Université de Lille
Laboratoire Paul Painlevé
École Doctorale SPI

Université de Lille

Laboratoire Paul Painlevé

Thèse de Doctorat
Mathématiques

Tours de Postnikov et Invariants de Postnikov pour les Opérades Simpliciales

Michaël Mienné

Thèse dirigée par Benoit Fresse
et soutenue le 14 Décembre 2018

Jury

Directeur de thèse :	Benoit Fresse	Université de Lille
Rapporteurs :	Clemens Berger	Université Nice-Sophia-Antipolis
	Paul Goerss	Northwestern University
Examinateurs :	Muriel Livernet	Université Paris-Didierot
	Antoine Touzé	Université de Lille

Remerciements

Mener de front cette thèse, ma vie professionnelle et ma vie familiale ne fut pas toujours aisé ; ce travail n'aurait pu aboutir sans l'aide de nombreuses personnes.

Je tiens tout d'abord à remercier mon directeur, Benoit Fresse, pour sa patience, ses conseils, sa gentillesse et la disponibilité dont il a su faire preuve, onze novembre inclus. Je suis très reconnaissant envers Paul Goerss et Clemens Berger, qui ont accepté d'être rapporteurs de cette thèse malgré l'ampleur de la tâche ; je remercie également les autres membres du jury, Muriel Livernet et Antoine Touzé.

Le soutien et l'enthousiasme d'Alexis Virelizier, d'Ivo Dell'Ambrogio, de Daniel Tanré, de David Chataur, de Mathieu Klimczak, de Najib Idrissi et de tous les topologues lillois ont été déterminants. Merci également aux membres et au personnel technique du laboratoire Painlevé et de l'UFR de mathématiques pour les conditions de travail dont j'ai pu bénéficier.

Je remercie mes anciens professeurs de mathématiques, avec une pensée toute particulière pour Eric Detrez, Robert Guergondey et Olivier Serman dont la passion fut contagieuse.

Je n'oublie pas mes collègues des collèges et lycées où j'ai travaillé, ainsi que les directions de ces établissements qui ont su m'aménager un emploi du temps compatible avec la reprise de mes études.

Merci à ma famille, à mes amis et à Marie qui m'ont tant apporté et m'ont encouragé dans les inévitables moments de doute. Je remercie de tout cœur Nina pour le soutien dans la dernière ligne droite.

Comme j'écris ces dernières lignes, mes pensées vont vers mon grand père Pierre, qui m'a fait découvrir enfant les mystères du ruban de Möbius et n'aura pu voir ce travail s'achever, ainsi que vers mon fils Tristan, né après que ce travail débute : cette thèse leur est dédiée.

Résumé. Nous adaptons la définition des sections de Postnikov et des tours de Postnikov des ensembles simpliciaux aux opérades simpliciales. Nous définissons ensuite des foncteurs de cotroncation afin de filtrer la tour de Postnikov d'une opérade simpliciale par les arités et former ainsi la double tour de Postnikov de cette opérade.

Nous introduisons un nouveau type d'opérade, les $\Gamma-$opérades, où Γ désigne une opérade dans les groupoïdes. Nous les utilisons pour modéliser l'action de l'opérade groupoïde fondamental d'une opérade simpliciale sur ses groupes d'homotopies et son revêtement universel. Nous munissons la catégorie des $\Gamma-$opérades d'ensembles simpliciaux d'une structure de catégorie modèle. D'autre part, nous montrons que les $\Gamma-$opérades dans la catégorie des groupes abéliens munie de la structure monoïdale induite par la somme directe forment une catégorie abélienne. Cette catégorie abélienne fournit les coefficients pour la cohomologie équivariante opéradique que nous étudions ensuite. Une version relative de cette cohomologie est également étudiée.

Nous définissons alors les invariants de Postnikov d'une opérade simpliciale : ce sont des classes de cohomologie équivariante opéradique qui permettent de reconstruire inductivement et à homotopie près une opérade simpliciale à l'aide de sa double tour. Ce processus de reconstruction est utilisé afin de développer une théorie de l'obstruction pour les opérades simpliciales : on peut étendre un morphisme d'opérades simpliciales le long d'une cofibration si et seulement une suite de classes de cohomologie équivariante opéradique relative définie inductivement est nulle.
Mots-clés : topologie algébrique, homotopie, opérades, obstruction (théorie de l')

Abstract. We adapt the definition of Postnikov sections and Postnikov towers of simplicial sets to simplicial operads. We then define cotruncation functors in order to filter the Postnikov tower of a simplicial operad by arity and form the Postnikov double tower of this operad.

We introduce a new kind of operad, the $\Gamma-$operads where Γ denotes a groupoid operad. We use them to modelize the action of the fundamental groupoid operad of a simplicial operad on its homotopy groups and its universal covering. We equip the category of $\Gamma-$operads in simplicial sets with a model structure. We also prove that the $\Gamma-$operads in the category of abelian groups equipped with the monoidal structure induced by the direct sum form an abelian category. This abelian category provides the coefficients for the operadic equivariant cohomology which we study afterwards. Furthermore, we study a relative version of this cohomology.

We thereafter define the Postnikov invariants of a simplicial operad : these are operadic equivariant cohomology classes which permit to reconstruct inductively and up to homotopy a simplicial operad by the mean of its double tower. This reconstruction process is used to develop an obstruction theory for simplicial operads : a simplicial operad morphism can be extended along a cofibration if an only if a sequence of relative operadic equivariant cohomology classes defined inductively vanishes.
English title : *Postnikov Towers and Postnikov Invariants for Simplicial Operads*
Keywords : algebraic topology, homotopy, operads, obstruction theory

Résumé

L'objectif de cette thèse est de définir les tours de Postnikov et les invariants de Postnikov d'une opérade simpliciale, puis de développer une théorie de l'obstruction pour les opérades simpliciales.

Les tours de Postnikov, introduites par M. Postnikov dans les années 50 [24], permettent de reconstruire à homotopie près un espace topologique à partir de ses groupes d'homotopies et de ses invariants de Postnikov. J.C. Moore a ensuite adapté les tours de Postnikov aux ensembles simpliciaux [23], qui modélisent de manière combinatoire les espaces topologiques "raisonnables". La théorie de l'obstruction est quant à elle un processus d'extension d'un morphisme de CW-complexes ou d'ensembles simpliciaux par récurrence sur la dimension ; on peut alternativement étudier la possibilité de relever un tel morphisme inductivement selon les étages d'une tour de Postnikov du codomaine [30, 28].

Les opérades furent introduites en théorie de l'homotopie pour modéliser des espaces de lacets itérés dans les années 70 [21, 3]. Elles ont connu un regain d'intérêt dans les années 90 après qu'on leur ait découvert des applications importantes dans de nombreux domaines des mathématiques et notamment, en théorie quantique des champs.

1. Introduction

Tour de Postnikov et invariants de Postnikov d'un ensemble simplicial

Tout ensemble simplicial (fibrant) X est isomorphe à la limite d'une tour de fibrations

$$\cdots \longrightarrow X\langle n\rangle \xrightarrow{p_n} X\langle n-1\rangle \longrightarrow \cdots \longrightarrow X\langle 0\rangle,$$

la *tour de Postnikov* de X. Les ensemble simpliciaux $X\langle n\rangle$ sont les *sections de Postnikov* de X et forment une filtration de X selon ses groupes d'homotopie : pour tout $0 \leqslant k \leqslant n$, on a $\pi_k(X\langle n\rangle) = \pi_k(X)$ et pour tout $k > n$, $\pi_k(X\langle n\rangle) = 0$. La fibre de p_n est donc un espace d'Eilenberg-MacLane $K(\pi_n(X), n)$, c'est-à-dire un ensemble simplicial dont les groupes d'homotopies sont tous nuls, sauf le $n-$ième qui est égal à $\pi_n(X)$.

On peut déduire à homotopie près la $n-$ième section de Postnikov de X de la $(n-1)-$ème section au moyen du carré cartésien homotopique

$$\begin{array}{ccc} X\langle n\rangle & \dashrightarrow & \operatorname{hocolim}_\Gamma L(\pi_n(X), n+1) \\ {\scriptstyle p_n}\downarrow & & \downarrow \\ X\langle n-1\rangle & \xrightarrow[k_n]{} & \operatorname{hocolim}_\Gamma K(\pi_n(X), n+1) \end{array}$$

où Γ est le groupoïde fondamental de X et k_n un morphisme qui représente une classe de cohomologie équivariante appelée *$n-$ième invariant de Postnikov* de X. L'ensemble simplicial $L(\pi_n(X), n+1)$ est contractile et tel qu'il existe une suite de fibration

$$K(\pi_n(X), n) \to L(\pi_n(X), n+1) \to K(\pi_n(X), n+1).$$

On peut donc reconstruire X à homotopie près à partir de ses groupes d'homotopie et de ses invariants de Postnikov ; autrement dit, le type d'homotopie d'un ensemble simplicial est entièrement déterminé pas ses groupes d'homotopies et ses invariants de Postnikov.

Opérades

Une *opérade* modélise les structures d'algèbres sur une catégorie monoïdale symétrique $(\mathtt{M}, \otimes)$: elle est constituée d'objets représentant des opérations comportant un nombre fini d'entrées et une sortie. Elle est de plus munie d'une structure indiquant comme composer de telles opérations.

L'exemple prototypique d'opérade est $\mathrm{End}(E) = \{\mathrm{Mor}_{\mathtt{Set}}(E^r, E)\}_{r \geqslant 1}$, l'opérade des endomorphismes d'un ensemble E. Les morphismes

$$f : \underbrace{E \times \cdots \times E}_{r} \to E$$

de $\mathrm{Mor}_{\mathtt{Set}}(E^r, E)$ représentent des opérations d'*arité* r, autrement dit, des opérations à r entrées. On peut composer ces opérations au moyen de morphismes

$$\circ_i : \quad \mathrm{Mor}_{\mathtt{Set}}(E^r, E) \times \mathrm{Mor}_{\mathtt{Set}}(E^s, E) \quad \longrightarrow \quad \mathrm{Mor}_{\mathtt{Set}}(E^{r+s-1}, E)$$

définis pour tout $r, s \geqslant 1$, $1 \leqslant i \leqslant r$ par

$$f \circ_i g(x_1, \cdots, x_{r+s-1}) = f\big(x_1, \cdots, x_{i-1}, g(x_i, \cdots, x_{i+s-1}), x_{i+s}, ..., x_{r+s-1}\big)$$

et on dispose de plus d'une action du groupe symétrique à r éléments sur $\mathrm{Mor}_{\mathtt{Set}}(E^r, E)$ qui correspond aux permutations des entrées des opérations.

Plus généralement, une opérade est une collection d'objets $O(r) \in \mathtt{M}$ pour tout $r \geqslant 1$, munie d'une action du groupe symétrique à r éléments sur $O(r)$, avec des morphismes de composition $\circ_i : O(r) \otimes O(s) \to O(r+s-1)$ et d'un morphisme d'unité $\mathbb{1} \to O(1)$ avec $\mathbb{1}$ l'unité monoïdale de $\mathtt{M}$ qui vérifient des axiomes d'équivariance, d'associativité et d'unité inspirés des relations présentes dans l'opérade des endomorphismes.

On munit un objet E de $\mathtt{M}$ d'une structure de O-algèbre en faisant agir l'opérade O sur E, c'est-à-dire, en se donnant un morphisme d'opérade de O dans $\mathrm{End}(E)$. Il existe par exemple une opérade Com dans la catégorie des modules telle qu'une $Com-$algèbre soit une algèbre commutative. Les opérades sont donc aux algèbres ce que les groupes sont aux représentations de groupes.

Une *opérade simpliciale* est une opérade dans la catégorie des ensembles simpliciaux.

2. Tour de Postnikov d'une opérade simpliciale

La définition des tours de Postnikov pour les opérades simplicales est aisée car la collection des n-ièmes sections de Postnikov $(P(r)\langle n \rangle)_{r \geqslant 1}$ d'une opérade simpliciale P hérite de la structure d'opérade de P.

Comme nous le verrons plus tard, la définition des invariants de Postnikov d'une opérade simpliciale et par conséquent son processus de reconstruction au moyen de sa tour de Postnikov présente des difficultés. On introduit donc des foncteurs de cotroncation $\mathrm{coar}_{\leqslant r} : \mathtt{sOp} \to \mathtt{sOp}$ pour tout $r \geqslant 1$, tels que

$$\mathrm{coar}_{\leqslant r} P(s) = \begin{cases} P(s) & \text{si } s \leqslant r, \\ pt & \text{sinon} \end{cases}$$

et où l'ensemble simplicial pt modélise le point. On utilise ensuite ces foncteurs pour filtrer la tour de Postnikov de P selon les arités et obtenir la double tour de Postnikov de P :

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & & & \vdots \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
\cdots & \longrightarrow & P\langle r+1, n+1\rangle & \xrightarrow{\mathrm{coar}_{\leqslant r}} & P\langle r, n+1\rangle & \longrightarrow & \cdots & \longrightarrow & P\langle 1, n+1\rangle \\
 & & \downarrow{\scriptstyle p_{r+1,n+1}} & & \downarrow{\scriptstyle p_{r,n+1}} & & & & \downarrow{\scriptstyle p_{1,n+1}} \\
\cdots & \longrightarrow & P\langle r+1, n\rangle & \xrightarrow{\mathrm{coar}_{\leqslant r}} & P\langle r, n\rangle & \longrightarrow & \cdots & \longrightarrow & P\langle 1, n\rangle \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 & & \vdots & & \vdots & & & & \vdots \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
\cdots & \longrightarrow & P\langle r+1, 0\rangle & \xrightarrow{\mathrm{coar}_{\leqslant r}} & P\langle r, 0\rangle & \longrightarrow & \cdots & \longrightarrow & P\langle 1, 0\rangle,
\end{array}
$$

qui nous permettra par la suite de contourner ce problème de reconstruction.

3. Invariants de Postnikov d'une opérade simpliciale

Rappelons brièvement la construction des invariants de Postnikov d'un ensemble simplicial X – pour plus de détail, on se reportera à [10]. Soit Γ le groupoïde fondamental de X et A un Γ-groupe abélien. On note $C^n_\Gamma(X, A)$ l'ensemble des applications α telles que

$$
\begin{array}{ccc}
X_n & \xrightarrow{\alpha} & \operatorname{hocolim}_\Gamma A_n = \coprod_{x_0 \to \ldots \to x_n \in B\Gamma_n} A(x_0) \\
 & {\scriptstyle \Phi_n}\searrow \quad \swarrow & \\
 & B\Gamma_n &
\end{array}
$$

commute, avec $B\Gamma$ le nerf du groupoïde fondamental Γ. La famille d'ensembles $C^\bullet_\Gamma(X, A)$ est munie d'une structure de groupe abélien cosimplicial ; la cohomologie associée $H^\bullet_\Gamma(X; A)$ est la cohomologie équivariante de X à coefficients dans A.

On peut également définir une cohomologie équivariante relative et obtenir une suite exacte longue associée. Si on considère en particulier la fibration $p_n : X\langle n\rangle \to X\langle n-1\rangle$ extraite de la tour de Postnikov de X, la suite exacte longue de cohomologie équivariante relative associée à p_n et à coefficients dans $\pi_n(X)$ se restreint à la suite exacte

$$
0 \longrightarrow H^n_\Gamma\big(X\langle n-1\rangle; \pi_n(X)\big) \xrightarrow{p_n^*} H^n_\Gamma\big(X\langle n\rangle; \pi_n(X)\big) \longrightarrow \mathrm{Mor}_{\mathtt{Ab}^\Gamma}\big(\pi_n(X), \pi_n(X)\big)
$$

$$
\xrightarrow{d} H^{n+1}_\Gamma\big(X\langle n-1\rangle; \pi_n(X)\big) \xrightarrow{p_n^*} H^{n+1}_\Gamma\big(X\langle n\rangle, \pi_n(X)\big)
$$

comme on a

$$H^k_\Gamma(X\langle n-1\rangle, X\langle n\rangle; \pi_n(X)) = 0$$

si $k \leqslant n$ et

$$H^{n+1}_\Gamma(X\langle n-1\rangle, X\langle n\rangle; \pi_n(X)) \simeq \mathrm{Mor}_{\mathtt{Ab}^\Gamma}\big(\pi_n(X), \pi_n(X)\big).$$

Remarquons au passage que $\pi_n(X)$ est munie d'une structure de $\Gamma-$groupe abélien par l'action usuelle du groupoïde fondamental sur les groupes d'homotopie. On définit alors le $n-$ième invariant de Postnikov de X comme l'image par d de $\mathrm{id} \in \mathrm{Mor}_{\mathtt{Ab}^\Gamma}\big(\pi_n(X), \pi_n(X)\big)$ dans $H^{n+1}_\Gamma\big(X\langle n-1\rangle; \pi_n(X)\big)$.

D'autre part, la cohomologie équivariante de X à coefficients dans A se ramène à la cohomologie à coefficients dans A de $X^\vee$, un $\Gamma-$diagramme de revêtements universels de X. On a de plus une adjonction de Quillen

$$-^\vee : \mathtt{sSet} \downarrow \mathtt{B}\Gamma \leftrightarrows \mathtt{sSet}^\Gamma : \mathrm{hocolim}_\Gamma$$

entre les foncteurs de revêtement universel $-^\vee$ et le foncteur $\mathrm{hocolim}_\Gamma$, avec $\mathtt{sSet} \downarrow \mathtt{B}\Gamma$ la catégorie des ensembles simpliciaux au dessus du nerf du groupoïde fondamental $B\Gamma$ et $\mathtt{sSet}^\Gamma$ la catégorie des $\Gamma-$ensembles simpliciaux. Le $n-$ième groupe de cohomologie équivariante $H^n_\Gamma(X; A)$ est donc isomorphe au groupe abélien formé par les classes d'homotopie de morphismes de X dans $\mathrm{hocolim}_\Gamma K(A, n)$. En particulier, on peut associer à k_n, le $n-$ème invariant de Postnikov de X, un morphisme de $X\langle n-1\rangle$ dans $\mathrm{hocolim}_\Gamma K(\pi_n(X), n+1)$ unique à homotopie près. C'est précisément ce morphisme que l'on utilise pour former le carré cartésien homotopique mentionné plus haut et qui permet de déduire la $n-$ième section de Postnikov de la $(n-1)-$ième à homotopie près.

Γ-opérades

Remarquons tout d'abord que la collection des groupoïdes fondamentaux $\pi(P) = (\pi(P(r)))_{r\geqslant 1}$ d'une opérade simpliciale P hérite d'une structure d'opérade dans les groupoïdes. Pour adapter la définition des invariants de Postnikov au contexte des opérades simpliciales, il nous faut être capable de décrire l'action de $\pi(P)$ sur les groupes d'homotopies de P et sur le revêtement universel de P.

Pour ce faire, on se donne plus généralement une opérade Γ dans les groupoïdes. Une $\Gamma-$*opérade* O dans une catégorie monoïdale $(\mathtt{M}, \otimes)$ est une collection de foncteurs $(O(r) : \Gamma(r) \to \mathtt{M})_{r\geqslant 1}$ munie de transformations naturelles qui décrivent sa structure de composition. Plus précisément, O est une collection d'objets $O(x)$ de M pour tout $x \in \Gamma(r)$, $r \geqslant 1$ et d'isomorphismes $O(\phi)$ de M pour tout $\phi \in \mathrm{Mor}_\Gamma(x, y)$, $x, y \in \Gamma(r)$, $r \geqslant 1$ avec de plus :

– des isomorphismes $\sigma_x : O(x) \to O(\sigma.x)$ pour tout $\sigma \in \Sigma(r)$, $x \in \Gamma(r)$, $r \geqslant 1$ décrivant l'action des groupes symétriques ;
- des produits de compositions

$$\circ_i : O(x) \otimes O(y) \to O(x \circ_i y)$$

pour tout $r, s \geqslant 1$, $1 \leqslant i \leqslant r$, $x \in \Gamma(r)$, $y \in \Gamma(s)$;
– une unité

$$\eta : \mathbb{1} \to O(1)$$

avec $\mathbb{1}$ l'unité monoïdale de $\mathtt{M}$ et 1 l'unité de Γ.

Les morphismes d'unité, de compositions et d'action des groupes symétriques doivent satisfaire des conditions de compatibilité avec les morphismes $P(\phi)$ ainsi que des axiomes d'équivariance, d'associativité et d'unité similaires aux axiomes d'une structure d'opérade usuelle. Signalons que si Γ est triviale, une Γ-opérade se ramène à une opérade usuelle.

Si l'on revient au cas d'une opérade simpliciale P, alors pour tout $k \geqslant 1$, la collection $\big(\pi_k(P)(x)\big)_{x\in P_0}$ est une $\pi(P)$-opérade dans $(\mathtt{Ab}, \oplus)$, la catégorie des groupes abéliens munie de la structure monoïdale induite par la somme directe. On montre que cette catégorie est abélienne et pourra donc bien fournir les coefficients pour la cohomologie équivariante opéradique.

D'autre part, la collection $(P^{\vee}(x))_{x \in P_0}$ des revêtements universels est également une $\pi(P)$-opérade et on a une adjonction

$$-^{\vee} : \mathtt{sOp} \downarrow \mathtt{B\Gamma} \rightleftarrows \mathtt{sOp}^{\Gamma} : \mathrm{hocolim}_{\Gamma},$$

avec $\mathtt{sOp}^{\Gamma}$ la catégorie des Γ−opérades dans les ensembles simpliciaux et $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$ la catégorie des opérades simpliciale au dessus de $B\Gamma$ – la collection $(B\Gamma(r))_{r \geqslant 1}$ des nerfs étant munie d'une structure d'opérade simpliciale héritée de la structure d'opérade dans les groupoïdes de Γ. On définit une structure de catégorie modèle sur $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$ où les équivalences faibles (respectivement, les fibrations) $f : P \to Q$ sont les morphismes tels que $f(x) : P(x) \to Q(x)$ est une équivalence faible (respectivement, une fibration) de $\mathtt{sSet}$ pour tout $x \in \Gamma(r)$, $r \geqslant 1$; les cofibrations sont les morphismes de $\mathtt{sOp}^{\Gamma}$ qui ont la propriété de relèvement à gauche relativement à la classe des fibrations acycliques. L'adjonction

$$-^{\vee} : \mathtt{sOp} \downarrow \mathtt{B\Gamma} \rightleftarrows \mathtt{sOp}^{\Gamma} : \mathrm{hocolim}_{\Gamma}$$

est une adjonction de Quillen pour cette structure de catégorie modèle et la structure de catégorie modèle de $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$ induite par la structure de catégorie modèle de $\mathtt{sOp}$ [8].

Cohomologie opéradique équivariante

On adapte la définition de la cohomologie équivariante d'un ensemble simplicial au cas des opérades simpliciales de la manière suivante : si P est une opérade simpliciale, Γ une opérade dans les groupoïdes et A une Γ-opérade dans $(\mathtt{Ab}, \oplus)$, on note $C^n_{\Gamma}(P, A)$ l'ensemble des morphismes d'opérades ensemblistes α tels que

$$\begin{array}{ccc} P_n & \xrightarrow{\alpha} & \mathrm{hocolim}_{\Gamma} A_n = \coprod_{x_0 \to \ldots \to x_n \in B\Gamma_n} A(x_0) \\ & {\scriptstyle \Phi_n} \searrow & \swarrow \\ & B\Gamma_n & \end{array}$$

commute, avec $B\Gamma$ le nerf de l'opérade dans les groupoïdes Γ. La famille d'ensembles $C^{\bullet}_{\Gamma}(P, A)$ est munie d'une structure de groupe abélien cosimplicial. La cohomologie associée $H^{\bullet}_{\Gamma}(P; A)$ est la cohomologie équivariante de P à coefficient dans A – en pratique, Γ sera l'opérade groupoïde fondamentale de P et A sera la Γ-opérade $\pi_k(P)$ des groupes d'homotopie supérieures de P pour $k \geqslant 2$.

Comme dans le cas des ensembles simpliciaux, il y a un isomorphisme de groupes abéliens cosimpliciaux

$$C^{\bullet}_{\Gamma}(P, A) \simeq \mathrm{Mor}_{\mathtt{SetOp}^{\Gamma}}(P^{\vee}{}_{\bullet}, A)$$

et à l'aide de l'adjonction de Quillen entre le foncteur de revêtement universel et le foncteur $\mathrm{hocolim}_{\Gamma}$, on établit l'existence d'isomorphismes

$$\mathrm{H}^n_{\Gamma}(P, A) \simeq \left[P, \mathrm{hocolim}_{\Gamma} K(A, n)\right]_{\mathtt{sOp}\downarrow\mathtt{B\Gamma}}$$

pour tout $n \geqslant 0$.

On définit également une version relative de la cohomologie équivariante opéradique et on montre qu'il existe une suite exacte longue associée.

Invariants de Postnikov opéradique

L'étude de la suite exacte longue de cohomologie équivariante associée à la fibration $p\langle n\rangle : P\langle n\rangle \to P\langle n-1\rangle$ entre les sections de Postnikov d'une opérade P montre qu'on ne peut définir dans le cas général les invariants de Postnikov de P comme dans le contexte des ensembles simpliciaux car on ne peut identifier $H^{n+1}_{\Gamma}(P\langle n-1\rangle, P\langle n\rangle; \pi_n(P))$ au groupe abélien des morphismes de $\pi_n(P)$ dans lui même.

Par contre, le problème ne se présente plus lorsque l'on suppose que les coefficients de la cohomologie équivariante sont concentrés en une arité car la cohomologie équivariante opéradique se restreint alors à la cohomologie des indécomposables de P. C'est notamment le cas si on considère le carré

$$\begin{array}{ccccc}
P\langle r,n\rangle & & \longrightarrow & & P\langle r-1,n\rangle \\
& \overset{\rho_{r,n}}{\searrow} & & \nearrow & \\
{\scriptstyle p_{r,n}}\downarrow & & Q & & \downarrow \\
& \swarrow & & & \\
P\langle r,n-1\rangle & & \longrightarrow & & P\langle r-1,n-1\rangle
\end{array}$$

extrait de la double tour de Postnikov de P avec Q le pullback

$$P\langle r,n-1\rangle \underset{P\langle r-1,n-1\rangle}{\times} P\langle r-1,n\rangle ;$$

notons $\pi_n(P(r))$ la Γ-opérade additive concentrée en arité r formée par la collection de groupes d'homotopies $(\pi_n(P(x)))_{x\in P(r)_0}$. La suite exacte longue de cohomologie opéradique équivariante relative associée à $\rho_{r,n}$ à coefficients dans $\pi_n(P(r))$ se restreint à

$$0 \longrightarrow H^n_{\Gamma}\big(Q;\pi_n(P(r))\big) \xrightarrow{p_n^*} H^n_{\Gamma}\big(P\langle r,n\rangle;\pi_n(P(r))\big) \longrightarrow \mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}\big(\pi_n(P(r)),\pi_n(P(r))\big)$$

$$\xrightarrow{\quad d \quad} H^{n+1}_{\Gamma}\big(Q,\pi_n(P(r))\big) \xrightarrow{p_n^*} H^{n+1}_{\Gamma}\big(P\langle r,n\rangle,\pi_n(P(r))\big).$$

On appelle alors (r,n)-ième invariant de Postnikov de P et on note $k_{r,n}$ l'image de $\mathrm{id} \in \mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}\big(\pi_n(P(r)),\pi_n(P(r))\big)$ dans $H^{n+1}_{\Gamma}\big(Q;\pi_n(P(r))\big)$ par d.

4. Théorie de l'obstruction

Si on considère de nouveau le carré extrait de la double tour de Postnikov de P, le théorème suivant permet de déduire $P\langle r,n\rangle$ des trois autres termes du carré à homotopie près :

Théorème A. *Soit P une opérade simpliciale fibrante. On note $\Gamma_{\leqslant r}$ la r-ème cotruncation de $\Gamma = \pi(P)$ et Q le pullback*

$$P\langle r,n-1\rangle \underset{P\langle r-1,n-1\rangle}{\times} P\langle r-1,n\rangle.$$

Pour tout $r,n \geqslant 2$, soit $\pi_n(P(r))$ la $\Gamma-$opérade additive concentrée en arité r. Il existe un remplacement cofibrant $\overline{Q}$ de

$$\overline{P\langle r,n-1\rangle} \underset{\overline{P\langle r-1,n-1\rangle}}{\times} \overline{P\langle r-1,n\rangle}$$

et un morphisme d'opérade $\overline{\rho_{r,n}}$ *tel que*

$$\begin{array}{ccc} P\langle r,n\rangle & \xrightarrow{\rho_{r,n}} & Q \\ {\scriptstyle\sim}\uparrow & & \uparrow{\scriptstyle\sim} \\ \overline{P\langle r,n\rangle} & \xrightarrow[\overline{\rho_{r,n}}]{} & \overline{Q} \end{array}$$

commute dans $\mathtt{sOp}\downarrow \mathtt{B}\Gamma_{\leqslant \mathtt{r}}$.

De plus, le carré dans $\mathtt{sOp}\downarrow \mathtt{B}\Gamma_{\leqslant \mathtt{r}}$

$$\begin{array}{ccc} \overline{P\langle r,n\rangle} & \dashrightarrow & \mathrm{hocolim}_{\Gamma_{\leqslant r}} L(\pi_n(P(r)), n+1) \\ {\scriptstyle\overline{\rho_{r,n}}}\downarrow & & \downarrow \\ \overline{Q} & \xrightarrow[k_{r,n}]{} & \mathrm{hocolim}_{\Gamma_{\leqslant r}} K(\pi_n(P(r)), n+1) \end{array}$$

est un carré cartésien homotopique.

Signalons que dans ce théorème, $\overline{P\langle r,n\rangle}$ désigne un remplacement cofibrant de $P\langle r,n\rangle$. Il est donc théoriquement possible sous les hypothèses de ce théorème de reconstruire à homotopie près une opérade simpliciale P inductivement selon les diagonales de sa double tour de Postnikov. En pratique, ce procédé de reconstruction permet de développer une théorie de l'obstruction pour les opérades simpliciales :

Théorème B. *Soit R une opérade simpliciale fibrante et notons Γ l'opérade groupoide fondamental $\pi(R)$ de R. Soit également $P, Q \in \mathtt{sOp}\downarrow\mathtt{B}\Gamma$ avec P cofibrant et considérons un morphisme $f : P \to R$ et une cofibration $i : P \to Q$ dans $\mathtt{sOp}\downarrow\mathtt{B}\Gamma$. Il existe un morphisme $f' : Q \to R$ tel que le diagramme*

$$\begin{array}{ccc} P & \xrightarrow{f} & R \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle f'} & \\ Q & & \end{array}$$

commute si et seulement si tous les termes d'une certaine suite de classe de cohomologie opéradique équivariante $w_{r,n} \in H^{n+1}_{\Gamma}(Q, P; \pi_n(R)(r))$ definie inductivement pour tout $n, r \geqslant 2$ sont nuls.

Contents

Remerciements i

Résumé/Abstract iii

Résumé v

1. Introduction . . . v
2. Tour de Postnikov d'une opérade simpliciale . . . vi
3. Invariants de Postnikov d'une opérade simpliciale . . . vii
4. Théorie de l'obstruction . . . x

Introduction 1

1 Conventions and background 5

1.1 Simplicial sets and related . . . 5
1.2 Operads . . . 8
1.3 Free operads, coproduct of an operad with a free operad . . . 10
1.4 Homotopy limits and colimits . . . 12

2 Postnikov double towers of simplicial operads 15

2.1 Postnikov sections of a simplicial operad . . . 15
2.2 Truncations and cotruncations . . . 15
2.3 Postnikov double towers of simplicial operads . . . 17

3 $\Gamma-$operads 19

3.1 $\Gamma-$operads . . . 19
3.2 $\Gamma-$symmetric sequences . . . 22
3.3 Free $\Gamma-$operads, coproduct with a free $\Gamma-$operad . . . 23

4 Model structures 27

4.1 Model structure on $\mathtt{sSeq} \downarrow \mathtt{B}\Gamma$ . . . 27
4.2 Model structure on $\mathtt{sOp} \downarrow \mathtt{B}\Gamma$ . . . 27
4.3 Model structure on $\mathtt{CSeq}^\Gamma$ . . . 28
4.4 Model structure on $\mathtt{COp}^\Gamma$ . . . 30
4.5 Quillen adjunctions and equivalences . . . 31

5 Covering $\Gamma-$operads of a simplicial operad 33

5.1 Covering $\Gamma-$symmetric sequence . . . 33
5.2 Covering $\Gamma-$operad . . . 35
5.3 Universal covering of a simplicial operad . . . 37

6 Additive $\Gamma-$operads, $\Gamma-$operadic derivations 39

6.1 Additive $\Gamma-$operads . . . 39
6.2 $\Gamma-$operadic derivations . . . 40

7 Operadic equivariant cohomology **41**
7.1 Definition . . . 41
7.2 Representation of the operadic equivariant cohomology . . . 42
7.3 Reduced operadic equivariant cohomology . . . 46
7.4 Relative operadic equivariant cohomology . . . 48
7.5 Indecomposables of a simplicial operad morphism . . . 50

8 Operadic Postnikov invariants **55**
8.1 Postnikov invariant of a simplicial operad morphism . . . 55
8.2 Postnikov invariants of a simplicial operad . . . 55

9 Comparison theorems **57**
9.1 Comparison of the equivariant cohomologies . . . 57
9.2 Comparison of the representations of the equivariant cohomologies . . . 59
9.3 Comparison of the Postnikov invariants . . . 62

10 Obstruction theory **67**
10.1 The main lemma . . . 67
10.2 Reconstruction of a simplicial operad . . . 69
10.3 Lift and operadic equivariant cohomology . . . 70
10.4 Obstruction theory . . . 76

Appendix A Universal coefficient theorem for operadic equivariant cohomology **83**

Appendix B The categories $\mathtt{AbSeq}^{\Gamma}$ and $\mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ are Grothendieck categories **89**

References **96**

Introduction

Postnikov towers, introduced by M. Postnikov in [24] (see also [30]) give a way to reconstruct up to homotopy a topological space inductively from its homotopy groups and some classes of equivariant cohomology, the Postnikov invariants or k-invariants. J.C. Moore has adapted Postnikov towers to simplicial sets (see [23], [20] or [10] for a modern account). An important application of this construction is an alternate way to develop obstruction theory (see [30] or [28]): instead of trying to extend a given simplicial morphism inductively on dimension, we can try to extend a simplicial morphism inductively through a Postnikov tower of its codomain.

The aim of this thesis is to define an analogous construction and an obstruction theory for simplicial operads.

We briefly remind the classic theory. A Kan complex X is isomorphic to the limit of a tower of fibrations

$$\cdots \longrightarrow X\langle n\rangle \xrightarrow{p_n} X\langle n-1\rangle \longrightarrow \cdots \longrightarrow X\langle 0\rangle,$$

the Postnikov tower of X. The simplicial sets $X\langle n\rangle$, the Postnikov sections of X, realize a filtration of X according to its homotopy groups : for all $0 \leqslant k \leqslant n$, we have $\pi_k(X\langle n\rangle) = \pi_k(X)$ and for all $k > n$, $\pi_k(X\langle n\rangle) = 0$. The fiber of p_n is an Eilenberg-MacLane space $K(\pi_n(X), n)$. The space X can be inductively reconstructed up to homotopy with the help of a homotopy pullback square

$$\begin{array}{ccc} X\langle n\rangle & \dashrightarrow & \operatorname{hocolim}_\Gamma L(\pi_n(X), n+1) \\ {\scriptstyle p_n}\downarrow & & \downarrow \\ X\langle n-1\rangle & \xrightarrow[k_n]{} & \operatorname{hocolim}_\Gamma K(\pi_n(X), n+1), \end{array}$$

where Γ is the fundamental groupoid of X, k_n is a morphism which represents an equivariant cohomology class, namely, the $n-$th Postnikov invariant of X. The space $L(\pi_n(X), n+1)$ is contractible and such that there is a fibration sequence

$$K(\pi_n(X), n) \to L(\pi_n(X), n+1) \to K(\pi_n(X), n+1).$$

Our goal is to extend these results to operads.

The definition of Postnikov towers for simplicial operads is easy since the Postnikov sections $P\langle n\rangle$ of a fibrant simplicial operad P inherit an operad structure. In contrast, the definition of the Postnikov invariants of a simplicial operad and therefore the reconstruction process of its Postnikov tower raises difficulties. To bypass this problem, we filter the Postnikov sections of P

by arity and construct a Postnikov double tower

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & & & \vdots \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
\cdots & \longrightarrow & P\langle r+1,n+1\rangle & \xrightarrow{\mathrm{coar}_{\leqslant r}} & P\langle r,n+1\rangle & \longrightarrow & \cdots & \longrightarrow & P\langle 1,n+1\rangle \\
 & & \downarrow{\scriptstyle p_{r+1,n+1}} & & \downarrow{\scriptstyle p_{r,n+1}} & & & & \downarrow{\scriptstyle p_{1,n+1}} \\
\cdots & \longrightarrow & P\langle r+1,n\rangle & \xrightarrow{\mathrm{coar}_{\leqslant r}} & P\langle r,n\rangle & \longrightarrow & \cdots & \longrightarrow & P\langle 1,n\rangle \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 & & \vdots & & \vdots & & & & \vdots \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
\cdots & \longrightarrow & P\langle r+1,0\rangle & \xrightarrow{\mathrm{coar}_{\leqslant r}} & P\langle r,0\rangle & \longrightarrow & \cdots & \longrightarrow & P\langle 1,0\rangle
\end{array}
$$

where

$$
P\langle r,n\rangle(k) = \begin{cases} P\langle n\rangle(k) & \text{if } k \leqslant r \\ pt & \text{else.} \end{cases}
$$

In the case of a simplicial set X, we have to keep track of the action of the fundamental groupoid $\Pi(X)$ on the homotopy group $\pi_k(X)$ and the universal covering $X^\vee$ of X. We therefore introduce $\Gamma-$operads to take into account the action of the fundamental groupoid operad Γ of a simplicial operad P on its homotopy groups and universal covering. The collection of its homotopy groups $\pi_k(P(x))$ for all point x in P form an additive $\Gamma-$operad with Γ the fundamental groupoid operad of P. We use this collection as coefficients for the operadic version of the equivariant cohomology which we define afterwards.

This operadic equivariant cohomology serves as receptacle for the Postnikov invariants. More precisely, if we consider the commutative square extracted from the Postnikov double tower :

$$
\begin{array}{ccc}
P\langle r,n\rangle & \longrightarrow & P\langle r-1,n\rangle \\
\downarrow & & \downarrow \\
P\langle r,n-1\rangle & \longrightarrow & P\langle r-1,n-1\rangle,
\end{array}
$$

the $(r,n)-$th Postnikov invariant of P is a class of relative operadic equivariant cohomology $k_{r,n}$ associated to the filling morphim

$$
\rho_{r,n} : P\langle r,n\rangle \to P\langle r,n-1\rangle \underset{P\langle r-1,n-1\rangle}{\times} P\langle r-1,n\rangle.
$$

We compare $k(f)$, the operadic Postnikov invariant of a simplicial operad morphism f, and $k(f(r))$, the Postnikov invariant of the simplicial set morphism $f(r)$. The obvious functor $\mathrm{ar}_r : \mathtt{sOp} \to \mathtt{sSet}$ which maps a simplicial operad P to the simplicial set $P(r)$ induces a morphism between equivariant cohomology groups, which maps $k(f)$ to $k(f(r))$.

Then our main result reads as follows. If we consider again the square previously extracted from the Postnikov double tower, the following theorem allows reconstructing up to homotopy $P\langle r,n\rangle$ from the three other simplicial sets in this square :

Theorem A. *Let P be a fibrant simplicial operad. We denote by $\Gamma_{\leqslant r}$ the $r-$cotruncation of Γ and by Q the pullback*

$$P\langle r,n-1\rangle \underset{P\langle r-1,n-1\rangle}{\times} P\langle r-1,n\rangle.$$

For all $r,n \geqslant 2$, let $\pi_n(P(r))$ be the additive $\Gamma-$operad concentrated in arity r. There is a cofibrant replacement $\overline{Q}$ of

$$\overline{P\langle r,n-1\rangle} \underset{\overline{P\langle r-1,n-1\rangle}}{\times} \overline{P\langle r-1,n\rangle}$$

and an operad morphism $\overline{\rho_{r,n}}$ so that

$$\begin{array}{ccc} P\langle r,n\rangle & \xrightarrow{\rho_{r,n}} & Q \\ \sim\uparrow & & \uparrow\sim \\ \overline{P\langle r,n\rangle} & \xrightarrow[\overline{\rho_{r,n}}]{} & \overline{Q} \end{array}$$

commutes in $\mathtt{sOp} \downarrow \mathtt{B}\Gamma_{\leqslant \mathtt{r}}$.

Moreover, the square in $\mathtt{sOp} \downarrow \mathtt{B}\Gamma_{\leqslant \mathtt{r}}$

$$\begin{array}{ccc} \overline{P\langle r,n\rangle} & \dashrightarrow & \operatorname{hocolim}_{\Gamma_{\leqslant r}} L(\pi_n(P(r)),n+1) \\ \overline{\rho_{r,n}}\downarrow & & \downarrow \\ \overline{Q} & \xrightarrow[k_{r,n}]{} & \operatorname{hocolim}_{\Gamma_{\leqslant r}} K(\pi_n(P(r)),n+1) \end{array}$$

is a homotopy pullback square.

It is therefore theoretically possible under the assumptions of this theorem to reconstruct up to homotopy the simplicial operads P inductively from its double tower, diagonal after diagonal. Practically, this reconstruction process permits to develop an obstruction theory for simplicial operads :

Theorem B. *Let $R \in \mathtt{sOp}$ be such that R is fibrant and $\Gamma = \pi(R)$. Let also $P,Q \in \mathtt{sOp} \downarrow \mathtt{B}\Gamma$ with P cofibrant. We consider a morphism $f : P \to R$ and a cofibration $i : P \to Q$ in $\mathtt{sOp} \downarrow \mathtt{B}\Gamma$. There is a morphism $f' : Q \to R$ such that the diagram*

$$\begin{array}{ccc} P & \xrightarrow{f} & R \\ i\downarrow & \nearrow_{f'} & \\ Q & & \end{array}$$

commutes if and only if a sequence of operadic equivariant cohomology class $w_{r,n} \in H^{n+1}_{\Gamma}(Q,P;\pi_n(R)(r))$ defined recursively for all $n,r \geqslant 2$ vanishes.

We first give a reminder in section 1 about simplicial sets, operads and homotopy limits. For our purpose, we notably review the definition of coverings of simplicial sets in connection with the fundamental groupoids. We explicitly describe the structure of operads in groupoids as well.

We define in section 2 the Postnikov sections of a simplicial operad, as well as its Postnikov tower and its Postnikov double tower.

The section 3 is devoted to the definition of $\Gamma-$operads.

We describe in section 4 the model structure on the new operad categories introduced in this thesis. We establish among other things that the category of $\Gamma-$operad in simplicial sets has a model structure where the weak equivalences (respectively, the fibrations) are the pointwise weak equivalences (respectively, the pointwise fibrations) in sSet.

The section 5 deals with the coverings of a simplicial operad. We define a covering functor from the category $\mathtt{sOp} \downarrow \mathtt{B}\Gamma$ of simplicial operads over the nerve of a groupoid operad Γ to the category $\mathtt{sOp}^{\Gamma}$ of simplicial $\Gamma-$operads. We prove that the covering functor is left adjoint to the homotopy colimit functor. Moreover, this adjunction is a Quillen one.

We establish in section 6 that the additive $\Gamma-$operad (explicitly, the $\Gamma-$operad in the category of abelian groups equipped with the monoidal structure provided by the direct sum) form an abelian category. This is important because, if we consider a simplicial operad P, the collection of its homotopy groups $\pi_k(P(x))$ for all point x in P form an additive $\Gamma-$operad, and we use this collection as coefficients for the operadic equivariant cohomology.

The purpose of section 7 is the definition of the operadic equivariant cohomology. After defining it, we demonstrate that $H^n_{\Gamma}(P;A)$ is isomorphic to the group of homotopy class of morphisms $[P^{\vee}, K(A,n)]_{\mathtt{sOp}^{\Gamma}}$ from $P^{\vee}$, the universal covering of P, to $K(A,n)$, an Eilenberg-MacLane $\Gamma-$operad. We also define a reduced and a relative version of the operadic equivariant cohomology, and give a universal coefficient theorem when the coefficients are concentrated in one arity r. Note that under this assumption, $H^n_{\Gamma}(P;A)$ reduces to the cohomology of the indecomposables of P in arity r with coefficients in the $\Gamma-$symmetric sequence in abelian groups $A(r)$.

We then define in section 8 the Postnikov invariant of a simplicial operad morphism and thereafter the Postnikov invariants of a simplicial operad P.

In section 9, we compare $k(f)$, the operadic Postnikov invariant of a simplicial operad morphism f, and $k(f(r))$, the Postnikov invariant of the simplicial set morphism $f(r)$.

We finally prove theorems A and B in section 10.

1. Conventions and background

We review classical results on simplicial sets, operads and homotopy limits. We also introduce the notation and terminology used throughout this thesis.

1.1. Simplicial sets and related

1.1.1. Simplicial sets. As usual, we denote by Δ the simplicial indexing category. We let Δ^- denote the subcategory of Δ with the surjective map as morphisms; equivalently, Δ^- is the subcategory whose morphisms are generated by the degeneracies.

We denote by $\mathtt{sSet}$ the category of simplicial sets, and more generally, if $\mathtt{C}$ is a category, we denote by $\mathtt{sC}$ the category of simplicial objects in $\mathtt{C}$ and $\mathtt{cC}$ the category of cosimplicial objects in $\mathtt{C}$. Formally, $\mathtt{sC}$ is the category of $\Delta^{op}-$diagrams. We also denote by $\mathtt{s^-C}$ the category of $\Delta^{-op}-$diagrams in $\mathtt{C}$. We adopt the notation $-_\flat : \mathtt{sC} \to \mathtt{s^-C}$ for the obvious forgetful functor.

We consider the category of differential non-negatively lower graded abelian group $\mathtt{dg_*Ab}$ (also know as the category of chain complexes) equipped with its usual model structure (see [13, 2.3]).

We use the notation

$$\mathrm{N}_* : \mathtt{sAb} \to \mathtt{dg_*Ab}$$

for the normalization functor, which is defined in degree n by the quotient over the image of the degeneracies

$$\mathrm{N}_n(X) = \frac{X_n}{s_0(X_{n-1}) + \dots + s_{n-1}(X_{n-1})},$$

for all $X \in \mathtt{sAb}$. The differential δ of $\mathrm{N}_*(X)$ is the alternating sum of the faces of X. The homology of the simplicial set X is defined by $\mathrm{H}_\bullet(X) = \mathrm{H}(\mathrm{N}_*(X))$.

We equip $\mathtt{sSet}$ with the usual Kan model structure (see [25, II.3, Theorem 3] or [10, Theorem I.11.3]).

The homotopy groups of a Kan complex X can be defined in a purely combinatorial way (see [10, I.7]), but this definition does not hold for a non-fibrant simplicial set. We therefore set $\pi_k(X, x)$, the $k-$th homotopy group of the simplicial set X at the basepoint x in X_0 as $\pi_k(|X|, x)$, the $k-$th homotopy group of the geometric realization $|X|$ of X at the basepoint x in $|X|$ (see [10, I.2]).

Similarly, we define the fundamental groupoid $\pi(X)$ of a simplicial set X as the full subcategory of the fundamental groupoid $\pi(|X|)$ of the geometric realization $|X|$ of X with $\mathrm{ob}\,(\pi(X)) = X_0$.

1.1.2. Cosimplicial abelian group. We denote by $\mathtt{cAb}$ the category of cosimplicial abelian groups. Formally, $\mathtt{cAb}$ is the category of $\Delta-$diagrams in $\mathtt{Ab}$.

We also consider the category of differential non-negatively upper graded abelian group $\mathtt{dg^*Ab}$ (also know as the category of cochain complexes), which is equipped with the following model structure :

- the weak equivalences are the morphisms which induce an isomorphism on cohomology;
- the cofibrations are the morphisms which are injective in positive degrees;
- the fibrations are the morphisms which are degreewise surjective with injective kernel.

We use the notation

$$\mathrm{N}^* : \mathtt{cAb} \to \mathtt{dg^*Ab}$$

for the conormalization functor, which is defined in degree n by the intersection of the kernel of the codegeneracies

$$\mathrm{N}^n(X) = \bigcap_{i=0}^{n-1} s_i(X^n)$$

for all $X \in \mathtt{cAb}$. The differential δ of $\mathrm{N}^*(X)$ is given by the alternate sum of the cofaces of X. The cohomology of the cosimplicial abelian group X is defined by $\mathrm{H}^\bullet(X) = \mathrm{H}^\bullet(\mathrm{N}^*(X))$.

There is a Dold-Kan equivalence between $\mathtt{dg^*Ab}$ and $\mathtt{cAb}$. We do not use it in this thesis, but we nevertheless mention it as it allows to transport the model structure on $\mathtt{dg^*Ab}$ to $\mathtt{cAb}$. Explicitly, in the category of cosimplicial abelian groups :
- the weak equivalences are the morphisms which induce an isomorphism on cohomology;
- a morphism $f : S \to T$ is a cofibration if $\mathrm{N}^* f : \mathrm{N}^* S \to \mathrm{N}^* T$ is injective in positive degrees;
- the fibrations are the morphisms which are surjective dimensionwise.

1.1.3. Universal covering system. We now describe universal coverings of simplicial sets by means of local coefficient systems – for a comprehensive account, see [10, VI]. We denote by $B : \mathtt{Grd} \to \mathtt{sSet}$ the nerve functor from the category of groupoids to the category of simplicial sets. Let $\Gamma \in \mathtt{Grd}$. The category $\mathtt{sSet}^\Gamma$ of $\Gamma-$simplicial sets and the category $\mathtt{sSet} \downarrow \mathtt{B}\Gamma$ of simplicial sets over $B\Gamma$ both admit a model structure induce by the Kan model structure on $\mathtt{sSet}$:
- a morphism $\phi \in \mathrm{Mor}_{\mathtt{sSet}^\Gamma}(X, Y)$ is a fibration (respectively, a weak equivalence) if, for all $x \in \mathrm{ob}\,(\Gamma)$, $\phi_x : X(x) \to Y(x)$ is a fibration (respectively, a weak equivalence) in $\mathtt{sSet}$; the cofibrations are the morphisms which have the left lifting property with respect to the acyclic fibrations.
- A morphism $\phi \in \mathrm{Mor}_{\mathtt{sSet}\downarrow\mathtt{B}\Gamma}(X, Y)$ is a fibration (respectively, a cofibration, a weak equivalence) if ϕ is a fibration (respectively, a cofibration, a weak equivalence) in $\mathtt{sSet}$.

There is a functor

$$\mathrm{hocolim}_\Gamma : \mathtt{sSet}^\Gamma \longrightarrow \mathtt{sSet} \downarrow \mathtt{B}\Gamma$$

with, for all $X \in \mathtt{sSet}^\Gamma$ and $n \geqslant 0$,

$$\mathrm{hocolim}_\Gamma X_n = \coprod_{x_0 \to \ldots \to x_n \in B\Gamma_n} X(x_0)_n.$$

This functor admits a left adjoint, the *universal covering system functor*

$$-^\vee : \mathtt{sSet} \downarrow \mathtt{B}\Gamma \longrightarrow \mathtt{sSet}^\Gamma$$

such that, for all $x \in \Gamma$, $X^\vee(x)$ is defined by the pullback

$$\begin{array}{ccc} X^\vee(x) & \dashrightarrow & B(\Gamma \downarrow x) \\ \vdots & & \downarrow \\ X & \longrightarrow & B\Gamma. \end{array}$$

In addition, the pair $(-^\vee, \mathrm{hocolim}_\Gamma)$ defines a Quillen adjunction.

Note that we can define in the very same way functors

$$\mathrm{hocolim}_\Gamma : \mathtt{s^-Set}^\Gamma \longrightarrow \mathtt{s^-Set} \downarrow \mathtt{B}\Gamma$$

and

$$-^\vee : \mathtt{s^-Set} \downarrow \mathtt{B}\Gamma \longrightarrow \mathtt{s^-Set}^\Gamma$$

such that there is an adjunction

$$-^{\vee} : \mathtt{s^{-}Set} \downarrow \mathrm{B}\Gamma \rightleftarrows \mathtt{s^{-}Set}^{\Gamma} : \mathrm{hocolim}_{\Gamma}.$$

Morever, we have for all $X \in \mathtt{sSet} \downarrow \mathrm{B}\Gamma$

$$(X^{\vee})_{\flat} = (X_{\flat})^{\vee}$$

and for all $Y \in \mathtt{sSet}^{\Gamma}$

$$(\mathrm{hocolim}_{\Gamma} Y)_{\flat} = \mathrm{hocolim}_{\Gamma}(Y_{\flat}).$$

1.1.4. The Dold-Kan equivalence. Recall that the simplices Δ^n are defined by the functors $\mathrm{Mor}_{\Delta}(-, \underline{n})$ for all $n \geqslant 0$ and form a cosimplicial object $\Delta^{\bullet}$ in the category of simplicial sets. The Yoneda lemma implies that $X_n \simeq \mathrm{Mor}_{\mathtt{sSet}}(\Delta^n, X)$ for all $n \geqslant 0$.

We denote by $[y] \in \mathrm{N}_* X$ the class of $y \in X$. There is an identity

$$\begin{array}{ccc} \int^{\underline{n}\in\Delta} X_n \otimes \mathrm{N}_*\Delta^n & \xrightarrow{\simeq} & \mathrm{N}_* X \\ x \otimes [\sigma : \underline{k} \to \underline{n}] \in (X_n \otimes \mathrm{N}_*\Delta^n)_k & \longmapsto & [\sigma^*(x)] \in \mathrm{N}_k X \end{array} \tag{1}$$

(see [8, II.5.0.11].)

Let

$$D_{\bullet} : \mathtt{dg_*Ab} \to \mathtt{sAb}$$

denotes the Dold-Kan functor. For all $Y \in \mathtt{dg_*Ab}$, there is a natural isomorphism

$$D_{\bullet} Y \simeq \mathrm{Mor}_{\mathtt{dg_*Ab}}(\mathrm{N}_*\Delta^{\bullet}, Y)$$

where the faces and the degeneracies of the simplicial abelian group $\mathrm{Mor}_{\mathtt{dg_*Ab}}(\mathrm{N}_*\Delta^{\bullet}, Y)$ are induced by the cofaces and the codegeneracies of the cosimplicial object $\Delta^{\bullet}$ (see [8, II.5.0.7]).

The functors $(N_*, D_{\bullet})$ are adjoint to each other:

$$\mathrm{N}_* : \mathtt{sAb} \rightleftarrows \mathtt{dg_*Ab} : \mathrm{D}_{\bullet}\,.$$

This adjunction relation actually gives an equivalence of categories (the Dold-Kan equivalence). The natural isomorphism of this adjunction

$$\rho : \mathrm{Mor}_{\mathtt{dg_*Ab}}(\mathrm{N}_* X, Y) \simeq \mathrm{Mor}_{\mathtt{dg_*Ab}}(X, \mathrm{Mor}_{\mathtt{sAb}}(\mathrm{N}_*\Delta^{\bullet}, Y))$$

is given by $\rho(f)(x)([\sigma]) = f([\sigma^* x])$ for all $f \in \mathrm{Mor}_{\mathtt{dg_*Ab}}(\mathrm{N}_* X, Y)$, $x \in X$ and $\sigma : \underline{k} \to \underline{n} \in \Delta^n_k$. In other words, $\rho(f)(x) \in \mathrm{Mor}_{\mathtt{dg_*Ab}}(N_*\Delta^{\bullet}, Y)$ is the composite

$$\begin{array}{ccccccc} \mathrm{N}_*\Delta^n & \longrightarrow & x \otimes \mathrm{N}_*\Delta^n & \xrightarrow{(1)} & \mathrm{N}_* X & \xrightarrow{f} & Y \\ [\sigma] & \longmapsto & x \otimes [\sigma] & \longmapsto & [\sigma^* x] & \longmapsto & f([\sigma^* x]). \end{array}$$

Moreover, this adjunction is a Quillen equivalence of categories when we equip $\mathtt{sAb}$ and $\mathtt{dg_*Ab}$ with their usual model structures (see [10, Theorem III.2.6 and III.2.11] and [13, 2.3]).

1.2. Operads

Operads were originally introduced to study iterated loop spaces by Boardman and Vogt in [3] and by May in [21] – for a comprehensive account about operads, see [19]. We mostly deal with operads in simplicial sets for the definition of the Postnikov towers. We however use operads in other base categories, so we define them in a general symmetric monoidal category $\mathtt{M}$. We use the classical simplicial set notation for colimits and limits in $\mathtt{M}$:
– $\varnothing$ is the initial object;
– $*$ is the terminal object;
– $\coprod$ denotes the coproduct.
in addition, $\otimes$ and $\mathbb{1}$ respectively denotes the tensor product and the monoidal unit in $\mathtt{M}$.

1.2.1. Operads. For us, an *operad* whitout further precision denotes a symmetric connected operad without arity zero term. Thus, an operad P in the base category $\mathtt{M}$ is a collection of objects $P(r) \in \mathtt{M}$ for all $r \geqslant 1$ equipped with :
– an action of the symmetric group on r elements $\Sigma(r)$ on $P(r)$ for all $r \geqslant 1$;
– composition products
$$\circ_i : P(r) \otimes P(s) \to P(r+s-1)$$
for all $r, s \geqslant 1$ and $1 \leqslant i \leqslant r$;
– a unit morphism $\eta : \mathbb{1} \to P(1)$.
Operads satisfy equivariance, associativity and unit axioms (see [8, I.22]).

In what follows, we assume in addition that our operads satisfy a connectedness condition $P(1) = \mathbb{1}$.

We denote by $\mathtt{MOp}$ the category of connected symmetric operads without arity zero term operads in the base category $\mathtt{M}$.

The initial object of $\mathtt{MOp}$ is the unit operad $\mathtt{I}$, such that $\mathtt{I}(1) = \mathbb{1}$ and $\mathtt{I}(r) = \varnothing$ for all $r \geqslant 2$. If $\mathtt{M}$ is a cartesian monoidal category, the terminal object of $\mathtt{MOp}$ is the operad $\mathtt{pt}$, such that $\mathtt{pt}(r) = \mathbb{1}$ for all $r \geqslant 1$. If $\mathtt{M} = (\mathtt{Ab}, \otimes)$, we rather denotes $\mathtt{Com}$ the operad in $\mathtt{Ab}^{\oplus}\mathtt{Op}$ such that $\mathtt{Ab}(r) = \mathbb{1}$ for all $r \geqslant 1$ since a $\mathtt{Com}-$algebra is a commutative algebra (whitout unity).

1.2.2. Simplicial and groupoid operads. There is an equivalence of categories between the category $\mathtt{sSetOp}$ of operads in simplicial sets and the category $\mathtt{s}(\mathtt{SetOp})$ of simplicial objects in set-based operads. We therefore call *simplicial operad* an operad in simplicial sets and simply denotes $\mathtt{sOp}$ the category of simplicial operads. The category $\mathtt{sOp}$ inherits a simplicial model structure from the usual Kan model structure on simplicial sets (see [8, II.8.2]). A morphism $\phi \in \mathrm{Mor}_{\mathtt{sOp}}(P, Q)$ is a fibration (respectively, a weak equivalence) if, for all $r \geqslant 1$, $\phi(r) : P(r) \to Q(r)$ is a fibration (respectively, a weak equivalence) in $\mathtt{sSet}$; cofibrations are morphisms which have the left lifting property with respect to the acyclic fibrations. A simplicial operad P is therefore fibrant if, for all $r \geqslant 1$, $P(r)$ is a Kan complex.

A *groupoid operad* Γ is an operad in $(\mathtt{Grd}, \times)$, the category of groupoids equipped with the cartesian product. Therefore, Γ is a collection of groupoids $\Gamma(r)$ for all $r \geqslant 1$. In this context,
– for all $r \geqslant 1$ and $\sigma \in \Sigma(r)$, the action of σ is a functor $\sigma : \Gamma(r) \to \Gamma(r)$;
– for all $r, s \geqslant 1$ and $0 \leqslant i \leqslant r$, the $i-$th composition product is a functor $\circ_i : \Gamma(r) \times \Gamma(s) \to \Gamma(r+s-1)$;
– We have $\Gamma(1) = pt$.
By abuse, we simply denote by Γ the set $\bigcup_{r \geqslant 1} \mathrm{ob}\ \Gamma(r)$ and we set $\mathrm{Mor}_{\Gamma}(x, y) = \mathrm{Mor}_{\Gamma(r)}(x, y)$ for short when we assume $x, y \in \Gamma(r)$.

The fundamental groupoid functor
$$\pi : \mathtt{sSet} \to \mathtt{Grd}$$

extends to a functor

$$\pi : \mathtt{sOp} \to \mathtt{GrdOp}$$

because, for any simplicial operad P, the collection $\big(\pi(P(r))\big)_{r\geqslant 1}$ of fundamental groupoids of P calculated aritywise inherits a structure of groupoid operads. Since the groupoids we consider in this paper are in a practical way fundamental groupoids, and since we assume that the simplicial sets we consider are connected, we assume by convention that all the groupoids we pick are connected.

If Γ is a groupoid operad, then the collection of the nerves $B\Gamma(r)$ is equipped with a structure of operads : the action of the symmetric groups is a consequence of the functoriality of the nerve. The nerve functor $B : \mathtt{Grd} \to \mathtt{sSet}$ is strongly monoidal, so there are natural isomorphisms $B\Gamma(m) \times B\Gamma(n) \simeq B(\Gamma(m) \times \Gamma(n))$. The composition products on the collection $B\Gamma(r)$ are the composites of these isomorphisms with the morphisms $B(\Gamma(m)\times\Gamma(n)) \to B\Gamma(m+n-1)$ induced by the composition products of the groupoid operad Γ. Explicitly, we have :
– for all $r \geqslant 1$, $n \geqslant 0$, $x_0 \to \ldots \to x_n \in B\Gamma(r)_n$ and $\sigma \in \Sigma_r$,

$$\sigma.(x_0 \to \ldots \to x_n) = \sigma.x_0 \to \ldots \to \sigma.x_n;$$

– for all $s \geqslant 1$, $1 \leqslant i \leqslant r$ and $y_0 \to .. \to y_n \in B\Gamma(s)_n$,

$$(x_0 \to \ldots \to x_n) \circ_i (y_0 \to .. \to y_n) = x_0 \circ_i y_0 \to \ldots \to x_n \circ_i y_n.$$

Therefore, the nerve functor

$$B : \mathtt{Grd} \to \mathtt{sSet}$$

extend to a functor

$$B : \mathtt{GrdOp} \to \mathtt{sOp}.$$

Note that $B\Gamma$ is fibrant because it is aritywise (see [10, Lemma 3.5]).

1.2.3. Symmetric sequences. In order to define free operads in the next subsection, we need to define the category $\mathtt{MSeq}$ of *(connected) symmetric sequences* in a base category $\mathtt{M}$. Let first denote by Σ the groupoid of finite ordinals $\underline{r} = \{1 < \ldots < r\}$ with $r \geqslant 2$ and such that

$$\mathrm{Mor}_{\Sigma}(\underline{r}, \underline{s}) = \begin{cases} \varnothing & \text{if } r \neq s, \\ \Sigma(r) & \text{otherwise.} \end{cases}$$

A (connected) symmetric sequence in $\mathtt{M}$ is a $\Sigma-$diagram in $\mathtt{M}$. In other terms, M is a collection of objects $M(r) \in \mathtt{M}$ with an action of the symmetric group $\Sigma(r)$ on $M(r)$ for all $r \geqslant 2$. Note that an operad can equivalently be defined as a symmetric sequence equipped with composition products.

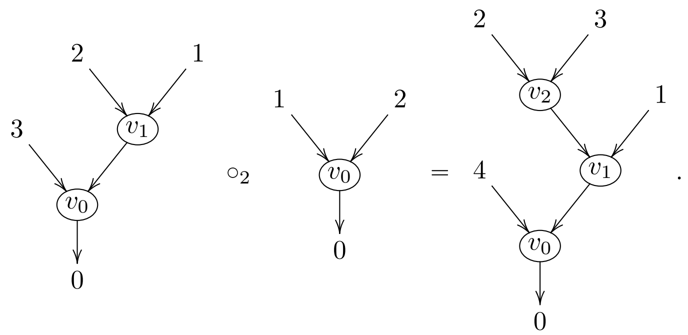
For all $r \geqslant 2$, we let $\mathtt{MSeq(r)}$ denotes the category of objects in M under an action of the symmetric group on r elements $\Sigma(r)$.

1.2.4. $\Gamma-$operads. In order to define operadic Postnikov invariants of an operad P, we will need another category of operads which takes into account the action of the fundamental operad groupoid Γ of P on the homotopy groups and the universal covering of P : this is the category $\mathtt{MOp}^{\Gamma}$ of $\Gamma-$operads in $\mathtt{M}$. We define this category in section 3.

1.3. Free operads, coproduct of an operad with a free operad

We now describe the free operads and the coproduct of an operad with a free operad by means of decorated trees. For a complete account about trees, free operads, semi-alternate two-colored trees and operad coproducts, see [8, Appendix A]. We assume in this subsection that all colimits exists in the symmetric monoidal category $(\mathtt{M}, \otimes, \mathbb{1})$ and that the tensor product distributes over colimits. We adopt the base set notation $\bigvee$ for the coproduct of operads as opposed to $\coprod$ used for the coproduct in $\mathtt{M}$.

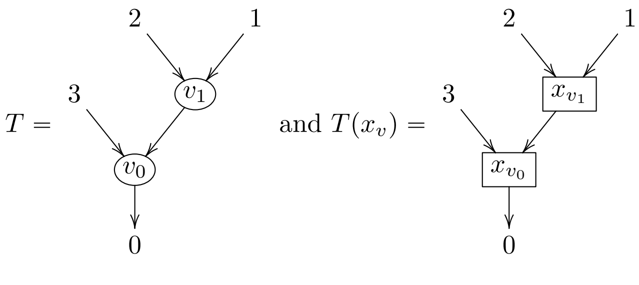
1.3.1. Trees. The operad $\mathtt{Tree} \in \mathtt{SetOp}$ is the operad of *trees*, where we consider oriented trees with one outgoing edge labelled by 0, *the root*, r ingoing edges labelled by integers from 1 to r, *the leaves*, and such that each vertex has at least two ingoing edges and one outgoing edge :
- The action of the symmetric group is provided by the reindexation of the leaves.
- The $i-$th composition product is provided by the grafting of the outgoing edge of one tree on the $i-$th leave of another tree up to relabeling the leaves of the resulting tree. For instance



- The unit is the tree $\overset{1}{\underset{0}{\downarrow}}$ without vertex and with one edge from its only leave to its root.

Let $T \in \mathtt{Tree}$; we denote $V(T)$ the set of vertices of T.

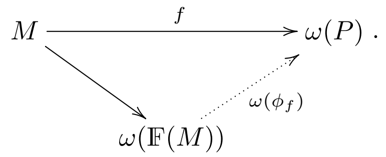
Let $M \in \mathtt{MSeq}$. A *tree* $T \in \mathtt{Tree}$ *decorated by objects in* M is a treewise tensor product of objects $(x_v)_{v \in V(T)} \in M$ such that $x_v \in M(r_v)$, where r_v is the number of ingoing edges of v. Moreover, if M is an operad, we denote by $\lambda_T(x_v)$ the element of M obtained by composing the objects $(x_v)_{v \in V(T)}$ in M according to T. For example, if



with $x_{v_0}, x_{v_1} \in M(2)$, then $\lambda_T(x_v) = (3\ 1).(x_{v_0} \circ_2 x_{v_1})$. We denote $T(M)$ the set of all possible decorations of T by objects in M.

We have the following theorem :

1.3.2. Theorem (S. Ginzburg and M. Kapranov [9]; see also [8, Appendix A]). *The forgetful functor* $\omega : \texttt{MOp} \to \texttt{MSeq}$ *which associates to an operad* P *the symmetric sequence* $(P(r))_{r\geqslant 2}$ *admits a left adjoint* $\mathbb{F}$*, the* free operad functor*, such that, for all* $P \in \texttt{MOp}$ *and* $f \in \mathrm{Mor}_{\texttt{MSeq}}(M, w(P))$*, there is one and only one* $\phi_f \in \mathrm{Mor}_{\texttt{MOp}}(\mathbb{F}(M), P)$ *that makes the following diagram commutes :*



Moreover, for all $r \geqslant 2$*, we have*

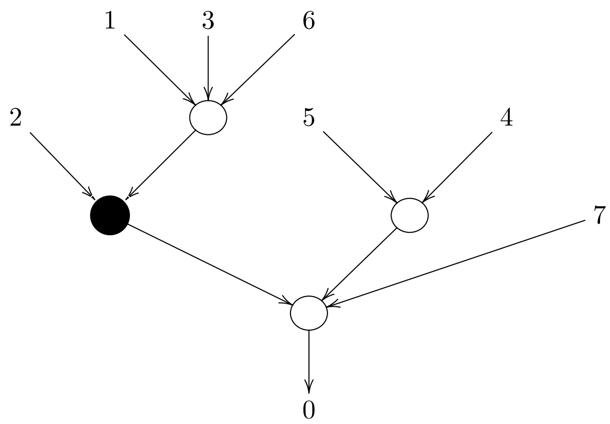
$$\mathbb{F}(M)(r) = \coprod_{T \in \texttt{Tree(r)}} T(M)$$

where the coproduct ranges over the trees with r *leaves.*

1.3.3. Corollary. *Let* $P \in \texttt{MOp}$*. The restriction of the free operad functor* $\mathbb{F} : \texttt{MSeq} \to \texttt{MOp}$ *to the overcategory* $\texttt{MSeq} \downarrow P$ *is left adjoint to the forgetful functor* $\omega : \texttt{MOp} \downarrow P \to \texttt{MSeq} \downarrow P$*.*

Proof. Straightforward consequence of the universal property of the free operad. ☐

1.3.4. Semi-alternate two-colored trees. A *semi-alternate two-colored tree* T is a tree with a partition of its vertices in two sets $V(T) = V_\circ(T) \coprod V_\bullet(T)$, the set of white vertices and the set of black vertices, such that there is no edge between two black vertices. For instance,



is a semi-alternate two-colored tree.

Let $M \in \texttt{MSeq}$ and $P \in \texttt{MOp}$. A *semi-alternate two-colored tree* T *decorated by objects* in P and M is a treewise tensor product of objects $\big((x_v)_{v\in V_\circ(T)}, (x_{v'})_{v'\in V_\bullet(T)}\big)$ over T such that $x_v \in M(r_v)$ if the white vertex v has r_v ingoing edges and $x'_v \in P(r_{v'})$ if the black vertex v' has $r_{v'}$ ingoing edges. We denote $T(P, M)$ the set of all possible decorations of T by objects in P and M.

1.3.5. Theorem (Fresse [8, Appendix A]). *For all* $P \in \texttt{MOp}$*,* $M \in \texttt{MSeq}$ *and* $r \geqslant 2$*,*

$$P \bigvee \mathbb{F}(M)(r) = \coprod_{T \in \texttt{Tree}_{\circ,\bullet}(r)} T(P, M)$$

where the colimit ranges over the semi-alternate two-colored trees with r *entries.*

1.4. Homotopy limits and colimits

We consider a simplicial model category $\mathtt{M}$. Recall that the *homotopy limit* $\operatorname{holim} X$ of a diagram $X \in \mathtt{M}^{\mathtt{I}}$ with $\mathtt{I}$ a small category is defined by the end $\int_{c \in \mathtt{I}} X(c)^{\mathtt{B}(\mathtt{I} \downarrow c)}$ (see [12, Definition 18.3.2 and Example 18.3.6]). The homotopy limit satisfies the homotopy invariance property (see [12, 18.5.1]) : if there is a natural transformation $f : X \to Y$ between two $\mathtt{I}-$diagram which is an objectwise fibration (respectively, an objectwise acyclic fibration), then the induced map $f_* : \operatorname{holim} X \to \operatorname{holim} Y$ is a fibration (respectively, an acyclic fibration). Moreover, the homotopy limit of an objectwise fibrant diagram is fibrant.

1.4.1. Homotopy pullback. The *homotopy pullback* $X \times^h_Z Y$ of a cospan $X \xrightarrow{f} Z \xleftarrow{g} Y$ in $\mathtt{M}$ is the homotopy limit of this cospan. Therefore, $X \times^h_Z Y$ is the limit of the diagram

$$X \xrightarrow{f} Z \xleftarrow{d_0} Z^{\Delta^1} \xrightarrow{d_1} Z \xleftarrow{g} Y, \qquad \text{(h)}$$

where Z^{Δ^1} denotes the function object provided by the simplicial model structure of $\mathtt{M}$. We now assume that Z is fibrant. A cone from $W \in \mathtt{M}$ to the diagram (h) hence defines a right homotopy between the composites $W \to X \xrightarrow{f} Z$ and $W \to Y \xrightarrow{g} Z$, which is also a left homotopy because Z is fibrant. In particular, the square

$$\begin{array}{ccc} X \times^h_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

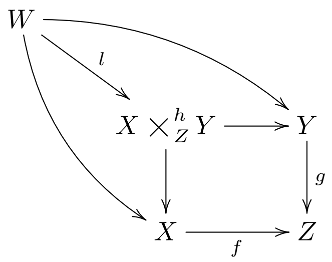
commutes up to homotopy.

In the converse direction, if W is cofibrant, a right homotopy between two composites $W \to X \xrightarrow{f} Z$ and $W \to Y \xrightarrow{g} Z$ defines a cone from W to h because the composites can be made homotopic by a homotopy defined on Z^{Δ^1} (see [12, Proposition 7.4.7]). We then have the following proposition :

1.4.2. Proposition. *Let $X \xrightarrow{f} Z \xleftarrow{g} Y$ be a cospan in $\mathtt{M}$ with Z fibrant and W a cofibrant object in $\mathtt{M}$. If the square*

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

commutes up to homotopy, then there is a morphism $l : W \to X \times^h_Z Y$ such that the triangles *in the diagram*

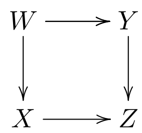


commutes.

The following proposition (see [12, Proposition 19.9.4]) permits to compare the ordinary pullback and the homotopy pullback of a given cospan when one of its morphisms is a fibration :

1.4.3. Proposition. *Let $X \xrightarrow{f} Z \xleftarrow{g} Y$ be a cospan of fibrant objects in $\mathtt{M}$. if f or g a fibration, then there is a natural weak equivalence from $X \times_Z Y$ to $X \times^h_Z Y$.*

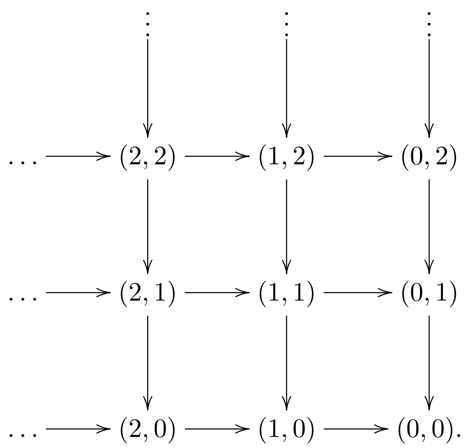
1.4.4. Homotopy pullback square. A commutative diagram



in $\mathtt{M}$ is *homotopy pullback square* if there is a weak equivalence $W \xrightarrow{\sim} X \times^h_Z Y$.

1.4.5. Homotopy pushout. The *homotopy pushout* is the categorical dual of the homotopy pullback. There are therefore dual propositions of Proposition 1.4.2 and Proposition 1.4.3 for homotopy pushouts. We denote by $X \coprod^h_Z Y$ (or by $X \bigvee^h_Z Y$ in a category of operads) the homotopy pushout of a span $X \leftarrow Z \rightarrow Y$.

1.4.6. Double towers. Let $\mathtt{T}$ be the small category



The category of *double towers* in $\mathtt{M}$ is the diagram category $\mathtt{M}^{\mathtt{T}}$.

1.4.7. Proposition. *The category* $\mathtt{M}^{\mathtt{T}}$ *admits a Reedy model structure (see [12, chapter 15]) such that :*
– *The weak equivalences are the objectwise weak equivalences in* $\mathtt{M}$.
– *The cofibrations are the objectwise cofibrations in* $\mathtt{M}$.
– *The fibrations are the natural transformations* $f : X \to Y$ *such that for every* $(i,j) \in \mathbb{N}^2$ *:*
  - $X_{(0,0)} \to Y_{(0,0)}$,
  - $X_{(i,0)} \to Y_{(i,0)} \times_{Y_{(i-1,0)}} X_{(i-1,0)}$ *if* $i > 0$,
  - $X_{(0,j)} \to Y_{(0,j)} \times_{Y_{(0,j-1)}} X_{(0,j-1)}$ *if* $j > 0$ *and*
  - $X_{(i,j)} \to Y_{(i,j)} \times_{\left(Y_{(i,j-1)} \times_{Y_{(i-1,j-1)}} Y_{(i-1,j-1)}\right)} \left(X_{(i,j-1)} \times_{X_{(i-1,j-1)}} X_{(i-1,j-1)}\right)$ *if* $i,j > 0$

  are fibrations in $\mathtt{M}$.

Therefore, a double tower X *is Reedy cofibrant if it is objectwise cofibrant, and Reedy fibrant if it is a tower of fibration with* $X_{(0,0)}$ *fibrant.*

Proof. The category $\mathtt{I}$ is a Reedy category (see [12, Definition 15.1.2]), where the degree of (i,j) is $\frac{(i+j)(i+j+1)}{2} + i$, the subcategory $\overrightarrow{\mathtt{I}}$ have only identity morphisms and $\overleftarrow{\mathtt{I}} = \mathtt{I}$. □

1.4.8. Proposition. *Let* X *in* $\mathtt{M}^{\mathtt{T}}$. *If* X *is Reedy fibrant, then there is a natural weak equivalence* $\lim X \underset{\sim}{\longrightarrow} \operatorname{holim} X$.

Proof. This a consequence of the fact that $\mathtt{T}$ has cofibrant constants (see [12, 19.9.1]). □

2. Postnikov double towers of simplicial operads

We first define the Postnikov tower of a simplicial operad. The reconstruction process of this tower raises difficulties (see appendix A). That is why we introduce the cotruncations of an operad in order to construct the double Postnikov tower of a simplicial operad.

2.1. Postnikov sections of a simplicial operad

Let X be a fibrant simplical set. Recall that for all $n \geqslant 0$, there is an equivalence relation $\sim_n$ defined on the simplices of X as follows : two $q-$simplices $x, y : \Delta_q \to X$ in X are equivalent if the restrictions of x and y to the $n-$skeleton of Δ^q agree. The $n-th$ *Postnikov section* of X is the coset $X\langle n\rangle = X/\sim_n$. Note that $X\langle n\rangle$ is a Kan complex.

2.1.1. Proposition. *Let P be a simplicial fibrant operad and $n \geqslant 0$. The collection $P\langle n\rangle = (P(r)\langle n\rangle)_{r\geqslant 1}$ inherits an operad structure from P. Moreover, $P\langle n\rangle$ is fibrant.*

Proof. For all $r \geqslant 1$, the action of $\Sigma(r)$ on $P(r)$ trivially induces an action on $P\langle n\rangle(r)$.

Let $r, s \geqslant 1$ and $1 \leqslant i \leqslant r$. Let also $x, x' \in P\langle n\rangle(r)$, $y, y' \in P\langle n\rangle(s)$ be such that $x \sim_n x'$ and $y \sim_n y'$. Then $x \circ_i y \sim_n x' \circ_i y'$ because the operadic composition is a simplicial morphism.

The equivariance, unity and associativity relations remains obviously satisfied in this quotient $P\langle n\rangle$. ☐

2.1.2. Proposition. *If $n > k$, then the morphisms $p_{n,k}(r) : P\langle n\rangle(r) \to P\langle k\rangle(r)$ define a fibration $p_{n,k}$. We simply denote $p_n = p_{n,n-1}$.*

Proof. Let us prove that $p_{n,k}$ is an operad morphism. Conditions about symmetric group actions and units are clearly fulfilled. Let $x \in P(r)$, $y \in P(s)$. Then $p_{n,k}(r+s-1)(x\langle n\rangle \circ_i y\langle n\rangle) = (x \circ_i y)\langle k\rangle = x\langle k\rangle \circ_i y\langle k\rangle = p_{n,k}(r)(x\langle n\rangle) \circ_i p_{n,k}(s)(y\langle n\rangle)$. The morphism $p_{k,n}$ is cleary a fibration because it is so aritywise.

☐

2.1.3. Postnikov tower. Let P be a fibrant simplical operad. We call *Postnikov tower* of P the diagram

$$\cdots \longrightarrow P\langle n+1\rangle \xrightarrow{p_{n+1}} P\langle n\rangle \longrightarrow \cdots \longrightarrow P\langle 0\rangle .$$

We have $P = \lim P\langle n\rangle$.

2.2. Truncations and cotruncations

We consider a monoidal category $(\mathtt{M}, \otimes, \mathbb{1})$. Recall that we denote by $\varnothing$ the initial object and by $*$ the terminal object of $\mathtt{M}$. We introduce now the cotrunctation functors that enable us to filter an operad by arity.

2.2.1. Truncations and cotruncations of symmetric sequences. Let $r \geqslant 2$ and consider the $r-$cotruncation functor $\mathrm{coar}_{\leqslant r} : \mathtt{MSeq} \to \mathtt{MSeq}$ such that

$$\mathrm{coar}_{\leqslant r} N(s) = \begin{cases} N(s) & \text{if } s \leqslant r, \\ * & \text{otherwise.} \end{cases}$$

It admits a left adjoint, namely, the $r-$truncation functor $\mathrm{ar}_{\leqslant r} : \mathtt{MSeq} \to \mathtt{MSeq}$ such that

$$\mathrm{ar}_{\leqslant r} M(s) = \begin{cases} M(s) & \text{if } s \leqslant r, \\ \varnothing & \text{otherwise.} \end{cases}$$

2.2.2. Cotruncations of an operad. The cotruncation of symmetric sequences extends to operads : for all $r \geqslant 1$, there is a functor $\mathrm{coar}_{\leqslant r} : \mathtt{MOp} \to \mathtt{MOp}$ such that

$$\mathrm{coar}_{\leqslant r} P(s) = \begin{cases} P(s) & \text{if } s \leqslant r, \\ * & \text{otherwise.} \end{cases}$$

The operadic composition of elements of $\mathrm{coar}_{\leqslant r}(P)$ of arity less than r are given by their composites in P if the composites have arity less than r because we assume that our operads have no component in arity zero.

In contrast, the truncation functor of symmetric sequences does not immediately extend to operads because the codomain of composition products could obviously not be the empty set. However, we can construct a left adjoint to the cotruncation functor in the operad setting. We assume that all colimits exits in the monoidal category $(\mathtt{M}, \otimes, \mathbb{1})$ and that the tensor product distributes over colimits to ensure that the free operad construction described in 1.3 works. We then have the following statement:

2.2.3. Theorem. *The functor* $\mathrm{coar}_{\leqslant r} : \mathtt{MOp} \to \mathtt{MOp}$ *admit a left adjoint* $\mathrm{ar}^{\sharp}_{\leqslant r} : \mathtt{MOp} \to \mathtt{MOp}$ *and if* $P \in \mathtt{MOp}$*, then*

$$P = \lim_{r} \mathrm{coar}_{\leqslant r}(P).$$

Moreover, if $\mathtt{M} = \mathtt{sSet}$ *and we assume that* $\mathtt{sOp}$ *is equipped with the model structure of 1.2.2, then*

$$\mathrm{ar}^{\sharp}_{\leqslant r} : \mathtt{sOp} \rightleftarrows \mathtt{sOp} : \mathrm{coar}_{\leqslant r}$$

is a Quillen adjunction.

Proof. We lift the adjunction from the symmetric sequence level to the operad level by adapting a general construction, the adjoint lifting theorem (see [15] and [4, section 4.5]). We first define the functor $\mathrm{ar}^{\sharp}_{\leqslant r}$ on the full subcategory of $\mathtt{MOp}$ generated by the free objects. Let $M \in \mathtt{MSeq}$ and set $\mathrm{ar}^{\sharp}_{\leqslant r} \mathbb{F}(M) = \mathbb{F}(\mathrm{ar}_{\leqslant r} M)$. For all $Q \in \mathtt{MOp}$, we have an isomorphism

$$\mathrm{Mor}_{\mathtt{MOp}}\,(\mathbb{F}(M), \mathrm{coar}_{\leqslant r} Q) \simeq \mathrm{Mor}_{\mathtt{MOp}}\,(\mathrm{ar}^{\sharp}_{\leqslant r} \mathbb{F}(M), Q)$$

provided by the adjunction between the free operad functor and the forgetful functor together with the adjunction between the endofunctors $\mathrm{coar}_{\leqslant r}$ and $\mathrm{ar}_{\leqslant r}$ in the category of simplicial symmetric sequences. Let now $\phi : \mathbb{F}(M) \to \mathbb{F}(N)$ be any morphism of free operads. We use the Yoneda lemma to obtain the existence of a morphism $\mathrm{ar}^{\sharp}_{\leqslant r}(\phi)$, associated to ϕ, such that we have a commutative diagram of morphism sets

$$\begin{array}{ccc} \mathrm{Mor}_{\mathtt{MOp}}\,(\mathbb{F}(N), \mathrm{coar}_{\leqslant r} Q) & \xrightarrow{\simeq} & \mathrm{Mor}_{\mathtt{MOp}}\,(\mathrm{ar}_{\leqslant r}(\mathbb{F}(N), Q) \\ \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \mathrm{ar}^{\sharp}_{\leqslant r}(\phi)^*} \\ \mathrm{Mor}_{\mathtt{MOp}}\,(\mathbb{F}(M), \mathrm{coar}_{\leqslant r} Q) & \xrightarrow{\simeq} & \mathrm{Mor}_{\mathtt{MOp}}\,(\mathrm{ar}_{\leqslant r}(\mathbb{F}(M), Q) \end{array}$$

fo all simplicial operads Q.

We now extend $\mathrm{ar}^{\sharp}_{\leqslant r}$ to the full category $\mathtt{MOp}$. For any $P \in \mathtt{MOp}$, we have

$$P = \mathrm{coeq}\left(\mathbb{F}(\mathbb{F}(P)) \underset{\mathbb{F}(\lambda_P)}{\overset{\lambda_{\mathbb{F}(P)}}{\rightrightarrows}} \mathbb{F}(P) \right), \quad \mathbb{F}(\iota_P) : \mathbb{F}(P) \to \mathbb{F}(\mathbb{F}(P))$$

where we consider the following morphisms of free operads :
– $\mathbb{F}(\lambda_P)$ is induced by the morphism associated to the identity of the object P;
– $\lambda_{\mathbb{F}(P)}$ is the morphism associated to the identity of the object $\mathbb{F}(P)$;
– $\mathbb{F}(\iota_P)$ is induced by the embedding $\iota_P : P \to \mathbb{F}(P)$.

We then define

$$\mathrm{ar}^\sharp_{\leqslant r}(P) = \mathrm{coeq}\left(\mathrm{ar}^\sharp_{\leqslant r}\,\mathbb{F}(\mathbb{F}(P)) \underset{\mathrm{ar}^\sharp_{\leqslant r}\,\mathbb{F}(\lambda_P)}{\overset{\mathrm{ar}^\sharp_{\leqslant r}\,\lambda_{\mathbb{F}(P)}}{\rightrightarrows}} \mathrm{ar}^\sharp_{\leqslant r}\,\mathbb{F}(P) \right), \quad \text{with } \mathrm{ar}^\sharp_{\leqslant r}\,\mathbb{F}(\iota_P) : \mathrm{ar}^\sharp_{\leqslant r}\,\mathbb{F}(P) \to \mathrm{ar}^\sharp_{\leqslant r}\,\mathbb{F}(\mathbb{F}(P)).$$

This functor fits the desired adjunction relation by construction:

$$\mathrm{ar}^\sharp_{\leqslant r} : \mathtt{MOp} \rightleftarrows \mathtt{MOp} : \mathrm{coar}_{\leqslant r}$$

and the relation $P = \lim_r \mathrm{coar}_r(P)$ is obvious.

If $\mathtt{M} = \mathtt{sSet}$, then this adjunction is a Quillen one because the functor $\mathrm{coar}_{\leqslant r}$ obviously preserves weak equivalences and fibrations. □

2.3. Postnikov double towers of simplicial operads

We use the cotruncation functors to filter the Postnikov tower of a simplicial operad P by arity and form the double Postnikov tower $\mathscr{T}(P)$ of P. In our applications, we will need that the objects of this double tower are cofibrant, so we define a cofibrant Postnikov double tower of P, which is simply a Reedy cofibrant replacement of $\mathscr{T}(P)$.

2.3.1. Postnikov double tower of a simplical operad. Let P be a fibrant simplicial operad. For all $r \geqslant 1$ and $n \in \mathbb{N}$, we denote $P\langle r,n\rangle = \mathrm{coar}_{\leqslant r}(P\langle n\rangle)$ and we consider the *Postnikov double tower*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & & & \vdots \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
\cdots & \to & P\langle r+1,n+1\rangle & \xrightarrow{\mathrm{coar}_{\leqslant r}} & P\langle r,n+1\rangle & \to & \cdots & \to & P\langle 1,n+1\rangle = \mathtt{pt} \\
 & & \downarrow{\scriptstyle p_{r+1,n+1}} & & \downarrow{\scriptstyle p_{r,n+1}} & & & & \downarrow{\scriptstyle p_{1,n+1}} \\
\cdots & \to & P\langle r+1,n\rangle & \xrightarrow{\mathrm{coar}_{\leqslant r}} & P\langle r,n\rangle & \to & \cdots & \to & P\langle 1,n\rangle = \mathtt{pt} \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 & & \vdots & & \vdots & & & & \vdots \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
\cdots & \to & P\langle r+1,0\rangle & \xrightarrow{\mathrm{coar}_{\leqslant r}} & P\langle r,0\rangle & \to & \cdots & \to & P\langle 1,0\rangle = \mathtt{pt}
\end{array}$$

which we denote by $\mathscr{T}(P)$. Of course, for all $r, n \geqslant 0$, $P\langle r,n\rangle$ is fibrant. Moreover, we have $P = \lim \mathscr{T}(P)$. Since the morphism

$$\rho_{r,n} : P\langle r,n\rangle \to P\langle r,n-1\rangle \underset{P\langle r-1,n-1\rangle}{\times} P\langle r-1,n\rangle$$

is such that

$$\rho_{r,n}(s) = \begin{cases} p_{r,n}(r) & \text{if } r = s, \\ \text{id} & \text{otherwise,} \end{cases}$$

we get that $\rho_{r,n}$ is a fibration and the double tower is Reedy fibrant (see Proposition 1.4.7). Thus, there is a weak equivalence $P = \lim \mathscr{T}(P) \to \operatorname{holim} \mathscr{T}(P)$ (see Proposition 1.4.8).

2.3.2. Cofibrant Postnikov double tower of a simplicial operad. We call *cofibrant Postnikov double tower* of a fibrant simplicial operad P a Reedy cofibrant replacement $\overline{\mathscr{T}}(P)$ of $\mathscr{T}(P)$, which is according to the model structure on $\mathtt{sOp}^{\mathtt{T}}$ a tower of fibrations

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & & & \vdots \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
\cdots & \longrightarrow & \overline{P\langle r+1,n+1\rangle} & \xrightarrow{\overline{\mathrm{coar}_{\leqslant r}}} & \overline{P\langle r,n+1\rangle} & \longrightarrow & \cdots & \longrightarrow & \overline{P\langle 1,n+1\rangle} = \mathtt{pt} \\
 & & \downarrow{\scriptstyle \overline{p_{r+1,n+1}}} & & \downarrow{\scriptstyle \overline{p_{r,n+1}}} & & & & \downarrow{\scriptstyle \overline{p_{1,n+1}}} \\
\cdots & \longrightarrow & \overline{P\langle r+1,n\rangle} & \xrightarrow{\overline{\mathrm{coar}_{\leqslant r}}} & \overline{P\langle r,n\rangle} & \longrightarrow & \cdots & \longrightarrow & \overline{P\langle 1,n\rangle} = \mathtt{pt} \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 & & \vdots & & \vdots & & & & \vdots \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
\cdots & \longrightarrow & \overline{P\langle r+1,0\rangle} & \xrightarrow{\overline{\mathrm{coar}_{\leqslant r}}} & \overline{P\langle r,0\rangle} & \longrightarrow & \cdots & \longrightarrow & \overline{P\langle 1,0\rangle} = \mathtt{pt}
\end{array}$$

of fibrant-cofibrant objects.

Moreover, if P is cofibrant, there is a weak equivalence $P \to \operatorname{holim} \overline{\mathscr{T}}(P)$ because there is an acyclic fibration $\operatorname{holim} \overline{\mathscr{T}}(P) \to \operatorname{holim} \mathscr{T}(P)$ induced by the acyclic fibration $\overline{\mathscr{T}}(P) \to \mathscr{T}(P)$ in $\mathtt{sOp}^{\mathtt{T}}$ (see [12, Theorem 19.7.2]).

3. $\Gamma-$operads

Let $(\mathtt{M}, \otimes, \mathbb{1})$ be a symmetric monoidal category with initial object $\varnothing$ and Γ an operad in the category of groupoids.

In order to define the Postnikov invariants of a simplicial operad P, we have to find a way to manage the change of base point for the homotopy groups and the universal covering operad of P : naively picking a point in each arity does not work because a basepoint system is not generally stable under the symmetric groups actions and the operadic composition. To handle this problem, we introduce the general notion of a $\Gamma-$operad (3.1), where Γ is a groupoid operad. The structure of a $\Gamma-$operad can precisely be used to handle the changes of base point in the construction of Postnikov invariants when we take for Γ the fundamental groupoid of our simplicial operad. This kind of operads is a generalization of local coefficient system operads, introduced in [8, I.9.2.6].

We then define the $\Gamma-$symmetric sequences : a $\Gamma-$symmetric sequence is the structure carried by the underlying collection of a $\Gamma-$operad when we forget about the composition products and the unity.

We finally give a construction of a free $\Gamma-$operad.

3.1. $\Gamma-$operads

We now define $\Gamma-$operads and give some basic examples of such a struc ture. Since we assume that operads are connected, we have $\Gamma(1) = pt$.

3.1.1. $\Gamma-$operads . A $\Gamma-$*operad* P in $\mathtt{M}$ is a sequence of functors $(P(r) : \Gamma(r) \to \mathtt{M})_{r\geqslant 1}$ together with natural transformations which describe its composition structure, compatible with the action of the groupoid operad Γ on P. More precisely, P is a collection of objects $(P(x))_{x\in\Gamma} \in \mathtt{M}$ together with :

– for all $x, y \in \Gamma$ and $\phi \in \mathrm{Mor}_\Gamma(x, y)$, isomorphisms $P(\phi) : P(x) \to P(y)$, such that $P(\mathrm{id}) = \mathrm{id}$ and for all $x, y, z \in \Gamma$, $\phi \in \mathrm{Mor}_\Gamma(x, y)$, $\psi \in \mathrm{Mor}_\Gamma(y, z)$, we have $P(\psi \circ \phi) = P(\psi) \circ P(\phi)$;

– for all $\sigma \in \Sigma(r)$ and $x \in \Gamma(r)$, isomorphisms $\sigma_x : P(x) \to P(\sigma.x)$, such that all possible diagrams

$$\begin{array}{ccc} P(x) & \xrightarrow{P(\phi)} & P(y) \\ \downarrow{\scriptstyle \sigma_x} & & \downarrow{\scriptstyle \sigma_y} \\ P(\sigma.x) & \xrightarrow[P(\sigma.\phi)]{} & P(\sigma.y) \end{array}$$

commutes;

- for all $r, s \geqslant 1$, $1 \leqslant i \leqslant r$, $x \in \Gamma(r)$ and $y \in \Gamma(s)$, compositions products

$$\circ_i : P(x) \otimes P(y) \to P(x \circ_i y)$$

such that, for all $\phi \in \mathrm{Mor}_\Gamma(x, z)$ and $\psi \in \mathrm{Mor}_\Gamma(y, t)$, the diagram

$$\begin{array}{ccc} P(x) \otimes P(y) & \xrightarrow{\circ_i} & P(x \circ_i y) \\ {\scriptstyle P(\phi)\otimes P(\psi)}\downarrow & & \downarrow{\scriptstyle P(\phi \circ_i \psi)} \\ P(z) \otimes P(t) & \xrightarrow{\circ_i} & P(z \circ_i t) \end{array}$$

commutes ;

– a unit
$$\eta : \mathbb{1} \to P(1)$$
where 1 is the unit of Γ.

These functors and natural transformations have to satisfy the following axioms :

(a) the diagram

$$\begin{array}{ccc} P(x) \otimes P(y) & \xrightarrow{\sigma \otimes \tau} & P(\sigma.x) \otimes P(\tau.y) \\ \downarrow{\scriptstyle \circ_i} & & \downarrow{\scriptstyle \circ_{\sigma(i)}} \\ P(x \circ_i y) & \xrightarrow[\sigma \circ_{\sigma(i)} \tau]{} & P(\sigma.x \circ_{\sigma(i)} \tau.y) \end{array}$$

commutes for all $r, s \geqslant 1$, $1 \leqslant i \leqslant r$, $x \in \Gamma(r)$, $y \in \Gamma(s)$, $\sigma \in \Sigma(r)$ and $\tau \in \Sigma(s)$.

(b) The diagram

$$\begin{array}{ccc} P(x) \otimes P(y) \otimes P(z) & \xrightarrow{\circ_i \otimes \mathrm{id}} & P(x \circ_i y) \otimes P(z) \\ & & \downarrow{\scriptstyle \circ_j} \\ \downarrow{\scriptstyle \mathrm{id} \otimes \circ_j} & & P\big((x \circ_i y) \circ_j z\big) \\ & & \downarrow{\scriptstyle =} \\ P(x) \otimes P(y \circ_j z) & \xrightarrow[\circ_{i+j-1}]{} & P\big(x \circ_{i+j-1} (y \circ_j z)\big) \end{array}$$

commutes for all $r, s, t \geqslant 1$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$, $x \in \Gamma(r)$, $y \in \Gamma(s)$ and $z \in \Gamma(t)$, where the equality in the right column follows from the associativity of the operad Γ. Similarly, the diagram

$$\begin{array}{ccc} P(x) \otimes P(y) \otimes P(z) & \xrightarrow{\simeq} & P(x) \otimes P(z) \otimes P(y) \\ \downarrow{\scriptstyle \circ_i \otimes \mathrm{id}} & & \downarrow{\scriptstyle \circ_j \otimes \mathrm{id}} \\ P\big(x \circ_i y\big) \otimes P(z) & & P(x \circ_j z) \otimes P(y) \\ \downarrow{\scriptstyle \circ_j \otimes \mathrm{id}} & & \downarrow{\scriptstyle \circ_{i+s-1}} \\ P\big(x \circ_i y) \circ_j z\big) & \xrightarrow[=]{} & P\big((x \circ_j z) \circ_{i+s-1} y\big) \end{array}$$

commutes for all $r, s, t \geqslant 0$, $1 \leqslant i < j \leqslant r$, $x \in \Gamma(r)$, $y \in \Gamma(s)$ and $z \in \Gamma(t)$.

(c) The diagrams

$$\begin{array}{ccc} \mathbb{1} \otimes P(y) & \xrightarrow{\eta \otimes \mathrm{id}} & P(1) \otimes P(y) \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \circ_1} \\ P(y) & \xrightarrow[=]{} & P(1 \circ_1 y) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} P(y) \otimes \mathbb{1} & \xrightarrow{\mathrm{id} \otimes \eta} & P(y) \otimes P(1) \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \circ_i} \\ P(y) & \xrightarrow[=]{} & P(y \circ_i 1) \end{array}$$

commute for all $r \geqslant 1$, $1 \leqslant i \leqslant n$ and $y \in \Gamma(r)$.

We assume that our $\Gamma-$operads P satisfy the operad connectedness condition $P(1) = \mathbb{1}$.

If Γ is trivial in each arity, a $\Gamma-$operad is nothing more than an operad.

A prime example of such a structure is given for all $P \in \mathtt{sOp}$ and $p \geqslant 2$ by $\pi_p(P)$, the $\pi(P)-$operad in $\mathtt{Ab}$ such that $\pi_p(P)(x) = \pi_p(P(x))$ for all $x \in \pi(P)$. Another example of a $\Gamma-$operad is given by the fiber of a fibration $f : P \to Q$ in the category of simplicial operads. Indeed, the collection $f(r)^{-1}(x) \in \mathtt{sSet}$, $x \in \text{ob}\,(\pi(Q(r))$, $r \geqslant 1$ is a simplicial $\pi(Q)-$operad.

3.1.2. The category of $\Gamma-$operads in $\mathtt{M}$. The $\Gamma-$operads in $\mathtt{M}$ form a category $\mathtt{MOp}^{\Gamma}$, where a morphism $f \in \text{Mor}_{\mathtt{MOp}^{\Gamma}}(P,Q)$ is a collection of morphisms $f_x \in \text{Mor}_{\mathtt{M}}(P(x),Q(x))$ for x in Γ such that :

– the diagrams

$$\begin{array}{ccc} P(x) & \xrightarrow{P(\phi)} & P(y) \\ {\scriptstyle f_x}\downarrow & & \downarrow{\scriptstyle f_y} \\ Q(x) & \xrightarrow[Q(\phi)]{} & Q(y) \end{array}$$

and

$$\begin{array}{ccc} P(x) & \xrightarrow{f_x} & Q(x) \\ {\scriptstyle \sigma_x}\downarrow & & \downarrow{\scriptstyle \sigma_x} \\ P(\sigma.x) & \xrightarrow[M(f_{\sigma.x})]{} & Q(\sigma.x) \end{array}$$

commute for all $x, y \in \Gamma$ and $\phi \in \text{Mor}_{\Gamma}(x,y)$;

– the diagram

$$\begin{array}{ccc} P(x) \otimes P(y) & \xrightarrow{\circ_i} & P(x \circ_i y) \\ {\scriptstyle f(x)\otimes f(y)}\downarrow & & \downarrow{\scriptstyle f(x \circ_i y)} \\ Q(x) \otimes Q(y) & \xrightarrow{\circ_i} & Q(x \circ_i y) \end{array}$$

commutes for all $r, s \geqslant 1$, $1 \leqslant i \leqslant r$, $x \in \Gamma(r)$ and $y \in \Gamma(s)$;

– the diagram

$$\begin{array}{ccc} & & P(1) \\ & \nearrow^{\eta_P} & \\ \mathbb{1} & & \downarrow{\scriptstyle f(1)} \\ & \searrow_{\eta_Q} & \\ & & Q(1) \end{array}$$

commutes. Since our $\Gamma-$operad are connected, in the case $\mathtt{M} = \mathtt{Set}$ or $\mathtt{sSet}$, this latter condition is automatically satisfied. The category $\mathtt{MOp}$ is the category of connected $\Gamma-$operads in $\mathtt{M}$.

3.1.3. The unit Γ−operad I. The unit Γ−operad $\mathtt{I}$ in $\mathtt{MOp}^\Gamma$ is such that $\mathtt{I}(x) = \mathbb{1}$ for all $x \in \Gamma(1)$ and $\mathtt{I}(x) = \varnothing$ for all $x \in \Gamma(r)$, $r \geqslant 2$, with $I(\phi) = \mathrm{id}$ for every $\phi \in \mathrm{Mor}_{\Gamma(1)}(x, y)$.

3.1.4. The Γ−operads pt and Com. If $\mathtt{M}$ is a cartesian closed monoidal category, the Γ−operad $\mathtt{pt}$ in $\mathtt{MOp}^\Gamma$ is such that, for all $x \in \Gamma$, $\mathtt{pt}(x) = \mathbb{1}$ and for all $\phi \in \mathrm{Mor}_\Gamma(x, y)$, $\mathtt{pt}(\phi) = \mathrm{id}$. If $\mathtt{M} = (\mathtt{Ab}, \otimes)$, we rather denote by $\mathtt{Com}$ the Γ−operad in $\mathtt{AbOp}^\Gamma$ such that, for all $x \in \Gamma$, $\mathtt{Com}(x) = \mathbb{Z}$ and for all $\phi \in \mathrm{Mor}_\Gamma(x, y)$, $\mathtt{Com}(\phi) = \mathrm{id}$.

3.1.5. Augmented Γ− operad. A Γ−operad $P \in \mathtt{MOp}^\Gamma$ is *augmented* over $\mathtt{Com}$ if there is an *augmentation morphism* $\epsilon \in \mathrm{Mor}_{\mathtt{AbOp}^\Gamma}(P, \mathtt{Com})$.

Note that an augmented Γ−operad in $\mathtt{M}$ in nothing more than an object in $\mathtt{MOp}^\Gamma \downarrow \mathtt{Com}$.

3.2. Γ−symmetric sequences

We introduce now the Γ−symmetric sequences. We need this structure among other things to define the free Γ−operads in the next subsection. We prove that the category $\mathtt{MSeq}^\Gamma$ of Γ−symmetric sequence in $\mathtt{M}$ is isomorphic to a functor category.

3.2.1. Γ−symmetric sequences. A (connected) Γ−*symmetric sequence* M in $\mathtt{M}$ is a collection of objects $(M(x)) \in \mathtt{M}$, $x \in \Gamma(r)$, $r \geqslant 2$ together with :

- for all $x, y \in \Gamma$ and $\phi \in \mathrm{Mor}_\Gamma(x, y)$, isomorphisms $M(\phi) : M(x) \to M(y)$, such that for all $x, y, z \in \Gamma$, $\phi \in \mathrm{Mor}_\Gamma(x, y)$ and $\psi \in \mathrm{Mor}_\Gamma(x, y)$, $M(\psi \circ \phi) = M(\psi) \circ M(\phi)$;
- for all $\sigma \in \Sigma(r)$ and $x \in \Gamma(r)$, isomorphisms $\sigma_x : M(x) \to M(\sigma.x)$, such that all possible diagrams

$$\begin{array}{ccc} M(x) & \xrightarrow{M(\phi)} & M(y) \\ \downarrow{\scriptstyle \sigma_x} & & \downarrow{\scriptstyle \sigma_y} \\ M(\sigma.x) & \xrightarrow[M(\sigma.\phi)]{} & M(\sigma.y) \end{array}$$

  commute.

3.2.2. The category of Γ−symmetric sequences. The Γ−symmetric sequences in $\mathtt{M}$ form a category $\mathtt{MOp}^\Gamma$, where a morphism $f \in \mathrm{Mor}_{\mathtt{MSeq}^\Gamma}(M, N)$ is a collection of morphisms $f_x : M(x) \to N(x)$ such that, for all $x, y \in \Gamma$ and $\phi \in \mathrm{Mor}_\Gamma(x, y)$, the diagrams

$$\begin{array}{ccc} M(x) & \xrightarrow{M(\phi)} & M(y) \\ \downarrow{\scriptstyle f_x} & & \downarrow{\scriptstyle f_y} \\ N(x) & \xrightarrow[N(\phi)]{} & N(y) \end{array}$$

and

$$\begin{array}{ccc} M(x) & \xrightarrow{f_x} & N(x) \\ \downarrow{\scriptstyle \sigma_x} & & \downarrow{\scriptstyle \sigma_x} \\ M(\sigma.x) & \xrightarrow[M(f_{\sigma.x})]{} & N(\sigma.x) \end{array}$$

commute.

3.2.3. Definition. Let $\Sigma \ltimes \Gamma$ the groupoid whose objects are the couples (x, r) (which we simply denote x) for all $r \geqslant 2$ and $x \in \Gamma(r)$. The set $\mathrm{Mor}_{\Sigma\ltimes\Gamma}(x, y)$ with $x \in \Gamma(r)$ and $y \in \Gamma(s)$ is the set of pairs (σ, α) with $\sigma \in \Sigma(r)$ and α a morphism in Γ such that

$$x \xrightarrow{\sigma_*} \sigma.x \overset{\alpha}{\dashrightarrow} y \ ;$$

thus $\mathrm{Mor}_{\Sigma\ltimes\Gamma}(x, y)$ is empty if $r \neq s$.

For all $(\sigma, \alpha) \in \mathrm{Mor}_{\Sigma\ltimes\Gamma}(x, y)$ and $(\tau, \beta) \in \mathrm{Mor}_{\Sigma\ltimes\Gamma}(y, z)$ we define

$$(\tau, \beta) \circ (\sigma, \alpha) = (\sigma \circ \tau, \beta \circ \tau_*(\alpha)).$$

This definition makes sense because τ is acting functorially. Moreover,

$$(\sigma, \alpha)^{-1} = \left(\sigma^{-1}, \sigma^{-1}.\alpha^{-1}\right)$$

and $\Sigma \ltimes \Gamma$ is indeed a groupoid.

3.2.4. Proposition. *The category* $\mathtt{MSeq}^{\Gamma}$ *is isomorphic to the functor category* $\mathtt{M}^{\Sigma\ltimes\Gamma}$.

Proof. Immediate. □

3.2.5. Augmented Γ−symmetric sequence. A Γ−symmetric sequence $M \in \mathtt{MSeq}^{\Gamma}$ is *augmented* over Com if there is an *augmentation morphism* $\epsilon \in \mathrm{Mor}_{\mathtt{MSeq}^{\Gamma}}(P, \mathtt{Com})$.

Note that an augmented Γ−symmetric sequence in M is nothing more than an object in $\mathtt{MSeq}^{\Gamma} \downarrow \mathtt{Com}$.

3.2.6. Proposition. *Let* $n \geqslant 0$. *If* $M \in \mathtt{dg}_*\mathtt{AbSeq}^{\Gamma}$, *then we have* $H_n(M) \in \mathtt{AbSeq}^{\Gamma}$ *with* $H_n(M)(x) = H_n(M(x))$ *for all* $x \in \Gamma$.

Proof. Immediate. □

3.3. Free Γ−operads, coproduct with a free Γ−operad

We give a construction of the free Γ−operad $\mathbb{F}(M) \in \mathtt{MOp}^{\Gamma}$ associated to a Γ−symmetric sequence $M \in \mathtt{MSeq}^{\Gamma}$. To ensure that the free Γ−operad construction works, we will now assume that colimits exist in M and that the tensor product $\otimes : \mathtt{M} \to \mathtt{M} \times \mathtt{M}$ distributes over colimits.

3.3.1. Proposition. *The forgetful functor* $\omega : \mathtt{MOp}^{\Gamma} \to \mathtt{MSeq}^{\Gamma}$, *such that* $\omega(P(x)) = P(x)$ *for all* $x \in \Gamma(r)$, $r \geqslant 2$ *admits a left adjoint, the* free Γ−operad *functor* $\mathbb{F} : \mathtt{MSeq}^{\Gamma} \to \mathtt{MOp}^{\Gamma}$. *Morover, for all* $M \in \mathtt{MSeq}^{\Gamma}$, *we have an embedding* $\iota_M : M \to \mathbb{F}(M)$.

Proof. Let $M \in \mathtt{sSeq}^\Gamma$ and $x \in \Gamma(r)$ with $r \geqslant 2$. We form a groupoid $\mathtt{Tree}(x)$ with

$$\left\{(T(x_v),\mu)\middle|T \in \mathtt{Tree}(r), (x_v)_{v\in V(T)} \in \Gamma, \mu \in \mathrm{Mor}_\Gamma(\lambda_T(x_v),x)\right\},$$

as object set. The morphisms of $\mathtt{Tree}(x)$ for all $(T(x_v),\mu)\,,(T(x'_v),\mu') \in \mathtt{Tree}(x)$ are the isomorphisms $T(f_v) : T(x_v) \to T(x'_v)$ with $f_v \in \mathrm{Mor}_\Gamma(x_v,x'_v)$, such that

$$\begin{array}{ccc} \lambda_T(x_v) & \xrightarrow{\mu} & x \\ {\scriptstyle \lambda_T(f_v)}\downarrow & \nearrow_{\mu'} & \\ \lambda_T(x'_v) & & \end{array}$$

commutes.

We set

$$\mathbb{F}(M)(x) = \operatorname*{colim}_{(T(x_v),\mu)\in\mathtt{Tree}(x)} T\left(M(x_v)\right)$$

and

$$\mathbb{F}(M)(1) = \mathbb{1}.$$

By construction, $\mathbb{F}(M)$ is a Γ−operad :

– The action of a permutation $\sigma \in \Sigma(r)$ on $F(M)(x)$ is given termwise by the obvious identity $T(M(x_v)) \xrightarrow{=} \sigma \cdot T(M(x_v))$ between the term $T(M(x_v))$ indexed by $(T(x_v),\mu) \in Tree(x)$ in our colimit and the term $\sigma \cdot T(M(x_v))$ indexed by $(\sigma \cdot T(x_v),\mu) \in Tree(\sigma \cdot x)$ in the expansion of the object $F(M)(\sigma x)$.

– The composition operation $\circ_i : F(M)(x) \otimes F(M)(y) \to F(M)(x \circ_i y)$ is given termwise by the obvious isomorphism $S(M(x_u)) \otimes T(M(y_v)) \simeq S \circ_i T(M(z_v))$, where, for any pair of terms $S(M(x_u))$ and $T(M(y_v))$ indexed by objects $(S(x_u),\mu) \in \mathtt{Tree}(x)$ and $(T(x_v),\mu) \in \mathtt{Tree}(y)$ in the expansion of $F(M)(x)$ and $F(M)(y)$, we consider the collection $(z_w)_{w\in V(S\circ_i T)}$ formed by the union of $(x_u)_{u\in V(S)}$ and $(y_v)_{v\in V(T)}$, by using the identity $V(S \circ_i T) = V(S) \cup V(T)$. We also consider the morphism $\mu \circ_i \nu : S \circ_i T(z_w) \to x \circ_i y$ given by the partial composite of $\mu : S(x_u) \to x$ and $\nu : T(y_v) \to y$ to associate an object of the category $Tree(x \circ_i y)$ to this collection $S \circ_i T(z_w)$.

The free Γ−operad functor is left adjoint to the forgetful functor. Indeed, there exists one and only one morphism of Γ−operads g such that

$$\begin{array}{ccc} M & \xrightarrow{f} & \omega(P) \\ & {\scriptstyle \iota_M}\searrow \quad \nearrow{\scriptstyle \omega(g)} & \\ & \omega(\mathbb{F}(M)) & \end{array}$$

commutes for all $f \in \mathrm{Mor}_{\mathtt{MSeq}^\Gamma}(M,P)$. For $x \in \Gamma$, the morphism g_x is induced by a collection of morphisms

$$g_{(T(x_v),\mu)} :\ T(M(x_v)) \xrightarrow{T(f_{x_v})} T(f_{x_v}M(x_v)) \xrightarrow{\lambda_T} P(T(x_v)) \xrightarrow{P(\mu)} P(x)$$

indexed by $\mathtt{Tree}(x)$.

□

3.3.2. Proposition. *Let $P \in \mathtt{MOp}^{\Gamma}$ and $M \in \mathtt{MSeq}^{\Gamma}$.*
For all $x \in \Gamma(r)$ with $r \geqslant 2$,

$$\left(P \bigvee \mathbb{F}(M)\right)(x) = \coprod_{T \in \mathtt{Tree}_{\circ,\bullet}(r)} \Big(T\big(M(x_v^o), P(x_{v'}^o)\big)\Big)_{K_T(x_v^o)}$$

with the notation of the previous theorem.

Proof. Similar to the proof of Proposition 3.3.1. □

4. Model structures

We already recalled the definition of the model structure on the usual categories of operads in simplical sets in section 1. We explain in this section the definition of model structures on the new categories of operads used in this thesis. We also describe model structures on the categories of symmetric sequences underlying our categories of operads. We generally consider operads in a base cofibrantly generated model category, which we denote by $\mathtt{C}$ throughout this section, and which is either :

– $\mathtt{C} = \mathtt{sSet}$, the model category of simplicial sets (see [25, II 3, Theorem 3] or [10, Theorem I.11.3]);
– $\mathtt{C} = \mathtt{sAb}$, the model category of simplicial abelian group (see [25, II.6] or [10, Theorem III.2.6 and III.2.11]);
– $\mathtt{C} = \mathtt{dg_*Ab}$, the model category of differential positively graded abelian group (see [25, II p. 4.11, Remark 5] or [13, 2.3]).

4.1. Model structure on $\mathtt{sSeq} \downarrow \mathtt{B\Gamma}$

There is a model structure on $\mathtt{sSeq}$ (see [8, II.8.1]). We just consider the category of objects over $B\Gamma$ associated to this category, where $B\Gamma$ is the nerve of a groupoid operad Γ and is therefore an operad in simplicial sets (see 1.2.2). Also remind that $-_\flat : \mathtt{sSet} \to \mathtt{s^-Set}$ is the obvious functor that forget about the degeneracies in the definition of a simplicial set (see 1.1.1).

4.1.1. Free objects. Let K be an object in a category $\mathtt{M}$ and $S \in \mathtt{S}$. The free $S-$object associated to K is the coproduct $K[S] = \coprod_{s \in S} K$. If $S = G$ is a group, then G acts on $K[G]$ by permutation of the summands. Moreover, if $M = \mathtt{Set}$ or $M = \mathtt{sSet}$, we can identify the free $G-$object $K[G]$ with the cartesian product $G \times K$.

4.1.2. Proposition. *The category $\mathtt{sSeq} \downarrow \mathtt{B\Gamma}$ has a cofibrantly generated model structure such that :*

– *the weak equivalences are the aritywise weak equivalences.*
– *The fibrations are the aritywise fibrations.*
– *The cofibrations are quasi-free extensions of symmetric sequences over $B\Gamma$, that is morphisms $f : K \to L$ such that $L_\flat = K_\flat \coprod (\Sigma \otimes S)$ and $f_\flat$ is the inclusion. We denote by S a collection of simplices $S(r) \in \mathtt{s^-Set} \downarrow \mathtt{B\Gamma(r)}$ for all $r \geqslant 1$ and by $(\Sigma \otimes S)(r)$ the free $\Sigma(r)$ object $S(r)[\Sigma(r)]$ associated to $S(r)$. Morever, $M \in \mathtt{sSeq} \downarrow \mathtt{B\Gamma}$ is cofibrant if and only if $\Sigma(r)$ operates freely on $M(r)$ for all $r \geqslant 1$.*

This model category is a simplicial one. For $K \in \mathtt{sSet}$ and $M \in \mathtt{sSeq} \downarrow \mathtt{B\Gamma}$, we set :

– $(M \otimes K)(r) = M(r) \times K$;
– $M^K(r) = M(r)^K$

for all $r \geqslant 1$.

Proof. We refer to the cited reference [8, II.8.1] for the definition of the model category on $\mathtt{sSeq}$, and to [11] for the general construction which gives the model structure of this proposition. □

4.2. Model structure on $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$

As for the symmetric sequences of simplicial sets, we have a model structure on the category $\mathtt{sOp}$ of simplicial operads(see [8, II.8.2]). We just consider the category of objects over $B\Gamma$ associated to this category, where $B\Gamma$ is the nerve of a groupoid operad Γ.

4.2.1. Proposition. *The category* $\mathtt{sOp} \downarrow \mathtt{B}\Gamma$ *has a cofibrantly generated model structure such that :*
- *The weak equivalences are the aritywise weak equivalences.*
- *The fibrations are the aritywise fibrations.*
- *The cofibrations are retracts of quasi-free operads* $f : P \to Q$ *such that* $Q_\flat = P_\flat \bigvee \mathbb{F}(\Sigma \otimes S)$ *with* S *a collection of object* $S(r) \in \mathtt{s^{-}Set} \downarrow \mathtt{B}\Gamma(\mathtt{r})$ *for all* $r \geqslant 1$*.*

This model category is a simplicial one. For $K \in \mathtt{sSet}$ *and* $P \in \mathtt{sOp} \downarrow \mathtt{B}\Gamma$*,* $K \in \mathtt{sSet}$*, we set :*
- $(P \otimes K)(r) = P(r) \times K$*;*
- $P^K(r) = P(r)^K$

for all $r \geqslant 1$*.*

Proof. We refer to the cited reference [8, II.8.2] for the definition of the model category on $\mathtt{sOp}$, and to [11] for the general construction which gives the model structure of this proposition. □

4.3. Model structure on $\mathtt{CSeq}^\Gamma$

We rely on the fact that $\mathtt{CSeq}^\Gamma$ is isomorphic to the category of diagrams $\mathtt{C}^{\Sigma \ltimes \Gamma}$ (see Definition 3.2.3 and Proposition 3.2.4) to define the model structure on $\mathtt{CSeq}^\Gamma$.

Notation. *For all* $K \in \mathtt{C}$ *and* $x_0, x \in \Gamma$*, for* $K \in C$ *and* $x_0 \in \Gamma$*, we use the notation* $K \otimes (\Sigma \ltimes \Gamma) F_{x_0}$ *for the* $\Gamma-$*symmetric sequence such that:*

$$K \otimes (\Sigma \ltimes \Gamma) F_{x_0}(x) = K[\mathrm{Mor}_{\Sigma \ltimes \Gamma}(x_0, x)],$$

for every $x \in \Gamma$*, where we again use the notation* $K[S] = \coprod_{s \in S} K$ *introduced in 4.1.1 for the free* $S-$*object associated to* K*.*

4.3.1. Proposition. *The category* $\mathtt{CSeq}^\Gamma$ *has a cofibrantly generated model structure such that :*
- *The weak equivalences are the pointwise weak equivalences in* $\mathtt{C}$*.*
- *The fibrations are the pointwise fibrations in* $\mathtt{C}$*.*
- *The cofibrations are the morphism wich have the left lifting property with respect to the acyclic fibrations.*

The generating cofibrations (respectively, generating acyclic cofibrations) are the natural transformations

$$K \otimes (\Sigma \ltimes \Gamma) F_{x_0} \to L \otimes (\Sigma \ltimes \Gamma) F_{x_0}$$

for all $x_0 \in \Gamma$ *and all generating cofibrations (respectively, generating acyclic cofibrations)* $K \to L$ *in our base category* $\mathtt{C}$*.*

Proof. We refer to [12, Theorem 11.6.1] for the general construction which gives the model structure of a diagram category on a base category equipped with a cofibrantly generated model structure. □

4.3.2. Proposition. *If* $\mathtt{C} = \mathtt{sSet}$*, then the cofibrations of* $\mathtt{sSeq}^\Gamma$ *are retracts of quasi-free extensions of* $\Gamma-$*symmetric sequences, that is, retracts of morphisms* $f : K \to L$ *such that :*
- $L_\flat = K_\flat \coprod (\Sigma \ltimes \Gamma) \otimes S$*;*
- $f_\flat$ *is the inclusion;*
- S *is a collection* $(S(r), x_0^r)$ *with* $S(r) \in \mathtt{s^{-}Set}$ *and* $x_0^r \in \Gamma(r)$*;*
- $(\Sigma \ltimes \Gamma) \otimes S(x) = S(r) \otimes (\Sigma \ltimes \Gamma) F_{x_0^r}(x)$ *for all* $r \geqslant 1$ *and* $x \in \Gamma(r)$*.*

The cofibrations if $\mathtt{C} = \mathtt{sAb}$ *are retracts of quasi-free extensions of simplicial abelian* $\Gamma-$*symmetric sequences, that is, morphisms* $f : K \to L$ *such that :*

- $L_\flat = K_\flat \oplus \mathbb{Z}\big((\Sigma \ltimes \Gamma) \otimes S\big)$;
- $f_\flat$ *is the inclusion;*
- S *is a collection* $(S(r), x_0^r)$ *with* $S(r) \in \mathtt{s}^-\mathtt{Set}$ *and* $x_0^r \in \Gamma(r)$.

Proof. Slight adaptation of the proof in the simplicial symmetric sequences setting (see [8, II.8.1]). □

4.3.3. Proposition. *If* $\mathtt{C} = \mathtt{sSet}$ *or* $\mathtt{sAb}$, *then the model structure is simplicial.*

If $\mathtt{C} = \mathtt{sSet}$, *we set :*
- $(M \otimes K)(x) = M(x) \times K$;
- $M^K(x) = M(x)^K$

for all $M \in \mathtt{sSeq}^\Gamma$, $K \in \mathtt{sSet}$ *and* $x \in \Gamma$.

If $\mathtt{sAb}$, *we set :*
- $(M \otimes K)(x) = M(x) \times \mathbb{Z}K$;
- $M^K(x) = M(x)^K$, *the abelian group structure of* M *providing* $M(x)^K$ *with a structure of simplicial abelian group.*

for all $M \in \mathtt{sAbSeq}^\Gamma$, $K \in \mathtt{sSet}$ *and* $x \in \Gamma$.

Proof. Straightforward consequence of the simplicial nature of model structures on $\mathtt{sSet}$ and $\mathtt{sAb}$ according to [12, Theorem 11.7.3]. □

We now describe the path objects and analyze homotopy classes of maps in $\mathtt{dg}_*\mathtt{AbSeq}^\Gamma$.

4.3.4. Proposition. *A path object* M^I *in* $\mathtt{dg}_*\mathtt{AbSeq}^\Gamma$ *is given for all* $M \in \mathtt{dg}_*\mathtt{AbSeq}^\Gamma$ *and* $n \geqslant 0$ *by*

$$M_n^I = M_{n+1} \oplus M_n \oplus M_n$$

with the differential d *such that, for all* $x \in \Gamma$, *for all* $(a, b, c) \in M_n^I(x)$,

$$d(a, b, c) = (-d^M(a) + b - c, d^M(b), d^M(c)).$$

The faces d_0 *and* d_1 *of this path object are given by*

$$d_0(a, b, c) = b$$

and

$$d_1(a, b, c) = c$$

for all $(a, b, c) \in M_n^I(x)$. *The degeneracy* s_1 *is given by*

$$s_1(a) = (0, a, a)$$

for all

$$a \in M_n(x).$$

The chain complex $M^I(x)$ *is a path object associated to* $M(x)$ *in* $\mathtt{dg}_*\mathtt{Ab}$.

If we assume that N *is fibrant and let* $f, g \in \mathrm{Mor}_{\mathtt{dg}_*\mathtt{AbSeq}^\Gamma}(M, N)$, *then* $f \sim g$ *if and only if it exists a sequence* $\big(h_n \in \mathrm{Mor}_{\mathtt{AbSeq}^\Gamma}(M_n, N_{n+1})\big)_{n \geqslant 1}$ *such that, for all* $n \geqslant 1$,

$$f_n - g_n = d_n^M \circ h_{n-1} + h_n \circ d_{n+1}^N.$$

Moreover, $f \sim g$ *implies that* $f(x) \sim g(x)$ *in* $\mathtt{dg}_*\mathtt{Ab}$ *for all* $x \in \Gamma$ *and* $f(r) \sim g(r)$ *in* $\mathtt{dg}_*\mathtt{Ab}^\Gamma(\mathtt{r})$ *for all* $r \geqslant 1$.

Proof. Simple check. □

4.4. Model structure on $\mathtt{COp}^\Gamma$

We now use the adjunction between the free $\Gamma-$operad functor and the forgetful functor to transfer the model structure of $\mathtt{CSeq}^\Gamma$ to $\mathtt{COp}^\Gamma$. The symmetric monoidal structure on $\mathtt{sAb}$ and $\mathtt{dg^*Ab}$ are given by the tensor product.

4.4.1. Theorem. *The category $\mathtt{COp}^\Gamma$ has a cofibrantly generated model structure such that :*
- *The weak equivalences are the pointwise weak equivalences in $\mathtt{C}$.*
- *The fibrations are the pointwise fibrations in $\mathtt{C}$.*
- *The cofibrations are the morphism wich have the left lifting property with respect to the acyclic fibrations.*

The generating cofibrations (respectively, generating acyclic cofibrations) are the natural transformations

$$\mathbb{F}\big(K \otimes (\Sigma \ltimes \Gamma) F_{x_0}\big) \to \mathbb{F}\big(L \otimes (\Sigma \ltimes \Gamma) F_{x_0}\big)$$

for $x_o \in \Gamma(r)$, $r \geqslant 1$ and all generating cofibrations (respectively, generating acyclic cofibrations) $K \to L$ in our base category $\mathtt{C}$.

Proof. A full proof can be written down by slightly adapting the proof of the simplicial operad case that can be found in [8, II.8.2].

In short, we use a Kan's theorem (see [12, Theorem 11.3.2] or [7, Theorem II.4.3.3]) to transfer the cofibrantly generated model structure on $\Gamma-$symmetric sequence (see Theorem 4.3.1) by means of the adjunction between the free $\Gamma-$operad functor and the forgetful functor (see Theorem 3.3.1). Note that a crucial argument is the fact that the domain of the generating cofibrations and acyclic cofibrations in $\mathtt{C}$ are small. □

4.4.2. Proposition. *Cofibrations in $\mathtt{C} = \mathtt{sSet}$ are retracts of quasi-free extension of simplicial $\Gamma-$operads, that is, retracts of morphisms $f : P \to Q$ such that $Q_\flat = P_\flat \bigvee \mathbb{F}((\Sigma \ltimes \Gamma) \otimes S)$ with S a collection of simplex $S(r) \in \mathtt{sSet}$ for all $r \geqslant 1$.*

Cofibrations in $\mathtt{C} = \mathtt{sAb}$ are retracts of quasi-free extensions of simplicial abelian $\Gamma-$operads, that is, morphisms $f : K \to L$ such that $L_\flat = K_\flat \oplus \mathbb{Z}\mathbb{F}\big((\Sigma \ltimes \Gamma) \otimes S\big)$ with S a collection of simplex $S(r) \in \mathtt{s^-Set}$ for all $r \geqslant 1$.

4.4.3. Proposition. *With the model structure provided by Theorem 4.4.1 and the function objects such that*

$$P^K = \{P(x)^K\}_{x \in \Gamma} = \{\mathrm{Map}_{\mathtt{sSet}}(K, P(x))\}_{x \in \Gamma}$$

for all $P \in \mathtt{sOp}^\Gamma$ (respectively, $P \in \mathtt{sAbOp}^\Gamma$) and $K \in \mathtt{sSet}$, the category $\mathtt{sOp}^\Gamma$ (respectively, $\mathtt{sAbOp}^\Gamma$) is a simplicial model category. We moreover have

$$\mathrm{Map}_{\mathtt{sOp}^\Gamma}(P, Q)_n = \mathrm{Mor}_{\mathtt{sOp}^\Gamma}(P, Q^{\Delta_n}).$$

Proof. We just give a sketch of this proof in $\mathtt{sOp}^\Gamma$. A full proof can be written down by slightly adapting the proof in the simplicial operad setting that can be found in [8, II.2.3]. The proof is the same in $\mathtt{sAbOp}^\Gamma$: in that case, P^K inherits an abelian group $\Gamma-$operad structure from the abelian group $\Gamma-$operad structure of P.

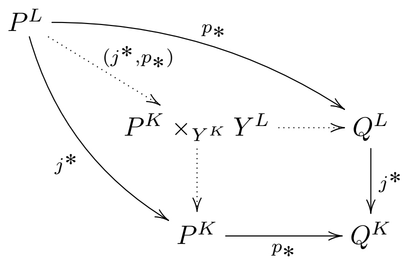
We first check that P^K is a simplicial $\Gamma-$operad, and $\mathrm{Map}_{\mathtt{sOp}^\Gamma}(-,-)$ is a bifunctor in simplicial set.

We then prove that this function object bifunctor provides $\mathtt{sOp}^\Gamma$ with a structure of category cotensored over $\mathtt{sSet}$. More precisely :
- the bifunctor $(P, K) \mapsto P^K$ carries colimits on the variable K to limits, and limits on the variable P to limits;

– there is a natural unit isomorphism $P^{pt} \simeq P$ for all $P \in \mathtt{sOp}^\Gamma$ and natural associativity isomorphisms ${P^K}^L \simeq P^{K\times L}$ for all $P \in \mathtt{sOp}^\Gamma$ and $K, L \in \mathtt{sSet}$, which satisfy natural coherence axioms with respect to the unit and associativity relations of the cartesian product of simplicial sets.

For all $K \in \mathtt{sSet}$ we check that the functor -^K admit as left adjoint a functor $\text{-}\otimes K$ by upgrading the adjunction at the $\Gamma-$symmetric sequence level to the $\Gamma-$operads level (the crucial point is the fact that -^K is a symmetric monoidal functor). Then $\mathtt{sOp}^\Gamma$ is tensored over $\mathtt{sSet}$.

Finally, we prove that if $p : P \to Q$ is a fibration in $\mathtt{sOp}^\Gamma$ and $j : K \to L$ is a cofibration of simplicial sets, then the morphism (j^*, p_*) in the pullback diagram



is a fibration which is acyclic when j or p is. □

4.5. Quillen adjunctions and equivalences

We upgrade the adjunction between the free abelian group functor and the forgetful functor at the $\Gamma-$symmetric sequence level and at the $\Gamma-$operad level. We also upgrade the classical Dold-Kan equivalence of categories at the $\Gamma-$symmetric sequence level.

4.5.1. Proposition. *We denote by $\mathbb{Z}[X]$ the free abelian group generated by the set X. We also denote by $\rho(G)$ the underlying set of the abelian group G. The adjunction*

$$\mathbb{Z}[-] : \mathtt{Set} \rightleftarrows \mathtt{Ab} : \rho$$

extends to adjunctions

$$\mathbb{Z} : \mathtt{SetSeq}^\Gamma \rightleftarrows \mathtt{AbSeq}^\Gamma : \rho$$

and

$$\mathbb{Z} : \mathtt{SetOp}^\Gamma \rightleftarrows \mathtt{AbOp}^\Gamma : \rho,$$

such that $\mathbb{Z}[M](x) = \mathbb{Z}[M(x)]$ and $\rho(N)(x) = \rho(N(x))$ for all $M \in \mathtt{SetSeq}^\Gamma$ or $M \in \mathtt{SetOp}^\Gamma$, $N \in \mathtt{AbSeq}^\Gamma$ or $N \in \mathtt{AbOp}^\Gamma$ and $x \in \Gamma$.

There is also Quillen adjunctions

$$\mathbb{Z} : \mathtt{sSeq}^\Gamma \rightleftarrows \mathtt{sAbSeq}^\Gamma : \rho$$

and

$$\mathbb{Z} : \mathtt{sOp}^\Gamma \rightleftarrows \mathtt{sAbOp}^\Gamma : \rho$$

such that $\mathbb{Z}[M](x)_n = \mathbb{Z}[M(x)_n]$ and $\rho(N)(x)_n = \rho(N(x)_n)$ for all $M \in \mathtt{sSeq}^\Gamma$ or $M \in \mathtt{sOp}^\Gamma$, $N \in \mathtt{sAbSeq}^\Gamma$ or $N \in \mathtt{sAbOp}^\Gamma$, $x \in \Gamma$ and $n \geqslant 0$.

Afterward, we simpler denote $\mathbb{Z}[M]$ by $\mathbb{Z}M$.

Proof. We easily check that $\mathbb{Z}$ and ρ preserves $\Gamma-$symmetric sequences and $\Gamma-$operad structures. Thus we have adjunctions.

The functor ρ obviously preserves weak equivalences and fibrations. Thus we have Quillen adjunctions. □

4.5.2. Proposition. *The equivalence of categories given by the Quillen adjunction*

$$\mathrm{N}_* : \mathtt{sAb} \rightleftarrows \mathtt{dg}_*\mathtt{Ab} : \mathrm{D}_\bullet$$

induces an equivalence of categories given by the Quillen adjunction

$$\mathrm{N}_* : \mathtt{sAbSeq}^\Gamma \rightleftarrows \mathtt{dg}_*\mathtt{AbSeq}^\Gamma : \mathrm{D}_\bullet$$

such that $\mathrm{N}_*(M)(x) = \mathrm{N}_*(M(x))$ *and* $\mathrm{D}_\bullet(N)(x) = \mathrm{D}_\bullet(N(x))$ *for all* $M \in \mathtt{sAbSeq}^\Gamma$, $N \in \mathtt{dg}_*\mathtt{AbSeq}^\Gamma$ *and* $x \in \Gamma$.

Moreover, for all $N \in \mathtt{dg}_*\mathtt{AbSeq}^\Gamma$*, there is a natural isomorphism*

$$D_\bullet N \simeq \mathrm{Mor}_{\mathtt{dg}_*\mathtt{Ab}}(\mathrm{N}_*\Delta^\bullet, N)$$

with $\mathrm{Mor}_{\mathtt{dg}_*\mathtt{Ab}}(\mathrm{N}_*\Delta^\bullet, N)(x) = \mathrm{Mor}_{\mathtt{dg}_*\mathtt{Ab}}(\mathrm{N}_*\Delta^\bullet, N(x))$.

Therefore, the adjunction

$$\mathrm{N}_* : \mathtt{sAbSeq}^\Gamma \rightleftarrows \mathtt{dg}_*\mathtt{AbSeq}^\Gamma : \mathrm{D}_\bullet$$

provides a natural isomorphism

$$\rho : \mathrm{Mor}_{\mathtt{sAbSeq}^\Gamma}(\mathrm{N}_*M, N) \simeq \mathrm{Mor}_{\mathtt{dg}_*\mathtt{AbSeq}^\Gamma}(M, \mathrm{Mor}_{\mathtt{dg}_*\mathtt{Ab}}(\mathrm{N}_*\Delta^\bullet, N))$$

given by $\rho(f)(y)([\sigma]) = f([\sigma^* y])$ *for all* $f \in \mathrm{Mor}_{\mathtt{sAbSeq}^\Gamma}(\mathrm{N}_*M, N)$, $y \in M(x)$ *and* $\sigma : \underline{k} \to \underline{n} \in \Delta^n_k$.

Proof. We easily check that N_* and $\mathrm{D}_\bullet$ preserves $\Gamma-$symmetric sequence structures, thus we have an equivalence of categories.

The functor N_* preserves weak equivalences, fibrations and induces a bijection between the set of weak equivalences in $\mathtt{sAb}$ and the set of weak equivalences in $\mathtt{dg}_*\mathtt{Ab}$ (see [10, Theorem III.2.6]). Thus we have a Quillen equivalence of categories.

The natural isomorphisms

$$D_\bullet Y \simeq \mathrm{Mor}_{\mathtt{dg}_*\mathtt{Ab}}(\mathrm{N}_*(\Delta^\bullet), Y)$$

and

$$\rho : \mathrm{Mor}_{\mathtt{sAb}}(\mathrm{N}_*X, Y) \simeq \mathrm{Mor}_{\mathtt{dg}_*\mathtt{Ab}}(X, \mathrm{Mor}_{\mathtt{dg}_*\mathtt{Ab}}(\mathrm{N}_*(\Delta^\bullet), Y))$$

defined in 1.1.4 for all $X \in \mathtt{sAb}$ and $Y \in \mathtt{dg}_*\mathtt{Ab}$ extend natural isomorphisms

$$D_\bullet N \simeq \mathrm{Mor}_{\mathtt{dg}_*\mathtt{Ab}}(\mathrm{N}_*(\Delta^\bullet), N)$$

and

$$\rho : \mathrm{Mor}_{\mathtt{sAbSeq}^\Gamma}(\mathrm{N}_*M, N) \simeq \mathrm{Mor}_{\mathtt{dg}_*\mathtt{AbSeq}^\Gamma}(M, \mathrm{Mor}_{\mathtt{dg}_*\mathtt{Ab}}(\mathrm{N}_*(\Delta^\bullet), N))$$

since N_* and $\mathrm{D}_\bullet$ are defined pointwise in the $\Gamma-$symmetric sequence setting.

□

5. Covering $\Gamma-$operads of a simplicial operad

We define now the coverings of a simplicial symmetric sequence as simplicial $\Gamma-$symmetric sequences, and the coverings of a simplicial operad as simplicial $\Gamma-$operads. We finally define the universal covering $\Gamma-$operad of a simplicial operad and prove its existence.

5.1. Covering $\Gamma-$symmetric sequence

Recall that $-^{\vee}$ denotes the covering functor, which is left adjoint to the homotopy colimit functor in the simplicial set setting (see 1.1.3).

We define the covering functor and the homotopy colimit functor at the symmetric sequence level aritywise, and then prove that these functors form a Quillen adjoint pair.

5.1.1. Proposition. *Let $M \in \mathtt{sSeq}^{\Gamma}$. The $\Gamma-$symmetric sequence structure on M induces a symmetric sequence structure over $B\Gamma$ on*

$$\mathrm{hocolim}_{\Gamma}(M) = \big(\mathrm{hocolim}_{\Gamma(r)} M(r)\big)_{r \geqslant 2}.$$

We therefore have a functor

$$\begin{array}{rccc} \mathrm{hocolim}_{\Gamma} : & \mathtt{sSeq}^{\Gamma} & \longrightarrow & \mathtt{sSeq} \downarrow \mathtt{B\Gamma} \\ & M & \longmapsto & \mathrm{hocolim}_{\Gamma} M. \end{array}$$

Proof. We first describe the $\Gamma-$symmetric structure on $\mathrm{hocolim}_{\Gamma} M$ dimensionwise. Let $n \geqslant 0$, $r \geqslant 2$, $\sigma \in \Sigma_r$, $(a, x_0 \to \dots \to x_n) \in \mathrm{hocolim}_{\Gamma(r)} M(r)$ with $x_0 \to \dots \to x_n \in B\Gamma(r)_n$ and $a \in M(x_o)$. The action of σ is given by

$$\sigma.\big(a, x_0 \to \dots \to x_n\big) = \big(\sigma.a, \sigma.x_0 \to \dots \to \sigma.x_n\big).$$

The sequences $\mathrm{hocolim}_{\Gamma}(M)_n$ are clearly symmetric sequences over $B\Gamma_n$ for all $n \geqslant 0$, and we easily check their compatibility with face and degeneracy morphisms. □

5.1.2. Proposition. *Let $M \in \mathtt{sSeq} \downarrow \mathtt{B\Gamma}$. The symmetric sequence structure on $B\Gamma$ and on M induce a $\Gamma-$symmetric sequence structure on the colllection*

$$M^{\vee} = (M(r)^{\vee})_{r \geqslant 2}.$$

We therefore have a functor

$$\begin{array}{rccc} -^{\vee} : & \mathtt{sSeq} \downarrow \mathtt{B\Gamma} & \longrightarrow & \mathtt{sSeq}^{\Gamma} \\ & M & \longmapsto & M^{\vee}. \end{array}$$

Proof. Let $x \in \Gamma(r)$, and let us remind that $M^{\vee}(x)$ is define by a pullback (see 5.3.1); the action of $\Sigma(r)$ is uniquely defined by the universal property of the pullback. □

In 1.1.3, we observe that that the universal covering and the homotopy colimit functors restrict to the category $\mathtt{s^{-}Set}$. We have a similar result in the context of symmetric sequences :

5.1.3. Theorem. *There are adjunctions*

$$-^{\vee}(r) : \mathtt{sSeq} \downarrow \mathtt{B\Gamma(r)} \rightleftarrows \mathtt{s}^{\mathtt{\Sigma \ltimes \Gamma(r)}} : \mathrm{hocolim}_{\Gamma(r)}$$

for all $r \geqslant 2$ that forms a Quillen adjunction

$$-^{\vee} : \mathtt{sSeq} \downarrow \mathtt{B\Gamma} \rightleftarrows \mathtt{sSeq}^{\Gamma} : \mathrm{hocolim}_{\Gamma}.$$

Proof. Let $r \geqslant 2$, $M \in \mathtt{sSeq(r)} \downarrow \mathtt{B\Gamma(r)}$, $N \in \mathtt{sSeq}^{\Gamma(\mathtt{r})}$ and

$$f \in \mathrm{Mor}_{\mathtt{sSeq(r)}\downarrow\mathtt{B\Gamma(r)}}(M(r), \mathrm{hocolim}_{\Gamma(r)} N(r)).$$

Since $M \in \mathtt{sSeq(r)} \downarrow \mathtt{B\Gamma(r)}$, we have a morphism $\psi : M \to B\Gamma(r)$ provided with M. For all $n \geqslant 0$, let

$$\begin{array}{rccc} f_n : & M_n & \longrightarrow & \mathrm{hocolim}_{\Gamma(r)} N_n \\ & y & \longmapsto & (y_0 \to \ldots \to y_n, z) \end{array}$$

with $y_0 \to \ldots \to y_n = \psi(y)$ and $z \in N(y_0)_n$.

The image of f by the adjunction on the simplicial set level is the map

$$g \in \mathrm{Mor}_{\mathtt{sSet}^{\Gamma(\mathtt{r})}}(M^{\vee}, N)$$

such that, for all $x \in \Gamma(r)$ and $n \geqslant 0$

$$\begin{array}{rccc} g(x)_n : & M^{\vee}(x)_n & \longrightarrow & N(x)_n \\ & \left(y,\ \begin{array}{ccccc} y_0 & \longrightarrow & \ldots & \longrightarrow & y_n \\ & \searrow^{\alpha} & & \swarrow & \\ & & x & & \end{array} \right) & \longmapsto & N(\alpha)(z) \end{array}$$

Since for all $\sigma \in \Sigma(r)$, $f_n(\sigma.y) = \sigma.f_n(y) = (\sigma.y_o \to \ldots \to \sigma.y_n, \sigma.z)$ and $\sigma.N(\alpha(z)) = N\big((\sigma.\alpha)(\sigma.z)\big)$, we have $g(\sigma.x)_n = \sigma.g(x)_n$ and therefore g is a morphism in $\mathtt{s}^{\Sigma\ltimes\Gamma(\mathtt{r})}$.

The adjunction is a Quillen one because $\mathrm{hocolim}_\Gamma$ preserves weak equivalences and fibrations aritywise.

□

5.1.4. Theorem. *We can define in the very same way functors*

$$\begin{array}{rccc} \mathrm{hocolim}_\Gamma : & \mathtt{s^{-}Seq}^{\Gamma} & \longrightarrow & \mathtt{s^{-}Seq} \downarrow \mathtt{B\Gamma} \\ & M & \longmapsto & \mathrm{hocolim}_\Gamma M \end{array}$$

and

$$\begin{array}{rccc} -^{\vee} : & \mathtt{s^{-}Seq} \downarrow \mathtt{B\Gamma} & \longrightarrow & \mathtt{s^{-}Seq}^{\Gamma} \\ & M & \longmapsto & M^{\vee} \end{array}$$

such that there are adjunctions

$$-^{\vee}(r) : \mathtt{s^{-}Set} \downarrow \mathtt{B\Gamma(r)} \rightleftarrows \mathtt{s^{-}Set}^{\Gamma(\mathtt{r})} : \mathrm{hocolim}_{\Gamma(r)}$$

for all $r \geqslant 2$ that forms an adjunction

$$-^{\vee} : \mathtt{s^{-}Seq} \downarrow \mathtt{B\Gamma} \rightleftarrows \mathtt{s^{-}Seq}^{\Gamma} : \mathrm{hocolim}_\Gamma .$$

Morever, we have for all $M \in \mathtt{sOp} \downarrow \mathtt{B\Gamma}$

$$(M^{\vee})_\flat = (M_\flat)^{\vee}$$

and for all $N \in \mathtt{sOp}^{\Gamma}$

$$(\mathrm{hocolim}_\Gamma N)_\flat = \mathrm{hocolim}_\Gamma(N_\flat).$$

Proof. Immediate. □

5.2. Covering $\Gamma-$operad

As in the symmetric sequence setting, the covering functor and the homotopy colimit functor in the operad setting can be defined aritywise and form a Quillen adjoint pair.

5.2.1. Proposition. *Let $P \in \mathtt{sOp}^{\Gamma}$. The sequence*

$$\mathrm{hocolim}_{\Gamma}(P) = \big(\mathrm{hocolim}_{\Gamma(r)} P(r)\big)_{r \geqslant 1}$$

forms an operad over $B\Gamma$. We therefore have a functor

$$\begin{array}{rccc} \mathrm{hocolim}_{\Gamma} : & \mathtt{sOp}^{\Gamma} & \longrightarrow & \mathtt{sOp} \downarrow \mathtt{B}\Gamma \\ & P & \longmapsto & \mathrm{hocolim}_{\Gamma} P. \end{array}$$

Proof. We first describe the operad structure on $\mathrm{hocolim}_{\Gamma} P$ dimensionwise. Let $n \geqslant 0$. We have $(\mathrm{hocolim}_{\Gamma} P)(1)_n = \mathrm{hocolim}_{pt}\, pt = pt$. Let $r, s \geqslant 1$, $1 \leqslant i \leqslant r$, $n \geqslant 0$, $(a, x_0 \to ... \to x_n) \in \mathrm{hocolim}_{\Gamma(s)} P(s)$ and $(b, y_0 \to ... \to y_n) \in \mathrm{hocolim}_{\Gamma(s)} P(s)$ with $x_0 \to ... \to x_n \in B\Gamma(r)_n$, $a \in P(x_o)$, $y_0 \to ... \to y_n \in B\Gamma(s)_n$ and $b \in P(y_o)$. The operadic compositions are given by

$$(a, x_0 \to ... \to x_n) \circ_i (b, y_0 \to ... \to y_n) = (a \circ_i b, x_o \circ_i y_0 \to ... \to x_n \circ_i y_n).$$

We easily check that this structure fullfill equivariance, composition and unit axioms so that $\mathrm{hocolim}_{\Gamma}(P)_n$ is an operad in $\mathtt{Set} \downarrow \mathtt{B}\Gamma_{\mathtt{n}}$. We easily prove the compatibility of this structure with face and degeneracy maps too.

□

5.2.2. Proposition. *The functor $-^{\vee} : \mathtt{sSet} \downarrow \mathtt{B}\Gamma \to \mathtt{sSet}^{\Gamma}$ satisfies the following properties :*
- *For the one-point set $pt \in \mathtt{sSet} \downarrow *$ and the trivial groupoid $\Gamma_{pt} = *$, we have the relation $pt^{\vee} \simeq pt$.*
- *For a pair of objects $X \in \mathtt{sSet} \downarrow \mathtt{B}\Gamma_{\mathtt{X}}$ and $Y \in \mathtt{sSet} \downarrow \mathtt{B}\Gamma_{\mathtt{Y}}$, we have a natural isomorphism $(X \times Y)^{\vee} \simeq X^{\vee} \times Y^{\vee}$ in the category $\mathtt{sSet}^{\Gamma_{\mathtt{X}} \times \Gamma_{\mathtt{Y}}}$, where we use the natural isomorphism of simplicial sets $B(\Gamma_X \times \Gamma_Y) \simeq B(\Gamma_X) \times B(\Gamma_Y)$ (see 1.2.2) to identify the cartesian product $X \times Y$ with an object of the category $\mathtt{sSet} \downarrow \mathtt{B}(\Gamma_{\mathtt{X}} \times \Gamma_{\mathtt{Y}})$. Furthermore, the isomorphisms that give these relations satisfy obvious unit, associativity and symmetry constraints.*

Proof. The first claim is obvious.

The proof of the second claim rely on the fact that in $\mathtt{sSet}$, the pullback of $X \times X' \to Y \times Y' \leftarrow Z \times Z'$ is the product of the pullback of $X \to Y \leftarrow Z$ by the pullback of $X' \to Y' \leftarrow Z'$.

We then check that the isomorphisms that give our relations satisfy obvious unit, associativity and symmetry constraints. □

5.2.3. Proposition. *Let $P \in \mathtt{sOp} \downarrow \mathtt{B}\Gamma$. The operad structure on $B\Gamma$ and P induce a $\Gamma-$operad structure on the collection*

$$P^{\vee} = (P(r)^{\vee}).$$

We therefore have a functor

$$\begin{array}{rccc} -^{\vee} : & \mathtt{sOp} \downarrow \mathtt{B}\Gamma & \longrightarrow & \mathtt{sOp}^{\Gamma} \\ & P & \longmapsto & P^{\vee}. \end{array}$$

Proof. We already know that $P^\vee$ is a $\Gamma-$symmetric sequence (see Proposition 5.1.2).

By the previous proposition, we have $P(1)^\vee = pt$.

Let $x \in \Gamma(r)$ and $y \in \Gamma(s)$. Remind that $P(r)^\vee(x)$, $P(s)^\vee(y)$ and $P(r+s-1)^\vee(x \circ_i y)$ are defined by pullbacks (see 5.3.1). Therefore, the composition products

$$\circ_i : P(r) \times P(s) \to P(r+s-1)$$

and

$$\circ_i : B\Gamma(r) \times B\Gamma(s) \to B\Gamma(r+s-1)$$

induce a morphism

$$(P(r) \times P(s))^\vee(x,y) \to P(r+s-1)^\vee(x \circ_i y).$$

We compose this morphism with the isomorphim

$$P(r)^\vee(x) \times P(s)^\vee(y) \to (P(r) \times P(s))^\vee(x,y)$$

provided by the previous proposition to form the composition product

$$\circ_i : P^\vee(x) \times P^\vee(y) \to P^\vee(x \circ_i y).$$

We then check that the axioms of the $\Gamma-$operad structure (see 3.1.1) are satisfied. □

5.2.4. Theorem. *There is a Quillen adjunction*

$$-^\vee : \mathtt{sOp} \downarrow \mathtt{B\Gamma} \rightleftarrows \mathtt{sOp}^\Gamma : \mathrm{hocolim}_\Gamma .$$

Moreover, there are adjunctions

$$-^\vee{}_n : \mathtt{SetOp} \downarrow \mathtt{B\Gamma_n} \rightleftarrows \mathtt{SetOp}^\Gamma : \mathrm{hocolim}_\Gamma(-)_n$$

for all $n \geqslant 0$.

Proof. We check that for every $P \in \mathtt{sOp} \downarrow \mathtt{B\Gamma}$, $Q \in \mathtt{sOp}^\Gamma$ the restriction to the set $\mathrm{Mor}_{\mathtt{sOp}^\Gamma}(P^\vee, Q)$ of the natural isomorphism

$$\mathrm{Mor}_{\mathtt{sSeq}^\Gamma}(P^\vee, Q) \simeq \mathrm{Mor}_{\mathtt{sSeq}\downarrow\mathtt{B\Gamma}}(P, \mathrm{hocolim}_\Gamma Q)$$

provided by the adjunction

$$-^\vee : \mathtt{sSeq} \downarrow \mathtt{B\Gamma} \rightleftarrows \mathtt{sSeq}^\Gamma : \mathrm{hocolim}_\Gamma$$

(see Theorem 5.1.3) induces an isomorphism

$$\mathrm{Mor}_{\mathtt{sOp}^\Gamma}(P^\vee, Q) \simeq \mathrm{Mor}_{\mathtt{sOp}\downarrow\mathtt{B\Gamma}}(P, \mathrm{hocolim}_\Gamma Q).$$

The adjunction at the operad level splits dimensionwise because it does at the simplicial sets level. □

5.2.5. Theorem. *We can define in the very same way functors*

$$\begin{array}{rccc} \mathrm{hocolim}_\Gamma : & \mathtt{s^-Op}^\Gamma & \longrightarrow & \mathtt{s^-Op} \downarrow \mathtt{B\Gamma} \\ & M & \longmapsto & \mathrm{hocolim}_\Gamma M. \end{array}$$

and

$$\begin{array}{rccc} -^\vee : & \mathtt{s^-Op} \downarrow \mathtt{B\Gamma} & \longrightarrow & \mathtt{s^-Op}^\Gamma \\ & M & \longmapsto & M^\vee \end{array}$$

such that there are adjunctions

$$-^{\vee}{}_n : \mathtt{Set}^{-}\mathtt{Op} \downarrow \mathtt{B}\Gamma_{\mathtt{n}} \rightleftarrows \mathtt{Set}^{-}\mathtt{Op}^{\Gamma} : \mathrm{hocolim}_{\Gamma}(-)_n$$

for all $n \geqslant 0$ *that forms an adjunction*

$$-^{\vee} : \mathtt{s}^{-}\mathtt{Op} \downarrow \mathtt{B}\Gamma \rightleftarrows \mathtt{s}^{-}\mathtt{Op}^{\Gamma} : \mathrm{hocolim}_{\Gamma}\,.$$

Morever, we have for all $P \in \mathtt{sOp} \downarrow \mathtt{B}\Gamma$

$$(P^{\vee})_{\flat} = (P_{\flat})^{\vee}$$

and for all $Q \in \mathtt{sOp}^{\Gamma}$

$$(\mathrm{hocolim}_{\Gamma}\, Q)_{\flat} = \mathrm{hocolim}_{\Gamma}(Q_{\flat}).$$

Proof. Straightforward consequence of the definition of the category $\mathtt{s}^{-}\mathtt{Set}$. □

5.3. Universal covering of a simplicial operad

We define the universal covering of a simplicial operad and prove the existence of such a covering.

If S is a simplicial set, there is a canonical fibration $\Psi_S : S \to B\pi(S)$ which associates to the simplex $\tau : \Delta_n \to S$ in S_n the object $\tau(0) \to \tau(1) \to ... \to \tau(n) \in B\pi(S)_n$ with $0 \to 1 \to ... \to n$ the string in the 1-skeleton of Δ^n. This fibration induces a isomorphism between the fundamental groups of S and $B\pi(S)$, and the long exact sequence of homotopy groups for a fibration implies that $S^{\vee}$ is a $\pi(S)-$diagram of simply connected space, hence a $\pi(S)-$diagram of universal covering of S (see [10, definition after Lemma VI.3.5]).

5.3.1. Universal covering Γ−operad. We say that a covering $\Gamma-$operad $P^{\vee}$ of an operad $P \in \mathtt{sOp} \downarrow \mathtt{B}\Gamma$ is *universal* if $\pi(P^{\vee}(x)) = 0$ for every $x \in \Gamma$.

5.3.2. Canonical fibration. Let $P \in \mathtt{sOp}$. There is a canonical fibration $\Psi_P : \mathtt{sOp} \to B\pi(P)$ such that, for every $x \in \pi(P)(r)$, $\Psi_P(x)$ associates to the simplex $\tau : \Delta_n \to P(r)$ in $P(r)_n$ the object $\tau(0) \to \tau(1) \to ... \to \tau(n) \in B\pi(P)(r)_n$.

5.3.3. Proposition. *Let* $P \in \mathtt{sOp} \downarrow \mathtt{B}\Gamma$*. If the augmentation* $\Psi_P : P \to B\Gamma$ *induces aritywise equivalences of categories between the associated groupoid operad, then* $P^{\vee}$ *is a universal covering* $\Gamma-$*operad.*

Proof. Immediate consequence of the definition of a universal covering $\Gamma-$operad. □

5.3.4. Corollary. *Every simplicial operad* P *admits a universal covering* $\Gamma-$*operad.*

Proof. Immediate consequence of the existence of the canonical fibration and the last proposition. □

6. Additive $\Gamma-$operads, $\Gamma-$operadic derivations

An additive $\Gamma-$operad is a $\Gamma-$operad in the category $(\mathtt{Ab}, \oplus)$ of abelian groups with the additive symmetric monoidal structure provided by the direct sum as tensor product operation. We mainly establish that the category of additive $\Gamma-$operads and the category of $\Gamma-$symmetric sequences in abelian groups are abelian.

We then define and study the abelian groups of $\Gamma-$operadic derivations, which are a straightforward adaptation of operadic derivations (see [8, III.2.1]) in the $\Gamma-$operad setting.

6.1. Additive $\Gamma-$operads

We prove that $\mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ and $\mathtt{AbSeq}^{\Gamma}$ are abelian categories. These categories are in fact Grothendieck categories : we do no need this result in this work, but we however prove it in appendix B.

We also prove that the cofibrant objects in $\mathtt{dg}_*\mathtt{AbSeq}^{\Gamma}$ (respectively, $\mathtt{sAbSeq}^{\Gamma}$) are degreewise (respectively, dimensionwise) projective in $\mathtt{AbSeq}^{\Gamma}$.

6.1.1. Additive $\Gamma-$operads. The category $\mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ of *additive* $\Gamma-$*operads* is the category of $\Gamma-$operads in $(\mathtt{Ab}, \oplus)$, where we take the direct sum $\oplus : \mathtt{Ab} \times \mathtt{Ab} \to \mathtt{Ab}$ as the tensor product operation of our symmetric monoidal category structure on $\mathtt{Ab}$. Note that, if $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$, $a \in A(x)$ and $b \in B(y)$, then $a \circ_i b = a \circ_i 0 + 0 \circ_i b$.

6.1.2. Proposition. *The categories* $\mathtt{AbSeq}^{\Gamma}$ *and* $\mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ *are abelian.*

Proof. Since $\mathtt{AbSeq}^{\Gamma} \simeq \mathtt{Ab}^{\Sigma \ltimes \Gamma}$ (Theorem 3.2.4), $\mathtt{AbSeq}^{\Gamma}$ is an abelian category such that kernels and cokernels are pointwise kernels and cokernels in $\mathtt{Ab}$.

The category $\mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ is enriched over $\mathtt{Ab}$ and has as zero object the $\Gamma-$operad 0 such that $0(x) = 0$ for all $x \in \Gamma$. The category $\mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ also has finite coproducts, which are calculated in $\mathtt{AbSeq}^{\Gamma}$.

We easily check that the kernel and the cokernel of a $\Gamma-$operad morphism considered as a morphism in $\mathtt{AbSeq}^{\Gamma}$ naturally possesses a $\Gamma-$operad structure and that the associated morphisms are $\Gamma-$operad morphisms. Moreover, the morphisms provided by the universal property of kernels and cokernels in $\mathtt{AbSeq}^{\Gamma}$ are $\Gamma-$operad morphisms and for all morphism f in $\mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$, the canonical morphism $\mathrm{Ker}(\mathrm{Coker}\, f) \to \mathrm{Coker}(\mathrm{Ker}(f))$ in $\mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ is an isomorphism. The category $\mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ is therefore abelian. □

6.1.3. Proposition. *The cofibrant objects in* $\mathtt{dg}_*\mathtt{AbSeq}^{\Gamma}$ *are degreeewise projective in* $\mathtt{AbSeq}^{\Gamma}$.

Proof. Let M a cofibrant object in $\mathtt{dg}_*\mathtt{AbSeq}^{\Gamma}$ and f an epimorphism in $\mathrm{Mor}_{\mathtt{AbSeq}^{\Gamma}}(A, B)$, that is, an aritywise and pointwise surjection. For all $n \geqslant 0$, $f\langle n+1\rangle : A\langle n+1\rangle \to B\langle n+1\rangle$ is an acyclic lift of f in $\mathtt{dg}_*\mathtt{AbSeq}^{\Gamma}$ such that

$$A\langle n+1\rangle_k = \begin{cases} A & \text{if } k = n, n+1 \\ 0 & \text{else,} \end{cases} \qquad B\langle n+1\rangle_k = \begin{cases} B & \text{if } k = n, n+1 \\ 0 & \text{else,} \end{cases}$$

$d_{n+1} = \mathrm{id}$ and $d_k = 0$ for all $k \neq n+1$ for both complexes.

Since M is cofibrant, there is a lift $g : M \to A\langle n+1\rangle$ of $f\langle n+1\rangle$ and $g_n : M_n \to A$ is a lift of f, thus A_n is projective. □

6.1.4. Proposition. *Cofibrant objects in* $\mathtt{sAbSeq}^{\Gamma}$ *are dimensionwise projective in* $\mathtt{AbSeq}^{\Gamma}$.

Proof. Let $M \in \mathtt{sAbSeq}^{\Gamma}$ cofibrant. According to Theorem 4.5.2 and Proposition 6.1.3, $\mathrm{N}_*(M)$ is cofibrant and thus projective degreewise. Moreover, for all $n \geqslant 0$, $M_n = \mathrm{D}_\bullet \mathrm{N}_*(M)_n$ is by definition a finite coproduct of $\Gamma-$symmetric sequences $N_*(M)_l$ with $l \leqslant n$, so M_n is projective. □

6.2. Γ−operadic derivations

The abelian group of Γ−operadic derivations $\mathrm{Der}_{\mathtt{AbOp}^\Gamma}(P, A)$ is a subgroup of $\mathrm{Mor}_{\mathtt{AbSeq}^\Gamma}(P, A)$ with P a Γ−operad in abelian groups augmented over Com and A an additive Γ−operad.

We study this group of derivations in the two cases we will have to deal with : when P is free abelian, and when P is free as a Γ−operad.

6.2.1. Γ−operadic derivations. Let $(P, \epsilon) \in \mathtt{AbOp}^\Gamma \downarrow \mathtt{Com}$ and $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^\Gamma$. We denote by $\mathrm{Der}_{\mathtt{AbOp}^\Gamma}(P, A)$ the abelian subgroup of $\mathrm{Mor}_{\mathtt{AbSeq}^\Gamma}(P, A)$ such that

$$f(p \circ_i q) = \epsilon(q).f(p) \circ_i 0 + 0 \circ_i \epsilon(p).f(q)$$

for all $f \in \mathrm{Der}_{\mathtt{AbOp}^\Gamma}(P, A)$, $r, s \geqslant 2$, $1 \leqslant i \leqslant r$, $x \in \Gamma(r)$, $y \in \Gamma(s)$, $p \in P(x)$ and $q \in P(y)$. Equivalently, we assume by additivity that

$$f(p \circ_i q) = \epsilon(q).f(p) \circ_i \epsilon(p).f(q).$$

6.2.2. Proposition. *Let $P \in \mathtt{SetOp}^\Gamma$. Then $\mathbb{Z}P = \mathbb{Z}[P] \in \mathtt{AbOp}^\Gamma \downarrow \mathtt{Com}$ with the augmentation ϵ such that $\epsilon(p) = 1$ for all $p \in P(x)$ and $x \in \Gamma$.*

Proof. Immediate. □

6.2.3. Proposition. *For all $P \in \mathtt{SetOp}^\Gamma$ and $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^\Gamma$, there is a natural isomorphism*

$$\mathrm{Der}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}P, A) \xrightarrow{\simeq} \mathrm{Mor}_{\mathtt{SetOp}^\Gamma}(P, A),$$

where we forgot the abelian group structure of A in $\mathrm{Mor}_{\mathtt{SetOp}^\Gamma}(P, A)$ and we regard A as a Γ−operad in sSet.

Proof. The adjunction

$$\mathbb{Z} : \mathtt{SetSeq}^\Gamma \rightleftarrows \mathtt{AbSeq}^\Gamma : \rho$$

provides a natural isomorphism

$$\mathrm{Mor}_{\mathtt{AbSeq}^\Gamma}(\mathbb{Z}(P), A) \to \mathrm{Mor}_{\mathtt{SetSeq}^\Gamma}(P, A).$$

We then check that the derivation relation for a morphism in $\mathrm{Mor}_{\mathtt{AbSeq}^\Gamma}(\mathbb{Z}P, A)$ is equivalent to the preservation of the operadic composition by the associated morphism in $\mathrm{Mor}_{\mathtt{SetSeq}^\Gamma}(P, A)$. □

6.2.4. Proposition. *Let $M \in \mathtt{AbSeq}^\Gamma \downarrow \mathtt{Com}$ and $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^\Gamma$. There is a natural isomorphism*

$$\iota^* : \mathrm{Mor}_{\mathtt{AbSeq}^\Gamma}(M, A) \xrightarrow{\simeq} \mathrm{Der}_{\mathtt{AbOp}^\Gamma}(\mathbb{F}(M), A)$$

induced by the canonical embedding $\iota_M : M \to \mathbb{F}(M)$.

Proof. Slight adaptation of the proof of [8, Theorem III.2.1.7]. □

7. Operadic equivariant cohomology

We introduce the operadic equivariant cohomology, which provides us with an efficient tool to define operadic Postnikov invariants in the next section.

We first define the operadic equivariant cohomology for a cofibrant simplicial operad an express it by means of derivations and covering $\Gamma-$operads.

We then establish that the $n-$th group of equivariant cohomology of a cofibrant simplicial operad P with coefficients in an additive operad A is isomorphic to a group of homotopy classes of morphisms from $P^\vee$, the universal covering operad of P, to $K(A,n)$, an Eilenberg-MacLane $\Gamma-$operad. We can therefore upgrade the definition of the operadic equivariant cohomology to non necessarily cofibrant simplicial operads.

We define a reduced and a relative version of this cohomology. We also etablish the existence of a long exact sequence in relative operadic equivariant cohomology.

We finally define the indecomposables of a simplicial morphism operad f, which forms a $\Gamma-$symmetric sequence $\mathrm{Indec}(f)$ in the simplicial abelian group setting. This permits us to prove a universal coefficient theorem for relative operadic equivariant cohomology when the coefficients are concentrated in one arity. A universal coefficient theorem also exists without this assumption (see Theorem A.1), but we will not need it to define the Postnikov invariants and prove our main results.

7.1. Definition

For a reminder about equivariant cohomology in the simplicial set setting, see [10, definition after Corollary VI.3.4 and Lemma VI.3.5]. We consider in this subsection $P \in \mathtt{sOp} \downarrow B\Gamma$ cofibrant with the augmentation $\Psi : P \to B\Gamma$ and $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$.

We first introduce the cosimplicial abelian group of operadic equivariant maps $C^{\bullet}_{\Gamma}(P,A)$, which associated cohomology is the operadic equivariant cohomology of P. We then prove that $C^{\bullet}_{\Gamma}(P,A)$ is isomorphic to the cosimplical abelian group of $\Gamma-$operadic derivations $\mathrm{Der}_{\mathtt{AbOp}^{\Gamma}}(\mathbb{Z}P^{\vee}{}_{\bullet}, A)$, with $\text{-}^{\vee}$ the covering functor.

7.1.1. Operadic equivariant maps. For all $n \geqslant 0$, we denote by $C^n_{\Gamma}(P,A)$ the set of *operadic equivariant maps*

$$\mathrm{Mor}_{\mathtt{SetOp}\downarrow(\mathtt{B}\Gamma)_{\mathtt{n}}}(P_n, \mathrm{hocolim}_{\Gamma} A_n)\,,$$

where we regard A as a discrete $\Gamma-$operad in $\mathtt{sSet}$ and use the functor $\mathrm{hocolim}_{\Gamma} : \mathtt{sOp}^{\Gamma} \to \mathtt{sOp} \downarrow \mathtt{B}\Gamma$ of Proposition 5.2.1. This set can also be described as the collection of the $\mathtt{Set}-$based operad morphisms $\alpha : P_n \to \mathrm{hocolim}_{\Gamma} A_n$ such that

$$\begin{array}{ccc} P_n & \xrightarrow{\ \alpha\ } & \mathrm{hocolim}_{\Gamma} A_n = \coprod_{x_0 \to \ldots \to x_n \in B\Gamma_n} A(x_0) \\ & {\scriptstyle \Psi_n} \searrow \quad \swarrow & \\ & B\Gamma_n & \end{array}$$

commutes, or equivalently, the collection of the $\mathtt{Set}-$based operad morphisms $\alpha : P_n \to \mathrm{hocolim}_{\Gamma} A_n$ such that, for all $r \geqslant 1$,

$$\begin{array}{cccc} \alpha : & P(r)_n & \longrightarrow & \mathrm{hocolim}_{\Gamma(r)} A(r)_n \\ & x & \longmapsto & y \in A(\Psi_n(x)_0) \end{array}$$

with $\Psi_n(x)_0 = z_0$ if $\Psi_n(x) = z_0 \to \cdots \to z_n \in B\Gamma(r)_n$. Note that the abelian group structure of A induces an abelian group structure on $C^{\Gamma}_n(P,A)$.

7.1.2. Proposition. *The sequence of abelian groups $C^\bullet_\Gamma(P,A)$ has a cosimplicial abelian group structure such that :*
– *let $f \in C^{n-1}_\Gamma(P,A)$ and $x \in P_n$. If $\Psi_n(x) = x_0 \xrightarrow{\nu} x_1 \to \dots \to x_n$, then*

$$d^i f(x) = \begin{cases} f(d_i(x)) & \text{for } i = 1, \cdots, n \\ A(\nu^{-1})(f(d_0 x)) & \text{for } i = 0. \end{cases}$$

– *let $f \in C^{n+1}_\Gamma(P,A)$ and $x \in P_n$. We set $s^j f(x) = f(s_j x)$ for $j = 0, \cdots, n$.*

Proof. Straighforward consequence of [10, Lemma IV.3.5]. □

7.1.3. Operadic equivariant cohomology. The *operadic equivariant cohomology* of a cofibrant simplicial $\Gamma-$operad P over $B\Gamma$ with coefficients in A is the graded abelian group

$$\mathrm{H}^\bullet_\Gamma(P;A) = \mathrm{H}^\bullet\, \mathrm{N}^* C^\bullet_\Gamma(P;A),$$

with $\mathrm{N}^* : \mathtt{cAb} \to \mathtt{dg}^*\mathtt{Ab}$ the conormalization functor.

The equivariant cohomology of a simplicial set is nothing more than the cohomology of its universal covering system (see [10, Corollary VI.3.8]). A similar link exists for operads but in order to take composition products into account, we have to use operadic derivations :

7.1.4. Lemma. *There are natural isomorphisms of cosimplicial abelian groups :*

$$\begin{aligned} C^\bullet_\Gamma(P,A) &= \mathrm{Mor}_{\mathtt{SetOp}\downarrow(\mathtt{B}\Gamma)_\bullet}(P_\bullet, (\mathrm{hocolim}_\Gamma A)_\bullet) \\ &\simeq \mathrm{Mor}_{\mathtt{SetOp}^\Gamma}(P^\vee{}_\bullet, A) \\ &\simeq \mathrm{Der}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}P^\vee{}_\bullet, A). \end{aligned}$$

Proof. The results of the Theorem 5.2.4 and the Proposition 6.2.3 imply we have such isomorphisms dimensionwise. We then check that the sequences of isomorphisms provided by this two propositions are compatible with the coface and the codegeneracy morphisms. □

7.2. Representation of the operadic equivariant cohomology

We define Eilenberg-MacLane $\Gamma-$symmetric sequences and Eilenberg-MacLane $\Gamma-$operads. We then establish that the operadic equivariant cohomology of a cofibrant simplicial operad P is isomorphic to a group of homotopy classes of morphisms from the universal covering $P^\vee$ of P to an Eilenberg-MacLane operad. This permits us to enlarge the definition of the operadic equivariant cohomology to all simplicial operads and prove that this cohomology is homotopy invariant.

7.2.1. Eilenberg-MacLane Γ−symmetric sequences and Γ−operads. Let $A \in \mathtt{Ab}^\oplus\mathtt{Op}^\Gamma$ (respectively, $A \in \mathtt{AbSeq}^\Gamma$), $n \geqslant 1$ and $A[n] \in \mathtt{dg}_*\mathtt{Ab}^\oplus\mathtt{Op}^\Gamma$ (respectively, $A[n] \in \mathtt{dg}_*\mathtt{AbSeq}^\Gamma$) such that, for all $k \geqslant 0$,

$$A[n]_k = \begin{cases} A & \text{if } k = n \\ 0 & \textit{else}. \end{cases}$$

We call *Eilenberg-MacLane Γ−operad* (respectively, *Eilenberg-MacLane Γ−symmetric sequence*)

$$K(A,n) = \mathrm{D}_\bullet\, A[n]$$

where the Dold-Kan functor $\mathrm{D}_\bullet$ is applied pointwise.

We then have for all $x \in \Gamma$

$$K(A,n)(x) = K(A(x),n),$$

where we consider on the right side the Eilenberg-MacLane space $K(\pi, n)$ associated to the abelian group $\pi = A(x)$.

Let also $A\langle n\rangle \in \mathtt{dg_*Ab^{\oplus}Op^{\Gamma}}$ (respectively, $A\langle n\rangle \in \mathtt{dg_*AbSeq^{\Gamma}}$) such that, for all $k \geqslant 0$,

$$A\langle n\rangle_k = \begin{cases} A & \text{if } k = n, n-1 \\ 0 & \textit{else} \end{cases}$$

with the differential d such that $d_n : A\langle n\rangle_n \to A\langle n\rangle_{n-1}$ is the identity. Observe that $A\langle n\rangle = A[n]^I \times_{A[n]} 0$ where $A[n]^I$ is the path object defined in the Proposition 4.3.4. We define

$$L(A,n) = D_\bullet A\langle n\rangle.$$

We still have

$$L(A,n)(x) = L(A(x),n).$$

The obvious morphism $A\langle n\rangle \to A[n]$ induces a natural fibration

$$q : L(A,n) \to K(A,n)$$

in $\mathtt{sOp}^{\Gamma}$. The fiber of $q(x)$ is $K(A(x), n-1)$. We moreover have the following lemma :

7.2.2. Lemma. *Let $A \in \mathtt{Ab^{\oplus}Op^{\Gamma}}$ and $n \geqslant 1$. The zero section $d_L : \mathtt{pt} \to L(A,n)$ of $t : L(A,n) \to \mathtt{pt}$ is a weak equivalence and the composite $d_L \circ t$ is right homotopic to the identity on $L(A,n)$. In other words, the zero section d_L is a homotopy equivalence.*

Moreover, the zero section $c_L : B\Gamma \to \mathrm{hocolim}_\Gamma L(A,n)$ of $\psi_L : L(A,n) \to B\Gamma$ is a weak equivalence and the composite $c_L \circ \Psi_L$ is right homotopic to the identity on $\mathrm{hocolim}_\Gamma L(A,n)$. In other words, the zero section c_L is a homotopy equivalence.

Proof. The zero section d_L is a weak equivalence because $d_L(x)$ is a weak equivalence for every $x \in \Gamma$.

There is a chain homotopy $h : A\langle n\rangle \to A\langle n\rangle$ in $\mathtt{dgAbSeq}^{\Gamma}$ between the identity and the zero morphism such that $h_m : A\langle n\rangle_m \to A\langle n\rangle_{m+1} = 0$ for every $m \neq n-1$ and $h_{n-1} = \mathrm{id}$. We then have a right homotopy $H : A\langle n\rangle \to A\langle n\rangle^I$ between the identity and zero, with $A\langle n\rangle^I$ the path object associated to $A\langle n\rangle$ (see Proposition 4.3.4 for the definition of chain homotopies and path objects in $\mathtt{dgAbSeq}^{\Gamma}$). Moreover, for every $m \geqslant 0$, h_m is a morphism in $\mathtt{Ab^{\oplus}Op^{\Gamma}}$ and $A\langle n\rangle^I$ inherits from A a $\Gamma-$operad structure in $\mathtt{dg^*Ab^{\oplus}}$. Thus, H is a morphism in $\mathtt{dg^*Ab^{\oplus}Op^{\Gamma}}$.

Since $D_\bullet(A\langle n\rangle^I)$ is a path object associated to $L(A,n)$ in $\mathtt{sOp}^{\Gamma}$, $D_\bullet(H)$ is a right homotopy between the morphism $D_\bullet(\mathrm{id}) = \mathrm{id}$ and $D_\bullet(0) = d_L \circ t$.

We deduce the remaining claims of this lemma from the previously proved ones since $B\Gamma = \mathrm{hocolim}_\Gamma \, \mathtt{pt}$ and since $\mathrm{hocolim}_\Gamma$ is a right Quillen adjoint, which therefore preserves weak equivalences between fibrant objects and right homotopies to fibrant objects. ☐

We now adapt [10, Theorem VI.3.10] in order to demonstrate that operadic equivariant cohomology is representable in an homotopy category.

7.2.3. Theorem (Representation of the operadic equivariant cohomology)**.** *Let $P \in \mathtt{sOp} \downarrow B\Gamma$ cofibrant, $A \in \mathtt{Ab^{\oplus}Op^{\Gamma}}$ and $n \geqslant 0$. There are natural isomorphisms in P*

$$\begin{aligned} \mathrm{H}^n_\Gamma(P,A) &\simeq \left[P^\vee, K(A,n)\right]_{\mathtt{sOp}^{\Gamma}} \\ &\simeq \left[P, \mathrm{hocolim}_\Gamma K(A,n)\right]_{\mathtt{sOp}\downarrow B\Gamma}. \end{aligned}$$

Proof. Recall that

$$C^\bullet_\Gamma(P, A) \simeq \mathrm{Der}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}P^\vee_\bullet, A)$$

(see Lemma 7.1.4) and that there is an identity

$$D_\bullet(X) = \mathrm{Mor}_{\mathtt{dg}_*\mathtt{Ab}}(\mathrm{N}_*(\Delta^\bullet), X)$$

(see Proposition 4.5.2). Let $Z^n\mathrm{N}^*\,\mathrm{Mor}_{\mathtt{AbSeq}^\Gamma}(\mathbb{Z}P^\vee_\bullet, A)$ be the abelian group of n-cocyles of the conormalized complex $\mathrm{N}^*\,\mathrm{Mor}_{\mathtt{AbSeq}^\Gamma}(\mathbb{Z}P^\vee_\bullet, A)$. We have isomorphisms

$$\begin{array}{c} Z^n\mathrm{N}^*\,\mathrm{Mor}_{\mathtt{AbSeq}^\Gamma}(\mathbb{Z}P^\vee_\bullet, A) \\ \tau \big\downarrow \simeq \\ \mathrm{Mor}_{\mathtt{dg}_*\mathtt{AbSeq}^\Gamma}(\mathrm{N}_*\mathbb{Z}P^\vee, A[n]) \\ \rho \big\downarrow \simeq \\ \mathrm{Mor}_{\mathtt{sAbSeq}^\Gamma}(\mathbb{Z}P^\vee, K(A,n)) \simeq \mathrm{Mor}_{\mathtt{sAbSeq}^\Gamma}\left(\mathbb{Z}P^\vee, \mathrm{Mor}_{\mathtt{dg}_*\mathtt{Ab}}(\mathrm{N}_*(\Delta^\bullet), A[n])\right) \\ \upsilon \big\downarrow \simeq \\ \mathrm{Mor}_{\mathtt{sSeq}^\Gamma}(P^\vee, K(A,n)) \end{array}$$

where :

- For $g \in Z^n\mathrm{N}^*\,\mathrm{Mor}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}P^\vee_\bullet, A)$ and $y \in \mathbb{Z}P^\vee(x)_m$, we set

  $$\tau(g)_m([y]) = \begin{cases} g(y) & \text{if } m = n \\ 0 & \textit{else} \end{cases}$$

  with $[y]$ the class of y in

  $$\mathrm{N}_*(\mathbb{Z}P^\vee)(x)_n = \frac{\mathbb{Z}P^\vee(x)_n}{s_0(\mathbb{Z}P^\vee(x)_{n-1}) + \cdots + s_{n-1}(\mathbb{Z}P^\vee(x)_{n-1})}.$$

  Note that $\tau(g)_m([y])$ does not depend of the choice of a class representative of $[y]$ because $g \in \bigcap_{i=0}^{n-1}\mathrm{Ker}(s^i)$. Note also that $\tau(g)$ is a morphism of chain complexes because g is a cocycle.
- The isomorphism ρ is described in Proposition 4.5.2. Recall that

  $$\rho(h)(y)([\sigma]) = h([\sigma^* y])$$

  for all $h \in \mathrm{Mor}_{\mathtt{sAbSeq}^\Gamma}(\mathbb{Z}P^\vee, D_\bullet A[n])$, $x \in \Gamma$, $y \in \mathbb{Z}P^\vee(x)_m$ and $\sigma \in \mathrm{Mor}_\Delta(\underline{k}, \underline{m}) \simeq \Delta^m_k$.
- The isomorphism υ is provided by the adjunction

  $$\mathbb{Z} : \mathtt{SetSeq}^\Gamma \rightleftarrows \mathtt{AbSeq}^\Gamma : \rho$$

  (see Proposition 4.5.1).

We therefore have

$$(\rho \circ \tau(g))(y)([\sigma]) = \tau(g)([\sigma^* y]) = \begin{cases} g(\sigma^* y), & \text{if } k = n, \\ 0, & \text{otherwise,} \end{cases}$$

with $g \in Z^n\mathrm{N}^*\,\mathrm{Mor}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}P^\vee_\bullet, A)$, $x \in \Gamma$, $y \in \mathbb{Z}P^\vee(x)_m$ and $\sigma \in \mathrm{Mor}_\Delta(\underline{k}, \underline{m})$.

The composite $\tau \circ \rho$ induces an isomorphism between $Z^n \mathrm{N}^* \operatorname{Der}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}P^\vee_\bullet, A)$ and the set of morphism in $\operatorname{Mor}_{\mathtt{sAbSeq}^\Gamma}(\mathbb{Z}P^\vee, K(A,n))$ that satisfy the derivation relation degreewise. Moreover, υ induces an isomorphism between this set of morphisms and $\operatorname{Mor}_{\mathtt{sOp}^\Gamma}(P^\vee, K(A,n)$ according to Proposition 6.2.3 : we simply check that the dimensionwise isomorphisms provided by this proposition form a simplicial isomorphism. Thus, the composite $\upsilon \circ \rho \circ \tau$ induces an isomorphism

$$Z^n \mathrm{N}^* \operatorname{Der}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}P^\vee_\bullet, A) \simeq \operatorname{Mor}_{\mathtt{sOp}^\Gamma}(P^\vee, K(A,n)).$$

Consider now a $n-$coboundary $\theta = \delta^* h$ in $N^* \operatorname{Der}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}P^\vee_\bullet, A)$. The morphism τ induces an obvious morphism which associates to this coboundary a morphism $H \in \operatorname{Mor}_{\mathtt{dg}_*\mathtt{AbSeq}^\Gamma}(\mathrm{N}_* \mathbb{Z}P^\vee_\bullet \bullet, A[n]^I)$ that satisfy the derivation relation degreewise and where $A[n]^I$ is the path object described in the Proposition 4.3.4. We have $d_0 H = \rho(h)$ and $d_1 H = 0$ where d_0 and d_1 are the face maps associated to the path object $A[n]^I$. Moreover, this correspondence is one-to-one. Because $D_\bullet(A[n]^I)$ is a path object associated to $K(A,n) = D_\bullet(A[n]) \in \mathtt{sOp}^\Gamma$, the composite $\upsilon \circ \rho$ induces a bijection between :

– the set of morphisms $H \in \operatorname{Mor}_{\mathtt{dg}_*\mathtt{AbSeq}^\Gamma}(\mathrm{N}_* \mathbb{Z}P^\vee_\bullet, A[n]^I)$ that satisfy the derivation relation degreewise and such that $d_1 H = 0$;
– the homotopy class of the zero morphism in $\operatorname{Mor}_{\mathtt{sOp}^\Gamma}(P^\vee, K(A,n))$.

Therefore, $\upsilon \circ \tau \circ \rho$ induces an isomorphism

$$\mathrm{H}^n_\Gamma(P, A) \simeq \left[P^\vee, K(A,n)\right]_{\mathtt{sOp}^\Gamma}.$$

Finally, there is an isomorphism

$$[P^\vee, K(A,n)]_{\mathtt{sOp}^\Gamma} \simeq [P, \operatorname{hocolim}_\Gamma K(A,n)]_{\mathtt{sOp}\downarrow \mathtt{B}\Gamma}$$

provided by the Quillen adjunction between the functors $\text{-}^\vee$ and $\operatorname{hocolim}_\Gamma$ since P is cofibrant. □

7.2.4. Operadic equivariant cohomology of a (non necessarily cofibrant) simplicial operad. Let $P \in \mathtt{sOp} \downarrow \mathtt{B}\Gamma$ and $A \in \mathtt{Ab}^\oplus\mathtt{Op}^\Gamma$. if $\overline{P}$ and $\overline{P}'$ are two cofibrant replacements of P, then we have $\overline{P} \sim \overline{P}'$ in the homotopy category of $\mathtt{sOp} \downarrow \mathtt{B}\Gamma$, and this equivalence gives a bijection on homotopy class sets

$$[\overline{P}, \operatorname{hocolim}_\Gamma K(A,n)]_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma} \simeq [\overline{P}', \operatorname{hocolim}_\Gamma K(A,n)]_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}$$

by general results of the theory of model categories. We thus have an isomorphism $\mathrm{H}^n_\Gamma(\overline{P}; A) \simeq \mathrm{H}^n_\Gamma(\overline{P}'; A)$.

Therefore, we define the operadic equivariant cohomology of a (non necessarily cofibrant) simplicial operad P as the operadic equivariant cohomology of a cofibrant replacement of P.

7.2.5. Proposition. *Let $f : P \to Q$ be a weak equivalence in $\mathtt{sOp} \downarrow \mathtt{B}\Gamma$ and $A \in \mathtt{Ab}^\oplus\mathtt{Op}^\Gamma$. We also consider $\overline{P}$ and $\overline{Q}$ in $\mathtt{sOp} \downarrow \mathtt{B}\Gamma$ so that we have acyclic fibrations $\overline{P} \to P$ and $\overline{Q} \to Q$. For every $n \geqslant 0$, the induced morphism $f^* : \mathrm{H}^n_\Gamma(Q; A) \to \mathrm{H}^n_\Gamma(P; A)$ is an isomorphism.*

Proof. Let $\overline{P}$ and $\overline{Q}$ be cofibrant replacements of P and Q. there is a lift $\overline{f}$ such that the square

$$\begin{array}{ccc} \overline{P} & \overset{\overline{f}}{\dashrightarrow} & \overline{Q} \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} \\ P & \underset{f}{\longrightarrow} & Q \end{array}$$

commutes. This morphism $\overline{f}$ is a weak equivalence by the two-out-of-three axiom. Since $\overline{P}$ and $\overline{Q}$ are cofibrant, $\overline{f}$ induces an isomorphism

$$\overline{f}^* : [\overline{Q}, \mathrm{hocolim}_\Gamma K(A,n)]_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma} \to [\overline{P}, \mathrm{hocolim}_\Gamma K(A,n)]_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}$$

for every $n \geqslant 0$, so we have an isomorphism $f^* : \mathrm{H}^n_\Gamma(Q;A) \to \mathrm{H}^n_\Gamma(P;A)$. □

7.3. Reduced operadic equivariant cohomology

We introduce the reduced operadic equivariant cohomology. We first define $\widetilde{C}^\bullet_\Gamma(P,A)$, the cosimplicial abelian group of reduced operadic equivariant cochains of a simplicial operad $P \in \mathtt{sOp} \downarrow \mathtt{B}\Gamma$, equipped with a section $c : B\Gamma \to P$ of its augmentation $P \to B\Gamma$, with coefficients an additive Γ-operad A.

In order to give an homotopy invariant definition of this reduced cohomology, we prove that the comma category $\mathtt{B}\Gamma \downarrow (\mathtt{sOp} \downarrow \mathtt{B}\Gamma)$ has a model structure inherited from the usual model structure on $\mathtt{sOp}$. Indeed, an operad in $\mathtt{sOp} \downarrow \mathtt{B}\Gamma$ equipped with a section of its augmentation morphism can be interpreted as an object in $\mathtt{B}\Gamma \downarrow (\mathtt{sOp} \downarrow \mathtt{B}\Gamma)$. We then establish that the functor $\widetilde{C}^\bullet_\Gamma(-,A) : \mathtt{B}\Gamma \downarrow (\mathtt{sOp} \downarrow \mathtt{B}\Gamma) \to \mathtt{cAb}$ preserves weak equivalences between cofibrants objects.

We accordingly define the reduced operadic equivariant cohomology of an augmented simplicial operad P over $B\Gamma$ equipped with a section of its augmentation as the cohomology of $\widetilde{C}^\bullet_\Gamma(\overline{P},A)$, with $\overline{P}$ a cofibrant replacement of P in $\mathtt{B}\Gamma \downarrow (\mathtt{sOp} \downarrow \mathtt{B}\Gamma)$.

7.3.1. Reduced operadic equivariant cochains. Let $P \in \mathtt{sOp} \downarrow \mathtt{B}\Gamma$ be such that the associated augmentation $\Psi_P : P \to B\Gamma$ admits a section c. Let also $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$. The *cosimplicial abelian group of reduced operadic equivariant maps of P with coefficients in A*, denoted by $\widetilde{C}^\bullet_\Gamma(P;A)$, is defined by

$$\widetilde{C}^n_\Gamma(P;A) = \{f \in C^n_\Gamma(P;A) | f \circ c = 0\}$$

for all $n \geqslant 0$, with the cosimplicial structure inherited from $C^\bullet_\Gamma(P,A)$.

7.3.2. Proposition. *let $P \in \mathtt{sOp} \downarrow \mathtt{B}\Gamma$ such that the associated augmentation $\Psi_P : P \to B\Gamma$ admits a section c. Let also $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$. There is a split short exact sequence*

$$0 \longrightarrow \widetilde{C}^\bullet_\Gamma(P;A) \longrightarrow C^\bullet_\Gamma(P;A) \underset{\Psi_P{}^*_\bullet}{\overset{c^*_\bullet}{\rightleftarrows}} C^\bullet_\Gamma(B\Gamma;A) \longrightarrow 0\,.$$

Observe that $\widetilde{C}^\bullet_\Gamma(P,A)$ does not depend on the choice of the section up to isomorphism since $\widetilde{C}^\bullet_\Gamma(P,A) = \mathrm{Coker}(\Psi_P)$.

Proof. Simple consequence of the definition of the reduced operadic equivariant maps. □

7.3.3. The model category $\mathtt{B}\Gamma \downarrow (\mathtt{sOp} \downarrow \mathtt{B}\Gamma)$. The mapping $P \mapsto \widetilde{C}^\bullet_\Gamma(P,A)$ given by the construction of section 7.3.1 actually defines a functor on the comma category $\mathtt{B}\Gamma \downarrow (\mathtt{sOp} \downarrow \mathtt{B}\Gamma)$ whose objects are operads $P \in \mathtt{sOp} \downarrow \mathtt{B}\Gamma$ equipped with a section $c : B\Gamma \to P$ of the augmentation $\Psi_P : P \to B\Gamma$.

We aim to establish that this functor satisfies good homotopy invariance properties. For this purpose, we use that the comma category $\mathtt{B}\Gamma \downarrow (\mathtt{sOp} \downarrow \mathtt{B}\Gamma)$ inherits a model structure by general results of the theory of model categories with as weak equivalences (respectively, cofibrations, fibrations) the morphisms which form a weak equivalence (respectively, cofibration, fibration) in the category $\mathtt{sOp} \downarrow \mathtt{B}\Gamma$. In particular, the cofibrant objects of this model category are the operads

$P \in \mathtt{sOp} \downarrow \mathtt{B\Gamma}$ equipped with a section of the augmentation $c : B\Gamma \to P$ that forms a cofibration of simplicial operads. This model category $\mathtt{B\Gamma} \downarrow (\mathtt{sOp} \downarrow \mathtt{B\Gamma})$ is also cofibrantly generated, with as generating (acyclic) cofibrations the morphisms $B\Gamma \bigvee i : B\Gamma \bigvee P \to B\Gamma \bigvee Q$ associated to the generating (acyclic) cofibrations of the model category $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$.

We study the categorical properties of the functor $\widetilde{C}^{\bullet}_{\Gamma}(-, A)$ first. We use the following observation about the functor of unreduced operadic equivariant maps.

7.3.4. Lemma. *We have the identity* $C^{\bullet}_{\Gamma}(\operatorname{colim}_{i\in I} P_i, A) \simeq \lim_{i\in I} C^{\bullet}_{\Gamma}(P_i, A)$ *for any diagram* $(P_i)_{i\in I}$ *in the category* $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$.

Proof. This result is an immediate consequence of the relation

$$C^{\bullet}_{\Gamma}(P, A) \simeq \operatorname{Mor}_{\mathtt{SetOp}\downarrow(\mathtt{B\Gamma})_{\bullet}}(P_{\bullet}, (\operatorname{hocolim}_{\Gamma} A)_{\bullet})$$

established in Lemma 7.1.4. □

7.3.5. Lemma. *We have the identity* $\widetilde{C}^{\bullet}_{\Gamma}(\operatorname{colim}_i P_i, A) \simeq \lim_i \widetilde{C}^{\bullet}_{\Gamma}(P_i, A)$ *for any diagram* $(P_i)_{i\in I}$ *in the category* $\mathtt{B\Gamma} \downarrow (\mathtt{sOp} \downarrow \mathtt{B\Gamma})$.

Proof. We use the result of the previous lemma and that the kernel

$$\widetilde{C}^{\bullet}_{\Gamma}(-, A) = \operatorname{Ker}(C^{\bullet}_{\Gamma}(-, A) \to C^{\bullet}_{\Gamma}(B\Gamma, A))$$

preserves limits. □

7.3.6. Lemma. *For an operad of the form* $Q = B\Gamma \bigvee P$ *in the category* $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$ *we have an isomorphism* $\widetilde{C}^{\bullet}_{\Gamma}(B\Gamma \bigvee P, A) \simeq C^{\bullet}_{\Gamma}(P, A)$.

Proof. This relation is an immediate consequence of the identity

$$C^{\bullet}_{\Gamma}(B\Gamma \bigvee P, A) \simeq C^{\bullet}_{\Gamma}(B\Gamma, A) \times C^{\bullet}_{\Gamma}(P, A)$$

given by the general result of Lemma 7.3.4. □

7.3.7. Lemma. *The functor* $\widetilde{C}^{\bullet}_{\Gamma}(-, A)$ *carries cofibrations (respectively, acyclic cofibrations) in the model category* $\mathtt{B\Gamma} \downarrow (\mathtt{sOp} \downarrow \mathtt{B\Gamma})$ *to fibrations (respectively, to acyclic fibrations) in the category of cosimplicial abelian groups.*

Proof. The observation of Lemma 7.3.5 implies that we can reduce the verification of this lemma to the case of a generating (acyclic) cofibration $B\Gamma \bigvee j : B\Gamma \bigvee P \to B\Gamma \bigvee Q$, where $j : P \to Q$ is a generating (acyclic) cofibration in the category $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$. For this purpose, we need that such a generating (acyclic) cofibration $j : P \to Q$, which the overcategory $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$ inherits from $\mathtt{sOp}$, is explicitly given by a morphism of free operads $\mathbb{F}(i) : \mathbb{F}(M) \to \mathbb{F}(N)$ induced by a generating (acyclic) cofibration $i : M \to N$ in the comma category of symmetric sequences $\mathtt{sSeq} \downarrow \mathtt{B\Gamma}$ (see [8, section II.8.1]).

By Lemma 7.3.6, the image of such a morphism $B\Gamma \bigvee j : B\Gamma \bigvee P \to B\Gamma \bigvee Q$ under the functor $\widetilde{C}^{\bullet}_{\Gamma}(-, A)$, is identified with the morphism $C^{\bullet}_{\Gamma}(Q, A) \to C^{\bullet}_{\Gamma}(P, A)$ associated to j on the cosimplicial abelian groups of unreduced operadic equivariant maps, and we moreover have $C^{\bullet}_{\Gamma}(P, A) \simeq \operatorname{Mor}_{\mathtt{SetOp}^{\Gamma}}(P^{\vee}{}_{\bullet}, A)$, $C^{\bullet}_{\Gamma}(Q, A) \simeq \operatorname{Mor}_{\mathtt{SetOp}^{\Gamma}}(Q^{\vee}{}_{\bullet}, A)$ by the result of Lemma 7.1.4. In the case $j = \mathbb{F}(i)$, we moreover have $\operatorname{Mor}_{\mathtt{SetOp}^{\Gamma}}(P^{\vee}{}_{\bullet}, A) \simeq \operatorname{Mor}_{\mathtt{SetSeq}^{\Gamma}}(M^{\vee}{}_{\bullet}, A)$ and $\operatorname{Mor}_{\mathtt{SetOp}^{\Gamma}}(Q^{\vee}{}_{\bullet}, A) \simeq \operatorname{Mor}_{\mathtt{SetSeq}^{\Gamma}}(N^{\vee}{}_{\bullet}, A)$ so that this morphism is identified with the map of morphism sets

$$i^* : \operatorname{Mor}_{\mathtt{SetSeq}^{\Gamma}}(N^{\vee}{}_{\bullet}, A) \to \operatorname{Mor}_{\mathtt{SetSeq}^{\Gamma}}(M^{\vee}{}_{\bullet}, A)$$

induced by $i^\vee : M^\vee \to N^\vee$, the image of i under the universal covering functor $-^\vee$. This morphism $i^\vee$ is a cofibration since $-^\vee$ is a left Quillen adjoint, and from the expression of the cofibrations of Γ-symmetric sequences in Proposition 4.3.2, we get that our map of morphism sets i^* is surjective in every degree, and hence, defines a fibration of cosimplicial abelian groups.

Let us mention that the source and target objects of the generating (acyclic) cofibration of the category of simplicial operads are cofibrant. We deduce from this observation and the result of Theorem 7.2.3 that the cohomology morphism $j^* : \mathrm{H}^n_\Gamma(Q, A) \to \mathrm{H}^n_\Gamma(P, A)$ induced by $j : P \to Q$ is identified with the map $j^* : [Q, \mathrm{hocolim}_\Gamma K(A, n)] \to [Q, \mathrm{hocolim}_\Gamma K(A, n)]$. In the case where j is acyclic, this map is a bijection for every n, so that $j^* : C^\bullet_\Gamma(Q, A) \to C^\bullet_\Gamma(P, A)$ is also a weak-equivalence, and hence defines an acyclic fibration of cosimplicial abelian groups. □

7.3.8. Theorem. *The functor $\widetilde{C}^\bullet_\Gamma(-, A)$ preserves the weak-equivalences between the cofibrant objects of the comma category* $\mathtt{B\Gamma} \downarrow (\mathtt{sOp} \downarrow \mathtt{B\Gamma})$.

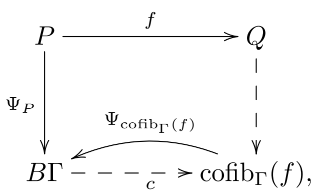
Proof. This theorem is a consequence of the result of the previous lemma and of the Brown lemma. □

7.3.9. Reduced operadic equivariant cohomology. Let $P \in \mathtt{sOp} \downarrow \mathtt{B\Gamma}$ be such that the associated augmentation $\Psi_P : P \to B\Gamma$ admits a section c. Let also $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$. The *reduced operadic equivariant cohomology of P with coefficients in A*, denoted by $\widetilde{H}^\bullet_\Gamma(P; A)$, is defined as $\mathrm{H}^\bullet\, \mathrm{N}^* \widetilde{C}^\bullet_\Gamma(\overline{P}; A)$ with $\overline{P}$ a cofibrant replacement of P in the comma category $\mathtt{B\Gamma} \downarrow (\mathtt{sOp} \downarrow \mathtt{B\Gamma})$. The previous theorem insures us that this definition does not depend up to isomorphism on the choice of $\overline{P}$.

7.4. Relative operadic equivariant cohomology

We define the relative operadic equivariant cohomology of a cofibration $f \in \mathrm{Mor}_{\mathtt{sOp}\downarrow\mathtt{B\Gamma}}(P, Q)$ as the reduced cohomology of the cofiber of f over $B\Gamma$. We then prove the existence of a long exact sequence associated to this cohomology.

7.4.1. Relative operadic equivariant cohomology. Let $f : P \to Q$ in $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$. By construction of the pushout square



there is a section c of the augmentation morphism $\Psi_{\mathrm{cofib}_\Gamma(f)}$.

If $f : P \to Q$ is a cofibration, we denote $\mathrm{H}^\bullet_\Gamma(Q, P; A)$ and call *relative operadic equivariant cohomology* the graded abelian group $\mathrm{H}^\bullet\, \mathrm{N}^* \widetilde{C}^\bullet_\Gamma(\mathrm{cofib}_\Gamma(f); A)$. Note that the section $B\Gamma \to \mathrm{cofib}_\Gamma(f)$ is a cofibration in that case. If this is not the case, we can pick a factorisation

$$P \rightarrowtail \overline{Q} \xrightarrow{\sim} Q$$

in $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$ and set $\mathrm{H}^\bullet_\Gamma(Q, P; A) = \mathrm{H}^\bullet_\Gamma(\overline{Q}, P; A)$.

We analyze the pushout square further when f is a quasi-free extension with $Q_\flat = P_\flat \bigvee \mathbb{F}(M)$, $M \in \mathtt{s^-Op} \downarrow \mathtt{B\Gamma}$: we then have $\mathrm{cofib}_\Gamma(f)_\flat = B\Gamma_\flat \vee \mathbb{F}(M)$. Faces in $\mathrm{cofib}_\Gamma(f)$

are defined vertex by vertex by means of the faces of $B\Gamma$ and M in Q : let $n \geqslant 1$, $r \geqslant 1$, $y \in M_n(r)$ and $0 \leqslant i \leqslant n$. The face $d_i(y)$ of y in $\mathrm{cofib}_\Gamma(f)$ is deduced from the face $d_i^Q(y)$ of y in Q by replacing the decorations in $d_i^Q(y)$ that are simplex in P by their images under Ψ_P. We use this analysis in Subsection 7.5.

7.4.2. Proposition. *Let $f : P \to Q$ be a cofibration in* $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$. *There is a natural short exact sequence*

$$0 \longrightarrow \widetilde{C}^\bullet_\Gamma(\mathrm{cofib}_\Gamma(f), A) \longrightarrow C^\bullet_\Gamma(Q, A) \longrightarrow C^\bullet_\Gamma(P, A) \longrightarrow 0.$$

Proof. Since $\mathrm{cofib}_\Gamma(f) = B\Gamma \bigvee_P Q$, we have

$$C^\bullet(\mathrm{cofib}_\Gamma(f), A) = C^\bullet_\Gamma(B\Gamma, A) \underset{C^\bullet_\Gamma(P,A)}{\times} C^\bullet_\Gamma(Q, A)$$

by Lemma 7.3.4. We therefore have morphisms $C^\bullet(\mathrm{cofib}_\Gamma(f), A) \to C^\bullet_\Gamma(B\Gamma, A)$ and $C^\bullet(\mathrm{cofib}_\Gamma(f), A) \to C^\bullet_\Gamma(Q, A)$ provided by the projections on the first and the secund factors. The short exact sequence of the proposition readily follows. □

7.4.3. Proposition. *Let $f : P \to Q$ in* $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$. *There is a natural long exact sequence*

$$\ldots \longrightarrow \mathrm{H}^n_\Gamma(Q, P; A) \longrightarrow \mathrm{H}^n_\Gamma(Q; A) \longrightarrow \mathrm{H}^n_\Gamma(P; A) \longrightarrow \mathrm{H}^{n+1}_\Gamma(Q, P; A) \longrightarrow \ldots .$$

Proof. Straightforward consequence of the previous propositions. □

7.4.4. Proposition. *Let $P \to Q \to R$ be a pair a morphism in* $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$. *There is a natural long exact sequence*

$$\ldots \longrightarrow \mathrm{H}^n_\Gamma(Q, P; A) \longrightarrow \mathrm{H}^n_\Gamma(R, P; A) \longrightarrow \mathrm{H}^n_\Gamma(R, Q; A) \longrightarrow \mathrm{H}^{n+1}_\Gamma(Q, P; A) \longrightarrow \ldots \cdot$$

Proof. Standard argument about cohomology of a triple. □

The cosimplicial abelian group of relative operadic equivariant maps can be expressed by means of derivations :

7.4.5. Proposition. *If $f : P \to Q$ is a morphism in* $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$, *then there is a cosimplicial abelian group isomorphism*

$$\widetilde{C}^\bullet_\Gamma(\mathrm{cofib}_\Gamma(f), A) \simeq \widetilde{\mathrm{Der}}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee_\bullet, A)$$

with

$$\widetilde{\mathrm{Der}}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee_n, A) = \left\{ g \in \mathrm{Der}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee_n, A) \middle| g(\mathbb{Z}B\Gamma^\vee_n) = 0 \right\}$$

for all $n \geqslant 0$.

Proof. According to Proposition 7.3.2 and Theorem 7.1.4,

$$\begin{aligned} \widetilde{C}^\bullet_\Gamma(\mathrm{cofib}_\Gamma(f), A) &= \mathrm{Ker}\left(C^\bullet_\Gamma(\mathrm{cofib}_\Gamma(f); A) \xrightarrow{c^*} C^\bullet_\Gamma(B\Gamma; A) \right) \\ &\simeq \mathrm{Ker}\left(\mathrm{Der}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee_\bullet, A) \xrightarrow{\mathbb{Z}c^{\vee *}} \mathrm{Der}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}B\Gamma^\vee_\bullet, A) \right) \end{aligned}$$

with c the section of $\Psi_{\mathrm{cofib}_\Gamma(f)}$.

Therefore, there is a one-to-one correspondence between $\widetilde{C}^n_\Gamma(\mathrm{cofib}_\Gamma(f), A)$ and the morphisms in $\mathrm{Der}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee_n, A)$ which are zero on the embedding of $\mathbb{Z}B\Gamma^\vee_n$ in $\mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee_n$. We then check that $\widetilde{\mathrm{Der}}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee_\bullet, A)$ inherits its cosimplicial abelian group structure from $\mathrm{Der}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee_\bullet, A)$.

□

7.5. Indecomposables of a simplicial operad morphism

In our applications of the operadic equivariant cohomology, we have to deal with coefficients $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^\Gamma$ concentrated in one arity r. In that case, the relative operadic equivariant cohomology of a simplicial operad morphism f reduce to the cohomology of the indecomposables with coefficients in $A(r) \in \mathtt{Ab}^{\Sigma \ltimes \Gamma(r)}$. We then prove a universal coefficient theorem in relative operadic equivariant cohomology when the coefficients are concentrated in one arity.

7.5.1. Indecomposable object functor . We define the *indecomposable object functor*

$$\mathrm{Indec} : \mathtt{sAbOp}^\Gamma \to \mathtt{sAbSeq}^\Gamma$$

such that, for all $r \geqslant 2$ and $x \in \Gamma(r)$,

$$\mathrm{Indec}\, S(x) = \mathrm{Coker}\left(\bigoplus_{(i,x_j,x_k,\phi)\in \mathrm{dec}(x)} S(x_j) \otimes S(x_k) \xrightarrow{S(\phi)\circ(\circ_i)} S(x) \right)$$

where $\mathrm{dec}(x)$ is the set of quadruplets $(i, x_j, x_k, \phi) \in \mathbb{N} \times \Gamma(j) \times \Gamma(k) \times \mathrm{Mor}_\Gamma(x_j \circ_i x_k, x)$ with $j, k \geqslant 2$, $1 \leqslant i \leqslant j$ and $\phi : x_j \circ_i x_k \to x$.

We also define the trivial $\Gamma-$operad functor -

$$_+ : \mathtt{sAbSeq}^\Gamma \to \mathtt{sAbOp}^\Gamma$$

such that $G_+(r) = G(r)$ for all $r \geqslant 2$, $G_+(1) = \mathbb{Z}$ and such that the composition products are zero.

7.5.2. Proposition. *The indecomposable object functor* Indec *is left-adjoint to the functor* -_+*. Moreover, this adjunction is a Quillen one.*

Proof. The functor -_+ obvisously preserves fibrations and acyclic fibrations. □

7.5.3. Indecomposables of a simplical operad morphism. Let $f : P \to Q$ be a morphism in $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$.

The *indecomposables* of f form the $\Gamma-$symmetric sequence

$$\mathrm{Indec}(f) = \mathrm{Indec}(\mathrm{cofib}(\mathbb{Z}f'^\vee)),$$

where f' is a replacement of f by a cofibration between cofibrant operads. The cofiber $\mathrm{cofib}(\mathbb{Z}f'^\vee)$ of the morphism $\mathbb{Z}f'^\vee$ is the homotopy pushout of the span

$$\mathtt{I} \longleftarrow \mathbb{Z}P^\vee \overset{\mathbb{Z}f^\vee}{\rightarrowtail} \mathbb{Z}Q^\vee.$$

Recall that the $\Gamma-$operad in simplicial abelian group $\mathtt{I}$ is such that, for every $x \in \Gamma(r)$, $\mathtt{I}(x) = 0$ if $r \geqslant 2$ and $\mathtt{I}(x) = \mathbb{Z}$ if $r = 1$.

If we consider another replacement f'' of f by a cofibration between cofibrant operads, the cofibrant objects $\mathrm{cofib}(\mathbb{Z}f')$ and $\mathrm{cofib}(\mathbb{Z}f'')$, which are pushouts of cofibrant objects along a cofibration are weakly equivalent (see [26, corollary after the Theorem B]). Thus, $\mathrm{Indec}(f)$ does not depend on the choice of the replacement of f up to a weak equivalence since the indecomposable object functor is a left Quillen adjoint and therefore preserves weak equivalences between cofibrant objects.

7.5.4. Proposition. *If $f : P \to Q$ is a cofibration in $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$ with P cofibrant and $Q_\flat = P_\flat \bigvee \mathbb{F}(M)$, $M \in \mathtt{s^-Seq} \downarrow \mathtt{B\Gamma}$, then $\mathrm{cofib}(\mathbb{Z}f^\vee)_\flat = \mathbb{F}(\mathbb{Z}M^\vee)$ and $\mathrm{Indec}(f)_\flat = \mathbb{Z}M^\vee$. Moreover, since $\mathbb{F}(\mathbb{Z}M^\vee)$ is a direct summand of $\mathbb{Z}Q_\flat^\vee$, the $i-$th face of $y \in \mathrm{cofib}(\mathbb{Z}f^\vee)$ is the projection of the $i-$th face of y in $\mathbb{Z}Q^\vee$ on $\mathbb{F}(\mathbb{Z}M^\vee)$; since $\mathbb{Z}M^\vee$ is a direct summand of $\mathbb{Z}Q_\flat^\vee$, the $i-$th face of $y \in \mathrm{Indec}(f)$ is the projection of the $i-$th face of y in $\mathbb{Z}Q^\vee$ on $\mathbb{Z}M^\vee$.*

Proof. We consider the pushout diagram

$$\begin{array}{ccc} \mathbb{Z}P^\vee & \overset{\mathbb{Z}f^\vee}{\rightarrowtail} & \mathbb{Z}Q^\vee \\ \downarrow & & \vdots \\ \mathtt{I} & \dashrightarrow & \mathrm{cofib}(f). \end{array}$$

We then have $\mathbb{Z}Q_\flat^\vee = \mathbb{Z}P_\flat^\vee \bigvee \mathbb{F}(\mathbb{Z}M^\vee)$ according to Theorem 5.2.5. Thus $\mathbb{Z}Q_\flat^\vee = \mathbb{F}(M) \oplus W$ where $W \in \mathtt{s^-AbSeq^\Gamma}$ is made of the semi-alternate two-colored trees decorated by at least one element in $\mathbb{Z}P^\vee$ and elements in $\mathbb{Z}M^\vee$. Therefore, $\mathrm{cofib}(\mathbb{Z}f^\vee)_\flat = \mathbb{F}(\mathbb{Z}M^\vee)$ and the $i-$th face of $y \in \mathrm{cofib}(f)$ is defined by the projection on $\mathbb{F}(\mathbb{Z}M^\vee)$ of the $i-$th face of y in $\mathbb{Z}Q^\vee$.

Moreover, $\mathrm{Indec}(f)_\flat = \mathbb{Z}M^\vee$ and the $i-$th face of $y \in \mathrm{Indec}(f)$ is defined by the projection on $\mathbb{Z}M^\vee$ of the $i-$face of y in $\mathbb{Z}Q_\flat^\vee = \mathbb{Z}M^\vee \oplus \mathbb{Z}P_\flat^\vee \oplus \mathbb{Z}Q_{\geqslant 2}^\vee$, where $\mathbb{Z}Q_{\geqslant 2}^\vee \in \mathtt{s^-AbSeq^\Gamma}$ is made of the semi-alternate two-colored trees with at least two vertices. ☐

7.5.5. Lemma. *Let $f \in \mathrm{Mor}_{\mathtt{sOp}\downarrow\mathtt{B\Gamma}}(P, Q)$ and $r \geqslant 2$. If $f(s)$ is a weak equivalence for every $s \neq r$, then there is a replacement of f by a cofibration between cofibrant operads $\overline{f} : \overline{P} \to \overline{Q}$, in other words, a commutative diagram*

$$\begin{array}{ccccc} & & P & \xrightarrow{f} & Q \\ & & \uparrow\!\!\uparrow \sim & & \uparrow\!\!\uparrow \sim \\ \mathtt{I} & \underset{\sim}{\twoheadrightarrow} & \overline{P} & \underset{\overline{f}}{\rightarrowtail} & \overline{Q} \end{array}$$

such that $\overline{Q}_\flat = \overline{P}_\flat \bigvee \mathbb{F}(M)$ and $M \in \mathtt{s^-Seq} \downarrow \mathtt{B\Gamma}$ with $M(s) = \varnothing$ for every $s < r$.

Proof. We assume that P if cofibrant, otherwise, we replace it by a cofibrant operad by using a factorization.

We pick a factorization

$$P(r) \overset{\widetilde{f(r)}}{\rightarrowtail} \overline{Q(r)} \underset{\sim}{\overset{\overline{f(r)}}{\twoheadrightarrow}} Q(r)$$

of $f(r)$ in $\mathtt{sSet^{\Sigma(r)}} \downarrow \mathtt{B\Gamma(r)}$. By construction of such factorizations we can assume that there is $M^0 \in \mathtt{s^-Set^{\Sigma(r)}} \downarrow \mathtt{B\Gamma_\flat(r)}$ so that $\overline{Q(r)}_\flat = P(r)_\flat \cup M^0$. Let $M^0[r] \in \mathtt{sSeq^-} \downarrow \mathtt{B\Gamma_\flat}$ be such that $M^0[r](s) = \varnothing$ if $s \neq r$ and $M^0[r](r) = M^0$. Let also $\overline{Q}^0 \in \mathtt{sOp} \downarrow \mathtt{B\Gamma}$ with $\overline{Q}_\flat^0 = P_\flat \bigvee \mathbb{F}(M^0[r])$

: the faces of $\overline{Q}^0$ are defined by the faces of P and the faces of M^0 in $Q(r)$. There is now a factorization of f

$$P \overset{\tilde{f}^0}{\rightarrowtail} \overline{Q}^0 \overset{\overline{f}^0}{\twoheadrightarrow} Q$$

such that $\overline{f}^0(s) = \mathrm{id}$ if $s < r$ and $\overline{f}^0(r) = \overline{f(r)}$. Note that $\overline{f}^0$ is not an acyclic fibration, but $\overline{f}^0(s)$ is in $\mathtt{sSet} \downarrow \mathtt{B\Gamma(s)}$ for all $s \leqslant r$ since $f(s)$ is a weak equivalence for $s < r$.

Repeating this process enables us to inductively construct a sequence of operads $(\overline{Q}^n)_{n\geqslant 0} \in \mathtt{sOp} \downarrow \mathtt{B\Gamma}$ and two sequences of morphisms $(\tilde{f}^n : \overline{Q}^{n-1} \hookrightarrow \overline{Q}^n)_{n\geqslant 0}$ with $\overline{Q}^{-1} = P$ and $(\overline{f}^n : \overline{Q}^n \to \overline{Q})_{n\geqslant 0}$ such that, for all $n \geqslant 0$:

- $\overline{Q}^n \overset{\tilde{f}^{n+1}}{\rightarrowtail} \overline{Q}^{n+1} \overset{\overline{f}^{n+1}}{\twoheadrightarrow} Q$ is a factorization of $\overline{f}^n$;
- there is $M^n \in \mathtt{s^{-}Set}^{\Sigma(\mathtt{r+n})} \downarrow \mathtt{B\Gamma}_\flat(\mathtt{r+n})$ such that $Q^n_\flat = P^n_\flat \bigvee \mathbb{F}(M^n[r+n])$;
- $\overline{f}^n(s)$ is an acyclic cofibration of $\mathtt{sSet} \downarrow \mathtt{B\Gamma(s)}$ for all $s \leqslant r+n$.

Finally, let

$$\overline{Q} = \operatorname{colim}\left(\overline{Q}^0 \overset{\tilde{f}^0}{\rightarrowtail} \overline{Q}^1 \overset{\tilde{f}^1}{\rightarrowtail} \ldots \right).$$

We have $\overline{Q}_\flat = \overline{P}_\flat \bigvee \mathbb{F}(M)$ with $M(s) = \varnothing$ if $s < r$ and $M(r+t) = M^t$ for all $t \geqslant 0$. □

7.5.6. Proposition. *Let $f : P \to Q$ be a fibration in* $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$ *and $r, n \geqslant 2$. If*

- *the augmentation $\Psi_Q : Q \to B\Gamma$ induces aritywise equivalences of categories between the groupoid operads $\pi(Q)$ and $\Gamma = \pi(B\Gamma)$,*
- *f induces aritywise isomorphisms of categories between the groupoid operads $\pi(P)$ and $\pi(Q)$,*
- *$f(s)$ is a weak equivalence for every $s \neq r$, and*
- *$f(r)$ is $n-$connected,*

then :

- $\mathrm{H}_k \operatorname{Indec}(f)(s) = 0$ *for all $k \geqslant 0$ and $s < r$,*
- $\mathrm{H}_k \operatorname{Indec}(f)(r) = 0$ *for all $k \leqslant n$, and*
- $\mathrm{H}_{n+1} \operatorname{Indec}(f)(r) \simeq \pi_n(\mathrm{fib}(f(r)))$ *in* $\mathtt{Ab}^{\Sigma \ltimes \Gamma(\mathtt{r})}$.

Proof. The previous lemma provides us with a replacement of f by a cofibration between cofibrant operads $\overline{f} : \overline{P} \to \overline{Q}$ such that $\overline{Q}_\flat = \overline{P}_\flat \bigvee \mathbb{F}(M)$, $M \in \mathtt{s^{-}Seq} \downarrow \mathtt{B\Gamma}$ with $M(s) = \varnothing$ for every $s < r$.

Therefore, $\mathbb{Z}\overline{f}^\vee : \mathbb{Z}\overline{P}^\vee \to \mathbb{Z}\overline{Q}^\vee$ is a cofibration between cofibrant operads since $\text{-}^\vee$ and $\mathbb{Z}$ are Quillen left adjoint : we have $\operatorname{Indec}(\overline{f})_\flat = \mathbb{Z}M^\vee$ (see 7.5.4), hence, $\operatorname{Indec}(f)(s) \sim 0$ for all $s < r$ and $\operatorname{Indec}(f)(x) \sim \operatorname{cofib}(\mathbb{Z}\overline{f}^\vee(x))$ for all $x \in \Gamma(r)$.

We now calcutate

$$\mathrm{H}_k \operatorname{Indec}(f)(x) = \mathrm{H}_k \operatorname{cofib}(\mathbb{Z}\overline{f}^\vee(x)) = \mathrm{H}_k(\overline{Q}^\vee(x), \overline{P}^\vee(x))$$

for all $x \in \Gamma(r)$ and $k \leqslant n+1$. The simplicial $\Gamma-$operad $\overline{Q}^\vee$ is a universal covering (see Proposition 5.3.3). Therefore, the morphism $\overline{f}^\vee(x)$ is $n-$connected. With the classical relative Hurewicz theorem (see [30, Theorem IV.7.2] or [10, Corollary III.3.12]), we get

$$\mathrm{H}_k \operatorname{Indec}(f)(x) = 0$$

for all $k \leqslant n$ and

$$\mathrm{H}_{n+1} \operatorname{Indec}(f)(x) \simeq \pi_{n+1}(\overline{Q}^\vee(x), \overline{P}^\vee(x)).$$

We do not indicate the basepoint in this relative homotopy group since $\overline{Q}^\vee(x)$ is a simple space. We then have

$$\mathrm{H}_{n+1}\,\mathrm{Indec}(f)(x) \simeq \pi_{n+1}(\overline{Q}(x), \overline{P}(x), x)$$

by the naturality of the long exact sequence of homotopy groups for a pair. Thus,

$$\mathrm{H}_{n+1}\,\mathrm{Indec}(f)(x) \simeq \pi_n(\mathrm{fib}(f(x)), x)$$

where $\mathrm{fib}(f(x))$ is the fiber of $f(r)$ over x and therefore a representative of the homotopy fiber of $\overline{f}(r)$ (see [10, proof of the Corollary III.3.12]). By hypothesis, f induces an isomorphism from $\pi(P)$ to $\pi(Q)$, thus, there is only one vertex in the fiber and there is no ambiguity in the choice of a base point. These isomorphisms are natural, so we have an isomorphism

$$\mathrm{H}_{n+1}\,\mathrm{Indec}(f)(r) \simeq \pi_n(\mathrm{fib}(f(r)))$$

in $\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}$.

□

7.5.7. Proposition. *Let $f \in \mathrm{Mor}_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}(P, Q)$, $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ and $r \geqslant 2$. If $A(s) = 0$ for all $s \neq r$, then $\mathrm{H}^\bullet_\Gamma(Q, P; A) \simeq \mathrm{H}^\bullet\,\mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(\mathrm{Indec}(f)(r)_\bullet, A(r))$.*

Proof. We can consider that f is a cofibration between cofibrant operads such that $Q_\flat = P_\flat \bigvee \mathbb{F}(M)$, $M \in \mathtt{s^-Seq}\downarrow\mathtt{B}\Gamma$ with $M(s) = \varnothing$ for every $s < r$ (otherwise we can get a factorization of f of that form by Lemma 7.5.5). Let

$$C^\bullet = \widetilde{\mathrm{Mor}}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(\mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee(r)_\bullet, A(r)) \in \mathtt{dg^*Ab}$$

with

$$\widetilde{\mathrm{Mor}}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(\mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee(r)_k, A(r)) =$$
$$\{g \in \mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(\mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee(r)_k, A(r)) \mid g(\mathbb{Z}B\Gamma^\vee(r)_k) = 0\}$$

for all $k \geqslant 0$.

Since A is concentrated in arity r, there is a cochain complex isomorphism (see Theorem 7.4.5)

$$C^\bullet \simeq \widetilde{C}^\bullet_\Gamma(\mathrm{cofib}_\Gamma(f), A).$$

There is also, for all $k \geqslant 0$, an abelian group isomorphism (see Proposition 6.2.4)

$$\phi_k^* : C^k \to \mathrm{Mor}_{\Sigma\mathtt{Ab}^r}(\mathrm{Indec}(f)(r)_k, A(r))$$

induced by the embedding $\phi : \mathrm{Indec}(f) \to \mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee$, but these isomorphisms do not induce a cochain complex isomorphism in general.

We consider $g \in C^k$ and $z \in \mathbb{Z}\,\mathrm{cofib}_\Gamma^\vee(x)_{k+1}$ with $x \in \Gamma(r)$. Like in the previous proof, $\mathrm{Indec}(f)(s) = 0$ for all $s < r$. We then have $\mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee(r) = \mathrm{Indec}(f)(r) \oplus \mathbb{Z}B\Gamma^\vee(r)$, hence $\delta(\phi(z)) = z^a + z^b$ with $z^a \in \mathrm{Indec}(f)(x)_{k+1}$, $z^b \in \mathbb{Z}B\Gamma(x)_{k+1}$. Moreover, $g(z^b) = 0$ so

$$\begin{aligned}\phi(\delta^*(g))(z) &= g(\delta(\phi(z)))\\ &= g(z^a)\\ &= (\phi(g))(\delta(z))\\ &= \delta^*(\phi(g))(z).\end{aligned}$$

and ϕ^* is a cochain complex isomorphism.

□

7.5.8. Theorem (Universal coefficient)**.** *Let* $f \in \mathrm{Mor}_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}(P,Q)$, $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ *and* $r \geqslant 2$*. If* $A(s) = 0$ *for all* $s \neq r$*, then there is a first quadrant cohomology spectral sequence associated to a bicomplex*

$$\mathrm{Ext}^{l}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(\mathrm{H}^{k}\,\mathrm{Indec}(f)(r), A(r)) \Rightarrow \mathrm{H}^{k+l}_{\Gamma}(Q,P;A).$$

Proof. We consider an injective resolution

$$I^{\bullet}(A(r)) = \{I^{0}(A(r)) \to I^{1}(A(r)) \to ...\}$$

of $A(r)$. Such a resolution exists because $\mathbb{Q}/\mathbb{Z}[\Sigma(r) \ltimes \Gamma(r)]$ is an injective cogenerator in the category $\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)} \simeq \mathtt{Ab}^{\Sigma(s)\ltimes\Gamma(r)}$. Let

$$B^{l,k} = \mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(\mathrm{Indec}(f)_{k}(r), I^{l}A(r))$$

be a bicomplex. The associated spectral sequence $B_{p}^{\bullet,\bullet}$ converges because this bicomplex is zero out of the first quadrant. On one hand, since $\mathrm{Indec}(f)$ is cofibrant, $\mathrm{Indec}(f)(r)_{k}$ is projective according to Proposition 6.1.4. As a consequence, the functor

$$\mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(\mathrm{Indec}(f)(r)_{k}, -) : \mathtt{dg}^{*}\mathtt{Ab}^{\Sigma\times\Gamma(r)} \to \mathtt{dg}^{*}\mathtt{Ab}$$

then preserves weak equivalence. Thus we have

$$\begin{cases} {}^{II}B_{1}^{l,k} & = 0 & \text{if } l \neq 0, \\ {}^{II}B_{1}^{0,k} & = \mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}\big(\mathrm{Indec}(f)(r)_{k}, A(r)\big). \end{cases}$$

Thereafter,

$$B_{p}^{l,k} \Rightarrow \mathrm{H}^{k+l}\,\mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(s)}}(\mathrm{Indec}(f)(r)_{\bullet}, A(r)) \simeq \mathrm{H}^{k+l}(Q,P;A)$$

by the previous proposition. □

Similarly, if $P \in \mathtt{sOp} \downarrow \mathtt{B}\Gamma$ is cofibrant and A is concentrated in arity r, then

$$\mathrm{H}^{\bullet}_{\Gamma}(P;A) \simeq \mathrm{H}^{\bullet}\,\mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(\mathrm{Indec}(\mathbb{Z}P^{\vee})(r)_{\bullet}, A(r))$$

and there is a first quadrant cohomology spectral sequence

$$\mathrm{Ext}^{l}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(\mathrm{H}^{k}\,\mathrm{Indec}(\mathbb{Z}P^{\vee})(r), A(r)) \Rightarrow \mathrm{H}^{k+l}_{\Gamma}(P;A),$$

but we do not need these results in this thesis.

8. Operadic Postnikov invariants

We introduce the Postnikov invariant of a simplicial operad morphism and the Postnikov invariants of a simplicial operad. In contrast with the Postnikov invariants of a simplicial set that classify the stage of a (simple) tower, the Postnikov invariants of a simplicial operads classify the stages of a double tower (see 1.4.6). We explain the necessity of this double tower in appendix A.

8.1. Postnikov invariant of a simplicial operad morphism

To define the Postnikov invariant of f, we first analyze the long exact sequence of relative operadic equivariant cohomology associated to f with coefficients concentrated in one arity. We take this assumption so that we can use the universal coefficient Theorem 7.5.8.

8.1.1. Proposition. *Let $f \in \mathrm{Mor}_{\mathtt{sOp}\downarrow B\Gamma}(P,Q)$ and $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ concentrated in an arity r with $r \geqslant 2$.*

If $\mathrm{H}_k \operatorname{Indec}(f)(r) = 0$ for all $k \leqslant n$, then $\mathrm{H}^k_\Gamma(P;A) \simeq \mathrm{H}^k_\Gamma(Q;A)$ for all $k < n$ and there is an exact sequence

$$0 \longrightarrow \mathrm{H}^n_\Gamma(Q;A) \xrightarrow{f^*} \mathrm{H}^n_\Gamma(P;A) \longrightarrow \mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(\mathrm{H}_{n+1}\operatorname{Indec}(f)(r), A(r))$$

$$\xrightarrow{d_f} \mathrm{H}^{n+1}_\Gamma(Q;A) \xrightarrow{f^*} \mathrm{H}^{n+1}_\Gamma(P,A)$$

natural in f.

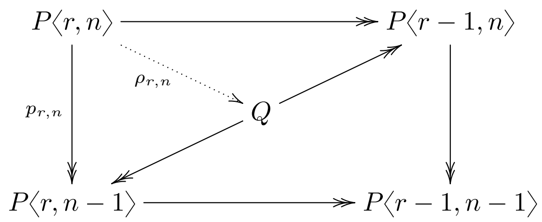
Proof. This proposition follows from Theorem 7.5.8 and of the long exact sequence in relative operadic cohomology (see Proposition 7.4.3). To get our exact sequence, we just use that the lines from 0 to n of the second page of the spectral sequence vanish and the term of coordinates $(0, n+1)$ is $\mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(H_{n+1}\operatorname{Indec}(f)(r), A(r))$. □

8.1.2. Postnikov invariant of a simplicial operad morphism. Let $f \in \mathrm{Mor}_{\mathtt{sOp}\downarrow B\Gamma}(P,Q)$, $r, n \geqslant 2$ be such that $\mathrm{H}_k \operatorname{Indec}(f)(r) = 0$ for all $k \leqslant n$ and let $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ concentrated in arity r with $A(r) = \mathrm{H}_{n+1}\operatorname{Indec}(f)(r)$. The *Postnikov invariant* (or $k-$invariant) of f is the image in $\mathrm{H}^{n+1}_\Gamma(Q;A)$ of $\mathrm{id} \in \mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(A(r), A(r))$ under the morphism d_f of Proposition 8.1.1. We denote this invariant by $k(f)$. If we moreover assume that Q is cofibrant, we also denote by $k(f)$ a representative of the homotopy class of morphisms in $\left[Q, \operatorname{hocolim}_\Gamma K(A, n+1)\right]_{\mathtt{sOp}\downarrow B\Gamma}$ associated to this cocycle by the isomorphism of Theorem 7.2.3.

8.2. Postnikov invariants of a simplicial operad

Let P be a fibrant operad in $\mathtt{sOp}$. Let also $r, n \geqslant 2$. We denote $\Gamma_{\leqslant r}$ the $r-$cotrunctation $\mathrm{coar}_{\leqslant r}(\Gamma)$ of the fundamental groupoid operad Γ of P. We have $\pi(\mathrm{coar}_{\leqslant r}(P)) = \mathrm{coar}_{\leqslant r}(\pi(P))$.

We examine the commutative diagram in $\mathtt{sOp}$



extracted from the double Postnikov tower of P. The simplicial operad Q is the pullback of the lower-right corner of this square with

$$Q(s) = \begin{cases} P\langle n\rangle(s) & \text{if } s<r, \\ P\langle n-1\rangle(r) & \text{if } s=r, \\ pt & \text{otherwise.} \end{cases}$$

The filling morphism $\rho_{r,n}$ of this pullback is such that

$$\rho_{r,n}(s) = \begin{cases} p_{r,n}(r) & \text{if } r = s, \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

and is therefore an $n-$connected fibration.

8.2.1. Proposition. *The morphism $\rho_{r,n}$ induces an isomorphism between the fundamental groupoid operads of $P\langle r,n\rangle$ and Q, which are both equals to $\Gamma_{\leqslant r}$. Therefore, $\rho_{r,n}$ is a morphism in $\mathtt{sOp} \downarrow \mathtt{B}\Gamma_{\leqslant \mathtt{r}}$ when we set the augmentation morphism $\Psi_Q : Q \to B\Gamma_{\leqslant r}$ as the canonical fibration (see Definition 5.3.2).*

Moreover, $\mathrm{H}_k(\mathrm{Indec}(\rho_{r,n})(r)) = 0$ *for every $k \leqslant n$.*

Proof. The first point is straighforward since $P\langle r,n\rangle$, Q and $\mathrm{coar}_{\leqslant r} P$ have the same 2-skeleton.

The second point is a direct consequence of Proposition 7.5.6 since $\pi_n(\mathrm{fib}(\rho_{r,n}(r))) = \pi_n(P(r))$. □

8.2.2. Postnikov invariant of a simplicial operad. The $(r,n)-$th *Postnikov invariant* of P is the Postnikov invariant of the filling morphism $\rho_{r,n} : P\langle r,n\rangle \to Q$ with

$$Q = P\langle r, n-1\rangle \underset{P\langle r-1,n-1\rangle}{\times} P\langle r-1, n\rangle.$$

Explicitly, it is the image in $\mathrm{H}^{n+1}_{\Gamma_{\leqslant r}}\big(Q, \pi_n(P(r))\big)$ of $\mathrm{id} \in \mathrm{Hom}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}\big(\pi_n(P(r)), \pi_n(P(r))\big)$ under the morphism $d_{\rho_{r,n}}$ of Lemma 8.1.1, where $A = \pi_n(P(r))$ is the additive $\Gamma-$operad concentrated in arity r. We denote this invariant by $k_{r,n}$.

If we replace $\rho_{r,n}$ by a morphism between cofibrant operads $\overline{\rho_{r,n}} : \overline{P} \to \overline{Q}$, we also denote by $k_{r,n}$ a representative of the homotopy class of morphisms in

$$\big[\overline{Q}, \mathrm{hocolim}_{\Gamma_{\leqslant r}} K(\pi_n(P(r)), n+1)\big]_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma_{\leqslant \mathtt{r}}}$$

associated to the cocyle $k_{r,n} \in H^{n+1}_{\Gamma_{\leqslant r}}(\overline{Q}, \pi(P)(r))$ by the isomorphism of Theorem 7.2.3.

9. Comparison theorems

We now compare the operadic equivariant cohomology of a given simplicial operad and its equivariant cohomology calculated aritywise. We also compare the representation of the operadic equivariant cohomology and the representation of the equivariant cohomology calculated aritywise. The results permit us to relate the operadic Postnikov invariants with the Postnikov invariants calculated in one arity, as well as their representations.

9.1. Comparison of the equivariant cohomologies

We establish the existence of a comparison morphism $\mathrm{ar}_r : \mathrm{H}^n_\Gamma(P;A) \to \mathrm{H}^n_\Gamma(P(r);A(r))$ for all $P \in \mathtt{sOp} \downarrow \mathtt{B\Gamma}$, $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$, $n \geqslant 0$ and $r \geqslant 1$. We also prove that there exists under technical assumptions a similar comparison morphism for relative equivariant cohomology.

9.1.1. Proposition. *Let $P \in \mathtt{sOp} \downarrow \mathtt{B\Gamma}$, $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ and $r \geqslant 1$. The functor*

$$\begin{array}{rccc} \mathrm{ar}_r : & \mathtt{sOp} \downarrow \mathtt{B\Gamma} & \longrightarrow & \mathtt{sSet} \downarrow \mathtt{B\Gamma(r)} \\ & P & \longmapsto & P(r) \end{array}$$

induces, for any $P \in \mathtt{sOp} \downarrow \mathtt{B\Gamma}$, a cosimplicial abelian group morphism

$$\begin{array}{rccc} \mathrm{ar}_r^C(P) : & C^\bullet_\Gamma(P;A) & \longrightarrow & C^\bullet_\Gamma(P(r);A(r)) \\ & f \in C^n_\Gamma(P;A) & \longmapsto & \mathrm{ar}_r(f) \end{array}.$$

This morphism induces in turn, for all $n \geqslant 0$, abelian group morphisms

$$\mathrm{ar}_r^{H^n}(P) : \quad \mathrm{H}^n_\Gamma(P;A) \quad \longrightarrow \quad \mathrm{H}^n_\Gamma(P(r);A(r))$$

so that ar_r^C is a natural transformation from the functor

$$C^\bullet_\Gamma(-;A) : \mathtt{sOp} \downarrow \mathtt{B\Gamma} \longrightarrow \mathtt{cAb}$$

to the functor

$$C^\bullet_{\Gamma(r)}(-;A) \circ \mathrm{ar}_r : \mathtt{sOp} \downarrow \mathtt{B\Gamma} \longrightarrow \mathtt{cAb}$$

and such that, for all $n \geqslant 0$, ar_r^H is a natural transformation from the functor

$$\mathrm{H}^n_\Gamma(-;A) : \mathtt{sOp} \downarrow \mathtt{B\Gamma} \longrightarrow \mathtt{Ab}$$

to the functor

$$\mathrm{H}^n_{\Gamma(r)}(-;A) \circ \mathrm{ar}_r : \mathtt{sOp} \downarrow \mathtt{B\Gamma} \longrightarrow \mathtt{Ab}.$$

To reduce the amount of notation, we denote by ar_r both natural transformations ar_r^C and $\mathrm{ar}_r^{H^n}$.

Proof. The morphism $\mathrm{ar}_r^C(P)$ is indeed a cosimplicial abelian group morphism since, for all $0 \leqslant i \leqslant n$, we have $d_i(\mathrm{ar}_r(f)) = \mathrm{ar}_r(d_i(f))$ and $s_i(\mathrm{ar}_r(f)) = \mathrm{ar}_r(s_i(f))$.

The rest of the proof is immediate. □

9.1.2. Proposition. *Let $f : P \to Q$ be a morphism in $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$ and $r \geqslant 2$ such that $f(s)$ is a weak equivalence for every $s \neq r$. There are morphisms*

$$\mathrm{ar}_r(Q,P) : \mathrm{H}^n_\Gamma(Q,P;A) \longrightarrow \mathrm{H}^n_{\Gamma(r)}(Q(r),P(r);A(r))$$

such that the diagram

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & \mathrm{H}^n_\Gamma(Q,P;A) & \longrightarrow & \mathrm{H}^n_\Gamma(Q;A) & \xrightarrow{f^*} & \mathrm{H}^n_\Gamma(P;A) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle \mathrm{ar}_r(Q,P)} & & \downarrow{\scriptstyle \mathrm{ar}_r(Q)} & & \downarrow{\scriptstyle \mathrm{ar}_r(P)} & & \\
\dots & \longrightarrow & \mathrm{H}^n_{\Gamma(r)}(Q(r),P(r);A(r)) & \longrightarrow & \mathrm{H}^n_{\Gamma(r)}(Q(r);A(r)) & \xrightarrow{f(r)^*} & \mathrm{H}^n_{\Gamma(r)}(P(r);A(r)) & \longrightarrow & \dots
\end{array}$$

commutes.

Proof. Lemma 7.5.5 provides us with a factorization of f

$$P \overset{\overline{f}}{\rightarrowtail} \overline{Q} \overset{\mu}{\underset{\sim}{\twoheadrightarrow}} Q$$

such that $\overline{Q}$ is cofibrant, $\overline{f}$ is a cofibration and $\overline{Q} = \overline{P} \bigvee \mathbb{F}(M)$, $M \in \mathtt{s^{-}Seq} \downarrow \mathtt{B\Gamma_\flat}$ with $M(s) = \varnothing$ for all $s < r$. We then have a factorization

$$P(r) \overset{\overline{f}(r)}{\rightarrowtail} \overline{Q}(r) \overset{\mu(r)}{\underset{\sim}{\twoheadrightarrow}} Q(r)$$

of $f(r)$. We use $\overline{f}$ to define $C^\bullet_\Gamma(Q,P;A)$ and $\overline{f}(r)$ to define $C^\bullet_{\Gamma(r)}(Q(r),P(r);A(r))$ (see [10, definition after Lemma VI.4.1]). By construction,

$$\mathrm{ar}_r\left(\begin{array}{ccc}
P & \overset{\overline{f}}{\rightarrowtail} & \overline{Q} \\
\downarrow & \overset{\Phi_{\mathrm{cofib}_\Gamma(\overline{f})}}{\swarrow} & \downarrow \\
B\Gamma & \overset{c}{\dashrightarrow} & \mathrm{cofib}_\Gamma(\overline{f})
\end{array}\right) = \begin{array}{ccc}
P(r) & \overset{\overline{f}(r)}{\rightarrowtail} & \overline{Q}(r) \\
\downarrow & \overset{\Phi_{\mathrm{cofib}_{\Gamma(r)}(\overline{f}(r))}}{\swarrow} & \downarrow \\
B\Gamma(r) & \overset{c(r)}{\dashrightarrow} & \mathrm{cofib}_{\Gamma(r)}(\overline{f}(r))
\end{array}.$$

The morphism

$$\mathrm{ar}_r\left(\mathrm{cofib}_\Gamma(\overline{f})\right) : C^\bullet_\Gamma\left(\mathrm{cofib}_\Gamma(\overline{f});A\right) \longrightarrow C^\bullet_{\Gamma(r)}\left(\mathrm{cofib}_{\Gamma(r)}(\overline{f}(r));A(r)\right)$$

induces a morphism

$$\alpha_r(\overline{f}) : \widetilde{C}^\bullet_\Gamma\left(\mathrm{cofib}_\Gamma(\overline{f});A\right) \longrightarrow \widetilde{C}^\bullet_{\Gamma(r)}\left(\mathrm{cofib}_{\Gamma(r)}(\overline{f}(r));A(r)\right)$$

because for all $g \in C^n_\Gamma(\mathrm{cofib}_\Gamma(\overline{f});A)$ such that $g \circ c = 0$, we have $\mathrm{ar}_r(g) \circ c(r) = 0$. Note that this morphism does not depend on the choice of the factorization of f up to homotopy and therefore induces for all $n \geqslant 0$ morphisms

$$\mathrm{ar}_r(Q,P) : H^n_\Gamma(Q,P;A) \longrightarrow H^n_{\Gamma(r)}(Q(r),P(r);A(r)).$$

We can therefore conclude since there is a short exact sequence morphism

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{C}^\bullet_\Gamma(\mathrm{cofib}_\Gamma(\overline{f});A) & \longrightarrow & C^\bullet_\Gamma(\overline{Q};A) & \xrightarrow{\overline{f}^*} & C^n_\Gamma(P;A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_r(\overline{f})} & & \downarrow{\scriptstyle \mathrm{ar}_r(\overline{Q})} & & \downarrow{\scriptstyle \mathrm{ar}_r(P)} & & \\
0 & \longrightarrow & \widetilde{C}^\bullet_{\Gamma(r)}(\mathrm{cofib}_{\Gamma(r)}(\overline{f}(r));A(r)) & \longrightarrow & C^\bullet_{\Gamma(r)}(\overline{Q}(r);A(r)) & \xrightarrow{\overline{f}(r)^*} & C^\bullet_{\Gamma(r)}(P(r);A(r)) & \longrightarrow & 0.
\end{array}$$

□

9.2. Comparison of the representations of the equivariant cohomologies

Recall that the $n-$th group of operadic equivariant cohomology of a cofibrant simplicial operad P with coefficients in an additive operad A is isomorphic to a group of homotopy classes of morphisms from $P^\vee$, the universal covering operad of P, to $K(A,n)$, an Eilenberg-MacLane $\Gamma-$operad (see Theorem 7.2.3). There is a similar result for the groups of equivariant cohomology of a simplicial set (see [10, Theorem VI.3.10]). We therefore compare the representation of the operadic equivariant cohomology and the representation of the equivariant cohomology calculated aritywise.

9.2.1. Lemma. *Let $P \in \mathtt{sOp} \downarrow \mathtt{B\Gamma}$ and $Q \in \mathtt{sOp}^\Gamma$. The diagram*

$$\begin{array}{ccc}
\mathrm{Mor}_{\mathtt{sOp}\downarrow\mathtt{B\Gamma}}(P, \mathrm{hocolim}_\Gamma Q) & \xrightarrow{\simeq} & \mathrm{Mor}_{\mathtt{sOp}^\Gamma}(P^\vee, Q) \\
\downarrow{\scriptstyle \mathrm{ar}_r} & & \downarrow{\scriptstyle \mathrm{ar}_r^\Gamma} \\
\mathrm{Mor}_{\mathtt{sSet}\downarrow\mathtt{B\Gamma(r)}}(P(r), \mathrm{hocolim}_{\Gamma(r)} Q(r)) & \xrightarrow[\simeq]{} & \mathrm{Mor}_{\mathtt{sSet}^{\Gamma(r)}}(P^\vee(r), Q(r))
\end{array}$$

commutes, where the morphism ar_r^Γ is induced by the obvious functor

$$\begin{array}{ccc}
\mathtt{sOp}^\Gamma & \longrightarrow & \mathtt{sSet}^{\Gamma(r)} \\
Q & \longmapsto & Q(r).
\end{array}$$

Proof. We identify the diagram of our lemma with the external of the diagram

$$\begin{array}{ccc}
\mathrm{Mor}_{\mathtt{sOp}\downarrow\mathtt{B\Gamma}}(P, \mathrm{hocolim}_\Gamma Q) & \xrightarrow{\simeq} & \mathrm{Mor}_{\mathtt{sOp}^\Gamma}(P^\vee, Q) \\
\downarrow & & \downarrow \\
\mathrm{Mor}_{\mathtt{sSeq}\downarrow\mathtt{B\Gamma}}(P, \mathrm{hocolim}_\Gamma Q) & \xrightarrow{\simeq} & \mathrm{Mor}_{\mathtt{sSeq}^\Gamma}(P^\vee, Q) \\
\downarrow & & \downarrow \\
\mathrm{Mor}_{\mathtt{sSet}\downarrow\mathtt{B\Gamma(r)}}(P(r), \mathrm{hocolim}_{\Gamma(r)} Q(r)) & \xrightarrow[\simeq]{} & \mathrm{Mor}_{\mathtt{sSet}^{\Gamma(r)}}(P^\vee(r), Q(r))
\end{array}$$

where the horizontal isomorphisms are the natural isomorphisms of the adjunctions between the functors $-^\vee$ and hocolim at the operad, the symmetric sequence and the simplicial set level. The top square obviously commutes (see proof of Theorem 5.2.4). We check that the bottom square commutes too (see Theorem 5.1.3 and [10, Lemma VI.3.6] for an explicit definition of the natural isomorphisms). □

9.2.2. Lemma. *Let $P \in \mathtt{sOp} \downarrow \mathtt{B\Gamma}$ cofibrant and $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$. The diagram in* $\mathtt{cAb}$

$$\begin{array}{ccc}
\mathrm{Mor}_{\mathtt{SetOp}\downarrow\mathtt{B\Gamma_\bullet}}(P_\bullet, \mathrm{hocolim}_\Gamma A_\bullet) & \xrightarrow[(1)]{\simeq} & \mathrm{Der}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}P_\bullet^\vee, A) \\
\downarrow{\scriptstyle \mathrm{ar}_r} & & \downarrow{\scriptstyle \mathrm{ar}_r^\Gamma} \\
\mathrm{Mor}_{\mathtt{Set}\downarrow\mathtt{B\Gamma_\bullet(r)}}(P_\bullet(r), \mathrm{hocolim}_{\Gamma(r)} A_\bullet(r)) & \xrightarrow[\simeq]{(2)} & \mathrm{Mor}_{\mathtt{Ab}^{\Gamma(r)}}(\mathbb{Z}P(r)_\bullet^\vee, A(r))
\end{array}$$

commutes. The isomorphism (1) *is defined in Lemma 7.1.4 and the isomorphism* (2) *is the counterpart of* (1) *in the non-operadic setting. Explicitly,* (2) *is the composite of the isomorphism*

$$\mathrm{Mor}_{\mathtt{Set}\downarrow\mathtt{B\Gamma_\bullet(r)}}(P_\bullet(r), \mathrm{hocolim}_{\Gamma(r)} A_\bullet(r)) \to \mathrm{Mor}_{\mathtt{Set}^{\Gamma(r)}}(P(r)_\bullet^\vee, A(r))$$

defined in [10, Lemma VI.3.10] and of the obvious isomorphism

$$\mathrm{Mor}_{\mathtt{Set}^{\Gamma(r)}}(P(r)_\bullet^\vee, A(r)) \to \mathrm{Mor}_{\mathtt{Ab}^{\Gamma(r)}}(\mathbb{Z}P(r)_\bullet^\vee, A(r)).$$

Proof. Similar to the proof of the previous lemma. □

9.2.3. Theorem (Comparison of the representations)**.** *Let $P \in \mathtt{sOp} \downarrow B\Gamma$ cofibrant, $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ and $n \geqslant 0$. The diagram*

$$\begin{array}{ccc}
\mathrm{H}^n_\Gamma(P, A) & \xrightarrow{\mathrm{ar}_r} & \mathrm{H}^n_{\Gamma(r)}(P(r), A(r)) \\
\downarrow{\scriptstyle \simeq} & (*) & \downarrow{\scriptstyle \simeq} \\
\big[P, \mathrm{hocolim}_\Gamma K(A,n)\big]_{\mathtt{sOp}\downarrow\mathtt{B\Gamma}} & \xrightarrow[\mathrm{Ho}\,\mathrm{ar}_r]{} & \big[P(r), \mathrm{hocolim}_\Gamma K(A(r),n)\big]_{\mathtt{sSet}\downarrow\mathtt{B\Gamma(r)}}
\end{array}$$

commutes.

Proof. We review the proof of Theorem 7.2.3 step by step to insure that this diagram commutes. We check that our vertical mapping correspond to each other when we forget about operad structures and focus on $\Gamma-$symmetric sequence maps as in Theorem 7.2.3. We check that this correspondence go through the restriction to operadic derivation.

The verification of the first step is given by the commutativity of the diagram

$$\begin{array}{ccc}
Z^n \mathrm{N}^* \operatorname{Mor}_{\mathtt{AbSeq}^\Gamma}(\mathbb{Z}P^\vee_\bullet, A) & \xrightarrow[\mathrm{ar}_r^\Gamma]{} & Z^n \mathrm{N}^* \operatorname{Mor}_{\mathtt{Ab}^{\Gamma(r)}}(\mathbb{Z}P^\vee(r)_\bullet, A) \\
\downarrow \simeq & (a) & \downarrow \simeq \\
\operatorname{Mor}_{\mathtt{dg}_*\mathtt{AbSeq}^\Gamma}(\mathrm{N}_*\mathbb{Z}P^\vee, A[n]) & \xrightarrow[\mathrm{ar}_r^\Gamma]{} & \operatorname{Mor}_{\mathtt{dg}_*\mathtt{Ab}^{\Gamma(r)}}(\mathrm{N}_*\mathbb{Z}P^\vee(r), A(r)[n]) \\
\downarrow \simeq & (b) & \downarrow \simeq \\
\operatorname{Mor}_{\mathtt{sAbSeq}^\Gamma}(\mathbb{Z}P^\vee, K(A,n)) & \xrightarrow[\mathrm{ar}_r^\Gamma]{} & \operatorname{Mor}_{\mathtt{sAb}^{\Gamma(r)}}(\mathbb{Z}P^\vee(r), K(A(r),n)) \\
\downarrow \simeq & (c) & \downarrow \simeq \\
\operatorname{Mor}_{\mathtt{sSeq}^\Gamma}(P^\vee, K(A,n)) & \xrightarrow[\mathrm{ar}_r^\Gamma]{} & \operatorname{Mor}_{\mathtt{sSet}^{\Gamma(r)}}(P^\vee(r), K(A(r),n)).
\end{array}$$

The isomorphisms in the left column of this diagram are described in the first step of the proof of Theorem 7.2.3. Since these isomorphisms are defined pointwise, we can define the isomorphisms in the right columns in the same way, so that the squares (a), (b) and (c) commutes.

The second step consist in the proof that the diagram

$$\begin{array}{ccc}
Z^n \mathrm{N}^* C^\bullet_\Gamma(P, A) & \xrightarrow{\mathrm{ar}_r} & Z^n \mathrm{N}^* C^\bullet_{\Gamma(r)}(P(r), A(r)) \\
\downarrow \simeq & (d) & \downarrow \simeq \\
Z^n \mathrm{N}^* \operatorname{Der}_{\mathtt{sAbOp}^\Gamma}(\mathbb{Z}P^\vee_\bullet, A) & \xrightarrow[\mathrm{ar}_r^\Gamma]{} & Z^n \mathrm{N}^* \operatorname{Mor}_{\mathtt{sAb}^{\Gamma(r)}}(\mathbb{Z}P^\vee(r)_\bullet, A(r)) \\
\downarrow \simeq & (e) & \downarrow \simeq \\
\operatorname{Mor}_{\mathtt{sOp}^\Gamma}(P^\vee, K(A,n)) & \xrightarrow[\mathrm{ar}_r^\Gamma]{} & \operatorname{Mor}_{\mathtt{sSet}^{\Gamma(r)}}\left(P^\vee(r), K(A(r),n)\right) \\
\downarrow \simeq & (f) & \downarrow \simeq \\
\operatorname{Mor}_{\mathtt{sOp}\downarrow B\Gamma}\left(P, \operatorname{hocolim}_\Gamma K(A,n)\right) & \xrightarrow[\mathrm{ar}_r]{} & \operatorname{Mor}_{\mathtt{sSet}\downarrow B\Gamma(r)}\left(P(r), \operatorname{hocolim}_\Gamma K(A(r),n)\right)
\end{array}$$

commutes. The squares (d) and (f) commute according to Lemma 9.2.1 and 9.2.2. The commutativity of the diagram in the first step of this proof and the second step of the proof of Theorem 7.2.3 imply that the square (e) commutes.

Finally, the previous diagram induces the diagram $(*)$ according to the third step of the proof of Theorem 7.2.3 and because the path object $A[n]^I$ is defined pointwise (see Proposition 4.3.4). Therefore, $(*)$ commutes. □

9.3. Comparison of the Postnikov invariants

We now prove the main result of this section : the comparison morphism ar_r between the operadic equivariant cohomology and the equivariant cohomology calculated in arity r (see 9.1.2) maps the operadic Postnikov invariant of a simplicial operad morphism f to the Postnikov invariant of the simplicial morphism $f(r)$. Moreover, this result holds for their representations.

9.3.1. Lemma. *Let $f \in \mathrm{Mor}_{\mathtt{sOp}\downarrow\mathtt{B\Gamma}}(P, Q)$, $r, n \geqslant 2$ such that $f(s)$ is a weak equivalence for every $r \neq s$ and $\mathrm{H}_k \operatorname{Indec}(f)(r) = 0$ for all $k \leqslant n$. Let also $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ concentrated in arity r with $A(r) = \mathrm{H}_{n+1} \operatorname{Indec}(f)(r)$. The diagram at page 66 commutes. The two lines are exact, and the bottom line is the counterpart of the top line in the simplicial set setting (see [10, Lemma VI.5.4]). The morphism w_r is induced by the forgetful functor $\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)} \to \mathtt{Ab}^{\Gamma(r)}$ and is injective.*

Proof. According to Proposition 9.1.2, we only have to prove that the square

$$\begin{array}{ccc}
\mathrm{Mor}_{\mathtt{AbSeq(r)}^{\Gamma(r)}}(A(r), A(r)) & \xrightarrow{\ w_r\ } & \mathrm{Mor}_{\mathtt{Ab}^{\Gamma(r)}}(A(r), A(r)) \\
\simeq \downarrow & & \downarrow \simeq \\
H^{n+1}_{\Gamma}(Q, P; A) & \xrightarrow{\ \mathrm{ar}_r(Q,P)\ } & H^{n+1}_{\Gamma}(Q(r), P(r); A(r))
\end{array}$$

commutes. We decompose this square into a diagram of the form

$$\begin{array}{ccc}
\mathrm{Mor}_{\mathtt{AbSeq(r)}^{\Gamma(r)}}(A(r), A(r)) & \xrightarrow{\ w_r\ } & \mathrm{Mor}_{\mathtt{Ab}^{\Gamma(r)}}(A(r), A(r)) \\
\simeq \downarrow & (a) & \downarrow \simeq \\
H_{n+1} \mathrm{Mor}_{\mathtt{AbSeq(r)}^{\Gamma(r)}}(\operatorname{Indec}(f)(r)_\bullet, A(r)) & \xrightarrow{\ \mathrm{H}_{n+1}(w_r)\ } & H_{n+1} \mathrm{Mor}_{\mathtt{Ab}^{\Gamma(r)}}(\operatorname{Indec}(f)(r)_\bullet, A(r)) \\
\downarrow \simeq & (b) & \downarrow \simeq \\
H^{n+1}_{\Gamma}(Q, P; A) & \xrightarrow{\ \mathrm{ar}_r(Q,P)\ } & H^{n+1}_{\Gamma}(Q(r), P(r); A(r))
\end{array}$$

and we check the commutativity of the subdiagrams (a) and (b).

Commutativity of (a) : We compare the spectral sequences associated to the double complexes

$$E^{k,l} = \mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(\operatorname{Indec} f(r)_k, I^l A(r))$$

and

$$F^{k,l} = \mathrm{Mor}_{\mathtt{Ab}^{\Gamma(\mathtt{r})}}(\mathrm{Indec}\, f(r)_k, I^l A(r))$$

with $k, l \geqslant 0$ and $I^\bullet A$ an injective resolution of $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ and $\phi : I^\bullet A \xrightarrow{\sim} A$. Note that $I^\bullet A(r)$ is an injective resolution in $\mathtt{Ab}^{\Sigma \ltimes \Gamma(\mathtt{r})}$ and $\mathtt{Ab}^{\Gamma(\mathtt{r})}$.

We first analyze the column filtration. For all $k \geqslant 0$ $\mathrm{Indec}(f)(r)_k$ is projective in $\mathtt{Ab}^{\Sigma \ltimes \Gamma(\mathtt{r})}$ since $\mathrm{Indec}(f)$ is cofibrant, and projective in $\mathtt{Ab}^{\Gamma(\mathtt{r})}$ because it is free abelian. Therefore, $\mathrm{Mor}_{\mathtt{Ab}^{\Sigma \ltimes \Gamma(\mathtt{r})}}(\mathrm{Indec}(f)(r)_k, -)$ and $\mathrm{Mor}_{\mathtt{Ab}^{\Gamma(\mathtt{r})}}(\mathrm{Indec}(f)(r)_k, -)$ are exact functors. The diagram

$$\begin{array}{ccc}
{}^{I}E_0^{\bullet,l} & \xrightarrow[\sim]{\phi_*(r)} & \mathrm{Mor}_{\mathtt{Ab}^{\Sigma \ltimes \Gamma(\mathtt{r})}}(\mathrm{Indec}(f)(r)_k, A(r)) \\
\downarrow{\scriptstyle w_r} & & \downarrow{\scriptstyle w_r} \\
{}^{I}F_0^{\bullet,l} & \xrightarrow[\sim]{w_r\phi_*(r)} & \mathrm{Mor}_{\mathtt{Ab}^{\Gamma(\mathtt{r})}}(\mathrm{Indec}(f)(r)_k, A(r))
\end{array}$$

commutes for all $l \geqslant 0$ in $\mathtt{dg}_*\mathtt{Ab}$ since w_r is natural and ϕ and w preserve differentials. The diagram

$$\begin{array}{ccc}
{}^{I}E_1^{k,l} & \xrightarrow[\simeq]{\mathrm{H}^*\,\phi_*(r)} & \delta_{0,l}\,\mathrm{Mor}_{\mathtt{Ab}^{\Sigma \ltimes \Gamma(\mathtt{r})}}(\mathrm{Indec}(f)(r)_\bullet, A(r)) \\
\downarrow{\scriptstyle \mathrm{H}^*\,w_r} & & \downarrow{\scriptstyle \mathrm{H}^*\,w_r} \\
{}^{I}F_0^{k,l} & \xrightarrow[\simeq]{H^*w_r\phi_*(r)} & \delta_{0,l}\,\mathrm{Mor}_{\mathtt{Ab}^{\Gamma(\mathtt{r})}}(\mathrm{Indec}(f)(r)_\bullet, A(r))
\end{array}$$

commutes in $\mathtt{Ab}$ for all $k, l \geqslant 0$ since the first page of the spectral sequences is zero out of the zeroth row and where $\delta_{0,l}M$ denotes the bigraded object concentrated in degree $(0, l)$ and equal to M. Thus,

$$\begin{array}{ccc}
\mathrm{H}^{n+1}\,\mathrm{Tot}\,E_{\bullet,\bullet} & \xrightarrow[\simeq]{\mathrm{H}^{n+1}\,\phi_*(r)} & \mathrm{Mor}_{\mathtt{Ab}^{\Sigma \ltimes \Gamma(\mathtt{r})}}(\mathrm{Indec}(f)(r)_\bullet, A(r)) \\
\downarrow{\scriptstyle \mathrm{H}^{n+1}\,w_r} & (a') & \downarrow{\scriptstyle \mathrm{H}^{n+1}\,w_r} \\
\mathrm{H}^{n+1}\,\mathrm{Tot}\,F_{\bullet,\bullet} & \xrightarrow[\simeq]{\mathrm{H}^{n+1}\,w_r\phi_*(r)} & \mathrm{Mor}_{\mathtt{Ab}^{\Gamma(\mathtt{r})}}(\mathrm{Indec}(f)(r)_\bullet, A(r))
\end{array}$$

commutes.

Wer now analyze the row filtration. $\mathrm{Mor}_{\mathtt{Ab}^{\Sigma \ltimes \Gamma(\mathtt{r})}}(-, I^l A(r))$ and $\mathrm{Mor}_{\mathtt{Ab}^{\Gamma(\mathtt{r})}}(-, I^l A(r))$ are exact since $I^l A(r)$ is injective in $\mathtt{Ab}^{\Sigma \ltimes \Gamma(\mathtt{r})}$ and $\mathtt{Ab}^{\Gamma(\mathtt{r})}$. We have the commutative diagram

$$\begin{array}{ccc}
{}^{II}E_1^{k,\bullet} = \mathrm{H}^k\,\mathrm{Mor}_{\mathtt{Ab}^{\Sigma \ltimes \Gamma(\mathtt{r})}}(\mathrm{Indec}(f)(r)_*, I^\bullet A(r)) & \xrightarrow[\simeq]{\nu} & \mathrm{Mor}_{\mathtt{Ab}^{\Sigma \ltimes \Gamma(\mathtt{r})}}(\mathrm{H}_k\,\mathrm{Indec}(f)(r)_*, I^\bullet A(r)) \\
\downarrow{\scriptstyle \mathrm{H}^k\,w_r} & & \downarrow{\scriptstyle w_r} \\
{}^{II}F_1^{k,\bullet} = \mathrm{H}^k\,\mathrm{Mor}_{\mathtt{Ab}^{\Gamma(\mathtt{r})}}(\mathrm{Indec}(f)(r)_*, I^\bullet A(r)) & \xrightarrow[\simeq]{\mu} & \mathrm{Mor}_{\mathtt{Ab}^{\Gamma(\mathtt{r})}}(\mathrm{H}_k\,\mathrm{Indec}(f)(r)_*, I^\bullet A(r))
\end{array}$$

such that, for all cohomology class $\tilde{g} \in \mathrm{H}^k \operatorname{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(\operatorname{Indec}(f)(r)_*, I^\bullet A(r))$ represented by $g : \operatorname{Indec}(f)(r)_k \to I^l A(r)$ with $d^*_{k+1} f = 0$, $\nu(g)$ is the restriction of g on $\operatorname{Ker} d_k$. We check that $\nu(g)$ does not depend on the choice of a representative of $\tilde{g}$ and that ν is a bijection, and we define μ in a similar way. The diagram

$$\begin{array}{ccc}
{}^{II}E_2^{k,l} & \xrightarrow[\simeq]{} & \operatorname{Ext}^l_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(\mathrm{H}_k \operatorname{Indec}(f)(r)_*, A(r)) \\
\Big\downarrow{\scriptstyle \mathrm{H}^l\,\mathrm{H}^k\, w_r} & & \Big\downarrow{\scriptstyle \mathrm{H}^l\, w_r} \\
{}^{II}F_2^{k,l} & \xrightarrow[\simeq]{} & \operatorname{Ext}^l_{\mathtt{Ab}^{\Gamma(r)}}(\mathrm{H}_k \operatorname{Indec}_{\mathbb{Z}} f(r)_*, A(r)).
\end{array}$$

commutes too. By hypothesis, $H_k \operatorname{Indec}(f) = 0$ if $k \leqslant n$ and $H_{n+1} \operatorname{Indec}(f)(r) = A(r)$, so the diagram

$$\begin{array}{ccc}
\mathrm{H}^{n+1} \operatorname{Tot} E_{\bullet,\bullet} & \xrightarrow[\simeq]{} & \operatorname{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(A(r), A(r)) \\
\Big\downarrow{\scriptstyle \mathrm{H}^{n+1}\, w_r} & (a'') & \Big\downarrow{\scriptstyle w_r} \\
\mathrm{H}^{n+1} \operatorname{Tot} F_{\bullet,\bullet} & \xrightarrow[\simeq]{} & \operatorname{Mor}_{\mathtt{Ab}^{\Gamma(r)}}(A(r), A(r))
\end{array}$$

commutes because the first n columns of the second page of the spectral sequences are zero. The commutativity of (a') and (a'') implies the commutativity of (a).

Commutativity of (b) *:* We have the following commutative diagram

$$\begin{array}{ccc}
\operatorname{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(\operatorname{Indec}(f)(r)_\bullet, A(r)) & \xrightarrow{w_r} & \operatorname{Mor}_{\mathtt{Ab}^{\Gamma(r)}}(\operatorname{Indec}(f)(r)_\bullet, A(r)) \\
\Big\downarrow{\scriptstyle \phi\ \simeq} & & \Big\downarrow{\scriptstyle \simeq\ \phi_r} \\
\widetilde{\operatorname{Der}}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}\operatorname{cofib}_\Gamma(f)^\vee_\bullet, A) & & \widetilde{\operatorname{Mor}}_{\mathtt{Ab}^{\Gamma(r)}}(\mathbb{Z}\operatorname{cofib}_{\Gamma(r)}(f(r))^\vee_\bullet, A(r)) \\
\Big\downarrow{\scriptstyle \beta\ \simeq} & & \Big\downarrow{\scriptstyle \simeq\ \beta_r} \\
\widetilde{C}^\bullet_\Gamma(\operatorname{cofib}_\Gamma f, A) & \xrightarrow[\alpha_r(Q,P)]{} & \widetilde{C}^\bullet_{\Gamma(r)}(\operatorname{cofib}_{\Gamma(r)}(f(r)), A(r))
\end{array}$$

where $\alpha_r(Q, P)$ is described in Proposition 9.1.2, ϕ in Proposition 7.5.7 and β is provided by the Proposition 7.4.5. The isomorphisms ϕ_r and β_r are the counterpart of ϕ and β in the simplicial set setting. Precisely, if $g : X \to Y$ is a cofibration of simplial sets over $B\Gamma$ with Γ a groupoid and G a $\Gamma-$simplical abelian group, then $\widetilde{C}^\bullet_\Gamma(\operatorname{cofib}_\Gamma(g); G) \simeq \widetilde{\operatorname{Mor}}_{\mathtt{Ab}^\Gamma}(\mathbb{Z}\operatorname{cofib}_\Gamma(g)^\vee_\bullet, G)$ with $\operatorname{cofib}_\Gamma(g) = B\Gamma \coprod_X Y$ and

$$\widetilde{\operatorname{Mor}}_{\mathtt{Ab}^\Gamma}(\mathbb{Z}\operatorname{cofib}_\Gamma(g)^\vee_\bullet, G) = \{h \in \operatorname{Mor}_{\mathtt{Ab}^\Gamma}(\mathbb{Z}\operatorname{cofib}_\Gamma(g)^\vee_\bullet, G) \mid h(\mathbb{Z}B\Gamma^\vee\ ) = 0\}$$

(see [10, Part VI] for a reminder about relative equivariant coholomogy for simplicial sets). Therefore, (b) commutes.

□

9.3.2. Theorem. *Let $f \in \mathrm{Mor}_{\mathtt{sOp}\downarrow\mathtt{B\Gamma}}(P,Q)$, $r,n \geqslant 2$ such that $f(s)$ is a weak equivalence for every $r \neq s$ and $\mathrm{H}_k\,\mathrm{Indec}(f)(r) = 0$ for all $k \leqslant n$. Let also $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ concentrated in arity r with $A(r) = \mathrm{H}_{n+1}\,\mathrm{Indec}(f)(r)$. Then $\mathrm{ar}_r(k(f)) = k(f(r))$, with $k(f)$ the Postnikov invariant of f and $k(f(r))$ the Postnikov invariant of the morphism $f(r) : P(r) \to Q(r)$ in $\mathtt{sSet} \downarrow \mathtt{B\Gamma(r)}$ (see [10, VI.5]).*

If we moreover assume that Q is cofibrant and if we keep on denoting by $k(f)$ a representative of the Postnikov invariant of f in

$$\left[Q, \mathrm{hocolim}_{\Gamma} K(A, n+1)\right]_{\mathtt{sOp}\downarrow\mathtt{B\Gamma}}$$

then $\mathrm{ar}_r(k(f))$ is a representative of the Postnikov invariant of $f(r)$ in

$$\left[Q(r), \mathrm{hocolim}_{\Gamma} K(A(r), n+1)\right]_{\mathtt{sOp}\downarrow\mathtt{B\Gamma(r)}}.$$

Proof. We simply chase $\mathrm{id} \in \mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(A(r), A(r))$ in the diagram of the previous lemma. The last statement of this theorem is a direct consequence of the Theorem 9.2.3. □

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{H}^n_\Gamma(Q;A) & \xrightarrow{f^*} & H^n_\Gamma(P;A) & \longrightarrow & \mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(\mathrm{H}_{n+1}\,\mathrm{Indec}(f)(r), A(r)) & \xrightarrow{d_f} & \mathrm{H}^{n+1}_\Gamma(Q,A) & \xrightarrow{f^*} & \mathrm{H}^{n+1}_\Gamma(P,A) \\
 & & \downarrow{\scriptstyle \mathrm{ar}_r(Q)} & & \downarrow{\scriptstyle \mathrm{ar}_r(P)} & & \downarrow{\scriptstyle w_r} & & \downarrow{\scriptstyle \mathrm{ar}_r(Q)} & & \downarrow{\scriptstyle \mathrm{ar}_r(P)} \\
0 & \longrightarrow & \mathrm{H}^n_{\Gamma(r)}(Q(r);A(r)) & \xrightarrow{f(r)^*} & H^n_{\Gamma(r)}(P(r);A(r)) & \longrightarrow & \mathrm{Mor}_{\mathtt{Ab}^{\Sigma(r)}}(\mathrm{H}_{n+1}\,\mathrm{Indec}(f)(r), A(r)) & \xrightarrow{d_{f(r)}} & \mathrm{H}^{n+1}_{\Gamma(r)}(Q(r),A(r)) & \xrightarrow{f(r)^*} & \mathrm{H}^{n+1}_{\Gamma(r)}(P(r),A(r))
\end{array}$$

Figure 1.

10. Obstruction theory

We now establish our main results. In this purpose, we prove in sections 10.1 and 10.3 technical lemmas and propositions. We give in section 10.2 a reconstruction process of a simplicial operad from its groupoid operad, its $\Gamma-$operads of homotopy groups and its Postnikov invariants. We finally use this reconstruction process to develop an obstruction theory for simplicial operads.

10.1. The main lemma

We demonstrate now a technical lemma straightforwardly adapts from [10, Theorem VI.5.9], which will allows us to prove Theorem A (10.2.2) in the next section.

Let Q be a cofibrant simplicial operad with Γ its fundamental groupoid operad. Let also $r, n \geqslant 2$, $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ concentrated in arity r and $x \in \mathrm{H}^{n+1}_{\Gamma}(Q; A)$. We form the pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & \mathrm{hocolim}_{\Gamma} L(A, n+1) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Q & \xrightarrow[x]{} & \mathrm{hocolim}_{\Gamma} K(A, n+1) \end{array}$$

where x denotes the morphism that represents the cohomology class x.

10.1.1. Proposition. *We have* $\mathrm{H}_k \operatorname{Indec}(q)(r) = 0$ *for all* $k \leqslant n$ *and* $\mathrm{H}_{n+1}(\operatorname{Indec}(q))(r) = A(r)$. *Moreover,* $k(q) = x \in \mathrm{H}^{n+1}_{\Gamma}(Q; A)$.

Proof. The pullback is calculated aritywise and A is concentrated in arity r, so $f(s)$ is a weak equivalence for every $r \neq s$. Proposition 7.5.6 insures us that $\mathrm{H}_k \operatorname{Indec}(q)(r) = 0$ for all $k \leqslant n$ and $\mathrm{H}_{n+1} \operatorname{Indec}(q)(r) = A(r)$. The Postnikov invariant $k(q)$ of q is therefore defined (see 8.1.2).

We have a commutative diagram

$$\begin{array}{ccccc} \mathrm{Mor}_{\mathtt{Ab}^{\Sigma \ltimes \Gamma(r)}}(A(r), A(r)) & \xrightarrow{d_q} & H^{n+1}_{\Gamma}(Q; A) & \xrightarrow{q^*} & H^{n+1}_{\Gamma}(R; A) \\ \downarrow{\scriptstyle w_r} & & \downarrow{\scriptstyle \mathrm{ar}_r(Q)} & & \downarrow{\scriptstyle \mathrm{ar}_r(R)} \\ \mathrm{Mor}_{\mathtt{Ab}^{\Gamma(r)}}(A(r), A(r)) & \xrightarrow[d_{q(r)}]{} & H^{n+1}_{\Gamma(r)}(Q(r); A(r)) & \xrightarrow[q(r)^*]{} & H^{n+1}_{\Gamma(r)}(Q(r); A(r)) \end{array}$$

where the two lines are exact (see Proposition 8.1.1). By [10, Lemma VI.5.8], $\mathrm{ar}_r(Q)(x) = k(q(r))$. Thus, $d_{q(r)}(\mathrm{id}) = \mathrm{ar}_r(Q)(x)$. We have $q^*(x) = 0$ because this cohomology class is representated by the composite

$$R \to \mathrm{hocolim}_{\Gamma} L(A, n+1) \to \mathrm{hocolim}_{\Gamma} K(A, n+1).$$

Thus, there is $y \in \mathrm{Mor}_{\mathtt{Ab}^{\Sigma \ltimes \Gamma(r)}}(A(r), A(r))$ such that $d_q(y) = x$ and $w_r(y) = \mathrm{id}$. Since w_r is injective, we have $y = \mathrm{id}$ and finally $x = k(q)$.

□

Observe that for every $s \geqslant 1$, $p(s)$ is a fiber bundle of fiber $\mathrm{hocolim}_{\Gamma} K(A, n)(s)$, hence a twisted cartesian product (see [20, §20 and §23]). The morphism $q(s)$ is also a twisted cartesian product since it is a pullback of $p(s)$.

10.1.2. Proposition. *The object* $\operatorname{hocolim}_\Gamma K(A,n) \in \mathtt{sOp} \downarrow \mathtt{B\Gamma}$ *is a group object over* $B\Gamma$. *Its group structure is given by the group structure of the Eilenberg-MacLane operad* $K(A,n)$*; the unit morphism is the zero section* $c_k : B\Gamma \to \operatorname{hocolim}_\Gamma K(A,n)$. *Moreover, there is an action of* $\operatorname{hocolim}_\Gamma K(A,n)$ *on* R *over* $B\Gamma$ *defined by*

$$\begin{array}{rccc} \mu: & \operatorname{hocolim}_\Gamma K(A,n) \times_{B\Gamma} R & \longrightarrow & R \\ & (k,(y,k')) & \longmapsto & (y, k+k') \end{array}$$

with $s \geqslant 1$, $y \in Q(s)$, $\Psi_Q(y) = z_0 \to \ldots$ *and* $k, k' \in K(A(z_0), n)(s)$. *This action commutes with the obvious action on* Q. *In other words, the diagram*

$$\begin{array}{ccc} \operatorname{hocolim}_\Gamma K(A,n) \times_{B\Gamma} R & \xrightarrow{\mu} & R \\ {\scriptstyle p_2}\downarrow & \swarrow{\scriptstyle q} & \\ Q & & \end{array}$$

commute with $p_2(k,z) = q(z)$.

Proof. The proof of this proposition follows from an immediate verification. □

10.1.3. Main lemma. *Let* $f \in \operatorname{Mor}_{\mathtt{sOp}\downarrow\mathtt{B\Gamma}}(P,Q)$ *be a fibration with* P *cofibrant, such that* $f(s)$ *is a weak equivalence for all* $s \neq r$ *and* $f(r)$ *is* $n-$*connected with* $\pi_n(\operatorname{fib}(f(r))) \simeq A(r)$. *We assume that* $x = k(f)$. *There is an operad morphism* ξ *so that*

$$\begin{array}{ccccc} & & R & \longrightarrow & \operatorname{hocolim}_\Gamma L(A,n+1) \\ & \overset{\xi}{\nearrow} & {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ P & \xrightarrow[f]{} & Q & \xrightarrow[k(f)]{} & \operatorname{hocolim}_\Gamma K(A,n+1) \end{array}$$

commutes and ξ *induces an isomorphism* $\xi_*(r) : \pi_n(\operatorname{fib}(f(r))) \to \pi_n(\operatorname{fib}(p(r)))$.

Proof. The composite $k(f) \circ f$ is represented by the zero cohomology class in $H^{n+1}_\Gamma(P;A)$ according to the long exact sequence of Proposition 8.1.1. Therefore, there is a lift of $k(f) \circ f$ over p (see remark after 7.2.3) which gives us in turn a lift ζ of f over q.

Note that $\zeta_*(r)$ is an epimorphism. Indeed, since $\pi_n(f(r))$ is an epimorphism and p a fibration, $\pi_n(\zeta(r))$ is an epimorphism. The commutative diagram

$$\begin{array}{ccccccc} \pi_{n+1}(Q(r)) & \longrightarrow & \pi_n(\operatorname{fib}(f(r))) & \longrightarrow & \pi_n(P(r)) & \longrightarrow & \pi_n(Q(r)) \\ {\scriptstyle =}\downarrow & & {\scriptstyle \zeta_*(r)}\downarrow & & {\scriptstyle \pi_n(\zeta(r))}\downarrow & & {\scriptstyle =}\downarrow \\ \pi_{n+1}(Q(r)) & \longrightarrow & \pi_{n+1}\operatorname{fib}(p(r)) & \longrightarrow & \pi_n(R(r)) & \longrightarrow & \pi_n(Q(r)) \end{array}$$

ensures us that $\zeta_*(r)$ is an epimorphism. By contrast, $\zeta_*(r)$ is not necessarily a monomorphism so we have to modify ζ.

The morphism ζ induces an operad morphism $\zeta' : P \to \operatorname{hocolim}_\Gamma K(A,n)$ such that $\zeta(y) = \big(f(y), \zeta'(y)\big)$ by using the twisted cartesian product structure. Therefore, ζ induces a cohomology class $\zeta' \in \mathrm{H}^n_\Gamma(P;A)$. Modifying ζ amounts to modifying ζ'. To be more precise, if we pick a cohomology class $g \in H^n_\Gamma(P;A)$, then we get an operad morphism

$$\begin{array}{rccccc} \xi: & P & \xrightarrow{g\times\zeta} & \operatorname{hocolim}_\Gamma K(A,n) \times_{B\Gamma} R & \xrightarrow{\mu} & R \\ & y & \longmapsto & \Big(g(y), \big(f(y), \zeta'(y)\big)\Big) & \longmapsto & \Big(f(y), \zeta'(y) + g(y)\Big). \end{array}$$

We consider the commutative diagram

$$\begin{array}{ccccc} H^n_\Gamma(R;A) & \longrightarrow & \mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(A,A) & \xrightarrow{d_q} & H^{n+1}_\Gamma(Q,A) \\ \downarrow{\scriptstyle H^n_\Gamma(\zeta)} & & \downarrow{\scriptstyle \zeta^*(r)} & & \downarrow{\scriptstyle =} \\ H^n_\Gamma(P;A) & \longrightarrow & \mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(r)}}(A,A) & \xrightarrow{d_f} & H^{n+1}_\Gamma(Q,A) \end{array}$$

given by the naturality of the long exact sequence in Proposition 8.1.1. Note that $d_f(\mathrm{id}_A) = k(f)$ by definition and $d_p(\mathrm{id}_A) = k(f)$ by Proposition 10.1.1. Moreover, $\zeta^*(r)(\mathrm{id}_A) = h$ with h the composite $A(r) \simeq \pi_n(\mathrm{fib}(f(r))) \to \pi_n(\mathrm{fib}(q(r))) = A(r)$. Since $d_f(\mathrm{id}_A - h) = k(f) - k(f) = 0$, there is a class $g \in \mathrm{H}^n_\Gamma(P;A)$ mapping to $\mathrm{id}_A - h$ and we let $\xi = \mu \circ g \times \zeta$.

We finally check that $\xi_*(r)$ is a monomorphism and therefore an isomorphism. □

10.2. Reconstruction of a simplicial operad

We explain the applications of the operadic Postnikov invariants defined in 8.1.2 to the study of Postnikov sections of a simplicial operad.

As a main outcome, we give a process to reconstruct a simplicial operad P up to homotopy from its fundamental groupoid operad $\Gamma = \pi(P)$, its $\Gamma-$operad of homotopy groups $\pi_k(P) \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ and its Postnikov invariants $k_{r,n}$ (see Definition 8.2.2). To do so, we inductively construct the cofibrant Postnikov double tower of P (see 2.3) from this bunch of data:

- Since P is connected as an operad, the right hand side of the tower is obvious.
- By Proposition 10.2.1, we can deduce from Γ the Postnikov section $\overline{P\langle r,1\rangle}$ and consequently the first line of the double tower.
- We can therefore proceed to the reconstruction diagonal after diagonal by Theorem A (10.2.2).

This reconstruction process will be use in the study of the obstruction theory (see proof of the theorem 10.4.4).

10.2.1. Proposition. *Let P be a fibrant simplicial operad, Γ be its fundamental groupoid operad and $r \geqslant 1$. We then have $\pi(\overline{P\langle r,1\rangle}) = \Gamma_{\leqslant r}$ with $\Gamma_{\leqslant r}$ the r-cotruncation of Γ. The canonical fibration $\Psi : \overline{P\langle r,1\rangle} \to B\Gamma_{\leqslant r}$ (see Definition 5.3.2) is a weak equivalence in* $\mathtt{sOp} \downarrow \mathtt{B}\Gamma_{\leqslant \mathtt{r}}$.

Proof. For every $s \geqslant 1$,

$$\Psi(s) : \overline{P\langle r,1\rangle}(s) \longrightarrow B\Gamma_{\leqslant r}(s)$$

induces an isomorphism between the fundamental groups. Since all the other homotopy groups of $B\Gamma_{\leqslant r}(s)$ and $\overline{P\langle r,1\rangle}(s)$ are trivial, $\Psi(s)$ is a weak equivalence in $\mathtt{sSet}$. Therefore, Ψ is a weak equivalence in $\mathtt{sOp} \downarrow \mathtt{B}\Gamma_{\leqslant \mathtt{r}}$. □

Remark. As well, we obviously have $\pi(P\langle r,1\rangle) = \Gamma_{\leqslant r}$ and the canonical fibration $\Psi : P\langle r,1\rangle \to B\Gamma_{\leqslant r}$ is a weak equivalence.

10.2.2. Theorem A. *Let P be a fibrant simplicial operad. Recall that we denote by $\Gamma_{\leqslant r}$ the $r-$cotruncation of Γ and by Q the pullback*

$$P\langle r,n-1\rangle \underset{P\langle r-1,n-1\rangle}{\times} P\langle r-1,n\rangle.$$

For all $r,n \geqslant 2$, let $\pi_n(P(r))$ be the additive $\Gamma-$operad concentrated in arity r. There is a cofibrant replacement $\overline{Q}$ of

$$\overline{P\langle r,n-1\rangle} \underset{\overline{P\langle r-1,n-1\rangle}}{\times} \overline{P\langle r-1,n\rangle}$$

and an operad morphism $\overline{\rho_{r,n}}$ so that

$$\begin{array}{ccc} P\langle r,n\rangle & \xrightarrow{\rho_{r,n}} & Q \\ \sim\uparrow & & \uparrow\sim \\ \overline{P\langle r,n\rangle} & \xrightarrow[\overline{\rho_{r,n}}]{} & \overline{Q} \end{array}$$

commutes in $\mathtt{sOp}\downarrow\mathtt{B}\Gamma_{\leqslant \mathtt{r}}$ *(see 8.2 for a definition of the morphism $\rho_{r,n}$).*

Moreover, the square in $\mathtt{sOp}\downarrow\mathtt{B}\Gamma_{\leqslant \mathtt{r}}$

$$\begin{array}{ccc} \overline{P\langle r,n\rangle} & \dashrightarrow & \operatorname{hocolim}_{\Gamma_{\leqslant r}} L(\pi_n(P(r)), n+1) \\ \overline{\rho_{r,n}}\downarrow & & \downarrow \\ \overline{Q} & \xrightarrow[k_{r,n}]{} & \operatorname{hocolim}_{\Gamma_{\leqslant r}} K(\pi_n(P(r)), n+1) \end{array}$$

is a homotopy pullback square.

Proof. By definition of the Postnikov double tower and of the cofibrant Postnikov double tower of P (see 1.4.6), we have a commutative square

$$\begin{array}{ccc} P\langle r,n\rangle & \xrightarrow{\rho_{r,n}} & P\langle r,n-1\rangle \times_{P\langle r-1,n-1\rangle} P\langle r-1,n\rangle \\ \sim\uparrow & & \uparrow\sim \\ \overline{P\langle r,n\rangle} & \xrightarrow[\rho'_{r,n}]{} & \overline{P\langle r,n-1\rangle} \times_{\overline{P\langle r-1,n-1\rangle}} \overline{P\langle r-1,n\rangle}. \end{array}$$

We then pick a factorization

$$\overline{P\langle r,n\rangle} \xrightarrow{\overline{\rho_{r,n}}} \overline{Q} \xrightarrow{\sim} \overline{P\langle r,n-1\rangle} \times_{\overline{P\langle r-1,n-1\rangle}} \overline{P\langle r-1,n\rangle}$$

of $\rho'_{r,n}$ to get the first commutative square of this theorem.

The main Lemma (10.1.3) directly implies that the last diagram of this theorem is a homotopy pullback square because the morphism ξ defined in the main lemma is a weak equivalence in this context. □

10.3. Lift and operadic equivariant cohomology

We prove that a fill-in morphism in a particular kind of commutative square in $\mathtt{sOp}\downarrow\mathtt{B}\Gamma$ exists if and only if a relative equivariant cohomology class associated to this square vanishes. This proposition will play a central role in the obstruction theory, which we study in the next subsection.

10.3.1. The reduced path fibration for Eilenberg-MacLane spaces. In section 7.2.1, we define the Eilenberg-MacLane operad $K(A,n)$ associated to an additive Γ-operad in abelian groups $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ by $K(A,n)(x) = D_{\bullet}A[n]$ and where $A[n] \in \mathtt{dg}_*\mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ is the additive Γ-operad in differential graded abelian groups given by A concentrated in degree n and $D_{\bullet}$ is the Dold functor. We also set $L(A,n) = D_{\bullet}A\langle n\rangle$, where $A\langle n\rangle$ is the operad in differential graded abelian groups given by A in degrees $n-1$ and n together with the identity map as diffential. We also observe that we have an identity $A\langle n\rangle = A[n]^I \times_{A[n]} 0$ where $A[n]^I$ is a natural path object associated to $A[n]$ (see Proposition 4.3.4 for the definition of this object).

We consider in this section another model for this additive Γ-operad $L(A,n)$ which is given by $L'(A,n) = \mathrm{Path}(K(A,n))$, where $\mathrm{Path}(K(A,n))$ is a reduced path object defined by a pullback of the form:

$$\mathrm{Path}(K(A,n)) = * \times_{K(A,n)} K(A,n)^{\Delta^1}$$

in the category of Γ-operads in simplicial sets. We then consider the (unreduced) path object $K(A,n)^{\Delta^1}$, canonically associated to $K(A,n)$, which we define by using the simplicial model structure of the category of Γ-operads in simplicial sets. Recall simply that this simplicial model structure is inherited from the base category of simplicial sets, so that the object $K(A,n)^{\Delta^1}$ is given by a mapping space of simplicial sets pointwise (see 1.2.2). To form our pullback, we just consider the zero map $* \to K(A,n)$ and the face map $d_0 : K(A,n)^{\Delta^1} \to K(A,n)$ induced by the 0-th coface $d^0 : \Delta^0 \to \Delta^1$ on the simplicial set Δ^1. The face map $d_1 : K(A,n)^{\Delta^1} \to K(A,n)$, induced by the other coface $d^1 : \Delta^0 \to \Delta^1$, restricts to a natural fibration $p : \mathrm{Path}(K(A,n)) \to K(A,n)$. For our purpose, we also use that this object $\mathrm{Path}(K(A,n)) = * \times_{K(A,n)} K(A,n)^{\Delta^1}$ inherits a canonical simplicial abelian group structure from $K(A,n)$, as well as $K(A,n)^{\Delta^1}$.

10.3.2. Proposition. *We have a natural comparison map*

$$\begin{array}{ccc} \mathrm{Path}(K(A,n)) & \overset{\sim}{\dashrightarrow} & L(A,n) \\ & \searrow \quad \swarrow & \\ & K(A,n) & \end{array}$$

between the additive Γ-operad $L(A,n) = D_{\bullet}A\langle n\rangle$ of section 7.2.1 and the reduced path object operad $L'(A,n) = \mathrm{Path}(K(A,n))$ of the previous paragraph.

Proof. We have $D_{\bullet}A\langle n\rangle = D_{\bullet}(0 \times_{A[n]} A[n]^I) \simeq D_{\bullet}(0) \times_{D_{\bullet}(A[n])} D_{\bullet}(A[n]^I)$ since the Dold functor defines a category equivalence (and hence, preserves cartesian products). We have therefore $L(A,n) = * \times_{K(A,n)} D_{\bullet}(A[n]^I)$.

The differential graded path object $A[n]^I$, given by the construction of Proposition 4.3.4, can also be defined by the pointwise formula

$$(A[n]^I)(x) = \mathrm{Hom}_{\mathtt{dg}_*\mathtt{Ab}}(N_*(\Delta^1), A[n](x)),$$

where $\mathrm{Hom}_{\mathtt{dg}_*\mathtt{Ab}}(-,-)$ denotes an internal hom-object functor on the category of differential graded abelian groups which is defined by using that this monoidal category is closed. Recall also that we have an identity $K(A,n)^{\Delta^1}(x) = K(A(x),n)^{\Delta^1}$ for each object x of our operad in groupoids Γ.

By the Dold-Kan equivalence, we have $A[n](x) = N_* D_{\bullet} A[n](x) = N_*(K(A(x),n))$. We have a natural comparison map

$$N_*(K(A(x),n)^{\Delta^1}) \to \mathrm{Hom}_{\mathtt{dg}_*\mathtt{Ab}}(N_*(\Delta^1), N_*(K(A(x),n))) = A[n]^I(x)$$

adjoint to the composite

$$N_*(K(A(x),n)^{\Delta^1}) \otimes N_*(\Delta^1) \to N_*(K(A(x),n)^{\Delta^1} \times \Delta^1) \to N_*(K(A(x),n))$$

where we take the Eilenberg-MacLane equivalence together with the natural evaluation map $K(A(x),n)^{\Delta^1} \times \Delta^1 \to K(A(x),n)$ in the category of simplicial sets. This comparison map is a weak-equivalence by [8, Proposition II.5.3.7]. Furthermore, we readily check, by using the naturality of the construction, that this map defines a morphism of additive Γ-operads $N_*(K(A,n)^{\Delta^1}) \to A[n]^I$ in the category of differential graded abelian groups. By the Dold-Kan equivalence, this morphism is equivalent to a morphism of additive Γ-operads in simplicial abelian groups $K(A,n)^{\Delta^1} \to D_\bullet A[n]^I$. Furthermore, we can use that our construction is functorial with respect to the simplicial set Δ^1, on the source of our mapping spaces, to check that this morphism of additive Γ-operads fits in a commutative diagram

$$\begin{array}{ccc} K(A,n)^{\Delta^1} & \overset{\sim}{\dashrightarrow} & D_\bullet A[n]^I \\ & {\scriptstyle (d_0,d_1)}\searrow \quad \swarrow{\scriptstyle (d_0,d_1)} & \\ & K(A,n) \times K(A,n) & \end{array},$$

where we consider the face operators (d_0, d_1) associated to the path object $A[n]^I$. Then we just take the image of this diagram under the pullback operation $* \times_{K(A,n)} -$ to get the result of the proposition. □

10.3.3. Cylinder objects and simplicial model structures. Recall that for every cofibrant operad $P \in \mathtt{sOp} \downarrow \mathtt{B\Gamma}$ there exists a (good) cylinder object $\mathrm{Cyl}(P) \in \mathtt{sOp} \downarrow \mathtt{B\Gamma}$ together with a cofibration $P \bigvee P \to \mathrm{Cyl}(P)$ and a weak equivalence $s : \mathrm{Cyl}(P) \to P$ such that the diagram

$$\begin{array}{ccccc} P & & \xrightarrow{\mathrm{id}} & & P \\ \downarrow & \overset{d^0}{\underset{\sim}{\searrow}} & & & \\ P \bigvee P & \rightarrowtail & \mathrm{Cyl}(P) & \overset{s^0}{\underset{\sim}{\longrightarrow}} & P \\ \uparrow & \overset{\sim}{\underset{d^1}{\nearrow}} & & & \\ P & & \xrightarrow[\mathrm{id}]{} & & P \end{array}$$

commutes. Note that the morphisms $d^0, d^1 : P \to \mathrm{Cyl}(P)$ are cofibrations since we assume that P is cofibrant.

In what follows, we consider a particular cylinder object $\mathrm{Cyl}(P) = P \otimes \Delta^1$ which we deduce from the simplicial model structure of the category of simplicial Γ-operads. The functor $- \otimes \Delta^1$ is left adjoint to the path object functor $-^{\Delta^1}$ which we use in the definition of the reduced path object operad $\mathrm{Path}(K(A,n))$ of the previous paragraph. We use this correspondence in the proof of the next lemma.

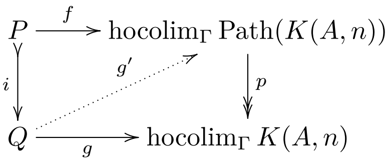
10.3.4. Lemma. *To every commutative diagram*

$$\begin{array}{ccc} P & \xrightarrow{f} & \mathrm{hocolim}_\Gamma \mathrm{Path}(K(A,n)) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ Q & \xrightarrow[g]{} & \mathrm{hocolim}_\Gamma K(A,n) \end{array}$$

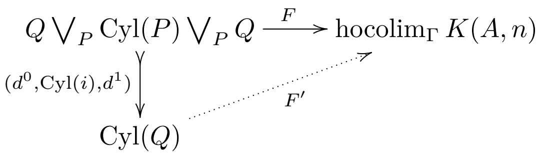
with P cofibrant, we can associate a morphism

$$F : Q \bigvee_P \mathrm{Cyl}(P) \bigvee_P Q \to \mathrm{hocolim}_\Gamma K(A,n)$$

in $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$*. Besides, a fill-in morphism g' in the diagram*



is equivalent to a solution of the extension problem



in the category $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$.

Proof. We set

$$K = \mathrm{hocolim}_\Gamma K(A,n)$$

for short. We readily see that the universal covering functor of section 1.1.3 satisfies $(X \times \Delta^1)^\vee(x) = X^\vee(x) \times \Delta^1$. By adjunction, this relation implies that the homotopy colimit functor commutes with the path object functor $-^{\Delta^1}$ on the category of simplicial sets, and the same result holds for simplicial Γ-operads, since we define homotopy colimits and path objects termwise in this category. We have in particular $\mathrm{hocolim}_\Gamma(K(A,n)^{\Delta^1}) = \mathrm{hocolim}_\Gamma(K(A,n))^{\Delta^1}$ for the Eilenberg-MacLane operad of our lemma $K(A,n)$. We now have

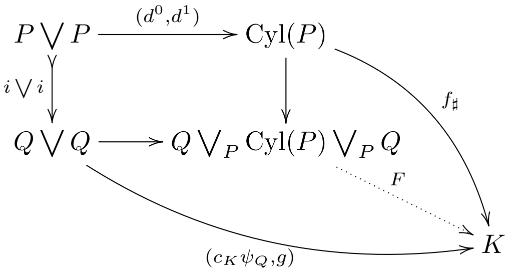
$$\mathrm{hocolim}_\Gamma \mathrm{Path}(K) = \mathrm{hocolim}_\Gamma(* \times_{K(A,n)} K(A,n)^{\Delta^1}) = B\Gamma \times_K K^{\Delta^1}$$

since $\mathrm{hocolim}_\Gamma(*) = B\Gamma$ and the homotopy colimit functor preserves cartesian products by adjunction. The morphism $B\Gamma \to K$ which occurs in this fiber product is identified with the zero section c_K of the morphism $\Psi_K : K \to B\Gamma$, while we consider the face operator $d_0 : K^{\Delta^1} \to K$ for the other factor K^{Δ^1}.

We then use the adjunction between the cylinder object functor $\mathrm{Cyl}(-) = - \otimes \Delta^1$ and the path object functor $-^{\Delta^1}$ to obtain that the morphism

$$f : P \to \mathrm{hocolim}_\Gamma \mathrm{Path}(K(A,n)) = B\Gamma \times_K K^{\Delta^1}$$

is equivalent to a morphism $f_\sharp : \mathrm{Cyl}(P) \to K$ such that $f_\sharp d^0 = c_K \Psi_P$. We moreover have $pf = gi \Leftrightarrow f_\sharp d^1 = gi$. We can therefore apply the universal property of the pushout to get a fill-in morphism



which gives the morphism F of our lemma.

We use the same adjunction process to associate a morphism $F' : \mathrm{Cyl}(Q) \to K$ such that $F'd^0 = c_K\psi_Q$ to any morphism $g' : Q \to \mathrm{hocolim}_\Gamma \mathrm{Path}(K(A,n)) = B\Gamma \times_K K^{\Delta^1}$ as in our lemma. We easily check, again, that the relations $g'i = f$ and $pg' = g$ are equivalent to $F'(d^0, \mathrm{Cyl}(i), d^1) = F$. □

We use the following observation:

10.3.5. Lemma. *The existence of a morphism F'' that makes the diagram*

$$\begin{array}{ccc} Q\bigvee_P \mathrm{Cyl}(P)\bigvee_P Q & \xrightarrow{F} & \mathrm{hocolim}_\Gamma K(A,n) \\ \downarrow & \overset{F''}{\nearrow} & \\ \mathrm{Cyl}(Q) & & \end{array}$$

commute up to homotopy in the category $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$ *implies the existence of a morphism F' which makes this diagram commute strictly, as in the statement of the previous lemma.*

Proof. We still set $K = K(A,n)$ for short and we consider a path object K^I associated to K in the category $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$. We fix a homotopy $G : Q\bigvee_P \mathrm{Cyl}(P)\bigvee_P Q \to K^I$ such that $d_0G = F$ and $d_1G = F''(d^0, \mathrm{Cyl}(i), d^1)$. We then have a commutative diagram:

$$\begin{array}{ccccc} Q\bigvee_P \mathrm{Cyl}(P)\bigvee_P Q & \xrightarrow{G} & K^I & \underset{\sim}{\xrightarrow{d_0}} & K\,. \\ {\scriptstyle (d^0,\mathrm{Cyl}(i),d^1)}\downarrow & \overset{G'}{\nearrow} & \downarrow{\scriptstyle d_1} & & \\ \mathrm{Cyl}(Q) & \xrightarrow[F'']{} & K & & \end{array}$$

We pick a fill-in morphism G' which exists by the general axiom of model categories, and we set $F' = d_0G'$. Then we trivially have

$$F'(d^0, \mathrm{Cyl}(i), d^1) = d_0G'(d^0, \mathrm{Cyl}(i), d^1) = d_0G = F.$$

□

10.3.6. Proposition. *Let $i : P \to Q$ be a cofibration in* $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$ *with P cofibrant. Also consider $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$, $n \geqslant 1$, a morphism $g : Q \to \mathrm{hocolim}_\Gamma K(A,n)$ and a morphism $f : P \to \mathrm{hocolim}_\Gamma \mathrm{Path}(K(A,n))$ with $n \geqslant 0$ such that the diagram*

$$\begin{array}{ccc} P & \xrightarrow{f} & \mathrm{hocolim}_\Gamma L(A,n) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ Q & \xrightarrow[g]{} & \mathrm{hocolim}_\Gamma K(A,n) \end{array}$$

commutes. There is a fill-in morphism $g' : Q \to \mathrm{hocolim}_\Gamma L(A,n)$ in the diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & \mathrm{hocolim}_\Gamma L(A,n) \\ {\scriptstyle i}\downarrow & \overset{g'}{\nearrow} & \downarrow{\scriptstyle p} \\ Q & \xrightarrow[g]{} & \mathrm{hocolim}_\Gamma K(A,n) \end{array}$$

if and only if a certain class of operadic equivariant cohomology $w \in \mathrm{H}^n_\Gamma(Q, P; A)$ associated to our square of maps is zero (see the proof for the explicit definition of this class).

Proof. We use the results of the previous lemmas to convert the problem of this proposition into an extension problem

$$\begin{array}{ccc} Q \bigvee_P \mathrm{Cyl}(P) \bigvee_P Q & \xrightarrow{F} & \mathrm{hocolim}_\Gamma K(A,n) \\ \downarrow & \nearrow & \\ \mathrm{Cyl}(Q) & & \end{array}$$

in the homotopy category of the category $\mathtt{sOp} \downarrow \mathtt{B\Gamma}$. We note $W = Q \bigvee_P \mathrm{Cyl}(P) \bigvee_P Q$ for short and form the diagram

$$\begin{array}{ccccc} [\mathrm{Cyl}(Q), \mathrm{hocolim}_\Gamma K(A,n)]_{\mathtt{sOp}\downarrow\mathtt{B\Gamma}} & \longrightarrow & [W, \mathrm{hocolim}_\Gamma K(A,n)]_{\mathtt{sOp}\downarrow\mathtt{B\Gamma}} & & \\ \simeq\downarrow & & \simeq\downarrow & & \\ \mathrm{H}^n_\Gamma(\mathrm{Cyl}(Q), A) & \longrightarrow & \mathrm{H}^n_\Gamma(W, A) & \longrightarrow & \mathrm{H}^{n+1}_\Gamma(\mathrm{Cyl}(Q), W, A), \end{array}$$

where the vertical bijections are given by the correspondence of Theorem 7.2.3, while the bottom horizontal line is an exact sequence, given by the result of Proposition 7.4.3. We deduce from this diagram that the existence of a solution of our homotopy lifting problem is equivalent to the vanishing the image in $\mathrm{H}^{n+1}_\Gamma(\mathrm{Cyl}(Q), W, A)$ of the class $w(F) \in \mathrm{H}^n_\Gamma(W; A)$ associated to the map F.

We now consider the long cohomology exact sequence associated to the pair of morphisms $Q \to W \to \mathrm{Cyl}(Q)$, where we consider the inclusion of the first summand Q in the coproduct $W = Q \bigvee_P \mathrm{Cyl}(P) \bigvee_P Q$ so that the composite of our morphisms is identified with the coface operator $d^0 : Q \to \mathrm{Cyl}(Q)$ of the cylinder object $\mathrm{Cyl}(Q)$. This long cohomology exact sequence has the form:

$$\ldots \longrightarrow \mathrm{H}^n_\Gamma(\mathrm{Cyl}(Q), W; A) \longrightarrow \mathrm{H}^n_\Gamma(\mathrm{Cyl}(Q), Q; A) \longrightarrow$$

$$\mathrm{H}^n_\Gamma(W, Q; A) \longrightarrow \mathrm{H}^{n+1}_\Gamma(\mathrm{Cyl}(Q), W; A) \longrightarrow \ldots$$

(see Proposition 7.4.4). We have $\mathrm{H}^n_\Gamma(\mathrm{Cyl}(Q), Q; A) = 0$, because the weak-equivalence $d^0 : Q \xrightarrow{\sim} \mathrm{Cyl}(Q)$ induces an isomorphism $\mathrm{H}^n_\Gamma(\mathrm{Cyl}(Q); A) \xrightarrow{\simeq} \mathrm{H}^n_\Gamma(Q; A)$ at the operadic cohomology level. Hence, we have an isomorphism $\mathrm{H}^{n+1}_\Gamma(\mathrm{Cyl}(Q), W; A) \simeq \mathrm{H}^n_\Gamma(W, Q; A)$. We then form the commutative square:

$$\begin{array}{ccc} P & \longrightarrow & Q \\ i\downarrow & & \downarrow j \\ Q & \longrightarrow & W, \end{array}$$

where j is the map consider in our pair $Q \to W \to \mathrm{Cyl}(Q)$. We have $\mathrm{cofib}_\Gamma(i) = B\Gamma \bigvee_P Q$ and $\mathrm{cofib}_\Gamma(j) = B\Gamma \bigvee_P \mathrm{Cyl}(P) \bigvee_P Q$. We get that the acyclic cofibration $d^0 : P \to \mathrm{Cyl}(P)$ induces a weak-equivalence $Q \simeq P \bigvee_P Q \xrightarrow{\sim} \mathrm{Cyl}(P) \bigvee_P Q$ (which is still a cofibration) by pushout along the morphism $i : P \to Q$, and a weak-equivalence yet again $B\Gamma \bigvee_P Q \xrightarrow{\sim} B\Gamma \bigvee_P \mathrm{Cyl}(P) \bigvee_P Q$ when we take the pushout of this map along the morphism $\psi_P : P \to B\Gamma$. We accordingly have a weak-equivalence $\mathrm{cofib}_\Gamma(i) \xrightarrow{\sim} \mathrm{cofib}_\Gamma(j)$ and we conclude from this result that we have an identity $\mathrm{H}^n_\Gamma(W, Q; A) \simeq \mathrm{H}^n_\Gamma(Q, P; A)$ at the relative cohomology level. The result of the proposition follows. □

10.4. Obstruction theory

We aim to extend a given simplicial operad morphism $f : P \to R$ to a morphism $f' : Q \to R$ under a given cofibration $i : P \to Q$. In other words, we want to find a morphism $f' : Q \to R$ so that the diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & R \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle f'} & \\ Q & & \end{array}$$

commutes in $\mathtt{sOp}$. We recursively define a sequence of operadic equivariant cohomology classes, the obstructions, whose vanishing is necessary and sufficient for the existence of such an extension. Note that if f' exists, then i, f and f' are morphisms over $B\pi(R)$ if we set the augmentation ψ_R as the canonical fibration (see Definition 5.3.2) and the augmentations Ψ_P and Ψ_R as the composites $\psi_R f$ and $\psi_R f'$. Therefore, we can assume that $P, Q \in \mathtt{sOp} \downarrow \mathtt{B}\pi(\mathtt{R})$, $f \in \mathrm{Mor}_{\mathtt{sOp}\downarrow\mathtt{B}\pi(\mathtt{R})}(P, R)$ and $i \in \mathrm{Mor}_{\mathtt{sOp}\downarrow\mathtt{B}\pi(\mathtt{R})}(P, Q)$.

We first go back to a broader context. We use that $\mathtt{sOp} \downarrow \mathtt{B}\Gamma$ is a simplicial model category. We accordingly have simplicial mapping spaces $\mathrm{Map}_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}(-,-)$ defined on this category, such that

$$\mathrm{Map}_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}(P, Q)_0 = \mathrm{Mor}_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}(P, Q).$$

If we moreover assume that P is cofibrant and Q is fibrant, then

$$\pi_0 \mathrm{Map}_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}(P, Q) = [P, Q]_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}.$$

10.4.1. Proposition. *A commutative diagram*

$$\begin{array}{ccc} P & \xrightarrow{f} & S \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ Q & \xrightarrow[g]{} & T \end{array}$$

in $\mathtt{sOp} \downarrow \mathtt{B}\Gamma$ *is equivalent to a vertex*

$$(f, g) \in \mathrm{Map}_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}(P, S) \underset{\mathrm{Map}_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}(P,T)}{\times} \mathrm{Map}_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}(Q, T).$$

Furthermore, a fill-in morphism h in

$$\begin{array}{ccc} P & \xrightarrow{f} & S \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle h} & \downarrow{\scriptstyle p} \\ Q & \xrightarrow[g]{} & T \end{array}$$

is equivalent to a preimage of this vertex under the pullback corner map

$$\mathrm{Map}_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}(Q,S) \to \mathrm{Map}_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}(P,S) \underset{\mathrm{Map}_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}(P,T)}{\times} \mathrm{Map}_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}(Q,T).$$

Proof. Immediate. □

Remark. The pullback corner map in the previous proposition is a fibration by general properties of simplicial model categories (see [12, 9.3.7]).

10.4.2. Lemma. *The existence of a strict preimage h in the previous proposition is equivalent to the existence of a preimage of the class of the pair (f,g) when we pass to the sets of connected component*

$$\pi_0 \,\mathrm{Map}_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}(Q,S) \to \pi_0 \left(\mathrm{Map}_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}(P,S) \underset{\mathrm{Map}_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}(P,T)}{\times} \mathrm{Map}_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}(Q,T)\right).$$

Proof. We use that any fibration $q : E \to B$ induces a surjection $q : E_x \to B_y$ for each pair of connected components such that $q(x) = y$ in $\pi_0 B$. The conclusion of the lemma follows. □

We now assume that P is cofibrant. We consider a commutative diagram in $\mathtt{sOp} \downarrow \mathtt{B}\Gamma$:

$$\begin{array}{ccccc} S & \xleftarrow{\sim} & S' & \longrightarrow & L \\ \downarrow & & \downarrow{\scriptstyle\sim} & & \downarrow \\ T & \xleftarrow{\sim} & T' & \longrightarrow & K. \end{array}$$

where the right square is a homotopy pullback as in the remark after the Theorem A (see 10.2.2), where $L = \mathrm{hocolim}_\Gamma L(A,n)$ and $K = \mathrm{hocolim}_\Gamma K(A,n)$. We also assume that T and T' are fibrant, and we pick a factorisation

$$S' \overset{\sim}{\rightarrowtail} S'' \twoheadrightarrow T$$

and fill-in morphisms to form the commutative diagram

$$\begin{array}{ccccc} S & \xleftarrow{\sim} & S' & \longrightarrow & L \\ \big\downarrow & \nwarrow{\scriptstyle\sim} & \downarrow{\scriptstyle\sim} & \nearrow & \big\downarrow \\ \big\downarrow & & S'' & & \big\downarrow \\ \big\downarrow & & \downarrow & (*) & \big\downarrow \\ T & \xleftarrow{\sim} & T' & \longrightarrow & K. \end{array}$$

We still have a weak equivalence $S'' \xrightarrow{\sim} T' \times_K L$, so that $(*)$ is still a homotopy pullback fibration. We once again consider the simplicial operad $L' = \mathrm{hocolim}_\Gamma \mathrm{Path}(K)$ of Proposition 10.3.2 : remind that we have a commutative diagram

$$\begin{array}{ccc} L' & \xrightarrow{\quad\sim\quad} & L \\ & \searrow \quad \swarrow & \\ & K. & \end{array}$$

We form :

$$\begin{array}{ccccc} S'' & \xrightarrow{\sim} & T' \times_K L & \xleftarrow{\sim} & T' \times_K L' \\ \downarrow & & \downarrow & & \downarrow \\ T' & \xrightarrow{=} & T' & \xleftarrow{=} & T' \end{array}$$

10.4.3. Theorem. *In this setting, the existence of a lifting h in*

$$\begin{array}{ccc} P & \xrightarrow{f} & S \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ Q & \xrightarrow[g]{} & T \end{array}$$

is equivalent to the vanishing of a obstruction class $w \in H^n_\Gamma(Q, P; A)$ associated to our diagram.

Proof. We now have a diagram of pullback corner maps :

$$\begin{array}{ccc} \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(Q, S) & \twoheadrightarrow & \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(P, S) \displaystyle\times_{\mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(P,T)} \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(Q, T) \\ \uparrow\sim & & \uparrow\sim \\ \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(Q, S'') & \twoheadrightarrow & \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(P, S'') \displaystyle\times_{\mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(P,T')} \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(Q, T') \\ \downarrow\sim & & \downarrow\sim \\ \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}\left(Q, T' \displaystyle\times_K L\right) & \twoheadrightarrow & \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}\left(P, T' \displaystyle\times_K L\right) \displaystyle\times_{\mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(P,T')} \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(Q, T') \\ \uparrow\sim & & \updownarrow\sim \\ \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}\left(Q, T' \displaystyle\times_K L'\right) & \twoheadrightarrow & \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}\left(P, T' \displaystyle\times_K L'\right) \displaystyle\times_{\mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(P,T')} \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(Q, T') \end{array}$$

where the vertical arrow are weak equivalences.

We deduce from this diagram that the existence of a homotopy preimage on the top is equivalent to the existence of a homotopy preimage of the class corresponding to our pair on the bottom. We use Lemma 10.4.2 again to get that this is equivalent to the existence of a strict preimage.

We have

$$\mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}\left(Q, T' \times_K L'\right) \simeq \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(Q, T') \times_{\mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(Q,K)} \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(Q, L')$$

and

$$\mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}\left(P, T' \times_K L'\right) \simeq \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(P, T') \times_{\mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(P,K)} \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(P, L')$$

by the preservation of limits by mapping spaces. A simple chase in the pullback square

$$\begin{array}{ccc}
\mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(Q,T') \underset{\mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(Q,K)}{\times} \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(Q,L') & \longrightarrow & \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(Q,L') \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(Q,T') \underset{\mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(P,K)}{\times} \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(P,L') & \longrightarrow & \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(Q,K) \underset{\mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(P,K)}{\times} \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(P,L')
\end{array}$$

then prove that our problem reduces to the problem of finding the preimage of a certain element under the map

$$\mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(Q,L') \to \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(P,L') \underset{\mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(P,K)}{\times} \mathrm{Map}_{\mathtt{sOp}\downarrow B\Gamma}(Q,K).$$

By Proposition 10.4.1, this is equivalent to a lifting problem of the form considered in section 10.3 :

$$\begin{array}{ccc}
P & \xrightarrow{f} & L' \\
{\scriptstyle i}\downarrow & \overset{g'}{\nearrow} & \downarrow{\scriptstyle p} \\
Q & \xrightarrow[g]{} & K,
\end{array}$$

which is in turn equivalent to the vanishing of a obstruction class $w \in H^n_\Gamma(Q,P;A)$ by Proposition 10.3.6.

□

10.4.4. Theorem B. *Let $R \in \mathtt{sOp}$ be such that R is fibrant and $\Gamma = \pi(R)$. Let also $P, Q \in \mathtt{sOp} \downarrow B\Gamma$ with P cofibrant. We consider a morphism $f : P \to R$ and a cofibration $i : P \to Q$ in $\mathtt{sOp} \downarrow B\Gamma$. There is a morphism $f' : Q \to R$ such that the diagram*

$$\begin{array}{ccc}
P & \xrightarrow{f} & R \\
{\scriptstyle i}\downarrow & \overset{f'}{\nearrow} & \\
Q & &
\end{array}$$

commutes if and only if a sequence of operadic equivariant cohomology class $w_{r,n} \in H^{n+1}_\Gamma(Q,P;\pi_n(R)(r))$ defined recursively for all $n, r \geqslant 2$ vanishes.

Proof. We proceed inductively through the double Postnikov tower of R :

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & & & \vdots \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
\cdots & \longrightarrow & R\langle r+1,n+1\rangle & \xrightarrow{\mathrm{coar}_{\leqslant r}} & R\langle r,n+1\rangle & \longrightarrow & \cdots & \longrightarrow & R\langle 1,n+1\rangle \\
 & & \downarrow{\scriptstyle p_{r+1,n+1}} & & \downarrow{\scriptstyle p_{r,n+1}} & & & & \downarrow{\scriptstyle p_{1,n+1}} \\
\cdots & \longrightarrow & R\langle r+1,n\rangle & \xrightarrow{\mathrm{coar}_{\leqslant r}} & R\langle r,n\rangle & \longrightarrow & \cdots & \longrightarrow & R\langle 1,n\rangle \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 & & \vdots & & \vdots & & & & \vdots \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
\cdots & \longrightarrow & R\langle r+1,1\rangle & \xrightarrow{\mathrm{coar}_{\leqslant r}} & R\langle r,1\rangle & \longrightarrow & \cdots & \longrightarrow & R\langle 1,1\rangle
\end{array}$$

diagonal after diagonal, following the reconstruction process of the operad R. We denote $g\langle r,n\rangle$ the composite $P \to R \to R\langle r,n\rangle$.

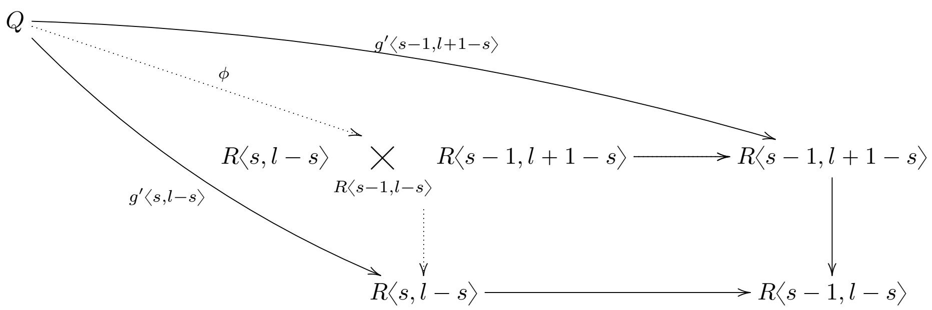
Since R is connected as an operad, $R\langle 1,n\rangle = \mathtt{I}$ for every $n \geqslant 1$. Thus, there are obvious extensions $g'\langle r,n\rangle$ of $g\langle r,n\rangle$ with respect to i for every (r,n) such that $r = 1$.

Let $\Psi : R\langle r,1\rangle \to B\Gamma_{\leqslant r}$ be the canonical fibration (see Definition 5.3.2), which is here acyclic (see remark after the Proposition 10.2.1). The diagram

$$\begin{array}{ccc}
P & \xrightarrow{g\langle r,1\rangle} & R\langle r,1\rangle \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \sim\ \Phi} \\
Q & \longrightarrow & B\Gamma_{\leqslant r}
\end{array}$$

commutes and there is an extension $g'\langle r,1\rangle$ of $g\langle r,1\rangle$ with respect to i.

Let now $l \geqslant 3$ and suppose that we have already extend $g\langle r,n\rangle$ to morphisms $g'\langle r,n\rangle : Q \to R\langle r,n\rangle$ with respect to i for all $r+n \leqslant l$. We want to extend g to the Postnikov sections on the diagonal which equation is $r+n = l+1$. Let $2 \leqslant s \leqslant l$ and assume that $g\langle 1+t,l-t\rangle$ have been successfully extended to $g'\langle 1+t,l-t\rangle$ for all $0 \leqslant t \leqslant s-2$. In other terms, assume that g have been successfully extended to the $(s-1)$−st top terms of the diagonal. We want now to lift the filling morphism ϕ :



in the commutative square in $\mathtt{sOp} \downarrow \mathtt{B\Gamma_{\leqslant s}}$

$$\begin{array}{ccc} P & \xrightarrow{g\langle s,l+1-s\rangle} & R\langle s, l+1-s\rangle \\ {\scriptstyle i}\downarrow & \overset{g'\langle s,l+1-s\rangle}{\nearrow} & \downarrow{\scriptstyle \rho'_{s,l+1-s}} \\ Q & \xrightarrow[\phi]{} & R\langle s, l-s\rangle \underset{R\langle s-1,l-s\rangle}{\times} R\langle s-1, l+1-s\rangle. \end{array}$$

By Theorem 10.4.3, this fill-in morphism exists if and only if a certain class

$$w_{s,l+1-s} \in H^{l+2-s}_{\Gamma_{\leqslant s}}(Q, P; \pi_{l+1-s}(R)(s)) = H^{l+2-s}_{\Gamma}(Q, P; \pi_{l+1-s}(R)(s))$$

vanishes. Since $R = \lim \mathscr{T}(R)$ (see 2.3.1), the proof of this theorem readily follows. □

Appendices

A. Universal coefficient theorem for operadic equivariant cohomology

We give in 7.5.8 a universal coefficient theorem in relative operadic cohomology when the coefficients are concentrated in one arity. We crucially rely on this result to prove Proposition 8.1.1, which permits us to define the $k-$invariant of a simplicial operad morphism and in particular, the Postnikov invariants of a simplicial operad.

A stronger version of this universal coefficient theorem however exists : we do not necessarily have to assume that the coefficients are concentrated in one arity. This statement permits us to establish Proposition 8.1.1 with less hypothesis. We can then understand the necessity of the arity filtration in the definition of the Postnikov invariants of a simplicial operad.

A.1. Theorem (Universal coefficient theorem in relative operadic equivariant cohomology)**.** *Let $f \in \mathrm{Mor}_{\mathtt{sOp}\downarrow \mathtt{B}\Gamma}(P, Q)$ and $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$. There is a cohomology spectral sequence $(B(f)_p^{\bullet,\bullet})$ associated to a decreasing filtration such that*

$$B(f)_1^{s,k} = H^{s+k}\,\mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(s)}}(\mathrm{Indec}(f)(s)_\bullet, A(s)) \Rightarrow\; H_\Gamma^{s+k}(Q, P; A)$$

and for all $s \geqslant 1$, a first quadrant cohomology spectral sequence associated to a bicomplex $(B_s(f)_p^{\bullet,\bullet})$ such that

$$B_s(f)_2^{l,k} = \mathrm{Ext}^l_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(s)}}(H_k\,\mathrm{Indec}(f)(s), A(s)) \Rightarrow \mathrm{H}^{k+l}\,\mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(s)}}(\mathrm{Indec}(f)(s)_\bullet, A(s)).$$

Proof. We assume that f is a quasi-free extension between cofibrant objects with $Q_\flat = P_\flat \bigvee \mathbb{F}(M)$, $M \in \mathtt{s}^-\mathtt{Seq}\downarrow \mathtt{B}\Gamma$ (otherwise we replace it by using a cofibrant replacement of P followed by a factorization).

Let

$$C^\bullet = \widetilde{\mathrm{Der}}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee_\bullet, A) \in \mathtt{dg}^*\mathtt{Ab}$$

with

$$\widetilde{\mathrm{Der}}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee{}_k, A) = \left\{g \in \mathrm{Der}_{\mathtt{AbOp}^\Gamma}(\mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee_k, A) \,\middle|\, g(\mathbb{Z}B\Gamma^\vee_k) = 0\right\}$$

for all $k \geqslant 0$ and with the alternate sum of the coface maps as differential. There is a cochain complex isomorphism (see Theorem 7.4.5)

$$C^\bullet \simeq \widetilde{C}^\bullet_\Gamma(\mathrm{cofib}_\Gamma(f), A).$$

There is also, for all $k \geqslant 0$, an abelian group isomorphism

$$\phi_k : C^k \to \mathrm{Mor}_{\mathtt{AbSeq}^\Gamma}(\mathrm{Indec}(f)_k, A)$$

induced by the composite of embeddings in $\mathtt{s}^-\mathtt{AbOp}^\Gamma$ (see Proposition 6.2.4)

$$\tau : \mathrm{Indec}(f) \to \mathrm{cofib}(\mathbb{Z}f^\vee) \to \mathrm{cofib}_\Gamma(\mathbb{Z}f^\vee),$$

but these isomorphisms do not necessarily induce a cochain complex isomorphism.

Let $B_\bullet = \mathrm{Mor}_{\mathtt{AbSeq}^\Gamma}(\mathrm{Indec}(f)_\bullet, A) \in \mathtt{dg}^*\mathtt{Ab}$ and $F_\bullet B^\bullet$ be the decreasing filtration defined for all $s, k \geqslant 0$ by

$$F_s B^k = \left\{f \in \mathrm{Mor}_{\mathtt{AbSeq}^\Gamma}(\mathrm{Indec}(f)_k, A) \mid \forall r < s, f(r) = 0\right\}.$$

Let also $F_\bullet C^\bullet$ be the decreasing filtration defined for all $s, k \geqslant 0$ by

$$F_s C^k = \phi_k^{-1}(F_s B^k).$$

These filtrations are compatible with the differentials, and if we consider the filtration of $\mathrm{Indec}(f)$ defined for all $k, s \geqslant 0$ by

$$F_s \,\mathrm{Indec}(f)_k(r) = \begin{cases} \mathrm{Indec}(f)_k(r) & \text{if } r < s, \\ 0 & \text{if } r \geqslant s, \end{cases}$$

then

$$F_s B^k = \mathrm{Mor}_{\Sigma\mathtt{Ab}^\Gamma}\left(\mathrm{Indec}(f)_k / F_s \,\mathrm{Indec}(f)_k, A\right)$$

and

$$\begin{aligned} G_s B^k &= F_s B^k / F_{s+1} B^k \\ &\simeq \mathrm{Mor}_{\mathtt{AbSeq}^\Gamma}\left(F_{s+1}\,\mathrm{Indec}(f)_k / F_s \,\mathrm{Indec}(f)_k, A\right) \\ &\simeq \mathrm{Mor}_{\mathtt{AbSeq}^\Gamma}\left(\mathrm{Indec}(f)_k(s), A\right) \\ &\simeq \mathrm{Mor}_{\mathtt{Ab}^{\Sigma \times \Gamma(s)}}\left(\mathrm{Indec}(f)_k(s), A(s)\right). \end{aligned}$$

Since $P(1) = Q(1)$, we have $\mathrm{Indec}(f)(1) = 0$ and the spectral sequence $(B(f)_p^{\bullet,\bullet})$ associated to the filtration of $B^\bullet$ is bounded in each arity. Therefore, this spectral sequence converges. Moreover,

$$\begin{aligned} G_s C^k &= \phi^{-1}(G_s B^k) \\ &= \left\{ g \in \mathrm{Der}_{\mathtt{AbOp}_\Gamma}\left(\mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee_k, A\right) \middle| g(\mathbb{Z}B\Gamma^\vee_k) = 0 \text{ and } g(\mathrm{Indec}(f)_k(r) = 0 \text{ for } r \neq s \right\}. \end{aligned}$$

We now prove that the morphisms ϕ_k induce a graded cochain complex isomorphism

$$G_\bullet C^\bullet \simeq G_\bullet B^\bullet.$$

To do so, we consider $g \in G_s C^k$ and $z \in \mathbb{Z}\,\mathrm{cofib}_\Gamma^\vee(x)_{k+1}$ with $x \in \Gamma(s)$. Recall that we have $\mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee_\flat \simeq \mathrm{Indec}(f)_\flat \oplus \mathbb{Z}B\Gamma^\vee_\flat \oplus \mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee_{\geqslant 2}$ where $\mathbb{Z}\,\mathrm{cofib}_\Gamma(f)^\vee_{\geqslant 2}$ is made of the decorated semi-alternate two-colored trees with at least two vertices. We have $\delta(\tau(z)) = z^a + z^b + z^c$ with $z^a \in \mathrm{Indec}(f)(x)_k$, $z^b \in \mathbb{Z}B\Gamma(x)_k$ and z^c a sum of semi-alternate two-colored decorated trees with at least two vertices. Since the decorating elements of such trees belong to $\mathbb{Z}B\Gamma^\vee(x_i)_k$ or to $\mathrm{Indec}(f)(x_i)_k$ with $x_i \in \Gamma(s_i)$ and $s_i < s$, $g(z^c) = 0$. Moreover, $g(z^b) = 0$, so $g(\delta(\tau(z)) = g(z^a)$ and

$$\begin{aligned} \phi(\delta^*(g))(z) &= g(\delta(\tau(z))) \\ &= g(z^a) \\ &= (\phi(g))(\delta(z)) \\ &= \delta^*(\phi(g))(z). \end{aligned}$$

Therefore, the diagram

$$\begin{array}{ccc} G_s C^{k+1} & \xrightarrow{\ \phi\ } & G_s B^{k+1} \\ {\scriptstyle \delta^*}\uparrow & & \uparrow{\scriptstyle \delta^*} \\ G_s C^k & \xrightarrow[\ \phi\ ]{} & G_s B^k \end{array}$$

commute for every $k \geqslant 0$ and

$$B(f)_p^{s,k} \Rightarrow \mathrm{H}^{s+k}\, C^\bullet = \mathrm{H}_\Gamma^{s+k}(Q, P; A).$$

We now want to calculate $B(f)_1^{s,k} = H^{s+k}\,\mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(s)}}(\mathrm{Indec}(f)(s), A(s))$. To do so, we fix the arity s and we consider an injective resolution

$$I^\bullet(A(s)) = \{I^0(A(s)) \to I^1(A(s)) \to ...\}$$

of $A(s)$. Such a resolution exists because $\mathbb{Q}/\mathbb{Z}[\Sigma(s) \ltimes \Gamma(s)]$ is an injective cogenerator in the category $\mathtt{Ab}^{\Sigma\ltimes\Gamma(s)} \simeq \mathtt{Ab}^{\Sigma(s)\ltimes\Gamma(s)}$. Let

$$B_s(f)^{l,k} = \mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(s)}}(\mathrm{Indec}(f)_k(s), I^l A(s))$$

be a bicomplex. The associated spectral sequence $B_s(f)_p^{\bullet,\bullet}$ converges because this bicomplex is zero out of the first quadrant. Since $\mathrm{Indec}(f)$ is cofibrant, $\mathrm{Indec}(f)_k$ is projective according to Proposition 6.1.4. The functor

$$\mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(s)}}(\mathrm{Indec}(f)_k, -) : \mathtt{dg}^*\mathtt{Ab}^{\Sigma\times\Gamma(s)} \to \mathtt{dg}^*\mathtt{Ab}$$

then preserves weak equivalence. Thus we have

$$\begin{cases} {}^{II}B_s(f)_1^{l,k} &= 0 \qquad\qquad\qquad \text{if } l \neq 0, \\ {}^{II}B_s(f)_1^{0,k} &= \mathrm{Mor}_{\mathtt{Ab}^{\Sigma\times\Gamma(s)}}\big(\mathrm{Indec}(f)_k, A(s)\big). \end{cases}$$

Thereafter,

$$B_s(f)_p^{l,k} \Rightarrow \mathrm{H}^{k+l}\,\mathrm{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(s)}}(\mathrm{Indec}(f)(s)_\bullet, A(s)).$$

□

A.2. Proposition. *Let $f : P \to Q$ be a morphism in $\mathtt{sOp} \downarrow \mathtt{B}\Gamma$ and $r, n \geqslant 2$. If the augmentations $\Psi_P : P \to B\Gamma$ and $\Psi_Q : Q \to B\Gamma$ induce equivalences of categories between the associated groupoid operads, then $\mathrm{H}_k(\mathrm{Indec}(f)) = 0$ for all $k \leqslant n$.*

Proof. We assume that f is a quasi-free extension between cofibrant operads with $Q_\flat = P_\flat \bigvee \mathbb{F}(M)$, $M \in \mathtt{s}^-\mathtt{Seq} \downarrow \mathtt{B}\Gamma$ (otherwise we replace it by a quasi-free extension by using a factorization).

We construct a spectral sequence associated to a filtration similar to the one defined in [6, Lemma 3.6.2]. Let $r \geqslant 2$, $x \in \Gamma(r)$ and

$$W(x) = \mathrm{Coker}(\mathbb{Z}P^\vee(x) \to \mathbb{Z}Q^\vee(x)) \in \mathtt{dg}_*\mathtt{Ab}$$

equipped with the differential δ given by the alternate sum of the face maps. Since the augmentations $\Psi_P : P \to B\Gamma$ and $\Psi_Q : Q \to B\Gamma$ induce isomorphisms between the associated groupoid operads, $f^\vee(r)$ is $n-$connected (see Proposition 5.3.3). The classical relative Hurewicz theorem (see [30, Theorem IV.7.2] or [10, Corollary III.3.12]) then implies that $\mathrm{H}_k\, W(r) = 0$ for all $k \leqslant n$.

We pick $x_r^0 \in \Gamma(r)$For all $r \geqslant 2$. For all $k \geqslant 0$, $W(x)_k$ is the direct sum of semi-alternate two-colored trees decorated by objects in $\mathbb{Z}P^\vee(x_v^0)_k$ for the black vertices and in $\mathrm{Indec}(f)(x_v^0)_k = \mathbb{Z}M^\vee(x_v^0)_k$ for the white vertices, except for semi-alternate two-colored trees with only one vertex decorated by an element in $\mathbb{Z}P^\vee(x_v^0)_k$. For all $j \geqslant 0$, $G_j(x)$ is the abelian subgroup of $W(x)$ obtained by restricting the direct sum which define $W(x)$ to the semi-alternate two-colored trees T with j vertices. We also consider the abelian group $F_i(x) = \bigoplus_{j\geqslant i} G_j(x)$.

The differential of a semi-alternate two-colored tree in $W(x)$ is calculated vertex by vertex :

– if the vertex v is black, thus decorated by $\zeta \in \mathbb{Z}P^\vee(x_v^0)$, then $\delta\zeta$ is a linear combination of elements in $\mathbb{Z}P^\vee(x_v^0)$;
– if the vertex v is white, thus decorated by $\zeta \in \mathbb{Z}M^\vee(x_v^0)$, then $\delta\zeta$ is a linear combination of elements in $\mathbb{Z}M^\vee(x_v^0)$ and of semi-alternate two-colored trees decorated by elements in $\mathbb{Z}P^\vee(x_v^0)$ and at least one element in $\mathbb{Z}M^\vee(x_v^0)$.

Therefore, $F_i(x)$ is stable under δ, and there is a bounded filtration

$$W(x) = F_1(x) \supset F_2(x) \supset ... \supset F_r(x) = 0.$$

We consider the homology spectral sequence $E_{s,t}(x)$ associated to this filtration so that

$$E^1_{s,t}(x) = \mathrm{H}_{s+t}\, G_{-s}(x)$$

and equipped with the differential $\delta^p_{s,t} : E^p_{s,t}(x) \to E^p_{s+p,t+p-1}(x)$. The differential $\delta^0_{s,t}$ on the zeroth page is induced by the alternate sum of the face maps in P and $\mathrm{Indec}(f)$ (see Proposition 7.5.4). Since the filtration is bounded, the spectral sequence converges to $H_*W(x)$. Moreover, we have $E^1_{-1,t} = H_{t-1}\,\mathrm{Indec}(f)(x)$.

We now prove by induction on the arity r that $\mathrm{H}_k\,\mathrm{Indec}(f)(r) = 0$ for all $k \leqslant n$. If $r = 2$, all the terms in the first page of the spectral sequence $E^1_{s,t}(x)$ are zero except for the one located in the column -1. We therefore have $\mathrm{H}_k\,\mathrm{Indec}(f)(2) = 0$ for all $k \leqslant n$.

Assume that $\mathrm{H}_k\,\mathrm{Indec}(f)(r') = 0$ for all $k \leqslant n$, $r' < r$ and let $x \in \Gamma(r)$. We then have $E^1_{s,t}(x) = 0$ for all (s,t) such that $s < -1$ and $t \leqslant n - s$ by the Künneth theorem. Indeed, an element in $E^1_{s,t}$ is a homology class of a composite including at least one component in $\mathrm{Indec}(f)(r')$, $r' < r$. Thus, $\mathrm{H}_k\,\mathrm{Indec}(f)(x) = 0$ for all $k \leqslant n$.

□

There is of course an absolute version of the universal coefficient theorem :

A.3. Theorem (Universal coefficient theorem in operadic equivariant cohomology)**.** *Let $P \in \mathtt{sOp} \downarrow \mathtt{B}\Gamma$ cofibrant and $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$. There is a cohomology spectral sequence $(B(P)^{\bullet;\bullet}_r)$ associated to a decreasing filtration such that*

$$B(P)^{s,k}_1 = \mathrm{H}^{s+k}\,\mathrm{Mor}_{\mathtt{Ab}^{\Sigma \ltimes \Gamma(s)}}(\mathrm{Indec}(\mathbb{Z}P^\vee)(s)_\bullet, A(s)) \Rightarrow\ \mathrm{H}^{s+k}_\Gamma(P;A)$$

and for all $s \geqslant 1$, a first quadrant cohomology spectral sequence associated to a bicomplex $(B_s(P)^{\bullet;\bullet}_r)$ such that

$$B_s(P)^{l,k}_2 = \mathrm{Ext}^l_{\mathtt{Ab}^{\Sigma \ltimes \Gamma(s)}}(\mathrm{H}_k\,\mathrm{Indec}(\mathbb{Z}P^\vee)(s), A(s)) \Rightarrow \mathrm{Mor}_{\mathtt{Ab}^{\Sigma \ltimes \Gamma(s)}}(\mathrm{Indec}(\mathbb{Z}P^\vee)(s)_\bullet, A(s)).$$

Proof. Straightforward adaptation of the previous proof. □

A.4. Proposition. *Let $f \in \mathrm{Mor}_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}(P,Q)$ and $A \in \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$. If $\mathrm{H}_k(\mathrm{Indec}(f)) = 0$ for all $1 \leqslant k \leqslant n$, then $\mathrm{H}^k_\Gamma(P;A) \simeq \mathrm{H}^k_\Gamma(Q;A)$ for all $k < n$ and there is an exact sequence*

$$0 \longrightarrow \mathrm{H}^n_\Gamma(Q;A) \xrightarrow{f^*} \mathrm{H}^n_\Gamma(P;A) \longrightarrow \mathrm{H}^{n+1}_\Gamma(Q,P;A)$$

$$\xrightarrow{d} \mathrm{H}^{n+1}_\Gamma(Q,A) \xrightarrow{f^*} \mathrm{H}^{n+1}_\Gamma(P,A)$$

natural in f and such that $\mathrm{H}^{n+1}_\Gamma(Q,P;A)$ is a graded abelian group with $G_s\,\mathrm{H}^{n+1}_\Gamma(Q,P;A) \subset \mathrm{Mor}_{\mathtt{Ab}^{\Sigma \ltimes \Gamma(s)}}(\mathrm{H}_{n+1}\,\mathrm{Indec}(f)(s), A(s))$ for all $s \geqslant 0$.

Proof. We use the spectral sequences provided by the universal coefficient Theorem A.1. According to our hypothesis on $\operatorname{Indec}(f)$, we have

$$\begin{cases} B_s(f)_1^{l,k} & = 0, & \text{if } k \leqslant n, \\ B_s(f)_1^{l,n+1} & = \operatorname{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(s)}}(\mathrm{H}_{n+1}\operatorname{Indec}(f)(s), I^l A(s)), & \text{otherwise.} \end{cases}$$

Thus :

$$\begin{cases} B_s(f)_2^{l,k} & = 0, & \text{if } k \leqslant n, \\ B_s(f)_2^{0,n+1} & = \operatorname{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(s)}}(\mathrm{H}_{n+1}\operatorname{Indec}(f)(s), A(s)), & \text{otherwise,} \end{cases}$$

from which we conclude that :

$$\begin{cases} B(f)_1^{s,k} = 0, & \text{if } s+k \leqslant n, \\ B(f)_1^{s,n+1} = \operatorname{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(s)}}(\mathrm{H}_{n+1}\operatorname{Indec}(f)(s), A(s)), & \text{if } s+k = n+1. \end{cases}$$

Thereafter,

$$\begin{cases} \mathrm{H}_\Gamma^k(Q,P;A) = 0, & \text{if } k \leqslant n, \\ G_s\,\mathrm{H}_\Gamma^{n+1}(Q,P;A) \subset \operatorname{Mor}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(s)}}(\mathrm{H}_{n+1}\operatorname{Indec}(f)(s), A(s)). \end{cases}$$

We then use the long exact sequence of operadic equivariant cohomology (see Proposition 7.3.2) to get the statement of this theorem. Moreover, $\operatorname{Indec}(f)$ does not depend up to a weak equivalence of the choice of a replacement of f and the long exact sequence of equivariant cohomology is natural. Hence, this result holds regardless of this choice. □

If f is the fibration $p\langle n\rangle : P\langle n\rangle \to P\langle n-1\rangle$ between the Postnikov sections (see Proposition 2.1.2), we have $H_k(\operatorname{Indec}(f)) = 0$ for all $k \leqslant n$ by Proposition A.2. We can therefore apply the previous proposition with $A = \mathrm{H}_{n+1}\operatorname{Indec}(f)$: there is not necessarily an element $\zeta \in \mathrm{H}_\Gamma^{n+1}(P\langle n-1\rangle, P\langle n\rangle; A)$ such that, for all $s \geqslant 0$, its component of degree s in the grading of $\mathrm{H}_\Gamma^{n+1}(P\langle n-1\rangle, P\langle n\rangle; A)$ is equal to $\mathrm{id} \in \operatorname{Hom}_{\mathtt{Ab}^{\Sigma\ltimes\Gamma(s)}}(A(s), A(s))$. Moreover, even if ζ exists, it is not uniquely defined. Thus, we can not necessarily find an operadic equivariant cohomology class $k_n \in \mathrm{H}_\Gamma^{n+1}\big(P\langle n-1\rangle, P\langle n\rangle; A\big)$ such that there is a homotopy pullback square

$$\begin{array}{ccc} \overline{P\langle n\rangle} & \dashrightarrow & \operatorname{hocolim}_\Gamma L(A,n+1) \\ {\scriptstyle \overline{p\langle n\rangle}}\Big\downarrow & & \Big\downarrow \\ \overline{P\langle n-1\rangle} & \xrightarrow[k_n]{} & \operatorname{hocolim}_\Gamma K(A,n+1) \end{array}$$

where $\overline{p\langle n\rangle} : \overline{P\langle n\rangle} \to \overline{P\langle n-1\rangle}$ is a replacement of $p\langle n\rangle$ by a cofibration between cofibrant operads and k_n also denotes a representative in

$$[P\langle n-1\rangle, \operatorname{hocolim}_\Gamma K(A,n+1)]_{\mathtt{sOp}\downarrow\mathtt{B}\Gamma}$$

of the cohomology class k_n. That is why we had to introduce the double Postnikov tower, beyond the fact that the arity filtration imply that the coefficients of the operadic equivariant cohomology groups are concentrated in one arity and this cohomology therefore restrain to the simpler cohomology of the indecomposable.

B. The categories $\texttt{AbSeq}^{\Gamma}$ and $\texttt{Ab}^{\oplus}\texttt{Op}^{\Gamma}$ are Grothendieck categories

Finding a set of projective generators in $\texttt{AbSeq}^{\Gamma}$ and then concluding that this category is a Grothendieck one raises no difficulties.

In order to extend this result to the category of additive $\Gamma-$operads $\texttt{Ab}^{\oplus}\texttt{Op}^{\Gamma}$, we want to exploit the adjunction between the free additive $\Gamma-$operad functor and the forgetful functor, but we can not use the construction given in Theorem 3.3.1 since the direct sum does not commute with the colimits. We therefore rely on the fact that an additive $\Gamma-$operad is a $\texttt{Com}-$bimodule and give a construction of the free $\texttt{Com}-$bimodules by the mean of composite trees.

B.1. Bimodule over a $\Gamma-$operad. Let $P \in \texttt{AbOp}^{\Gamma}$. A $P-$*bimodule* is a $\Gamma-$symmetric sequence $M \in \texttt{AbSeq}^{\Gamma}$ equipped with an action of the operad P given by composition operations

$$o_i : P(x) \otimes M(y) \to M(x \circ_i y)$$

for all $x \in \Gamma(r)$, $y \in \Gamma(s)$, $r \geqslant 1$, $s \geqslant 2$, $1 \leqslant i \leqslant n$, and

$$o_i : M(x) \otimes P(y) \to M(x \circ_i y)$$

for all $x \in \Gamma(r)$, $y \in \Gamma(s)$, $r \geqslant 2$, $s \geqslant 1$, $1 \leqslant i \leqslant n$. These compositions operations satisfy equivariance, asssociativity and unit axioms obtained by replacing one factor P by M in the domain of the equivariance, associativity and unit axioms of the structure of a $\Gamma-$operad.

B.2. Proposition. *The additive $\Gamma-$operads are equivalent to $\texttt{Com}-$bimodules where $\texttt{Com}$ is the $\Gamma-$operad defined in 3.1.4.*

Proof. There is an equivalence between the additive composition operation

$$o_i : P(x) \oplus P(y) \to P(x \otimes_i y)$$

of an additive $\Gamma-$operad $P \in \texttt{Ab}^{\oplus}\texttt{Op}^{\Gamma}$ and the composition operations of an action of the operad $\texttt{Com}$

$$o_i : \texttt{Com}(x) \otimes P(y) \to P(x \circ_i y),$$
$$o_i : P(x) \otimes \texttt{Com}(y) \to P(x \circ_i y)$$

because $\texttt{Com}(x) = \mathbb{Z}$ for all $x \in \Gamma$. □

B.3. Proposition. *The category $\texttt{AbSeq}^{\Gamma}$ admits $\{G_{x_0}\}_{x_0 \in \Gamma}$ as a set of projective generators, where*

$$G_{x_0}(x) = \mathbb{Z} \otimes (\Sigma \ltimes \Gamma)F_{x_0}(x) = \mathbb{Z}[\mathrm{Mor}_{\Sigma \ltimes \Gamma}(x_0, x)]$$

for all $x \in \Gamma$.

Proof. For all $M \in \texttt{AbSeq}^{\Gamma}$,

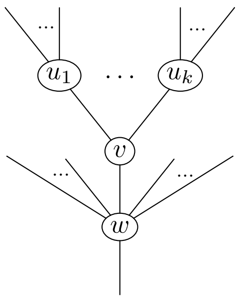
$$\mathrm{Mor}_{\texttt{AbSeq}^{\Gamma}}(G_{x_0}, M) \simeq \mathrm{Mor}_{\texttt{SetSeq}^{\Gamma}}(\mathrm{Mor}_{\Sigma \ltimes \Gamma}(x_0, -), M) \simeq M(x_0).$$

Therefore, $\{G_{x_0}\}_{x_o \in \Gamma}$ is a set of projective generators of $\texttt{Abseq}^{\Gamma}$. □

We now want to find a set of projective generators in $\texttt{Ab}^{\oplus}\texttt{Op}^{\Gamma}$. To do so, we have to consider free additive $\Gamma-$operads, but we can not use the construction given in Theorem 3.3.1 since the direct sum does not commute with the colimits. We will equivalently give a construction of free $\texttt{Com}-$bimodules by the mean of composite trees.

B.4. Composite trees. A *composite tree* T is a tree equipped with a partition of its vertices in three sets $V(T) = V_0(T) \coprod V_1(T) \coprod V_2(T)$ such that :
– There is one and only one vertex $v \in V_1(T)$.
– There is at least one vertex in $V_0(T)$. The ingoing edges of every $u \in V_0(T)$ are entries of the trees, the outgoing edge of u is an ingoing edge of v. Morever, every ingoing edge of v is the outgoing edge of a vertex in $V_1(T)$.
– There is one an only one vertex $w \in V_2(T)$. One of its ingoing edge is the outgoing edge of v, the other ingoing edges of w are entries of the tree. The outgoing edge of w is the root of the tree.

For instance,



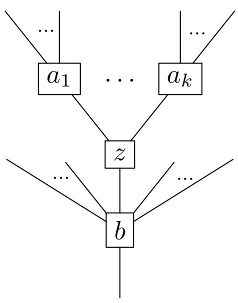
is a composite tree.

B.5. Proposition. *The forgetful functor* $\omega : \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma} \to \mathtt{AbSeq}^{\Gamma}$ *admits a left adjoint, the* free additive $\Gamma-$operad *functor* $\mathbb{F}^{\oplus} : \mathtt{AbSeq}^{\Gamma} \to \mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$.

Proof. Let $M \in \mathtt{AbSeq}^{\Gamma}$ and $x \in \Gamma(r)$ with $r \geqslant 2$. We form a groupoid $\mathtt{Tree}_3(x)$ with the object set

$$\{(T(b, z, a_1, .., a_k), \mu) | T \in \mathtt{Tree}_3, a_1, ...a_k, b, z \in \Gamma, \mu \in \mathrm{Mor}_{\Gamma}(\lambda_T(x_v), x)\},$$

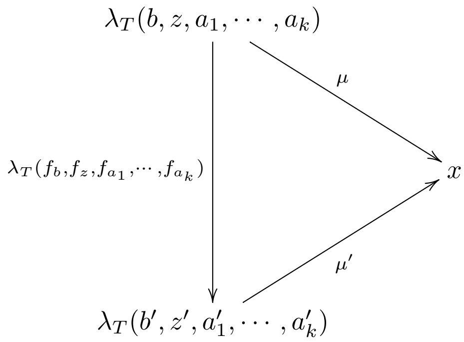
where $T(b, z, a_1, ...a_k)$ denotes a composite tree T decorated by elements $a_1, \cdots, a_k, b, z \in \Gamma$. The vertex $u_1, ..u_k$ in $V_0(T)$ are decorated by $a_1, ..., a_k$, the vertex $v \in V_1(T)$ is decorated by z and the vertex $w \in V_2(T)$ is decorated by b. For example,



is a composite tree decorated by elements in Γ. The morphisms of $\mathtt{Tree}_3(x)$ are the isomorphisms

$$T(f_b, f_z, f_{a_1}, \cdots, f_{a_k}) : T(b, z, a_1, \cdots, a_k) \to T(b', z', a'_1, \cdots, a'_k,)$$

such that



commutes, with $f_{a_i} \in \mathrm{Mor}_{\Gamma}(a_i, a'_i)$, $f_b \in \mathrm{Mor}_{\Gamma}(b, b')$ and $f_z \in \mathrm{Mor}_{\Gamma}(z, z')$, for all $(T(b, z, a_1, \cdots, a_k), \mu)$, $(T(b', z', a'_1, \cdots, a'_k), \mu') \in \mathtt{Tree}_3(x)$.

We set

$$\mathbb{F}^{\oplus}(M)(x) = \underset{T(b,z,a_1,\cdots,a_k)\in\mathtt{Tree}_3(x)}{\mathrm{colim}} T\big(\mathtt{Com}(b), M(z), \mathtt{Com}(a_1), \cdots, \mathtt{Com}(a_k)\big).$$

By construction, $\mathbb{F}^{\oplus}(M)$ is a $\mathtt{Com}-$bimodule. The action of $\sigma \in \Sigma(r)$ on the term

$$T\big(\mathtt{Com}(b), M(z), \mathtt{Com}(a_1), \cdots, \mathtt{Com}(a_k)\big)$$

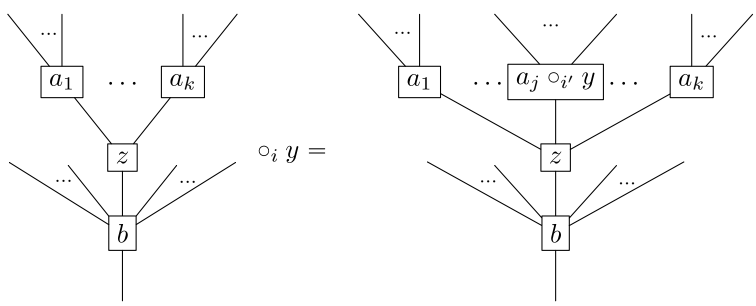
indexed by $(T(b, z, a_1, \cdots, a_k), \mu) \in \mathtt{Tree}_3(x)$ with $x \in \Gamma(r)$ is given by

$$\sigma.T\big(\mathtt{Com}(b), M(z), \mathtt{Com}(a_1), \cdots, \mathtt{Com}(a_k)\big)$$

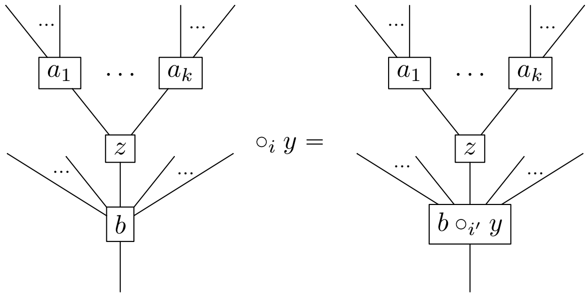
indexed by $(\sigma.(b, z, a_1, \cdots, a_k), \sigma.\mu) \in \mathtt{Tree}_3(\sigma.x)$. To describe the right action of $\mathtt{Com}$ on $\mathbb{F}^{\oplus}(M)$, we first describe the right action of Γ on the composite trees decorated by elements in Γ. Let $x \in \Gamma(r)$, $y \in \Gamma$ and $1 \leqslant i \leqslant r$. There is a functor

$$\circ_i y : \quad \mathtt{Tree}_3(x) \quad \longrightarrow \quad \mathtt{Tree}_3(x \circ_i y)$$

such that



if the $i-$th entry of the tree in the domain is the ingoing edge numbered by i' of the vertice decorated by a_j and

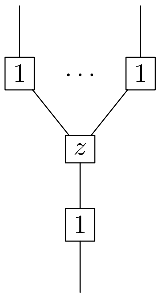


if the $i-$th entry of the tree in the domain is the ingoing edge numbered by $i'$of the vertice decorated by b. These functors induce a right action of $\mathtt{Com}$ on $\mathbb{F}^{\oplus}(M)$ by composition operations

$$\circ_i : \mathbb{F}^{\oplus}(M)(x) \oplus \mathtt{Com}(y) \to \mathbb{F}^{\oplus}(M)(x \circ_i y).$$

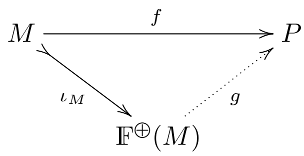
We define in a similar way the left action of $\mathtt{Com}$ on $\mathbb{F}^{\oplus}(M)$.

There is a natural embedding $\iota_M : M \to \mathbb{F}^{\oplus}(M)$. It associates to $z \in M(x)$ the decorated tree



$$\in T\big(\mathtt{com}(pt), M(z), \mathtt{com}(pt), \cdots, \mathtt{com}(pt)\big)$$

in the colimit which defines $\mathbb{F}^{\oplus}(M)(x)$, where $1 \in \mathtt{Com}(pt)$ and pt is the operad unit of Γ and we consider the operadic unit $1 \in \mathtt{Com}(e)$. Finally, since an additive $\Gamma-$operad is equivalent to a $\mathtt{Com}-$bimodule (see Proposition B.2), there exists one and only one morphism of additive $\Gamma-$operads g such that



commutes for all $f \in \mathrm{Mor}_{\mathtt{AbSeq}^{\Gamma}}(M, P)$. □

B.6. Proposition. *The category* $\mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$ *admits* $\{\mathbb{F}^{\oplus}(G_{x_0})\}_{x_0 \in \Gamma}$ *as a set of projective generators (see Proposition B.3 for a definition of* G_{x_0}*).*

Proof. According to the proof of the Proposition B.3 and the universal property of the free additive $\Gamma-$operad,

$$\mathrm{Mor}_{\mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}}(\mathbb{F}^{\oplus}(G_{x_0}), P) \simeq \mathrm{Mor}_{\mathtt{AbSeq}^{\Gamma}}(G_{x_0}, P) \simeq \mathrm{Mor}_{\mathtt{SetSeq}^{\Gamma}}(\mathrm{Mor}_{\Sigma \ltimes \Gamma}(x_0, -), P) \simeq P(x_0)$$

for all $x_0 \in \Gamma$. Therefore, $\{\mathbb{F}^{\oplus}(G_{x_0})\}_{x_0 \in \Gamma}$ is a set of projective generators of $\mathtt{Ab}^{\oplus}\mathtt{Op}^{\Gamma}$. □

B.7. Corollary. *The categories* $\mathtt{AbSeq}^\Gamma$ *and* $\mathtt{Ab}^\oplus\mathtt{Op}^\Gamma$ *are Gothendieck categories. They therefore have enough injectives. Moreover, they have enough projectives.*

Proof. All small colimits exists in $\mathtt{AbSeq}^\Gamma$ and $\mathtt{Ab}^\oplus\mathtt{Op}^\Gamma$ and filtered colimits are exact as they are calculated pointwise. Morever, thes categories admit a set of generators so they are Grothendieck categories. □

References

[1] M. Batanin and C. Berger, *Homotopy Theory for Algebras over Polynomial Monads*, Theory Appl. Categ., 32 (2017), pp. 148–253.

[2] C. Berger and I. Moerdijk, *Axiomatic homotopy theory for operads*, Comment. Math. Helv, 78 (2002), pp. 020609–4.

[3] J. Boardman and R. Vogt, *Homotopy Invariant Algebraic Structures on Topological Spaces*, Lecture Notes in Mathematics, Springer, 2006.

[4] F. Borceux, *Handbook of Categorical Algebra: Volume 2, Categories and Structures*, Cambridge Studies in Philosophy, Cambridge University Press, 1994.

[5] A. Bousfield and D. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Mathematics, Springer, 1972.

[6] B. Fresse, *Koszul Duality of Operads and Homology of Partition Posets. Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-Theory*, American Mathematical Society, 346 (2004), pp. 115–215, Contemp. Math.

[7] ——, *Modules over Operads and Functors*, Lecture Notes in Mathematics, Springer, 2009.

[8] ——, *Homotopy of Operads and Grothendieck-Teichmuller Groups*, Mathematical Surveys and Monographs, American Mathematical Society, 2017.

[9] V. Ginzburg and M. Kapranov, *Koszul Duality for Operads*, Duke Math. J., 76 (1994), pp. 203–272.

[10] P. Goerss and J. Jardine, *Simplicial Homotopy Theory*, Progress in Mathematics, Birkhäuser, 1999.

[11] P. Hirschhorn, *Overcategories and Undercategories of Model Categories*. http://www-math.mit.edu/~psh/undercat.pdf.

[12] ——, *Model Categories and Their Localizations*, Mathematical surveys and monographs, American Mathematical Society, 2009.

[13] M. Hovey, *Model Categories*, Mathematical surveys and monographs, American Mathematical Society, 2007.

[14] M. Hovey, J. Palmieri, and N. Strickland, *Axiomatic Stable Homotopy Theory*, no. 610 in American Mathematical Society: Memoirs of the American Mathematical Society, American Mathematical Society, 1997.

[15] P. T. Johnstone, *Adjoint Lifting Theorems for Categories of Algebras*, Bulletin of the London Mathematical Society, 7 (1975), pp. 294–297.

[16] G. M. Kelly, *Basic Concepts of Enriched Category Theory*, Repr. Theory Appl. Categ., (2005), pp. vi+137. Reprint of the 1982 original.

[17] S. MacLane, *Homology*, Classics in Mathematics, Springer, 1995.

[18] S. MacLane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics, Springer, 1998. Second edition.

[19] M. Markl, S. Shnider, and J. Stasheff, *Operads in Algebra, Topology and Physics*, Mathematical surveys and monographs, American Mathematical Society, 2007.

[20] J. May, *Simplicial Objects in Algebraic Topology*, Chicago Lectures in Mathematics, University of Chicago Press, 1992.

[21] ———, *The Geometry of Iterated Loop Spaces*, Lecture Notes in Mathematics, Springer, 2006.

[22] J. McCleary, B. Bollobas, and W. Fulton, *A User's Guide to Spectral Sequences*, A User's Guide to Spectral Sequences, Cambridge University Press, 2001.

[23] J. Moore, *Semi-Simplicial Complexes and Postnikov Systems*, Symposium Internacional de Topologia Algebraica, Mexico City, Universidad Nacional Autónoma de México y la UNESCO, 1958.

[24] M. M. Postnikov, *Determination of the Homology Groups of a Space by Means of the Homotopy Invariants*, Doklady Akad. Nauk SSSR, 76 (1951), pp. 359–362.

[25] D. G. Quillen, *Homotopical Algebra*, Springer, 1967.

[26] C. Reedy, *Homotopy Category of Model Categories*. http://www-math.mit.edu/~psh/reedy.pdf, 1974.

[27] C. A. Robinson, *Moore-Postnikov Systems for Non-Simple Fibrations*, Illinois J. Math., 16 (1972), pp. 234–242.

[28] E. Spanier, *Algebraic Topology*, Springer, 2012. Corrected reprint.

[29] C. A. Weibel, *An Introduction to Homological Algebra*, vol. 38 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1994.

[30] G. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics, Springer, 2012.

