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Spécialité : Automatique, Génie Informatique, Traitement du Signal et Image

par

Zohra Kader

Doctorat délivré par l'Université Lille 1-Sciences et Technologies

Commande et observation des systèmes affines à commutations

Soutenue le 18 septembre 2017 devant le jury composé de:

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Thèse préparée au Centre de Recherche en Informatique, Signal et Automatique de Lille
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École Doctorale SPI 072 (Lille I, Lille III, Artois, ULCO, UVHC, EC LILLE)
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Serial N^o : | | | |

Université Lille 1-Sciences et Technologies

THESIS

Presented to obtain the degree of

DOCTOR

Speciality : Control theory, computer science, signal processing and image

by

Zohra Kader

PhD awarded by University Lille 1

Control and observation of switched affine systems

Defended on September 18, 2017 in presence of the committee:

Chairman:	Antoine Girard	CNRS Research Director at LSS
Examiner:	Pierre Riedinger	Professor at Université de Lorraine
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Acronyms

- BMI - Bilinear Matrix Inequality.
- LMI - Linear Matrix Inequality.
- LTI - Linear Time-Invariant.
- LTV - Linear Time-Varying.

Notations

Notations concerning sets:

- $\mathbb{R}_+$ is the set $\{\lambda \in \mathbb{R}, \lambda \geq 0\}$.

- $\mathbb{R}^*$ is the set $\{\lambda \in \mathbb{R}, \lambda \neq 0\}$.

- $\mathbb{R}^{n\times m}$ denotes the set of real $n \times m$ matrices.

- $\mathbb{R}^{n\times n}$ denotes the set of real $n \times n$ matrices.

- $\mathrm{Conv}\{\mathcal{S}\}$ indicates the convex hull of the set $\mathcal{S}$.

- $\overline{\mathrm{Conv}}\{\mathcal{S}\}$ denotes the closed convex hull of a set $\mathcal{S}$.

- $\mathrm{int}\{\mathcal{S}\}$ refers to the interior of a set $\mathcal{S}$.

- $\overline{\mathcal{S}}$ denotes the closure of a set $\mathcal{S}$.

- $\mathrm{Vert}\{\mathcal{S}\}$ indicates the set of vertices of a set $\mathcal{S}$.

- $\mathrm{card}(\mathcal{S})$ refers to the cardinal of a set $\mathcal{S}$.

- For a positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a positive scalar γ, $\mathcal{E}(P,\gamma)$ indicates the ellipsoid

$$\mathcal{E}(P,\gamma) = \{x \in \mathbb{R}^n : x^T P x \leq \gamma\}.$$

- For all positive scalar r, $\mathcal{B}(0,r)$ denotes the closed ball of radius $\sqrt{r}$:

$$\mathcal{B}(0,r) = \mathcal{E}(I,r) = \{x \in \mathbb{R}^n : x^T x \leq r\}.$$

- Let $\mathcal{S} \subset \mathbb{R}^m$ be a finite set of vectors. The minimum argument of a given function $f : \mathcal{S} \longrightarrow \mathbb{R}$ is noted by

$$\arg\min_{x\in\mathcal{S}} f(x) = \{y \in \mathcal{S} : f(y) \leq f(z), \forall z \in \mathcal{S}\}.$$

- For a positive integer N, we denote by $\mathcal{I}_N$ the set $\{1, \ldots, N\}$.

- Δ_N stands for the unit simplex

$$\Delta_N = \left\{ \beta = \begin{bmatrix} \beta_{(1)}, & \ldots, & \beta_{(N)} \end{bmatrix}^T \in \mathbb{R}^N : \sum_{i=1}^{N} \beta_{(i)} = 1, \beta_{(i)} \geq 0, i \in \mathcal{I}_N \right\}.$$

Notations concerning matrices:

- M^{-1} stands for inverse of a non-singular square matrix M

- M^T refers to the transpose of a matrix M.

- $M \succeq 0$ (resp. $M \preceq 0$) means that the square matrix M is positive (resp. negative) semi-definite.

- $M \succ 0$ (resp. $M \prec 0$) means that the square matrix M is positive definite (resp. negative definite).

- $\text{eig}(M)$ denotes the eigenvalues of the square matrix M.

- $\text{eig}_{min}(M)$ and $\text{eig}_{max}(M)$ are used to refer to the minimum and maximum eigenvalue respectively of a square matrix M.

- I stands for the identity matrix of appropriate dimension.

- $M_{(i)}$ refers to the i-th row of a matrix M.

- $*$ denotes the elements that can be deduced by symmetry in a symmetric matrix.

- $M_{(i,j)}$ refers to the element of the i-th row and the j-th column of a matrix M.

- 0 indicates the null scalar or the null matrix of appropriate dimension.

- $\text{Re}(\text{eig}_j(M))$ denotes the real part of the j-th eigenvalue of the square matrix M.

Notations concerning vectors:

- $x_{(i)}$ refers to the i-th row of a vector x.

- x^T refers to the transpose of a vector x.

- $\|.\|$ denotes the Euclidean norm for a vector.

- A vector $v \in \mathbb{R}^m$ is said to be strictly positive if for all $i \in \mathcal{I}_m$ $v_{(i)} > 0$.

Notations concerning functions:

- A class $\mathcal{K}$ function is a function $f : [0, a) \to [0, +\infty)$ which is strictly increasing, and such that $f(0) = 0$.

- A class $\mathcal{K}^\infty$ is a class $\mathcal{K}$ function such that $a = +\infty$ and $\lim_{t \to +\infty} f(t) = +\infty$.

- A C^1 function is a continuously differentiable function.

Notations concerning scalars:

- $|a|$ denotes the absolute value of a scalar a.

- sign(x) refers to the sign function of $x \in \mathbb{R}$.

Résumé

Cette thèse est dédiée au problème de la stabilisation des systèmes affines à commutation. L'objectif est de concevoir des lois de commutation dépendantes de l'état qui stabilisent le système en boucle fermée. Premièrement, un aperçu de quelque résultat existant dans la littérature est présenté. Ensuite, un résultat général permettant la synthèse de lois de commutations pour la stabilisation des systèmes nonlinéaires affines en entrée est proposé. La particularisation de ce résultat aux cas des systèmes affines à commutation et des systèmes linéaires à temps invariant avec une commande à relais a permis de synthétiser des lois de commutations garantissant leur stabilité asymptotique locale ou globale en boucle fermée. Des surfaces de commutations linéaires et nonlinéaires sont proposées en utilisant des fonctions de Lyapunov quadratiques et non-quadratiques. Grâce à l'utilisation des fonctions de Lyapunov commutées une méthode numérique basée sur des LMIs permettant la conception de surfaces de commutations nonlinéaires est proposée. Une méthode permettant la synthèse de lois de commutations robustes vis-à-vis des perturbations sur les mesures est également développée. Enfin, le problème de la synthèse de lois de commutations basée-observateur est considéré. Des conditions de stabilisation asymptotique locale et globale sont développées. Les lois de commutations conçues dépendent de l'état reconstruit en utilisant un observateur du type Luenberger. De plus, le principe de séparation est démontré pour les systèmes affines à commutation ainsi que pour les systèmes linéaires temps invariant avec une commande à relais.

Mots-clés: Systèmes affines à commutation, commande basée-observateur, lois de commutation dépendantes de l'état, stabilité de lyapunov, commande à relais, robustesse.

Abstract

This thesis is dedicated to the stabilization problem of switched affine systems with state-dependent switching laws. First, an overview of some existing results is proposed. In order to define the closed-loop system's solutions and to analyse its behaviour over the switching surfaces, the Filippov formalism is used. The stabilization problem is addressed using a Lyapunov approach which allows deriving numerical approaches based on LMIs. Second, a general framework for the design of a switching control in the case of nonlinear input-affine systems is proposed. The application of the obtained results to switched affine systems and LTI systems with a relay controller led to the design of full state-dependent switching controllers which ensure either local or global asymptotic stability of the closed-loop systems. Thanks to switching (Lur'e type) Lyapunov functions, a numerical approach based on LMIs that allows to derive a nonlinear stabilizing switching law and to provide an estimation of a non ellipsoidal domain of attraction is proposed. Third, a design approach of robust state-dependent switching laws for switched affine systems stabilization is proposed. The robustness property is studied with respect to bounded exogenous disturbances that affect the state measurements which are used for the design of the switching laws. Finally, observer-based switching controllers are designed to guarantee both local and global asymptotic stability of the closed-loop system. Using both quadratic and non-quadratic Lyapunov functions, linear and nonlinear switching surfaces are designed. The derived switching surfaces depend on the estimated state which is computed by a Luenberger observer. For both switched affine systems and LTI systems with relay controller, the separation principle is proved.

Keywords: Switched affine systems, relay feedback control , LTI systems, observer-based switching laws, state-dependent switching laws.

General introduction

Hybrid systems represent dynamical systems that exhibit simultaneously continuous and discrete dynamics [4], [43], [44], [46], [52]. Switched systems represent a popular class of hybrid systems [54], [77], [76], [78], [102], [103], [107], [109]. They consist of a family of continuous-time subsystems and a rule [1] orchestrating the switching among them. The design of the switching rule is one of the most important problems in the switched system community. Even thought a large variety of results addressed this problem for the case when the family of subsystems shares a common equilibrium point, few results cover the case when each of the subsystem presents a different fixed point. In the present work, we are interested in the switching rule design problem for the case of switched affine systems [1], [16], [26], [51], [87], [95]. This class of switched systems has attracted the interest of the research community since its study has direct applications in various domains: power electronics (for example DC/DC and AC/DC converters), electromechanical systems, etc..

However, the control of switched affine systems presents some difficulties: fast switching, presence of non standard equilibrium points (the different subsystems do not necessarily share a common equilibrium), sliding dynamics, zeno behaviour, etc. Such problems have been studied in mathematics, in the more general context of discontinuous dynamical systems [5], [6], [8], [21], [84] . Their study is quite delicate since for such systems it is required to use more general concepts of solutions than the classical one, in order to take into account the dynamics over the discontinuities.

[1] also called switching law, representing the discrete dynamic.

Goals

The main objective of this thesis is to provide new methods for the design of state-dependent switching laws that guarantee the stability of the closed-loop switched affine systems at the origin. We intend to propose stabilization criteria for the challenging case when the individual subsystems are not stable and do not share a Hurwitz convex combination of their evolution matrices. The classical relay control for LTI systems is also considered as a particular case. Using Filippov's formalism, we will handle both sliding dynamics and *non-standard equilibrium points* which spring out by fast switching. Both quadratic and non-quadratic Lyapunov functions will be utilized in order to provide design approaches for linear and nonlinear switching surfaces that ensure the local asymptotic stability of the closed-loop switched affine system. Ellipsoidal and non-ellipsoidal estimations of the domains of attraction will also be investigated. Furthermore, we will study under which conditions the global stabilization may be achieved. The robustness with respect to exogenous perturbations will be analysed. Finally, we will also consider the stabilization of switched affine systems using observer-based switching laws. An investigation concerning the separation principles for both LTI systems with relay and switched affine systems will be led.

Structure of the thesis

The thesis is organized as follows:

Chapter 1

The first chapter provides an overview of the recent results on the stabilization of switched affine systems with state-dependent switching laws. First, the concept of solutions, the notion of equilibrium point and the stability concepts necessary for the comprehension are provided while considering the Filippov framework. Then, the problems that can be encountered in the study of the class of switched affine systems are presented using pedagogical examples. Finally, recent results concerning the design of stabilizing state-dependent switching for this class of systems are presented. Using the Filippov formalism some of these results are reproved. The advantages and disadvantages of each approach are analysed in order to point out the problems that are still open.

Chapter 2

In the second chapter, a new approach for the design of state-dependent switching laws is proposed. Thanks to the use of non-quadratic Lyapunov functions a method allowing the design of nonlinear switching surfaces and the enlargement of the domain of attraction with respect to the approach provided in [55] is developed. LMI criteria are given in order to design the switching law and provide an estimation of a non-ellipsoidal domain of attraction. Using the properties of Lur'e type Lyapunov functions, a constructive approach based on LMIs is provided in order to derive state-dependent switching laws ensuring the global asymptotic stability of the closed-loop switched affine systems. The approach is then particularised to the stabilization of the simpler class of LTI systems with relay controllers. Numerical methods allowing the design of a relay controller have been provided. Moreover, a general framework for the design of a relay control for the class of nonlinear input-affine systems is provided.

Chapter 3

The third chapter provides a method for the stabilization of switched affine systems with perturbed state-dependent switching laws. The method considers the perturbation in the states measurements. Qualitative conditions for stability are developed. In addition, a numerical approach based on LMIs which allows to derive state-dependent switching laws and to enlarge the domain of attraction or diminish the size of the chattering zone is given. The results are particularized to provide a method to design a relay control for LTI systems stabilization with perturbed measurements.

Chapter 4

In the last chapter, the problem of observer-based switching laws design is considered to ensure local asymptotic stability of the origin of switched affine systems. A Luenberger observer is used to design both linear and nonlinear switching surfaces dependent on the estimated state. LMI conditions are proposed in order to allow a numerical implementation of the results. Considering a particular property of the switching Lyapunov function, LMI conditions ensuring the global asymptotic stability of the closed-loop switched affine system to the origin are equally provided. The proposed approaches are then particularized to the case of stabilization of LTI systems by an observer-based relay feedback controller. It has been equally shown that for both LTI systems with a relay controller and switched affine systems the separation principle holds.

Personal publications

The research exposed in this thesis can be found in the following publications:

Journals

- Zohra Kader, Christophe Fiter, Laurentiu Hetel, and Lotfi Belkoura, «Non-quadratic stabilization by a relay control», provisionally accepted as a regular paper, with minor revisions in Automatica.
- Zohra Kader, Christophe Fiter, Laurentiu Hetel, and Lotfi Belkoura, «Stabilization of switched affine systems with disturbed state-dependent switching laws», International Journal of Robust and Nonlinear Control, 2017, `https://doi.org/10.1002/rnc.3887`.

International conferences

- Zohra Kader, Christophe Fiter, Laurentiu Hetel, and Lotfi Belkoura, «Non-quadratic stabilization of switched affine systems », 20th IFAC World Congress, Toulouse, France, 2017.
- Zohra Kader, Christophe Fiter, Laurentiu Hetel, and Lotfi Belkoura, «Stabilization of LTI systems by relay feedback controller with disturbed measurements», American Control Conference, Boston, USA, 2016.

- Zohra Kader, Christophe Fiter, Laurentiu Hetel, and Lotfi Belkoura, «Observer-based relay feedback controller design for LTI systems», European Control Conference, Aalborg, Denmark, 2016.

Chapter 1

State of the art

Over this chapter we will consider the following class of *switched affine systems*:

$$\begin{aligned} \dot{x} &= A_{\sigma(x)}x + b_{\sigma(x)}, \\ y &= C_{\sigma(x)}x, \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the vector of the state variables, $y \in \mathbb{R}^p$ is the vector of the outputs, $A_i \in \mathbb{R}^{n\times n}$, $C_i \in \mathbb{R}^{p\times n}$ and $b_i \in \mathbb{R}^{n\times 1}$ are the matrices describing the N subsystems, and $\sigma : \mathbb{R}^n \longrightarrow \mathcal{I}_N = \{1, \dots, N\}$ represents the switching law. We are interested in the design of state-dependent switching laws stabilizing system (1.1). More exactly we are interested in the design of state-dependent switching laws σ as piecewise constant functions. In the literature the closed loop system (1.1) is called piecewise affine system [65], [66], [82]. However, in this case σ is assumed to be given, while in our case we have to design it.

System (1.1) is a particular class of *discontinuous dynamical systems* [21]. Indeed, switched systems with state-dependent switching laws are differential equations with discontinuous right-hand side. For this class of systems classical solutions or solutions in the sense of Carathéodory [7] may not exist. For differential equations with discontinuous right-hand side, several concepts of solutions can be found [10], [21], [48], [49]. Over this thesis the concept of solutions which was initially proposed by Filippov in the sixties will be used. Before focusing on some basic results for the stabilizability of switched systems, the next section aims to introduce the concept of solutions, the notion of equilibrium points, and the stability concepts for discontinuous dynamical systems.

1.1 Solution concept and stability notions

For a given function $\sigma(x)$, system (1.1) is a subclass of the more general class of dynamical systems with discontinuous right-hand side given as

$$\dot{x} = \mathcal{X}(t, x), \tag{1.2}$$

with $\mathcal{X} : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ locally bounded and discontinuous.

Since the sixties, the study of discontinuous systems in the form (1.2) received a wide interest from many researchers in mathematics and in control theory. This interest is motivated by the fact that (1.2) can represent different classes of physical systems in various application domains, where the discontinuity can be connected to the physical nature of the system : power electronic converters [9], [29], [56], mechanical systems with dry frictions [20], [116], aerospace [39], [40], ..., etc. It can also be introduced intentionaly in the feedback loop in order to ensure desirable properties such as in sliding mode [32], [89], [115], relay [30], [59], [91], [114], or adaptive control [18], [73]. From the mathematical point of view, discontinuous systems can present several difficulties and challenges, among which we can mention the problem of existence and uniqueness of the solutions, the presence of non-linear phenomena such as sliding motion, limit cycles, chattering and zeno behaviour - see for instance [32], [45], [63], [64], [115], and the references therein. Among the existing frameworks the formalism developed by Filippov is well suited for their modelling and analysis, since it takes into account the discontinuities and the behaviour of the systems over them - see for instance the tutorial of Cortès [21]. In the following, we present the notions of solutions, equilibrium points, and stability of discontinuous systems. In fact, it seems convenient to start by the exposition of these notions for the class of time-invariant systems which is simple. These notions are then provided for the case of time-varying systems.

1.1.1 Time-invariant discontinuous systems

In order to highlight the most important concepts involved in our study and to present the main contribution of this work, we consider the class of time-invariant discontinuous systems. The notions of stability and equilibrium points are defined as well as the notion of solutions of discontinuous differential equations.

Solution concept

Let us consider the following time-invariant dynamical system

$$\dot{x} = \mathcal{X}(x), \tag{1.3}$$

with $\mathcal{X} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ locally bounded and discontinuous. The Filippov's formalism associates to the differential equation (1.3) a set-valued map taking into account all the possible values of the derivative. This is possible by considering the convex hull of the possible trajectories over the discontinuities. Then, solutions are constructed using differential inclusions [5]. The differential inclusion associated to the differential equation (1.3) is given by

$$\dot{x}(t) \in \mathcal{F}[\mathcal{X}](x), \tag{1.4}$$

The set-valued map $\mathcal{F}[\mathcal{X}](x)$ is computed from the differential equation (1.3) using the construction given in [36] such that

$$\mathcal{F}[\mathcal{X}](x) = \bigcap_{\delta>0} \bigcap_{\mu(\mathcal{S})=0} \mathrm{Conv}\{\mathcal{X}(\breve{\mathcal{B}}(x,\delta) \setminus \mathcal{S})\}, \forall x \in \mathbb{R}^n. \tag{1.5}$$

Here, we denote by Conv the closed convex hull, $\breve{\mathcal{B}}(x,\delta)$ is the open ball centered at x of radius $\sqrt{\delta}$, and finally $\mathcal{S}$ is a set of measure $\mu(\mathcal{S}) = 0$ (in the sense of Lebesgue). In the sequel, we note $\mathcal{F}[\mathcal{X}](x)$ the set-valued map associated to the system (1.3). We consider the solutions of (1.3) (or of (1.4)) in the sense of Filippov given as follows :

Definition 1 (Filippov solution [36])**:** Consider system (1.3) and its associated differential inclusion (1.4). A *Filippov solution* of the discontinuous systems (1.3) over the interval $[t_a, t_b] \subset [0, \infty)$ is an absolutely continuous mapping $\varsigma : [t_a, t_b] \longrightarrow \mathbb{R}^n$ satisfying:

$$\dot{\varsigma}(t) \in \mathcal{F}[\mathcal{X}](\varsigma(t)), \text{ for all most all } t \in \Big[t_a, t_b\Big] \tag{1.6}$$

with $\mathcal{F}[\mathcal{X}](x)$ given by (1.5).

The existence of at least one solution for some initial condition of the differential inclusion (1.4) is guaranteed for all $x \in \mathbb{R}^n$ if $\mathcal{F}$ is locally bounded and takes nonempty, compact and convex values [36].

The use of Filippov solutions is motivated by the fact that there exist differential equations with discontinuous right-hand side which do not admit classical solutions.

Example 1: Switched affine system which does not admit classical solutions

Consider the following discontinuous system

$$\dot{x} = \mathcal{X}(x) = \begin{cases} -1, \text{ if } x > 0, \\ 1, \text{ if } x \leq 0 \end{cases} \tag{1.7}$$

which is discontinuous at $x = 0$. Note that this example represents a basic switched affine system (1.1) with $A_1 = A_2 = 0$ and $b_1 = -b_2 = -1$. Suppose that there exists a classical solution of the differential equation (1.7) : a continuously differentiable function $x : [0,\ t_1] \to \mathbb{R}$ which satisfies (1.7) with $x(0) = 0$. Therefore, $\dot{x}(0) = \mathcal{X}(0) = 1$ which implies that for all t sufficiently small, $x(t) > 0$ and hence $\dot{x}(0) = \mathcal{X}(x) = -1$ which contradicts the fact that $t \to \dot{x}$ is continuous. Therefore, no classical solution starting from 0 exists. On the other hand, to the discontinuous equation (1.7) we may associate the following differential inclusion

$$\dot{x} \in \mathcal{F}[\mathcal{X}](x) = \begin{cases} -1, \text{ if } x > 0, \\ \text{Conv}\{-1, 1\}, \text{ if } x = 0 \\ 1, \text{ if } x < 0. \end{cases} \tag{1.8}$$

- if $x(0) = 0$ then the function $x : [0,\ \infty) \to \mathbb{R}$ of (1.7), with $x(t) = 0$ satisfies the differential inclusion (1.8), that it represents a Filippov solution according to Definition 1.

  Depending on the initial condition $x(0)$, the Filippov solutions of system (1.7) can be constructed as follows:

- if $x(0) > 0$, then $x : [0, \infty) \to \mathbb{R}$ is given by

$$x(t) = \begin{cases} x(0) - t, \text{ if } t \leq x(0), \\ 0, \text{ if } t \geq x(0); \end{cases}$$

- if $x(0) < 0$, then $x : [0, \infty) \to \mathbb{R}$ is given by

$$x(t) = \begin{cases} x(0) + t, \text{ if } t \leq -x(0), \\ 0, \text{ if } t \geq -x(0). \end{cases}$$

The following example illustrates different phenomena that can be encountered in the case of switched affine systems and it shows how the Filippov formalism is applied to the case of switched affine systems with two modes and a quadratic partitioning of the state space.

Example 2: Sliding modes occurrence in switched affine systems

Let us consider the following switched affine system

$$\begin{cases} \dot{x} = A_1 x + b_1, \text{ if } x^T \Gamma x < 0, \\ \dot{x} = A_2 x + b_2, \text{ if } x^T \Gamma x \geq 0, \end{cases} \tag{1.9}$$

where $x = \begin{bmatrix} x_{(1)} & x_{(2)} \end{bmatrix}^T \in \mathbb{R}^2$,

$$A_1 = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}, \; A_2 = \begin{bmatrix} -1 & -3 \\ -1 & 0 \end{bmatrix}, \; b_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \text{and } b_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

Here, $\Gamma = \begin{bmatrix} 10 & -20 \\ 1 & -0.2 \end{bmatrix}$ is the matrix characterizing the switching surfaces.

Let us define the following regions $\mathcal{R}_i$, $i \in \{1, 2\}$

$$\mathcal{R}_1 = \{x \in \mathbb{R}^2 : x^T \Gamma x < 0\},$$

and

$$\mathcal{R}_2 = \{x \in \mathbb{R}^2 : x^T \Gamma x \geq 0\}.$$

We have

$$\begin{cases} \mathcal{R}_1 \cap \mathcal{R}_2 = \emptyset, \\ \bar{\mathcal{R}}_1 \cup \bar{\mathcal{R}}_2 = \mathbb{R}^2. \end{cases} \tag{1.10}$$

Using these relations, the differential inclusion associated to system (1.9) is given as follows

$$\dot{x} \in \mathcal{F}[\mathcal{X}](x) = \begin{cases} A_1 x + b_1, \text{ if } x \in \mathcal{R}_1, \\ A_2 x + b_2, \text{ if } x \in \mathcal{R}_2, \\ \text{Conv}\{A_1 x + b_1, A_2 x + b_2\}, \text{ if } x \in \bar{\mathcal{R}}_1 \cap \bar{\mathcal{R}}_2. \end{cases} \tag{1.11}$$

Figure 1.1 shows the phase plot of system (1.9). From Figure 1.1 we can observe the two situations which appear when the trajectories of system (1.11) reach switching surfaces. For example, the trajectory originating at $x = \begin{bmatrix} 4 & 4 \end{bmatrix}^T$ (represented with a black line)

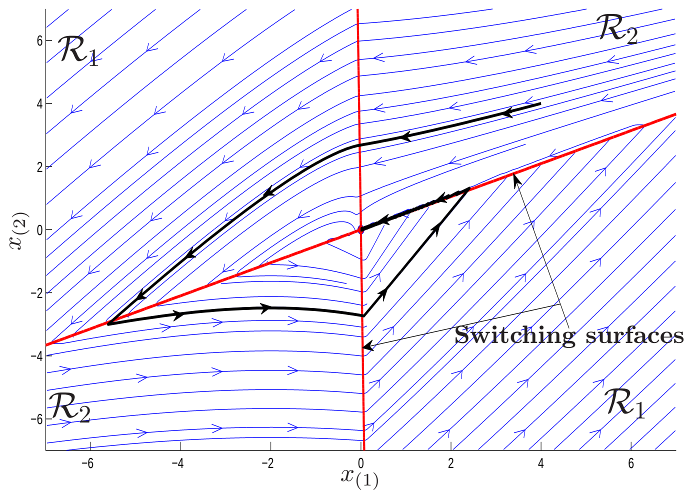

Figure 1.1: Sliding mode-Example 2

passes through the switching surfaces several times. When the trajectory reaches the switching surfaces at the point $\begin{bmatrix} 2.41 & 1.31 \end{bmatrix}^T$, it evolves according to a *sliding mode*: it stays on the switching surface and it follows a convex combination of the neighbouring vector fields. Contrary to the classical concept of solution, solutions in the sense of Filippov take into account the presence of sliding modes. For every $x(0)$ starting on the switching surfaces, there exists $\alpha(x) > 0$ such that the instantaneous velocity of x is given by

$$\dot{x} = \alpha(x)(A_1x + b_1) + (1 - \alpha(x))(A_2x + b_2), \tag{1.12}$$

and x is an absolutely continuous function satisfying the differential inclusion (1.11).

Equilibrium points

Considering solutions in the sense of Filippov, we provide extensions of the notions of equilibrium points and stability in the case of differential inclusions in the following section.

Definition 2 (Equilibrium point)**:** The point $\bar{x} \in \mathbb{R}^n$ is said to be an *equilibrium point* of the differential inclusion (1.4) if $0 \in \mathcal{F}[\mathcal{X}](\bar{x})$.

Applying this definition we will see that an equilibrium point of a subsystem of a

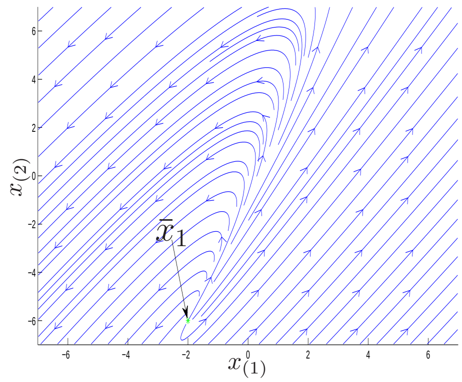

Figure 1.2: Phase plot of the subsystem $\dot{x} = A_1x + b_1$-Example 3

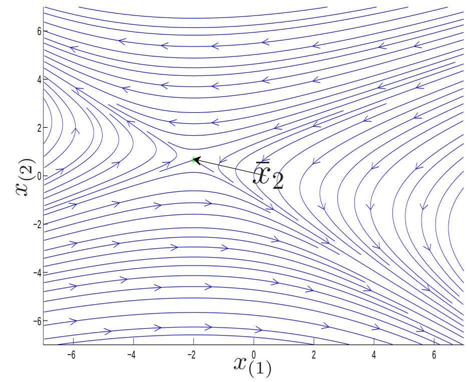

Figure 1.3: Phase plot of the subsystem $\dot{x} = A_2x + b_2$-Example 3

switched system is not necessarily an equilibrium point of the switched system and vice versa. The following example illustrates this phenomena.

Example 3: Equilibrium points of switched affine systems

In order to illustrate the notion of *equilibrium points* which are obtained by switching, let us reconsider Example 2. The phase plots of the subsystems $\dot{x} = A_1x + b_1$ and $\dot{x} = A_2x + b_2$ of system (1.9) are respectively depicted in Figures 1.2 and 1.3. Figure 1.4 shows the phase plot of system (1.9) and its equilibrium point.

From Figure 1.4 we can see that the switched affine system (1.9) has an equilibrium point $\bar{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ (red point) obtained by switching, since $0 \in \text{Conv}\{b_1, b_2\}$. Note that $x = 0$ is not an equilibruim for $\dot{x} = A_1x + b_1$ or for $\dot{x} = A_2x + b_2$. We can also note that the equilibrium points of the subsystems $\bar{x}_1 = -A_1^{-1}b_1$ and $\bar{x}_2 = -A_2^{-1}b_2$ (represented with green stars) do not constitute equilibrium points of the switched affine system since they are «hidden»by the state space partition.

Considering the above definitions we are now able to provide the stability concepts used in this work. Without loss of generality, we consider the case where the equilibrium point is the origin, because of the fact that any equilibrium point $x^* \neq 0$ can be shifted to the origin by using the change of coordinates $\bar{x} = x - x^*$. This coordinate transformation will be shown for switched affine systems in Section 1.3.

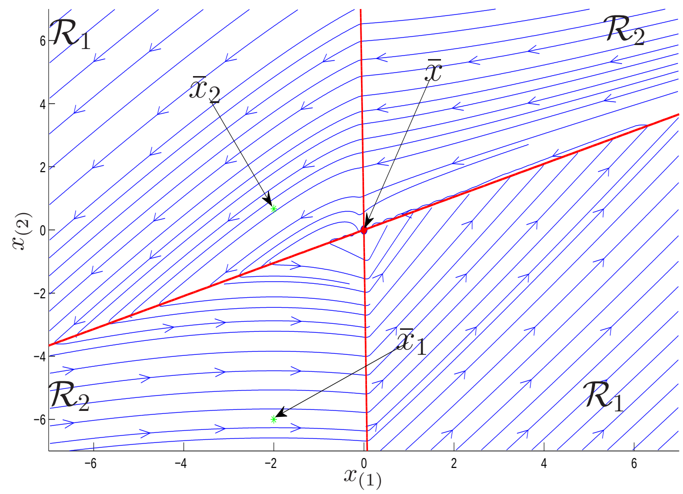

Figure 1.4: Phase plot of the switched affine system (1.9) illustrating its equilibrium point-Example 3

Stability notions

Definition 3: The equilibrium point $\bar{x} = 0$ of the differential inclusion (1.4) is said to be :

1. **stable**, if for any $\epsilon > 0$, there exists $\delta > 0$ such that for all Filippov solutions $x(t)$ of (1.4), $\|x(0)\| < \delta \Longrightarrow \|x(t)\| < \epsilon$, $\forall t \geq 0$,

2. **locally asymptotically stable**, if it is stable and there exists a set $\mathcal{D} \subset \mathbb{R}^n$, $0 \in \text{int}\{\mathcal{D}\}$, such that for all Filippov solutions $x(t)$ of (1.4) with $x(0) \in \mathcal{D}$, $x(t) \longrightarrow 0$ when $t \longrightarrow \infty$,

3. **locally exponentially stable with a decay rate** α (or **locally** α**-stable**) if there exist a set $\mathcal{D} \subset \mathbb{R}^n$, $0 \in \text{int}\{\mathcal{D}\}$, and strictly positive scalars κ and α such that for all Filippov solutions $x(t)$ of (1.4) with $x(0) \in \mathcal{D}$,

$$\|x(t)\| \leq \kappa e^{-\alpha t} \|x(0)\|, \forall t \geq 0. \tag{1.13}$$

A set $\mathcal{D}$ satisfying one of these properties is usually called an estimation of the domain of attraction.

The *Lyapunov stability approach* is the most used stability tool for linear and nonlinear systems. This method has been generalised by Filippov to the case of discontinuous dynamical systems (see for instance Theorem 1, page 153 in [36]). The principle of the Lyapunov stability approach remains the same in the case of discontinuous systems. The Lyapunov stability approach uses a function $V : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ which depends on the system's state to analyse the stability of an equilibrium point. This function is called a *candidate Lyapunov function.* There exists a large variety of books addressing the Lyapunov stability analysis of nonlinear systems - see for instance [8], [74], etc. Here, we recall the sufficient conditions for stability of discontinuous systems provided in [36] and [8].

Theorem 1: Let $\bar{x} = 0$ be an equilibrium point of system (1.4) and $\mathcal{D}$ a domain such that $0 \in \text{int}\{\mathcal{D}\}$. Let $V : \mathcal{D} \longrightarrow \mathbb{R}$ be a continuously differentiable function such that

$$V(0) = 0,\ V(x) > 0, \forall x \in \mathcal{D} \setminus \{0\} \tag{1.14}$$

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x}\varsigma \leq 0, \forall x \in \mathcal{D} \setminus \{0\}. \tag{1.15}$$

Then, $\bar{x} = 0$ is locally stable. Furthermore, if

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x}\varsigma < 0, \forall x \in \mathcal{D} \setminus \{0\}, \tag{1.16}$$

then, $\bar{x} = 0$ is locally asymptotically stable. Moreover, if there exists a positive scalar α such that

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x}\varsigma \leq -2\alpha V(x), \forall x \in \mathcal{D}, \tag{1.17}$$

then, the equilibrium point $\bar{x} = 0$ is locally α-stable with a decay rate α. Finally, $\bar{x} = 0$ is said to be unstable if it is not stable.

Conditions of global stability have been provided in [8] and are reported in the following theorem.

Theorem 2: Consider system (1.4). Assume that there exists a strict Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ such that for some functions $a, b, c \in \mathcal{K}^\infty$,

$$a(\|x\|) \leq V(x) \leq b(\|x\|), \forall x \in \mathbb{R}^n, \tag{1.18}$$

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x}\varsigma \leq -c(\|x\|), \forall x \in \mathbb{R}^n, \tag{1.19}$$

then the origin of system (1.4) is *globally asymptotically stable.*

1.1.2 Time-varying discontinuous systems

In this document we will also consider time-varying systems with discontinuous right-hand side given by

$$\dot{x} = \bar{\mathcal{X}}(t, x),\ x(t_0) = x_0, \tag{1.20}$$

where $\bar{\mathcal{X}} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is locally bounded and discontinuous.

To this system we associate the following time-varying differential inclusion

$$\dot{x} \in \mathcal{F}[\bar{\mathcal{X}}](t, x),\ x(t_0) = x_0, \tag{1.21}$$

with

$$\mathcal{F}[\bar{\mathcal{X}}](t, x) = \bigcap_{\delta>0} \bigcap_{\mu(\mathcal{S})=0} \mathrm{Conv}\{\bar{\mathcal{X}}(t, \breve{\mathcal{B}}(x, \delta) \setminus \mathcal{S})\}, \forall x \in \mathbb{R}^n, t \in \mathbb{R}_+. \tag{1.22}$$

The notion of Filippov solutions of a time-varying differential inclusion is recalled hereafter.

Solution concept

Definition 4 (Filippov solution)**:** Consider the differential inclusion (1.21). A *Filippov solution* of the discontinuous system (1.21) over the interval $[t_a, t_b] \subset [0, \infty)$ is an absolutely continuous mapping $\varsigma(t) : [t_a, t_b] \longrightarrow \mathbb{R}^n$ satisfying:

$$\dot{\varsigma}(t) \in \mathcal{F}[\bar{\mathcal{X}}](t, \varsigma(t)), \quad \text{for almost all} \quad t \in [t_a, t_b], \tag{1.23}$$

with $\mathcal{F}[\bar{\mathcal{X}}](t, x)$ given by (1.22).

A differential inclusion has at least one solution if the set-valued map $\mathcal{F}[\bar{\mathcal{X}}](t, x)$ is nonempty, locally bounded, closed, convex, and $\mathcal{F}$ is upper semicontinuous on x, t [5], [8], [21], [36].

Equilibrium point

For the time-varying system (1.21), the notion of equilibrium points is given as follows:

Definition 5 (Equilibrium point)**:** $\bar{x}$ is said to be an equilibrium point of the differential inclusion (1.21) if $0 \in \mathcal{F}[\bar{\mathcal{X}}](t, \bar{x})$ for all $t \geq t_0 \geq 0$.

Hereafter the notions of stability of the origin of time-varying discontinuous systems are introduced.

Stability notions

The stability notions of the origin of system (1.21) depends on the initial time t_0 and are defined in the following.

Definition 6 (Stability concepts)**:** The equilibrium point $\bar{x} = 0$ of the differential inclusion (1.21) is said to be :

1. **uniformly stable**, if for any $\epsilon > 0$, there exists $\delta > 0$, independent of t_0, such that for all Filippov solutions $x(t)$ of (1.21), $\|x(t_0)\| < \delta \Longrightarrow \|x(t)\| < \epsilon$, $\forall t \geq t_0 \geq 0$,

2. **locally uniformly asymptotically stable**, if it is uniformly stable and there exists a set $\mathcal{D} \subset \mathbb{R}^n$, $0 \in \text{int}\{\mathcal{D}\}$, such that for all Filippov solutions $x(t)$ of (1.21) with $x(t_0) \in \mathcal{D}$, $x(t) \longrightarrow 0$ when $t \longrightarrow \infty$,

3. **locally uniformly exponentially stable with a decay rate** α (or **locally uniformly** α**-stable**), if there exist a set $\mathcal{D} \subset \mathbb{R}^n$, $0 \in \text{int}\{\mathcal{D}\}$, and strictly positive scalars κ independent of t_0, and α such that for all Filippov solutions $x(t)$ of (1.21) with $x(t_0) \in \mathcal{D}$,
$$\|x(t)\| \leq \kappa e^{-\alpha(t-t_0)} \|x(t_0)\|, \forall t \geq t_0 \geq 0. \tag{1.24}$$

A set $\mathcal{D}$ satisfying one of these properties is usually called an estimation of the domain of attraction.

Let us recall that sufficient conditions for the local uniform asymptotic stability of the origin of systems modelled by a differential equation with a discontinuous right hand side $\dot{x} = \bar{\mathcal{X}}(t, x)$ have been given in [36].

Theorem 3: Let $\bar{x} = 0$ be an equilibrium point of system (1.21) and a domain $\mathcal{D}$ such that $0 \in \text{int}\{\mathcal{D}\}$. Let W_1, W_2, and W_3 be continuous positive definite functions and $V : \mathcal{D} \times \mathbb{R}_+ \to \mathbb{R}$ a strict Lyapunov function such that

$$V(t, 0) = 0,\ V(t, x) \geq 0,\ \forall x \neq 0,\ t \geq t_0 \geq 0, \tag{1.25}$$

$$W_1(x) \leq V(t, x) \leq W_2(x), \tag{1.26}$$

and

$$\sup_{\varsigma \in \mathcal{F}(t,x)} \left\{ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \varsigma \right\} \leq -W_3(x), \forall x \in \mathcal{D} \setminus \{0\}, \forall t \geq 0. \tag{1.27}$$

Then, $\bar{x} = 0$ is locally uniformly asymptotically stable.

For the case of uniform exponential stability the same conditions have to be verified while considering a particular form of W_3 such that $W_3(x) = 2\alpha V(t, x)$ where α is a positive scalar.

We may remark that in the literature there exist results about stability of discontinuous systems at the origin, when considering non-smooth Lyapunov functions - see for instance [8] and the references therein. Here we only consider the case where the Lyapunov functions are C^1.

1.2 Stabilization of switched systems: difficulties and challenges

Over this thesis we are concerned with the study of switched affine systems where the switching laws are available for control and used to stabilize the switched system. More precisely, we are interested in the design of state-dependent switching laws ensuring the stability of switched systems, along with the construction of Lyapunov functions that prove it. This problem is very challenging. Several difficulties must be considered. As we will see further, depending on the switching law the switched systems can be stable or unstable (Example 4). Moreover, even when the individual subsystems of the switched system are all stable, switching among them can have a destabilizing effect (See Examples 5 and 7). Finally, the stability of the sliding modes must be considered when designing the switching laws (see Example 6).

Example 4: Switching results in either stable or unstable systems

Consider the following switched affine system

$$\begin{cases} \dot{x} = A_1 x + b_1, \text{ if } x^T \Gamma x > 0, \\ \dot{x} = A_2 x + b_2, \text{ if } x^T \Gamma x \leq 0, \end{cases} \tag{1.28}$$

where $x \in \mathbb{R}^2$

$$A_1 = \begin{bmatrix} 1 & 1 \\ -1 & -4 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & 5 \\ -1 & -1 \end{bmatrix}, \ b_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \text{ and } b_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{1.29}$$

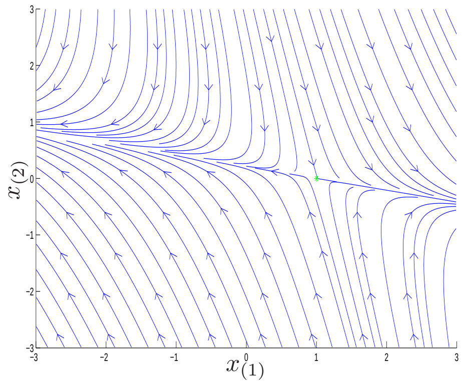

Figure 1.5: Phase plot of the subsystem $\dot{x} = A_1x + b_1$-Example 4

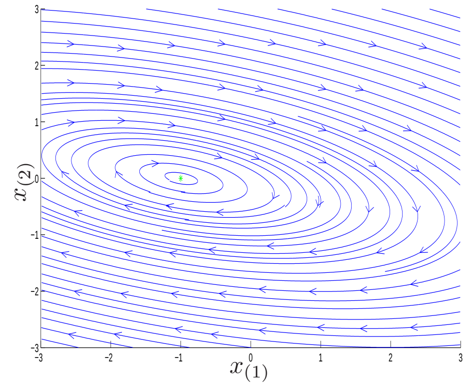

Figure 1.6: Phase plot of the subsystem $\dot{x} = A_2x + b_2$-Example 4

The phase plot of the subsystems are represented in Figures 1.6-1.5.

Here Γ is a matrix which characterizes the switching surfaces. The simulations are performed for two values of Γ:

$$\Gamma = \Gamma_1 = \begin{bmatrix} 1 & -6 \\ -200 & -10 \end{bmatrix}, \text{ and } \Gamma_2 = -\Gamma_1.$$

Let us define the regions $\mathcal{R}_i$, $i \in \{1, 2\}$ for the switched affine system (1.28) such that

$$\mathcal{R}_1 = \{x \in \mathbb{R}^2 : x^T \Gamma x > 0\},$$

and

$$\mathcal{R}_2 = \{x \in \mathbb{R}^2 : x^T \Gamma x \leq 0\}.$$

We have

$$\begin{cases} \mathcal{R}_1 \cap \mathcal{R}_2 = \emptyset, \\ \bar{\mathcal{R}}_1 \cup \bar{\mathcal{R}}_2 = \mathbb{R}^2. \end{cases} \tag{1.30}$$

The phase portraits of the switched affine system (1.28) with $\Gamma = \Gamma_1$, and $\Gamma = \Gamma_2$ are reported in Figures 1.7 and 1.8, respectively. From Figures 1.7-1.8, we can observe that depending on the switching law, we can obtain either a system which seems *asymptotically stable* (Figure 1.8) or a system which seems *unstable* (Figure 1.7) at the origin.

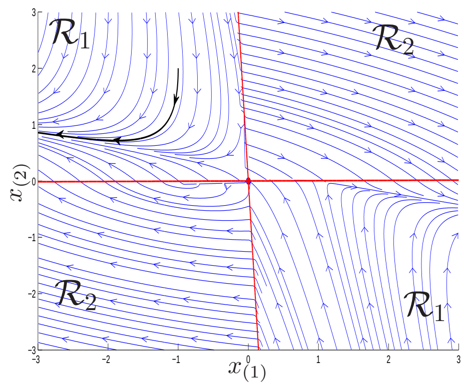

Figure 1.7: Phase plot of the switched affine system (1.28), (1.32) with $\Gamma = \Gamma_1$- Example 4

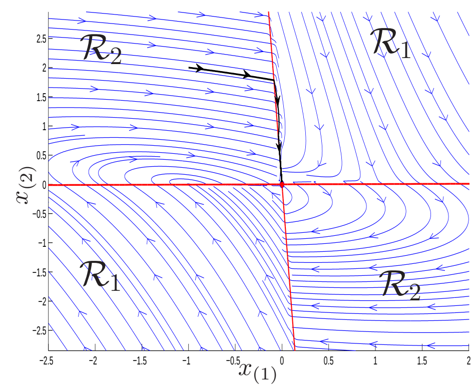

Figure 1.8: Phase plot of the switched affine system (1.28), (1.32) with $\Gamma = \Gamma_2$- Example 4

Example 5: Unstable switched affine system obtained by switching among stable systems

Consider the following switched affine system

$$\begin{cases} \dot{x} = A_1 x + b_1, \text{ if } x_{(1)}x_{(2)} \leq 0, \\ \dot{x} = A_2 x + b_2, \text{ if } x_{(1)}x_{(2)} > 0, \end{cases} \tag{1.31}$$

where $x = \begin{bmatrix} x_{(1)} & x_{(2)} \end{bmatrix}^T \in \mathbb{R}^2$

$$A_1 = \begin{bmatrix} -5 & -30 \\ 4 & -5 \end{bmatrix}, \; A_2 = \begin{bmatrix} -5 & -4 \\ 30 & -5 \end{bmatrix}, \; b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ and } b_2 = -b_1. \tag{1.32}$$

The phase portraits of the subsystems $\dot{x} = A_1 x + b_1$ and $\dot{x} = A_2 x + b_2$ are depicted in Figures 1.9 and 1.10 while Figure 1.11 represents the phase plot of the resulting switched system.

Let us define the regions $\mathcal{R}_i$, $i \in \{1, 2\}$ for the switched affine system (1.31) such that

$$\mathcal{R}_1 = \{x \in \mathbb{R}^2 : x_{(1)}x_{(2)} \leq 0\},$$

and

$$\mathcal{R}_2 = \{x \in \mathbb{R}^2 : x_{(1)}x_{(2)} > 0\}.$$

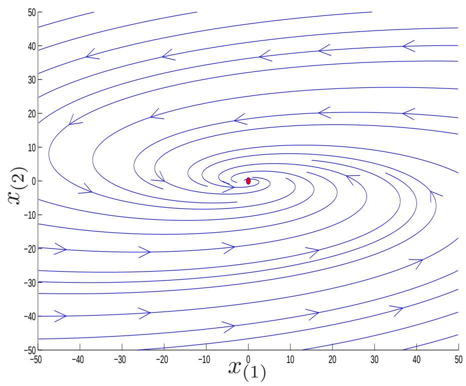

Figure 1.9: Phase plot of the subsystem $\dot{x} = A_1x + b_1$-Example 5

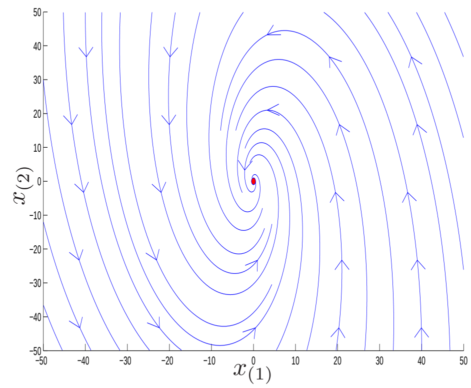

Figure 1.10: Phase plot of the subsystem $\dot{x} = A_2x + b_2$-Example 5

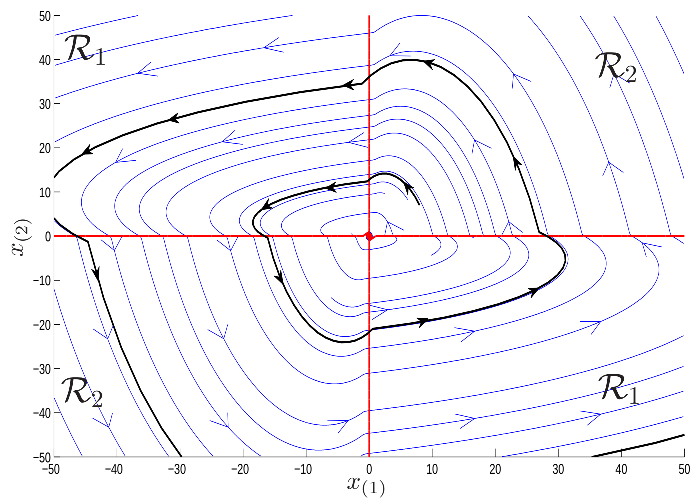

Figure 1.11: Phase plot of the switched system (1.31)-Example 5

From Figure 1.11 we can observe that even if the subsystems are stable, the switched systems seems to be unstable at the origin. Therefore, if the switching law is not chosen in an appropriate manner the system solution may diverge.

The following example shows a case where the two subsystems are unstable, however by taking into account sliding dynamics we may obtain a switched affine system which seems to be stable at the origin.

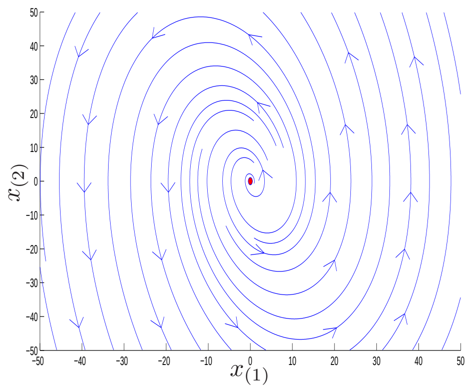

Figure 1.12: Phase plot of the subsystem $\dot{x} = A_1 x + b_1$-Example 6

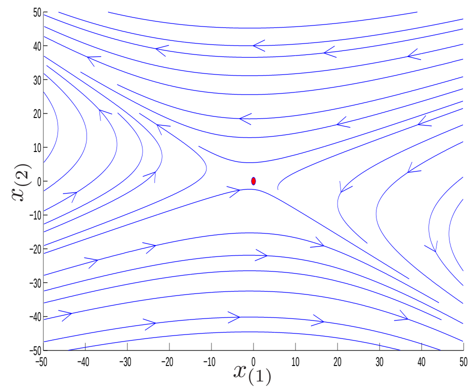

Figure 1.13: Phase plot of the subsystem $\dot{x} = A_2 x + b_2$-Example 6

Example 6: Stabilization to an equilibrium point obtained by sliding modes

Let us consider the switched affine system (1.28) with

$$A_1 = \begin{bmatrix} 0 & -1 \\ 4 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} -1 & -3 \\ -1 & 0 \end{bmatrix},\ b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ b_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix},\ \text{and } \Gamma = \Gamma_3 = \begin{bmatrix} -1 & 6 \\ 200 & 10 \end{bmatrix}. \tag{1.33}$$

We can observe that both subsystems $\dot{x} = A_1 x + b_1$ and $\dot{x} = A_2 x + b_2$ are *unstable* (see Figures 1.12-1.13). Note that the origin is not a common equilibrium point of the subsystems. However, $0 \in \text{Conv}\{b_1, b_2\}$. Therefore, it represents an equilibrium point of the differential inclusion. Moreover, the phase plot given in Figure 1.14 seems to indicate that the origin is asymptotically stable: for any initial condition the system state reaches the surface $x = 0$ and slides on it towards the origin.

The following example illustrates the fact that depending on the switching law and considering sliding modes the trajectory of the obtained switched system can be stable or unstable at the origin.

Example 7: Switching results on unstable sliding modes

Let us consider the switched affine system

$$\begin{cases} \dot{x} = A_1 x + b_1, \text{ if } x^T \Gamma x > 0, \\ \dot{x} = A_2 x + b_2, \text{ if } x^T \Gamma x \leq 0, \end{cases} \tag{1.34}$$

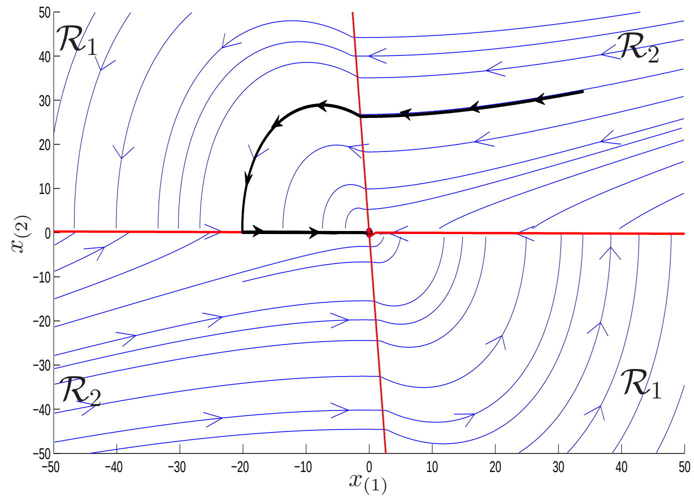

Figure 1.14: Phase plot of the switched system (1.28), (1.33) with $\Gamma = \Gamma_3$-Example 6

where

$$A_1 = \begin{bmatrix} -1 & 1.6 \\ -15 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} -1 & -15 \\ 1.6 & -1 \end{bmatrix},$$
$$b_1 = \begin{bmatrix} 28 \\ -21 \end{bmatrix},\ b_2 = -b_1,\ \text{and } \Gamma = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}. \tag{1.35}$$

Let us define the regions $\mathcal{R}_i$, $i \in \{1, 2\}$ for the switched affine system (1.34) such that

$$\mathcal{R}_1 = \{x \in \mathbb{R}^2 : x^T \Gamma x > 0\},$$

and

$$\mathcal{R}_2 = \{x \in \mathbb{R}^2 : x^T \Gamma x \leq 0\}.$$

We have

$$\begin{cases} \mathcal{R}_1 \cap \mathcal{R}_2 = \emptyset, \\ \bar{\mathcal{R}}_1 \cup \bar{\mathcal{R}}_2 = \mathbb{R}^2. \end{cases} \tag{1.36}$$

We can observe that both subsystems $\dot{x} = A_1 x + b_1$ and $\dot{x} = A_2 x + b_2$ are *stable* (see Figures 1.15-1.16). Note that the origin is not a common equilibrium point of the subsystems. However, $0 \in \text{Conv}\{b_1, b_2\}$. Therefore, it represents an equilibrium point of the differential inclusion. The phase plot given in Figure 1.17 seems to indicate that the origin is unstable: the trajectories starting (or reaching) the switching surfaces can either

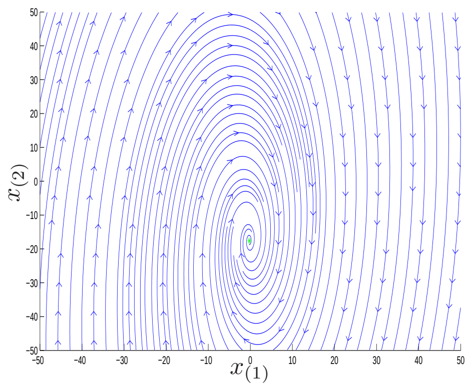

Figure 1.15: Phase plot of the subsystem $\dot{x} = A_1x + b_1$-Example 7

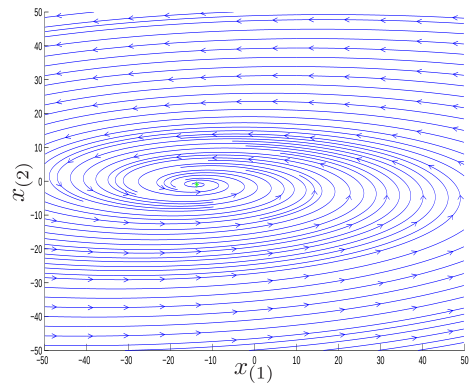

Figure 1.16: Phase plot of the subsystem $\dot{x} = A_2x + b_2$-Example 7

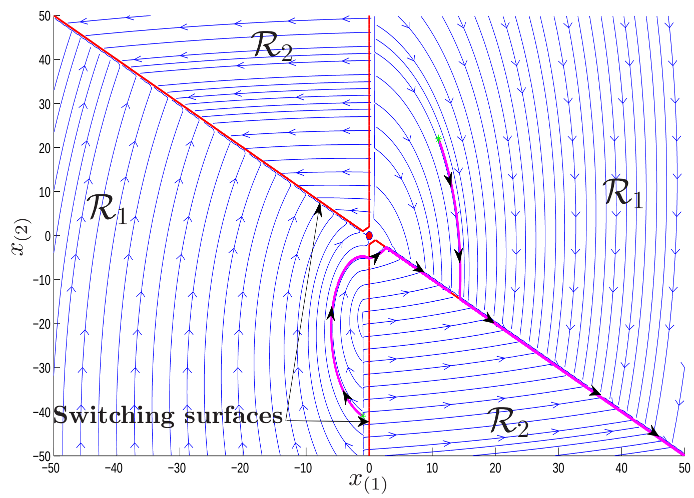

Figure 1.17: Phase plot of the switched system (1.34), (1.35)-Example 7

switch to the regions $\mathcal{R}_1$ and $\mathcal{R}_2$ or slide on the surfaces away from the origin. Therefore, in order to avoid these situations the design approaches of switching laws must take into account the behaviour of the switched affine system over the switching surfaces (sliding modes) which is allowed by the Filippov formalism.

The above examples seem to indicate that it is possible to stabilize a switched system to the origin by designing a suitably constrained switching law even if all the subsystems are unstable and do not share a common equilibrium point. However, a particular attention has to be given to sliding dynamics. Here, we intend to address this switching law design problem in a formal manner. Using the Filippov formalism and the Lyapunov theory we will focus on designing stabilizing switching laws. Before presenting our results we provide as follows a short overview of some of the existing results about stabilizing state-dependent switching laws design.

1.3 Some stabilization conditions for switched affine systems

Let us consider the class of switched affine systems given as follows:

$$\dot{\tilde{x}} = \tilde{A}_{\sigma(\tilde{x})}\tilde{x} + \tilde{b}_{\sigma(\tilde{x})} = \tilde{\mathcal{X}}(\tilde{x}), \tag{1.37}$$

with $\tilde{x} \in \mathbb{R}^n$ is the vector of the state variables, $\tilde{A}_i, i \in \{1, 2\}$ are the evolution matrices of the subsystems, $\tilde{b}_i \in \mathbb{R}^n$ represent the affine terms, and $\sigma(\tilde{x})$ is the switching law.

The majority of the existing results in the literature considers the case of autonomous systems, in which the state of the switched system must be steered to a common equilibrium point of all the component subsystems (or at least of some of them). In what follows and in our results we consider the case where the equilibrium point is obtained by switching among the subsystems. Very few results were concerned with this case. The first methods for the stabilization of this class of systems have been proposed by [16] while the cases of linear switched systems and nonlinear switched systems with common equilibria have been extensively studied in the literature [76], [78], [103], [106].

Consider the differential inclusion associated to (1.37)

$$\dot{\tilde{x}} = \mathcal{F}[\tilde{\mathcal{X}}](\tilde{x}), \tag{1.38}$$

with

$$\begin{aligned} \mathcal{F}[\tilde{\mathcal{X}}](\tilde{x}) &= \text{Conv}\{\tilde{A}_i\tilde{x} + \tilde{b}_i, i \in \mathcal{I}_N\} \\ &= \sum_{i=1}^{N}(\tilde{A}_i\tilde{x} + \tilde{b}_i)\beta_{(i)}. \end{aligned} \tag{1.39}$$

From Definition 2, a necessary condition for $\tilde{x}$ to be an equilibrium is that there exists

$\beta^* \in \Delta_N$ such that

$$\mathcal{F}[\tilde{\mathcal{X}}](\tilde{x}^*) = \sum_{i=1}^{N}(\tilde{A}_i\tilde{x}^* + \tilde{b}_i)\beta^*_{(i)} = 0. \tag{1.40}$$

Therefore, when the matrix $\sum_{i=1}^{N} \tilde{A}_i\beta^*_{(i)}$ is invertible the equilibrium point $\tilde{x}^*$ of system (1.37) must satisfy $\tilde{x}^* = -(\sum_{i=1}^{N} \tilde{A}_i\beta^*_{(i)})^{-1}\sum_{i=1}^{N} \beta^*_{(i)}b_i$. The Filippov formalism allows the description of both the equilibrium points of the subsystems ($\beta^* \in \text{Vert}(\Delta_N)$) and the equilibrium points induced by fast switching ($\beta^* \in \Delta_N, \beta^* \notin \text{Vert}(\Delta_N)$). Without loss of generality, we consider the case where the equilibrium point is the origin. Note that any prescribed equilibrium point can be shifted to the origin via a change of coordinate $x = \tilde{x} - \tilde{x}^*$. Then, system (1.37) becomes

$$\dot{x} = A_{\sigma(x)}x + b_{\sigma(x)} = \mathcal{X}_{aff}(x), \tag{1.41}$$

with $A_i = \tilde{A}_i$, $b_i = \tilde{A}_i\tilde{x}^* + \tilde{b}_i, \forall i \in \mathcal{I}_N$ and $\sum_{i=1}^{N} \beta_{(i)}b_i = 0$ i.e. $x^* = 0$ satisfies the necessary condition to be an equilibrium point in the origin.

Considering these notions, next we intend to presents an overview of some recent results on the stabilization of switched affine systems at the origin.

1.3.1 Existence of a Hurwitz convex combination

Case of switched linear systems

Before presenting this approach for the class of switched affine systems, we present the idea for the simpler class of linear switched systems, while considering the concept of Filippov solutions.

In the sequel, for $f_i \in \mathbb{R}$ with $i \in \mathcal{I}_N = \{1, 2, \ldots, N\}$, we note

$$\arg\min_{i \in \mathcal{I}_N}\{f_i\} = \{i \in \mathcal{I}_N, f_i \leq f_j, \forall j \in \mathcal{I}_N\},$$

and we define the set

$$\Delta_N = \left\{\beta = \begin{bmatrix}\beta_{(1)} & \ldots & \beta_{(N)}\end{bmatrix}^T \in \mathbb{R}^N : \beta_{(i)} \in \begin{bmatrix}0, & 1\end{bmatrix}, \sum_{i=1}^{N}\beta_{(i)} = 1\right\}.$$

Let us consider the linear switched system given by

$$\dot{x} = A_{\sigma(x)}x = \mathcal{X}(x), \tag{1.42}$$

with $x \in \mathbb{R}^n$ the states variables. The matrices $A_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{I}_N$, are the evolution matrices of the N subsystems. We suppose that they are not Hurwitz. Then, they satisfy the relation

$$\max_{j\in\{1,2,\ldots,n\}} \mathrm{Re}(\mathrm{eig}_j(A_i)) \geq 0, \forall i \in \mathcal{I}_N. \tag{1.43}$$

The function $\sigma : \mathbb{R}^n \longrightarrow \mathcal{I}_N$ is the stabilizing switching law.

Using these notations and the definitions provided in the first section, in what follows we present pioneering results based on the Lyapunov stability theory for the design of state-dependent switching laws in order to stabilize switched linear systems. This constructive method of stabilizing state-dependent switching laws for the linear switched system (4.119) has been provided in [119]. It has been shown in [119] that the existence of a Hurwitz convex combination of the matrices A_i, $i \in \mathcal{I}_2$ implies the existence of a stabilizing state-dependent switching law. A quadratic Lyapunov function proves the stability of the origin of the closed-loop switched system. In the following, the main results given in [119] are presented by considering the solutions of the closed-loop system in the sense of Filippov (defined in Section 1).

Theorem 4: (adapted from [119]) Consider the system (4.119) with $N = 2$. Assume that there exists $\beta^* \in [0, 1]$ such that the convex combination $A(\beta^*) := \beta^* A_1 + (1-\beta^*)A_2$ is Hurwitz.

Therefore, the switching law

$$\sigma(x) \in \arg\min_{i\in\{1,\ 2\}} \left\{x^T(A_i^T P + PA_i)x\right\}, \tag{1.44}$$

with $P^T = P \succ 0$ the matrix satisfying

$$A^T(\beta^*)P + PA(\beta^*) \prec 0, \tag{1.45}$$

stabilizes globally asymptotically the switched linear system (4.119) at the origin.

Proof. For system (4.119) with the switching law (1.44), we define the regions $\mathcal{R}_i$, $i \in \{1, 2\}$ such that

$$\mathcal{R}_i = \{x \in \mathbb{R}^n : x^T(A_i^T P + PA_i)x < x^T(A_j^T P + PA_j)x, \forall j \in \{1, 2\} \setminus \{i\}\}.$$

We have

$$\begin{cases} \mathcal{R}_1 \cap \mathcal{R}_2 = \emptyset, \\ \bar{\mathcal{R}}_1 \cup \bar{\mathcal{R}}_2 = \mathbb{R}^n. \end{cases}$$

With these relations, the differential inclusion (1.4) and the set-valued map defined in (1.5) associated to the closed-loop system (4.119), (1.44), are given by:

$$\dot{x} \in \mathcal{F}[\mathcal{X}](x) = \begin{cases} \{A_1 x\}, & \text{if} \quad x \in \mathcal{R}_1, \\ \{A_2 x\}, & \text{if} \quad x \in \mathcal{R}_2, \\ \text{Conv}\{A_1 x, A_2 x\}, & \text{if} \quad x \in \bar{\mathcal{R}}_1 \cap \bar{\mathcal{R}}_2. \end{cases} \tag{1.46}$$

For all $x \in \mathbb{R}^n$ we define

$$\mathcal{I}^*(x) = \left\{ i \in \{1,\ 2\} : x^T(A_i^T P + PA_i)x \leq x^T(A_j^T P + PA_j)x, \forall j \in \{1,2\} \right\}.$$

We have then:

$$\mathcal{I}^*(x) = \begin{cases} \{1\}, & \text{if} \quad x \in \mathcal{R}_1, \\ \{2\}, & \text{if} \quad x \in \mathcal{R}_2, \\ \{1,2\}, & \text{if} \quad x \in \bar{\mathcal{R}}_1 \cap \bar{\mathcal{R}}_2. \end{cases}$$

Using this notation, we have

$$\mathcal{F}[\mathcal{X}](x) = \underset{i \in \mathcal{I}^*(x)}{\text{Conv}} \{A_i x\}. \tag{1.47}$$

The set-valued map (1.47) is locally bounded, has a nonempty, compact and convex values. Therefore, the differential inclusion (1.46) admits at least one solution $x(t)$.

Consider the candidate Lyapunov function $V(x) = x^T P x$, with $P = P^T \succ 0$ satisfying (1.45). We want to show that the closed-loop system is stabilized by the switching law (1.44). In order to show this, it is sufficient to prove that

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma < 0 \tag{1.48}$$

for all $x \neq 0$.

From (1.45) we have:

$$\beta^* x^T(A_1^T P + PA_1)x + (1-\beta^*)x^T(A_2^T P + PA_2)x < 0, \quad \forall x \neq 0. \tag{1.49}$$

Since $\beta^* > 0$ and $1 - \beta^* > 0$, we deduce that

$$\forall x \neq 0, \quad \exists i \in \mathcal{I}_N, \quad x^T(A_i^T P + PA_i)x < 0, \tag{1.50}$$

and then

$$\forall x \neq 0, \quad \forall i \in \mathcal{I}^*(x), \quad x^T(A_i^T P + PA_i)x < 0. \tag{1.51}$$

It comes:

- if $x \in \mathcal{R}_1$, $\mathcal{I}^*(x) = \{1\}$ and $\mathcal{F}[\mathcal{X}](x) = \{A_1 x\}$. From (1.51), we obtain

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x}\varsigma = x^T(A_1^T P + PA_1)x < 0, \tag{1.52}$$

  and then (1.48) is verified for all $x \in \mathcal{R}_1$.

- If $x \in \mathcal{R}_2$, $\mathcal{I}^*(x) = \{2\}$ and $\mathcal{F}[\mathcal{X}](x) = \{A_2 x\}$. From (1.51), we obtain

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x}\varsigma = x^T(A_2^T P + PA_2)x < 0, \tag{1.53}$$

  and then (1.48) is verified for all $x \in \mathcal{R}_2$.

- If $x \in \bar{\mathcal{R}}_1 \cap \bar{\mathcal{R}}_2$, we have $\mathcal{I}^*(x) = \{1, 2\}$ and

$$\mathcal{F}[\mathcal{X}](x) = \text{Conv}\{A_1 x, A_2 x\} = \big\{\beta A_1 x + (1-\beta)A_2 x : \beta \in [0,1]\big\}.$$

  Consequently, we have

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x}\varsigma = \sup_{\beta \in [0,1]} \{x^T(A^T(\beta)P + PA(\beta))x\}. \tag{1.54}$$

  Since the set $[0, 1]$ is compact, we have

$$\sup_{\beta \in [0,1]} \{x^T(A^T(\beta)P + PA(\beta))x\} = \max_{\beta \in [0,1]} \{x^T(A^T(\beta)P + PA(\beta))x\}, \tag{1.55}$$

  and then, for all $x \neq 0$, the inequality

$$\sup_{\beta \in [0,1]} \{x^T(A^T(\beta)P + PA(\beta))x\} < 0 \tag{1.56}$$

  is verified, if and only if

$$x^T(A^T(\beta)P + PA(\beta))x < 0 \tag{1.57}$$

  is verified for all $\beta \in [0, 1]$.

Since $\mathcal{I}^*(x) = \{1, 2\}$ we deduce from (1.51) that

$$x^T(A_i^T P + PA_i)x < 0, \quad \forall i \in \{1, 2\}. \tag{1.58}$$

Thus, since $\beta > 0$ and $1 - \beta > 0$, we obtain

$$\beta x^T(A_1^T P + PA_1)x + (1 - \beta)x^T(A_2^T P + PA_2)x < 0. \tag{1.59}$$

The equation (1.57) is then verified for all $\beta \in [0, 1]$. Therefore, the relation (1.48) is verified for all $x \neq 0$, and thus the origin of the closed-loop system (4.119), (1.44) is globally asymptotically stable.

□

Remark 1: The method proposed in Theorem 4 is based on the existence of a quadratic Lyapunov function. When a stabilizing switching law exists while ensuring the decay of a quadratic Lyapunov function, we usually say that the system is quadratically stable.

The feasibility of (1.45) allows the design of the switching law (1.44). We may remark that (1.45) is a Bilinear Matrix Inequality (BMI) with variables β and P. In order to solve (1.45) we have to proceed in two steps. Firstly, we need to find the vector β^* such that $A(\beta^*)$ is Hurwitz. Secondly, for $\beta = \beta^*$ we have to solve the Linear Matrix Inequality (LMI) (1.45) with the variable P using the existing solvers on Matlab or another programming language.

In the following we provide an illustrative example of this method.

Example 8: Consider the linear system (4.119) with matrices

$$A_1 = \begin{bmatrix} 0 & -1 \\ 4 & 1 \end{bmatrix}, \quad \text{and} \quad A_2 = \begin{bmatrix} -1 & -3 \\ -1 & 0 \end{bmatrix}. \tag{1.60}$$

One can verify that the open-loop linear systems are unstable (the eigenvalues of the matrices A_1 and A_2 are $0.5 \pm 1.9365i$ and $-2.3, 1.3$ respectively). Considering $\beta^* = \frac{1}{2}$, such that the convex combination of the matrices $A_i, i \in \mathcal{I}_2$ is Hurwitz, We obtain the following solution of (1.45)

$$P = \begin{bmatrix} 95.23 & 65.86 \\ 65.86 & 338.1 \end{bmatrix}. \tag{1.61}$$

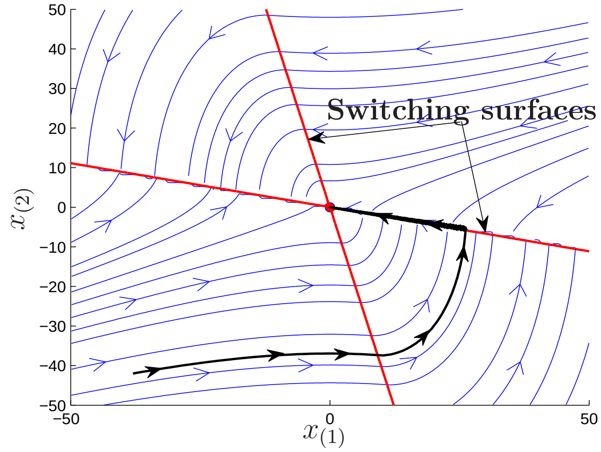

Figure 1.18: Phase plot

Figure 1.18 shows the phase plot of the closed-loop switched system. From Figure 1.18 we can observe that starting from any point in the state space, the trajectories of the closed-loop switched system (4.119), (1.60), (1.44), (1.61) converge asymptotically to the origin.

The result of Theorem 4 has been extended in [33] to show that for the case $N = 2$, the existence of a Hurwitz convex combination of the matrices A_1 and A_2 is a necessary and sufficient condition of the existence of a stabilizing state-dependent switching law which ensure the decay of a quadratic Lyapunov function. The result has been proved by using the S-procedure Lemma. Feron [33] also extends the result to the case where the state of the switched system (4.119) is not fully available to measurement.

A sliding mode controller has also been proposed by Wicks and coauthors in [118] using the same hypothesis of existence of a Hurwitz convex combination of matrices $A_i, i \in \mathcal{I}_2$. A hybrid controller has been proposed in the same paper and provides a chattering-free switching strategy by using the *hysteresis loop behaviour*: the basic idea consists on defining two new switching surfaces, $\mathcal{R}_1'$ near the boundary of $\mathcal{R}_1$ and $\mathcal{R}_2'$ near the boundary of $\mathcal{R}_2$ and selecting the switching law to ensure that switching occurs only when the state attempts to cross the boundary $\mathcal{R}_i', i \in \{1, 2\}$ in a direction leaving the cone. State trajectories crossing the boundary upon entering the cone do not cause switching.

The result in Theorem 4 has been also generalized in [90] to the case of the switched systems having N subsystems. Hereafter, we present this result while considering Filippov solutions of the closed-loop switched linear system.

Theorem 5: (adapted from [90]) Consider the system (4.119) with $\sigma(x) \in \mathcal{I}_N$. Assume that there exists $\beta^* \in \Delta_N$ such that the convex combination

$$A(\beta^*) = \sum_{i=1}^{N} \beta^*_{(i)} A_i \tag{1.62}$$

is Hurwitz.

Then, the switching law

$$\sigma(x) \in \arg\min_{i \in \mathcal{I}_N} \left\{ x^T (A_i^T P + P A_i) x \right\}, \tag{1.63}$$

with a matrix $P^T = P \succ 0$ such that

$$\sum_{i=1}^{N} \beta^*_{(i)} (A_i^T P + P A_i) \prec 0, \tag{1.64}$$

stabilizes globally asymptotically system (4.119) at the origin.

Proof. Form the closed-loop system (4.119), (1.63), we define the regions $\mathcal{R}_i$ for $i \in \mathcal{I}_N$ as

$$\mathcal{R}_i = \left\{ x \in \mathbb{R}^n : x^T (A_i^T P + P A_i) x < x^T (A_j^T P + P A_j) x, \forall j \in \mathcal{I}_N \setminus \{i\} \right\}. \tag{1.65}$$

We have

$$\begin{cases} \mathcal{R}_i \cap \mathcal{R}_j = \emptyset, i \neq j \\ \bigcup_{i \in \mathcal{I}_N} \bar{\mathcal{R}}_i = \mathbb{R}^n. \end{cases}$$

In addition, for all $x \in \mathbb{R}^n$ we define

$$\mathcal{I}^*(x) = \left\{ i \in \mathcal{I}_N : x^T (A_i^T P + P A_i) x \leq x^T (A_j^T P + P A_j) x, \forall j \in \mathcal{I}_N \right\}. \tag{1.66}$$

The set-valued map for all $x \in \mathbb{R}^n$ and $i \in \mathcal{I}^*(x)$ is given by

$$\mathcal{F}^*[\mathcal{X}](x) = \operatorname*{Conv}_{i \in \mathcal{I}^*(x)} \{A_i x\}. \tag{1.67}$$

We can remark that the following relation is satisfied

$$\mathcal{F}[\mathcal{X}](x) \subseteq \mathcal{F}^*[\mathcal{X}](x). \tag{1.68}$$

The set-valued map $\mathcal{F}^*[\mathcal{X}](x)$ is locally bounded, takes nonempty, compact and convex values, then for each initial condition $x(0)$, the differential inclusion $\dot{x} \in \mathcal{F}^*[\mathcal{X}](x)$ admits at least one solution. This is also true for the differential inclusion $\dot{x} \in \mathcal{F}[\mathcal{X}](x)$ by considering the relation (1.68).

Consider the candidate Lyapunov function $V(x) = x^T P x$ with $P = P^T \succ 0$ satisfying (1.64). We want to prove that the origin of the closed-loop system (4.119), (1.63) is globally asymptotically stable. In order to show this, it is sufficient to prove that

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x}\varsigma < 0, \quad \forall x \neq 0 \tag{1.69}$$

is satisfied.

From (1.68), to prove (1.69), it is sufficient to show that

$$\sup_{\varsigma \in \mathcal{F}^*[\mathcal{X}](x)} \frac{\partial V}{\partial x}\varsigma < 0, \tag{1.70}$$

for all $x \neq 0$.

From (1.64), we have:

$$\forall x \neq 0, \quad \sum_{i=1}^{N} \beta^*_{(i)} x^T (A_i^T P + P A_i) x < 0. \tag{1.71}$$

Since $\beta^* \in \Delta_N$, it comes

$$\forall x \neq 0, \exists i \in \mathcal{I}_N, x^T (A_i^T P + P A_i) x < 0, \tag{1.72}$$

then

$$x^T (A_i^T P + P A_i) x < 0, \forall i \in \mathcal{I}^*(x), \forall x \neq 0. \tag{1.73}$$

From this last equation, we obtain (for all $x \neq 0$):

- if $\mathcal{I}^*(x) = \{i\}$, $x \in \mathcal{R}_i$ and $\mathcal{F}^*[\mathcal{X}](x) = \mathcal{F}[\mathcal{X}](x) = \{A_i x\}$.

  From (1.73), we have

  $$\sup_{\varsigma \in \mathcal{F}^*[\mathcal{X}](x)} \frac{\partial V}{\partial x}\varsigma = x^T (A_i^T P + P A_i) x < 0, \tag{1.74}$$

  and then (1.70) is verified when $\text{card}(\mathcal{I}^*(x)) = 1$.

- If $\text{card}(\mathcal{I}^*(x)) > 1$, we define the set of vectors β such that

$$\Delta^*_{N(x)} = \{\beta \in \Delta_N : \beta_{(i)} = 0, i \notin \mathcal{I}^*(x)\}.$$

We have

$$\mathcal{F}^*[\mathcal{X}](x) = \text{Conv}\{A_i x, i \in \mathcal{I}^*(x)\} = \left\{\sum_{i=1}^{N} \beta_{(i)} A_i x : \beta \in \Delta^*_{N(x)}\right\} \tag{1.75}$$

consequently, we get

$$\sup_{\varsigma \in \mathcal{F}^*[\mathcal{X}](x)} \frac{\partial V}{\partial x}\varsigma = \sup_{\beta \in \Delta^*_{N(x)}} \left\{\sum_{i=1}^{N} \beta_{(i)} x^T (A_i^T P + P A_i) x\right\}. \tag{1.76}$$

Since the set $\Delta^*_{N(x)}$ is compact we have

$$\sup_{\beta \in \Delta^*_{N(x)}} \left\{\sum_{i=1}^{N} \beta_{(i)} x^T (A_i^T P + P A_i) x\right\} = \max_{\beta \in \Delta^*_{N(x)}} \left\{\sum_{i=1}^{N} \beta_{(i)} x^T (A_i^T P + P A_i) x\right\}. \tag{1.77}$$

Consider then $\beta \in \Delta^*_{N(x)}$. Using (1.75), since for all $i \in \mathcal{I}_N \setminus \mathcal{I}^*(x)$ we obtain $\beta_{(i)} = 0$, then from (1.73) for all $i \in \mathcal{I}^*(x)$ we have

$$\sum_{i=1}^{N} \beta_{(i)} x^T (A_i^T P + P A_i) x = \sum_{i \in \mathcal{I}^*(x)} \beta_{(i)} x^T (A_i^T P + P A_i) x < 0. \tag{1.78}$$

Equation (1.78) is verified for all vector $\beta \in \Delta_{N^*(x)}$, from (1.76), and (1.77) we have

$$\sup_{\varsigma \in \mathcal{F}^*[\mathcal{X}](x)} \frac{\partial V}{\partial x}\varsigma < 0, \tag{1.79}$$

for all $x \neq 0$ and $\text{card}(\mathcal{I}^*(x)) > 1$.

Then, the origin of the closed-loop system (4.119), (1.63) is globally asymptotically stable.

□

One should remark that for the case of switched systems with N subsystems Theorem 5 provides sufficient only stabilizability criteria.

The results presented above are based on the existence of a stable convex combination $A(\beta^*)$. However, in general, finding a stable convex combination is an NP-hard problem [14], [105]. There are classes of systems for which no stable convex combination exists and a stabilizing switching law may be obtained [76].

Case of switched affine systems

The pioneering results in the stabilization of switched affine systems have been provided in [16]. Based on the existence of a Hurwitz convex combination of the matrices A_i, this approach can be considered as a direct extension of the proposed methods in [33], [119] to the case of switched affine systems. This result is presented in the following while the solutions are understood in the sense of Filippov.

Theorem 6: (adapted from [16]) Consider system (1.41) with the switching law $\sigma(x) \in \mathcal{I}_N$ and the notations:

$$\begin{aligned} A(\beta) = & \sum_{i=1}^{N} \beta_{(i)} A_i, \\ b(\beta) = & \sum_{i=1}^{N} \beta_{(i)} b_i, \end{aligned} \tag{1.80}$$

with $\beta \in \Delta_N$. Suppose that there exists $\beta^* \in \Delta_N$ such that $A(\beta^*)$ is Hurwitz and $b(\beta^*) = 0$.

Then, the switching law

$$\sigma(x) \in \arg\min_{i \in \mathcal{I}_N} \{x^T(A_i^T P + PA_i)x + 2b_i^T Px\}, \tag{1.81}$$

with $P = P^T \succ 0$ a matrix satisfying

$$\sum_{i=1}^{N} \beta^*_{(i)}(A_i^T P + PA_i) < 0, \tag{1.82}$$

stabilizes globally asymptotically the origin $\bar{x} = 0$ of the switched system (1.41).

Proof. For the closed-loop system (1.41), (1.81), we define the regions $\mathcal{R}_i$ for $i \in \mathcal{I}_N$ by

$$\mathcal{R}_i = \{x \in \mathbb{R}^n : x^T(A_i^T P + PA_i)x + 2b_i^T Px < x^T(A_j^T P + PA_j)x + 2b_j^T Px, \forall j \in \mathcal{I}_N \setminus \{i\}\}. \tag{1.83}$$

We have

$$\begin{cases} \mathcal{R}_i \cap \mathcal{R}_j = 0, i \neq j, \\ \bigcup_{i \in \mathcal{I}_N} \bar{\mathcal{R}}_i = \mathbb{R}^n. \end{cases}$$

For all $x \in \mathbb{R}^n$ let us also define the set of minimizers where the switching controller (1.81) takes values as follows:

$$\mathcal{I}^*(x) = \{i \in \mathcal{I}_N : x^T(A_i^T P + PA_i)x + 2b_i^T Px \leq x^T(A_j^T P + PA_j)x + 2b_j^T Px, \forall j \in \mathcal{I}_N\}. \tag{1.84}$$

Then, the set-valued map $\mathcal{F}^*[\mathcal{X}_{aff}](x)$ for all $x \in \mathbb{R}^n$ and $i \in \mathcal{I}^*(x)$ is given by

$$\mathcal{F}^*[\mathcal{X}_{aff}](x) = \underset{i\in\mathcal{I}^*(x)}{\text{Conv}}\{A_i x + b_i\}. \tag{1.85}$$

We can remark that the set-valued map associated to the system (1.41) $\mathcal{F}[\mathcal{X}_{aff}](x)$ (computed using (1.5) and $\mathcal{F}^*[\mathcal{X}_{aff}](x)$ satisfy the following relation

$$\mathcal{F}[\mathcal{X}_{aff}](x) \subseteq \mathcal{F}^*[\mathcal{X}_{aff}](x). \tag{1.86}$$

The set-valued map $\mathcal{F}^*[\mathcal{X}_{aff}](x)$ is locally bounded, takes nonempty, compact and convex values. Hence, considering the inclusion (1.86), we have that both $\dot{x} \in \mathcal{F}^*[\mathcal{X}_{aff}](x)$ and $\dot{x} \in \mathcal{F}[\mathcal{X}_{aff}](x)$ admit at least one solution .

Let us consider the quadratic Lyapunov function $V(x) = x^T P x$ with $P = P^T \succ 0$ satisfying (1.82). We want to show that the origin of the closed-loop system (1.41), (1.81) is globally asymptotically stable. To show this, it is sufficient to prove that

$$\sup_{\varsigma\in\mathcal{F}[\mathcal{X}_{aff}](x)} \frac{\partial V}{\partial x}\varsigma < 0, \forall x \neq 0. \tag{1.87}$$

Considering (1.86), in order to prove (1.87), it is sufficient to show that

$$\sup_{\varsigma\in\mathcal{F}^*[\mathcal{X}_{aff}](x)} \frac{\partial V}{\partial x}\varsigma < 0, \forall x \neq 0. \tag{1.88}$$

From (1.82), we have:

$$\forall x \neq 0, \sum_{i=1}^{N} \beta^*_{(i)} x^T (A_i^T P + P A_i) x < 0. \tag{1.89}$$

Considering the condition $b(\beta^*) = \sum_{i=1}^{N} \beta^*_{(i)} b_i = 0$, we obtain

$$\forall x \neq 0, \sum_{i=1}^{N} \beta^*_{(i)} (x^T (A_i^T P + P A_i) x + 2b_i^T P x) < 0. \tag{1.90}$$

Since $\beta^* \in \Delta_N$, we deduce that

$$\forall x \neq 0, \exists i \in \mathcal{I}_N, x^T (A_i^T P + P A_i) x + 2b_i^T P x < 0, \tag{1.91}$$

and then

$$\forall i \in \mathcal{I}^*(x), \forall x \neq 0, x^T (A_i^T P + P A_i) x + 2b_i^T P x < 0. \tag{1.92}$$

From the last inequalities, we get (for all $x \neq 0$):

- if $\mathcal{I}^*(x) = \{i\}$, $x \in \mathcal{R}_i$ and $\mathcal{F}^*[\mathcal{X}_{aff}](x) = \mathcal{F}[\mathcal{X}_{aff}](x) = \{A_i x + b_i\}$. From (1.92) we have

$$\sup_{\varsigma \in \mathcal{F}^*[\mathcal{X}_{aff}](x)} \frac{\partial V}{\partial x}\varsigma = x^T(A_i^T P + PA_i)x + 2b_i^T Px < 0, \tag{1.93}$$

therefore (1.88) is verified in the case where $\text{card}(\mathcal{I}^*(x)) = 1$.

- if $\text{card}(\mathcal{I}^*(x)) > 1$, let us define the set $\Delta^*_{N(x)}$ of vectors β, such that

$$\Delta^*_{N(x)} = \{\beta \in \Delta_N : \beta_{(i)} = 0, i \notin \mathcal{I}^*(x)\}.$$

Using this we obtain

$$\mathcal{F}^*[\mathcal{X}_{aff}](x) = \text{Conv}\{A_i x + b_i, i \in \mathcal{I}^*(x)\} = \left\{\sum_{i=1}^{N} \beta_{(i)}\big(A_i x + b_i\big) : \beta \in \Delta^*_{N(x)}\right\}. \tag{1.94}$$

This leads to

$$\sup_{\varsigma \in \mathcal{F}^*[\mathcal{X}_{aff}](x)} \frac{\partial V}{\partial x}\varsigma = \sup_{\beta \in \Delta^*_{N(x)}} \left\{\sum_{i=1}^{N} \beta_{(i)}(x^T(A_i^T P + PA_i)x + 2b_i^T Px)\right\}. \tag{1.95}$$

Thanks to the fact that $\Delta^*_{N(x)}$ is compact, we have

$$\begin{aligned} &\sup_{\beta \in \Delta^*_{N(x)}} \left\{\sum_{i=1}^{N} \beta_{(i)}(x^T(A_i^T P + PA_i)x + 2b_i^T Px)\right\} \\ &= \max_{\beta \in \Delta^*_{N(x)}} \left\{\sum_{i=1}^{N} \beta_{(i)}(x^T(A_i^T P + PA_i)x + 2b_i^T Px)\right\}. \end{aligned} \tag{1.96}$$

with $\beta \in \Delta^*_N$. Since $\beta_{(i)} = 0$ for all $i \in \mathcal{I}_N \setminus \mathcal{I}^*(x)$, from (1.92) and using (1.94), for all $i \in \mathcal{I}^*(x)$ we have

$$\sum_{i=1}^{N} \beta_{(i)}(x^T(A_i^T P + PA_i)x + 2b_i^T Px) = \sum_{i \in \mathcal{I}^*(x)} \beta_{(i)}(x^T(A_i^T P + PA_i)x + 2b_i^T Px) < 0. \tag{1.97}$$

Since equation (1.97) is verified for all vector $\beta \in \Delta^*_{N(x)}$, from (1.95), and (1.96) we have

$$\sup_{\varsigma \in \mathcal{F}^*[\mathcal{X}_{aff}](x)} \frac{\partial V}{\partial x}\varsigma < 0, \tag{1.98}$$

for all $x \neq 0$ and $\text{card}(\mathcal{I}^*(x)) > 1$. Then, the origin of the closed-loop system (1.41), (1.81) is globally asymptotically stable.

□

Note that just as for Theorem 5, the conditions in Theorem 6 are only sufficient. Necessary and sufficient conditions for quadratic stabilization have been also provided for the case of switched affine systems with two subsystems in [16]. The article equaly presents a hybrid approach for the design of switching laws where the occurrence of sliding modes is avoided. This approach has been extended in [57] to the case of switched affine systems stabilization with a sampled-data switching law. Based on the use of Lyapunov-Krasovskii functionals, this method ensures the robustness with respect to sampling, implementation imperfections like jitters, and uncertainties. LMI conditions have been provided in order to optimize the Lyapunov function choice. An alternative approach for stabilization of switched affine systems is proposed in [113]. However, this approach also depends on the existence of a Hurwitz convex combination of the matrices A_i, $i \in \mathcal{I}_N$.

Note that, when there are more than two subsystems, it is possible to search a pair of subsystems satisfying the hypothesis of Theorem 6. However, the switching law depends on the equilibrium point selected by the operator. The matrix P must be computed for every selected equilibrium point. An approach which avoids this problem will be presented in the next section.

1.3.2 Existence of a common quadratic Lyapunov function

This approach is inspired from the theory of stability analysis of switched systems with arbitrary switching laws based on the existence of a common quadratic Lyapunov fuction [24], [77], [83]. It is also related to the existing results in the stability analysis and stabilization of linear differential inclusions [17]. The main idea is that if the individual subsystems of the switched system share a common quadratic Lyapunov function guaranteeing their stability then, there exists a state-dependent switching law which stabilizes globally asymptotically the switched system at the origin. The result is presented in the following while considering the solutions of the switched system in the sense of Filippov.

Theorem 7: (adapted from [26]) Consider system (1.41) with the switching law $\sigma \in \mathcal{I}_N$ and the notation

$$b(\beta) = \sum_{i=1}^{N} \beta_{(i)} b_i, \tag{1.99}$$

with $\beta \in \Delta_N$. Suppose that there exists $\beta^* \in \Delta_N$ such that $b(\beta^*) = 0$. Then, the origin of system (1.41) with the switching law

$$\sigma(x) \in \arg\min_{i \in \mathcal{I}_N} x^T P b_i, \tag{1.100}$$

with $P = P^T \succ 0$ satisfying

$$A_i^T P + P A_i \prec 0, \forall i \in \mathcal{I}_N, \tag{1.101}$$

is globally asymptotically stable.

Proof. For the closed-loop system (1.41), (1.100), we define the regions $\mathcal{R}_i$ for $i \in \mathcal{I}_N$ by

$$\mathcal{R}_i = \{x \in \mathbb{R}^n : x^T P b_i < x^T P b_j, \forall j \in \mathcal{I}_N \setminus \{i\}\}. \tag{1.102}$$

We have

$$\begin{cases} \mathcal{R}_i \cap \mathcal{R}_j = 0, i \neq j, \\ \bigcup\limits_{i \in \mathcal{I}_N} \bar{\mathcal{R}}_i = \mathbb{R}^n. \end{cases}$$

For all $x \in \mathbb{R}^n$ let us also define the set of minimizers where the switching controller (1.100) takes values as follows:

$$\mathcal{I}^*(x) = \{i \in \mathcal{I}_N : x^T P b_i \leq x^T P b_j, \forall j \in \mathcal{I}_N\}. \tag{1.103}$$

Then, the set-valued map $\mathcal{F}^*[\mathcal{X}_{aff}](x)$ for all $x \in \mathbb{R}^n$ and $i \in \mathcal{I}^*(x)$ is given by

$$\mathcal{F}^*[\mathcal{X}_{aff}](x) = \underset{i \in \mathcal{I}^*(x)}{\mathrm{Conv}} \{A_i x + b_i\}. \tag{1.104}$$

We can remark that the set-valued map associated to the system (1.41) $\mathcal{F}[\mathcal{X}_{aff}](x)$ (computed using (1.5) and $\mathcal{F}^*[\mathcal{X}_{aff}](x)$ satisfy the following relation

$$\mathcal{F}[\mathcal{X}_{aff}](x) \subseteq \mathcal{F}^*[\mathcal{X}_{aff}](x). \tag{1.105}$$

The set-valued map $\mathcal{F}^*$ is locally bounded and takes nonempty, compact and convex values then, we can deduce that both $\dot{x} \in \mathcal{F}^*[\mathcal{X}_{aff}](x)$ and $\dot{x} \in \mathcal{F}[\mathcal{X}_{aff}](x)$ admit at least one solution by considering relation (1.105).

Consider the quadratic Lyapunov function $V(x) = x^T P x$ with $P = P^T \succ 0$ satisfying (1.101). We want to show that the origin of the closed-loop system (1.41), (1.100) is globally asymptotically stable. To prove this, it is sufficient to show that

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}_{aff}](x)} \frac{\partial V}{\partial x}\varsigma < 0, \forall x \neq 0. \tag{1.106}$$

To prove this, considering (1.105), it is sufficient to show that

$$\sup_{\varsigma \in \mathcal{F}^*[\mathcal{X}_{aff}](x)} \frac{\partial V}{\partial x}\varsigma < 0, \forall x \neq 0. \tag{1.107}$$

From (1.101), for all $x \neq 0$ we have

$$x^T(A_i^T P + P A_i)x < 0, \forall i \in \mathcal{I}_N. \tag{1.108}$$

Considering the condition $b(\beta^*) = \sum_{i=1}^N \beta^*_{(i)} b_i = 0$, we obtain

$$\forall x \neq 0, \sum_{i=1}^N \beta^*_{(i)}(x^T(A_i^T P + P A_i)x + 2b_i^T P x) < 0. \tag{1.109}$$

Since $\beta^* \in \Delta_N$, we deduce that

$$\forall x \neq 0, \exists i \in \mathcal{I}_N, x^T(A_i^T P + P A_i)x + 2b_i^T P x < 0, \tag{1.110}$$

and then

$$\forall i \in \mathcal{I}^*(x), \forall x \neq 0, x^T(A_i^T P + P A_i)x + 2b_i^T P x < 0. \tag{1.111}$$

Let us define the set $\Delta^*_{N(x)}$ of vectors β, such that

$$\Delta^*_{N(x)} = \{\beta \in \Delta_N : \beta_{(i)} = 0, i \notin \mathcal{I}^*(x)\}$$

Using this we obtain

$$\mathcal{F}^*[\mathcal{X}_{aff}](x) = \text{Conv}\{A_i x + b_i, i \in \mathcal{I}^*(x)\} = \left\{\sum_{i=1}^N \beta_{(i)}(A_i x + b_i) : \beta \in \Delta^*_{N(x)}\right\}. \tag{1.112}$$

This leads to

$$\sup_{\varsigma \in \mathcal{F}^*[\mathcal{X}_{aff}](x)} \frac{\partial V}{\partial x}\varsigma = \sup_{\beta \in \Delta^*_{N(x)}} \left\{\sum_{i=1}^N \beta_{(i)}(x^T(A_i^T P + P A_i)x + 2b_i^T P x)\right\}. \tag{1.113}$$

Thanks to the fact that $\Delta^*_{N(x)}$ is compact, we have

$$\begin{aligned} &\sup_{\beta\in\Delta^*_{N(x)}}\left\{\sum_{i=1}^{N}\beta_{(i)}(x^T(A_i^TP+PA_i)x+2b_i^TPx)\right\} \\ &=\max_{\beta\in\Delta^*_{N(x)}}\left\{\sum_{i=1}^{N}\beta_{(i)}(x^T(A_i^TP+PA_i)x+2b_i^TPx)\right\}. \end{aligned} \tag{1.114}$$

with $\beta\in\Delta^*_N$. Since $\beta_{(i)}=0$ for all $i\in\mathcal{I}_N\setminus\mathcal{I}^*(x)$, from (1.111) and using (1.112), for all $i\in\mathcal{I}^*(x)$ we have

$$\sum_{i=1}^{N}\beta_{(i)}(x^T(A_i^TP+PA_i)x+2b_i^TPx)=\sum_{i\in\mathcal{I}^*(x)}\beta_{(i)}(x^T(A_i^TP+PA_i)x+2b_i^TPx)<0. \tag{1.115}$$

Since (1.115) is verified for all vector $\beta\in\Delta^*_{N(x)}$, from (1.113) and (1.114), we have

$$\sup_{\varsigma\in\mathcal{F}^*[\mathcal{X}_{aff}](x)}\frac{\partial V}{\partial x}\varsigma<0, \tag{1.116}$$

for all $x\neq 0$ and $\mathrm{card}(\mathcal{I}^*(x))>1$. Then, the origin of the closed-loop system (1.41), (1.100) is globally asymptotically stable. □

Remark 2: The switching law (1.100) presents some advantages :

- the switching surfaces are simple and amenable for practical purposes;
- the matrix $P=P^T\succ 0$ does not depend on the vector β^* associated to the equilibrium point $\tilde{x}^*$ selected by the designer. Then, the same P (when one exists) and therefore the same switching law can be used to stabilise any selected equilibrium point.

However, the existence of the switching law proposed in Theorem 7 depends on the feasibility of the set of N LMIs (1.101) with respect to the variable $P=P^T\succ 0$, which induces some conservatism. In addition, even if the equilibrium point of the switched affine system is a stable common equilibrium point of all the individual subsystems a common quadratic Lyapunov function satisfying (1.101) may not exist [19], [24]. In fact this condition is more conservative then the existence of a Hurwitz convex combination of the matrices A_i, $i\in\mathcal{I}_N$.

The switching law proposed in Theorem 7 allows the appearance of sliding modes. Therefore, in order to avoid this behaviour and ensure the implementability of this switching controller in real system applications, an extension of this method to the design of hybrid switching law has been provided in [1]. More precisely, in [1] the switched affine system has been modelled in the context of a hybrid dynamic system [43] and sliding modes are precluded by introducing hysteresis. The method has been equally extended to the case of stabilizing observer-based switching laws design in [120]. In the same sperit, assuming that the switched affine system is incrementally stable, a numerical approach based on the construction of symbolic abstraction have been provided in [42]. This approach allows the use of the techniques developed in the areas of supervisory control of discrete-event systems for the design of stabilizing state-dependent switching laws for switched affine systems.

1.3.3 Existence of a continuous stabilizer

The results presented above are based either on the existence of a Hurwitz convex combination of the matrices A_i or on the stability of all subsystems component of the switched system which are assumptions difficult to satisfy. In addition, the existing results treat the global stabilization case. However, there exist switched systems where only local stabilization can be possible. Recently, a design method of state-dependent switching laws allowing the local stabilization of switched affine systems at the origin even when no Hurwitz convex combination of the matrices A_i exists has been proposed in [55] and it is presented in the following.

It has been shown in [55] that the stabilization of switched affine systems can be reformulated as a classical stabilization problem of a nonlinear input-affine system with an input constrained to take values in a finite set of vectors. The result is presented in the following.

Proposition 1: [55] Consider the switched affine system (1.41), $\beta^* \in \Delta_N$ a vector satisfying

$$\sum_{i=1}^{N} \beta^*_{(i)} b_i = 0, \qquad (1.117)$$

and a set of vectors $\mathcal{V}(\beta^*) = \{v_i, i \in \mathcal{I}_N\} \subset \mathbb{R}^m$, where $m = N-1$, $v_i = M(\psi_i - \beta^*)$, with $i \in \mathcal{I}_N$, and $M = \begin{bmatrix} I_{m\times m} \ , \ 0_{m\times 1}\end{bmatrix}$. Then, system (1.41) can be rewritten as an interconnection of the nonlinear input-affine system of the form

$$\dot{x} = f(x) + G(x)u, u \in \mathbb{R}^m, \tag{1.118}$$

with the discontinuous controller

$$u = k^d(x), k^d : \mathbb{R}^n \longrightarrow \mathcal{V}(\beta^*) \tag{1.119}$$

where

$$k^d(x) = v_{\sigma(x)}, \tag{1.120}$$

and the functions $f(x) = \sum_{i=1}^N \beta^*_{(i)} A_i x = A(\beta^*)x$, $G(x) = \begin{bmatrix} g_1(x) & , \dots, & g_m(x)\end{bmatrix}$ with $g_j(x) = (A_j - A_N)x + (b_j - b_N), j \in \mathcal{I}_m$.

This result means that designing a switching law σ stabilizing the switched affine system (1.41) at the origin leads to constructing a discontinuous controller k^d such that the nonlinear input-affine system (1.118) is locally asymptotically stable at the origin.

Using the reformulation of the switched affine system as a nonlinear input-affine system and the existence of a continuous controller stabilizing (locally or globally) the nonlinear system, it has been shown in [55] that a state-dependent switching law can be derived. In other words: the existence of a continuous controller implies the existence of a switching law which can be designed by embedding locally the behaviour of a continuous controller.

Theorem 8: [55] Consider system (1.41) (or equivalently system (1.118)). Suppose that

1. There exists $\beta^* \in \Delta_N$, $\beta^*_{(i)} > 0$, $\forall i \in \mathcal{I}_N$ such that $\sum_{i=1}^N \beta^*_{(i)} b_i = 0$.

2. System (1.118) is locally asymptotically stabilizable at the origin by the continuous controller $u(x(t)) = k(x)$, with $k(0) = 0$.

Then there exists a $\mathcal{C}^\infty$ function defined in the set $\mathcal{B}(0, r)$, $r > 0$, $V(0) = 0$, $V(x) > 0$, $\forall x \neq 0$, a positive scalar $\gamma \in (0, r]$, and a switching law

$$\sigma(x) \in \arg\min_{i\in\mathcal{I}_N} 2x^T P(A_i x + b_i). \tag{1.121}$$

such that
$$\max_{y \in \mathcal{F}(x)} \frac{\partial V}{\partial x} y < 0, \tag{1.122}$$
is verified for all $x \in \mathcal{B}(0, \gamma) \setminus \{0\}$.

Then, the switched system (1.41) with the switching law (1.121) is locally asymptotically stable at the origin.

This methodology can be interpreted as an approach based on the existence of a state-dependent convex combination of the matrices A_i. The restriction concerning the existence of a Hurwitz convex combination can be relaxed and easily avoided. However, the approach can only guarantee local asymptotic stabilization.

A constructive method based on LMIs that allows the design of the switching law and the estimation of the domain of attraction has also been proposed in [55]. Using the result of Proposition 1, system (1.41) can be rewritten as follows:
$$\dot{x} = A(\beta^*)x + Bu + D(u)x, \tag{1.123}$$
with $B = \Big[b_1 - b_N, \ldots, b_{N-1} - b_N\Big]$, $D(u) = (G(x) - B)u$ and $u \in \mathcal{V}(\beta^*)$. A direct consequence of the model (1.123) is the fact that if the pair $(A(\beta^*), B)$ is stabilizable then the switched system is locally stabilizable at the origin.

Using the fundamental theorem of polytopes [122], the set $\mathrm{Conv}\{\mathcal{V}(\beta^*)\}$ is a convex polytope described by a finite number n_l of vectors $l_i \in \mathbb{R}^m$, $i \in \mathcal{I}_{n_l}$ such that
$$\mathrm{Conv}\{\mathcal{V}(\beta^*)\} = \{u \in \mathbb{R}^N : l_i^T u \leq 1, i \in \mathcal{I}_{n_l}\}. \tag{1.124}$$
Assuming that the pair $(A(\beta^*), B)$ is stabilizable and considering the term $D(u)$ as a perturbation, an LMI approach allowing the design of state-dependent switching laws which ensures the local exponential stability of the closed loop switched affine system at the origin and providing an ellipsoidal estimation of the domain of attraction is presented in the following proposition:

Proposition 2: [55] Consider the switched affine system (1.41) and (1.118) with u taking values in $\mathcal{V}(\beta^*)$, and satisfying (1.124). If there exist a matrix $Q^{-1} = P = P^T \succ 0$ and a positive scalar $\chi > 0$ such that
$$(A(\beta^*) + D(v_i))^T Q + Q(A(\beta^*) + D(v_i)) - \chi BB^T \prec -2\alpha Q, \forall i \in \mathcal{I}_N, \tag{1.125}$$

$$\begin{bmatrix} rI & I \\ I & Q \end{bmatrix} \succ 0, \tag{1.126}$$

and

$$\begin{bmatrix} 1 & \frac{\chi}{2} l_j^T B^T \\ \frac{\chi}{2} B l_j & Q \end{bmatrix} \succ 0, \forall j \in \mathcal{I}_{n_l} \tag{1.127}$$

where $\alpha > 0$ and $r > 0$ are positive scalars. Then, the switched affine system (1.41) with the switching law

$$\sigma(x) \in \arg\min_{i \in \mathcal{I}_N} x^T Q^{-1}(A_i x + b_i), \tag{1.128}$$

is locally asymptotically stable at the origin. Moreover, the estimated domain of attraction

$$\mathcal{E}(P,1) = \{x \in \mathbb{R}^n : x^T P x \leq 1\}, \tag{1.129}$$

includes the ball $\mathcal{B}(0, \frac{1}{\sqrt{r}})$ and there exists a positive scalar κ such that

$$|x(t)|^2 \leq \kappa e^{-2\chi t} |x(0)|, \forall x(0) \in \mathcal{E}(P,1). \tag{1.130}$$

The feasibility of the LMI conditions (1.125) and (1.127) ensures that the switching law (1.128) emulates the behaviour of the stabilizing continuous controller $k(x) = Kx$ with $K = -\frac{\chi}{2} B^T Q^{-1}$ in the domain of attraction $\mathcal{E}(P,1)$. i.e, the derivative of the quadratic Lyapunov function along the Filippov solutions of the closed-loop switched affine system (1.41), (1.128) is negative for all $x \in \mathcal{E}(P,1) \subset \text{Conv}\{\mathcal{V}(\beta*)\}$. The LMI (1.126) implies the inclusion $\mathcal{B}(0, \frac{1}{\sqrt{r}}) \subset \mathcal{E}(P,1)$ and can be used to optimize the domain of attraction by solving the standard convex optimization problem

$$\inf r \quad \text{subject to (1.125), (1.126), (1.127).}$$

In addition, the feasibility of the set of LMIs (1.125), (1.126), (1.127) ensures that the trajectories of the system starting in the domain of attraction $\mathcal{E}(P,1)$ converges to the origin with a decay rate χ equal to the decay rate guaranteed by the continuous controller $k(x) = Kx$. Finally, we may remark that these conditions are only sufficient.

1.4 Further notes on stability and stabilization of switched systems

The closed-loop switched affine system represents a piecewise affine system. There exist elegant results on the stability analysis and stabilization of this class of systems - see for instance [37], [65], [88], [94], [96], [97], [98], and the references therein. However, it is not clear how they can be used in a constructive manner for designing stabilizing switching laws.

The literature concerning the particular case of switched linear systems is quite extensive. We can point to the surveys of [28], [76], [78], [103], [106] and the references therein for detailed treatment of this particular case. We mention however some recent results that have not been included in these surveys. When the system admits Caratheodory solutions (no sliding mode is possible) we point to the results based on Lyapunov Metzler inequalities given in [41]. However, it is not clear if the results hold when all types of sliding dynamics are represented [53], [75]. We can equaly mention the work in [22], [35], [34], and [67] for the discrete-time case and also the work in [38] and [62] based on sliding modes conditions.

Next to the results presented here, some methods have been provided in the literature for specific devices where the underlying model is a switched affine system. See for example the sliding mode approach presented in [104] for power converters. Other control design methodologies using discrete time models [51], model predictive control (MPC) [11], or optimal control [12], [87], [93], [101] can equally be interesting for the reader.

1.5 Conclusion

The goal of this chapter is to provide an overview of the existing stabilization methods of switched affine systems. Some previous results did not specify the concept of solutions that they use, we have then reproved them using the Filippov formalism. The research on switched affine systems is still widely open. In this thesis we will particularly focus on methods using the nonlinear affine formulation presented in Section 1.3.3. More precisely, in Chapter 2, a switching controller is designed in order to ensure the local asymptotic stability of the closed-loop switched affine system at the origin. A numerical method based on LMI conditions allowing the design of nonlinear switching surfaces is provided by using non-quadratic Lur'e type Lyapunov functions. Local asymptotic stability of the closed-loop system at the origin is guaranteed in a non-ellipsoidal domain of attraction. Furthermore, sufficient conditions for the global asymptotic stability of switched affine systems are provided even though no Hurwitz convex combination of the evolution matrices of the individual subsystems exist. Moreover, a general result is proposed for the class of nonlinear input-affine systems. Finally, the developed method is particularized to the simpler case of LTI systems stabilization with a relay controller. In Chapter 3, we study the stabilization problem for the class of switched affine systems with a disturbed state-dependent switching law. Since the states measurements are in general subject to perturbations and noises, we propose a robust switching law design method. Qualitative conditions for the stability of the closed-loop switched system are given. A constructive method based on LMIs allowing the design of a stabilizing switching law and the optimization of the size of the domain of attraction or of the chattering zone is provided. Since, Linear Time Invariant (LTI) systems with relay controllers are a simpler class of switched affine systems, the obtained results are then particularized to deal with their stabilization problem. Since the state variables in real systems are not always fully available to measurements, Chapter 4 deals with the stabilization problem by an observer-based switching control. A general result is proposed for the case of switched affine systems. An observer-based switching controller is designed in order to ensure the local asymptotic stability of the closed-loop system. Both quadratic and non-quadratic Lyapunov functions are used to derive linear and nonlinear switching surfaces dependent on the estimated state while using a Luenberger observer. Constructive methods based on LMI conditions are given in order to allow a numerical implementation of the proposed approaches. Estimations of ellipsoidal and non-ellipsoidal domains of attraction are provided. Moreover, LMI conditions which allow the design of nonlinear switching surfaces dependent on the estimated state ensuring the global asymptotic stability of the closed-loop switched affine system at the origin are provided. Finally, the result is applied to the particular case of LTI systems with an observer-based relay feedback control. Finally, we provide a separation principle for both LTI systems with relay controllers and switched affine systems while, to the best of our knowledge, the separation principle exists only for systems with continuous controller [99] and thus can not be applied to these classes of systems.

Chapter 2

Non-quadratic stabilization

In this chapter, we consider the stabilization problem of switched affine systems using non-quadratic Lyapunov functions. A general result is proposed for the case of nonlinear input-affine systems. A switching controller is designed in order to ensure the local asymptotic stability of the closed-loop system at the origin. Then, the result is applied for the class of switched affine systems. A numerical method based on LMI conditions allowing the design of nonlinear switching surfaces is provided by using non-quadratic Lur'e type Lyapunov functions. Local asymptotic stability of the closed-loop system at the origin is guaranteed in a non-ellipsoidal domain of attraction. Moreover, sufficient conditions for the global stability of switched affine systems are provided even though no Hurwitz convex combination of the evolution matrices of the individual subsystems exist. Finally, the developed method is particularized to the simpler case of LTI systems stabilization with a relay controller.

2.1 Preliminaries and problem statement

Consider the following system

$$\dot{x} = Ax + \sum_{k=1}^{m} (\mathcal{N}_k x + b_k) u_{(k)}, \tag{2.1}$$

with $x \in \mathbb{R}^n$ and $u_{(k)}$ the k-th component of the input u. The input u is only allowed to take values in the set $\mathcal{V} = \{v_1, \ldots, v_N\} \subset \mathbb{R}^m$. $A \in \mathbb{R}^{n\times n}$, $B = \begin{bmatrix} b_1, & \ldots, & b_m \end{bmatrix} \in \mathbb{R}^{n\times m}$, and $\mathcal{N}_k \in \mathbb{R}^{n\times n}$ are the matrices describing the system. It has been demonstrated in [55] (see Proposition 1 in Chapter 1) that the class of switched system (2.1) is quite general in the sense that any switched affine system

$$\begin{aligned} &\dot{x} = \tilde{A}_{\sigma(x)}x + \tilde{b}_{\sigma(x)}, \\ &\sigma \in \mathcal{I}_N, \end{aligned} \tag{2.2}$$

with $\tilde{A}_i \in \mathbb{R}^{n\times n}$ and $\tilde{b}_i \in \mathbb{R}^{n\times 1}$ the matrices describing the subsystems, can be represented in the form (2.1) (see Proposition 1 in [55]).

In the sequel we assume that:

A-1 The pairs $(A(v_i), B)$, for all $i \in \mathcal{I}_N$ with $A(v_i) = A + \sum_{k=1}^{m} \mathcal{N}_k v_{i(k)}$ are simultaneously quadratically stabilizable by a linear state feedback $k(x) = Kx$. This means that there exist matrices K and $P = P^T \succ 0$ and a positive scalar α such that

$$A(v_i)_{cl}^T P + PA(v_i)_{cl} \leq -2\alpha P, \forall i \in \mathcal{I}_N, \tag{2.3}$$

with $A(v_i)_{cl} = A(v_i) + BK$.

A-2 The set $\text{int}\{\text{Conv}\{\mathcal{V}\}\}$ is nonempty and the null vector is contained inside ($0 \in \text{int}\{\text{Conv}\{\mathcal{V}\}\}$).

Note that for any finite set $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ there exists a finite number n_l of vectors $l_i \in \mathbb{R}^{1\times m}$, $i \in \mathcal{I}_{n_l}$ such that

$$\text{Conv}\{\mathcal{V}\} = \{u \in \mathbb{R}^m : l_i u \leq 1, \forall i \in \mathcal{I}_{n_l}\}. \tag{2.4}$$

Note also that typical control sets $\mathcal{V}$ are often of the form

$$\mathcal{V} = \text{Vert}\{\mathcal{P}(c)\}, \tag{2.5}$$

where the hyper-rectangle $\mathcal{P}(c)$, with c a strictly positive vector, is given by

$$\mathcal{P}(c) = \left\{ u = \begin{bmatrix} u_{(1)} \\ \vdots \\ u_{(m)} \end{bmatrix} \in \mathbb{R}^m : \left|u_{(k)}\right| \leq c_{(k)}, \forall k \in \mathcal{I}_m \right\}. \tag{2.6}$$

However, we want to keep the problem formulation as general as possible. Note that since A.2 holds, even for more general sets $\mathcal{V}$ there exists a vector $c \in \mathbb{R}^m$ such that the hyper-rectangle satisfies $\mathcal{P}(c) \subseteq \text{Conv}\{\mathcal{V}\}$. In the sequel, we will consider such a vector c and use the notation (2.6) to prove the results.

This section deals with the stabilization problem of system (2.1). We consider a controller given by

$$u(x) \in \arg\min_{v \in \mathcal{V}} \Gamma(x, v), \tag{2.7}$$

where the mapping $\Gamma : \mathbb{R}^n \times \mathcal{V} \longrightarrow \mathbb{R}$ characterizes the switching surfaces.

This formulation encompasses the classical sign function in the classical relay feedback controller of LTI systems. Indeed, if $\mathcal{V} = \{-v, v\}$ with $v > 0$ and $\Gamma(x, v) = x^T \Psi v$ for some $\Psi \in \mathbb{R}^{n \times m}$, we get

$$u(x) \in -v\text{sign}(\Psi^T x) = \begin{cases} \{v\} & \text{if} \quad \Psi^T x < 0, \\ \{-v, v\} & \text{if} \quad \Psi^T x = 0, \\ \{-v\} & \text{if} \quad \Psi^T x > 0. \end{cases} \tag{2.8}$$

The interconnection (2.1), (2.7) can be rewritten as

$$\dot{x} = \mathcal{X}(x) = Ax + \sum_{k=1}^{m} \mathcal{N}_k x u_{(k)} + Bu(x). \tag{2.9}$$

Note that this is a differential equation with a discontinuous right hand side [21], [36]. Therefore, considering the solutions in the sense of Filippov, we are interested in the study of the following problem:

Problem 1. Given system (2.1) under Assumptions A-1 and A-2 and the set $\mathcal{V}$, design a switching law (2.7) such that the closed loop system is locally asymptotically stable in some domain $\mathcal{D}$.

In [55] (see Proposition 2, Section 1.3.3), assuming A-2 and

A-1' There exist a positive definite matrix Q and positive scalars χ and α such that

$$A(v_i)Q + QA(v_i)^T - \chi BB^T \preceq -2\alpha Q, \forall i \in \mathcal{I}_N, \tag{2.10}$$

a switching law ensuring the local exponential stability of the closed-loop system at the origin with a decay rate α is derived by embedding locally the behaviour of the continuous controller $k(x) = Kx$ with $K = -\frac{\chi}{2} B^T Q^{-1}$.

Note that A-1' is equivalent to A-1 (this is similar to what is presented in [17] page 112). An ellipsoidal estimation of the domain of attraction is equally given using a quadratic Lyapunov function $V(x) = x^T Px$ with $P = Q^{-1}$: $\mathcal{E}(P, \gamma)$ where γ is computed such that $\mathcal{E}(P, \gamma)$ does not cross the convex hull

$$\mathcal{C}_v(K) = \{x \in \mathbb{R}^n : l_i Kx \leq 1, \forall i \in \mathcal{I}_{n_l}\}, \tag{2.11}$$

where $l_i, i \in \mathcal{I}_{n_l}$ are vectors defined in (2.4), which leads to $\gamma \leq \min_{i\in\mathcal{I}_{n_l}} (l_i KQK^T l_i^T)^{-1}$. Nevertheless, considering a quadratic Lyapunov function and an ellipsoidal estimation of the domain of attraction introduces some conservatism in the proposed method [13].

Here we would like to provide a more general design procedure using non-quadratic Lyapunov functions to compute nonlinear switching surfaces and larger non-ellipsoidal domains of attraction. Moreover, a numerical approach based on LMI conditions is developed in order to derive state-dependent switching laws ensuring the global asymptotic stability of switched affine systems at the origin even though no Hurwitz convex combination of the matrices $\tilde{A}_i, i \in \mathcal{I}_N$ exist i.e, the matrix A of system (2.1) is not Hurwitz. As it is explained in this section and in Section 1.3.3 there is an explicit relation between nonlinear input-affine systems with discontinuous controller and switched affine systems. Therefore, in the next section a general theoretical result on the stabilization of this class of systems is provided.

2.2 A general theoretical result

Before considering the case of switched affine systems, here we provide a general result for input-affine nonlinear systems stabilization with a relay controller.

The method uses the existence of a locally stabilizing continuous control to design the switching controller.

Consider the following nonlinear input-affine system

$$\dot{x} = f(t, x) + g(t, x)u(t, x) = \bar{\mathcal{X}}(t, x), \tag{2.12}$$

where $f : \mathbb{R}_+ \times \mathcal{D} \longrightarrow \mathbb{R}^n$, $g : \mathbb{R}_+ \times \mathcal{D} \longrightarrow \mathbb{R}^{n\times m}$ are Lipschitz functions in x on $\mathbb{R}_+ \times \mathcal{D}$ and piecewise continuous in t, and where $\mathcal{D} \subset \mathbb{R}^n$ is a domain such that $0 \in \text{int}\{\mathcal{D}\}$. The input u takes values in the set $\mathcal{V} = \{v_1, \ldots, v_N\} \subset \mathbb{R}^m$.

Using the definitions presented in Chapter 1, we are now able to provide sufficient conditions for the stabilization of the nonlinear time-varying system (2.12) by relay controller.

Theorem 9: Consider the nonlinear system (2.12) and Assumption A-2. Assume that there exists a continuous controller $k(t,x)$ such that $k(t,x) \in \text{Conv}\{\mathcal{V}\}$ and $g(t,x)k(t,x)$ is Lipschitz in x on $\mathbb{R}_+ \times \mathcal{D}$ and piecewise continuous in t, for all $x \in \mathcal{D}$ and for all $t \geq 0$. Assume that there exists a continuously differentiable function $V : \mathbb{R}_+ \times \mathcal{D} \longrightarrow \mathbb{R}$ such that

$$W_1(x) \leq V(t,x) \leq W_2(x), \tag{2.13}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\Big\{f(t,x) + g(t,x)k(t,x)\Big\} \leq -W_3(x), \forall t \geq 0, \forall x \in \mathcal{D}, \tag{2.14}$$

where W_1, W_2 and W_3 are continuous positive definite functions on $\mathcal{D}$. Then, the origin of system (2.12) with the discontinuous controller

$$u(t,x) = k^d(t,x) \in \arg\min_{v \in \mathcal{V}} \frac{\partial V}{\partial x} g(t,x)v, \forall t \geq 0, \forall x \in \mathcal{D}, \tag{2.15}$$

is locally uniformly asymptotically stable. Moreover, the set $\mathcal{L}_V(\eta^*) = \{x \in \mathbb{R}^n : V(t,x) \leq \eta^*, \forall t \geq 0\} \subseteq \mathcal{D}$ with $\eta^* = \max\{\eta > 0 : \mathcal{L}_V(\eta) \subseteq \mathcal{D}\}$ is an estimation of the domain of attraction.

Proof. We would like to prove that the origin of the closed-loop system

$$\dot{x} = f(t,x) + g(t,x)k^d(t,x) \tag{2.16}$$

is locally uniformly asymptotically stable when solutions are understood in the sense of Filippov. Since $\frac{\partial V}{\partial x}$ and $g(t,x)$ are continuous, then $\text{Conv}\Big\{\arg\min_{v \in \mathcal{V}} \frac{\partial V}{\partial x} g(t,x)v\Big\}$ is upper semicontinuous. Therefore, we consider the following differential inclusion [36], [47], [85]

$$\dot{x} \in \mathcal{F}(t,x), \tag{2.17}$$

with

$$\mathcal{F}(t,x) = \text{Conv}\left\{f(t,x) + g(t,x)u^d : u^d \in \arg\min_{v \in \mathcal{V}} \frac{\partial V}{\partial x} g(t,x)v\right\}. \tag{2.18}$$

The origin of the differential inclusion (2.17) is locally uniformly asymptotically stable if for a given Lyapunov function $V(t,x)$ we have

$$\sup_{\varsigma \in \mathcal{F}(t,x)} \left\{\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\varsigma\right\} \leq -W_3(x), \forall t \geq 0, \forall x \in \mathcal{D}. \tag{2.19}$$

Thanks to the fact that $k(t,x) \in \text{Conv}\{\mathcal{V}\}$ for all $t \geq 0$ and for all $x \in \mathcal{D}$, there exist N scalars $\rho_i(t,x) \geq 0$, $i \in \mathcal{I}_N$ with $\sum_{i=1}^{N} \rho_i(t,x) = 1$ such that

$$k(t,x) = \sum_{i=1}^{N} \rho_i(t,x) v_i. \tag{2.20}$$

Considering (2.20) and replacing $k(t,x)$ by $\sum_{i=1}^{N} \rho_i(t,x) v_i$, in (2.14) we obtain

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\left(f(t,x) + g(t,x)\sum_{i=1}^{N} \rho_i(t,x) v_i\right) + W_3(x) \leq 0. \tag{2.21}$$

Then, using the fact that $\sum_{i=1}^{N} \rho_i(t,x) = 1$, we get

$$\sum_{i=1}^{N} \rho_i(t,x)\left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\Big(f(t,x) + g(t,x) v_i\Big) + W_3(x)\right) \leq 0, \tag{2.22}$$

for all $t \geq 0$ and for all $x \in \mathcal{D}$.

Let us define the function

$$F(t,x,v_i) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\Big(f(t,x) + g(t,x) v_i\Big) + W_3(x). \tag{2.23}$$

Since $\rho_i(t,x) \geq 0, \forall i \in \mathcal{I}_N$ and from inequality (2.22), it can be inferred that the inequality

$$F(t,x,v_i) \leq 0, \forall x \in \mathcal{D}, \forall t \geq 0 \tag{2.24}$$

holds at least for one index $i(t,x) \in \mathcal{I}_N$. We can then define the switching controller as follows

$$k^d(t,x) \in \arg\min_{v_i \in \mathcal{V}} F(t,x,v_i) = \arg\min_{v_i \in \mathcal{V}} \frac{\partial V}{\partial x} g(t,x) v_i. \tag{2.25}$$

Let us define the set of minimizers corresponding to the controller (2.25) as follows

$$\mathcal{I}^*(t,x) = \left\{ i \in \mathcal{I}_N : \frac{\partial V}{\partial x} g(t,x)(v_j - v_i) \geq 0, \forall j \in \mathcal{I}_N \right\}. \tag{2.26}$$

Consider the set valued map given by

$$\mathcal{F}^*(t,x) = \text{Conv}\{f(t,x) + g(t,x) v_i : i \in \mathcal{I}^*(t,x)\}. \tag{2.27}$$

Since

$$\arg\min_{v \in \mathcal{V}} \left\{ \frac{\partial V}{\partial x} g(t,x) v \right\} \subseteq \{v_i : i \in \mathcal{I}^*(t,x)\} \tag{2.28}$$

is satisfied, according to the definition of $\mathcal{I}^*(t,x)$ in (2.26), one can show that

$$\mathcal{F}(t,x) \subseteq \mathcal{F}^*(t,x), \tag{2.29}$$

with $\mathcal{F}(t,x)$ defined in (2.18) and $\mathcal{F}^*(t,x)$ defined in (2.27).

Considering the relation (2.29), in order to prove (2.19), it is sufficient to show that

$$\sup_{\varsigma \in \mathcal{F}^*(t,x)} \left\{ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\varsigma \right\} \leq -W_3(x),\ \forall t \geq 0,\ \forall x \in \mathcal{D}. \tag{2.30}$$

Let us define the following set of vectors

$$\Delta^*(t,x) = \{\beta \in \Delta_N : \beta_{(i)} = 0, \forall i \in \mathcal{I}_N \setminus \mathcal{I}^*(t,x)\}. \tag{2.31}$$

We have then

$$\begin{aligned} \mathcal{F}^*(t,x) &= \mathrm{Conv}\{f(t,x) + g(t,x)v_i : i \in \mathcal{I}^*(t,x)\} \\ &= \left\{ f(t,x) + g(t,x)\sum_{i=1}^{N} \beta_{(i)} v_i : \beta \in \Delta^*(t,x) \right\}. \end{aligned} \tag{2.32}$$

Consequently, since $\Delta^*(t,x)$ is compact, we obtain

$$\begin{aligned} \sup_{\varsigma \in \mathcal{F}(t,x)} \left\{ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\varsigma \right\} &\leq \sup_{\varsigma \in \mathcal{F}^*(t,x)} \left\{ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\varsigma \right\} \\ &= \sup_{\beta \in \Delta^*(t,x)} \left\{ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\left(f(t,x) + g(t,x)\sum_{i=1}^{N} \beta_{(i)} v_i \right) \right\} \\ &= \max_{\beta \in \Delta^*(t,x)} \left\{ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\left(f(t,x) + g(t,x)\sum_{i=1}^{N} \beta_{(i)} v_i \right) \right\}. \end{aligned} \tag{2.33}$$

Therefore, to prove (2.19) it is sufficient to show that

$$\max_{\beta \in \Delta^*(t,x)} \left\{ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\left(f(t,x) + g(t,x)\sum_{i=1}^{N} \beta_{(i)} v_i \right) \right\} \leq -W_3(x), \tag{2.34}$$

for all $t \geq 0$ and for all $x \in \mathcal{D}$.

Let $\beta \in \Delta^*(t,x)$. Since $\beta_{(i)} = 0$ for all $i \in \mathcal{I}_N \setminus \mathcal{I}^*(t,x)$, from (2.23), (2.24) (which is verified at least for one $i \in \mathcal{I}_N$, and then for all $i \in \mathcal{I}^*(t,x)$), and (2.26), we can deduce that

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\left(f(t,x) + g(t,x)\sum_{i=1}^{N} \beta_{(i)} v_i \right) &= \sum_{i=1}^{N} \beta_{(i)} \left\{ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\Big(f(t,x) + g(t,x)v_i \Big) \right\} \\ &\leq -W_3(x). \end{aligned} \tag{2.35}$$

Therefore, (2.34) (and thus (2.19)) is satisfied, and the origin of system (2.12) with the controller (2.15) is locally uniformly asymptotically stable.

In addition, the level set $\mathcal{L}_V(\eta) = \{x \in \mathbb{R}^n : V(t,x) \leq \eta, \forall t \in \mathbb{R}_+\}$ of the Lyapunov function V can be considered as an inner estimation of the domain of attraction if η is such that $\mathcal{L}_V(\eta) \subseteq \mathcal{D}$. □

The control principle given in Theorem 9 can be used to provide constructive methods of stabilizing state-dependent switching laws design. Using a non-quadratic Lyapunov function, a tractable LMI approach providing an estimation of the domain of attraction and stabilizing nonlinear switching surfaces is given in the following for switched affine systems.

2.3 Stabilization of switched affine systems

As follows we particularise the result of Theorem 9 to the case of switched affine systems. Our first objective is to enlarge the domain of attraction with respect to the result provided in [55] by using non-quadratic Lyapunov functions. The second goal of this section consists on providing a constructive approach of state-dependent switching laws ensuring the global asymptotic stability of the class of switched affine systems at the origin.

2.3.1 Switching law design for local stabilization

In this section we provide numerical tools for nonlinear switching surfaces design using switching Lyapunov functions of the form

$$V(x) = x^T P x - 2\sum_{k=1}^{m} \int_0^{K_{(k)}x} \phi_{(k)}(s)\Omega_{(k,k)} ds, \tag{2.36}$$

with K satisfying (2.10), $P \in \mathbb{R}^{n\times n}$ a symmetric positive definite function, Ω a diagonal positive definite matrix, and $\phi : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ a nonlinear function defined for all $y \in \mathbb{R}^m$ as $\phi(y) = \begin{bmatrix}\phi_{(1)}(y_{(1)}), & \ldots, & \phi_{(m)}(y_{(m)})\end{bmatrix}^T \in \mathbb{R}^m$, with

$$\phi_{(k)}(\sigma) = \begin{cases} c_{(k)} - \sigma & \text{if} \quad \sigma > c_{(k)}, \\ 0 & \text{if} \quad -c_{(k)} \leq \sigma \leq c_{(k)}, \quad \forall \sigma \in \mathbb{R}. \\ -c_{(k)} - \sigma & \text{if} \quad \sigma < -c_{(k)}. \end{cases} \tag{2.37}$$

This Lyapunov class of functions has been used for various nonlinearity types (see for instance [17], [74], [110] and references therein). They have the following properties.

Lemma 1: [111] Consider $w_1 \in \mathbb{R}^m$ and $w_2 \in \mathbb{R}^m$. If $(w_1 - w_2) \in \mathcal{P}(c)$, with $\mathcal{P}(c)$ defined in (2.6), then

$$\phi(w_1)^T M(\phi(w_1) + w_2) \leq 0, \tag{2.38}$$

for any diagonal positive definite matrix $M \in \mathbb{R}^{m\times m}$.

Lemma 2: [60] The Lur'e function (2.36) satisfies the inequality

$$x^T Px \leq V(x) \leq x^T(P + K^T \Omega K)x, \forall x \in \mathbb{R}^n. \tag{2.39}$$

Lemma 3: For any $y \in \mathbb{R}^m$, $y + \phi(y) \in \mathcal{P}(c)$, with $\mathcal{P}(c)$ defined in (2.6).

Proof. Let $y \in \mathbb{R}^m$ and $i \in \mathcal{I}_m$. Three cases arise:

1. If $y_{(i)} > c_{(i)}$, then $y_{(i)} + \phi_{(i)}(y_{(i)}) = c_{(i)}$.
2. If $-c_{(i)} \leq y_{(i)} \leq c_{(i)}$, then $y_{(i)} + \phi_{(i)}(y_{(i)}) = y_{(i)}$.
3. If $y_{(i)} < -c_{(i)}$, then $y_{(i)} + \phi_{(i)}(y_{(i)}) = -c_{(i)}$.

Therefore, for any $i \in \mathcal{I}_m$, $|y_{(i)} + \phi_{(i)}(y_{(i)})| \leq c_{(i)}$, and thus $y + \phi(y) \in \mathcal{P}(c)$. □

Considering these properties we are able to develop the following result.

Theorem 10: Consider system (2.1) and assume that A-1' (or equivalently A-1) and A-2 hold. If there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, two diagonal positive definite matrices $\Omega \in \mathbb{R}^{m\times m}$ and $M \in \mathbb{R}^{m\times m}$, a matrix $\Upsilon \in \mathbb{R}^{m\times n}$, and a strictly positive vector $\tau \in \mathbb{R}^m$ such that for all $i \in \mathcal{I}_N$

$$\begin{bmatrix} A(v_i)_{cl}^T P + PA(v_i)_{cl} & PB - \Upsilon^T - A(v_i)_{cl}^T K^T \Omega \\ * & -2M - \Omega KB - (\Omega KB)^T \end{bmatrix} \prec 0, \tag{2.40}$$

and

$$\begin{bmatrix} P & M_{(k,k)}K_{(k)}^T - \Upsilon_{(k)}^T \\ M_{(k,k)}K_{(k)} - \Upsilon_{(k)} & \tau_{(k)} c_{(k)}^2 \end{bmatrix} \succeq 0, \forall k \in \mathcal{I}_m, \tag{2.41}$$

where $K = -\frac{\chi}{2} B^T Q^{-1}$ and $A(v_i)_{cl} = A + \sum_{k=1}^{m} \mathcal{N}_k v_{i(k)} + BK$, then system (2.1) with the switching law

$$u(x) \in \arg\min_{v\in\mathcal{V}} (x^T P - \phi(Kx)^T \Omega K)Bv \tag{2.42}$$

is locally asymptotically stable at the origin.

An estimation of the domain of attraction is given by

$$\mathcal{L}_V(r^{-1}) = \{x \in \mathbb{R}^n : V(x) \leq r^{-1}\}, \tag{2.43}$$

where V is a Lur'e candidate Lyapunov function defined in (2.36) and $r \geq \max_{k \in \mathcal{I}_m} \left\{ \frac{\tau_{(k)}}{M^2_{(k,k)}} \right\} > 0$.

Proof. The idea of the proof is to show that if A-1' and A-2 hold and the LMIs (2.40) and (2.41) are feasible then the decay of the function V in the domain $\mathcal{L}_V(r^{-1})$ is ensured by switching among the elements of the set $\mathcal{V}$. This will be shown in three steps. In the first step, we associate a differential inclusion to system (2.1), (2.7) and provide some sufficient conditions for local asymptotic stability. In the second step, we show that the feasibility of LMI (2.40) ensures the decay of the Lyapunov function in a domain $\tilde{\mathcal{D}}$ around the origin. Finally, we will show that if the LMI (2.41) is feasible then the Lyapunov function decreases in the positive invariant domain $\mathcal{L}_V(r^{-1}) \subseteq \tilde{\mathcal{D}}$ which constitutes an inner estimation of the domain of attraction.

First, to the closed-loop system (2.9), we associate the differential inclusion

$$\dot{x} \in \mathcal{F}[\mathcal{X}](x), \tag{2.44}$$

with $\mathcal{F}[\mathcal{X}](x)$ is a set valued map, which can be computed using the construction proposed in [36] as

$$\mathcal{F}[\mathcal{X}](x) = \bigcap_{\delta>0} \bigcap_{\mu(\mathcal{S})=0} \mathrm{Conv}\{\mathcal{X}(\breve{\mathcal{B}}(x,\delta)) \setminus \mathcal{S}\}, \forall x \in \mathbb{R}^n, \tag{2.45}$$

where $\breve{\mathcal{B}}(x,\delta)$ is the open ball centred on x with radius δ, and $\mathcal{S}$ is a set of measure (in the sense of Lebesgue) $\mu(\mathcal{S}) = 0$.

From Lemma 2, one can see that the Lur'e candidate Lyapunov function V is positive definite.

In order to prove the local asymptotic stability of system (2.1), (2.42) in the domain $\mathcal{L}_V(r^{-1})$, it is sufficient to demonstrate that

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x}\varsigma < 0, \forall x \in \mathcal{L}_V(r^{-1}) \setminus \{0\}, \tag{2.46}$$

where $\mathcal{F}[\mathcal{X}](x)$ is defined in (2.45) and $\mathcal{L}_V(r^{-1})$ in (2.43).

Let us define for all $x \in \mathbb{R}^n$ the set of indexes $\mathcal{I}^*(x)$ corresponding to the set of minimizers in which the controller (2.42) takes values:

$$\mathcal{I}^*(x) = \{i \in \mathcal{I}_N : (x^T P - \phi(Kx)^T \Omega K) B(v_j - v_i) \geq 0, \forall j \in \mathcal{I}_N\}. \tag{2.47}$$

We associate to this set of indexes the set $\Delta^*(x)$ of vectors defined for all $x \in \mathbb{R}^n$ as

$$\Delta^*(x) = \{\beta \in \Delta_N : \beta_{(i)} = 0, \forall i \in \mathcal{I}_N \setminus \mathcal{I}^*(x)\}. \tag{2.48}$$

Using (2.47) and (2.48) the set valued map $\mathcal{F}[\mathcal{X}](x)$ in (2.45) satisfies

$$\mathcal{F}[\mathcal{X}](x) \subseteq \mathcal{F}^*[\mathcal{X}](x), \tag{2.49}$$

with

$$\begin{aligned} \mathcal{F}^*[\mathcal{X}](x) &= \underset{i \in \mathcal{I}^*(x)}{\mathrm{Conv}} \left\{ Ax + \sum_{k=1}^{m} \mathcal{N}_k x v_{i(k)} + B v_i \right\} \\ &= \left\{ Ax + \sum_{k=1}^{m} \mathcal{N}_k x v_{(k)}(\beta) + B v(\beta) : \beta \in \Delta^*(x) \right\}, \end{aligned} \tag{2.50}$$

where $v(\beta) = \sum_{i=1}^{N} \beta_{(i)} v_i$.

From (2.49) and (2.50), and using the fact that $\Delta^*(x)$ is compact, we have

$$\begin{aligned} \sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x}\varsigma \leq \sup_{\varsigma \in \mathcal{F}^*[\mathcal{X}](x)} \frac{\partial V}{\partial x}\varsigma &= \sup_{\beta \in \Delta^*(x)} \left\{ \frac{\partial V}{\partial x} \left\{ Ax + \sum_{k=1}^{m} \mathcal{N}_k x v_{(k)}(\beta) + Bv(\beta) \right\} \right\} \\ &= \max_{\beta \in \Delta^*(x)} \left\{ \frac{\partial V}{\partial x} \left\{ Ax + \sum_{k=1}^{m} \mathcal{N}_k x v_{(k)}(\beta) + Bv(\beta) \right\} \right\}. \end{aligned} \tag{2.51}$$

Thus, in order to show (2.46), it is sufficient to prove that we have

$$\max_{\beta \in \Delta^*(x)} \left\{ \frac{\partial V}{\partial x} \left\{ Ax + \sum_{k=1}^{m} \mathcal{N}_k x v_{(k)}(\beta) + Bv(\beta) \right\} \right\} < 0, \forall x \in \mathcal{L}_V(r^{-1}) \setminus \{0\}. \tag{2.52}$$

The LMI (2.40) is equivalent to

$$z^T \begin{bmatrix} A(v_i)_{cl}^T P + P A(v_i)_{cl} & PB - \Upsilon^T - A(v_i)_{cl}^T K^T \Omega \\ B^T P - \Upsilon - \Omega K A(v_i)_{cl} & -2M - \Omega K B - (\Omega K B)^T \end{bmatrix} z < 0, \tag{2.53}$$

for all $z \in \mathbb{R}^{n+m} \setminus \{0\}$ and $i \in \mathcal{I}_N$.

Considering the vector $z^T = \begin{bmatrix} x^T & \phi(Kx)^T \end{bmatrix}$ and $G = M^{-1}\Upsilon$, inequality (2.53) leads to

$$\begin{aligned} & x^T\Big(\big(A(v_i)+BK\big)^T P + P\big(A(v_i)+BK\big)\Big)x \\ & + \phi(Kx)^T\Big(B^T P - MG - \Omega K\big(A(v_i)+BK\big)\Big)x \\ & + x^T\Big(PB - G^T M^T - \big(A(v_i)+BK\big)^T K^T\Omega\Big)\phi(Kx) \\ & + \phi(Kx)^T\Big(-2M - \Omega KB - B^T K^T \Omega\Big)\phi(Kx) < 0, \forall x \in \mathbb{R}^n \setminus \{0\}, \forall i \in \mathcal{I}_N. \end{aligned} \tag{2.54}$$

Let us consider the notation $\kappa(x) = Kx + \phi(Kx)$. According to Lemma 3, for any $x \in \mathbb{R}^n$, we have $\kappa(x) \in \mathcal{P}(c) \subseteq \text{Conv}\{\mathcal{V}\}$. Therefore, there exist N positive scalars $\rho_j(x)$, $\sum_{j=1}^N \rho_j(x) = 1$ such that

$$\kappa(x) = Kx + \phi(Kx) = \sum_{j=1}^N \rho_j(x) v_j. \tag{2.55}$$

Using this property in (2.54), we obtain

$$\begin{aligned} & 2x^T P\Big(A(v_i)x + B\sum_{j=1}^N \rho_j(x)v_j\Big) - 2\phi(Kx)^T\Omega K\Big(A(v_i)x + B\sum_{j=1}^N \rho_j(x)v_j\Big) \\ & - 2\phi(Kx)^T M\Big(\phi(Kx) + Gx\Big) < 0, \forall x \in \mathbb{R}^n \setminus \{0\}, \forall i \in \mathcal{I}_N. \end{aligned} \tag{2.56}$$

From (2.47), for any $x \in \mathbb{R}^n$ and $i \in \mathcal{I}^*(x)$ we have

$$\Big(x^T P - \phi(Kx)^T\Omega K\Big)B\big(v_j - v_i\big) \geq 0, \forall j \in \mathcal{I}_N. \tag{2.57}$$

By adding and subtracting the term $2\sum_{j=1}^N \rho_j(x)(x^T P - \phi(Kx)^T\Omega K)B(v_j - v_i)$ to (2.56), we obtain

$$\begin{aligned} & 2x^T P\Big(A(v_i)x + Bv_i\Big) - 2\phi(Kx)^T\Omega K\Big(A(v_i)x + Bv_i\Big) - 2\phi(Kx)^T M\Big(\phi(Kx) + Gx\Big) \\ & + 2\sum_{j=1}^N \rho_j(x)\Big(x^T P - \phi(Kx)^T\Omega K\Big)B\big(v_j - v_i\big) < 0, \forall x \in \mathbb{R}^n \setminus \{0\}, \forall i \in \mathcal{I}_N. \end{aligned} \tag{2.58}$$

Then, multiplying (2.58) by $\beta_{(i)}$ with $i \in \mathcal{I}_N$, and summing the N elements, we obtain

$$\begin{aligned} & 2x^T P\Big(A(v(\beta))x + Bv(\beta)\Big) - 2\phi(Kx)^T\Omega K\Big(A(v(\beta))x + Bv(\beta)\Big) \\ & - 2\phi(Kx)^T M\Big(\phi(Kx) + Gx\Big) \\ & + 2\sum_{j=1}^N \rho_j(x)\Big(x^T P - \phi(Kx)^T\Omega K\Big)B\big(v_j - v(\beta)\big) < 0, \forall x \in \mathbb{R}^n \setminus \{0\}. \end{aligned} \tag{2.59}$$

with

$$\Big(x^T P - \phi(Kx)^T \Omega K\Big) B\big(v_j - v(\beta)\big) \geq 0, \forall j \in \mathcal{I}_N, \forall \beta \in \Delta^*(x). \tag{2.60}$$

Applying Lemma 1, with $w_1 = Kx$ and $w_2 = Gx$, and using the definition of $\mathcal{P}(c)$ in (2.6), we have

$$\phi(Kx)^T M\big(\phi(Kx) + Gx\big) \leq 0, \forall x \in \mathcal{A}, \tag{2.61}$$

with

$$\mathcal{A} = \{x \in \mathbb{R}^n : \big|(K_{(k)} - G_{(k)})x\big| \leq c_{(k)}, \forall k \in \mathcal{I}_m\}. \tag{2.62}$$

Note that $\frac{\partial V}{\partial x} = 2x^T P - 2\phi(Kx)^T \Omega K$. Therefore, taking this into account, as well as (2.59) and (2.60), we deduce that

$$\begin{aligned}
\max_{\beta \in \Delta^*(x)} \left\{ \frac{\partial V}{\partial x} \Big(A(v(\beta))x + Bv(\beta) \Big) \right\} \leq & \max_{\beta \in \Delta^*(x)} \left\{ \frac{\partial V}{\partial x} \Big(A(v(\beta))x + Bv(\beta) \Big) \right\} \\
& - 2\phi(Kx)^T M\big(\phi(Kx) + Gx\big) \\
& + 2\sum_{j=1}^{N} \rho_j(x) \Big(x^T P - \phi(Kx)^T \Omega K\Big) B\big(v_j - v(\beta)\big) \\
& < 0, \forall x \in \mathcal{A} \setminus \{0\}.
\end{aligned} \tag{2.63}$$

In order to show (2.52) (and thus (2.46)), we will now prove that $\mathcal{L}_V(r^{-1}) \subseteq \mathcal{A}$.

By multiplying (2.41) from both sides by $\begin{bmatrix} I & 0 \\ 0 & (M_{(k,k)})^{-1} \end{bmatrix}$ and considering again $G = M^{-1}\Upsilon$, we obtain

$$\begin{bmatrix} P & K_{(k)}^T - G_{(k)}^T \\ K_{(k)} - G_{(k)} & \frac{\tau_{(k)}}{M_{(k,k)}^2} c_{(k)}^2 \end{bmatrix} \succeq 0, \forall k \in \mathcal{I}_m. \tag{2.64}$$

Considering a scalar $r \geq \max\limits_{k \in \mathcal{I}_m} \{\frac{\tau_{(k)}}{M_{(k,k)}^2}\} > 0$, from (2.64), we obtain

$$\begin{bmatrix} P & K_{(k)}^T - G_{(k)}^T \\ K_{(k)} - G_{(k)} & r c_{(k)}^2 \end{bmatrix} \succeq 0, \forall k \in \mathcal{I}_m. \tag{2.65}$$

This last inequality leads to

$$x^T \big(K_{(k)} - G_{(k)}\big)^T (c_{(k)}^2)^{-1} \big(K_{(k)} - G_{(k)}\big) x \leq x^T \frac{P}{r^{-1}} x, \forall x \in \mathbb{R}^n, \tag{2.66}$$

which is equivalent to the inclusion

$$\mathcal{E}(P, r^{-1}) \subseteq \mathcal{A}. \tag{2.67}$$

Applying Lemma 2, we also have

$$x^T Px \leq V(x) \leq x^T(P + K^T \Omega K)x. \tag{2.68}$$

Which leads to the double inclusion

$$\mathcal{E}(P + K^T \Omega K, r^{-1}) \subseteq \mathcal{L}_V(r^{-1}) \subseteq \mathcal{E}(P, r^{-1}). \tag{2.69}$$

Thus, from (2.69) and (2.67) we obtain

$$\mathcal{L}_V(r^{-1}) \subseteq \mathcal{A}. \tag{2.70}$$

Therefore, using (2.70) and (2.63), we have shown (2.52), and therefore (2.46) is verified, which ends the proof. □

Remark 3: Theorem 10 provides LMI conditions for the stabilization of system (2.1) instead of using the existence of a Hurwitz convex combination the appraoch requires that the pairs $(A(v_i), B)$ are all simultaneously stabilizable. The method generalizes the conditions of Proposition 2 provided in Chapter 1 using a switched Lyapunov function that allows to enlarge the estimation of the domain of attraction. It will be shown in the following example that the estimation of the domain of attraction always encompasses the one provided with the approach in [55]. With respect to the result provided in Proposition 2 which uses the existence of a continuous stabilizer Theorem 10 is based on the existence of the continuous controller (2.55) which corresponds to a more general saturation of linear controller.

Example 9: In order to illustrate the performance of the proposed control method, we consider the switched affine system (2.1) with matrices

$$A = \begin{bmatrix} 1 & 2 \\ 4 & -5 \end{bmatrix},\ B = \begin{bmatrix} 15 & 1 \\ -1 & -5 \end{bmatrix},\ \mathcal{N}_1 = \begin{bmatrix} 1 & -5 \\ 0.5 & 2 \end{bmatrix},\ \mathcal{N}_2 = \begin{bmatrix} -1 & 5 \\ -0.5 & -2 \end{bmatrix},$$

and the controller u which takes values in the set

$$\mathcal{V} = \left\{ \begin{bmatrix} 30 \\ 30 \end{bmatrix}, \begin{bmatrix} 30 \\ -30 \end{bmatrix}, \begin{bmatrix} -30 \\ 30 \end{bmatrix}, \begin{bmatrix} -30 \\ -30 \end{bmatrix} \right\}.$$

One can verify that the open-loop linear system is unstable (the eigenvalues of the matrix A are 2.12 and -6.12). Choosing a decay rate $\alpha = 0.5$, we design the linear

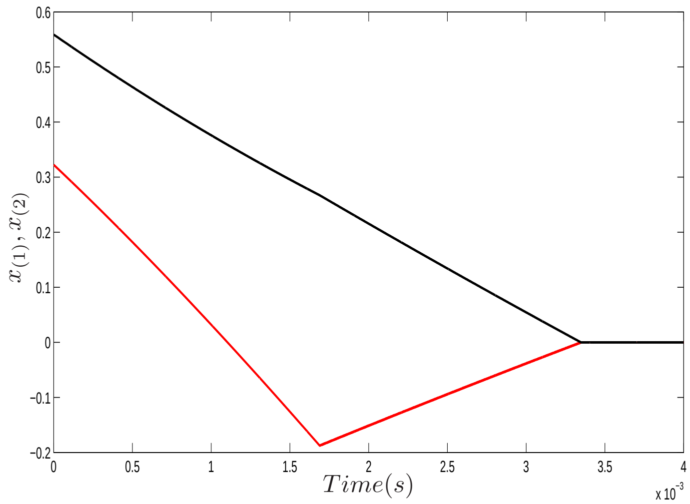

Figure 2.1: Evolution of the state variables of system (2.1), (2.42)-Example 9

switching law proposed in [58] in order to stabilize the system to the origin. We obtain the following solutions of (2.10)

$$Q = \begin{bmatrix} 0.0629 & -0.0068 \\ -0.0068 & 0.0042 \end{bmatrix} \tag{2.71}$$

and $\chi = 0.1$.

We deduce then

$$K = \begin{bmatrix} 12.61 & -8.843 \\ 6.73 & 69.34 \end{bmatrix} \tag{2.72}$$

with an estimation of the ellipsoidal domain of attraction $\mathcal{E}(P, \gamma)$ where $\gamma = 54.1602$ and $P = Q^{-1}$. Based on Theorem 10, we design a nonlinear switching law by solving LMIs (2.40) and (2.41) for $P = Q^{-1}$ and K as given in (2.71)-(2.72) and a vector c satisfying (2.6) such that $c_{(1)} = c_{(2)} = 30$.

We obtain

$$\Omega = \begin{bmatrix} 1.075 & 0 \\ 0 & 0.34 \end{bmatrix} \text{ and } r^{-1} = 209.85.$$

As we can see from Figure 2.2, the obtained trajectories starting in the domain of attraction $\mathcal{L}_V(r^{-1})$ converge to the origin. We can also note that, the domain of attraction

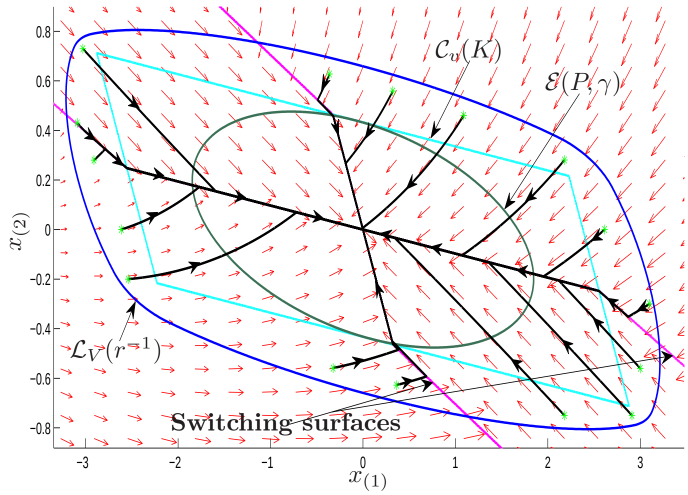

Figure 2.2: Phase plot of system (2.1), (2.42)-Example 9

$\mathcal{L}_V(r^{-1})$ is larger than the ellipsoidal domain of attraction $\mathcal{E}(P,\gamma)$ obtained by the method proposed in [55]. The nonlinear switching surfaces and the convex hull $\mathcal{C}_v(K)$ defined in (2.11), which limits the domain of attraction in the approach in [55], are equally represented. The evolution of the state variables starting at $x(0) = [0.32, 0.56]^T$ are presented in Figure 2.1. We can observe that the state converges to zero.

In this section a constructive method based on LMIs allowing the design of nonlinear switching surfaces is proposed. A larger non-ellipsoidal estimation of the domain of attraction is given. In what follows, LMI conditions allowing the computation of state-dependent switching laws which ensures the global asymptotic stability of closed-loop switched affine systems at the origin is provided.

2.3.2 Switching law design for global stabilization

In order to provide LMI conditions for global stabilization of system (2.1), we use the following property of the nonlinearity ϕ.

Lemma 4: [110] Consider $w_1 = w_2 = w \in \mathbb{R}^m$, the nonlinearity $\phi(w)$ defined in (2.37) satisfies the inequality

$$\phi(w)^T M\left(\phi(w) + w\right) \leq 0, \tag{2.73}$$

for any vector $w \in \mathbb{R}^m$ and any matrix $M \in \mathbb{R}^{m\times m}$ such that

$$c_{(k)}M_{(k,k)} \geq \sum_{k\neq j, j=1}^{m} c_{(j)}|M_{(k,j)}|, \forall k \in \mathcal{I}_m. \tag{2.74}$$

Remark 4: Following the method proposed in [23], the condition (2.74) can be rewritten as an LMI given by

$$c_{(k)}M^+_{(k,k)} \geq \sum_{k\neq j, j=1}^{m} c_{(j)}\left(M^+_{(k,j)} + M^-_{(j,k)}\right), \forall k \in \mathcal{I}_m, \tag{2.75}$$

with $M = M^+ - M^-$, $M^+_{(k,j)} = M^+_{(j,k)} \geq 0$, $M^-_{(k,j)} = M^-_{(j,k)} \geq 0$ and $M^-_{(k,k)} = 0$.

Considering this property, we are able to state the following result.

Theorem 11: Assume that A-1 and A-2 hold. Consider system (2.1). If there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, a diagonal positive definite matrix $\Omega \in \mathbb{R}^{m\times m}$, and symmetric matrices M^+ and M^- with non-negative entries such that LMI (2.40) and

$$c_{(k)}M^+_{(k,k)} \geq \sum_{k\neq j, j=1}^{m} c_{(j)}\left(M^+_{(k,j)} + M^-_{(j,k)}\right), \forall k \in \mathcal{I}_m. \tag{2.76}$$

are satisfied with $\Upsilon = MK$, $M = M^+ - M^-$ and $M^-_{(k,k)} = 0, \forall k \in \mathcal{I}_m$, then system (2.1) with the switching law (2.42) is globally asymptotically stable.

Proof. Considering $\Upsilon = MK$ and $M = M^+ - M^-$, LMI (2.40) becomes

$$\begin{bmatrix} A(v_i)_{cl}^T P + PA(v_i)_{cl} & PB - ((M^+ - M^-)K)^T - A(v_i)_{cl}^T K^T\Omega \\ * & -2(M^+ - M^-) - \Omega KB - (\Omega KB)^T \end{bmatrix} \prec 0, \forall i \in \mathcal{I}_N. \tag{2.77}$$

Following the same steps as in Theorem 10, one can show that the feasibility of LMI (2.77) together with (2.76) ensures that

$$\begin{aligned} &\max_{\beta\in\Delta^*(x)} \left\{\frac{\partial V}{\partial x}\Big(A(v(\beta))x + Bv(\beta)\Big)\right\} \\ &\leq \max_{\beta\in\Delta^*(x)} \left\{\frac{\partial V}{\partial x}\Big(A(v(\beta))x + Bv(\beta)\Big)\right\} - 2\phi(Kx)^T\big(M^+ - M^-\big)\big(\phi(Kx) + Kx\big) \\ &+ 2\sum_{j=1}^{N} \rho_j(x)\Big(x^TP - \phi(Kx)^T\Omega K\Big)B\big(v_j - v(\beta)\big) < 0, \forall x \in \mathbb{R}^n. \end{aligned} \tag{2.78}$$

It follows immediately from the last inequality that

$$\max_{\beta\in\Delta^*(x)} \left\{\frac{\partial V}{\partial x}\Big(A(v(\beta))x + Bv(\beta)\Big)\right\} < 0, \forall x \in \mathbb{R}^n. \tag{2.79}$$

which end the proof. □

Remark 5: In the case where $w_1 = w_2 = Kx$ and M is a diagonal matrix, condition (2.73) is globally satisfied (see for instance [110], page 41). In this case the existence of a positive diagonal matrix M, a symmetric positive definite matrix P and a diagonal positive definite matrix Ω satisfying LMI (2.40) suffices to conclude on the global asymptotic stability of the closed-loop system (2.1), (2.42).

Example 10: In order to illustrate the performance of the proposed control method, we consider the switched affine system (2.2) with matrices

$$\tilde{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1.9 \end{bmatrix},\ \tilde{A}_2 = \begin{bmatrix} 0.5 & -1 \\ 0.5 & -1 \end{bmatrix},\ \tilde{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \text{and } \tilde{b}_2 = \begin{bmatrix} -10 \\ 10 \end{bmatrix}. \tag{2.80}$$

Choosing $\beta^* = \frac{1}{3}$ and following the method proposed in Proposition 1, system (2.2), (2.80) is rewritten in the form (2.1) with

$$A = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & 1.3 \end{bmatrix},\ B = \begin{bmatrix} 31.53 \\ -28.47 \end{bmatrix},\ \mathcal{N} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & -0.6 \end{bmatrix}, \tag{2.81}$$

and the controller u which takes values in the set

$$\mathcal{V} = \left\{-\frac{1}{3}, \frac{2}{3}\right\}.$$

One can verify that the matrix A is not Hurwitz (the eigenvalues of the matrix A are 0.55 and -1.18). Therefore, the existing method in the literature [16], [26], [113] can not be used to stabilise the system. Choosing a decay rate $\alpha = 3.7$, we obtain the following solution of (2.10)

$$Q = \begin{bmatrix} 1.89 & -2.57 \\ -2.57 & 3.76 \end{bmatrix} \tag{2.82}$$

and $\chi = 0.021$.

We deduce then

$$K = \begin{bmatrix} -0.9 & -0.53 \end{bmatrix}. \tag{2.83}$$

Based on Theorem 11, we design a nonlinear switching law by solving LMIs (2.76) and (2.77) for K as given in (2.83) and a vector c satisfying (2.6) such that $c_{(1)} = c_{(2)} = \frac{1}{3}$.

We obtain

$$\Omega = 3.83,\ P = \begin{bmatrix} 0.67 & 0.43 \\ 0.43 & 0.43 \end{bmatrix},\ \text{and } M = 1.61.$$

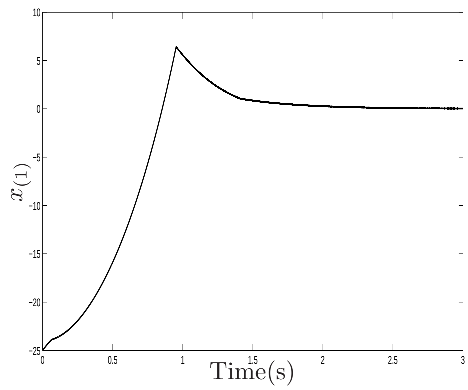

Figure 2.3: Evolution of the state variable $x_{(1)}$ of system (2.1), (2.42)-Example 10

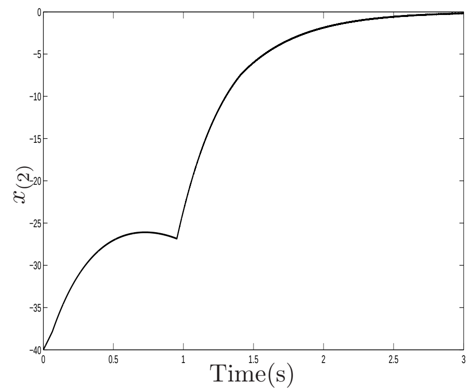

Figure 2.4: Evolution of the state variable $x_{(2)}$ of system (2.1), (2.42)-Example 10

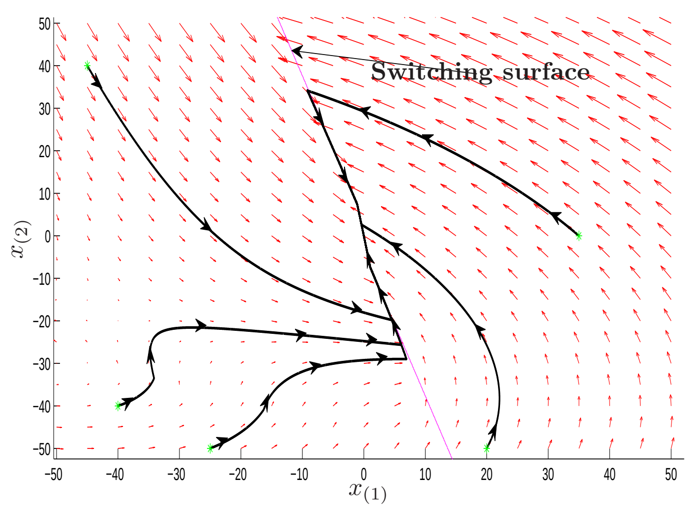

Figure 2.5: Phase plot of system (2.1), (2.42)-Example 10

Simulations are performed for different initial conditions $x(0) = \begin{bmatrix} 35 & 0 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} -45 & 40 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} 20 & -50 \end{bmatrix}^T$, and $x(0) = \begin{bmatrix} -25 & -40 \end{bmatrix}^T$. The different trajectories are reported in the phase plot given in Figure 2.5 together with the nonlinear switching surface. The evolution of the state variables starting at $x_{(1)}(0) = -25$ and $x_{(2)}(0) = -40$ are depicted in Figures 2.3-2.4.

As we can see from Figures 2.5-2.4, the obtained trajectories starting from different initial conditions in the state space converge to the origin.

The results provided in this section can be directly applied to the problem of relay feedback control design for LTI systems. Indeed, relay systems can be considered as switched affine systems where the matrices $\tilde{A}_i, i \in \mathcal{I}_N$ do not switch. This idea is developed in the next section.

2.4 LTI systems with relay control

As follows we particularize the proposed method to the case of LTI systems with relay control.

Consider the linear system with a relay feedback control given as follows

$$\dot{x} = Ax + Bu, \tag{2.84}$$

with $x \in \mathbb{R}^n$, and an input u which takes values in the set $\mathcal{V} = \{v_1, \ldots, v_N\} \subset \mathbb{R}^m$. $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the matrices describing the system.

This class of systems presents the simplest class of switched control systems [76]. They are widely used in different application fields - see for instance [50], [79], [117], [121]. They are motivated by their use in simple electrical (DC-DC converters) [29], [31], electromechanical [3], [114], and aerospace applications [39], [40]. They are also used for quantization errors modelling in digital control [76], [86], delta-sigma modulator design in signal processing [100], and controllers auto-tuning [50].

Relay systems are known for being simple, efficient and robust [32], [59], [92]. Various approaches have been proposed for relay feedback control design in the literature, both in the space domain and in the frequency domain - see for instance [15], [59], [91], and [92]. However, the problem of relay feedback control design is still widely open.

Just as for the case of switched affine systems, in what follows we assume that:

A-3 The pair (A, B) is stabilizable. This means that there exists a matrix K such that the closed-loop matrix $A_{cl} = A + BK$ is Hurwitz.

A-4 The set $\text{int}\{\text{Conv}\{\mathcal{V}\}\}$ is nonempty and the null vector is contained inside ($0 \in \text{int}\{\text{Conv}\{\mathcal{V}\}\}$).

Recently, in [58] a constructive method for a relay feedback controller design is given. Assuming A-4 and

A-3' There exist a positive definite matrix Q and positive scalars χ and α such that

$$AQ + QA^T - \chi BB^T \preceq -2\alpha Q, \tag{2.85}$$

it is proved that system (2.84) with a switching law (2.7) is locally exponentially stable with a decay rate α. Note that A-3' is equivalent to A-3. In that paper ([58]), a linear switching function is considered: $\Gamma(x,v) = -\frac{2}{\chi}x^T K^T v$ with $K = -\frac{\chi}{2}B^T Q^{-1}$. An ellipsoidal estimation of the domain of attraction is equally given using a quadratic Lyapunov function. Nevertheless, considering a quadratic Lyapunov function, linear switching surfaces and an ellipsoidal estimation of the domain of attraction introduces some conservatism in the proposed method [13].

Here by applying the developed approach in Theorem 10 to the class of LTI systems, we would like to provide a design procedure of relay control using non-quadratic Lyapunov functions. This approach allows to compute nonlinear switching surfaces and to provide non-ellipsoidal estimations of the domain of attraction. Moreover, A constructive method based on LMI criteria allowing the design of nonlinear state-dependent switching laws ensuring the global asymptotic stability of the closed-loop system at the origin is provided.

Since relay systems represent a simpler class of switched affine systems, the results in Theorem 10 can be directly applied for deriving a relay feedback control stabilizing locally the closed-loop system at the origin with a larger estimation of a non-ellipsoidal domain of attraction. The result is as follows:

Corollary 1: Consider system (2.84) and assume that A-3' (or equivalently A-3) and A-4 hold. If there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, two diagonal positive definite matrices $\Omega \in \mathbb{R}^{m\times m}$ and $M \in \mathbb{R}^{m\times m}$, a matrix $\Upsilon \in \mathbb{R}^{m\times n}$, and a strictly positive vector $\tau \in \mathbb{R}^m$ such that

$$\begin{bmatrix} A_{cl}^T P + PA_{cl} & PB - \Upsilon^T - A_{cl}^T K^T \Omega \\ * & -2M - \Omega KB - (\Omega KB)^T \end{bmatrix} \prec 0 \tag{2.86}$$

and

$$\begin{bmatrix} P & M_{(i,i)} K_{(i)}^T - \Upsilon_{(i)}^T \\ M_{(i,i)} K_{(i)} - \Upsilon_{(i)} & \tau_{(i)} c_{(i)}^2 \end{bmatrix} \succeq 0, \forall i \in \mathcal{I}_m, \tag{2.87}$$

where $K = -\frac{\chi}{2}B^TQ^{-1}$ and $A_{cl} = A + BK$, then the origin of system (2.84) with the switching law

$$u(x) \in \arg\min_{v\in\mathcal{V}}(x^TP - \phi(Kx)^T\Omega K)Bv \tag{2.88}$$

is locally asymptotically stable.

An estimation of the domain of attraction is given by

$$\mathcal{L}_V(r^{-1}) = \{x \in \mathbb{R}^n : V(x) \leq r^{-1}\}, \tag{2.89}$$

with V a Lur'e candidate Lyapunov function given by

$$V(x) = x^TPx - 2\sum_{j=1}^{m}\int_0^{K_{(j)}x}\phi_{(j)}(\sigma)\Omega_{(j,j)}d\sigma, \tag{2.90}$$

and $r \geq \max_{i\in\mathcal{I}_m}\left\{\frac{\tau_{(i)}}{M^2_{(i,i)}}\right\} > 0$.

Remark 6: Note that since K is known and satisfies (2.85), inequalities (2.86) and (2.87) are affine in the matrix A. Then, the approach can be directly extended to the case of LTV systems with A varying in a convex polytope. In this case the condition should only be checked on the vertices of the polytope.

Example 11: In order to illustrate the performance of the proposed control method, we consider the linear system (2.84) with matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ -1 & 0.5 \end{bmatrix},$$

and the controller u which takes values in the set

$$\mathcal{V} = \left\{ \begin{bmatrix} 25 \\ 25 \end{bmatrix}, \begin{bmatrix} 25 \\ -25 \end{bmatrix}, \begin{bmatrix} -25 \\ 25 \end{bmatrix}, \begin{bmatrix} -25 \\ -25 \end{bmatrix} \right\}.$$

One can verify that the open-loop linear system is unstable (the eigenvalues of the matrix A are -1 and 1). Choosing a decay rate $\alpha = 2.5$, we design the linear switching law proposed in [58] in order to stabilize the system to the origin. We obtain the following solutions of (2.85)

$$Q = \begin{bmatrix} 0.101 & 0.073 \\ 0.073 & 0.172 \end{bmatrix} \tag{2.91}$$

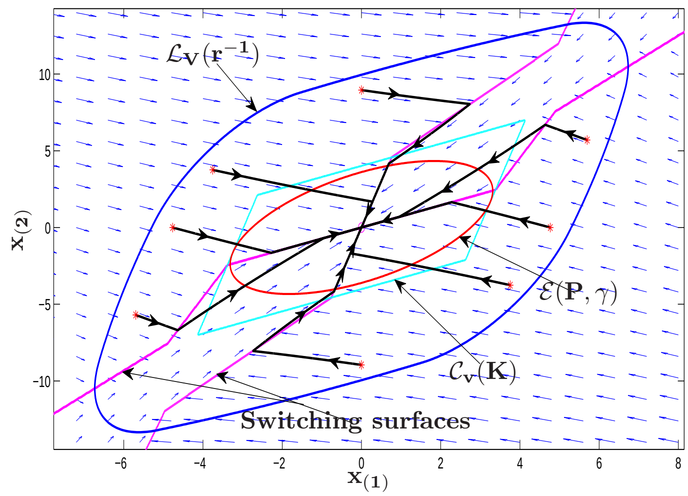

Figure 2.6: Phase plot of system (2.84), (2.88)-Example 11

and $\chi = 1.6$. We deduce then

$$K = \begin{bmatrix} -4.5 & 6.2 \\ -8.4 & 1.4 \end{bmatrix} \tag{2.92}$$

with an estimation of the ellipsoidal domain of attraction $\mathcal{E}(P,\gamma)$ where $\gamma = 109.2$ and $P = Q^{-1}$. Based on Theorem 10, we design a nonlinear switching law by solving LMIs (2.86) and (2.87) for $P = Q^{-1}$ and K as given in (2.91)-(2.92) and a vector c satisfying (2.6) such that $c_{(1)} = c_{(2)} = 25$. We obtain

$$\Omega = \begin{bmatrix} 0.9 & 0 \\ 0 & 4.34 \end{bmatrix} \text{ and } r^{-1} = 2.05 \times 10^3.$$

As we can see from Figure 2.6, the obtained trajectories starting in the domain of attraction $\mathcal{L}_V(r^{-1})$ converge to the origin. We can also note that, the domain of attraction $\mathcal{L}_V(r^{-1})$ is larger than the ellipsoidal domain of attraction $\mathcal{E}(P,\gamma)$ obtained by the method proposed in [58]. The nonlinear switching surfaces and the convex hull $\mathcal{C}_v(K)$ defined in (2.11), which limits the domain of attraction in the approach in [58], are equally represented.

Assume now that the state matrix is affected by polytopic uncertainties :

$$A(t) \in \text{Conv}\{A_1, A_2\},\ \forall t \geq 0, \tag{2.93}$$

with

$$A_1 = \begin{bmatrix} 0 & 1.5 \\ 1 & 0 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0.5 \end{bmatrix}.$$

First, considering $\alpha = 1.5$ and the method in [58], we solve the LMI (2.85) to obtain a common quadratic Lyapunov function for both subsystems 1 and 2. We obtain the following parameters of the linear switching law :

$$Q = \begin{bmatrix} 8.66 & 0.22 \\ 0.22 & 10.8 \end{bmatrix}, \ \chi = 54.25, \text{ and } K = \begin{bmatrix} -0.064 & 2.52 \\ -3.1 & -1.194 \end{bmatrix}.$$

An estimation of the ellipsoidal domain of attraction is given by $\mathcal{E}(Q^{-1}, \gamma)$ with $\gamma = 6.23$.

Next we solve the LMIs (2.86) and (2.87) simultaneously for all system vertices, with the same matrices $P = Q^{-1}$, K and $c_{(1)} = c_{(2)} = 25$. Then, we compute a Lur'e Lyapunov function for the subsystems 1 and 2 (A_1 and A_2) using (2.90). We design the nonlinear switching law (2.88) with

$$\Omega = \begin{bmatrix} 0.11 & 0 \\ 0 & 0.18 \end{bmatrix}.$$

An estimation of the domain of attraction (2.89) is obtained with $r^{-1} = 42.72$. For our simulations we consider

$$A(t) = \left(\frac{\sin(x_{(1)}(t) + x_{(2)}(t)) + 1}{2} \right) A_1 + \left(1 - \frac{\sin(x_{(1)}(t) + x_{(2)}(t)) + 1}{2} \right) A_2.$$

Figure 2.7 shows the trajectories of the closed-loop system in the phase plot for different initial conditions together with the non-ellipsoidal domain of attraction $\mathcal{L}_V(r^{-1})$ including the domain $\mathcal{E}(P, \gamma)$ obtained with the method proposed in [58].

Here, we have proposed a constructive LMI-based method to design nonlinear state-dependent switching laws stabilizing locally asymptotically the LTI system to the origin. A larger non-ellipsoidal estimation of the domain of attraction has also been provided. The result of global stabilization provided in Theorem 11 can equally be adapted to the stabilization problem of the class of LTI systems with a relay controller as follows.

Corollary 2: Assume that A-3 and A-4 hold. Consider system (2.84). If there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$, a diagonal positive definite matrix $\Omega \in$

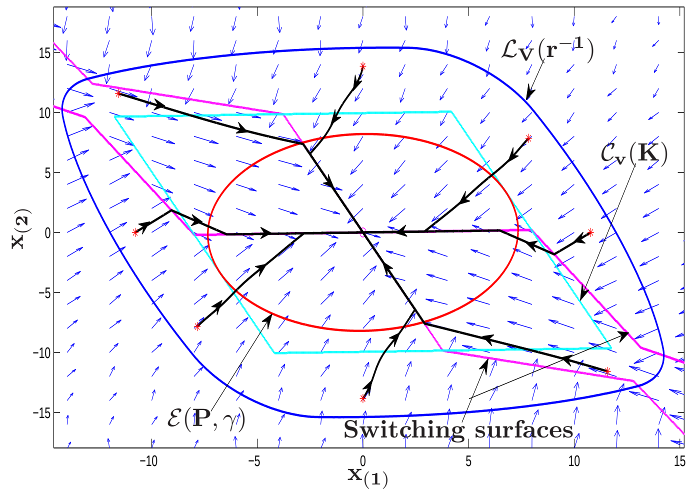

Figure 2.7: Phase plot of system (2.84), (2.88) with polytopic uncertainties-Example 12

$\mathbb{R}^{m\times m}$, and symmetric matrices M^+ and M^- with non-negative entries such that

$$\begin{bmatrix} A_{cl}^T P + PA_{cl} & PB - ((M^+ - M^-)K)^T - A_{cl}^T K^T \Omega \\ * & -2(M^+ - M^-) - \Omega KB - (\Omega KB)^T \end{bmatrix} \prec 0, \forall i \in \mathcal{I}_N. \tag{2.94}$$

and

$$c_{(k)} M^+_{(k,k)} \geq \sum_{k\neq j, j=1}^{m} c_{(j)} \left(M^+_{(k,j)} + M^-_{(j,k)} \right), \forall k \in \mathcal{I}_m. \tag{2.95}$$

are satisfied with $\Upsilon = MK$, $M = M^+ - M^-$ and $M^-_{(k,k)} = 0, \forall k \in \mathcal{I}_m$, then the origin of system system (2.84) with the switching law (2.88) is globally asymptotically stable.

Example 12: Let us consider the buck converter [9] shown in Figure 2.8.

The state-space model for the state vector $\bar{x} = \begin{bmatrix} i_L & v_c \end{bmatrix}^T$ (i_L the inductor current and v_c the capacitor voltage) is described by :

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u} \tag{2.96}$$

with

$$\bar{A} = \begin{bmatrix} 0 & \frac{1}{L} \\ \frac{1}{C_c} & \frac{-1}{RC_c} \end{bmatrix}, \ \bar{B} = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \text{ and } \bar{u} \in \bar{\mathcal{V}} = \{0, E\}.$$

Here we consider the numerical values L = 2mH, C_c = 470μF, E = 15V, and R = 10Ω. One can note that the eigenvalues of the open loop system are purely imaginary

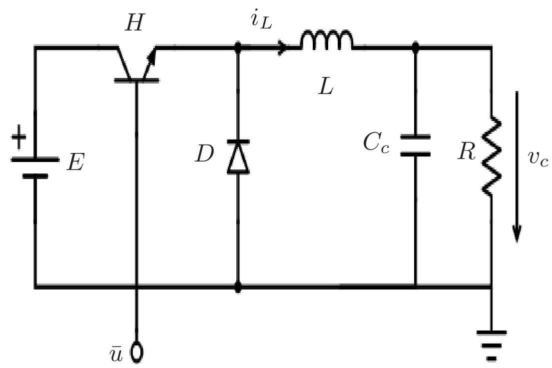

Figure 2.8: Buck converter

$(\pm 10^3 \times 1.03i)$. We want to stabilize the system to the equilibrium point $\bar{x}^* = -\bar{A}^{-1}\bar{B}\beta^*$ which correspond to $i_L = 0.16\mu A$ and $v_c = 7.5V$. Using the transformation from [55] (see Proposition 1 in Chapter 1) and the change of coordinates $x = \bar{x} - \bar{x}^*$, system (2.96) becomes

$$\dot{x} = Ax + Bu, \tag{2.97}$$

with $A = \bar{A}$, $u \in= \{-\frac{1}{2}, \frac{1}{2}\}$, and $B = \begin{bmatrix} \frac{E}{L} \\ 0 \end{bmatrix}$. We can remark that system (2.97) satisfies Assumptions A-3 and A-4. Therefore, we can design a relay feedback controller. Considering a decay rate $\alpha = 3.55$, the LMI (2.85) is feasible with

$$K = \begin{bmatrix} -3.2 & 0.4078 \end{bmatrix}.$$

Considering the obtained matrices K and $c = \frac{1}{2}$, we can design a nonlinear switching law. The set of LMIs (2.94)-(2.95) is feasible with :

$$P = \begin{bmatrix} 1.52 & 0.2 \\ 0.2 & 0.56 \end{bmatrix}, \ \Omega = 0.09, \text{ and } M = 357.93.$$

Figures 2.9-2.10 show the evolution of the state variables starting at $\begin{bmatrix} 0.5, & 27.5 \end{bmatrix}^T$. We can remark that the trajectories converge to the equilibrium point $\bar{x}^*$.

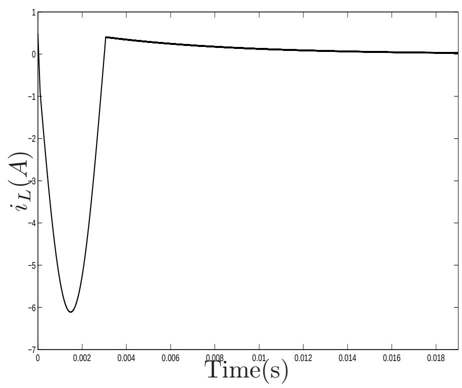

Figure 2.9: Evolution of the state variable i_L-Example 12

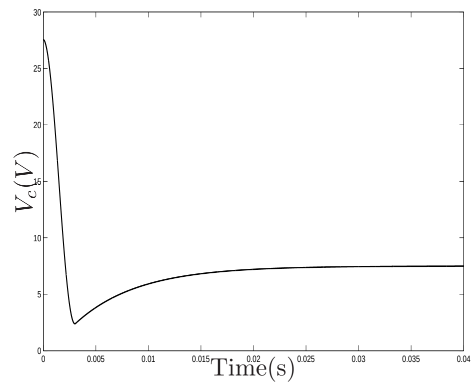

Figure 2.10: Evolution of the state variable V_c-Example 12

2.5 Conclusion

This chapter presented a new approach for the design of state-dependent switching laws. In the case of switched affine systems, non-quadratic Lyapunov functions have been used to develop a method allowing the computation of nonlinear switching surfaces and the enlargement of the domain of attraction. LMI criteria have been given in order to design the switching law and provide an estimation of a non-ellipsoidal domain of attraction. Using the properties of the Lur'e type Lyapunov function, an LMI-based approach has been developed in order to derive state-dependent switching laws ensuring the global asymptotic stability of the closed-loop switched affine systems. The approach have been then particularised to the stabilization of the simpler class of LTI systems with relay controllers. Numerical methods allowing the design of a relay controller have been provided. Moreover, a general framework for the design of a relay control in the class of nonlinear input-affine systems has been provided.

Chapter 3

Stabilization of switched affine systems with disturbed state-dependent switching laws

In this chapter we investigate the stabilization problem for a class of switched affine systems with a state-dependent switching law. Since the states measurements are in general subject to perturbations and noises, we propose a robust switching law design method. Qualitative conditions for the stability of the closed-loop switched system are given. A constructive method based on BMIs is provided. This method allows the design of the switching surfaces, the enlargement of the domain of attraction or the minimization of the size of the chattering zone. The results are then particularized to the stabilization of Linear Time Invariant (LTI) systems by a relay controller.

3.1 Preliminaries and problem statement

Consider the following system

$$\dot{x} = Ax + \sum_{k=1}^{m} (\mathcal{N}_k x + b_k) u_{(k)}, \tag{3.1}$$

with $x \in \mathbb{R}^n$ and $u_{(k)}$ the k-th component of the input u. The input u is only allowed to take values in the set $\mathcal{V} = \{v_1, \ldots, v_i, \ldots, v_N\} \subset \mathbb{R}^m$, with $i \in \mathcal{I}_N$. $A \in \mathbb{R}^{n\times n}$,

$B = \begin{bmatrix} b_1, & \ldots, & b_m \end{bmatrix} \in \mathbb{R}^{n\times m}$, and $\mathcal{N}_k \in \mathbb{R}^{n\times n}$ are the matrices describing the system.

In the sequel we assume that:

A-1 The pair (A, B) is controllable, which implies that there exists a matrix K such that the closed-loop matrix $A_{cl} = A + BK$ is Hurwitz.

A-2 The set $\text{int}\{\text{Conv}\{\mathcal{V}\}\}$ is nonempty, and the null vector is contained inside ($0 \in \text{int}\{\text{Conv}\{\mathcal{V}\}\}$).

This chapter deals with the stabilization of system (3.1) in the case of a disturbed switching law. We consider a controller inspired from the min-switching strategies in [25], [61], [90] given by

$$u(x + e(t)) \in \arg\min_{v\in\mathcal{V}} (x + e(t))^T \Gamma v, \tag{3.2}$$

where e is an exogenous unknown disturbance considered as a measurable and bounded function from $\mathbb{R}_+$ to $\mathbb{R}^n$ satisfying

$$e(t)^T e(t) \leq \bar{e}, \tag{3.3}$$

with $\bar{e}$ its upper bound. The matrix $\Gamma \in \mathbb{R}^{n\times m}$ characterizes the switching hyperplanes of the control and will take a particular form, in the sequel, obtained using the Lyapunov theory.

The closed-loop system (3.1), (3.2) is modeled by a differential equation with discontinuous right hand-side [21]. Consequently, in order to study the stability of the system we will consider the Filippov solutions [36] based on the use of differential inclusions [5].

The interconnection (3.1), (3.2) is the closed-loop system modeled by a discontinuous differential equation of the form

$$\dot{x} = Ax + \sum_{k=1}^{m} (\mathcal{N}_k x + b_k)\tilde{u}_{(k)}(t, x) = f(t, x), \tag{3.4}$$

where $\tilde{u}_{(k)}(t, x)$ is the k-th component of $\tilde{u}(t, x) = \begin{bmatrix} \tilde{u}_{(1)}(t, x) \\ \vdots \\ \tilde{u}_{(m)}(t, x) \end{bmatrix} = u(x + e(t))$.

To the discontinuous closed-loop system (3.4) we associate the differential inclusion

$$\dot{x} \in \mathcal{F}(t, x), \tag{3.5}$$

with $\mathcal{F}(t,x)$ a set-valued map which can be computed using the construction given in [8], [36] such that

$$\mathcal{F}(t,x) = \bigcap_{\delta>0} \bigcap_{\mu(\mathcal{S})=0} \text{Conv}\{f(t, \breve{\mathcal{B}}(x,\delta) \setminus \mathcal{S})\}, \forall x \in \mathbb{R}^n, t \in \mathbb{R}_+, \tag{3.6}$$

where Conv is the closed convex hull, $\breve{\mathcal{B}}(x,\delta)$ is the open ball centered on x with radius $\sqrt{\delta}$, and $\mathcal{S} \subset \mathbb{R}^n$ with $\mu(\mathcal{S})$ its measure in the sense of Lebesgue. Hereafter, we call $\mathcal{F}(t,x)$ the set-valued map associated to the discontinuous system (3.4).

Over this chapter the following notion of stability will be used.

Definition 7 ($R\epsilon$-stability)**:** Consider positive scalars R and ϵ, such that $\epsilon < R$. Assume that there exists a matrix $P = P^T \succ 0$ such that for all Filippov solutions $x(.)$ of system (3.4) with $x(0) \in \mathcal{E}(P,R)$, the value of the state $x(t)$ converges to $\mathcal{E}(P,\epsilon)$ as t goes to infinity. Then, system (3.4) is said to be $R\epsilon$-stable from $\mathcal{E}(P,R)$ to $\mathcal{E}(P,\epsilon)$.

This notion of stability is adapted from [92], where it was used for relay systems with input delays. Considering this notions, in this chapter we are interested in studying the following problem:

Problem 1. Given system (3.1) under Assumptions A.1 and A.2, the set $\mathcal{V}$, and a perturbation e satisfying (3.3), design a robust state-dependent switching law (3.2) such that the closed-loop system is $R\epsilon$-stable in a domain $\mathcal{D}$.

The closed-loop system (3.1), (3.2), (3.3) is affected by a bounded disturbance in the actuator. First, in Section 3.2, we show that there exists a switching control law such that system (3.1) is $R\epsilon$-stable for small enough perturbations. Then, in Section 3.3, we provide BMI-based stability conditions that allow the design of a stabilizing switching law, the maximization of the size of the domain of attraction or the minimization of the size of the chattering zone. Finally, in Section 3.4, since relay systems are a simpler subclass of switched affine systems, the obtained results are particularized to the case of LTI systems stabilization by a relay controller.

3.2 Qualitative conditions for the robust stabilization of switched affine systems

This section deals with the $R\epsilon$-stabilization of system (3.1) under bounded input disturbances. Assumptions A.1 and A.2 are used to prove that there exists a switching control law such that the system is $R\epsilon$-stable.

Theorem 12: Assume that A.1 and A.2 hold. Then there exist positive scalars R, ϵ, a matrix $P = P^T \succ 0$, and a switching control law $u(x + e(t))$ such that system (3.1) is $R\epsilon$-stable from $\mathcal{E}(P, R)$ to $\mathcal{E}(P, \epsilon)$, for an input perturbation e satisfying (3.3) with a sufficiently small bound $\bar{e}$.

Proof. Since the pair (A, B) is controllable then for any $\alpha > 0$, there exists a gain K and a matrix $P = P^T \succ 0$ such that

$$A_{cl}^T P + P A_{cl} \preceq -2\alpha P, \tag{3.7}$$

with $A_{cl} = A + BK$ is Hurwitz. Consider system (3.1) and the switching law

$$u(x + e(t)) \in \arg\min_{v \in \mathcal{V}} \mathcal{G}(x, e(t), v), \tag{3.8}$$

with $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{V} \to \mathbb{R}$ defined as $\mathcal{G}(x, e(t), v) = (x+e(t))^T \sum_{k=1}^{m} (\mathcal{N}_k(x+e(t))+b_k)v_{(k)}$. Just as we did in (3.5) we associate to the closed-loop system (3.1), (3.8) a differential inclusion

$$\dot{x} \in \tilde{\mathcal{F}}(t, x). \tag{3.9}$$

Consider the function V such that $V(x) = x^T P x$. We want to prove that there exists $\bar{e}$ such that if (3.3) is satisfied then

$$\sup_{y \in \tilde{\mathcal{F}}(t,x)} \frac{\partial V}{\partial x} y \leq -2\chi V(x), \tag{3.10}$$

for some $\chi > 0$ in a domain $\mathcal{D} \subset \mathbb{R}^n$ which will be determined.

In order to prove these results, we rewrite the system (3.1) as

$$\dot{x} = Ax + Bu + w(x, u), \tag{3.11}$$

with $w(x, u) = \sum_{j=1}^{m} \mathcal{N}_j x u_{(j)}$.

The main idea of the proof of Theorem 12 is to show that for a sufficiently small and bounded perturbation e satisfying (3.3), the decay of the function V in some domain $\mathcal{D}$ can be ensured by switching among the elements of the set $\mathcal{V}$. This will be proved in the following in three steps. In Step 1, we associate a differential inclusion to system (3.4) and provide some sufficient conditions for $R\epsilon$-stability. Then, in Step 2, for a decay rate α and a static gain K satisfying (3.7) stability conditions of system (3.11) are given, considering the term $w(x, Kx)$ as a perturbation. Finally, in Step 3, based on the results from Steps 1 and 2, we use the property of existence of a static linear stabilizer to design the switching surface such that the Lyapunov function V decreases over the domain of attraction $\mathcal{E}(P, \gamma)$ until its reaches the chattering ball $\mathcal{E}(P, \epsilon)$, which proves the $R\epsilon$-stability of system (3.1), (3.8), (3.3).

Step 1 : *Explicit stability condition based on Filippov differential inclusions*

We define for any $z \in \mathbb{R}^n$ the set of indexes $\mathcal{I}^*(z)$ such that

$$\mathcal{I}^*(z) = \{i \in \mathcal{I}_N : z^T \sum_{k=1}^{m} (\mathcal{N}_k z + b_k)(v_{j(k)} - v_{i(k)}) \geq 0, \quad \forall j \in \mathcal{I}_N\}, \tag{3.12}$$

which corresponds to the set of minimizers of $u(z)$ defined in (3.8).

To $\mathcal{I}^*(z)$ we associate for all $z \in \mathbb{R}^n$ the set $\Delta^*(z)$ of vectors defined by

$$\Delta^*(z) = \{\beta \in \Delta_N : \beta_{(i)} = 0, \forall i \in \mathcal{I}_N \setminus \mathcal{I}^*(z)\}. \tag{3.13}$$

Using (3.12) and (3.13), the set valued map $\tilde{\mathcal{F}}(t, x)$ in (3.9) satisfies

$$\tilde{\mathcal{F}}(t, x) \subseteq \tilde{\mathcal{F}}^*(t, x), \tag{3.14}$$

with

$$\begin{aligned} \tilde{\mathcal{F}}^*(t, x) &= \underset{i \in \mathcal{I}^*(\bar{x}_e(t))}{\text{Conv}} \{Ax + Bv_i + w(x, v_i)\} \\ &= \{Ax + Bv(\beta) + w(x, v(\beta)) : \beta \in \Delta^*(\bar{x}_e(t))\}, \end{aligned} \tag{3.15}$$

$v(\beta) = \sum_{i=1}^{N} \beta_{(i)} v_i$, and $\bar{x}_e(t) = x + e(t)$.

Therefore, from (3.14) and in order to show (3.10), it is sufficient to prove that for some positive scalar χ we have

$$\sup_{y \in \tilde{\mathcal{F}}^*(t,x)} \frac{\partial V}{\partial x} y \leq -2\chi V(x), \tag{3.16}$$

in some domain $\mathcal{D} \subset \mathbb{R}^n$ to be determined.

From (3.15), and using the fact that the set $\Delta^*(z)$ is compact for all $z \in \mathbb{R}^n$, we have

$$\begin{aligned} \sup_{y \in \tilde{\mathcal{F}}^*(t,x)} \frac{\partial V}{\partial x} y &= \sup_{\beta \in \Delta^*(\bar{x}_e(t))} \left\{ \frac{\partial V}{\partial x} \Big(Ax + Bv(\beta) + w(x, v(\beta)) \Big) \right\} \\ &= \max_{\beta \in \Delta^*(\bar{x}_e(t))} \left\{ \frac{\partial V}{\partial x} \Big(Ax + Bv(\beta) + w(x, v(\beta)) \Big) \right\}. \end{aligned} \tag{3.17}$$

Then, showing (3.16) is equivalent to proving that for some $\chi > 0$

$$\max_{\beta \in \Delta^*(\bar{x}_e(t))} \left\{ \frac{\partial V}{\partial x} \Big(Ax + Bv(\beta) + w(x, v(\beta)) \Big) \right\} \leq -2\chi V(x), \tag{3.18}$$

in a domain $\mathcal{D} \subset \mathbb{R}^n$ to be determined below.

Step 2: *Stability properties with a continuous controller*

Here we will study the robustness properties of system (3.11) with linear state feedback $\bar{u} = Kx$ based on the linearized model. These properties will be useful for redesigning a switching controller. Before showing (3.18), it is useful to prove that

$$\frac{\partial V}{\partial x}(A_{cl}x + w(x, Kx)) \leq -\alpha V(x), \tag{3.19}$$

is satisfied in a neighbourhood of the origin where A_{cl} satisfies (3.7) for some α and P, and $w(x, Kx) = \sum_{j=1}^{m} \mathcal{N}_j x(Kx)_{(j)}$.

From inequality (3.7), we obtain

$$\frac{\partial V}{\partial x}(A_{cl}x) \leq -2\alpha V(x), \forall x \in \mathbb{R}^n.$$

Using this, (3.19) becomes

$$\frac{\partial V}{\partial x}(A_{cl}x + w(x, Kx)) \leq -2\alpha V(x) + 2x^T P w(x, Kx), \forall x \in \mathbb{R}^n, \tag{3.20}$$

Furthermore, for some scalar $\varrho > 0$ we have

$$\|w(x, Kx)\| = \left\| \sum_{k=1}^{m} \mathcal{N}_k x(Kx)_{(k)} \right\| \leq \varrho \|x\|^2, \ \forall x \in \mathbb{R}^n.$$

Therefore, we can show that

$$\forall \hat{\varrho} > 0, \ \exists \tilde{\varrho} > 0 \ : \ \|w(x, Kx)\| \leq \hat{\varrho} \|x\|, \ \forall \|x\| < \tilde{\varrho}.$$

Then, from Cauchy-Schwarz inequality, we get

$$x^T P w(x, Kx) \leq \left\| x^T P \right\| \|w(x, Kx)\| \leq \hat{\varrho} \|P\| \|x\|^2, \ \forall \|x\| < \tilde{\varrho}.$$

Thus for all $\|x\| < \tilde{\varrho}$, we have

$$\begin{aligned} \frac{\partial V}{\partial x}(A_{cl}x + w(x, Kx)) &\leq -2\alpha V(x) + 2x^T P w(x, Kx) \\ &\leq -(2\alpha V(x) - 2\hat{\varrho} \|P\| \|x\|^2). \end{aligned} \tag{3.21}$$

Therefore, (3.19) is satisfied for all x such that $\|x\| < \tilde{\varrho}$ if we can ensure that

$$-(2\alpha V(x) - 2\hat{\varrho} \|P\| \|x\|^2) \leq -\alpha V(x),$$

which is verified if

$$\hat{\varrho} \leq \frac{\alpha \text{eig}_{min}(P)}{2\|P\|}. \tag{3.22}$$

Note that for given α and P satisfying inequality (3.7), $\hat{\varrho}$ can always be chosen as small as possible (this only constrains x in a neighbourhood $\mathcal{B}(0, \tilde{\varrho}^2)$ of the origin) such that (3.22) is verified.

In the next step we will see how the property (3.22) is useful for the design of a switching law.

Step 3: *Switching controller reconfiguration*

Note that, since Assumption A-2 holds, then there exists a neighbourhood of the origin $\mathcal{E}(P, \gamma)$ with $\gamma > 0$ such that for all $x \in \mathcal{E}(P, \gamma)$ we have

$$Kx \in \text{Conv}\{\mathcal{V}\}.$$

Therefore, for all $x \in \mathcal{E}(P, \gamma)$ there exist positive scalars $\rho_j(x)$, $j \in \mathcal{I}_N$, such that $\sum_{j=1}^{N} \rho_j(x) = 1$ and

$$Kx = \sum_{j=1}^{N} \rho_j(x) v_j. \tag{3.23}$$

In the development that follows we consider that $\mathcal{E}(P, \gamma) \subseteq \mathcal{B}(0, \tilde{\varrho}^2)$ (we can always choose a constant γ satisfying this inclusion). We can also consider the case where (3.19) and

(3.23) are verified (i.e. for all $x \in \mathcal{E}(P,\gamma) \subseteq \mathcal{B}(0,\tilde{\varrho}^2)$). From (3.12), for all $i \in \mathcal{I}^*(\bar{x}_e(t))$ with $\bar{x}_e(t) = x + e(t)$, we have

$$\Big(x+e(t)\Big)^T P \sum_{k=1}^{m} \Big(\mathcal{N}_k(x+e(t)) + b_k\Big)\Big(v_{j(k)} - v_{i(k)}\Big) \geq 0, \forall j \in \mathcal{I}_N.$$

Then, for any $\beta \in \Delta^*(\bar{x}_e(t))$ we have

$$\Big(x+e(t)\Big)^T P \sum_{k=1}^{m} \Big(\mathcal{N}_k(x+e(t)) + b_k\Big)\Big(v_{j(k)} - v(\beta)_{(k)}\Big) \geq 0, \forall j \in \mathcal{I}_N.$$

Multiplying this last inequality by $\rho_j(x)$ for $j \in \mathcal{I}_N$ ($\rho_j(x)$ defined in (3.23)) and summing the N elements, we obtain

$$2\Big(x+e(t)\Big)^T P \sum_{k=1}^{m} \Big(\mathcal{N}_k(x+e(t)) + b_k\Big)\Big((Kx)_{(k)} - v(\beta)_{(k)}\Big) \geq 0.$$

Adding this to the left part of (3.18), it comes

$$\begin{aligned}
&\max_{\beta\in\Delta^*(\bar{x}_e(t))} \left\{\frac{\partial V}{\partial x}\Big(Ax + Bv(\beta) + w(x,v(\beta))\Big)\right\} \\
&\leq \max_{\beta\in\Delta^*(\bar{x}_e(t))} \Bigg\{x^T\Big(A_{cl}^T P + PA_{cl}\Big)x + 2x^T Pw(x,Kx) + 2x^T P\sum_{k=1}^{m}\mathcal{N}_k e(t)\Big((Kx)_{(k)} - v(\beta)_{(k)}\Big) \\
&\quad +2e^T P\sum_{k=1}^{m}\mathcal{N}_k x\Big((Kx)_{(k)} - v(\beta)_{(k)}\Big) + 2e^T P\sum_{k=1}^{m}\mathcal{N}_k e(t)\Big((Kx)_{(k)} - v(\beta)_{(k)}\Big) \\
&\quad +2e^T PB\Big(Kx - v(\beta)\Big)\Bigg\}.
\end{aligned} \tag{3.24}$$

Then, using (3.19) we get

$$\begin{aligned}
&\max_{\beta\in\Delta^*(\bar{x}_e(t))} \left\{\frac{\partial V}{\partial x}\Big(Ax + Bv(\beta) + w(x,v(\beta))\Big)\right\} \\
&\leq \max_{\beta\in\Delta^*(\bar{x}_e(t))} \Bigg\{-\alpha V(x) + 2x^T P\sum_{k=1}^{m}\mathcal{N}_k e(t)\Big((Kx)_{(k)} - v(\beta)_{(k)}\Big) + 2e^T PB\Big(Kx - v(\beta)\Big) \\
&\quad +2e^T P\sum_{k=1}^{m}\mathcal{N}_k x\Big((Kx)_{(k)} - v(\beta)_{(k)}\Big) + 2e^T P\sum_{k=1}^{m}\mathcal{N}_k e(t)\Big((Kx)_{(k)} - v(\beta)_{(k)}\Big)\Bigg\}.
\end{aligned} \tag{3.25}$$

Let us define $\bar{v}_{(k)} = \max\limits_{x\in\mathcal{E}(P,\gamma)} \left\{\Big(\sum_{j=1}^{N}\rho_j(x)v_j\Big)_{(k)} - \max\limits_{\beta\in\Delta^*(\bar{x}_e(t))}\big\{v(\beta)_{(k)}\big\}\right\}$. From (3.23), we have

$$\max_{\beta\in\Delta^*(\bar{x}_e(t))}\big\{(Kx)_{(k)} - v(\beta)_{(k)}\big\} = \max_{\beta\in\Delta^*(\bar{x}_e(t))}\left\{\Big(\sum_{j=1}^{N}\rho_j(x)v_j\Big)_{(k)} - v(\beta)_{(k)}\right\} \leq \bar{v}_{(k)}, \forall k \in \mathcal{I}_m. \tag{3.26}$$

Thus, using (3.26), from (3.25) we obtain that for all $x \in \mathcal{E}(P,\gamma) \subseteq \mathcal{B}(0,\tilde{\varrho}^2)$,

$$\max_{\beta\in\Delta^*(\bar{x}_e(t))} \left\{ \frac{\partial V}{\partial x}\Big(Ax + Bv(\beta) + w(x, v(\beta))\Big)\right\}$$
$$\leq -\alpha V(x) + 2x^T P \sum_{k=1}^{m} \mathcal{N}_k \bar{v}_{(k)} e(t) + 2e^T PB\bar{v} + 2e^T P \sum_{k=1}^{m} \mathcal{N}_k \bar{v}_{(k)} x + 2e^T P \sum_{k=1}^{m} \mathcal{N}_k \bar{v}_{(k)} e(t). \tag{3.27}$$

Recall [17] that for any positive number θ

$$2a^T b \leq \frac{1}{\theta} a^T a + \theta b^T b, \forall a, b \in \mathbb{R}^n. \tag{3.28}$$

Applying (3.28) to the terms $2e^T\Xi x$, $2x^T\Xi e$ with $\Xi = P\sum_{k=1}^{m}\mathcal{N}_k\bar{v}_{(k)}$ and $2e^TPB\bar{v}$, with

$$\theta_1 = \eta, \quad a_1 = e, \quad b_1 = \Xi x, \qquad \theta_2 = \eta^{-1}, \quad a_2 = \Xi^T x, \quad b_2 = e,$$
$$\text{and} \quad \theta_3 = \eta, \quad a_3 = e, \quad b_3 = PB\bar{v},$$

we obtain the following inequality

$$-\alpha V(x) + 2x^T P \sum_{k=1}^{m} \mathcal{N}_k \bar{v}_{(k)} e(t) + 2e^T PB\bar{v} + 2e^T P \sum_{k=1}^{m} \mathcal{N}_k \bar{v}_{(k)} x + 2e^T P \sum_{k=1}^{m} \mathcal{N}_k \bar{v}_{(k)} e(t)$$
$$\leq -\alpha V(x) + 3\eta^{-1} e^T e + 2\eta x^T \Xi^T \Xi x + \eta \bar{v}^T B^T PPB\bar{v} + 2e^T \Xi e. \tag{3.29}$$

Let us define $\zeta_{\max} = \text{eig}_{max}(\Xi^T\Xi)\tilde{\varrho}^2$, $\kappa_{max} = \sqrt{\text{eig}_{max}(\Xi^T\Xi)}$, and $\xi_{\max} = \bar{v}^T B^T PPB\bar{v}$. We have

$$x^T\Xi^T\Xi x \leq \zeta_{max}, \forall x \in \mathcal{B}(0,\tilde{\varrho}^2),$$
$$e^T\Xi e \leq \kappa_{max} e^T e, \forall e \in \mathbb{R}^n.$$

Then, considering $\bar{e}$ a positive scalar satisfying (3.3), we obtain from (3.29)

$$-\alpha V(x) + 2x^T P \sum_{k=1}^{m} \mathcal{N}_k \bar{v}_{(k)} e(t) + 2e^T PB\bar{v} + 2e^T P \sum_{k=1}^{m} \mathcal{N}_k \bar{v}_{(k)} x + 2e^T P \sum_{k=1}^{m} \mathcal{N}_k \bar{v}_{(k)} e(t)$$
$$\leq -\alpha V(x) + (3\eta^{-1} + \kappa_{max})\bar{e} + 2\eta\zeta_{max} + \eta\xi_{max} \tag{3.30}$$

for all $x \in \mathcal{E}(P,\gamma) \subseteq \mathcal{B}(0,\tilde{\varrho}^2)$.

From (3.27) and (3.30) , in order to prove (3.18), it is sufficient to show that there exists $\chi > 0$ such that in some domain $\mathcal{D} := \mathcal{E}(P,\gamma) \setminus \mathcal{E}(P,\epsilon) \subseteq \mathcal{B}(0,\tilde{\varrho}^2)$ we have

$$-\alpha V(x) + (3\eta^{-1} + \kappa_{max})\bar{e} + 2\eta\zeta_{max} + \eta\xi_{max} \leq -2\chi V(x). \tag{3.31}$$

Therefore, we can show that if we take $0 < \chi < \frac{\alpha}{2}$, and sufficiently small $\bar{e}$ and η, we have (3.31) (and thus (3.10)) which is satisfied for all $x \in \mathcal{E}(P,\gamma) \setminus \mathcal{E}(P,\epsilon) \subseteq \mathcal{B}(0,\tilde{\varrho}^2)$, with

$$\epsilon := \frac{(3\eta^{-1} + \kappa_{max})\bar{e} + \eta\xi_{max} + 2\eta\zeta_{max}}{\alpha - 2\chi} < \gamma. \tag{3.32}$$

In conclusion, system (3.1), (3.8) with perturbation (3.3) is $R\epsilon$-stable from $\mathcal{E}(P,R)$ to $\mathcal{E}(P,\epsilon)$ with $R = \gamma$ and ϵ given in (3.32), if the bound $\bar{e}$ on the perturbation is small enough. □

Remark 7: From the proof of Theorem 12, (see equation (3.32)), we can note that the size of the level set $\mathcal{E}(P,\epsilon)$ depends on the upper bound of the disturbance. Furthermore, it depends also on the upper bound of the control value (the terms ξ_{max}, ζ_{max}, and κ_{max} depend on $\bar{v}$ defined in (3.26)). Then, the size of the chattering ball $\mathcal{E}(P,\epsilon)$ increases with the amplitude of the control vector.

In Theorem 12 we have shown that there exists a switching law such that the system is $R\epsilon$-stable for a small enough perturbation. In the following, we propose a constructive numerical implementation based in BMIs that allows to design such a controller.

3.3 A constructive method using controller redesign

The first result (Theorem 12) has a qualitative nature. As we have seen in the proof of Theorem 12, it is possible to use the property of existence of a linear static feedback to redesign switching surfaces for system (3.1), (3.2), (3.3). In practice, it is useful to find a constructive procedure which, for desired domain of attraction and chattering ball, provides a switching law which ensures the $R\epsilon$-stability. Here we would like to ensure that the domain of attraction contains the ball $\mathcal{B}(0,r_\gamma)$ of radius $\sqrt{r_\gamma}$ and that the chattering zone is included in a ball $\mathcal{B}(0,r_c)$ of radius $\sqrt{r_c}$. In this section a numerical approach to deal with the design problem is given, ensuring this property. A BMI solution is proposed hereafter. In order to express the result, note that for any finite set of vectors $\mathcal{V} = \{v_1,\ldots,v_N\} \subset \mathbb{R}^m$, there exists a finite number n_h of vectors $h_i \in \mathbb{R}^{1\times m}$, $i \in \mathcal{I}_{n_h}$, such that

$$\mathrm{Conv}\{\mathcal{V}\} = \{u \in \mathbb{R}^m : h_i u \le 1, i \in \mathcal{I}_{n_h}\}.$$

To provide constructive numerical conditions, we rewrite the system (3.1) as

$$\dot{x} = Ax + \Pi(u)x + Bu, \tag{3.33}$$

with $\Pi(u) = \sum_{i=1}^{m} \mathcal{N}_i u_{(i)}$.

Theorem 13: Assume that A.2 holds. Consider the closed-loop system (3.1), (3.2), (3.3) with $\Gamma = PB$, P a design parameter, and positive scalars r_c and r_γ such that $r_c < r_\gamma$ and $\chi > 0$. Consider K such that $A_{cl} = A+BK$ is Hurwitz. If there exist $\epsilon_1 > 0, \epsilon_2 > 0, \epsilon_3 > 0$, $c > 0$, $\gamma > 0$ and $P = P^T \succ 0$ such that

1.
$$\mathcal{M}(v_i) = \begin{bmatrix} M_i & PBK & 0 \\ * & -\epsilon_3 I & -PBv_i \\ * & * & -\psi \end{bmatrix} \preceq 0, \forall i \in \mathcal{I}_N, \tag{3.34}$$

   with
$$\begin{aligned} M_i &= (A_{cl} + \Pi(v_i))^T P + P(A_{cl} + \Pi(v_i)) + (2\chi + \epsilon_1 - \epsilon_2)P, \\ \psi &= -\epsilon_1 c + \epsilon_3 \bar{e} + \epsilon_2 \gamma, \end{aligned} \tag{3.35}$$

2.
$$P - K^T h_i^T \gamma h_i K \succ 0, \forall i \in \mathcal{I}_{n_h}, \tag{3.36}$$

3.
$$P \preceq \frac{\gamma}{r_\gamma} I, \tag{3.37}$$

4.
$$P \succeq \frac{c}{r_c} I, \tag{3.38}$$

5.
$$c < \gamma, \tag{3.39}$$

then the system (3.1), (3.2) is $R\epsilon$-stable from $\mathcal{E}(P,\gamma)$ to $\mathcal{E}(P,c)$. Furthermore, $\mathcal{B}(0,r_\gamma) \subseteq \mathcal{E}(P,\gamma)$ and $\mathcal{E}(P,c) \subseteq \mathcal{B}(0,r_c)$.

Proof. We want to prove that if the set of inequalities (3.34)-(3.39) is feasible, then the closed-loop system (3.1), (3.2) is $R\epsilon$-stable from $\mathcal{E}(P,\gamma)$ to $\mathcal{E}(P,c)$. It is sufficient to prove that the function V defined by $V(x) = x^T P x$ satisfies

$$\sup_{y \in \mathcal{F}^*(t,x)} \frac{\partial V}{\partial x} y \leq -2\chi V(x), \forall x \in \mathcal{E}(P,\gamma) \setminus \mathcal{E}(P,c),$$

with $\mathcal{F}^*(t,x)$ defined in (3.15).

Let us define the set $\mathcal{C}_v(K)$ as

$$\mathcal{C}_v(K) = \{x \in \mathbb{R}^n : h_i K x < 1, i \in \mathcal{I}_N\}.$$

From (3.36) we have

$$x^T K^T h_i^T h_i K x < x^T \frac{P}{\gamma} x, \forall i \in \mathcal{I}_{n_h}, \forall x \in \mathbb{R}^n,$$

and thus, the ellipsoid $\mathcal{E}(P,\gamma)$ satisfies

$$\mathcal{E}(P,\gamma) \subset \mathcal{C}_v(K). \tag{3.40}$$

Inequality (3.34) is equivalent to

$$z^T \mathcal{M}(v_i) z \leq 0, \forall z \in \mathbb{R}^{2n+1}.$$

Considering the vector $z^T = (x, e, 1)^T$, this leads to

$$\begin{aligned} &x^T((A_{cl} + \Pi(v_i))^T P + P(A_{cl} + \Pi(v_i)))x + (2\chi - \epsilon_2 + \epsilon_1)x^T P x + 2e^T PBKx \\ &- 2e^T PBv_i - \epsilon_3 e^T e - \epsilon_1 c + \epsilon_2 \gamma + \epsilon_3 \bar{e} \leq 0, \forall x \in \mathbb{R}^n, \forall e \in \mathbb{R}^n, \forall i \in \mathcal{I}_N. \end{aligned} \tag{3.41}$$

Note that for all $x \in \mathcal{C}_v(K)$, there exist N positive scalars $\rho_j(x)$, $j \in \mathcal{I}_N$, $\sum_{j=1}^N \rho_j(x) = 1$ such that

$$Kx = \sum_{j=1}^N \rho_j(x) v_j. \tag{3.42}$$

Since the constraint (3.40) is satisfied, then using (3.41) and (3.42), we obtain

$$\begin{aligned} &x^T((A + \Pi(v_i))^T P + P(A + \Pi(v_i)))x + (2\chi - \epsilon_2 + \epsilon_1)x^T P x + 2e^T PB \sum_{j=1}^N \rho_j(x) v_j \\ &+ 2x^T PB \sum_{j=1}^N \rho_j(x) v_j - 2e^T PBv_i - \epsilon_3 e^T e - \epsilon_1 c + \epsilon_2 \gamma + \epsilon_3 \bar{e} \leq 0, \forall x \in \mathcal{E}(P,\gamma), \forall e \in \mathbb{R}^n, \end{aligned}$$

$\forall i \in \mathcal{I}_N$, which leads to

$$\sum_{j=1}^{N} \rho_j(x)\{2x^T P(A+\Pi(v_i))x + 2x^T PBv_j + (2\chi - \epsilon_2 + \epsilon_1)x^T Px - \epsilon_3 e^T e$$
$$+ 2e^T PB(v_j - v_i) - \epsilon_1 c + \epsilon_2 \gamma + \epsilon_3 \bar{e}\}$$
$$\leq 0, \forall x \in \mathcal{E}(P,\gamma), \forall e \in \mathbb{R}^n, \forall i \in \mathcal{I}_N.$$

By adding and subtracting the term $2\sum_{j=1}^{N} \rho_j(x)x^T PB(v_j - v_i)$, we get

$$\begin{aligned}&\sum_{j=1}^{N} \rho_j(x)\{2x^T P(A+\Pi(v_i))x + 2x^T PBv_i + (2\chi - \epsilon_2 + \epsilon_1)x^T Px \\ &+ 2(x+e(t))^T PB(v_j - v_i) - \epsilon_3 e^T e - \epsilon_1 c + \epsilon_2\gamma + \epsilon_3\bar{e}\} \\ &\leq 0, \forall x \in \mathcal{E}(P,\gamma), \forall e \in \mathbb{R}^n, \forall i \in \mathcal{I}_N.\end{aligned} \tag{3.43}$$

For $z \in \mathbb{R}^n$, we define $\mathcal{I}^*(z)$ and $\Delta^*(z)$ as in (3.12), (3.13). By construction, we see that

$$\forall i \in \mathcal{I}^*(x+e(t)), \forall j \in \mathcal{I}_N, (x+e(t))^T PB(v_j - v_i) \geq 0. \tag{3.44}$$

Furthermore, inequality (3.39) guarantees the fact that $\mathcal{E}(P,c) \subset \mathcal{E}(P,\gamma)$. Then, for $x \in \mathcal{E}(P,\gamma) \setminus \mathcal{E}(P,c)$, it is clear that $x^T Px > c$, and $x^T Px \leq \gamma$. Therefore, taking this into account, as well as (3.44) and the fact that $e^T e \leq \bar{e}$ (according to (3.3)), we can deduce from (3.43) that

$$\frac{\partial V}{\partial x}(Ax + Bv_i + \Pi(v_i)x) \leq -2\chi V(x), \forall i \in \mathcal{I}^*(x+e(t)), \forall x \in \mathcal{E}(P,\gamma) \setminus \mathcal{E}(P,c).$$

Then, using the same arguments as in Theorem 12, we can show that

$$\sup_{y \in \mathcal{F}(t,x)} \frac{\partial V}{\partial x} y \leq \max_{\beta \in \Delta^*(\bar{x}_e(t))} \frac{\partial V}{\partial x}(Ax + Bv(\beta) + \Pi(v(\beta))x) \leq -2\chi V(x), \forall x \in \mathcal{E}(P,\gamma) \setminus \mathcal{E}(P,c), \tag{3.45}$$

with $v(\beta) = \sum_{i=1}^{N} \beta_{(i)} v_i$, which ends the proof. Note that inequality (3.37) is equivalent to the constraint

$$\mathcal{B}(0, r_\gamma) \subset \mathcal{E}(P,\gamma),$$

where $\mathcal{B}(0, r_\gamma)$ is the ball of radius $\sqrt{r_\gamma}$, and inequality (3.38) is equivalent to the constraint

$$\mathcal{E}(P,c) \subset \mathcal{B}(0, r_c),$$

where $\sqrt{r_c}$ is the radius of the ball $\mathcal{B}(0, r_c)$. These constraints are used for the optimization of either the chattering ball $\mathcal{E}(P,c)$ or the domain of attraction $\mathcal{E}(P,\gamma)$ (see Remark 8). □

Remark 8: The method uses the property of the existence of linear state feedback with gain K in order to design the switching surfaces Γ. Designing a static gain K for systems as (3.1) is a classical problem. Numerical methods have been proposed in [2], [112]. To solve the conditions of Theorem 13 as an LMI problem for a given gain K such that $A_{cl} = A + BK$ is Hurwitz P, c, and γ are taken as LMI variables. The parameter χ in Theorem 13 corresponds to the system's decay rate from $\mathcal{E}(P,\gamma)$ to $\mathcal{E}(P,c)$. A line search can be used to find r_c and r_γ, and parameters ϵ_1, ϵ_2, and ϵ_3. An optimization algorithm can then be used to maximize r_γ or minimize r_c. Note that when the conditions of Theorem 13 are satisfied, the continuous controller $\bar{u} = Kx$ ensures a decay rate of at least $2\chi + \epsilon_1 - \epsilon_2$. In this paper we only provide sufficient conditions for the design of a switching controller while the measurements are affected by perturbation. The proposed conditions are not necessary. What we show is that, if the LMIs are satisfied for a given gain K and a decay rate χ, then a locally stabilizing switching controller is obtained with guaranteed estimations of the domain of attraction and chattering zone. The approach is based on the inclusion $\tilde{\mathcal{F}}(t,x) \subseteq \tilde{\mathcal{F}}^*(t,x)$ (equations (3.14), (3.15) used in (3.45)) which might be a source of conservatism. The polytopic modelling of the bilinear term can also introduce some conservatism.

In Sections 3.2 and 3.3, stability conditions of the closed-loop system have been given. In order to show the efficiency of the developed method, numerical implementations have been performed. The results are reported in the following.

Example 13: Single-input system

Consider the switched affine system

$$\begin{aligned} &\dot{x} = \tilde{A}_\sigma x + \tilde{b}_\sigma, \\ &\sigma \in \mathcal{I}_2 = \{1,2\}, \end{aligned} \tag{3.46}$$

with

$$\tilde{A}_1 = \begin{bmatrix} 0.3 & 1 \\ 1 & 0.3 \end{bmatrix}, \ \tilde{A}_2 = \begin{bmatrix} -0.1 & 1 \\ -\frac{1}{3} & -0.1 \end{bmatrix}, \ \tilde{b}_1 = \begin{bmatrix} 1.5 \\ 6 \end{bmatrix}, \text{ and } \tilde{b}_2 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}.$$

Considering the result in [55] (see Chapter 1, Section 1.3.3), this system can be reformulated as a bilinear system (3.1) with

$$u \in \mathcal{V} = \{v_1, v_2\} = \{0.75, -0.25\},\ \mathcal{N} = \begin{bmatrix} 0.4 & 0 \\ 1.3333 & 0.4 \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ \text{and } B = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

Note that the proposed methods in the literature can not be used to stabilize the system since there exist no convex combination of the matrices A_1 and A_2 [16], [26]. We consider a gain $K = \begin{bmatrix} -0.4, & -0.44 \end{bmatrix}$ computed by pole assignment such that the eigenvalues of $A + BK$ are $\{-1, -2\}$ ($A + BK$ is Hurwitz) and a desired decay rate $\chi = 0.15$. Using the results from Theorem 13, we can design a switching law (3.2) that makes the system (3.1), (3.2), (3.3) $R\epsilon$-stable for a given bound $\bar{e}$ on the perturbation. Then, implementing an optimization algorithm as discussed in Remark 8 allows the minimization of the ball $\mathcal{B}(0, r_c)$ containing the chattering zone. The results obtained for various values of $\bar{e}$ are given in Table 3.1. For $\bar{e} = 10^{-4}$ for example, we find that the LMIs from Theorem 13 are feasible for

$$r_\gamma = 1.5, \quad r_c = 0.14, \quad \gamma = 0.0589, \quad c = 0.0027, \quad P = \begin{bmatrix} 0.03 & 0.0091 \\ 0.0091 & 0.03 \end{bmatrix},$$

and with the parameters $\epsilon_1 = 0.71$, $\epsilon_2 = 0.1$, and $\epsilon_3 = 1.1980 \times 10^2$. Simulations are performed for different initial conditions $x(0) = \begin{bmatrix} -0.92, & 1.28 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} -0.12, & 1.4 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} -1.28, & -0.08 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} 0.32, & -1.4 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} 1.12, & -1.08 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} 1.36, & -0.12 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} 0.84, & 0.84 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} -0.68, & -0.96 \end{bmatrix}^T$ and a perturbation $e(t) = \sqrt{\bar{e}} \begin{bmatrix} \sin(3t), & \cos(3t) \end{bmatrix}^T$. The different trajectories are reported in the phase plot given in Figure 3.1, while the state variables starting at $x(0) = \begin{bmatrix} -0.92, & 1.28 \end{bmatrix}^T$ are presented in Figure 3.2.

Using the above design method, a switching law is derived and the domain of attraction is successfully estimated. From the simulation results, it can be seen that for a sufficiently small perturbation, the states variables starting in the domain of attraction $\mathcal{E}(P, \gamma)$ converge to a small neighbourhood $\mathcal{E}(P, c)$ of the origin and oscillate indefinitely around it. This confirms the theoretical results. Note that, in Figure 3.1 is reported the sliding surface of the system free of perturbation ($x^T\Gamma = 0$). We can remark that the sliding surface in the presence of the perturbation is variable since its depends on the perturbation dynamics and variations ($(x + e)^T\Gamma = 0$).

Table 3.1: Chattering ball radius r_c obtained for different values of $\bar{e}$

$\bar{e}$	5×10^{-5}	10^{-4}	5×10^{-4}	10^{-3}	5×10^{-3}	10^{-2}
r_c	0.1	0.14	0.55	0.69	1.1	1.4

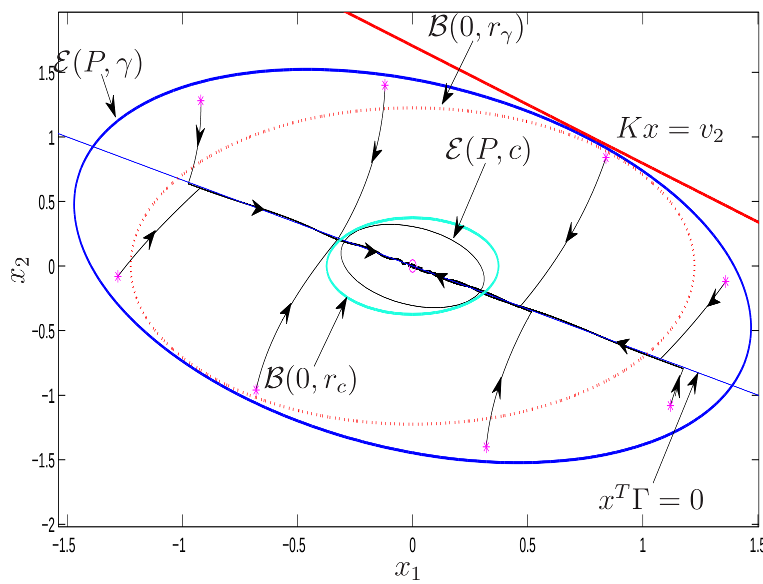

Figure 3.1: Phase portrait of the closed-loop system with $\mathcal{E}(P,\gamma)$ an estimation of the domain of attraction, and $\mathcal{E}(P,c)$ the chattering zone-Example 13.

Example 14: Multi-input system

Consider system (3.1) with

$$A = \begin{bmatrix} 8 & 1.5 \\ 1.5 & -3 \end{bmatrix},\ B = \begin{bmatrix} 8 & -3 \\ 2 & 1 \end{bmatrix},\ \mathcal{N}_1 = \begin{bmatrix} 0 & 0 \\ 0.1 & 0.2 \end{bmatrix},\ \mathcal{N}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0.2 \end{bmatrix},$$

and

$$u \in \mathcal{V} = \left\{ \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ -5 \end{bmatrix}, \begin{bmatrix} -5 \\ 5 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \end{bmatrix} \right\}.$$

The eigenvalues of A are $\{-3.2, 8.2\}$ hence the open-loop system (3.1) is unstable. Note the requirements from [26] and [16] are not satisfied since the matrix A is not Hurwitz. Therefore, the methods in these articles can not be used to stabilize the system. Consider

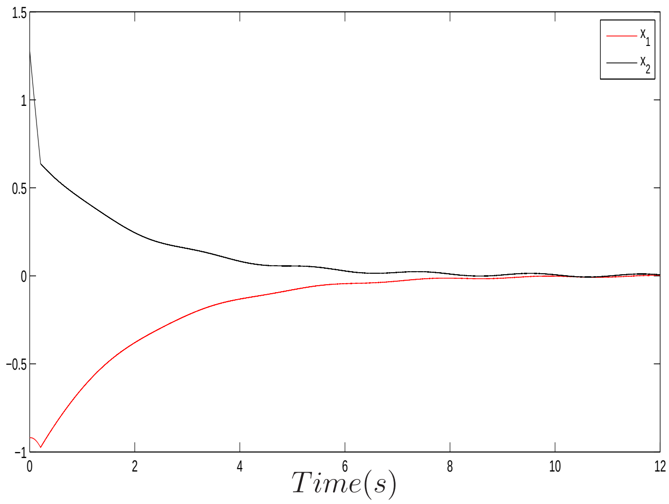

Figure 3.2: State variables x_1 and x_2-Example 13

the gain

$$K = \begin{bmatrix} -1.0714 & -0.6429 \\ 0.6429 & -1.2143 \end{bmatrix}$$

such that the matrix $A + BK$ is Hurwitz (the gain K is computed by pole assignment such that the eigenvalues of $A + BK$ are { -5.5, -2.5}). Applying the method developed above with a decay rate $\chi = 2$, a switching law is designed to stabilize the system in the presence of a perturbation bounded by $\bar{e} = 10^{-4}$. An algorithm of minimization of r_c with a line search to find the parameters ϵ_1, ϵ_2, ϵ_3, r_γ and r_c is implemented.

Using our algorithm of optimization we find that the LMIs are feasible for

$$r_\gamma = 13.2, \quad r_c = 1.6, \quad \gamma = 0.2728, \quad c = 0.0324, \quad P = \begin{bmatrix} 0.02 & -0.0001 \\ -0.0001 & 0.021 \end{bmatrix},$$

and with parameters $\epsilon_1 = 0.7250$, $\epsilon_2 = 0.1$, and $\epsilon_3 = 1.1980 \times 10^2$.

Simulations are performed for different initial conditions:

$x(0) = \begin{bmatrix} 1.3, & 3.2 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} -3.4, & -0.8 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} 0, & 3.5 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} 3, & 1.5 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} 3.5, & -0.1 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} -3.1, & 1.4 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} 3.4, & 0.9 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} -3.5, & 0.1 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} -3, & 2 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} -1.7, & 3 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} 1.9, & -0.6 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} 1.74, & 2 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} 1.5, & -3.1 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} -2.97, & -1.9536 \end{bmatrix}^T$, $x(0) = \begin{bmatrix} 2.83, & -1.93 \end{bmatrix}^T$, and a perturbation

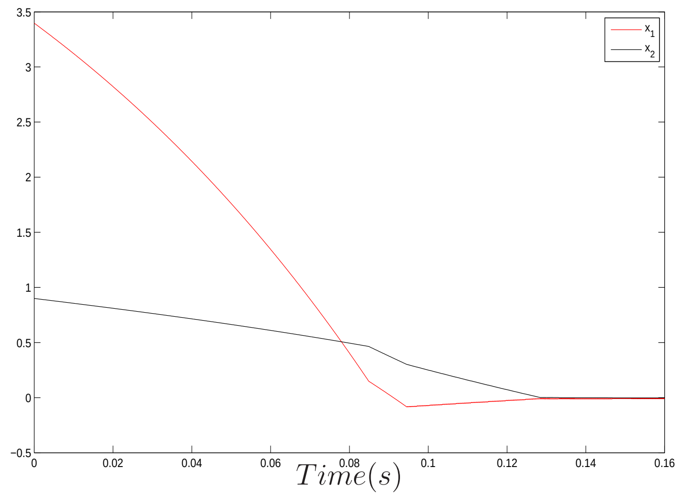

Figure 3.3: State variables x_1 and x_2-Example 14

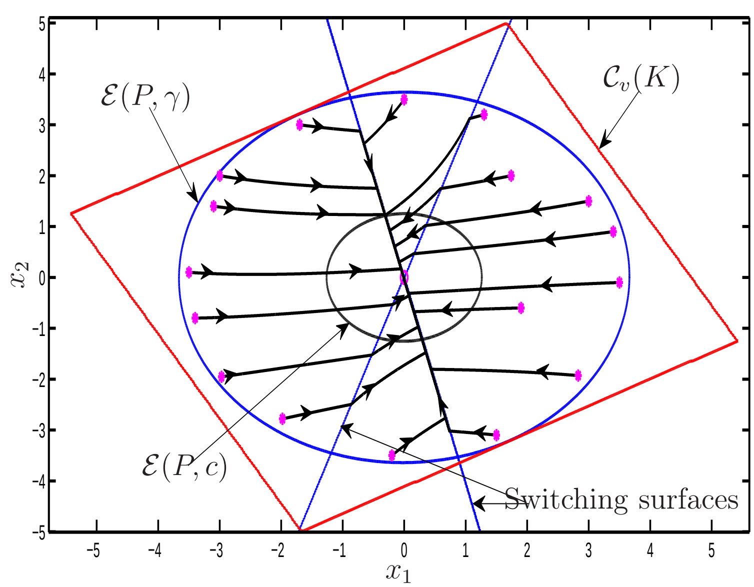

Figure 3.4: Phase portrait of the closed-loop system with $\mathcal{E}(P,\gamma)$ an estimation of the domain of attraction, and $\mathcal{E}(P,c)$ the chattering zone-Example 14.

$e(t) = \sqrt{\bar{e}}\begin{bmatrix}\sin(3t), & \cos(3t)\end{bmatrix}^T$. The different trajectories are reported in the phase plot given in Figure 3.4, while the state variables starting at $x(0) = \begin{bmatrix}3.4, & 0.9\end{bmatrix}^T$ are presented in Figure 3.6.

It is obvious that if the bilinear term vanishes, system (3.1) will model an LTI system with a relay controller. Then, the obtained results in this section can be directly particularized to derive stabilizing relay controllers for LTI systems when the measurements are disturbed. This idea is developed in the next section.

3.4 Robust LTI systems stabilization by a relay feedback control

Let us consider the linear system

$$\dot{x} = Ax + Bu, \tag{3.47}$$

with $x \in \mathbb{R}^n$ and an input u which takes only values in the set $\mathcal{V} = \{v_1, \dots, v_N\} \subset \mathbb{R}^m$. $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$ are the matrices describing the system. We assume A-2 and

A-3 the pair (A, B) is stabilizable.

The results in Theorem 12 can be applied to derive a stabilizing relay controller of the form (3.2) for the class of LTI systems when the measurements are affected by disturbances satisfying (3.3). Sufficient conditions for the existence of such a controller are given in the following.

Corollary 3: Assume that A.2 and A.3 hold. Then there exist positive scalars R and ϵ and matrices $P = P^T \succ 0$ and $\Gamma = PB$ such that the system (3.47) with control (3.2) is $R\epsilon$-stable from $\mathcal{E}(P, R)$ to $\mathcal{E}(P, \epsilon)$ for a perturbation (3.3) with a sufficiently small bound $\bar{e}$.

Proof. Since the pair (A, B) is stabilizable then there exists a gain K such that $A_{cl} = A + BK$ is Hurwitz. Furthermore, for all $\delta > 0$ there exists a matrix $P = P^T \succ 0$ satisfying

$$A_{cl}^T P + PA_{cl} \preceq -2\delta P. \tag{3.48}$$

Consider the closed-loop system (3.47), (3.2), (3.3) and the associated differential inclusion (1.4).

We want to prove that for $\Gamma = PB$ there exists $\bar{e}$ such that if $e^T e \leq \bar{e}$ then

$$\sup_{y\in\mathcal{F}(t,x)} \frac{\partial V}{\partial x} y \leq -2\alpha V(x), \tag{3.49}$$

for some $\alpha > 0$ in a domain $\mathcal{D} \subset \mathbb{R}^n$ which will be determined.

We define for any $z \in \mathbb{R}^n$ the set of index $\mathcal{I}^*(z)$ such that

$$\mathcal{I}^*(z) = \{i \in \mathcal{I}_N : z^T PB(v_j - v_i) \geq 0, \quad \forall j \in \mathcal{I}_N\}. \tag{3.50}$$

To $\mathcal{I}^*(z)$ we associate for all $z \in \mathbb{R}^n$ the set $\Delta^*(z)$ of vectors defined by

$$\Delta^*(z) = \{\beta \in \Delta_N : \beta_i = 0, \forall i \in \mathcal{I}_N \setminus \mathcal{I}^*(z)\}. \tag{3.51}$$

Using (3.50) and (3.51), the set valued map $\mathcal{F}(t, x)$ in (1.5) satisfies

$$\mathcal{F}(t, x) \subseteq \mathcal{F}^*(t, x), \tag{3.52}$$

with

$$\begin{aligned}\mathcal{F}^*(t, x) &= \underset{i\in\mathcal{I}^*(\bar{x}_e(t))}{\text{Conv}} \{Ax + Bv_i\} \\ &= \{Ax + Bv(\beta) : \ \beta \in \Delta^*(\bar{x}_e(t))\},\end{aligned} \tag{3.53}$$

$v(\beta) = \sum_{i=1}^N \beta_i v_i$, and $\bar{x}_e(t) = x + e(t)$.

Therefore, in order to show (3.49), it is sufficient to prove that for some positive scalar α we have

$$\sup_{y\in\mathcal{F}^*(t,x)} \frac{\partial V}{\partial x} y \leq -2\alpha V(x), \tag{3.54}$$

in some domain $\mathcal{D} \subset \mathbb{R}^n$ to be determined.

From (3.53), and using the fact that the set $\Delta^*(z)$ is compact for all $z \in \mathbb{R}^n$, we have

$$\begin{aligned}\sup_{y\in\mathcal{F}^*(t,x)} \frac{\partial V}{\partial x} y &= \sup_{\beta\in\Delta^*(\bar{x}_e(t))} \left\{\frac{\partial V}{\partial x}(Ax + Bv(\beta))\right\} \\ &= \max_{\beta\in\Delta^*(\bar{x}_e(t))} \left\{\frac{\partial V}{\partial x}(Ax + Bv(\beta))\right\}.\end{aligned} \tag{3.55}$$

Then, showing (3.54) is equivalent to prove that for some $\alpha > 0$

$$\max_{\beta\in\Delta^*(\bar{x}_e(t))} \left\{\frac{\partial V}{\partial x}(Ax + Bv(\beta))\right\} \leq -2\alpha V(x), \tag{3.56}$$

in a domain $\mathcal{D} \subset \mathbb{R}^n$ to be determined below.

From inequality (3.48), we obtain

$$\frac{\partial V}{\partial x}(A_{cl}x) \leq -2\delta V(x), \forall x \in \mathbb{R}^n. \tag{3.57}$$

Note that, since the set $\text{Conv}\{\mathcal{V}\}$ is nonempty and the null vector is contained in its interior ($0 \in \text{int}\{\text{Conv}\{\mathcal{V}\}\}$), then there exists a neighborhood of the origin $\mathcal{E}(P,\gamma)$ with $\gamma > 0$ such that for all $x \in \mathcal{E}(P,\gamma)$ we have

$$Kx \in \text{Conv}\{\mathcal{V}\}. \tag{3.58}$$

Therefore for all $x \in \mathcal{E}(P,\gamma)$ there exist positive scalars $\rho_j(x)$, $j \in \mathcal{I}_N$, such that $\sum_{j=1}^{N} \rho_j(x) = 1$ and

$$Kx = \sum_{j=1}^{N} \rho_j(x) v_j. \tag{3.59}$$

In the development that follows, we consider the case where (3.57) and (3.59) are verified (i.e. for all $x \in \mathcal{E}(P,\gamma)$). From (3.50), for all $i \in \mathcal{I}^*(\bar{x}_e(t))$ we have

$$(x + e(t))^T PB(v_j - v_i) \geq 0, \forall j \in \mathcal{I}_N. \tag{3.60}$$

Then, for any $\beta \in \Delta^*(\bar{x}_e(t))$ we have

$$(x + e(t))^T PB(v_j - v(\beta)) \geq 0, \forall j \in \mathcal{I}_N. \tag{3.61}$$

Multiplying this last inequality by $\rho_j(x)$ for $j \in \mathcal{I}_N$ ($\rho_j(x)$ defined in (3.59)) and summing the N elements, we obtain

$$(x + e(t))^T PB(Kx - v(\beta)) \geq 0. \tag{3.62}$$

Adding this to the left part of (3.56), it comes

$$\begin{aligned} \max_{\beta \in \Delta^*(\bar{x}_e(t))} \left\{ \frac{\partial V}{\partial x}(Ax + Bv(\beta)) \right\} \leq \max_{\beta \in \Delta^*(\bar{x}_e(t))} \{ & x^T(A_{cl}^T P + PA_{cl})x \\ & + 2e^T PBKx - 2e^T PBv(\beta)\}. \end{aligned} \tag{3.63}$$

Then, using (3.57), we get

$$\begin{aligned} & \max_{\beta \in \Delta^*(\bar{x}_e(t))} \{x^T(A_{cl}^T P + PA_{cl})x + 2e^T PBKx - 2e^T PBv(\beta)\} \\ & \leq \max_{\beta \in \Delta^*(\bar{x}_e(t))} \left\{-2\delta V(x) + 2e^T PBKx - 2e^T PBv(\beta)\right\}. \end{aligned} \tag{3.64}$$

Then, in order to prove (3.56) in some domain $\mathcal{D} \subseteq \mathcal{E}(P, \gamma)$, from (3.64) it is sufficient to show that for some $\alpha > 0$

$$\max_{\beta \in \Delta^*(\bar{x}_e(t))} \left\{-2\delta V(x) + 2e^T PBKx - 2e^T PBv(\beta)\right\} \leq -2\alpha V(x). \tag{3.65}$$

Recall that for any positive number θ

$$2a^T b \leq \frac{1}{\theta} a^T a + \theta b^T b, \forall a, b \in \mathbb{R}^n. \tag{3.66}$$

Applying (3.66) to the terms $2e^T PBKx$ and $-2e^T PBv(\beta)$, with

$$\theta = \eta, \quad a_1 = e, \quad b_1 = PBKx, \tag{3.67}$$

and

$$\theta = \eta, \quad a_2 = e, \quad b_2 = -PBv(\beta), \tag{3.68}$$

we obtain the following inequality

$$\begin{aligned}\max_{\beta \in \Delta^*(\bar{x}_e(t))} \left\{-2\delta V(x) + 2e^T PBKx - 2e^T PBv(\beta)\right\} \leq & -2\delta V(x) + 2\eta^{-1} e^T e \\ & + \eta x^T K^T B^T PPBKx \\ & + \eta \max_{\beta \in \Delta^*(\bar{x}_e(t))} \left\{v^T(\beta) B^T PPBv(\beta)\right\}.\end{aligned} \tag{3.69}$$

Note that there exist $\zeta_{max} > 0$ and $\xi_{max} > 0$ such that

$$\begin{aligned} x^T K^T B^T PPBKx \leq \zeta_{max} V(x), \forall x \in \mathbb{R}^n, \\ \max_{\beta \in \Delta^*(\bar{x}_e(t))} \left\{v(\beta)^T B^T PPBv(\beta)\right\} \leq \xi_{max}. \end{aligned} \tag{3.70}$$

Also, with $\bar{e}$ a positive scalar satisfying (3.3), thus from (3.69) we obtain

$$\begin{aligned}\max_{\beta \in \Delta^*(\bar{x}_e(t))} \left\{-2\delta V(x) + 2e^T PBKx - 2e^T PBv(\beta)\right\} \leq & -2\delta V(x) + 2\eta^{-1} \bar{e} \\ & + \eta(\zeta_{max} V(x) + \xi_{max}).\end{aligned} \tag{3.71}$$

Then, (3.71) is verified (and consequently (3.49)) if there exists $\alpha > 0$ such that

$$-2\delta V(x) + 2\eta^{-1}\bar{e} + \eta(\zeta_{max} V(x) + \xi_{max}) \leq -2\alpha V(x), \tag{3.72}$$

which is satisfied if

$$\begin{cases} -\delta V(x) + 2\eta^{-1}\bar{e} + \eta \xi_{max} \leq 0, \\ -\delta + \eta \zeta_{max} + 2\alpha \leq 0. \end{cases} \tag{3.73}$$

Therefore, if we take $0 < \alpha < \frac{\delta}{2}$ and

$$0 < \eta \leq \frac{-2\alpha + \delta}{\zeta_{max}}, \tag{3.74}$$

then for a sufficiently small $\bar{e}$ and η, we have (3.72) (and thus (3.49)) is satisfied for all $x \in \mathcal{D} := \mathcal{E}(P, \gamma) \setminus \mathcal{E}(P, c(\bar{e}))$, with

$$x^T P x \leq c(\bar{e}) := \frac{2\eta^{-1}\bar{e} + \eta\xi_{max}}{\delta} < \gamma. \tag{3.75}$$

Therefore, system (3.47), (3.2), (3.3) is $R\epsilon$-stable from $\mathcal{E}(P, \gamma)$ to $\mathcal{E}(P, c)$ with $R = \gamma$ and $\epsilon = c(\bar{e})$. □

Remark 9: From the proof of Corollary 3, equation (3.75), we can note that the size of the level set $\mathcal{E}(P, \epsilon)$ depends on the upper bound of the disturbance. Furthermore, it depends also on the upper bound of the control value. Then, the size of the chattering ball $\mathcal{E}(P, \epsilon)$ increases with the amplitude of the control vector.

Remark 10: In Corollary 3, the upper bound $\bar{e}$ of the disturbance is assumed to be sufficiently small in order to ensure the resolvability of the stabilization problem. The existence of solutions may be preserved despite the large value of $\bar{e}$.

A constructive method allowing the numerical implementation of this result can be derived as in the previous section by reformulating the stability conditions as BMIs. The result in Theorem 13 is particularized to relay controllers design for LTI systems stabilization when the state variables are affected by disturbances in the following.

Corollary 4: Assume that A.2 holds. Consider the linear closed-loop system (3.47), (3.2), (3.3) with $\Gamma = PB$, P a design parameter, and positive scalars c, r_c, r_γ, γ such that $r_c < r_\gamma$ and $\alpha > 0$. The matrix $A + BK$ is Hurwitz. If there exist $\epsilon_1 > 0, \epsilon_2 > 0, \epsilon_3 > 0$ and $P = P^T \succ 0$ such that

1.

$$\mathcal{M}(v_i) = \begin{bmatrix} M & PBK & 0 \\ * & -\epsilon_3 I & -PBv_i \\ * & * & -\psi \end{bmatrix} \preceq 0, \forall i \in \mathcal{I}_N, \tag{3.76}$$

with

$$M = A_{cl}^T P + P A_{cl}$$
$$+(2\alpha + \epsilon_1 - \epsilon_2)P,$$
$$\psi = -\epsilon_1 c + \epsilon_3 \bar{e} + \epsilon_2 \gamma,$$

2.
$$P - K^T h_i^T \gamma h_i K \succ 0, \forall i \in \mathcal{I}_{n_h}, \quad (3.77)$$

3.
$$P \preceq \frac{\gamma}{r_\gamma} I, \quad (3.78)$$

4.
$$P \succeq \frac{c}{r_c} I, \quad (3.79)$$

5.
$$c < \gamma, \quad (3.80)$$

are feasible, then the system (3.47), (3.2) is $R\epsilon$-stable from $\mathcal{E}(P, \gamma)$ to $\mathcal{E}(P, c)$ for a perturbation e satisfying (3.3). Furthermore, $\mathcal{B}(0, r_\gamma) \subseteq \mathcal{E}(P, \gamma)$ and $\mathcal{E}(P, c) \subseteq \mathcal{B}(0, r_c)$.

Remark 11: To compute the LMI solution, for a given gain K such that (A, B) is stabilizable, P, c, and γ are taken as LMIs variables. A line search can be used to find the radius r_c and r_γ and a gridding to find the parameters ϵ_1, ϵ_2, and ϵ_3. Optimization algorithm can then be used to maximize r_γ or minimize r_c.

Remark 12: In Corollary 4 the controller gain K is supposed to be given such that the closed-loop continuous system matrix $A + BK$ is Hurwitz. The choice of the controller gain K has an influence on the size of the domain of attraction $\mathcal{E}(P, \gamma)$ and the chattering domain $\mathcal{E}(P, c)$. Although the main contribution here is the switching law design, one can use the results in Corollary 4 to co-design the controller gain K and the switching hyperplane characterizing matrix Γ (and domains $\mathcal{E}(P, c)$ and $\mathcal{E}(P, \gamma)$) by using recursive LMI optimization algorithms for example the one used in [110].

The following example illustrates the efficiency of the proposed method.

Example 15: Consider the linear system (3.47) with

$$u \in \mathcal{V} = \{-v, v\} = \{-3, 3\},$$

and matrices

$$A = \begin{bmatrix} -1 & 0.3 \\ 0.5 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Consider the static gain $K = \begin{bmatrix} 0.2295, & -2.7897 \end{bmatrix}$ computed using non robust approach given in [60]. The eigenvalues of A are -1.0724, and 1.0724 hence the open-loop linear system is unstable. Applying the method developed above, a relay feedback controller is designed to stabilize the system in the presence of a bounded perturbation $e(t) = \sqrt{\bar{e}} \times \begin{bmatrix} \sin(5t), & \cos(5t) \end{bmatrix}^T$ with $\bar{e} = 0.5 \times 10^{-4}$ and considering a decay rate $\alpha = 0.4$. An algorithm of minimization of c with a line search to find the parameters ϵ_1, ϵ_2, ϵ_3, r_γ and r_c is implemented.

The LMIs are feasible for

$$r_\gamma = 1.1,\ r_c = 0.1,\ \gamma = 3.8112,\ c = 0.0922,\ P = \begin{bmatrix} 1.0397 & -0.5053 \\ * & 3.3555 \end{bmatrix},$$

$$\text{and with parameters } \epsilon_1 = 1.4216,\ \epsilon_2 = 0.02,\ \text{and} \epsilon_3 = 1.1980 \times 10^3.$$

The computer simulations are realized for an initial condition $x(0) = \begin{bmatrix} 1.5, & 0.4 \end{bmatrix}^T$, and are reported in Figures 3.5-3.6.

As we can see from Figures 3.5 and 3.6, the states starting in the domain of attraction converge to a neighborhood of the origin and remain in it. In other words, from Figure 3.5 we can see that the states starting in the largest level set $\mathcal{E}(P, \gamma)$ contained in the convex $\mathcal{C}_v(K)$ evolve until reaching the smallest level set $\mathcal{E}(P, c)$ surrounding the origin. The states stay bounded and oscillate around the origin indefinitely as it can be seen from Figure 3.6 and this confirms the provided results.

3.5 Conclusion

This chapter has provided a method for the stabilization of switched affine systems with perturbed state-dependent switching laws. The method considers the perturbation in the states measurements. Qualitative conditions for stability have been developed. In addition, a numerical approach which allows to design of the switching surfaces and to enlarge or diminish the size of the chattering zone has been provided. The results

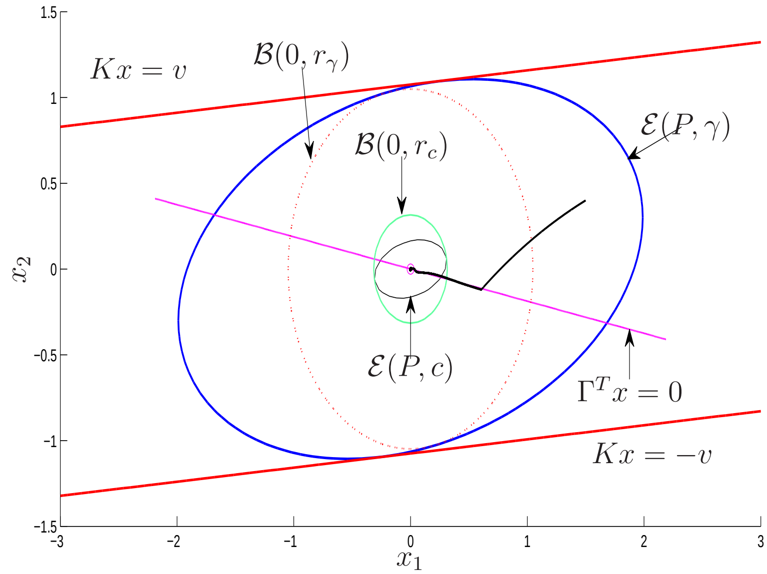

Figure 3.5: Phase plant of the closed-loop system with $\mathcal{E}(P, \gamma)$ an estimation of the domain of attraction, and $\mathcal{E}(P, c)$ the chattering zone - Example 15

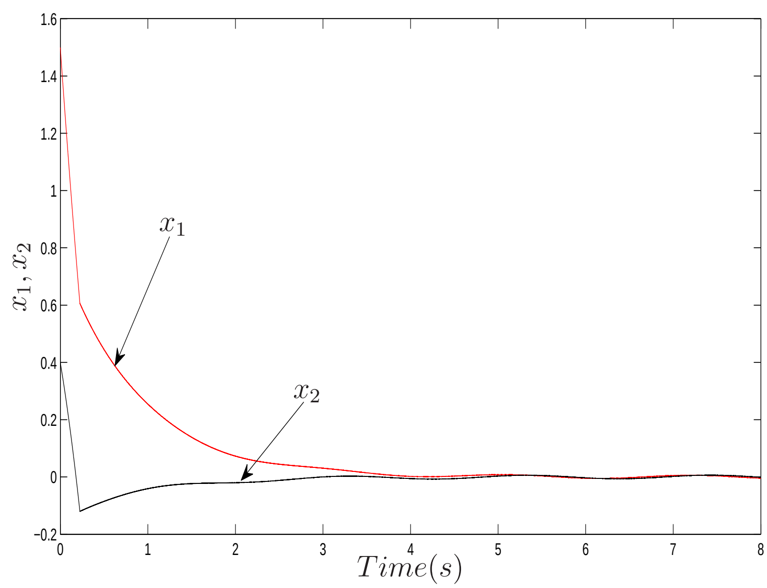

Figure 3.6: State variables - Example 15

have been particularized to provide a method for relay control design for LTI systems stabilization. The efficiency of the proposed method has been shown using numerical examples and computer simulations.

In the future, the approach may be improved in various manners. For example, it is of interest to considerer the bilinear terms in the switching law design method. In order to reduce the chattering, we can try to extend the approach to the case of min-switching stategy with min-dwell time condition. In this direction we can follow the approach in [58] or the one proposed in [27].

Chapter 4

Observer-based state-dependent switching law design

In this chapter we consider the stabilization problem by an observer-based switching control. First, a general result is proposed for the case of switched affine systems. An observer-based switching controller is designed in order to ensure the local stability of the closed-loop system. Both quadratic and non-quadratic Lyapunov functions are used to derive linear and nonlinear switching surfaces dependent on the estimated state while using a Luenberger observer. A constructive method based on LMI conditions is given in order to allow a numerical implementation of the proposed approach. Estimations of ellipsoidal and non-ellipsoidal domains of attraction are provided. In addition, we propose a numerical approach based on LMIs which allows the design of nonlinear switching surfaces dependent on the estimated state that guarantee the global asymptotic stability of the closed-loop switched affine systems at the origin. Moreover, the result is applied to the particular case of LTI systems with an observer-based relay feedback control. A separation principle is proved for both LTI systems with relay controller and switched affine systems when, to the best of our knowledge, no separation principle exist for these classes of systems in the literature. Finally, illustrative examples are provided in order to show the efficiency of the proposed methods and simulations are performed for a Buck converter structure.

4.1 Preliminaries and problem statement

Consider the following system

$$\dot{x} = Ax + \sum_{k=1}^{m} (\mathcal{N}_k x + b_k) u_{(k)}, \tag{4.1}$$

with $x \in \mathbb{R}^n$ and $u_{(k)}$ the k-th component of the input u. The input u is only allowed to take values in the set $\mathcal{V} = \{v_1, \ldots, v_N\} \subset \mathbb{R}^m$. $A \in \mathbb{R}^{n\times n}$, $B = \begin{bmatrix} b_1, & \ldots, & b_m \end{bmatrix} \in \mathbb{R}^{n\times m}$, and $\mathcal{N}_k \in \mathbb{R}^{n\times n}$ are the matrices describing the system. In this chapter, we consider the case where only a part of the state is measured, and the output is defined as

$$y = Cx. \tag{4.2}$$

We assume that:

A-1 The pairs $(A(v_i), B)$, for all $i \in \mathcal{I}_N$ with $A(v_i) = A + \sum_{k=1}^{m} \mathcal{N}_k v_{i(k)}$ are simultaneously quadratically stabilizable by a linear state feedback $k(x) = Hx$. This means that there exist matrices H and $P_1 = P_1^T \succ 0$ and a positive scalar α_H such that

$$A_{cl}(v_i)^T P_1 + P_1 A_{cl}(v_i) \leq -2\alpha_H P_1, \forall i \in \mathcal{I}_N, \tag{4.3}$$

with $A_{cl}(v_i) = A(v_i) + BH$.

A-2 The set $\text{int}\{\text{Conv}\{\mathcal{V}\}\}$ is nonempty and the null vector is contained inside ($0 \in \text{int}\{\text{Conv}\{\mathcal{V}\}\}$).

A-3 The pairs $(A(v_i), C)$, for all $i \in \mathcal{I}_N$ with $A(v_i) = A + \sum_{k=1}^{m} \mathcal{N}_k v_{i(k)}$ are simultaneously quadratically detectable. This means that there exist matrices L and $P_2 = P_2^T$ and a positive scalar α_o such that

$$A_o(v_i)^T P_2 + P_2 A_o(v_i) \preceq -2\alpha_o P_2, \forall i \in \mathcal{I}_N, \tag{4.4}$$

with $A_o(v_i) = A(v_i) + LC$.

Recall that for any finite set $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ there exists a finite number n_l of vectors $l_i \in \mathbb{R}^{1\times m}$, $i \in \mathcal{I}_{n_l}$ such that

$$\text{Conv}\{\mathcal{V}\} = \{u \in \mathbb{R}^m : l_i u \leq 1, \forall i \in \mathcal{I}_{n_l}\}. \tag{4.5}$$

Note also that typical control sets $\mathcal{V}$ are often of the form

$$\mathcal{V} = \mathrm{Vert}\{\mathcal{P}(c)\}, \tag{4.6}$$

where the hyperrectangle $\mathcal{P}(c)$, with c a strictly positive vector, is given by

$$\mathcal{P}(c) = \left\{ u = \begin{bmatrix} u_{(1)} \\ \vdots \\ u_{(m)} \end{bmatrix} \in \mathbb{R}^m : \left|u_{(k)}\right| \leq c_{(k)}, \forall k \in \mathcal{I}_m \right\}. \tag{4.7}$$

As in Chapter 2, we can consider a more general set $\mathcal{V}$ for which there exists a vector c such that $\mathcal{P}(c) \subseteq \mathrm{Conv}\{\mathcal{V}\}$. In the sequel, we will consider such a vector c and use the notation (4.7) to prove the results.

In this case we provide a method for the stabilization of system (4.1) by an observer-based switching controller given by

$$u(\hat{x}) \in \arg\min_{v\in\mathcal{V}} \Gamma(\hat{x}, v). \tag{4.8}$$

The estimated state $\hat{x} \in \mathbb{R}^n$ is computed by the full-order Luenberger state observer [80], [81]

$$\begin{cases} \dot{\hat{x}} = A(v_i)\hat{x} + Bu + L(\hat{y} - y), \\ \hat{y} = C\hat{x} \end{cases} \tag{4.9}$$

with $A(v_i) = A + \sum_{k=1}^{m} \mathcal{N}_k v_{i(k)}$.

Our objective is to provide conditions which guarantee the existence of a mapping $\Gamma(\hat{x}, v)$ (which characterizes the switching surfaces of the control law) and of a matrix L (the observer gain) such that the closed-loop system

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A(v_i) & 0 \\ -LC & A(v_i) + LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} u \tag{4.10}$$

with the control law (4.8) is locally exponentially stable at the origin.

Using the augmented state

$$\xi = \begin{bmatrix} \hat{x} \\ e \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \tag{4.11}$$

where $e = \hat{x} - x$ is the estimation error, the interconnection (4.1), (4.9) can be written as the augmented closed-loop system

$$\begin{cases} \dot{\xi} &= \begin{bmatrix} A(v_i) & LC \\ 0 & A(v_i) + LC \end{bmatrix} \xi + \begin{bmatrix} B \\ 0 \end{bmatrix} u(\hat{x}), \\ y &= \begin{bmatrix} C & -C \end{bmatrix} \xi, \end{cases} \tag{4.12}$$

which leads to

$$\begin{cases} \dot{\xi} &= \tilde{A}(v_i)\xi + \tilde{B}\bar{u}(\xi) = \tilde{\mathcal{X}}(\xi), \\ y &= \begin{bmatrix} C & -C \end{bmatrix} \xi, \end{cases} \tag{4.13}$$

where $\tilde{A}(v_i) = \begin{bmatrix} A(v_i) & LC \\ 0 & A(v_i) + LC \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$, and

$$\bar{u}(\xi) = u\left(\begin{bmatrix} I & 0 \end{bmatrix} \xi\right) = u(\hat{x}) \in \arg\min_{v \in \mathcal{V}} \Gamma\left(\begin{bmatrix} I & 0 \end{bmatrix} \xi, v\right). \tag{4.14}$$

In a similar way as in the full state feedback case we associate a differential inclusion

$$\dot{\xi} \in \mathcal{F}[\tilde{\mathcal{X}}](\xi), \tag{4.15}$$

to the system (4.13), (4.14) with the set valued map $\mathcal{F}[\tilde{\mathcal{X}}](\xi)$ designed as

$$\mathcal{F}[\tilde{\mathcal{X}}](\xi) = \bigcap_{\delta > 0} \bigcap_{\mu(\mathcal{S})=0} \overline{\text{Conv}}\{\tilde{\mathcal{X}}(\breve{\mathcal{B}}(\xi, \delta)) \setminus \mathcal{S}\}, \forall \xi \in \mathbb{R}^{2n}, \tag{4.16}$$

where $\breve{\mathcal{B}}(\xi, \delta)$ is the open ball centred on ξ with radius δ, and $\mathcal{S}$ is a set of measure (in the sense of Lebesgue) $\mu(\mathcal{S}) = 0$.

In this chapter we study the following problem:

Problem. Considering system (4.1), (4.2) and given a set $\mathcal{V}$, and Assumptions A-1, A-2 and A-3, design an observer-based switching controller such that the closed-loop system is asymptotically stable in some domain $\mathcal{D}$.

Since the state variables in real systems are not always fully available to measurements, here we generalize the approach proposed in Chapter 2 to observer-based controller design. In Section 4.2, qualitative conditions for the existence of a stabilizing switching law dependent on the estimated state are provided while using a Luenberger observer. Constructive methods based on LMIs allowing the design of linear and nonlinear switching surfaces and the estimation of the domain of attraction are provided. Using the properties of Lur'e type Lyapuvov functions, a numerical approach based on LMIs is also developed. This method allows the design of nonlinear switching surfaces that ensure the global asymptotic stability of the closed-loop switched affine system at the origin. In Section 4.3, we particularize the results to the case of LTI systems with an observer-based relay controller. Among other notable results, we prove that a separation principle holds for both LTI and switched affine systems.

4.2 Observer-based switching law design

In this section, we first provide conditions for the existence of a switching law dependent on the estimated state which ensures the local asymptotic stability of the closed-loop system to the origin. Then, we propose constructive methods based on LMIs allowing the design of the switching surfaces. Estimations of the domains of attraction (ellipsoidal and non-ellipsoidal) are provided by using quadratic and non-quadratic Lyapunov functions. Finally, a numerical approach allowing the global asymptotic stability of the closed-loopsystem at the origin is developed.

4.2.1 Qualitative existence conditions

This section deals with the local exponential stabilization of system (4.13), (4.14) and equivalently with the local exponential stabilization of system (4.1), (4.2), (4.9) by the switching law (4.8). Assumptions A.1, A.2 and A.3 are used to prove that there exist a switching mapping Γ and an observer gain L such that the system is locally exponentially stable. The results are given in the following.

Theorem 14: Assume that A.1, A.2, and A.3 hold. Then there exist a mapping $\Gamma(\hat{x}, v) = \hat{x}^T \Psi v$ (characterizing the switching hyperplanes) and a matrix L (the observer gain) such that system (4.13), (4.14) (or equivalently the closed-loop system (4.1), (4.2), (4.8), (4.9)) is locally exponentially stable at the origin.

Proof. Since all the pairs $(A(v_i), B), i \in \mathcal{I}_N$ are simultaneously stabilizable by a linear state feedback, then there exist a static gain H, a scalar $\alpha_H > 0$ and a symmetric positive definite matrix P_1 such that

$$A_{cl}(v_i)^T P_1 + P_1 A_{cl}(v_i) \preceq -2\alpha_H P_1, \forall i \in \mathcal{I}_N, \tag{4.17}$$

where $A_{cl}(v_i) = A(v_i) + BH$ with $A(v_i) = A + \sum_{k=1}^{m} \mathcal{N}_k v_{i(k)}$. Likewise, since all the pairs $\big(A(v_i), C\big), i \in \mathcal{I}_N$ are simultaneously quadratically detectable, then there exist an observer gain L, a scalar $\alpha_o > 0$ and a symmetric positive definite matrix P_2 such that

$$A_o(v_i)^T P_2 + P_2 A_o(v_i) \preceq -2\alpha_o P_2, \forall i \in \mathcal{I}_N, \tag{4.18}$$

where $A_o(v_i) = A(v_i) + LC$ with $A(v_i) = A + \sum_{k=1}^{m} \mathcal{N}_k v_{i(k)}$.

We want to prove that the system (4.13), (4.14) is locally exponentially stable in some domain $\mathcal{D}$ around the origin. Let us consider the quadratic Lyapunov function

$$V(\xi) = \xi^T P \xi \tag{4.19}$$

with the $2n \times 2n$ matrix

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & \lambda P_2 \end{bmatrix} \tag{4.20}$$

with a scaling term $\lambda > 0$ to be determined later. We want to show then that taking $\Gamma(\hat{x}, v) = \hat{x}^T \Psi v$ in (4.8), such that

$$\Psi = P_1 B = \begin{bmatrix} I & 0 \end{bmatrix} P \tilde{B} \tag{4.21}$$

with $\tilde{B}$ defined in (4.13) and for some positive scalar α we have

$$\sup_{y \in \mathcal{F}[\tilde{\mathcal{X}}](\xi)} \frac{\partial V}{\partial \xi} y \leq -2\alpha V(\xi), \tag{4.22}$$

in a domain $\mathcal{D} \subset \mathbb{R}^{2n}$ to be determined.

For each $\hat{x} \in \mathbb{R}^n$ we define the set of minimizers in which the control (4.8) takes values (with $\Gamma(\hat{x}, v) = \hat{x}^T \Psi v$). This corresponds to defining minimizers in which the control (4.14) takes values. We may remark that

$$\hat{x}^T \Psi v = \hat{x}^T P_1 B v = \xi^T \begin{bmatrix} I \\ 0 \end{bmatrix} P \tilde{B} v. \tag{4.23}$$

Therefore, we only need to look for minimizers of the right hand term in (4.23).

We define for any $z \in \mathbb{R}^{2n}$ the set of indexes $\mathcal{I}^*(z)$ such that

$$\mathcal{I}^*(z) = \left\{ i \in \mathcal{I}_N : z^T P \tilde{B}(v_j - v_i) \geq 0, \forall j \in \mathcal{I}_N \right\}, \tag{4.24}$$

with $\tilde{B}$ defined in (4.13). To $\mathcal{I}^*(z)$ we associate for all $z \in \mathbb{R}^{2n}$ the set $\Delta^*(z)$ of vectors defined by:

$$\Delta^*(z) = \left\{ \beta \in \Delta_N : \beta_i = 0, \forall i \in \mathcal{I}_N \setminus \mathcal{I}^*(z) \right\}. \tag{4.25}$$

Using (4.24) and (4.25), the set valued map $\mathcal{F}[\tilde{\mathcal{X}}](\xi)$ in (4.16) satisfies

$$\mathcal{F}[\tilde{\mathcal{X}}](\xi) \subseteq \mathcal{F}^*[\tilde{\mathcal{X}}](\xi) \tag{4.26}$$

with

$$\begin{aligned} \mathcal{F}^*[\tilde{\mathcal{X}}](\xi) &= \overline{\underset{i \in \mathcal{I}^*(\xi)}{\mathrm{Conv}}} \{ \tilde{A}(v_i)\xi + \tilde{B}v_i \} \\ &= \{ \tilde{A}(v(\beta))\xi + \tilde{B}v(\beta) : \beta \in \Delta^*(\xi) \}, \end{aligned} \tag{4.27}$$

with $v(\beta) = \sum_{i=1}^{N} \beta_i v_i$.

Consider the gain H satisfying (4.17). From (4.26) and (4.27) and using the fact that $\Delta^*(\xi)$ is compact, we have

$$\begin{aligned} \sup_{y \in \mathcal{F}[\tilde{\mathcal{X}}](\xi)} \frac{\partial V}{\partial \xi} y &\leq \sup_{y \in \mathcal{F}^*[\tilde{\mathcal{X}}](\xi)} \frac{\partial V}{\partial \xi} y \\ &= \sup_{\beta \in \Delta^*(\xi)} \left\{ \frac{\partial V}{\partial \xi} \left\{ \tilde{A}(v(\beta))\xi + \tilde{B}v(\beta) \right\} \right\} \\ &= \max_{\beta \in \Delta^*(\xi)} \left\{ \frac{\partial V}{\partial \xi} \left\{ \tilde{A}(v(\beta))\xi + \tilde{B}v(\beta) \right\} \right\}. \end{aligned} \tag{4.28}$$

Thus, in order to show (4.22), it is sufficient to prove that for some scalar $\alpha > 0$ we have

$$\max_{\beta \in \Delta^*(\xi)} \left\{ \frac{\partial V}{\partial \xi} \left\{ \tilde{A}(v(\beta))\xi + \tilde{B}v(\beta) \right\} \right\} \leq -2\alpha V(\xi), \tag{4.29}$$

in a domain $\mathcal{D}$ to be determined.

Note that, since Assumption A-2 holds, then there exists a neighborhood of the origin $\mathcal{E}(P, \gamma) \subset \mathbb{R}^{2n}$, with $\gamma > 0$ such that for all $\xi = \begin{bmatrix} \hat{x} \\ e \end{bmatrix} \in \mathcal{E}(P, \gamma)$, we have

$$H\hat{x} = \mathcal{H}\xi \in \mathrm{Conv}\{\mathcal{V}\}, \tag{4.30}$$

with $\mathcal{H} = \begin{bmatrix} H & 0 \end{bmatrix}$.

Therefore, for all $\xi \in \mathcal{E}(P, \gamma)$ there exist scalars $\rho_j(\xi)$, $j \in \mathcal{I}_N$ such that $\sum_{j=1}^{N} \rho_j(\xi) = 1$ and

$$\mathcal{H}\xi = \sum_{j=1}^{N} \rho_j(\xi) v_j. \tag{4.31}$$

From (4.24), for all $i \in \mathcal{I}^*(\xi)$ we have

$$\xi^T P \tilde{B}(v_j - v_i) \geq 0, \forall j \in \mathcal{I}_N. \tag{4.32}$$

Then, for any $\beta \in \Delta^*(\xi)$, we have

$$\xi^T P \tilde{B}(v_j - v(\beta)) \geq 0, \forall j \in \mathcal{I}_N. \tag{4.33}$$

Then, considering (4.31), and multiplying the last inequalities by $\rho_j(\xi)$ and summing the N elements we obtain

$$\xi^T P \tilde{B}(\mathcal{H}\xi - v(\beta)) \geq 0. \tag{4.34}$$

Adding this to the left part of (4.29), it comes

$$\begin{aligned}
&\max_{\beta \in \Delta^*(\xi)} \left\{ \frac{\partial V}{\partial \xi} \left\{ \tilde{A}(v(\beta))\xi + \tilde{B}v(\beta) \right\} \right\} \\
&\leq \max_{\beta \in \Delta^*(\xi)} \left\{ 2\xi^T P \{ \tilde{A}(v(\beta))\xi + \tilde{B}v(\beta) \} \right\} + 2\xi^T P \tilde{B}(\mathcal{H}\xi - v(\beta)) \\
&= 2\frac{\partial V}{\partial \xi} \begin{bmatrix} A_{cl}(v(\beta)) & LC \\ 0 & A_o \end{bmatrix} \xi = 2\frac{\partial V}{\partial \xi}(\tilde{A}_{cl}(v(\beta))\xi) = 2\xi^T P \tilde{A}_{cl}(v(\beta))\xi.
\end{aligned} \tag{4.35}$$

Thus, in order to show (4.22), it is sufficient to prove that

$$2\xi^T P \tilde{A}_{cl}(v(\beta))\xi \leq -2\alpha V(\xi), \forall \xi \in \mathcal{E}(P, \gamma), \tag{4.36}$$

which holds if

$$\tilde{A}_{cl}^T(v(\beta))P + P\tilde{A}_{cl}(v(\beta)) \preceq -2\alpha P. \tag{4.37}$$

Note that

$$\begin{aligned}
&\tilde{A}_{cl}^T(v(\beta))P + P\tilde{A}_{cl}(v(\beta)) + 2\alpha P = \\
&\begin{bmatrix} A_{cl}(v(\beta))^T + P_1 A_{cl}(v(\beta)) + 2\alpha P_1 & P_1 LC \\ (LC)^T P_1 & \lambda(A_o(v(\beta))^T P_2 + P_2 A_o(v(\beta)) + 2\alpha P_2) \end{bmatrix}.
\end{aligned} \tag{4.38}$$

Applying the Schur complement, the matrix (4.38) is negative if

$$A_o(v(\beta))^T P_2 + P_2 A_o(v(\beta)) + 2\alpha P_2 \preceq 0 \tag{4.39}$$

and

$$\begin{aligned} &(A_{cl}(v(\beta))^T P_1 + P_1 A_{cl}(v(\beta)) + 2\alpha P_1) \\ &- \frac{1}{\lambda} P_1 LC[2\alpha P_2 + A_o(v(\beta))^T P_2 + P_2 A_o(v(\beta))]^{-1}(LC)^T P_1 \preceq 0. \end{aligned} \tag{4.40}$$

Since (4.17) and (4.18) are satisfied and $\beta \in \Delta^*(x)$, it is obvious that if we take $\alpha \leq \min(\alpha_H, \alpha_o)$, and λ large enough both inequalities are verified. □

In Theorem 14 we have shown that there exists a switching law depending on the estimated state such that the closed-loop switched affine system is locally exponentially stable at the origin. In the following we provide a constructive method based on LMIs that allows to design such a controller and provide an estimation of the domain of attraction. In addition, using switching Lyapunov functions, LMI conditions allowing the design of nonlinear switching surfaces when the state is not fully available for measurement is proposed while using a Luenberger observer.

4.2.2 A constructive method for observer-based nonlinear switching surfaces design for switched affine systems stabilization

Here we are interested in finding a constructive procedure providing a mapping Γ and an observer gain L such that the closed-loop system (4.13), (4.14) (or equivalently (4.1), (4.2), (4.8), (4.9)) is locally exponentially stable at the origin. We would also like to provide an estimation of the domain of attraction. In what follows, a numerical approach to deal with the design problem is given. An LMI solution is proposed hereafter.

For $\mathcal{H} \in \mathbb{R}^{m \times 2n}$, let us define the set $\mathcal{C}_v(\mathcal{H})$ as follows

$$\mathcal{C}_v(\mathcal{H}) = \{\xi \in \mathbb{R}^{2n} : l_i \mathcal{H} \xi \leq 1, \forall i \in \mathcal{I}_{n_l}\}, \tag{4.41}$$

where l_i is given in (4.5).

Theorem 15: Assume that A.1, A.2, and A.3 hold. Consider a tuning parameter $\alpha > 0$.

1. If there exist positive definite matrices $Q_1 \in \mathbb{R}^{n\times n}$ and $P_2 \in \mathbb{R}^{n\times n}$, and scalars $\theta_1 > 0$ and $\theta_2 > 0$ such that

$$Q_1 A(v_i)^T + A(v_i)Q_1 - \theta_1 BB^T \preceq -2\alpha Q_1, \forall i \in \mathcal{I}_N, \tag{4.42}$$

$$A(v_i)^T P_2 + P_2 A(v_i) - \theta_2 C^T C \preceq -2\alpha P_2, \forall i \in \mathcal{I}_N \tag{4.43}$$

then, system (4.1), (4.2), (4.9) with the switching law

$$u(\hat{x}) \in \arg\min_{v\in\mathcal{V}} \hat{x}^T \Psi v, \tag{4.44}$$

is locally α-stable with $\Psi = Q_1^{-1}B$ and $L = -\frac{\theta_2}{2}P_2^{-1}C^T$.

2. If in addition we consider $\lambda > 0$ such that

$$\begin{bmatrix} A_{cl}(v_i)^T P_1 + P_1 A_{cl}(v_i) + 2\alpha P_1 & P_1 LC \\ * & \lambda(A_o(v_i)^T P_2 + P_2 A_o(v_i) + 2\alpha P_2) \end{bmatrix} \preceq 0, \forall i \in \mathcal{I}_N, \tag{4.45}$$

with $A_{cl}(v_i) = A(v_i) - \frac{\theta_1}{2}BB^T Q_1^{-1}$ and $A_o(v_i) = A(v_i) + LC$, then an estimation of the domain of attraction is given by $\mathcal{E}(P, \gamma^*)$ with $P = \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & \lambda P_2 \end{bmatrix}$,

$$\gamma^* \leq \min_{i\in\mathcal{I}_{n_l}} (l_i \mathcal{H} P^{-1} \mathcal{H}^T l_i^T)^{-1}, \tag{4.46}$$

and $\mathcal{H} = \begin{bmatrix} H & 0 \end{bmatrix}$, with $H = -\frac{\theta_1}{2}B^T Q_1^{-1}$.

3. If there exist positive definite matrices $P_1 \in \mathbb{R}^{n\times n}$, $Q_2 \in \mathbb{R}^{n\times n}$, and positive scalars θ_1 and θ_2 such that the LMIs (4.42), (4.43) are feasible for some $\alpha > 0$, and if there exist two symmetric positive definite matrices $\tilde{P}_1 \in \mathbb{R}^{n\times n}$ and $\tilde{P}_2 \in \mathbb{R}^{n\times n}$, two diagonal positive definite matrices $\tilde{\Omega} \in \mathbb{R}^{m\times m}$ and $\tilde{M} \in \mathbb{R}^{m\times m}$, a matrix $\tilde{\Upsilon} \in \mathbb{R}^{m\times n}$ and a strictly positive vector $\tilde{\tau} \in \mathbb{R}^m$ such that

$$\begin{bmatrix} A_{cl}(v_i)^T \tilde{P}_1 + \tilde{P}_1 A_{cl}(v_i) & \tilde{P}_1 LC & \tilde{P}_1 B - \tilde{\Upsilon}^T - A_{cl}(v_i)^T H^T \tilde{\Omega} \\ * & A_o(v_i)^T \tilde{P}_2 + \tilde{P}_2 A_o(v_i) & -C^T L^T H^T \tilde{\Omega} \\ * & * & -2\tilde{M} - \tilde{\Omega} HB - (\tilde{\Omega} HB)^T \end{bmatrix} \prec 0, \forall i \in \mathcal{I}_N, \tag{4.47}$$

and

$$\begin{bmatrix} \tilde{P}_1 & \tilde{M}_{(k,k)}H_{(k)}^T - \tilde{\Upsilon}_{(k)}^T \\ * & \tilde{\tau}_{(k)}c_{(k)}^2 \end{bmatrix} \succeq 0, \forall k \in \mathcal{I}_m, \tag{4.48}$$

with $L = -\frac{\theta_2}{2}P_2^{-1}C^T$, $A_{cl}(v_i) = A(v_i) - \frac{\theta_1}{2}BB^TQ_1^{-1}$, $H = -\frac{\theta_1}{2}B^TQ_1^{-1}$, and $A_o(v_i) = A(v_i) + LC$,

then system (4.1), (4.2), (4.9) with the switching law

$$u(\hat{x}) \in \arg\min_{v \in \mathcal{V}}(\hat{x}^T\tilde{P}_1 - \phi(H\hat{x})^T\tilde{\Omega}H)Bv \tag{4.49}$$

is locally asymptotically stable.

An estimation of the domain of attraction is given by

$$\mathcal{L}_V(\eta^{-1}) = \{\xi \in \mathbb{R}^{2n} : V(\xi) \leq \eta^{-1}\}, \tag{4.50}$$

with

$$V(\xi) = \xi^T\tilde{P}\xi - 2\sum_{k=1}^{m}\int_0^{\mathcal{H}_{(k)}\xi}\phi_{(k)}(\sigma)\tilde{\Omega}_{(k,k)}d\sigma, \tag{4.51}$$

$\xi = \begin{bmatrix} \hat{x} \\ e \end{bmatrix}$, $\tilde{P} = \begin{bmatrix} \tilde{P}_1 & 0 \\ 0 & \tilde{P}_2 \end{bmatrix}$, $\mathcal{H} = \begin{bmatrix} H & 0 \end{bmatrix}$, and $\eta \geq \max_{k \in \mathcal{I}_m}\left\{\frac{\tilde{\tau}_{(k)}}{\tilde{M}_{(k,k)}^2}\right\}$.

Proof. 1. Consider positive definite matrices Q_1, P_2, and positive scalars θ_1, θ_2 such that (4.42), (4.43) hold. Then, we want to prove that the closed-loop system (4.13), (4.44), with $\Psi = Q_1^{-1}B$ and $L = -\frac{\theta_2}{2}P_2^{-1}C^T$, is locally α-stable in some domain $\mathcal{D} \subset \mathbb{R}^{2n}$ around the origin.

We can remark that the feasibility of (4.42) implies that the inequality

$$A_{cl}(v_i)^TP_1 + P_1A_{cl}(v_i) \preceq -2\alpha P_1, \forall i \in \mathcal{I}_N \tag{4.52}$$

is verified with $P_1 = Q_1^{-1}$, and $A_{cl}(v_i) = A(v_i) - \frac{\theta_1}{2}BB^TQ_1^{-1}$ (see for instance [17]).

Similarly, the feasibility of LMI (4.43) implies that the inequality

$$A_o(v_i)^TP_2 + P_2A_o(v_i) \preceq -2\alpha P_2, \forall i \in \mathcal{I}_N \tag{4.53}$$

is verified with $L = -\frac{\theta_2}{2}P_2^{-1}C^T$ and $A_o(v_i) = A(v_i) + LC$. Let us consider the quadratic Lyapunov function $V(\xi) = \xi^TP\xi$, with $P = \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & \lambda P_2 \end{bmatrix}$ and $\lambda > 0$.

Following the same steps as in the proof of Theorem 14 we can show that if we take λ large enough, the Lyapunov function V ensures the stability of the closed-loop system (4.1), (4.9), (4.44) in the domain $\mathcal{E}(P,\gamma)$.

2. It is now required to estimate the domain of attraction for system (4.13) with the switching law (4.44). Thus, we want to determine a scalar γ^* characterizing the ellipsoid $\mathcal{E}(P,\gamma^*)$ such that

$$\sup_{\varsigma\in\mathcal{F}[\tilde{\mathcal{X}}](\xi)} \frac{\partial V}{\partial \xi}\varsigma \leq -2\alpha V(\xi), \forall \xi \in \mathcal{E}(P,\gamma^*). \tag{4.54}$$

For a given decay rate α and if LMIs (4.42) and (4.43) are feasible, then from the result above, there exist at least one scalar λ satisfying the inequality (4.45).

Considering such a scalar λ, our objective here is to provide an estimation $\mathcal{E}(P,\gamma^*)$ of the domain of attraction such that

$$\mathcal{E}(P,\gamma^*) \subseteq \mathcal{C}_v(\mathcal{H}), \tag{4.55}$$

with $\mathcal{C}_v(\mathcal{H})$ defined in (4.41). Note that, if the set $\mathcal{E}(P,\gamma^*)$ satisfies (4.55), then according to (4.5), one will have that for all $\xi \in \mathcal{E}(P,\gamma^*)$, $\mathcal{H}\xi \in \text{Conv}\{\mathcal{V}\}$.

For this inclusion to hold it is both necessary and sufficient that none of the hyperplanes $l_i\mathcal{H}\xi = 1, i \in \mathcal{I}_{n_l}$ crosses the level set $\mathcal{E}(P,\gamma^*)$. Note that for any $i \in \mathcal{I}_{n_l}$, the minimum of V along the hyperplane $\{\xi \in \mathbb{R}^{2n} : l_i\mathcal{H}\xi = 1\}$ is given as (see [17])

$$\min_{l_i\mathcal{H}\xi=1} \xi^T P \xi = \min_{i\in\mathcal{I}_{n_l}} (l_i\mathcal{H}P^{-1}\mathcal{H}^T l_i^T)^{-1}. \tag{4.56}$$

We can remark that by taking γ^* as

$$\gamma^* \leq \min_{l_i\mathcal{H}\xi=1} \xi^T P \xi = (l_i\mathcal{H}P^{-1}\mathcal{H}^T l_i^T)^{-1}, \tag{4.57}$$

the inclusion (4.55) is verified and $\mathcal{E}(P,\gamma^*)$ is thus an estimation of the domain of attraction.

3. The aim here is to show that if there exist positive definite matrices Q_1, P_2, and positive scalars θ_1 and θ_2 satisfying (4.42) and (4.43) and the LMIs (4.47) and (4.48) are feasible with $H = -\frac{\theta_1}{2}B^T Q_1^{-1}$ and $L = -\frac{\theta_2}{2}P_2^{-1}C^T$, then the closed-loop system (4.1), (4.2), (4.9), (4.49) is locally asymptotically stable in the domain of attraction $\mathcal{L}_V(\eta^{-1})$.

Recall that we associate to the closed-loop system (4.13), (4.49) the differential inclusion (4.27), which is locally asymptotically stable in the domain $\mathcal{L}_V(\eta^{-1})$ if

$$\sup_{\varsigma\in\mathcal{F}[\tilde{\mathcal{X}}](\xi)} \frac{\partial V}{\partial \xi}\varsigma < 0, \forall \xi \in \mathcal{L}_V(\eta^{-1}) \setminus \{0\}. \tag{4.58}$$

Thanks to the particular structure of the matrices $\tilde{P} = \begin{bmatrix} \tilde{P}_1 & 0 \\ 0 & \tilde{P}_2 \end{bmatrix}$, $\mathcal{H} = \begin{bmatrix} H & 0 \end{bmatrix}$, and $\tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$, we can show that

$$(\hat{x}^T\tilde{P}_1 - \phi(H\hat{x})^T\tilde{\Omega}H)Bv = (\xi^T\tilde{P} - \phi(\mathcal{H}\xi)^T\tilde{\Omega}\mathcal{H})\tilde{B}v. \tag{4.59}$$

Let us first define for all $\zeta \in \mathbb{R}^{2n}$ the set of minimizers in which the controller (4.49) takes values:

$$\begin{aligned} \mathcal{I}^*(\zeta) = \{ & i \in \mathcal{I}_N : \\ & (\zeta^T\tilde{P} - \phi(\mathcal{H}\zeta)^T\tilde{\Omega}\mathcal{H})\tilde{B}(v_j - v_i) \geq 0, \forall j \in \mathcal{I}_N \Big\}. \end{aligned} \tag{4.60}$$

To this set of indexes, we associate the set of vectors $\Delta^*(\zeta)$ defined for all $\zeta \in \mathbb{R}^{2n}$ as

$$\Delta^*(\zeta) = \{\beta \in \Delta_N : \beta_{(i)} = 0, \forall i \in \mathcal{I}_N \setminus \mathcal{I}^*(\zeta)\}. \tag{4.61}$$

Using (4.60) and (4.61) we obtain that the set valued map $\mathcal{F}[\tilde{\mathcal{X}}](\xi)$ satisfies the following relation

$$\mathcal{F}[\tilde{\mathcal{X}}](\xi) \subseteq \mathcal{F}^*[\tilde{\mathcal{X}}](\xi), \tag{4.62}$$

with

$$\begin{aligned} \mathcal{F}^*[\tilde{\mathcal{X}}](\xi) &= \text{Conv}\{\tilde{A}(v_i)\xi + \tilde{B}v_i : i \in \mathcal{I}^*(\xi)\} \\ &= \left\{\tilde{A}(v(\beta))\xi + \tilde{B}v(\beta) : \beta \in \Delta^*(\xi)\right\}, \end{aligned} \tag{4.63}$$

where $v(\beta) = \sum_{i=1}^N \beta_{(i)}v_i$. Thus, using the same argument as in the first part of proof we can show that to prove (4.58) it is sufficient to show that

$$\max_{\beta\in\Delta^*(\zeta)} \left\{\frac{\partial V}{\partial \xi}\left(\tilde{A}(v(\beta))\xi + \tilde{B}v(\beta)\right)\right\} < 0, \forall \xi \in \mathcal{L}_V(\eta^{-1}) \setminus \{0\}. \tag{4.64}$$

Considering the particular structure of matrices $\mathcal{H} = \begin{bmatrix} H & 0 \end{bmatrix}$, $\tilde{P} = \begin{bmatrix} \tilde{P}_1 & 0 \\ 0 & \tilde{P}_2 \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$, and $\tilde{\mathcal{Y}} = \begin{bmatrix} \tilde{\Upsilon} & 0 \end{bmatrix}$, we can show that (4.47) is equivalent to

$$\begin{bmatrix} \tilde{A}_{cl}(v_i)^T \tilde{P} + \tilde{P}\tilde{A}_{cl}(v_i) & \tilde{P}\tilde{B} - \tilde{\mathcal{Y}}^T - \tilde{A}_{cl}(v_i)^T \mathcal{H}^T \tilde{\Omega} \\ * & -2\tilde{M} - \tilde{\Omega}\mathcal{H}\tilde{B} - (\tilde{\Omega}\mathcal{H}\tilde{B})^T \end{bmatrix} \prec 0, \forall i \in \mathcal{I}_N. \tag{4.65}$$

From (4.65) we have for all $\Xi \in \mathbb{R}^{2n+m} \setminus \{0\}$

$$\Xi^T \begin{bmatrix} \tilde{A}_{cl}(v_i)^T \tilde{P} + \tilde{P}\tilde{A}_{cl}(v_i) & \tilde{P}\tilde{B} - \tilde{\mathcal{Y}}^T - \tilde{A}_{cl}(v_i)^T \mathcal{H}^T \tilde{\Omega} \\ * & -2\tilde{M} - \tilde{\Omega}\mathcal{H}\tilde{B} - (\tilde{\Omega}\mathcal{H}\tilde{B})^T \end{bmatrix} \Xi < 0, \forall i \in \mathcal{I}_N. \tag{4.66}$$

Considering $\Xi = \begin{bmatrix} \xi \\ \phi(\mathcal{H}\xi) \end{bmatrix}$ with $\xi \in \mathbb{R}^{2n} \setminus \{0\}$ and $\mathcal{G} = \tilde{M}^{-1}\tilde{\mathcal{Y}}$, (4.66) leads to

$$\begin{aligned} &(2\xi^T \tilde{P} - 2\phi(\mathcal{H}\xi)^T \tilde{\Omega}\mathcal{H})((\tilde{A}(v_i) + \tilde{B}\mathcal{H})\xi + \tilde{B}\phi(\mathcal{H}\xi)) - 2\phi(\mathcal{H}\xi)^T \tilde{M}(\phi(\mathcal{H}\xi) + \mathcal{G}\xi) \\ &< 0, \forall \xi \in \mathbb{R}^{2n} \setminus \{0\}, \forall i \in \mathcal{I}_N. \end{aligned} \tag{4.67}$$

Let us consider the notation $k(\xi) = \mathcal{H}\xi + \phi(\mathcal{H}\xi)$. According to Lemma 3, for any $\xi \in \mathbb{R}^{2n}$, we have $k(\xi) \in \mathcal{P}(c) \subseteq \text{Conv}\{\mathcal{V}\}$. Therefore, there exist N positive scalars $\rho_j(\xi)$, $\sum_{j=1}^N \rho_j(\xi) = 1$ such that

$$k(\xi) = \mathcal{H}\xi + \phi(\mathcal{H}\xi) = \sum_{j=1}^N \rho_j(\xi) v_j. \tag{4.68}$$

Using this property in (4.67), we get

$$\begin{aligned} &(2\xi^T \tilde{P} - 2\phi(\mathcal{H}\xi)^T \tilde{\Omega}\mathcal{H})(\tilde{A}(v_i)\xi + \tilde{B}\sum_{j=1}^N \rho_j(\xi) v_j) - 2\phi(\mathcal{H}\xi)^T \tilde{M}(\phi(\mathcal{H}\xi) + \mathcal{G}\xi) \\ &< 0, \forall \xi \in \mathbb{R}^{2n} \setminus \{0\}, \forall i \in \mathcal{I}_N. \end{aligned} \tag{4.69}$$

From (4.60), for all $i \in \mathcal{I}^*(\xi)$ we have

$$(\xi^T \tilde{P} - \phi(\mathcal{H}\xi)^T \tilde{\Omega}\mathcal{H})\tilde{B}(v_j - v_i) \geq 0, \forall j \in \mathcal{I}_N. \tag{4.70}$$

Therefore, by adding and subtracting the term
$2\sum_{j=1}^N \rho_j(\xi)(\xi^T \tilde{P} - \phi(\mathcal{H}\xi)^T \tilde{\Omega}\mathcal{H})\tilde{B}(v_j - v_i)$ to (4.69), we obtain

$$\begin{aligned} &(2\xi^T \tilde{P} - 2\phi(\mathcal{H}\xi)^T \tilde{\Omega}\mathcal{H})(\tilde{A}(v_i)\xi + \tilde{B}v_i) - 2\phi(\mathcal{H}\xi)^T \tilde{M}(\phi(\mathcal{H}\xi) + \mathcal{G}\xi) \\ &+ 2\sum_{j=1}^N \rho_j(\xi)(\xi\tilde{P} - \phi(\mathcal{H}\xi)^T \tilde{\Omega}\mathcal{H})(v_j - v_i) < 0, \forall \xi \in \mathbb{R}^{2n} \setminus \{0\}, \forall i \in \mathcal{I}_N. \end{aligned} \tag{4.71}$$

Thus, for all $\beta \in \Delta^*(\xi)$ we obtain

$$\begin{aligned}&(2\xi^T\tilde{P} - 2\phi(\mathcal{H}\xi)^T\tilde{\Omega}\mathcal{H})(\tilde{A}(v(\beta))\xi + \tilde{B}v(\beta)) - 2\phi(\mathcal{H}\xi)^T\tilde{M}(\phi(\mathcal{H}\xi) + \mathcal{G}\xi)\\&+ 2\sum_{j=1}^{N}\rho_j(\xi)(\xi\tilde{P} - \phi(\mathcal{H}\xi)^T\tilde{\Omega}\mathcal{H})(v_j - v(\beta)) < 0, \forall\xi \in \mathbb{R}^{2n} \setminus \{0\},\end{aligned} \tag{4.72}$$

with

$$(\xi^T\tilde{P} - \phi(\mathcal{H}\xi)^T\tilde{\Omega}\mathcal{H})\tilde{B}(v_j - v(\beta)) \geq 0, \forall j \in \mathcal{I}_N. \tag{4.73}$$

Applying Lemma 1 with $w_1 = \mathcal{H}\xi$, $w_2 = \mathcal{G}\xi$ and using the definition of $\mathcal{P}(c)$ in (4.7), we have

$$\phi(\mathcal{H}\xi)^T\tilde{M}(\phi(\mathcal{H}\xi) + \mathcal{G}\xi) \leq 0, \forall\xi \in \tilde{\mathcal{A}}, \tag{4.74}$$

with

$$\tilde{\mathcal{A}} = \{\xi \in \mathbb{R}^{2n} : \left|(\mathcal{H}_{(k)} - \mathcal{G}_{(k)})\xi\right| \leq c_{(k)}, \forall k \in \mathcal{I}_m\}. \tag{4.75}$$

Note that $\frac{\partial V}{\partial \xi} = 2\xi^T\tilde{P} - 2\phi(\mathcal{H}\xi)^T\tilde{\Omega}\mathcal{H}$. Therefore, taking this into account, as well as (4.72) and (4.74), we obtain

$$\begin{aligned}\max_{\beta\in\Delta^*(\xi)}\left\{\frac{\partial V}{\partial \xi}\left(\tilde{A}(v(\beta))\xi + \tilde{B}v(\beta)\right)\right\} \leq & \max_{\beta\in\Delta^*(\xi)}\left\{\frac{\partial V}{\partial \xi}\left(\tilde{A}(v(\beta))\xi + \tilde{B}v(\beta)\right)\right\}\\&- 2\phi(\mathcal{H}\xi)^T\tilde{M}(\phi(\mathcal{H}\xi) + \mathcal{G}\xi)\\&+ 2\sum_{j=1}^{N}\rho_j(\xi)(\xi\tilde{P} - \phi(\mathcal{H}\xi)^T\tilde{\Omega}\mathcal{H})(v_j - v(\beta))\\&< 0, \forall\xi \in \tilde{\mathcal{A}}.\end{aligned} \tag{4.76}$$

In order to show (4.64) (and thus (4.58)), we will now prove that $\mathcal{L}_V(\eta^{-1}) \subseteq \tilde{\mathcal{A}}$.

Considering the same arguments as in Theorem 10 we can show that (4.48) is equivalent to

$$\begin{bmatrix}\tilde{P}_1 & H_{(k)}^T - \tilde{G}_{(k)}^T\\ H_{(k)} - G_{(k)} & \eta c_{(k)}^2\end{bmatrix} \succeq 0, \forall k \in \mathcal{I}_m. \tag{4.77}$$

Applying the Schur complement to (4.77), we obtain

$$\frac{\tilde{P}_1}{\eta^{-1}} - (H_{(k)} - \tilde{G}_{(k)})^T(c_{(k)}^2)^{-1}(H_{(k)} - G_{(k)}) \succeq 0, \forall k \in \mathcal{I}_m. \tag{4.78}$$

Since $\tilde{P}_2 \succ 0$ and $\eta > 0$ then (4.78) leads to

$$\begin{bmatrix} \frac{\tilde{P}_1}{\eta^{-1}} - (H_{(k)} - \tilde{G}_{(k)})^T (c_{(k)}^2)^{-1} (H_{(k)} - G_{(k)}) & 0 \\ 0 & \frac{\tilde{P}_2}{\eta^{-1}} \end{bmatrix} \succeq 0, \forall k \in \mathcal{I}_m, \tag{4.79}$$

which is equivalent to

$$\begin{bmatrix} \frac{\tilde{P}_1}{\eta^{-1}} & 0 \\ 0 & \frac{\tilde{P}_2}{\eta^{-1}} \end{bmatrix} - \begin{bmatrix} (H_{(k)} - \tilde{G}_{(k)})^T \\ 0 \end{bmatrix} (c_{(k)}^2)^{-1} \begin{bmatrix} H_{(k)} - \tilde{G}_{(k)} & 0 \end{bmatrix} \succeq 0, \tag{4.80}$$

for all $k \in \mathcal{I}_m$.

For all $\xi \in \mathbb{R}^{2n}$ and $\mathcal{G} = \tilde{M}^{-1}\tilde{\mathcal{Y}} = \begin{bmatrix} \tilde{G} & 0 \end{bmatrix}$, (4.80) leads to

$$\xi^T \frac{\tilde{P}}{\eta^{-1}} \xi - \xi^T (\mathcal{H}_{(k)} - \mathcal{G}_{(k)})^T (c_{(k)}^2)^{-1} (\mathcal{H}_{(k)} - \mathcal{G}_{(k)}) \xi \geq 0, \forall k \in \mathcal{I}_m. \tag{4.81}$$

From this we obtain the inclusion

$$\mathcal{E}(\tilde{P}, \eta^{-1}) \subseteq \tilde{\mathcal{A}}. \tag{4.82}$$

In addition, according to Lemma 2, we have

$$\xi^T \tilde{P} \xi \leq V(\xi) \leq \xi^T \bar{P} \xi, \tag{4.83}$$

where $\bar{P} = \begin{bmatrix} \tilde{P}_1 + H^T \Omega H & 0 \\ 0 & \tilde{P}_2 \end{bmatrix}$.

This leads to

$$\mathcal{E}(\bar{P}, \eta^{-1}) \subseteq \mathcal{L}_V(\eta^{-1}) \subseteq \mathcal{E}(\tilde{P}, \eta^{-1}). \tag{4.84}$$

Thus, from (4.84) and (4.82), we have

$$\mathcal{L}_V(\eta^{-1}) \subseteq \tilde{\mathcal{A}}. \tag{4.85}$$

From this and using (4.76) we have shown that

$$\begin{aligned} \sup_{\varsigma \in \mathcal{F}[\tilde{\mathcal{X}}](\xi)} \frac{\partial V}{\partial \xi} \varsigma &\leq \max_{\beta \in \Delta^*(\xi)} \left\{ \frac{\partial V}{\partial \xi} \left(\tilde{A}(v(\beta))\xi + \tilde{B}v(\beta) \right) \right\} \\ &< 0, \forall \xi \in \mathcal{L}_V(\eta^{-1}) \setminus \{0\}, \end{aligned} \tag{4.86}$$

which ends the proof. □

Remark 13: In Theorem 15, we have provided LMI conditions for the stabilization of system (4.1) when the state variables are not fully available to measurements. Instead of the existence of a Hurwitz convex combination of the evolution matrices of the individual subsystem of the augmented switched affine system, the proposed approach here requires that the pairs $(A(v_i), B)$ are all simultaneously quadratically stabilizable and the pairs $(A(v_i), C)$ are simultaneously quadratically detectable. The approach generalizes the method proposed in the Proposition 2 provided in Chapter 1 to the case of observer-based state-dependent switching laws design. The method proposed in Chapter 2 using non-quadratic Lyapunov functions to enlarge the domain of attraction is equally extended to the design of nonlinear switching surfaces dependent on the estimated state.

Remark 14: We can remark that (4.42) and (4.43) do not share cross LMI variables, thus they can be solved separately. Therefore, the control law matrix Ψ and the observer gain L designed independently ensures the stabilization of the closed-loop system at the origin. This shows that the separation principle holds in the case of switched affine systems.

Remark 15: The feasibility of the set of conditions (4.42)-(4.43) allows the design of the matrix Ψ for the switching law and the observer gain L separately, and equations (4.45) and (4.46) provide an estimation $\mathcal{E}(P, \gamma^*)$ of the domain of attraction such that any solution of the closed-loop system (4.13), (4.44) starting in the domain of attraction $\mathcal{E}(P, \gamma^*)$ converges to the origin exponentially with a decay rate α.

The numerical implementation can be done in two steps. First, LMIs (4.42)-(4.43) are solved to find the matrices P_1, θ_1, P_2, θ_2, L, and Ψ. In the second step, λ is computed from (4.45) and then the estimation of the domain of attraction can be computed using the equation (4.46). An optimization of the domain of attraction can be done by using recursive LMI algorithms.

In addition the feasibility of LMIs (4.47), (4.48) allows the design of nonlinear switching laws dependent on the estimated states ensuring the local asymptotic stability of the origin in a larger non-ellipsoidal domain of attraction.

Remark 16: The feasibility of the LMIs (4.42)-(4.43) is guaranteed for a sufficiently small α since Assumptions A-4 and A-5 hold. Thus, for a sufficiently small decay rate α, since system (4.1) is stabilizable, there exist a gain H such that $A(v_i) + BH$ are Hurwitz and a symmetric positive definite matrix $P = Q^{-1}$ satisfying (4.42). Likewise, since system (4.1) is detectable then there exist a gain L such that $A(v_i)+LC$ are Hurwitz and a symmetric positive definite matrix P_2 verifying (4.43) for a sufficiently small decay rate α.

Example 16: Consider system (4.1), (4.2) with

$$A = \begin{bmatrix} 1 & 3 \\ 4 & -5 \end{bmatrix},\ B = \begin{bmatrix} 15 & 1 \\ -1 & -5 \end{bmatrix},\ \mathcal{N}_1 = \begin{bmatrix} 1 & -5 \\ 0.5 & 2 \end{bmatrix},\ \mathcal{N}_2 = \begin{bmatrix} -1 & 5 \\ -0.5 & -2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and

$$u \in \mathcal{V} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}.$$

One can easily check that the open-loop system is unstable. In order to illustrate the result in Theorem 15, both observer-based linear and nonlinear switching laws are designed.

Linear switching law

Choosing a decay rate $\alpha = 3.5$, LMIs (4.42) and (4.43) and inequalities (4.45) and (4.46) are feasible with

$$Q_1 = \begin{bmatrix} 33.19 & -13.33 \\ -13.33 & 19.73 \end{bmatrix},\ P_2 = \begin{bmatrix} 3.38 & 2.19 \\ 2.19 & 5.79 \end{bmatrix},\ \theta_2 = 6.43 \times 10^2,\ \theta_1 = 4.1,$$
$$L = 10^2 \times \begin{bmatrix} -1.25, & 0.47 \end{bmatrix}^T,\ \lambda = 1.05 \times 10^4,\ \gamma^* = 2.3.$$

Simulations of the behaviour of the augmented closed-loop system (4.13), (4.44) are performed for the initial conditions $x(0) = \begin{bmatrix} -2 & 1 \end{bmatrix}^T$, and $\hat{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and Figures 4.1-4.2 report the obtained results.

We can see from Figures 4.1-4.2 that the trajectory originating in the domain of attraction of the closed-loop system converges to the origin and remains therein. The linear switching surfaces dependent on the real state and the phase plot of the closed-loop system are represented in Figure 4.1. Figure 4.2 represents the phase plot of the observer and the linear switching surfaces dependent on the estimated state. We can observe that the trajectory starting at the origin hit the switching surfaces several times and evolves until reaching the origin. Comparing Figure 4.2 to Figure 4.1 we can remark that the linear switching surfaces dependent on the real state do not coincide with the

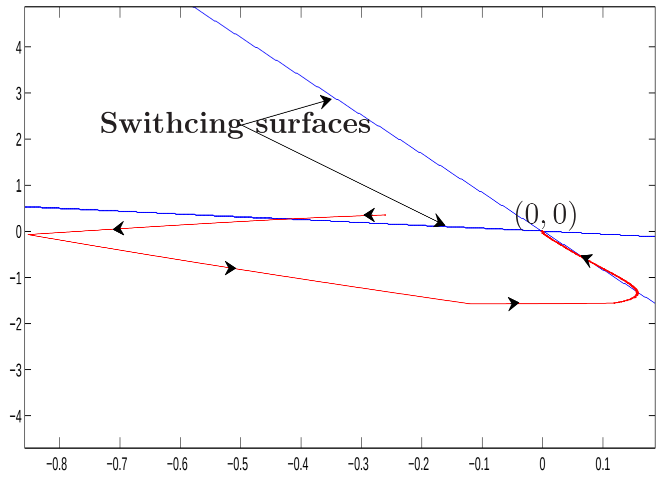

Figure 4.1: $x_{(1)}$ and $x_{(2)}$ in the phase plot-Example 16

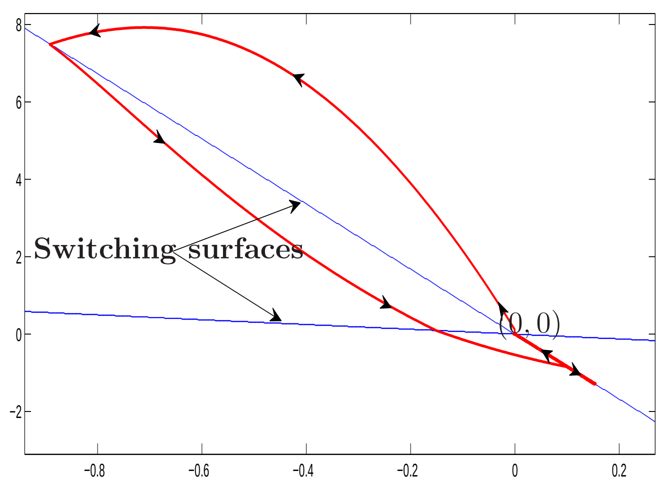

Figure 4.2: $\hat{x}_{(1)}$ and $\hat{x}_{(2)}$ in the phase plot-Example 16

linear switching surfaces dependent on the estimated state. This is due to the fact that $\hat{x}$ converges to x exponentially.

Nonlinear switching law

In order to design the nonlinear switching law (4.49) stabilizing the system (4.13), we solve the LMIs (4.47) and (4.48) for the same matrices H and L already obtained in the

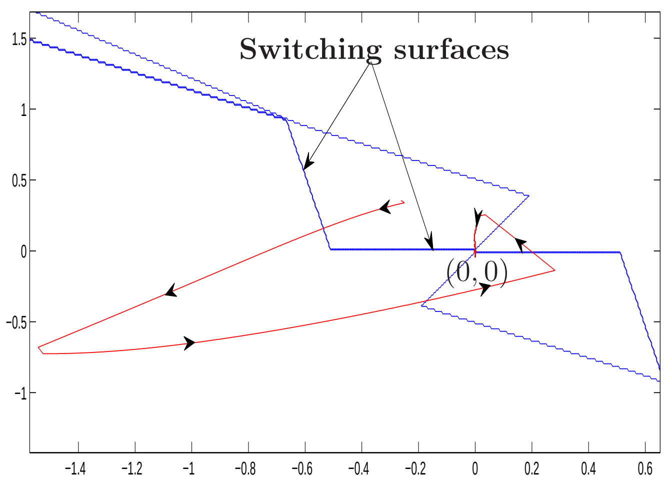

Figure 4.3: $x_{(1)}$ and $x_{(2)}$ in the phase plot-Example 16

first part of the example and $c_{(1)} = c_{(2)} = 1$. We obtain

$$\tilde{P}_1 = \begin{bmatrix} 12.4 & 4.43 \\ 4.43 & 7.5 \end{bmatrix}, \ \tilde{P}_2 = 10^8 \times \begin{bmatrix} 2.7 & 2.9 \\ 2.9 & 7.5 \end{bmatrix}, \ \tilde{M} = 10^3 \times \begin{bmatrix} 0.19 & 0 \\ 0 & 8.9 \end{bmatrix},$$

$$\tilde{\Omega} = 10^2 \times \begin{bmatrix} 0.024 & 0 \\ 0 & 7.2 \end{bmatrix}, \tilde{\Upsilon} = 10^3 \times \begin{bmatrix} 0.35 & 0.19 \\ 1.7 & 5.7 \end{bmatrix}, \ \tilde{\tau} = 10^4 \times \begin{bmatrix} 2.87 \\ 0.12 \end{bmatrix}, \text{ and } \eta^{-1} = 1.2.$$

Simulations are performed for the same initial conditions as in the linear switching law case. The results are reported in Figures 4.3-4.4.

From Figures 4.3-4.4 we can observe that the trajectories originating in the domain of attraction of the closed-loop system converges to the origin and remains therein. The nonlinear switching surfaces dependent on the real state and the phase plot of the closed-loop system are depicted in Figure 4.3. Figure 4.4 represents the phase plot of the observer and the nonlinear switching surfaces dependent on the estimated state. We can observe from Figure 4.4 that the trajectory starting at the origin hit the nonlinear switching surfaces dependent on the estimated state several times and evolves until reaching the origin. Comparing Figure 4.4 to Figure 4.3 we can remark that the nonlinear switching surfaces dependent on the real state do not coincide with nonlinear switching surfaces dependent on the estimated state. This is due to the fact that $\hat{x}$ converges to x asymptotically ($\hat{x}$ converges to x when t converges to infinity).

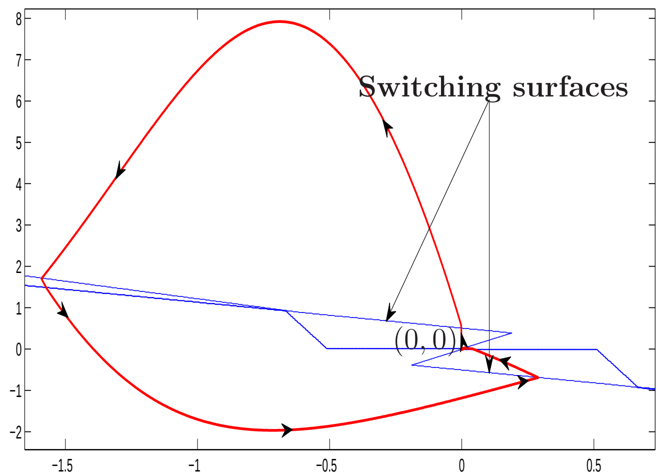

Figure 4.4: $\hat{x}_{(1)}$ and $\hat{x}_{(2)}$ in the phase plot-Example 16

The approach proposed in this section ensures only local stability of the closed-loop system at the origin. In the next section, using the properties of the Lur'e type Lyapunov functions, a constructive method of nonlinear switching surfaces dependent on the estimated state and ensuring the global asymptotic stability of the closed-loop system at the origin are provided.

4.2.3 Global stabilization

Here we provide an LMI approach for the design of nonlinear switching surfaces that ensure the global asymptotic stability of the closed-loop system at the origin. The switching law depends on the estimated state computed using a Luenberger observer.

Theorem 16: Assume that A-1, A-2 and A-3 hold. If LMIs (4.42)-(4.43) are feasible and there exist a symmetric positive definite matrix $\tilde{P} \in \mathbb{R}^{2n\times 2n}$, a diagonal positive definite matrix $\tilde{\Omega} \in \mathbb{R}^{m\times m}$, and symmetric matrices $\tilde{M}^+$ and $\tilde{M}^-$ with non-negative entries such that LMI

$$\begin{bmatrix} A_{cl}(v_i)^T\tilde{P}_1 + \tilde{P}_1 A_{cl}(v_i) & \tilde{P}_1 LC & \tilde{P}_1 B - \tilde{\Upsilon}^T - A_{cl}(v_i)^T H^T \tilde{\Omega} \\ * & A_o(v_i)^T\tilde{P}_2 + \tilde{P}_2 A_o(v_i) & -C^T L^T H^T \tilde{\Omega} \\ * & * & -2\tilde{M} - \tilde{\Omega} HB - (\tilde{\Omega} HB)^T \end{bmatrix} \prec 0, \forall i \in \mathcal{I}_N \tag{4.87}$$

and

$$c_{(k)}\tilde{M}^{+}_{(k,k)} \geq \sum_{k\neq j, j=1}^{m} c_{(j)}\left(\tilde{M}^{+}_{(k,j)} + \tilde{M}^{-}_{(j,k)}\right), \forall k \in \mathcal{I}_m. \tag{4.88}$$

are satisfied with $\tilde{\Upsilon} = \tilde{M}H$, $\tilde{M} = \tilde{M}^{+} - \tilde{M}^{-}$ and $\tilde{M}^{-}_{(k,k)} = 0, \forall k \in \mathcal{I}_m$, then system (4.1), (4.2), (4.9) with the switching law (4.49) is globally asymptotically stable at the origin.

Proof. The proof of this theorem follows the same steps as the one of Theorem 15. For $\mathcal{G}\xi = \mathcal{H}\xi$, thanks to the feasibility of LMI (4.88) the inequality (4.74) becomes

$$\phi(\mathcal{H}\xi)^T\tilde{M}(\phi(\mathcal{H}\xi) + \mathcal{H}\xi) \leq 0, \forall \xi \in \mathbb{R}^{2n}. \tag{4.89}$$

Therefore, if (4.87) is feasible, using (4.89) and the same arguments as in the proof of Theorem 15, inequality (4.76) becomes:

$$\begin{aligned}\max_{\beta\in\Delta^*(\xi)}\left\{\frac{\partial V}{\partial \xi}\left(\tilde{A}(v(\beta))\xi + \tilde{B}v(\beta)\right)\right\} \leq &\max_{\beta\in\Delta^*(\xi)}\left\{\frac{\partial V}{\partial \xi}\left(\tilde{A}(v(\beta))\xi + \tilde{B}v(\beta)\right)\right\}\\ &- 2\phi(\mathcal{H}\xi)^T\tilde{M}(\phi(\mathcal{H}\xi) + \mathcal{H}\xi)\\ &+ 2\sum_{j=1}^{N}\rho_j(\xi)(\xi\tilde{P} - \phi(\mathcal{H}\xi)^T\tilde{\Omega}\mathcal{H})(v_j - v(\beta))\\ &< 0, \forall \xi \in \mathbb{R}^{2n}.\end{aligned} \tag{4.90}$$

From this last inequality, we have shown that

$$\begin{aligned}\sup_{\varsigma\in\mathcal{F}[\tilde{\mathcal{X}}](\xi)} \frac{\partial V}{\partial \xi}\varsigma \leq &\max_{\beta\in\Delta^*(\xi)}\left\{\frac{\partial V}{\partial \xi}\left(\tilde{A}(v(\beta))\xi + \tilde{B}v(\beta)\right)\right\}\\ &< 0, \forall \xi \in \mathbb{R}^{2n},\end{aligned} \tag{4.91}$$

which ends the proof. □

Since LTI systems with relay controllers are a particular class of switched affine systems, in the following we particularize the results of this section to the case of LTI systems stabilization with an observer-based relay feedback control.

4.3 Observer-based relay control for LTI systems

Consider the LTI system given by

$$\dot{x} = Ax + Bu \tag{4.92}$$

In this section, we consider the case where the state variables are not fully available to measurements. The output is defined as

$$y = Cx. \tag{4.93}$$

For this particular case Assumptions A-1 and A-3 are reduced to

A-4 The pair (A, B) is stabilizable. This means that there exists a matrix H such that $A_{cl} = A + BH$ is Hurwitz.

A-5 The pair (A, C) is detectable. This means that there exists a matrix L such that $A_o = A + LC$ is Hurwitz.

As for the case of switched affine systems, in this case we provide a method for the stabilization of system (4.92), (4.93) by an observer-based relay feedback controller given by

$$u(\hat{x}) \in \arg\min_{v\in\mathcal{V}} \Gamma(\hat{x}, v). \tag{4.94}$$

Similar to the switched affine systems case, the estimated state $\hat{x} \in \mathbb{R}^n$ is computed by the full-order Luenberger state observer [80], [81]

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L(\hat{y} - y), \\ \hat{y} = C\hat{x}. \end{cases} \tag{4.95}$$

Our goal consists first in providing conditions that guarantee the existence of a mapping $\Gamma(\hat{x}, v)$ (which characterizes the switching surfaces of the control law) and a matrix L (the observer gain) as well as a numerical approach allowing the design of an observer-based relay controller such that the closed-loop system

$$\begin{cases} \dot{\xi} &= \tilde{A}\xi + \tilde{B}\bar{u}(\xi) = \tilde{\mathcal{X}}(\xi), \\ y &= \begin{bmatrix} C & -C \end{bmatrix} \xi, \end{cases} \tag{4.96}$$

where $\xi = \begin{bmatrix} \hat{x} \\ e \end{bmatrix}$ with $e = \hat{x} - x$, $\tilde{A} = \begin{bmatrix} A & LC \\ 0 & A + LC \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$, and

$$\bar{u}(\xi) = u\left(\begin{bmatrix} I & 0 \end{bmatrix} \xi\right) = u(\hat{x}) \in \arg\min_{v\in\mathcal{V}} \Gamma\left(\begin{bmatrix} I & 0 \end{bmatrix} \xi, v\right), \tag{4.97}$$

is locally asymptotically stable at the origin.

4.3.1 Local stabilization

We particularize the result provided in Theorem 14 to the case of observer-based relay controller design for LTI systems stabilization. Qualitative conditions for the existence of a mapping Γ (characterizing the switching surface) and of an observer gain L such that system (4.96), (4.97) (or equivalently (4.92), (4.93), (4.94), (4.95)) is locally exponentially stable at the origin are provided. The result is reported in the following.

Corollary 5: Assume that A-2, A-4, and A-5 hold. Then there exist a mapping $\Gamma(\hat{x}, v) = \hat{x}^T \Psi v$ (characterizing the switching hyperplanes) and a matrix L (the observer gain) such that the origin of system (4.96), (4.97) (or equivalently of the closed-loop system (4.92), (4.93), (4.94), (4.95)) is locally exponentially stable.

Here we want to provide a constructive procedure to derive a mapping Γ and an observer gain L such that the origin of the closed-loop system (4.96), (4.97) (or equivalently (4.92),(4.93), (4.94), (4.95)) is locally exponentially stable. We would also like to provide an estimation of the domain of attraction. The application of the result in Theorem 15 to the particular case of relay feedback controller reveals an interesting property of relays systems. Indeed, in this section we will show that the separation principle holds in the case of observer-based relay controller design for LTI systems stabilization. To the best of our knowledge, no separation principal exists in the literature for the case of relay systems.

Corollary 6: Assume that A-2, A-4, and A-5 hold. Consider a tuning parameter $\alpha > 0$.

1. If there exist positive definite matrices $Q_1 \in \mathbb{R}^{n\times n}$ and $P_2 \in \mathbb{R}^{n\times n}$, and scalars $\theta_1 > 0$ and $\theta_2 > 0$ such that

$$Q_1 A^T + A Q_1 - \theta_1 B B^T \preceq -2\alpha Q_1, \tag{4.98}$$

$$A^T P_2 + P_2 A - \theta_2 C^T C \preceq -2\alpha P_2, \tag{4.99}$$

then, the origin of system (4.92), (4.93), (4.95) with the switching law

$$u(\hat{x}) \in \arg\min_{v\in\mathcal{V}} \hat{x}^T \Psi v, \tag{4.100}$$

is locally α-stable with $\Psi = Q_1^{-1} B$ and $L = -\frac{\theta_2}{2} P_2^{-1} C^T$.

2. If in addition we consider $\lambda > 0$ such that

$$\begin{bmatrix} A_{cl}^T P_1 + P_1 A_{cl} + 2\alpha P_1 & P_1 LC \\ * & \lambda(A_o^T P_2 + P_2 A_o + 2\alpha P_2) \end{bmatrix} \preceq 0, \tag{4.101}$$

with $A_{cl} = A - \frac{\theta_1}{2} BB^T Q_1^{-1}$ and $A_o = A + LC$, then an estimation of the domain of attraction is given by $\mathcal{E}(P, \gamma^*)$ with $P = \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & \lambda P_2 \end{bmatrix}$,

$$\gamma^* \leq \min_{i \in \mathcal{I}_{n_l}} (l_i \mathcal{H} P^{-1} \mathcal{H}^T l_i^T)^{-1}, \tag{4.102}$$

and $\mathcal{H} = \begin{bmatrix} H & 0 \end{bmatrix}$ with $H = -\frac{\theta_1}{2} B^T Q_1^{-1}$.

3. If there exist positive definite matrices $P_1 \in \mathbb{R}^{n \times n}$, $Q_2 \in \mathbb{R}^{n \times n}$, and positive scalars θ_1 and θ_2 such that the LMIs (4.98), (4.99) are feasible for some $\alpha > 0$, and if there exist two symmetric positive definite matrices $\tilde{P}_1 \in \mathbb{R}^{n \times n}$ and $\tilde{P}_2 \in \mathbb{R}^{n \times n}$, two diagonal positive definite matrices $\tilde{\Omega} \in \mathbb{R}^{m \times m}$ and $\tilde{M} \in \mathbb{R}^{m \times m}$, a matrix $\tilde{\Upsilon} \in \mathbb{R}^{m \times n}$ and a strictly positive vector $\tilde{\tau} \in \mathbb{R}^m$ such that

$$\begin{bmatrix} A_{cl}^T \tilde{P}_1 + \tilde{P}_1 A_{cl} & \tilde{P}_1 LC & \tilde{P}_1 B - \tilde{\Upsilon}^T - A_{cl}^T H^T \tilde{\Omega} \\ * & A_o^T \tilde{P}_2 + \tilde{P}_2 A_o & -C^T L^T H^T \tilde{\Omega} \\ * & * & -2\tilde{M} - \tilde{\Omega} HB - (\tilde{\Omega} HB)^T \end{bmatrix} \prec 0 \tag{4.103}$$

and

$$\begin{bmatrix} \tilde{P}_1 & \tilde{M}_{(i,i)} H_{(i)}^T - \tilde{\Upsilon}_{(i)}^T \\ * & \tilde{\tau}_{(i)} c_{(i)}^2 \end{bmatrix} \succeq 0, \forall i \in \mathcal{I}_m, \tag{4.104}$$

with $L = -\frac{\theta_2}{2} P_2^{-1} C^T$, $A_{cl} = A - \frac{\theta_1}{2} BB^T Q_1^{-1}$, $H = -\frac{\theta_1}{2} B^T Q_1^{-1}$, and $A_o = A + LC$, then the origin of system (4.92), (4.93), (4.95) with the switching law

$$u(\hat{x}) \in \arg\min_{v \in \mathcal{V}} (\hat{x}^T \tilde{P}_1 - \phi(H\hat{x})^T \tilde{\Omega} H) Bv \tag{4.105}$$

is locally asymptotically stable.

An estimation of the domain of attraction is given by

$$\mathcal{L}_V(\eta^{-1}) = \{\xi \in \mathbb{R}^{2n} : V(\xi) \leq \eta^{-1}\}, \tag{4.106}$$

with

$$V(\xi) = \xi^T \tilde{P} \xi - 2 \sum_{j=1}^{m} \int_0^{\mathcal{H}_{(j)}\xi} \phi_{(j)}(\sigma) \tilde{\Omega}_{(j,j)} d\sigma, \tag{4.107}$$

$$\xi = \begin{bmatrix} \hat{x} \\ e \end{bmatrix}, \tilde{P} = \begin{bmatrix} \tilde{P}_1 & 0 \\ 0 & \tilde{P}_2 \end{bmatrix}, \text{ and } \eta \geq \max_{i \in \mathcal{I}_m} \left\{ \frac{\tilde{\tau}_{(i)}}{\tilde{M}^2_{(i,i)}} \right\}.$$

Remark 17: We can observe that (4.98), (4.99) do not share cross variables, thus they can be solved separately. Therefore, the control law matrix Ψ and the observer gain L can be designed independently, which shows that the separation principle holds in the case of LTI systems with relay controllers.

Example 17: Numerical example: single input system

Consider the linear system (4.92) with

$$u \in \mathcal{V} = \{-v, v\} = \{-5, 5\}, \tag{4.108}$$

and matrices

$$A = \begin{bmatrix} -1.6 & 1.7 \\ 1.5 & 2 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1, 0 \end{bmatrix}. \tag{4.109}$$

The eigenvalues of A are -2.2, and 2.6 thus the open-loop linear system is unstable. Considering a decay rate $\alpha = 5.5$ an observer-based relay feedback controller is designed to stabilize the system to the origin.

After the implementation of the set of LMIs (4.98)-(4.99), we find that they are feasible for

$$\begin{aligned} \theta = 486.8634, \ Q_1 &= \begin{bmatrix} 1.4334 & -4.7050 \\ -4.7050 & 28.6001 \end{bmatrix}, \\ \mu = 348.4742, \ P_2 &= \begin{bmatrix} 37.3918 & -5.4278 \\ -5.4278 & 1.0098 \end{bmatrix}. \end{aligned} \tag{4.110}$$

Then, we compute the observer gain

$$L = \begin{bmatrix} -21.2 & -113.98 \end{bmatrix}^T, \tag{4.111}$$

and the matrix characterizing the switching hyperplanes

$$\Psi = \begin{bmatrix} 0.25 & 0.076 \end{bmatrix}^T. \tag{4.112}$$

The computer simulations are performed for the initial conditions $x(0) = \begin{bmatrix} 1, & 0.5 \end{bmatrix}^T$, and $\hat{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ ($\xi^T = \begin{bmatrix} 0 & 0 & -1 & -0.5 \end{bmatrix}^T$) and the results are reported in Figures 4.5-4.9.

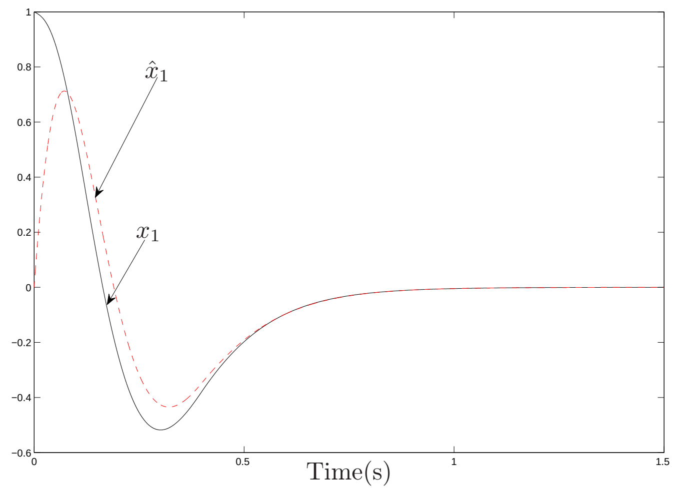

Figure 4.5: Real state x_1 and its estimate $\hat{x}_1$-Example 17

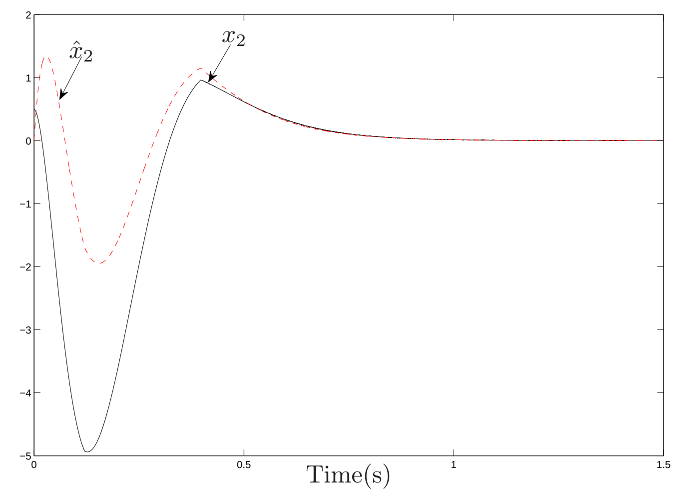

Figure 4.6: Real state x_2 and its estimate $\hat{x}_2$-Example 17

As we can see from Figures 4.5 and 4.6, the states are exactly estimated and they converge to the origin and remain therein. From Figure 4.7, we can remark that the observation errors converge to zero exponentially, and then the estimated states converge to the real states. In Figure 4.9 the observer's phase portrait is presented together with the switching hyperplane $\hat{x}^T \Psi v = 0$. We can observe that the trajectory initialized at zero evolves until reaching the switching hyperplane and it slides over it. The hyperplane

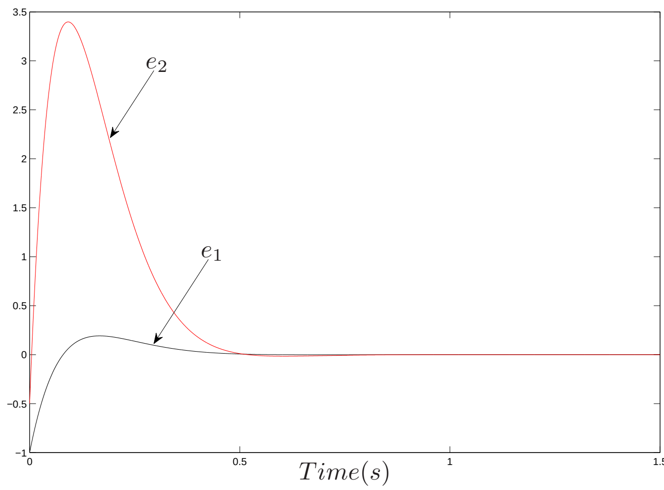

Figure 4.7: Observation errors $e = \hat{x} - x$-Example 17

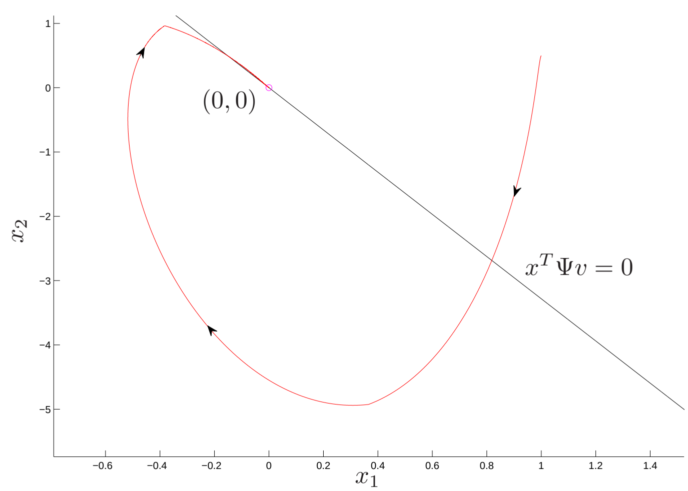

Figure 4.8: x_1 and x_2 in the phase plot-Example 17

$x^T\Psi v = 0$ and the phase plot of the closed-loopsystem (4.96), (4.97) are presented in Figure 4.8. Comparing Figure 4.8 and Figure 4.9, we can see that the hyperplane $x^T\Psi v = 0$ does not coincide exactly with the hyperplane $\hat{x}^T\Psi v = 0$. This is due to the fact that $\hat{x}$ converges to x when t tends to infinity. In simulations, the trajectory of the closed-loop system reaches first the hyperplane $\hat{x}^T\Psi v = 0$ which tends to $x^T\Psi v = 0$ as t goes to infinity and slides over it until reaching the origin.

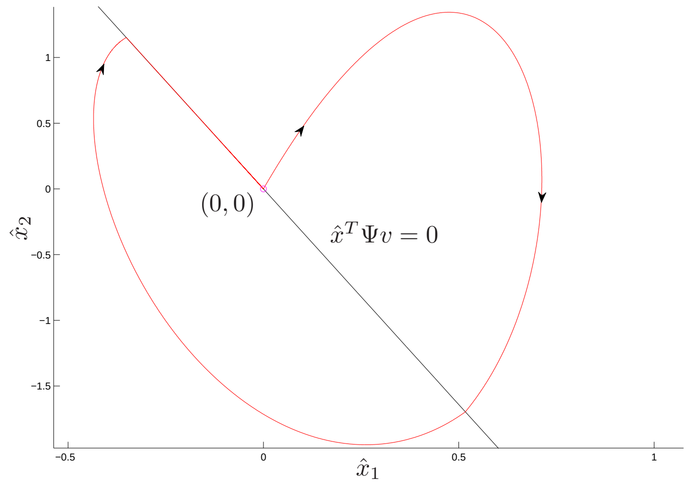

Figure 4.9: $\hat{x}_1$ and $\hat{x}_2$ in the phase plot-Example 17

Corollary 6 only provides sufficient conditions for local stabilization of LTI system by an observer-based relay control. In the next section, we apply the results provided in Theorem 16 to the case of relay control. LMI conditions which allows the design of nonlinear switching surfaces dependent on the estimated state that ensures the global asymptotic stability of the closed-loop system at the origin are provided.

4.3.2 Global stabilization

Here we particularize the result of Theorem 16 to the case of observer-based relay controller design for LTI systems stabilization. LMI conditions allowing the design of nonlinear switching surfaces that ensure the global asymptotic stability of the closed-loop system at the origin are provided. The estimations of the state variables used in the design of the switching surfaces are computed using a Luenberger observer.

Corollary 7: Assume that A-2, A-4 and A-5 hold. Consider system (4.92). If LMIs (4.98)-(4.99) are feasible and there exist a symmetric positive definite matrix $\tilde{P} \in \mathbb{R}^{2n\times 2n}$, a diagonal positive definite matrix $\tilde{\Omega} \in \mathbb{R}^{m\times m}$, and symmetric matrices $\tilde{M}^+$ and $\tilde{M}^-$ with

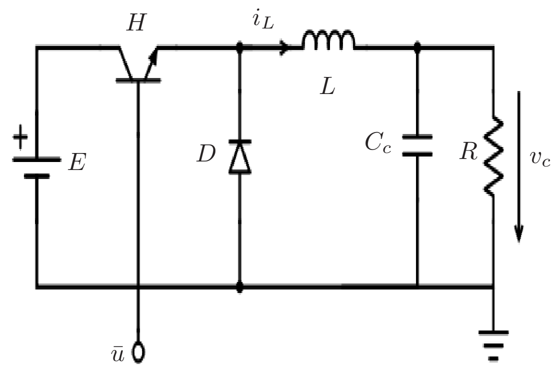

Figure 4.10: Buck converter

non-negative entries such that LMI

$$\begin{bmatrix} A_{cl}^T \tilde{P}_1 + \tilde{P}_1 A_{cl} & \tilde{P}_1 LC & \tilde{P}_1 B - \tilde{\Upsilon}^T - A_{cl}^T H^T \tilde{\Omega} \\ * & A_o^T \tilde{P}_2 + \tilde{P}_2 A_o & -C^T L^T H^T \tilde{\Omega} \\ * & * & -2\tilde{M} - \tilde{\Omega} HB - (\tilde{\Omega} HB)^T \end{bmatrix} \prec 0 \tag{4.113}$$

and

$$c_{(k)} \tilde{M}^+_{(k,k)} \geq \sum_{k \neq j, j=1}^{m} c_{(j)} \left(\tilde{M}^+_{(k,j)} + \tilde{M}^-_{(j,k)} \right), \forall k \in \mathcal{I}_m. \tag{4.114}$$

are satisfied with $\tilde{\Upsilon} = \tilde{M}H$, $\tilde{M} = \tilde{M}^+ - \tilde{M}^-$ and $\tilde{M}^-_{(k,k)} = 0, \forall k \in \mathcal{I}_m$, then system (4.92), (4.93), (4.95) with the switching law (4.105) is globally asymptotically stable at the origin.

The feasibility of (4.113)-(4.114) ensures the decay of the Lur'e Lyapunov function in each point of the state space. The LMI (4.114) is obtained as in the case of full state feedback provided in Chapter 2, by using the property of the nonlinearity ϕ given in Lemma 4.

Example 18: Numerical example: Buck converter

Let us consider the buck converter [9] shown in Figure 4.10. The state-space model for the state vector $\bar{x} = \begin{bmatrix} i_L & v_c \end{bmatrix}^T$ (i_L the inductor current and v_c the capacitor voltage) is described by :

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u} \tag{4.115}$$

with

$$\bar{A} = \begin{bmatrix} 0 & \frac{1}{L} \\ \frac{1}{C_c} & \frac{-1}{RC_c} \end{bmatrix},\ \bar{B} = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix},\ \text{and } \bar{u} \in \bar{\mathcal{V}} = \{0, E\}.$$

Here we consider the numerical values $L = 2\text{mH}$, $C_c = 470\mu\text{F}$, $E = 15\text{V}$, and $R = 10\Omega$. One can note that the eigenvalues of the open loop system are purely imaginary ($\pm 10^3 \times 1.03i$). Here we want to stabilize the system to the equilibrium point $\bar{x}^* = -\bar{A}^{-1}\bar{B}\beta^*$ which correspond to $i_L = 0.16\mu A$ and $v_c = 7.5V$. Using the transformation from [55] and the change of coordinates $x = \bar{x} - \bar{x}^*$, system (4.115) becomes

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{4.116}$$

with $A = \bar{A}$, $u \in= \{-\frac{1}{2}, \frac{1}{2}\}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} \frac{E}{L} \\ 0 \end{bmatrix}$. We can remark that system (4.116) satisfies Assumptions A-1, A-2 and A-3. Therefore, we can design an observer-based relay feedback controller.

First, using Corollary 6, we can show that the system (4.116) with a relay controller is locally asymptotically stable. Considering a decay rate $\alpha = 2.55$, LMIs (4.98)-(4.99) are feasible with

$$Q_1 = \begin{bmatrix} 1.33 & -7.79 \\ -7.79 & 55.81 \end{bmatrix},\ P_2 = \begin{bmatrix} 0.005 & -0.002 \\ -0.002 & 0.0015 \end{bmatrix},\ \theta_1 = 20.1,\ \theta_2 = 4.6 \times 10^6,$$

$$\Psi = 10^3 \times \begin{bmatrix} 2.05 \\ 0.3 \end{bmatrix},\ L = 10^3 \times \begin{bmatrix} -3.378 \\ -7 \end{bmatrix},\ \text{and } H = \begin{bmatrix} -0.6 & 0.98 \end{bmatrix}.$$

Considering the obtained matrices H and L and $c = \frac{1}{2}$, we can design a nonlinear switching law depending in the estimated states (4.105). We obtain :

$$\tilde{P}_1 = \begin{bmatrix} 1.47 & 0.21 \\ 0.21 & 0.037 \end{bmatrix},\ \tilde{P}_2 = 10^8 \times \begin{bmatrix} 5.41 & -2.07 \\ -2.07 & 5.27 \end{bmatrix},$$

$$\tilde{\Omega} = 0.35,\ \tilde{\Upsilon} = 10^2 \times \begin{bmatrix} -0.72 & 1.075 \end{bmatrix},\ \tilde{M} = 1.11 \times 10^2,\ \tilde{\tau} = 1.24,$$

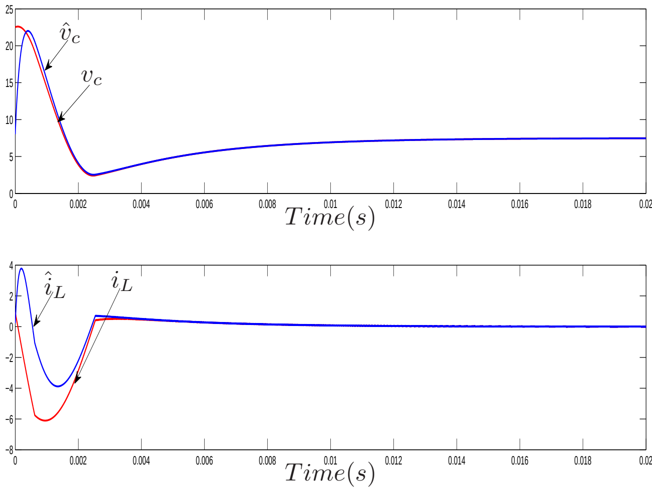

Figure 4.11: Evolution of the state variables and their estimates-Example 18

with an estimation of the domain of attraction (4.106) given by $\eta^{-1} = 9.94 \times 10^3$. Simulations are performed for the initial conditions $\hat{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and $x = \begin{bmatrix} 1 & 15 \end{bmatrix}^T$. The results are reported in Figure 4.11, which shows that the estimated state converges to the real state and they both converge to the equilibrium point.

Second, one can also verify that the LMIs (4.113)-(4.114) from Corollary 7 are feasible for the same values of L and H. This means that the asymptotic stability is global.

4.4 Conclusion

This chapter has considered the problem of observer-based switching law design to ensure local asymptotic stability of the origin of switched affine systems. A Luenberger observer is used to design both linear and nonlinear switching surfaces dependent on the estimated state. LMI conditions are proposed in order to allow a numerical implementation of the results. Considering a particular property of the switching Lyapunov functions, LMI conditions ensuring the global asymptotic stability of the closed-loop switched affine system at the origin have been also provided. The proposed approaches have been then particularized to the case of stabilization of LTI systems by an observer-based relay feedback controller. For both LTI systems with a relay controller and switched affine systems it has been equally shown that the separation principle holds.

General conclusion

This thesis was devoted to the study of the stabilization problem of switched affine systems with state-dependent switching laws. A particular interest was given to the stabilization of switched affine systems with unstable component subsystems which do not share a Hurwitz convex combination of their evolution matrices. The Filippov formalism has been used to define the solutions of the closed-loop system, and it allows the analysis of the system's behaviour on the switching surfaces. To solve the stabilization problem, a Lyapunov-based approach which enables to derive numerical approaches based on LMIs has been proposed.

First, the state feedback stabilization problem of switched affine systems has been considered. In this context, the existing methods, using quadratic Lyapunov functions, only allow the local stabilization of the closed-loop switched affine system in a guaranteed ellipsoidal domain of attraction. Here, using a general framework for the class of nonlinear input-affine systems, a full state-dependent switching controller has been designed in order to ensure both the local and global asymptotic stability of the closed-loop system. Thanks to switching (Lur'e type) Lyapunov functions, a numerical approach based on LMIs has been developed. This approach allows to derive a nonlinear stabilizing switching law. In what concerns the local stabilization, the method allows to compute a larger non-ellipsoidal estimation of the domain of attraction, compared to the ellipsoidal estimation obtained using quadratic Lyapunov functions. The results have been particularized to the stabilization of LTI systems with a relay feedback control.

Second, the design problem of robust state-dependent switching laws for switched affine systems stabilization has been investigated. The robustness property has been studied with respect to bounded exogenous disturbances that affect the state measurements which are used for the design of the switching laws. A constructive method based on LMIs which allows to derive the robust switching laws and to maximize the size of the domain of attraction or minimize the size of the chattering zone have been proposed. The results have been equally particularized to the case of LTI systems for which a design method of robust relay controllers has been provided.

Finally, the stabilization issue of the class of switched affine systems by an observer-based switching laws has been considered. An observer-based switching controller has been designed to guarantee the local asymptotic stability of the closed-loop system. Using both quadratic and non-quadratic Lyapunov functions, linear and nonlinear switching surfaces have been designed. The derived switching surfaces depend on the estimated state which is computed by a Luenberger observer. Numerical approaches based on LMIs have been proposed in order to allow the design of the stabilizing switching laws and the estimation of the domains of attraction. Both conditions for local and global asymptotic stabilization of switched affine systems have been provided. The obtained results have been then applied to the particular class of LTI systems with observer-based relay controllers. Along with the notable proposed results we have shown that the separation principle holds for both LTI systems with relay controllers and switched affine systems.

The results proposed in this thesis can be extended in several directions, which makes the perspectives that emerge from this work multiple.

First, in order to take into account the potential implementations imperfections such as jitters, uncertainties, etc, and to reduce chattering, one can extend the proposed approaches to the case of min-switching stategy with dwell time condition. In this direction one can follow the approach in [58] or the ones in [1] and [27]. The approach proposed in [42] based on the construction of finite abstraction [108] can be also used for the same purposes.

Second, even though numerical results have been provided, there is still place for improvement. The conditions provided in this work could be relaxed by taking into account in a more accurate manner the underlying bilinear model, using nonlinear control methods. Furthermore, piecewise quadratic Lyapunov functions [65] could be useful for the same goal. In this case, a particular attention must be given to the differentiability of the Lyapunov function on the switching surfaces and its decay after each switching instant [8], [36].

As another research direction, in the case of disturbed switching laws, one can also consider an extended observer to estimate the perturbation and derive a robust stabilizing approach. In this context, other types of observers can be used instead of the Luenberger observer.

Finally, in this thesis, we have considered the problem of stabilizing state-dependent switching laws design for the class of switched affine systems. The extension of the proposed methods to the more general class of switched nonlinear systems with autonomous and controlled switching would be of a great interest. For example, the study of such systems could be useful in Networked Control Systems, for instance in the internet congestion problem. This constitutes a challenging issue.

Résumé étendu en français

Introduction générale

Les systèmes dynamiques hybrides ont la particularité de présenter des dynamiques continues et discrètes simultanément [4], [43], [44], [46], [52]. Les systèmes à commutation constituent une classe populaire des systèmes hybrides [54], [77], [76], [78], [102], [103], [107], [109]. Ils sont composés d'une famille de sous-systèmes continus et d'une loi qui orchestre la commutation entre eux. La synthèse de la loi de commutation est un problème très important dans la communauté des systèmes à commutation. Bien que le cas où les différents sous-systèmes partagent le même point d'équilibre soit très largement étudié et qu'une grande variété de résultats a été déjà publiée, très peu de résultats existent dans le cas où les sous-systèmes n'ont pas le même point d'équilibre [1], [16], [26], [51], [87], [95]. Dans ce travail nous sommes intéressés par la synthèse de lois de commutation pour les systèmes affines à commutation. L'étude de cette classe de systèmes est motivée par leur présence dans différents domaines d'applications : l'électronique de puissance (convertisseurs DC/DC, AC/DC,...), l'électromécanique, etc. Toutefois, la commande des systèmes affines à commutation exhibe plusieurs difficultés : commutations rapides, présence de points d'équilibres non-standards (les points d'équilibres des différents sous-systèmes ne sont pas forcément un point d'équilibre du système à commutation), dynamiques de glissement, phénomène de Zénon, etc. Ces problèmes ont été étudiés en mathématiques dans le cas plus général des systèmes dynamiques discontinus [5], [6], [8], [21], [84]. Leur étude est très délicate puisqu'elle requièrte l'utilisation d'un concept de solution plus général que le concept classique des solutions pour permettre la prise en considération des dynamiques du système sur les surfaces de commutations.

Objectifs

L'objectif principal de cette thèse est de proposer de nouvelles approches pour la synthèse de lois de commutation dépendantes de l'état garantissant la stabilité à l'origine des systèmes affines à commutation en boucle fermée. Tout au long de cette thèse nous allons développer des critères de stabilisation pour les systèmes affines à commutation dont les sous-systèmes ne sont pas stables, ne partagent pas le même point d'équilibre, et ne possèdent pas de combinaison convexe stable de leurs matrices d'évolutions. Le cas particulier des systèmes linéaires invariants dans le temps avec commande à relais sera également traité. Tout d'abord, en utilisant le formalisme de Filippov, les dynamiques de glissements ainsi que les points d'équilibres non standard seront considérés. Des fonctions de Lyapunov quadratique et non-quadratique seront utilisées pour proposer des méthodes constructives de surfaces de commutation linéaires et non linéaires assurant la stabilité locale des systèmes affines à commutation en boucle fermée. Le problème de l'estimation de domaines d'attractions ellipsoïdal et non-ellipsoïdal sera également abordé. Ensuite, nous nous intéresserons à l'étude des conditions sous lesquelles la stabilité globale des systèmes affines à commutation peut être assurée par une loi de commutation dépendante de l'état. De plus, la robustesse vis-à-vis des perturbations externes sera analysée. Enfin, le problème de synthèse de lois de commutation basées-observateurs est considéré. Une étude concernant le principe de séparation dans les cas des systèmes affines à commutation et des systèmes linéaires invariants dans le temps avec commande à relais sera aussi menée.

Structure de la thèse

Le document est organisé comme suit:

Chapitre 1

Le premier chapitre présente un aperçu des résultats récents sur la stabilisation des systèmes affines à commutation par une loi de commutations dépendante de l'état. Tout d'abord, le concept de solutions, la notion de points d'équilibre ainsi que les concepts de stabilité nécessaires à la compréhension de ce manuscrit sont présentés en considérant le formalisme de Filippov. Ensuite, les problèmes qui peuvent être rencontrés dans l'étude des systèmes affines à commutation sont montrés à l'aide d'exemples illustratifs. Enfin, les résultats récents concernant la synthèse de lois de commutation stabilisantes dépendantes de l'état sont exposés. En utilisant le formalisme de Filippov, certains de ces résultats sont redémontrés. Les avantages ainsi que les inconvénients de chacune des différentes approches sont détaillés afin de mettre en évidence les problèmes qui restent ouverts.

Chapitre 2

Dans le deuxième chapitre, nous considérons le problème de stabilisation des systèmes affines à commutation définis par

$$\dot{x} = Ax + \sum_{k=1}^{m} (\mathcal{N}_k x + b_k) u_{(k)}, \tag{4.117}$$

avec $x \in \mathbb{R}^n$ est le vecteur d'état et $u_{(k)}$ le k-ième élément du vecteur d'entrée u. L'entrée u est restreinte à prendre des valeurs dans l'ensemble fini de vecteurs $\mathcal{V} = \{v_1, \dots, v_N\} \subset \mathbb{R}^m$. $A \in \mathbb{R}^{n\times n}$, $B = \begin{bmatrix} b_1, & \dots, & b_m \end{bmatrix} \in \mathbb{R}^{n\times m}$, et $\mathcal{N}_k \in \mathbb{R}^{n\times n}$ sont les matrices décrivant la dynamique du système.

Grâce à l'utilisation de fonctions de Lyapunov non quadratiques (de type Lur'e) données sous la forme

$$V(x) = x^T P x - 2 \sum_{k=1}^{m} \int_0^{K_{(k)}x} \phi_{(k)}(s) \Omega_{(k,k)} ds, \tag{4.118}$$

avec $\phi(.) \in \mathbb{R}^m$ une fonction discontinue (voir Figure 4.12), une nouvelle approche de synthèse de lois de commutation dépendantes de l'état qui stabilisent le système affine à commutation en boucle fermée est proposée. Cette méthode permet la construction de surfaces de commutation non linéaires et l'élargissement du domaine d'attraction par rapport au cas de la stabilisation quadratique. Des critères du type LMI sont proposés afin de construire la loi de commutation et de fournir une estimation non ellipsoïdalle du domaine d'attraction. De plus, en utilisant les propriétés des fonctions de Lyapunov du type Lur'e, des conditions LMIs sont développées pour permettre la synthèse de lois de commutation dépendantes de l'état qui assurent la stabilisation globale du système affine à commutation en boucle fermée. Finalement, les résultats sont particularisés au cas des systèmes linéaires invariants dans le temps avec commande à relais definis par

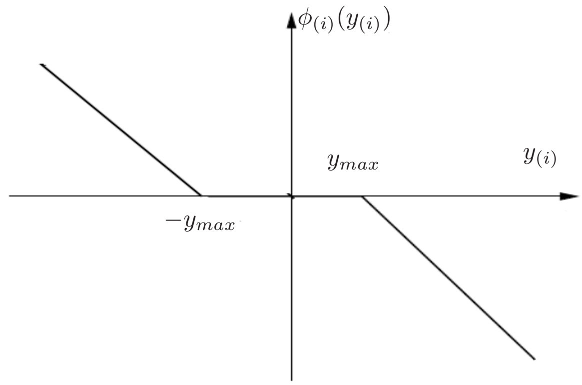

Figure 4.12: Nonlinéarité sector

$$\dot{x} = Ax + Bu,$$
$$u \in \mathcal{V} = \{v_1, \ldots, v_N\} \subset \mathbb{R}^m. \tag{4.119}$$

Une nouvelle méthode de synthèse de commande à relais est proposée. De plus, un résultat général de stabilisation des systèmes non linéaires affines en l'entrée avec une commande à relais definis par

$$\dot{x} = f(t, x) + g(t, x)u(t, x),$$
$$u \in \mathcal{V} = \{v_1, \ldots, v_N\} \subset \mathbb{R}^m \tag{4.120}$$

est développé.

Chapitre 3

Le troisième chapitre est consacré à la synthèse de lois de commutation dépendantes de l'état perturbé. La méthode proposée considère des perturbations externes additives aux variables d'état mesurés. Des conditions qualitatives de stabilité du système affine en boucle fermée sont développées. De plus, une méthode numérique basée sur des LMIs est proposée et permet la synthèse de lois de commutation robuste ainsi que l'élargissement du domaine d'attraction ou la diminution de la zone de recouvrement. Le résultat est finalement particularisé au cas des systèmes linéaires invariants dans le temps avec une commande à relais robuste.

Chapitre 4

Ce dernier chapitre est dédié à l'étude du problème de la synthèse de lois de commutation dépendantes de l'état estimé pour assurer la stabilité asymptotique locale des systèmes affines à commutation en boucle fermée. Un observateur du type Luenberger est utilisé pour concevoir des lois de commutation linéaires et non linéaires dépendantes des variables d'état estimées. Ensuite, des conditions LMIs sont proposées pour permettre l'implémentation numérique des résultats. De plus, en se basant sur les propriétés des fonctions de Lyapunov commutées, des conditions LMIs sont proposées pour permettre la construction de lois de commutation dépendantes de l'état estimé et qui assure la stabilité asymptotique globale du système affine à commutation en boucle fermée. Les résultats obtenus sont par la suite particularisés au cas des systèmes linéaires temps invariant avec une commande à relais basée-observateur. Enfin, le principe de séparation est prouvé pour le cas des systèmes affines à commutation ainsi que pour le cas particulier des systèmes linéaires invariants dans le temps avec commande à relais.

Conclusions et perspectives

Cette thèse a été dédiée à l'étude du problème de stabilisation des systèmes affinés à commutation avec des lois de commutation dépendantes de l'état. Un intérêt particulier a été accordé à la stabilisation des systèmes affines ayant des sous-systèmes instables qui ne partagent pas de combinaison convexe stable de leurs matrices d'évolution. Le formalisme de Filippov a été utilisé pour définir les solutions du système en boucle fermée, et permet l'analyse du comportement du système sur les surfaces de commutation. Pour résoudre le problème de stabilisation, une approche basée sur la théorie de stabilité de Lyapunov qui permet de dériver des approches numériques basées sur des LMIs a été proposé.

Tout d'abord, le problème de stabilisation des systèmes affines à commutation a été pris en considération dans le cas où toutes les variables d'état sont accessibles à la mesure.

Dans ce contexte, les méthodes existantes, basées sur l'utilisation des fonctions de Lyapunov quadratiques, ne permettent que la stabilisation locale du système affine à commutation en boucle fermée dans un domaine d'attraction ellipsoïdal. Dans ce travail, nous proposons un résultat général pour la stabilisation d'une classe de systèmes non-linéaire affines en l'entrée. De plus, une loi de commutation dépendante de l'état a été conçue pour assurer la stabilité asymptotique locale ou la stabilité globale des systèmes affines à commutation en boucle fermée. Grâce aux fonctions de Lyapunov (de type Lur'e), une approche numérique basée sur des LMIs a été développé. Cette approche permet de concevoir une loi de commutation non linéaire stabilisante. En ce qui concerne la stabilisation locale, la méthode permet de calculer une plus grande estimation du domaine d'attraction, par rapport à l'estimation ellipsoïdale obtenu à l'aide des fonctions de Lyapunov quadratiques. Les résultats ont été particularisés au cas de la stabilisation des systèmes LTI avec une commande à relais.

Ensuite, le problème de la synthèse de lois de commutation robustes dépendantes de l'état qui assurent la stabilité des systèmes affines à commutations a été étudié. La propriété de robustesse a été étudiée par rapport aux perturbations externes qui affectent les mesures de l'état utilisés dans la conception des lois de commutation. Une méthode constructive basée sur des conditions LMI a été proposée. Cette approche permet de synthétiser les lois de commutation robustes et de maximiser la taille du domaine d'attraction ou de minimiser la taille de la zone de chattering. Les résultats ont été également particularisés aux cas des systèmes LTI pour lesquels une méthode de conception de contrôleurs à relais robustes a été fournie.

Enfin, la question de la stabilisation de la classe des systèmes affines à commutations par une loi de commutation basée-observateur a été prise en considération. Dans ce contexte, une loi de commutation basée-observateur a été conçue pour garantir la stabilité asymptotique locale du système en boucle fermée. En utilisant des fonctions de Lyapunov quadratiques et non quadratiques, des surfaces de commutations linéaires et non linéaires ont été synthétisées. Les lois de commutation développées dépendent des variables d'état estimées qui sont calculées par un observateur de type Luenberger. Des approches numériques basées sur des LMIs ont été proposées pour permettre la conception des lois de commutations stabilisantes et l'estimation des domaines d'attraction. Des conditions de stabilisation asymptotique locale et globale des systèmes affines à commutation ont été fournies. Les résultats obtenus ont ensuite été appliqués à la classe particulière des systèmes LTI avec une commande à relais basée observateur. Enfin, nous avons montré que le principe de séparation est vérifié dans les deux classes de systèmes LTI avec contrôleurs à relais et systèmes affines à commutation.

Les résultats proposés dans cette thèse peuvent être étendus dans plusieurs directions, ce qui rend les perspectives qui émergent de ce travail multiples.

Premièrement, afin de prendre en compte les imperfections d'implémentations potentielles telles que les incertitudes, la perte de paquets de communication etc., et pour réduire les commutations à très hautes fréquences, on peut étendre les approches proposées au cas de commutation avec un temps de séjour minimum. Dans ce contexte on peut suivre l'approche proposée dans [58] ou celles dans [1] et [27]. L'approche proposée dans [42], basée sur la construction d'abstraction finie [108], peut également être utilisée pour le même objectif.

Deuxièmement, même si des résultats numériques ont été fournis, il existe encore des points à améliorer. Les conditions proposées dans ce travail pourraient être assouplies en prenant en compte de manière plus précise le modèle bilinéaire sous-jacent et en utilisant des méthodes issues de la commande des systèmes non linéaires. En outre, les fonctions de Lyapunov quadratiques par morceaux [65] pourraient être utiles pour le même objectif. Dans ce cas, une attention particulière doit être accordée à la différentiabilité de la fonction Lyapunov sur les surfaces de commutation et sa décroissance après chaque instant de commutation [8], [36].

Une autre direction de recherche intéressante, dans le cas de lois de commutation perturbées, serait de considérer un observateur étendu pour estimer la perturbation et dériver une approche de stabilisation robuste. Dans ce contexte, d'autres types d'observateurs peuvent être utilisés à la place de l'observateur de type Luenberger.

Enfin, dans cette thèse, nous avons étudié le problème de la conception des lois de commutation dépendantes de l'état permettant la stabilisation de la classe des systèmes affines à commutation. L'extension des méthodes proposées à la classe la plus générale de systèmes non linéaires à commutation autonomes serait d'un grand intérêt. Par exemple, l'étude de tels systèmes pourrait être utile dans les systèmes de contrôle en réseau, tel que le problème de congestion sur Internet.

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