Exponential sums, cell decomposition and p-adic integration

Saskia Chambille

Supervisors:
Prof. dr. R. Cluckers
Prof. dr. W. Veys

Dissertation presented in partial fulfillment of the requirements for the degree of Doctor of Science (PhD): Mathematics

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Examination committee:
Prof. dr. G. Bhowmik, chair
(Université de Lille)
Prof. dr. R. Cluckers, supervisor
(Université de Lille & KU Leuven)
Prof. dr. W. Veys, supervisor
(KU Leuven)
Prof. dr. L. Darnière
(Université d'Angers)
Prof. dr. I. Halupczok
(Heinrich-Heine-Universität Düsseldorf)
Prof. dr. P. Igodt
(KU Leuven)
Prof. dr. D. Macpherson
(Leeds University)
Prof. dr. F. Point
(Université de Mons)

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Uitgegeven in eigen beheer, Saskia Chambille, Celestijnenlaan 200B box 2400, B-3001 Leuven (Belgium)

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Abstract

In this thesis we study p-adic exponential sums and integrals using ideas from model theory and geometry. The first part of this thesis deals with families of exponential sums in P-minimal fields. The second part discusses estimates for the asymptotic behaviour of exponential sums over p-adic fields.

Our work on P-minimal fields starts with the proof of a cell decomposition theorem that holds in all P-minimal fields, i.e., independently of the existence of definable Skolem functions. For P-minimal fields that lack these functions, we introduce the notion of *regular clustered cells*. This notion is close to the classical notion of p-adic cells, that was introduced by Denef. Our cell decomposition uses both classical cells and regular clustered cells.

Next, we extend the notion of exponential-constructible functions, already defined in the semi-algebraic and subanalytic setting, to all P-minimal fields. We do this by enlarging the algebras of constructible functions with families of exponential sums. Using our cell decomposition theorem we prove that exponential-constructible functions are stable under integration. This means that the act of integrating an exponential-constructible function over some of its variables produces an exponential-constructible function in the other variables.

In our work on estimates for the asymptotic behaviour of exponential sums we prove the Igusa, Denef-Sperber and Cluckers-Veys conjectures for polynomials with log-canonical threshold at most one half. These conjectures predict uniform upper bounds for the absolute values of certain exponential sums that depend on a polynomial. We give two different proofs, one using motivic integration and cell decomposition, and the other one using the Igusa zeta functions.

Beknopte samenvatting

In deze thesis bestuderen we p-adische exponentiële sommen en integralen met behulp van modeltheorie en meetkunde. In het eerste deel van deze thesis behandelen we families van exponentiële sommen over P-minimale velden. Het tweede deel is gewijd aan afschattingen van het asymptotisch gedrag van exponentiële sommen over p-adische velden.

Het deel over P-minimale velden begint met het bewijs van een celdecompositie stelling die geldt voor alle P-minimale velden, dus onafhankelijk van de aanwezigheid van definieerbare Skolem functies. Voor P-minimale velden waarin deze functies ontbreken, introduceren we het begrip van *reguliere geclusterde cellen*. Deze cellen lijkt op de klassieke p-adische cellen, die door Denef gedefinieerd zijn. Onze celdecompositie maakt gebruik van zowel klassieke cellen, als reguliere geclusterde cellen.

Vervolgens beschouwen we de notie van exponentieel-construeerbare functies, gedefinieerd voor semi-algebraïsche en subanalytische structuren, en breiden deze uit naar alle P-minimale velden. We doen dit door families van exponentiële sommen toe te voegen aan de algebra's van construeerbare functies. Met behulp van onze celdecompositie bewijzen we dat de exponentieel-construeerbare functies stabiel zijn onder integratie. Dit betekent dat het integreren van een dergelijke functie over sommige van haar variabelen weer een exponentieel-construeerbare functie oplevert in de overige variabelen.

In het deel over afschattingen van het asymptotisch gedrag van exponentiële sommen bewijzen we de vermoedens van Igusa, Denef-Sperber en Cluckers-Veys voor veeltermen met log-canonieke drempel hoogstens een half. Deze vermoedens voorspellen uniforme bovengrenzen voor de absolute waardes van bepaalde exponentiële sommen die afhangen van een veelterm. We geven twee verschillende bewijzen; het ene bewijs maakt gebruik van motivische integratie en celdecompositie en het andere van de Igusa zeta functies.

Résumé

Dans cette thèse nous étudions des sommes exponentielles et des intégrales p-adiques, en utilisant la théorie des modèles et la géométrie. La première partie traite des familles de sommes exponentielles dans des corps P-minimaux. La deuxième partie examine le comportement asymptotique des sommes exponentielles sur les corps p-adiques.

Dans la première partie nous commençons par démontrer une théorème de décomposition cellulaire pour tous les corps P-minimaux, c'est-à-dire indépendamment de l'existence des fonctions de Skolem définissables. En l'absence de ces fonctions nous introduisons les *cellules en grappe régulières*, inspirés par la notion classique de cellule p-adique de Denef. Notre décomposition cellulaire utilise les cellules classiques et les cellules en grappe régulières.

Ensuite nous étendons la notion de fonction constructible exponentielle, déjà définie pour les structures semi-algébriques et sous-analytiques, à tous les corps P-minimaux. Pour cela nous ajoutons des familles de sommes exponentielles aux algèbres des fonctions constructibles. En utilisant notre décomposition cellulaire, nous démontrons que les fonctions constructibles exponentielles sont stables dans le contexte d'intégration. Cela signifie que l'intégration d'une telle fonction sur certaines de ses variables produit une fonction constructible exponentielle dans les autres variables.

Dans la deuxième partie nous démontrons les conjectures d'Igusa, Denef-Sperber et Cluckers-Veys sur le comportement asymptotique des sommes exponentielles pour les polynômes dont le seuil log-canonique ne dépasse pas un demi. Ces conjectures prédisent des bornes supérieures uniformes pour les valeurs absolues de certaines sommes exponentielles, dépendantes d'un polynôme. Nous apportons deux démonstrations ; l'une utilise l'intégration motivique et une décomposition cellulaire et l'autre les fonctions zêtas d'Igusa.

Contents

Introduction

This thesis consists of two separate projects that are both centered around the study of exponential sums over p-adic fields. Exponential sums are a fundamental object in mathematics and their study dates back to Gauss, who determined the values of the quadratic Gauss sums.

We will study sums over the p-adic numbers $\mathbb{Q}_p$ or over a p-adic field K, i.e., a finite field extension of $\mathbb{Q}_p$. We fix some additive character $\psi\colon K \to \mathbb{C}^\times$ whose values will be p^m-roots of unity in $\mathbb{C}$ for some $m \in \mathbb{Z}$. The exponential sums that we study are of the form

$$\sum_{z \in A} \psi(z),$$

where A is a finite set. Depending on the context the character ψ and the set A are of a specific form.

These exponential sums exhibit oscillatory behaviour. Certain values appearing in the sum might cancel each other out. The simplest form of complete cancellation is the elementary fact that the p-roots of unity in $\mathbb{C}$ add up to zero. This phenomenon plays an important role in our proofs.

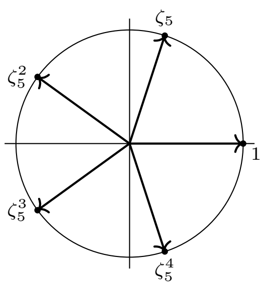


Figure 1: The fifth roots of unity in $\mathbb{C}$

The study of exponentials sums in this thesis will be restricted to the following two settings.

- We consider families of exponential sums over sets that are definable in a P-minimal structure on a p-adic field. By adding such sums to the algebras of constructible functions one obtains *exponential-constructible functions*. We look at integration properties of such functions.
- A polynomial f with integer coefficients determines an exponential sum modulo p^m for a prime number p and an integer m. We study the asymptotic behaviour of these sums, when p and m go to infinity. There exist several conjectures that predict uniform upper bounds for the absolute values of these sums.

In both these projects the exponential sums are strongly linked with p-adic integrals. We use different techniques and ideas coming from model theory and geometry. In particular cell decomposition theorems are essential in several of the proofs.

Part I: exponential-constructible functions

The first project, which is joint work with Pablo Cubides Kovacsics and Eva Leenknegt, is inspired by the work of Denef on families of p-adic integrals. It starts with the following Poincaré series

$$P(T) := \sum_{m=0}^{\infty} P_m T^m,$$

where $P_m := \#\{x \in (\mathbb{Z}/p^m\mathbb{Z})^n \mid f(x) \equiv 0 \bmod p^m\}$ for some polynomial $f \in \mathbb{Z}[x]$ in n variables $x = (x_1, \ldots, x_n)$. The fact that these Poincaré series are rational functions in T was shown first by Igusa [Igu74a] and later by Denef [Den84]. These series can be easily related to p-adic integrals by realising that $f(x) \equiv 0 \bmod p^m$ if and only if the p-adic valuation of $f(x)$ is at least m. Hence

$$P_m = \int_{\{x\in\mathbb{Z}_p^n \mid \mathrm{ord}_p(f(x)) \geqslant m\}} p^{nm} |dx|,$$

where we integrate with respect to the Haar measure on $\mathbb{Q}_p$, normalised such that $\mathbb{Z}_p$ has measure 1. The numbers P_m form a family of p-adic integrals, depending on the parameter m. Understanding the way in which the integrals depend on the parameter is important for proving the rationality of $P(T)$. This has led to the study of more general families of p-adic integrals over $\mathbb{Q}_p$ and also over general p-adic fields K.

Denef considered integrals where the domain of integration is a semi-algebraic set, i.e., a set that is definable in the ring language $\mathcal{L}_{\text{ring}} = \{+, -, \cdot, 0, 1\}$. Let $S \subseteq K^m$ be a parameter set, $X \subseteq S \times K^n$ and $f\colon X \to K$ a function, all of them semi-algebraic. We denote by $\operatorname{ord}\colon K \to \mathbb{Z} \cup \{\infty\}$ the valuation map on K and by q_K its residue characteristic. Then

$$\int_{X_s} \operatorname{ord}(f(x))|dx| \qquad \text{and} \qquad \int_{X_s} q_K^{-\operatorname{ord}(f(x))}|dx|$$

are families of integrals, parametrized by the set S. Denef proved in [Den85] that these families of integrals can be written as a $\mathbb{Q}$-linear combination of products of functions of the form

$$s \mapsto \operatorname{ord}(a(s)) \qquad \text{and} \qquad s \mapsto q_K^{\operatorname{ord}(b(s))},$$

where $a, b\colon S \to K$ are semi-algebraic functions. We call the $\mathbb{Q}$-algebra of such functions on S the algebra of *($\mathcal{L}_{\text{ring}}$-)constructible functions on S* and denote it by $\mathcal{C}(S)$.

Moreover, Denef showed that if $G(s, x)$ is an $\mathcal{L}_{\text{ring}}$-constructible function on $X \subseteq S \times K^n$, integrable over the fibers X_s, then

$$s \mapsto \int_{X_s} G(s, x)|dx|$$

is an $\mathcal{L}_{\text{ring}}$-constructible function on S. We say that the $\mathcal{L}_{\text{ring}}$-constructible functions are *base-stable under integration.*

There are also interesting families of p-adic integrals that can only be defined by adding more symbols to the semi-algebraic language. The best studied example is the subanalytic language

$$\mathcal{L}_{\text{an}} := \mathcal{L}_{\text{ring}} \cup \{\ ^{-1}, \cup_{r \geqslant 1} K\{x_1, \ldots, x_r\}\},$$

where $^{-1}$ denotes the multiplicative inverse with $0^{-1} := 0$, and where $\sum a_i x^i \in K\{x_1, \ldots, x_r\}$ denotes a restricted formal power series that converges on the valuation ring $\mathcal{O}_K^r$, and is set to 0 outside, i.e.,

$$K^r \to K\colon x \mapsto \begin{cases} \sum a_i x^i & \text{if } x \in \mathcal{O}_K^r; \\ 0 & \text{otherwise.} \end{cases}$$

Cluckers [Clu04] showed that the $\mathcal{L}_{\text{an}}$-constructible functions are also base-stable under integration.

The proofs of these results are based on a good understanding of the shape of $\mathcal{L}_{\text{ring}}$- and $\mathcal{L}_{\text{an}}$-definable sets and functions. The interesting thing about these two languages is that the $\mathcal{L}_{\text{an}}$-definable subsets of K are the same as the $\mathcal{L}_{\text{ring}}$-definable subsets of K. This is not the case for the definable subsets of K^n for $n \geqslant 2$, but still there exists some common description of these definable sets by means of cell decomposition theorems. This means that an $\mathcal{L}_{\text{ring}}/\mathcal{L}_{\text{an}}$-definable set can be partitioned into finitely many 'simple' definable sets of a certain form, called *cells*.

The idea to study a class of languages for which the one-variable definable subsets are of a certain form, comes from real geometry. A totally ordered field $\mathcal{R}$ with a language $\mathcal{L} \supseteq \{<\}$ is called *o-minimal* if the $\mathcal{L}$-definable subsets of $\mathcal{R}$ are exactly the finite unions of intervals and points. From this condition Knight, Pillay and Steinhorn [PS86, KPS86] proved a cell decomposition theorem for all *o*-minimal structures.

The successful development of *o*-minimal structures inspired Haskell and Macpherson [HM97] to define an analogous concept for p-adic fields, which they named P-minimality. A frequently used reformulation of their definition states that a p-adically closed field K with a language $\mathcal{L} \supseteq \mathcal{L}_{\text{ring}}$ is called *P-minimal* if the $\mathcal{L}$-definable subsets of K are exactly the $\mathcal{L}_{\text{ring}}$-definable subsets of K and if the same holds for any $\mathcal{L}$-elementary equivalent field K'.

Denef defined a notion of p-adic cells for semi-algebraic structures and proved that each $\mathcal{L}_{\text{ring}}$-definable set partitions into finitely many of these cells [Den84, Den86]. This cell decomposition theorem can be used to show the rationality of the above Poincaré series and to study families of p-adic integrals of constructible functions. His cell decomposition was adapted to subanalytic structures by Cluckers [Clu04] and to all P-minimal structures that have definable sections (also called *definable Skolem function*) by Mourgues [Mou09]. Moreover, she showed that satisfying a cell decomposition with Denef-type cells is equivalent to having definable Skolem functions.

For some time it was unknown whether there exist P-minimal structures that do not have definable Skolem functions. Recently an example of such a structure was provided by Cubides-Kovacsics and Nguyen [CN17b]. Therefore the notion of cell as introduced by Denef has to be broadened if one wants to obtain a cell decomposition theorem that is valid in all P-minimal structures. A first version of such a decomposition was given by Cubides-Kovacsics and Leenknegt [CL16]. In this thesis we improve this result using Denef-type cells and *regular clustered cells*. A regular clustered cell is a finite disjoint union of sets that look like they are Denef-type cells, but the lack of definable Skolem functions makes it impossible to actually describe these sets as such cells.

This cell decomposition was developed with the idea to study families of p-adic integrals. When the domain of integration is a definable set in some P-minimal structure, then we can partition this set into cells. Restricting to families of integrals over cells can make the calculations easier. Cubides-Kovaciscs and Leenknegt [CL16] applied this idea to prove a generalisation of the results of Denef and Cluckers, namely that in any P-minimal structure the constructible functions are base-stable under integration.

In [CL10] Cluckers and Loeser introduced a natural extension of the algebra of constructible functions for the semi-algebraic and subanalytic languages by adding exponential sums of the form

$$s \mapsto \sum_{j=1}^{r} \psi(f_j(s)),$$

for some additive character $\psi\colon K \to \mathbb{C}^\times$ and $f_j\colon S \to K$ definable. They named these functions *exponential-constructible functions* and showed that they are base-stable under integration. The goal of the first part of this thesis is to generalise this result to all P-minimal structures.

Part II: upper bounds for p-adic exponential sums

In the second project, which is joint work with Kien Nguyen, we study exponential sums that depend on a nonconstant polynomial $f \in \mathbb{Z}[x]$ in n variables $x = (x_1, \ldots, x_n)$ for which $f(0) = 0$. These sums depend on the values of f modulo p^m for a prime number p and an integer $m \geqslant 1$. There is a global exponential sum:

$$E_f(m,p) := \frac{1}{p^{nm}} \sum_{\overline{x} \in (\mathbb{Z}/p^m\mathbb{Z})^n} \exp\left(\frac{2\pi i f(x)}{p^m}\right),$$

and for each $y \in \mathbb{Z}^n$ a local sum around y:

$$E_f^y(m,p) := \frac{1}{p^{nm}} \sum_{\overline{x} \in \overline{y} + (p\mathbb{Z}/p^m\mathbb{Z})^n} \exp\left(\frac{2\pi i f(x)}{p^m}\right),$$

where $\overline{y} + (p\mathbb{Z}/p^m\mathbb{Z})^n = \{\overline{x} \in (\mathbb{Z}/p^m\mathbb{Z})^n \mid \forall i : x_i \equiv y_i \bmod p\}$. The goal is to find good upper bounds that are uniform in p and m for the absolute value of these sums.

These sums can be written as p-adic integrals and were introduced as such by Weil [Wei65] in a more general context. There is a formula that relates these exponential sums to the Igusa zeta functions, which are themselves related to the Poincaré series mentioned in Part I. From this formula and results by Igusa, Denef and Veys, one can easily obtain upper bounds for p fixed and $m \to \infty$.

More precisely, for each prime number p, there exists a constant $C_p > 0$ such that for all $m \geqslant 1$ and for all $y \in \mathbb{Z}^n$, we have

$$|E_f(m,p)|_{\mathbb{C}} \leqslant C_p m^{n-1} p^{-ma(f)}; \tag{0.0.1}$$

$$|E_f^y(m,p)|_{\mathbb{C}} \leqslant C_p m^{n-1} p^{-ma_{y,p}(f)}, \tag{0.0.2}$$

where $a(f)$ and $a_{y,p}(f)$ are invariants that depend on geometric properties of f, in particular the critical points of f. There exist other versions with different invariants, sometimes giving stronger upper bounds.

Igusa [Igu78] studied the global exponential sum $E_f(m,p)$ for homogeneous polynomials f, and related its asymptotic behaviour to the validity of a certain Poisson summation formula. He conjectured that in the upper bound (0.0.1) the constant C_p can be taken independently of p, which gives an upper bound that is uniform in both m and p. Several cases of his conjecture have been proved by himself [Igu74b], Denef and Sperber [DS01], Cluckers [Clu08a, Clu08b, Clu10], Wright [Wri12], and Lichtin [Lic13, Lic16]; but the general conjecture still remains open.

The behaviour of the local sum $E_f^0(m,p)$ around 0 can provide information on the global sum. The first ones to study upper bounds for this local sum were Denef and Sperber [DS01]. They conjectured that for all polynomials f, i.e., not necessarily homogeneous, there exists a uniform upper bound of the form (0.0.2) with the constant C_p independent of p. They proved their own conjecture for nondegenerate polynomials under an extra condition and later Cluckers [Clu10] showed that this condition can be omitted.

In some of the cases where the Igusa conjecture has been proven, the proofs were not limited to homogeneous polynomials. This motivates the investigation of a more general version of this conjecture. Recently, Cluckers and Veys [CV16] formulated a conjecture that generalises the Igusa conjecture to all polynomials and the Denef-Sperber conjecture to all local sums in a uniform way, under the condition that $m \geqslant 2$. In other words, they conjecture that when $m \geqslant 2$, the constant C_p from (0.0.1) and (0.0.2) can be taken independently of p. They prove this conjecture for 'small' values of m. The nondegenerate case is proved by Castryk and Nguyen [CN18].

In the second part of this thesis we will prove the Cluckers-Veys conjecture for polynomials that have log-canonical threshold at most one half, which means that the polynomials have very 'singular' singularities. We will give one proof that uses a cell decomposition theorem and some other results from the Cluckers-Loeser theory of motivic integration [CL05, CL08], and another proof that uses the relation between the exponential sums and the Igusa zeta functions.

Outline

In the first chapter we give the details and background on cell decomposition in P-minimal structures, constructible and exponential-constructible functions, Cluckers-Loeser motivic integration, Igusa zeta functions and their connection with exponential sums.

The second chapter is dedicated to proving our cell decomposition theorem that holds in all P-minimal structures.

In chapter three we use this cell decomposition theorem to prove that the algebras of exponential-constructible functions are base-stable under integration.

In the fourth chapter we present two different proofs of the Cluckers-Veys conjecture on exponential sums for polynomials with log-canonical threshold at most one half.

Chapter 1

Preliminaries

In this chapter we will give some background on the questions that we are trying to answer in this thesis. All of these questions concern certain classes of valued fields. For a general background on valued fields, see [EP05].

Definition 1.0.1. Let K be a field and let Γ_K be a totally ordered, abelian group. A *valuation* on K is a surjective map $\mathrm{ord}\colon K \to \Gamma_K \cup \{\infty\}$, that satisfies, for all $x, y \in K$,

(i) $\mathrm{ord}(x) = \infty$ if and only if $x = 0$;

(ii) $\mathrm{ord}(xy) = \mathrm{ord}(x) + \mathrm{ord}(y)$;

(iii) $\mathrm{ord}(x + y) \geqslant \min\{\mathrm{ord}(x), \mathrm{ord}(y)\}$.

A *valued field* (K, ord) has a *value group* Γ_K and a *valuation ring*

$$\mathcal{O}_K := \{x \in K \mid \mathrm{ord}(x) \geqslant 0\},$$

which has a unique maximal ideal

$$\mathcal{M}_K := \{x \in K \mid \mathrm{ord}(x) > 0\}.$$

The quotient field

$$k_K := \mathcal{O}_K / \mathcal{M}_K$$

is called the *residue field*. We denote the image of $x \in \mathcal{O}_K$ under the quotient map $\mathcal{O}_K \to k_K$ by $\overline{x}$. An *angular component map modulo* $\mathcal{M}_K$ is a multiplicative map $\overline{\mathrm{ac}}\colon K^\times \to k_K^\times$ satisfying $\overline{\mathrm{ac}}(x) = \overline{x}$ for all x with $\mathrm{ord}(x) = 0$. It can be extended to K by putting $\overline{\mathrm{ac}}(0) = 0$.

A valuation induces a topology on K. Let $x \in K$ and $\gamma \in \Gamma_K$, then the open ball around x of valuation radius γ is

$$B_\gamma^\circ(x) := \{y \in K \mid \operatorname{ord}(y-x) > \gamma\},$$

and the closed ball around x of valuation radius γ is

$$B_\gamma(x) := \{y \in K \mid \operatorname{ord}(y-x) \geqslant \gamma\}. \tag{1.0.1}$$

The points $x \in K$ are sometimes denoted as $B_\infty(x)$. Let $\mathbb{B}$ be the set of closed balls in K, including the (closed) points, i.e.,

$$\mathbb{B} := \{B_\gamma(x) \mid x \in K, \gamma \in \Gamma_K \cup \{\infty\}\},$$

and for a fixed $\gamma \in \Gamma_K$,

$$\mathbb{B}_\gamma := \{B_\gamma(x) \mid x \in K\}$$

is the set of closed balls in K with valuation radius γ. Usually, when we will speak of *balls*, we will mean *closed balls*.

Definition 1.0.2. Let $X \subseteq K$ and let $B \subseteq X$ be a ball such that for all balls $B' \subseteq X$, one has that $B \subseteq B' \Rightarrow B = B'$. Then we call B a *maximal ball of X*. This property will be denoted as $B \sqsubseteq X$.

In this thesis we will often study valued fields from a model theoretic point of view. For a general introduction to model theory, see [Mar02]. The languages that we will encounter, are the following.

(i) Languages for the valued field:

(a) The ring language $\mathcal{L}_{\text{ring}} := \{+, -, \cdot, 0, 1\}$. The $\mathcal{L}_{\text{ring}}$-definable sets are called *semi-algebraic sets*.

(b) The Macintyre language $\mathcal{L}_{\text{Mac}} := \mathcal{L}_{\text{ring}} \cup \{P_n\}_{n \geqslant 2}$, where $P_n := \{x \in K \mid \exists y \in K : x = y^n\}$. The $\mathcal{L}_{\text{Mac}}$-definable sets are the same as the $\mathcal{L}_{\text{ring}}$-definable sets, but the upshot of adding these predicates is the result from [PR84] on quantifier elimination for p-adically closed fields in the language $\mathcal{L}_{\text{Mac}}$.

(c) The subanalytic language $\mathcal{L}_{\text{an}} := \mathcal{L}_{\text{Mac}} \cup \{\,^{-1}, \cup_{r \geqslant 1} K\{x_1, \ldots, x_r\}\}$ for a p-adic field K. Here $^{-1}$ denotes the multiplicative inverse with $0^{-1} := 0$ and $\sum a_i x^i \in K\{x_1, \ldots, x_r\}$ denotes a restricted formal power series that converges on $\mathcal{O}_K^r$, and is set to 0 outside, i.e.,

$$K^r \to K : x \mapsto \begin{cases} \sum a_i x^i & \text{if } x \in \mathcal{O}_K^r; \\ 0 & \text{otherwise.} \end{cases}$$

The $\mathcal{L}_{\text{an}}$-definable sets are called *subanalytic sets.* In [DvdD88] Denef and Van den Dries proved that the $\mathcal{L}_{\text{an}}$-theory of K admits quantifier elimination.

(ii) Languages for the value group:

(d) The language $\mathcal{L}_{\text{oag}} = \{+, <\}$ of ordered abelian groups.

(e) The Presburger language $\mathcal{L}_{\text{Pres}} = \{+, <, 0, 1, \{\equiv_n\}_{n \geqslant 1}\}$ for certain value groups, for example $\mathbb{Z}$. The symbol $\equiv_n$ expresses congruence modulo n. The $\mathcal{L}_{\text{Pres}}$-theory of $\mathbb{Z}$ admits elimination of quantifiers in the Presburger language [Pre30].

(iii) The two-sorted language $\mathcal{L}_2 = (\mathcal{L}, \mathcal{L}_{\text{Pres}}, \text{ord})$, where $\mathcal{L}$ is any expansion of the ring language used for the valued field sort VF and $\mathcal{L}_{\text{Pres}}$ is the language for the value group sort VG. The symbol ord, going from the nonzero elements of VF to VG, denotes the valuation.

(iv) The three-sorted Denef-Pas language $\mathcal{L}_{\text{DP}} = (\mathcal{L}_{\text{ring}}, \mathcal{L}_{\text{ring}}, \mathcal{L}_{\text{oag}}, \text{ord}, \overline{\text{ac}})$. The first sort VF is for the valued field K, the second sort RF for the residue field k_K and the third sort VG is for the value group Γ_K. The valuation map ord is the same as before and $\overline{\text{ac}}$ from VF to RF is an angular component map modulo $\mathcal{M}_K$.

If K is a valued field and $\mathcal{L}$ is a language whose symbols have an interpretation in K, then, informally, the *$\mathcal{L}$-definable sets* are the subsets of K^n, for $n \geqslant 1$, that can be described using symbols from the language $\mathcal{L}$, elements from K and the standard first-order symbols $\vee, \wedge, \neg, \to, \forall, \exists, =, (,)$, and variables $x, y, z, \ldots$. We say that a function $f\colon K^n \to K^m$ is $\mathcal{L}$-definable if its graph is an $\mathcal{L}$-definable set. The same notions are used for other mathematical structures with an appropriate language. Whenever it is clear from the context which language $\mathcal{L}$ we are working with, we will simply write *definable* instead of *$\mathcal{L}$-definable.* For us *definable* will always mean *definable with parameters.* A definable set in a many-sorted structure can contain variables from the different sorts.

Below we list some examples of valued fields that will play a role in this thesis.

The p-adic numbers: for each prime number p, $\mathbb{Q}_p$ denotes the field of p-adic numbers with the p-adic valuation $\text{ord}_p\colon \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$. The valuation ring is the ring of p-adic integers, denoted by $\mathbb{Z}_p$, and the maximal ideal is $p\mathbb{Z}_p$. The residue field is the finite field $\mathbb{F}_p$ with p elements.

The p-adic fields: let K be a finite field extension of $\mathbb{Q}_p$, then the p-adic valuation ord_p extends uniquely to $K \to \frac{1}{e}\mathbb{Z} \cup \{\infty\}$ for some $e \in \mathbb{N} \setminus \{0\}$. Since it is more convenient to work with value group $\mathbb{Z}$ than $\frac{1}{e}\mathbb{Z}$, we will rescale the valuation, i.e., $\text{ord}\colon K \to \mathbb{Z} \cup \{\infty\}\colon x \mapsto e \cdot \text{ord}_p(x)$, for all $x \neq 0$. Any element

ϖ with $\mathrm{ord}(\varpi) = 1$ is called a *uniformizer*. After fixing a uniformizer ϖ, we can define an *angular component map*

$$\mathrm{ac} : K \to \mathcal{O}_K : x \mapsto \begin{cases} x\varpi^{-\mathrm{ord}(x)} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0, \end{cases}$$

which induces naturally an angular component map $\overline{\mathrm{ac}}$ modulo $\mathcal{M}_K$. The residue field of K is a finite field $\mathbb{F}_{q_K}$, with $q_K = p^f$ elements for some $f \in \mathbb{N}\backslash\{0\}$. The p-adic fields are sometimes also referred to as *non-archimedean local fields of characteristic zero.*

The valuation induces a norm on a p-adic field K:

$$|x| := q_K^{-\mathrm{ord}(x)},$$

which gives the same open and closed balls as we defined above. For p-adic fields K the closed and open balls are actually the same, since $B_\gamma^\circ(x) = B_{\gamma+1}(x)$. With the operation of addition, K can be seen as a locally compact, Hausdorff topological group. So there exist a unique countable additive, translation invariant measure μ on K for which $\mu(\mathcal{O}_K) = 1$. This measure is called a *Haar measure* on K. The measurable subsets of K are the sets in the σ-algebra generated by the open subsets of K. At some point we will encounter Lebesgue integrals of certain measurable functions from K^n to $\mathbb{C}$, which are an important object of study in this thesis.

The Henselian valued fields: when we apply the quotient map $\mathcal{O}_K \to k_K$ to the coefficients of a polynomial $f \in \mathcal{O}_K[x]$, we obtain a polynomial in $k_K[x]$ that we denote by $\overline{f}$. When studying valued fields, it can be useful to be able to lift polynomial roots of $\overline{f}$ to roots of f.

Definition 1.0.3. Let K be a valued field. We say that K is *Henselian* if each polynomial $f \in \mathcal{O}_K[x]$ satisfies the following property: if $\overline{x} \in k_K$ is a simple zero of $\overline{f}$, i.e., $\overline{f}(\overline{x}) = 0$ and $\overline{f}'(\overline{x}) \neq 0$, then there exists a unique $y \in \mathcal{O}_K$, such that $f(y) = 0$ and $\overline{y} = \overline{x}$.

The p-adic numbers and p-adic fields are examples of Henselian valued fields.

The p-adically closed fields: an important class of Henselian valued fields in this thesis is the class of p-adically closed fields. They are the p-adic analogue of real closed fields and can be defined in a model theoretic way.

Definition 1.0.4. A valued field K is called *p-adically closed* if it is $\mathcal{L}_{\mathrm{ring}}$-elementary equivalent to a p-adic field.

It follows from this definition that the value group Γ_K of a p-adically closed field is a $\mathbb{Z}$-group, i.e., $\mathcal{L}_{\mathrm{Pres}}$-elementary equivalent to $\mathbb{Z}$. This implies that p-adically closed fields have uniformizing elements ϖ. Furthermore, their residue field is finite of characteristic p. Another way of defining p-adically closed fields is the following: a valued field (K_1, ord_1) is p-adically closed if it is a (not necessarily algebraic) field extension of a p-adic field (K, ord), such that ord_1 extends ord, uniformizers of K remain uniformizers of K_1, the residue fields k_{K_1} and k_K are isomorphic and there exists no proper algebraic field extension (K_2, ord_2) of (K_1, ord_1) with these properties.

We will mostly be interested in structures $(K; \mathcal{L})$, that consist of a p-adically closed field K and a language $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{ring}}$, that satisfy some *minimality condition.*

Before we introduce the context of the specific topics of this thesis, we will fix some general notation. If S and Y are any kinds of sets, $X \subseteq S \times Y$ and $s \in S$, then

$$X_s := \{y \in Y \mid (s, y) \in X\}$$

denotes the fiber over s and $\pi_S \colon X \to S$ the projection onto S. In particular, if $X \subset K^n$, then the projection onto the first $n-1$ coordinates is denoted by π_{n-1}. The topological interior and closure of X, when defined, are denoted by $\mathrm{Int}(X)$ and $\mathrm{Cl}(X)$, respectively.

1.1 P-minimal fields and cell decomposition

In a large part of this thesis we will be working with p-adically closed fields with a P-minimal structure on them. In particular, we are interested in integrals of certain complex valued functions over definable subsets of such fields. Hence, a good understanding of the definable sets in P-minimals fields is necessary. Cell decomposition theorems provide a tool to better understand the structure of these sets.

1.1.1 o-minimality

The notion of P-minimality was inspired by the one of o-minimality. In general, o-minimal structures do not necessarily have to be fields, but since we are only interested in fields, we will restrict to that case. For more background on o-minimal structures (not only fields), we refer to [vdD98].

Definition 1.1.1. Let $\mathcal{L} \supseteq \{<\}$ and let $\mathcal{R}$ be a totally ordered field. A structure $(\mathcal{R}; \mathcal{L})$ is called *o-minimal* if every $\mathcal{L}$-definable subset of $\mathcal{R}$ is a finite union of intervals and points. Recall that for us 'definable' means 'definable with parameters'.

Another way of phrasing the condition of o-minimality is saying that every $\mathcal{L}$-definable subset of $\mathcal{R}$ is in fact $\{<\}$-definable. In a *real closed field* $\mathcal{R}$, i.e., a field that is $\mathcal{L}_{\text{ring}}$-elementary equivalent to the real numbers $\mathbb{R}$, it is possible to define a total order using only the ring language $\mathcal{L}_{\text{ring}}$:

$$< := \{(x, y) \in \mathcal{R}^2 \mid \exists z : z \neq 0 \wedge x + z^2 = y\}.$$

One can prove that the $\mathcal{L}_{\text{ring}}$-definable subsets of $\mathcal{R}$ are exactly the $\{<\}$-definable subsets of $\mathcal{R}$. This observation will be essential for the analogy between o-minimal fields and P-minimal fields, because for p-adically closed fields we cannot use the language $\{<\}$, but we can use $\mathcal{L}_{\text{ring}}$.

An example of an o-minimal field is $(\mathbb{R}; \mathcal{L}_{\text{ring}} \cup \{<, \exp\})$ (see [Wil96]), where $\exp\colon \mathbb{R} \to \mathbb{R}$ is the usual exponential function. Other examples can be found in [Wil99]. The fact that there exist many examples of o-minimal structures makes the theory of o-minimality interesting for real geometers and has provided an important application of model theory. Moreover, o-minimal structures share many nice geometric properties. In particular their geometry does not exhibit any 'wild' behaviour and is therefore often referred to as 'tame geometry'. One of the results that has been crucial to the development of o-minimality, is the cell decomposition theorem. Informally, this theorem states that from the assumption that we have on the definable subsets of $\mathcal{R}$, we can deduce a description of the definable subsets of $\mathcal{R}^n$, for all $n \geqslant 1$. More precisely, any definable subset of $\mathcal{R}^n$ can be partitioned as a finite union of some specific definable sets, called *cells*.

Definition 1.1.2. Let $\mathcal{R}$ be an ordered field and $\mathcal{L} \supseteq \{<\}$. For $n \in \mathbb{N} \setminus \{0\}$ and $(i_1, \dots, i_n) \in \{0,1\}^n$ we define the notion of $(i_1, \dots, i_n)$-*cell* $C \subseteq \mathcal{R}^n$ by induction on n.

- For $n = 1$ a (0)-cell is a point $\{r\} \subseteq \mathcal{R}$ and a (1)-cell is an open interval $(a, b) \subseteq \mathcal{R}$.
- For $n > 1$ and $(i_1, \dots, i_{n-1}) \in \{0,1\}^{n-1}$, an $(i_1, \dots, i_{n-1}, 0)$-cell is a set of the form
$$\{(s, t) \in C \times \mathcal{R} \mid t = f(s)\},$$
and an $(i_1, \dots, i_{n-1}, 1)$-cell is a set of the form
$$\{(s, t) \in C \times \mathcal{R} \mid f(s) \ \square_1 \ t \ \square_2 \ g(s)\},$$
where $C \subseteq \mathcal{R}^{n-1}$ is an $(i_1, \dots, i_{n-1})$-cell, $f, g\colon C \to \mathcal{R}$ are $\mathcal{L}$-definable functions and $\square_1, \square_2$ are either $<$ or 'no condition', which means that either $f(s) < t$ (resp. $t < g(s)$) or there is no lower (resp. upper) bound on t.

The cell decomposition theorem presented here is a reformulation of some of the results from [PS86] and [KPS86].

Theorem 1.1.3. *Let $(\mathcal{R};\mathcal{L})$ be an o-minimal field. Any $\mathcal{L}$-definable set $X \subseteq \mathcal{R}^n$ can be partitioned into a finite number of cells. Moreover, if $f\colon X \to \mathcal{R}^m$ is an $\mathcal{L}$-definable function, then there exists a partition of X into cells C_i, such that the restrictions $f_{|C_i}\colon C_i \to \mathcal{R}^m$ are continuous.*

1.1.2 P-minimality

The promising developments in o-minimality motivated a whole bunch of minimality concepts, including v-minimality and C-minimality. The one we are interested in, P-minimality, was introduced by Haskell and Macpherson in [HM97] as a p-adic analogue of o-minimality. Originally, their definition used a sightly different language than the ring language and was not restricted to the class of p-adically closed fields. They defined the notion of P-minimality for any valued field of characteristic 0 with value group a $\mathbb{Z}$-group and finite residue field of characteristic p, Next, they showed that if such a field is P-minimal, then it must be Henselian, hence p-adically closed. Therefore the following, equivalent formulation is now often taken as the definition for P-minimality.

Definition 1.1.4. Let $\mathcal{L} \supseteq \mathcal{L}_{\text{ring}}$ and let K be a p-adically closed field. A structure $(K;\mathcal{L})$ is called *P-minimal* if, for every structure $(K';\mathcal{L})$ elementary equivalent to $(K;\mathcal{L})$, the $\mathcal{L}$-definable subsets of K' coincide with the $\mathcal{L}_{\text{ring}}$-definable subsets of K'.

It was proven in [KPS86] that any structure $(\mathcal{R}';\mathcal{L})$ elementary equivalent to an o-minimal structure $(\mathcal{R};\mathcal{L})$, is also o-minimal. Therefore the definition of o-minimality looks slightly simpler than the one of P-minimality.

For any p-adically closed field K the structures $(K;\mathcal{L}_{\text{ring}})$ and $(K;\mathcal{L}_{\text{an}})$ are P-minimal structures. As a consequence all intermediate structures $(K;\mathcal{L})$, where $\mathcal{L}_{\text{ring}} \subseteq \mathcal{L} \subseteq \mathcal{L}_{\text{an}}$, are P-minimal as well. Only some of these intermediate structures have been studied and unfortunately, no other examples are known yet. Therefore, finding new structures would be an important development for P-minimality. In this thesis, however, we will focus on investigating properties shared by all P-minimal structures, in particular a cell decomposition theorem.

As in the o-minimal setting, it is essential for the development of P-minimality to have a (strong enough) cell decomposition theorem for definable sets. The first person to prove such a theorem in a specific case, was Denef, whose work was inspired by the work of Cohen [Coh69]. Before the notion of P-minimality was introduced, Denef developed a cell decomposition theorem for semi-algebraic subsets of p-adic fields, i.e., definable sets in $(K; \mathcal{L}_{\text{ring}})$, or, equivalently, in $(K; \mathcal{L}_{\text{Mac}})$. He used this theorem to prove the rationality of certain Poincaré series [Den84] and to give a new proof of the result of Prestel and Roquette that the theory of p-adically closed fields has quantifier elimination for the language $\mathcal{L}_{\text{Mac}}$ [Den86]. The cells that Denef introduced, were of the form

$$\left\{(s,t) \in S \times K \;\middle|\; \begin{array}{l} \operatorname{ord}(a(s)) \,\square_1\, \operatorname{ord}(t - c(s)) \,\square_2\, \operatorname{ord}(b(s)) \wedge \\ t - c(s) \in \lambda P_n \end{array} \right\}, \tag{1.1.1}$$

where S is a definable set, $a, b, c \colon S \to K$ are definable functions, $\square_1, \square_2$ are either $<$ or 'no condition', $\lambda \in K$ and $n \geqslant 1$. The function $c \colon S \to K$ is called the *center* of the cell. If $\lambda = 0$, we call this cell a 0-cell and if $\lambda \neq 0$, a 1-cell.

When working with cells, where the set S and the functions a, b, c are definable in the subanalytic language $\mathcal{L}_{\text{an}}$, Cluckers obtained a cell decomposition for subanalytic sets [Clu04]. An important generalisation of these two p-adic cell decomposition results was given by Mourgues in [Mou09]. She gave a necessary and sufficient condition for a P-minimal structure to satisfy a cell decomposition result with cells of the form (1.1.1). To understand her result, we first need the following definition.

Definition 1.1.5. A structure $(K; \mathcal{L})$ has *definable Skolem functions*, if for each definable set $X \subset K^{n+1}$, there exists a definable function $f \colon \pi_n(X) \to K$, such that for all $x \in X$, $(\pi_n(x), f(\pi_n(x))) \in X$. Such a function is sometimes also called a *definable section.*

Theorem 1.1.6 ([Mou09])**.** *Let $(K; \mathcal{L})$ be a P-minimal structure. Then the following are equivalent.*

(i) Each definable set partitions as a finite union of sets of the form (1.1.1)*.*

(ii) The structure $(K; \mathcal{L})$ has definable Skolem functions.

When Mourgues proved this theorem, it was not known whether all P-minimal fields have definable Skolem functions. An o-minimal structure where the language contains at least the symbols $<$ and $+$, always has definable Skolem functions [vdD98, Section 6.1]. However, choosing points in a definable way may be easier in o-minimal fields than in P-minimal fields. In $\mathbb{R}$ it is easy to describe a point in an interval (a, b) by taking the midpoint $\frac{a+b}{2}$. There is however no natural way of picking a point from a ball in $\mathbb{Q}_p$, since any point in such a ball can be seen as the center of the ball. Indeed, by now we know, by a result of Cubides and Nguyen [CN17b], that there exist P-minimal fields that do not admit definable Skolem functions. Hence Theorem 1.1.6 implies that there exist P-minimal fields in which not all definable sets can be partitioned into finitely many cells of the form (1.1.1). Hence, we need to adapt the definition of cells of Denef to get a cell decomposition theorem valid in all P-minimal structures.

A first adaptation was introduced by Leenknegt. She showed that one can slightly change the notion of cells to one that is easier to work with, by choosing a uniformizer $\varpi \in K$ and replacing the predicate P_n by

$$Q_{n,m} := \{x \in K^\times \mid \mathrm{ord}(x) \equiv 0 \bmod n \ \wedge \ \mathrm{ac}_m(x) = 1\},$$

for $n, m \in \mathbb{N} \setminus \{0\}$, and where $\mathrm{ac}_m \colon K^\times \to (\mathcal{O}_K/\varpi^m\mathcal{O}_K)^\times$ is the unique group homomorphism that satisfies $\mathrm{ac}_m(\varpi) = 1$ and $\mathrm{ac}_m(x) = (x \bmod \varpi^m)$ for every unit $x \in \mathcal{O}_K^\times$. Such a function exists in any p-adically closed field [CL12, Lemma 1.3]. In the case that K is a p-adic field, we have $\mathrm{ac}_m(x) = (\mathrm{ac}(x) \bmod \varpi^m)$.

Secondly, we will work in two-sorted structures $(K, \Gamma_K; \mathcal{L}_2)$, where we also consider a sort for the value group Γ_K. As explained before the language $\mathcal{L}_2$ consists of a language $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{ring}}$ on K, the Presburger language $\mathcal{L}_{\mathrm{Pres}}$ on Γ_K and a map $\mathrm{ord} : K^\times \to \Gamma_K$ that connects the two sorts. The definition of P-minimality in the two-sorted context is motivated by the result below by Cluckers, which shows that in a P-minimal structure $(K; \mathcal{L})$, with the valuation map ord one can define exactly the $\mathcal{L}_{\mathrm{Pres}}$-definable subsets of Γ_K. So it is natural to take the Presburger language for the value group sort. Furthermore Cluckers' result implies that the $\mathcal{L}_2$-definable subsets of K^n are exactly the $\mathcal{L}$-definable subsets.

Theorem 1.1.7 ([Clu03], Lemma 2 and Theorem 6)**.** *Let $(K; \mathcal{L})$ be a P-minimal field.*

(i) Let $X \subseteq (K^\times)^n$ be an $\mathcal{L}$-definable set, then

$$\mathrm{ord}(X) := \{(\mathrm{ord}(x_1), \dots, \mathrm{ord}(x_n)) \in \Gamma_K^n \mid (x_1, \dots, x_n) \in X\}$$

is $\mathcal{L}_{\mathrm{Pres}}$-definable.

(ii) Let $W \subseteq \Gamma_K^n$ be an $\mathcal{L}_{\mathrm{Pres}}$-definable set, then

$$\mathrm{ord}^{-1}(W) := \{(x_1, \dots, x_n) \in (K^\times)^n \mid \mathrm{ord}(x) \in W\}$$

is $\mathcal{L}_{\mathrm{ring}}$-definable.

The definition of a P-minimal field in the two-sorted context is the following.

Definition 1.1.8. Let $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{ring}}$ and $\mathcal{L}_2 = (\mathcal{L}, \mathcal{L}_{\mathrm{Pres}}, \mathrm{ord})$ and let K be a p-adically closed field with value group Γ_K. A structure $(K, \Gamma_K; \mathcal{L}_2)$ is called *P-minimal* if the underlying structure $(K; \mathcal{L})$ is P-minimal.

Having variables from two different sorts means that we will need two kinds of cells: K-cells where the last variable is from the VF-sort and Γ-cells where the last variable is from the VG-sort. In [CL16] Cubides and Leenknegt introduce a notion of Γ-cells and they prove that for a definable parameter set $S \subseteq K^{r_1} \times \Gamma_K^{r_2}$, a definable set $X \subseteq S \times \Gamma_K$ can be partitioned into finitely many Γ-cells. They even prove that there exists a cell decomposition of X such that a definable function $f\colon X \to \Gamma_K$ is of a specific form on each of the Γ-cells in the partition. This part of the result is sometimes referred to as *function preparation.*

Theorem 1.1.9 ([CL16], Proposition 2.4)**.** *Let $f\colon X \subseteq S \times \Gamma_K \to \Gamma_K$ be definable in a P-minimal structure $(K, \Gamma_K; \mathcal{L}_2)$. Then there exists a finite partition of X into Γ-cells of the form*

$$C = \{(s, \gamma) \in D \times \Gamma_K \mid \alpha(s) \,\square_1\, \gamma \,\square_2\, \beta(s) \,\wedge\, \gamma \equiv k \bmod n\},$$

where D is a definable subset of S, $\alpha, \beta\colon D \to \Gamma_K$ are definable functions, $\square_1$ and $\square_2$ are either $<$ or 'no condition' and $k, n \in \mathbb{N}$. Furthermore, on each cell C, the function f has the form

$$f_{|C}(s, \gamma) = a\left(\frac{\gamma - k}{n}\right) + \delta(s),$$

where $a \in \mathbb{Z}$ and $\delta\colon D \to \Gamma_K$ is a definable function.

Finding a good notion of K-cells is the more tricky part. The idea is to stay close to the cells introduced by Denef, but without assuming that the structure $(K, \Gamma_K; \mathcal{L}_2)$ has definable Skolem functions. Without this assumption it is not always possible to find a definable function $c\colon S \to K$ for the center of a K-cell.

A first version of a general K-cell decomposition was proposed by Cubides and Leenknegt in [CL16]. An improvement of this result was provided in [CCL17a], which uses simpler cells and gives a better understanding of the geometric structure of definable sets in general P-minimal fields. The content of this paper is presented in Chapter 2.

There are other interesting results on definable sets in P-minimal structures, that take different approaches than the one we follow. Darnière and Halupczok [DH17] define the class of p-optimal fields, which is a subclass of the P-minimal fields with definable Skolem functions. For these fields they prove a function preparation result. Cubides, Darnière and Leenknegt [CDL17] prove a cell decomposition theorem for P-minimal structures where the cells are more topological in nature.

1.2 Constructible functions

The cell decomposition techniques that Denef developed for the semi-algebraic language, were motivated by questions about the rationality of certain Poincaré series [Den84]. In the study of these Poincaré series, p-adic integrals appear naturally. As explained before, one can use cell decomposition techniques to partition the domain of integration into cells. Evaluating p-adic integrals over cells is sometimes easier to do than over general definable sets and if some preparation theorem is available, then the integrand might simplify.

Studying certain families of p-adic integrals Denef introduced a class of functions called *(p-adic) constructible functions*. These functions (and some other types of constructible functions as well) were generalised by Cluckers and Loeser to the motivic setting. By introducing additive characters they extended the constructible functions to the exponential-constructible functions. This opened the way to doing Fourier transformations in a very general setting and brought exponential sums into the picture.

In order to do integration we have to fix some measure. Remember that there is a unique Haar measure on a p-adic field K such that $\mathcal{O}_K$ has measure 1. We will integrate with respect to this measure on K and the counting measure on $\mathbb{Z}$. If S and Y are sets containing both K-variables and Γ_K-variables, $X \subseteq S \times Y$ and $f\colon X \to \mathbb{C}$, then the *locus of integrability of f with respect to Y* is

$$\mathrm{Int}(f, Y) := \{s \in S \mid f(s, \cdot\,) \text{ is measurable and integrable over } X_s\}.$$

1.2.1 Constructible functions over p-adic fields

Let K be a p-adic field. We start with the following natural question. Let S and $X \subseteq S \times K$ be $\mathcal{L}_{\mathrm{ring}}$-definable sets and $f\colon X \to K$ an $\mathcal{L}_{\mathrm{ring}}$-definable function. What do the functions

$$s \mapsto \int_{X_s} \mathrm{ord}(f(x))|dx| \qquad \text{and} \qquad s \mapsto \int_{X_s} q_K^{-\mathrm{ord}(f(x))}|dx|$$

look like?

In [Den85] Denef showed that these families of integrals are functions on S of a specific form. He gave them the name *constructible functions*. Although he formulated this notion only for the semi-algebraic language, it can easily be extended to any P-minimal structure.

Definition 1.2.1. Let $(K, \mathbb{Z}; \mathcal{L}_2)$ be a P-minimal structure and X a definable set. The algebra $\mathcal{C}(X)$ of *$\mathcal{L}_2$-constructible functions on X* is the $\mathbb{Q}$-algebra generated by the constant functions and the functions of the form

$$\alpha\colon X \to \mathbb{Z} \qquad \text{and} \qquad X \to \mathbb{Q}\colon x \mapsto q_K^{\beta(x)},$$

where $\alpha, \beta\colon X \to \mathbb{Z}$ are definable.

In the same paper [Den85] Denef showed that the algebras of $\mathcal{L}_{\mathrm{ring},2}$-constructible functions are *base-stable under integration of K-variables*. We will define this notion and a more general notion, where integration over $\mathbb{Z}$-variables (i.e., summation) is also allowed, for general classes of functions.

Definition 1.2.2. Let $(K, \mathbb{Z}; \mathcal{L}_2)$ be a P-minimal structure and $\mathcal{H}$ a class of $\mathbb{C}$-valued functions on definable sets. We say that $\mathcal{H}$ is *base-stable under integration* if, for all definable sets S and $X \subseteq S \times Y$ and for every $f\colon X \to \mathbb{C}$ with $f \in \mathcal{H}$ and $\mathrm{Int}(f, Y) = S$, there exists $g\colon S \to \mathbb{C}$, such that $g \in \mathcal{H}$ and for all $s \in S$,

$$g(s) = \int_{X_s} f(s,x)|dx|.$$

We say that $\mathcal{H}$ is *base-stable under integration of K-variables* if the above condition holds for all definable sets $X \subseteq S \times K^m$, for all $m \in \mathbb{N}$.

Denef uses semi-algebraic cell decomposition and function preparation to show the stability under integration of K-variables. In [Clu04], where Cluckers generalises Denef's cell decomposition to subanalytic structures, he also shows that the algebras of $\mathcal{L}_{\mathrm{an},2}$-constructible functions are base-stable under integration of K-variables.

Cubides and Leenknegt [CL16] have generalised the results of Denef and Cluckers to all P-minimal fields, i.e., without assuming the existence of definable Skolem functions. They used (and proved) a cell decomposition theorem that served as the basis for our cell decomposition.

Theorem 1.2.3 ([CL16], Theorem 4.1)**.** *Let K be a p-adic field and $(K, \mathbb{Z}; \mathcal{L}_2)$ a P-minimal structure. Let S and $X \subseteq S \times Y$ be definable sets and $f \in \mathcal{C}(X)$ with* $\mathrm{Int}(f, Y) = S$*. Then there exists $g \in \mathcal{C}(S)$ such that for all $s \in S$,*

$$g(s) = \int_{X_s} f(s,x)|dx|.$$

1.2.2 Constructible motivic functions

Inspired by the work on p-adic constructible functions and p-adic integration, Cluckers and Loeser transformed these ideas to the motivic setting. Their theory of motivic integration generalises the theory of Kontsevich by allowing the integrals to depend on parameters. Furthermore, their constructible motivic functions specialize to p-adic constructible functions and motivic integration specializes to p-adic integration. In this subsection we will give some definitions and results from the theory of motivic integration as developed by Cluckers and Loeser. For more background we refer to [CL05] and [CL08].

We will use the Denef-Pas language

$$\mathcal{L}_{\mathrm{DP}} = (\mathcal{L}_{\mathrm{ring}}, \mathcal{L}_{\mathrm{ring}}, \mathcal{L}_{\mathrm{oag}}, \mathrm{ord}, \overline{\mathrm{ac}}).$$

We recall that this language consists of three sorts. For the VF-sort and the RF-sort we use the ring language $\mathcal{L}_{\mathrm{ring}}$ and for the VG-sort we use the ordered abelian group language $\mathcal{L}_{\mathrm{oag}}$. The map ord connects the VF-sort to the VG-sort and the map $\overline{\mathrm{ac}}$ connect the VF-sort to the RF-sort. Structures of the Denef-Pas language are of the form $(K, k_K, \Gamma_K, \mathrm{ord}, \overline{\mathrm{ac}})$, where K is a valued field with residue field k_K, value group Γ_K, valuation map $\mathrm{ord}\colon K^\times \to \Gamma_K$ and some angular component map (modulo $\mathcal{M}_K$) $\overline{\mathrm{ac}}\colon K \to k_K$.

An important result is the elimination of valued field quantifiers in the language $\mathcal{L}_{\mathrm{DP}}$. Denote by $H_{\overline{\mathrm{ac}},0}$ the $\mathcal{L}_{\mathrm{DP}}$-theory of the above described three-sorted structures, whose valued field is Henselian and whose residue field is of characteristic zero. Then the theory $H_{\overline{\mathrm{ac}},0}$ admits elimination of quantifiers in the valued field sort, as stated in the following theorem. This theorem is proven by using a cell decomposition theorem [Pas89, Theorem 3.2]. A more general version of this cell decomposition theorem will be stated further on.

Theorem 1.2.4 ([Pas89], Theorem 4.1)**.** *The theory $H_{\overline{\mathrm{ac}},0}$ admits elimination of quantifiers in the valued field sort. More precisely, every $\mathcal{L}_{\mathrm{DP}}$-formula $\varphi(x, \xi, \alpha)$ (without parameters), with x denoting variables in the* VF*-sort, ξ variables in the* RF*-sort and α variables in the* VG*-sort, is $H_{\overline{\mathrm{ac}},0}$-equivalent to a finite disjunction of formulas of the form*

$$\psi\big(\overline{\mathrm{ac}}(f_1(x)), \ldots, \overline{\mathrm{ac}}(f_r(x)), \xi\big) \wedge \vartheta\big(\mathrm{ord}(f_1(x)), \ldots, \mathrm{ord}(f_r(x)), \alpha\big),$$

where ψ is an $\mathcal{L}_{\mathrm{ring}}$-formula, ϑ an $\mathcal{L}_{\mathrm{oag}}$-formula and $f_1, \ldots, f_r$ are polynomials in $\mathbb{Z}[X]$.

This theorem implies the following, useful corollary for $\mathcal{L}_{\mathrm{DP}}$-formulas with parameters in the VF-sort. Let (K, k_K, Γ_K) be a model of the theory $H_{\overline{\mathrm{ac}},0}$. For a subring R of K, we denote by $\mathcal{L}_{\mathrm{DP}} \cup R$ the language that is obtained from $\mathcal{L}_{\mathrm{DP}}$ by adding a constant symbol for each element of R to the language $\mathcal{L}_{\mathrm{ring}}$ for the VF-sort. Then we take T_R to be the atomic diagram of R, i.e., the set of atomic $\mathcal{L}_{\mathrm{DP}} \cup R$-sentences and negations of atomic sentences φ such that $R \models \varphi$, and we take $H_R := H_{\overline{\mathrm{ac}},0} \cup T_R$.

Corollary 1.2.5 ([CL08], Corollary 2.1.2)**.** *Let (K, k_K, Γ_K) be a model of the theory $H_{\overline{\mathrm{ac}},0}$ and R a subring of K. Then Theorem 1.2.4 holds with $H_{\overline{\mathrm{ac}},0}$ replaced by H_R, $\mathcal{L}_{\mathrm{DP}}$ replaced by $\mathcal{L}_{\mathrm{DP}} \cup R$, and $\mathbb{Z}[X]$ replaced by $R[X]$.*

It is important to remark that by logical compactness, this theorem and its corollary are still true for the Henselian valued fields $\mathbb{Q}_p$ for p sufficiently large (where the lower bound on p depends on the formulas involved).

Next we will explain rather informally the notion of *constructible motivic function.* For all the details we refer to [CL08]. We will need several expansions of the Denef-Pas language.

- If k is a field of characteristic zero, then we denote by $\mathcal{L}_{\mathrm{DP},k}$ the language $\big(\mathcal{L}_{\mathrm{ring}} \cup k((t)), \mathcal{L}_{\mathrm{ring}} \cup k, \mathcal{L}_{\mathrm{oag}}, \mathrm{ord}, \overline{\mathrm{ac}}\big)$, i.e., in the VF-sort we add constant symbols for the elements of $k((t))$ and in the RF-sort for the elements of k.
- If $\mathcal{O}$ is the ring of integers of some number field k, then we denote by $\mathcal{L}_{\mathcal{O}}$ the language $\big(\mathcal{L}_{\mathrm{ring}} \cup \mathcal{O}[[t]], \mathcal{L}_{\mathrm{ring}} \cup \mathcal{O}, \mathcal{L}_{\mathrm{oag}}, \mathrm{ord}, \overline{\mathrm{ac}}\big)$.

For a fixed field k of characteristic zero, we denote by Field_k the category of fields that contain k. For any field $\mathcal{K} \in \mathrm{Field}_k$ we have an $\mathcal{L}_{\mathrm{DP},k}$-structure $(\mathcal{K}((t)), \mathcal{K}, \mathbb{Z})$. There is unfortunately no first-order theory whose models are exactly the valued fields of the form $\mathcal{K}((t))$ with $\mathcal{K}$ a field containing k. Therefore talk about these structures in a categorical way. For $m, n, r \in \mathbb{N}$, we put

$$h[m, n, r](\mathcal{K}) := \mathcal{K}((t))^m \times \mathcal{K}^n \times \mathbb{Z}^r,$$

and for an $\mathcal{L}_{\mathrm{DP},k}$-formula φ with m VF-variables, n RF-variables and r VG-variables, we denote by $h_\varphi(\mathcal{K})$ the definable subset of $h[m, n, r](\mathcal{K})$ consisting of points that satisfy φ. The category Def_k has

- as objects the *k-definable subassignments* $\mathcal{K} \mapsto h_\varphi(\mathcal{K})$ from Field_k to Sets, induced by the $\mathcal{L}_{\mathrm{DP},k}$-formulas φ;
- as morphisms $f\colon Y \to Z$ the k-definable subassignments h_φ that assign to each $\mathcal{K} \in \mathrm{Field}_k$, the graph $h_\varphi(\mathcal{K}) \subseteq Y(\mathcal{K}) \times Z(\mathcal{K})$ of a definable function $f(\mathcal{K})\colon Y(\mathcal{K}) \to Z(\mathcal{K})$.

If $Y, Z \in \mathrm{Def}_k$, such that $Y(\mathcal{K}) \subseteq Z(\mathcal{K})$ for all $\mathcal{K} \in \mathrm{Field}_k$, then we write $Y \subseteq Z$ and we say that Y is a *k-definable subassignment of Z*. For $m, n, r \in \mathbb{N}$ we put

$$Z[m, n, r] := Z \times h[m, n, r].$$

We write Def_Z for the category whose objects are morphisms $Y \to Z$ in Def_k and whose morphisms f are given by commutative triangles

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr] & & Y' \arrow[dl] \\ & Z & \end{tikzcd}$$

in Def_k. We denote RDef_Z for the full subcategory of Def_Z on the objects $Y \to Z$, where $Y \subseteq Z[0, n, 0]$ for some $n \in \mathbb{N}$ and where $Y \to Z$ is induced by the projection $Z[0, n, 0] \to Z$.

The Cluckers-Loeser theory of motivic integration is not only inspired by the p-adic constructible functions from Definition 1.2.1 with p-adic integration, but also constructible functions over the real numbers with integration along the Euler characteristic as in [Vir88]. That is why we introduce the following Grothendieck ring, that is an analogue of the Grothendieck ring of varieties.

Definition 1.2.6. Let $Z \in \mathrm{Def}_k$. The *Grothendieck group* $K_0(\mathrm{RDef}_Z)$ is the quotient of the free abelian group on the symbols $[Y \to Z]$, for $Y \to Z$ objects of RDef_Z, by the relations

(i) $[Y \to Z] = [Y' \to Z]$, if $Y \to Z$ is isomorphic with $Y' \to Z$,

(ii) $[Y \cup Y' \to Z] + [Y \cap Y' \to Z] = [Y \to Z] + [Y' \to Z]$, where $Y, Y' \subseteq Z[0, n, 0]$ for some $n \in \mathbb{N}$.

The Cartesian fiber product over Z induces a natural ring structure on $K_0(\mathrm{RDef}_Z)$ by

$$[Y \to Z] \times [Y' \to Z] := [Y \times_Z Y' \to Z].$$

Informally, this Grothendieck ring will specialize, over a p-adic field K, to counting points of fibers of projections $Y(K) \subseteq Z(K) \times k_K^n \to Z(K)$.

The next part consists of a generalisation of the constructible functions from Definition 1.2.1. We consider a formal symbol $\mathbb{L}$ and the ring

$$\mathbb{A} := \mathbb{Z}\Big[\mathbb{L}, \mathbb{L}^{-1}, \Big(\frac{1}{1 - \mathbb{L}^{-i}}\Big)_{i>0}\Big].$$

If Z is a k-definable subassignment, then the points of Z are tuples $z = (z_0, \mathcal{K})$, where $z_0 \in Z(\mathcal{K})$ and $\mathcal{K} \in \mathrm{Field}_k$. We denote the set of points of Z by $|Z|$.

Definition 1.2.7. Let Z be in Def_k. The ring $\mathcal{P}(Z)$ of *constructible Presburger functions on Z* is defined as the subring of the ring of functions $|Z| \to \mathbb{A}$, generated by

- the constant functions $|Z| \to \mathbb{A}$;
- the functions $\widehat{\alpha}\colon |Z| \to \mathbb{Z}$ that correspond to a definable morphism $\alpha\colon Z \to h[0,0,1]$;
- the functions $\mathbb{L}^{\widehat{\beta}}\colon |Z| \to \mathbb{A}$ that correspond to a definable morphism $\beta\colon Z \to h[0,0,1]$.

There is some subring $\mathcal{P}^0(Z)$ of $\mathcal{P}(Z)$ consisting of functions that can be viewed as elements of $K_0(\mathrm{RDef}_Z)$ as well. We will write $\mathbb{L}_Z$ for the class $[Z \times h[0,1,0] \to Z]$ and $\mathbb{L}_Z - 1$ for the class $[Z \times (h[0,1,0] \setminus \{0\}) \to Z]$ in $K_0(\mathrm{RDef}_Z)$.

Definition 1.2.8. Let Z be in Def_k. We denote by $\mathcal{P}^0(Z)$ the subring of $\mathcal{P}(Z)$, generated by

- the functions $\mathbb{1}_Y$, for all $Y \subseteq Z$, which take the value 1 on $|Y|$ and 0 on $|Z \setminus Y|$;
- the constant function $\mathbb{L} - 1$.

Then there is a canonical ring morphism $\mathcal{P}^0(Z) \to K_0(\mathrm{RDef}_Z)$ sending $\mathbb{1}_Y$ to the class of the inclusion morphism $[i\colon Y \to Z]$ and $\mathbb{L} - 1$ to $\mathbb{L}_Z - 1$.

Now we are ready to define constructible motivic functions. For technical reasons related to the purpose of integration, dimensions have to be taken into account when developing the notion of motivic integration. That is why Cluckers and Loeser also introduced *constructible motivic Functions.*

Definition 1.2.9. Let Z be in Def_k. We define the ring $\mathcal{C}(Z)$ of *constructible motivic functions on* Z as

$$\mathcal{C}(Z) := K_0(\mathrm{RDef}_Z) \otimes_{\mathcal{P}^0(Z)} \mathcal{P}(Z).$$

Let Y be a subassignment of $h[m,n,r]$, for some $m,n,r \in \mathbb{N}$. We denote by $\dim Y$ the dimension of the Zariski closure of $\pi(Y)$ for π the projection $h[m,n,r] \to h[m,0,0]$. For a natural number d, we denote by $\mathcal{C}^{\leqslant d}(Z)$ the ideal of $\mathcal{C}(Z)$ generated by all elements of the form $\mathbb{1}_Y$, with $Y \subseteq Z$, such that $\dim Y \leqslant d$. We set

$$\mathcal{C}^d(Z) := \mathcal{C}^{\leqslant d}(Z)/\mathcal{C}^{\leqslant d-1}(Z) \quad \text{and} \quad \boldsymbol{C}(Z) := \bigoplus_{d \geqslant 0} \mathcal{C}^d(Z).$$

The elements of $\boldsymbol{C}(Z)$ are called *constructible motivic Functions.*

In [CL08] Cluckers and Loeser construct, for S in Def_k and Z in Def_S, a graded subgroup $I_S\boldsymbol{C}(Z)$ of $\boldsymbol{C}(Z)$ of *S-integrable Functions* and for each morphism $f\colon Y \to Z$ in Def_S a map $f_!\colon I_S\boldsymbol{C}(Y) \to I_S\boldsymbol{C}(Z)$, satisfying a list of axioms, that represent natural condition for integration. When $S = h[0,0,0]$ and $f\colon Y \to h[0,0,0]$, then the map $f_!\colon I_S\boldsymbol{C}(Y) \to \boldsymbol{C}(h[0,0,0])$ is exactly the same as taking the integral over Y.

An essential element in proving the above statements is the following cell decomposition theorem, which is a generalisation of [Pas89, Theorem 3.2].

Definition 1.2.10. Let S be in Def_k and $C \subseteq S$. Let α, ξ, c be morphisms $\alpha\colon C \to h[0,0,1]$, $\xi\colon C \to h[0,1,0]$ and $c\colon C \to h[1,0,0]$ in Def_k. The 0-*cell* $Z_{C,c}$ with *basis* C and *center* c, is the definable subassignment of $S[1,0,0]$, defined by the formula

$$y \in C \;\wedge\; z = c(y),$$

where y belongs to S and z to $h[1,0,0]$. Similarly, the 1-*cell* $Z_{C,\alpha,\xi,c}$ with *basis* C, *order* α, *angular component* ξ and *center* c, is the definable subassignment of $S[1,0,0]$, defined by the formula

$$y \in C \;\wedge\; \mathrm{ord}(z - c(y)) = \alpha(y) \;\wedge\; \overline{\mathrm{ac}}(z - c(y)) = \xi(y),$$

A definable subassignment Z of $S[1,0,0]$ will be called a 0-*cell*, resp. 1-*cell*, if there exists a definable isomorphism

$$\lambda\colon Z \to Z_C = Z_{C,c} \subseteq S[1,s,0],$$

resp. a definable isomorphism

$$\lambda\colon Z \to Z_C = Z_{C,\alpha,\xi,c} \subseteq S[1,s,r],$$

for some $r, s \geqslant 0$, some basis $C \subseteq S[0,s,0]$, resp. $S[0,s,r]$, and some 0-cell $Z_{C,c}$, resp. 1-cell $Z_{C,\alpha,\xi,c}$, such that the morphism $\pi \circ \lambda$, with π the projection $Z_C \to S[1,0,0]$, is the identity on Z. The data $(\lambda, Z_{C,c})$, resp. $(\lambda, Z_{C,\alpha,\xi,c})$, will be called a *presentation of the cell Z* and denoted for short by (λ, Z_C).

Theorem 1.2.11 ([CL08], Theorem 7.2.1)**.** *Suppose that k is a field of characteristic 0. Let X be a definable subassignment of $S[1,0,0]$ with S in Def_k. Then the following statements hold.*

(i) The subassignment X can be written as a finite disjoint union of cells.

(ii) For every $\varphi \in \mathcal{C}(X)$, there exists a finite partition of X into cells Z_i with presentation (λ_i, Z_{C_i}), such that $\varphi_{|Z_i} = \lambda_i^* p_i^*(\psi_i)$, with $\psi_i \in \mathcal{C}(C_i)$ and $p_i\colon Z_{C_i} \to C_i$ the projection. Similar statements hold for φ in $\mathcal{P}(X)$ and in $K_0(\mathrm{RDef}_X)$.

Corollary 1.2.12. *Let $\mathcal{O}$ be the ring of integers of a number field k. Then Theorem 1.2.11 still holds, if we work with the language $\mathcal{L}_{\mathcal{O}}$ instead of $\mathcal{L}_{\mathrm{DP},k}$.*

Proof. The proof is the same as for Theorem 1.2.11, but we replace $\mathcal{L}_{\mathrm{DP},k}$ by $\mathcal{L}_{\mathcal{O}} \subseteq \mathcal{L}_{\mathrm{DP},k}$. □

1.2.3 Specialization

We have already informally explained the similarities between constructible motivic functions and p-adic constructible functions. Now we will make this more precise. Essentially it comes down to choosing a uniformizer ϖ for a p-adic field and 'replacing t by ϖ'.

Let k be a number field with $\mathcal{O}$ its ring of integers. We denote by $\mathcal{F}_{\mathcal{O}}$ the set of all p-adic completions of k and of finite field extensions of k. All the fields in $\mathcal{F}_{\mathcal{O}}$ can be equipped with the structure of an $\mathcal{O}$-algebra. For $N \in \mathbb{N}$ we denote by $\mathcal{F}_{\mathcal{O},N}$ the set of all fields $K \in \mathcal{F}_{\mathcal{O}}$ whose residue field k_K has characteristic at least N.

We will work with the language $\mathcal{L}_{\mathcal{O}}$. The category $\mathrm{Def}_{\mathcal{L}_{\mathcal{O}}}$ consists of the definable subassignments that can be defined in the language $\mathcal{L}_{\mathcal{O}}$. If $S \in \mathrm{Def}_{\mathcal{L}_{\mathcal{O}}}$, then the ring of constructible motivic functions $\mathcal{C}(S, \mathcal{L}_{\mathcal{O}})$ consists of the constructible motivic functions on S that can be formed with formulas in the language $\mathcal{L}_{\mathcal{O}}$.

Let $K \in \mathcal{F}_{\mathcal{O}}$. For each choice of a uniformizing element ϖ_K of $\mathcal{O}_K$, there is a unique angular component map (modulo $\mathcal{M}_K$) $\overline{\mathrm{ac}}_{\varpi_K} : K^\times \to k_K^\times$, which extends the map 'reduction modulo $\mathcal{M}_K$': $\mathcal{O}_K^\times \to k_K^\times$ and sends ϖ_K to 1. Then $(K, k_K, \mathbb{Z})$ is an $\mathcal{L}_{\mathrm{DP}}$-structure with respect to ϖ_K. Moreover K can be equipped with the structure of an $\mathcal{O}[[t]]$-algebra via the morphism

$$\lambda_{\varpi_K} : \mathcal{O}[[t]] \to K\colon \sum_{i \geqslant 0} a_i t^i \mapsto \sum_{i \geqslant 0} a_i \varpi_K^i.$$

By interpreting $a \in \mathcal{O}[[t]]$ as $\lambda_{\varpi_K}(a)$, the structure $(K, k_K, \mathbb{Z})$ becomes an $\mathcal{L}_{\mathcal{O}}$-structure. An $\mathcal{L}_{\mathcal{O}}$-formula φ defines, for each $K \in \mathcal{F}_{\mathcal{O}}$ (and a choice of uniformizer ϖ_K) a definable subset $\varphi(K)$ of $K^m \times k_K^n \times \mathbb{Z}^r$, for some $m, n, r \in \mathbb{N}$. If we have two $\mathcal{L}_{\mathcal{O}}$-formulas φ_1 and φ_2, which define the same subassignment of $h[m, n, r]$ from Field_k to Sets, then, by logical compactness, there exists $N_0 \in \mathbb{N}$, such that for all $N > N_0$, we have $\varphi_1(K) = \varphi_2(K)$, for all $K \in \mathcal{F}_{\mathcal{O},N}$ (and for any choice of ϖ_K).

Let $Y \in \mathrm{Def}_{\mathcal{L}_{\mathcal{O}}}$ be defined by a formula φ in the language $\mathcal{L}_{\mathcal{O}}$. Even though there could be other $\mathcal{L}_{\mathcal{O}}$-formulas defining the same subassignment Y, we will usually just choose one. For any $K \in \mathcal{F}_{\mathcal{O}}$ and uniformizer ϖ_K, we denote the interpretation of Y in K by $Y_{K,\varpi_K} := \varphi(K)$. We will often simplify the notation to Y_K and use the notation Y_{K,ϖ_K} (or Y_{ϖ_K}) only when the dependence on ϖ_K is important. Similarly, each morphism $f\colon Y \to Z$ in $\mathrm{Def}_{\mathcal{L}_{\mathcal{O}}}$ has an interpretation $f_K\colon Y_K \to Z_K$.

Now we will explain how a constructible motivic function $\theta \in \mathcal{C}(S, \mathcal{L}_{\mathcal{O}})$ *specializes* to a constructible function $\theta_K \in \mathcal{C}(S_K)$ (as in Definition 1.2.1) over a field $K \in \mathcal{F}_{\mathcal{O}}$.

- For the elements of $\mathcal{P}(S)$, we replace $\mathbb{L}$ by q_K and definable functions $\alpha\colon S \to h[0, 0, 1]$ by their interpretations $\alpha_K\colon S_K \to \mathbb{Z}$.
- The elements $\theta = [Y \xrightarrow{f} S]$ in $K_0(\mathrm{RDef}_{S,\mathcal{L}_{\mathcal{O}}})$, with $f\colon Y \to S$ a morphism in $\mathrm{Def}_{\mathcal{L}_{\mathcal{O}}}$, are interpreted in K by putting
$$\theta_K(s) := \#(f^{-1}(s)),$$
for all $s \in S_K$.

Of course these interpretations can depend on the choice of formulas needed to define θ.

In [CL05, Theorem 6.9] Cluckers and Loeser show that specialization commutes with integration, in the sense that for an integrable motivic function, the specialization of its motivic integral equals the p-adic integral of its specialization for almost all prime numbers p.

1.2.4 Exponential-constructible functions

Exponential-constructible functions were first introduced in the motivic setting by Cluckers and Loeser in [CL10]. They are obtained by adding additive characters on the VF- and RF-variables to the constructible motivic functions. This is done by extending the category RDef_Z to the category $\mathrm{RDef}_Z^{\exp}$. The Grothendieck ring $K_0(\mathrm{RDef}_Z^{\exp})$ is obtained by quotienting the free abelian group on the elements of $\mathrm{RDef}_Z^{\exp}$ by some relations. The rest of the construction is similar to the construction of the constructible motivic functions.

Cluckers-Loeser also give the definition of the p-adic analogue of the exponential-constructible functions for the semi-algebraic and subanalytic languages and show that they can be obtained from the motivic ones by specialization. Furthermore, they show that the p-adic exponential-constructible functions are base-stable under integration over K-variables (see Definition 1.2.2) under an extra condition. We will formulate their definition and result for two sorted languages and put it in the context of general P-minimal structures on p-adic fields.

Let K be a p-adic field and $\psi\colon K \to \mathbb{C}^\times$ an additive character, such that $\psi_{|\mathcal{M}_K} = 1$ and $\psi_{|\mathcal{O}_K} \neq 1$. An example of such a function on $\mathbb{Q}_p$ is

$$\psi(x) = \exp\left(\frac{2\pi i x'}{p}\right),$$

where $x' \in \mathbb{Z}\left[\frac{1}{p}\right] \cap (x + \mathcal{M}_K)$.

Definition 1.2.13. Let $(K, \mathbb{Z}; \mathcal{L}_2)$ be a P-minimal structure and X a definable set. The algebra $\mathcal{C}_{\exp,\psi}(X)$ of *$\mathcal{L}_2$-exponential-constructible functions on X* is the $\mathbb{Q}$-algebra generated by the functions in $\mathcal{C}(X)$ and the functions of the form $\psi \circ f$, where $f\colon X \to K$ is definable.

We will write $\mathcal{C}_{\exp}(X)$ rather than $\mathcal{C}_{\exp,\psi}(X)$ when no confusion is possible.

Theorem 1.2.14 ([CL10], Proposition 8.6.1)**.** *Let $\mathcal{L}$ be either $\mathcal{L}_{\text{ring}}$ or $\mathcal{L}_{\text{an}}$, S and $X \subseteq S \times K^n$ definable sets and $f \in \mathcal{C}_{\exp}(X)$ with*

$$f(s,x) = \sum_{i=1}^{m} h_i(s,x)\psi(f_i(s,x)), \tag{1.2.1}$$

where, for each $i \in \{1, \dots, m\}$, $h_i \in \mathcal{C}(X)$ with $\mathrm{Int}(h_i, K^n) = S$ and f_i is a definable function. Then there exists $g \in \mathcal{C}_{\exp}(S)$ such that for all $s \in S$,

$$g(s) = \int_{X_s} f(s,x)|dx|.$$

In a subsequent paper, Cluckers, Gordon and Halupczok managed to remove the condition (1.2.1) on the form of f, thereby showing that for $\mathcal{L}_{\text{ring}}$ and $\mathcal{L}_{\text{an}}$, the algebras of exponential-constructible functions are always base-stable under integration over K-variables [CGH14, Theorem 3.2.1]. Moreover, they also managed to remove the condition on the locus of integrability, for both constructible and exponential-constructible functions.

It is natural to ask whether the result of Cubides and Leenknegt for constructible functions in general P-minimal structures (Theorem 1.2.3) can be extended to exponential-constructible functions and under what conditions. In Chapter 3 we will discuss [CCL18] in which such an extension is proposed.

1.3 Igusa zeta functions and exponential sums

In this section we will discuss the relation between certain exponential sums over p-adic fields and the Igusa zeta functions. Furthermore, we will study the asymptotic behaviour of these sums, when the residue field characteristic goes to infinity.

We fix a number field k and a nonconstant polynomial $f \in k[x]$ in n variables $x = (x_1, \dots, x_n)$. Let $\mathfrak{p}$ be a prime ideal of the ring of integers $\mathcal{O}$ of k. We denote the completions of $\mathcal{O}$ and k with respect to $\mathfrak{p}$ by $\mathcal{O}_\mathfrak{p}$ and $K_\mathfrak{p}$ respectively. The field $K_\mathfrak{p}$ is a p-adic field for some prime number $p \in \mathfrak{p}$, with valuation ring $\mathcal{O}_\mathfrak{p}$, maximal ideal $\mathfrak{p}\mathcal{O}_\mathfrak{p}$, value group $\mathbb{Z}$, residue field $k_\mathfrak{p}$ (with $q = p^f$ elements) and a choice of uniformizer ϖ induces an angular component map $\mathrm{ac}\colon K_\mathfrak{p} \to \mathcal{O}_\mathfrak{p}$. Furthermore we will encounter the following functions.

- $\Phi\colon K_\mathfrak{p}^n \to \mathbb{C}$ denotes a *Schwartz-Bruhat function*, that is, a locally constant function with compact support, denoted by $\mathrm{Supp}(\Phi)$. We say that Φ is *residual* if $\mathrm{Supp}(\Phi) \subseteq \mathcal{O}_\mathfrak{p}^n$ and if $\Phi(x)$ only depends on ($x \bmod \mathfrak{p}\mathcal{O}_\mathfrak{p}^n$). In this case Φ induces a function $\overline{\Phi}\colon k_\mathfrak{p}^n \to \mathbb{C}$.
- $\chi\colon \mathcal{O}_\mathfrak{p}^\times \to \mathbb{C}^\times$ denotes a multiplicative character, i.e., a multiplicative group homomorphism with finite image. We denote $d(\chi)$ for $\#(\mathrm{Im}(\chi))$ and $c(\chi)$ for the *conductor* of χ, that is, the smallest $c \geqslant 1$ such that χ is constant on $\mathfrak{p}^c\mathcal{O}_\mathfrak{p}$. Moreover, we put $\chi(0) = 0$.
- $\Psi\colon K_\mathfrak{p} \to \mathbb{C}^\times$ denotes the standard additive character on $K_\mathfrak{p}$, i.e., $\Psi(x) = \exp(2\pi i \mathrm{Tr}_{K_\mathfrak{p}/\mathbb{Q}_p}(x))$, where $\mathrm{Tr}_{K_\mathfrak{p}/\mathbb{Q}_p}(x)$ is the trace of the map 'multiplication by x', seen as a $\mathbb{Q}_p$-linear map on $K_\mathfrak{p}$.

1.3.1 Igusa zeta functions

The Igusa zeta functions are closely related to the Poincaré series that we saw in the introduction and whose coefficients count the number of solutions of congruences modulo p^m. We consider these zeta functions for their relation to certain exponential sums.

Definition 1.3.1. To a nonconstant polynomial $f \in k[x_1, \dots, x_n]$, a prime ideal $\mathfrak{p}$, a multiplicative character χ and a Schwartz-Bruhat function Φ we can associate the *Igusa local zeta function*, which is a function in the complex variable $s \in \mathbb{C}$, with $\mathrm{Re}(s) > 0$:

$$Z_f^\Phi(s, \mathfrak{p}, \chi) := \int_{K_\mathfrak{p}^n} \Phi(x)\chi\big(\mathrm{ac}(f(x))\big)|f(x)|^s |dx|,$$

where $\mathrm{ac}(f(x)) \in \mathcal{O}_{\mathfrak{p}}^{\times} \cup \{0\}$, thus $\chi\big(\mathrm{ac}(f(x))\big)$ is well-defined.

When the polynomial f is clear from the context, we will write simply $Z^{\Phi}(s, \mathfrak{p}, \chi)$. If we see $Z^{\Phi}(s, \mathfrak{p}, \chi)$ as a function in the variable $t = q^{-s}$, then it is a rational function in t, which was shown by both Igusa [Igu75] and Denef [Den84]. As a consequence there exists a meromorphic continuation of $Z^{\Phi}(s, \mathfrak{p}, \chi)$ to all of $\mathbb{C}$.

In what follows, we will mention some results of Denef that give more explicit formulas for the Igusa zeta functions, expressed in terms of the numerical data associated to a resolution of the singularities of $f^{-1}(0)$. A nice overview of these results can be found in [Den91b]. Let us fix some notation. Let k^{alg} be an algebraic closure of k. Fix any embedded resolution $h\colon Y \to (k^{\mathrm{alg}})^n$ of $f^{-1}(0)$. Let E be a prime divisor on Y, then we denote the multiplicities of E in the divisors $f \circ h$ and $h^*(dx_1 \wedge \ldots \wedge dx_n)$ by N and $\nu - 1$ respectively. We denote by $\{E_i \mid i \in T\}$ the set of prime divisors from $(f \circ h)^{-1}(0)$ and the corresponding multiplicities $\{(N_i, \nu_i) \mid i \in T\}$ are called the *numerical data* of the resolution (Y, h).

We will use the following notation. For any $I \subseteq T$ we denote $E_I := \cap_{i\in I} E_i$. So in particular, $E_{\emptyset} = Y$. Moreover, we will use $C_f \subseteq (k^{\mathrm{alg}})^n$ to denote the critical locus of f, i.e., the points where all partial derivatives of f vanish, and $V_f \subseteq k^{\mathrm{alg}}$ to denote the set of *critical values of* f, i.e., $V_f := f(C_f)$. The set V_f is a finite set.

For a closed subscheme Z of Y there exists the notion of *the reduction modulo* $\mathfrak{p}$ of Z, denoted by $\overline{Z}$. For a precise definition see [Shi55]. In what follows it is necessary that the resolution (Y, h) behaves well under reduction modulo $\mathfrak{p}$. What we mean by this is explained in the following definition.

Definition 1.3.2. We say that an embedded resolution (Y, h) of $f^{-1}(0)$ has *good reduction modulo* $\mathfrak{p}$ if $\overline{Y}$ and $\overline{E_i}$ are smooth for all $i \in T$, $\cup_{i\in T}\overline{E_i}$ has only normal crossings and $\overline{E_i}$ and $\overline{E_j}$ have no common components for all $i, j \in T$ with $i \neq j$.

We say that a resolution (Y, h) has *tame good reduction modulo* $\mathfrak{p}$ if it has good reduction modulo $\mathfrak{p}$ and furthermore $N_i \notin \mathfrak{p}$, for all $i \in T$.

If (Y, h) has good reduction modulo $\mathfrak{p}$, then $\overline{E_I} = \cap_{i\in I}\overline{E_i}$, for any $I \subseteq T$. We will write

$$\overline{E_I}^{\circ} := \overline{E_I} \setminus \bigcup_{j\in T\setminus I} \overline{E_j}.$$

Furthermore, in the local ring of $\overline{Y}$ at $a \in \overline{E_I}^{\circ}$ we can write

$$\overline{f} \circ \overline{h} = \overline{u} \prod_{i\in I} \overline{y_i}^{N_i},$$

where $\overline{u}$ is a unit and $\{\overline{y_i}\}_{i\in I}$ is part of a regular system of parameters of the local ring.

A resolution (Y, h) has (tame) good reduction modulo $\mathfrak{p}$ for all but finitely many prime ideals $\mathfrak{p}$ [Den87, Theorem 2.4]. Also $f \in \mathcal{O}_{\mathfrak{p}}[x]$ and $\overline{f} \neq 0$ for all but finitely many prime ideals $\mathfrak{p}$. When we exclude these finitely many prime ideals, we can give more explicit formulas for the Igusa zeta functions. There are two cases to be distinguished. In the first case we consider a character χ with conductor $c(\chi) = 1$. Such a character induces a character $\overline{\chi}$ on $k_{\mathfrak{p}}^{\times}$.

Theorem 1.3.3 ([Den91a], Theorem 2.2)**.** *Let $f \in k[x_1, \ldots, x_n]$ be nonconstant and let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$, such that $f \in \mathcal{O}_{\mathfrak{p}}[x]$, $\overline{f} \neq 0$ and the resolution (Y, h) has good reduction modulo $\mathfrak{p}$. Take Φ a residual Schwartz-Bruhat function and χ a multiplicative character with conductor $c(\chi) = 1$. Then*

$$Z^{\Phi}(s, \mathfrak{p}, \chi) = q^{-n} \sum_{\substack{I\subseteq T, \\ \forall i\in I: d(\chi)\mid N_i}} c_{I,\chi,\Phi} \prod_{i\in I} \frac{q-1}{q^{N_i s+\nu_i}-1},$$

where

$$c_{I,\chi,\Phi} = \sum_{a\in \overline{E_I}^{\circ}(k_{\mathfrak{p}})} \overline{\Phi}(\overline{h}(a))\Omega_{\overline{\chi}}(a).$$

If $d(\chi) \mid N_i$ for all $i \in I$ and $a \in \overline{E_I}^{\circ}(k_{\mathfrak{p}})$, then we define $\Omega_{\overline{\chi}}(a) := \overline{\chi}(\overline{u}(a))$ and if not, then $\Omega_{\overline{\chi}}(a) := 0$.

Remark 1.3.4. For characters χ with conductor $c(\chi) = 1$ and such that for all $i \in T$, $d(\chi) \nmid N_i$, we have $Z^{\Phi}(s, \mathfrak{p}, \chi) = 0$. In particular this is the case when $d(\chi) > \max\{N_i \mid i \in T\}$. It is important to note that there are only finitely many characters χ with $c(\chi) = 1$ and $d(\chi) \leqslant \max\{N_i \mid i \in T\}$.

The second case in which we know an explicit formula for $Z^{\Phi}(s, \mathfrak{p}, \chi)$ concerns characters χ with conductor $c(\chi) > 1$.

Theorem 1.3.5 ([Den91a], Theorem 2.1)**.** *Let $f \in k[x_1, \ldots, x_n]$ be nonconstant and let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$, such that $f \in \mathcal{O}_{\mathfrak{p}}[x]$, $\overline{f} \neq 0$ and the resolution (Y, h) has tame good reduction modulo $\mathfrak{p}$. Take Φ a residual Schwartz-Bruhat function for which $C_{\overline{f}} \cap \mathrm{Supp}(\overline{\Phi}) \subseteq \overline{f}^{-1}(0)$, and χ a multiplicative character with conductor $c(\chi) > 1$. Then $Z^{\Phi}(s, \mathfrak{p}, \chi) = 0$.*

1.3.2 Exponential sums

The exponential sums that we are interested in were introduced by Weil [Wei48] and can be written as p-adic integrals:

$$E_f^{\Phi}(z,\mathfrak{p}) := \int_{K_{\mathfrak{p}}^n} \Phi(x)\Psi(zf(x))|dx|,$$

where $z \in K_{\mathfrak{p}}$. The case $z = 0$ is not so interesting, so we will usually write z as $z_0\varpi^{-m}$, where $m = -\mathrm{ord}(z) \in \mathbb{Z}$ and $z_0 = \mathrm{ac}(z) \in \mathcal{O}_{\mathfrak{p}}^{\times}$. Also for any $m \geqslant 2$ and any $y \in \mathcal{O}_{\mathfrak{p}}$, we will denote by $\overline{y}^{(m)}$ the image of y under the quotient map $\mathcal{O}_{\mathfrak{p}} \to \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^m\mathcal{O}_{\mathfrak{p}}$ and if no confusion is possible we will simply write $\overline{y}$.

For prime ideals $\mathfrak{p}$ such that $f \in \mathcal{O}_{\mathfrak{p}}[x]$, we will consider two special cases of these sums, namely the global sum E_f, where we take $\Phi = \mathbb{1}_{\mathcal{O}_{\mathfrak{p}}^n}$ (the characteristic function on $\mathcal{O}_{\mathfrak{p}}^n$),

$$\begin{aligned} E_f(z_0\varpi^{-m},\mathfrak{p}) &:= \int_{\mathcal{O}_{\mathfrak{p}}^n} \Psi\left(\frac{z_0 f(x)}{\varpi^m}\right)|dx| \\ &= \frac{1}{q^{mn}} \sum_{\overline{x}\in(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^m\mathcal{O}_{\mathfrak{p}})^n} \Psi\left(\frac{z_0 f(x)}{\varpi^m}\right), \end{aligned}$$

and the local sums E_f^y around a point $y \in \mathcal{O}_{\mathfrak{p}}^n$, where $\Phi = \mathbb{1}_{y+\mathfrak{p}\mathcal{O}_{\mathfrak{p}}^n}$,

$$\begin{aligned} E_f^y(z_0\varpi^{-m},\mathfrak{p}) &:= \int_{y+\mathfrak{p}\mathcal{O}_{\mathfrak{p}}^n} \Psi\left(\frac{z_0 f(x)}{\varpi^m}\right)|dx| \\ &= \frac{1}{q^{mn}} \sum_{\overline{x}\in\overline{y}+(\mathfrak{p}\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^m\mathcal{O}_{\mathfrak{p}})^n} \Psi\left(\frac{z_0 f(x)}{\varpi^m}\right), \end{aligned}$$

where $\overline{y} + (\mathfrak{p}\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^m\mathcal{O}_{\mathfrak{p}})^n = \{\overline{x} \in (\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^m\mathcal{O}_{\mathfrak{p}})^n \mid \forall 1 \leqslant i \leqslant n : x_i - y_i \in \mathfrak{p}\mathcal{O}_{\mathfrak{p}}\}$.

The relation between these exponential sums and the Igusa zeta functions is given in the following proposition.

Proposition 1.3.6 ([Den91b], Proposition 1.4.4)**.** *Let $f \in k[x_1,\dots,x_n]$ be a nonconstant polynomial, $\mathfrak{p}$ a prime ideal of $\mathcal{O}$, Φ a Schwartz-Bruhat function, $m \in \mathbb{Z}$ and $z_0 \in \mathcal{O}_{\mathfrak{p}}^{\times}$, then*

$$\begin{aligned} E_f^{\Phi}(z_0\varpi^{-m},\mathfrak{p}) = Z^{\Phi}(0,\mathfrak{p},\chi_{\mathrm{triv}}) &+ \mathrm{Coeff}_{t^{m-1}} \frac{(t-q)Z^{\Phi}(s,\mathfrak{p},\chi_{\mathrm{triv}})}{(q-1)(1-t)} \\ &+ \sum_{\chi\neq\chi_{\mathrm{triv}}} g_{\chi^{-1}}\chi(z_0)\mathrm{Coeff}_{t^{m-c(\chi)}} Z^{\Phi}(s,\mathfrak{p},\chi), \end{aligned}$$

where g_χ denotes the Gaussian sum

$$g_\chi := \frac{q^{1-c(\chi)}}{q-1} \sum_{v \in (\mathcal{O}_\mathfrak{p}/p^{c(\chi)}\mathcal{O}_\mathfrak{p})^\times} \chi(v)\Psi\left(\frac{v}{\varpi^{c(\chi)}}\right).$$

Using this proposition we can express the sums E_f^Φ in terms of the poles of Z^Φ. The corollary below is similar to Corollary 1.4.5 from [Den91b] (see also [Igu75, Theorem 2]), but under slightly different assumptions.

Corollary 1.3.7. *Let $f \in k[x_1, \ldots, x_n]$ be a nonconstant polynomial, $\mathfrak{p}$ a prime ideal of $\mathcal{O}$ and Φ a Schwartz-Bruhat function, such that $f \in \mathcal{O}_\mathfrak{p}[x]$, $\overline{f} \neq 0$, the resolution (Y, h) has tame good reduction modulo $\mathfrak{p}$, Φ is residual and $C_{\overline{f}} \cap \operatorname{Supp}(\overline{\Phi}) \subseteq \overline{f}^{-1}(0)$. Then, for $m \in \mathbb{Z}$ big enough and $z_0 \in \mathcal{O}_\mathfrak{p}^\times$, $E_f^\Phi(z_0\varpi^{-m}, \mathfrak{p})$ is a finite $\mathbb{C}$-linear combination of expressions of the form*

$$\chi(z_0)m^\beta q^{m\lambda},$$

where λ is a pole of either $(q^{s+1}-1)Z^\Phi(s, \mathfrak{p}, \chi_{\mathrm{triv}})$ or $Z^\Phi(s, \mathfrak{p}, \chi)$, for some $\chi \neq \chi_{\mathrm{triv}}$ such that $d(\chi) \mid N_i$ for some $i \in T$, and where $\beta \in \mathbb{N}$ with $\beta \leqslant$ (multiplicity of pole λ) $- 1$.

Proof. We analyse the different components of the formula given in Proposition 1.3.6. We know that $Z^\Phi(s, \mathfrak{p}, \chi) = 0$ if $c(\chi) > 1$ by Theorem 1.3.5, or if $c(\chi) = 1$ and $d(\chi) \nmid N_i$ for all $i \in T$, by Theorem 1.3.3. So there are only finitely many characters that contribute to $E_f^\Phi(z_0\varpi^{-m}, \mathfrak{p})$.

Take such a nontrivial character $\chi \neq \chi_{\mathrm{triv}}$ and let $\{\lambda_j \mid j \in J\}$ be the set of poles of $Z^\Phi(s, \mathfrak{p}, \chi)$ with multiplicities $m(\lambda_j)$. Then we can decompose $Z^\Phi(s, \mathfrak{p}, \chi)$ into a constant term plus a sum of partial fractions over $\mathbb{C}$ and we can write each of these partial fractions as a product of power series:

$$Z^\Phi(s, \mathfrak{p}, \chi) = C + \sum_{j \in J} \sum_{b=1}^{m(\lambda_j)} \frac{c_{j,b}(q)}{(1 - q^{\lambda_j}t)^b} = C + \sum_{j \in J} \sum_{b=1}^{m(\lambda_j)} c_{j,b}(q) \Big(\sum_{\ell \geqslant 0} q^{\lambda_j \ell} t^\ell \Big)^b.$$

Hence

$$\operatorname{Coeff}_{t^{m-1}} Z^\Phi(s, \mathfrak{p}, \chi) =$$

$$\sum_{j \in J} \sum_{b=1}^{m(\lambda_j)} c_{j,b}(q) q^{\lambda_j(m-1)} \#\{(\ell_1, \ldots, \ell_b) \in \mathbb{N}^b \mid \ell_1 + \ldots + \ell_b = m-1\} =$$

$$\sum_{j\in J} q^{\lambda_j m} \sum_{b=1}^{m(\lambda_j)} c_{j,b}(q) q^{-\lambda_j} \binom{m+b-2}{b-1}.$$

Because $c_{j,m(\lambda_j)}(q) \neq 0$, the expression $\sum_{b=1}^{m(\lambda_j)} c_{j,b}(q)q^{-\lambda_j}\binom{m+b-2}{b-1}$ can be seen as some polynomial in m of degree $m(\lambda_j)-1$ with complex coefficients, that can depend on q. Hence $g_{\chi^{-1}}\chi(z_0)\mathrm{Coeff}_{t^{m-1}} Z^{\Phi}(s,\mathfrak{p},\chi)$ is a sum of expressions of the form $\chi(z_0)a_{j,\beta}(q)m^{\beta}q^{m\lambda_j}$, where $\beta \leqslant m(\lambda_j)-1$ and $a_{j,\beta}(q)$ are complex coefficients that depend on q, but not on m.

For the trivial character one can reason similarly by writing the expression

$$\frac{(t-q)Z^{\Phi}(s,\mathfrak{p},\chi_{\mathrm{triv}})}{(q-1)(1-t)} = \frac{(q^{s+1}-1)Z^{\Phi}(s,\mathfrak{p},\chi_{\mathrm{triv}})}{(q-1)(1-q^s)}$$

in partial fractions. □

This corollary and [Den91b, Corollary 1.4.5] tell us something about the growth of $|E_f^{\Phi}(z_0\varpi^{-m},\mathfrak{p})|_{\mathbb{C}}$ when $m\to\infty$ and $\mathfrak{p}$ is fixed. There are several invariants that can be used to describe this growth. From the corollaries one observes that the real parts of the poles of the zeta functions play a role. From Theorem 1.3.3 it follows that these real parts are of the form $-\frac{\nu_i}{N_i}$ for some $i\in T$. Thus the real part of any of the poles λ will be at most $-\min_{i\in T}\{\frac{\nu_i}{N_i}\}$, a constant that is the opposite of the well-known *log-canonical threshold*.

Definition 1.3.8. Let k be a field of characteristic 0 and $f\in k[x_1,\dots,x_n]\setminus k$. Fix an embedded resolution (Y,h) of $f^{-1}(0)$ and let $y\in (k^{\mathrm{alg}})^n$ such that $f(y)=0$. We call

$$c_y(f) := \min_{i\in T: y\in h(E_i)} \left\{\frac{\nu_i}{N_i}\right\}$$

the *log-canonical threshold of f at y* and

$$c(f) := \inf_{y\in f^{-1}(0)} c_y(f) = \min_{i\in T}\left\{\frac{\nu_i}{N_i}\right\}$$

the *log-canonical threshold of f*.

Remark 1.3.9. Since the prime divisors of the strict transform of $f^{-1}(0)$ have numerical data $(N,1)$, we have $c_y(f)\leqslant 1$, for all $y\in f^{-1}(0)$, and $c(f)\leqslant 1$. Even though the set of numerical data depends on the choice of the resolution (Y,h), the log-canonical thresholds do not depend on this choice.

Igusa observed that not all the numerical data play a role in the asymptotic behaviour of $|E_f^{\Phi}(z_0\varpi^{-m},\mathfrak{p})|_{\mathbb{C}}$. Therefore he introduced the *essential numerical data* of a resolution (Y,h) in [Igu78]. These data are a subset of the set

$\{(N_i,\nu_i) \mid i \in T\}$ of numerical data, namely, let $J \subseteq T$ be the set of indices j such that $(N_j,\nu_j) = (1,1)$ and E_j does not intersect any other E_i with $(N_i,\nu_i) = (1,1)$. Now put $\tilde{T} := T\backslash J$, then the set of essential numerical data is the set $\{(N_i,\nu_i) \mid i \in \tilde{T}\}$. Similarly as for the log-canonical threshold, we take the minimum:

$$\tilde{c}(h) := \min_{i\in\tilde{T}} \left\{\frac{\nu_i}{N_i}\right\}.$$

The notation $\tilde{c}(h)$ indicates that this invariant does depend on the choice of the resolution (Y,h) of f, contrary to the log-canonical threshold $c(f)$. If $\tilde{c}(h) \leqslant 1$, then $\tilde{c}(h) = c(f)$, but $\tilde{c}(h)$ could also be bigger than 1, in which case $c(f) = 1$. In this case one can obtain another resolution (Y',h') by adding more blow-ups to (Y,h) in a well chosen manner, such that $1 < \tilde{c}(h') < \tilde{c}(h)$ and $\tilde{c}(h')$ is arbitrarily close to 1 (see [Clu08b, Remark 2.1]).

The asymptotic behaviour of these exponential sums when *both* $m \to \infty$ *and* $q \to \infty$ has been the subject of several conjectures and results. In what follows we will give an overview of these result. For convenience we will formulate the results for $k = \mathbb{Q}$, $\varpi = p$ a prime number, $z_0 = 1$ and $f \in \mathbb{Z}[x_1,\dots,x_n]$, but they also have a variant for other number fields. In particular we will denote the exponential sums from now on by $E_f^\Phi(m,p)$.

Note that for any $b \in \mathbb{Q}$ we have

$$\begin{aligned}
\left|E_{f+b}^\Phi(m,p)\right|_{\mathbb{C}} &= \left| \int_{\mathbb{Q}_p^n} \Phi(x)\exp\left(\frac{2\pi i(f(x)+b)}{p^m}\right)|dx|\right|_{\mathbb{C}} \\
&= \left|\exp\left(\frac{2\pi i b}{p^m}\right)\right|_{\mathbb{C}} \left| \int_{\mathbb{Q}_p^n} \Phi(x)\exp\left(\frac{2\pi i f(x)}{p^m}\right)|dx|\right|_{\mathbb{C}} \\
&= |E_f^\Phi(m,p)|_{\mathbb{C}}.
\end{aligned}$$

This implies that we can assume that f does not have a constant term, i.e., $f(0) = 0$. Furthermore we have assumed f to be nonconstant (since constant polynomials do not give interesting exponential sums), hence $f \not\equiv 0$.

In [Igu78] Igusa only studied the behaviour of the global exponential sum E_f for *homogeneous* polynomials f. Note that a homogeneous polynomial never has any other critical values than 0. Hence the condition $C_f \cap \mathrm{Supp}(\Phi) \subseteq f^{-1}(0)$ from [Den91b, Corollary 1.4.5] is satisfied automatically. Therefore, for each prime number p, there exists a constant $C_p > 0$, such that for all $m \geqslant 1$, we have

$$|E_f(m,p)|_{\mathbb{C}} \leqslant C_p m^{n-1} p^{-m\tilde{c}(h)}.$$

In studying certain criteria for the validity of a Poisson summation formula, Igusa conjectured that the constants C_p could be chosen independently of p.

Conjecture 1.3.10 (Igusa, [Igu78])**.** *Let $f \in \mathbb{Z}[x_1, \dots, x_n] \backslash \mathbb{Z}$ be a homogeneous polynomial. Then there exists a constant $C > 0$, such that for all primes p and for all $m \geqslant 1$, we have*

$$|E_f(m,p)|_{\mathbb{C}} \leqslant Cm^{n-1}p^{-m\tilde{c}(h)}.$$

This conjecture has been proven in several cases. Igusa showed it for homogeneous polynomials with an isolated singularity [Igu74b]. For homogeneous polynomials that are nondegenerate with respect to their Newton polyhedron, the conjecture has been proved under an extra assumption, by Denef and Sperber [DS01] and in general by Cluckers [Clu08a]. In [Clu10] Cluckers generalised this result to weighted homogeneous polynomials. Results have also been obtained by fixing the number of variables n. For $n = 2$ different proofs of the conjecture, for both homogeneous and weighted homogeneous polynomials, have been given by Wright [Wri12] and Lichtin [Lic13]. For $n = 3$ Lichtin [Lic16] proved the conjecture for homogeneous polynomials with singular locus of dimension at most 1.

In [Clu08b] Cluckers introduced an invariant $\alpha(f)$, called the *motivic oscillation index of f*, which sometimes gives even stronger upper bounds when replacing $\tilde{c}(h)$ by $-\alpha(f)$. Cluckers proved this version of Igusa's conjecture when $m = 1$ for all quasi-homogeneous polynomials [Clu08b, Clu10], and when $m = 2$ for all polynomials [Clu08b].

Another one of the exponential sums has been studied by Denef and Sperber [DS01], namely the local sum around zero $E_f^0(m,p)$, in particular for nondegenerate polynomials. Under an extra assumption they proved that these local sums are uniformly bounded by an upper bound that uses $\tilde{c}(h)$ for a toric resolution h. Cluckers showed in [Clu10] that the extra assumption can be omitted. Denef and Sperber conjectured that the assumption of nondegenerateness could be removed, when replacing the invariant $\tilde{c}(h)$ by the *complex oscillation index* $\beta_0(f)$ of f around 0 (see [AVGZ86] for a definition).

Conjecture 1.3.11 (Denef-Sperber, [DS01])**.** *Let $f \in \mathbb{Z}[x_1, \dots, x_n] \setminus \mathbb{Z}$. Then there exists a constant $C > 0$, such that for all primes p and for all $m \geqslant 1$, we have*

$$|E_f^0(m,p)|_{\mathbb{C}} \leqslant Cm^{n-1}p^{m\beta_0(f)}.$$

Since the assumption of (weighted) homogeneousness does not seem to be necessary for the local sum around 0, it is natural to wonder whether this assumption can also be omitted for the global sum. It is important to note that without this assumption f can have more critical values than only 0. Thus the conditions from Corollary 1.3.7 and [Den91b, Corollary 1.4.5] are not automatically satisfied. Proposition 2.7 from [DV95] indicates that all the critical values of f have to be taken into account to give a good estimate for the global exponential sum $|E_f(m,p)|_{\mathbb{C}}$. That is why the following two invariants were introduced.

Definition 1.3.12. Let k be a number field and $f \in k[x_1, \dots, x_n] \setminus k$. For any $b \in V_f \cup \{0\}$ we fix a resolution $h_b \colon Y_b \to (k^{\mathrm{alg}})^n$ of $f^{-1}(b) = (f-b)^{-1}(0)$ and we denote by h the tuple $(h_b)_{b \in V_f \cup \{0\}}$ of resolutions. We define the following two invariants:

$$a(f) := \min_{b \in V_f \cup \{0\}} c(f-b);$$

$$\tilde{a}(h) := \min_{b \in V_f \cup \{0\}} \tilde{c}(h_b).$$

Clearly $a(f) \leqslant 1$, $\tilde{a}(h) \leqslant 1$ implies $\tilde{a}(h) = a(f)$, and else $a(f) = 1 < \tilde{a}(h)$.

It follows from [DV95, Proposition 2.7] that for each prime number p, there exists a constant $C_p > 0$, such that for all $m \geqslant 1$, we have

$$|E_f(m,p)|_{\mathbb{C}} \leqslant C_p m^{n-1} p^{-m\tilde{a}(h)}. \tag{1.3.1}$$

A generalisation of Igusa's conjecture for all polynomials has been formulated by Cluckers and Veys. The case $m = 1$ does not generalise, hence they assume that $m \geqslant 2$. Note that they formulate the conjecture using the invariant $a(f)$, but they conjecture that stronger formulations with $\tilde{a}(h)$ or $-\alpha(f)$ will also hold. Moreover, they generalise the Denef-Sperber conjecture to all local sums $E_f^y(m,p)$, uniformly in $y \in \mathbb{Z}^n$. The invariant that they use in these upper bounds is

$$a_{y,p}(f) := \inf_{y' \in y + p\mathbb{Z}_p^n} c_{y'}(f - f(y')), \tag{1.3.2}$$

where $y \in \mathbb{Z}^n$ and p is a prime number. Resolutions are taken over an algebraic closure $(\mathbb{Q}_p)^{\mathrm{alg}}$ of the p-adic numbers. From Corollary 1.3.7 we can deduce that for each prime number p, there exists a constant $C_p > 0$, such that for all $m \geqslant 1$ and for all $y \in \mathbb{Z}^n$, we have

$$|E_f^y(m,p)|_{\mathbb{C}} \leqslant C_p m^{n-1} p^{-m a_{y,p}(f)}. \tag{1.3.3}$$

Conjecture 1.3.13 (Cluckers-Veys, [CV16])**.** *Let $f \in \mathbb{Z}[x_1, \dots, x_n] \setminus \mathbb{Z}$. Then there exists a constant $C > 0$, such that for all primes p, for all $m \geqslant 2$ and for all $y \in \mathbb{Z}^n$, we have*

$$|E_f(m,p)|_{\mathbb{C}} \leqslant C m^{n-1} p^{-m a(f)}; \tag{1.3.4}$$

$$|E_f^y(m,p)|_{\mathbb{C}} \leqslant C m^{n-1} p^{-m a_{y,p}(f)}. \tag{1.3.5}$$

In [CV16] Cluckers and Veys prove that their conjecture holds for all $m \leqslant 4$ and, depending on some orders of vanishing of f, for some more small values of m. Castryk and Nguyen [CN18] prove this conjecture for nondegenerate polynomials. In Chapter 4 we will discuss two different proofs of this conjecture for polynomials that have log-canonical threshold at most one half. The Igusa and Denef-Sperber conjectures for the same class of polynomials follow from this (see [CN17a]).

Chapter 2

Clustered cell decomposition in P-minimal structures

This chapter consists of the paper [CCL17a] and some parts of the additional note [CCL17b]. Both are joint work with Pablo Cubides Kovacsics and Eva Leenknegt.

The goal of this chapter is to obtain a cell decomposition that is valid in all P-minimal fields. As discussed in Section 1.1, the classical cell decomposition results of Denef [Den84, Den86] and Cluckers [Clu04] cannot be extended to all P-minimal fields, as follows from the work of Mourges [Mou09] and Cubides-Nguyen [CN17b]. In particular, the absence of definable Skolem functions makes it impossible to obtain a decomposition in which the cells have a *definable* function as their center. Therefore, a less restrictive definition of 'cells' is necessary. The decomposition result of Cubides-Leenknegt [CL16] serves as a starting point, to which step-by-step improvements are added, leading up to Theorem 2.7.1. Informally, this theorem states that in any P-minimal field, any definable set can be partitioned as a finite union of *classical cells* (cells with a definable center) and *regular clustered cells*. These clustered cells look geometrically like a finite union of the usual cells, but instead of having definable functions as centers, there is a definable set of *potential centers*.

Throughout this chapter we will always work with a p-adically closed field K with value group Γ_K, valuation map $\operatorname{ord}\colon K \to \Gamma_K \cup \{\infty\}$, valuation ring $\mathcal{O}_K$, maximal ideal $\mathcal{M}_K$, and quotient field k_K with q_K elements. After fixing a uniformizer ϖ there exists, for each $m \in \mathbb{N} \setminus \{0\}$, a unique group homomorphism $\mathrm{ac}_m\colon K^\times \to (\mathcal{O}_K/\varpi^m\mathcal{O}_K)^\times$ that satisfies $\mathrm{ac}_m(\varpi) = 1$ and

$\mathrm{ac}_m(x) = (x \bmod \varpi^m)$ for every unit $x \in \mathcal{O}_K^\times$. In this chapter we will study two sorted P-minimal structures $(K, \Gamma_K; \mathcal{L}_2)$ on p-adically closed fields (Definitions 1.1.8 and 1.1.4). Since there already exist a decomposition of definable sets $X \subseteq S \times \Gamma_K$ into Γ-cells (Theorem 1.1.9), we will only work with definable sets $X \subseteq S \times K$.

This chapter is structured as follows. In Section 2.1 we will introduce a different way of looking at cells. Furthermore we will fix some notation and introduce vocabulary to describe cells and their centers.

In Section 2.2, we will revisit semi-algebraic cell decomposition for subsets of P-minimal fields K, and show that every definable set $X \subseteq K$ admits a so-called *admissible* cell decomposition. Such a decomposition imposes some technical restrictions on the way centers can appear as elements of a cell, and controlling this will be crucial in later proofs.

A first strengthening of the decomposition result from [CL16] is proven in Section 2.3. This intermediate result allows us to decompose a definable set into finitely many classical cells and objects called *cell arrays*. Roughly speaking, a cell array is a definable set which geometrically has the structure of a finite union of cells (which may not be definable individually), possibly involving multiple cell conditions.

In Section 2.4, we prove a finiteness result for centers. We will use this in Section 2.6 to partition cell arrays into classical and regular clustered cells (where only a single cell condition is involved). The regularity condition, which is explored in Section 2.5, imposes further restrictions on the set of centers.

The full cell decomposition theorem (Theorem 2.7.1) will be presented in Section 2.7. This last section also includes some additional remarks and open questions.

2.1 Cells

In order to broaden the notion of cells to the non-Skolem setting, we have adopted a different way of looking at cells. In our view, a cell has two major ingredients: its center (which we will discuss further on), and the formula C defining the cell.

Definition 2.1.1 (K-cell condition)**.** A *K-cell condition* over $S \subseteq K^{d_1} \times \Gamma_K^{d_2}$ is a formula of the form

$$C(s, c, t) := s \in S \ \wedge \ \alpha(s) \ \square_1 \ \mathrm{ord}(t - c) \ \square_2 \ \beta(s) \ \wedge \ t - c \in \lambda Q_{n,m},$$

where t and c are variables over K, $\alpha, \beta \colon S \to \Gamma_K$ are definable functions, $\square_1$ and $\square_2$ may denote either $<$ or $\emptyset$ (i.e., 'no condition'), $\lambda \in K$ and $n, m \in \mathbb{N} \setminus \{0\}$. The variable c is called the *center* of the K-cell condition. Recall the definition of the sets $Q_{n,m}$:

$$Q_{n,m} := \{x \in K^\times \mid \mathrm{ord}(x) \equiv 0 \bmod n \wedge \mathrm{ac}_m(x) = 1\}.$$

A K-cell condition C is called a 0-*cell condition*, resp. a 1-*cell condition* if $\lambda = 0$, resp. $\lambda \neq 0$.

Since this chapter does not discuss Γ-cell conditions, we will often omit the K and simply speak of *cell conditions*.

Remark 2.1.2. We will use the following notational convention. Capital C will always denote a cell condition over some set of parameters S for which the symbols $\alpha, \beta, \lambda, \square_1, \square_2, n, m$ are fixed as in the previous definition. In particular, the letters α and β will only be used to denote the functions picking the lower and upper bounds in a cell condition C. If multiple cell conditions are discussed at the same time, say $C_1, \ldots, C_r$, the same index will be applied to the symbols in the associated formula. Thus, α_i and β_i denote the functions picking the lower and upper bounds of a cell condition C_i, and the use of $\square_{i1}, \square_{i2}, \lambda_i, n_i, m_i$ follows similar conventions.

Let C be a cell condition over S and $\sigma \colon S \to K$ a function (not necessarily definable). Using this function as the center for C, we get the induced set

$$C^\sigma := \{(s,t) \in S \times K \mid C(s, \sigma(s), t)\}.$$

When there is no dependence on parameters (i.e., if C is a cell condition over $S = \Gamma^0 \times K^0$), a function $\sigma \colon S \to K$ will be identified with a point $\sigma \in K$. Sets of the form C^σ will be informally called *cells over* S (or simply *cells*, when the parameter set S is clear from the context). The reader will probably be most familiar with *classical cells*, that is, cells C^σ for which the function σ is definable. For instance, one may think of semi-algebraic or subanalytic cells, where the center σ is a semi-algebraic, resp. a subanalytic function (see [Den86, Clu03]).

We will denote the fiber of a cell C^σ over $s \in S$ by

$$C^{\sigma(s)} := \{t \in K \mid C(s, \sigma(s), t)\}.$$

When C is a 0-, resp. a 1-cell condition, we will call C^σ a 0-*cell*, resp. a 1-*cell*.

Definition 2.1.3. Let C be a K-cell condition over S and $\sigma \colon S \to K$ a function. The *leaf of* $C^{\sigma(s)}$ *at height* γ corresponds to the ball

$$C^{\sigma(s),\gamma} := \{t \in C^{\sigma(s)} \mid \mathrm{ord}(t - \sigma(s)) = \gamma\}.$$

The fibers $C^{\sigma(s)}$ of a cell C^σ can be visualised in the following way. Here we adopt the perspective used also in [HM94, HM97], representing elements and basic subsets of valued fields by trees (see more in Section 6).

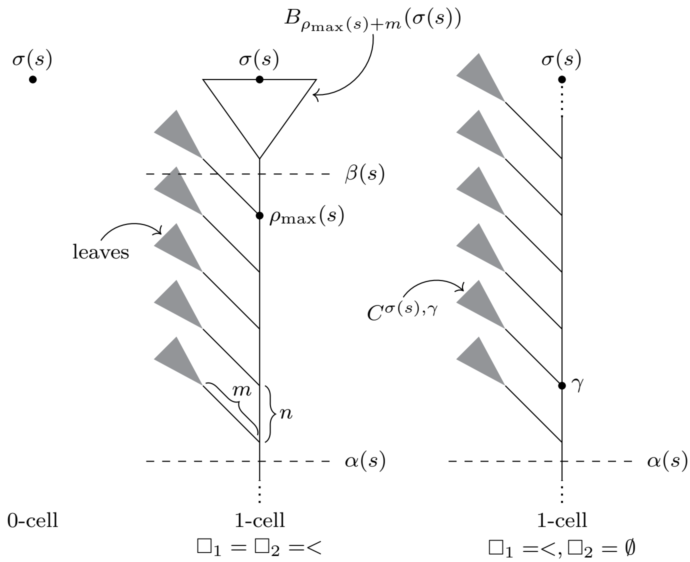


Figure 2.1: Different configurations of cells

When C is a 0-cell condition, fibers correspond to points: $C^{\sigma(s)} = \{\sigma(s)\}$. When C is a 1-cell condition, the fiber $C^{\sigma(s)}$ is the disjoint union of its leaves $C^{\sigma(s),\gamma}$. One can check that a leaf at height γ corresponds to a ball of radius $\gamma + m$. Note that $\sigma(s) \notin C^{\sigma(s)}$, and that $\sigma(s) \in \mathrm{Cl}(C^{\sigma(s)})$ if and only if $\square_2 = \emptyset$.

When $\square_2$ denotes $<$, the center of a cell C^σ is not unique. Indeed, write $\rho_{\max}(s)$ for the height of the top leaf of $C^{\sigma(s)}$ (so $\beta(s) - n \leqslant \rho_{\max}(s) \leqslant \beta(s) - 1$). Note that $\rho_{\max}\colon S \to \Gamma_K$ is a definable function which only depends on the cell condition, and not on the choice of the center. It is easy to see that one still gets the exact same fiber $C^{\sigma(s)}$, if $\sigma(s)$ is replaced by any other element of the ball $B_{\rho_{\max}(s)+m}(\sigma(s))$ (see (1.0.1) for notation). Hence, it is reasonable to consider the set

$$\Sigma = \{(s, c) \in S \times K \mid c \in B_{\rho_{\max}(s)+m}(\sigma(s))\}$$

as the set of centers for C^σ. In P-minimal structures without definable Skolem functions, it might happen that Σ itself is a definable set, yet no section of Σ is definable. Nevertheless, even when σ is a non-definable section of Σ, the cell C^σ will still be definable (as a set), since we have the equality

$$C^\sigma = \{(s, t) \in S \times K \mid \exists c : c \in \Sigma_s \wedge C(s, c, t)\}.$$

It is therefore natural to consider the following notion.

Definition 2.1.4. Let C be a cell condition and $\Sigma \subseteq S \times K$ be a definable set. The set $C^\Sigma \subseteq S \times K$ is defined as

$$C^\Sigma := \{(s,t) \in S \times K \mid \exists c : c \in \Sigma_s \wedge C(s,c,t)\}.$$

Every (not necessarily definable) section $\sigma\colon S \to K$ of Σ is called a *potential center* of C^Σ. We call the induced sets C^σ a *potential cells.*

Let us stress that, given two different sections σ and σ' of Σ, the induced cells C^σ and $C^{\sigma'}$ may be very different (possibly even disjoint) subsets of C^Σ, since we have not yet imposed any conditions on Σ. If we want sets C^Σ to be useful building blocks in our cell decomposition, we will have to significantly restrict the type of set that can occur for Σ. Indeed, every definable set $X \subseteq S \times K$ is already of the form C^Σ if we were to take $\Sigma = X$, and C a 0-cell condition over S.

We will show that it is sufficient to consider certain definable sets $\Sigma \subseteq S \times K$ for which there is $k \in \mathbb{N}$ such that every fiber Σ_s is the disjoint union of k balls. For such a Σ, the corresponding set C^Σ will have the following structure.

Let $\sigma_1, \ldots, \sigma_k$ be sections of Σ such that for every $s \in S$, the set $\{\sigma_1(s), \ldots, \sigma_k(s)\}$ contains representatives of each of the k disjoint balls covering Σ_s. For any such choice, C^Σ partitions as

$$C^\Sigma = C^{\sigma_1} \cup \ldots \cup C^{\sigma_k}.$$

Note that C^Σ is definable even when no section σ_i is definable. Such sets C^Σ are what we will call *clustered cells* (for a formal definition, see Definitions 2.3.4 and 2.6.2). The main theorem of this chapter essentially states that any definable set can be partitioned as a finite union of classical and clustered cells.

2.2 Semi-algebraic cell decomposition revisited

Since every ball is the disjoint union of q_K smaller balls, semi-algebraic sets $X \subseteq K$ admit infinitely many different cell decompositions. A decomposition $\mathcal{C}$ consists of the following data: a finite set I and, for each $i \in I$, a cell condition C_i and a center $\sigma_i \in K$. We denote this as $\mathcal{C} = \{C_i^{\sigma_i} \mid i \in I\}$. Note that since all cells are subsets of K, the center σ of every cell C^σ is an element of K rather than a function. We will also use the notation

$$\mathcal{C}(K) := \bigcup_{i \in I} C_i^{\sigma_i} \qquad \text{and} \qquad \text{Centers}(\mathcal{C}) := \{\sigma_i \mid i \in I\}.$$

Two decompositions $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if they define the same set, that is, if $\mathcal{C}(K) = \mathcal{D}(K)$.

In this section we will define a collection of so-called *admissible decompositions* and show that every semi-algebraic set $X \subseteq K$ admits a decomposition from this collection. First we need to introduce some further notation. See Definition 1.0.2 for a recall of the notation $\sqsubseteq$.

Definition 2.2.1. Let $\mathcal{C} = \{C_i^{\sigma_i} \mid i \in I\}$ be a decomposition. Define the subset of cells $\mathcal{C}^* \subseteq \mathcal{C}$ as

$$\mathcal{C}^* := \{C_i^{\sigma_i} \mid \sigma_i \neq 0 \wedge \square_{i1} = \square_{i2} = <\}.$$

We define the set $W(\mathcal{C})$ as the following subset of centers in $\mathcal{C}^*$:

$$W(\mathcal{C}) := \Big\{\sigma \in \mathrm{Centers}(\mathcal{C}^*) \mid \exists \gamma \in \Gamma_K : B_\gamma(\sigma) \sqsubseteq \mathcal{C}^*(K) \wedge \bigwedge_{C_i^{\sigma_i} \in \mathcal{C}^*} B_\gamma(\sigma) \not\subseteq C_i^{\sigma_i}\Big\}.$$

In words, $W(\mathcal{C})$ consists of those centers in Centers($\mathcal{C}^*$) which are in $\mathcal{C}^*(K)$, but where the biggest ball in $\mathcal{C}^*(K)$ around this center is not contained within a single cell of $\mathcal{C}^*$. We are now able to define what admissible decompositions are.

Definition 2.2.2. A decomposition $\mathcal{C} = \{C_i^{\sigma_i} \mid i \in I\}$ is called *pre-admissible* if it satisfies the following properties:

(a) For every 0-cell $C_i^{\sigma_i}$, if $\sigma_i \neq 0$, then $\sigma_i \in X \setminus \mathrm{Int}(X)$.

(b) For every 1-cell $C_i^{\sigma_i}$, if $\sigma_i \neq 0$ and $\square_{i1} = <$, then $\mathrm{ord}(\sigma_i) \leqslant \alpha_i$.

(c) For every 1-cell $C_i^{\sigma_i}$ in which $\square_{i1} = \emptyset$, it holds that $\sigma_i = 0$.

It is called *admissible*, if it moreover satisfies

(d) $W(\mathcal{C}) = \emptyset$.

Condition (a) ensures that elements defined by 0-cells different from $\{0\}$, are isolated points. Condition (c) will later imply that cells for which $\square_1 = \emptyset$, will always be centered at 0. Conditions (b) and (d), which might seem arbitrary at this point, will be needed for technical reasons in later proofs.

The goal of this section is to prove the following theorem.

Theorem 2.2.3. *Every semi-algebraic set $X \subseteq K$ has an admissible cell decomposition.*

We split the proof of Theorem 2.2.3 into two steps: we first show (in the next lemma) that semi-algebraic sets always have a pre-admissible decomposition. The second step will then be to prove that every pre-admissible decomposition can be modified into an admissible one.

Lemma 2.2.4. *Every semi-algebraic set $X \subseteq K$ has a pre-admissible decomposition.*

Proof. Let $\mathcal{C} = \{C_i^{\sigma_i} \mid i \in I\}$ be a cell decomposition of X. Let $a(\mathcal{C})$ (resp. $b(\mathcal{C})$ and $c(\mathcal{C})$) be the number of cells in $\mathcal{C}$ which are counterexamples of part (a) of Definition 2.2.2 (resp. of (b) and (c)). If $a(\mathcal{C}) > 0$ (resp. $b(\mathcal{C}) > 0$, $c(\mathcal{C}) > 0$), we will show how to produce a cell decomposition $\widehat{\mathcal{C}}$ of X such that $a(\widehat{\mathcal{C}}) \leqslant a(\mathcal{C}) - 1$ (and similarly for $b(\mathcal{C})$ and $c(\mathcal{C})$). By iterating this process a finite number of times, one can then obtain a cell decomposition satisfying (a) (resp. (b) and (c)).

Fix an index $j \in I$ such that $\sigma_j \neq 0$, $C_j^{\sigma_j}$ is the 0-cell $C_j^{\sigma_j} = \{\sigma_j\}$ and $\sigma_j \in \mathrm{Int}(X)$. Write $\mathcal{C} = \{C_j^{\sigma_j}\} \cup \mathcal{C}_1 \cup \mathcal{C}_2$, where

$$\mathcal{C}_1 := \{C_i^{\sigma_i} \in \mathcal{C} \mid i \neq j \wedge \sigma_j \in \mathrm{Cl}(C_i^{\sigma_i})\},$$

and $\mathcal{C}_2 := \mathcal{C} \setminus (\{C_j^{\sigma_j}\} \cup \mathcal{C}_1)$. Let X' be the set $X' := C_j^{\sigma_j} \cup \mathcal{C}_1(K)$. Let $\gamma \in \Gamma_K$ be minimal such that $B_\gamma(\sigma_j)$ is contained in X'. If no minimal γ exists, set $\gamma := \mathrm{ord}(\sigma_j)$. Note that this case only occurs when $X' = K$. Indeed, by Theorem 1.1.7, P-minimal definable subsets of Γ_K are Presburger-definable. From this it follows that every definable subset of Γ_K without a minimal element must be unbounded from below, hence X' contains arbitrarily large balls. Let $\zeta \in K$ be such that $\mathrm{ord}(\sigma_j - \zeta) = \gamma - 1$ and let D^ζ be the cell

$$D^\zeta := \{t \in K \mid \mathrm{ord}(t - \zeta) = \gamma - 1 \wedge t - \zeta \in \lambda Q_{1,1}\},$$

where we have chosen $\lambda \in K$ such that $D^\zeta = B_\gamma(\sigma_j)$. For every 1-cell $C_i^{\sigma_i} \in \mathcal{C}_1$, let $D_i^{\sigma_i}$ be the 1-cell obtained from $C_i^{\sigma_i}$ by replacing $\square_{i2}$ by $<$ and making γ the upper bound. Then the set of cells $\widehat{\mathcal{C}}$ formed by

$$\{D^\zeta\} \cup \{D_i^{\sigma_i} \mid C_i^{\sigma_i} \in \mathcal{C}_1\} \cup \mathcal{C}_2$$

is a cell decomposition of X. Clearly, $a(\widehat{\mathcal{C}}) \leqslant a(\mathcal{C}) - 1$.

Suppose that $\mathcal{C}$ satisfies (a). Let $C_j^{\sigma_j} \in \mathcal{C}$ be a 1-cell centered at $\sigma_j \neq 0$ for which either $\alpha_j < \mathrm{ord}(\sigma_j)$, or $\square_{j1} = \emptyset$. We need to consider two cases, depending on whether $\mathrm{ord}(\sigma_j) < \beta_j$ or $\beta_j \leqslant \mathrm{ord}(\sigma_j)$. We will only discuss the first case in detail, as the second one is completely similar. If $\mathrm{ord}(\sigma_j) < \beta_j$, first partition the cell $C_j^{\sigma_j}$ further as

$$D^{\sigma_j} := \{t \in K \mid \mathrm{ord}(\sigma_j) < \mathrm{ord}(t - \sigma_j) \ \square_{j2} \ \beta_j \wedge t - \sigma_j \in \lambda_j Q_{n_j,m_j}\},$$

$$E := \{t \in K \mid \alpha_j \ \square_{j1} \ \mathrm{ord}(t - \sigma_j) < \mathrm{ord}(\sigma_j) + 1 \ \wedge \ t - \sigma_j \in \lambda_j Q_{n_j,m_j}\}.$$

To prove our claim, we need to show how the cell E can be partitioned as a finite union of 1-cells centered at 0. Put $M_j := \min\{m_j, \mathrm{ord}(\sigma_j) - \alpha_j\}$ (or just $M_j = m_j$ if $\square_{j1} = \emptyset$). We will first partition E further as $E' \cup E_0 \cup \ldots \cup E_{M_j-1}$, where

$$E' := \{t \in K \mid \alpha_j \ \square_{j1} \ \mathrm{ord}(t - \sigma_j) < \mathrm{ord}(\sigma_j) - m_j + 1 \ \wedge \ t - \sigma_j \in \lambda_j Q_{n_j,m_j}\},$$

$$E_i := \{t \in K \mid \mathrm{ord}(t - \sigma_j) = \mathrm{ord}(\sigma_j) - i \ \wedge \ t - \sigma_j \in \lambda_j Q_{n_j,m_j}\},$$

Note that most of these sets are actually already cells centered at zero (and some might be empty). Indeed, for E' we can rewrite the description of the set as

$$E' = \{t \in K \mid \alpha_j \ \square_{j1} \ \mathrm{ord}(t) < \mathrm{ord}(\sigma_j) - m_j + 1 \ \wedge \ t \in \lambda_j Q_{n_j,m_j}\}.$$

Similarly, for $1 \leqslant i \leqslant M_j - 1$, we have that

$$E_i = \{t \in K \mid \mathrm{ord}(t) = \mathrm{ord}(\sigma_j) - i \ \wedge \ t \in \mu_i Q_{n_j,m_j}\},$$

where $\mu_i \in K$ is chosen in such a way as to assure that $t - \sigma_j \in \lambda_j Q_{n_j,m_j}$.

When $i = 0$, we need to do a bit more work. A further partitioning will be necessary. For $0 \leqslant k < m_j$, let $E_{0,k}$ be the set

$$E_{0,k} := \{t \in E_0 \mid \mathrm{ord}(t) = \mathrm{ord}(\sigma_j) + k\},$$

and we write $E_{0,>}$ for the set

$$E_{0,>} := \{t \in E_0 \mid \mathrm{ord}(t) \geqslant \mathrm{ord}(\sigma_j) + m_j\}.$$

Then clearly, if they are non-empty, the sets $E_{0,k}$ are cells centered at zero, since for a suitably chosen value $\mu_{0,k} \in K$, they can be rewritten as

$$E_{0,k} = \{t \in K \mid \mathrm{ord}(t) = \mathrm{ord}(\sigma_j) + k \ \wedge \ t \in \mu_{0,k} Q_{n_j,m_j-k}\}.$$

Finally, consider the set $E_{0,>}$. First note that this set is empty unless $-\sigma_j \in \lambda_j Q_{n_j,m_j}$, as for elements of this set it holds that

$$t - \sigma_j \in \lambda_j Q_{n_j,m_j} \Leftrightarrow -\sigma_j \in \lambda_j Q_{n_j,m_j}.$$

Moreover, if $E_{0,>}$ is non-empty, it equals the ball $B_{\mathrm{ord}(\sigma_j)+m_j}(0)$. In this case, we will partition $E_{0,>}$ into cells $\{F_0, \ldots, F_{q_K-1}\}$ as follows. Put $F_0 := \{0\}$ and for $1 \leqslant r \leqslant q_K - 1$, define

$$F_r := \{t \in K \mid \mathrm{ord}(\sigma_j) + m_j - 1 < \mathrm{ord}(t) \ \wedge \ t \in \hat{\mu}_r Q_{1,1}, \}$$

where $\hat{\mu}_1, \ldots, \hat{\mu}_{q_K-1} \in K$ are representatives such that $\mathrm{ac}_1(\{\hat{\mu}_1, \ldots, \hat{\mu}_{q_K-1}\}) = \mathrm{ac}_1(K^\times)$. To summarize, we obtain the following decomposition of E_0, which we will denote as $\mathcal{E}_0$. Put

$$\mathcal{E}_0 := \begin{cases} \{E_{0,k} \mid 0 \leqslant k < m_j\} \cup \{F_r \mid 0 \leqslant r \leqslant q_K - 1\} & \text{if } -\sigma_j \in \lambda_j Q_{n_j,m_j}, \\ \{E_{0,k} \mid 0 \leqslant k < m_j\} & \text{otherwise.} \end{cases}$$

Now let $\widehat{\mathcal{C}}$ be the decomposition obtained by replacing $C_j^{\sigma_j}$ by the cells in $\{D^{\sigma_j}, E'\} \cup \{E_i \mid 1 \leqslant i \leqslant M_j - 1\} \cup \mathcal{E}_0$. If $C_j^{\sigma_j}$ was a cell contradicting (b), (resp. (c)), then $\widehat{\mathcal{C}}$ is a cell decomposition of X for which $b(\widehat{\mathcal{C}}) = b(\mathcal{C}) - 1$ and $c(\widehat{\mathcal{C}}) \leqslant c(\mathcal{C})$ (resp. $c(\widehat{\mathcal{C}}) = c(\mathcal{C}) - 1$ and $b(\widehat{\mathcal{C}}) \leqslant b(\mathcal{C})$). Moreover, no new 0-cells that are not centered at 0, were added during this process, so $\widehat{\mathcal{C}}$ still satisfies property (a). Repeating this partitioning process for a finite number of cells then yields the lemma. $\square$

It remains to show that every pre-admissible decomposition allows an equivalent admissible decomposition. We need to introduce some additional notations first. Given a cell C^σ with $\square_1 = \square_2 =<$, and an interval (α', β'), we put

$$C^\sigma_{|(\alpha',\beta')} := \{t \in K \mid \widetilde{\alpha} < \mathrm{ord}(t - \sigma) < \widetilde{\beta} \wedge t - \sigma \in \lambda Q_{n,m}\}, \tag{2.2.1}$$

where $(\widetilde{\alpha}, \widetilde{\beta}) = (\alpha, \beta) \cap (\alpha', \beta')$.

Lemma 2.2.5. *Let $\mathcal{C}$ be a pre-admissible decomposition. Then there exists an equivalent decomposition $\mathcal{D}$ which is admissible.*

Proof. We use induction on l, for $0 \leqslant l \leqslant L = |W(\mathcal{C})|$, to show that there exist equivalent pre-admissible decompositions $\mathcal{D}_l$ such that

1. $\mathcal{D}_0 = \mathcal{C}$;
2. if $W(\mathcal{D}_l) \neq \emptyset$ then $|W(\mathcal{D}_{l+1})| \leqslant |W(\mathcal{D}_l)| - 1$.

The result will then follow by putting $\mathcal{D} := \mathcal{D}_L$. For $l = 0$, there is nothing to prove. Suppose that $\mathcal{D}_l := \{C_j^{\sigma_j} \mid j \in J\}$ has already been constructed. If $W(\mathcal{D}_l) = \emptyset$, we set $\mathcal{D}_{l+1} = \mathcal{D}_l$ and there is again nothing to prove.

Otherwise, let $J^* \subseteq J$ be the set $J^* := \{j \in J \mid C_j^{\sigma_j} \in \mathcal{D}_l^*\}$. Choose an element $j_0 \in J^*$ such that $\sigma_{j_0} \in W(\mathcal{D}_l)$. By the definition of $W(\mathcal{D}_l)$, $\sigma_{j_0} \neq 0$ and there is $\rho \in \Gamma_K$ such that $B_\rho(\sigma_{j_0}) \sqsubseteq \mathcal{D}_l^*(K)$ and $B_\rho(\sigma_{j_0})$ is not contained in a single cell $C_j^{\sigma_j}$ of $\mathcal{D}_l^*$. Let $J' \subset J^*$ be minimal such that

$$B_\rho(\sigma_{j_0}) \subseteq \bigcup_{j \in J'} C_j^{\sigma_j}.$$

Note that $|J'| \geqslant 2$. For each $j \in J'$, let Y_j be the subset of Γ_K defined by

$$Y_j := \{\gamma \in \Gamma_K \mid B_\rho(\sigma_{j_0}) \cap C_j^{\sigma_j,\gamma} \neq \emptyset\}.$$

Then we have that

$$B_\rho(\sigma_{j_0}) = \bigcup_{j \in J'} \bigcup_{\gamma \in Y_j} C_j^{\sigma_j,\gamma}.$$

Let $\gamma_{j,1} := \min\{\gamma \mid \gamma \in Y_j\}$ and $\gamma_{j,2} := \max\{\gamma \mid \gamma \in Y_j\}$.

Claim 2.2.6. *The following equality holds*

$$Y_j = \{\gamma \in \Gamma_K \mid \gamma_{j,1} \leqslant \gamma \leqslant \gamma_{j,2} \ \wedge\ \gamma \equiv \mathrm{ord}(\lambda_j) \bmod n_j\}.$$

The inclusion from left to right is trivial. For the remaining inclusion let $\gamma \in \Gamma_K$ be an element of the right-hand set. Since for $k = 1, 2$ the leaves $C_j^{\sigma_j,\gamma_{j,k}}$ are subsets of $B_\rho(\sigma_{j_0})$, the ball $B_\rho(\sigma_{j_0})$ must contain the smallest ball containing both leaves. Clearly such a ball contains $C_j^{\sigma_j,\gamma}$, which proves the claim.

By Claim 2.2.6, we have that

$$\bigcup_{j \in J'} C_j^{\sigma_j} = B_\rho(\sigma_{j_0}) \cup \bigcup_{j \in J'} C^{\sigma_j}_{j|(\alpha_j,\gamma_{j,1})} \cup C^{\sigma_j}_{j|(\gamma_{j,2},\beta_j)}. \tag{2.2.2}$$

Note that some of these cells might be empty. We will now need to distinguish between three cases, indexed as $d = 1, 2, 3$. For each case, one can define a decomposition $\mathcal{E}_d$ such that $\mathcal{E}_d(K) = B_\rho(\sigma_{j_0})$ as follows:

Case $d = 1$**:** Suppose that $0 \in B_\rho(\sigma_{j_0})$. We will partition this ball as a union of cells D_i^0 which are centered at 0. Let D_0^0 be the 0-cell $\{0\}$. Choose representatives $\mu_1, \ldots, \mu_{q_K-1} \in K$ such that $\mathrm{ac}_1(K^\times) = \mathrm{ac}_1(\{\mu_1, \ldots, \mu_{q_K-1}\})$. For $1 \leqslant i \leqslant q_K - 1$, we define the cells D_i^0 as follows:

$$D_i^0 := \{t \in K \mid \rho - 1 < \mathrm{ord}(t) \ \wedge\ t \in \mu_i Q_{1,1}\}.$$

Now put $\mathcal{E}_1 := \{D_i^0 \mid i \in \{0, \ldots, q_K - 1\}\}$. One can check that $\mathcal{E}_1(K) = B_\rho(\sigma_{j_0})$.

Case $d = 2$**:** Suppose that $0 \notin B_\rho(\sigma_{j_0})$, and that there exists $m \in \mathbb{N} \setminus \{0\}$ such that $\mathrm{ord}(\sigma_{j_0}) = \rho - m$. Let $\lambda \in K$ be such that $B_\rho(\sigma_{j_0})$ is equal to the cell centered at zero

$$E^0 := \{t \in K \mid \mathrm{ord}(t) = \rho - m \ \wedge\ t \in \lambda Q_{1,m}\}.$$

If we put $\mathcal{E}_2 = \{E^0\}$, then clearly it holds that $\mathcal{E}_2(K) = B_\rho(\sigma_{j_0})$.

Case $d = 3$**:** Suppose that $0 \notin B_\rho(\sigma_{j_0})$ and $\rho - \mathrm{ord}(\sigma_{j_0}) > m$ for all $m \in \mathbb{N}$. Since $B_\rho(\sigma_{j_0}) \sqsubseteq \mathcal{D}_l^*(K)$, there exists $\zeta \in B_{\rho-1}(\sigma_{j_0}) \setminus \mathcal{D}_l^*(K)$. In this case we have that

$$\mathrm{ord}(\zeta) = \mathrm{ord}(\sigma_{j_0}) < \rho - m, \tag{2.2.3}$$

for every $m \in \mathbb{N}$, so in particular $\zeta \neq 0$. Let $\lambda \in K$ be such that $B_\rho(\sigma_{j_0})$ is equal to the cell

$$D^\zeta := \{t \in K \mid \operatorname{ord}(t-\zeta) = \rho - 1 \wedge t - \zeta \in \lambda Q_{1,1}\}.$$

Define $\mathcal{E}_3 = \{D^\zeta\}$, which again clearly satisfies $\mathcal{E}_3(K) = B_\rho(\sigma_{j_0})$.

Finally define $\mathcal{D}_{l+1}$ as

$$\mathcal{D}_{l+1} := \bigcup_{j \in J \setminus J'} \{C_j^{\sigma_j}\} \cup \bigcup_{j \in J'} \{C^{\sigma_j}_{j|(\alpha_j,\gamma_{j,1})}, C^{\sigma_j}_{j|(\gamma_{j,2},\beta_j)}\} \cup \mathcal{E}_d,$$

where $d = 1, 2, 3$ depending on the previous case distinction.

The identity (2.2.2) shows that in all three cases, $\mathcal{D}_{l+1}$ is equivalent to $\mathcal{D}_l$. Let us now discuss why $\mathcal{D}_{l+1}$ is pre-admissible. First note that, if a cell $C_j^{\sigma_j}$ satisfies conditions (a)-(c) from Definition 2.2.2, then any restriction $C^{\sigma_j}_{j|(\alpha'_j,\beta'_j)}$ will also satisfy these conditions. Therefore, since $\mathcal{D}_l$ is pre-admissible, by the definition of $\mathcal{D}_{l+1}$ it suffices to check that the cells in $\mathcal{E}_d$ also satisfy conditions (a)-(c). Suppose first that $d = 1$ or $d = 2$. In both cases, all cells in $\mathcal{E}_d$ are centered at 0, so they satisfy these conditions by default. Now consider the remaining case, $\mathcal{E}_3 = \{D^\zeta\}$. Since D^ζ is not a 0-cell and $\square_1 \neq \emptyset$, conditions (a) and (c) are trivially satisfied. For condition (b) one needs to check that $\operatorname{ord}(\zeta) \leqslant \rho - 2$, but this follows immediately from (2.2.3). Hence, $\mathcal{D}_{l+1}$ is pre-admissible.

It remains to show that $|W(\mathcal{D}_{l+1})| \leqslant |W(\mathcal{D}_l)| - 1$.

Claim 2.2.7. *$W(\mathcal{D}_{l+1}) \subseteq W(\mathcal{D}_l)$.*

Let $\sigma \in W(\mathcal{D}_{l+1})$, and let $\delta \in \Gamma_K$ be such that $B_\delta(\sigma) \sqsubseteq \mathcal{D}^*_{l+1}(K)$ and $B_\delta(\sigma)$ is not contained in a single cell of $\mathcal{D}^*_{l+1}$. We split in cases:

Case $d = 1$ and $d = 2$: In both cases, $\mathcal{E}_d$ only consists of cells centered at 0. Therefore, $\mathcal{D}^*_{l+1} = (\mathcal{D}_{l+1} \setminus \mathcal{E}_d)^*$, which implies that

$$B_\delta(\sigma) \subseteq \bigcup_{j \in J^* \setminus J'} C_j^{\sigma_j} \cup \bigcup_{j \in J'} C^{\sigma_j}_{j|(\alpha_j,\gamma_{j,1})} \cup \bigcup_{j \in J'} C^{\sigma_j}_{j|(\gamma_{j,2},\beta_j)}. \tag{2.2.4}$$

Suppose first that there exists a single $j \in J'$ such that

$$B_\delta(\sigma) \subseteq C^{\sigma_j}_{j|(\alpha_j,\gamma_{j,1})} \cup C^{\sigma_j}_{j|(\gamma_{j,2},\beta_j)}. \tag{2.2.5}$$

Our assumption on $B_\delta(\sigma)$ implies that $B_\delta(\sigma)$ intersects both cells on the right hand-side of (2.2.5). This situation cannot occur, since $B_\delta(\sigma)$ would then necessarily intersect leaves $C_j^{\sigma_j,\gamma}$ with $\gamma_{j,1} \leqslant \gamma \leqslant \gamma_{j,2}$ as well, but these are not part of the union on the right hand-side of (2.2.5). Hence, the ball $B_\delta(\sigma)$ must have non-zero intersection with at least two cells that already occurred in the decomposition $\mathcal{D}_l^*$, which means that $\sigma \in W(\mathcal{D}_l)$. This completes this case.

Case $d = 3$: By construction, we have that

$$\mathrm{Centers}(\mathcal{D}_{l+1}^*) \subseteq \mathrm{Centers}(\mathcal{D}_l^*) \cup \{\zeta\}.$$

Note that $\zeta \notin \mathcal{D}_l^*(K) = \mathcal{D}_{l+1}^*(K)$, where the equality holds since we only added or altered cells with non-zero centers, for which $\square_1 = \square_2 =<$. Therefore we must have that $\sigma \neq \zeta$, hence $\sigma \in \mathrm{Centers}(\mathcal{D}_l^*)$. It suffices to show that $B_\delta(\sigma) \cap D^\zeta = \emptyset$. Indeed, if this intersection is empty, then the inclusion (2.2.4) will hold since $B_\delta(\sigma) \sqsubseteq \mathcal{D}_{l+1}^*(K)$, and we can conclude as in case 1. Suppose for a contradiction that $B_\delta(\sigma) \cap D^\zeta \neq \emptyset$. Recall that by construction, $D^\zeta = B_\rho(\sigma_{j_0})$ is a ball. Therefore, since no cell in $\mathcal{D}_{l+1}^*$ contains $B_\delta(\sigma)$ as a subset, we must have that $D^\zeta \subsetneq B_\delta(\sigma)$. This in turn implies that $B_{\rho-1}(\sigma_{j_0}) \subseteq B_\delta(\sigma)$. Now since $\zeta \in B_{\rho-1}(\sigma_{j_0})$, the previous inclusion contradicts that $\zeta \notin \mathcal{D}_{l+1}^*(K)$. This completes the claim.

It follows from Claim 2.2.7 that $|W(\mathcal{D}_{l+1})| \leqslant |W(\mathcal{D}_l)|$. We show that $\sigma_{j_0} \notin W(\mathcal{D}_{l+1})$, which will imply that $|W(\mathcal{D}_{l+1})| \leqslant |W(\mathcal{D}_l)| - 1$, since by assumption $\sigma_{j_0} \in W(\mathcal{D}_l)$. Again we split in cases. Suppose first that $d = 1$ or $d = 2$. In both cases, σ_{j_0} is contained in a cell of $\mathcal{E}_d$, and hence cannot be contained in a cell of $\mathcal{D}_{l+1}^*$. For case $d = 3$, suppose towards a contradiction that there is some $\delta \in \Gamma_K$ witnessing that $\sigma_{j_0} \in W(\mathcal{D}_{l+1})$. If $\delta \geqslant \rho$, then the ball $B_\delta(\sigma_{j_0})$ would be contained in D^ζ, and since $D^\zeta \in \mathcal{D}_{l+1}^*$, this contradicts the assumption that $\sigma_{j_0} \in W(\mathcal{D}_{l+1})$. If $\delta < \rho$, then $\zeta \in B_\delta(\sigma_{j_0}) \sqsubseteq \mathcal{D}_{l+1}^*(K)$, which contradicts that $\zeta \notin \mathcal{D}_{l+1}^*(K)$. □

Proof of Theorem 2.2.3. This is an immediate consequence of Lemmas 2.2.4 and 2.2.5. □

2.3 Refinement of the decomposition

In [CL16], Cubides and Leenknegt proved a weak, but unconditional version of cell decomposition for P-minimal fields. The building blocks used in that theorem are closely related to (classical) cells, but have a far more complex structure. As a first step towards the main result of this chapter, we will restate this version (using slightly different terminology) and consider some refinements of it, which will lead to Theorem 2.3.7. This theorem will be used as a basis for further improvements in later sections, where we will step by step reduce the complexity of the sets involved. We first need the following notation and definitions.

Let S be a parameter set and $\Sigma \subseteq S \times K^r$ be a definable set. For each $i = 1, \ldots, r$, we write $\Sigma^{(i)}$ for the projection

$$\Sigma^{(i)} := \{(s, c) \in S \times K \mid \exists \zeta_k : (s, \zeta_1, \ldots, \zeta_{i-1}, c, \zeta_{i+1}, \ldots, \zeta_r) \in \Sigma\},$$

and $\Sigma_s^{(i)}$ for its fibers $\left(\Sigma^{(i)}\right)_s$.

Definition 2.3.1. Let $C_1, \ldots, C_r$ be cell conditions, $\{C_1, \ldots, C_r\}$ a set with a prescribed ordering, and $\Sigma \subseteq S \times K^r$ be a definable set. The pair $\mathcal{A} = (\{C_i\}_{1 \leqslant i \leqslant r}, \Sigma)$ is called a *multi-cell* if the following conditions hold:

(i) Every section $\sigma \colon s \mapsto (\sigma_1(s), \ldots, \sigma_r(s))$ of Σ induces the same set X, where

$$X = C_1^{\sigma_1} \cup \ldots \cup C_r^{\sigma_r}.$$

We say that X is *the set defined or induced by* $\mathcal{A}$, and we also denote it by $\mathcal{A}(K)$.

(ii) For every section σ of Σ, the induced potential cells $C_1^{\sigma_1}, \ldots, C_r^{\sigma_r}$ are all disjoint.

The multi-cell $\mathcal{A}$ is called *admissible*, if for every section σ of Σ and every $s \in S$, the fibers $C_i^{\sigma_i(s)}$ form an admissible decomposition of X_s.

We want to stress that the partition in part (i) of the above definition depends on the choice of section, and that different sections of Σ will in general induce different partitions of X. If $\mathcal{A}$ is a multi-cell and $X = \mathcal{A}(K)$, then clearly X is a definable subset of $S \times K$. As is common practice in model theory, we will also refer to the set X *itself* as a multi-cell, by which we mean that there exists some multi-cell $\mathcal{A}$ such that $X = \mathcal{A}(K)$.

The cell decomposition theorem from [CL16] can then be stated in the following way:

Theorem 2.3.2. *Let $X \subseteq S \times K$ be a definable set. There exists a finite partition of X into multi-cells.*

We can now state a first refinement of Theorem 2.3.2. Its proof is a word-for-word analogue of the proof of Theorem 2.3.2, in which we replace each semi-algebraic cell decomposition by an admissible decomposition using Theorem 2.2.3.

Theorem 2.3.3. *Let $X \subseteq S \times K$ be a definable set. There exists a finite partition of X into admissible multi-cells.*

Theorem 2.3.7 will be a refinement of the above theorem. In order to state it, we need the following definitions first.

Definition 2.3.4. A set $X \subseteq S \times K$ is a *classical cell*, if there exist a cell condition C over S, and a definable function $\sigma \colon S \to K$ such that $X = C^{\sigma}$.

The set X is a *clustered cell* if there exist a cell condition C over S, and a definable set $\Sigma \subseteq S \times K$ such that $X = C^{\Sigma}$ (see Definition 2.1.4) and the following holds:

(i) C is a 1-cell condition over S, and both $\square_1$ and $\square_2$ denote $<$.

(ii) For any potential center $\sigma \colon S \to K$, the condition $\mathrm{ord}(\sigma(s)) \leqslant \alpha(s)$ holds for all $s \in S$.

(iii) If $\sigma, \sigma' \colon S \to K$ are potential centers, then $\mathrm{ord}(\sigma(s)) = \mathrm{ord}(\sigma'(s))$.

(iv) Whenever $c \in \Sigma_s$, the set Σ_s also contains all $c' \in K$ such that

$$\forall t : C(s, c, t) \leftrightarrow C(s, c', t).$$

Note that a clustered cell $X = C^{\Sigma}$ may also be a classical cell, provided that Σ has a definable section. Further, remark that conditions (i) and (ii) imply that the potential cells C^{σ} induced by C^{Σ} satisfy the conditions outlined in the definition of pre-admissibility.

Another remark is that, even though the above definition includes some conditions on Σ, it still leaves the structure of the set Σ quite unspecified. Condition (iv) imposes that each Σ_s is a union of balls, but at this point we do not yet require this to be a finite union. In Section 2.4, the structure of this set will be discussed in more detail.

Remark 2.3.5. Let $X = C^{\Sigma}$ be a clustered cell and $\sigma \colon S \to K$ a section of Σ. The condition that $\mathrm{ord}(\sigma(s)) \leqslant \alpha(s)$ enforces that $\mathrm{ord}(t - \sigma(s)) > \min\{\mathrm{ord}(t), \mathrm{ord}(\sigma(s))\}$, and hence that

$$\mathrm{ord}(t) = \mathrm{ord}(\sigma(s))$$

for all $t \in C^{\sigma(s)}$.

Definition 2.3.6. Let $\mathcal{A} = (\{C_i\}_{1 \leqslant i \leqslant r}, \Sigma)$ be a multi-cell with induced set $X = \mathcal{A}(K)$. We say that $\mathcal{A}$ is a *cell array* if the following additional properties hold:

(i) For every $i = 1, \ldots, r$, the set $C_i^{\Sigma^{(i)}}$ is a clustered cell.

(ii) For every section $\sigma = (\sigma_1, \dots, \sigma_r)$ of Σ and all $s \in S$, we have that $\operatorname{ord}(\sigma_i(s)) = \operatorname{ord}(\sigma_j(s))$ for $1 \leqslant i \leqslant j \leqslant r$.

(iii) All centers are non-zero, i.e., $0 \notin \Sigma_s^{(i)}$ for any $1 \leqslant i \leqslant r$ and $s \in S$.

(iv) For every $i = 1, \dots, r$, let $\rho_{i,\max}(s)$ denote the height of the top leaf of C_i. For any section σ_i of $\Sigma^{(i)}$, any $s \in S$ and any ball $B \subseteq X_s$ such that $\sigma_i(s) \in B$, it holds that $B \subseteq B_{\rho_{i,\max}(s)+1}(\sigma_i(s))$.

The last condition in this definition is a slight weakening of the admissibility condition (d) from Definition 2.2.2 in the previous section. This condition will play an important role in our proofs in later sections. The connection between both notions will be explained further in the proof of Theorem 2.3.7.

Similar to the case of multi-cells, we will refer to both $\mathcal{A}$ and its induced set $X = \mathcal{A}(K)$ as cell arrays.

The following notation will be used for both multi-cells and arrays. Let $\mathcal{A} = (\{C_i\}_{i \in I}, \Sigma)$ be a multi-cell over S and $S_1, \dots, S_l$ a partition of S. For each $1 \leqslant j \leqslant l$, we define $\mathcal{A}_{|S_j}$ to be the multi-cell over S_j defined by $\mathcal{A}_{|S_j} := (\{C_i,\}_{i \in I}, \Sigma_{|S_j})$, where $\Sigma_{|S_j} := \{(s,c) \in \Sigma \mid s \in S_j\}$. It is not hard to check that each $\mathcal{A}_{|S_j}$ is still a multi-cell, and that admissibility is preserved as well. Similarly, if $\mathcal{A}$ is a cell array, then so is $\mathcal{A}_{|S_j}$.

Note that the cell conditions of $\mathcal{A}_{|S_j}$ are the same as the ones in the original array $\mathcal{A}$, and that no new potential centers were introduced in this procedure. Moreover, the sets $\mathcal{A}_{|S_j}(K)$ form a partition of $\mathcal{A}(K)$, and if $\mathcal{A}(K) = X$, then $\mathcal{A}_{|S_j}(K) = X_{|S_j}$.

We will now state the main theorem of this section.

Theorem 2.3.7. *Let $X \subseteq S \times K$ be a definable set. There exists a partition of X into sets $X_1, \dots, X_n$ such that each X_i is either a classical cell or a cell array.*

2.3.1 Splitting multi-cells

For the remainder of the chapter we will write $(\{C_i\}_i, \Sigma)$ as shorthand for $(\{C_i\}_{1 \leqslant i \leqslant r}, \Sigma)$ whenever r is clear from the context. Multi-cells will be assumed to be admissible unless otherwise stated.

Definition 2.3.8. Let Σ be a definable subset of $S \times K^r$, and let $1 \leqslant k < r$. Define the following coordinate projections of Σ:

$$\Sigma^{(1,\dots,k)} := \{(s,c) \in S \times K^k \mid \exists \zeta_i \in K : (s, c, \zeta_{k+1}, \dots, \zeta_r) \in \Sigma\};$$

$$\Sigma^{(k+1,\ldots,r)} := \{(s,c) \in S \times K^{r-k} \mid \exists \zeta_i \in K : (s,\zeta_1,\ldots,\zeta_k,c) \in \Sigma\}.$$

Let $\mathcal{A} = (\{C_i\}_i, \Sigma)$ be a multi-cell with $\mathcal{A}(K) = X$. If the sets

$$X^{(1,\ldots,k)} := \bigcup_{\substack{\sigma \text{ section} \\ \text{of } \Sigma^{(1,\ldots,k)}}} C_1^{\sigma_1} \cup \ldots \cup C_k^{\sigma_k} \text{ and}$$

$$X^{(k+1,\ldots,r)} := \bigcup_{\substack{\sigma' \text{ section} \\ \text{of } \Sigma^{(k+1,\ldots,r)}}} C_{k+1}^{\sigma'_{k+1}} \cup \ldots \cup C_r^{\sigma'_r} \text{ are disjoint,} \tag{2.3.1}$$

then we say that $\mathcal{A}$ *can be split at k (by projection)*; if we consider the multi-cells

$$\mathcal{A}^{(1,\ldots,k)} := (\{C_1,\ldots,C_k\}, \Sigma^{(1,\ldots,k)}),$$

$$\mathcal{A}^{(k+1,\ldots r)} := (\{C_{k+1},\ldots,C_r\}, \Sigma^{(k+1,\ldots,r)}),$$

then the sets $\mathcal{A}^{(1,\ldots,k)}(K)$ and $\mathcal{A}^{(k+1,\ldots,r)}(K)$ form a partition of $\mathcal{A}(K)$.

Note that in the above definition, condition (2.3.1) ensures that $\mathcal{A}^{(1,\ldots,k)}$ and $\mathcal{A}^{(k+1,\ldots,r)}$ are multi-cells. Further, remark that $\mathcal{A}^{(1,\ldots,k)}(K) = X^{(1,\ldots,k)}$ and $\mathcal{A}^{(k+1,\ldots,r)}(K) = X^{(k+1,\ldots,r)}$.

For example, if for every section σ of $\Sigma^{(1)}$, C_1^{σ} defines the same set, then $\mathcal{A}$ splits at 1.

Definition 2.3.9. We say that a multi-cell $\mathcal{A} = (\{C_i\}_i, \Sigma)$ *splits at k by definable choice*, if there exists a definable section $\sigma_k \colon S \to K$ of $\Sigma^{(k)}$. Then $\mathcal{A}(K)$ partitions as the union of the classical cell $C_k^{\sigma_k}$ and the multi-cell $(\{C_1,\ldots,C_{k-1},C_{k+1},\ldots,C_r\}, \Sigma')$, where Σ' equals the set

$$\{(s,\zeta_1,\ldots,\zeta_{k-1},\zeta_{k+1},\ldots,\zeta_r) \in S \times K^{r-1} \mid (s,\zeta_1,\ldots,\zeta_{k-1},\sigma_k(s),\zeta_{k+1},\ldots,\zeta_r) \in \Sigma\}.$$

Note that both these splitting procedures preserve admissibility for multi-cells. The same procedures can also be applied to cell arrays, to obtain a partitioning in smaller cell arrays (and classical cells).

The following lemma, which has exactly the same proof as the original lemma by Denef for semi-algebraic sets, will be used in later proofs.

Lemma 2.3.10 ([Den84], Lemma 7.1)**.** *Let $(K;\mathcal{L}_2)$ be a P-minimal structure. Let $X \subseteq S \times K^l$ be a definable set and k a positive integer, such that for every $s \in S$, the fiber X_s has less than k elements. Then there exists a definable section $g \colon S \to K^l$ of X, that is, $g(s) \in X_s$ for all $s \in S$.*

We will now show how a multi-cell can be split into smaller parts where the cell conditions involved satisfy further properties.

Lemma 2.3.11. *Let $\mathcal{A} = (\{C_i\}_{1\leqslant i\leqslant r}, \Sigma)$ be a multi-cell over S. There exists a partition of $\mathcal{A}(K)$ as $Y_1 \cup Y_2$, such that*

(i) Y_1 can be partitioned as a finite union of classical cells;

(ii) there exist multi-cells $\mathcal{A}' = (\{C'_i\}_{i\in I}, \Sigma')$ over definable sets $S' \subseteq S$, such that the sets $\mathcal{A}'(K)$ form a finite partition of Y_2, and

(a) all cell conditions C'_i are 1-cell conditions and have $\square_1 = \square_2 =<$;

(b) for all $s \in S'$ and $i \in I$, we have that $0 \notin (\Sigma')^{(i)}_s$.

Proof. Let $X := \mathcal{A}(K)$. We will prove the lemma by sequentially partitioning off parts of X. We begin by isolating those cell fibers for which 0 is a potential center. Consider the following inductive procedure. First, put

$$S_0 := \{s \in S \mid 0 \in \Sigma^{(1)}_s\},$$

and $S_1 := S \setminus S_0$. This induces a partition of X with respect to the multi-cells $\mathcal{A}_{|S_l}$ for $l = 0, 1$.

Now, $\mathcal{A}_{|S_0}$ admits a split at 1 by definable choice, using the constant function $\sigma_1 \colon S_0 \to K \colon s \mapsto 0$. Write $\mathcal{A}'_{|S_0} = (\{C_i\}_{2\leqslant i\leqslant r}, \Sigma')$ for the multi-cell that remains after the split. The multi-cell $\mathcal{A}_{|S_1}$ already has the property that $0 \notin \left(\Sigma^{(1)}_{|S_1}\right)_s$ for any $s \in S_1$. Repeating a similar procedure for all components of $\mathcal{A}'_{|S_0}$ and $\mathcal{A}_{|S_1}$ will yield a finite number of classical cells, and a finite number of multi-cells for which 0 is not in any of the sets $\Sigma^{(i)}_s$. Hence, we may as well assume from now on that $\mathcal{A}$ itself is a multi-cell satisfying this property.

As a next step, we will consider the 0-cell conditions. Without loss of generality, we may assume that there exists a $k \in \{1, \ldots, r\}$ such that all cell conditions C_i with $1 \leqslant i \leqslant k$ are 0-cell conditions and all cell conditions C_i with $i > k$ are 1-cell conditions. We need to show that X splits at k (by projection), i.e., that $X_1 := X^{(1\ldots,k)}$ and $X^{(k+1,\ldots,r)}$ are disjoint sets. Recall that $(\{C_i\}_i, \Sigma)$ is assumed to be an admissible multi-cell. Now part (a) of Definition 2.2.2 implies the following. If $(s,t) \in X_1 \cap X^{(k+1,\ldots,r)}$, then $t \in X_s \setminus \mathrm{Int}(X_s)$, since $(s,t) \in X_1$. However, $t \in \mathrm{Int}(X_s)$ since $(s,t) \in X^{(k+1,\ldots,r)}$, which is a contradiction. Using Lemma 2.3.10, the set X_1 can be partitioned into a finite number of classical cells.

For the next part we work with $X \setminus X_1$ (which we will still call X, since we may as well assume that X_1 is empty). After reordering if necessary, there exists

$k \in \{0, 1, \ldots, r\}$ such that all cell conditions C_i with $1 \leqslant i \leqslant k$ are precisely those cell conditions for which $\square_1 = \emptyset$. Note that part (c) of Definition 2.2.2 implies that $\Sigma^{(1,\ldots,k)} = S \times \{(0, \ldots, 0)\}$, which actually implies that $k = 0$, since we had assumed that all potential centers for X were non-zero.

After reordering if necessary, we can find $k \in \{1, \ldots, r\}$ such that all cell conditions C_i with $1 \leqslant i \leqslant k$ are precisely those cell conditions for which $\square_2 = \emptyset$. Let $\sigma = (\sigma_1, \ldots, \sigma_r)$ and $\theta = (\theta_1, \ldots, \theta_r)$ be two sections of Σ.

First note that for any $1 \leqslant j \leqslant k$, we have that $\theta_j(s) \notin X_s$. Indeed, suppose for a contradiction that $\theta_j(s) \in X_s$. Because the multi-cell for X does not contain any 0-cell conditions, X_s can be written as a finite disjoint union of open cell fibers $C_i^{\theta_i(s)}$. Note that $\theta_j(s) \in \mathrm{Cl}\left(C_j^{\theta_j(s)}\right) \setminus C_j^{\theta_j(s)}$, and hence there must be some $i \neq j$ such that $\theta_j(s) \in C_i^{\theta_i(s)}$. Since this cell fiber $C_i^{\theta_i(s)}$ is open, it must contain a ball $B_\gamma(\theta_j(s))$. But this implies that $C_i^{\theta_i(s)} \cap C_j^{\theta_j(s)} \neq \emptyset$, which is a contradiction, so we conclude that $\theta_j(s) \notin X_s$.

We will show that for every s, the sets $\{\theta_1(s), \ldots, \theta_k(s)\}$ and $\{\sigma_1(s), \ldots, \sigma_k(s)\}$ contain the same elements. If this were not the case, there would exist $s \in S$ and $1 \leqslant j \leqslant k$ such that $\theta_j(s) \neq \sigma_i(s)$ for all $1 \leqslant i \leqslant k$.

Since $\theta_j(s) \in \mathrm{Cl}(X_s) \setminus X_s$, the set X_s contains elements $t \in K$ which are arbitrarily close to $\theta_j(s)$. But since $\theta_j(s) \neq \sigma_i(s)$ for all $1 \leqslant i \leqslant k$, such a t cannot belong to $\bigcup_{i=1}^k C_i^{\sigma_i(s)}$. Hence, for any such element t, there must exist some $i_0 > k$ such that $t \in C_{i_0}^{\sigma_{i_0}(s)}$. But since t is arbitrarily close to $\theta_j(s)$, and the cell condition C_{i_0} has $\square_2 =<$, this implies that $\theta_j(s) \in C^{\sigma_{i_0}(s)}$, which is a contradiction.

We have now shown that for $1 \leqslant i \leqslant k$, the sets $\Sigma_s^{(i)}$ contain at most k elements. By Lemma 2.3.10, there is a definable way to choose an element from these sets uniformly in s. In particular, there exists a function $\sigma_1 \colon S \to K$ such that X splits by definable choice into $C_1^{\sigma_1}$ and $(\{C_2, \ldots, C_r\}, \Sigma')$, where Σ' is as in Definition 2.3.9. Applying this procedure k times shows that we can split off k classical cells and be left with a multi-cell satisfying the conditions of (ii). $\square$

In the next lemma, we will show that one can definably fix the order of the potential centers for every component.

Lemma 2.3.12. *Let $\mathcal{A} = (\{C_i\}_i, \Sigma)$ be a multi-cell satisfying the conditions in part (ii) of Lemma 2.3.11. There exists a multicell $\mathcal{A}' = (\{C_i\}_i, \Sigma')$ with $\Sigma' \subseteq \Sigma$, such that*

(i) $\mathcal{A}(K) = \mathcal{A}'(K)$;

(ii) for all $s \in S$, all $\sigma(s) = (\sigma_1(s), \ldots, \sigma_r(s)), \theta(s) = (\theta_1(s), \ldots, \theta_r(s)) \in \Sigma'_s$, and all $1 \leqslant j \leqslant r$, it holds that

$$\mathrm{ord}(\sigma_j(s)) = \mathrm{ord}(\theta_j(s)).$$

Proof. Use induction to define a chain of sets $\Sigma_l \subseteq S \times K^r$ for $0 \leqslant l \leqslant r$, with $\Sigma_0 := \Sigma$. Write $(s, \sigma) = (s, \sigma_1(s), \ldots, \sigma_r(s))$ for elements of Σ. Assuming Σ_{l-1} has been defined, set

$$\Sigma_l := \{(s, \sigma) \in \Sigma_{l-1} \mid \forall (s, \sigma') \in \Sigma_{l-1} : \mathrm{ord}(\sigma'_l(s)) \leqslant \mathrm{ord}(\sigma_l(s))\}.$$

Note that this is well-defined, as by condition (b) of pre-admissibility, $\alpha_l(s)$ is an upper bound for $\mathrm{ord}(\sigma_l(s))$, since $\sigma_l(s) \neq 0$ for the multi-cells we consider in this lemma. Moreover, by Theorem 1.1.7, P-minimal definable subsets of Γ_K are Presburger-definable, and every such set has a maximal element if it is bounded.

We leave it to the reader to check that for each l, $\mathcal{A}_l := (\{C_i\}_i, \Sigma_l)$ is indeed a multi-cell. Also, for each l, $\mathcal{A}_l(K) = \mathcal{A}(K)$ since the only thing we do in every step is to put restrictions on which centers we allow for each of the components: Σ_1 will fix the order of $\sigma_1(s)$, then Σ_2 will pick a subset from Σ_1 where $\mathrm{ord}(\sigma_2(s))$ is fixed, and so on. Note that at no point in the induction, Σ_l will be empty. Setting $\mathcal{A}' := \mathcal{A}_r$ completes the proof. □

Lemma 2.3.13. *Let $\mathcal{A} = (\{C_i\}_i, \Sigma)$ be a multi-cell as obtained in Lemma 2.3.12 with $X = \mathcal{A}(K)$. There exists a finite partitioning of X into sets $X_j \subseteq S_j \times K$ (where the S_j are definable subsets of S), such that each part X_j can be written as a finite disjoint union of multi-cells $\mathcal{A}_{jk} = (\{C_{jk,i}\}_i, \Sigma_{jk})$ over S_j, and*

$$\mathrm{ord}(\sigma_1(s)) = \ldots = \mathrm{ord}(\sigma_{r_{jk}}(s))$$

for all $(s, \sigma_1(s), \ldots, \sigma_{r_{jk}}(s)) \in \Sigma_{jk}$.

Proof. Assume that the refinements of Lemma 2.3.12 have been applied. Let Perm be the set consisting of all tuples $\Delta = (\triangle_k)_k$ of length $\binom{r}{2}$, where each $\triangle_k$ is an element of the set $\{<, >, =\}$, and $k \in \{(k_1, k_2) \mid 1 \leqslant k_1 < k_2 \leqslant r\}$. Now partition S into sets

$$S_\Delta := \{s \in S \mid \forall (s, \sigma) \in \Sigma : \mathrm{ord}(\sigma_{k_1}) \triangle_k \mathrm{ord}(\sigma_{k_2})\}.$$

Since Perm is a finite set, this gives us a finite partitioning of S, which in turn induces a partitioning of X into multi-cells $(\{C_{\delta(i)}\}_i, \Sigma_\Delta)$. Here δ is a permutation of $\{1, \ldots, r\}$ and Σ_Δ is obtained from $\Sigma \subseteq S \times K^r$ by restricting S to S_Δ, and reordering the components, such that they are ordered by valuation.

That is, for each multi-cell there is a tuple $(\square_k)_{k<r}$ where each $\square_k$ is either $<$ or $=$ such that, for every section σ of Σ_Δ,

$$\operatorname{ord}(\sigma_k(s)) \ \square_k \ \operatorname{ord}(\sigma_{k+1}(s)), \text{ for all } s \in S_\Delta \text{ and all } 1 \leqslant k < r.$$

We will now focus on one such multi-cell over a set S_Δ (which we will denote again as $(\{C_i\}_i, \Sigma)$ for simplicity), and show how it can be split by projection to obtain the lemma. Let $k \in \{1, \dots, r-1\}$ be such that for all $(s, \sigma_1, \dots, \sigma_r) \in \Sigma$, we have that

$$\operatorname{ord}(\sigma_1) = \ldots = \operatorname{ord}(\sigma_k) < \operatorname{ord}(\sigma_{k+1}).$$

If no such k exists, we are done. Otherwise, it suffices to show that $(\{C_i\}_i, \Sigma)$ splits at k. For if it does, $X^{(1,\dots,k)}$ is a multi-cell satisfying the condition stated in the lemma, and we can iterate the process for $X^{(k+1,\dots,r)}$. This process must stop because we are decreasing the ambient dimension of Σ (indeed, $\Sigma^{(k+1,\dots,r)} \subseteq S \times K^{r-k}$).

Let us now show that one can indeed split X at k: if $(s,t) \in X^{(1,\dots,k)} \cap X^{(k+1,\dots,r)}$, there are $(s, \sigma_1, \dots, \sigma_k) \in \Sigma^{(1,\dots,k)}$, $(s, \theta_1, \dots, \theta_{r-k}) \in \Sigma^{(k+1,\dots,r)}$ and some $1 \leqslant j \leqslant r-k$ such that by Remark 2.3.5,

$$\operatorname{ord}(t) = \operatorname{ord}(\sigma_1) < \operatorname{ord}(\theta_j) = \operatorname{ord}(t),$$

which is a contradiction. $\square$

We have now done all the preparatory work to prove Theorem 2.3.7.

Proof of Theorem 2.3.7. By Theorem 2.3.3, we may suppose that X is an admissible multi-cell. Using Lemmas 2.3.11, 2.3.12 and 2.3.13, X can be partitioned as a finite union of classical cells and multi-cells $(\{C_i\}_i, \Sigma)$ satisfying conditions (ii) and (iii) of Definition 2.3.6. Moreover, each $C_i^{\Sigma^{(i)}}$ satisfies condition (i)-(iii) of Definition 2.3.4.

All operations used in the previous lemmas preserve admissibility, so it can assumed that each multi-cell $(\{C_i\}_i, \Sigma)$ is admissible. Without loss of generality, we may suppose that X is defined by one such multi-cell $(\{C_i\}_{1 \leqslant i \leqslant r} \Sigma)$.

To ensure condition (i) from Definition 2.3.6, it remains to show that each $C_i^{\Sigma^{(i)}}$ satisfies condition (iv) of Definition 2.3.4. To obtain this condition, it may be that we have to add extra elements to Σ. Consider the set Σ' defined as

$$\left\{ (s, x_1, \dots, x_r) \in S \times K^r \;\middle|\; \bigwedge_{i=1}^{r} \left[\exists c : (s,c) \in \Sigma^{(i)} \wedge x_i \in B_{\rho_{i,\max}(s) + m_i}(c) \right] \right\}.$$

The set Σ' is obtained from the original set Σ by adding, for every $c \in \Sigma_s^{(i)}$, all elements in the ball $B_{\rho_{i,\max}(s)+m_i}(c)$. This ensures that each $C_i^{\Sigma'^{(i)}}$ now satisfies condition (iv) of Definition 2.3.4. It is easy to check that $(\{C_i\}_i, \Sigma')$ still defines the same set X, and still satisfies conditions (i)-(iii) of Definition 2.3.6.

Before we can discuss condition (iv) of Definition 2.3.6, we need to introduce the following notion. Let σ_i be a potential center contained in $\Sigma^{(i)}$. We say that $\sigma_i(s)$ is an *admissible center* (for some $s \in S$), if it does not violate condition (d) of the definition of admissibility (Definition 2.2.2). More precisely, we mean the following. Let B be the maximal ball in X_s that contains $\sigma_i(s)$. Then $\sigma_i(s)$ is an admissible center if, for any section σ of Σ that has σ_i as a component, the ball B is contained within a single cell of the decomposition of X_s induced by $\sigma(s)$.

When replacing the original set Σ by Σ', we may have added centers which are not admissible (the reader can check that the conditions of pre-admissibility will never be violated). Yet, note that by construction, any ball in ${\Sigma'}_s^{(i)}$ of size $\rho_{i,\max}(s) + m_i$ still contains at least one admissible center.

Let us now show that this implies condition (iv) from Definition 2.3.6. Without loss of generality, we can take $i = 1$. Consider all possible sections of Σ' which are of the form $(\sigma_1(s), \zeta_2(s), \ldots, \zeta_r(s))$. Each such section induces a partition

$$X_s = C_1^{\sigma_1(s)} \cup C_2^{\zeta_2(s)} \cup \ldots \cup C_r^{\zeta_r(s)}.$$

Now consider the maximal ball B around $\sigma_1(s)$. We need to distinguish between two cases. It may be that this ball does not contain any admissible centers. However, in that case the ball must have a valuation radius strictly bigger than $\rho_{1,\max}(s) + m_1$, in which case condition (iv) holds. If the ball does contain an admissible center, we may as well assume that $\sigma_1(s)$ itself is admissible. Hence, there should be a single cell in the decomposition that contains the maximal ball B around $\sigma_1(s)$. This has to be one of the cells $C_j^{\zeta_j(s)}$ (since $\sigma_1(s) \notin C^{\sigma_1(s)}$).

Let us assume that $B \subset C_2^{\zeta_2(s)}$. Note that, if the ball B were strictly bigger than the ball $B_{\rho_{1,\max}(s)+1}(\sigma_1(s))$, then the cells $C_1^{\sigma_1(s)}$ and $C_2^{\zeta_2(s)}$ would have non-empty intersection, which is a contradiction. □

2.4 On the structure of the trees of potential centers

Let C^Σ be a clustered cell. As we have observed before, there may exist different sections σ, σ' of Σ such that the potential cells C^σ and $C^{\sigma'}$ do not define the same set. To formalize this observation, let us introduce the following equivalence relation.

Definition 2.4.1. Let C^Σ be a clustered cell. For $s \in S$, elements $c, c' \in \Sigma_s$ are said to be (C, Σ_s)-*equivalent* if they define the same cell fiber over s, that is, if

$$\forall t : (C(s, c, t) \leftrightarrow C(s, c', t)).$$

Given sections $\sigma, \sigma' \colon S \to K$ of Σ, then σ and σ' are (C, Σ_s)-*equivalent* if $\sigma(s)$ and $\sigma'(s)$ are (C, Σ_s)-equivalent, that is, if $C^{\sigma(s)} = C^{\sigma'(s)}$.

We will sometimes write *equivalent* rather than (C, Σ_s)-equivalent, when the meaning is clear from the context.

The main goal of this section is to prove the following proposition.

Proposition 2.4.2. *Let $(\{C_i\}_i, \Sigma)$ be a cell array. There exists a uniform bound $N \in \mathbb{N}$, such that for all $s \in S$ and all $1 \leqslant i \leqslant r$, the number of $\big(C_i, \Sigma_s^{(i)}\big)$-equivalence classes is at most N.*

The proof of Proposition 2.4.2 will rely on the combinatorial structure of the set Σ. Let us first introduce some notions which will be used in the proof.

We start by noting that, given a clustered cell C^Σ, a section σ of Σ and $s \in S$, the (C, Σ_s)-equivalence class of σ corresponds to the ball of radius $\rho_{\max}(s) + m$ centered at $\sigma(s)$ (recall that $\rho_{\max}$ and m only depend on the cell condition C). This follows from the definition of clustered cell (condition (iv) of Definition 2.3.4). If no confusion arises, we will use the abbreviated notation $B(\sigma(s))$ for such balls of equivalent centers, i.e.,

$$B(\sigma(s)) := B_{\rho_{\max}(s)+m}(\sigma(s)).$$

Figure 2.2 further illustrates this concept. Here we have drawn the leaves of the cell fiber $C^{\sigma_3(s)}$, and the leaves for the fibers $C^{\sigma_1(s)}$ and $C^{\sigma_2(s)}$ could be depicted similarly.

Note that the cell fibers $C^{\sigma_2(s)}$ and $C^{\sigma_3(s)}$ are disjoint, whereas $C^{\sigma_1(s)}$ and $C^{\sigma_2(s)}$ are not. To study possible intersection between potential cell fibers, it will be important to consider *branching heights* ($\gamma_1(s)$ and $\gamma_2(s)$) in the picture), as they determine whether an intersection could possibly be nonempty.

Definition 2.4.3. Let C^Σ be a clustered cell. For $s \in S$, we call $\gamma \in \Gamma_K$ a *branching height* of Σ_s, if there exist sections σ, σ' of Σ which are *not* (C, Σ_s)-equivalent, and for which $\operatorname{ord}(\sigma(s) - \sigma'(s)) = \gamma$.

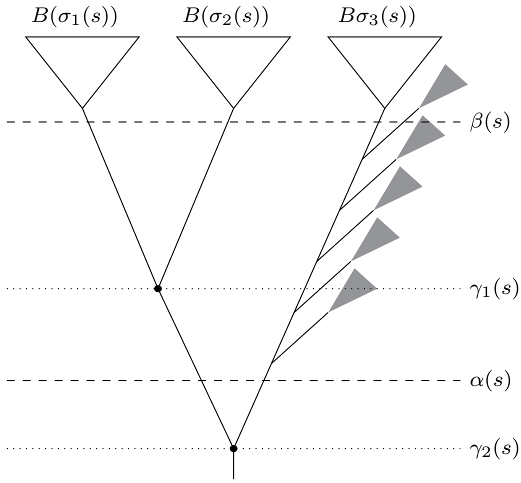


Figure 2.2: Equivalence classes and branching heights of a cell fiber

Recall that $\mathbb{B}$ denotes the set of balls of K, that is

$$\mathbb{B} := \{B_\gamma(a) \mid a \in K, \gamma \in \Gamma_K \cup \{\infty\}\}.$$

The set $\mathbb{B}$, equipped with the reversed inclusion relation $\supseteq$, forms a meet semi-lattice tree. The meet of two balls B_1 and B_2, denoted by $\inf(B_1, B_2)$, corresponds to the smallest ball $B \in \mathbb{B}$ containing both B_1 and B_2. This structure is interpretable in K. Note that K can be identified with the set of maximal elements of $\mathbb{B}$: elements of K are in definable bijection with balls of radius ∞ in $\mathbb{B}$, which are maximal balls with respect to reverse inclusion.

Let C^Σ be a clustered cell. To each Σ_s we associate a subtree $T(\Sigma_s)$ of $\mathbb{B}$ (the set of all balls) generated by the (C, Σ_s)-equivalence classes, i.e.,

$$T(\Sigma_s) := \{B \in \mathbb{B} \mid B = \inf(B(\sigma(s)), B(\sigma'(s))), \text{where } \sigma, \sigma' \text{ are sections of } \Sigma\}.$$

Let $Y \subseteq S \times \Gamma_K$ be such that for each $s \in S$, Y_s denotes the set of all branching heights of Σ_s. Each set Y_s is bounded above by $\beta(s) + m$ and is uniformly definable in s. For each non-zero $l \in \mathbb{N}$, we can inductively define a function $\gamma_l \colon S \to \Gamma_K \cup \{-\infty\}$ as follows: let $\gamma_1(s)$ denote the biggest element of Y_s and put

$$\gamma_{l+1}(s) := \begin{cases} \sup(Y_s \setminus \{\gamma_1(s), \dots, \gamma_l(s)\}) & \text{if } Y_s \setminus \{\gamma_1(s), \dots, \gamma_l(s)\} \neq \emptyset, \\ -\infty & \text{otherwise.} \end{cases}$$

Both Y_s and the functions γ_l depend on the ambient clustered cell C^Σ we are working in.

Let $\gamma \in Y_s$ be a branching height, and σ a section of Σ such that $B_\gamma(\sigma(s))$ is a node of $T(\Sigma_s)$. By the *successors* of $B_\gamma(\sigma(s))$ in $T(\Sigma_s)$, we will mean those balls $B \in T(\Sigma_s)$ with $B \subsetneq B_\gamma(\sigma(s))$, for which there does not exist any ball $B' \in T(\Sigma_s)$ with $B \subsetneq B' \subsetneq B_\gamma(\sigma(s))$. If $B_\gamma(\sigma(s))$ is a node of $T(\Sigma_s)$, then the number of successors of $B_\gamma(\sigma(s))$ must be an integer k between 2 and q_K. We denote the following first order formula, which expresses that $B_\gamma(\sigma(s))$ has exactly k successors by, $\phi_k(\sigma(s), \gamma)$:

$$\exists c_1, \ldots, c_k \in \Sigma_s \ \forall \zeta \in \Sigma_s : \begin{pmatrix} \sigma(s) = c_1 \ \wedge \ \bigwedge_{i \neq j} \operatorname{ord}(c_i - c_j) = \gamma \wedge \\ \bigwedge_{i \neq j} [c_i \text{ and } c_j \text{ are not } (C, \Sigma_s)\text{-equivalent}] \wedge \\ \left[\operatorname{ord}(\zeta - c_1) = \gamma \to \bigvee_{i \neq 1} \operatorname{ord}(\zeta - c_i) > \gamma\right] \end{pmatrix}.$$

One should be aware that for some $\gamma \in Y_s$ and some sections σ of Σ, the ball $B_\gamma(\sigma(s))$ may not necessarily be a node of $T(\Sigma_s)$. We express this situation by the following first-order formula $\phi_1(\sigma(s), \gamma)$:

$$\phi_1(\sigma(s), \gamma) := \sigma(s) \in \Sigma_s \ \wedge \ (\forall \zeta \in \Sigma_s : \operatorname{ord}(\sigma(s) - \zeta) \neq \gamma).$$

The previous discussion implies that given any $\gamma \in Y_s$ and any section σ of Σ, there exists a unique $k \in \{1, \ldots, q_k\}$ such that $\phi_k(\sigma(s), \gamma)$ holds.

Definition 2.4.4. Let $d \in \mathbb{N} \setminus \{0\}$, C^Σ a clustered cell and σ a section of Σ. For $s \in S$, the *d-signature* of $\sigma(s)$ is the tuple $(k_1, \ldots, k_d) \in \{1, \ldots, q_K, -\infty\}^d$, where for $i \in \{1, \ldots, d\}$,

$$k_i = \begin{cases} k & \text{if } \gamma_i(s) \neq -\infty \text{ and } \phi_k(\sigma(s), \gamma_i(s)) \text{ holds,} \\ -\infty & \text{if } \gamma_i(s) = -\infty. \end{cases}$$

Hence, if some $k_i > 1$, then the ball $B_{\gamma_i(s)}(\sigma(s))$ is a node of the tree $T(\Sigma_s)$ with k_i successors. On the other hand, if $k_i = 1$, then $B_{\gamma_i(s)}(\sigma(s))$ is not a

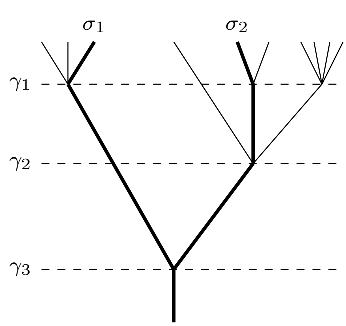


Figure 2.3: In this tree σ_1 has 3-signature $(3, 1, 2)$ and σ_2 has 3-signature $(2, 3, 2)$. The 4-signature of σ_1 is $(3, 1, 2, -\infty)$.

node of $T(\Sigma_s)$. The d-signature $(k_1, \ldots, k_d)$ of $\sigma(s)$ also encodes information about the number of branching heights: if $k_i \neq -\infty$ for all $1 \leqslant i \leqslant d$, then Σ_s has at least d branching heights. If the tree $T(\Sigma_s)$ has depth $i_0 < d$ (that is, the tree has i_0 branching heights), then $i_0 + 1$ will be the least index such that $k_{i_0+1} = -\infty$.

We will now show that, if the tree associated to some $\Sigma_s^{(i)}$ is infinite, then it can be assumed to be dense, in the following sense.

Lemma 2.4.5. *Let $(\{C_i\}_{1\leqslant i\leqslant r}, \Sigma)$ be a cell array defining a set X. Assume that there exists $s_0 \in S$ for which there are infinitely many $\left(C_1, \Sigma_{s_0}^{(1)}\right)$-equivalence classes. Let $R > r$ be an integer. Then there exists a definable set $\Sigma' \subseteq \Sigma$, such that $(\{C_i\}_{1\leqslant i\leqslant r}, \Sigma')$ is a cell array defining the same set X, such that all elements of $(\Sigma')_{s_0}^{(1)}$ have R-signature $(q_K, \ldots, q_K)$.*

Proof. Let $s_0 \in S$ be such that there are infinitely many $\left(C_1, \Sigma_{s_0}^{(1)}\right)$-equivalence classes. For κ an infinite cardinal number, let $\{\sigma_j \mid j < \kappa\}$ be a set of sections of $\Sigma^{(1)}$ such that

(i) each $\left(C_1, \Sigma_{s_0}^{(1)}\right)$-equivalence class is represented by some $\sigma_j(s_0)$;

(ii) for $j < j' < \kappa$, σ_j and $\sigma_{j'}$ are not $\left(C_1, \Sigma_{s_0}^{(1)}\right)$-equivalent.

Let $\gamma_l(s_0)$ be the l^{th}-branching height of $\Sigma_{s_0}^{(1)}$.

Claim 2.4.6. *For any $d \in \mathbb{N} \setminus \{0\}$, there exists a finite set of ordinals W_d such that for all $j < \kappa$ with $j \notin W_d$, the d-signature of $\sigma_j(s_0)$ equals $(q_K, \ldots, q_K)$.*

Suppose that the claim is false, and let $d \in \mathbb{N} \setminus \{0\}$ be the smallest integer witnessing this. Let $(q_K, \ldots, q_k, k_d)$ be a d-signature with $k_d < q_K$ such that the set

$$J := \{j < \kappa \mid \sigma_j(s_0) \text{ has signature } (q_K, \ldots, q_k, k_d)\},$$

is infinite in κ. The set

$$Z := \bigcup_{j\in J} B_{\gamma_{d-1}(s)}(\sigma_j(s_0))$$

is a definable subset of K which is the union of infinitely many balls of radius $\gamma_{d-1}(s_0)$ (here we put $\gamma_0(s_0)$ equal to the radius of the equivalence classes of $\Sigma_{s_0}^{(1)}$, i.e., $\gamma_0(s_0) := \rho_{\max}(s_0) + m_1$, where $\rho_{\max}(s_0)$ is the height of the top leaves for C_1), which are maximal with respect to inclusion in Z. By semi-algebraic cell decomposition, this situation cannot occur in a P-minimal field, which shows the claim.

Let r be the number of cell conditions in the cell array (counted with multiplicity). By our claim, we know that, whenever we fix an integer $R > r$, we can assume that the R-signature of $\sigma_j(s_0)$ will be $(q_K, q_K, \ldots, q_K)$ for all $j < \kappa$, except for a finite set of indices W_R. Now define a set $\widetilde{W_R}$ as follows:

$$\widetilde{W_R} := \Big\{ c \in \Sigma_{s_0}^{(1)} \;\Big|\; \bigvee_{j \in W_R} \operatorname{ord}(c - \sigma_j(s_0)) \geqslant \gamma_R(s_0) \Big\}.$$

Let $\Sigma' \subseteq \Sigma$ be the set obtained by removing the following fibers from Σ_{s_0}:

$$\{(c, \zeta_2, \ldots, \zeta_r) \in \Sigma_{s_0} \mid c \in \widetilde{W_R}\}.$$

The array $(\{C_i\}_i, \Sigma')$ still defines X and moreover, all elements of $(\Sigma')_{s_0}^{(1)}$ have the same R-signature $(q_K, \ldots, q_K)$. □

We are now ready to prove Proposition 2.4.2.

Proof of Proposition 2.4.2. Permuting the cell conditions if necessary, it suffices to show the result for $\Sigma^{(1)}$. Suppose towards a contradiction that such a uniform bound does not exist. By logical compactness, possibly working over an elementary extension, let $s \in S$ be such that there are infinitely many $\big(C_1, \Sigma_s^{(1)}\big)$-equivalence classes. Fix some sufficiently large value of R, such that at least $R > \max\{r, m_1\}$. Applying Lemma 2.4.5, we may assume that all elements of $\Sigma_s^{(1)}$ have the same R-signature $(q_K, \ldots, q_K)$.

We need to fix some notations first. We write σ_j for potential centers in $\Sigma^{(1)}$. The top leaf of a potential cell fiber $C_1^{\sigma_j(s)}$ will be denoted by $\Theta_{\sigma_j(s)}$. Note that for $j \neq j'$, the leaves $\Theta_{\sigma_j(s)}$ and $\Theta_{\sigma_{j'}(s)}$ are disjoint (this follows from the assumption that σ_j and $\sigma_{j'}$ are non-equivalent at s) and all of these leaves are subsets of the set X that is defined by the cell array $(\{C_i\}_i, \Sigma)$.

Fix a cell condition C_i from the description of the array, together with a center ζ from $\Sigma^{(i)}$. Write $\rho(s)$ for the height where $\zeta(s)$ branches off from the tree of $\Sigma_s^{(1)}$, i.e., put

$$\rho(s) := \max_{c \in \Sigma_s^{(1)}} \{\operatorname{ord}(\zeta(s) - c)\}.$$

Note that $\rho(s) \in \Gamma_K \cup \{\infty\}$. We want to know in what ways leaves of $C_i^{\zeta(s)}$ can intersect with balls $\Theta_{\sigma_j(s)}$. Note that the following always holds if $t \in$

$C_i^{\zeta(s),\gamma} \cap \Theta_{\sigma_j(s)}$ (see Definition 2.1.3 for the notation $C_i^{\zeta(s),\gamma}$). For such a t, $\operatorname{ord}(t-\zeta(s)) = \gamma$ and $\operatorname{ord}(t-\sigma_j(s)) = \rho_{1,\max}(s)$. Hence, one has that

$$\begin{aligned}\operatorname{ord}(\zeta(s)-\sigma_j(s)) &= \operatorname{ord}\big((\zeta(s)-t)+(t-\sigma_j(s))\big)\\ &\geqslant \min\left\{\operatorname{ord}(\zeta(s)-t), \operatorname{ord}(t-\sigma_j(s))\right\}\\ &= \min\{\gamma, \rho_{1,\max}(s)\}.\end{aligned}$$

We will now first consider the leaves $C_i^{\zeta(s),\gamma}$ for which $\gamma \geqslant \rho_{1,\max}(s)$. For these leaves we have the following claim.

Claim 2.4.7. *There exist at most $q_K^{m_1}$ leaves $\Theta_{\sigma_j(s)}$ (with $\sigma_j(s) \in \Sigma_s^{(1)}$), for which*

$$\Big(\bigcup_{\gamma \geqslant \rho_{1,\max}(s)} C_i^{\zeta(s),\gamma}\Big) \cap \Theta_{\sigma_j(s)} \neq \emptyset.$$

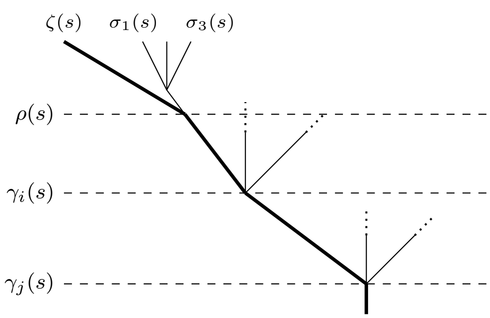
Note that the above intersection will be empty unless $\rho(s) \geqslant \rho_{1,\max}(s)$. Now, if $C_i^{\zeta(s),\gamma} \cap \Theta_{\sigma_j(s)}$ is nonempty for some center $\sigma_j(s)$ and some $\gamma \geqslant \rho_{1,\max}(s)$, then it must hold that

$$\operatorname{ord}(\zeta(s)-\sigma_j(s)) \geqslant \rho_{1,\max}(s).$$

Moreover, there can at most be $q_K^{m_1}$ non-equivalent centers with this property. Our claim follows immediately from this observation.

For the remaining leaves of $C_i^{\zeta(s)}$, one has the following.

Claim 2.4.8. *Let $\gamma < \rho_{1,\max}(s)$. If there exists $\sigma_j \in \Sigma^{(1)}$ such that $C_i^{\zeta(s),\gamma} \cap \Theta_{\sigma_j(s)}$ is nonempty, then either γ is a branching height of $\Sigma_s^{(1)}$, or $\gamma = \rho(s)$.*



Since $\gamma < \rho_{1,\max}(s)$, we must have that

$$\operatorname{ord}(\zeta(s)-\sigma_j(s)) = \gamma.$$

Note that by the definition of $\rho(s)$, we have that $\rho(s) \geqslant \gamma$. Now if $\rho(s) > \gamma$, there exists $c \in \Sigma_s^{(1)}$ such that $\operatorname{ord}(\zeta(s) - c) > \gamma$. We have to show that in this case γ is a branching point. This holds since

$$\begin{aligned}\operatorname{ord}(c - \sigma_j(s)) &= \operatorname{ord}\big((c - \zeta(s)) + (\zeta(s) - \sigma_j(s))\big)\\ &\geqslant \min\big(\operatorname{ord}(c - \zeta(s)), \operatorname{ord}(\zeta(s) - \sigma_j(s)\big)\\ &= \gamma.\end{aligned}$$

Again, since $\operatorname{ord}(c - \zeta(s)) > \gamma = \operatorname{ord}(\zeta(s) - \sigma_j(s))$, this must be an equality. Therefore, c and $\sigma_j(s)$ are nonequivalent centers of $\Sigma_s^{(1)}$ that branch at height γ. This proves the claim.

We will also need to use the following.

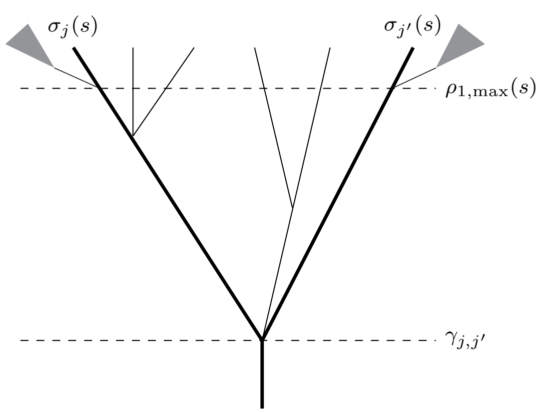
Claim 2.4.9. *Let $\gamma < \rho_{1,\max}(s)$. Then a leaf $C_i^{\zeta(s),\gamma}$ can intersect at most $q_K^{m_1}$ balls $\Theta_{\sigma_j(s)}$.*

Fix some $\gamma < \rho_{1,\max}(s)$ for which there are at least two non-equivalent centers $\sigma_j(s), \sigma_{j'}(s)$ such that

$$C_i^{\zeta(s),\gamma} \cap \Theta_{\sigma_j(s)} \neq \emptyset \quad \text{and} \quad C_i^{\zeta(s),\gamma} \cap \Theta_{\sigma_{j'}(s)} \neq \emptyset \tag{2.4.1}$$

(for other values of γ there is nothing to prove). Let $B_{j,j'}$ denote the smallest ball containing both $\Theta_{\sigma_j(s)}$ and $\Theta_{\sigma_{j'}(s)}$. Since $\Theta_{\sigma_j(s)}$ and $\Theta_{\sigma_{j'}(s)}$ are disjoint, (2.4.1) implies that $B_{j,j'} \subseteq C_i^{\zeta(s),\gamma}$.

Put $\gamma_{j,j'} := \operatorname{ord}(\sigma_j(s) - \sigma_{j'}(s))$, and note that $\gamma_{j,j'}$ is a branching height of $\Sigma_s^{(1)}$. We need to consider the location of this branching height $\gamma_{j,j'}$ versus $\rho_{1,\max}(s)$.

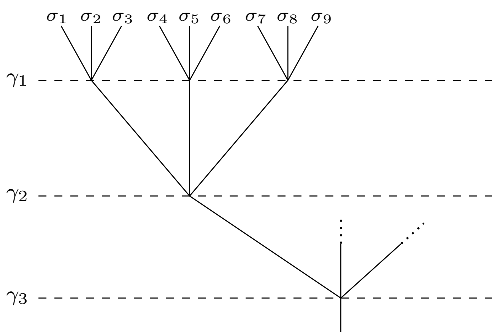


First suppose that $\gamma_{j,j'} \leqslant \rho_{1,\max}(s)$. In this situation, we find that $B_{j,j'} = B_{\gamma_{j,j'}}(\sigma_j(s))$. Since $B_{j,j'}$ contains centers and $B_{j,j'} \subseteq C_i^{\zeta(s),\gamma} \subseteq X_s$, but $\gamma_{j,j'} \leqslant \rho_{1,\max}(s)$, we obtain a contradiction to condition (iv) from the definition of cell array (Definition 2.3.6). Hence, condition (2.4.1) can never be satisfied in this case.

Now consider the case where $\gamma_{j,j'} > \rho_{1,\max}(s)$. This condition expresses that $\sigma_j(s)$ and $\sigma_{j'}(s)$ branch above $\rho_{1,\max}(s)$. There can be at most m_1 such branching heights, and hence the leaf $C_i^{\zeta(s),\gamma}$ can intersect at most $q_K^{m_1}$ balls $\Theta_{\sigma_j}(s)$. This proves the claim.

After a possible reordering, we can assume that the elements $\sigma_j(s) \in \Sigma_s^{(1)}$ are ordered in such a way that for each $l \leqslant R$, the potential centers $\sigma_1(s), \dots, \sigma_{q_k^l}(s)$ generate a finite tree of depth l.

The picture shows an example for $q_K = 3$ and $l = 2$.



Now consider, for $m_1 < l < R$, the depth l subtree of $T(\Sigma_s^{(1)})$ defined above. Combining the claims above, we can conclude that a single cell $C_i^{\zeta(s)}$ can never intersect more than $q_K^{m_1} + (l+1)q_K^{m_1} = (l+2)q_K^{m_1}$ top leaves $\Theta_{\sigma_j(s)}$ from this subtree (and a more careful count would probably show that this upper bound is too high). Since, for the given tree of depth $l < R$, there exist q_K^l disjoint leaves $\Theta_{\sigma_j(s)}$, we can conclude that at least $\frac{q_K^{l-m_1}}{l+2}$ cell conditions are required to account for all top leaves. Hence, we obtain a contradiction when l is sufficiently big, given that there is only a fixed number of cell conditions. We conclude that there cannot exist $s \in S$ for which the number of non-equivalent centers for $\Sigma_s^{(1)}$ is not bounded. $\square$

2.5 Regularity

The main purpose of this section is to prove Proposition 2.5.8, which establishes that a cell array can be partitioned into finitely many *regular* cell arrays. A formal definition will be given in Subsection 2.5.2 (see Definition 2.5.4). We start with some preliminaries needed to prove Proposition 2.5.8.

2.5.1 Repartitionings

Let $(\{C_i\}_{i\in I}, \Sigma)$ be a cell array defining a set X. In this subsection we describe three procedures to obtain a new cell array $(\{C_i'\}_{i\in I'}, \Sigma')$ that defines the same set X. These procedures are called *repartitionings of* $(\{C_i\}_{i\in I}, \Sigma)$ and will be used often in what follows. Some care is needed to make sure that the new pair $(\{C_i'\}_{i\in I'}, \Sigma')$ still satisfies all conditions from the definition of a cell array (Definition 2.3.6). The details are given in the following lemma-definition.

Lemma-Definition 2.5.1. Let $\mathcal{A} = (\{C_i\}_{1\leqslant i\leqslant r}, \Sigma)$ be a cell array over S defining a set X.

(a) Let $\delta\colon S \to \Gamma_K$ be a definable function. Given a cell condition C_i, there exists a definable set $\Sigma' \subseteq S \times K^{r+1}$ such that

$$\mathcal{A}' := (\{C_1, \dots, C_{i-1}, C_{i|(\alpha_i,\delta)}, C_{i|(\delta-1,\beta_i)}, C_{i+1}, \dots, C_r\}, \Sigma')$$

is a cell array defining the same set X. For the notation $C_{i|(\alpha_i,\delta)}$, $C_{i|(\delta-1,\beta_i)}$ see (2.2.1).

(b) Given a cell condition C_i, and $\ell \in \mathbb{N}\setminus\{0\}$, let $C_{i,j}$, for $0 \leqslant j < \ell$, be the cell condition

$$C_{i,j}(s,c,t) := \alpha_i(s) < \mathrm{ord}(t-c) < \beta_i(s) \ \wedge\ t-c \in \varpi^{jn_i}\lambda Q_{\ell n_i, m_i}.$$

There exists a definable set $\Sigma' \subseteq S \times K^{r+\ell-1}$ such that

$$\mathcal{A}' := (\{C_1, \dots, C_{i-1}, C_{i,0}, \dots, C_{i,\ell-1}, C_{i+1}, \dots, C_r\}, \Sigma')$$

is a cell array defining the same set X.

(c) Given a cell condition C_i, and $\ell' \in \mathbb{N}$, let $C_{i,j}$ denote the cell condition

$$C_{i,j}(s,c,t) := \alpha_i(s) < \mathrm{ord}(t-c) < \beta_i(s) \ \wedge\ t-c \in \lambda_j Q_{n_i, m_i+\ell'},$$

where the elements λ_j are representatives of each of the $q_K^{\ell'}$ disjoint subballs of size $(\mathrm{ord}(\lambda)+m+\ell')$ of $B_{\mathrm{ord}(\lambda)+m}(\lambda)$. Put $r' := q_K^{\ell'}$. There exists a definable set $\Sigma' \subseteq S \times K^{r+r'-1}$ such that the repartitioning

$$\mathcal{A}' := (\{C_1, \dots, C_{i-1}, C_{i,1}, \dots, C_{i,r'}, C_{i+1}, \dots, C_r\}, \Sigma')$$

is a cell array defining the same set X.

Proof. First consider part (a). We will show how to define a set Σ' such that conditions (i) and (iv) from the definition of cell array are still satisfied for the repartitioning. Conditions (ii) and (iii) are left to the reader (but they should be rather obvious). Write $\rho_{(\alpha_i,\delta),\max}(s)$ for the height of the top leaf for fibers of $C_{i|(\alpha_i,\delta)}$. First put

$$D_{i,s} := \{c \in K \mid \exists c' \in \Sigma_s^{(i)} : \mathrm{ord}(c-c') \geqslant \rho_{(\alpha_i,\delta),\max}(s) + m_i\}.$$

Now, put $\zeta := (\zeta_1, \dots, \zeta_{i-1}, \zeta', \zeta_i, \dots, \zeta_r)$, and let Σ' be the set

$$\Sigma' := \left\{(s,\zeta) \in S \times K^{r+1} \;\middle|\; \bigwedge_{j=1}^{r} \zeta_j \in \Sigma_s^{(j)} \wedge \zeta' \in D_{i,s} \wedge \phi(s,\zeta) = X_s\right\},$$

where $\phi(s,\zeta)$ is the formula expressing that the centers ζ induce a partition of X_s:

$$\phi(s,\zeta) := \Big[\bigcup_{j\neq i} C_j^{\zeta_j} \;\cup C_{i|(\alpha_i,\delta)}^{\zeta'} \cup C_{i|(\delta-1,\beta_i)}^{\zeta_i} = X_s\Big].$$

It should be clear that with this set Σ', the repartitioning still defines the same set X, and that condition (i) still holds.

It remains to check condition (iv). Note that there is only something to prove for the cell condition $C_{i|(\alpha_i,\delta)}$. Fix an $s \in S$. The set of centers for the clustered cell fiber associated to s and $C_{i|(\alpha_i,\delta)}$ is then $D_{i,s}$. Suppose towards a contradiction that (iv) is not satisfied for some $c \in D_{i,s}$, i.e., that X_s contains a ball $B_\gamma(c)$, for some $\gamma \leqslant \rho_{(\alpha_i,\delta),\max}(s)$. By construction, there exists $\zeta_i \in \Sigma_s^{(i)}$ such that $\mathrm{ord}(c-\zeta_i) \geqslant \rho_{(\alpha_i,\delta),\max}(s) + m_i$. However, this implies that $\zeta_i \in B_\gamma(c)$. But since ζ_i was already a potential center for the clustered cell $C_i^{\Sigma^{(i)}}$, induced by the original cell array, this contradicts condition (iv) for the original cell array.

For case (b), we will assume that $i = 1$ to ease the notation, but the same idea can obviously be applied for other components. For $0 \leqslant j < \ell$, let $\rho_{1j,\max}(s)$ denote the height of the top leaf for fibers of $C_{1,j}$. Let $D_{j,s}$ be the set

$$D_{j,s} := \{c_j \in K \mid \exists c' \in \Sigma_s^{(1)} : \mathrm{ord}(c_j - c') \geqslant \rho_{1j,\max}(s) + m_1\},$$

and put $\zeta := (c_0, \dots, c_{\ell-1}, \zeta_2, \dots, \zeta_r)$. Now, let Σ' be the set

$$\Sigma' := \left\{(s,\zeta) \in S \times K^{r+\ell-1} \;\middle|\; \bigwedge_{j=0}^{\ell-1} c_j \in D_{j,s} \wedge \bigwedge_{i=2}^{r} \zeta_i \in \Sigma_s^{(i)} \wedge \phi(s,\zeta)\right\},$$

where $\phi(s, \zeta)$ is the formula

$$\phi(s,\zeta) := \left[\bigcup_{j=0}^{\ell-1} C_{1,j}^{c_j} \cup \bigcup_{i=2}^{r} C_i^{\zeta_i} = X_s\right].$$

We leave it to the reader to check that all conditions are satisfied in this case.

For (c), the set Σ' can be defined in a similar way. Note that in this case, the potential centers for the new cells $C_{i,j}$ are the same ones as for the old C_i, but each equivalence class splits in $q_K^{\ell'}$ smaller equivalence classes. Since there are no 'new' centers, and the value of $\rho_{i,\max}$ does not change, condition (iv) from the definition of cell array will be preserved. $\square$

2.5.2 Regular cell arrays

In order to give the formal definition of regularity we need the following definitions first.

Definition 2.5.2. A clustered cell C^Σ over S is said to have *uniform tree structure* if for all $s, s' \in S$, the trees $T(\Sigma_s)$ and $T(\Sigma_{s'})$ are isomorphic.

Here, a function $f\colon T_1 \to T_2$ between trees T_1 and T_2 is a tree isomorphism if f is a bijection and both f and f^{-1} are order preserving. We will also need the following additional definitions for types of clustered cells.

Definition 2.5.3. Let C^Σ be a clustered cell. Then C^Σ is said to be

- *large (M-large)*, if there exists $M \in \mathbb{N}$ with $M > 1$, such that $|\alpha(s) - \beta(s)| > M$ for all $s \in S$;
- *uniformly bounded (M-bounded)*, if there exists $M \in \mathbb{N}$ with $M \geqslant 1$, such that $|\alpha(s) - \beta(s)| \leqslant M$ for all $s \in S$;
- *small*, if there exists a definable function $\gamma\colon S \to \Gamma_K$, such that for any potential center $\sigma\colon S \to K$, C^σ is of the form

$$C^\sigma = \{(s,t) \in S \times K \mid \mathrm{ord}(t - \sigma(s)) = \gamma(s) \wedge t - \sigma(s) \in \lambda Q_{n,m}\}.$$

We are now ready to define regular cell arrays.

Definition 2.5.4. A cell array $(\{C_i\}_{i\in I}, \Sigma)$ is said to be *regular* if it satisfies the following conditions:

(R1) There exists $n, m \in \mathbb{N} \setminus \{0\}$ such that all cell conditions are described using the same set $Q_{n,m}$.

(R2) For $i, i' \in I$, either $(\alpha_i(s), \beta_i(s)) \cap (\alpha_{i'}(s), \beta_{i'}(s)) = \emptyset$ for all $s \in S$, or $(\alpha_i(s), \beta_i(s)) = (\alpha_{i'}(s), \beta_{i'}(s))$ for all $s \in S$; cell conditions $C_i, C_{i'}$ that share the same interval will be called *parallel*.

(R3) There is a natural ordering on the cell conditions, that is, either two cells are parallel, or, for any two non-parallel cells C_i and $C_{i'}$, we have that either C_i lies *on top of* $C_{i'}$ (if $\beta_{i'}(s) \leqslant \alpha_i(s) + 1$) or C_i lies *below* $C_{i'}$ (if $\beta_i(s) \leqslant \alpha_{i'}(s) + 1$).

(R4) If C_i and $C_{i'}$ are copies of the same cell condition, then $\Sigma^{(i)} = \Sigma^{(i')}$.

(R5) For each $i \in I$, the clustered cell $C_i^{\Sigma^{(i)}}$ has uniform tree structure.

(R6) If C_i is large and $\gamma(s)$ is a branching height of $\Sigma_s^{(i)}$, then $\gamma(s) \leqslant \alpha_i(s)$.

Remark 2.5.5. For $x = \{1, \dots, 6\}$, let $\mathcal{A} = (\{C_i\}_{i \in I}, \Sigma)$ be a cell array satisfying condition (Rx) from Definition 2.5.4. If S is partitioned into sets $S_1, \dots, S_l$, then each cell array $\mathcal{A}_{|S_j}$ also satisfies condition (Rx). In particular, if $\mathcal{A}$ is a regular cell array, then so are the arrays $\mathcal{A}_{|S_j}$.

Lemma 2.5.6. *Let $\mathcal{A} = (\{C_i\}_i, \Sigma)$ be a cell array. There is a definable partition of S into sets $S_1, \dots, S_l$ such that for each $j \in \{1, \dots, l\}$, each clustered cell in $\mathcal{A}_{|S_j}$ has uniform tree structure.*

Proof. By Proposition 2.4.2, there exist only finitely many tree isomorphism types for the trees $T(\Sigma_s^{(i)})$, for all $s \in S$ and all $1 \leqslant i \leqslant r$. Since the tree isomorphism type of the finite tree $T(\Sigma_s^{(i)})$ is a definable condition, the result follows by a straightforward partitioning of S. $\square$

Lemma 2.5.7. *Let $X \subseteq S \times K$ be a set defined by a cell array $\mathcal{A} = (\{C_i\}_i, \Sigma)$. There exist cell arrays $\mathcal{A}_j$, satisfying conditions (R1) - (R5), such that the induced sets $\mathcal{A}_j(K)$ form a finite partition of X.*

Proof. Condition (R1) is obtained through a repartitioning of the original array $(\{C_i\}_i, \Sigma)$. Put $n := \mathrm{lcm}_i\{n_i\}$ and $m := \max_i\{m_i\}$. By applying procedures (b) and (c) outlined in Lemma-Definition 2.5.1 to each cell C_i with respect to $\ell_i := \frac{n}{n_i}$ (for procedure (b)) and $\ell_i' := m - m_i$ (for procedure (c)), one obtains a repartitioning where all cell conditions are defined using the same set $Q_{n,m}$. We may therefore assume without loss of generality that $X = (\{C_i\}_i, \Sigma)$ already satisfies condition (R1).

Let us now first give the main ideas for a procedure to achieve conditions (R2) and (R3). We want to cut up the intervals in pieces such that there is never any overlap between them. If there were no parameter s involved, one could simply do the following. If C_1, C_2 were cell conditions for which, say

$$\alpha_2 < \alpha_1 < \beta_2 < \beta_1,$$

we would split both conditions: replace C_1 by a condition $C_{1,1}$ with interval (α_1, β_2) and a condition $C_{1,2}$ with interval $(\beta_2 - 1, \beta_1)$. Similarly, split C_2 in a condition $C_{2,1}$ with interval $(\alpha_2, \alpha_1 + 1)$ and a condition $C_{2,2}$ with interval (α_1, β_2). Each split will induce a new array representation of the set. Repeating this until there is no more overlap between intervals would achieve the first condition of the lemma.

In order to do this uniformly in s, one needs to make sure that the *interval structure* is the same for all $s \in S$. This means that we need to first do a partitioning of S to ensure that all the boundary points $\alpha_i(s), \beta_i(s)$ are ordered in the same way for all $s \in S$. Since this is a finite set, this can be done by a finite partition, so let $S_1, \ldots, S_l$ be such a partition. By Remark 2.5.5, each cell array $\mathcal{A}_{|S_j}$ still satisfies condition (R1). Finally, we apply the above idea to cut the intervals of each cell array $\mathcal{A}_{|S_j}$ using a repartitioning as in (a) of Lemma-Definition 2.5.1. Note that this new cell array satisfies both conditions (R2) and (R3). Moreover, the repartitioning (a) does not change the values of n or m used in $Q_{n,m}$ for any of the cell conditions, so the new cell arrays still satisfy condition (R1). Hence, without loss of generality we may suppose that $X = (\{C_i\}_i, \Sigma)$ already satisfies conditions (R1)-(R3).

For condition (R4), suppose that C_i and C_j are the same cell condition for $i \neq j$. At this point, there need not be any connection between the sets $\Sigma^{(i)}$ and $\Sigma^{(j)}$. However, we can replace both $\Sigma^{(i)}$ and $\Sigma^{(j)}$ by $\Sigma^{(i)} \cup \Sigma^{(j)}$, and propagate this to Σ itself in the obvious way: if $\sigma_i \in \Sigma^{(i)}$, $\sigma_j \in \Sigma^{(j)}$, and $(s, \ldots, \sigma_i, \ldots, \sigma_j, \ldots)$ is contained in Σ, then add $(s, \ldots, \sigma_j, \ldots, \sigma_i, \ldots)$ to Σ if necessary. This ensures condition (R4). In addition, since we did not change any cell condition, conditions (R1)-(R3) are still satisfied.

Finally, by Lemma 2.5.6 and Remark 2.5.5 each cell array satisfying (R1)-(R4) can be partitioned into finitely many cell arrays satisfying (R1)-(R5). $\square$

Proposition 2.5.8. *Let $\mathcal{A} = (\{C_i\}_{i \in I}, \Sigma)$ be a cell array with $\mathcal{A}(K) = X$. There exist regular cell arrays $\mathcal{A}_j$, such that the induced sets $\mathcal{A}_j(K)$ form a finite partition of X.*

Proof. By Lemma 2.5.7, we can assume that $\mathcal{A}$ already satisfies conditions (R1)-(R5), so it remains to show how to obtain condition (R6).

For $i \in I$ we can assume, after a finite partitioning of S, that there exists $N \in \mathbb{N}$ such that $C_i^{\Sigma^{(i)}}$ is a large clustered cell for which each fiber $\left(C_i^{\Sigma^{(i)}}\right)_s$ has exactly N branching heights $\gamma_1(s) > \cdots > \gamma_N(s)$. Put $I' := \{i \in I \mid C_{i'}$ is parallel to $C_i\}$. In the next steps of the proof, we will always apply the same repartitionings to each of the cell conditions in $\{C_i\}_{i \in I'}$, simultaneously. By condition (R5), the partitioning process described below can be carried out in a definable way, uniformly in s.

Consider the set

$$\Delta(s) := \{\gamma_j(s) + k \mid 1 \leqslant j \leqslant N, -m \leqslant k \leqslant m\},$$

where m is the integer value in the set $Q_{n,m}$ used to describe all cell conditions (such an m exists by (R1)). Partitioning S into finitely many parts if necessary (which is allowed by Remark 2.5.5), we may assume that the set $\{\alpha_1(s), \beta_1(s)\} \cup \Delta(s)$ is ordered in the same way for all $s \in S$ (with respect to the ordering $<$). Write $\delta_1(s) < \delta_2(s) < \ldots < \delta_L(s)$ for the elements of $\Delta(s) \cap (\alpha_1(s), \beta_1(s))$, and put $\delta_0(s) := \alpha_1(s) + 1$, $\delta_{L+1}(s) := \beta_1(s)$. We now apply a repartitioning as in (a) of Lemma-Definition 2.5.1, with respect to each function $\delta_j(s)$ and each cell C_i for $i \in I'$. That is, we replace each cell condition C_i by cell conditions

$$C_{i,j} := C_{i|(\delta_j - 1, \delta_{j+1})},$$

for each $1 \leqslant j \leqslant L$. Note that some of these conditions may induce empty sets (in which case we will drop the corresponding cell condition).

The value of m and n does not change in these new cell conditions, so (R1) is preserved. The fact that the repartitioning is applied for all parallel cells simultaneously preserves both (R2) and (R3). The same is true for (R4). Indeed, if C_1 and C_2 are copies of the same cell condition (in the original array), then the above procedure produces cell conditions $C_{1,j}$, resp. $C_{2,j}$ such that for each j, $C_{1,j} = C_{2,j}$. Because condition (R4) holds for the original array, one has that $\Sigma^{(1)} = \Sigma^{(2)}$. This equality is preserved when applying repartitioning (a) of Lemma-Definition 2.5.1 to both cell conditions. Since this is the only way to obtain multiple copies of the same cell condition, condition (R4) must be preserved. By Lemma 2.5.6 and Remark 2.5.5, we can assume (R5) is also satisfied.

Let us now explain how this partitioning will ensure (R6). Consider again the large cell condition C_i from the original array, and its set of potential centers $\Sigma^{(i)}$. By the repartitioning, this cell condition was replaced by smaller cell conditions $C_{i,j}$. The set of potential centers for each part $C_{i|(\delta_j - 1, \delta_{j+1})}$ (which we will denote as $\Sigma^{(i,j)}$), is defined from the set of potential centers for C_i, by procedure (a) outlined in Lemma-Definition 2.5.1. In that procedure, either equivalence classes are preserved, or it may be that some equivalence classes merge, and are replaced by a ball containing both original classes: indeed, any two centers in $\Sigma_s^{(i)}$ whose branching height is above $\delta_{j+1}+m$, are equivalent with respect to $C_{i|(\delta_j-1,\delta_{j+1})}$. So the tree $T\big(\Sigma_s^{(i,j)}\big)$ associated to any of the cell conditions $C_{i,j}$ can have at most the same number of branching heights as the tree of C_i (and will probably have less).

Moreover, for *large* cell conditions $C_{i,j}$ (deduced from C_i or a copy of C_i), our construction assures there are no branching heights between δ_j and $\delta_{j+1}+m$, which indeed leaves us with a cell condition for which no branching heights are bigger than the lower bound of the cell.

A similar procedure should be repeated for the remaining parallel, large cell conditions. Note that this indeed ends after a finite number of steps, since the number of branching heights possibly contradicting (R6) only decreases at each step. $\square$

The following lemma gives a property of regular cell arrays that will be used often.

Lemma 2.5.9. *Let $(\{C_i\}_{i\in I},\Sigma)$ be a regular cell array and $i\in I$. If $\sigma_1(s),\sigma_2(s)\in\Sigma_s^{(i)}$ are non-equivalent centers, then $C_i^{\sigma_1(s)}\cap C_i^{\sigma_2(s)}=\emptyset$.*

Proof. Assume that C_i is a large cell condition, as otherwise there is nothing to prove. If $\sigma_1(s),\sigma_2(s)\in\Sigma_s^{(i)}$ are non-equivalent centers, then condition (R6) implies that $\operatorname{ord}(\sigma_1(s)-\sigma_2(s))\leqslant\alpha_i(s)$. Hence, for $(s,t)\in C_i^{\sigma_1}$ we have that

$$\operatorname{ord}(t-\sigma_2(s))=\operatorname{ord}\big((t-\sigma_1(s))+(\sigma_1(s)-\sigma_2(s))\big)\leqslant\alpha_i(s),$$

which means that $(s,t)\notin C_i^{\sigma_2}$. $\square$

2.6 Separating cell arrays

In this section, we will need to keep track of the multiplicity with which a given cell condition occurs in a cell array. Since in a regular array, the associated set of potential centers is the same for each copy of a given cell condition, we will regroup this information, and, in the proofs that follow, whenever convenient adopt the following notation for regular cell arrays. The notation

$$\big(\{C_i^{\langle k_i\rangle}\}_{1\leqslant i\leqslant l},\langle\Sigma\rangle\big),$$

with $\langle\Sigma\rangle\in S\times K^l$, will denote an array where the cell condition C_i occurs with multiplicity k_i. The associated set of potential centers for C_i will be denoted as $\langle\Sigma\rangle^{(i)}$, and corresponds to the projection of the fibers of $\langle\Sigma\rangle$ onto the i-th coordinate. Given a set $\langle\Sigma\rangle$, it should be clear to the reader how this set can be *expanded* to the set $\Sigma \subseteq S \times K^{k_1+\dots+k_l}$ used in the standard notation. We will only use this condensed notation for regular arrays.

Our goal in this section is to show that, possibly after further partitioning or applying certain transformations, one can definably split a cell array into clustered cells $C_i^{\langle\Sigma\rangle^{(i)}}$. Since these clustered cells are derived from regular cell arrays, they will inherit certain properties of regularity. The following terminology will be useful.

Definition 2.6.1. Let $k > 0$ be an integer. A set $H \subseteq S \times K$ is called a *multi-ball of order k* over S, if every fiber H_s (for $s \in S$) is a union of k disjoint balls of the same radius.

Definition 2.6.2. A clustered cell C^Σ is called *regular of order k* if it is regular (when considered as a cell array) and Σ is a multi-ball of order k, where the k balls coincide with the k different (C, Σ_s)-equivalence classes.

In particular, the regularity condition (R6) implies that if two sections σ, σ' of Σ are not (C, Σ_s)-equivalent, then $C^{\sigma(s)} \cap C^{\sigma'(s)} = \emptyset$, and hence for every $s \in S$, we have that, if $\sigma_1, \dots, \sigma_k$ are sections of Σ for which $\{\sigma_1(s), \dots, \sigma_k(s)\}$ are representatives of the k equivalence classes in Σ_s, then

$$C^{\sigma_1(s)} \cup C^{\sigma_2(s)} \cup \dots \cup C^{\sigma_k(s)}$$

is a partition of $(C^\Sigma)_s$.

Remark 2.6.3. The splitting procedures outlined in Definitions 2.3.8 and 2.3.9 can also be used for regular cell arrays, and the regularity condition is preserved under splits by projection. We leave it to the reader to check that, in particular, condition (R5) about uniformity in the tree structure is preserved. When applying a split by definable choice, condition (R5) might get lost initially, but this can always be restored by a further finite partitioning (as described in Lemma 2.5.6) if necessary.

Let us start by considering the cases where a clustered cell can be split off without modifying the array first. Here we use the terminology and notations of Definition 2.3.8.

Lemma 2.6.4. *Let $\mathcal{A} = \left(\{C_i^{\langle k_i\rangle}\}_{1\leqslant i\leqslant l}, \langle\Sigma\rangle\right)$ be a regular cell array, with $\mathcal{A}(K) = X$ and $l > 1$, for which $C_1^{\langle\Sigma\rangle^{(1)}}$ is a regular clustered cell of order k_1. Then $\mathcal{A}$ can be partitioned as the union of $C_1^{\langle\Sigma\rangle^{(1)}}$ and the regular cell array $\left(\{C_i^{\langle k_i\rangle}\}_{2\leqslant i\leqslant l}, \langle\Sigma\rangle^{(2,\dots,l)}\right)$.*

Proof. The suggested split is a split at k_1 (by projection). The regularity claim follows from Remark 2.6.3. Note that $C_1^{\langle\Sigma\rangle^{(1)}} = X^{(1,\ldots,k_1)}$. What needs to be checked is whether

$$C_1^{\langle\Sigma\rangle^{(1)}} \cap X^{(k_1+1,\ldots,\sum k_i)} = \emptyset.$$

The reason that this intersection is empty is as follows. For any section $\sigma = (\sigma_{1,1},\ldots,\sigma_{1,k_1},\sigma_{2,1},\ldots,\sigma_{l,k_l})$ of Σ, we get a partition

$$X_s = \bigcup_{i=1}^{k_1} C_1^{\sigma_{1,i}(s)} \cup \left[\bigcup_{i=1}^{k_2} C_2^{\sigma_{2,i}(s)} \cup \ldots \cup \bigcup_{i=1}^{k_l} C_l^{\sigma_{l,i}(s)}\right], \tag{2.6.1}$$

where the elements $\sigma_{1,i}(s)$ are k_1 distinct (i.e., non-equivalent) elements of $\langle\Sigma\rangle_s^{(1)}$. However, by our assumption, this set only consists of k_1 equivalence classes. Hence, for any possible choice of σ, $\bigcup_{i=1}^{k_1} C_1^{\sigma_{1,i}(s)}$ is the same set, so a nonempty intersection would imply the existence of a σ that contradicts the fact that (2.6.1) gives a partition of X_s. □

Given a regular cell array $\left(\{C_i^{\langle k_i\rangle}\}_{1\leqslant i\leqslant l}, \langle\Sigma\rangle\right)$, let us now consider a cell condition C_1 for which $\langle\Sigma\rangle^{(1)}$ is a multi-ball of order strictly bigger than k_1. In this case, the reasoning in the previous proof implies that there exists some center $\widehat{\sigma}$ in $\langle\Sigma\rangle^{(1)}$, and a section $\sigma' = (\sigma'_{1,1},\ldots,\sigma'_{1,k_1},\sigma'_{2,1},\ldots,\sigma'_{l,k_l})$ of Σ such that for every s,

$$C_1^{\widehat{\sigma}(s)} \cap \left[\bigcup_{i=1}^{k_2} C_2^{\sigma'_{2,i}(s)} \cup \ldots \cup \bigcup_{i=1}^{k_l} C_l^{\sigma'_{l,i}(s)}\right] \neq \emptyset$$

(and hence obviously $\widehat{\sigma}(s)$ is not equivalent to any of the elements of $\{\sigma'_{1,1}(s),\ldots,\sigma'_{1,k_1}(s)\}$). We will refer to this situation by saying that $\widehat{\sigma}(s)$ admits *external exchange*. The following lemma shows that the property of external exchange has consequences for the size of a large cell.

Lemma 2.6.5. *Let $\mathcal{A} = \left(\{C_i^{\langle k_i\rangle}\}_i, \langle\Sigma\rangle\right)$ be a regular cell array with $\mathcal{A}(K) = X$, and C_j a large cell condition for which $\langle\Sigma\rangle^{(j)}$ is a multi-ball with order $k > k_j$. Then there exists $M \in \mathbb{N}$ such that C_j is M-bounded.*

Proof. Fix a large cell condition from the cell array, which will be denoted as C_λ:

$$C_\lambda(s,c,t) = \alpha(s) < \operatorname{ord}(t-c) < \beta(s) \wedge t - c \in \lambda Q_{n,m}.$$

We write k_λ for its multiplicity and $\langle\Sigma\rangle^{(\lambda)}$ for its set of potential centers. By assumption, $\langle\Sigma\rangle^{(\lambda)}$ is a multiball of order $k > k_\lambda$. Let $\widehat{\sigma}$ be as in the discussion preceding this lemma. Hence, there exists a section $\sigma =$

$(\sigma_1, \dots, \sigma_{k_\lambda}, \zeta_1, \dots, \zeta_{r_1+r_2})$, such that, for all $s \in S$, $\widehat{\sigma}(s)$ is not $\big(C_\lambda, \langle\Sigma\rangle_s^{(\lambda)}\big)$-equivalent to any of the elements of $\{\sigma_1(s), \dots, \sigma_{k_\lambda}(s)\}$. We write the corresponding decomposition of X_s as

$$X_s = \left[C_\lambda^{\sigma_1(s)} \cup \dots \cup C_\lambda^{\sigma_{k_\lambda}(s)}\right] \cup \left[\bigcup_{i=1}^{r_1} C_i^{\zeta_i(s)} \cup \bigcup_{i=1}^{r_2} D_i^{\zeta_{r_1+i}(s)}\right],$$

where the cells C_i are parallel to C_λ and the cells D_i are non-parallel to C_λ. We allow that $C_i = C_j$ for $i \neq j$ and similarly for D_i. Note that by Lemma 2.5.9, the intersections $C_\lambda^{\widehat{\sigma}(s)} \cap C_\lambda^{\sigma_i(s)}$ are all empty, and hence

$$C_\lambda^{\widehat{\sigma}(s)} \subset \left[\bigcup_{i=1}^{r_1} C_i^{\zeta_i(s)} \cup \bigcup_{i=1}^{r_2} D_i^{\zeta_{r_1+i}(s)}\right].$$

We will show that there exists a fixed bound $N \in \mathbb{N}$ such that, for any $s \in S$, each of the intersections $C_\lambda^{\widehat{\sigma}(s)} \cap C_i^{\zeta_i(s)}$, resp. $C_\lambda^{\widehat{\sigma}(s)} \cap D_i^{\zeta_{r_1+i}(s)}$ can contain points of at most N leaves of $C_\lambda^{\widehat{\sigma}(s)}$. The statement of the lemma follows from this, since clearly this implies that the larger the interval $(\alpha(s), \beta(s))$ in the description of C_λ gets, the more cells will be involved in this exchange process, yet the decomposition is finite.

Let us first consider the non-parallel cells D_i.

Claim 2.6.6. *For every $s \in S$, and any $1 \leqslant i \leqslant r_2$, at most one leaf of $C_\lambda^{\widehat{\sigma}(s)}$ can intersect the cell fiber $D_i^{\zeta_{r_1+i}(s)}$.*

Write $(\alpha(s), \beta(s))$ for the interval associated to C_λ, and $(\alpha_i(s), \beta_i(s))$ for the interval associated to D_i. By assumption, these intervals have empty intersection. First consider the case where D_i lies *above* C_λ (i.e., $\beta(s) \leqslant \alpha_i(s)+1$). Suppose that $D_i^{\zeta_{r_1+i}(s)}$ contains a point t from a leaf $C_\lambda^{\widehat{\sigma}(s),\gamma}$. Then $\mathrm{ord}(t - \zeta_{r_1+i}(s)) > \alpha_i(s)$, and hence $\mathrm{ord}(\zeta_{r_1+i}(s) - \widehat{\sigma}(s)) = \gamma$. But this implies that the cell fiber $D_i^{\zeta_{r_1+i}(s)}$ cannot possibly contain points from other leaves of $C_\lambda^{\widehat{\sigma}(s)}$. Hence, at most one leaf of $C_\lambda^{\widehat{\sigma}(s)}$ can intersect with $D_i^{\zeta_{r_1+i}(s)}$.

On the other hand, when D_i lies *below* C_λ (i.e., $\beta_i(s) \leqslant \alpha(s)+1$), a cell fiber $D_i^{\zeta_{r_1+i}(s)}$ can contain at most a single leaf of $C_\lambda^{\widehat{\sigma}(s)}$ (or no leaf at all). Indeed, if $D_i^{\zeta_{r_1+i}(s)}$ contained points from more than one leaf of $C_\lambda^{\widehat{\sigma}(s)}$, then $D_i^{\zeta_{r_1+i}(s)}$ would contain a ball $B_r(\widehat{\sigma}(s))$ which contains those leaves. It is easy to check that this ball $B_r(\widehat{\sigma}(s))$ would have radius $r < \rho_{\max}(s)$, which contradicts condition (iv) of the definition of cell array (Definition 2.3.6).

Claim 2.6.7. *For every $s \in S$, and any $1 \leqslant i \leqslant r_1$, at most $2m$ leaves of $C_\lambda^{\widehat{\sigma}(s)}$ can intersect the cell fiber $C_i^{\zeta_i(s)}$.*

Consider a cell fiber $C_i^{\zeta_i(s)}$ for which $C_i^{\zeta_i(s)} \cap C_\lambda^{\widehat{\sigma}(s)} \neq \emptyset$. Put $\gamma_0(s) := \mathrm{ord}(\widehat{\sigma}(s) - \zeta_i(s))$. It is sufficient to show that $C_i^{\zeta_i(s)} \cap C_\lambda^{\widehat{\sigma}(s)} \subseteq C_{\lambda|(\gamma_0(s)-m,\gamma_0(s)+m)}^{\widehat{\sigma}(s)}$, as this set cannot contain more than $2m$ leaves.

Suppose that the intersection contains some $t \in K$ for which $\mathrm{ord}(t - \widehat{\sigma}(s)) \geqslant \gamma_0(s) + m$. Note that this implies that $\gamma_0(s) + m \leqslant \rho_{\max}(s)$. One can check that for such a t to exist, $C_i^{\zeta_i(s)}$ needs to contain the whole ball $B_{\gamma_0(s)+m}(\widehat{\sigma}(s))$, which would again contradict condition (iv) of Definition 2.3.6, since it would mean that X_s contains a ball $B_r(\widehat{\sigma}(s))$ with radius $r < \rho_{\max}(s) + 1$.

Finally, suppose the intersection contains some $t \in K$ for which $\mathrm{ord}(t - \widehat{\sigma}(s)) \leqslant \gamma_0(s) - m$. In this case, we would have that $\mathrm{ord}(t - \widehat{\sigma}(s)) = \mathrm{ord}(t - \zeta_i(s)) \leqslant \gamma_0(s) - m$, and hence the fact that $(t - \widehat{\sigma}(s)) \in \lambda Q_{n,m}$ would imply that also $(t - \zeta_i(s)) \in \lambda Q_{n,m}$. However, this contradicts the assumption that $t \in C_i^{\zeta_i(s)}$, since C_i is a parallel cell condition different from C_λ (and hence $\mathrm{ac}_m(\lambda_i) \neq \mathrm{ac}_m(\lambda)$.) $\square$

A consequence of this lemma is the following.

Proposition 2.6.8. *Let $\mathcal{A} = \left(\{C_i^{\langle k_i \rangle}\}_{1 \leqslant i \leqslant l}, \langle \Sigma \rangle\right)$ be a regular cell array defining a set X. There exists a finite partition of $\mathcal{A}$ into arrays $(\mathcal{A}_j)_{j \in J}$, such that for each $j \in J$, $\mathcal{A}_j$ is either a regular clustered cell, or a regular cell array only containing small cell conditions.*

Proof. Let C_i be a large cell condition and assume that $\langle \Sigma \rangle^{(i)}$ is a multi-ball of order l_i. If $k_i = l_i$, then by Lemma 2.6.4, the clustered cell $C_i^{\langle \Sigma \rangle^{(i)}}$ can be split off. Moreover, since $\mathcal{A}$ is regular, so is $C_i^{\langle \Sigma \rangle^{(i)}}$.

Now if $l_i > k_i$, by Lemma 2.6.5 there exists $M \in \mathbb{N}$ such that $C_i^{\langle \Sigma \rangle^{(i)}}$ is M-bounded. Partitioning S if necessary (and using Remark 2.5.5), we may assume that for all $s \in S$, the interval $(\alpha_i(s), \beta_i(s))$ contains exactly M' elements for some $M' \leqslant M$. Define functions $\delta_1 < \ldots < \delta_{M'}$, such that for each $s \in S$, $(\alpha_i(s), \beta_i(s)) = \{\delta_1(s), \ldots, \delta_{M'}(s)\}$. Let $\mathcal{A}'$ be the cell array one obtains by applying repartitioning (a) of Lemma-Definition 2.5.1 simultaneously to all cell conditions parallel to C_i, with respect to the functions δ_i. That is, $\mathcal{A}'$ is obtained from $\mathcal{A}$ by replacing the cell condition C_i (and each cell condition parallel to C_i) by M' small cell conditions (and adjusting Σ accordingly).

Note that $\mathcal{A}'$ still satisfies all properties of regularity except possibly (R5), but by Lemma 2.5.6 and Remark 2.5.5, there exists a definable partition of S into sets S_j such that each array $\mathcal{A}'_{|S_j}$ is regular. Moreover, each such array has at least one large cell condition less than the original cell array $\mathcal{A}$. Iterating the process for the remaining large cell conditions on each $\mathcal{A}'_{|S_j}$ completes the proof. □

2.6.1 Dealing with the remaining small cell arrays

Let us now have a closer look at the remaining small cell arrays, and how their structure can be simplified.

We will do some normalizations first, to ensure that small cell conditions only differ in their height functions $\gamma(s)$. These normalizations will not change the actual cells that partition $\mathcal{A}(K)$, in the sense that, if C was a cell condition from $\mathcal{A}$, and σ a corresponding potential center, then, if the normalization replaces C by C', there will exist a corresponding center σ' such that $C^\sigma = (C')^{\sigma'}$. In particular, the original cell condition C will be replaced by a condition C' in which $\mathrm{ac}_m(t - \sigma'(s))$ will always be equal to 1.

Unfortunately, it is not obvious whether the normalization procedure described in Lemma 2.6.10 does preserve all properties of regular cell arrays. The definition below (of small regular multi-cells) lists those properties that will still be relevant for subsequent proofs. Other properties may or may not be preserved, but we will pay no further attention to them.

Definition 2.6.9. A multi-cell $\mathcal{A} = (\{C_{\gamma_j}\}_{1\leqslant j\leqslant r}, \Sigma)$ is called a *small regular multi-cell* if the following properties hold:

(S1) All cell conditions C_{γ_j} are small cell conditions of the form

$$\mathrm{ord}(t - \sigma(s)) = \gamma_j(s) \wedge \mathrm{ac}_m(t - \sigma(s)) \equiv 1 \bmod \varpi^m,$$

for some $m \in \mathbb{N}$ independent of j. Also, for all $s \in S$ it holds that

$$\gamma_1(s) < \ldots < \gamma_r(s).$$

(S2) Each $C_{\gamma_j}^{\Sigma^{(j)}}$ is a clustered cell.

(S3) For any $1 \leqslant i, j \leqslant r$, and any $\sigma_i \in \Sigma^{(i)}, \sigma_j \in \Sigma^{(j)}$, it holds that $\mathrm{ord}(\sigma_i(s)) = \mathrm{ord}(\sigma_j(s))$ for all $s \in S$.

(S4) If C_{γ_i} and C_{γ_j} are copies of the same cell condition, then $\Sigma^{(i)} = \Sigma^{(j)}$.

(S5) Each clustered cell $C_{\gamma_j}^{\Sigma^{(j)}}$ has uniform tree structure.

The listed conditions correspond to condition (i) and (ii) in the definition of cell array, and conditions (R1)-(R5) in the definition of regularity, specialized to the case where all cell conditions have the form specified in the above definition. Condition (R6) is no longer relevant since all cell conditions are assumed to be small. Note that by condition (S4) we can use the condensed notation that we introduced at the beginning of the section and write small regular multi-cells in the form $\big(\{C_{\gamma_j}^{\langle k_j\rangle}\}_{1\leqslant j\leqslant r}, \langle\Sigma\rangle\big)$.

In the proof of Lemma 2.6.10 below, we will show how to transform regular cell arrays with only small cell conditions into small regular multi-cells.

Lemma 2.6.10. *Let $\mathcal{A}$ be a regular cell array, where all cell conditions are small. There exists a finite partition of $\mathcal{A}$ into small regular multi-cells $\mathcal{B}_i$.*

Proof. Given a small cell condition, we may as well assume that it has the form $C_{\gamma,\lambda}$, where

$$C_{\gamma,\lambda}^{\sigma} := \{(s,t)\in S\times K \mid \operatorname{ord}(t-\sigma(s)) = \gamma(s) \,\wedge\, \operatorname{ac}_m(t-\sigma(s)) = \operatorname{ac}_m(\lambda)\},$$

and $\lambda\in K$ with $\operatorname{ord}(\lambda)=0$. Indeed, the condition that $\operatorname{ord}(t-\sigma(s))\equiv k \bmod n$ can in this case be expressed as a condition on $\gamma(s)$, and thus on S. Hence, after a finite partitioning of S, this last condition is either obvious, or the set is empty.

Now let $\mathcal{A} = (\{C_{\gamma,\lambda}\}_{\gamma,\lambda},\Sigma)$ be a regular cell array where each cell condition has the form described above. We will show how to define small regular multicells $\mathcal{B}_k = \big(\{C_{\gamma_i}^{\langle k_i\rangle}\}_i, \langle\Sigma_k\rangle\big)$ such that the sets $\mathcal{B}_k(K)$ form a partition of $\mathcal{A}(K) =: X$.

Fix a cell condition $C_{\gamma,\lambda}$ from the description of the array, and write $\Sigma^{(\gamma,\lambda)}$ for its set of potential centers. Put $r := \operatorname{ord}(\lambda-1)$, and note that we may suppose that $r<m$, since otherwise we would have that $\operatorname{ac}_m(\lambda)=1$, in which case there is nothing to prove. Now let $\delta_\lambda\colon \Gamma_K\to\Gamma_K$ be the function defined by

$$\delta_\lambda(\gamma) := \gamma + r.$$

Hence, δ_λ is simply the constant function $\gamma\mapsto\gamma$ when $\operatorname{ac}_1(\lambda)\neq 1$. When $\operatorname{ac}_1(\lambda)=1$, we write λ_1 for the element of $\mathcal{O}_K\setminus\mathcal{M}_K$ satisfying $\lambda = 1+\varpi^r\lambda_1$. Define a function $\Lambda\colon K\to K$ by putting

$$\Lambda(\lambda) := \begin{cases} \lambda-1 & \text{if } \operatorname{ac}_1(\lambda)\neq 1;\\ \lambda_1 & \text{otherwise.}\end{cases}$$

Let $T^{(\gamma,\lambda)}$ be the following set:

$$T^{(\gamma,\lambda)} := \{(s,b) \in S \times K \mid \mathrm{ord}(b) = \delta_\lambda(\gamma(s)) \wedge \mathrm{ac}_m(b) = \mathrm{ac}_m(\Lambda(\lambda))\}.$$

We will write $\Sigma^{(\gamma,\lambda)} + T^{(\gamma,\lambda)}$ for the set $\{(s, b_1 + b_2) \mid (s, b_1) \in \Sigma^{(\gamma,\lambda)} \wedge (s, b_2) \in T^{(\gamma,\lambda)}\}$, and for any section σ of $\Sigma^{(\gamma,\lambda)}$, the set $\sigma + T^{(\gamma,\lambda)}$ is defined similarly. Our claim is now that

Claim 2.6.11. $C_{\gamma,\lambda}^{\Sigma^{(\gamma,\lambda)}} = C_{\gamma,1}^{\Sigma^{(\gamma,\lambda)}+T^{(\gamma,\lambda)}}$.

For this it is sufficient to show that, for any section σ of $\Sigma^{(\gamma,\lambda)}$, it holds that

$$C_{\gamma,\lambda}^{\sigma} = C_{\gamma,1}^{\sigma+T^{(\gamma,\lambda)}}. \tag{2.6.2}$$

Fix a section σ, and some $s \in S$. Choose $b \in K$ such that $(s,b) \in T^{(\gamma,\lambda)}$, and put $\zeta(s) := \sigma(s) + b$. We will prove the inclusion $\subseteq$ in (2.6.2), by checking that $C_{\gamma,\lambda}^{\sigma(s)} \subseteq C_{\gamma,1}^{\zeta(s)}$. Take $t \in C_{\gamma,\lambda}^{\sigma(s)}$. Then we have that

$$\mathrm{ord}(t - \zeta(s)) = \mathrm{ord}(t - (\sigma(s) + b)) = \mathrm{ord}((t - \sigma(s)) - b) = \mathrm{ord}(t - \sigma(s)),$$

since either $\mathrm{ord}(t - \sigma(s)) = \mathrm{ord}(b)$ and $\mathrm{ac}_1(t - \sigma(s)) \neq \mathrm{ac}_1(b)$, or else $\mathrm{ord}(t - \sigma(s)) < \mathrm{ord}(b)$ (when $\mathrm{ac}_1(\lambda) = 1$). We also find that, if $\mathrm{ac}_1(\lambda) \neq 1$, then

$$\mathrm{ac}_m(t - \zeta(s)) = \mathrm{ac}_m(t - \sigma(s)) - \mathrm{ac}_m(b) = \mathrm{ac}_m(\lambda) - \mathrm{ac}_m(\Lambda(\lambda)) = 1,$$

and

$$\mathrm{ac}_m(t - \zeta(s)) \equiv \mathrm{ac}_m(t - \sigma(s)) - \varpi^r \mathrm{ac}_m(b) \equiv \lambda - \varpi^r \lambda_1 \equiv 1 \bmod \varpi^m$$

if $\mathrm{ac}_1(\lambda) = 1$. This proves the inclusion $\subseteq$. The other inclusion can be proven in a similar way.

In order to show that this procedure will give us a multi-cell with the desired properties, we need the following further observation.

Claim 2.6.12. *Every equivalence class-ball in the multi-ball $\Sigma^{(\gamma_i,\lambda_{ij})}$ is translated to a ball with the same radius and with the same valuation.*

Indeed, $\Sigma^{(\gamma_i,\lambda_{ij})}$ is a multi-ball where all the balls have radius $\gamma_i(s) + m$. The set $T^{(\gamma_i,\lambda_{ij})}$ is a multi-ball of order 1 for which the radius of the balls is at least $\gamma_i(s) + m$. This means that, if B is one of the balls of radius $\gamma_i(s) + m$ from $\Sigma^{(\gamma_i,\lambda_{ij})}$, then $B + T_s^{(\gamma_i,\lambda_{ij})}$ will again be a ball of radius $\gamma_i(s) + m$. Hence, we are just translating $\Sigma_s^{(\gamma_i,\lambda_{ij})}$ without changing the tree structure. Furthermore, the elements of $T^{(\gamma_i,\lambda_{ij})}$ have valuation at least $\gamma_i(s)$, while the elements of

B have valuation at most $\gamma_i(s) - 1$ (by condition (ii) from Definition 2.3.4). Therefore, the translation will preserve the valuation of the elements of $\Sigma_s^{(\gamma_i,\lambda_{ij})}$.

The multi-cells $\mathcal{B}_k$ can now be defined as follows. For any fixed height function γ_i, we replace all cell conditions $C_{\gamma_i,\lambda_{ij}}$ by $C_{\gamma_i} := C_{\gamma_i,1}$, so the multiplicity k_i is given by the number of cell conditions of the form $C_{\gamma_i,\lambda_{ij}}$ occurring in the description of $\mathcal{A}$.

A set $\widehat{\Sigma}$ can then be defined in the following way. Let $\gamma_1, \ldots, \gamma_l$ be the height functions occurring in the cell conditions $C_{\gamma_i,\lambda_{ij}}$ from $\mathcal{A}$. Put $c := (c_{1,1}, \ldots, c_{1,k_1}, \ldots, c_{l,1}, \ldots, c_{l,k_l})$, and write $\phi(s,c)$ for the formula expressing that the cell fibers $C_{\gamma_i}^{c_{i,j}}$ form a partition of X_s. Then put

$$\widehat{\Sigma} := \{(s,c) \in S \times K^{k_1+\ldots+k_l} \mid c_{i,j} \in \Sigma^{(\gamma_i,\lambda_{ij})} + T^{(\gamma_i,\lambda_{ij})} \wedge \phi(s,c)\}.$$

Now, the pair $\big(\{C_{\gamma_i}^{\langle k_i\rangle}\}_i, \langle\widehat{\Sigma}\rangle\big)$ is a multi-cell defining the set $\mathcal{A}(K) = X$. We leave it to the reader to check that conditions (S1)-(S3) from Definition 2.6.9 follow from the above claim.

However, note that projections $\widehat{\Sigma}^{(i,j_1)}$ and $\widehat{\Sigma}^{(i,j_2)}$ need not be equal in general, even though the corresponding cell condition is C_{γ_i} in both cases. Hence, we will need to repeat the procedure described in the proof of Lemma 2.5.7 to obtain condition (S4). Applying this procedure to $\widehat{\Sigma}$ will yield a set Σ', and the reader can check that the multi-cell $\big(\{C_{\gamma_i}^{\langle k_i\rangle}\}_i, \langle\Sigma'\rangle\big)$ still satisfies conditions (S1)-(S3). A further partitioning of S into sets S_k, like in Lemma 2.5.6, will then yield small regular multi-cells $\mathcal{B}_k := \big(\{C_{\gamma_i}^{\langle k_i\rangle}\}_i, \langle\Sigma'_{|S_k}\rangle\big)$, such that the sets $\mathcal{B}_k(K)$ partition $\mathcal{A}(K)$. $\square$

Lemma 2.6.13. *Let $\mathcal{A} = \big(\{C_{\gamma_i}^{\langle k_i\rangle}\}_{1\leqslant i\leqslant l}, \langle\Sigma\rangle\big)$ be a regular array consisting only of small cells C_{γ_i}. There exists a definable, finite partition of S into sets S_j, and, for each $\mathcal{A}_{|S_j}(K)$, a finite partition into regular clustered cells.*

Proof. Applying Lemma 2.6.10, we may as well assume that $\mathcal{A}$ is a small regular multi-cell. Let $\gamma_1(s) < \ldots < \gamma_l(s)$ be the height functions for the cell conditions in $\mathcal{A}$, and write $\Sigma^{(\gamma_i)}$ for the set of potential centers of the clustered cell associated to C_{γ_i}. Put $\mathcal{A}(K) := X$. We will first focus on the cells with the smallest leaves, i.e., the cells at height $\gamma_l(s)$. As discussed before, we may assume that $\Sigma^{(\gamma_l)}$ contains centers that admit external exchange.

For a center σ in $\Sigma^{(\gamma_l)}$ to admit external exchange, there must exist a center ζ for a lower level γ_j (with $j < l$), such that $C_{\gamma_l}^{\sigma(s)} \subseteq C_{\gamma_j}^{\zeta(s)}$. Now consider a decomposition of X_s that contains the potential cell $C_{\gamma_l}^{\sigma(s)}$ as one of its components. This decomposition cannot contain the ball $B := C_{\gamma_j}^{\zeta(s)}$ as a single leaf at height $\gamma_j(s)$, nor as a subset of a leaf at a lower height $\gamma_{j'}$ (for $j' < j$). Indeed, the presence of the ball $C_{\gamma_l}^{\sigma(s)}$ means that such a decomposition could never be a partition.

Hence, in order to represent the points of the ball B, we will need a union of smaller balls (small potential cell fibers of heights strictly bigger that $\gamma_j(s)$), where clearly the number of balls one can use is bounded by the sum of the multiplicities of the cell conditions $C_{\gamma_{j+1}}, \ldots, C_{\gamma_l}$. Note that this implies that, if there is exchange possible between two heights $\gamma_i(s)$ and $\gamma_j(s)$, then necessarily the distance $|\gamma_j(s) - \gamma_i(s)|$ is finite (as otherwise one would need infinitely many balls). Moreover, there exists a uniform upper bound for this distance (depending on the respective multiplicities of C_{γ_i} and C_{γ_j}).

Since we are working with a small regular multi-cell, the tree structure for each $\Sigma_s^{(\gamma_i)}$ is independent of s and therefore the number of nonequivalent potential centers at each height is independent of s as well. However, as the tree structure does not fix the distance between the height functions $\gamma_i(s)$, we still need to be a bit careful.

What the above discussion shows is that, if a center $\sigma(s)$ in $\Sigma_s^{(\gamma_l)}$ admits external exchange, then this implies that X_s must contain a ball B' of radius $\gamma_{l-1}(s)+m$, such that $C_{\gamma_l}^{\sigma(s)} \subseteq B'$. We will now rewrite the array so that such balls B' can be represented as small cells at height $\gamma_{l-1}(s)$.

Note that the number of potential centers of $\Sigma_s^{(\gamma_l)}$ that are involved in this, will depend on the distance between $\gamma_l(s)$ and $\gamma_{l-1}(s)$, a number which may vary with s. Hence, in order to work uniformly, we will need to partition the set S. Put $n_k := q_K^k$ and let $\phi_k(s)$ be the definable condition stating that

$$n_k < k_l \ \wedge \ \exists c_1, \ldots, c_{n_k} \in \Sigma_s^{(\gamma_l)} : \cup_{i=1}^{n_k} C_{\gamma_l}^{c_i} \text{ is a ball of radius } \gamma_{l-1}(s) + m.$$

Now partition S into sets S_k defined as

$$S_k := \{s \in S \mid |\gamma_l(s) - \gamma_{l-1}(s)| = k \text{ and } \phi_k(s) \text{ holds}\}.$$

Clearly, this gives a partition of S, since by assumption there is exchange between C_{γ_l} and lower heights. Also, the partition must be finite, since we had already remarked that there exists a uniform upper bound for k.

Each such set can then be further partitioned as a finite union of sets $S_{k,r}$, where r is the number of disjoint balls of radius $\gamma_{l-1}(s)+m$ that can be formed for a given s using leaves $C_{\gamma_l}^{\sigma_i(s)}$. This number r is finite since the number of non-equivalent potential centers is finite.

Now fix one such set $S_{k,r}$. The given partition of S naturally induces a partition of $\mathcal{A}$ into small regular multi-cells $\mathcal{A}_{k,r} := \mathcal{A}_{|S_{k,r}}$, with $X_{k,r} := \mathcal{A}_{k,r}(K)$ (where all properties are preserved by Remark 2.5.5). To unburden notation below, we will simply denote $\mathcal{A}_{k,r}$ as $\big(\{C_{\gamma_i}^{\langle k_i\rangle}\}_i, \langle\Sigma\rangle\big)$.

Because of the way $\mathcal{A}_{k,r}$ was defined, we know that there must exist r disjoint sets, each consisting of n_k non-equivalent centers $\{\sigma_1,\ldots,\sigma_{n_k}\}$ in $\langle\Sigma\rangle^{(\gamma_l)}$, such that for each s, the union

$$\bigcup_{i=1}^{n_k} C_{\gamma_l}^{\sigma_i(s)} \tag{2.6.3}$$

equals a single ball $B'(s)$ of radius $\gamma_{l-1}(s)+m$. Note that it is possible that $\langle\Sigma\rangle^{(\gamma_{l-1})}$ currently does not contain a center $\zeta'(s)$ such that $B'(s) = C_{\gamma_{l-1}}^{\zeta'(s)}$. However, it is possible to definably extend $\langle\Sigma\rangle^{(\gamma_{l-1})}$ to include such a center. Indeed, put

$$\widetilde{\Sigma}_{l-1} := \Big\{(s,\zeta(s)) \in S\times K \mid \exists c_1,\ldots,c_{n_k} \in \Sigma_s^{(\gamma_k)} : C_{\gamma_{l-1}}^{\zeta(s)} = \bigcup_i C_{\gamma_l}^{c_i}\Big\}.$$

This gives us a set whose fibers consist of centers $\zeta(s)$ such that $C_{\gamma_{l-1}}^{\zeta(s)}$ is equal to one of the balls $B'(s)$. We will now replace $\mathcal{A}_{k,r}$ by $\mathcal{A}'_{k,r} := \big(\{C_{\gamma_i}^{\langle k'_i\rangle}\}, \langle\Sigma'\rangle\big)$, where

$$k'_i := \begin{cases} k_i & \text{if } i < l-1;\\ k_i + r & \text{if } i = l-1;\\ k_i - rn_k & \text{if } i = l, \end{cases}$$

replacing cell conditions at height γ_l by a concurrent number of cell conditions at height γ_{l-1}. The potential centers can be adjusted accordingly: if we put

$$c := (c_{11},\ldots,c_{1k'_1},\ldots,c_{l1},\ldots,c_{lk'_l}),$$

then Σ' can be defined as $\Sigma' := \{(s,c) \in S_{k,r}\times K^{\sum k'_i} \mid \psi_{k,r}(s,c)\}$, where $\psi_{k,r}(s,c)$ is the formula

$$\bigwedge_{i\neq l-1,j} c_{ij} \in \langle\Sigma\rangle_s^{(i)} \wedge \bigwedge_j c_{l-1,j} \in \langle\Sigma\rangle_s^{(l-1)} \cup (\widetilde{\Sigma}_{l-1})_s \wedge \bigcup_{i,j} C_{\gamma_i}^{c_{ij}} = (X_{k,r})_s.$$

It should be clear that $\mathcal{A}'_{k,r}$ still satisfies conditions (S1)-(S4), and that $\mathcal{A}_{k,r}(K) = \mathcal{A}'_{k,r}(K)$. It may be that (S5) no longer holds, but this can be remedied by a further partitioning of S if necessary. Moreover, we claim that after this transformation, there is no further exchange possible between cells C_{γ_l} and cells at lower heights. The reason is simply that the condition for exchange is no longer satisfied, as the original leaves $C_{\gamma_l}^{\sigma}$ that were part of a bigger ball are now represented inside a bigger leaf at height γ_{l-1}. Hence, since there is no more exchange, the remaining cell conditions C_{γ_l} can now be split off definably.

Repeating the same procedure $l-2$ more times for the remaining small regular multi-cells will result in a union of regular clustered cells. $\square$

2.7 A decomposition into regular clustered cells

We are now ready to state a full, detailed version of our cell decomposition theorem.

Theorem 2.7.1 (Clustered cell decomposition)**.** *Let $X \subseteq S \times K$ be a set definable in a P-minimal structure $(K, \Gamma_K; \mathcal{L}_2)$. Then there exist $n, m \in \mathbb{N} \setminus \{0\}$ and a finite partition of X into definable sets $X_i \subseteq S_i \times K$ of one of the following forms:*

(i) Classical cells

$$X_i = \left\{ (s,t) \in S_i \times K \;\middle|\; \begin{array}{l} \alpha_i(s) \;\square_1\; \mathrm{ord}(t - c_i(s)) \;\square_2\; \beta_i(s) \wedge \\ t - c_i(s) \in \lambda_i Q_{n,m} \end{array} \right\},$$

where α_i, β_i are definable functions $S_i \to \Gamma_K$, the squares $\square_1, \square_2$ may denote either $<$ or $\emptyset$ (i.e., 'no condition'), and $\lambda_i \in K$. The center $c_i \colon S_i \to K$ is a definable function (which may not be unique).

(ii) Regular clustered cells $X_i = C_i^{\Sigma_i}$ of order k_i.

Let $\sigma_1, \ldots, \sigma_{k_i}$ be (non-definable) sections of the definable multi-ball $\Sigma_i \subseteq S_i \times K$, such that for each $s \in S_i$, the set $\{\sigma_1(s), \ldots, \sigma_{k_i}(s)\}$ contains representatives of all k_i disjoint balls covering $(\Sigma_i)_s$. Then X_i partitions as

$$X_i = C_i^{\sigma_1} \cup \ldots \cup C_i^{\sigma_{k_i}},$$

where each set $C_i^{\sigma_l}$ is of the form

$$C_i^{\sigma_l} = \left\{ (s,t) \in S_i \times K \;\middle|\; \begin{array}{l} \alpha_i(s) < \mathrm{ord}(t - \sigma_l(s)) < \beta_i(s) \wedge \\ t - \sigma_l(s) \in \lambda_i Q_{n,m} \end{array} \right\}.$$

Here α_i, β_i are definable functions $S_i \to \Gamma_K$, $\lambda_i \in K \setminus \{0\}$, and $\mathrm{ord}(\alpha_i(s)) \geqslant \mathrm{ord}(\sigma_l(s))$ for all $s \in S_i$. Finally, we may suppose that $C_i^{\Sigma_i}$ satisfies the following two conditions.

(a) The set Σ_i does not admit any definable sections.

(b) There exists $d_i \in \mathbb{N}$, such that for every $s \in S_i$, $(\Sigma_i)_s$ had exactly d_i branching heights and when $d_i \geqslant 1$, there exist $k_1, \ldots, k_{d_i} \in \mathbb{N}$, such that all elements of Σ_i have d_i-signature $(k_1, \ldots, k_{d_i})$.

Proof. By Theorem 2.3.7, there is a partition of X into classical cells and cell arrays $(\{C_j\}_j, \Sigma)$. If different values of m_i, n_i occur for different cell conditions in the partition, put $m := \max_i\{m_i\}$ and $n := \mathrm{lcm}_i\{n_i\}$. The classical cells in the decomposition can be partitioned in a straightforward way to obtain cells described using the set $Q_{n,m}$.

By Proposition 2.4.2, we know that there exists a uniform upper bound N for the number of $\left(C_j, \Sigma_s^{(j)}\right)$-equivalence classes. This allows us to obtain Proposition 2.5.8, where we show that any cell array can be partitioned as a finite union of regular cell arrays. Moreover, recall that the first step in this proof uniformises the value of n and m within an array, and we can use the procedure described there to make sure that the same n and m are used uniformly for all cell arrays in the partition of X. Later steps in the proof will never need to modify the values of n and m again.

In Proposition 2.6.8 and Lemma 2.6.13, we show how to split a regular cell array into a finite union of regular clustered cells of finite order. If for one of the clustered cells in our partition, the corresponding set Σ_i admits a definable section, then the splitting procedure from Definition 2.3.9 can be used to partition off one or more classical cells, until no more definable sections remain. So we can indeed suppose that all regular clustered cells appearing in the partition, satisfy condition (a).

Now let $C_i^{\Sigma_i}$ be a regular clustered cell appearing in the partition. By regularity the tree structures of all the fibers of Σ_i are isomorphic, hence we can, after a finite partitioning of S, assume that for each $s, s' \in S$, $(\Sigma_i)_s$ and $(\Sigma_i)_{s'}$ essentially look the same. What we mean by this is that the number of branching heights is the same, say d_i, and if we were to pick representatives for equivalence classes of $(\Sigma_i)_s$ and $(\Sigma_i)_{s'}$, then there would exist a bijection between these sets of representatives that preserves all d_i-signatures. This already establishes the existence of d_i from condition (b).

Now if $k_i = 1$, then condition (b) is automatically satisfied, so we may assume that $k_i > 1$, which in turn implies that $d_i \geqslant 1$. For each $l \in \mathbb{N}$, write $(k_1(c), \ldots, k_l(c))$ for the l-signature of $c \in (\Sigma_i)_s$. If $C_i^{\Sigma_i}$ does not yet satisfy condition (b), then there exists some $s \in S$ for which the d_i-signature is not fixed on $(\Sigma_i)_s$, hence for every $s \in S$, $(\Sigma_i)_s$ will contain elements with at least two different signatures. In this case we will give an explicit decomposition of $C_i^{\Sigma_i}$ into regular clustered cells that satisfy both conditions.

First, partition (Σ_i) in sets $\Sigma_{i,(l_1)}$, for $l_1 \in \{1, \ldots, q_K\}$, which are defined as

$$\Sigma_{i,(l_1)} := \{(s, c) \in \Sigma_i \mid k_1(c) = l_1\}.$$

Note that some of these sets may be empty. This induces a partition of $C_i^{\Sigma_i}$ as the union of the regular clustered cells $C_i^{\Sigma_{i,(l_1)}}$. (It should be clear that the uniformity of the tree structure is preserved. Further, since the tree of $\left(\Sigma_{i,(l_1)}\right)_s$

is a pruning of the original tree of $(\Sigma_i)_s$, and no new branching heights are introduced, we still have that all branching happens below $\alpha_i(s)$.)

This process can now be repeated inductively. If we fix a clustered cell $C_i^{\Sigma_{i,(l_1)}}$, the 1-signature is fixed. This clustered cell can now be partitioned into cells $C_i^{\Sigma_{i,(l_1,l_2)}}$, where $C_i^{\Sigma_{i,(l_1,l_2)}}$ is defined as

$$\Sigma_{i,(l_1,l_2)} := \{(s,c) \in \Sigma_{i,(l_1)} \mid k_2(c) = l_2\},$$

again for $l_2 \in \{1,\ldots,q_K\}$. We can repeat the process until we have a partition of $C_i^{\Sigma_i}$ into regular clustered cells $C_i^{\Sigma_{i,(l_1,\ldots,l_{d_i})}}$ that satisfy conditions (a) and (b). □

Readers familiar with other cell decomposition theorems may have noticed that in both Theorem 2.7.1 and 1.1.9, no further conditions are imposed on the parameter set S (besides definability). In many similar-style theorems, cells are defined inductively, in the sense that the set S is required to be a cell as well, and similarly for its consecutive projections. We have not insisted on this, but we are however convinced that such an inductive cell decomposition theorem can be derived quite easily, when taking into account both K-cell and Γ-cell decomposition.

2.7.1 Clustered cells of minimal order and open questions

In looking for further simplifications of Theorem 2.7.1, we considered the following open question.

Question 2.7.2. *Can every regular clustered cell of finite order be decomposed into finitely many regular clustered cells of order* 1*?*

If this question were to have an affirmative answer, this would imply a significant simplification of Theorem 2.7.1. Moreover, this would mean that, at least in spirit, such a generalized cell decomposition theorem stays very close to the spirit of *classical* (Denef-type) cell decomposition: for a clustered cell of order 1, the set Σ can still be seen as the graph of a definable function $c : S \to \mathbb{B}$, where $\mathbb{B}$ denotes the set of balls in K. Unfortunately, it may not be possible to achieve this.

To investigate this question, we introduce regular clustered cells of *minimal order.*

Definition 2.7.3. A regular clustered cell C^{Σ} of order k is of *minimal order* if it cannot be partitioned as a finite union of regular clustered cells $C_i^{\Sigma_i}$ of order $k_i < k$.

Some remarks are in order here. In this definition we allow for the option that, given a regular clustered cell C^{Σ} of order k, there may exist a cell condition C_1 and a multi-ball Σ_1 such that $C^{\Sigma} = C_1^{\Sigma_1}$, but the order of $C_1^{\Sigma_1}$ is strictly lower than the order of C^{Σ}. Also in more general cases there need not be a direct connection between the original C and Σ and the C_i and Σ_i occurring in the partition.

Lemma 2.7.4. *Every regular clustered cell of finite order can be partitioned into regular clustered cells of minimal order.*

Proof. Let k be the minimal integer for which there exists a regular clustered cell C^{Σ} of order k that cannot be partitioned as a finite union of regular clustered cells of minimal order. In particular, C^{Σ} is not of minimal order, hence it can be partitioned into finitely many regular clustered cells $C_1^{\Sigma_1}, \ldots, C_n^{\Sigma_n}$ each of order $k_i < k$. But by the minimality of k, each $C_i^{\Sigma_i}$ can be partitioned into regular clustered cells of minimal order, which provides a decomposition of C^{Σ}, contradicting the assumption. □

Note that there is no canonicity here: it may well be that, by making different choices in each step of the induction, one can obtain different partitions of the same set where the number of cells in the decomposition and their specific orders k_i may differ.

This lemma allows us to formulate an alternative clustered cell decomposition theorem. However, some of the statements of Theorem 2.7.1 need to be relaxed slightly. Specifically, we can no longer require that all clustered cells occurring in the decomposition, are described using the same set $Q_{n,m}$.

Theorem 2.7.5. *Let $X \subseteq S \times K$ be a definable set. Then X partitions as a finite union of classical cells and regular clustered cells of minimal order. Moreover, no regular clustered cell has a definable section.*

Proof. By Theorem 2.7.1 it suffices to show the result for a regular clustered cell C^{Σ}. By Lemma 2.7.4, C^{Σ} can be decomposed into finitely many regular clustered cells of minimal order $C_1^{\Sigma_1}, \ldots, C_n^{\Sigma_n}$. It remains to check whether these cells admit a definable section. Note that if Σ_i is of order $k_i > 1$, then it cannot contain a definable section. Indeed, if such a section were to exist, this would contradict the minimality of k_i, since it would be possible to definably split $C_i^{\Sigma_i}$ into a regular clustered cell of order 1 and a cell of order $k_i - 1$. Hence, if Σ_i has a definable section it must be of order 1. Put $I := \{1, \ldots, n\}$ and let $I_0 := \{i \in I \mid \Sigma_i \text{ has a definable section } \sigma_i\}$. Then C^{Σ} is decomposed into

$$\bigcup_{i \in I_0} C_i^{\sigma_i} \cup \bigcup_{i \in I \setminus I_0} C_i^{\Sigma_i},$$

which shows the result. □

Note that the uniformity for n, m from Theorem 2.7.1 is lost here because a priori, there is no guarantee that the cell conditions C_i in the above proof are defined using the same set $Q_{n,m}$. In the proof of Theorem 2.7.1, this uniformity was obtained through a further partitioning of the cell conditions C_i. Unfortunately, the cost of this (especially for m) is that the order of the associated multi-balls Σ_i may increase, and hence we risk losing the minimality. The proof of the following lemma illustrates that this can indeed happen.

Lemma 2.7.6. *Let C^Σ be a regular clustered cell of minimal order over S. Then for every $s \in S$, every ball B which is an equivalence class of (C, Σ_s), is maximally contained in Σ_s.*

Proof. Note that, if C^Σ is defined by a large cell condition, then Σ already satisfies the conclusion of this lemma. Indeed, regularity implies that all branching heights are below $\alpha(s)$, and hence the equivalence classes are always maximal balls.

Suppose now that C^Σ is a small clustered cell of (minimal) order $k > 1$, and that the maximal balls of Σ_s contain more than one (C, Σ_s)-equivalence class. We need to show that C^Σ cannot be of minimal order. We will do this by showing explicitly how to decompose C^Σ as a finite union of regular clustered cells of order strictly smaller than k.

So let C^Σ be a clustered cell associated to a small cell condition C with its leaf at height $\gamma(s)$. By the similar reasoning as in the proof of condition (b) from Theorem 2.7.1 we may assume that all elements of $(\Sigma)_s$ have the same N-signature for all $N \in \mathbb{N}$, uniformly in S. Hence, we can assume that there exists some $\ell \in \mathbb{Z}$ with $\ell < m$, such that for all $s \in S$, Σ_s consists of maximal balls of the same size $\gamma(s) + \ell$ (here we also use our assumption that maximal balls contain more than one equivalence class).

First consider the case where $\ell \leqslant 0$. Consider a maximal ball B in Σ_s. We claim that $C^B = B$. Take $b \in B$. Then B contains an element c with $\mathrm{ord}(c-b) = \gamma(s)$ and $\mathrm{ac}_m(c-b) = \lambda$, and hence $B \subset C^B$. The other inclusion is proven similarly. Hence, both Σ_s and C^{Σ_s} consist of $k' := k/q_K^{m-\ell}$ maximal balls. We will rewrite both the cell condition and the set of centers, such that these k' balls become the leaves of the new small cell fibers. Write $\rho(s) = \gamma(s) + \ell$ for the size of the maximal balls in Σ. First put

$$\widetilde{\Sigma} := \{(s, c') \in S \times K \mid \exists (s, c) \in \Sigma : \mathrm{ord}(c' - c) = \rho(s) - 1 \wedge c - c' \in Q_{1,1}\},$$

and let $\widetilde{C}$ be the cell condition

$$\widetilde{C}(s,c,t) := s \in S \wedge \rho(s) - 1 = \mathrm{ord}(t-c) \wedge t - c \in Q_{1,1}.$$

Then clearly, $\widetilde{C^{\widetilde{\Sigma}}}$ is a regular clustered cell defining the same set as C^{Σ}, yet having strictly smaller order.

For $0 < \ell < m$, we can apply the inverse operation of the repartitioning of Lemma-Definition 2.5.1 part (c). There we observed that, when increasing the value of m (in the set $Q_{n,m}$), the effect was that a single equivalence class was split in smaller equivalence classes. In our case, we will replace the original condition $\mathrm{ac}_m(t-c) \in \lambda Q_{n,m}$ in C by a condition $\mathrm{ac}_\ell(t-c) \in \lambda Q_{n,\ell}$, and call the resulting cell condition $\widehat{C}$. Then $\widehat{C}^{\Sigma}$ will be a clustered cell where the maximal balls of Σ_s coincide with the $(\widehat{C}, \Sigma_s)$-equivalence classes. Since moreover, the order of $\widehat{C}^{\Sigma}$ is smaller than the order of C^{Σ}, this completes the proof. $\square$

Using clustered cells of minimal order, we can reformulate the open Question 2.7.2.

Definition 2.7.7. Let H be a multi-ball of order k over S.

- We say that H is *maximal* if for every $s \in S$, every ball B among the k balls whose union is Σ_s, satisfies $B \sqsubseteq \Sigma_s$.
- We say that a maximal multi-ball Σ *admits finite Skolem functions* if there exists a definable function $f : S \to \mathbb{B}$ such that for all $s \in S$, $f(s)$ is a maximal ball of Σ_s.

When we say that a function $f : S \to \mathbb{B}$ is definable, we simply mean that its graph should correspond to a definable set $A \subseteq S \times K$, such that A_s is a ball for all $s \in S$.

Lemma 2.7.8. *The following questions are equivalent:*

1. *Can every regular clustered cell of finite order be partitioned into finitely many regular clustered cells of order 1?*
2. *Is every regular clustered cell of minimal order of order 1?*
3. *Does every maximal multi-ball admit finite Skolem functions?*

Proof. We first show that Questions 1 and 2 are equivalent. By Lemma 2.7.4, if the answer to Question 2 is yes, then Question 1 has a positive answer as well. Now suppose that Question 1 has an affirmative answer and let C^{Σ} be a regular clustered cell of minimal order $k > 1$. By assumption, it is equal to a finite union of regular clustered cells of order 1, contradicting the minimality of the order of C^{Σ}. Hence Question 2 has affirmative answer too.

Now let us show that Questions 2 and 3 are equivalent as well. Suppose that Question 3 has an affirmative answer. Let C^{Σ} be a regular clustered cell of minimal order $k > 1$. By Lemma 2.7.6, we may assume that Σ is maximal. Pick a finite Skolem function for Σ, say with graph $H \subseteq S \times K$. Then we have that C^{Σ} is equal to the union of C^H and $C^{\Sigma \setminus H}$. Since C^H has order 1 and $C^{\Sigma\setminus H}$ has order $k-1$, this contradicts the minimality of k.

Now suppose that every regular clustered cell of minimal order has order 1 (i.e., Question 2 has an affirmative answer) and let Σ be a maximal multi-ball of order $k > 1$ over S. Note that by the definition of multi-balls, these k maximal balls have the same radius $\gamma(s)$ for every $s \in S$. Using Theorem 2.7.5, the set Σ can be partitioned as a finite union of classical cells D_i and regular clustered cells $C_i^{\Sigma_i}$ of minimal order, i.e.,

$$\Sigma = \bigcup_{i=1}^{r_1} D_i \cup \bigcup_{i=1}^{r_2} C_i^{\Sigma_i}.$$

By our assumptions, the cells $C_i^{\Sigma_i}$ all have order 1. Without loss of generality, we may also assume that all cells in the decomposition are over S. Now put

$$\gamma_{D_i}(s) := \min\{\gamma \in K \mid (D_i)_s \text{ contains a ball of radius } \gamma\},$$

$$\gamma_{C_i}(s) := \min\{\gamma \in K \mid (C_i^{\Sigma_i})_s \text{ contains a ball of radius } \gamma\}.$$

We will first explain why $k > 1$ implies that $r_1+r_2 > 1$. Suppose that $r_1+r_2 = 1$. We will only consider the case $(r_1, r_2) = (0, 1)$, but the other case is completely similar. In this case, we have that $\Sigma_s = (C_1^{\Sigma_1})_s$ for all $s \in S$. However, note that Σ_s is the union of $k > 1$ disjoint maximal balls of the same size, while the fiber $(C_1^{\Sigma_1})_s$ can only contain a single ball of any given radius. This gives a contradiction, and hence it must be the case that $r_1 + r_2 > 1$.

Moreover, since Σ is maximal, we may assume that $\gamma(s) \leqslant \gamma_{D_i}(s)$ and $\gamma(s) \leqslant \gamma_{C_i}(s)$. Write $D_i^{\gamma_{D_i}}$, resp. $C_i^{\gamma_{C_i}}$ for the subset of D_i, resp. $C_i^{\Sigma_i}$ whose fibers are the (unique) maximal balls of $(D_i)_s$, resp. $(C_i^{\Sigma_i})_s$. If $r_1 \neq 0$, we can define f as

$$f(s) = B \Leftrightarrow B \text{ is a ball, maximally contained in } \Sigma_s \text{ and } B \cap D_1^{\gamma_{D_1}} \neq \emptyset,$$

otherwise we put

$$f(s) = B \Leftrightarrow B \text{ is a ball, maximally contained in } \Sigma_s \text{ and } B \cap C_1^{\gamma_{C_1}} \neq \emptyset.$$

Then f provides a finite Skolem function for Σ. $\square$

An indication that the above questions may well have a negative answer comes from Remark 4.8 of [HMRC15]. In this remark, it is shown that there exist elementary extensions K of $\mathbb{Q}_p$ (for the language of rings), in which the set of balls is not rigid. In particular, the authors show that there exists an automorphism σ of K and a ball B such that the orbit of B under σ has size p. This seems to imply that the answer to the questions in Lemma 2.7.8 would be 'no', at least if there exists such a set of p balls which is also maximal, i.e., this set of p balls does not cover a bigger ball of K.

This observation could serve as a basis for constructing an example showing that the question in its third form has a negative answer. The basic idea is to try building a parametrized family of subsets having fibers that are such sets of p balls. However, actually constructing such a higher-order multi-ball (and proving that no finite Skolem function exists), appears to be a rather non-trivial exercise.

Part of the complication, especially for the one-sorted case, lies in the fact that one needs to work within a structure that does not admit definable Skolem functions. However, such structures have not been studied in much detail as yet, given that a first concrete example was only very recently constructed [CN17b]. In this paper Cubides and Nguyen introduce the following multi-ball of order 1:

$$A := \{(x, y) \in K^2 \mid \operatorname{ord}(f(x) - y) > \rho\},$$

where $(K; \mathcal{L}_{\text{an}})$ is a nonstandard elementary extension of $(\mathbb{Q}_p; \mathcal{L}_{\text{an}})$, $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ a transcendental convergent power series, and $\rho \in \Gamma_K$ such that $\rho > n$ for all $n \in \mathbb{Z}$. They prove that the structure $(K; \mathcal{L}_A)$, where $\mathcal{L}_A := \mathcal{L}_{\text{ring}} \cup \{A\}$, is P-minimal, but that there exists no definable Skolem function for the set A. Modifying the multi-ball A into a multi-ball of higher order could be a possible direction for constructing a multi-ball that does not have finite Skolem functions.

Chapter 3

Exponential-constructible functions in P-minimal structures

This chapter is based on joint work with Pablo Cubides-Kovacsics and Eva Leenknegt [CCL18].

In this chapter we will try to generalize Theorem 1.2.14 to the wider context of P-minimal structures on p-adic fields. Part of the difficulty lies in considering P-minimal expansions which do not have definable Skolem functions. As we shall explain later, in situations where such functions do not exist, the classical definition of exponential-constructible functions (Definition 1.2.13) will not satisfy the desired stability-under-integration result. Therefore we will introduce the *exponential*-constructible functions*, which are a refinement of the exponential-constructible functions. We will then show that the exponential*-constructible functions are base-stable under integration under one extra condition that is similar to Condition (1.2.1) from Theorem 1.2.14.

Since we will need to distinguish between P-minimal fields with or without definable Skolem functions, we will adopt the following convention. We will refer to the *Skolem setting* when working over P-minimal fields that admit definable Skolem functions. The term *non-Skolem setting* refers to situations where we explicitly assume we are working over P-minimal fields that do not admit such definable Skolem functions. When we make no assumptions either way, we will refer to the *P-minimal setting.* Also, when referring to *general P-minimal structures*, this means that we are considering not only P-minimal expansions of p-adic fields but rather P-minimal expansions of arbitrary p-adically closed fields. We will always work with the two sorted version of P-minimality, i.e., structures $(K, \Gamma_K; \mathcal{L}_2)$, where $\mathcal{L}_2 = (\mathcal{L}, \mathcal{L}_{\mathrm{Pres}}, \mathrm{ord})$.

3.1 Introduction: from $\mathcal{C}_{\exp}$ to $\mathcal{C}^*_{\exp}$

In the first part of this introduction we will introduce and motivate our refinement of the class of exponential-constructible functions and in the second part we will state our main results.

3.1.1 The algebra $\mathcal{C}^*_{\exp}$ of exponential*-constructible functions

In this section K will be a p-adic field. We denote by $\psi\colon K \to \mathbb{C}^\times$ an additive character that satisfies $\psi_{|\mathcal{M}_K} = 1$ and $\psi_{|\mathcal{O}_K} \neq 1$. We recall the definitions of the constructible and exponential-constructible functions.

Definition 3.1.1. Let $(K, \mathbb{Z}; \mathcal{L}_2)$ be a P-minimal structure and X a definable set.

(i) The algebra $\mathcal{C}(X)$ of *constructible functions on* X is the $\mathbb{Q}$-algebra generated by the constant functions and the functions of the form

$$\alpha\colon X \to \mathbb{Z} \qquad \text{and} \qquad X \to \mathbb{Q}\colon x \mapsto q_K^{\beta(x)},$$

where $\alpha, \beta\colon X \to \mathbb{Z}$ are definable.

(ii) The algebra $\mathcal{C}_{\exp}(X)$ of *exponential-constructible functions on* X is the $\mathbb{Q}$-algebra generated by the functions in $\mathcal{C}(X)$ and the functions of the form $\psi \circ f$, where $f\colon X \to K$ is definable.

Before providing the definition of $\mathcal{C}^*_{\exp}$, let us informally explain why the algebra $\mathcal{C}_{\exp}$ will need to be adapted to suit our purposes. We will need the following notation and definitions, some of which we have seen already in the previous chapters.

The set consisting of all balls in K with a given valuation radius γ is denoted as

$$\mathbb{B}_\gamma := \{B_\gamma(x) \mid x \in K\}.$$

Definition 3.1.2. Let $X \subseteq K$. Let $B \subseteq X$ be a ball such that for all balls $B' \subseteq X$, one has that $B \subseteq B' \Rightarrow B = B'$. Then we call B a *maximal ball* of X. This property will be denoted as $B \sqsubseteq X$.

Definition 3.1.3. Let $k \in \mathbb{N} \setminus \{0\}$.

- A set $A \subseteq S \times K$ is called a *multi-ball of order k* over S, if every fiber A_s is a union of k disjoint balls of the same radius.
- A multi-ball A of order k is said to be *on* $\mathbb{B}_\gamma$ if, for each fiber A_s, the k balls B contained in A_s all satisfy $B \sqsubseteq A_s$ and $B \in \mathbb{B}_\gamma$.

We may not always explicitly mention the order of a multi-ball, but even in such cases, the order will always be assumed finite.

In the next example we will compute the integral of a very simple exponential-constructible function to illustrate the type of difficulties one may encounter when definable Skolem functions are not available. Recall that we integrate with respect to the normalised Haar measure on K and the counting measure on $\mathbb{Z}$. If S and Y are sets containing both K- and Γ_K-variables, $X \subseteq S \times Y$ and $f\colon X \to \mathbb{C}$, then

$$\mathrm{Int}(f, Y) := \{s \in S \mid f(s, \cdot\,) \text{ is measurable and integrable over } X_s\}.$$

Example 3.1.4. Let $k \in \mathbb{N} \setminus \{0\}$ and let $A \subseteq S \times K$ be a definable multi-ball of order k on $\mathbb{B}_1$. Consider the exponential-constructible function $f\colon A \to \mathbb{C}\colon (s, x) \mapsto \psi(x)$. For any $B \in \mathbb{B}_1$ and for all $x, y \in B$ we have $x - y \in \mathcal{M}_K$, hence $\psi(x) = \psi(x - y)\psi(y) = \psi(y)$. We denote this value by $\psi(B)$ and then we can calculate that

$$\int_{A_s} f(s, x)|dx| = \int_{A_s} \psi(x)|dx| = \sum_{B \sqsubseteq A_s} \psi(B) \cdot \mathrm{Vol}(B) = q_K^{-1} \cdot \sum_{B \sqsubseteq A_s} \psi(B).$$

If a structure admits definable Skolem functions, there exist definable sections of A, say $f_1, \ldots, f_k\colon S \to K$, such that

$$\sum_{B \sqsubseteq A_s} \psi(B) = \sum_{j=1}^{k} \psi(f_j(s)). \tag{3.1.1}$$

Hence, the function $s \mapsto \int_{A_s} f(s, x)|dx|$ will be an element of $\mathcal{C}_{\exp}(S)$. Note that if a structure does not admit Skolem functions, one cannot always make this type of substitution.

The next example shows that the character of a definable function can always be written as a character sum over a multi-ball.

Example 3.1.5. Let $f\colon X \to K$ be a definable function. For each $x \in X$, let A_x be the ball $A_x := f(x) + \mathcal{M}_K$. Then the set $A := \{(x, y) \in X \times K \mid y \in A_x\}$ is a definable multi-ball of order 1 on $\mathbb{B}_1$. Moreover, for all $x \in X$ one has that

$$\psi(f(x)) = \sum_{B \sqsubseteq A_x} \psi(B).$$

To summarize, Example 3.1.4 shows that the algebras of exponential-constructible functions will need to be extended if one wants base-stability under integration (Definition 1.2.2) in the P-minimal setting, and Example 3.1.5 indicates that working with functions of the form $x \mapsto \sum_{B \sqsubseteq A_x} \psi(B)$, rather than $x \mapsto \psi(f(x))$, yields a very natural generalization to a wider setting. This motivates us to propose the following definition.

Definition 3.1.6. Let $(K, \mathbb{Z}; \mathcal{L}_2)$ be a P-minimal structure and X be a definable set. The algebra $\mathcal{C}^*_{\exp,\psi}(X)$ of $\mathcal{L}_2$-exponential*-constructible functions on X is the $\mathbb{Q}$-algebra generated by functions in $\mathcal{C}(X)$ and functions of the form $x \mapsto \sum_{B \sqsubseteq A_x} \psi(B)$, where $A \subseteq X \times K$ is a definable multi-ball on $\mathbb{B}_1$.

We will write $\mathcal{C}^*_{\exp}$ rather than $\mathcal{C}^*_{\exp,\psi}$ whenever ψ is clear from the context.

Remark 3.1.7. Note that in the Skolem setting, the identity (3.1.1) holds, hence any exponential*-constructible function is also exponential-constructible for such structures.

3.1.2 Overview of main results

We are now ready to state the main theorems of this chapter. We will always work in a P-minimal structure $(K, \mathbb{Z}; \mathcal{L}_2)$ unless explicitly stated otherwise.

Theorem 3.1.8. *Let $X \subseteq S \times \mathbb{Z}^n$ be a definable set. Let $f \in \mathcal{C}^*_{\exp}(X)$ be such that* $\mathrm{Int}(f, \mathbb{Z}^n) = S$. *Then there exists $g \in \mathcal{C}^*_{\exp}(S)$ such that for all $s \in S$,*

$$g(s) = \int_{X_s} f(s, \gamma)|d\gamma|.$$

Let $X \subseteq S \times K^n$ be a definable set and $f \in \mathcal{C}^*_{\exp}(X)$ be such that $\mathrm{Int}(f, K^n) = S$. The main obstruction to deriving the stability under integration for f is related to integrability conditions on some of the functions used to define f. To formalize this let us introduce some terminology.

Definition 3.1.9. For $n \geqslant 1$, let $X \subseteq S \times K^n$ be a definable set. We say that a function $f \in \mathcal{C}^*_{\exp}(X)$ can be written in *n-normal form*, if there exists a definable partition $X = \cup_{w \in W} X_w$ (where W is a finite index set), such that on each X_w the following holds:

(i) There exist $m \geqslant 1$ and functions $f_1, \ldots, f_m \in \mathcal{C}^*_{\exp}(X_w)$ such that $f_{|X_w} = \sum_{i=1}^m f_i$,

(ii) Each function f_i can be further expanded as

$$f_i(s,x) = h_i(s,x) \sum_{B \sqsubseteq A^i_{s,x}} \psi(B), \quad \text{where}$$

(a) $h_i \in \mathcal{C}(X_w)$ and $\mathrm{Int}(h_i, K^n) = \pi_S(X_w)$,

(b) A^i is a definable multi-ball over X_w of order k_i on $\mathbb{B}_1$,

(c) for each $s \in \pi_S(X_w)$, $x \mapsto \sum_{B \sqsubseteq A^i_{s,x}} \psi(B)$ is a measurable function in x.

Theorem 3.1.10. *Let $X \subseteq S \times K^n$ be a definable set and $f \in \mathcal{C}^*_{\exp}(X)$. If f can be written in n-normal form, then there exists $g \in \mathcal{C}^*_{\exp}(S)$ such that, for all $s \in S$,*

$$g(s) = \int_{X_s} f(s,x)|dx|.$$

Note that our notion of n-normal form is similar in nature to the assumptions on the form of f made in Theorem 1.2.14. Now consider the following conjecture.

Conjecture 3.1.11. *Let $X \subseteq S \times K$ be a definable set and $f \in \mathcal{C}^*_{\exp}(X)$ such that* $\mathrm{Int}(f, K) = S$. *Then f can be written in* 1*-normal form.*

Under the above conjecture, Theorems 3.1.8 and 3.1.10 imply that the algebras of exponential*-constructible functions are base-stable under integration. As the proof is rather short, we will include it here in the introduction for the reader's convenience.

Theorem 3.1.12. *Suppose Conjecture 3.1.11 holds. Then the algebras of exponential*-constructible functions are base-stable under integration.*

Proof. Let $X \subseteq S \times Y$ be a definable set and $f \in \mathcal{C}^*_{\exp}(X)$ be such that $\mathrm{Int}(f, Y) = S$. By Fubini, it suffices to consider the cases where $Y = K$ or $Y = \mathbb{Z}$. The case $Y = \mathbb{Z}$ follows from Theorem 3.1.8. For the case $Y = K$, the conjecture implies that there exists a finite partition $X = \cup_{w \in W} X_w$ such that for each X_w, we can write $f_{|X_w}(s,x) = \sum_{i=1}^m f_i(s,x)$, where the functions f_i satisfy condition (ii) of Definition 3.1.9. This implies that for all $s \in S_w := \pi_S(X_w)$, each $f_i(s, \cdot\,)$ is integrable over $(X_w)_s$, hence

$$\int_{(X_w)_s} f(s,x)|dx| = \int_{(X_w)_s} \sum_{i=1}^m f_i(s,x)|dx| = \sum_{i=1}^m \int_{(X_w)_s} f_i(s,x)|dx|.$$

By Theorem 3.1.10 there exists, for each $i \in \{1, \dots, m\}$, a function $g_i \in \mathcal{C}^*_{\exp}(S_w)$ such that $g_i(s) = \int_{(X_w)_s} f_i(s,x)|dx|$. Remark 3.1.13 below allows us to extend the g_i to functions in $\mathcal{C}^*_{\exp}(S)$. Putting $g_w(s) := \sum_{i=1}^m g_i(s)$ and summing over all $w \in W$ completes the proof. □

Remark 3.1.13. If $U \subseteq X$ are definable sets, then for any $f \in \mathcal{C}^*_{\exp}(U)$, there exists $f_X \in \mathcal{C}^*_{\exp}(X)$, such that

$$f_X(x) = \begin{cases} f(x) & \text{if } x \in U; \\ 0 & \text{if not.} \end{cases}$$

We will often abuse notation and simply write f rather than f_X. This trick will be used when partitioning the domain X of an exponential*-constructible function, as it allows us to extend functions on one of the sets in the partition to functions on X. We may not always explicitly mention this.

It is worth noting that in [CGH14] they showed (by proving a variation on the above conjecture) that from Theorem 1.2.14 the Assumption (1.2.1) on the form of f in the case of $\mathcal{L}_{\text{ring}}$ or $\mathcal{L}_{\text{an}}$ can be removed. The proof made use of the Jacobian Property [CGH14, Proposition 3.3.5]. At the time of writing, it is still an open question as to whether (a version of) that property holds in the P-minimal setting or even in the Skolem setting. Currently only a local version is known [KL14]. Note that Conjecture 3.1.11 is open in the Skolem setting as well.

The remainder of this chapter is organised as follows. Sections 3.2 and 3.3 contain several auxiliary results that will be needed in the later sections. The Theorems 3.1.8 and 3.1.10 will be proven in Sections 3.4 and 3.5.

3.2 Auxiliary results on multi-balls over the value group

In this section $(K, \Gamma_K; \mathcal{L}_2)$ will be a general P-minimal structure. The main result of this section is Proposition 3.2.11, which holds under the assumption of relative P-minimality.

Definition 3.2.1. A P-minimal structure $(K, \Gamma_K; \mathcal{L}_2)$ is called *relative P-minimal* if for all $n \geqslant 0$, every $\mathcal{L}_2$-definable subset of $K \times \Gamma_K^n$ is definable in $\mathcal{L}_{\text{ring},2}$.

By definition any relative P-minimal structure is P-minimal. The converse is true under the assumption of the extreme value property.

Definition 3.2.2. A structure $(K, \Gamma_K; \mathcal{L}_2)$ has the *extreme value property*, if for every closed and bounded definable subset $U \subseteq K$ and every definable continuous function $f\colon U \to \Gamma_K$, $f(U)$ admits a maximal value.

The following is a reformulation of Theorem 4.1 from Darnière and Halupczok.

Theorem 3.2.3 ([DH17])**.** *Assume that $(K, \Gamma_K; \mathcal{L}_2)$ is P-minimal and satisfies the extreme value property. Then $(K, \Gamma_K; \mathcal{L}_2)$ is relative P-minimal.*

Remark 3.2.4. Note that every P-minimal expansion of a p-adic field satisfies the extreme value property. This follows from the fact that in a p-adic field a closed and bounded set U is contained in a compact ball and hence U itself is compact. Therefore the image of U under a continuous function is a compact subset of $\mathbb{Z}$, which admits a maximal value. Therefore, all the results proven in this section for relative P-minimal structures, will hold in particular for P-minimal expansions of p-adic fields.

Moreover, one can easily prove that any relative P-minimal structure has the extreme value property. Since this is a property that is preserved under elementary equivalence, one can see that if a structure $(K, \Gamma_K; \mathcal{L}_2)$ is elementary equivalent to a relative P-minimal structure, then the structure $(K, \Gamma_K; \mathcal{L}_2)$ is also relative P-minimal. This allows us to use logical compactness arguments.

3.2.1 A finiteness result

The purpose of this subsection is to show the following theorem.

Theorem 3.2.5. *Let $(K, \Gamma_K; \mathcal{L}_2)$ be a relative P-minimal structure and let $A \subseteq S \times \Gamma_K \times K$ be a definable multi-ball of order k with fibers of the form*

$$A_{s,\gamma} = \text{union of } k \text{ disjoint balls in } \mathbb{B}_1.$$

Then there exists a uniform bound $N \in \mathbb{N}$, such that for every $s \in S$,

$$\#\{B \in \mathbb{B}_1 \mid \exists \gamma \in \Gamma_K : B \subseteq A_{s,\gamma}\} < N.$$

Remark 3.2.6. This theorem holds also for multi-balls for which the k balls in each fiber are in $\mathbb{B}_\eta$ for some $\eta \in \Gamma_K$.

Before we can give the proof of Theorem 3.2.5, we will need some preliminary results. The following lemma is due to Haskell and Macpherson.

Lemma 3.2.7 ([HM97], Remark 3.4)**.** *Let $(K, \Gamma_K; \mathcal{L}_2)$ be a P-minimal structure and let $g\colon D \subseteq K \to \Gamma_K$ be a definable function. Then there exists a finite set D' such that g is locally constant on $D \setminus D'$.*

Lemma 3.2.8. *Let $(K, \Gamma_K; \mathcal{L}_2)$ be a P-minimal structure and let $f\colon \Gamma_K \to K$ be a definable function. Then f has finite image.*

Proof. We apply Γ-cell decomposition (Theorem 1.1.9) to the inverted graph of f, that is, to the set

$$\{(x, \gamma) \in K \times \Gamma_K \mid f(\gamma) = x\}.$$

It is sufficient to show that on each Γ-cell C of such a decomposition, the projection onto the first coordinate is finite. Let C be a given Γ-cell of the decomposition,

$$C := \{(x, \gamma) \in D \times \Gamma_K \mid \alpha(x) \mathrel{\square_1} \gamma \mathrel{\square_2} \beta(x) \wedge \gamma \equiv k \bmod n\},$$

where D is a definable subset of K, α, β are definable functions $D \to \Gamma_K$ and $k, n \in \mathbb{N}$. We may furthermore assume that D is a K-cell over $\emptyset$. From assuming that D is a 1-cell, we will derive a contradiction. This will show that D has to be a 0-cell, hence consisting of only one element. There are two cases to consider.

Case 1: Suppose that $\square_1$ (resp. $\square_2$) equals 'no condition'. Pick distinct $x, y \in D$ and $\gamma \in \Gamma_K$ such that $\gamma \equiv k \bmod n$ and $\gamma \in C_x \cup C_y$. Because of the assumption, we can do this by taking γ small enough (resp. big enough), i.e., $\gamma < \min\{\beta(x), \beta(y)\}$ (resp. $\gamma > \max\{\alpha(x), \alpha(y)\}$). This contradicts the assumption that f is a function, since it implies that γ would have two images x and y.

Case 2: Suppose that both $\square_1$ and $\square_2$ are '$<$'. By Lemma 3.2.7, there exist distinct $x, y \in D$ such that $\alpha(x) = \alpha(y)$ and $\beta(x) = \beta(y)$. As before, this contradicts the assumption that f is a function, since any γ such that $\alpha(x) < \gamma < \beta(x)$ will have both x and y as images. □

Lemma 3.2.9. *Let $(K, \Gamma_K; \mathcal{L}_2)$ be a relative P-minimal structure and let $\alpha\colon D \subseteq K \to \Gamma_K$ be a definable function. Then there exists a finite set D' and constants $c_1, \ldots, c_l \in K$, and a partition of $D \setminus D'$ into l 1-cells $C_i^{c_i}$, such that the function α is constant on each of the leaves $C_i^{c_i,\gamma}$.*

Proof. By relative P-minimality, the inverted graph of α is an $\mathcal{L}_{\mathrm{ring},2}$-definable set, which can be partitioned as a finite union of classical cells of the form

$$C := \left\{ (\gamma, x) \in \Gamma_K \times K \;\middle|\; \begin{array}{l} a(\gamma) \mathrel{\square_{11}} \operatorname{ord}(x - c(\gamma)) \mathrel{\square_{12}} b(\gamma) \wedge \\ x - c(\gamma) \in \lambda Q_{n,m} \wedge \\ c_1 \mathrel{\square_{21}} \gamma \mathrel{\square_{22}} c_2 \;\wedge\; \gamma \equiv k \bmod n' \end{array} \right\},$$

where $c\colon \Gamma_K \to K$ is an $\mathcal{L}_{\mathrm{ring},2}$-definable function. Moreover, by Lemma 3.2.8 we know that c has finite image, hence we may as well assume that $c(\gamma)$ is in fact constant on each cell. Note that, if $\lambda = 0$ for some cell C, then C only contains a single point $(\alpha(c), c)$. We take D' to be the set consisting of these values $c \in K$.

Let us show that the projection of a cell C with constant center $c(\gamma) = c$ and $\lambda \neq 0$, onto the second variable, can be written as a finite union of cells $C_i^c \subseteq D \setminus D'$ (that is, all cells are centered at c). Let Z denote the projection of C onto the second variable, and consider the set $Y := \{\mathrm{ord}(x-c) \mid x \in Z\}$. The set Y can be partitioned into finitely many Γ-cells $Y_1, \ldots, Y_r$. The reader can check that for $i \in \{1, \ldots, r\}$, the sets

$$C_i^c := \{x \in K \mid \mathrm{ord}(x-c) \in Y_i \ \wedge \ x - c \in \lambda Q_{n,m}\}$$

form a cell decomposition of Z with cells centered at c.

Doing this for all cells for which $\lambda \neq 0$, gives a partition of $D \setminus D'$. If $x \in C_i^c$, then the value of $\alpha(x)$ equals the unique γ for which $a(\gamma) \square_{11} \mathrm{ord}(x-c) \square_{12} b(\gamma)$, $c_1 \square_{21} \gamma \square_{22} c_2$ and $\gamma \equiv k \bmod n'$. Hence, $\alpha(x)$ is constant on leaves of C_i^c. $\square$

Lemma 3.2.10. *Let $(K, \Gamma_K; \mathcal{L}_2)$ be a relative P-minimal structure and let $\alpha_1, \alpha_2 \colon D \subseteq K \to \Gamma_K$ be two definable functions. Then there exists a finite set D', finitely many constants $c_1, \ldots, c_l \in K$ and a partition of $D \setminus D'$ into l 1-cells $C_j^{c_j}$, such that on each cell, both α_1 and α_2 have constant value on each of the leaves $C_j^{c_j,\gamma}$.*

Proof. Applying Lemma 3.2.9 yields two partitions of D:

$$D = D_1' \cup \bigcup_{i=1}^{l} D_i^{d_i} = D_2' \cup \bigcup_{\iota=1}^{l'} E_\iota^{e_\iota},$$

such that the D_j' are finite sets, and $D_i^{d_i}$, resp. $E_\iota^{e_\iota}$ are 1-cells such that α_1, resp. α_2 have constant value on leaves of $D_i^{d_i}$, resp. $E_\iota^{e_\iota}$.

A refinement of both partitions can be found by considering intersections $D_i^{d_i} \cap E_\iota^{e_\iota}$, $D_1' \cap E_\iota^{e_\iota}$ and $D_2' \cap D_i^{d_i}$. The intersection of a point and a cell can either be empty or a point. The intersection of two 1-cells $D_i^{d_i} \cap E_\iota^{e_\iota}$ is either empty, or a definable set that can once again be partitioned as a finite union of points and 1-cells $C_j^{c_j}$. In order to finish the proof, we need to check that, for any $\gamma_0 \in \Gamma_K$, there exist $\gamma, \gamma' \in \Gamma_K$ such that

$$C_j^{c_j,\gamma_0} \subseteq D_i^{d_i,\gamma} \cap E_\iota^{e_\iota,\gamma'}. \tag{3.2.1}$$

Indeed, if this holds then α_1 and α_2 will have constant value on the leaves of $C_j^{c_j}$ as required.

We will show that there exists $\gamma \in \Gamma_K$, such that $C_j^{c_j,\gamma_0} \subseteq D_i^{d_i,\gamma}$. The same argument will allow us to find $\gamma' \in \Gamma_K$ such that $C_j^{c_j,\gamma_0} \subseteq E_\iota^{e_\iota,\gamma'}$. These two statements together imply (3.2.1).

Since $C_j^{c_j,\gamma_0} \subseteq D_i^{d_i}$, we know that there must exist at least one leaf of $D_i^{d_i}$ that has nonempty intersection with $C_j^{c_j,\gamma_0}$. Let $L := \{\rho \in \Gamma_K \mid D_i^{d_i,\rho} \cap C_j^{c_j,\gamma_0} \neq \emptyset\}$ be the set listing the heights of such leaves. We need to check that L cannot contain more than one element. Note that, if L has more than one element, then $C_j^{c_j,\gamma_0}$ contains elements from at least two different leaves of $D_i^{d_i}$, hence $C_j^{c_j,\gamma_0}$ also contains the smallest ball that contains these elements. Such a ball will always contain the center d_i, hence $d_i \in C_j^{c_j,\gamma_0}$, but $d_i \notin D_i^{d_i}$. This contradicts $C_j^{c_j,\gamma_0} \subseteq D_i^{d_i}$, and therefore we can conclude that L can only have a single element γ, which implies that $C_j^{c_j,\gamma_0} \subseteq D_i^{d_i,\gamma}$. $\square$

Proof of Theorem 3.2.5. By logical compactness, it suffices to show that for every $s \in S$, there exists $N_s \in \mathbb{N}$, and balls $B_1, \ldots, B_{N_s}$ from $\mathbb{B}_1$, such that

$$\bigcup_{\gamma} A_{s,\gamma} = B_1 \cup \ldots \cup B_{N_s}.$$

Fix $s \in S$, and consider the fiber $A_s \subseteq \Gamma_K \times K$. Reversing the order of the variables and applying Γ-cell decomposition, this set can be partitioned as a finite union of cells of the form

$$C := \{(x, \gamma) \in D \times \Gamma_K \mid \alpha_1(x) \,\square_{11}\, \gamma \,\square_{12}\, \alpha_2(x) \ \wedge\ \gamma \equiv \kappa \bmod n'\},$$

where D is a semi-algebraic cell of the form

$$D := \{x \in K \mid \gamma_1 \,\square_{21}\, \mathrm{ord}(x - c) \,\square_{22}\, \gamma_2 \ \wedge\ x - c \in \lambda Q_{n,m}\},$$

and $\gamma_i \in \Gamma_K$, $\lambda, c \in K$, $m, n, n', \kappa \in \mathbb{N}$ and the $\alpha_i \colon D \to \Gamma_K$ are definable functions. By Lemma 3.2.10, there is a finite set D', finitely many constants $c_1, \ldots, c_l \in K$ and a partition of $D \setminus D'$ into l 1-cells $C_j^{c_j}$, such that on each cell, both α_1 and α_2 have constant value on each leaf $C_j^{c_j,\gamma}$.

Choose r such that $k < q_K^r$. Fix one of the cells $C_j^{c_j}$, and consider a leaf $C_j^{c_j,\gamma_0}$ for some $\gamma_0 < 1 - m_j - r$, where m_j is as in the set Q_{n_j,m_j}, appearing in the cell condition C_j. Note that this leaf is the union of at least q_K^r disjoint balls from $\mathbb{B}_1$. Take some $x \in C_j^{c_j,\gamma_0}$, and choose γ such that

$$\alpha_1(x) \,\square_{11}\, \gamma \,\square_{12}\, \alpha_2(x) \ \wedge\ \gamma \equiv \kappa \bmod n'.$$

Then $x \in A_{s,\gamma}$. Lemma 3.2.10 implies that $\alpha_i(x) = \alpha_i(x')$ for any other $x' \in C_j^{c_j,\gamma_0}$, and hence $C_j^{c_j,\gamma_0} \subseteq A_{s,\gamma}$. This means that $A_{s,\gamma}$ must contain at least $q_K^r > k$ balls from $\mathbb{B}_1$, which contradicts our assumption that $A_{s,\gamma}$ consists of k balls.

The only way this contradiction can be avoided is if the cells $C_j^{c_j}$ have no leaves $C_j^{c_j,\gamma_0}$ for which $\gamma_0 < 1 - m_j - r$. This in turn implies that D can only intersect a finite number of disjoint balls from $\mathbb{B}_1$, since for $\gamma \geqslant 1$, all leaves $C_j^{c_j,\gamma}$ of the cell $C_j^{c_j}$ are contained within a single ball of $\mathbb{B}_1$. Hence, the theorem follows. $\square$

3.2.2 Multi-balls over the value group

In this section we show that in relative P-minimal structures, definable multi-balls on $\mathbb{B}_1$ over definable sets of the form $S \times \Gamma_K$ can be partitioned into finitely many definable sets which are multi-balls over S.

Proposition 3.2.11. *Let $(K, \Gamma_K; \mathcal{L}_2)$ be a relative P-minimal structure and let $X \subseteq S \times \Gamma_K$ be a definable set and $A \subseteq X \times K$ a definable multi-ball of order k on $\mathbb{B}_1$. Then there is a finite set W and a definable partition $X = \bigcup_{w \in W} X_w$ with $S_w := \pi_S(X_w)$, such that for every $s \in S_w$, A_s has constant fibers over $(X_w)_s$ (i.e., $A_{s,\gamma_1} = A_{s,\gamma_2}$ for all $\gamma_1, \gamma_2 \in (X_w)_s$).*

Proof. By Theorem 3.2.5, there is an integer N_A such that for every s, there exists $N_s < N_A$, and balls $\{B_{1,s}, \ldots, B_{N_s,s}\} =: \mathcal{B}_s$ from $\mathbb{B}_1$ (depending on s!), such that

$$\bigcup_{\gamma \in X_s} A_{s,\gamma} = B_{1,s} \cup \ldots \cup B_{N_s,s}.$$

By partitioning S into finitely many definable pieces, without loss of generality we may assume that the cardinality of $\mathcal{B}_s$ is constant and equal to N for all $s \in S$. For every $s \in S$, there are $\binom{N}{k}$ possible values for the fiber $A_{s,\gamma}$.

Recall that a definably well-ordering $\lhd$ on Γ_K is a linear ordering satisfying that every definable subset $Y \subseteq \Gamma_K$ has a $\lhd$-minimal element. Let $\lhd$ be the definably well-ordering on Γ_K defined by

$$x \lhd y \Leftrightarrow |x| < |y| \,\vee\, x = y \,\vee\, (-x = y \wedge y \geqslant 0).$$

To see that $\lhd$ is a definably well-ordering on Γ_K, note first that on $\mathbb{Z}$, $\lhd$ defines the well-ordering

$$0 \lhd -1 \lhd 1 \lhd -2 \lhd 2 \lhd \cdots ,$$

so, in particular, every $\mathcal{L}_{\mathrm{Pres}}$-definable subset of $\mathbb{Z}$ has a $\lhd$-minimal element. Since $(\Gamma_K, \mathcal{L}_{\mathrm{Pres}}) \equiv (\mathbb{Z}, \mathcal{L}_{\mathrm{Pres}})$, the fact that every definable subset of Γ_K is $\mathcal{L}_{\mathrm{Pres}}$-definable implies the desired property.

Let $\delta_1 \colon S \to \Gamma_K$ be the definable function sending s to $\min_{\lhd} X_s$, the minimal element with respect to the ordering $\lhd$. Setting $Z_1 := X$, we inductively define sets $Z_{i+1} \subseteq Z_i$ and functions $\delta_{i+1} \colon S \to \Gamma_K \cup \{\infty\}$ for $i \geqslant 1$ as follows:

$$Z_{i+1} := \{(s,\gamma) \in Z_i \mid A_{s,\gamma} \neq A_{s,\delta_i(s)}\};$$

$$\delta_{i+1}(s) := \begin{cases} \min_{\lhd}(Z_{i+1})_s & \text{if } (Z_{i+1})_s \neq \emptyset; \\ \infty & \text{otherwise.} \end{cases}$$

The idea is to order the different configurations of $\mathcal{B}_s$ that appear as fibers $A_{s,\gamma}$ with respect to their minimal representatives in X_s. Doing this uniformly in S and grouping the elements defining the same fiber is the idea behind the sets Z_i. Note that

$$X = Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_{\binom{N}{k}} \supseteq Z_{\binom{N}{k}+1} = \emptyset.$$

Moreover, for $s \in S$, if $\delta_i(s) \neq \infty$, the same holds for all $i' < i$. Therefore, the sets $S_1, \ldots, S_{\binom{N}{k}}$ with

$$S_i := \{s \in S \mid \delta_i(s) \neq \infty \,\wedge\, \delta_{i+1}(s) = \infty\},$$

form a definable partition of S, which induces a partition of X as well (some of the S_i might be empty). By restricting to $S = S_m$ for some $m \in \left\{1, \ldots, \binom{N}{k}\right\}$, we find that $A_{s,\delta_1(s)}, \ldots, A_{s,\delta_m(s)}$ are all the multi-balls that appear as fibers of A_s over X_s. To conclude the proof we use the functions $\delta_1, \ldots, \delta_m$ to partition X as follows. For $1 \leqslant i \leqslant m$ let

$$X_i := \{(s,\gamma) \in X \mid A_{s,\gamma} = A_{s,\delta_i(s)}\}.$$

We clearly have that the sets $\{X_1, \ldots, X_m\}$ form a partition of X. Moreover, by construction, for every $s \in S$ the fibers of A_s over $(X_i)_s$ are constant and equal to $A_{s,\delta_i(s)}$. □

3.3 Auxiliary results on the form of the elements of $\mathcal{C}^*_{\exp}(X)$

For the rest of this chapter K will be a p-adic field. In the following lemma we state an elementary yet important result on integration of the character ψ over balls of valuation radius at most 0.

Lemma 3.3.1. *Let $a \in K$ and $\gamma \in \mathbb{Z}$ with $\gamma \leqslant 0$, then*

$$\int_{B_\gamma(a)} \psi(x)|dx| = 0.$$

Proof. Since $\psi_{|\mathcal{O}_K} \neq 1$, there exists $g \in \mathcal{O}_K$ such that $\psi(g) \neq 1$. The ball $B_\gamma(a)$ is a disjoint union of $q_K^{-\gamma+1}$ balls from $\mathbb{B}_1$ and it is easy to see that the map $x \mapsto x + g$ permutes these balls. So if we take R to be a set of representatives from each of those balls, then

$$B_\gamma(a) = \bigcup_{b\in R} B_1(b) = \bigcup_{b\in R} B_1(b+g).$$

The character ψ is constant on each of the balls $B_1(b)$, which have volume q_K^{-1}, so

$$\int_{B_\gamma(a)} \psi(x)|dx| = q_K^{-1}\sum_{b\in R}\psi(b) = q_K^{-1}\sum_{b\in R}\psi(b+g) = q_K^{-1}\psi(g)\sum_{b\in R}\psi(b). \quad (3.3.1)$$

Since $\psi(g) \neq 1$, (3.3.1) can only hold if $\sum_{b\in R}\psi(b) = 0$, which implies that $\int_{B_\gamma(a)} \psi(x)|dx| = 0$. $\square$

Recall that in the Definition 3.1.6, the character ψ is summed over multi-balls that consist purely of *maximal* balls from $\mathbb{B}_1$. The above lemma explains why it makes sense to impose that restriction.

The following lemma gives a useful description of the form of exponential*-constructible functions.

Lemma 3.3.2. *Let X be a definable set. Then $\mathcal{C}^*_{\exp}(X) = W(X)$, where*

$$W(X) := \left\{ \sum_{i\in I} h_i(x) \sum_{B\sqsubseteq A^i_x} \psi(B) \;\middle|\; \begin{array}{l} I \text{ finite set, } h_i \in \mathcal{C}(X), A^i \subseteq X\times K \\ \text{definable multi-ball of order } k_i \text{ on } \mathbb{B}_1 \end{array} \right\}.$$

Proof. The inclusion $W(X) \subseteq \mathcal{C}^*_{\exp}(X)$ is clear from the definition of $\mathcal{C}^*_{\exp}(X)$. Furthermore, $W(X)$ contains the generators of $\mathcal{C}^*_{\exp}(X)$ and is closed under addition and scalar multiplication by elements of $\mathbb{Q}$. Hence, it remains to show that $W(X)$ is closed under multiplication. Now consider

$$\Big(\sum_{i\in I} h_i(x) \sum_{B\sqsubseteq A^i_x} \psi(B)\Big)\cdot\Big(\sum_{j\in J}\tilde{h}_j(x)\sum_{\tilde{B}\sqsubseteq\tilde{A}^j_x}\psi(\tilde{B})\Big) =$$

$$\sum_{(i,j)\in I\times J} h_i(x)\tilde{h}_j(x)\Big(\sum_{\substack{B\sqsubseteq A^i_x,\\ \tilde{B}\sqsubseteq\tilde{A}^j_x}}\psi(B+\tilde{B})\Big).$$

The reader can check that $B+\tilde{B}$ is again a ball in $\mathbb{B}_1$. For each $i \in I$, $j \in J$ and $r \geqslant 1$, there exist definable sets

$$D^{(i,j,\geqslant r)} := \left\{ (x,b) \in X \times K \;\middle|\; \begin{array}{l} \exists b_1,\ldots,b_r \in A_x^i \;\exists \tilde{b}_1,\ldots,\tilde{b}_r \in \tilde{A}_x^j : \\ \bigwedge_{k,l=1}^{r} \big(\mathrm{ord}(b_k-b_l) \leqslant 0 \wedge \mathrm{ord}(\tilde{b}_k-\tilde{b}_l) \leqslant 0\big) \\ \wedge \;\bigwedge_{k=1}^{r} (b = b_k+\tilde{b}_k) \end{array} \right\};$$

$$D^{(i,j,r)} := D^{(i,j,\geqslant r)} \backslash D^{(i,j,\geqslant r+1)};$$

$$E^{(i,j,r)} := \Big\{ (x,b) \in D^{(i,j,r)} \mid \exists b' \in K : \mathrm{ord}(b-b') \geqslant 0 \,\wedge\, b' \notin D_x^{(i,j,r)} \Big\}.$$

Each fiber $D_x^{(i,j,r)}$ consists of the balls from $\mathbb{B}_1$ that can be written in exactly r ways as the sum of a ball $B \sqsubseteq A_x^i$ and a ball $\tilde{B} \sqsubseteq \tilde{A}_x^j$. Now $E_x^{(i,j,r)}$ contains only those balls in $D_x^{(i,j,r)}$, that are maximal in $D_x^{(i,j,r)}$. Remark that for $r > k_{(i,j)} := \min\{k_i, \tilde{k}_j\}$ (where k_i is the order of A^i and $\tilde{k}_j$ is the order of $\tilde{A}^j$), $D_x^{(i,j,r)}$ and $E_x^{(i,j,r)}$ are empty, so we can write

$$\sum_{(i,j)\in I\times J} h_i(x)\tilde{h}_j(x)\Big(\sum_{\substack{B\sqsubseteq A_x^i,\\ \tilde{B}\sqsubseteq \tilde{A}_x^j}} \psi(B+\tilde{B})\Big) =$$

$$\sum_{(i,j)\in I\times J} \sum_{r=1}^{k_{(i,j)}} r h_i(x)\tilde{h}_j(x)\Big(\sum_{\substack{B\subseteq D_x^{(i,j,r)},\\ B\in\mathbb{B}_1}} \psi(B)\Big) =$$

$$\sum_{(i,j)\in I\times J} \sum_{r=1}^{k_{(i,j)}} r h_i(x)\tilde{h}_j(x)\Big(\sum_{B\sqsubseteq E_x^{(i,j,r)}} \psi(B)\Big).$$

Here we have used the fact that $\int_{D_x^{(i,j,r)}\backslash E_x^{(i,j,r)}} \psi(b)|db| = 0$ which is a consequence of Lemma 3.3.1.

Unfortunately, the set $E^{(i,j,r)}$ is not necessarily a multi-ball, because different fibers might contain a different number of maximal balls. However, we do know that for $1 \leqslant r \leqslant k_{(i,j)}$, each fiber $E_x^{(i,j,r)}$ contains at most $k_i\tilde{k}_j$ maximal balls from $\mathbb{B}_1$. Hence we can partition the set X into definable sets $X_t^{(i,j,r)}$, for $1 \leqslant t \leqslant k_i\tilde{k}_j$, such that $E_x^{(i,j,r)}$ contains exactly t maximal balls, for each $x \in X_t^{(i,j,r)}$. Remark that for each $E^{(i,j,r)}$ we might have to consider a different partition of X.

Now, for each $1 \leqslant t \leqslant k_i\tilde{k}_j$, we fix a set $H_t \subseteq K$ of t maximal balls from $\mathbb{B}_1$. Then we define (with parameters) a subset of $X \times K$, of which each fiber consists of t maximal balls from $\mathbb{B}_1$:

$$E^{(i,j,r,t)} := \{(x,b) \in E^{(i,j,r)} \mid x \in X_t^{(i,j,r)}\} \cup (X \setminus X_t^{(i,j,r)}) \times H_t.$$

Then we find

$$\Big(\sum_{i\in I} h_i(x) \sum_{B\sqsubseteq A_x^i} \psi(B)\Big) \cdot \Big(\sum_{j\in J} \tilde{h}_j(x) \sum_{\tilde{B}\sqsubseteq \tilde{A}_x^j} \psi(\tilde{B})\Big) =$$

$$\sum_{(i,j)\in I\times J} \sum_{r=1}^{k_{(i,j)}} \sum_{t=1}^{k_i\tilde{k}_j} r h_i(x)\tilde{h}_j(x)\mathbb{1}_{X_t^{(i,j,r)}}(x) \sum_{B\sqsubseteq E_x^{(i,j,r,t)}} \psi(B),$$

where $\mathbb{1}_{X_t^{(i,j,r)}} : X \to \mathbb{Z}$ denotes the characteristic function of the set $X_t^{(i,j,r)}$. The last expression is clearly an element of $W(X)$, since $rh_i\tilde{h}_j\mathbb{1}_{X_t^{(i,j,r)}} \in \mathcal{C}(X)$ and $E^{(i,j,r,t)}$ is a definable multi-ball of order t on $\mathbb{B}_1$. □

Remark 3.3.3. There are of course several ways of writing a function $f \in \mathcal{C}^*_{\exp}(X)$ as an element of $W(X)$. The main difficulty in proving Conjecture 3.1.11 lies in showing that for an integrable function f, at least one of those ways uses only constructible functions h_i that are *integrable* themselves.

3.4 Integration of $\mathbb{Z}$-variables

Let us now prove Theorem 3.1.8, which we restate here for the reader's convenience.

Theorem 3.4.1. *Let $X \subseteq S \times \mathbb{Z}^n$ be a definable set. Let $f \in \mathcal{C}^*_{\exp}(X)$ be such that* $\mathrm{Int}(f, \mathbb{Z}^n) = S$. *Then there exists $g \in \mathcal{C}^*_{\exp}(S)$ such that for all $s \in S$,*

$$g(s) = \int_{X_s} f(s,\gamma)|d\gamma|.$$

Proof. First of all, note that by Fubini it suffices to show the result for $n = 1$. By Lemma 3.3.2 and Definition 3.1.1 (i) on the generators of $\mathcal{C}(X)$ as a $\mathbb{Q}$-algebra, we may suppose that f has the following form

$$f(s,\gamma) = \sum_{i=1}^{m} c_i q_K^{\alpha_i(s,\gamma)} \prod_{k=1}^{r_i} \beta_{ik}(s,\gamma) \sum_{B\sqsubseteq A^i_{s,\gamma}} \psi(B), \tag{3.4.1}$$

where the c_i are non-zero rational constants, α_i, β_{ik} are definable functions from X to $\mathbb{Z}$ and A^i are definable multi-balls on $\mathbb{B}_1$ over X. Let I denote the set $I := \{1,\ldots,m\}$.

By iterating Proposition 3.2.11, there is a finite definable partition of X into sets $\{X_w\}_{w\in W}$ such that for each $s \in S_w := \pi_S(X_w)$, and for each $i \in I$, the multi-ball A^i_s has constant fibers over $(X_w)_s$. Without loss of generality, suppose from now on that X is one such piece X_w. Therefore, for all $s \in S$ the function

$$e_i \colon (s,\gamma) \mapsto \sum_{B \sqsubseteq A^i_{s,\gamma}} \psi(B)$$

does not depend on γ. Note that this function is an element of $\mathcal{C}^*_{\exp}(S)$. Indeed, the set $E^i \subseteq S \times K$, defined by having fibers $E^i_s := \bigcup_{\gamma\in X_s} A^i_{s,\gamma}$, is a multi-ball on $\mathbb{B}_1$ over S, which shows that the function $e_i \colon s \mapsto \sum_{B\sqsubseteq E^i_s} \psi(B)$ is in $\mathcal{C}^*_{\exp}(S)$. By multiplying e_i by the constant c_i, we may omit such constants and rewrite equation (3.4.1) as

$$f(s,\gamma) = \sum_{i\in I} e_i(s) q_K^{\alpha_i(s,\gamma)} \prod_{k=1}^{r_i} \beta_{ik}(s,\gamma). \tag{3.4.2}$$

By Theorem 1.1.9 we may further suppose that X is a Γ-cell of the form

$$X = \{(s,\gamma) \in S \times \mathbb{Z} \mid \theta_1(s) \square_1 \gamma \square_2 \theta_2(s) \wedge \gamma \equiv l \bmod M\}, \tag{3.4.3}$$

for θ_1, θ_2 definable functions from S to $\mathbb{Z}$ and $l, M \in \mathbb{N}$ with $M > 0$, and that for all $s \in S$, the functions $\alpha_i(s,\cdot)$ and $\beta_{ik}(s,\cdot)$ are linear in $\frac{\gamma-l}{M}$. From now on, we will denote $\frac{\gamma-l}{M}$ by ζ. Writing products of linear terms as polynomials in ζ, we have that, for each $i \in I$,

$$q_K^{\alpha_i(s,\gamma)} \prod_{k=1}^{r_i} \beta_{ik}(s,\gamma) = q_K^{a_i\zeta+\delta_i(s)} \sum_{k=0}^{r_i} d_{ik}(s)\zeta^k, \tag{3.4.4}$$

where $a_i \in \mathbb{Z}$, δ_i is a definable function from S to $\mathbb{Z}$ and $d_{ik} \in \mathcal{C}(S)$. Since $q_K^{\delta_i(s)} \in \mathcal{C}^*_{\exp}(S)$, we may assume that $\delta_i(s) = 0$ by merging this factor into the functions $e_i(s)$. Therefore we have that

$$f(s,\gamma) = \sum_{i\in I} e_i(s) q_K^{a_i\zeta} \sum_{k=0}^{r_i} d_{ik}(s)\zeta^k. \tag{3.4.5}$$

After merging terms with the same factor $q_K^{a_i\zeta}$, we may suppose that $a_i \neq a_j$ for all $i, j \in I$ such that $i \neq j$. Set

$$h_i(s,\gamma) := q_K^{a_i\zeta} \left(\sum_{k=0}^{r_i} d_{ik}(s)\zeta^k \right).$$

Claim 3.4.2. *If for each $s \in S$ and $i \in I$ the functions $h_i(s,\cdot)$ are integrable over X_s, then there is $g \in \mathcal{C}^*_{\exp}(S)$ such that $g(s) = \int_{X_s} f(s,\gamma)|d\gamma|$.*

Note that for each $i \in I$, h_i is $\mathcal{L}_2$-constructible. Therefore, by Theorem 1.2.3, letting $g_i(s) = \int_{X_s} h_i(s,\gamma)|d\gamma|$ we have that

$$\int_{X_s} f(s,\gamma)|d\gamma| = \int_{X_s} \left(\sum_{i\in I} e_i(s) h_i(s,\gamma) \right) |d\gamma|$$
$$= \sum_{i\in I} e_i(s) \int_{X_s} h_i(s,\gamma)|d\gamma| = \sum_{i\in I} e_i(s) g_i(s),$$

which is a function in $\mathcal{C}^*_{\exp}(S)$. This completes the proof of the claim.

We will finish the argument by splitting in cases depending on the possible values of $\square_1$ and $\square_2$.

Case 1: Suppose that $\square_1 = \square_2 =$ '$<$'. In this case the set X_s is finite for each $s \in S$, hence the functions $h_i(s,\cdot)$ are integrable over X_s. The result follows now by Claim 3.4.2.

Case 2: Suppose that $\square_1 =$ '$<$' and $\square_2 =$ 'no condition'. Let $j \in I$ be such that $a_j = \max_{i\in I} a_i$. In this case, if $a_j \geqslant 0$, then $e_j(s)\sum_{k=1}^{r_j} d_{jk}(s)\zeta^k = 0$ for all $(s,\gamma) \in X$, since $f(s,\cdot)$ must be integrable over X_s for each $s \in S$. If this holds, then

$$f(s,\gamma) = \sum_{i\in I, i\neq j} e_i(s) q_K^{a_i\zeta} \left(\sum_{k=0}^{r_i} d_{ik}(s)\zeta^k \right).$$

By induction on $\#I$, either f is identically 0 or we may suppose that $a_j < 0$. In the first case, the theorem clearly holds. In the second case, we have that each function $h_i(s,\cdot)$ is integrable over X_s and we conclude again by Claim 3.4.2. A similar argument handles the case $\square_2 =$ '$<$' and $\square_1 =$ 'no condition'. Finally, the case $\square_1 = \square_2 =$ 'no condition' also reduces to this case by partitioning X into X_1 and X_2 where

$$X_1 := \{(s,\gamma) \in X \mid 0 \leqslant \gamma\} \quad \text{and} \quad X_2 := \{(s,\gamma) \in X \mid \gamma < 0\}. \qquad \square$$

3.5 Integration of K-variables

This section is dedicated to the proof of Theorem 3.1.10. In the first subsection we deal with a special instance of the theorem.

3.5.1 Integrating characters over a definable subset of K

The goal of this section is to show that functions of the form $s \mapsto \int_{X_s} \psi(x)|dx|$, where X is a definable set, are always exponential*-constructible. Note that this constitutes a generalization of our observations in Example 3.1.4.

Proposition 3.5.1. *Let $X \subseteq S \times K$ be a definable set with bounded fibers X_s, i.e., for each $s \in S$, there exists $M \in \mathbb{Z}$ such that $\mathrm{ord}(x) \geqslant M$ for all $x \in X_s$. Then the function*

$$s \mapsto \int_{X_s} \psi(x)|dx| \tag{3.5.1}$$

is an element of $\mathcal{C}^*_{\exp}(S)$.

For the proof of this proposition we will use the clustered cell decomposition from Theorem 2.7.1. This theorem states that one can partition the set X into a finite union of classical cells and regular clustered cells. Recall that a regular clustered cell C^Σ of order k can be written as a disjoint union $C^{\sigma_1} \cup \ldots \cup C^{\sigma_k}$ of k sets C^{σ_i} that can (non-definably) be described as

$$C^{\sigma_i} = \{(s,t) \in S \times K \mid \alpha(s) < \mathrm{ord}(t - \sigma_i(s)) < \beta(s) \wedge t - \sigma_i(s) \in \lambda Q_{n,m}\}.$$

Let us take a moment to explore the case where $X = C^\Sigma$. By Lemma 3.3.1 we may assume that the leaves of any fiber X_s are balls of valuation radius at least 1, since leaves with smaller valuation radius will contribute nothing to the integral $\int_{X_s} \psi(x)|dx|$. Hence, we may assume that $\alpha(s) \geqslant -m$, for all $s \in S$. Note that this assumption will not affect the tree structure $T(\Sigma_s)$ of the fibers of C^Σ. Furthermore, it implies that there exists a uniform bound on the number of balls in $\mathbb{B}_1$ that have non-empty intersection with X_s. For fixed s, the union of all these balls is equal to the corresponding fiber of the definable set

$$\mathcal{B} := \{(s,x) \in S \times K \mid \exists y \in X_s : \mathrm{ord}(x-y) \geqslant 1\}.$$

In the proof of Proposition 3.5.1 we will partition this set into a finite number of definable multi-balls $\mathcal{B}^1, \ldots, \mathcal{B}^{i_0}$ of orders $k_1, \ldots, k_{i_0}$ on $\mathbb{B}_1$, such that

$$\int_{X_s} \psi(x)|dx| = \sum_{j=1}^{i_0} g_j(s) \sum_{B \sqsubseteq \mathcal{B}^j_s} \psi(B), \tag{3.5.2}$$

for certain $g_1, \ldots, g_{i_0} \in \mathcal{C}(S)$. These constructible functions g_i will denote the volumes of the fibers of certain definable subsets of $S \times K$.

Proof of Proposition 3.5.1. We will first consider the case where X is a large, regular clustered cell C^Σ of order k, using the notation from the previous discussion. Recall that for such cells, all of the branching heights of Σ_s occur below $\alpha(s)$, which follows from the definition of regularity. As mentioned before, we may also assume that $\alpha(s) \geqslant -m$.

For each ball $B \subseteq \mathcal{B}_s$ from $\mathbb{B}_1$ we want to analyse the set $B \cap X_s$ and its volume. In most cases this set will consist of exactly one leaf from one of the sets $C^{\sigma_i(s)}$, but in some cases several leaves (possibly from different sets $C^{\sigma_{i'}(s)}$) could be contained within the ball B. We will have to distinguish between these two cases. Let σ_i be a (non-definable) section of Σ and $C^{\sigma_i(s),\gamma}$ the leaf of $C^{\sigma_i(s)}$ at height γ. This leaf has volume $q_K^{-(\gamma+m)}$, which is at most q_K^{-1}, since $-(\gamma+m) \leqslant -\gamma+\alpha(s) \leqslant -1$. Depending on the height γ, two cases may occur.

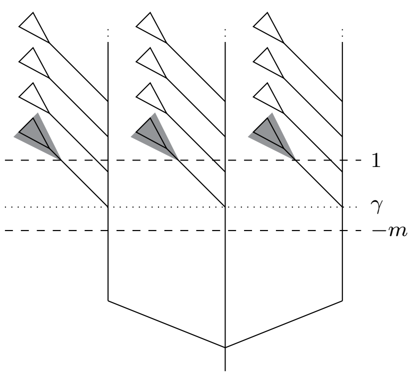


Figure 3.1: leaves of type (1)

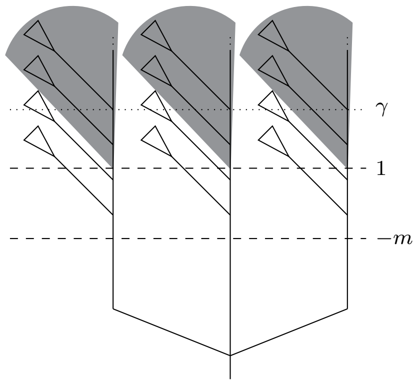


Figure 3.2: leaves of type (2)

(1) If $\alpha(s) < \gamma \leqslant 0$, then the unique ball $B \subseteq \mathcal{B}_s$ from $\mathbb{B}_1$ which contains $C^{\sigma_i(s),\gamma}$, contains no other leaves of $C^{\sigma_i(s)}$. Since all the branching heights of Σ_s occur below $\alpha(s)$, hence below 0, B does not intersect any of the other $k-1$ sets $C^{\sigma_{i'}(s)}$. This is the situation depicted in Figure 3.1.

(2) If $\gamma > 0$, then $C^{\sigma_i(s),\gamma}$ is contained in the ball $B_1(\sigma_i(s)) \subseteq \mathcal{B}_s$ from $\mathbb{B}_1$. For any other (non-definable) section ζ for which $\sigma_i(s)$ and $\zeta(s)$ are (C,Σ_s)-equivalent, we know that $\mathrm{ord}(\sigma_i(s)-\zeta(s)) > \alpha(s)+m \geqslant 0$, hence $B_1(\sigma_i(s)) = B_1(\zeta(s))$. Thus $\mathcal{B}_s$ contains at most k of these balls. This is the situation depicted in Figure 3.2.

As we have already mentioned, we want to partition the set $\mathcal{B}$ in a definable way. For each γ of type (1), we will define a set consisting of all the balls that contain a leaf at height γ. The balls that contain a leaf of type (2) will be collected in an additional definable set. We will now explain how to define these sets uniformly in s. For this, note that the leaves of X_s are also the maximal balls of X_s, since all the branching heights of Σ_s occur below $\alpha(s)$.

We inductively define, for each $j \geqslant 1$, a definable set $X^{(j)}$ and a definable function $d_j \colon S \to \mathbb{Z} \cup \{\infty\}$ as follows. For each $s \in S$, the set $X_s^{(j)}$ is the set containing the leaves with the largest volume in $X_s \setminus \left(\cup_{l=1}^{j-1} X_s^{(l)}\right)$. The function d_j is such that for each $s \in S$, the volume of the leaves in $X_s^{(j)}$ equals $q_K^{-d_j(s)}$ when the set $X_s^{(j)}$ is not empty, and $d_j(s) = \infty$ otherwise. Note that one always has $d_j(s) \geqslant 1$. Furthermore, $d_j(s) \leqslant m$ if and only if the leaves in $X_s^{(j)}$ are leaves of type (1).

Now let $i_0(s)$ be the smallest positive integer for which $d_{i_0(s)}(s) > m$. This integer can depend on S, but in any case, $i_0(s)$ is uniformly bounded on S, by $m+1$. Thus there exists a finite definable partition of S, such that $i_0(s)$ is constant on each of the sets in the partition. By restricting our clustered cell to any set in this partition, we may assume that i_0 is constant on S. This means that for all $s \in S$ there are $i_0 - 1$ definable sets of leaves of type (1), $X_s^{(1)}, \ldots, X_s^{(i_0-1)}$. Note that each of these sets contains exactly k leaves, one for each equivalence class of centers in Σ_s.

For each $1 \leqslant j \leqslant i_0 - 1$, define

$$\mathcal{B}^j := \{(s,x) \in S \times K \mid \exists y \in X_s^{(j)} : \operatorname{ord}(x-y) \geqslant 1\}$$

to be the definable set whose fibers $\mathcal{B}_s^j$ contain all balls in $\mathbb{B}_1$ that have a nonempty intersection with $X_s^{(j)}$. With this definition, the sets $\mathcal{B}^j$ are disjoint subsets of $\mathcal{B}$ and each of them is a multi-ball of order k on $\mathbb{B}_1$, since all the branching heights of Σ_s occur below $\alpha(s)$.

For each $1 \leqslant j \leqslant i_0 - 1$, we now have that

$$\begin{aligned}\int_{X_s^{(j)}} \psi(x)|dx| &= \sum_{B \sqsubseteq \mathcal{B}_s^j} \psi(B) \cdot \operatorname{Vol}(B \cap X_s) \qquad (3.5.3)\\ &= q_K^{-d_j(s)} \cdot \sum_{B \sqsubseteq \mathcal{B}_s^j} \psi(B) \in \mathcal{C}^*_{\exp}(S),\end{aligned}$$

where we use the fact that $B \cap X_s$ is one leaf of X_s with volume $q_K^{-d_j(s)}$.

For all $j \geqslant i_0$ the leaves in $X_s^{(j)}$ are leaves of type (2). The union of these leaves is the definable set

$$X_s' := X_s \setminus \left(\cup_{l=1}^{i_0-1} X_s^{(l)}\right).$$

The balls from $\mathbb{B}_1$ in $\mathcal{B}_s$ that have nonempty intersection with X_s' make up the fibers of the definable set

$$\begin{aligned}\mathcal{B}^{i_0} &:= \{(s,x) \in S \times K \mid \exists y \in X_s' : \operatorname{ord}(x-y) \geqslant 1\}\\ &= \mathcal{B} \setminus \left(\cup_{l=1}^{i_0-1} \mathcal{B}^l\right).\end{aligned}$$

Note that if Σ_s has branching heights above 1, then even for certain $\sigma_i, \sigma_{i'}$ not equivalent at s, we will have $B_1(\sigma_i(s)) = B_1(\sigma_{i'}(s))$. Therefore it could happen that $\mathcal{B}_s$ contains strictly less than k of these balls. Let us denote the number of such balls in $\mathcal{B}_s^{i_0}$ by $k_0(s)$. Since this number may change with s, the set $\mathcal{B}^{i_0}$ is not necessarily a multi-ball. To make it into one, we will have to partition S, using the procedure described below.

It could happen that the balls from $\mathbb{B}_1$ in $\mathcal{B}_s^{i_0}$ are not maximal balls. This happens exactly if Σ_s has a branching height at 0 with q_K branches. Let $S' \subseteq S$ be the definable set of s for which this happens, then for each $s \in S'$,

$$\int_{X_s'} \psi(x)|dx| = \sum_{\substack{B \subseteq \mathcal{B}_s^{i_0},\\ B \in \mathbb{B}_1}} \psi(B) \cdot \operatorname{Vol}(X_s' \cap B) = 0,$$

where we have used Lemma 3.3.1 and the fact that the volume of $X_s' \cap B$ is the same for each $B \subseteq \mathcal{B}_s^{i_0}$ with $B \in \mathbb{B}_1$, by condition (b) of Theorem 2.7.1. The constant function 0 is clearly an exponential*-constructible function on S'.

From now on we may assume without loss of generality that $S' = \emptyset$. Since we have $k_0(s) \leqslant k$ for all $s \in S$, we can (definably) partition S further and reduce to the case where for all $s \in S$, $\mathcal{B}_s^{i_0}$ contains exactly k_0 maximal balls from $\mathbb{B}_1$. By condition (b) from Theorem 2.7.1, the tree associated to Σ_s is highly symmetric, and hence the sets $X_s' \cap (B_1(c))$ all have the same volume, for each $c \in \Sigma_s$. There are k_0 of these sets, hence each of them has volume $\frac{1}{k_0} \cdot \operatorname{Vol}(X_s')$, which is a constructible function on S. We can conclude that

$$\int_{X_s'} \psi(x)|dx| = \frac{1}{k_0} \cdot \operatorname{Vol}(X_s') \sum_{B \sqsubseteq \mathcal{B}_s^{i_0}} \psi(B) \in \mathcal{C}^*_{\exp}(S). \tag{3.5.4}$$

The equations (3.5.3) and (3.5.4) give us the form of (3.5.2).

Now consider the case where X is a small clustered cell. For such a cell, each fiber only has leaves at a single height, but we can no longer assure that the branching heights will necessarily occur below $\alpha(s)$. Still, the reader can check that this case can be proven similarly as the case of large cells, by partitioning S in the same way as for case (2) above. The proof for classical cells follows the same structure as the proof for large cells and will be left to the reader. This concludes the proof of this theorem. $\square$

3.5.2 The proof of Theorem 3.1.10

We are ready to prove Theorem 3.1.10, which we restate for the reader's convenience.

Theorem 3.5.2. *Let $X \subseteq S \times K^n$ be a definable set and $f \in \mathcal{C}^*_{exp}(X)$. If f can be written in n-normal form, then there exists $g \in \mathcal{C}^*_{\mathrm{exp}}(S)$ such that, for all $s \in S$,*

$$g(s) = \int_{X_s} f(s,x)|dx|.$$

Proof. Since f can be written in n-normal form, by additivity of integration we can reduce to proving the case where f is of the form

$$f(s,x) := h(s,x) \sum_{B \sqsubseteq A_{s,x}} \psi(B),$$

with $A \subseteq X \times K$ a definable multi-ball on $\mathbb{B}_1$, $\mathrm{Int}(h, K^n) = S$ and $\mathrm{Int}(f, K^n) = S$.

Put $Y := \{(s,b,x) \in S \times K^{n+1} \mid b \in A_{s,x}\}$, which can have empty fibers $Y_{s,b}$ for some $(s,b) \in S \times K$. Since $h(s,\cdot)$ is integrable over X_s for each $s \in S$, we can apply Fubini to change the order of integration. Hence

$$\begin{aligned}\int_{X_s} h(s,x) \cdot \sum_{B \sqsubseteq A_{s,x}} \psi(B)|dx| &= q_K \int_{X_s} \int_{A_{s,x}} h(s,x)\psi(b)|db||dx| \\ &= q_K \int_K \int_{Y_{s,b}} h(s,x)\psi(b)|dx||db| \\ &= q_K \int_K \psi(b)\Big(\int_{Y_{s,b}} h(s,x)|dx|\Big)|db|. \qquad (3.5.5)\end{aligned}$$

The set $Y_{s,b}$ is a definable subset of K^n, so by Theorem 1.2.3 the function

$$S \times K \to \mathbb{Q} \colon (s,b) \mapsto \int_{Y_{s,b}} h(s,x)|dx|$$

is a constructible function. This means it is composed of generators of the $\mathbb{Q}$-algebra $\mathcal{C}(S \times K)$: there exist definable functions $\alpha_i \colon S \times K \to \mathbb{Z}$, $\beta_{ij} \colon S \times K \to \mathbb{Z}$ and constants $c_i \in \mathbb{Q}$ such that, for all $s \in S$, $b \in K$,

$$\int_{Y_{s,b}} h(s,x)|dx| = \sum_{i=1}^{m} c_i q_K^{\alpha_i(s,b)} \prod_{j=1}^{r} \beta_{ij}(s,b). \qquad (3.5.6)$$

For each $s \in S$ and $b \in K$ we denote

$$\overline{\alpha}(s,b) := \big(\alpha_1(s,b), \ldots, \alpha_m(s,b)\big);$$

$$\overline{\beta}(s,b) := \big(\beta_{11}(s,b), \ldots, \beta_{mr}(s,b)\big),$$

and for each $\gamma = (\gamma_i)_i \in \mathbb{Z}^{m(r+1)}$ we denote by C_γ the rational number $\sum_{i=1}^m c_i q_K^{\gamma_i} \prod_{j=1}^r \gamma_{jm+i}$. We consider the definable set

$$D := \{(s,\gamma,b) \in S \times \mathbb{Z}^{m(r+1)} \times K \mid \big(\overline{\alpha}(s,b), \overline{\beta}(s,b)\big) = \gamma\}.$$

Note that for a fixed $s \in S$, the fibers $D_{s,\gamma}$ partition K. We can definably distinguish between the fibers $D_{s,\gamma}$ that are bounded and the ones that are not:

$$G := \{(s,\gamma) \in S \times \mathbb{Z}^{m(r+1)} \mid \exists \delta \in \mathbb{Z} : D_{s,\gamma} \subseteq B_\delta(0)\}.$$

Combining (3.5.5) and (3.5.6) gives us, for each $s \in S$,

$$\begin{aligned}
g(s) &= \int_{X_s} h(s,x) \cdot \sum_{B \sqsubseteq A_{s,x}} \psi(B)|dx| \\
&= q_K \cdot \int_K \psi(b) \sum_{i=1}^m c_i q_K^{\alpha_i(s,b)} \prod_{j=1}^r \beta_{ij}(s,b)|db| \\
&= q_K \sum_{\gamma \in \mathbb{Z}^{m(r+1)}} \int_{D_{s,\gamma}} C_\gamma \psi(b)|db| \\
&= q_K \sum_{\gamma \in G_s} \int_{D_{s,\gamma}} C_\gamma \psi(b)|db| + q_K \sum_{\gamma \in \mathbb{Z}^{m(r+1)} \setminus G_s} \int_{D_{s,\gamma}} C_\gamma \psi(b)|db| \\
&= q_K \int_{G_s} \Big(C_\gamma \int_{D_{s,\gamma}} \psi(b)|db| \Big) |d\gamma|,
\end{aligned}$$

where the last equality follows from the fact that each of the integrals $\int_{Ds,\gamma} C_\gamma \psi(b)|db|$ must exist, which means that if $\gamma \notin G_s$, then $C_\gamma = 0$. Using Proposition 3.5.1 we see that the function

$$l \colon (s,\gamma) \mapsto C_\gamma \int_{D_{s,\gamma}} \psi(b)|db|$$

is an exponential*-constructible function on $G \subseteq S \times \mathbb{Z}^{m(r+1)}$. By applying Theorem 3.1.8 to l we can conclude that $g \colon s \mapsto q_K \int_{G_s} l(s,\gamma)|d\gamma|$ is an exponential*-constructible function on S. $\square$

Chapter 4

The Cluckers-Veys conjecture on exponential sums for polynomials with log-canonical threshold at most a half

This chapter is based on [CN17a], which is joint work with Kien Huu Nguyen.

The goal of this chapter is to proof the Cluckers-Veys conjecture 1.3.13 for polynomials with log-canonical threshold at most one half. This result will imply the Igusa conjecture 1.3.10 and the Denef-Sperber conjecture 1.3.11 under the same restrictions on the log-canonical threshold.

In these conjectures $f \in \mathbb{Z}[x_1, \dots, x_n]$ is a nonconstant polynomial and without loss of generality we can assume that $f(0) = 0$. We are interested in the global sum $E_f(m,p)$ and the local sums $E_f^y(m,p)$ around points $y \in \mathbb{Z}^n$, both depending on an integer $m \in \mathbb{Z}$ and a prime number p. Recall that

$$E_f(m,p) := \frac{1}{p^{mn}} \sum_{\overline{x} \in (\mathbb{Z}/p^m\mathbb{Z})^n} \exp\left(\frac{2\pi i f(x)}{p^m}\right);$$

$$E_f^y(m,p) := \frac{1}{p^{mn}} \sum_{\overline{x}\in\overline{y}+(p\mathbb{Z}/p^m\mathbb{Z})^n} \exp\left(\frac{2\pi i f(x)}{p^m}\right).$$

In the conjectures of Igusa, Denef-Sperber and Cluckers-Veys the asymptotic behaviour of $|E_f(m,p)|_{\mathbb{C}}$ and $|E_f^y(m,p)|_{\mathbb{C}}$ is expressed in terms of log-canonical thresholds. We recall their definitions, that were introduced in Definitions 1.3.8 and 1.3.12 and Equation (1.3.2).

Definition 4.0.1. Let k be a field of characteristic 0 and $f \in k[x_1,\dots,x_n]$ a nonconstant polynomial. We fix an embedded resolution (Y,h) of $f^{-1}(0)$ with $\{E_i \mid i \in T\}$ the set of prime divisors of $(f\circ h)^{-1}(0)$ and $\{(N_i,\nu_i) \mid i\in T\}$ the corresponding set of numerical data. For $y \in (k^{\mathrm{alg}})^n$ such that $f(y)=0$, we call

$$c_y(f) := \min_{i\in T: y\in h(E_i)} \left\{\frac{\nu_i}{N_i}\right\}$$

the log-canonical threshold of f at y and

$$c(f) := \inf_{y\in f^{-1}(0)} c_y(f) = \min_{i\in T}\left\{\frac{\nu_i}{N_i}\right\}$$

the log-canonical threshold of f.

Definition 4.0.2. Let $f \in \mathbb{Z}[x_1,\dots,x_n]$ be a nonconstant polynomial. Recall that $V_f \subseteq \mathbb{Q}^{\mathrm{alg}}$ denotes the set of critical values of f. Fix $y\in\mathbb{Z}^n$ and p a prime number. Then we define

$$a(f) := \min_{b\in V_f\cup\{0\}} c(f-b);$$

$$a_{y,p}(f) := \inf_{y'\in y+p\mathbb{Z}_p^n} c_{y'}(f-f(y')).$$

The exact theorem that we are proving in this chapter is the following.

Theorem 4.0.3. *Let $f\in\mathbb{Z}[x_1,\dots,x_n]$ be a nonconstant polynomial with $f(0)=0$. Put $\sigma := \min\left\{a(f),\frac{1}{2}\right\}$ and $\sigma_{y,p} := \min\left\{a_{y,p}(f),\frac{1}{2}\right\}$ for all $y\in\mathbb{Z}^n$ and all primes p. Then there exists a constant $C>0$ (that may depend on the polynomial f), such that for all $m\geqslant 2$, for all primes p and for all $y\in\mathbb{Z}^n$, we have*

$$|E_f(m,p)|_{\mathbb{C}} \leqslant Cm^{n-1}p^{-m\sigma}, \tag{4.0.1}$$

$$|E_f^y(m,p)|_{\mathbb{C}} \leqslant Cm^{n-1}p^{-m\sigma_{y,p}}. \tag{4.0.2}$$

Note that the homogeneous polynomials f in two variables that are not yet covered by Igusa's proof of his own conjecture [Igu78], satisfy that $a(f) \leqslant \frac{1}{2}$. For these polynomials Igusa's conjecture was already proven by Lichtin [Lic13] and by Wright [Wri12], hence our results can be considered a generalisation of theirs to more variables.

In order to prove Theorem 4.0.3, we will first prove an upper bound for the local sum around 0, $E_f^0(m,p)$. It is a consequence of Corollary 1.3.7 that for each prime number p, there exists a constant $C_p > 0$, such that for all $m \geqslant 1$, we have

$$|E_f^0(m,p)|_{\mathbb{C}} \leqslant C_p m^{n-1} p^{-mc_0(f)}. \tag{4.0.3}$$

We will show that if $c_0(f) \leqslant \frac{1}{2}$, then the constant C_p can be taken independent of p, for p big enough. This is the content of the following theorem.

Theorem 4.0.4. *Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be a nonconstant polynomial with $f(0) = 0$. Put $\sigma_0 := \min\left\{c_0(f), \frac{1}{2}\right\}$. Then there exist a constant $C > 0$ and a natural number N, such that for all $m \geqslant 1$ and for all primes $p > N$, we have*

$$|E_f^0(m,p)|_{\mathbb{C}} \leqslant C m^{n-1} p^{-m\sigma_0}.$$

Combining this theorem and Equation (4.0.3) shows that this theorem even holds for $N = 1$.

Remark 4.0.5. If $m = 1$, then $E_f^0(m,p) = \frac{1}{p^n}$, hence Theorem 4.0.4 is trivially satisfied. Therefore we only need to prove it for $m \geqslant 2$.

We will give two proofs of Theorem 4.0.4. Both of these proofs contain ideas that could be useful for later developments around this conjecture. The first approach, in Section 4.1, makes use of the Cluckers-Loeser motivic integration theory. By considering certain well-chosen motivic functions and specializing them to p-adic functions, we can deduce information about our exponential sums in a uniform way for all but finitely many prime numbers p. To get strong enough upper bounds, we need to prove that the specializations of the motivic functions do not depend on the choice of a uniformizer in $\mathbb{Q}_p$, but only on the angular component of the chosen uniformizer. Hence, when varying uniformizers, we obtain orbits of points on which these functions are constant. In fact, these orbits depend on actions of the group $\mu_{p-1}(\mathbb{Q}_p)$, the group of $(p-1)^{\text{th}}$ roots of unity of $\mathbb{Q}_p$, on the set of uniformizers of $\mathbb{Q}_p$ and on $\mathbb{Q}_p$. This idea could link the Cluckers-Loeser theory of motivic integration to the one of Hrushovski-Kazhdan (see [HK06]). The latter makes use of the action of the group $\hat{\mu} = \varprojlim_d \mu_d$ on the set of uniformizers and hence on the residue field.

The second approach, in Section 4.2, is based on existing results on the Igusa zeta functions and the Lang-Weil estimates ([LW54]) for the number of points of varieties over finite fields. Improving certain parts of this approach could lead to the proof of other cases of the Cluckers-Veys conjecture.

In Section 4.3 we will deduce the global Cluckers-Veys upper bound (4.0.1) and in Section 4.4 we will prove the uniform local Cluckers-Veys upper bounds (4.0.2) for all $y \in \mathbb{Z}^n$.

We remark that our results can be extended to the ring of integers of any number field, but we will only work with $\mathbb{Z}$ and $\mathbb{Q}$ to simplify notation.

4.1 The model theoretic approach

The proof of Theorem 4.0.4 that we give in this section, uses some results from the Cluckers-Loeser theory of motivic integration. The results that we need here, can be found in Section 1.2. The idea of the proof is to split the exponential sum $E_f^0(m,p)$ into three subsums and to give estimates for each of these subsums. Some of these estimates will depend on the value of m, but when we combine them with Corollary 1.3.7, we can obtain an estimate that is uniform in m. The decomposition of $E_f^0(m,p)$ that we consider, is the following:

$$E_f^0(m,p) = \frac{1}{p^{nm}} \sum_{\substack{\overline{x} \in (p\mathbb{Z}/p^m\mathbb{Z})^n, \\ \mathrm{ord}_p(f(x)) \leqslant m-2}} \exp\left(\frac{2\pi i f(x)}{p^m}\right) +$$

$$\frac{1}{p^{nm}} \sum_{\substack{\overline{x} \in (p\mathbb{Z}/p^m\mathbb{Z})^n, \\ \mathrm{ord}_p(f(x)) = m-1}} \exp\left(\frac{2\pi i f(x)}{p^m}\right) + \frac{1}{p^{nm}} \sum_{\substack{\overline{x} \in (p\mathbb{Z}/p^m\mathbb{Z})^n, \\ \mathrm{ord}_p(f(x)) \geqslant m}} \exp\left(\frac{2\pi i f(x)}{p^m}\right).$$

In three different lemmas we will analyse each of these sums.

For the first subsum we will introduce a constructible function G, that expresses, for a certain input $z \in \mathbb{Z}_p$ with $\mathrm{ord}_p(z) \leqslant m-2$, how many $x \in p\mathbb{Z}_p^n$ are mapped close to z by f. We will apply a cell decomposition theorem to G and with some further techniques like elimination of quantifiers, we will show that certain values z of f occur equally often. In the exponential sum these values will cancel out.

Lemma 4.1.1. *Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be a nonconstant polynomial with $f(0) = 0$. Then there exists a natural number N such that, for all $m \geqslant 1$ and for all primes $p > N$, we have*

$$\sum_{\substack{\overline{x} \in (p\mathbb{Z}/p^m\mathbb{Z})^n, \\ \mathrm{ord}_p(f(x)) \leqslant m-2}} \exp\left(\frac{2\pi i f(x)}{p^m}\right) = 0.$$

Proof. The statement is obvious when $m = 1$ or $m = 2$, so we can assume that $m > 2$. Let φ be the $\mathcal{L}_{\mathbb{Z}}$-formula given by

$$\varphi(x_1, \dots, x_n, z, m) := \bigwedge_{i=1}^{n} (\operatorname{ord}(x_i) \geqslant 1) \wedge \big(\operatorname{ord}(z) \leqslant m - 2\big) \wedge$$
$$\big(\operatorname{ord}(z - f(x_1, \dots, x_n)) \geqslant m\big),$$

where x_i, z are in the VF-sort and m is in the VG-sort. To shorten notation we set $x = (x_1, \dots, x_n)$. For each prime p and each uniformizer ϖ_p of $\mathbb{Q}_p$, φ defines a definable set $X_p \subseteq p\mathbb{Z}_p^n \times \mathbb{Z}_p \times \mathbb{Z}$. More precisely, we have

$$X_p := \{(x, z, m) \in p\mathbb{Z}_p^n \times \mathbb{Z}_p \times \mathbb{Z} \mid \operatorname{ord}_p(f(x) - z) \geqslant m \wedge \operatorname{ord}_p(z) \leqslant m - 2\}.$$

It is obvious that X_p does not depend on ϖ_p.

We denote by $X \subseteq h[n + 1, 0, 1]$ the definable subassignment defined by φ. Let $F := \mathbb{1}_X \in I_{h[0,0,1]}\boldsymbol{C}(h[n + 1, 0, 1])$ be the characteristic function on X and π the projection from $h[n + 1, 0, 1]$ to $h[1, 0, 1]$. Then we have $G := \pi_!(F) \in I_{h[0,0,1]}\boldsymbol{C}(h[1, 0, 1])$. For each prime p and each uniformizer ϖ_p of $\mathbb{Q}_p$, we can interpret F and G in $\mathbb{Q}_p$:

$$F_p = \mathbb{1}_{X_p},$$

and if $\operatorname{ord}_p(z) \leqslant m - 2$, then

$$G_p(z, m) = \int_{X_{p,z,m}} |dx| = p^{-mn} \#\{\overline{x} \in (p\mathbb{Z}/p^m\mathbb{Z})^n \mid f(x) \equiv z \bmod p^m\},$$

where $X_{p,z,m}$ is the fiber of X_p over (z, m). If $\operatorname{ord}_p(z) \geqslant m - 1$, then

$$G_p(z, m) = 0.$$

We can see that both $F_p(x, z, m)$ and $G_p(z, m)$ do not depend on ϖ_p.

Now we use Corollary 1.2.12 for $G \in \mathcal{C}(h[1, 0, 1])$. This means that there exists a finite partition of $h[1, 0, 1]$ into cells Z_i (for i in some finite index set I) with presentation $(\lambda_i, Z_{C_i,\alpha_i,\xi_i,c_i})$, such that $G_{|Z_i} = \lambda_i^* p_i^*(G_i)$ with $G_i \in \mathcal{C}(C_i)$ and $p_i \colon Z_{C_i,\alpha_i,\xi_i,c_i} \to C_i$ the projection. Note that $C_i \subseteq h[0, r_i, s_i + 1]$ for some $r_i, s_i \in \mathbb{N}$. Several observations related to this cell decomposition can be made.

First of all, we will show that the functions $c_i \colon C_i \to h[1, 0, 0]$ have finite image. We denote by $\theta_i(z, \eta, \gamma, m)$ the $\mathcal{L}_{\mathbb{Z}}$-formula defining the graph of $c_i \colon C_i \to h[1, 0, 0]$, where $z \in h[1, 0, 0]$, $\eta \in h[0, r_i, 0]$, $\gamma \in h[0, 0, s_i]$ and $m \in h[0, 0, 1]$. By elimination of quantifiers in the VF-sort (Corollary 1.2.5), there exist polynomials $f_1, \dots, f_r$ in one variable z with coefficients in $\mathbb{Z}[[t]]$, such that

$\theta_i(z,\eta,\gamma,m)$ is equivalent to the formula

$$\bigvee_j \Big(\zeta_{ij}\big(\overline{\mathrm{ac}}(f_1(z)),\dots,\overline{\mathrm{ac}}(f_r(z)),\eta\big)\,\wedge\,\nu_{ij}\big(\mathrm{ord}(f_1(z)),\dots,\mathrm{ord}(f_r(z)),\gamma,m\big)\Big),$$

where the ζ_{ij} are $\mathcal{L}_{\mathrm{ring}}$-formulas and the ν_{ij} are $\mathcal{L}_{\mathrm{oag}}$-formulas. Since c_i is a function, we know that, for each $(\eta,\gamma,m)\in C_i$, there exists a unique $z = c_i(\eta,\gamma,m)$ such that $\theta_i(z,\eta,\gamma,m)$ holds.

We claim that there exists $1\leqslant \ell\leqslant r$ such that $f_\ell(z)=0$. Indeed, if $f_\ell(z)\neq 0$, for all ℓ, then there exists a small open neighbourhood V of z and an index j, such that, for all $y\in V$, the formula

$$\zeta_{ij}\big(\overline{\mathrm{ac}}(f_1(y)),\dots,\overline{\mathrm{ac}}(f_r(y)),\eta\big)\,\wedge\,\nu_{ij}\big(\mathrm{ord}(f_1(y)),\dots,\mathrm{ord}(f_r(y)),\gamma,m\big)$$

is satisfied. Since this would contradict the uniqueness of z, we must have that $f_\ell(z)=0$ for some ℓ. We conclude that the set of centers is a finite set. More precisely,

$$A := \{c_i(\eta,\gamma,m)\in h[1,0,0]\mid i\in I,\ (\eta,\gamma,m)\in C_i\}\subseteq \bigcup_{\ell=1}^{r} Z(f_\ell),$$

where $Z(f_\ell)$ denotes the zero set of f_ℓ. Now, for each $m>2$, we set

$$D_m := A\cap\{z\in h[1,0,0]\mid m-2\geqslant \mathrm{ord}(z)\geqslant 1\};$$

$$U_m := \{y\in h[1,0,0]\mid \forall z\in D_m : \mathrm{ord}(y-z)<m-1\}.$$

Hence U_m is a union of balls of radius $m-1$. Since $f(0)=0$, we can see that $G(\,\cdot\,,m)$ is zero on the set $\{z\in h[1,0,0]\mid \mathrm{ord}(z)\leqslant 0\}$, when $m>2$.

Claim 4.1.2. *Let $m>2$ and $z\in U_m$, with $\mathrm{ord}(z)\geqslant 1$, then $G(\,\cdot\,,m)$ is constant on the ball $B_{m-1}(z)$ (the ball with center z and valuation radius $m-1$).*

From the cell decomposition that we applied to $h[1,0,1]$, we know that there exist $i\in I$ and $(\eta,\gamma)\in h[0,r_i,s_i]$, such that $(z,\eta,\gamma,m)\in Z_{C_i,\alpha_i,\xi_i,c_i}$. Hence $(\eta,\gamma,m)\in C_i$ and z belongs to the ball

$$B=\left\{ y\in h[1,0,0] \;\middle|\; \begin{array}{l} \mathrm{ord}(y-c_i(\eta,\gamma,m))=\alpha_i(\eta,\gamma,m),\\ \overline{\mathrm{ac}}(y-c_i(\eta,\gamma,m))=\xi_i(\eta,\gamma,m)\end{array}\right\}.$$

It follows from the definition of G_i that $G_i(\eta,\gamma,m)=G(y,m)$ for all $y\in B$, hence $G(\,\cdot\,,m)$ is constant on B.

To prove the claim we will distinguish three cases, depending on the value of $c_i(\eta,\gamma,m)$.

- If $c_i(\eta,\gamma,m) \in D_m$, then we see that $\alpha_i(\eta,\gamma,m) = \mathrm{ord}(z - c_i(\eta,\gamma,m)) < m-1$. Therefore the ball B will contain the ball $B_{m-1}(z)$, hence $G(\,\cdot\,,m)$ will be constant on $B_{m-1}(z)$.

- If $\mathrm{ord}(c_i(\eta,\gamma,m)) \leqslant 0$, then we see that $\mathrm{ord}(z) \geqslant 1$ implies that $\alpha_i(\eta,\gamma,m) \leqslant 0 < m-1$ so we have the same situation as the first case.

- If $\mathrm{ord}(c_i(\eta,\gamma,m)) \geqslant m-1$, then the case $\alpha_i(\eta,\gamma,m) < m-1$ has already been treated above. Hence we can assume that $\alpha_i(\eta,\gamma,m) \geqslant m-1$, in which case we have $B_{m-1}(z) = B_{m-1}(0)$. By definition of G we have $G(\,\cdot\,,m)_{|B_{m-1}(0)} = 0$.

This proves the claim.

Now for p big enough we can interpret all of the above discussion in $\mathbb{Q}_p$. More precisely, there exists $N_0 \in \mathbb{N}$ (independent of m) such that for all primes $p > N_0$ and any choice of uniformizer ϖ_p, there exists an interpretation of the above by applying the map λ_{ϖ_p} to the coefficients of the polynomials $f_1,\dots,f_r$ and by taking the angular component map $\overline{\mathrm{ac}}_{\varpi_p}$ with respect to the uniformizer ϖ_p. The interpretation $U_{m,\varpi_p} \subseteq \mathbb{Q}_p$ of U_m is an $\{m,\varpi_p\}$-definable set in the language $\mathcal{L}_{\mathrm{DP}}$, which can vary when changing ϖ_p. Therefore we set $\mathcal{U}_{m,p} := \cup_{\varpi_p} U_{m,\varpi_p}$ with ϖ_p running over the set of all uniformizers of $\mathbb{Q}_p$. Then $\mathcal{U}_{m,p}$ is $\{m\}$-definable by an $\mathcal{L}_{\mathrm{DP}}$-formula.

Claim 4.1.3. *There exists $N \in \mathbb{N}$, such that $\mathcal{U}_{m,p} = \mathbb{Q}_p$, for all $m > 2$ and for all primes $p > N$.*

It follows from the definition of U_{m,ϖ_p} that $\mathcal{V}_{m,p} := \mathbb{Q}_p \setminus \mathcal{U}_{m,p}$ is a union of $d_{m,p}$ balls of radius $m-1$, contained in $p\mathbb{Z}_p$, for some $d_{m,p} \leqslant \sum_{\ell=1}^r \deg f_\ell$. Moreover, $\mathcal{V}_{\_,p}$ is given by a $\mathcal{L}_{\mathrm{DP}}$-formula. We use elimination of quantifiers (Theorem 1.2.4) for this formula. Hence there exist polynomials $g_1,\dots,g_s$ in one variable z with coefficients in $\mathbb{Z}$ and formulas φ_j in $\mathcal{L}_{\mathrm{ring}}$ and ν_j in $\mathcal{L}_{\mathrm{oag}}$, such that $\mathcal{V}_{m,p}$ contains exactly the elements z that satisfy

$$\bigvee_j \Big(\varphi_j\big(\overline{\mathrm{ac}}_{\varpi_p}(g_1(z)),\dots,\overline{\mathrm{ac}}_{\varpi_p}(g_s(z))\big) \wedge \nu_j\big(\mathrm{ord}_p(g_1(z)),\dots,\mathrm{ord}_p(g_s(z)),m\big)\Big),$$

for any $p > N_0$ (after enlarging N_0 if necessary) and any uniformizer ϖ_p.

We note that if $z \in \mathcal{V}_{m,p}$, then $\mathrm{ord}_p(z) \geqslant 1$. Since g_i has coefficients in $\mathbb{Z}$, we can assume that $\overline{\mathrm{ac}}_{\varpi_p}(g_i(z))$ only depends on $\overline{\mathrm{ac}}_{\varpi_p}(z)$ and $\mathrm{ord}_p(g_i(z))$ only depends on $\mathrm{ord}_p(z)$, for any $p > N_0$ after possibly enlarging N_0 again. This follows from the t-adic version of this statement by a compactness argument. Therefore, if z_1 and z_2 satisfy

- $\mathrm{ord}_p(z_1) = \mathrm{ord}_p(z_2) \geqslant 1$,
- there exist two uniformizers $\varpi_{1,p}$ and $\varpi_{2,p}$, such that $\overline{\mathrm{ac}}_{\varpi_{1,p}}(z_1) = \overline{\mathrm{ac}}_{\varpi_{2,p}}(z_2)$,

then $z_1 \in \mathcal{V}_{m,p}$ if and only if $z_2 \in \mathcal{V}_{m,p}$. This implies that the set $\overline{\mathcal{V}_{m,p}} := \overline{\mathrm{ac}}_{\varpi_p}(\mathcal{V}_{m,p})$ is independent of ϖ_p, for any $p > N_0$. In particular, since $B_{m-1}(0) \nsubseteq \mathcal{V}_{m,p}$, we see that the number of elements in $\overline{\mathcal{V}_{m,p}}$ is at most $\sum_{\ell=1}^r \deg f_\ell$.

In what follows we will show that if $\mathcal{V}_{m,p}$ were not empty, then the set $\overline{\mathcal{V}_{m,p}}$ would grow with p. This gives a contradiction which proves the claim. We set

$$D_\infty := A \cap \{z \in h[1,0,0] \mid \infty > \mathrm{ord}(z) \geqslant 1\} \subseteq \bigcup_{\ell=1}^r Z(f_\ell),$$

thus D_∞ is a finite set with $0 \notin D_\infty$ and $D_m \subseteq D_\infty$ for all $m > 2$. Looking at the order of the coefficients of f_ℓ we see that there exists $M \in \mathbb{N}$ such that $\mathrm{ord}_p(z) \leqslant M$ for all $z \in \cup_{\ell=1}^r Z(f_{\ell,\varpi_p}) \setminus \{0\}$, $p > N_0$ and uniformizers ϖ_p. So $\mathrm{ord}_p(z) \leqslant M$ for all $z \in D_{\infty,\varpi_p}$, for any ϖ_p.

It follows that $\mathrm{ord}_p(z) \leqslant M$ for all $z \in \mathcal{V}_{m,p}$, for all $m > 2$ and $p > N_0$. Indeed, since $B_{m-1}(0) \nsubseteq \mathcal{V}_{m,p}$ we have $\mathrm{ord}_p(z) < m-1$ for all $z \in \mathcal{V}_{m,p}$, so the claim clearly holds if $m-1 \leqslant M$. On the other hand, if $m-1 > M$, then for each $z \in \mathcal{V}_{m,p}$ and each uniformizer ϖ_p, there exists $z_0 \in D_{\infty,\varpi_p}$ such that $\mathrm{ord}_p(z-z_0) \geqslant m-1 > M \geqslant \mathrm{ord}_p(z_0)$, thus $\mathrm{ord}_p(z) = \mathrm{ord}_p(z_0) \leqslant M$.

Now put $N := \max\left\{N_0, 1 + M\sum_{\ell=1}^r \deg f_\ell\right\}$. Suppose for a contradiction, that for some $p > N$, there exists $z \in \mathcal{V}_{p,m}$. Then $\overline{\mathrm{ac}}_{\varpi_p}(z) \in \overline{\mathcal{V}_{m,p}}$, for every uniformizer ϖ_p, and so $\{\overline{\mathrm{ac}}_{\varpi_p}(z) \mid \mathrm{ord}_p(\varpi_p) = 1\} \subseteq \overline{\mathcal{V}_{m,p}}$. Suppose that $\overline{\mathrm{ac}}_p(\varpi_p) = u$, then $u^{\mathrm{ord}_p(z)}\overline{\mathrm{ac}}_{\varpi_p}(z) = \overline{\mathrm{ac}}_p(z)$, so we have $\{\overline{\mathrm{ac}}_{\varpi_p}(z) \mid \mathrm{ord}_p(\varpi_p) = 1\} = \{u^{-\mathrm{ord}_p(z)}\overline{\mathrm{ac}}_p(z) \mid u \in \mathbb{F}_p^\times\}$. Therefore $\#\{u^{-\mathrm{ord}_p(z)}\overline{\mathrm{ac}}_p(z) \mid u \in \mathbb{F}_p^\times\} \leqslant \sum_{\ell=1}^r \deg f_\ell$. However,

$$\#\{u^{-\mathrm{ord}_p(z)}\overline{\mathrm{ac}}_p(z) \mid u \in \mathbb{F}_p^\times\} = \frac{p-1}{\gcd(\mathrm{ord}_p(z), p-1)} \geqslant \frac{p-1}{\mathrm{ord}_p(z)} \geqslant \frac{p-1}{M},$$

where $\gcd(a,b)$ is the greatest common divisor of a and b. Then we have $p-1 \leqslant M\sum_{\ell=1}^r \deg f_\ell \leqslant N-1$. This is a contradiction, since $p > N$, so this proves the claim.

We know from Claim 4.1.2 that if $m > 2$ and $z \in U_{m,\varpi_p}$ such that $1 \leqslant \mathrm{ord}_p(z) \leqslant m-2$, then $G_p(\,\cdot\,, m)$ will be constant on the ball $B_{m-1}(z)$. So for any $\overline{y} \in p\mathbb{Z}/p^m\mathbb{Z}$ with $y \equiv z \bmod p^{m-1}$, we have $\#\{\overline{x} \in (p\mathbb{Z}/p^m\mathbb{Z})^n \mid f(x) \equiv$

$y \bmod p^m\} = \#\{\overline{x} \in (p\mathbb{Z}/p^m\mathbb{Z})^n \mid f(x) \equiv z \bmod p^m\}$. Hence

$$\sum_{\substack{\overline{x}\in(p\mathbb{Z}/p^m\mathbb{Z})^n,\\ f(x)\equiv z \bmod p^{m-1}}} p^{-mn} \exp\left(\frac{2\pi i f(x)}{p^m}\right) =$$

$$G_p(z,m) \sum_{\substack{\overline{y}\in p\mathbb{Z}/p^m\mathbb{Z},\\ y\equiv z \bmod p^{m-1}}} \exp\left(\frac{2\pi i y}{p^m}\right) = 0,$$

since the values of $\exp\left(\frac{2\pi i y}{p^m}\right)$ cancel out. This implies that

$$\sum_{\substack{\overline{x}\in(p\mathbb{Z}/p^m\mathbb{Z})^n,\\ \overline{f(x)}\in\overline{\mathcal{U}_{m,p}}}} p^{-mn} \exp\left(\frac{2\pi i f(x)}{p^m}\right) = 0,$$

where $\overline{\mathcal{U}_{m,p}} := \{\overline{z} \in p\mathbb{Z}/p^{m-1}\mathbb{Z} \mid z \in \mathcal{U}_{m,p},\ m-2 \geqslant \mathrm{ord}_p(z) \geqslant 1\}$. For all $m > 2$ and $p > N$, we have $\mathcal{U}_{m,p} = \mathbb{Q}_p$, so $\overline{\mathcal{U}_{m,p}} = \{\overline{z} \in p\mathbb{Z}/p^{m-1}\mathbb{Z} \mid \mathrm{ord}_p(z) \leqslant m-2\}$. Thus

$$\sum_{\substack{\overline{x}\in(p\mathbb{Z}/p^m\mathbb{Z})^n,\\ \mathrm{ord}_p(f(x))\leqslant m-2}} p^{-mn} \exp\left(\frac{2\pi i f(x)}{p^m}\right) = 0.$$

□

To estimate the other two subsums we need the following theorem of Mustaţă that relates the log-canonical threshold to the dimensions of arc spaces and jet spaces.

Theorem 4.1.4 ([Mus02], Corollaries 0.2 and 3.6)**.** *Let k be an algebraically closed field of characteristic* 0 *and $f \in k[x_1, \ldots, x_n]$ a nonconstant polynomial. For any $m \in \mathbb{N}$ we set*

$$\mathrm{Cont}^{\geqslant m}(f) := \{x \in k[[t]]^n \mid f(x) \equiv 0 \bmod t^m\},$$

$$\mathrm{Cont}_0^{\geqslant m}(f) := \{x \in tk[[t]]^n \mid f(x) \equiv 0 \bmod t^m\}.$$

Let p_m denote the quotient map $k[[t]]^n \to \big(k[t]/(t^m)\big)^n$. The codimensions of $p_m(\mathrm{Cont}^{\geqslant m}(f))$ and $p_m(\mathrm{Cont}_0^{\geqslant m}(f))$ in $\big(k[t]/(t^m)\big)^n \cong k^{nm}$ are denoted by $\mathrm{codim}\ \mathrm{Cont}^{\geqslant m}(f)$ *and* $\mathrm{codim}\ \mathrm{Cont}_0^{\geqslant m}(f)$*, respectively. Then the log-canonical threshold of f equals*

$$c(f) = \inf_{m\geqslant 1} \frac{\mathrm{codim}\ \mathrm{Cont}^{\geqslant m}(f)}{m}$$

and if $f(0) = 0$, then the log-canonical threshold of f at 0 equals

$$c_0(f) = \inf_{m \geqslant 1} \frac{\operatorname{codim} \operatorname{Cont}_0^{\geqslant m}(f)}{m}.$$

In the proof of the following lemma we will introduce again a constructible function G, similar to the one from the previous lemma. For this exponential sum the different values z of f do not cancel out completely. By using the Lang-Weil estimate [LW54] and the above theorem we obtain the following upper bound for the second subsum.

Lemma 4.1.5. *Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be a nonconstant polynomial with $f(0) = 0$. Put $\sigma_0 := \min\left\{c_0(f), \frac{1}{2}\right\}$. Then for each integer $m \geqslant 2$, there exist a natural number N_m and a constant $D_m > 0$, such that, for all primes $p > N_m$, we have*

$$\Bigg| \sum_{\substack{\overline{x} \in (p\mathbb{Z}/p^m\mathbb{Z})^n, \\ \operatorname{ord}_p(f(x)) = m-1}} p^{-mn} \exp\left(\frac{2\pi i f(x)}{p^m}\right) \Bigg|_{\mathbb{C}} \leqslant D_m p^{-m\sigma_0}.$$

Proof. Let φ and $\overline{\varphi}$ be two $\mathcal{L}_{\mathbb{Z}}$-formulas given by

$$\varphi(x_1, \ldots, x_n, z, m) := \bigwedge_{i=1}^{n} (\operatorname{ord}(x_i) \geqslant 1) \wedge \big(\operatorname{ord}(z) = m-1\big) \wedge$$

$$\big(\operatorname{ord}(z - f(x_1, \ldots, x_n)) \geqslant m\big),$$

$$\overline{\varphi}(x_1, \ldots, x_n, \xi, m) := \bigwedge_{i=1}^{n} (\operatorname{ord}(x_i) \geqslant 1) \wedge \big(\operatorname{ord}(f(x_1, \ldots, x_n) = m-1\big) \wedge$$

$$\big(\overline{\operatorname{ac}}(f(x_1, \ldots, x_n)) = \xi\big),$$

where x_i, z are in the VF-sort, m is in the VG-sort and ξ is in the RF-sort. To shorten notation we set $x = (x_1, \ldots, x_n)$. For each prime p and each uniformizer ϖ_p of $\mathbb{Q}_p$, φ and $\overline{\varphi}$ define definable sets $X_p \subseteq p\mathbb{Z}_p^n \times \mathbb{Z}_p \times \mathbb{Z}$ and $\overline{X}_{\varpi_p} \subseteq p\mathbb{Z}_p^n \times \mathbb{F}_p \times \mathbb{Z}$. More precisely, we have

$$X_p := \{(x, z, m) \in p\mathbb{Z}_p^n \times \mathbb{Z}_p \times \mathbb{Z} \mid \operatorname{ord}_p(f(x) - z) \geqslant m \wedge \operatorname{ord}_p(z) = m-1\},$$

$$\overline{X}_{\varpi_p} := \{(x, \xi, m) \in p\mathbb{Z}_p^n \times \mathbb{F}_p \times \mathbb{Z} \mid \operatorname{ord}_p(f(x)) = m-1 \wedge \overline{\operatorname{ac}}_{\varpi_p}(f(x)) = \xi\}.$$

It is obvious that X_p does not depend on ϖ_p.

We denote by $X \subseteq h[n+1, 0, 1]$, resp. $\overline{X} \subseteq h[n, 1, 1]$, the definable subassignments defined by φ, resp. $\overline{\varphi}$. Let $F := \mathbb{1}_X \in I_{h[0,0,1]}\boldsymbol{C}(h[n+1, 0, 1])$

be the characteristic function on X and π the projection from $h[n+1,0,1]$ to $h[1,0,1]$. Then we have $G := \pi_!(F) \in I_{h[0,0,1]}\boldsymbol{C}(h[1,0,1])$. For each prime p and each uniformizer ϖ_p of $\mathbb{Q}_p$, we can interpret F and G in $\mathbb{Q}_p$:

$$F_p = \mathbb{1}_{X_p},$$

and if $\mathrm{ord}_p(z) = m-1$, then

$$G_p(z,m) = \int_{X_{p,z,m}} |dx| = p^{-mn}\#\{\overline{x} \in (p\mathbb{Z}/p^m\mathbb{Z})^n \mid f(x) \equiv z \bmod p^m\},$$

where $X_{p,z,m}$ is the fiber of X_p over (z,m). If $\mathrm{ord}_p(z) \neq m-1$, then

$$G_p(z,m) = 0.$$

We can see that both $F_p(x,z,m)$ and $G_p(z,m)$ do not depend on ϖ_p. The idea is to partition $p^{m-1}\mathbb{Z}_p \setminus p^m\mathbb{Z}_p$ into sets on which $G_p(\,\cdot\,,m)$ is constant. First of all, we can see that $G_p(\,\cdot\,,m)$ is constant on balls of the form

$$\{z \in \mathbb{Z}_p \mid \mathrm{ord}_p(z) = m-1 \,\wedge\, \overline{\mathrm{ac}}_{\varpi_p}(z) = \xi_0\},$$

with $\xi_0 \in \mathbb{F}_p^\times$. Now we will look more closely on which of these balls $G_p(\,\cdot\,,m)$ takes the same value. In what follows we will show is that for p big enough, if ϖ_p and ϖ_p' are two uniformizers, then $G_p(\,\cdot\,,m)$ will take the same value on the sets

$$\{z \in \mathbb{Z}_p \mid \mathrm{ord}_p(z) = m-1 \;\wedge\; \overline{\mathrm{ac}}_{\varpi_p}(z) = \xi_0\}$$

and

$$\{z \in \mathbb{Z}_p \mid \mathrm{ord}_p(z) = m-1 \;\wedge\; \overline{\mathrm{ac}}_{\varpi_p'}(z) = \xi_0\}.$$

From this it will follow that $G_p(\,\cdot\,,m)$ is constant on the orbits of an action of the group $\mu_{p-1}(\mathbb{Q}_p)$ (the group of $(p-1)$-roots of unity) on $\mathbb{Q}_p$.

Let $\overline{F} := \mathbb{1}_{\overline{X}} \in I_{h[0,0,1]}\boldsymbol{C}(h[n,1,1])$ be the characteristic function on $\overline{X}$ and $\overline{\pi}$ the projection from $h[n,1,1]$ to $h[0,1,1]$. Then we have $\overline{G} := \overline{\pi}_!(\overline{F}) \in I_{h[0,0,1]}\boldsymbol{C}(h[0,1,1])$. For each prime p and each uniformizer ϖ_p of $\mathbb{Q}_p$, we can interpret $\overline{F}$ and $\overline{G}$ in $\mathbb{Q}_p$:

$$\begin{aligned}
\overline{F}_{\varpi_p} &= \mathbb{1}_{\overline{X}_{\varpi_p}},\\
\overline{G}_{\varpi_p}(\xi,m) &= \int_{\overline{X}_{\varpi_p,\xi,m}} |dx| \\
&= p^{-mn}\#\left\{\overline{x} \in (p\mathbb{Z}/p^m\mathbb{Z})^n \;\middle|\; \begin{array}{l} \mathrm{ord}_p(f(x)) = m-1, \\ \overline{\mathrm{ac}}_{\varpi_p}(f(x)) = \xi \end{array}\right\},
\end{aligned}$$

where $\overline{X}_{\varpi_p,\xi,m}$ is the fiber of $\overline{X}_{\varpi_p}$ over (ξ, m).

Since $\overline{G} \in \mathcal{C}(h[0,1,1])$, we can write $\overline{G}$ in the form

$$\overline{G}(\xi, m) = \sum_{i \in I} n_i \alpha_i(\xi, m) \mathbb{L}^{\beta_i(\xi,m)}[V_i],$$

where $n_i \in \mathbb{Z}$, α_i, β_i are $\mathcal{L}_{\mathbb{Z}}$-definable functions from $h[0,1,1]$ to $h[0,0,1]$ and $[V_i] \in K_0(\mathrm{RDef}_{h[0,1,1],\mathcal{L}_{\mathbb{Z}}})$. We apply elimination of VF-quantifiers (Corollary 1.2.5) to the formulas defining α_i, β_i and V_i and we interpret everything in $\mathbb{Q}_p$. This means that there exist $(\mathcal{L}_{\mathrm{ring}} \cup \mathbb{Z})$-formulas ϕ_{ij}, θ_{ij}, ς_{ij} and $(\mathcal{L}_{\mathrm{oag}} \cup \mathbb{Z})$-formulas η_{ij}, ν_{ij}, τ_{ij}, and a natural number N, such that for all primes $p > N$ and all uniformizers ϖ_p of $\mathbb{Q}_p$, we have

$$\alpha_{i,\varpi_p}(\xi, m) = a \Leftrightarrow \vee_{j\in J}(\phi_{ij}(\xi) \wedge \eta_{ij}(a, m));$$

$$\beta_{i,\varpi_p}(\xi, m) = b \Leftrightarrow \vee_{j\in J}(\theta_{ij}(\xi) \wedge \nu_{ij}(b, m));$$

$$(\xi, m, \zeta) \in V_{i,\varpi_p} \Leftrightarrow \vee_{j\in J}(\varsigma_{ij}(\zeta, \xi) \wedge \tau_{ij}(m)).$$

From these formulas we can see that the interpretations α_{i,ϖ_p}, β_{i,ϖ_p}, V_{i,ϖ_p} and hence $\overline{G}_{\varpi_p}(\xi, m)$ are independent of the choice of the uniformizer ϖ_p, for $p > N$. Therefore we will write $\overline{G}_p(\xi, m)$ instead of $\overline{G}_{\varpi_p}(\xi, m)$.

By definition of G and $\overline{G}$ we can see that $G_p(z, m) = G_{\varpi_p}(z, m) = \overline{G}_{\varpi_p}(\xi, m) = \overline{G}_p(\xi, m)$, if $\mathrm{ord}_p(z) = m - 1$ and $\overline{\mathrm{ac}}_{\varpi_p}(z) = \xi$ for some uniformizer ϖ_p. Hence, for all $m \geqslant 2$, $p > N$ and $z_1, z_2 \in \mathbb{Z}_p$ for which $\mathrm{ord}_p(z_1) = \mathrm{ord}_p(z_2) = m - 1$, we have $G_p(z_1, m) = G_p(z_2, m)$, if there exist two uniformizers $\varpi_{1,p}, \varpi_{2,p}$ such that $\overline{\mathrm{ac}}_{\varpi_{1,p}}(z_1) = \overline{\mathrm{ac}}_{\varpi_{2,p}}(z_2) \in \mathbb{F}_p^\times$. Let $d := \gcd(m-1, p-1)$, then by the same reasoning as in Lemma 4.1.1 we see that $G_p(\,\cdot\,, m)$ is constant on the sets

$$\left\{ z \in \mathbb{Z}_p \;\middle|\; \mathrm{ord}_p(z) = \mathrm{ord}_p(z_0) = m - 1 \wedge \overline{\mathrm{ac}}_p\left(\frac{z}{z_0}\right)^{\frac{p-1}{d}} = 1 \right\},$$

for any $z_0 \in \mathbb{Z}_p$ with $\mathrm{ord}_p(z_0) = m - 1$. Now $L_m := p^{m-1}\mathbb{Z}_p \setminus p^m\mathbb{Z}_p$ partitions into d of these sets, each of them consisting of $\frac{p-1}{d}$ disjoint balls of volume p^{-m} and $G_p(\,\cdot\,, m)$ is constant on these sets. We denote these sets by $Y_1, \ldots, Y_d$ and the values of $G_p(\,\cdot\,, m)$ on these sets by $G_1, \ldots, G_d$ respectively. We remark that if $\mathrm{ord}_p(z) = m - 1$, then

$$\exp\left(\frac{2\pi i z}{p^m}\right) = \exp\left(\frac{2\pi i\ \overline{\mathrm{ac}}_p(z)}{p}\right),$$

so

$$\left| \sum_{\overline{y} \in Y_i / p^m\mathbb{Z}_p} \exp\left(\frac{2\pi i y}{p^m}\right) \right|_{\mathbb{C}} = \left| \sum_{\xi \in \overline{\mathrm{ac}}_p(Y_i)} \exp\left(\frac{2\pi i \xi}{p}\right) \right|_{\mathbb{C}}$$

$$= \left| \sum_{u \in \mathbb{F}_p^\times} \exp\left(\frac{2\pi i u^d \xi_0}{p}\right) \right|_{\mathbb{C}}$$

for any $\xi_0 \in \overline{\mathrm{ac}}_p(Y_i)$. Since $d < p$ we can apply Weil's bound, i.e., the last result from [Wei48], so we have

$$\left| \sum_{u \in \mathbb{F}_p^\times} \exp\left(\frac{2\pi i u^d \xi_0}{p}\right) \right|_{\mathbb{C}} = \left| \sum_{u \in \mathbb{F}_p} \exp\left(\frac{2\pi i u^d \xi_0}{p}\right) - 1 \right|_{\mathbb{C}} \leqslant (d-1)p^{\frac{1}{2}} + 1 \leqslant dp^{\frac{1}{2}},$$

hence

$$\left| \sum_{\substack{\overline{x} \in (p\mathbb{Z}/p^m\mathbb{Z})^n, \\ \mathrm{ord}_p(f(\overline{x}))=m-1}} p^{-mn} \exp\left(\frac{2\pi i f(x)}{p^m}\right) \right|_{\mathbb{C}} = \left| \sum_{\overline{z} \in L_m/p^m\mathbb{Z}_p} G_p(z,m) \exp\left(\frac{2\pi i z}{p^m}\right) \right|_{\mathbb{C}}$$

$$= \left| \sum_{i=1}^d G_i \sum_{\overline{y} \in Y_i/p^m\mathbb{Z}_p} \exp\left(\frac{2\pi i y}{p^m}\right) \right|_{\mathbb{C}} \leqslant \sum_{i=1}^d G_i d p^{\frac{1}{2}}.$$

We also have

$$\sum_{i=1}^d \frac{p-1}{d} G_i = \sum_{\overline{z} \in L_m/p^m\mathbb{Z}_p} G_p(z,m)$$

$$= p^{-mn} \#\{\overline{x} \in (p\mathbb{Z}/p^m\mathbb{Z})^n \mid \mathrm{ord}_p(f(x)) = m-1\} = p^{-mn} \# A_{p,m},$$

where $A_{p,m} := \{\overline{x} \in (p\mathbb{Z}_p/p^m\mathbb{Z}_p)^n \mid \mathrm{ord}_p(f(x)) = m-1\}$. When we view $A_{p,m}$ as a constructible subset of $\mathbb{F}_p^{mn}$, then, by the Lang-Weil estimate [LW54], there exists a constant D'_m, not depending on p, such that

$$\# A_{p,m} \leqslant D'_m p^{\dim_{\mathbb{F}_p}(A_{p,m})}.$$

By Theorem 4.1.4 we have

$$c_0(f) \leqslant \frac{(m-1)n - \dim_{\mathbb{F}_p}(\tilde{A}_{p,m})}{m-1},$$

where $\tilde{A}_{p,m}$ is the image of $A_{p,m}$ under the projection $p_{m-1} \colon (\mathbb{Z}_p/p^m\mathbb{Z}_p)^n \to (\mathbb{Z}_p/p^{m-1}\mathbb{Z}_p)^n$, viewed as a constructible subset of $\mathbb{F}_p^{mn-n}$. Then we have

$$\dim_{\mathbb{F}_p}(A_{m,p}) \leqslant n + \dim_{\mathbb{F}_p}(\tilde{A}_{m,p}) \leqslant mn - (m-1)c_0(f).$$

Now we finish the proof by showing that for all p big enough,

$$\left| \sum_{\substack{\overline{x} \in (p\mathbb{Z}/p^m\mathbb{Z})^n, \\ \mathrm{ord}_p(f(x))=m-1}} p^{-mn} \exp\left(\frac{2\pi i f(x)}{p^m}\right) \right|_{\mathbb{C}} \leqslant \sum_{i=1}^d G_i d p^{\frac{1}{2}} =$$

$$d^2\frac{p^{-mn+\frac{1}{2}}}{p-1}\#A_{p,m} \leqslant 2d^2p^{-mn-\frac{1}{2}}D'_m p^{mn-(m-1)c_0(f)} \leqslant D_m p^{-m\sigma_0},$$

because $\sigma_0 = \min\left\{c_0(f), \frac{1}{2}\right\}$. Here $D_m = 2(m-1)^2D'_m$. □

The last subsum can be easily estimated using the Lang-Weil estimate and Theorem 4.1.4.

Lemma 4.1.6. *Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be a nonconstant polynomial with $f(0) = 0$. Put $\sigma_0 := \min\left\{c_0(f), \frac{1}{2}\right\}$. Then for each integer $m \geqslant 2$, there exist a natural number N_m and a constant $D_m > 0$, such that, for all primes $p > N_m$, we have*

$$\Bigg|\sum_{\substack{\overline{x}\in(p\mathbb{Z}/p^m\mathbb{Z})^n,\\ \mathrm{ord}_p(f(x))\geqslant m}} p^{-mn}\exp\left(\frac{2\pi i f(x)}{p^m}\right)\Bigg|_{\mathbb{C}} \leqslant D_m p^{-m\sigma_0}.$$

Proof. If $\mathrm{ord}_p(f(x)) \geqslant m$, then $\exp\left(\frac{2\pi i f(x)}{p^m}\right) = 1$, so we have

$$\Bigg|\sum_{\substack{\overline{x}\in(p\mathbb{Z}/p^m\mathbb{Z})^n,\\ \mathrm{ord}_p(f(x))\geqslant m}} p^{-mn}\exp\left(\frac{2\pi i f(x)}{p^m}\right)\Bigg|_{\mathbb{C}} = p^{-mn}\#B_{p,m},$$

where $B_{p,m} := \{\overline{x} \in (p\mathbb{Z}_p/p^m\mathbb{Z}_p)^n \mid \mathrm{ord}_p(f(x)) \geqslant m\}$. We can view $B_{p,m}$ as a subvariety of $\mathbb{F}_p^{mn}$. Then, by the Lang-Weil estimate [LW54], there exist a natural number N_m and a constant $D_m > 0$, such that, for all primes $p > N_m$, we have

$$\#B_{p,m} \leqslant D_m p^{\dim_{\mathbb{F}_p}(B_{m,p})}.$$

By Theorem 4.1.4 we have

$$c_0(f) \leqslant \frac{mn - \dim_{\mathbb{F}_p}(B_{m,p})}{m},$$

so $\dim_{\mathbb{F}_p}(B_{m,p}) \leqslant mn - mc_0(f)$. Hence, for all $p > N_m$,

$$\begin{aligned}\Bigg|\sum_{\substack{\overline{x}\in(p\mathbb{Z}/p^m\mathbb{Z})^n,\\ \mathrm{ord}_p(f(x))\geqslant m}} p^{-mn}\exp\left(\frac{2\pi i f(x)}{p^m}\right)\Bigg|_{\mathbb{C}} &\leqslant p^{-mn}D_m p^{mn-mc_0(f)}\\ &\leqslant D_m p^{-m\sigma_0}.\end{aligned}$$

□

We will now put the three lemmas together to prove the Denef-Sperber conjecture for polynomials with log-canonical threshold at most one half (Theorem 4.0.4). The essential ingredient in this proof is the expression that was obtained in Corollary 1.3.7.

Proof of Theorem 4.0.4. From the Lemmas 4.1.1, 4.1.5 and 4.1.6 it follows that, for each $m \geqslant 2$, there exist a natural number N_m and a positive constant C_m, such that for all $p > N_m$, we have

$$|E_f^0(m,p)|_{\mathbb{C}} \leqslant C_m p^{-m\sigma_0}. \tag{4.1.1}$$

We apply Corollary 1.3.7 (with $\mathrm{Supp}(\overline{\Phi}) = \{0\}$) to $E_f^0(m,p)$. For the poles λ of the Igusa zeta functions that occur in this expression, we can write

$$p^{-m\lambda} = p^{-m\rho} \exp\left(\frac{2\pi i \ell m}{N}\right),$$

where ρ is the real part of λ (hence of the form $\frac{\nu}{N}$) and ℓ is some integer. The angular component of this expression can be merged with the complex coefficients from Corollary 1.3.7, as long as the value of $m \bmod N$ is taken into consideration, for example by means of a Presburger-definable set. More precisely, there exist constants $s, M', N' \in \mathbb{N}$, and for each $1 \leqslant i \leqslant s$, there exist constants $\beta_i \in \mathbb{N}$, $\rho_i \in \mathbb{Q}$ and an $\mathcal{L}_{\mathrm{Pres}}$-definable set $A_i \subseteq \mathbb{N}$, such that for all $p > N'$ and for all $1 \leqslant i \leqslant s$, there exists $a_{i,p} \in \mathbb{C}$, such that for all $m > M'$, we have

$$E_f^0(m,p) = \sum_{i=1}^{s} a_{i,p} \mathbb{1}_{A_i}(m) m^{\beta_i} p^{-m\rho_i}. \tag{4.1.2}$$

Moreover, $0 \leqslant \beta_i \leqslant n-1$ and $\sigma_0 \leqslant c_0(f) \leqslant \rho_i$ for all $1 \leqslant i \leqslant s$. After removing finitely many small elements from the sets A_i and enlarging M', we can assume that for each subset $I \subseteq \{1, \ldots, s\}$, the set $\cap_{i\in I} A_i \setminus \cup_{i\notin I} A_i$ is either empty or infinite and that (4.1.2) holds for all $m > M'$. Furthermore, for each $m > M'$ there is a unique subset $I \subseteq \{1, \ldots, s\}$, such that $m \in \cap_{i\in I} A_i \setminus \cup_{i\notin I} A_i$.

Claim 4.1.7. *There exist $\tilde{M} > M'$, $\tilde{N} > N'$ and a constant $\tilde{C} > 0$, such that for all $m > \tilde{M}$, $p > \tilde{N}$ and $1 \leqslant i \leqslant s$, we have*

$$|a_{i,p} \mathbb{1}_{A_i}(m) p^{-m\rho_i}|_{\mathbb{C}} \leqslant \tilde{C} p^{-m\sigma_0}.$$

Since there are only finitely many subsets $I \subseteq \{1, \ldots, s\}$, it suffices to fix a subset I and prove the claim for m restricted to the set $\cap_{i\in I} A_i \setminus \cup_{i\notin I} A_i$ and i restricted to the set I. Without loss of generality, we can assume that $I = \{1, \ldots, r\}$ for some $r \leqslant s$. Since the claim is trivial for $r = 0$, we can assume moreover that $r \geqslant 1$. Now for $p > N'$, $m \in \cap_{i\in I} A_i \setminus \cup_{i\notin I} A_i$ and $m > M'$, we have

$$E_f^0(m,p) = \sum_{i=1}^{r} a_{i,p} m^{\beta_i} p^{-m\rho_i}.$$

From Equation (4.1.1) we can see that, for such m and for all $p > \max\{N', N_m\}$, we have

$$\left| \sum_{i=1}^{r} a_{i,p} m^{\beta_i} p^{-m\rho_i} \right|_{\mathbb{C}} = |E_f^0(m,p)|_{\mathbb{C}} \leqslant C_m p^{-m\sigma_0}.$$

This implies that

$$\left| \sum_{i=1}^{r} a_{i,p} m^{\beta_i} p^{m(\sigma_0-\rho_i)} \right|_{\mathbb{C}} \leqslant C_m.$$

Since the set $\cap_{i\in I} A_i \backslash \cup_{i\notin I} A_i$ is infinite, there exist $m_1, \ldots, m_r \in \cap_{i\in I} A_i \backslash \cup_{i\notin I} A_i$, all bigger than M', such that all of the determinants of the size r and $r-1$ submatrices of the matrix

$$B_p = \left(m_j^{\beta_i} p^{(\sigma_0-\rho_i)m_j} \right)_{1\leqslant j,i\leqslant r}$$

are different from zero for every $p > N_I := \max\{N', N_{m_1}, \ldots, N_{m_r}\}$. We take the determinant of a matrix of size 0 to be 1. We set

$$C_I := \max\{C_{m_i} \mid 1 \leqslant i \leqslant r\};$$

$$u_{j,p} := \sum_{i=1}^{r} a_{i,p} m_j^{\beta_i} p^{(\sigma_0-\rho_i)m_j}, \text{ for } 1 \leqslant j \leqslant r;$$

$$D_p := \det(B_p);$$

$$D_{k,l,p} := (-1)^{k+l} \det\left((m_j^{\beta_i} p^{(\sigma_0-\rho_i)m_j})_{j\neq k, i\neq l} \right), \text{ for } 1 \leqslant k,l \leqslant r.$$

If we write $x_p := (a_{1,p}, \ldots, a_{r,p})^T$ and $u_p := (u_{1,p}, \ldots, u_{r,p})^T$, then x_p is a solution of the equation $B_p X = u_p$. By our assumption on $m_1, \ldots, m_r$ we see that $D_p \neq 0$ and $D_{k,l,p} \neq 0$ for every $1 \leqslant k,l \leqslant r$ and $p > N_I$. Using Cramer's rule we have

$$a_{i,p} = \frac{\sum_{j=1}^{r} u_{j,p} D_{j,i,p}}{D_p},$$

for all $1 \leqslant i \leqslant r$ and $p > N_I$. We remark that $|u_{j,p}|_{\mathbb{C}} \leqslant C_I$, for all $p > N_I$. This gives us

$$|a_{i,p}|_{\mathbb{C}} \leqslant \frac{\sum_{j=1}^{r} |u_{j,p} D_{j,i,p}|_{\mathbb{C}}}{|D_p|_{\mathbb{C}}} \leqslant C_I \frac{\sum_{j=1}^{r} |D_{j,i,p}|_{\mathbb{C}}}{|D_p|_{\mathbb{C}}},$$

for all $1 \leqslant i \leqslant r$ and $p > N_I$. When $p \to \infty$, then the determinants $|D_{j,i,p}|_{\mathbb{C}}$, resp. $|D_p|_{\mathbb{C}}$ behave like $d_{j,i} p^{\gamma_{j,i}}$, resp. dp^{γ}. So there exists α such that, for $1 \leqslant i \leqslant r$ and $p > N_I$, we have $|a_{i,p}|_{\mathbb{C}} \leqslant p^{\alpha}$.

Take $1 \leqslant i \leqslant r$. To finish the proof of the claim we will distinguish between two cases.

- If $\rho_i > \sigma_0$, then there exists $\tilde{M}_i > M'$ such that, for every $m > \tilde{M}_i$ and $p > N_I$ we have

$$|a_{i,p} p^{-m\rho_i}|_{\mathbb{C}} \leqslant p^{\alpha - m\rho_i} \leqslant p^{-m\sigma_0}.$$

- If $\rho_i = \sigma_0$, we observe that

$$D_p = \sum_{j=1}^{r} m_j^{\beta_i} p^{(\sigma_0 - \rho_i)m_j} D_{j,i,p} = \sum_{j=1}^{r} m_j^{\beta_i} D_{j,i,p}.$$

  This means that $\gamma = \max\{\gamma_{j,i} \mid 1 \leqslant j \leqslant r\}$. Thus there exist $\tilde{C}_i > 0$ and $\tilde{N}_i > N_I$, such that

$$|a_{i,p}|_{\mathbb{C}} \leqslant C_I \frac{\sum_{j=1}^{r} |D_{j,i,p}|_{\mathbb{C}}}{|D_p|_{\mathbb{C}}} \leqslant \tilde{C}_i,$$

  for all $p > \tilde{N}_i$. Thus

$$|a_{i,p} p^{-m\rho_i}|_{\mathbb{C}} \leqslant \tilde{C}_i p^{-m\sigma_0},$$

  for all $p > \tilde{N}_i$.

This proves the claim. Hence we have

$$|E_f^0(m,p)|_{\mathbb{C}} = \left| \sum_{i=1}^{s} a_{i,p} \mathbb{1}_{A_i}(m) m^{\beta_i} p^{-m\rho_i} \right|_{\mathbb{C}} \leqslant s\tilde{C} m^{n-1} p^{-m\sigma_0},$$

for all $m > \tilde{M}$, $p > \tilde{N}$. By Equation (4.1.1) we also have, for each $2 \leqslant m \leqslant \tilde{M}$, an upper bound for $|E_f^0(m,p)|_{\mathbb{C}}$ in terms of some constant C_m. Now let $N := \max\{\tilde{N}, N_2, \ldots, N_{\tilde{M}}\}$ and $C := \max\{s\tilde{C}, C_2, \ldots, C_{\tilde{M}}\}$, then we have

$$|E_f^0(m,p)|_{\mathbb{C}} = \left| \sum_{i=1}^{s} a_{i,p} \mathbb{1}_{A_i}(m) m^{\beta_i} p^{-m\rho_i} \right|_{\mathbb{C}} \leqslant C m^{n-1} p^{-m\sigma_0},$$

for all $m \geqslant 2$ and $p > N$. □

4.2 The geometric approach

In this section we will give another proof of Theorem 4.0.4, using the results from Section 1.3 on Igusa zeta functions and their relation to exponential sums. We will work over the number field $\mathbb{Q}$ with ring of integers $\mathbb{Z}$, $f \in \mathbb{Z}[x_1, \ldots, x_n]$

a nonconstant polynomial with $f(0)=0$ and (Y,h) a fixed embedded resolution of the singularities of $f^{-1}(0)$. For any prime number p, we will write Φ_p for the Schwartz-Bruhat function $\mathbb{1}_{p\mathbb{Z}_p^n}$, $Z(s,p,\chi)$ for the zeta function $Z^{\Phi_p}(s,p,\chi)$ and $c_{I,\chi}$ for c_{I,χ,Φ_p}.

We will need the following lemma several times in our calculations.

Lemma 4.2.1. *Let $f\in\mathbb{Z}[x_1,\ldots,x_n]\setminus\mathbb{Z}$. Then there exists $\tilde{N}\in\mathbb{N}$, such that for all $p>\tilde{N}$ and for all $y\in\mathbb{Z}_p^n$ for which $f(y)=0$, we have*

$$c_y(f)=\min_{i\in T:\overline{y}\in\overline{h}(\overline{E_i}(\mathbb{F}_p))}\left\{\frac{\nu_i}{N_i}\right\}.$$

Proof. There exists $\tilde{N}\in\mathbb{N}$ such that (Y,h) has good reduction modulo p for all $p>\tilde{N}$. If $p>\tilde{N}$, $y\in f^{-1}(0)\cap\mathbb{Z}_p^n$ and E is an irreducible component of $(f\circ h)^{-1}(0)$, such that $\overline{y}\in\overline{h}(\overline{E})$, then $h(E)\cap(y+p\mathbb{Z}_p^n)\neq\emptyset$. Remark that h is proper, so $h(E)$ is a closed subvariety of $\mathbb{A}^n$. Therefore, after possibly enlarging $\tilde{N}$ if necessary, we can assume that for all $p>\tilde{N}$, if $y\notin h(E)$, then $h(E)\cap(y+p\mathbb{Z}_p^n)=\emptyset$. Hence, for $p>\tilde{N}$, $\overline{y}\in\overline{h}(\overline{E})$ implies $y\in h(E)$. So the map $E\mapsto\overline{E}$ is a bijection between

$$\{E_i\mid i\in T,\ y\in h(E_i)\}\qquad\text{and}\qquad\{\overline{E_i}\mid i\in T,\ \overline{y}\in\overline{h}(\overline{E_i})\}.$$

This completes the proof. □

To prove Theorem 4.0.4, we use Proposition 1.3.6 with $z_0=1$, $\varpi=p$ and $m\geqslant 2$. This proposition tells us that $E_f^0(m,p)$ is equal to

$$Z(0,p,\chi_{\mathrm{triv}})+\mathrm{Coeff}_{t^{m-1}}\frac{(t-p)Z(s,p,\chi_{\mathrm{triv}})}{(p-1)(1-t)}+\sum_{\chi\neq\chi_{\mathrm{triv}}}g_{\chi^{-1}}\mathrm{Coeff}_{t^{m-1}}Z(s,p,\chi).$$

We will split this expression in two parts and in the following two lemmas we will prove estimates of these two parts.

Lemma 4.2.2. *There exist a constant $C>0$ and a natural number N, such that for all $m\geqslant 2$ and for all primes $p>N$, we have*

$$\left|Z(0,p,\chi_{\mathrm{triv}})+\mathrm{Coeff}_{t^{m-1}}\frac{(t-p)Z(s,p,\chi_{\mathrm{triv}})}{(p-1)(1-t)}\right|_{\mathbb{C}}\leqslant Cm^{n-1}p^{-mc_0(f)}.$$

Proof. We use Theorem 1.3.3 which tells us that there exists a natural number N', such that for all $p>N'$,

$$Z(0,p,\chi_{\mathrm{triv}})=\sum_{I\subseteq T}c_{I,\chi_{\mathrm{triv}}}\frac{(p-1)^{\#I}}{p^n}\prod_{i\in I}\frac{p^{-\nu_i}}{1-p^{-\nu_i}};\tag{4.2.1}$$

$$Z(s,p,\chi_{\mathrm{triv}}) = \sum_{I\subseteq T} c_{I,\chi_{\mathrm{triv}}} \frac{(p-1)^{\#I}}{p^n} \prod_{i\in I} \frac{t^{N_i}p^{-\nu_i}}{1-t^{N_i}p^{-\nu_i}}.$$

Since $\frac{(t-p)}{(p-1)(1-t)} = -\frac{1}{p-1} - \frac{1}{1-t}$, we need to consider

$$\mathrm{Coeff}_{t^{m-1}} \frac{Z(s,p,\chi_{\mathrm{triv}})}{p-1} = \sum_{I\subseteq T} c_{I,\chi_{\mathrm{triv}}} \frac{(p-1)^{\#I}}{p^n(p-1)} \mathrm{Coeff}_{t^{m-1}} \prod_{i\in I} \frac{t^{N_i}p^{-\nu_i}}{1-t^{N_i}p^{-\nu_i}} \tag{4.2.2}$$

and

$$\mathrm{Coeff}_{t^{m-1}} \frac{Z(s,p,\chi_{\mathrm{triv}})}{1-t} = \sum_{I\subseteq T} c_{I,\chi_{\mathrm{triv}}} \frac{(p-1)^{\#I}}{p^n} \mathrm{Coeff}_{t^{m-1}} \frac{1}{1-t} \prod_{i\in I} \frac{t^{N_i}p^{-\nu_i}}{1-t^{N_i}p^{-\nu_i}}. \tag{4.2.3}$$

In what follows, we will give upper bounds for (4.2.2) and for the difference of (4.2.1) and (4.2.3). By writing the fractions $\frac{t^{N_i}p^{-\nu_i}}{1-t_i^{N}p^{-\nu_i}}$ as power series in t, we get

$$\mathrm{Coeff}_{t^{m-1}} \prod_{i\in I} \frac{t^{N_i}p^{-\nu_i}}{1-t^{N_i}p^{-\nu_i}} = \sum_{(\ell_i)_{i\in I}\in J_{m-1}} p^{-\sum_{i\in I}\nu_i(\ell_i+1)}; \tag{4.2.4}$$

$$\mathrm{Coeff}_{t^{m-1}} \frac{1}{1-t} \prod_{i\in I} \frac{t^{N_i}p^{-\nu_i}}{1-t^{N_i}p^{-\nu_i}} = \sum_{(\ell_i)_{i\in I}\in J_{\leqslant m-1}} p^{-\sum_{i\in I}\nu_i(\ell_i+1)}, \tag{4.2.5}$$

where

$$J_{m-1} := \Big\{(\ell_i)_{i\in I} \in \mathbb{N}^{\#I} \mid \sum_{i\in I} N_i(\ell_i+1) = m-1\Big\};$$

$$J_{\leqslant m-1} := \Big\{(\ell_i)_{i\in I} \in \mathbb{N}^{\#I} \mid \sum_{i\in I} N_i(\ell_i+1) \leqslant m-1\Big\}.$$

Note that if $I\subseteq T$, such that $\overline{E_I}^{\circ} \cap \overline{h}^{-1}(0) = \overline{E_I}^{\circ} \cap \overline{h}^{-1}(\mathrm{Supp}(\overline{\Phi_p})) = \emptyset$, then $c_{I,\chi_{\mathrm{triv}}} = 0$. Hence we can restrict the Equations (4.2.1), (4.2.2) and (4.2.3) to sums over $I\subseteq T$ for which $\overline{E_I}^{\circ} \cap \overline{h}^{-1}(0) \neq \emptyset$. For such $I\subseteq T$, we have $\#I \leqslant n$. Moreover, by Lemma 4.2.1, there exists $\tilde{N}\in\mathbb{N}$, such that for all $p>\tilde{N}$, $I\subseteq T$ with $\overline{E_I}^{\circ} \cap \overline{h}^{-1}(0) \neq \emptyset$ and $(\ell_i)_{i\in I}\in J_{m-1}$, we have

$$-\sum_{i\in I}\nu_i(\ell_i+1) \leqslant -\sum_{i\in I} N_i(\ell_i+1)c_0(f) = -(m-1)c_0(f),$$

because $\frac{\nu_i}{N_i} \geqslant c_0(f)$ for all $i \in I$. By using Equation (4.2.4), we get

$$\mathrm{Coeff}_{t^{m-1}} \prod_{i\in I} \frac{t^{N_i}p^{-\nu_i}}{1-t^{N_i}p^{-\nu_i}} \leqslant \#(J_{m-1})p^{-(m-1)c_0(f)} \leqslant m^{n-1}p^{-(m-1)c_0(f)}, \tag{4.2.6}$$

for all $p > \tilde{N}$.

Next, we want to find an upper bound for the difference of (4.2.1) and (4.2.3). Using Equation (4.2.5) we can see that we need to analyse the expression

$$\begin{aligned}
&\prod_{i\in I} \frac{p^{-\nu_i}}{1-p^{-\nu_i}} - \sum_{(\ell_i)_{i\in I}\in J_{\leqslant m-1}} p^{-\sum_{i\in I}\nu_i(\ell_i+1)} = \\
&\sum_{(\ell_i)_{i\in I}\in \mathbb{N}^{\#I}} p^{-\sum_{i\in I}\nu_i(\ell_i+1)} - \sum_{(\ell_i)_{i\in I}\in J_{\leqslant m-1}} p^{-\sum_{i\in I}\nu_i(\ell_i+1)} = \\
&\sum_{(\ell_i)_{i\in I}\in J_{\geqslant m}} p^{-\sum_{i\in I}\nu_i(\ell_i+1)} = \qquad (4.2.7)\\
&\sum_{(\ell_i)_{i\in I}\in \overline{J}_{\geqslant m}} p^{-\sum_{i\in I}\nu_i(\ell_i+1)} + \sum_{(\ell_i)_{i\in I}\in J_{\geqslant m}\setminus\overline{J}_{\geqslant m}} p^{-\sum_{i\in I}\nu_i(\ell_i+1)},
\end{aligned}$$

where

$$\begin{aligned}
J_{\geqslant m} &:= \Big\{(\ell_i)_{i\in I} \in \mathbb{N}^{\#I} \mid \sum_{i\in I} N_i(\ell_i+1) \geqslant m\Big\};\\
m_I &:= m + \max\{N_i \mid i \in I\};\\
\overline{J}_{\geqslant m} &:= \Big\{(\ell_i)_{i\in I} \in \mathbb{N}^{\#I} \mid m \leqslant \sum_{i\in I} N_i(\ell_i+1) \leqslant m_I\Big\}.
\end{aligned}$$

Suppose $(\ell_i)_{i\in I} \in J_{\geqslant m} \setminus \overline{J}_{\geqslant m}$, then $\sum_{i\in I} N_i(\ell_i+1) > m_I$. Hence, there exists a tuple $(l_i)_{i\in I} \in \mathbb{N}^{\#I}$, such that for each $i \in I$, $l_i \leqslant \ell_i$, and $m \leqslant \sum_{i\in I} N_i(\ell_i - l_i + 1) \leqslant m_I$. In other words, $(\ell_i - l_i)_{i\in I} \in \overline{J}_{\geqslant m}$. Now we can write

$$\begin{aligned}
p^{-\sum_{i\in I}\nu_i(\ell_i+1)} &= p^{-\sum_{i\in I}\nu_i l_i} \cdot p^{-\sum_{i\in I}\nu_i(\ell_i-l_i+1)}\\
&= \Big(\prod_{i\in I} p^{-\nu_i l_i}\Big) p^{-\sum_{i\in I}\nu_i(\ell_i-l_i+1)}.
\end{aligned}$$

This implies that

$$\sum_{(\ell_i)_{i\in I}\in J_{\geqslant m}} p^{-\sum_{i\in I}\nu_i(\ell_i+1)} \leqslant \Big(1+\prod_{i\in I}\frac{1}{1-p^{-\nu_i}}\Big) \sum_{(\ell_i)_{i\in I}\in \overline{J}_{\geqslant m}} p^{-\sum_{i\in I}\nu_i(\ell_i+1)}. \tag{4.2.8}$$

If $I\subseteq T$ with $\overline{E_I}^\circ\cap\overline{h}^{-1}(0)\neq\emptyset$, $p>\tilde{N}$ and $(\ell_i)_{i\in I}\in\overline{J}_{\geqslant m}$, then, we can use Lemma 4.2.1 again for the following estimate:

$$-\sum_{i\in I}\nu_i(\ell_i+1)\leqslant-\sum_{i\in I}N_i(\ell_i+1)c_0(f)\leqslant -mc_0(f).$$

Combining this with the Equations (4.2.5), (4.2.7) and (4.2.8), we find that, for all $I\subseteq T$ with $\overline{E_I}^\circ\cap\overline{h}^{-1}(0)\neq\emptyset$, and for all $p>\tilde{N}$, we have,

$$\begin{aligned}\prod_{i\in I}\frac{p^{-\nu_i}}{1-p^{-\nu_i}}-\mathrm{Coeff}_{t^{m-1}}\frac{1}{1-t}\prod_{i\in I}\frac{t^{N_i}p^{-\nu_i}}{1-t^{N_i}p^{-\nu_i}} &\leqslant (1+2^{\#I})\#(\overline{J}_{\geqslant m})p^{-mc_0(f)}\\ &\leqslant C_I m^{n-1}p^{-mc_0(f)},\end{aligned} \tag{4.2.9}$$

where C_I is a constant which does not depend on m and p, for example we can take $C_I=(1+2^{\#I})(\max\{N_i\mid i\in I\}+1)$.

We also have to estimate the constants $c_{I,\chi_{\mathrm{triv}}}$. By the Lang-Weil estimate [LW54], there exist, for each $I\subseteq T$, a constant D_I and a natural number N_I, depending only on I, such that for all $p>N_I$, we have

$$c_{I,\chi_{\mathrm{triv}}}=\sum_{a\in\overline{E_I}^\circ\cap\overline{h}^{-1}(0)}\Omega_{\chi_{\mathrm{triv}}}(a)=\#\big(\overline{E_I}^\circ\cap\overline{h}^{-1}(0)\big)\leqslant\#\overline{E_I}\leqslant D_Ip^{n-\#I}. \tag{4.2.10}$$

From the Equations (4.2.1), (4.2.2), (4.2.3), (4.2.6), (4.2.9) and (4.2.10), it follows that there exists a natural number $N>\max\{N',\tilde{N},(N_I)_{I\subseteq T}\}$, such that for all $p>N$, we have

$$\begin{aligned}&\left|Z(0,p,\chi_{\mathrm{triv}})+\mathrm{Coeff}_{t^{m-1}}\frac{(t-p)Z(s,p,\chi_{\mathrm{triv}})}{(p-1)(1-t)}\right|_{\mathbb{C}}\\ &\leqslant\sum_{\substack{I\subseteq T,\\ \overline{E_I}^\circ\cap\overline{h}^{-1}(0)\neq\emptyset}}c_{I,\chi_{\mathrm{triv}}}\frac{(p-1)^{\#I}}{p^n}m^{n-1}\left(\frac{p^{-(m-1)c_0(f)}}{p-1}+C_Ip^{-mc_0(f)}\right)\\ &\leqslant\sum_{I\subseteq T}D_Ip^{n-\#I}\frac{(p-1)^{\#I}}{p^n}m^{n-1}\left(\frac{p^{-mc_0(f)+c_0(f)}}{p-1}+C_Ip^{-mc_0(f)}\right)\end{aligned}$$

$$\leqslant \sum_{I\subseteq T} D_I(C_I+2)m^{n-1}p^{-mc_0(f)} \leqslant Cm^{n-1}p^{-mc_0(f)},$$

where $C = \sum_{I\subseteq T} D_I(C_I+2)$ is a constant that is independent of p and m and where we have used the fact that $c_0(f) \leqslant 1$. □

Lemma 4.2.3. *There exist a constant $C > 0$ and a natural number N, such that for all $m \geqslant 2$ and for all primes $p > N$, we have*

$$\Big| \sum_{\chi\neq\chi_{\mathrm{triv}}} g_{\chi^{-1}} \mathrm{Coeff}_{t^{m-1}} Z(s,p,\chi)\Big|_{\mathbb{C}} \leqslant Cm^{n-1}p^{-m\sigma_0}.$$

Proof. Denote by Υ_p the set of non-trivial multiplicative characters χ on $\mathbb{Z}_p^\times$ for which $Z(s,p,\chi) \neq 0$. By Theorem 1.3.5 and Remark 1.3.4 there exist $M, N_0 \in \mathbb{N}$, such that for all $p > N_0$, the set Υ_p has at most M elements. Moreover any such character has conductor $c(\chi) = 1$.

Just as in the proof of Lemma 4.2.2, we use Theorem 1.3.3. Hence, there exists a natural number N', such that for all $p > N'$ and for all characters χ with conductor $c(\chi) = 1$, we have

$$Z(s,p,\chi) = \sum_{\substack{I\subseteq T,\\ \forall i\in I: d(\chi)|N_i}} c_{I,\chi}\frac{(p-1)^{\#I}}{p^n}\prod_{i\in I}\frac{t^{N_i}p^{-\nu_i}}{1-t^{N_i}p^{-\nu_i}}.$$

We can assume as well that we only sum over $I \subseteq T$ for which $\overline{E_I}^\circ \cap \overline{h}^{-1}(0) \neq \emptyset$. For any such I, and for any $p > \tilde{N}$ (coming from Lemma 4.2.1), Equation (4.2.6) holds:

$$\mathrm{Coeff}_{t^{m-1}} \prod_{i\in I}\frac{t^{N_i}p^{-\nu_i}}{1-t^{N_i}p^{-\nu_i}} \leqslant m^{n-1}p^{-(m-1)c_0(f)}.$$

We use the Lang-Weil estimate [LW54] again, so for each $I \subseteq T$, there exist a constant D_I and a natural number N_I, depending only on I, such that for all $p > N_I$, we have

$$|c_{I,\chi}|_{\mathbb{C}} = \Big|\sum_{a\in\overline{E_I}^\circ\cap\overline{h}^{-1}(0)} \Omega_\chi(a)\Big|_{\mathbb{C}} \leqslant \sum_{a\in\overline{E_I}^\circ\cap\overline{h}^{-1}(0)} 1$$

$$= \#\big(\overline{E_I}^\circ \cap \overline{h}^{-1}(0)\big) \leqslant \#\overline{E_I} \leqslant D_I p^{n-\#I}.$$

If we take $N > \max\{N_0, N', \tilde{N}, (N_I)_{I\subseteq T}\}$, then for all $p > N$ and for any character χ with $c(\chi) = 1$, we have

$$\begin{aligned}
|\mathrm{Coeff}_{t^{m-1}} Z(s,p,\chi)|_{\mathbb{C}} &\leqslant \sum_{\substack{I\subseteq T,\\ \forall i\in I: \overline{d}(\chi)|N_i,\\ \overline{E_I}^{\circ}\cap \overline{h}^{-1}(0)}} c_{I,\chi}\frac{(p-1)^{\#I}}{p^n} m^{n-1}p^{-(m-1)c_0(f)}\\
&\leqslant \sum_{I\subseteq T} D_I p^{n-\#I}\frac{(p-1)^{\#I}}{p^n} m^{n-1}p^{-(m-1)c_0(f)}\\
&\leqslant \sum_{I\subseteq T} D_I m^{n-1}p^{-(m-1)c_0(f)}\\
&\leqslant C' m^{n-1}p^{-(m-1)c_0(f)},
\end{aligned}$$

where $C' := \sum_{I\subseteq T} D_I$. Furthermore, by a standard result on Gauss sums, we can see that, if $\chi \neq \chi_{\mathrm{triv}}$, then $|g_{\chi^{-1}}|_{\mathbb{C}} \leqslant Dp^{-\frac{1}{2}}$, for some constant D, that does not depend on χ and p. So for any $p > N$, we have

$$\begin{aligned}
\Big|\sum_{\chi\neq\chi_{\mathrm{triv}}} g_{\chi^{-1}}\mathrm{Coeff}_{t^{m-1}} Z(s,p,\chi)\Big|_{\mathbb{C}} &= \Big|\sum_{\chi\in\Upsilon_p} g_{\chi^{-1}}\mathrm{Coeff}_{t^{m-1}} Z(s,p,\chi)\Big|_{\mathbb{C}}\\
&\leqslant \sum_{\chi\in\Upsilon_p} Dp^{-\frac{1}{2}}C'm^{n-1}p^{-(m-1)c_0(f)}\\
&\leqslant \sum_{\chi\in\Upsilon_p} DC'm^{n-1}p^{-(m-1)\sigma_0-\sigma_0}\\
&\leqslant Cm^{n-1}p^{-m\sigma_0},
\end{aligned}$$

where $C = MDC'$ is a constant that is independent of p and m and where we have used the fact that $\sigma_0 = \min\left\{c_0(f), \frac{1}{2}\right\}$. □

Proof of Theorem 4.0.4. The proof follows by combining the Proposition 1.3.6 with the two Lemmas 4.2.2 and 4.2.3 and using the fact that $\sigma_0 \leqslant c_0(f)$. □

Remark 4.2.4. The proofs in this section work for a wider range of exponential sums. If we take, for each prime p, instead of $\mathbb{1}_{p\mathbb{Z}_p^n}$, Φ_p to be any residual Schwartz-Bruhat function, for which $C_{\overline{f}}\cap\mathrm{Supp}(\overline{\Phi_p}) \subseteq \overline{f}^{-1}(0)$, then the Lemmas 4.2.2 and 4.2.3 still hold, when we replace $c_0(f)$ by $c(f)$. In the proofs we have to replace $\overline{E_I}^{\circ} \cap \overline{h}^{-1}(0)$ by $\overline{E_I}^{\circ} \cap \overline{h}^{-1}(\mathrm{Supp}(\overline{\Phi_p}))$. The constant C and the natural number N that are found in these proofs, do not depend on the tuple of functions $(\Phi_p)_p$. They do depend however on f and on the embedded resolution (Y,h) of f. So for all $m \geqslant 1$ and for all primes $p > N$, we have

$$\left|E_f^{\Phi_p}(m,p)\right|_{\mathbb{C}} \leqslant Cm^{n-1}p^{-m\sigma'},$$

where $\sigma' := \min\left\{c(f), \frac{1}{2}\right\}$.

4.3 Proof of the global Cluckers-Veys upper bound

In this section we will prove the global upper bound (4.0.1) from the Cluckers-Veys conjecture for $\sigma := \min\left\{a(f), \frac{1}{2}\right\}$ by adapting the proofs from Section 4.2. We saw in Equation (1.3.1) that there exists, for each prime number p, a constant $C_p > 0$, such that for all $m \geqslant 2$, we have

$$|E_f(m,p)|_{\mathbb{C}} \leqslant C_p m^{n-1}p^{-ma(f)}.$$

Therefore, in order to prove the global Cluckers-Veys upper bound, it is enough to prove it for all primes p big enough. More precisely, we will prove that there exists a constant $C > 0$ and a natural number N, such that for all $m \geqslant 2$ and for all primes $p > N$, we have

$$|E_f(m,p)|_{\mathbb{C}} \leqslant Cm^{n-1}p^{-m\sigma}. \tag{4.3.1}$$

First, we need the following lemma.

Lemma 4.3.1. *Let $f \in \mathbb{Z}[x_1,\ldots,x_n]$ and $V_{f,p}$ be the set of critical values of f in $\mathbb{Q}_p$. Then $\#(V_{f,p})$ has an upper bound d, that does not depend on p. Furthermore, there exists $N' \in \mathbb{N}$, such that for all $p > N'$, the following holds:*

(i) for all $z \in V_{f,p}$, we have $\operatorname{ord}_p(z) = 0$*;*

(ii) for any two distinct points z_1, z_2 in $V_{f,p}$, we have $\operatorname{ord}_p(z_1 - z_2) = 0$*;*

(iii) if $x \in \mathbb{Z}_p^n$ such that $\operatorname{ord}_p(f(x) - z) = 0$ *for all $z \in V_{f,p}$, then x, resp. $\overline{x}$, is a regular point of f, resp. $\overline{f}$.*

Proof. Remark that we can uniquely extend the valuation ord_p from $\mathbb{Q}_p$ to $(\mathbb{Q}_p)^{\mathrm{alg}}$ (an algebraic closure of $\mathbb{Q}_p$). We denote by

$$\mathcal{O}_p = \{z \in (\mathbb{Q}_p)^{\mathrm{alg}} \mid \operatorname{ord}_p(z) \geqslant 0\}$$

the ring of integers of $(\mathbb{Q}_p)^{\mathrm{alg}}$ and by

$$\mathcal{M}_p = \{z \in (\mathbb{Q}_p)^{\mathrm{alg}} \mid \mathrm{ord}_p(z) > 0\}$$

its maximal ideal.

The set of critical values V_f of f is a definable set in $\mathcal{L}_{\mathrm{ring}}$ given by

$$z \in V_f \Leftrightarrow \exists y : f(y) = z \wedge \frac{\partial f}{\partial x_1}(y) = 0 \wedge \ldots \wedge \frac{\partial f}{\partial x_n}(y) = 0.$$

By elimination of quantifiers in the ACF_0-theory, i.e., the theory of algebraically closed fields of characteristic 0, and the fact that V_f is a finite set, there exist nonzero polynomials $T(z) \in \mathbb{Z}[z]$ and $R(z) \in \mathbb{Q}^{\mathrm{alg}}[z]$, such that

$$V_f = Z(R) \subseteq Z(T) \subseteq \mathbb{Q}^{\mathrm{alg}},$$

where $Z(R)$ and $Z(T)$ denote the zero sets of R, resp. T. Moreover, we can assume that $T(z)$ and $R(z)$ only have simple roots in $\mathbb{Q}^{\mathrm{alg}}$. Since $V_{f,p} \subseteq V_f$, we have

$$\#(V_{f,p}) \leqslant \#(V_f) = \deg(R) =: d,$$

for all primes p. Because $Z(T) \subseteq \mathbb{Q}^{\mathrm{alg}}$ is a finite set of algebraic numbers, there exists $N'' \in \mathbb{N}$, such that for all $p > N''$, $Z(T)$ satisfies the conditions (i) and (ii). This implies that these two conditions also hold for $Z(R)$ and for $V_{f,p}$.

To prove condition (iii) we put

$$\overline{T} := (T \bmod p) \in \mathbb{F}_p[z] \quad \text{and} \quad \overline{R} := (R \bmod \mathcal{M}_p) \in (\mathbb{F}_p)^{\mathrm{alg}}[z].$$

By logical compactness, there exists $N' > N''$, such that for all $p > N'$, the polynomials $\overline{T}(z)$ and $\overline{R}(z)$ also only have simple roots in $(\mathbb{F}_p)^{\mathrm{alg}}$ and $V_{\overline{f}} = Z(\overline{R}) \subseteq Z(\overline{T}) \subseteq (\mathbb{F}_p)^{\mathrm{alg}}$.

Now take $p > N'$ and $x \in \mathbb{Z}_p^n$ such that $\mathrm{ord}_p(f(x) - z) = 0$ for all $z \in V_{f,p}$. Then $f(x) \notin V_{f,p}$, so x is a regular point of f. Suppose, for a contradiction, that $\overline{x}$ is a critical point of $\overline{f}$, then $\xi := \overline{f}(\overline{x}) \in V_{\overline{f}} = Z(\overline{R}) \subseteq Z(\overline{T})$. From the facts that $\overline{T}$ has only simple roots in $(\mathbb{F}_p)^{\mathrm{alg}}$, $\xi \in \mathbb{F}_p$ and $\overline{T}(\xi) = 0$, it follows by Hensel's lemma that there exists $z_1 \in \mathbb{Z}_p$ such that $T(z_1) = 0$ and $\overline{z_1} = \xi$. Hence $\mathrm{ord}_p(f(x) - z_1) > 0$, and therefore $z_1 \notin V_{f,p}$. On the other hand, $\overline{R}$ also has only simple roots in $(\mathbb{F}_p)^{\mathrm{alg}}$ and $\overline{R}(\xi) = 0$. Thus, again by Hensel's lemma, there exists $z_2 \in \mathcal{O}_p$ such that $R(z_2) = 0$ and $\overline{z_2} = \xi$. From the facts that z_1 and z_2 are both roots of T, $\overline{z_1} = \xi = \overline{z_2}$ and $Z(T)$ satisfies the conditions (i) and (ii), it follows that $z_1 = z_2$. Hence $z_1 \in Z(R) = V_f$, and we knew already that $z_1 \in \mathbb{Z}_p$ so $z_1 \in V_{f,p}$. This is a contradiction and thus condition (iii) also holds. $\square$

Now we are ready to prove the global upper bound from Theorem 4.0.3.

Proof of (4.3.1). Let N' and d be as in Lemma 4.3.1 and write $V_f = \{z_1, \ldots, z_d\}$. For each $1 \leqslant j \leqslant d$, we define $f_j(x) := f(x) - z_j \in \mathbb{Z}[z_j][x_1, \ldots, x_n]$, which is a polynomial with coefficients in the ring $\mathbb{Z}[z_j]$, and we fix an embedded resolution $h_j \colon Y_j \to (\mathbb{Q}^{\mathrm{alg}})^n$ of $f_j^{-1}(0)$.

If we fix $p > N'$, then $V_{f,p} \subseteq \{z_1, \ldots, z_d\}$. We define, for each $1 \leqslant j \leqslant d$, a Schwartz-Bruhat function on $\mathbb{Q}_p^n$ as follows:

$$\Phi_j := \begin{cases} \mathbb{1}_{\{x \in \mathbb{Z}_p^n \mid \mathrm{ord}_p(f_j(x)) > 0\}} & \text{if } z_j \in V_{f,p}; \\ 0 & \text{if not.} \end{cases}$$

Since $f \in \mathbb{Z}[x_1, \ldots, x_n]$ and by the conditions (i) and (ii) from Lemma 4.3.1 we see that Φ_j is residual, for all $1 \leqslant j \leqslant d$, and that $\mathrm{Supp}(\Phi_i) \cap \mathrm{Supp}(\Phi_j) = \emptyset$, if $i \neq j$. Furthermore, by definition of Φ_j, we have $C_{\overline{f_j}} \cap \mathrm{Supp}(\overline{\Phi_j}) \subseteq \overline{f_j}^{-1}(0)$. This means that we can apply Remark 4.2.4 to the exponential sums $E_{f_j}^{\Phi_j}(m,p)$. Thus for each $1 \leqslant j \leqslant d$, there exist a constant C_j and a natural number $N_j > N'$, only depending on f and on the chosen resolution (Y_j, h_j) of $f_j^{-1}(0)$, such that, for all $m \geqslant 1$ and for all $p > N_j$, we have

$$|E_{f_j}^{\Phi_j}(m,p)|_{\mathbb{C}} \leqslant C_j m^{n-1} p^{-m\sigma_j},$$

where $\sigma_j := \min\left\{c(f_j), \frac{1}{2}\right\}$.

For any $p > N'$, we take $\Phi_0 := \mathbb{1}_{\mathbb{Z}_p^n} - \sum_{j=1}^d \Phi_j$. Then Φ_0 is also residual and by condition (iii) from Lemma 4.3.1, we have $C_{\overline{f}} \cap \mathrm{Supp}(\overline{\Phi_0}) = \emptyset$. It is well known that this implies that $E_f^{\Phi_0}(m,p) = 0$ for all $m \geqslant 2$ (see [Den91b, Remark 4.5.3]).

Therefore, for any $m \geqslant 2$ and for any $p > N := \max\{N', N_1, \ldots, N_d\}$, we have

$$\begin{aligned} |E_f(m,p)|_{\mathbb{C}} &= \Big| \int_{\mathbb{Z}_p^n} \exp\left(\frac{2\pi i f(x)}{p^m}\right) |dx| \Big|_{\mathbb{C}} \\ &\leqslant \sum_{j=0}^d \Big| \int_{\mathbb{Z}_p^n} \Phi_j(x) \exp\left(\frac{2\pi i f(x)}{p^m}\right) |dx| \Big|_{\mathbb{C}} \\ &\leqslant \sum_{j=1}^d \Big| \int_{\mathbb{Z}_p^n} \Phi_j(x) \exp\left(\frac{2\pi i f(x)}{p^m}\right) |dx| \Big|_{\mathbb{C}} + |E_f^{\Phi_0}(m,p)|_{\mathbb{C}} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{d} \Big| \int_{\mathbb{Z}_p^n} \Phi_j(x) \exp\left(\frac{2\pi i z_j}{p^m}\right) \exp\left(\frac{2\pi i (f(x)-z_j)}{p^m}\right) |dx| \Big|_{\mathbb{C}} \\
&\leqslant \sum_{j=1}^{d} \Big| \exp\left(\frac{2\pi i z_j}{p^m}\right) \Big|_{\mathbb{C}} |E_{f_j}^{\Phi_j}(m,p)|_{\mathbb{C}} \\
&\leqslant \sum_{j=1}^{d} C_j m^{n-1} p^{-m\sigma_j} \\
&\leqslant C m^{n-1} p^{-m\sigma},
\end{aligned}
$$

where $C := \sum_{j=1}^{d} C_j$ and $\sigma = \min\left\{a(f), \frac{1}{2}\right\} \leqslant \min\left\{c(f_j), \frac{1}{2}\right\} = \sigma_j$ for all $1 \leqslant j \leqslant d$. $\square$

4.4 Proof of the uniform local Cluckers-Veys upper bounds

In this section we will prove the local upper bounds (4.0.2) from the Cluckers-Veys conjecture for $\sigma_{y,p} := \min\left\{a_{y,p}(f), \frac{1}{2}\right\}$, for each $y \in \mathbb{Z}^n$, by adapting the proofs from the Sections 4.1 and 4.2. For each $y \in \mathbb{Z}^n$ and for each prime p we denote by $\Phi_{y,p}$ the Schwarz-Bruhat function $\mathbb{1}_{y+p\mathbb{Z}_p^n}$ and by $Z^y(s,p,\chi)$ the zeta function $Z^{\Phi_{y,p}}(s,p,\chi)$.

For each prime p we will partition $\mathbb{Z}^n$, according to the location of the critical values of f. Let N' and d be as in Lemma 4.3.1 and write $V_f = \{z_1, \ldots, z_d\}$. For each $1 \leqslant j \leqslant d$, we define $f_j(x) := f(x) - z_j$ and we fix an embedded resolution $h_j \colon Y_j \to (\mathbb{Q}^{\mathrm{alg}})^n$ of $f_j^{-1}(0)$. Let N_j' be a natural number, such that for all $p > N_j'$, (Y_j, h_j) has good reduction modulo p. We put $N_0 := \max\{N', N_1', \ldots, N_d'\}$ and for each $p > N_0$, we consider the following partition of $\mathbb{Z}^n$:

$$
\mathbb{Z}^n = \bigcup_{j=1}^{d} A_{j,p} \cup \bigcup_{j=1}^{d} B_{j,p} \cup W_p,
$$

where

$$
A_{j,p} := \{y \in \mathbb{Z}^n \mid \mathrm{ord}_p(f_j(y)) > 0 \text{ and } f \text{ has a critical point in } y + p\mathbb{Z}_p^n\},
$$

$$
B_{j,p} := \{y \in \mathbb{Z}^n \mid \mathrm{ord}_p(f_j(y)) > 0 \text{ and } f \text{ has no critical points in } y + p\mathbb{Z}_p^n\},
$$

for $1 \leqslant j \leqslant d$, and

$$W_p := \mathbb{Z}^n \setminus \bigcup_{j=1}^{d} (A_{j,p} \cup B_{j,p}).$$

In different lemmas we will analyse the local sums $E_f^y(m,p)$ for y in each of the sets in this partition.

Lemma 4.4.1. *For all $m \geqslant 2$, $p > N_0$ and $y \in W_p$ we have $E_f^y(m,p) = 0$.*

Proof. We observe that for all $p > N_0$ and $y \in W_p$, we have $\operatorname{ord}_p(f(y) - z_j) \leqslant 0$ for all $1 \leqslant j \leqslant d$. In particular, $\operatorname{ord}_p(f(y) - z_j) = 0$ for all $\underline{z_j} \in V_f \cap \mathbb{Z}_p = V_{f,p}$. So by condition (iii) from Lemma 4.3.1, $\overline{y}$ is a regular point of $\overline{f}$, hence the condition $C_{\overline{f}} \cap \operatorname{Supp}(\overline{\Phi_{y,p}}) = C_{\overline{f}} \cap \{\overline{y}\} = \emptyset$ is satisfied. Now it follows from [Den91b, Remark 4.5.3] that $E_f^y(m,p) = 0$, for all $m \geqslant 2$, $p > N_0$ and $y \in W_p$. □

Lemma 4.4.2. *Let $j \in \{1, \ldots, d\}$. For all $m \geqslant 2$, $p > N_0$ and $y \in B_{j,p}$ we have $E_{f_j}^y(m,p) = 0$.*

Proof. If $1 \leqslant j \leqslant d$, $p > N_0$, and $y \in B_{j,p}$, then f_j has no critical points in $y + p\mathbb{Z}_p^n$, i.e., $C_{f_j} \cap \operatorname{Supp}(\Phi_{y,p}) = \emptyset$. So by (1.4.1) from [Den91b], we have $E_{f_j}^y(m,p) = 0$, for m large enough. Using Corollary 1.4.5 from [Den91b], we see that

$$(p^{s+1} - 1) Z_{f_j}^y(s,p,\chi_{\text{triv}}) \quad \text{and} \quad Z_{f_j}^y(s,p,\chi), \text{ with } \chi \neq \chi_{\text{triv}},$$

cannot have any poles in $\mathbb{C}$.

For all $p > N_0$, the resolution (Y_j, h_j) of $f_j^{-1}(0)$ has good reduction modulo p and $C_{\overline{f_j}} \cap \operatorname{Supp}(\overline{\Phi_{y,p}}) \subseteq \overline{f_j}^{-1}(0)$, for any $y \in B_{j,p}$, hence Theorem 1.3.5 applies. By combining this information with Proposition 1.3.6, we get that for all $m \geqslant 1$, $p > N_0$ and $y \in B_{j,p}$, the sum $E_{f_j}^y(m,p)$ equals

$$Z_{f_j}^y(0,p,\chi_{\text{triv}}) + \operatorname{Coeff}_{t^{m-1}} \frac{(t-p) Z_{f_j}^y(s,p,\chi_{\text{triv}})}{(p-1)(1-t)} \qquad (4.4.1)$$

$$+ \sum_{\substack{\chi \neq \chi_{\text{triv}},\\ c(\chi)=1}} g_{\chi^{-1}} \operatorname{Coeff}_{t^{m-1}} (Z_{f_j}^y(s,p,\chi)).$$

Since $Z_{f_j}^y(s,p,\chi)$ does not have any poles for $\chi \neq \chi_{\text{triv}}$, we can see that, for m big enough, $\operatorname{Coeff}_{t^{m-1}}(Z_{f_j}^y(s,p,\chi))$ does not depend on m. Also the total expression (4.4.1) is independent of m, for m big enough (because it is equal to 0). Therefore the part $\operatorname{Coeff}_{t^{m-1}} \frac{(t-p) Z_{f_j}^y(s,p,\chi_{\text{triv}})}{(p-1)(1-t)}$ must be independent of m

as well, for m big enough. This can only be the case if $\frac{(t-p)Z^y_{f_j}(s,p,\chi_{\mathrm{triv}})}{(p-1)(1-t)}$, as a function in t, has at most two poles, one pole at $t=1$ of order 1 and one pole at $t=0$. However, the explicit formula of $Z^y_{f_j}(s,p,\chi_{\mathrm{triv}})$ implies that it can not have poles at $t=0$. So $\frac{(t-p)Z^y_{f_j}(s,p,\chi_{\mathrm{triv}})}{(p-1)(1-t)}$ has at most one pole, and this pole (if it exists) must be of order 1 at $t=1$.

According to (4.1.1) from [Den91b], the degree of $Z^y_{f_j}(s,p,\chi)$ is at most 0 (as a rational function in t), for all $p > N_0$ and all characters χ with conductor $c(\chi) = 1$. This implies that $\frac{(t-p)Z^y_{f_j}(s,p,\chi_{\mathrm{triv}})}{(p-1)(1-t)}$ is of the form $c + \frac{d}{1-t}$, for certain $c, d \in \mathbb{C}$, and that $Z^y_{f_j}(s,p,\chi)$ is equal to a constant function, for $\chi \neq \chi_{\mathrm{triv}}$. Now we can easily see that for all $m \geqslant 2$, $\mathrm{Coeff}_{t^{m-1}} \frac{(t-p)Z^y_{f_j}(s,p,\chi_{\mathrm{triv}})}{(p-1)(1-t)}$ and $\mathrm{Coeff}_{t^{m-1}} Z^y_{f_j}(s,p,\chi)$, for $\chi \neq \chi_{\mathrm{triv}}$, are independent of m. We conclude that $E^y_{m,p}(f_j) = 0$, for all $m \geqslant 2$, $p > N_0$ and $y \in B_{j,p}$. □

Lemma 4.4.3. *Let $j \in \{1, \ldots, d\}$. There exists a constant $C_j > 0$ and a natural number N_j, such that for all $m \geqslant 2$, $p > N_j$ and $y \in A_{j,p}$ we have*

$$|E^y_{f_j}(m,p)|_{\mathbb{C}} \leqslant C_j m^{n-1} p^{-m\sigma_{y,p}}.$$

We will give two proofs of this lemma. One proof will be based on techniques from Section 4.2 and the other on techniques from Section 4.1. First we will show how one concludes the local Cluckers-Veys upper bounds (4.0.2) from these three lemmas. For any $1 \leqslant j \leqslant d$ we have $|E^y_{f_j}(m,p)|_{\mathbb{C}} = |E^y_f(m,p)|_{\mathbb{C}}$. Take $C := \max\{C_j \mid 1 \leqslant j \leqslant d\}$ and $N := \max\{N_j \mid 0 \leqslant j \leqslant d\}$, then for all $m \geqslant 2$, $p > N$ and $y \in \mathbb{Z}^n$, we have

$$|E^y_f(m,p)|_{\mathbb{C}} \leqslant C m^{n-1} p^{-\sigma_{y,p}}.$$

Together with Equation (1.3.3) the local Cluckers-Veys upper bounds follow.

4.4.1 Adapting the geometric proof

We will prove the following variant of Lemma 4.2.1.

Lemma 4.4.4. *Let $f \in \mathbb{Z}[x_1, \ldots, x_n] \setminus \mathbb{Z}$ and $j \in \{1, \ldots, d\}$. Then there exists $\tilde{N}_j \in \mathbb{N}$, such that for all $p > \tilde{N}_j$ and $y \in A_{j,p}$,*

$$a_{y,p}(f) = \min_{E : \overline{y} \in \overline{h_j}(\overline{E}(\mathbb{F}_p))} \left\{ \frac{\nu}{N} \right\}.$$

Proof. We remark that if $y' \in y + p\mathbb{Z}_p^n$ is not a critical point of f, then $c_{y'}(f(x) - f(y')) = 1$. If, on the other hand, $y' \in y + p\mathbb{Z}_p^n$ is a critical point of f, then we know by Lemma 4.3.1 that $f(y') = z_j$, because we assumed that $y \in A_{j,p}$. Thus $f_j(y') = f(y') - z_j = 0$. Now it follows from Lemma 4.2.1 that there exists $\tilde{N}_j$, independent of $y' \in f_j^{-1}(0)$, such that for all $p > \tilde{N}_j$,

$$c_{y'}(f(x) - f(y')) = c_{y'}(f_j) = \min_{E:\overline{y'} \in \overline{h_j}(\overline{E}(\mathbb{F}_p))} \left\{ \frac{\nu}{N} \right\} = \min_{E:\overline{y} \in \overline{h_j}(\overline{E}(\mathbb{F}_p))} \left\{ \frac{\nu}{N} \right\} \leqslant 1.$$

If $y \in A_{j,p}$, then $y + p\mathbb{Z}_p^n$ contains at least one critical point of f, in which proves the claim. $\square$

Now for any $y \in A_{j,p}$, we can apply Remark 4.2.4 to f_j with $\Phi_p = \mathbb{1}_{y+p\mathbb{Z}_p^n}$ and $\sigma' = \min\left\{a_{y,p}, \frac{1}{2}\right\} = \sigma_{y,p}$. The constant C_j and the natural number $N_j > \tilde{N}_j$ that we find, are independent of $y \in A_{j,p}$. This proves Lemma 4.4.3.

4.4.2 Adapting the model theoretic proof

For $1 \leqslant j \leqslant d$ and $y \in A_{j,p}$, we will split the exponential sum $E_{f_j}^y(m,p)$ into three subsums in exactly the same way as in Section 4.1. In each of the Lemmas 4.1.1, 4.1.5, 4.1.6 and in the proof of Theorem 4.0.4 we will need to make some changes.

Lemma 4.4.5. *Let $f \in \mathbb{Z}[x_1, \ldots, x_n] \setminus \mathbb{Z}$ and $j \in \{1, \ldots, d\}$. Then there exists a natural number N_j, such that for all $m \geqslant 1$, for all primes $p > N_j$ and for all $y \in A_{j,p}$, we have*

$$\sum_{\substack{\overline{x} \in \overline{y} + (p\mathbb{Z}/p^m\mathbb{Z})^n, \\ \mathrm{ord}_p(f_j(x)) \leqslant m-2}} \exp\left(\frac{2\pi i f_j(x)}{p^m}\right) = 0.$$

Remark that if $A_{j,p} \neq \emptyset$, then $z_j \in V_{f,p} \subseteq \mathbb{Z}_p$, so $\exp\left(\frac{2\pi i f_j(x)}{p^m}\right)$ is well-defined.

To prove this lemma, we adapt the proof of Lemma 4.1.1 as follows. We replace the formula φ by

$$\varphi_j(x_1, \ldots, x_n, z, \xi_1, \ldots, \xi_n, m) = \bigwedge_{i=1}^{n} (\overline{x_i} = \xi_i) \wedge \big(\mathrm{ord}(z - z_j) \leqslant m - 2\big) \wedge \big(\mathrm{ord}(z - f(x_1, \ldots, x_n)) \geqslant m\big),$$

where x_i, z are in the VF-sort, ξ_i are in the RF-sort and m is in the VG-sort. This is an $\mathcal{L}_{\mathbb{Z}} \cup \{z_j\}$-formula, with z_j a constant symbol in the VF-sort. We remark that the function $\mathcal{O}_K \to k_K : x \mapsto \overline{x} = (x \bmod \mathcal{M}_K)$ is definable in $\mathcal{L}_{\mathrm{DP}}$.

Now φ_j induces a definable subassignment $X_j \subset h[n+1,n,1]$ and constructible functions $F_j := \mathbb{1}_{X_j}$ and $G_j := \pi_!(F_j)$, where $\pi : h[n+1,n,1] \to h[1,n,1]$ is the projection onto the last $n+2$ coordinates. For each prime p, for each uniformizer ϖ_p of $\mathbb{Q}_p$ and for each $y \in A_{j,p}$, we have the following interpretation of G_j in $\mathbb{Q}_p$:

$$G_{j,\varpi_p}(z,\overline{y},m) = \#\{\overline{x}^{(m)} \in \overline{y}^{(m)} + (p\mathbb{Z}/p^m\mathbb{Z})^n \mid f(x) \equiv z \bmod p^m\}$$

if $\mathrm{ord}_p(z - z_j) \leqslant m-2$ and $G_{j,\varpi_p}(z,\overline{y},m) = 0$ if $\mathrm{ord}_p(z-z_j) \geqslant m-1$. Here the notation $\overline{x}^{(m)}$ means the class of $(x \bmod p^m)$. Note however that G_{j,ϖ_p} actually only depends on $(y \bmod p)$, i.e., on $\overline{y}$. We remark that if $A_{j,p} \neq \emptyset$, then $z_j \in \mathbb{Z}_p$, which makes it possible to interpret $\mathrm{ord}(z - z_j)$ (and other formulas that contain the symbol z_j) in $\mathbb{Q}_p$.

We apply Corollary 1.2.12 to G_j to obtain a cell decomposition where the centers c_i are given by $\mathcal{L}_{\mathbb{Z}} \cup \{z_j\}$-formulas $\theta_i(z,\xi,\eta,\gamma,m)$. By elimination of quantifiers, θ_i is equivalent to the formula

$$\bigvee_{\ell} \Big(\zeta_{i\ell}\big(\overline{\mathrm{ac}}(g_1(z)),\ldots,\overline{\mathrm{ac}}(g_s(z)),\xi,\eta\big) \wedge \nu_{i\ell}\big(\mathrm{ord}(g_1(z)),\ldots,\mathrm{ord}(g_s(z)),\gamma,m\big)\Big),$$

where $\zeta_{i\ell}$ is an $\mathcal{L}_{\mathrm{ring}}$-formula, $\nu_{i\ell}$ an $\mathcal{L}_{\mathrm{oag}}$-formula, and $g_1,\ldots,g_s \in (\mathbb{Z}[z_j][[t]])[z]$. The rest of the proof of Lemma 4.1.1 still applies if we replace $\mathrm{ord}(z)$ by $\mathrm{ord}(z - z_j)$ everywhere. By going over the proof, we can see that the natural number N_j that is obtained in the proof, only depends on j.

Lemma 4.4.6. *Let $f \in \mathbb{Z}[x_1,\ldots,x_n] \setminus \mathbb{Z}$ and $j \in \{1,\ldots,d\}$. Then for each integer $m \geqslant 2$, there exists a natural number N_m and a constant $D_m > 0$, such that for all primes $p > N_m$ and for all $y \in A_{j,p}$, we have*

$$\left| \sum_{\substack{\overline{x} \in \overline{y} + (p\mathbb{Z}/p^m\mathbb{Z})^n,\\ \mathrm{ord}_p(f_j(x)) = m-1}} p^{-mn} \exp\left(\frac{2\pi i f_j(x)}{p^m}\right) \right|_{\mathbb{C}} \leqslant D_m p^{-m\sigma_{y,p}}.$$

To prove this lemma, we adapt the proof of Lemma 4.1.5 as follows. We replace the formulas φ and $\overline{\varphi}$ by

$$\varphi_j(x_1,\ldots,x_n,z,\xi_1,\ldots,\xi_n,m) = \bigwedge_{i=1}^{n} (\overline{x_i} = \xi_i) \wedge \big(\mathrm{ord}(z - z_j) = m-1\big)$$

$$\wedge\ \big(\mathrm{ord}(z - f(x_1,\dots,x_n)) \geqslant m\big),$$

$$\overline{\varphi}_j(x_1,\dots,x_n,\xi,\xi_1,\dots,\xi_n,m) = \bigwedge_{i=1}^{n}(\overline{x_i}=\xi_i)\ \wedge\ \big(\overline{\mathrm{ac}}(f(x_1,\dots,x_n)-z_j)=\xi\big)$$

$$\wedge\ \big(\mathrm{ord}(f(x_1,\dots,x_n)-z_j) = m-1\big),$$

where x_i, z are in the VF-sort, ξ_i, ξ are in the RF-sort and m is in the VG-sort. These are also $\mathcal{L}_{\mathbb{Z}} \cup \{z_j\}$-formulas. Most of the other modifications in the proof of Lemma 4.1.5 are the same as we discussed above for Lemma 4.4.5.

The only moment that we have to be more careful, is when estimating $\#\{\overline{x}^{(m)} \in \overline{y}^{(m)} + (p\mathbb{Z}_p/p^m\mathbb{Z}_p)^n \mid \mathrm{ord}_p(f_j(x)) = m-1\}$. From the proof of Lemma 4.4.4 we know that there exists a natural number $\tilde{N}_j$, such that if $p > \tilde{N}_j$ and $y \in A_{j,p}$, then there exists a critical point $\tilde{y} \in y + (p\mathbb{Z}_p)^n$, such that $a_{y,p}(f) = c_{\tilde{y}}(f_j)$. By [Mus02, Corollary 3.6] we have

$$a_{y,p}(f) = c_{\tilde{y}}(f_j) \leqslant \frac{(m-1)n - \dim_{\mathbb{F}_p}(\tilde{A}_{p,m,y})}{m-1},$$

where $A_{p,m,y} := \{\overline{x}^{(m)} \in \overline{y}^{(m)} + (p\mathbb{Z}_p/p^m\mathbb{Z}_p)^n \mid \mathrm{ord}_p(f_j(x)) = m-1\}$, viewed as a constructible subset of $\mathbb{F}_p^{mn}$, and where $\tilde{A}_{p,m,y}$ is the image of $A_{p,m,y}$ under the projection $p_m\colon (\mathbb{Z}_p/p^m\mathbb{Z}_p)^n \to (\mathbb{Z}_p/p^{m-1}\mathbb{Z}_p)^n$, viewed as a constructible subset of $\mathbb{F}_p^{mn-n}$. Then $\#A_{p,m,y} \leqslant \#\tilde{A}_{p,m,y} \cdot p^n$. By the Lang-Weil estimate, there exists a constant $D'_{m,y}$, not depending on p, such that

$$\#\tilde{A}_{p,m,y} \leqslant D'_{m,y} p^{\dim_{\mathbb{F}_p}(\tilde{A}_{p,m,y})}.$$

By looking at the arc space of $Z(f_j)$, we can see that, for each m, there are finitely many schemes $Z_1^{(m)},\dots,Z_{\ell_m}^{(m)}$, such that for all p and y, $\tilde{A}_{p,m,y} \cong Z_i^{(m)}(\mathbb{F}_p)$ for some $i \in \{1,\dots,\ell_m\}$. This means that the constant $D'_{m,y}$, which we know already to be independent of p, only depends on the set of schemes $\{Z_1^{(m)},\dots,Z_{\ell_m}^{(m)}\}$. Hence there exists a constant $D'_{m,j}$, such that $D'_{m,j} \geqslant D'_{m,y}$ for all $y \in A_{j,p}$. By going over the rest of the proof of Lemma 4.1.5, we can see that the natural number N_m and the constant D_m, that are obtained in the proof, only depend on m and j.

We need to make similar adjustments in the proof of Lemma 4.1.6, to obtain the following lemma.

Lemma 4.4.7. *Let $f \in \mathbb{Z}[x_1,\dots,x_n] \setminus \mathbb{Z}$ and $j \in \{1,\dots,d\}$. Then for each integer $m \geqslant 2$, there exists a natural number N_m and a constant $D_m > 0$, such* *that for all primes $p > N_m$ and for all $y \in A_{j,p}$, we have*

$$\Big| \sum_{\substack{\overline{x} \in \overline{y} + (p\mathbb{Z}/p^m\mathbb{Z})^n, \\ \mathrm{ord}_p(f_j(x)) \geqslant m}} p^{-mn} \exp\left(\frac{2\pi i f_j(x)}{p^m}\right) \Big|_{\mathbb{C}} \leqslant D_m p^{-m\sigma_{y,p}}.$$

The final step after these three lemmas, is to modify the proof of Theorem 4.0.4 at the end of Section 4.1. According to Corollary 1.3.7 and its proof, there exist natural numbers s_j, M_j', N_j'', such that for all $p > N_j''$, $m > M_j'$ and $y \in A_{j,p}$, we have

$$E^y_{m,p}(f_j) = \sum_{i=1}^{s_j} a_{i,p,y} \mathbb{1}_{A_{ij}}(m) m^{\beta_{ij}} p^{-m\rho_{ij}}. \tag{4.4.2}$$

We can easily see that β_{ij}, ρ_{ij} and A_{ij} only depend on f_j and not on y. By going through the proof of Claim 2.2.7 we obtain a constant $\tilde{C}_j$ and natural numbers $\tilde{M}_j, \tilde{N}_j$ (that depend on β_{ij}, ρ_{ij} and A_{ij}, but not on $a_{i,p,y}$), such that for all $m > \tilde{M}_j$, $p > \tilde{N}_j$, $y \in A_{j,p}$ and $1 \leqslant i \leqslant s_j$, we have

$$|a_{i,p,y} \mathbb{1}_{A_{ij}} p^{-m\rho_{ij}}|_{\mathbb{C}} \leqslant \tilde{C} p^{-m\sigma_{y,p}}.$$

This proves Lemma 4.4.3.

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Index

FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
ALGEBRA SECTION
Celestijnenlaan 200B box 2400
B-3001 Leuven