begin{cases} H^0(\widetilde{X}) = \mathbf{Q} & k = 0 \\ H^k(X) \cong \ker j^k \oplus \operatorname{coker} j^{k-1} & 1 \leqslant k < p+1 \\ H^k(X) \oplus \operatorname{im} H^k(X_{reg}) \to H^k(L) \cong \operatorname{coker} \gamma^k_{\vert C_{\overline{p}}} \oplus \operatorname{coker} j^{k-1} \oplus \ker \gamma^{k+1} & k = p+1 \\ H^k(X_{reg}) \cong \ker \gamma^{k+1} \oplus \operatorname{coker} \gamma^k & k > p+1 \end{cases}$$

3.5.1.2 *Remark on* $\operatorname{coker} j^0$

It is important to note here that the values of $\ker j^s$ and $\operatorname{coker} j^s$ recorded in the array of the EI_2 term above start with $s = 1$, meaning we don't take into account $\ker j^0$ and $\operatorname{coker} j^0$, this is intended.

Indeed, $\operatorname{coker} j^0$ accounts for the number of loops created when the intersection spaces are defined as a homotopy pushout over a single point, like in the original definition of [6], this not the definition we use.

As a consequence, when we have multiple isolated singularities, the generalised Poincaré duality of the intersection spaces fails for $\widetilde{HI}^1_{\overline{p}}(X) \cong \widetilde{HI}^{n-1}_{\overline{q}}(X)$.

3.5.1.3 *Remark on the zero perversity*

The intersection space for the zero perversity is by definition 3.2.1.2 the regular part X_{reg} of the complex projective variety $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$ involved. The isomorphism given above by the EI_2 term gives

$$HI^1_{\overline{0}}(X) = \operatorname{coker} \gamma^1_{\vert C_{\overline{0}}} \oplus \ker \gamma^2.$$

Let's see that this coincides with $H^1(X_{reg})$.

Consider the term $\operatorname{coker} \gamma^1_{\vert C_{\overline{0}}}$, by definition $C_{\overline{0}}$ is defined as the image of a section of $j^0_\sharp \colon H^{-1}(D) = 0 \to H^1(D)$. So we have $C_{\overline{0}} = 0$, and we then have $\operatorname{coker} \gamma^1_{\vert C_{\overline{0}}} \cong \operatorname{coker} \gamma^1$.

We then have what we wanted

$$HI^1_{\overline{0}}(X) = \operatorname{coker} \gamma^1 \oplus \ker \gamma^2 = H^1(X_{reg}).$$

© 2016 Tous droits réservés.

Thèse de Mathieu Klimczak, Lille 1, 2016

3.5.2 $(\overline{p}, r)$-*purity implies* $(\overline{p}, r)$-*formality*

Let $f\colon A \to B$ be a morphism of cdga's, f is called an r-quasi-isomorphism if the induced morphism

$$H^i(f)\colon H^i(A) \longrightarrow H^i(B)$$

is an isomorphism for all $i \leqslant r$ and a monomorphism for $i = r+1$. We extend this definition to the case of a morphism of coperverse cdga's to prove a result of "purity implies formality".

Definition 3.5.3.2. *Let $0 \leqslant r \leqslant \infty$ be an integer and $\overline{p}$ a perversity. A morphism of coperverse cdga's $f_{\overline{\bullet}}\colon A_{\overline{\bullet}} \to B_{\overline{\bullet}}$ is a $(\overline{p}, r)$-quasi-isomorphism if for all perversities $\overline{s} \leqslant \overline{p}$ in $\mathcal{P}^{op}$ the map $f_{\overline{s}}$ is an r-quasi-isomorphism.*

Definition 3.5.3.3.
1. *A coperverse cdga $(A_{\overline{\bullet}}, d)$ over $\mathbf{k}$ is said to be $(\overline{p}, r)$-formal if there exists a string of $(\overline{p}, r)$-quasi-isomorphisms from $(A_{\overline{\bullet}}, d)$ to its cohomology $(H_{\overline{\bullet}}(A, \mathbf{k}), 0)$ seen as a coperverse cdga with zero differential.*
2. *Let $X \in \mathrm{Super}\mathcal{V}_{\mathbf{C}}$, $I^{\overline{\bullet}}X$ is said to be $(\overline{p}, r)$-formal if its coperverse rational model $AI_{\overline{\bullet}}(X)$ is $(\overline{p}, r)$-formal.*
3. *Let $X \in \mathrm{Super}\mathcal{V}_{\mathbf{C}}$, $I^{\overline{\bullet}}X$ is said to be $(\overline{p}, r)$-pure if the weight filtration $HI^k_{\overline{s}}(X)$ is pure of weight k for all $k \leqslant r$ and for all perversities $\overline{s} \leqslant \overline{p}$ in $\mathcal{P}^{op}$.*

Theorem 3.5.4. *Let $X \in \mathrm{Super}\mathcal{V}_{\mathbf{C}}$ of dimension n with only isolated singularities. Let $r \geqslant 0$ be an integer and $\overline{p}$ a perversity. Suppose that $I^{\overline{\bullet}}X$ is $(\overline{p}, r)$-pure, then $I^{\overline{\bullet}}X$ is $(\overline{p}, r)$-formal.*

Proof. By theorem 3.5.2, we need to define a string of $(\overline{p}, r)$-quasi-isomorphisms of differential bigraded algebras from

$$(EI^{i,j}_{1,\overline{s}}(X), d^{i,j}_{\overline{s}}) \longleftarrow * \longrightarrow (EI^{i,j}_{2,\overline{s}}(X), 0)$$

for $i + j \leqslant r$ and $\overline{s} \leqslant \overline{p}$ in $\widehat{\mathcal{P}_n}^{op}$.

Given $X \in \mathrm{Super}\mathcal{V}_{\mathbf{C}}$ of dimension n with only isolated singularities, the terms EI_1 and EI_2 of the spectral sequence look like.

$j=5$	$\vdots$	$\vdots$	$\vdots$
$j=4$	$EI^{-1,4}_{1,\overline{\bullet}}(X)$	$EI^{0,4}_{1,\overline{\bullet}}(X)$	$EI^{1,4}_{1,\overline{\bullet}}(X)$
$j=3$	$EI^{-1,3}_{1,\overline{\bullet}}(X)$	$EI^{0,3}_{1,\overline{\bullet}}(X)$	$EI^{1,3}_{1,\overline{\bullet}}(X)$
$j=2$	$EI^{-1,2}_{1,\overline{\bullet}}(X)$	$EI^{0,2}_{1,\overline{\bullet}}(X)$	$EI^{1,2}_{1,\overline{\bullet}}(X)$
$j=1$	$EI^{-1,1}_{1,\overline{\bullet}}(X)$	$EI^{0,1}_{1,\overline{\bullet}}(X)$	$EI^{1,1}_{1,\overline{\bullet}}(X)$
$j=0$	0	$EI^{0,0}_{1,\overline{\bullet}}(X)$	$EI^{1,0}_{1,\overline{\bullet}}(X)$
$EI^{i,j}_{1,\overline{\bullet}}(X)$	$i=-1$	$i=0$	$i=1$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

$j=5$	$\vdots$	$\vdots$	$\vdots$
$j=4$	$gr_4^W(HI^3_{\bullet}(X))$	$gr_4^W(HI^4_{\bullet}(X))$	$gr_4^W(HI^5_{\bullet}(X))$
$j=3$	$gr_3^W(HI^2_{\bullet}(X))$	$gr_3^W(HI^3_{\bullet}(X))$	$gr_3^W(HI^4_{\bullet}(X))$
$j=2$	$gr_2^W(HI^1_{\bullet}(X))$	$gr_2^W(HI^2_{\bullet}(X))$	$gr_2^W(HI^3_{\bullet}(X))$
$j=1$	$gr_1^W(HI^0_{\bullet}(X))$	$gr_1^W(HI^1_{\bullet}(X))$	$gr_1^W(HI^2_{\bullet}(X))$
$j=0$	0	$gr_0^W(HI^0_{\bullet}(X))$	$gr_0^W(HI^1_{\bullet}(X))$
$EI^{i,j}_{2,\bullet}(X)$	$i=-1$	$i=0$	$i=1$

The $(\overline{p}, r)$-purity assumption implies that $gr_j^W(HI^{j-1}_{\overline{s}}(X)) = 0$ for all $j \leqslant r+1$ and $gr_j^W(HI^{j+1}_{\overline{s}}(X)) = 0$ for all $j \leqslant r-1$. This means that $\ker d^{-1,j}_{\overline{s}} = 0$ for all $j \leqslant r+1$ and $\operatorname{im} d^{0,j}_{\overline{s}} = EI^{1,j}_{1,\overline{s}}(X)$ for all $j \leqslant r-1$.

Denote by $FI^{i,j}_{\overline{s}}(X)$ the bigraded differential algebra defined by, for all $\overline{s} \leqslant \overline{p}$ in $\widehat{\mathcal{P}_n}^{op}$

$$\begin{cases} FI^{-1,j}_{\overline{s}}(X) := EI^{-1,j}_{1,\overline{s}}(X) & j \leqslant r+1, \\ FI^{-1,j}_{\overline{s}}(X) := 0 & j > r+1, \\ FI^{0,j}_{\overline{s}}(X) := \ker d^{0,j}_{\overline{s}} & \forall j, \\ FI^{1,j}_{\overline{s}}(X) := 0 & \forall j. \end{cases}$$

The differential being $d^{i,j}_{\overline{s}}$.

The bigraded differential algebra $FI^{*,*}_{\overline{s}}(X)$ has the following product structure

$$\begin{cases} FI^{-1,j}_{\overline{s}}(X) \times FI^{-1,j}_{\overline{s}}(X) \longrightarrow 0 & \forall j, \\ FI^{-1,j}_{\overline{s}}(X) \times FI^{0,j'}_{\overline{s}}(X) \longrightarrow FI^{-1,j+j'}_{\overline{s}}(X) & \forall j, j', \\ FI^{0,j}_{\overline{s}}(X) \times FI^{0,j'}_{\overline{s}}(X) \longrightarrow FI^{0,j+j'}_{\overline{s}}(X) & \forall j, j'. \end{cases}$$

which is well defined and is compatible with $d^{i,j}_{\overline{s}}$ and the poset maps $EI_1(\varphi_{\overline{s+1},\overline{s}})$ for all $\overline{s} \leqslant \overline{p}$.

We then clearly have a inclusion $(FI^{i,j}_{\overline{s}}(X), d^{i,j}_{\overline{s}}) \hookrightarrow (EI^{i,j}_{1,\overline{s}}(X), d^{i,j}_{\overline{s}})$, the map $(FI^{i,j}_{\overline{s}}(X), d^{i,j}_{\overline{s}}) \to (EI^{i,j}_{2,\overline{s}}(X), 0)$ is defined by the following commutative diagram where the dashed arrows are the zero map.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

$$\begin{tikzcd}
FI^{-1,j}_{\overline{s+1}}(X) \arrow[rr, hook, "d^{-1,j}_{\overline{s+1}}"] \arrow[dr, "EI_1(\varphi_{\overline{s+1},\overline{s}})"'] \arrow[dd, dashed] & & \ker d^{0,j}_{1,\overline{s+1}} \arrow[dr, "EI_1(\varphi_{\overline{s+1},\overline{s}})"] \arrow[dd, two heads, "p"] & \\
& FI^{-1,j}_{\overline{s}}(X) \arrow[rr, hook, "d^{-1,j}_{\overline{s}}"] \arrow[dd, dashed] & & \ker d^{0,j}_{1,\overline{s}} \arrow[dd, two heads, "p"] \\
gr^W_j(HI^{j-1}_{\overline{s+1}}(X)) \arrow[rr, dashed] \arrow[dr, "EI_2(\varphi_{\overline{s+1},\overline{s}})"'] & & gr^W_j(HI^{j}_{\overline{s+1}}(X)) \arrow[dr, "EI_2(\varphi_{\overline{s+1},\overline{s}})"'] & \\
& gr^W_j(HI^{j-1}_{\overline{s}}(X)) \arrow[rr, dashed] & & gr^W_j(HI^{j}_{\overline{s}}(X))
\end{tikzcd}$$

The string $(EI^{i,j}_{1,\overline{s}}(X), d^{i,j}_{\overline{s}}) \longleftarrow (FI^{i,j}_{\overline{s}}(X), d^{i,j}_{\overline{s}}) \to (EI^{i,j}_{2,\overline{s}}(X), 0)$ then defines a $(\overline{p}, r)$-quasi-isomorphism. □

Regardless of the perversity. The two cases of special interest here are the cases where $r = 1$ and $r = \infty$.

The case $r = 1$, the 1-formality, implies that the rational Malcev completion of $\pi_1(I^{\overline{p}}X)$ can be computed directly from the cohomology group $HI^1_{\overline{p}}(X)$, together with the cup product $HI^1_{\overline{p}}(X) \otimes HI^1_{\overline{p}}(X) \to HI^2_{\overline{p}}(X)$. We then say that $\pi_1(I^{\overline{p}}X)$ is 1-formal.

The case $r = \infty$ implies the formality of $I^{\overline{p}}X$ in the usual sense, which in the cases where $I^{\overline{p}}X$ is simply-connected or nilpotent implies that the rational homotopy groups $\pi_i(I^{\overline{p}}X) \otimes \mathbf{Q}$ can be directly computed from the cohomology ring $HI^*_{\overline{p}}(X)$. We note that formality implies 1-formality.

Suppose now $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$ with only normal isolated singularities, that is

$$HI^k_{\overline{\infty}}(X) = H^k(\overline{X}) = H^k(X)$$

then by the Van-Kampen theorem and by definition 3.2.1.2 for any perversity $\overline{p}$ we have

$$\pi_1(X) = \pi_1(I^{\overline{p}}X) = \pi_1(X_{reg}).$$

Morevover, whether $\overline{p} = \overline{0}$ or $\overline{p} \neq \overline{0}$ we have the two following commutative diagrams.

$$\begin{tikzcd}
HI^1_{\overline{0}}(X) \otimes HI^1_{\overline{0}}(X) \arrow[r, "- \cup -"] \arrow[d, "="'] & HI^2_{\overline{0}}(X) \arrow[d, "="] \\
H^1(X_{reg}) \otimes H^1(X_{reg}) \arrow[r, "- \cup -"'] & H^2(X_{reg})
\end{tikzcd}$$

© 2016 Tous droits réservés.

Thèse de Mathieu Klimczak, Lille 1, 2016

$$\begin{tikzcd}
H^1(X)\otimes H^1(X) \arrow[r, "-\cup-"] \arrow[d, "\cong"'] & H^2(X) \arrow[d, hook] \\
HI^1_{\overline{p}}(X)\otimes HI^1_{\overline{p}}(X) \arrow[r, "-\cup-"'] & HI^2_{\overline{p}}(X)
\end{tikzcd}$$

Which means that if X is 1-formal then we can compute the rational Malcev completion of $\pi_1(I^{\overline{p}}X)$ by computing the one from $\pi_1(X)$. It is a result from [3] that when considering normal projective varieties the fundamental group is always 1-formal, see also [13, Corollary 3.8] for the isolated singularities case. We can then deduce the following result

Proposition 3.5.4.1. *Let $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$ with only normal isolated singularities. Then for any perversity $\overline{p}$ $\pi_1(I^{\overline{p}}X)$ is 1-formal.*

We also highlight the case $r = \infty$.

Corollary 3.5.4.1. *Let $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$ with only isolated singularities. If $I^{\overline{\bullet}}X$ is $(\overline{p},\infty)$-pure then $I^{\overline{\bullet}}X$ is $(\overline{p},\infty)$-formal.*

Let $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$ with only normal isolated singularities. For any perversity $\overline{p}$ we have a map $MI_{\overline{\infty}}(X) \longrightarrow MI_{\overline{p}}(X)$. By construction of the intersection spaces this map is a p-quasi-isomorphism for all perversities $\overline{p}$. Moreover, this map is also a morphism of mixed Hodge structures by theorem 3.4.1. We then have the following proposition.

Proposition 3.5.4.2. *Let $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$ with only normal isolated singularities. If the weight filtration on $H^k(X)$ is pure of weight k for all $k \leqslant r$, then $I^{\overline{\bullet}}X$ is $(\overline{r},r)$-pure.*

Corollary 3.5.4.2. *Let $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$ with only normal isolated singularities. If the weight filtration on $H^k(X)$ is pure of weight k for all $k \leqslant r$, then $I^{\overline{\bullet}}X$ is $(\overline{r},r)$-formal.*

Remark 3.5.5. *The question of the purity of the weight filtration is also considered in intersection cohomology, where a similar result of "purity implies formality" exists [12, corollary 3.13]. It must be pointed out that the purity of $X \in \mathsf{Super}\mathcal{V}_{\mathbf{C}}$ in intersection cohomology does not imply the purity of $I^{\overline{\bullet}}X$. For example the Kummer surface of section 3.7.2, it is a $\mathbf{Q}$-homology manifold and as such $IH^k_{\overline{p}}(X)$ is pure of weight k for any perversities and then is intersection formal. This is not the case of the corresponding intersection space for the middle perversity $I^{\overline{1}}X$ since $gr^W_4(HI^3_{\overline{1}}(X)) \neq 0$.*

Another and more involved example. It is a consequence of Gabber's purity theorem and the decomposition theorem of intersection homology (see [43]) that for projective varieties X with isolated singularities and for the middle perversity, the weight filtration W on $IH^k_{\overline{m}}(X)$ is pure of weight k for all $k \geqslant 0$, this is not the case for the Calabi-Yau 3folds treated in the last parts as we see that the weight filtration W on $HI^k_{\overline{m}}(X)$ isn't pure.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

3.6 formality of intersection spaces for 3-folds

3.6.1 *Preparatory work*

Let X be a complex projective algebraic 3-fold with isolated singularities and denote by $\Sigma = \{\sigma_1, \dots, \sigma_v\}$ the singular locus of X. Assume that there is a resolution of singularities $f\colon \widetilde{X} \to X$ such that the exceptional divisor $D := f^{-1}(\Sigma)$ is smooth and the link L_i of σ_i in X, for all $\sigma_i \in \Sigma$ is simply connected.

First we recall and collect the different properties we will need. We state them in the case of a space of complex dimension 3 but they are completely general and holds for any complex projective variety of complex dimension n with only isolated singularities by replacing 3 by n. The proofs can be found in [12].

Lemma 3.6.0.1. *We have the following Poincaré duality isomorphisms for all* $0 \leqslant s \leqslant 3$,

$$\operatorname{coker} \gamma^{3+s} \cong (\ker j^{3-s})^\vee \quad \ker \gamma^{3+s} \cong (\operatorname{coker} j^{3-s})^\vee$$

Recall that since $\dim(\Sigma) = 0$, the weight filtration on the cohomology of the link is semi-pure, meaning :

- the weights on $H^k(L)$ are less than or equal to k for $k < 3$,
- the weights on $H^k(L)$ are greater or equal to $k+1$ for $k \geqslant 3$.

We have the following results.

Lemma 3.6.0.2. *With the previous notations we have :*

1. *The map* $j_\sharp^k\colon H^{k-2}(D) \to H^k(D)$ *is injective for* $k \leqslant 3$ *and surjective for* $k \geqslant 3$.
2. *The Gysin map* $\gamma^k\colon H^{k-2}(D) \to H^k(\widetilde{X})$ *is injective for* $k \leqslant 3$ *and* γ^6 *is surjective.*
3. *The restriction morphism* $j^k\colon H^k(\widetilde{X}) \to H^k(D)$ *is surjective for* $k \geqslant 3$.

Lemma 3.6.0.3. *With the assumption on the links* L, *we have the following :*

1. *The map* $j_\sharp^2\colon H^0(D) \to H^2(D)$ *in injective, the map* $j_\sharp^4\colon H^2(D) \to H^4(D)$ *is surjective,* $j_\sharp^k\colon H^{k-2}(D) \to H^k(D)$ *is an isomorphism for* $k = 1, 3, 5$.
2. *The map* $\gamma^k\colon H^{k-2}(D) \to H^k(\widetilde{X})$ *is injective for all* $k \neq 4, 6$ *and the map* $j^k\colon H^k(\widetilde{X}) \to H^k(D)$ *is surjective for all* $k \neq 0, 2$.

Lemma 3.6.0.4. *With the above assumptions we have the following :*

1. $H^k(\widetilde{X}) \cong \ker j^k \oplus \operatorname{im} \gamma^k$ *for* $k = 1, 3, 5$.
2. $\ker j^2 \cap \operatorname{im} \gamma^2 = 0$.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

With the lemmas above the second term of the spectral sequences for the regular part and the links are given by

$E_2^{r,s}(X_{reg})$		
$s=6$	$\ker\gamma^6$	0
$s=5$	0	$\operatorname{coker}\gamma^5$
$s=4$	$\ker\gamma^4$	$\operatorname{coker}\gamma^4$
$s=3$	0	$\operatorname{coker}\gamma^3$
$s=2$	0	$\operatorname{coker}\gamma^2$
$s=1$	0	$\operatorname{coker}\gamma^1$
$s=0$	0	$H^0(\widetilde{X})$
	$r=-1$	$r=0$

$E_2^{r,s}(L)$		
$s=6$	$H^4(D)$	0
$s=5$	0	0
$s=4$	$\ker j_\sharp^4$	0
$s=3$	0	0
$s=2$	0	$\operatorname{coker} j_\sharp^2$
$s=1$	0	0
$s=0$	0	$H^0(D)$
	$r=-1$	$r=0$

The computation of the cohomology of the intersection spaces involve a choice of complementary subspace $C_{\overline{p}}$, we detail here the choice we make.

- For the perversity $\overline{1}$, the map $j_\sharp^2$ is injective by lemma 3.6.0.2, we then have $C_{\overline{1}} = H^0(D)$ and $\operatorname{coker}\gamma^2_{|C_{\overline{1}}} = \operatorname{coker}\gamma^2$.
- For the perversity $\overline{2}$, the map $j_\sharp^3$ is an isomorphism by lemma 3.6.0.3, we then also have $C_{\overline{2}} = H^1(D)$ and $\operatorname{coker}\gamma^3_{|C_{\overline{2}}} = \operatorname{coker}\gamma^3$.
- For the perversity $\overline{3}$, there is no assumption on $j_\sharp^4$ and we chose a complementary subspace of $\ker j_\sharp^4$ which we denote by $C_{\overline{3}}$.
- For the perversity $\overline{4}$, the map $j_\sharp^5$ is an isomorphism by lemma 3.6.0.3, we then also have $C_{\overline{4}} = H^3(D)$ and $\operatorname{coker}\gamma^5_{|C_{\overline{4}}} = \operatorname{coker}\gamma^5$.

Since the links of the singularities are simply connected five dimensional manifolds, by definition of the intersection spaces we have $I^{\overline{0}}X \simeq I^{\overline{1}}X$ and $I^{\overline{3}}X \simeq I^{\overline{4}}X$. Thus the second terms of the corresponding spectral sequences

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

must be isomorphic, for now the corresponding second term for the associated spectral sequences are the following.

	$EI^{r,s}_{2,\bar{0}}(X)$			$EI^{r,s}_{2,\bar{1}}(X)$		
$s=6$	$\ker\gamma^6$	0	0	$\ker\gamma^6$	0	0
$s=5$	0	$\operatorname{coker}\gamma^5$	0	0	$\operatorname{coker}\gamma^5$	0
$s=4$	$\ker\gamma^4$	$\operatorname{coker}\gamma^4$	0	$\ker\gamma^4$	$\operatorname{coker}\gamma^4$	0
$s=3$	0	$\operatorname{coker}\gamma^3$	0	0	$\operatorname{coker}\gamma^3$	0
$s=2$	0	$\operatorname{coker}\gamma^2$	0	0	$\operatorname{coker}\gamma^2$	0
$s=1$	0	$\operatorname{coker}\gamma^1$	0	0	$\ker j^1$	0
$s=0$	0	$H^0(\widetilde{X})$	0	0	$H^0(\widetilde{X})$	0
	$r=-1$	$r=0$	$r=1$	$r=1$	$r=0$	$r=1$

	$EI^{r,s}_{2,\bar{3}}(X)$			$EI^{r,s}_{2,\bar{4}}(X)$		
$s=6$	$\ker\gamma^6$	0	0	$\ker\gamma^6$	0	0
$s=5$	0	$\operatorname{coker}\gamma^5$	0	0	$\operatorname{coker}\gamma^5$	0
$s=4$	0	$\operatorname{coker}\gamma^4_{\vert C_{\bar{3}}}$	0	0	$\ker j^4$	0
$s=3$	0	$\ker j^3$	0	0	$\ker j^3$	0
$s=2$	0	$\ker j^2$	$\operatorname{coker} j^2$	0	$\ker j^2$	$\operatorname{coker} j^2$
$s=1$	0	$\ker j^1$	0	0	$\ker j^1$	0
$s=0$	0	$H^0(\widetilde{X})$	0	0	$H^0(\widetilde{X})$	0
	$r=-1$	$r=0$	$r=1$	$r=1$	$r=0$	$r=1$

We have to show that $EI^{r,s}_{2,\bar{0}}(X) \cong EI^{r,s}_{2,\bar{1}}(X)$ and $EI^{r,s}_{2,\bar{3}}(X) \cong EI^{r,s}_{2,\bar{4}}(X)$. The first isomorphism is given by the isomorphism

$$H^1(\widetilde{X}) \cong \ker j^1 \oplus \operatorname{im}\gamma^1$$

from the lemma 3.6.0.4, we then have $\operatorname{coker}\gamma^1 \cong \ker j^1$.

For the second isomorphism we need to show that

$$\operatorname{coker}\gamma^4_{\vert C_{\bar{3}}} \cong \ker j^4.$$

Which is given by the following lemma

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Lemma 3.6.0.5. *We have the following isomorphism*

$$H^4(\widetilde{X}) \cong \ker j^4 \oplus \operatorname{im} \gamma^4_{|C_{\overline{3}}}.$$

Proof. Denote by $(\ker j^4)^\perp$ a complementary subspace of $\ker j^4 \subset H^4(\widetilde{X})$. The maps $j^4_\sharp$ and j^4 are surjective by lemma 3.6.0.2. We then have the following commutative diagram

$$\begin{tikzcd}
H^2(D) \cong \ker j^4_\sharp \oplus C_{\overline{3}} \arrow[r, "\gamma^4"] \arrow[dr, two heads, "j^4_\sharp"'] & H^4(\widetilde{X}) \cong \ker j^4 \oplus (\ker j^4)^\perp \arrow[d, two heads, "j^4"] \\
& H^4(D)
\end{tikzcd}$$

By definition of $C_{\overline{3}}$ we have $\gamma^4_|\colon C_{\overline{3}} \to (\ker j^4)^\perp$. The commutative diagram restricts then to the following commutative diagram where the restrictions $j^4_{\sharp|}$ and $j^4_|$ are isomorphisms. Which finishes the proof.

$$\begin{tikzcd}
C_{\overline{3}} \arrow[r, "\gamma^4_|"] \arrow[dr, "j^4_{\sharp|}"'] & (\ker j^4)^\perp \arrow[d, "j^4_|"] \\
& H^4(D)
\end{tikzcd}$$

□

The second terms of the spectral sequences of $EI^{r,s}_{2,\overline{p}}(X)$ for $\overline{p} \in \{\overline{0}, \overline{2}, \overline{4}\}$ are finally.

	$EI^{r,s}_{2,\overline{0}}(X)$			$EI^{r,s}_{2,\overline{2}}(X)$			$EI^{r,s}_{2,\overline{4}}(X)$		
$s=6$	$\ker \gamma^6$	0	0	$\ker \gamma^6$	0	0	$\ker \gamma^6$	0	0
$s=5$	0	$\operatorname{coker} \gamma^5$	0	0	$\operatorname{coker} \gamma^5$	0	0	$\operatorname{coker} \gamma^5$	0
$s=4$	$\ker \gamma^4$	$\operatorname{coker} \gamma^4$	0	$\ker \gamma^4$	$\operatorname{coker} \gamma^4$	0	0	$\ker j^4$	0
$s=3$	0	$\operatorname{coker} \gamma^3$	0	0	$\operatorname{coker} \gamma^3$	0	0	$\ker j^3$	0
$s=2$	0	$\operatorname{coker} \gamma^2$	0	0	$\ker j^2$	$\operatorname{coker} j^2$	0	$\ker j^2$	$\operatorname{coker} j^2$
$s=1$	0	$\operatorname{coker} \gamma^1$	0	0	$\ker j^1$	0	0	$\ker j^1$	0
$s=0$	0	$H^0(\widetilde{X})$	0	0	$H^0(\widetilde{X})$	0	0	$H^0(\widetilde{X})$	0
	$r=-1$	$r=0$	$r=1$	$r=-1$	$r=0$	$r=1$	$r=-1$	$r=0$	$r=1$

We are now ready to state the following theorem.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

3.6.2 *Statement and proof*

In [14, Theorem E] it is proved that any nodal hypersurface X in $\mathbf{C}P^4$ is GM-intersection-formal, meaning there is a zig-zag of quasi-isomorphisms between their perverse rational models and their intersection cohomology algebras

$$IA_{\overline{\bullet}}(X) \longleftarrow * \longrightarrow IH^*_{\overline{\bullet}}(X).$$

This result is extended in [12, theorem 4.5] to the case of complex projective varieties of dimension n with only isolated singularities and $(n-2)$-connected links using mixed Hodge structures.

We follow these methods and show that for X a complex projective algebraic 3-fold with isolated singularities and simply connected links, the intersection spaces are formal topological spaces.

Theorem 3.6.1. *Let X be a complex projective algebraic 3-fold with isolated singularities and denote by $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ the singular locus of X. Assume that there is a resolution of singularities $f\colon \widetilde{X} \to X$ such that the exceptional divisor $D := f^{-1}(\Sigma)$ is smooth and the link L_i of σ_i in X, for all $\sigma_i \in \Sigma$, is simply connected. Then $I^{\bullet}X$ is $(\overline{2}, \infty)$-formal over $\mathbf{C}$. Moreover, if $\Sigma = \{\sigma\}$ is given by a unique normal isolated singularity, then $I^{\bullet}X$ is $(\overline{0}, \infty)$-formal over $\mathbf{C}$*

By theorem 3.5.2 there is a string of quasi-isomorphisms of coperverse cdga's from $AI_{\overline{\bullet}}(X) \otimes \mathbf{C}$ to $EI_{1,\overline{\bullet}}(X) \otimes \mathbf{C}$. Moreover we have $EI^{*,*}_{2,\overline{\bullet}}(X) \cong HI^*_{\overline{\bullet}}(X)$. We follow this pattern

1. We define a bigraded differential algebra $(FI^{r,s}_{\overline{p}}(X), \partial^{r,s}_{\overline{p}})$ step by step for the perversities $\overline{4}$, $\overline{2}$ and $\overline{0}$.
   - When needed, we then define the poset map $\varphi_{\overline{p},\overline{q}}\colon FI^{r,s}_{\overline{p}}(X) \to FI^{r,s}_{\overline{q}}(X)$ and show its compatibility with the product and the differential.
2. We define the quasi-isomorphisms
$$(EI^{r,s}_{1,\overline{p}}(X), d^{r,s}_{\overline{p}}) \xleftarrow{\psi^{r,s}_{\overline{p}}} (FI^{r,s}_{\overline{p}}(X), \partial^{r,s}_{\overline{p}}) \xrightarrow{\phi^{r,s}_{\overline{p}}} (EI^{r,s}_{2,\overline{p}}(X), 0)$$
   and check their compatibility with the products and differentials.
   - When needed, we then check the compatibility of the maps $\psi^{*,*}_{\overline{\bullet}}$ and $\phi^{*,*}_{\overline{\bullet}}$ with the poset map $\varphi_{\overline{p},\overline{q}}\colon FI^{r,s}_{\overline{p}}(X) \to FI^{r,s}_{\overline{q}}(X)$.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

3.6.2.1 *The top perversity*

We begin with the top perversity $\overline{t} = \overline{4}$. We define the bigraded differential algebra $(FI^{r,s}_{\overline{4}}(X), \partial^{r,s}_{\overline{4}})$.

$s=6$	$H^4(D)$	$H^6(\widetilde{X})$	0
$s=5$	$H^3(D)$	$H^5(\widetilde{X})$	0
$s=4$	0	$\ker j^4$	0
$s=3$	0	$\ker j^3$	0
$s=2$	0	$\ker j^2$	$(\ker\gamma^4)^\vee \otimes dt$
$s=1$	0	$\ker j^1$	0
$s=0$	0	$H^0(\widetilde{X})$	0
$FI^{r,s}_{\overline{4}}(X)$	$r=-1$	$r=0$	$r=1$

The only non-trivial differentials are $\partial^{-1,s}_{\overline{4}}\colon H^{s-2}(D) \to H^s(\widetilde{X})$ given by $\partial^{-1,s}_{\overline{4}} = \gamma^s$ for $s = 5, 6$. The algebra structure is defined by $FI^{0,s}_{\overline{4}}(X) \times FI^{0,s'}_{\overline{4}}(X) \to FI^{0,s+s'}_{\overline{4}}(X)$.

Let's now define the map $\psi^{*,*}_{\overline{4}}\colon FI^{*,*}_{\overline{4}}(X) \to EI^{*,*}_{1,\overline{4}}(X)$. Recall that we have the following first term for the weight spectral sequence.

$s \geqslant 5$	$H^{s-2}(D)\otimes \mathbf{Q}[t]$	$\to$	$\mathcal{I}^s_0 \oplus H^{s-2}(D)\otimes\mathbf{Q}[t]dt$	$\to$	$H^s(D)\otimes\mathbf{Q}[t]dt$
$1 \leqslant s \leqslant 4$	$H^{s-2}(D)\otimes \mathbf{Q}[t]t$	$\to$	$\mathcal{I}^s_1 \oplus H^{s-2}(D)\otimes\mathbf{Q}[t]dt$	$\to$	$H^s(D)\otimes\mathbf{Q}[t]dt$
$s=0$	0		$\mathcal{I}^0_0$	$\to$	$H^0(D)\otimes\mathbf{Q}[t]dt$
$EI^{r,s}_{1,\overline{4}}(X)$	$r=-1$		$r=0$		$r=1$

For $r = s = 0$, the map $\psi^{0,0}_{\overline{4}}$ is the identity map.For $r = 0$, $s > 0$, the map $\psi^{0,s}_{\overline{4}}\colon FI^{0,s}_{\overline{4}}(X) \to EI^{0,s}_{1,\overline{4}}(X)$ is defined to be

$$\psi^{0,s}_{\overline{4}}(x) := (x, j^s(x)).$$

For $r = -1$, $\psi^{-1,s}_{\overline{4}}$ is defined to be the canonical inclusion.

By lemma 3.6.0.1 we have $(\ker\gamma^4)^\vee \cong \operatorname{coker} j^2 \subset H^2(D)$, we then define $\psi^{1,2}_{\overline{4}}$ to be the injective map

$$\psi^{1,2}_{\overline{4}}\colon (\ker\gamma^4)^\vee \otimes dt \longrightarrow EI^{1,2}_{1,\overline{4}}(X) = H^2(D)\otimes\mathbf{Q}[t]dt.$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

By definition $\mathcal{I}^s_k$, $k \in \{0,1\}$, is the vector space given by the following pullback square.

$$\begin{tikzcd}
\mathcal{I}^s_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & H^s(D) \otimes \mathbf{Q}[t]t^k \arrow[d, "\delta_1"] \\
H^s(\widetilde{X}) \arrow[r, "j^s"] & H^s(D)
\end{tikzcd}$$

We have $\mathcal{I}^s_1 \subset \mathcal{I}^s_0$, the map $\psi^{0,s}_{\overline{4}}(x) := (x, j^s(x))$ is then compatible with the algebra structure of $FI^{*,*}_{\overline{4}}(X)$. The commutativity of the following diagrams

$$\begin{tikzcd}
FI^{-1,s}_{\overline{4}}(X) \arrow[r, "\partial^{-1,s}_{\overline{4}} = \gamma^s"] \arrow[d, hook, "\psi^{-1,s}_{\overline{4}}"'] & FI^{0,s}_{\overline{4}}(X) \arrow[d, "\psi^{0,s}_{\overline{4}}"] \\
EI^{-1,s}_{1,\overline{4}}(X) \arrow[r, "d^{-1,s}_{\overline{4}}"'] & EI^{0,s}_{1,\overline{4}}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\ker j^2 \arrow[r, "0"] \arrow[d, "\psi^{0,2}_{\overline{4}}"'] & (\ker \gamma^4)^\vee \otimes dt \arrow[d, "\psi^{1,2}_{\overline{4}}"] \\
EI^{0,2}_{1,\overline{4}}(X) \arrow[r, "d^{-1,s}_{\overline{4}}"'] & EI^{1,2}_{1,\overline{4}}(X)
\end{tikzcd}$$

concludes that we have a quasi-isomorphism $\psi^{*,*}_{\overline{4}} \colon FI^{*,*}_{\overline{4}}(X) \to EI^{*,*}_{1,\overline{4}}(X)$.

We now detail the map $\phi^{*,*}_{\overline{4}} \colon FI^{*,*}_{\overline{4}}(X) \to EI^{*,*}_{2,\overline{4}}(X)$.

For $r = -1$, $\phi^{-1,s}_{\overline{4}}$ is non zero only for $s = 6$ where it is the projection $H^4(D) \twoheadrightarrow \ker \gamma^6$.

For $r = 0$, since $FI^{0,s}_{\overline{4}}(X) = \ker d^{0,s}_{\overline{4}}$ for all s, we define the map $\phi^{0,s}_{\overline{4}}$ to be the surjection $\phi^{0,s}_{\overline{4}} \colon \ker d^{0,s}_{\overline{4}} \twoheadrightarrow EI^{0,s}_{2,\overline{4}}(X)$.

For $r = 1$, the assignation $(\ker \gamma^4)^\vee \otimes dt \mapsto \operatorname{coker} j^2$ defines $\phi^{1,2}_{\overline{4}}$ and $\phi^{1,s}_{\overline{4}}$ is zero for any other s.

Since we have $\ker d^{0,s}_{\overline{4}} \times \ker d^{0,s'}_{\overline{4}} \to \ker d^{0,s+s'}_{\overline{4}}$, the map $\phi^{*,*}_{\overline{4}}$ is compatible with the algebra structure of $FI^{*,*}_{\overline{4}}(X)$.

The map $\phi^{*,*}_{\overline{4}}$ is also compatible with the two non zero differentials of $FI^{*,*}_{\overline{4}}(X)$ since the two following diagrams are commutative.

$$\begin{tikzcd}
H^4(D) \arrow[r, "\gamma^6"] \arrow[d, two heads, "\phi^{-1,6}_{\overline{4}}"'] & H^6(\widetilde{X}) \arrow[d] \\
\ker \gamma^6 \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
H^3(D) \arrow[r, hook, "\gamma^5"] \arrow[d] & H^5(\widetilde{X}) \arrow[d, two heads, "\phi^{0,5}_{\overline{4}}"] \\
0 \arrow[r] & EI^{0,5}_{2,\overline{4}}(X) = \operatorname{coker} \gamma^5
\end{tikzcd}$$

We then have a quasi-isomorphism of algebras

$$(EI^{r,s}_{1,\overline{4}}(X), d^{r,s}_{\overline{4}}) \xleftarrow{\psi^{r,s}_{\overline{4}}} (FI^{r,s}_{\overline{4}}(X), \partial^{r,s}_{\overline{4}}) \xrightarrow{\phi^{r,s}_{\overline{4}}} (EI^{r,s}_{2,\overline{4}}(X), 0).$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

3.6.2.2 *The middle perversity*

We define the bigraded differential algebra $(FI^{r,s}_{\overline{2}}(X), \partial^{r,s}_{\overline{2}})$ as the sub-algebra of $(EI_{1,\overline{2}}(X), d^{r,s}_{\overline{2}})$ given by

$s=6$	$H^4(D)$	$H^6(\widetilde{X})$	0
$s=5$	$H^3(D)$	$H^5(\widetilde{X})$	0
$s=4$	$H^2(D)$	$H^4(\widetilde{X})$	0
$s=3$	0	$\ker j^3$	0
$s=2$	0	$\ker j^2$	$(\ker \gamma^4)^\vee \otimes dt$
$s=1$	0	$\ker j^1$	0
$s=0$	0	$H^0(\widetilde{X})$	0
$FI^{r,s}_{\overline{2}}(X)$	$r=-1$	$r=0$	$r=1$

Where $(EI_{1,\overline{2}}(X), d^{r,s}_{\overline{2}})$ is given by

$s \geqslant 3$	$H^{s-2}(D)\otimes \mathbf{Q}[t]$	$\rightarrow$	$\mathcal{J}^s_0 \oplus H^{s-2}(D)\otimes \mathbf{Q}[t]dt$	$\rightarrow$	$H^s(D)\otimes \mathbf{Q}[t]dt$
$1 \leqslant s \leqslant 2$	$H^{s-2}(D)\otimes \mathbf{Q}[t]t$	$\rightarrow$	$\mathcal{J}^s_1 \oplus H^{s-2}(D)\otimes \mathbf{Q}[t]dt$	$\rightarrow$	$H^s(D)\otimes \mathbf{Q}[t]dt$
$s=0$	0		$\mathcal{J}^0_0$	$\rightarrow$	$H^0(D)\otimes \mathbf{Q}[t]dt$
$EI^{r,s}_{1,\overline{2}}(X)$	$r=-1$		$r=0$		$r=1$

Compared to $FI^{*,*}_{\overline{4}}(X)$, we added $H^2(D)$ in bidegree $(-1,4)$ and replaced $\ker j^4$ by $H^4(\widetilde{X})$ in bidegree $(0,4)$, both are related by a new non-trivial differential $\partial^{-1,4}_{\overline{2}} = \gamma^4$.

The algebra structure is still non-trivial only for $r=0$, with

$$FI^{0,s}_{\overline{2}}(X) \times FI^{0,s'}_{\overline{2}}(X) \rightarrow FI^{0,s+s'}_{\overline{2}}(X).$$

The map $\varphi_{\overline{4},\overline{2}}\colon FI^{*,*}_{\overline{4}}(X) \rightarrow FI^{*,*}_{\overline{2}}(X)$ is then the canonical inclusion, which is clearly compatible with the differential and the algebra structure.

To construct $\psi^{*,*}_{\overline{2}}\colon FI^{*,*}_{\overline{2}}(X) \rightarrow EI^{*,*}_{1,\overline{2}}(X)$, we extend $\psi^{*,*}_{\overline{4}}$, meaning that $\psi^{-1,s}_{\overline{2}}$ is the inclusion, $\psi^{0,s}_{\overline{2}}(x) = (x, j^s(x))$ and $\psi^{1,s}_{\overline{2}} = \psi^{1,s}_{\overline{4}}$. The algebra structure is preserved by $\psi^{0,s}_{\overline{2}}$ and the following diagram commutes

$$\begin{tikzcd}
H^2(D) \arrow[r, "\partial^{-1,4}_{\overline{2}} = \gamma^4"] \arrow[d, hook, "\psi^{-1,4}_{\overline{2}}"'] & H^4(\widetilde{X}) \arrow[d, "\psi^{0,4}_{\overline{2}}"] \\
EI^{-1,4}_{1,\overline{2}}(X) \arrow[r, "d^{-1,4}_{\overline{2}}"'] & EI^{0,4}_{1,\overline{2}}(X)
\end{tikzcd}$$

© 2016 Tous droits réservés.
lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

The rest being the same as for the top perversity, we have the following quasi-isomorphism

$$\psi_{\overline{2}}^{*,*}\colon\ FI_{\overline{2}}^{*,*}(X) \longrightarrow EI_{1,\overline{2}}^{*,*}(X).$$

We now construct $\phi_{\overline{2}}^{*,*}\colon\ FI_{\overline{2}}^{*,*}(X) \to EI_{2,\overline{2}}^{*,*}(X)$.

First of all nothing changes for $r=1$ and $\phi_{\overline{2}}^{1,s} = \phi_{\overline{4}}^{1,s}$.

For $r=-1$, $\phi_{\overline{2}}^{-1,s}$ is non zero only for $s=4,6$ where it is the projection $H^{s-2}(D) \twoheadrightarrow \ker\gamma^s$.

For $r=0$, since $FI_{\overline{2}}^{0,s}(X) = \ker d_{\overline{2}}^{0,s}$ for all $s \neq 3$, we define the map $\phi_{\overline{2}}^{0,s}$ to be the surjection $\phi_{\overline{2}}^{0,s}\colon\ \ker d_{\overline{2}}^{0,s} \twoheadrightarrow EI_{2,\overline{2}}^{0,s}(X)$. For $s=3$, by lemma 3.6.0.4 we have $\ker j^3 \cong \operatorname{coker}\gamma^3$, this isomorphism defines $\phi_{\overline{2}}^{0,3}$.

For $s=4,6$ or $s=5$, the following diagrams commute

$$\begin{tikzcd}
H^{s-2}(D) \arrow[r, "\gamma^s"] \arrow[d, two heads, "\phi_{\overline{2}}^{-1,s}"'] & H^s(\widetilde{X}) \arrow[d, two heads, "\phi_{\overline{2}}^{0,s}"] \\
\ker\gamma^s \arrow[r, "0"'] & \operatorname{coker}\gamma^s
\end{tikzcd}
\qquad
\begin{tikzcd}
H^3(D) \arrow[r, hook, "\gamma^5"] \arrow[d] & H^5(\widetilde{X}) \arrow[d, two heads, "\phi_{\overline{2}}^{0,5}"] \\
0 \arrow[r] & EI_{2,\overline{2}}^{0,5}(X) = \operatorname{coker}\gamma^5
\end{tikzcd}$$

So $\phi_{\overline{2}}^{*,*}$ is compatible with the differential.

To see its compatibility with the algebra structure of $FI_{\overline{2}}^{*,*}(X)$ we have to check the commutativity of the following diagram

$$\begin{tikzcd}
\ker j^1 \times \ker j^2 \arrow[r] \arrow[d] & \ker j^3 \arrow[d, "\cong"] \\
\operatorname{coker}\gamma^1 \times \operatorname{coker}\gamma^2 \arrow[r] & \operatorname{coker}\gamma^3
\end{tikzcd}$$

We then have a quasi-isomorphism of algebras

$$(EI_{1,\overline{2}}^{r,s}(X), d_{\overline{2}}^{r,s}) \xleftarrow{\psi_{\overline{2}}^{r,s}} (FI_{\overline{2}}^{r,s}(X), \partial_{\overline{2}}^{r,s}) \xrightarrow{\phi_{\overline{2}}^{r,s}} (EI_{2,\overline{2}}^{r,s}(X), 0).$$

We now check the commutativity of the following diagram

$$\begin{tikzcd}
(EI_{1,\overline{4}}^{r,s}(X), d_{\overline{4}}^{r,s}) \arrow[d, "EI_1(\varphi_{\overline{4},\overline{2}})"'] & (FI_{\overline{4}}^{r,s}(X), \partial_{\overline{4}}^{r,s}) \arrow[l, "\psi_{\overline{4}}^{r,s}"'] \arrow[r, "\phi_{\overline{4}}^{r,s}"] \arrow[d, hook, "\varphi_{\overline{4},\overline{2}}"] & (EI_{2,\overline{4}}^{r,s}(X), 0) \arrow[d, "EI_2(\varphi_{\overline{4},\overline{2}})"] \\
(EI_{1,\overline{2}}^{r,s}(X), d_{\overline{2}}^{r,s}) & (FI_{\overline{2}}^{r,s}(X), \partial_{\overline{2}}^{r,s}) \arrow[l, "\psi_{\overline{2}}^{r,s}"] \arrow[r, "\phi_{\overline{2}}^{r,s}"'] & (EI_{2,\overline{2}}^{r,s}(X), 0)
\end{tikzcd}$$

The only differences between $EI_{i,\overline{4}}^{r,s}(X)$ and $EI_{i,\overline{2}}^{r,s}(X)$, $i=1,2$, arise for $s=3,4$. We then only check these cases.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

The only square that does not trivially commutes for $s = 3$ is the following

$$\begin{tikzcd}
\mathcal{I}^3_1 \oplus H^1(D) \otimes \mathbf{Q}[t]dt \arrow[d, hook, "EI_1(\varphi_{\overline{4},\overline{2}})"'] & \ker j^3 \arrow[l, "\psi^{0,3}_{\overline{4}}"'] \arrow[r, "\phi^{0,3}_{\overline{4}}"] \arrow[d, "="', "\varphi_{\overline{4},\overline{2}}"] & \ker j^3 \arrow[d, "EI_2(\varphi_{\overline{4},\overline{2}})"] \\
\mathcal{I}^3_0 \oplus H^1(D) \otimes \mathbf{Q}[t]dt & \ker j^3 \arrow[l, "\psi^{0,3}_{\overline{2}}"] \arrow[r, "\phi^{0,3}_{\overline{2}}"'] & \operatorname{coker} \gamma^3
\end{tikzcd}$$

The left hand square commutes because $\operatorname{im} \psi^{0,3}_{\overline{4}} \subset \mathcal{I}^3_1$, $\operatorname{im} \psi^{0,3}_{\overline{2}} \subset \mathcal{I}^3_0$ and the fact that $\mathcal{I}^3_1 \subset \mathcal{I}^3_0$. The right hand square commutes because of the isomorphism $\ker j^3 \cong \operatorname{coker} \gamma^3$.

For $s = 4$, the only square that does not trivially commutes is the following

$$\begin{tikzcd}
\mathcal{I}^4_1 \oplus H^2(D) \otimes \mathbf{Q}[t]dt \arrow[d, hook, "EI_1(\varphi_{\overline{4},\overline{2}})"'] & \ker j^4 \arrow[l, "\psi^{0,4}_{\overline{4}}"'] \arrow[r, "\phi^{0,4}_{\overline{4}}"] \arrow[d, hook, "\varphi_{\overline{4},\overline{2}}"] & \ker j^4 \arrow[d, "EI_2(\varphi_{\overline{4},\overline{2}})"] \\
\mathcal{I}^4_0 \oplus H^2(D) \otimes \mathbf{Q}[t]dt & H^4(\widetilde{X}) \arrow[l, "\psi^{0,4}_{\overline{2}}"] \arrow[r, "\phi^{0,4}_{\overline{2}}"'] & \operatorname{coker} \gamma^4
\end{tikzcd}$$

The left hand square commutes for the same reason that for $s = 3$. We then consider the right hand square. By lemma 3.6.0.5 we have $H^4(\widetilde{X}) \cong \ker j^4 \oplus \operatorname{im} \gamma^4_{|C_{\overline{3}}}$, moreover we have $\operatorname{im} \gamma^4_{|C_{\overline{3}}} \subset \operatorname{im} \gamma^4$, this implies that

$$\ker j^4 \cap \operatorname{im} \gamma^4 \neq \{0\}.$$

We may then find a direct sum decomposition

$$\ker j^4 = (\ker j^4 \cap \operatorname{im} \gamma^4) \oplus C$$

and defines a map $\ker j^4 \to C$ by projection on the second summand. We then have

$$H^4(\widetilde{X}) \cong (\ker j^4 \cap \operatorname{im} \gamma^4) \oplus C \oplus \operatorname{im} \gamma^4_{|C_{\overline{3}}},$$

the maps $EI_2(\varphi_{\overline{4},\overline{2}})$ and $\phi^{0,4}_{\overline{2}}$ then send the summand $(\ker j^4 \cap \operatorname{im} \gamma^4) \oplus \operatorname{im} \gamma^4_{|C_{\overline{3}}}$ to zero and C to its class in $\operatorname{coker} \gamma^4$. Which makes the right hand square commute.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

3.6.2.3 *The infinite perversity*

We define the bigraded differential algebra $(FI^{r,s}_{\overline{\infty}}(X), \partial^{r,s}_{\overline{\infty}})$ as the sub-algebra of $(EI_{1,\overline{\infty}}(X), d^{r,s}_{\overline{\infty}})$ given by

$s=6$	0	$H^6(\widetilde{X})$	0
$s=5$	0	$H^5(\widetilde{X})$	0
$s=4$	0	$\ker j^4$	0
$s=3$	0	$\ker j^3$	0
$s=2$	0	$\ker j^2$	$(\ker \gamma^4)^\vee \otimes dt$
$s=1$	0	$\ker j^1$	0
$s=0$	0	$H^0(\widetilde{X})$	0
$FI^{r,s}_{\overline{\infty}}(X)$	$r=-1$	$r=0$	$r=1$

There is no non trivial differentials. The algebra structure is as always concentrated in $r=0$. The map $\varphi_{\overline{\infty},\overline{4}}$ is the canonical inclusion and is compatible the algebra structure.

The maps $\psi^{*,*}_{\overline{\infty}}$ and $\phi^{*,*}_{\overline{\infty}}$ are clear from the previous computations for the top perversity.

We then have a quasi-isomorphism of algebras

$$(EI^{r,s}_{1,\overline{\infty}}(X), d^{r,s}_{\overline{\infty}}) \xleftarrow{\psi^{r,s}_{\overline{\infty}}} (FI^{r,s}_{\overline{\infty}}(X), \partial^{r,s}_{\overline{\infty}}) \xrightarrow{\phi^{r,s}_{\overline{\infty}}} (EI^{r,s}_{2,\overline{\infty}}(X), 0).$$

We then define the coperverse cdga $FI^{*,*}_{\bullet}(X)$ to be

$$FI^{*,*}_{\bullet}(X) = \begin{cases} FI^{*,*}_{\overline{\infty}}(X) & \overline{p} = \infty, \\ FI^{*,*}_{\overline{4}}(X) & \overline{p} \in \{\overline{3}, \overline{4}\}, \\ FI^{*,*}_{\overline{2}}(X) & \overline{p} = \overline{2}, \end{cases}$$

We then have a quasi-isomorphism of coperverse cdga's.

$$(EI^{*,*}_{1,\bullet}(X), d^{*,*}_{\bullet}) \xleftarrow{\psi^{*,*}_{\bullet}} (FI^{*,*}_{\bullet}(X), \partial^{*,*}_{\bullet}) \xrightarrow{\phi^{*,*}_{\bullet}} (EI^{*,*}_{2,\bullet}(X), 0).$$

Then $I^{\overline{\bullet}}X$ is $(\overline{2}, \infty)$-formal.

3.6.2.4 *The zero perversity*

Suppose that X has only one normal isolated singularity. Then $\ker \gamma^6 = 0$ and the EI_2-term of the weight spectral sequence is

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

	$EI^{r,s}_{2,\overline{0}}(X)$		
$s=6$	0	0	0
$s=5$	0	$\operatorname{coker}\gamma^5$	0
$s=4$	$\ker\gamma^4$	$\operatorname{coker}\gamma^4$	0
$s=3$	0	$\operatorname{coker}\gamma^3$	0
$s=2$	0	$\operatorname{coker}\gamma^2$	0
$s=1$	0	$\operatorname{coker}\gamma^1$	0
$s=0$	0	$H^0(\widetilde{X})$	0
	$r=-1$	$r=0$	$r=1$

We define the bigraded differential algebra $(FI^{r,s}_{\overline{0}}(X), \partial^{r,s}_{\overline{0}})$ as the sub-algebra of $(EI_{1,\overline{0}}(X), d^{r,s}_{\overline{0}})$ given by

$s=6$	$H^4(D)$	$H^6(\widetilde{X})$	0
$s=5$	$H^3(D)$	$H^5(\widetilde{X})$	0
$s=4$	$H^2(D)$	$H^4(\widetilde{X})$	0
$s=3$	0	$\ker j^3$	0
$s=2$	0	$(\ker j^4)^\vee \oplus (\ker\gamma^4)^\vee \otimes t$	$(\ker\gamma^4)^\vee \otimes dt$
$s=1$	0	$\ker j^1$	0
$s=0$	0	$H^0(\widetilde{X})$	0
$FI^{r,s}_{\overline{0}}(X)$	$r=-1$	$r=0$	$r=1$

Where $(EI_{1,\overline{0}}(X), d^{r,s}_{\overline{0}})$ is given by

$s\geqslant 1$	$H^{s-2}(D)\otimes\mathbf{Q}[t]$	$\to$	$\mathcal{I}^s_{\overline{0}} \oplus H^{s-2}(D)\otimes\mathbf{Q}[t]dt$	$\to$	$H^s(D)\otimes\mathbf{Q}[t]dt$
$s=0$	0		$\mathcal{I}^0_{\overline{0}}$	$\to$	$H^0(D)\otimes\mathbf{Q}[t]dt$
$EI^{r,s}_{1,\overline{0}}(X)$	$r=-1$		$r=0$		$r=1$

Compared to $FI^{*,*}_{\overline{2}}(X)$, we added $(\ker\gamma^4)^\vee\otimes t$ and replaced $\ker j^2$ by $(\ker j^4)^\vee$ in bidegree $(0,2)$. There is also a new differential

$$\partial^{0,2}_{\overline{0}}\colon (\ker\gamma^4)^\vee\otimes t \to (\ker\gamma^4)^\vee\otimes dt$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

which is differentiation with respect to t.

The algebra structure is non trivial only for $r = 0$ where we have

$$\begin{cases} ((\ker\gamma^4)^\vee \otimes t) \times FI^{0,s}_{\overline{0}}(X) \longrightarrow 0 & \forall\, s, \\ FI^{0,s}_{\overline{0}}(X) \times FI^{0,s'}_{\overline{0}}(X) \longrightarrow FI^{0,s+s'}_{\overline{0}}(X) & \text{otherwise.} \end{cases}$$

We now define $\varphi_{\overline{2},\overline{0}}\colon FI^{*,*}_{\overline{2}}(X) \to FI^{*,*}_{\overline{2}}(X)$. For $s \geqslant 3$, there is no changes and $\varphi_{\overline{2},\overline{0}}$ is the identity, same if $s = 0, 1$. For $s = 2$, by lemma 3.6.0.4 we have $\ker j^2 \cap \operatorname{im}\gamma^2 = 0$ so we have the inclusion $\ker j^2 \to (\ker j^4)^\vee$. The map $\varphi_{\overline{2},\overline{0}}$ is then an inclusion and is compatible with the differential and the algebra structure.

We now construct $\psi^{*,*}_{\overline{0}}\colon FI^{*,*}_{\overline{0}}(X) \to EI^{*,*}_{1,\overline{0}}(X)$. Since we have $(\ker j^4)^\vee \oplus (\ker\gamma^4)^\vee \otimes t \subset \mathcal{I}^2_{\overline{0}}$ there is no difference between $\psi^{*,*}_{\overline{0}}$ and $\psi^{*,*}_{\overline{2}}$ and the definition is the same. We then have a quasi-isomorphism

$$\psi^{*,*}_{\overline{0}}\colon FI^{*,*}_{\overline{0}}(X) \to EI^{*,*}_{1,\overline{0}}(X).$$

We define $\phi^{*,*}_{\overline{0}}\colon FI^{*,*}_{\overline{0}}(X) \to EI^{*,*}_{2,\overline{0}}(X)$, for $s \geqslant 3$ there is no difference with the middle perversity. If $s = 2$ then we define $\phi^{0,2}_{\overline{0}}$ by $(\ker j^4)^\vee \mapsto \operatorname{coker}\gamma^2$ and $(\ker\gamma^4)^\vee \mapsto 0$, we then have the following commutative diagram.

$$\begin{tikzcd} (\ker j^4)^\vee \oplus (\ker\gamma^4)^\vee \otimes t \arrow[r, "\partial^{0,2}_{\overline{0}}"] \arrow[d, "\phi^{0,2}_{\overline{0}}"'] & (\ker\gamma^4)^\vee \otimes dt \arrow[d] \\ EI^{0,2}_{2,\overline{0}}(X) = \operatorname{coker}\gamma^2 \arrow[r] & 0 \end{tikzcd}$$

If $s = 1$, the isomorphism $\ker j^1 \cong \operatorname{coker}\gamma^1$ defines $\phi^{0,1}_{\overline{0}}$.

We then have a quasi-isomorphism of algebras

$$(EI^{r,s}_{1,\overline{0}}(X), d^{r,s}_{\overline{0}}) \xleftarrow{\psi^{r,s}_{\overline{0}}} (FI^{r,s}_{\overline{0}}(X), \partial^{r,s}_{\overline{0}}) \xrightarrow{\phi^{r,s}_{\overline{0}}} (EI^{r,s}_{2,\overline{0}}(X), 0).$$

We now check the commutativity of the following diagram

$$\begin{tikzcd} (EI^{r,s}_{1,\overline{2}}(X), d^{r,s}_{\overline{2}}) \arrow[d, "EI_1(\varphi_{\overline{2},\overline{0}})"'] & (FI^{r,s}_{\overline{2}}(X), \partial^{r,s}_{\overline{2}}) \arrow[l, "\psi^{r,s}_{\overline{2}}"'] \arrow[r, "\phi^{r,s}_{\overline{2}}"] \arrow[d, hook, "\varphi_{\overline{2},\overline{0}}"] & (EI^{r,s}_{2,\overline{2}}(X), 0) \arrow[d, "EI_2(\varphi_{\overline{2},\overline{0}})"] \\ (EI^{r,s}_{1,\overline{0}}(X), d^{r,s}_{\overline{0}}) & (FI^{r,s}_{\overline{0}}(X), \partial^{r,s}_{\overline{0}}) \arrow[l, "\psi^{r,s}_{\overline{0}}"] \arrow[r, "\phi^{r,s}_{\overline{0}}"'] & (EI^{r,s}_{2,\overline{0}}(X), 0) \end{tikzcd}$$

The only differences between $EI^{r,s}_{i,\overline{2}}(X)$ and $EI^{r,s}_{i,\overline{0}}(X)$, $i = 1, 2$, arise for $s = 1, 2$. We then only check these cases. For $s = 1$, there is nothing to check and everything commutes. For $s = 2$, the only thing to check is the commutativity of the square

© 2016 Tous droits réservés. lilliad.univ-lille.fr

$$\begin{tikzcd}
\mathcal{I}^2_{\overline{1}} \oplus H^0(D)\otimes \mathbf{Q}[t]dt \arrow[d, hook, "EI_1(\varphi_{\overline{2},\overline{0}})"'] & \ker j^2 \arrow[l, "\psi^{0,2}_{\overline{2}}"'] \arrow[r, "\phi^{0,2}_{\overline{2}}"] \arrow[d, hook, "\varphi_{\overline{2},\overline{0}}"] & \ker j^2 \arrow[d, hook, "EI_2(\varphi_{\overline{2},\overline{0}})"] \\
\mathcal{I}^2_{\overline{0}} \oplus H^0(D)\otimes \mathbf{Q}[t]dt & (\ker j^4)^\vee \oplus (\ker\gamma^4)^\vee \otimes t \arrow[l, "\psi^{0,2}_{\overline{0}}"] \arrow[r, "\phi^{0,2}_{\overline{0}}"'] & \operatorname{coker}\gamma^2
\end{tikzcd}$$

Which is clear by the previous computations.

We then define the coperverse cdga $FI^{*,*}_{\bullet}(X)$ to be

$$FI^{*,*}_{\bullet}(X) = \begin{cases} FI^{*,*}_{\overline{\infty}}(X) & \overline{p} = \infty, \\ FI^{*,*}_{\overline{4}}(X) & \overline{p} \in \{\overline{3}, \overline{4}\}, \\ FI^{*,*}_{\overline{2}}(X) & \overline{p} = \overline{2}, \\ FI^{*,*}_{\overline{0}}(X) & \overline{p} \in \{\overline{0}, \overline{1}\}. \end{cases}$$

Then $I^{\bullet}X$ is $(\overline{0}, \infty)$-formal.

If X has more than one normal isolated singularity, then $\ker\gamma^6 \neq 0$ and the EI_2-term of the spectral sequence has a non-trivial product outside of the column $r = 0$ given by

$$EI^{0,2}_{2,\overline{0}}(X) \times EI^{-1,4}_{2,\overline{0}}(X) \longrightarrow EI^{-1,6}_{2,\overline{0}}(X)$$

with

$$\operatorname{coker}\gamma^2 \times \ker\gamma^4 \longrightarrow \ker\gamma^6.$$

This implies the following diagram does not commutes is $x \notin \ker\gamma^4 \subset H^2(D)$. This gives an obstruction to the $(\overline{0}, \infty)$-formality.

$$\begin{tikzcd}
FI^{0,2}_{2,\overline{0}}(X) \times FI^{-1,4}_{2,\overline{0}}(X) \arrow[r] \arrow[d] & FI^{-1,6}_{2,\overline{0}}(X) \arrow[d] \\
EI^{0,2}_{2,\overline{0}}(X) \times EI^{-1,4}_{2,\overline{0}}(X) \arrow[r] & EI^{-1,6}_{2,\overline{0}}(X)
\end{tikzcd}
=
\begin{tikzcd}
(\ker j^4)^\vee \times H^2(D) \arrow[r] \arrow[d] & H^4(D) \arrow[d] \\
\operatorname{coker}\gamma^2 \times \ker\gamma^4 \arrow[r] & \ker\gamma^6
\end{tikzcd}$$

3.7 EXAMPLES AND APPLICATIONS

We use the following conventions in the rest of this section :

- When needed, we will denote by $\{1_i, E_i\}$ a basis of $H^*(\mathbf{C}P^1_{(i)})$, we complete it into a basis $\{1_i, E_i, \mathcal{E}_i, \Lambda_i\}$ of $H^*(\mathbf{C}P^1_{(i)} \times \mathbf{C}P^1_{(i)})$ with $|E_i| = |\mathcal{E}_i| = 2$, $|\Lambda_i| = 4$ and where $\mathcal{E}_i E_i = \Lambda_i$.
- even if we do not take into account the loops in the first cohomology group (see subsection 3.5.1.2), we mark them in red

© 2016 Tous droits réservés.

Thèse de Mathieu Klimczak, Lille 1, 2016

3.7.1 *Projective cone over a K3 surface*

Definition 3.7.0.1. *A K3 surface S is a simply connected compact smooth complex surface such that its canonical bundle K_S is trivial.*

Denote by S a K3 surface, for example a nonsingular degree 4 hypersurface in $\mathbf{C}P^3$, such as the Fermat quartic

$$S = \{[z_0 : z_1 : z_2 : z_3] \in \mathbf{C}P^3 \ : \ z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}.$$

In fact every K3 surface over $\mathbf{C}$ is diffeomorphic to this example, see [35]. The Hodge diamond of a K3 surface is completely determined and is given by the following.

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Which means that we have the following cohomology.

s	0	1	2	3	4
$H^s(S)$	$\mathbf{Q}$	0	$\mathbf{Q}^{22}$	0	$\mathbf{Q}$

Denote by $\mathbb{P}_{\mathbf{C}}S \subset \mathbf{C}P^4$ the projective cone over the K3 surface. This is a simply connected hypersurface of complex dimension 3 with only one isolated singularity which is the cone point and defined by the same equation but in $\mathbf{C}P^4$

$$\mathbb{P}_{\mathbf{C}}S = \{[z_0 : z_1 : z_2 : z_3 : z_4] \in \mathbf{C}P^4 \ : \ z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}.$$

The cohomology of $\mathbb{P}_{\mathbf{C}}S$ is given by (see [19, p.169])

$$H^k(\mathbb{P}_{\mathbf{C}}S) = H^{k-2}(S) \, \forall \, k \geqslant 2.$$

By Hironaka's Theorem on resolution of singularities there exists a cartesian diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \widetilde{P} \arrow[d, "f"] \\ * \arrow[r, hook] & \mathbb{P}_{\mathbf{C}}S \end{tikzcd}$$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

where the exceptional divisor is the K3 surface S and $\widetilde{P}$ is a smooth projective variety of complex dimension 3. We then have the following Mayer-Vietoris sequence

$$\cdots \to H^k(\mathbb{P}_{\mathbf{C}}S) \to H^k(\widetilde{P}) \oplus H^k(*) \to H^k(S) \to \cdots$$

which gives the following cohomology for $\widetilde{P}$.

s	0	1	2	3	4	5	6
$H^s(\widetilde{P})$	$\mathbf{Q}$	0	$\mathbf{Q} \oplus \mathbf{Q}^{22}$	0	$\mathbf{Q} \oplus \mathbf{Q}^{22}$	0	$\mathbf{Q}$

We compute the intersection space for the perversities $\{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$.

First of all the intersection space for the zero perversity is by definition the regular part, which is computed by the following spectral sequence

	$E_1^{r,s}((\mathbb{P}_{\mathbf{C}}S)_{reg})$		$E_2^{r,s}((\mathbb{P}_{\mathbf{C}}S)_{reg})$	
$s=6$	$\mathbf{Q}$	$\mathbf{Q}$	0	0
$s=5$	0	0	0	0
$s=4$	$\mathbf{Q}^{22}$	$\mathbf{Q} \oplus \mathbf{Q}^{22}$	0	$\mathbf{Q}$
$s=3$	0	0	0	0
$s=2$	$\mathbf{Q}$	$\mathbf{Q} \oplus \mathbf{Q}^{22}$	0	$\mathbf{Q}^{22}$
$s=1$	0	0	0	0
$s=0$	0	$\mathbf{Q}$	0	$\mathbf{Q}$
	$r=-1$	$r=0$	$\ker \gamma^s$	$\operatorname{coker} \gamma^s$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Now we need the cohomology of the link, which is given by the spectral sequence defined by $j^s_\sharp\colon H^{s-2}(D) \to H^s(D)$, as in the section 3.5.1.

	$E_1^{r,s}(L)$		$E_2^{r,s}(L)$	
$s=6$	$\mathbf{Q}$	0	$\mathbf{Q}$	0
$s=5$	0	0	0	0
$s=4$	$\mathbf{Q}^{22}$	$\mathbf{Q}$	$\mathbf{Q}^{21}$	0
$s=3$	0	0	0	0
$s=2$	$\mathbf{Q}$	$\mathbf{Q}^{22}$	0	$\mathbf{Q}^{21}$
$s=1$	0	0	0	0
$s=0$	0	$\mathbf{Q}$	0	$\mathbf{Q}$
	$r=-1$	$r=0$	$\ker j^s_\sharp$	$\operatorname{coker} j^s_\sharp$

We then have

$$\begin{cases} H^0(L) &= H^5(L) = \mathbf{Q}, \\ H^1(L) &= H^4(L) = 0, \\ H^2(L) &= H^3(L) = \mathbf{Q}^{21}. \end{cases}$$

By the E_2 term of the previous spectral sequence we see that the only sections of $j^s_\sharp$ for which the image won't be zero correspond to the perversities $\overline{1}$ and $\overline{3}$. Each times the image of the section is equal to $\mathbf{Q}$, we then have the two following map

$$\gamma^2_{|C_{\overline{1}}}\colon\ C_{\overline{1}} = \mathbf{Q} \longrightarrow H^2(\widetilde{P}) = \mathbf{Q} \oplus \mathbf{Q}^{22},$$

$$\gamma^4_{|C_{\overline{3}}}\colon\ C_{\overline{3}} = \mathbf{Q} \longrightarrow H^4(\widetilde{P}) = \mathbf{Q} \oplus \mathbf{Q}^{22}.$$

and $\operatorname{coker} \gamma^2_{|C_{\overline{1}}} \cong \operatorname{coker} \gamma^4_{|C_{\overline{3}}} \cong \mathbf{Q}^{22}$.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

The last map we need to know is $j^s\colon H^s(\widetilde{P}) \to H^s(S)$, the map induced by the inclusion $S \hookrightarrow \widetilde{P}$.

$s=6$	$\mathbf{Q}$	0
$s=5$	0	0
$s=4$	$\mathbf{Q}^{22}$	0
$s=3$	0	0
$s=2$	$\mathbf{Q}$	0
$s=1$	0	0
$s=0$	0	0
	$\ker j^s$	$\operatorname{coker} j^s$

We recall the EI_2 term of the spectral sequence of $I^{\overline{p}}X$.

$s>p+1$	$\ker\gamma^s$	$\operatorname{coker}\gamma^s$	0
$s=p+1$	0	$\operatorname{coker}\gamma^s_{\mid C_{\overline{p}}}$	0
$1\leqslant s<p+1$	0	$\ker j^s$	$\operatorname{coker} j^s$
$s=0$	0	$H^0(\widetilde{P})$	0
$EI^{r,s}_{2,\overline{p}}(X)$	$r=-1$	$r=0$	$r=1$

We then have the following results.

$EI^{r,s}_{2,\overline{1}}(\mathbb{P}_{\mathbf{C}}S)$	
$s\geqslant 5$	0
$s=4$	$\mathbf{Q}$
$s=3$	0
$s=2$	$\mathbf{Q}^{22}$
$s=1$	0
$s=0$	$H^0(\widetilde{P})$
	$r=0$

$EI^{r,s}_{2,\overline{2}}(\mathbb{P}_{\mathbf{C}}S)$	
$s\geqslant 5$	0
$s=4$	$\mathbf{Q}$
$s=3$	0
$s=2$	$\mathbf{Q}$
$s=1$	0
$s=0$	$H^0(\widetilde{P})$
	$r=0$

$EI^{r,s}_{2,\overline{3}}(\mathbb{P}_{\mathbf{C}}S)$	
$s\geqslant 5$	0
$s=4$	$\mathbf{Q}^{22}$
$s=3$	0
$s=2$	$\mathbf{Q}$
$s=1$	0
$s=0$	$H^0(\widetilde{P})$
	$r=0$

$EI^{r,s}_{2,\overline{4}}(\mathbb{P}_{\mathbf{C}}S)$	
$s\geqslant 5$	0
$s=4$	$\mathbf{Q}^{22}$
$s=3$	0
$s=2$	$\mathbf{Q}$
$s=1$	0
$s=0$	$H^0(\widetilde{P})$
	$r=0$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

Note that for complementary perversities, such as $\overline{1}$ and $\overline{3}$ or $\overline{0}$ and $\overline{4}$, and for $s \neq 0$ the EI_2 term gives back the generalized Poincaré duality between the various intersection spaces such as proved in [6, theorem 2.12]. The middle perversity here is $\overline{2}$ and we also get back the self-duality of the space $I^{\overline{2}}\mathbb{P}_{\mathbf{C}}S$.

For any perversity $\overline{p}$ the weight filtration is pure, so by the theorem 3.5.4 we get the following proposition.

Proposition 3.7.0.1. *Let S be a K3-surface, the intersection space $I^{\overline{\bullet}}\mathbb{P}_{\mathbf{C}}S$ is $(\overline{0}, \infty)$-formal.*

3.7.2 *Kummer quartic surface*

Let K be a Kummer quartic surface, that is an irreducible surface of degree 4 in $\mathbf{C}P^3$ with 16 ordinary double points, which is the maximum for such surfaces.

From the algebraic topologist point of view, a Kummer surface is constructed in the following way. Let's consider a 4-dimensional torus

$$\mathbf{T} = S^1 \times S^1 \times S^1 \times S^1$$

endowed with the complex involution $\tau\colon z \mapsto \bar{z}$ action. This action has 16 fixed point and we define the Kummer surface to be the quotient complex surface

$$K := \mathbf{T}/\tau.$$

We have the following cohomology for K.

s	0	1	2	3	4
$H^s(K)$	$\mathbf{Q}$	0	$\mathbf{Q}^6$	0	$\mathbf{Q}$

The link of each singularity is then a projective space $\mathbf{R}P^3$. These singularities are quotients singularities so by [21] K admits a resolution where the exceptional set consists of curves of genus zero and self-intersection -2. Which means we have the following resolution diagram

$$\begin{tikzcd}
\bigsqcup_{i=1}^{16} \mathbf{C}P^1_{(i)} \arrow[r] \arrow[d] & \widetilde{K} \arrow[d, "f"] \\
\bigsqcup_{i=1}^{16} *_{(i)} \arrow[r, hook] & K
\end{tikzcd}$$

The Mayer-Vietoris sequence gives the following cohomology for $\widetilde{K}$.

s	0	1	2	3	4
$H^s(\widetilde{K})$	$\mathbf{Q}$	0	$\mathbf{Q}^6 \oplus \bigoplus_{i=1}^{16} \mathbf{Q}E_i$	0	$\mathbf{Q}$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

We have the fairly easy following spectral sequence for the links.

$E_1^{r,s}(L)$			$E_2^{r,s}(L)$	
$s=4$	$\bigoplus_{i=1}^{16} \mathbf{Q}E_i$	0	$\bigoplus_{i=1}^{16} \mathbf{Q}E_i$	0
$s=3$	0	0	0	0
$s=2$	$\bigoplus_{i=1}^{16} \mathbf{Q}1_i$	$\bigoplus_{i=1}^{16} \mathbf{Q}E_i$	0	0
$s=1$	0	0	0	0
$s=0$	0	$\bigoplus_{i=1}^{16} \mathbf{Q}1_i$	0	$\bigoplus_{i=1}^{16} \mathbf{Q}1_i$
	$r=-1$	$r=0$	$\ker j_\sharp^s$	$\operatorname{coker} j_\sharp^s$

The rational cohomology of link of each singularities is then a 3-sphere, which is the rationalization of $\mathbf{R}P^3$.

The only interesting perversity here is the middle perversity $\bar{1}$. We need a $C_{\bar{1}}$ for the computation, we have here

$$C_{\bar{1}} = \bigoplus_{i=1}^{16} \mathbf{Q}1_i$$

and $\gamma^2_{|C_{\bar{1}}} = \gamma^2$.

The following spectral sequence computes the regular part and the second array is the restriction map j^s.

$E_1^{r,s}(K_{reg})$	$\gamma^s\colon\ H^{s-2}(D) \longrightarrow H^s(\widetilde{K})$		$E_2^{r,s}(K_{reg})$	
$s=4$	$\bigoplus_{i=1}^{16} \mathbf{Q}E_i$	$\mathbf{Q}$	$\bigoplus_{i=1}^{15} \mathbf{Q}E_i$	0
$s=3$	0	0	0	0
$s=2$	$\bigoplus_{i=1}^{16} \mathbf{Q}1_i$	$\mathbf{Q}^6 \oplus \bigoplus_{i=1}^{16} \mathbf{Q}E_i$	0	$\mathbf{Q}^6$
$s=1$	0	0	0	0
$s=0$	0	$\mathbf{Q}$	0	$\mathbf{Q}$
	$r=-1$	$r=0$	$\ker\gamma^s$	$\operatorname{coker}\gamma^s$

© 2016 Tous droits réservés. lilliad.univ-lille.fr

$s=4$	$\mathbf{Q}$	0	$\mathbf{Q}$	0
$s=3$	0	0	0	0
$s=2$	$\mathbf{Q}^6 \oplus \bigoplus_{i=1}^{16} \mathbf{Q}E_i$	$\bigoplus_{i=1}^{16} \mathbf{Q}E_i$	$\mathbf{Q}^6$	0
$s=1$	0	0	0	0
$s=0$	$\mathbf{Q}$	$\bigoplus_{i=1}^{16} \mathbf{Q}1_i$	0	$\bigoplus_{i=1}^{15} \mathbf{Q}1_i$
	$H^s(\widetilde{K})$	$H^s(D)$	$\ker j^s$	$\operatorname{coker} j^s$

The cohomology of the middle perversity intersection space of a Kummer surface is then given by the following array. Note that the cohomology obtained isn't pure.

$s=4$	$\bigoplus_{i=1}^{15} \mathbf{Q}E_i$	0	0	0
$s=3$	0	0	0	$\bigoplus_{i=1}^{15} \mathbf{Q}E_i$
$s=2$	0	$\mathbf{Q}^6$	0	$\mathbf{Q}^6$
$s=1$	0	0	0	$\bigoplus_{i=1}^{15} \mathbf{Q}1_i$
$s=0$	0	$H^0(\widetilde{K})$	$\bigoplus_{i=1}^{15} \mathbf{Q}1_i$	$H^0(\widetilde{K})$
$EI_{2,\bar{1}}^{r,s}(K)$	$r=-1$	$r=0$	$r=1$	$HI_{\bar{1}}^{s}(K)$

3.7.3 *The Calabi-Yau generic quintic 3-fold*

Let $Y \subset \mathbf{C}P^4$ the singular hypersurface given by the equation

$$Y := \{[z_0 : z_1 : z_2 : z_3 : z_4] \in \mathbf{C}P^4 \;:\; z_3 g(z_0, \dots, z_4) + z_4 h(z_0, \dots, z_4) = 0\}$$

where g and h are generic homogeneous polynomials of degree 4. Y is the Calabi-Yau generic quintic 3-fold containing the plane

$$\pi := \{z_3 = z_4 = 0\} \cong \mathbf{C}P^2.$$

The singular locus

$$\Sigma := \{[x] \in \mathbf{C}P^4 \;:\; z_3 = z_4 = g(z) = h(z) = 0\} \subset \mathbf{C}P^2$$

is given by 16 ordinary double points. That is the link of each singularity $\sigma \in \Sigma$ is topologically equal to $L_\sigma = S^2 \times S^3$.

We have the following cohomology for Y.

s	0	1	2	3	4	5	6
$H^s(Y)$	$\mathbf{Q}$	0	$\mathbf{Q}$	$\mathbf{Q}^{189}$	$\mathbf{Q}^2$	0	$\mathbf{Q}$

© 2016 Tous droits réservés.

Thèse de Mathieu Klimczak, Lille 1, 2016

We consider the following commutative diagram of resolutions

$$\begin{tikzcd}
\bigsqcup_{i=1}^{16} \mathbf{C}P^1_{(i)} \times \mathbf{C}P^1_{(i)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{Blow up}"] & \bigsqcup_{i=1}^{16} \mathbf{C}P^1_{(i)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{small res.}"] & \bigsqcup_{i=1}^{16} *_{(i)} \arrow[d] \\
\overline{Y} \arrow[r, "\mathcal{B}\ell"'] & \widetilde{Y} \arrow[r, "f"'] & Y
\end{tikzcd}$$

The first square is a simultaneous small resolution of the 16 singularities obtained by blowing up $\mathbf{C}P^4$ along the plane $\pi \cong \mathbf{C}P^2$. The exceptionnal divisor of this blow-up is a $\mathbf{C}P^1$-bundle over $\pi \cong \mathbf{C}P^2$.

For the second square $\mathcal{B}\ell$ is a blow-up along the $\mathbf{C}P^1_{(i)}$'s.

Denote by Ψ the generator of $H^2(Y)$.

By using twice the Mayer-Vietoris long exact sequence, we get the following cohomology for $\overline{Y}$.

$$\begin{cases} H^0(\overline{Y}) = H^6(\overline{Y}) = \mathbf{Q}, \\ H^1(\overline{Y}) = H^5(\overline{Y}) = 0, \\ H^2(\overline{Y}) = \mathbf{Q}\Psi \oplus \mathbf{Q}E_1 \oplus \bigoplus_{i=1}^{16} \mathbf{Q}\Lambda_i^\vee, \\ H^4(\overline{Y}) = \mathbf{Q}\Psi^\vee \oplus \mathbf{Q}E_1^\vee \oplus \bigoplus_{i=1}^{16} \mathbf{Q}\Lambda_i, \\ H^3(\overline{Y}) = \mathbf{Q}^{174}. \end{cases}$$

The cohomology of the links of the singularities is given by the spectral sequence

	$E_1^{r,s}(L)$		$E_2^{r,s}(L)$	
$s=6$	$\bigoplus_{i=1}^{16} \mathbf{Q}\Lambda_i$	0	$\bigoplus_{i=1}^{16} \mathbf{Q}\Lambda_i$	0
$s=5$	0	0	0	0
$s=4$	$\bigoplus_{i=1}^{16}(\mathbf{Q}E_i \oplus \mathbf{Q}\mathcal{E}_i)$	$\bigoplus_{i=1}^{16} \mathbf{Q}\Lambda_i$	$\bigoplus_{i=1}^{16} \mathbf{Q}\mathcal{E}_i$	0
$s=3$	0	0	0	0
$s=2$	$\bigoplus_{i=1}^{16} \mathbf{Q}1_i$	$\bigoplus_{i=1}^{16}(\mathbf{Q}E_i \oplus \mathbf{Q}\mathcal{E}_i)$	0	$\bigoplus_{i=1}^{16} \mathbf{Q}E_i$
$s=1$	0	0	0	0
$s=0$	0	$\bigoplus_{i=1}^{16} \mathbf{Q}1_i$	0	$\bigoplus_{i=1}^{16} \mathbf{Q}1_i$
	$r=-1$	$r=0$	$\ker j_\sharp^s$	$\operatorname{coker} j_\sharp^s$

We here follow the section 3.6 and do the computations for the top, middle and zero perversity. The spectral sequence of the regular part is given by

© 2016 Tous droits réservés. lilliad.univ-lille.fr

$E_1^{r,s}(Y_{reg})$	$\gamma^s\colon H^{s-2}(D) \longrightarrow H^s(\overline{Y})$		$E_2^{r,s}(Y_{reg})$	
$s=6$	$\bigoplus_{i=1}^{16} \mathbf{Q}\Lambda_i$	$\mathbf{Q}$	$\bigoplus_{i=1}^{15} \mathbf{Q}\Lambda_i$	0
$s=5$	0	0	0	0
$s=4$	$\bigoplus_{i=1}^{16}(\mathbf{Q}E_i \oplus \mathbf{Q}\mathcal{E}_i)$	$\mathbf{Q}\Psi^\vee \oplus \mathbf{Q}E_1^\vee \oplus \bigoplus_{i=1}^{16} \mathbf{Q}\Lambda_i$	$\bigoplus_{i=1}^{15} \mathbf{Q}\mathcal{E}_i$	$\mathbf{Q}\Psi^\vee$
$s=3$	0	$\mathbf{Q}^{174}$	0	$\mathbf{Q}^{174}$
$s=2$	$\bigoplus_{i=1}^{16} \mathbf{Q}1_i$	$\mathbf{Q}\Psi \oplus \mathbf{Q}E_1 \oplus \bigoplus_{i=1}^{16} \mathbf{Q}\Lambda_i^\vee$	0	$\mathbf{Q}\Psi \oplus \mathbf{Q}E_1$
$s=1$	0	0	0	0
$s=0$	0	$\mathbf{Q}$	0	$\mathbf{Q}$
	$r=-1$	$r=0$	$\ker\gamma^s$	$\operatorname{coker}\gamma^s$

Finally we also need the restriction morphism j^s.

$s=6$	$\mathbf{Q}$	0	$\mathbf{Q}$	0
$s=5$	0	0	0	0
$s=4$	$\mathbf{Q}\Psi^\vee \oplus \mathbf{Q}E_1^\vee \oplus \bigoplus_{i=1}^{16} \mathbf{Q}\Lambda_i$	$\bigoplus_{i=1}^{16} \mathbf{Q}\Lambda_i$	$\mathbf{Q}\Psi^\vee \oplus \mathbf{Q}E_1^\vee$	0
$s=3$	$\mathbf{Q}^{174}$	0	$\mathbf{Q}^{174}$	0
$s=2$	$\mathbf{Q}\Psi \oplus \mathbf{Q}E_1 \oplus \bigoplus_{i=1}^{16} \mathbf{Q}\Lambda_i^\vee$	$\bigoplus_{i=1}^{16}(\mathbf{Q}E_i \oplus \mathbf{Q}\mathcal{E}_i)$	$\mathbf{Q}\Psi$	$\bigoplus_{i=1}^{15} \mathbf{Q}E_i$
$s=1$	0	0	0	0
$s=0$	$\mathbf{Q}$	$\bigoplus_{i=1}^{16} \mathbf{Q}1_i$	0	$\bigoplus_{i=1}^{15} \mathbf{Q}1_i$
	$H^s(\overline{Y})$	$H^s(D)$	$\ker j^s$	$\operatorname{coker} j^s$

We then get the following tables for the perversities $\overline{0}, \overline{2}, \overline{4}$. Note here that the generalized Poincaré duality is only partial as we explained in the subsection 3.5.1.2 since we do not take into accounts the loops of $\operatorname{coker} j^0$ (marked in red in the arrays).

© 2016 Tous droits réservés.

Thèse de Mathieu Klimczak, Lille 1, 2016

$s=6$	$\bigoplus_{i=1}^{15} \mathbf{Q}\Lambda_i$	0	0	0
$s=5$	0	0	0	$\bigoplus_{i=1}^{15} \mathbf{Q}\Lambda_i$
$s=4$	$\bigoplus_{i=1}^{15} \mathbf{Q}\mathcal{E}_i$	$\mathbf{Q}\Psi^\vee$	0	$\mathbf{Q}\Psi^\vee$
$s=3$	0	$\mathbf{Q}^{174}$	0	$\mathbf{Q}^{189}$
$s=2$	0	$\mathbf{Q}\Psi \oplus \mathbf{Q}E_1$	0	$\mathbf{Q}\Psi \oplus \mathbf{Q}E_1$
$s=1$	0	0	0	$\bigoplus_{i=1}^{15} \mathbf{Q}1_i$
$s=0$	0	$H^0(\overline{Y})$	$\bigoplus_{i=1}^{15} \mathbf{Q}1_i$	$H^0(\overline{Y})$
$EI_{2,\overline{0}}^{r,s}(Y)$	$r=-1$	$r=0$	$r=1$	$HI_{\overline{0}}^{s}(Y)$

Note here the partial duality for the values $s=2,3,4$ for the perversities $\overline{0}$ and $\overline{4}$.

$s=6$	$\bigoplus_{i=1}^{15} \mathbf{Q}\Lambda_i$	0	0	0
$s=5$	0	0	0	$\bigoplus_{i=1}^{15} \mathbf{Q}\Lambda_i$
$s=4$	0	$\mathbf{Q}\Psi^\vee \oplus \mathbf{Q}E_1^\vee$	0	$\mathbf{Q}\Psi^\vee \oplus \mathbf{Q}E_1^\vee$
$s=3$	0	$\mathbf{Q}^{174}$	0	$\mathbf{Q}^{189}$
$s=2$	0	$\mathbf{Q}\Psi$	$\bigoplus_{i=1}^{15} \mathbf{Q}E_i$	$\mathbf{Q}\Psi$
$s=1$	0	0	0	$\bigoplus_{i=1}^{15} \mathbf{Q}1_i$
$s=0$	0	$H^0(\overline{Y})$	$\bigoplus_{i=1}^{15} \mathbf{Q}1_i$	$H^0(\overline{Y})$
$EI_{2,\overline{4}}^{r,s}(Y)$	$r=-1$	$r=0$	$r=1$	$HI_{\overline{4}}^{s}(Y)$

For the perversity $\overline{2}$ we retrieve the values of the smooth deformation as in [8], unless for $s=1$.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

$s=6$	$\bigoplus_{i=1}^{15}\mathbf{Q}\Lambda_i$ $\quad 0$	0		0
$s=5$	0	0	0	$\bigoplus_{i=1}^{15}\mathbf{Q}\Lambda_i$
$s=4$	$\bigoplus_{i=1}^{15}\mathbf{Q}\mathcal{E}_i$	$\mathbf{Q}\Psi^\vee$	0	$\mathbf{Q}\Psi^\vee$
$s=3$	0	$\mathbf{Q}^{174}$	0	$\mathbf{Q}^{204}$
$s=2$	0	$\mathbf{Q}\Psi$	$\bigoplus_{i=1}^{15}\mathbf{Q}E_i$	$\mathbf{Q}\Psi$
$s=1$	0	0	0	$\bigoplus_{i=1}^{15}\mathbf{Q}1_i$
$s=0$	0	$H^0(\overline{Y})$	$\bigoplus_{i=1}^{15}\mathbf{Q}1_i$	$H^0(\overline{Y})$
$EI^{r,s}_{\overline{2},\overline{2}}(Y)$	$r=-1$	$r=0$	$r=1$	$HI^s_{\overline{2}}(Y)$

3.7.4 *The Quintic*

Let ψ be a complex number and consider the variety

$$X_\psi := \left\{[z_0 : z_1 : z_2 : z_3 : z_4] \in \mathbf{CP}^4 \;:\; z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 z_3 z_4 = 0\right\},$$

which is Calabi-Yau. It is smooth for small $\psi \neq 1$ and becomes singular when $\psi = 1$, denote by X the singular degeneration $X_{\psi=1}$.

The singular locus Σ of X is here composed of 125 ordinary double points. That is the link of each singularity $\sigma \in \Sigma$ is topologically equal to $L_\sigma = S^2 \times S^3$, just like before.

We get the following cohomology for X.

s	0	1	2	3	4	5	6
$H^s(X)$	$\mathbf{Q}$	0	$\mathbf{Q}$	$\mathbf{Q}^{103}$	$\mathbf{Q}^{25}$	0	$\mathbf{Q}$

Using the same method of resolution that before

$$\begin{tikzcd}
\bigsqcup_{i=1}^{125}\mathbf{CP}^1_{(i)} \times \mathbf{CP}^1_{(i)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{Blow up}"] & \bigsqcup_{i=1}^{125}\mathbf{CP}^1_{(i)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{small res.}"] & \bigsqcup_{i=1}^{125} *_{(i)} \arrow[d] \\
\overline{X} \arrow[r, "\mathcal{B}\ell"'] & \widetilde{X} \arrow[r, "f"'] & X
\end{tikzcd}$$

© 2016 Tous droits réservés.

Thèse de Mathieu Klimczak, Lille 1, 2016

With the Mayer-Vietoris long exact sequence, we get the following cohomology for $\overline{X}$, we still denote by Ψ the generator of $H^2(X)$.

6	$\mathbf{Q}$
5	0
4	$\mathbf{Q}\Psi^\vee \oplus \bigoplus_{i=1}^{24} E_i^\vee \oplus \bigoplus_{i=1}^{125} \mathbf{Q}\Lambda_i$
3	$\mathbf{Q}^2$
2	$\mathbf{Q}\Psi \oplus \bigoplus_{i=1}^{24} \mathbf{Q}E_i \oplus \bigoplus_{i=1}^{125} \mathbf{Q}\Lambda_i^\vee$
1	0
0	$\mathbf{Q}$
s	$H^s(\overline{X})$

The spectral sequences of the regular part if given by

	$E_1^{r,s}(X_{reg})$	$\gamma^s\colon H^{s-2}(D) \longrightarrow H^s(\overline{X})$	$E_2^{r,s}(X_{reg})$	
$s=6$	$\bigoplus_{i=1}^{125} \mathbf{Q}\Lambda_i$	$\mathbf{Q}$	$\bigoplus_{i=1}^{124} \mathbf{Q}\Lambda_i$	0
$s=5$	0	0	0	0
$s=4$	$\bigoplus_{i=1}^{125}(\mathbf{Q}E_i \oplus \mathbf{Q}\mathcal{E}_i)$	$\mathbf{Q}\Psi^\vee \oplus \bigoplus_{i=1}^{24} E_i^\vee \oplus \bigoplus_{i=1}^{125} \mathbf{Q}\Lambda_i$	$\bigoplus_{i=1}^{101} \mathbf{Q}\mathcal{E}_i$	$\mathbf{Q}\Psi^\vee$
$s=3$	0	$\mathbf{Q}^2$	0	$\mathbf{Q}^2$
$s=2$	$\bigoplus_{i=1}^{125} \mathbf{Q}1_i$	$\mathbf{Q}\Psi \oplus \bigoplus_{i=1}^{24} \mathbf{Q}E_i \oplus \bigoplus_{i=1}^{125} \mathbf{Q}\Lambda_i^\vee$	0	$\mathbf{Q}\Psi \oplus \bigoplus_{i=1}^{24} \mathbf{Q}E_i$
$s=1$	0	0	0	0
$s=0$	0	$\mathbf{Q}$	0	$\mathbf{Q}$
	$r=-1$	$r=0$	$\ker\gamma^s$	$\operatorname{coker}\gamma^s$

The formulas for the restriction morphism are

© 2016 Tous droits réservés. lilliad.univ-lille.fr

$s=6$	$\mathbf{Q}$	0	$\mathbf{Q}$	0
$s=5$	0	0	0	0
$s=4$	$\mathbf{Q}\Psi^\vee \oplus \bigoplus_{i=1}^{24} E_i^\vee \oplus \bigoplus_{i=1}^{125} \mathbf{Q}\Lambda_i$	$\bigoplus_{i=1}^{125} \mathbf{Q}\Lambda_i$	$\mathbf{Q}\Psi^\vee \oplus \bigoplus_{i=1}^{24} \mathbf{Q}E_i^\vee$	0
$s=3$	$\mathbf{Q}^2$	0	$\mathbf{Q}^2$	0
$s=2$	$\mathbf{Q}\Psi \oplus \bigoplus_{i=1}^{24} \mathbf{Q}E_i \oplus \bigoplus_{i=1}^{125} \mathbf{Q}\Lambda_i^\vee$	$\bigoplus_{i=1}^{125} (\mathbf{Q}E_i \oplus \mathbf{Q}\mathcal{E}_i)$	$\mathbf{Q}\Psi$	$\bigoplus_{i=1}^{101} \mathbf{Q}E_i$
$s=1$	0	0	0	0
$s=0$	$\mathbf{Q}$	$\bigoplus_{i=1}^{125} \mathbf{Q}1_i$	0	$\bigoplus_{i=1}^{124} \mathbf{Q}1_i$
	$H^s(\overline{X})$	$H^s(D)$	$\ker j^s$	$\operatorname{coker} j^s$

We let the reader fill in the arrays for the top and zero perversities, we here give the result for the middle perversity $\overline{2}$.

$s=6$	$\bigoplus_{i=1}^{124} \mathbf{Q}\Lambda_i$	0	0	0
$s=5$	0	0	0	$\bigoplus_{i=1}^{124} \mathbf{Q}\Lambda_i$
$s=4$	$\bigoplus_{i=1}^{101} \mathbf{Q}\mathcal{E}_i$	$\mathbf{Q}\Psi^\vee$	0	$\mathbf{Q}\Psi^\vee$
$s=3$	0	$\mathbf{Q}^2$	0	$\mathbf{Q}^{204}$
$s=2$	0	$\mathbf{Q}\Psi$	$\bigoplus_{i=1}^{101} \mathbf{Q}E_i$	$\mathbf{Q}\Psi$
$s=1$	0	0	0	$\bigoplus_{i=1}^{124} \mathbf{Q}1_i$
$s=0$	0	$H^0(\overline{X})$	$\bigoplus_{i=1}^{124} \mathbf{Q}1_i$	$H^0(\overline{X})$
$EI_{2,\overline{2}}^{r,s}(X)$	$r=-1$	$r=0$	$r=1$	$HI_{\overline{2}}^{s}(X)$

© 2016 Tous droits réservés.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

BIBLIOGRAPHY

[1] S. Akbulut and H. King. "The topology of real algebraic sets with isolated singularities." In: *Annals of Mathematics* 113 (1981), pp. 425–446.

[2] S. Akbulut and H. King. *Topology of Real Algebraic Sets*. Ed. by Springer. 1991.

[3] D. Arapura, A. Dimca, and R. Hain. "On the fundamental groups of normal varieties." In: *Communications in Contemporary Mathematics* (2015).

[4] V. Navarro Aznar. "Sur la théorie de Hodge-Deligne." In: *Inv. Math.* 90 (1987), pp. 11–76.

[5] M. Banagl. *Extending Intersection homology type invariants to non-Witt spaces*. Memoirs Amer. Math. Soc. 160. 2002.

[6] M. Banagl. *Intersection spaces, spatial homology truncation and string theory*. Vol. 1997. Lecture Notes in Mathematics. Springer, 2010.

[7] M. Banagl and E. Hunsicker. "Hodge theory for intersection space cohomology." submitted preprint. 2015. URL: https://www.mathi.uni-heidelberg.de/~banagl/pdfdocs/BanaglHunsicker.pdf.

[8] M. Banagl and L. Maxim. "Deformation of Singularities and the Homology of Intersection Spaces." In: *J. Topol. Anal.* 4.4 (2012), pp. 413–448.

[9] J. Barge. "Structures différentiables sur les types d'homotopie rationnelle simplements connexes." In: *Ann. Scient. Ec. Norm. Sup.* 9 (1976), pp. 469–501.

[10] G. Bredon. *Geometry and Topology*. Vol. 139. Graduate Texts in Mathematics. Springer, 1997.

[11] W. Browder. *Surgery on simply-connected manifolds*. Springer, 1972.

[12] D. Chataur and J. Cirici. "Mixed Hodge structures on the intersection homotopy type of complex varieties with isolated singularities." URL: http://arxiv.org/abs/1603.09125.

[13] D. Chataur and J. Cirici. "Rational homotopy of complex projective varieties with normal isolated singularities." to appear in Forum. Math. 2015. URL: http://arxiv.org/abs/1503.05347.

[14] D. Chataur, M. Saralegi, and D. Tanré. "Intersection cohomology. simplicial blow-up and rational homotopy." Preprint. 2012. URL: http://arxiv.org/abs/1205.7057.

[15] J. Cirici and F. Guillén. "E_1-formality of complex algebraic variety." In: *Algebr. Geom. Topol.* 14.5 (2014), pp. 3049–3079.

[16] P. Deligne. "Théorie de Hodge : II." In: *Publications mathématiques de l'IHES* 40 (1971), pp. 5–47.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

[17] P. Deligne. "Théorie de Hodge : III." In: *Publications Mathématiques de l'IHES* 44 (1974), pp. 5–77.

[18] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan. "Real homotopy theory of Kähler manifolds." In: *Inventiones Mathematicae* 29.3 (1975), pp. 245–274.

[19] A. Dimca. *Singularities and topology of hypersurfaces*. Springer, 1992.

[20] A. H. Durfee and R. M. Hain. "Mixed Hodge structures on the homotopy of links." In: *Math. Ann.* 280.1 (1988), pp. 69–83.

[21] A. Durfee. "Fifteen characterizations of rational double points and simple critical points." In: *Enseign. Math.* 25 (1979), pp. 131–163.

[22] A. Durfee. "Neighborhoods of algebraic sets." In: *Transactions A.M.S.* 276 (1983), pp. 517–530.

[23] M. Dyer. "Rational homology and Whitehead products." In: *Pacific J. Math.* 40 (1972), pp. 59–71.

[24] Y. Félix, S. Halperin, and J.-C. Thomas. *Rational homotopy theory*. Vol. 205. Graduate Texts in Mathematics. Springer-Verlag, 2001.

[25] Y. Félix, J. Oprea, and D. Tanré. *Algebraic models in geometry*. Oxford graduate texts in mathemathics. Oxford University Press, 2008.

[26] M. Goresky and R. MacPherson. "Intersection homology theory." In: *Topology* 19 (1980), pp. 135–162.

[27] M. Goresky and R. MacPherson. "Intersection homology II." In: *Invent. Math.* 71 (1983), pp. 77–129.

[28] R. Hain. "The de Rham homotopy of complex algebraic varieties I, II." In: *K-theory* 1 (1987), 271–324, resp. 481–497.

[29] R. Hain. "The geometry of the mixed Hodge structure on the fundamental group." In: *Proc. Symp. Pure Math. A.M.S.* 6-2 (1987), pp. 247–282.

[30] F. Hirzebruch. *Topological Methods in Algebraic Geometry.* Springer Verlag, 1966.

[31] W. Hodge. *The Theory and Applications of Harmonic Integrals*. Cambridge University Press, 1947.

[32] M. Hovey. *Model Categories*. American Mathematical Society, 1999.

[33] M. Kapovich and J. Kollár. "Fundamental groups of links of isolated singularities." In: *J. Amer. Math. Soc.* 27.4 (2014), pp. 929–952.

[34] S. Klaus and M. Kreck. "A quick proof of the rational Hurewicz theorem and a computation of the rational homotopy groups of spheres." In: *Math. Proc. Camb. Phil. Soc.* 136 (2004), pp. 617–623.

[35] K. Kodaira. "On the Structure of Compact Complex Analytic Surfaces, I." In: *American Journal of Mathematics* 86.4 (1964), pp. 751–798.

[36] J.W. Morgan. "The algebraic topology of smooth algebraic varieties." In: *Publications mathématiques de l'IHES* 1.48 (1978), pp. 137–204.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

[37] C. A.M. Peters and J. H.M. Steenbrink. *Mixed Hodge Structures*. Springer, 2007.

[38] H. Poincaré. "Analysis situs." In: *Journal de l'Ecole Polytechinque* 1 (1895). (Reprinted in Oeuvres de H. Poincaré VI, 193-288), pp. 1–121.

[39] H. Poincaré. "Complément à l'Analysis situs." In: *Rend. Circ. Matem. Palermo* 13.285-343 (1899). (Reprinted in Oeuvres de H. Poincaré VI, 290-337).

[40] C. de Seguins Pazzis. *Invitation aux formes quadratiques*. Calvage et Mounet, 2010.

[41] C. Simpson. "Local systems on proper algebraic V-manifolds." In: *Pure Appl. Math. Q.* 7.4 (2011). Special Issue : In memory of Eckart Vieweg, pp. 1675–1759.

[42] J. Stasheff. "Rational Poincaré duality spaces." In: *Illinois journal of mathematics* 27 (1983), pp. 104–109.

[43] J. H. Steenbrink. "Mixed Hodge structures associated with isolated singularities." In: *Singularities, Part 2 (Arcata, Calif., 1981)*. Ed. by Amer. Math. Soc. Vol. 40. Proc. Sympos. Pure Math. 1983, pp. 513–536.

[44] J. Strom. *Modern classical homotopy theory*. Vol. 127. Graduate Studies in Mathematics. American Mathematical Society, 2011.

[45] D. Sullivan. "Infinitesimal computations in topology." In: *Inst. Hautes Etudes Sci. Publ. Math.* 47 (1977), pp. 269–331.

[46] R. Thom. "Quelques propriétés globales des variétés différentiables." In: *Comm. Math. Helv.* 28 (1954), pp. 17–86.

© 2016 Tous droits réservés. lilliad.univ-lille.fr

© 2016 Tous droits réservés. lilliad.univ-lille.fr

INDEX

© 2016 Tous droits réservés. lilliad.univ-lille.fr

Thèse de Mathieu Klimczak, Lille 1, 2016

© 2016 Tous droits réservés.
lilliad.univ-lille.fr