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## STATE SUM INVARIANTS OF COMBED 3-MANIFOLDS

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## Introduction

Quantum topology is an area of mathematics and theoretical physics founded by Jones and Witten in the 1980s. This subject is a modern tool used for studying problems of low-dimensional topology via so-called quantum invariants of topological objects such as knots, links, manifolds, homeomorphisms, etc. Quantum invariants are constructed using an ingredient of algebraic nature (for example the category of representations of a quantum group) and via a combinatorial description of the studied objects.

A fundamental example of a quantum invariant of compact oriented 3-manifolds is due to Turaev and Viro in 1992, see [TV]. Their construction is closely related to the Ponzano-Regge quantum gravity state sum model. This approach (in its general form due to Barrett and Westbury, see $[\overline{B W}]$ ) uses a spherical fusion category as the main ingredient and consists in a state sum on skeletons of 3 -manifolds whose vertices are colored by the $6 j$-symbols of the category. Recall that a pivotal fusion category is a finitely semisimple monoidal linear category endowed with a left duality and a right duality which are monoidally equivalent. A spherical fusion category is a pivotal fusion category whose left and right dimensions of objects are equal.

The goal of the present thesis is to extend the Turaev-Viro construction to combed 3 -manifolds. A combed 3 -manifold is a compact oriented 3 -manifold endowed with a nowhere-zero vector field. The initial ingredient we use to construct this extension is a pivotal fusion category (not necessarily spherical). The additional data of the vector field on the 3 -manifold allows us to remove the hypothesis of sphericity of the category. Our construction consists in a state sum on branched spines of combed 3 -manifolds, which are a combinatorial presentation of combed 3 -manifolds developed by Ishii, Benedetti, and Petronio.

This monograph comprises five chapters and one appendix. Chapter 1 is devoted to monoidal categories, with particular attention to those that are pivotal and fusion. We describe the Penrose graphical calculus which allows to replace lengthy algebraic computations by elementary topological arguments.

In Chapter 2, we review an invariant of colored planar graphs which takes values in tensor products of multiplicity modules. This invariant generalizes $6 j$-symbols. Also, we study duality pairings for colored graphs and their associated contraction vectors. The invariant of colored graphs and the contraction vectors are the main tool in our topological constructions.

In Chapter 3, we review the theory of branched spines and the theory of o-graphs. The o-graphs are enhanced graphs that encode specific branched
spines. In particular, we explain how branched spines and o-graphs represent combed 3-manifolds.

In Chapter 4, we construct of a state sum invariant of combed 3-manifolds which generalizes the Turaev-Viro construction. More precisely, we associate to any pivotal fusion category $\mathcal{C}$ a scalar topological invariant $\mathrm{I}_{\mathcal{C}}(M, \nu)$ of a combed 3-manifold ( $M, \nu$ ), see Theorem 4.1. This invariant is defined in terms of a state sum on a branched spine of $(M, \nu)$. If the category $\mathcal{C}$ is spherical, then $\mathrm{I}_{\mathcal{C}}(M, \nu)$ does not depend on the vector field $\nu$ and is equal to the TuraevViro invariant $\mathrm{TV}_{\mathcal{C}}(M)$ of the 3-manifold $M$ defined using $\mathcal{C}$. We also give an algorithm to compute $\mathrm{I}_{\mathcal{C}}(M, \nu)$ starting from o-graphs (see Theorem 4.2).

In Chapter 5, we focus on the case of a specific pivotal fusion category: the category $G_{\mathrm{k}}^{d}$ associated with a character $d$ of a finite group $G$. We study in detail the invariant $\mathrm{I}_{G_{\mathrm{k}}^{d}}$ of combed 3-manifolds defined with this category. In particular, we prove (by examples) that $I_{G_{\mathrm{k}}^{d}}$ is non-trivial and does depend on the vector field: it may distinguish two non-homotopic vector fields on the same 3-manifold (see Theorem 5.2). Finally, we give an interpretation of the state sum invariant $\mathrm{I}_{G_{\mathrm{k}}^{d}}(M, \nu)$ in terms of classical topological invariants: we prove that it corresponds to the evaluation by the character $d$ on the Euler class of a real vector bundle of rank 2 associated to the vector field $\nu$ (see Theorem (5.5).

We end with an appendix on the unordered tensor products of modules.

## CHAPTER 1

## Pivotal fusion categories

In this chapter, we review the notions of a monoidal category (Section 1.1) and of a pivotal category (Section (1.2), with particular attention to the case of a fusion category (Section (1.4). We also discuss a way to represent morphisms: the graphical calculus (Section 1.3).

### 1.1. Monoidal categories

We discuss some basics on monoidal categories. We also study non-degenerate pairings in monoidal categories.

### 1.1.1. Categories. A category $\mathcal{C}$ consists of the following data:

- a class $\operatorname{Ob}(\mathcal{C})$, whose elements are called objects of $\mathcal{C}$;
- for any $X, Y \in \operatorname{Ob}(\mathcal{C})$, a set $\operatorname{Hom}_{\mathcal{C}}(X, Y)$, whose elements are called morphisms from $X$ to $Y$ and represented by arrows $X \rightarrow Y$;
- for any $X, Y, Z \in \operatorname{Ob}(\mathcal{C})$, a map

$$
\circ: \operatorname{Hom}_{\mathcal{C}}(Y, Z) \times \operatorname{Hom}_{\mathcal{C}}(X, Y) \rightarrow \operatorname{Hom}_{\mathcal{C}}(X, Z)
$$

called composition. The image of a pair $(g, f)$ under this map is denoted $g \circ f$ or just $g f$;

- for every $X \in \operatorname{Ob}(\mathcal{C})$, a morphism $\operatorname{id}_{X} \in \operatorname{Hom}_{\mathcal{C}}(X, X)$, called the identity of $X$.
It is required that the composition is associative and unitary in the following sense:

$$
(h \circ g) \circ f=h \circ(g \circ f) \quad \text { and } \quad f \circ \operatorname{id}_{X}=f=\operatorname{id}_{Y} \circ f
$$

for all morphisms $f: X \rightarrow Y, g: Y \rightarrow Z, h: Z \rightarrow T$ with $X, Y, Z, T \in \operatorname{Ob}(\mathcal{C})$.
Given a morphism $f: X \rightarrow Y$ in a category $\mathcal{C}$, the object $X$ is called the source and the object $Y$ the target of $f$. Two morphisms $g, f$ in $\mathcal{C}$ are composable if the source of $g$ coincides with the target of $f$. For $X \in \operatorname{Ob}(\mathcal{C})$, the set $\operatorname{Hom}_{\mathcal{C}}(X, X)$ is denoted by $\operatorname{End}_{\mathcal{C}}(X)$, and its elements are called endomorphisms of $X$. The set $\operatorname{End}_{\mathcal{C}}(X)$ is a monoid with product $g f=g \circ f$ for any $f, g \in \operatorname{End}_{\mathcal{C}}(X)$ and unit $\mathrm{id}_{X}$. A morphism $f: X \rightarrow Y$ in $\mathcal{C}$ is an isomorphism if there exists a morphism $g: Y \rightarrow X$ in $\mathcal{C}$ such that $g f=\mathrm{id}_{X}$ and $f g=\operatorname{id}_{Y}$. Such a $g$ is uniquely determined by $f$, is called the inverse of $f$ and denoted $f^{-1}$. Two objects $X, Y$ of $\mathcal{C}$ are isomorphic if there exists an isomorphism $X \rightarrow Y$. Isomorphism of objects is an equivalence relation on $\mathrm{Ob}(\mathcal{C})$ denoted by $\simeq$.
1.1.2. Functors and natural transformations. Functors are morphisms of categories and natural transformations are morphisms of functors. More precisely, a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ assigns to each object $X$ of $\mathcal{C}$ an object $F(X)$ of $\mathcal{D}$ and to each morphism $f: X \rightarrow Y$ in $\mathcal{C}$ a morphism $F(f): F(X) \rightarrow F(Y)$ in $\mathcal{D}$ so that

$$
F(g f)=F(g) F(f) \quad \text { and } \quad F\left(\mathrm{id}_{X}\right)=\operatorname{id}_{F(X)}
$$

for all composable morphisms $g, f$ in $\mathcal{C}$ and all $X \in \operatorname{Ob}(\mathcal{C})$. For example, the identity functor $1_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$ carries every object/morphism in $\mathcal{C}$ to itself. The composition of two functors $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{E}$ is defined in the obvious way and yields a functor $G F: \mathcal{C} \rightarrow \mathcal{E}$.

A natural transformation $F \rightarrow G$ between two functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$ is a family

$$
\varphi=\left\{\varphi_{X}: F(X) \rightarrow G(X)\right\}_{X \in \mathrm{Ob}(\mathcal{C})}
$$

of morphisms in $\mathcal{D}$ such that

$$
\varphi_{Y} F(f)=G(f) \varphi_{X}
$$

for all morphisms $f: X \rightarrow Y$ in $\mathcal{C}$. A natural transformation $\varphi: F \rightarrow G$ is invertible if $\varphi_{X}$ is an isomorphism for all $X \in \operatorname{Ob}(\mathcal{C})$. Then the family of morphisms

$$
\left\{\varphi_{X}^{-1}: G(X) \rightarrow F(X)\right\}_{X \in \operatorname{Ob}(\mathcal{C})}
$$

is a natural transformation $G \rightarrow F$ called the inverse of $\varphi$ and denoted by $\varphi^{-1}$. Invertible natural transformations of functors are called natural isomorphisms. Clearly, the inverse of a natural isomorphism is a natural isomorphism. Two functors $\mathcal{C} \rightarrow \mathcal{D}$ are isomorphic if there is a natural isomorphism between them.
1.1.3. Isomorphisms and equivalences of categories. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is an isomorphism if there is a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $G F=1_{\mathcal{C}}$ and $F G=1_{\mathcal{D}}$. Such a functor $G$ is uniquely determined by $F$, is an isomorphism, and is called the inverse of $F$. Two categories are isomorphic if there is an isomorphism between them.

A quasi-inverse of a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that there are natural isomorphisms $G F \simeq 1_{\mathcal{C}}$ and $F G \simeq 1_{\mathcal{D}}$. A functor is an equivalence if it has a quasi-inverse. Note that any quasi-inverse of an equivalence is an equivalence and the composition of two composable equivalences is an equivalence. Two categories are equivalent if there is an equivalence between them. It is clear from the definitions that isomorphisms of categories are equivalences and isomorphic categories are equivalent.

Any equivalence of categories $F: \mathcal{C} \rightarrow \mathcal{D}$ is essentially surjective in the sense that each object of $\mathcal{D}$ is isomorphic to $F(X)$ for some $X \in \operatorname{Ob}(\mathcal{C})$ and fully faithful in the sense that for all $X, Y \in \operatorname{Ob}(\mathcal{C})$, the map

$$
\operatorname{Hom}_{\mathcal{C}}(X, Y) \rightarrow \operatorname{Hom}_{\mathcal{D}}(F(X), F(Y)), \quad f \mapsto F(f)
$$

is bijective. If one assumes the axiom of choice, then all essentially surjective and fully faithful functors are equivalences.
1.1.4. Monoidal categories. A monoidal category is a category $\mathcal{C}$ endowed with

- a functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, called the monoidal product (or tensor prod$u c t$ );
- an object $\mathbb{1} \in \mathrm{Ob}(\mathcal{C})$, called the unit object;
- a family of isomorphisms

$$
a=\left\{a_{X, Y, Z}:(X \otimes Y) \otimes Z \rightarrow X \otimes(Y \otimes Z)\right\}_{X, Y, Z \in \mathrm{Ob}(\mathcal{C})}
$$

called the associativity constraint;

- a family of isomorphisms $l=\left\{l_{X}: \mathbb{1} \otimes X \rightarrow X\right\}_{X \in \mathrm{Ob}(\mathcal{C})}$, called the left unitality constraint;
- a family of isomorphisms $r=\left\{r_{X}: X \otimes \mathbb{1} \rightarrow X\right\}_{X \in \operatorname{Ob}(\mathcal{C})}$, called the right unitality constraint.
It is required that:
(i) (Pentagon coherence) For all objects $X, Y, Z, W$ of $\mathcal{C}$, the following diagram commutes:

(ii) (Triangle coherence) For all objects $X, Y$ of $\mathcal{C}$, the following diagram commutes:

(iii) The associativity constraint $a$ is a natural isomorphism from the functor $\otimes\left(\otimes \times 1_{\mathcal{C}}\right)$ to the functor $\otimes\left(1_{\mathcal{C}} \times \otimes\right)$.
(iv) The left unitality constraint $l$ is a natural isomorphism from the functor $\mathbb{1} \otimes-: \mathcal{C} \rightarrow \mathcal{C}$ to the functor $1_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$.
(iv) The right unitality constraint $r$ is a natural isomorphism the functor $-\otimes \mathbb{1}: \mathcal{C} \rightarrow \mathcal{C}$ to $1_{\mathcal{C}}$.

Here, the functors $\mathbb{1} \otimes-$ et $-\otimes \mathbb{1}$ are defined by

$$
\begin{array}{ll}
(\mathbb{1} \otimes-)(X)=\mathbb{1} \otimes X, & (-\otimes \mathbb{1})(X)=X \otimes \mathbb{1}, \\
(\mathbb{1} \otimes-)(f)=\operatorname{id}_{\mathbb{1}} \otimes f, & (-\otimes \mathbb{1})(f)=f \otimes \mathrm{id}_{\mathbb{1}},
\end{array}
$$

for any $X \in \operatorname{Ob}(\mathcal{C})$ and any morphism $f$ in $\mathcal{C}$.
Each monoidal category $\mathcal{C}=(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ gives rise to three opposite monoidal categories:

$$
\begin{aligned}
\mathcal{C}^{\mathrm{op}} & =\left(\mathcal{C}^{\mathrm{op}}, \otimes, \mathbb{1}, a^{\mathrm{op}}, l^{\mathrm{op}}, r^{\mathrm{op}}\right), \\
\mathcal{C}^{\otimes \mathrm{op}} & =\left(\mathcal{C}, \otimes^{\mathrm{op}}, \mathbb{1}, a^{\otimes \mathrm{op}}, l^{\otimes \mathrm{op}}, r^{\otimes \mathrm{op}}\right), \\
\mathcal{C}^{\mathrm{rev}} & =\left(\mathcal{C}^{\mathrm{op}}, \otimes^{\mathrm{op}}, \mathbb{1}, a^{\mathrm{rev}}, l^{\mathrm{rev}}, r^{\mathrm{rev}}\right) .
\end{aligned}
$$

Here, $\mathcal{C}^{\text {op }}$ is the category opposite to $\mathcal{C}$ defined by $\mathrm{Ob}\left(\mathcal{C}^{\mathrm{op}}\right)=\mathrm{Ob}(\mathcal{C})$ and $\operatorname{Hom}_{\mathcal{C}}$ op $(X, Y)=\operatorname{Hom}_{\mathcal{C}}(Y, X)$ for all $X, Y \in \operatorname{Ob}(\mathcal{C})$ with composition oop defined by $g \circ^{\mathrm{op}} f=f g$. The functor $\otimes^{\mathrm{op}}: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is the opposite monoidal product of $\mathcal{C}$ defined by $X \otimes{ }^{\text {op }} Y=Y \otimes X$ for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ and similarly for morphisms. The above associativity and unitality constraints are given for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ by

$$
\begin{array}{lrl}
\left(a^{\mathrm{op}}\right)_{X, Y, Z}=\left(a_{X, Y, Z}\right)^{-1}, & \left(l^{\mathrm{op}}\right)_{X}=\left(l_{X}\right)^{-1}, & \left(r^{\mathrm{op}}\right)_{X}=\left(r_{X}\right)^{-1}, \\
\left(a^{\otimes \mathrm{op}}\right)_{X, Y, Z}=\left(a_{Z, Y, X}\right)^{-1}, & \left(l^{\otimes \mathrm{op}}\right)_{X}=r_{X}, & \left(r^{\otimes \mathrm{op}}\right)_{X}=l_{X}, \\
\left(a^{\mathrm{rev}}\right)_{X, Y, Z}=a_{Z, Y, X}, & \left(l^{\mathrm{rev}}\right)_{X}=\left(r_{X}\right)^{-1}, & \left(r^{\mathrm{rev}}\right)_{X}=\left(l_{X}\right)^{-1} .
\end{array}
$$

The transformations $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}, \mathcal{C} \mapsto \mathcal{C}^{\otimes \mathrm{op}}$, and $\mathcal{C} \mapsto \mathcal{C}^{\text {rev }}$ are involutive, commute with each other, and each of them is the composition of the other two. In particular, $\mathcal{C}^{\mathrm{rev}}=\left(\mathcal{C}^{\otimes \mathrm{op}}\right)^{\mathrm{op}}=\left(\mathcal{C}^{\mathrm{op}}\right)^{\otimes \mathrm{op}}$.
1.1.5. Actions of the ground monoid. A monoidal category $\mathcal{C}=(\mathcal{C}, \otimes$, $\mathbb{1}, a, l, r)$ determines a commutative monoid $\operatorname{End}_{\mathcal{C}}(\mathbb{1})$, called the ground monoid of $\mathcal{C}$. Its product is the composition of morphisms and its unit is $\operatorname{id}_{\mathbb{1}}$. For any $X, Y \in \operatorname{Ob}(\mathcal{C})$, the set $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ carries left and right actions of the monoid $\operatorname{End}_{\mathcal{C}}(\mathbb{1})$ defined by

$$
\alpha \cdot f=l_{Y}(\alpha \otimes f) l_{X}^{-1} \quad \text { and } \quad f \cdot \alpha=r_{Y}(f \otimes \alpha) r_{X}^{-1}
$$

for any $\alpha \in \operatorname{End}_{\mathcal{C}}(\mathbb{1})$ and $f \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$. The left and right actions of $\operatorname{End}_{\mathcal{C}}(\mathbb{1})$ on itself are given by the monoid product in $\operatorname{End}_{\mathcal{C}}(\mathbb{1})$.

The actions of $\operatorname{End}_{\mathcal{C}}(\mathbb{1})$ on the sets of morphisms are compatible with monoidal product of morphisms in the following sense: for any $\alpha \in \operatorname{End}_{\mathcal{C}}(\mathbb{1})$ and any morphisms $f, g$ in $\mathcal{C}$, we have

$$
\alpha \cdot(f \otimes g)=(\alpha \cdot f) \otimes g \quad \text { and } \quad(f \otimes g) \cdot \alpha=f \otimes(g \cdot \alpha) .
$$

1.1.6. Pure categories. A monoidal category $\mathcal{C}$ is pure if the left and right actions of $\operatorname{End}_{\mathcal{C}}(\mathbb{1})$ on the sets of morphisms in $\mathcal{C}$ coincide. Thus, $\mathcal{C}$ is pure if $\alpha \cdot f=f \cdot \alpha$ for any $\alpha \in \operatorname{End}_{\mathcal{C}}(\mathbb{1})$ and any morphism $f$ in $\mathcal{C}$. In fact, it suffices to require that $\alpha \cdot \mathrm{id}_{X}=\mathrm{id}_{X} \cdot \alpha$ for any $\alpha \in \operatorname{End}_{\mathcal{C}}(\mathbb{1})$ and any
$X \in \operatorname{Ob}(\mathcal{C})$. Indeed, this condition implies that for any morphism $f: X \rightarrow Y$ in $\mathcal{C}$, we have

$$
\alpha \cdot f=\alpha \cdot\left(f \circ \mathrm{id}_{X}\right)=f \circ\left(\alpha \cdot \mathrm{id}_{X}\right)=f \circ\left(\mathrm{id}_{X} \cdot \alpha\right)=\left(f \circ \mathrm{id}_{X}\right) \cdot \alpha=f \cdot \alpha
$$

For a pure monoidal category $\mathcal{C}$, hold the following identities:

$$
\alpha \cdot(f \otimes g)=(\alpha \cdot f) \otimes g=f \otimes(\alpha \cdot g)
$$

for all $\alpha \in \operatorname{End}_{\mathcal{C}}(\mathbb{1})$ and all morphisms $f, g$ in $\mathcal{C}$.
1.1.7. Conventions. Mac Lane's coherence theorem asserts that every diagram in a monoidal category made up of the associativity and unitality constraints commutes, see [ML1, ML2]. In the sequel we suppress in our formulas the associativity and unitality constraints of monoidal categories. This does not lead to ambiguity because by Mac Lane's coherence theorem, all legitimate ways of inserting these constraints give the same results. For any objects $X_{1}, \ldots, X_{n}$ of a monoidal category with $n \geq 2$, we set

$$
X_{1} \otimes X_{2} \otimes \cdots \otimes X_{n}=\left(\ldots\left(\left(X_{1} \otimes X_{2}\right) \otimes X 3\right) \otimes \cdots \otimes X_{n-1}\right) \otimes X_{n}
$$

and similarly for morphisms.
1.1.8. Monoidal functors and natural transformations. Let $\mathcal{C}=$ $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ and $\mathcal{D}=\left(\mathcal{D}, \otimes^{\prime}, \mathbb{1}^{\prime}, a^{\prime}, l^{\prime}, r^{\prime}\right)$ be monoidal categories. A monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ endowed with a morphism $F_{0}: \mathbb{1}^{\prime} \rightarrow F(\mathbb{1})$ in $\mathcal{D}$ and with a natural transformation

$$
F_{2}=\left\{F_{2}(X, Y): F(X) \otimes^{\prime} F(Y) \rightarrow F(X \otimes Y)\right\}_{X, Y \in \mathrm{Ob}(\mathcal{C})}
$$

between the functors $F \otimes^{\prime} F=\otimes^{\prime}(F \times F): \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{D}$ and $F \otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{D}$ such that for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, the following three diagrams commute:

$$
\begin{aligned}
& \left(F(X) \otimes^{\prime} F(Y)\right) \otimes^{\prime} F(Z) \xrightarrow{a_{F(X), F(Y), F(Z)}^{\prime}} F(X) \otimes^{\prime}\left(F(Y) \otimes^{\prime} F(Z)\right) \\
& F_{2}(X, Y) \otimes^{\prime} \operatorname{id}_{F(Z)} \downarrow \downarrow \operatorname{id}_{F(X)} \otimes^{\prime} F_{2}(Y, Z) \\
& F(X \otimes Y) \otimes^{\prime} F(Z) \quad F(X) \otimes^{\prime} F(Y \otimes Z) \\
& \begin{array}{c}
F_{2}(X \otimes Y, Z) \downarrow \\
F((X \otimes Y) \otimes Z) \xrightarrow{\downarrow\left(a_{X, Y, Z}\right)} \xrightarrow{\downarrow} F(X \otimes(Y \otimes Z)),
\end{array}
\end{aligned}
$$



The morphisms $F_{0}$ and $F_{2}$ are called the monoidal constraints associated with $F$. Recall that the naturality of $F_{2}$ means that for arbitrary morphisms $f: X \rightarrow X^{\prime}$ and $g: Y \rightarrow Y^{\prime}$ in $\mathcal{C}$, the following diagram commutes:


The composition of two monoidal functors $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{E}$ is the monoidal functor $G F: C \rightarrow \mathcal{E}$ with

$$
(G F)_{0}=G\left(F_{0}\right) G_{0} \quad \text { and } \quad(G F)_{2}(X, Y)=G\left(F_{2}(X, Y)\right) G_{2}(F(X), F(Y))
$$

for all $X, Y \in \operatorname{Ob}(\mathcal{C})$. The composition of monoidal functors is associative with identity functors being the units.

A monoidal functor $\left(F, F_{2}, F_{0}\right)$ from a monoidal category $\mathcal{C}$ to a monoidal category $\mathcal{D}$ is strict if $F_{0}$ and $F_{2}(X, Y)$ are identity morphisms for all $X, Y \in$ $\mathrm{Ob}(\mathcal{C})$. For example, the identity functor $1_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$ is strict.

A monoidal functor $\left(F, F_{2}, F_{0}\right)$ is strong if $F_{0}$ and $F_{2}(X, Y)$ are isomorphisms for all $X, Y \in \operatorname{Ob}(\mathcal{C})$. Clearly, all strict monoidal functors are strong. The composition of two strict (respectively, strong) monoidal functors is strict (respectively, strong). A strong monoidal functor $\left(F, F_{2}, F_{0}\right)$ from $\mathcal{C}$ to $\mathcal{D}$ induces a morphism of monoids $\operatorname{End}_{\mathcal{C}}(\mathbb{1}) \rightarrow \operatorname{End}_{\mathcal{D}}\left(\mathbb{1}^{\prime}\right)$ by $\alpha \mapsto F_{0}^{-1} F(\alpha) F_{0}$ for all $\alpha \in \operatorname{End}_{\mathcal{C}}(\mathbb{1})$.

Each monoidal functor $F: \mathcal{C} \rightarrow \mathcal{D}$ induces a monoidal functor $F^{\otimes o \mathrm{op}}: \mathcal{C}^{\otimes \mathrm{op}} \rightarrow$ $\mathcal{D}^{\otimes \text { op }}$, which is the same functor $F$ with monoidal constraints

$$
\left(F^{\otimes \mathrm{op}}\right)_{0}=F_{0} \quad \text { and } \quad\left(F^{\otimes \mathrm{op}}\right)_{2}(X, Y)=F_{2}(Y, X)
$$

for all $X, Y \in \operatorname{Ob}(\mathcal{C})$. A strong monoidal functor $F: \mathcal{C} \rightarrow \mathcal{D}$ induces strong monoidal functors $F^{\mathrm{op}}: \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{D}^{\mathrm{op}}$ and $F^{\mathrm{rev}}: \mathcal{C}^{\mathrm{rev}} \rightarrow \mathcal{D}^{\mathrm{rev}}$. Both are equal to $F$ as functors and have the following monoidal constraints:

$$
\left(F^{\mathrm{op}}\right)_{0}=\left(F^{\mathrm{rev}}\right)_{0}=F_{0}^{-1}
$$

and for all $X, Y \in \mathrm{Ob}(\mathcal{C})$,

$$
\left(F^{\mathrm{op}}\right)_{2}(X, Y)=F_{2}(X, Y)^{-1} \quad \text { and } \quad\left(F^{\mathrm{rev}}\right)_{2}(X, Y)=F_{2}(Y, X)^{-1}
$$

Note that $F^{\mathrm{rev}}=\left(F^{\otimes \mathrm{op}}\right)^{\mathrm{op}}=\left(F^{\mathrm{op}}\right)^{\otimes \mathrm{op}}$.

A natural transformation $\varphi$ from a monoidal functor $F: \mathcal{C} \rightarrow \mathcal{D}$ to a monoidal functor $G: \mathcal{C} \rightarrow \mathcal{D}$ is monoidal if

$$
\varphi_{\mathbb{1}} F_{0}=G_{0} \quad \text { and } \quad \varphi_{X \otimes Y} F_{2}(X, Y)=G_{2}(X, Y)\left(\varphi_{X} \otimes \varphi_{Y}\right)
$$

for all $X, Y \in \operatorname{Ob}(\mathcal{C})$. If the map $\varphi_{X}: F(X) \rightarrow G(X)$ is an isomorphism for all $X \in \operatorname{Ob}(\mathcal{C})$, then such a $\varphi$ is a monoidal natural isomorphism. The functors $F$ and $G$ are monoidally isomorphic if there is a monoidal natural isomorphism $F \rightarrow G$.
1.1.9. Example. Consider the category $\operatorname{Mod}_{\mathfrak{k}}$ of $\mathbb{k}$-modules and $\mathbb{k}$-linear homomorphisms. It is equipped with the usual tensor product $\otimes_{\mathfrak{k}}$, the unit object $\mathbb{k}$. For all $\mathbb{k}$-modules $X, Y, Z$, the monoidal constraints are given by

$$
\begin{gathered}
a_{X, Y, Z}((x \otimes y) \otimes z)=x \otimes(y \otimes z), \\
l_{X}(\lambda \otimes x)=\lambda x=r_{X}(x \otimes \lambda),
\end{gathered}
$$

where $x \in X, y \in Y, z \in Z$ and $\lambda \in \mathbb{k}$. Then $\operatorname{Mod}_{\mathbb{k}}$ is a monoidal category.
1.1.10. Pairings. Let $\mathcal{C}=(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category. A pairing between two objects $X, Y$ of $\mathcal{C}$ is a morphism $X \otimes Y \rightarrow \mathbb{1}$ in $\mathcal{C}$. A pairing $\omega: X \otimes Y \rightarrow \mathbb{1}$ is non-degenerate if there is a morphism $\Omega: \mathbb{1} \rightarrow Y \otimes X$ in $\mathcal{C}$ such that

$$
\begin{equation*}
\left(\operatorname{id}_{Y} \otimes \omega\right)\left(\Omega \otimes \operatorname{id}_{Y}\right)=\operatorname{id}_{Y} \quad \text { and } \quad\left(\omega \otimes \operatorname{id}_{X}\right)\left(\mathrm{id}_{X} \otimes \Omega\right)=\mathrm{id}_{X} \tag{1.1}
\end{equation*}
$$

The morphism $\Omega$ is called the inverse of $\omega$ and is uniquely determined by $\omega$. Indeed, if we suppose that $\Omega^{\prime}: \mathbb{1} \rightarrow Y \otimes X$ is another morphism in $\mathcal{C}$ with the same property of $\Omega$, then

$$
\begin{aligned}
\Omega^{\prime} & =\operatorname{id}_{Y \otimes X} \Omega^{\prime}=\left(\operatorname{id}_{Y} \otimes \operatorname{id}_{X}\right) \Omega^{\prime}=\left(\operatorname{id}_{Y} \otimes\left(\omega \otimes \operatorname{id}_{X}\right)\left(\operatorname{id}_{X} \otimes \Omega\right)\right) \Omega^{\prime} \\
& =\left(\operatorname{id}_{Y} \otimes \omega \otimes \operatorname{id}_{X}\right)\left(\Omega^{\prime} \otimes \Omega\right)=\left(\left(\operatorname{id}_{Y} \otimes \omega\right)\left(\Omega^{\prime} \otimes \operatorname{id}_{Y}\right) \otimes \operatorname{id}_{X}\right) \Omega \\
& =\left(\operatorname{id}_{Y} \otimes \operatorname{id}_{X}\right) \Omega=\operatorname{id}_{Y \otimes X} \Omega=\Omega .
\end{aligned}
$$

1.1.11. Pairings in $\operatorname{Mod}_{k}$. By Sections 1.1.9 and 1.1.10, a pairing $\omega$ between $\mathbb{k}$-modules $X$ and $Y$ is a $\mathbb{k}$-linear homomorphism $\omega: X \otimes_{\mathbb{k}} Y \rightarrow \mathbb{k}$. The pairing $\omega$ is non-degenerate if there exists a $\mathbb{k}$-linear homomorphism $\Omega: \mathbb{1} \rightarrow Y \otimes_{\mathbb{k}} X$ satisfying (1.1). In this case, the vector

$$
*_{\omega}=\Omega\left(1_{\mathfrak{k}}\right) \in Y \otimes_{\mathfrak{k}} X
$$

is called the contraction vector of $\omega$.
Recall that the dual of a $\mathbb{k}$-module $X$ is the $\mathbb{k}$-module $X^{\star}=\operatorname{Hom}_{\mathbb{k}}(X, \mathbb{k})$ consisting of all $\mathbb{k}$-linear homomorphisms $X \rightarrow \mathbb{k}$ with the $\mathbb{k}$-module structure given by $(k f)(x)=k f(x)$ for all $k \in \mathbb{k}, f \in X^{\star}, x \in X$. A $\mathbb{k}$-module is projective of finite type if it is a direct summand of a free $\mathbb{k}$-module of finite rank. The next lemma reformulates the non-degeneracy condition of a pairing between $\mathbb{k}$-modules in terms of dual modules and matrices.

Lemma 1.1 ( $[\mathbf{T V i} \mid)$. Let $\omega: X \otimes_{\mathbb{k}} Y \rightarrow \mathbb{k}$ be a pairing in $\operatorname{Mod}_{\mathfrak{k}}$ between $\mathbb{k}$-modules $X$ and $Y$. The following three conditions are equivalent:
(a) $\omega$ is non-degenerate;
(b) $X$ is a projective $\mathbb{k}$-module of finite type and the homomorphism $Y \rightarrow$ $X^{\star}$ adjoint to $\omega$ a is an isomorphism;
(c) $Y$ is a projective $\mathbb{k}$-module of finite type and the homomorphism $X \rightarrow$ $Y^{\star}$ adjoint to $\omega$ a is an isomorphism.

Assume now that the $\mathbb{k}$-modules $X$ and $Y$ are free. Then the pairing $\omega$ is nondegenerate if and only if $X$ and $Y$ have the same finite rank $n$ and for some bases $\left(x_{i}\right)_{i=1}^{n}$ of $X$ and $\left(y_{j}\right)_{j=1}^{n}$ of $Y$, the matrix $\left[\omega\left(x_{i} \otimes_{\mathfrak{k}} y_{j}\right)\right]_{i, j=1}^{n}$ is invertible. If such is the case, the contraction vector of $\omega$ is then computed by

$$
*_{\omega}=\sum_{i, j=1}^{n} \Omega_{i, j} y_{j} \otimes_{\mathfrak{k}} x_{i} \in Y \otimes_{\mathbb{k}} X,
$$

where $\left[\Omega_{i, j}\right]_{i, j=1}^{n}$ is the inverse of the matrix $\left[\omega\left(x_{i} \otimes_{\mathbb{k}} y_{j}\right)\right]_{i, j=1}^{n}$.

### 1.2. Pivotal categories

In this section, we recall the notion of a pivotal category. We also discuss traces of endomorphisms and dimensions of objects in pivotal categories.
1.2.1. Rigid categories. A left dual of an object $X$ of a monoidal category $\mathcal{C}$ is a pair $\left({ }^{\vee} X, \mathrm{ev}_{X}\right)$, where ${ }^{\vee} X$ is an object of $\mathcal{C}$ and $\mathrm{ev}_{X}:{ }^{\vee} X \otimes X \rightarrow \mathbb{1}$ is a non-degenerate pairing. The pairing $\mathrm{ev}_{X}$ is called the left evaluation and its inverse $\operatorname{coev}_{X}: \mathbb{1} \rightarrow X \otimes^{\vee} X$ the left coevaluation. A left dual of the object $X$, if it exists, is unique up to a unique isomorphism preserving the evaluation pairing. More precisely, if $(Y, e: Y \otimes X \rightarrow \mathbb{1})$ is another left dual of $X$, then

$$
\left(e \otimes \operatorname{id}_{X}\right)\left(\operatorname{id}_{Y} \otimes \operatorname{coev}_{X}\right): Y \rightarrow{ }^{\vee} X
$$

is the unique isomorphism $a: Y \rightarrow{ }^{\vee} X$ such that $e=\operatorname{ev}_{X}\left(a \otimes \operatorname{id}_{X}\right)$.
A left duality in a monoidal category $\mathcal{C}$ is a family $\left\{\left({ }^{\vee} X, \mathrm{ev}_{X}\right)\right\}_{X \in \operatorname{Ob}(\mathcal{C})}$ where, for every $X \in \operatorname{Ob}(\mathcal{C})$, the pair $\left({ }^{\vee} X, \mathrm{ev}_{X}\right)$ is a left dual of X . A left rigid category is a monoidal category admitting a left duality. A left rigid category with distinguished left duality is a left rigid category endowed with a left duality.

Similarly, a right dual of $X \in \operatorname{Ob}(\mathcal{C})$ is a pair $\left(X^{\vee}, \widetilde{\mathrm{ev}}_{X}\right)$ where $X^{\vee} \in \mathrm{Ob}(\mathcal{C})$ and $\widetilde{\mathrm{ev}}_{X}: X \otimes X^{\vee} \rightarrow \mathbb{1}$ is a non-degenerate pairing. The pairing $\widetilde{\mathrm{ev}}_{X}$ is called the right evaluation and its inverse $\widetilde{\operatorname{coev}}_{X}: \mathbb{1} \rightarrow X^{\vee} \otimes X$ the right coevaluation. A right dual of an object of $\mathcal{C}$, if it exists, is unique up to a unique isomorphism preserving the evaluation pairing. A right duality in a monoidal category $\mathcal{C}$ is a family $\left\{\left(X^{\vee}, \widetilde{\mathrm{ev}}_{X}\right)\right\}_{X \in \mathrm{Ob}(\mathcal{C})}$ where, for every $X \in \operatorname{Ob}(\mathcal{C})$, the pair $\left(X^{\vee}, \widetilde{\mathrm{ev}}_{X}\right)$ is a right dual of $X$. A right rigid category is a monoidal category admitting a right duality. A right rigid category with distinguished right duality is a right rigid category endowed with a right duality.

A rigid category is a monoidal category which is both left rigid and right rigid, that is, which admits both a left duality and a right duality. A rigid category with distinguished duality is a rigid category endowed with a left duality and a right duality.
1.2.2. Dual functors. A left duality in a left rigid category $\mathcal{C}$ determines a functor

$$
{ }^{v_{?}}: \mathcal{C}^{\mathrm{rev}}=\left(\mathcal{C}^{\mathrm{op}}, \otimes^{\mathrm{op}}, \mathbb{1}\right) \rightarrow \mathcal{C}
$$

which carries each $X \in \operatorname{Ob}(\mathcal{C})=\mathrm{Ob}\left(\mathcal{C}^{\text {rev }}\right)$ to ${ }^{\vee} X$ and carries each morphism $f: X \rightarrow Y$ in $\mathcal{C}$ (that is a morphism $Y \rightarrow X$ in $\mathcal{C}^{\text {rev }}$ ) to its left dual

$$
{ }^{\vee} f=\left(\mathrm{ev}_{Y} \otimes \operatorname{id}_{v_{X}}\right)\left(\operatorname{id}_{v_{Y}} \otimes f \otimes{\operatorname{id} \vee_{X}}\right)\left({\operatorname{id} v_{Y}} \otimes \operatorname{coev}_{X}\right):{ }^{\vee} Y \rightarrow{ }^{\vee} X .
$$

The functor ${ }^{\vee}$ ? is strong monoidal with monoidal constraints ${ }^{\vee}{ }^{?} ?_{0}=$ coev : $\mathbb{1} \rightarrow$ ${ }^{\vee} \mathbb{1}$ and ${ }^{\vee}{ }^{\vee}{ }_{2}(X, Y):{ }^{\vee} X \otimes{ }^{\vee} Y \rightarrow{ }^{\vee}(Y \otimes X)$ defined by
${ }^{v_{?}}{ }_{2}(X, Y)=\left(\mathrm{ev}_{X} \otimes \operatorname{id}_{v_{(Y \otimes X)}}\right)\left(\operatorname{id}_{v_{X}} \otimes \mathrm{ev}_{Y} \otimes \operatorname{id}_{X \otimes}{ }^{\vee}(Y \otimes X)\right)\left(\mathrm{id}_{v_{X} \otimes^{\vee} Y} \otimes \operatorname{coev}_{Y \otimes X}\right)$.
The functor ${ }^{\vee}$ ? is called the left dual functor associated with the given left duality. The uniqueness of the left duals of objects implies that the left dual functors associated with different left dualities are monoidally isomorphic in a canonical way.

A right duality in a right rigid category $\mathcal{C}$ determines a functor $?^{\vee}: \mathcal{C}^{\mathrm{rev}} \rightarrow \mathcal{C}$ carrying each object $X$ of $\mathcal{C}$ to $X^{\vee}$ and each morphism $f: X \rightarrow Y$ in $\mathcal{C}$ to its right dual

$$
f^{\vee}=\left(\mathrm{id}_{X^{\vee}} \otimes \widetilde{\mathrm{ev}}_{Y}\right)\left(\mathrm{id}_{X^{\vee}} \otimes f \otimes \operatorname{id}_{Y^{\vee}}\right)\left({\widetilde{\operatorname{coev}_{X}}}_{X} \otimes \operatorname{id}_{Y^{\vee}}\right): Y^{\vee} \rightarrow X^{\vee}
$$

The functor $?^{\vee}$ is strong monoidal, with monoidal constraints $?_{0}^{\vee}={\widetilde{\operatorname{coev}_{1}}}_{1}: \rightarrow$ $\mathbb{1}^{\vee}$ and $?_{2}^{\vee}(X, Y): X^{\vee} \otimes Y^{\vee} \rightarrow(Y \otimes X)^{\vee}$ defined by $?_{2}^{\vee}(X, Y)=\left(\mathrm{id}_{(Y \otimes X)^{\vee}} \otimes \widetilde{\mathrm{ev}}_{Y}\right)\left(\mathrm{id}_{(Y \otimes X)^{\vee} \otimes Y} \otimes \widetilde{\mathrm{ev}}_{X} \otimes \mathrm{id}_{Y^{\vee}}\right)\left({\widetilde{\operatorname{coev}_{Y \otimes X}}}^{2} \otimes \mathrm{id}_{X^{\vee} \otimes Y^{\vee}}\right)$.

The functor ? ${ }^{\vee}$ is called the right dual functor associated with the given right duality. The right dual functors associated with different right dualities are monoidally isomorphic in a canonical way.

For a rigid category $\mathcal{C}$ with distinguished duality, the left and right dual functors ${ }^{\vee}$ ?: $\mathcal{C}^{\text {rev }} \rightarrow \mathcal{C}$ and $?^{\vee}: \mathcal{C}^{\text {rev }} \rightarrow \mathcal{C}$ are strong monoidal equivalences
 $X \in \operatorname{Ob}(\mathcal{C})$, the corresponding monoidal natural isomorphisms ${ }^{\vee}\left(X^{\vee}\right) \simeq X \simeq$ $\left({ }^{\vee} X\right)^{\vee}$ are

$$
\begin{aligned}
& \left(\widetilde{\mathrm{ev}}_{X} \otimes \operatorname{id}_{\vee}\left(X^{\vee}\right)\right)\left(\mathrm{id}_{X} \otimes \operatorname{coev}_{X^{\vee}}\right): X \rightarrow{ }^{\vee}\left(X^{\vee}\right), \\
& \left(\operatorname{id}_{\left(\vee_{X}\right)^{\vee}} \otimes \mathrm{ev}_{X}\right)\left(\widetilde{\operatorname{coev} \vee_{X}} \otimes \operatorname{id}_{X}\right): X \rightarrow\left({ }^{\vee} X\right)^{\vee} .
\end{aligned}
$$

1.2.3. Duality and monoidal functor. Note that a strong monoidal functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between monoidal categories carries any object having a left (respectively, right) dual to an object having a left (respectively, right) dual. Indeed, consider an object $X$ of $\mathcal{C}$ with left dual $\left({ }^{\vee} X, \mathrm{ev}_{X}\right)$. By [TVi, Lemma 1.5], the non-degeneracy of $\mathrm{ev}_{X}$ implies the non-degeneracy of the pairing

$$
\left(\mathrm{ev}_{X}\right)^{F}=F_{0}{ }^{-1} F\left(\mathrm{ev}_{X}\right) F_{2}\left({ }^{\vee} X, X\right): F\left({ }^{\vee} X\right) \otimes F(X) \rightarrow \mathbb{1} .
$$

Thus $\left(F\left({ }^{\vee} X\right),\left(\mathrm{ev}_{X}\right)^{F}\right)$ is a left dual of $F(X)$. Similarly, if $X \in \operatorname{Ob}(\mathcal{C})$ has a right dual $\left(X^{\vee}, \widetilde{\mathrm{ev}}_{X}\right)$, then $\left(F\left(X^{\vee},\left(\widetilde{\mathrm{ev}}_{X}\right)^{F}\right)\right.$ is a right dual of $F(X)$.

A strong monoidal functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between left rigid categories with distinguished left duality determines a monoidal natural isomorphism

$$
F^{l}=\left\{F^{l}(X): F\left({ }^{\vee} X\right) \rightarrow{ }^{\vee} F(X)\right\}_{X \in \mathrm{Ob}(\mathcal{C})}
$$

from the functor $F^{\vee}$ ?: $\mathcal{C}^{\text {rev }} \rightarrow \mathcal{D}$ to the functor ${ }^{\vee} ? F^{\text {rev }}: \mathcal{C}^{\text {rev }} \rightarrow \mathcal{D}$. It is defined as follows. For each $X \in \operatorname{Ob}(\mathcal{C})$, both $\left(F\left({ }^{\vee} X\right),\left(\mathrm{ev}_{X}\right)^{F}\right)$ and $\left({ }^{\vee} F(X), \mathrm{ev}_{F(X)}\right)$ are left duals of $F(X)$. By the uniqueness of a left dual, there is a unique isomorphism

$$
F^{l}(X): F\left({ }^{\vee} X\right) \rightarrow{ }^{\vee} F(X)
$$

preserving the evaluation pairing, i.e., such that

$$
\left(\mathrm{ev}_{X}\right)^{F}=\operatorname{ev}_{F(X)}\left(F^{l}(X) \otimes \operatorname{id}_{F(X)}\right) .
$$

The isomorphism $F^{l}(X)$ is computed by

$$
F^{l}(X)=\left(\left(\mathrm{ev}_{X}\right)^{F} \otimes \operatorname{id}_{v_{F(X)}}\right)\left(\operatorname{id}_{F\left(\vee_{X}\right)} \otimes \operatorname{coev}_{F(X)}\right) .
$$

Likewise, a strong monoidal functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between right rigid categories with distinguished right duality determines a monoidal natural isomorphism

$$
F^{r}=\left\{F^{r}(X): F\left(X^{\vee}\right) \rightarrow F(X)^{\vee}\right\}_{X \in \mathrm{Ob}(\mathcal{C})}
$$

from $F ?^{\vee}: \mathcal{C}^{\text {rev }} \rightarrow \mathcal{D}$ to $?^{\vee} F^{\mathrm{rev}}: \mathcal{C}^{\mathrm{rev}} \rightarrow \mathcal{D}$. It is computed by

$$
F^{r}(X)=\left(\operatorname{id}_{F(X)^{\vee}} \otimes\left(\widetilde{\mathrm{ev}}_{X}\right)^{F}\right)\left(\widetilde{\operatorname{coev}}_{F(X)} \otimes \operatorname{id}_{F\left(X^{\vee}\right)}\right)
$$

for any $X \in \operatorname{Ob}(\mathcal{C})$.
1.2.4. Pivotal categories. A pivotal category is a rigid category with distinguished duality such that the induced left and right dual functors coincide as monoidal functors. In other words, a pivotal category is a monoidal category $\mathcal{C}$ endowed with a pivotal duality, that is, a family of triples

$$
\left\{\left(X^{*}, \mathrm{ev}_{X}, \widetilde{\mathrm{ev}}_{X}\right)\right\}_{X \in \mathrm{Ob}(\mathcal{C})},
$$

where

- $X^{*}$ is an object of $\mathcal{C}$ called the dual of $X$;
- $\mathrm{ev}_{X}: X^{*} \otimes X \rightarrow \mathbb{1}$ is a non-degenerate pairing in $\mathcal{C}$;
- $\widetilde{\mathrm{ev}}_{X}: X \otimes X^{*} \rightarrow \mathbb{1}$ is a non-degenerate pairing in $\mathcal{C}$;
such that the left dual functor associated with the left duality $\left\{\left(X^{*}, \operatorname{ev}_{X}\right)\right\}_{X \in \operatorname{Ob}(\mathcal{C})}$ and the right dual functor associated with the right duality $\left\{\left(X^{*}, \widetilde{\mathrm{ev}}_{X}\right)\right\}_{X \in \mathrm{Ob}(\mathcal{C})}$ coincide as monoidal functors. The pairings $\mathrm{ev}_{X}$ and $\widetilde{\mathrm{ev}}_{X}$ are called the left evaluation and the right evaluation, respectively. Let $\operatorname{coev}_{X}: \mathbb{1} \rightarrow X \otimes X^{*}$ and $\widetilde{\operatorname{coev}}_{X}: \mathbb{1} \rightarrow X^{*} \otimes X$ be the inverses of these pairings. These two morphisms are called respectively the left coevaluation and the right coevaluation. The equality of the left and right dual functors means that:
(i) for any morphism $f: X \rightarrow Y$ in $\mathcal{C}$,

$$
\begin{aligned}
f^{*} & =\left(\mathrm{ev}_{Y} \otimes \operatorname{id}_{X^{*}}\right)\left(\operatorname{id}_{Y^{*}} \otimes f \otimes \operatorname{id}_{X^{*}}\right)\left(\operatorname{id}_{Y^{*}} \otimes \operatorname{coev}_{X}\right) \\
& =\left(\operatorname{id}_{X^{*}} \otimes \widetilde{\mathrm{ev}}_{Y}\right)\left(\operatorname{id}_{X^{*}} \otimes f \otimes \operatorname{id}_{Y^{*}}\right)\left(\widetilde{\operatorname{cov}}_{X} \otimes \operatorname{id}_{Y^{*}}\right): Y^{*} \rightarrow X^{*} ;
\end{aligned}
$$

(ii) $\operatorname{coev}_{\mathbb{1}}={\widetilde{\operatorname{coev}_{\mathbb{1}}}}: \mathbb{1} \rightarrow \mathbb{1}^{*}$;
(iii) for all $X, Y \in \operatorname{Ob}(\mathcal{C})$, we have the following equality of morphisms from $X^{*} \otimes Y^{*} \rightarrow(Y \otimes X)^{*}:$

$$
\begin{aligned}
& \left(\mathrm{ev}_{X} \otimes \operatorname{id}_{(Y \otimes X)^{*}}\right)\left(\operatorname{id}_{X^{*}} \otimes \mathrm{ev}_{Y} \otimes \operatorname{id}_{X \otimes(Y \otimes X)^{*}}\right)\left(\operatorname{id}_{X^{*} \otimes Y^{*}} \otimes \operatorname{coev}_{Y \otimes X}\right) \\
= & \left.\operatorname{id}_{(Y \otimes X)^{*}} \otimes \widetilde{\mathrm{ev}}_{Y}\right)\left(\operatorname{id}_{(Y \otimes X)^{*} \otimes Y} \otimes \widetilde{\mathrm{ev}}_{X} \otimes \operatorname{id}_{Y^{*}}\right)\left({\left.\widetilde{\operatorname{coev}_{Y \otimes X}} \otimes \operatorname{id}_{X^{*} \otimes Y^{*}}\right) .} .\right.
\end{aligned}
$$

Clearly, a pivotal category is, in particular, a rigid category with distinguished duality. The left and right dual functors form a single functor $?^{*}: \mathcal{C}^{\text {rev }} \rightarrow \mathcal{C}$ called the dual functor of $\mathcal{C}$. It carries any object $X \in \operatorname{Ob}\left(\mathcal{C}^{\text {rev }}\right)=$ $\mathrm{Ob}(\mathcal{C})$ to $X^{*}$ and any morphism $f: X \rightarrow Y$ in $\mathcal{C}$ (that is a morphism $Y \rightarrow X$ in $\mathcal{C}^{\text {rev }}$ ) to its dual $f^{*}: Y^{*} \rightarrow X^{*}$ defined as the left-hand side (or the right-hand side) of the equality in ( $i$ ) above. The monoidal constraints

$$
?_{0}^{*}: \mathbb{1} \rightarrow \mathbb{1}^{*} \quad \text { and } \quad ?_{2}^{*}(X, Y): X^{*} \otimes Y^{*} \rightarrow(Y \otimes X)^{*}
$$

of the dual functor ?* are the morphisms defined by $(i i)$ and (iii), respectively. The duality identities

$$
\begin{aligned}
\left(\mathrm{id}_{X} \otimes \mathrm{ev}_{X}\right)\left(\operatorname{coev}_{X} \otimes \operatorname{id}_{X}\right) & =\mathrm{id}_{X}=\left(\widetilde{\mathrm{ev}}_{X} \otimes \operatorname{id}_{X}\right)\left(\mathrm{id}_{X} \otimes \widetilde{\operatorname{coev}_{X}}\right), \\
\left(\mathrm{ev}_{X} \otimes \mathrm{id}_{X^{*}}\right)\left(\mathrm{id}_{X^{*}} \otimes \operatorname{coev}_{X}\right) & =\operatorname{id}_{X^{*}}=\left(\mathrm{id}_{X^{*}} \otimes \widetilde{\mathrm{ev}}_{X}\right)\left(\widetilde{\operatorname{coev}}_{X} \otimes \operatorname{id}_{X^{*}}\right)
\end{aligned}
$$

imply that $\left(?_{0}^{*}\right)^{-1}=\mathrm{ev}_{\mathbb{1}}=\widetilde{\mathrm{ev}}_{\mathbb{1}}: \mathbb{1}^{*} \rightarrow \mathbb{1}$ and

$$
\begin{gathered}
\left(?_{2}^{*}(X, Y)\right)^{-1} \\
=\left(\mathrm{ev}_{Y \otimes X} \otimes \operatorname{id}_{X^{*} \otimes Y^{*}}\right)\left(\operatorname{id}_{(Y \otimes X)^{*} \otimes Y} \otimes \operatorname{coev}_{X} \otimes \operatorname{id}_{Y^{*}}\right)\left(\operatorname{id}_{(X \otimes Y)^{*}} \otimes \operatorname{coev}_{Y}\right) \\
=\left(\operatorname{id}_{X^{*} \otimes Y^{*}} \otimes \widetilde{\mathrm{ev}}_{Y \otimes X}\right)\left(\operatorname{id}_{X^{*}} \otimes \widetilde{\left.\operatorname{coev}_{Y} \otimes \operatorname{id}_{X \otimes(Y \otimes X)^{*}}\right)\left(\widetilde{\operatorname{coev}_{X}} \otimes \operatorname{id}_{(Y \otimes X)^{*}}\right) .}\right.
\end{gathered}
$$

If $\mathcal{C}$ is a pivotal category, then the opposite monoidal categories

$$
\mathcal{C}^{\mathrm{op}}=\left(\mathcal{C}^{\mathrm{op}}, \otimes, \mathbb{1}\right), \quad \mathcal{C}^{\otimes \mathrm{op}}=\left(\mathcal{C}, \otimes^{\mathrm{op}}, \mathbb{1}\right), \quad \mathcal{C}^{\mathrm{rev}}=\left(\mathcal{C}^{\mathrm{op}}, \otimes^{\mathrm{op}}, \mathbb{1}\right)
$$

are pivotal in a canonical way. The dual objects in them are the same as in $\mathcal{C}$ and the evaluation morphisms are

$$
\begin{array}{lll}
\mathrm{ev}_{X}^{\mathrm{op}}=\widetilde{\operatorname{coev}_{X}}, & \mathrm{ev}_{X}^{\otimes \mathrm{op}}=\widetilde{\mathrm{ev}}_{X}, & \mathrm{ev}_{X}^{\mathrm{rev}}=\operatorname{coev}_{X}, \\
\widetilde{\mathrm{ev}}_{X}^{\mathrm{op}}=\operatorname{coev}_{X}, & \widetilde{\mathrm{ev}}_{X}^{\otimes \mathrm{op}}=\mathrm{ev}_{X}, & \widetilde{\mathrm{ev}}_{X}^{\mathrm{rev}}=\widetilde{\operatorname{coev}_{X}} .
\end{array}
$$

For each $X \in \operatorname{Ob}(\mathcal{C})$, we set $X^{* *}=\left(X^{*}\right)^{*}$ and consider a morphism $\psi_{X}: X \rightarrow X^{* *}$ by

$$
\psi_{X}=\left(\widetilde{\mathrm{ev}}_{X} \otimes \operatorname{id}_{X^{* *}}\right)\left(\mathrm{id}_{X} \otimes \operatorname{coev}_{X^{*}}\right)
$$

The pivotal structure is the following monoidal natural isomorphism:

$$
\psi=\left\{\psi_{X}: X \rightarrow X^{* *}\right\}_{X \in \operatorname{Ob}(\mathcal{C})} .
$$

The expressions given above for the dual $f^{*}: Y^{*} \rightarrow X^{*}$ of a morphism $f: X \rightarrow Y$ in $\mathcal{C}$ and the duality identities imply the dual morphism identities:

$$
\begin{array}{ll}
\mathrm{ev}_{X}\left(f^{*} \otimes \operatorname{id}_{X}\right)=\operatorname{ev}_{Y}\left(\operatorname{id}_{Y^{*}} \otimes f\right), & \left(\operatorname{id}_{Y} \otimes f^{*}\right) \operatorname{coev}_{Y}=\left(f \otimes \operatorname{id}_{X^{*}}\right) \operatorname{coev}_{X}, \\
\widetilde{\mathrm{ev}}_{X}\left(\operatorname{id}_{X} \otimes f^{*}\right)=\widetilde{\mathrm{ev}}_{Y}\left(f \otimes \operatorname{id}_{Y^{*}}\right), & \left(f^{*} \otimes \operatorname{id}_{Y}\right) \widetilde{\operatorname{cov}}_{Y}=\left(\operatorname{id}_{X^{*}} \otimes f\right) \widetilde{\operatorname{coev}}_{X} .
\end{array}
$$

Lemma 1.2. Let $\phi=\left\{\phi_{X}: X \rightarrow X\right\}_{X \in \mathrm{Ob}(\mathcal{C})}$ be a monoidal natural endomorphism of the identity functor $1_{\mathcal{C}}$ of $\mathcal{C}$. Then $\phi$ is an automorphism and

$$
\begin{equation*}
\phi_{X^{*}}=\left(\phi_{X}^{*}\right)^{-1}=\left(\phi_{X}^{-1}\right)^{*} \tag{1.2}
\end{equation*}
$$

for all $X \in \operatorname{Ob}(\mathcal{C})$
Proof. For any object $X$ of $\mathcal{C}$ consider the left evaluation pairing

$$
\mathrm{ev}_{X}: X^{*} \otimes X \rightarrow \mathbb{1}
$$

$$
\mathrm{ev}_{X} \stackrel{(i)}{=} \phi_{\mathbb{1}} \operatorname{ev}_{X} \stackrel{(i i)}{=} \operatorname{ev}_{X} \phi_{X^{*} \otimes X} \stackrel{(i i i)}{=} \mathrm{ev}_{X}\left(\phi_{X^{*}} \otimes \phi_{X}\right) \stackrel{(i v)}{=} \mathrm{ev}_{X}\left(\phi_{X}^{*} \phi_{X^{*}} \otimes \operatorname{id}_{X}\right)
$$

here (ii) follows from the naturality of $\phi,(i)$ and (iii) from the monoidality of $\phi$ and ( $i v$ ) from dual morphism identities. Since ev ${ }_{X}$ is invertible, $\phi_{X}^{*} \phi_{X^{*}}=\mathrm{id}_{X^{*}}$ and so $\phi_{X}$ is an isomorphism and $\phi_{X^{*}}=\left(\phi_{X}^{*}\right)^{-1}$. One proves similarly the other equality.
1.2.5. Remark. A pivotal category may be equivalently defined as a left rigid category $\mathcal{C}$ with distinguished left duality $\left\{\left({ }^{\vee} X, \mathrm{ev}_{X}\right)\right\}_{X \in \mathrm{Ob}(\mathcal{C})}$ and distinguished monoidal natural isomorphism $\psi: 1_{\mathcal{C}} \rightarrow{ }^{\vee}{ }^{\vee}$ ? where ${ }^{\vee V}$ ?: $\mathcal{C} \rightarrow \mathcal{C}$ is the strong monoidal functor defined by ${ }^{\vee}{ }^{\vee} ?={ }^{\vee}$ ? $\circ\left({ }^{\mathrm{V}} \text { ? }\right)^{\mathrm{rev}}$. Indeed, this data turns $\mathcal{C}$ into a pivotal category (in the sense of Section 1.2.4) with pivotal duality

$$
\left\{\left(X^{*}={ }^{\vee} X, \mathrm{ev}_{X}, \widetilde{\mathrm{ev}}_{X}=\operatorname{ev} \vee_{X}\left(\psi_{X} \otimes \operatorname{id}^{\prime}\right): X \otimes X^{*} \rightarrow \mathbb{1}\right)\right\}_{X \in \mathrm{Ob}(\mathcal{C})} .
$$

1.2.6. Pivotal functor. Let $\mathcal{C}$ and $\mathcal{D}$ be pivotal categories. A pivotal functor from $\mathcal{C}$ to $\mathcal{D}$ is a strong monoidal functor $F: \mathcal{C} \rightarrow \mathcal{D}$ such that the associated monoidal natural isomorphisms $F^{l}$ and $F^{r}$ defined in Section 1.2.3 are equal. Set then $F^{1}=F^{l}=F^{r}$. The composition of two pivotal functors is pivotal. If $F$ is a pivotal functor, then so are $F^{\otimes \mathrm{op}}, F^{\mathrm{op}}, F^{\mathrm{rev}}$, see Section 1.1.8,

A strictly pivotal functor from $\mathcal{C}$ to $\mathcal{D}$ is a pivotal functor $F: \mathcal{C} \rightarrow \mathcal{D}$ such that $F^{1}$ is the identity, that is, $F\left(X^{*}\right)=F(X)^{*}$ and $F^{1}(X)=\operatorname{id}_{F(X)^{*}}$ for all $X \in \operatorname{Ob}(\mathcal{C})$. For example, given a pivotal category $\mathcal{C}$, the identity functor $1_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$ and the dual functor $?^{*}: \mathcal{C}^{\text {rev }} \rightarrow \mathcal{C}$ are strictly pivotal. Note that a strict monoidal functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between pivotal categories is strictly pivotal if and only if $F\left(X^{*}\right)=F(X)^{*}, F\left(\mathrm{ev}_{X}\right)=\mathrm{ev}_{F(X)}$, and $F\left(\widetilde{\mathrm{ev}}_{X}\right)=\widetilde{\mathrm{ev}}_{F(X)}$ for all $X \in \operatorname{Ob}(\mathcal{C})$.

Two pivotal categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there is a pivotal equivalence $\mathcal{C} \rightarrow \mathcal{D}$, that is, a pivotal functor $\mathcal{C} \rightarrow \mathcal{D}$ which is an equivalence of the underlying categories. For example, for any pivotal category $\mathcal{C}$, the dual functor ?*: $\mathcal{C}^{\text {rev }} \rightarrow \mathcal{C}$ is a pivotal equivalence. Consequently, the pivotal categories $\mathcal{C}$ and $\mathcal{C}^{\text {rev }}$ are pivotal equivalent and so are the pivotal categories $\mathcal{C}^{\otimes \mathrm{op}}=\left(\mathcal{C}^{\mathrm{op}}\right)^{\mathrm{rev}}$ and $\mathcal{C}^{\mathrm{op}}$.
1.2.7. Trace and dimensions. Let $\mathcal{C}$ be a pivotal category. Recall that the monoid $\operatorname{End}_{\mathcal{C}}(\mathbb{1})$ is commutative. For an endomorphism $f$ in $\mathcal{C}$ of an object $X$ of $\mathcal{C}$ are defined the left trace $\operatorname{tr}_{l}(f) \in \operatorname{End}_{\mathcal{C}}(\mathbb{1})$ and the right trace $\operatorname{tr}_{r}(f) \in \operatorname{End}_{\mathcal{C}}(\mathbb{1})$ in the following way:

$$
\operatorname{tr}_{l}(f)=\operatorname{ev}_{X}\left(\operatorname{id}_{X^{*}} \otimes f\right) \widetilde{\operatorname{coev}}_{X} \quad \text { and } \quad \operatorname{tr}_{r}(f)=\widetilde{\operatorname{ev}}_{X}\left(f \otimes \operatorname{id}_{X^{*}}\right) \operatorname{coev}_{X} .
$$

Both traces are symmetric, that is for any morphisms $g: X \rightarrow Y$ and $h: Y \rightarrow$ $X$ in $\mathcal{C}$ we have:

$$
\operatorname{tr}_{l}(g h)=\operatorname{tr}_{l}(h g) \quad \text { and } \quad \operatorname{tr}_{r}(g h)=\operatorname{tr}_{r}(h g)
$$

We denote the left/right actions of the ground monoid $\operatorname{End}_{\mathcal{C}}(\mathbb{1})$ with a dot. Furthermore, for any $\alpha \in \operatorname{End}_{\mathcal{C}}(\mathbb{1})$ and for any endomorphism $f, g$ in $\mathcal{C}$ of an object $X$ of $\mathcal{C}$ we have that:

$$
\begin{aligned}
& \operatorname{tr}_{l}(\alpha)= \operatorname{tr}_{r}(\alpha)=\alpha, \quad \operatorname{tr}_{l}(f \cdot \alpha)=\alpha \operatorname{tr}_{l}(f), \quad \operatorname{tr}_{r}(\alpha \cdot f)=\alpha \operatorname{tr}_{r}(f), \\
& \operatorname{tr}_{l}(f \otimes g)=\operatorname{tr}_{l}\left(\operatorname{tr}_{l}(f) \cdot g\right), \quad \operatorname{tr}_{l}(f)=\operatorname{tr}_{r}\left(f^{*}\right), \\
& \operatorname{tr}_{r}(f \otimes g)=\operatorname{tr}_{r}\left(f \cdot \operatorname{tr}_{r}(g)\right), \quad \operatorname{tr}_{r}(f)=\operatorname{tr}_{l}\left(f^{*}\right) .
\end{aligned}
$$

These formulas imply the identities

$$
\operatorname{tr}_{l}(f)=\operatorname{tr}_{l}\left(f^{* *}\right) \quad \text { and } \quad \operatorname{tr}_{r}(f)=\operatorname{tr}_{r}\left(f^{* *}\right)
$$

If $\mathcal{C}$ is pure (see 1.1.6), then the traces $\operatorname{tr}_{l}$ and $\operatorname{tr}_{r}$ are $\otimes$-multiplicative:

$$
\operatorname{tr}_{l}(f \otimes g)=\operatorname{tr}_{l}(f) \operatorname{tr}_{l}(g) \quad \text { and } \quad \operatorname{tr}_{r}(f \otimes g)=\operatorname{tr}_{r}(f) \operatorname{tr}_{r}(g)
$$

for all endomorphisms $f$ and $g$ of objects of $\mathcal{C}$.
The left dimension and right dimension of an object $X$ of $\mathcal{C}$ is defined by

$$
\operatorname{dim}_{l}(X)=\operatorname{tr}_{l}\left(\mathrm{id}_{X}\right) \quad \text { and } \quad \operatorname{dim}_{r}(X)=\operatorname{tr}_{r}\left(\mathrm{id}_{X}\right) .
$$

We observe that $\operatorname{dim}_{l}(\mathbb{1})=\operatorname{dim}_{r}(\mathbb{1})=\mathrm{id}_{\mathbb{1}}$. Clearly, we have that if $\mathcal{C}$ is pure then the dimensions are $\otimes$-multiplicative, i.e., for any $X, Y \in \mathrm{Ob}(\mathcal{C})$

$$
\operatorname{dim}_{l}(X \otimes Y)=\operatorname{dim}_{l}(X) \operatorname{dim}_{l}(Y) \quad \text { and } \quad \operatorname{dim}_{r}(X \otimes Y)=\operatorname{dim}_{r}(X) \operatorname{dim}_{r}(Y)
$$

1.2.8. Spherical categories. A spherical category is a pivotal category whose left and right traces are equal, that is, $\operatorname{tr}_{l}(f)=\operatorname{tr}_{r}(f)$ for every endomorphism $f$ in the category. Then

$$
\operatorname{tr}(f)=\operatorname{tr}_{l}(f)=\operatorname{tr}_{r}(f)
$$

is the trace of $f$. In a spherical category, the left and right dimensions of any object $X$ are equal. Then $\operatorname{dim}(X)=\operatorname{dim}_{l}(X)=\operatorname{dim}_{r}(X)$ is the dimension of $X$. The properties of the traces imply that in any spherical category, $\operatorname{tr}$ and $\operatorname{dim}$ are $\otimes$-multiplicative. Indeed, for any endomorphisms $f, g \in \operatorname{End}_{\mathcal{C}}(X)$,
$\operatorname{tr}(f \otimes g)=\operatorname{tr}_{l}(f \otimes g)=\operatorname{tr}_{l}\left(\operatorname{tr}_{l}(f) \cdot g\right)=\operatorname{tr}_{r}\left(\operatorname{tr}_{l}(f) \cdot g\right)=\operatorname{tr}_{l}(f) \operatorname{tr}_{r}(g)=\operatorname{tr}(f) \operatorname{tr}(g)$.

### 1.3. Graphical calculus

In this section, we briefly discuss a method firstly suggested by Penrose $\mathbf{P e}$ that allows to represent morphisms in categories by diagrams. We focus on the case of pivotal categories.
1.3.1. Pictorical representation. We present morphisms in a pivotal category $\mathcal{C}$ by plane diagrams called Penrose diagrams, that must be read from the bottom to the top. The diagrams are made of two elements:

- oriented arcs, each of them colored with an object of $\mathcal{C}$,
- boxes, each of them colored with a morphism of $\mathcal{C}$.

The arcs connect the boxes and have no self or mutual intersections. We represent the identity $\mathrm{id}_{X}$ of an object $X$ of $\mathcal{C}$, a morphism $f: X \rightarrow Y$, and the composition of two morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ as follows:

$$
\mathrm{id}_{X}=\left.\right|_{\mathrm{X}}, \quad f=\frac{\downarrow_{Y}}{\frac{f}{f^{\prime}}}, \quad \text { and } \quad g \circ f=\frac{\begin{array}{c}
\dagger^{Z} \\
g \\
\dagger_{Y} \\
\frac{f_{X}}{f}
\end{array} .}{.} .
$$

The monoidal product of two morphisms $f: U \rightarrow V$ and $g: W \rightarrow Z$ is represented by juxtaposition of the diagrams:

$$
f \otimes g=\frac{t_{v}}{\dagger_{v}} \frac{t_{z}}{\dagger^{g}} .
$$

We also use boxes with several arcs attached to their horizontal sides, for example a morphism $f: A \otimes B \otimes C \rightarrow A^{\prime} \otimes B^{\prime} \otimes C^{\prime}$ in $\mathcal{C}$ can be represented in various ways:

The dual of an object is encoded by the orientation of the arc colored by that object. That is, an arc colored with $X \in \operatorname{Ob}(\mathcal{C})$ and oriented downward contributes X to the source/target of morphisms. An arc colored with $X \in$ $\mathrm{Ob}(\mathcal{C})$ and oriented upward contributes $X^{*}$ to the source/target of morphisms. For example, $\mathrm{id}_{X^{*}}$ and a morphism $f: X^{*} \otimes Y \rightarrow A^{*} \otimes B \otimes C^{*}$ in $\mathcal{C}$ can be represented as:

$$
\mathrm{id}_{X^{*}}=\left.\right|_{x^{*}}=\left.\right|_{x} \quad \text { and } \quad f=\frac{\hat{t}_{A} t_{B} \dagger_{C}}{f} \hat{f}_{x} \dagger_{Y}
$$

The left/right evaluations and the left/right coevaluations for an object $X$ of $\mathcal{C}$, are depicted as follows:


The dual $f^{*}: Y^{*} \rightarrow X^{*}$ of a morphism $f: X \rightarrow Y$ in $\mathcal{C}$ are graphically presented as follows:

$$
f^{*}=\overbrace{Y} \quad \uparrow^{X}=\uparrow^{X}
$$

The fact that $\operatorname{coev}_{X}$ and $\widetilde{\operatorname{coev}}_{X}$ are the inverses of pairings $\mathrm{ev}_{X}$ and $\widetilde{\mathrm{ev}}_{X}$ for $X \in \mathrm{Ob}(\mathcal{C})$ is graphically expressed by the following identities:


For the dual object $X^{*}$ of $X$ we have:


The previous relations are called duality identities. The dual morphism identities may be represented graphically as:


The left and right traces of a morphism $g: X \rightarrow X$ are depicted as follows:


In the particular case in which $g=\mathrm{id}_{X}$, the left/right dimensions of $X$ are represented as follows:

$$
\operatorname{dim}_{l}(X)=\text { and }_{X} \operatorname{dim}_{r}(X)=
$$

The following theorem is due to Joyal and Street [JS1, JS2].
THEOREM 1.3. If $\mathcal{C}$ is a pivotal category, then the morphism represented by a Penrose diagram $\mathcal{P}$ is invariant under isotopies of $\mathcal{P}$ in the 2 -dimensional plane.
1.3.2. Signed objects. A signed object of a pivotal category $\mathcal{C}$ is a pair $(X, \varepsilon)$ where $X \in \operatorname{Ob}(\mathcal{C})$ and $\varepsilon \in\{+,-\}$. The corresponding object in $\mathcal{C}$ of pair $(X, \varepsilon)$ is noted by $X^{\varepsilon}$ and defined as follow:

$$
X^{\varepsilon}= \begin{cases}X & \text { if } \varepsilon=+, \\ X^{*} & \text { if } \varepsilon=-.\end{cases}
$$

We extend, for $n \geq 1$, this notation to any tuple

$$
S=\left(\left(X_{1}, \varepsilon_{1}\right), \ldots,\left(X_{n}, \varepsilon_{n}\right)\right)
$$

of signed objects of $\mathcal{C}$, we set

$$
X_{S}=X_{1}^{\varepsilon_{1}} \otimes \cdots \otimes X_{n}^{\varepsilon_{n}} \in \operatorname{Ob}(\mathcal{C})
$$

For an empty tuple of signed objects $S=\emptyset$, we set $X_{\emptyset}=\mathbb{1}$. The dual of a tuple $S$ of signed objects of $\mathcal{C}$ is

$$
S^{*}=\left(\left(X_{n},-\varepsilon_{n}\right), \ldots,\left(X_{1},-\varepsilon_{1}\right)\right) .
$$

1.3.3. Generalized evaluations. For any tuple $S=\left(\left(X_{1}, \varepsilon_{1}\right), \ldots,\left(X_{n}, \varepsilon_{n}\right)\right)$ of a signed object we consider the following pairing

$$
\begin{equation*}
\mathrm{ev}_{S}: X_{S^{*}} \otimes X_{S} \rightarrow \mathbb{1}: \tag{1.3}
\end{equation*}
$$

called evaluation. Let

$$
\begin{equation*}
\operatorname{coev}_{S}: \mathbb{1} \rightarrow X_{S} \otimes X_{S^{*}} \tag{1.4}
\end{equation*}
$$

be a morphism in $\mathcal{C}$ called coevaluation. They are respectively represented by the following Penrose diagrams:

and


Here the arc labeled with $X_{i}$ is oriented toward the right endpoint if $\varepsilon_{i}=+$ and toward the left endpoint if $\varepsilon_{i}=-$.

Using graphical calculus, we prove that

$$
\left(\mathrm{id}_{X_{S}} \otimes \mathrm{ev}_{S}\right)\left(\operatorname{coev}_{S} \otimes \operatorname{id}_{X_{S}}\right)=\operatorname{id}_{X_{S}}
$$

and

$$
\left(\mathrm{ev}_{S} \otimes \operatorname{id}_{X_{S^{*}}}\right)\left(\operatorname{id}_{X_{S^{*}}} \otimes \operatorname{coev}_{S}\right)=\operatorname{id}_{X_{S^{*}}} .
$$

Thus, the pairing $\mathrm{ev}_{S}$ is non-degenerate with inverse $\operatorname{coev}_{S}$. By definition, for $n=0$, we have: $\emptyset^{*}=\emptyset$ and $\mathrm{ev}_{\emptyset}=\operatorname{coev}_{\emptyset}=\operatorname{id}_{\mathbb{1}}$. The tuple $S$ also determines an isomorphism

$$
\Psi_{S}: X_{S} \rightarrow\left(X_{S^{*}}\right)^{*} .
$$

For $n=0$, we set $\Psi_{\emptyset}=\operatorname{coev}_{\mathbb{1}}=\widetilde{\operatorname{coev}_{\mathbb{1}}}: \mathbb{1} \rightarrow \mathbb{1}^{*}$. For $n=1$ and $X \in \operatorname{Ob}(\mathcal{C})$, set

$$
\Psi_{(X,-)}=\operatorname{id}_{X^{*}}: X^{*} \rightarrow X^{*} \quad \text { and } \quad \Psi_{(X,+)}=\psi_{X}: X \rightarrow X^{* *}
$$

where

$$
\psi_{X}=\left(\widetilde{\mathrm{ev}}_{X} \otimes \operatorname{id}_{X^{* *}}\right)\left(\mathrm{id}_{X} \otimes \operatorname{coev}_{X^{*}}\right)
$$

For $n \geq 2$, we define $\Psi_{S}$ as the composition of the isomorphism
$\Psi_{\left(X_{1}, \varepsilon_{1}\right)} \otimes \cdots \otimes \Psi_{\left(X_{n}, \varepsilon_{n}\right)}: X_{S}=X_{1}^{\varepsilon_{1}} \otimes \cdots \otimes X_{n}^{\varepsilon_{n}} \rightarrow\left(X_{1}^{-\varepsilon_{1}}\right)^{*} \otimes \cdots \otimes\left(X_{n}^{-\varepsilon_{n}}\right)^{*}$ with the isomorphism

$$
\left(X_{1}^{-\varepsilon_{1}}\right)^{*} \otimes \cdots \otimes\left(X_{n}^{-\varepsilon_{n}}\right)^{*} \simeq\left(X_{n}^{-\varepsilon_{n}} \otimes \cdots \otimes X_{1}^{-\varepsilon_{1}}\right)^{*}=\left(X_{S^{*}}\right)^{*} .
$$

By [TVi, Lemma 2.4], for any tuple $S$ of signed objects of $\mathcal{C}$,

$$
\begin{gathered}
\operatorname{ev}_{X_{S}}\left(\Psi_{S^{*}} \otimes \operatorname{id}_{X_{S}}\right)=\operatorname{ev}_{S}=\widetilde{\operatorname{ev}}_{X_{S^{*}}}\left(\operatorname{id}_{X_{S^{*}}} \otimes \Psi_{S}\right), \\
\left(\operatorname{id}_{X_{S}} \otimes \Psi_{S^{*}}{ }^{-1}\right) \operatorname{coev}_{X_{S}}=\operatorname{coev}_{S}=\left(\Psi_{S}^{-1} \otimes \operatorname{id}_{X_{S^{*}}}\right) \widetilde{\operatorname{coev}_{X_{S^{*}}}}
\end{gathered}
$$

### 1.4. Fusion categories

In this section we recall some basics on linear and fusion categories. We recall that the symbol $\mathbb{k}$ is used for a non-zero commutative ring.
1.4.1. Linear categories. A category $\mathcal{C}$ is $\mathbb{k}$-linear if for all objects $X, Y$ of $\mathcal{C}$, the set $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ is endowed with a structure of a left $\mathbb{k}$-module so that the composition of morphisms in $\mathcal{C}$ is $\mathbb{k}$-bilinear. For shortness, $\mathbb{k}$-linear categories are called $\mathbb{k}$-categories. A functor $F: C \rightarrow D$ between $\mathbb{k}$-categories is $\mathbb{k}$-linear if its action on the Hom-sets is $\mathbb{k}$-linear, that is, if for all $X, Y \in$ $\operatorname{Ob}(\mathcal{C})$, the map

$$
\operatorname{Hom}_{\mathcal{C}}(X, Y) \rightarrow \operatorname{Hom}_{\mathcal{D}}(F(X), F(Y)), \quad f \mapsto F(f)
$$

is $\mathbb{k}$-linear. For example, the identity functor of a $\mathbb{k}$-category is $\mathbb{k}$-linear. Clearly, the composition of $\mathbb{k}$-linear functors is a $\mathbb{k}$-linear functor. By a monoidal (respectively, left/right rigid, rigid, pivotal, spherical) $\mathbb{k}$-category, we mean a $\mathbb{k}$ category which is monoidal (respectively, left/right rigid, rigid, pivotal, spherical) and such that monoidal product of morphisms is $\mathbb{k}$-bilinear. Clearly, any monoidal subcategory of a monoidal $\mathbb{k}$-category is a monoidal $\mathbb{k}$-category. If $\mathcal{C}$ is a monoidal $\mathbb{k}$-category, then so are $\mathcal{C}^{\mathrm{op}}, \mathcal{C}^{\otimes \mathrm{op}}$, and $\mathcal{C}^{\text {rev }}$ (see Section 1.1.4).

Equivalences of monoidal/pivotal $\mathbb{k}$-categories are always required to be $\mathbb{k}$-linear. In particular, two pivotal $\mathbb{k}$-categories are equivalent if there is a $\mathbb{k}$ linear pivotal functor between them which is an equivalence of the underlying categories.

It follows from the definitions that all left/right dual functors of a left/right rigid $\mathbb{k}$-category are $\mathbb{k}$-linear. In particular, given a pivotal $\mathbb{k}$-category $\mathcal{C}$, the dual functor ?*: $\mathcal{C}^{\text {rev }} \rightarrow \mathcal{C}$ is $\mathbb{k}$-linear, and so $\mathcal{C}^{\text {rev }}$ and $\mathcal{C}$ are equivalent pivotal $\mathbb{k}$-categories (see Section 1.2.6). Consequently, $\mathcal{C}^{\otimes \mathrm{op}}=\left(\mathcal{C}^{\mathrm{op}}\right)^{\mathrm{rev}}$ and $\mathcal{C}^{\mathrm{op}}$ are equivalent pivotal $\mathbb{k}$-categories.
1.4.2. Direct sum. Let $\left(X_{\alpha}\right)_{\alpha \in A}$ be finite family of objects in a pivotal $\mathbb{k}$-category $\mathcal{C}$. An object $X \in \operatorname{Ob}(\mathcal{C})$ is a direct sum of the family $\left(X_{\alpha}\right)_{\alpha \in A}$ if there is a family $\left(p_{\alpha}, q_{\alpha}\right)_{\alpha \in A}$ of morphisms in $\mathcal{C}$ with:

$$
p_{\alpha}: X \rightarrow X_{\alpha} \text { and } q_{\alpha}: X_{\alpha} \rightarrow X
$$

for all $\alpha \in A$, such that

$$
\operatorname{id}_{X}=\sum_{\alpha \in A} q_{\alpha} p_{\alpha} \quad \text { and } \quad p_{\alpha} q_{\beta}=\delta_{\alpha, \beta} \operatorname{id}_{X_{\alpha}} \quad \text { for all } \alpha, \beta \in A,
$$

where $\delta_{\alpha, \beta}$ is the Kronecker symbol. If such $X$ exists, it is unique, up to a unique isomorphism commuting with $p_{\alpha}$ and $q_{\alpha}$. We denote $X$ as $\bigoplus_{\alpha \in A} X_{\alpha}$.
1.4.3. Simple objects in linear categories. Let $\mathcal{C}$ be a $\mathbb{k}$-category. An object $X$ of $\mathcal{C}$ is simple if the map $\mathbb{k} \rightarrow \operatorname{End}_{\mathcal{C}}(X)$ that sends $k \mapsto k \operatorname{id}_{X}$ is an isomorphism of $\mathbb{k}$-modules. Let $X$ be an object of $\mathcal{C}$, then the following conditions are equivalent:
(i) $X$ is simple;
(ii) the map $\mathbb{k} \rightarrow \operatorname{End}_{\mathcal{C}}(X)$ that sends $k \mapsto k \operatorname{id}_{X}$ is an isomorphism of $\mathbb{k}$-modules;
(iii) the $\mathbb{k}$-algebra $\operatorname{End}_{\mathcal{C}}(X)$ is isomorphic to $\mathbb{k}$;
(iv) the $\mathbb{k}$-module $\operatorname{End}_{\mathcal{C}}(X)$ is free of rank 1.

The $\mathbb{k}$-bilinearity of the composition of morphisms in $\mathcal{C}$ implies that all objects of $\mathcal{C}$ isomorphic to a simple object are simple. Any monoidal $\mathbb{k}$-category whose unit object $\mathbb{1}$ is simple is pure (see Section 1.1.6). The left and right traces of endomorphisms in a pivotal $\mathbb{k}$-linear category are $\mathbb{k}$-linear. This follows from the $\mathbb{k}$-linearity of the monoidal product and composition of morphisms.
1.4.4. Non-degenerate categories. Let $\mathcal{C}$ be a monoidal $\mathbb{k}$-category. Any pairing $e: X \otimes Y \rightarrow \mathbb{1}$ between objects $X$ and $Y$ of $\mathcal{C}$ induces a $\mathbb{k}$-linear homomorphism

$$
\operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, X) \otimes_{\mathbb{k}} \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, Y) \rightarrow \operatorname{End}_{\mathcal{C}}(\mathbb{1}), \quad \alpha \otimes_{\mathbb{k}} \beta \mapsto e(\alpha \otimes \beta)
$$

If the unit object $\mathbb{1}$ of $\mathcal{C}$ is simple we identify $\operatorname{End}_{\mathcal{C}}(\mathbb{1})=\mathbb{k}$ (see Section 1.4.3) and so we get a pairing in $\operatorname{Mod}_{\mathfrak{k}}$

$$
\begin{equation*}
\operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, X) \otimes_{\mathbb{k}} \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, Y) \rightarrow \mathbb{k}, \quad \alpha \otimes_{\mathfrak{k}} \beta \mapsto e(\alpha \otimes \beta) . \tag{1.5}
\end{equation*}
$$

A monoidal $\mathbb{k}$-category $\mathcal{C}$ is non-degenerate if its unit object is simple and for each non-degenerate pairing $e: X \otimes Y \rightarrow \mathbb{1}$ in $\mathcal{C}$, the induced pairing (1.5) is non-degenerate in the monoidal category $\operatorname{Mod}_{\mathfrak{k}}$.

Lemma 1.4. Let $\mathcal{C}$ be a non-degenerate pivotal $\mathbb{k}$-category. Then the left and right dimensions of any simple object of $\mathcal{C}$ are invertible in $\mathfrak{k}$.

Proof. Let $i$ be a simple object of $\mathcal{C}$. Consider the right evaluation of $i$ given by $\widetilde{\mathrm{ev}}_{i}: i \otimes i^{*} \rightarrow \mathbb{1}$. By the following bijection

$$
\operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, i^{*} \otimes i\right) \rightarrow \operatorname{Hom}_{\mathcal{C}}\left(i^{*}, i^{*}\right), \quad \alpha \mapsto\left(\mathrm{id}_{i^{*}} \otimes \widetilde{\mathrm{ev}}_{i}\right)\left(\alpha \otimes \mathrm{id}_{i^{*}}\right)
$$

whose inverse is given by the map that sends any $\beta \in \operatorname{End}_{\mathcal{C}}\left(i^{*}\right)$ to

$$
\beta \mapsto\left(\beta \otimes \operatorname{id}_{i}\right){\widetilde{\operatorname{coev}_{i}}}_{i}
$$

we have that $\operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, i^{*} \otimes i\right) \simeq \operatorname{Hom}_{\mathcal{C}}\left(i^{*}, i^{*}\right)$. Since $i^{*}$ is simple, $\operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, i^{*} \otimes i\right)$ is a free $\mathbb{k}$-module of rank 1 with basis vector $\widetilde{\operatorname{coev}}_{i}: \mathbb{1} \rightarrow i^{*} \otimes i$. Consider now the pairing in the category $\operatorname{Mod}_{k}$ :

$$
\omega_{i}: \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, i^{*} \otimes i\right) \otimes_{\mathbb{k}} \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, i^{*} \otimes i\right) \rightarrow \mathbb{k}
$$

given by


The non-degeneracy of $\omega_{i}$ and Lemma 1.1 imply that $\operatorname{dim}_{l}(i)$ is invertible in $\mathbb{k}$. Using a similar argument for the pairing $\mathrm{ev}_{i}: i^{*} \otimes i \rightarrow \mathbb{1}$ we deduce the same result for $\operatorname{dim}_{r}(i)$.
1.4.5. Fusion categories. A fusion $\mathbb{k}$-category is a rigid $\mathbb{k}$-category $\mathcal{C}$ such that there is a finite set $I$ of simple objects of $\mathcal{C}$ satisfying the following conditions:
(a) the unit object $\mathbb{1} \in \mathrm{Ob}(\mathcal{C})$ belongs to $I$;
(b) $\operatorname{Hom}_{\mathcal{C}}(i, j)=0$ for any distinct $i, j \in I$, ;
(c) every object of $\mathcal{C}$ is a direct sum of a finite family of elements of $I$.

Such a set $I$ is called a representative set of simple objects of $\mathcal{C}$.
Let $\mathcal{C}$ be a fusion $\mathbb{k}$-category and let $I$ be a representative set of simple objects of $\mathcal{C}$. Condition $(a)$ implies that $\mathcal{C}$ is pure and $\operatorname{End}_{\mathcal{C}}(\mathbb{1}) \simeq \mathbb{k}$. Condition $(c)$ implies that for each object $X$ of $\mathcal{C}$, there is a finite family of morphisms

$$
\left(p_{\alpha}: X \rightarrow i_{\alpha}, q_{\alpha}: i_{\alpha} \rightarrow X\right)_{\alpha \in A}
$$

such that

$$
i_{\alpha} \in I, \quad \operatorname{id}_{X}=\sum_{\alpha \in A} q_{\alpha} p_{\alpha} \quad \text { and } \quad p_{\alpha} q_{\beta}=\delta_{\alpha, \beta} \operatorname{id}_{i_{\alpha}} \quad \text { for all } \alpha, \beta \in A .
$$

We call such a family an $I$-partition of $X$.
Given a simple object $i$ of $\mathcal{C}$, an $i$-partition of $X \in \operatorname{Ob}(\mathcal{C})$ is a family of morphisms $\left(p_{\alpha}: X \rightarrow i, q_{\alpha}: i \rightarrow X\right)_{\alpha \in A^{\prime}}$ such that $\left(p_{\alpha}\right)_{\alpha \in A^{\prime}}$ is a basis of $\operatorname{Hom}_{\mathcal{C}}(X, i),\left(q_{\alpha}\right)_{\alpha \in A^{\prime}}$ is a basis of $\operatorname{Hom}_{\mathcal{C}}(i, X)$, and $p_{\alpha} q_{\beta}=\delta_{\alpha, \beta} \operatorname{id}_{i}$ for all $\alpha, \beta \in A^{\prime}$. Note that the cardinality of the set $A^{\prime}$ is equal to the number of simple objects isomorphic to $i$ in a $I$-partition of $X$. For any $I$-partition $\left(p_{\alpha}: X \rightarrow i_{\alpha}, q_{\alpha}: i_{\alpha} \rightarrow X\right)_{\alpha \in A}$ of $X$ and any $i \in I$, the family $\left(p_{\alpha}, q_{\alpha}\right)_{\alpha \in A_{i}}$ is an $i$-partition of $X$, where $A_{i}=\left\{\alpha \in A \mid i_{\alpha}=i\right\}$. Conversely, a union of $i$-partitions of $X$ over all $i \in I$ is an $I$-partition of $X$.

Let $\mathcal{C}$ be a pivotal fusion $\mathbb{k}$-category. Since $\mathcal{C}$ is pure (because $\mathbb{1}$ is simple), the traces of endomorphisms and the dimensions of objects are $\otimes$-multiplicative. By [TVi, Lemma 4.3], $\mathcal{C}$ is non-degenerate. Then it follows from Lemma 1.4 that the left/right dimensions of any simple object of $\mathcal{C}$ are invertible in $\operatorname{End}_{\mathcal{C}}(\mathbb{1}) \simeq \mathbb{k}$.

Two pivotal fusion $\mathbb{k}$-categories are equivalent if there is a $\mathbb{k}$-linear pivotal equivalence between them. If $\mathcal{C}$ is a pivotal fusion $\mathbb{k}$-category, then so are its opposites

$$
\mathcal{C}^{\mathrm{op}}=\left(\mathcal{C}^{\mathrm{op}}, \otimes, \mathbb{1}\right), \quad \mathcal{C}^{\otimes \mathrm{op}}=\left(\mathcal{C}, \otimes^{\mathrm{op}}, \mathbb{1}\right), \quad \mathcal{C}^{\mathrm{rev}}=\left(\mathcal{C}^{\mathrm{op}}, \otimes^{\mathrm{op}}, \mathbb{1}\right) .
$$

By Section 1.2.6, $\mathcal{C}^{\text {rev }}$ is equivalent to $\mathcal{C}$, and $\mathcal{C}^{\mathrm{op}}$ is equivalent to $\mathcal{C}^{\otimes \mathrm{op}}$.
1.4.6. Enriched graphical calculus. Let $\mathcal{C}$ be a pivotal fusion $\mathbb{k}$-category. Consider a simple object $i$ of $\mathcal{C}$ and an $i$-partition $\left(p_{\alpha}: X \rightarrow i, q_{\alpha}: i \rightarrow X\right)_{\alpha \in A}$ of an object $X$ of $\mathcal{C}$. Consider a (finite) formal sum of $\mathcal{C}$-colored Penrose diagrams

where the area outside the dotted line represents a part of these diagrams independent of $\alpha \in A$ and, in particular, not involving $\left(p_{\alpha}, q_{\alpha}\right)$. By the Penrose graphical calculus and the $\mathbb{k}$-linearity of $\mathcal{C}$, the sum (1.6) represents a morphism in $\mathcal{C}$. Using changes of basis, we obtain that the tensor

$$
\begin{equation*}
\sum_{\alpha \in A} p_{\alpha} \otimes q_{\alpha} \in \operatorname{Hom}_{\mathcal{C}}(X, i) \otimes_{\mathfrak{k}} \operatorname{Hom}_{\mathcal{C}}(i, X) \tag{1.7}
\end{equation*}
$$

does not depend on the choice of the $i$-partition of $X$. The morphism (1.6) in $\mathcal{C}$ also does not depend on this choice. Therefore we can eliminate the $\mathcal{C}$-colors $p_{\alpha}, q_{\alpha}$ of the two boxes, keeping in mind only the order of the boxes and the fact that they jointly stand for the tensor (1.7). We will graphically represent this pair of boxes by two curvilinear boxes (a semi-disk and a compressed rectangle) standing respectively for $p_{\alpha}$ and $q_{\alpha}$ where $\alpha$ runs over $A$ :


The area outside the dotted line in the picture are the same as above. We will also use similar notation obtained from (1.7) by reorienting the $X$-labeled arcs upward and replacing $\left(p_{\alpha}, q_{\alpha}\right)_{\alpha \in A}$ with an $i$-partition of $X^{*}$, or by reorienting the $i$-labeled arcs upward and replacing $\left(p_{\alpha}, q_{\alpha}\right)_{\alpha \in A}$ with an $i^{*}$-partition of $X$. We will allow several arcs to be attached to the bottom of the semi-disk and to
the top of the compressed rectangle in (1.7). We will allow to erase $i$-labeled arcs for $i=\mathbb{1}$. In particular,

where $\left(p_{\alpha}, q_{\alpha}\right)_{\alpha \in A}$ is any $\mathbb{1}$-partition of $X$.
1.4.7. Properties. For any object $X$ of a pivotal fusion $\mathbb{k}$-category $\mathcal{C}$ and any simple object $i$ of $\mathcal{C}$, we have

where $N_{X}^{i}$ is the rank of the free $\mathbb{k}$-modules $\operatorname{Hom}_{\mathcal{C}}(X, i)$ and $\operatorname{Hom}_{\mathcal{C}}(i, X)$. This equality follows from the fact that given an $i$-partition $\left(p_{\alpha}, q_{\alpha}\right)_{\alpha \in A}$ of $X$, we have $p_{\alpha} q_{\alpha}=\operatorname{id}_{i}$ for all $\alpha \in A$ and $\operatorname{card}(A)=N_{X}^{i}$. Next, pick a representative set $I$ of simple objects of $\mathcal{C}$. Since the union of $i$-partitions of $X \in \operatorname{Ob}(\mathcal{C})$ over all $i \in I$ is an $I$-partition of $X$, we have


This formula and the fact that $\operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, i)=0=\operatorname{Hom}_{\mathcal{C}}(i, \mathbb{1})$ for all $i \in I \backslash\{\mathbb{1}\}$ imply that for any $f \in \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, X)$ and $g \in \operatorname{Hom}_{\mathcal{C}}(X, \mathbb{1})$,


Finally, for any object $X$ of $\mathcal{C}$ and any simple object $i$ of $\mathcal{C}$, we have:


This equality follows from the fact that if $\left(p_{\alpha}, q_{\alpha}\right)_{\alpha \in A}$ is a $\mathbb{1}$-partition of $i^{*} \otimes X$, then $\left(P_{\alpha}, Q_{\alpha}\right)_{\alpha \in A}$ is a $i$-partition of $X$, where

$$
P_{\alpha}=\left.\operatorname{dim}_{l}(i)\right|_{\left.\right|_{X}} ^{\sqrt[p_{\alpha}]{p_{\alpha}}} \quad \text { and } \quad Q_{\alpha}=\overbrace{q_{\alpha}}^{\underbrace{}_{i}} .
$$

Similarly, we have:


## CHAPTER 2

## Invariants of colored graphs

In this chapter, we associate with each linear pivotal category a family of modules called multiplicity modules (Section 2.1). Then we review an invariant of colored planar graphs which takes values in tensor products of multiplicity modules (Section 2.2). Finally, we study in detail duality pairings for colored graphs and their associated contraction vectors (Section 2.3). The invariant of colored graphs and the contraction vectors will be our main tools in the topological constructions of Chapter 4.

### 2.1. Multiplicity modules

In this section we associate with each linear pivotal category a family of modules called multiplicity modules.
2.1.1. Cyclic sets. A cyclic $\mathcal{C}$-set is a triple $(E, c, \varepsilon)$ consisting of a nonempty finite set $E$ endowed with a cyclic order and two maps $c: E \rightarrow \operatorname{Ob}(\mathcal{C})$ and $\varepsilon: E \rightarrow\{+,-\}$. In other words, a cyclic $\mathcal{C}$-set is a nonempty cyclically ordered finite set whose elements are equipped with a signed object of $\mathcal{C}$. For shortness, we will often write $E$ for $(E, c, \varepsilon)$.

An isomorphism between two cyclic $\mathcal{C}$-sets $E$ and $E^{\prime}$ is a bijection $E \rightarrow$ $E^{\prime}$ preserving the cyclic order and commuting with the maps to $\mathrm{Ob}(\mathcal{C})$ and $\{+,-\}$. More generally, a weak isomorphism between cyclic $\mathcal{C}$-sets $(E, c, \varepsilon)$ and $\left(E^{\prime}, c^{\prime}, \varepsilon^{\prime}\right)$ is a pair $\phi=(\rho, \varphi)$ consisting of a bijection $\rho: E \rightarrow E^{\prime}$ preserving the cyclic order and a family of isomorphisms in $\mathcal{C}$

$$
\varphi=\left\{\varphi_{e}: c(e)^{\varepsilon(e)} \rightarrow c^{\prime}(\rho(e))^{\varepsilon^{\prime}(\rho(e))}\right\}_{e \in E} .
$$

2.1.2. Permutation maps. For $X, Y \in \operatorname{Ob}(\mathcal{C})$ we define the permutation map

$$
\pi_{X, Y}: \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, X \otimes Y) \rightarrow \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, Y \otimes X)
$$

to be the map carrying any element $\alpha \in \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, X \otimes Y)$ to

$$
\pi_{X, Y}(\alpha)=\left(\mathrm{ev}_{X} \otimes \operatorname{id}_{Y \otimes X}\right)\left(\operatorname{id}_{X^{*}} \otimes \alpha \otimes \operatorname{id}_{X}\right) \widetilde{\operatorname{coev}}_{X}
$$

Note that, using the isotopy invariance of the graphical calculus, we have:

$$
\pi_{X, Y}(\alpha)=\left(\operatorname{id}_{Y \otimes X} \otimes \widetilde{\mathrm{ev}}_{Y}\right)\left(\mathrm{id}_{Y} \otimes \alpha \otimes \mathrm{id}_{Y^{*}}\right) \operatorname{coev}_{Y} .
$$

The permutation maps are $\mathbb{k}$-linear isomorphisms and for any $X, Y, Z \in \operatorname{Ob}(\mathcal{C})$ have the following properties:
(a) $\pi_{X, Y}{ }^{-1}=\pi_{Y, X}$;
(b) $\pi_{X, \mathbb{1}}=\pi_{\mathbb{1}, X}=\operatorname{id}_{\operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, X)}$;
(c) $\pi_{X \otimes Y, Z}=\pi_{Y, Z \otimes X} \pi_{X, Y \otimes Z}$ and $\pi_{X, Y \otimes Z}=\pi_{Z \otimes X, Y} \pi_{X \otimes Y, Z}$.
2.1.3. Multiplicity modules. Let $E=(E, c, \varepsilon)$ be a cyclic $\mathcal{C}$-set, we derive from this data a $\mathbb{k}$-module $H(E)$. For $e \in E$, set

$$
H_{e}(E)=\operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, c\left(e_{1}\right)^{\varepsilon\left(e_{1}\right)} \otimes c\left(e_{2}\right)^{\varepsilon\left(e_{2}\right)} \otimes \cdots \otimes c\left(e_{n}\right)^{\varepsilon\left(e_{n}\right)}\right)
$$

where $n$ is the cardinality of $E$ and $e=e_{1}<e_{2}<\cdots<e_{n}$ are the elements of $E$ in the given cyclic order starting from $e$. If $f \in E \backslash\{e\}$, then $f=e_{k}$ for some integer $k \in\{2, \ldots, n\}$. Set

$$
[e, f)=c\left(e_{1}\right)^{\varepsilon\left(e_{1}\right)} \otimes c\left(e_{2}\right)^{\varepsilon\left(e_{2}\right)} \otimes \cdots \otimes c\left(e_{k-1}\right)^{\varepsilon\left(e_{k-1}\right)}
$$

and

$$
[f, e)=c\left(e_{k}\right)^{\varepsilon\left(e_{k}\right)} \otimes c\left(e_{k+1}\right)^{\varepsilon\left(e_{k+1}\right)} \otimes \cdots \otimes c\left(e_{n}\right)^{\varepsilon\left(e_{n}\right)}
$$

Clearly

$$
H_{e}(E)=\operatorname{Hom}_{\mathcal{C}}(\mathbb{1},[e, f) \otimes[f, e)) \quad \text { and } \quad H_{f}(E)=\operatorname{Hom}_{\mathcal{C}}(\mathbb{1},[f, e) \otimes[e, f)) .
$$

Define $p_{e, f}: H_{e}(E) \rightarrow H_{f}(E)$ by

$$
p_{e, f}= \begin{cases}\pi_{[e, f),[f, e)} & \text { if } e \neq f \\ \operatorname{id}_{H_{e}} & \text { if } e=f\end{cases}
$$

The properties of the permutation maps imply that $p_{e, f}$ is a $\mathbb{k}$-linear isomorphism and that $p_{f, g} p_{e, f}=p_{e, g}$ for all $e, f, g \in E$. Thus the family

$$
\left(\left\{H_{e}(E)\right\}_{e \in E},\left\{p_{e, f}\right\}_{e, f \in E}\right)
$$

is a projective system of $\mathbb{k}$-modules and $\mathbb{k}$-linear isomorphisms. The multiplicity module $H(E)$ is the projective limit of this system:

$$
H(E)=\lim _{\leftrightarrows} H_{e}(E) .
$$

The $\mathbb{k}$-module $H(E)$ depends only on $E$ and it is endowed with a family of $\mathbb{k}$-linear isomorphisms

$$
\left\{\tau_{e}^{E}: H(E) \rightarrow H_{e}(E)\right\}_{e \in E}
$$

such that $p_{e, f} \tau_{e}^{E}=\tau_{f}^{E}$ for all $e, f \in E$. We call $\tau_{e}^{E}$ the cone isomorphism and the family $\left\{\tau_{e}^{E}\right\}_{e \in E}$ the universal cone.

An isomorphism $\phi=(\rho, \varphi)$ between two cyclic $\mathcal{C}$-sets $E$ and $E^{\prime}$ induces a family of $\mathbb{k}$-linear isomorphisms

$$
\left\{\varphi_{e}: H_{e}(E) \rightarrow H_{\rho(e)}\left(E^{\prime}\right)\right\}_{e \in E}
$$

which commute with the maps $p_{e, f}$ as above. These isomorphisms induce a $\mathbb{k}$-linear isomorphism $H(E) \rightarrow H\left(E^{\prime}\right)$ denoted $H(\phi)$.

### 2.2. An invariant of colored planar graphs

In this section, we define an invariant of colored planar graphs. Throughout this section, we orient the plane $\mathbb{R}^{2}$ counterclockwise and $\mathcal{C}$ is a $\mathbb{k}$-linear pivotal category.
2.2.1. Graphs. By a graph we mean a topological space $G$ obtained from a finite number of disjoint copies of the closed interval $[0,1]$ by identification of certain endpoints. The images of the copies of $[0,1]$ in $G$ are called edges of $G$. The endpoints of the edges of $G$ (that is, the images of $0,1 \in[0,1]$ ) are called vertices of $G$. Each edge of $G$ connects two (possibly, coinciding) vertices, and each vertex of $G$ is incident to at least one edge. By half-edges of $G$, we mean the images of the closed intervals $[0,1 / 2] \subset[0,1]$ and $[1 / 2,1] \subset[0,1]$ in $G$. The number of half-edges of $G$ incident to a vertex $v$ of $G$ is called the valence of $v$ and for any vertex is greater then or equal to 1 . A graph is oriented if all its edges are oriented. An half edge incident to $v$ is said to be incoming if it is oriented towards and outgoing otherwise. The empty set is viewed as an oriented graph with no vertices and no edges.
2.2.2. Colored graphs. A $\mathcal{C}$-colored graph is an oriented graph such that each edge is endowed with an object of $\mathcal{C}$ called the color of this edge. Let $\Sigma$ be an oriented surface. A $\mathcal{C}$-colored graph in $\Sigma$ is a graph embedded in $\Sigma$. For shortness, by a $\mathcal{C}$-colored planar graph we mean a $\mathcal{C}$-colored graph embedded in an oriented plane (i.e., an oriented surface homeomorphic to $\mathbb{R}^{2}$ ).
2.2.3. The $\mathbb{k}$-module associated to $\mathcal{C}$-colored graphs. Let $\Sigma$ be an oriented surface and let $G$ be a $\mathcal{C}$-colored graph in $\Sigma$. A vertex $v$ of $G$ determines a cyclic $\mathcal{C}$-set $E_{v}=\left(E_{v}, c_{v}, \varepsilon_{v}\right)$ as follows: $E_{v}$ is the set of half-edges of $G$ incident to $v$ with the cyclic order induced by the opposite orientation of $\Sigma$, the map $c_{v}: E_{v} \rightarrow \operatorname{Ob}(\mathcal{C})$ assigns to a half-edge $e \in E_{v}$ the color of the edge of $G$ containing $e$ and the map $\varepsilon_{v}: E_{v} \rightarrow\{+,-\}$ assigns to $e \in E_{v}$ the sign + if $e$ is oriented towards $v$ and - otherwise. Note that the cardinality of $E_{v}$ is equal to the valence of $v$. Let $H_{v}(G)=H\left(E_{v}\right)$ be the multiplicity module of $E_{v}$, and set

$$
H(G)=\bigotimes_{v} H_{v}(G)
$$

where $\otimes$ is the unordered tensor product of $\mathbb{k}$-modules, that run over all vertices $v$ of $G$. By definition, for $G=\emptyset$, we have $H(G)=\mathbb{k}$.

For a vertex $v$ of $G$, the $\mathbb{k}$-module $H_{v}(G)$ can be described as follows. Let $n \geq 1$ be the valence of $v$ and let $e_{1}<e_{2}<\cdots<e_{n}<e_{1}$ be the half-edges of $G$ incident to $v$ with cyclic order induced by the opposite orientation of $\Sigma$. Then we have the cone isomorphism

$$
\tau_{e_{1}}^{E_{v}}: H_{v}(G) \xrightarrow{\simeq} \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, c_{v}\left(e_{1}\right)^{\varepsilon_{v}\left(e_{1}\right)} \otimes \cdots \otimes c_{v}\left(e_{n}\right)^{\varepsilon_{v}\left(e_{n}\right)}\right) .
$$

By definition of $H_{v}(G)$, the cone isomorphism determined by different elements of $E_{v}$ are related via composition with the permutation maps. For example,
the trivalent vertex $v$ of the following $\mathcal{C}$-colored graph:

with $i, j, k \in \operatorname{Ob}(\mathcal{C})$, give rise to the $\mathbb{k}$-module $H_{v}(G)$ isomorphic, via the cone isomorphism, to the $\mathbb{k}$-modules

$$
\operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, i \otimes j^{*} \otimes k\right) \simeq \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, j^{*} \otimes k \otimes i\right) \simeq \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, k \otimes i \otimes j^{*}\right)
$$

For any disjoint $\mathcal{C}$-colored graphs $G_{1}$ and $G_{2}$ in $\Sigma$, there is a canonical $\mathbb{k}$-linear isomorphism between the $\mathbb{k}$-modules

$$
H\left(G_{1} \sqcup G_{2}\right) \simeq H\left(G_{1}\right) \otimes H\left(G_{2}\right)
$$

2.2.4. The invariant $\mathbb{F}_{\mathcal{C}}$. Let $G$ be a $\mathcal{C}$-colored graph in $\mathbb{R}^{2}$. For each vertex $v$ of $G$, pick a half-edge $e_{v} \in E_{v}$ and deform $G$ near $v$ so that the halfedges incident to $v$ lie above $v$ with respect to the second coordinate on $\mathbb{R}^{2}$ and $e_{v}$ is the leftmost of them. Pick any $\alpha_{v} \in H_{v}(G)$ and replace $v$ by a box colored with $\tau_{e_{v}}^{E_{v}}\left(\alpha_{v}\right)$, where $\tau^{E_{v}}$ is the universal cone of $H_{v}(G)$ :


This transforms $G$ into a $\mathcal{C}$-colored Penrose diagram without free ends.
Let $\mathbb{F}_{\mathcal{C}}(G)\left(\otimes_{v} \alpha_{v}\right) \in \operatorname{End}_{\mathcal{C}}(\mathbb{1})$ be the associated morphism computed via the Penrose graphical calculus. This extends by linearity to a $\mathbb{k}$-linear homomorphism

$$
\mathbb{F}_{\mathcal{C}}(G): H(G)=\otimes_{v} H_{v}(G) \rightarrow \operatorname{End}_{\mathcal{C}}(\mathbb{1})
$$

By definition, for $G=\emptyset$, the map $\mathbb{F}_{\mathcal{C}}(G): H(G)=\mathbb{k} \rightarrow \operatorname{End}_{\mathcal{C}}(\mathbb{1})$ is the $\mathbb{k}$-linear homomorphism carrying $1_{\mathbb{k}}$ to $\operatorname{id}_{\mathbb{1}}$.

By TVil Lemma 12.2], the homomorphism $\mathbb{F}_{\mathcal{C}}(G): H(G) \rightarrow \operatorname{End}_{\mathcal{C}}(\mathbb{1})$ is a well-defined isotopy invariant of the $\mathcal{C}$-colored graph $G$ in $\mathbb{R}^{2}$.
2.2.5. Example. Consider the following $\mathcal{C}$-colored planar graph with four vertices $a, b, c, d$ and four edges colored by $X, Y, Z, T \in \operatorname{Ob}(\mathcal{C})$ :


The half-edges $e_{1}^{a}, e_{2}^{a}$ incident to $a, e_{1}^{b}$ and $e_{2}^{b}$ incident to $b, e_{1}^{c}, e_{2}^{c}, e_{3}^{c}$ incident to $c$ and $e_{1}^{d}$ incident to $d$ are reported below:


The total order compatible with the cyclic order on $E_{a}=\left\{e_{1}^{a}, e_{2}^{a}\right\}$ is $e_{1}^{a}<e_{2}^{a}$, on $E_{b}=\left\{e_{1}^{b}, e_{2}^{b}\right\}$ is $e_{1}^{b}<e_{2}^{b}$ and on $E_{c}=\left\{e_{1}^{c}, e_{2}^{c}, e_{3}^{c}\right\}$ is $e_{1}^{c}<e_{2}^{c}<e_{3}^{c}$. There are several cone isomorphisms associated with each vertex:

$$
\begin{aligned}
& \tau_{e_{1}^{a}}^{E_{a}}: H_{a}(G) \rightarrow \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X \otimes T^{*}\right), \\
& \tau_{e_{1}^{b}}^{E_{1}}: H_{b}(G) \rightarrow \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X^{*} \otimes Y\right), \\
& \tau_{e_{1}^{c_{c}}}^{E_{c}}: H_{c}(G) \rightarrow \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, T \otimes Y^{*} \otimes Z\right), \\
& \tau_{e_{1}^{d}}^{E_{d}}: H_{d}(G) \rightarrow \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, Z^{*}\right) .
\end{aligned}
$$

These isomorphisms are related to each other via composition with the permutation maps, see Section 2.1.3.

By definition, $H(G)=H_{a}(G) \otimes H_{b}(G) \otimes H_{c}(G) \otimes H_{d}(G)$. For any $\alpha \in$ $H_{a}(G), \beta \in H_{b}(G), \gamma \in H_{c}(G)$ and $\delta \in H_{d}(G)$ we have

2.2.6. Properties of $\mathbb{F}_{\mathcal{C}}$. We state some properties of the invariant $\mathbb{F}_{\mathcal{C}}$ of $\mathcal{C}$-colored graphs in $\mathbb{R}^{2}$.
(A) Let $G^{\prime}$ be the $\mathcal{C}$-colored graph in $\mathbb{R}^{2}$ obtained from a $\mathcal{C}$-colored graph $G \subset \mathbb{R}^{2}$ by replacing the color $X$ of an edge $e$ by an isomorphic object $X^{\prime}$ of $\mathcal{C}$. Any isomorphism $X^{\prime} \simeq X$ induces a weak isomorphism between the cyclic $\mathcal{C}$-sets (see Section 2.1.1) associated with the endpoints of $e$ in $G$ and $G^{\prime}$, and the latter induces a $\mathbb{k}$-linear isomorphism $\Phi: H\left(G^{\prime}\right) \rightarrow H(G)$. Then

$$
\mathbb{F}_{\mathcal{C}}\left(G^{\prime}\right)=\mathbb{F}_{\mathcal{C}}(G) \Phi
$$

We call this property the naturality of $\mathcal{C}$.
(B) If an edge $e$ of a $\mathcal{C}$-colored graph $G$ in $\mathbb{R}^{2}$ is colored with $\mathbb{1}$ and the endpoints of $e$ are also endpoints of other edges of $G$, then $G^{\prime}=$ $G \backslash \operatorname{Int}(e) \subset \mathbb{R}^{2}$ inherits from $G$ the structure of a $\mathcal{C}$-colored graph, there is a canonical $\mathbb{k}$-linear isomorphism $\Delta: H\left(G^{\prime}\right) \rightarrow H(G)$, and

$$
\mathbb{F}_{\mathcal{C}}\left(G^{\prime}\right)=\mathbb{F}_{\mathcal{C}}(G) \Delta
$$

Indeed, by the Penrose calculus, an edge colored with $\mathbb{1}$ can be deleted without changing the associated morphism.
(C) If $G, G^{\prime}$ are disjoint $\mathcal{C}$-colored graphs in $\mathbb{R}^{2}$ lying on different sides of a straight line, then

$$
\mathbb{F}_{\mathcal{C}}\left(G \amalg G^{\prime}\right)=\mu\left(\mathbb{F}_{\mathcal{C}}(G) \otimes \mathbb{F}_{\mathcal{C}}\left(G^{\prime}\right)\right) \Theta
$$

where $\Theta: H\left(G \amalg G^{\prime}\right) \rightarrow H(G) \otimes H\left(G^{\prime}\right)$ is the canonical isomorphism and $\mu$ is multiplication in $\operatorname{End}_{\mathcal{C}}(\mathbb{1})$. We call this property the $\otimes$ multiplicativity of $\mathbb{F}_{\mathcal{C}}$.
(D) If $\mathcal{C}$ is pure, then

where the $\mathcal{C}$-colored graphs on the left and on the right coincide outside the big rectangles and the small rectangles on both sides stand for the same $\mathcal{C}$-colored graph.
2.2.7. The case of a pivotal fusion $\mathbb{k}$-category. Suppose that $\mathcal{C}$ is a pivotal fusion $\mathbb{k}$-category. Recall that $\operatorname{End}_{\mathcal{C}}(\mathbb{1}) \simeq \mathbb{k}$. For any $\mathcal{C}$-colored graph $G$ in $\mathbb{R}^{2}$, the $\mathbb{k}$-module $H(G)$ is free of finite rank and

$$
\mathbb{F}_{\mathcal{C}}(G) \in H(G)^{\star}=\operatorname{Hom}_{\mathbb{k}}(H(G), \mathbb{k})
$$

For any non-isomorphic simple objects $i$ and $j$ of $\mathcal{C}$ we have

where the white box stands for any piece of a $\mathcal{C}$-colored graph with one input and one output as in the picture. Formula (2.1) holds because in a fusion category for non-isomorphic simple objects we have $\operatorname{Hom}_{\mathcal{C}}(i, j)=0$.

Lemma 2.1. For any simple object $i$ of $\mathcal{C}$, the following equalities hold:

and


In the above equalities, the small white boxes represent pieces of $\mathcal{C}$-colored planar graphs which are the same in both sides.

Proof. Since $i$ is a simple object of $\mathcal{C}$, any endomorphism $\varphi \in \operatorname{Hom}_{\mathcal{C}}(i, i)$ expands as $\varphi=\lambda \operatorname{id}_{i}$ with $\lambda \in \mathbb{k}$. The $\mathbb{k}$ - linearity of the trace implies $\operatorname{tr}_{l}(\varphi)=$ $\lambda \operatorname{tr}_{l}\left(\mathrm{id}_{i}\right)=\lambda \operatorname{dim}_{l}(i)$ and $\operatorname{tr}_{r}(\varphi)=\lambda \operatorname{tr}_{r}\left(\mathrm{id}_{i}\right)=\lambda \operatorname{dim}_{r}(i)$. Since $\operatorname{dim}_{l}(i)$ and $\operatorname{dim}_{r}(i)$ are invertible by Lemma 1.4, we deduce $\lambda=\operatorname{dim}_{l}(i)^{-1} \operatorname{tr}_{l}(\varphi)$ and $\lambda=\operatorname{dim}_{r}(i)^{-1} \operatorname{tr}_{r}(\varphi)$. We obtain $\varphi=\operatorname{dim}_{l}(i)^{-1} \operatorname{tr}_{l}(\varphi) \operatorname{id}_{i}$ and $\varphi=$ $\operatorname{dim}_{r}(i)^{-1} \operatorname{tr}_{r}(\varphi) \operatorname{id}_{i}$. The statement follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ (see Section 2.2.6).

### 2.3. Duality and contraction vectors

In this section, we define contraction vectors associated to edges of colored graphs.
2.3.1. Duality pairings. Let $\mathcal{C}$ be a pivotal $\mathbb{k}$-category. Every tuple $S$ of signed objects of $\mathcal{C}$ (see Section 1.3.2) gives rise to a pairing in $\operatorname{Mod}_{\mathfrak{k}}$

$$
\omega_{S}: \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S^{*}}\right) \otimes_{\mathfrak{k}} \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S}\right) \rightarrow \operatorname{End}_{\mathcal{C}}(\mathbb{1})
$$

called duality pairing and defined by

$$
\omega_{S}\left(\alpha \otimes_{\mathfrak{k}} \beta\right)=\operatorname{ev}_{S}\left(\alpha \otimes_{\mathfrak{k}} \beta\right)
$$

for all $\alpha \in \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S^{*}}\right)$ and $\beta \in \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S}\right)$, where $\mathrm{ev}_{S}$ is the generalized evaluation discussed in Section 1.3.3. By the isotopy invariance of the graphical calculus,

$$
\omega_{S}\left(\alpha \otimes_{\mathfrak{k}} \beta\right)=\omega_{S^{*}}\left(\beta \otimes_{\mathfrak{k}} \alpha\right)
$$

for all $S, \alpha, \beta$, where $S^{*}$ is the dual of $S$ (see Section 1.3.2).
The dual of a cyclic $\mathcal{C}$-set $(E, c, \varepsilon)$ is the cyclic $\mathcal{C}$-set $\left(E^{\text {op }}, c,-\varepsilon\right)$ where $E^{\text {op }}$ is the set $E$ endowed with the opposite cyclic order of $E$. For each element $e$ in a cyclic $\mathcal{C}$-set $E=(E, c, \varepsilon)$, we define $S_{e}^{E}$ to be the tuple of signed objects of $\mathcal{C}$ obtained by enumerating the elements of $E$ in the given cyclic order starting with $e$ and recording the value of $c$ and $\varepsilon$. Let $e^{*}$ be the element in ( $E^{\mathrm{op}}, c,-\varepsilon$ ) preceding $e$ in the given cyclic order on $E$. In this way, by construction we have that

$$
S_{e^{*}}^{E^{\mathrm{op}}}=\left(S_{e}^{E}\right)^{*} .
$$

For each $e \in E$, set

$$
\tilde{\omega}_{E}^{e}=\omega_{S_{e}^{E}}\left(\tau_{e^{*}}^{E^{\mathrm{op}}} \otimes \tau_{e}^{E}\right): H\left(E^{\mathrm{op}}\right) \otimes_{\mathfrak{k}} H(E) \rightarrow \operatorname{End}_{\mathcal{C}}(\mathbb{1}),
$$

where

$$
\tau_{e}^{E}: H(E) \rightarrow \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S_{e}^{E}}\right) \quad \text { and } \quad \tau_{e^{*}}^{E^{\mathrm{op}}}: H\left(E^{\mathrm{op}}\right) \rightarrow \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{\left(S_{e}^{E}\right)^{*}}\right) .
$$

are the cone isomorphisms. The pairings $\tilde{\omega}_{E}^{e}$ and

$$
\tilde{\omega}_{E^{e^{*} \mathrm{p}}}: H(E) \otimes_{\mathfrak{k}} H\left(E^{\mathrm{op}}\right) \rightarrow \operatorname{End}_{\mathcal{C}}(\mathbb{1})
$$

are equal up to permutation of tensor factors. Consequently, they induce a $\mathbb{k}$-bilinear pairing

$$
\omega_{E}^{e}: H\left(E^{\mathrm{op}}\right) \otimes H(E) \rightarrow \operatorname{End}_{\mathcal{C}}(\mathbb{1}) .
$$

where $\otimes$ is the unordered tensor product (see Appendix (A) of $\mathbb{k}$-modules. Notice that it follows from the definition that

$$
\omega_{E}^{e^{*}}{ }^{e^{*}}=\omega_{E}^{e} .
$$

In general, the pairing $\omega_{E}^{e}$ does depend on the choice of $e \in E$. If the category $\mathcal{C}$ is spherical, then the pairing $\omega_{E}^{e}$ does not depend on the choice of $e \in E$ (see [TVi, Lemma 12.4]).
2.3.2. Contraction vectors. Let $\mathcal{C}$ be a non-degenerate pivotal $\mathbb{k}$-category. Then all pairings considered in Section 2.3.1 take values in $\operatorname{End}_{\mathcal{C}}(\mathbb{1}) \simeq \mathbb{k}$. Let $E$ be a cyclic $\mathcal{C}$-set and $e \in E$. The contraction vectors

$$
*_{\tilde{\omega}_{E}^{e}} \in H(E) \otimes_{\mathbb{k}} H\left(E^{\mathrm{op}}\right) \quad \text { and } \quad *_{\omega_{E}^{e \circ p}}^{e^{*}} \in H\left(E^{\mathrm{op}}\right) \otimes_{\mathbb{k}} H(E)
$$

(see Section 1.1.11) of the pairings

$$
\tilde{\omega}_{E}^{e}: H\left(E^{\mathrm{op}}\right) \otimes_{\mathbb{k}} H(E) \rightarrow \mathbb{k} \quad \text { and } \quad \tilde{\omega}_{E \circ \mathrm{p}}^{e^{*}}: H(E) \otimes_{\mathbb{k}} H\left(E^{\mathrm{op}}\right) \rightarrow \mathbb{k}
$$

are equal up to permutation of the tensor factors. Consequently they determine a vector

$$
*_{E}^{e} \in H(E) \otimes H\left(E^{\mathrm{op}}\right)
$$

2.3.3. Duality pairing for $\mathcal{C}$-colored graphs. Let $\mathcal{C}$ be a pivotal $\mathbb{k}$ category. Recall that an element $e$ of a cyclic $\mathcal{C}$-set $E$ determines a tuple $S_{e}^{E}$ of signed objects of $\mathcal{C}$ (see Section 2.3.1). Also recall the dual $S^{*}$ of a tuple $S$ of signed objects of $\mathcal{C}$ (see Section 1.3.2).

Let $G$ and $G^{\prime}$ be $\mathcal{C}$-colored graphs in the oriented surfaces $\Sigma$ and $\Sigma^{\prime}$. Let $u$ be a vertex of $G$ and $v$ be a vertex of $G^{\prime}$. A duality between $u$ and $v$ consists in an half-edge $e_{u}$ incident to $u$ and an half-edge $e_{v}$ incident to $v$ such that

$$
S_{e_{v}}^{E_{v}\left(G^{\prime}\right)}=\left(S_{e_{u}}^{E_{u}(G)}\right)^{*} .
$$

Here $E_{u}(G)$ and $E_{v}\left(G^{\prime}\right)$ are the cyclic $\mathcal{C}$-sets associated with the vertices $u$ and $v$ (see Section (2.2.3). We say that $u$ and $v$ are in duality if there is a duality between $u$ and $v$.

A duality between $u$ and $v$ induces a $\mathbb{k}$-bilinear pairing

$$
\omega_{u, v}: H_{u}(G) \otimes H_{v}\left(G^{\prime}\right) \rightarrow \operatorname{End}_{\mathcal{C}}(\mathbb{1})
$$

defined as follows. The composition of the cone isomorphism

$$
\tau_{e_{v}}^{E_{v}\left(G^{\prime}\right)}: H_{v}\left(G^{\prime}\right)=H\left(E_{v}\left(G^{\prime}\right)\right) \rightarrow \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S_{e_{v}}^{E_{v}\left(G^{\prime}\right)}}\right)
$$

with the inverse of the cone isomorphism

$$
\tau_{e_{u}}^{E_{u}(G)^{\mathrm{op}}}: H\left(\left(E_{u}(G)\right)^{\mathrm{op}}\right) \rightarrow \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S_{e_{u}}^{\left(E_{u}(G)\right)^{\mathrm{op}}}}\right)
$$

induce a $\mathbb{k}$-linear isomorphism

$$
\varphi_{u, v}: H_{v}\left(G^{\prime}\right) \rightarrow H\left(E_{u}(G)^{\mathrm{op}}\right) .
$$

The pairing

$$
\omega_{E_{u}(G)}^{e_{u}}: H\left(E_{u}(G)^{\mathrm{op} \mathrm{p}}\right) \otimes H\left(E_{u}(G)\right) \rightarrow \operatorname{End}_{\mathcal{C}}(\mathbb{1}) .
$$

from Section 2.3.1 induces a pairing

$$
\omega_{u, v}=\omega_{E_{u}(G)}^{e_{u}}\left(\varphi_{u, v} \otimes \operatorname{id}_{H_{u}(G)}\right): H_{v}\left(G^{\prime}\right) \otimes H_{u}(G) \rightarrow \operatorname{End}_{\mathcal{C}}(\mathbb{1})
$$

It follows from the definition that $\omega_{v, u}=\omega_{u, v}$.
If $\mathcal{C}$ is non-degenerate, then the contraction vector

$$
*_{E_{u}(G)}^{e_{u}} \in H\left(E_{u}(G)\right) \otimes H\left(E_{u}(G)^{\mathrm{op}}\right)
$$

from Section 2.3.2 induces a contraction vector

$$
*_{u, v}=\left(\operatorname{id}_{H_{u}(G)} \otimes \varphi_{u, v}^{-1}\right)\left(*_{E_{u}(G)}^{e_{u}}\right) \in H_{u}(G) \otimes H_{v}\left(G^{\prime}\right)
$$

Note that this vector does depend on the duality between $u$ and $v$. It follows from the definition that $*_{v, u}=*_{u, v}$.
2.3.4. Graphical representation of evaluations. Let $\mathcal{C}$ be a non-degenerate pivotal $\mathbb{k}$-category. Consider two $\mathcal{C}$-colored planar graphs $G$ and $G^{\prime}$. Consider a duality between a vertex $u$ of $G$ and a vertex $v$ of $G^{\prime}$. Recall that it consists in an half-edge $e_{u}$ incident to $u$ and an half-edge $e_{v}$ incident to $v$ satisfying some condition (see Section 2.3.3). We represent the evaluation

$$
\left(\mathbb{F}_{\mathcal{C}}(G) \otimes \mathbb{F}_{\mathcal{C}}\left(G^{\prime}\right)\right)\left(*_{u, v}\right)=\mathbb{F}_{\mathcal{C}}\left(G \sqcup G^{\prime}\right)\left(*_{u, v}\right)
$$

by adding to a diagram of $G \sqcup G^{\prime}$ a red arc whose endpoints determine the duality. This means that the endpoints of this added arc are points near $u$ and $v$ such that by starting from these points and following the opposite orientation of the plane, the first encountered half-edges are $e_{u}$ and $e_{v}$. If there are several evaluations, we graphically represent them with several red arcs (one for each evaluation). For example:

2.3.5. The case of a fusion category. Let $\mathcal{C}$ be a pivotal fusion $\mathbb{k}$ category. Recall that $\mathcal{C}$ is non-degenerate (see Section 1.4.5).

Lemma 2.2. Let $S$ be a tuple of signed objects of $\mathcal{C}$ and let

$$
\omega_{S}: \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S^{*}}\right) \otimes_{\mathbb{k}} \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S}\right) \rightarrow \operatorname{End}_{\mathcal{C}}(\mathbb{1}) \simeq \mathbb{k}
$$

be the pairing defined by $S$ in Section 2.3.1. Then the contraction vector

$$
*_{w_{S}} \in \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S}\right) \otimes_{\mathfrak{k}} \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S^{*}}\right)
$$

of $\omega_{S}$ is computed by

$$
*_{w_{S}}=\stackrel{L}{n}_{S}^{S} \otimes \sqrt{ }_{S^{*}}^{\cdots}
$$

where the arcs are colored and oriented so that $S$ is the tuple of signed objects determined by the horizontal side of the curvilinear boxes, and where the notation of Section 1.4 .6 is used for a $\mathbb{1}$-partition of $X_{S}$.

Proof. Let $*_{S} \in \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S}\right) \otimes_{\mathfrak{k}} \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S^{*}}\right)$ be the vector defined in the right-hand side of the equality above. For any $f \in \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S}\right)$,

Here, ( $i$ ) follows from the definitions of $\omega_{S}$ and $*_{S}$, (ii) from the isotopy invariance of the graphical calculus, and (iii) from formula (1.10). Similarly, we have that:

$$
\left(\omega_{S} \otimes_{\mathbb{k}} \operatorname{id}_{\operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S^{*}}\right)}\right)\left(g \otimes *_{S}\right)=g
$$

This prove that $*_{S}$ is the contraction vector of $\omega_{S}$.
Lemma 2.3. Let I be a representative set of simple objects of $\mathcal{C}$. Then:

(b)

where white box stands for a piece of a $\mathcal{C}$-colored graph (the same on the left-hand and right hand side).
Proof. To prove the lemma we only need to compare the contributions to $\mathbb{F}_{\mathcal{C}}$ of the depicted pieces of $\mathcal{C}$-colored graphs for both expressions (a) and (b). Let $S=\left(\left(X_{1}, \varepsilon_{1}\right), \ldots\left(X_{n}, \varepsilon_{n}\right)\right)$ be the tuple of signed objects of $\mathcal{C}$ determined by the left-hand side of the equality (a). Consider the dual tuple $S^{*}=\left(\left(X_{n},-\varepsilon_{n}\right), \ldots,\left(X_{1},-\varepsilon_{1}\right)\right)$ and the morphisms
$\operatorname{ev}_{S^{*}}: X_{S} \otimes X_{S^{*}} \rightarrow \mathbb{1}, \quad \operatorname{coev}_{S^{*}}: \mathbb{1} \rightarrow X_{S^{*}} \otimes X_{S} \quad$ and $\quad \Psi_{S^{*}}: X_{S^{*}} \rightarrow X_{S}^{*}$ defined in Section 1.3.3. For $i \in I$, set $S_{i}=\left((i,-),\left(X_{1}, \varepsilon_{1}\right), \ldots\left(X_{n}, \varepsilon_{n}\right)\right)$. Then $S_{i}^{*}=\left(\left(X_{n},-\varepsilon_{n}\right), \ldots,\left(X_{1},-\varepsilon_{1}\right),(i,+)\right) \quad X_{S_{i}}=i^{*} \otimes X_{S} \quad$ and $\quad X_{S_{i}^{*}}=X_{S^{*}} \otimes i$.
Next, consider the non-degenerate pairing

$$
\omega_{i}: \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S_{i}^{*}}\right) \otimes_{\mathbb{k}} \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S_{i}}\right) \rightarrow \mathbb{k}
$$

Let $*_{i} \in \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S_{i}}\right) \otimes_{\mathbb{k}} \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S_{i}^{*}}\right)$ be the associated contraction vector. Consider the following isotopy between $\mathcal{C}$-colored graphs:


Using the definitions of $\mathbb{F}_{\mathcal{C}}$ and of the contraction vector $*_{u, v}$ between the vertices $u$ and $v$, we reduce assertion (a) to the following claim: for some expansion

$$
*_{i}=\sum_{\alpha} e_{i, \alpha} \otimes_{\mathbb{k}} f_{i, \alpha}
$$

with $e_{i, \alpha} \in \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S_{i}}\right)$ and $f_{i, \alpha} \in \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{S_{i}^{*}}\right)$ we have

where the top (respectively bottom) free ends of the Penrose diagram are colored and oriented so that the corresponding tuple of signed objects is $S$ (respectively $S^{*}$ ). We verify (2.2) for the expansion

provided by Lemma 2.2. The left-hand side of (2.2) is equal to


Here (i) follows from the isotopy invariance of graphical calculus, (ii) from formula (1.11) and (iii) from formula (1.9). This proves formula (a).

To prove formula (b), we proceed as follows. For $i \in I$, set

$$
\tilde{S}_{i}=\left(\left(X_{1}, \varepsilon_{1}\right), \ldots\left(X_{n}, \varepsilon_{n}\right),(i,-)\right)
$$

Then
$\tilde{S}_{i}^{*}=\left((i,+),\left(X_{n},-\varepsilon_{n}\right), \ldots,\left(X_{1},-\varepsilon_{1}\right)\right) \quad X_{\tilde{S}_{i}}=X_{S} \otimes i^{*} \quad$ and $\quad X_{\tilde{S}_{i}^{*}}=i \otimes X_{S^{*}}$.
Consider the non-degenerate pairing

$$
\tilde{\omega}_{i}: \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{\tilde{S}_{i}^{*}}\right) \otimes_{\mathbb{k}} \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{\tilde{S}_{i}}\right) \rightarrow \mathbb{k}
$$

Let $\tilde{\star}_{i} \in \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{\tilde{S}_{i}}\right) \otimes_{\mathbb{k}} \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{\tilde{S}_{i}^{*}}\right)$ be the associated contraction vector. Consider the following isotopy between $\mathcal{C}$-colored graphs:


Using the definition of $\mathbb{F}_{\mathcal{C}}$ and the contraction vector $*_{\tilde{u}, \tilde{v}}$, between the vertices $\tilde{u}$ and $\tilde{v}$ of the pairing $\tilde{\omega}_{i}$ we reduce the lemma to the following claim: for some expansion

$$
\tilde{※}_{i}=\sum_{\beta} \tilde{e}_{i, \alpha} \otimes_{\mathbb{k}} \tilde{f}_{i, \alpha}
$$

with $\tilde{e}_{i, \alpha} \in \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{\tilde{S}_{i}}\right)$ and $\tilde{f}_{i, \alpha} \in \operatorname{Hom}_{\mathcal{C}}\left(\mathbb{1}, X_{\tilde{S}_{i}^{*}}\right)$ we have

where the top (respectively bottom) free ends of the Penrose diagram are colored and oriented so that the corresponding tuple of signed objects is $S$. We verify (2.3) for the expansion

provided by Lemma 2.2. The left-hand side of (2.3) is equal to


Here (i) follows from the isotopy invariance of graphical calculus, (ii) from formula (1.12) and (iii) from formula (1.9). This proves formula (b).

Next we prove formula (c). This equality follows from the previous points since $\operatorname{dim}_{l}(\mathbb{1})=\operatorname{dim}_{r}(\mathbb{1})=1_{\mathbf{k}}$ and $\operatorname{Hom}_{\mathcal{C}}(i, \mathbb{1})=0$ for all $i \in I$ different from $\mathbb{1}$.

## CHAPTER 3

## Combed 3-manifolds

This chapter is devoted to the theory of combed 3-dimensional manifolds, which are 3-manifolds endowed with a non-vanishing vector field. Branched spines have been firstly considered by Gillman and Rolfsen [GR1, GR2] and more explicitly by Ishii [Is1, Is2, Is3]. In [BP1, BP2], Benedetti and Petronio, besides having given substantial contributions to the theory of branched spines, introduce and develop the theory of o-graphs which encode a special kind of branched spines.

In Section [3.1, we review the theory of spines of 3-manifolds. Then, we discuss the presentation of combed 3-manifolds via branched spines in Section 3.2 and via o-graphs in Section 3.3,

### 3.1. Spines of 3-manifolds

In this section, we review the theory of spines of 3 -manifolds. The main contributors to this theory are Casler, Matveev, and Piergallini.
3.1.1. Manifolds. For $n \geq 1$, by a $n$-manifold, we mean a manifold of dimension $n$ with or without boundary. The boundary $\partial M$ of a manifold $M$ is then a $(n-1)$-manifold without boundary. If $M$ is oriented, then its boundary $\partial M$ is oriented in such a way that at any point of $\partial M$, the orientation of $M$ is given by a direction away from $M$ followed by the orientation of $\partial M$. A closed manifold is a compact manifold with empty boundary. The empty set $\emptyset$ is considered as a closed oriented manifold of arbitrary dimension.

It is a well-known result that all compact 3 -manifolds have a smooth structure unique up to ambient isotopy, therefore every time that we need the hypothesis of smoothness we refer implicitly to that one.
3.1.2. Polyhedra. A 2 -polyhedron is a compact topological space $P$ that can be triangulated using a finite number of simplices of dimension $\leq 2$ so that all 0 -simplices and 1 -simplices are faces of 2 -simplices. For a 2 -polyhedron $P$, denote by $\operatorname{Int}(P)$ the subspace of $P$ consisting of all points having a neighborhood homeomorphic to $\mathbb{R}^{2}$. By the definition of a 2-polyhedron, the surface $\operatorname{Int}(P)$ is dense in $P$. A stratification of a 2-polyhedron $P$ is an (unoriented) graph $P^{(1)}$ embedded in $P$ so that $P \backslash \operatorname{Int}(P) \subset P^{(1)}$. The vertices and edges of $P^{(1)}$ are called respectively the vertices and edges of $P$. We denote the set of vertices of $P$ as $P^{(0)}$. To specify a stratification of $P$ it suffices to specify the edges of $P$ because the vertices of $P$ are just the endpoints of the edges. Note that any 2-polyhedron can be endowed with a stratification. A stratified polyhedron is a 2-polyhedron endowed with a stratification.

Cutting a stratified polyhedron $P$ along the graph $P^{(1)} \subset P$ we obtain a compact surface $\tilde{P}$ with interior $P \backslash P^{(1)}$. The 2-polyhedron $P$ can be recovered by gluing $\tilde{P}$ to $P^{(1)}$ along a surjective map $\pi: \partial \tilde{P} \rightarrow P^{(1)}$. The set

$$
\pi^{-1}\left(P^{(0)}\right) \subset \partial \tilde{P}
$$

is closed and discrete, and therefore is finite. The points of this set split $\partial \tilde{P}$ into arcs whose interiors are mapped by $\pi$ homeomorphically onto the interiors of edges of $P$. The connected components of $\tilde{P}$ are called the regions of $P$. Each component of $P \backslash P^{(1)} \subset \tilde{P}$ is the interior of a unique region. We let $\operatorname{Reg}(P)$ be the finite set of all regions of $P$.

A branch of a stratified 2-polyhedron $P$ at a vertex $x$ of $P$ is a germ at $x$ of an adjacent region. More formally, a branch of $P$ at $x$ is a homotopy class of paths $[0,1] \rightarrow P$ starting in $x$ and carrying $(0,1]$ to $P \backslash P^{(1)}$. The number of branches of $P$ at $x$ is equal to $\operatorname{card}\left(\pi^{-1}(x)\right)$, where $\pi: \partial \tilde{P} \rightarrow P^{(1)}$ is the map above. Similarly, a branch of $P$ at an edge $e$ of $P$ is a germ at $e$ of an adjacent region. Formally, a branch of $P$ at $e$ is the homotopy class of paths $[0,1] \rightarrow P$ starting in the interior of $e$ and carrying $(0,1]$ to $P \backslash P^{(1)}$. There is an obvious bijective correspondence between the branches of $P$ at $e$ and the connected components of $\pi^{-1}$ (interior of $e$ ). The set of branches of $P$ at $P$ is denoted $P_{e}$. This set is finite and non-empty. The number of elements of $P_{e}$ is called the valence of $e$.

An orientation of a region $r$ of $P$ induces an orientation for each edge $e$ of $P$ adjacent to $r$ in the following way: the orientation of $e$ followed by a vector at a point of $e$ directed inside $r$ is the given orientation of $r$.

An orientation of a stratified polyhedron $P$ is an orientation of the surface $P \backslash P^{(1)}$. To orient $P$, one must orient all its regions. An oriented polyhedron is a stratified polyhedron endowed with an orientation.
3.1.3. Simple polyhedra. Let $\mathcal{S}$ be the following subset of $\mathbb{R}^{3}$ :
$\mathcal{S}=\left\{\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3} \mid x_{3}=0\right.$, or $x_{1}=0$ and $x_{3}>0$, or $x_{2}=0$ and $\left.x_{3}<0\right\}$ that is


A point $v$ of a topological space is said to be special if there is a homeomorphism of $\mathcal{S}$ onto a neighborhood of $v$ carrying the origin $(0,0,0)$ to $v$.

A simple polyhedron is an oriented connected polyhedron $P$ with at least one special point such that each point of $P$ has a neighborhood homeomorphic to an open subset of $\mathcal{S}$.

A simple polyhedron $P$ has a canonical stratification given by the graph $P^{(1)}=P \backslash \operatorname{Int}(P)$ whose vertices are the special points of $P$. Note that all
edges of $P$ have valence $\geq 2$. In what follows, we endow any simple polyhedron with this canonical stratification.
3.1.4. Standard polyhedra. A standard polyhedron is a simple polyhedron such that its regions are disks. Any standard polyhedron has 6 branches at every vertex, 3 branches at every edge, and an empty boundary.
3.1.5. Spines of 3-manifolds. A spine of a compact connected 3-manifold $M$ is a simple polyhedron $P$ embedded in $M$ such that $M \backslash P$ is homeomorphic to an open 3 -ball if $\partial M=\emptyset$ or to $\partial M \times[0,1)$ if $\partial M \neq \emptyset$. Note that if $P$ is a spine of a closed connected 3 -manifold $M$, then $P$ is a spine of $M \backslash \operatorname{Int}\left(B^{3}\right)$, where $B^{3}$ is a 3 -ball embedded in $M$. A spine of $M$ is standard if the underlying polyhedron is standard.

A result due to Casler [Ca, Matveev [Ma1, and Piergallini [Pi] asserts that any compact connected 3 -manifold has a standard spine.

Let $P$ be a spine of a compact connected 3-manifold $M$. Any vertex $x$ of $P$ has a closed ball neighborhood $B_{x} \subset M$ such that $\Delta_{x}=P \cap \partial B_{x}$ is a nonempty graph and $P \cap B_{x}$ is the cone over $\Delta_{x}$ with vertex $x$. The vertices of $\Delta_{x}$ are the intersection points of the 2 -sphere $\partial B_{x}$ with the edges of $P$ incident to $x$. The edges of $\Delta_{x}$ are the intersections of $\partial B_{x}$ with the branches of $P$ at $x$. Since all edges of $P$ have valence $\geq 2$, so do all vertices of $\Delta_{x}$. We call $B_{x}$ a $P$-cone neighborhood of $x$ and call $\Delta_{x} \subset \partial B_{x}$ the link graph of $x$.
3.1.6. Moves on spines. Let $M$ be a compact connected oriented 3manifold. We define two local transformations (moves) on a spine $P$ of $M$ transforming $P$ into a new spine of $M$. Each of these moves modifies $P$ inside a closed 3-ball in $M$.

The move MP $(0,2)$ (also called lune move) pushes a branch of $P$ at an edge of $P$ through an edge of $P$ :


This move increases the number of vertices of $P$ by 2 , increases the number of edges of $P$ by 4 , and increases the number of regions of $P$ by 2 . The new region created is a disk. This move keeps the orientations of the regions and, the new region created is arbitrarily oriented. The inverse move MP $(0,2)^{-1}$ is allowed only when the orientations of two regions united under this move are compatible.

The move MP $(2,3)$ pushes a branch of $P$ at a vertex of $P$ through another vertex of $P$ :


This move increases the number of vertices of $P$ by 1 , increases the number of edges of $P$ by 2 , and increases the number of regions of $P$ by 1 . The new region created is a disk. This move keeps the orientations of the regions, and the new region created is arbitrarily oriented. The inverse move $\operatorname{MP}(2,3)^{-1}$ can always be applied.

By Matveev-Piergallini moves or MP-moves on spines of $M$, we mean ambient isotopies of spines in $M$ together with the moves $\operatorname{MP}(0,2), \operatorname{MP}(2,3)$, and their inverses. Note that all MP-moves transform standard spines into standard spines.

Theorem 3.1 ([ $\mathbf{M a 2}, \mathbf{P i} \mathbf{]})$. Any two standard spines of a compact connected oriented 3-manifold are related by a finite sequence of MP-moves.

### 3.2. Combed 3 -manifolds via branched spines

In this section, we review the theory of combed 3-manifolds and their presentation via branched spines. For more details, we refer to Is1, Is2, BP1, BP2, BP3.
3.2.1. Combed 3-manifolds. A combing on a 3 -manifold $M$ is a vector field $\nu$ on $M$ (that is, a section $\nu: M \rightarrow T M$ of the tangent bundle of $M$ ) such that:
(i) $\nu$ is always nonzero;
(ii) $\nu$ is tangent to $\partial M$ exactly at the points of a compact 1-dimensional submanifold $\gamma \subset \partial M$;
(iii) $\nu$ is never tangent to $\gamma$;
(iv) if $\partial M \neq \emptyset$, then the orbits of $\nu$ are closed intervals.

This definition agrees with that of a concave traversing vector field given in [BP2, Definition 4.1.8]. Note that if $M$ is closed, then a combing on $M$ is just a nowhere-zero vector field on $M$.

A combed 3-manifold is a pair $(M, \nu)$ where $M$ is a compact oriented connected 3 -manifold and $\nu$ a combing on $M$.

Two combed 3-manifolds $(M, \nu)$ and ( $M^{\prime}, \nu^{\prime}$ ) are equivalent if there is an orientation-preserving diffeomorphism $\phi: M \rightarrow M^{\prime}$ such that the combings $\phi_{*} \circ \nu \circ \phi^{-1}$ and $\nu^{\prime}$ are homotopic within the class of combings on $M^{\prime}$. Here, the map $\phi_{*}: T M \rightarrow T M^{\prime}$ is induced by $\phi$. We write $(M, \nu) \sim\left(M^{\prime}, \nu^{\prime}\right)$. Note that $\sim$ is an equivalence relation on the class of combed 3 -manifolds.
3.2.2. Branched polyhedra. Each edge $e$ of a standard polyhedron $P$ carries three orientations, each of them being induced by the orientation of the branch of $P$ at $e$ as in Section 3.1.2.

A branched polyhedron is a standard polyhedron $P$ such that at any edge of $P$, two of the three induced orientations are opposite to the third one.

Branched polyhedra, can be viewed as the smoothed version of standard polyhedra. Let $P$ be a branched polyhedron. Consider an edge $e$ of $P$ :


The orientation on $P$ allows to define a tangent plane at every point of $P$ and we represent this as follows:


With this convention, there are two possible configurations for a vertex of $P$ :

3.2.3. Combed 3 -manifolds associated to branched polyhedra. Following [BP2, Section 2.1], to every branched polyhedron $P$ is associated a combed 3-manifold ( $M_{P}, \nu_{P}$ ) with boundary such that:
(i) $P$ is a spine of $M_{P}$;
(ii) the combing $\nu_{P}$ is positively transverse to $P$. This means that $\nu_{P}$ is transverse to each region $r$ of $P$ and the orientation of $r$ together with the orientation of $\nu_{P}$ gives the orientation of $M_{P}$.
The 3-manifold $M_{P}$ is defined as follows. Replace each region of $P$ (which is a disk) by the piece:


Replace each edge of $P$ by the piece:


Replace each vertex of $P$ by one of the following two pieces according to their possible configuration (see Section 3.2.2):


Then the 3-manifold $M_{P}$ is obtained by gluing these pieces along the grey sides by respecting the smoothing of Section 3.2.2. The boundary of $M_{P}$ is then the union of the white sides of the above pieces. Note that $P$ is a spine of $M_{P}$.

The vector field $\nu_{P}$ is defined to be transverse to $P$ as follows:



and is extended to $M_{P}$ as follows:


Finally, we orient $M_{P}$ so that its orientation is given by the orientation of any region together with the orientation of $\nu_{P}$ at this region (such an orientation exists since $P$ is branched).
3.2.4. Branched spines of combed 3 -manifolds with boundary. A branched spine of a combed 3-manifold ( $M, \nu$ ) with boundary is a branched polyhedron $P$ such that $\left(M_{P}, \nu_{P}\right)$ is equivalent to $(M, \nu)$ in the sense of Section 3.2.1.

Theorem 3.2 ( $\widehat{\mathbf{B P} 3}$, Theorem 4.3.1]). Any combed 3-manifold with boundary has a branched spine.
3.2.5. Combed 3-manifolds with trivial spherical boundary. We say that a combed 3-manifold $(M, \nu)$ has trivial spherical boundary if
(i) the boundary $\partial M$ of $M$ is a 2-dimensional sphere;
(ii) the compact 1-dimensional submanifold of $\partial M$ where $\nu$ is tangent (see Section 3.2.1) is a circle.
This tangency circle splits $\partial M$ into two disks. The vector field is positively transverse to one disk and negatively transverse to the other one.

For example, consider the 3-ball

$$
B^{3}=\left\{(x, y, z) \in \mathbb{R}^{3} \mid x^{2}+y^{2}+z^{2} \leq 1\right\}
$$

This is a compact connected 3 -manifold with boundary the 2 -sphere

$$
S^{2}=\partial B^{3}=\left\{(x, y, z) \in \mathbb{R}^{3} \mid x^{2}+y^{2}+z^{2}=1\right\} .
$$

We endow $B^{3}$ with the orientation induced by the right-hand orientation of $\mathbb{R}^{3}$. Consider the vector field $\nu_{\text {triv }}$ on $B^{3}$ which is constant equal to $(0,0,1)$ :


Then the pair $\left(B^{3}, \nu_{\text {triv }}\right)$ is a combed 3 -manifold with trivial spherical boundary.

Any combed 3-manifold ( $M, \nu$ ) with trivial spherical boundary gives rise to a closed combed 3-manifold

$$
(\widehat{M}, \widehat{\nu})=(M, \nu) \cup_{f}\left(B^{3}, \nu_{\text {triv }}\right)
$$

Here, the gluing is defined by an orientation reversing diffeomorphism $f: \partial M \rightarrow$ $S^{2}=\partial B^{3}$ preserving the tangency circle. The following lemma is straightforward.

Lemma 3.3. (a) Any closed combed 3-manifold is equivalent to $(\widehat{M}, \widehat{\nu})$ for some combed 3-manifold ( $M, \nu$ ) with trivial spherical boundary.
(b) Let $(M, \nu)$ and $\left(M^{\prime}, \nu^{\prime}\right)$ be combed 3-manifolds with trivial spherical boundary. Then $(\widehat{M}, \widehat{\nu})$ and $\left(\widehat{M^{\prime}}, \widehat{\nu^{\prime}}\right)$ are equivalent if and only if $(M, \nu)$ and $\left(M^{\prime}, \nu^{\prime}\right)$ are equivalent.
3.2.6. Branched spines of closed combed 3 -manifolds. By a closed branched polyhedron, we mean a branched polyhedron $P$ such that its associated combed 3-manifold $\left(M_{P}, \nu_{P}\right)$ (see Section 3.2.3) is a combed 3-manifold with trivial spherical boundary.

A branched spine of a closed combed 3-manifold $(M, \nu)$ is a closed branched polyhedron $P$ such that $\left(\widehat{M_{P}}, \widehat{\nu_{P}}\right)$ is equivalent to $(M, \nu)$.

The following result is a direct consequence of Theorem 3.2] and Lemma3.3(a).
Theorem 3.4. Any closed combed 3-manifold has a branched spine.
3.2.7. Moves on branched spines. Let $(M, \nu)$ be a combed 3-manifold. We define moves on a branched spine $P$ of $(M, \nu)$ transforming $P$ into a new branched spine of $(M, \nu)$. These moves are the branched versions of the moves on standard spines of Section 3.1.6. Each of these moves modifies $P$ inside a closed 3-ball in $M$.

The moves $\operatorname{BMP}(2,3)$ are branched versions of the move MP $(2,3)$. They are the moves $M_{1}, \ldots, M_{5}$ depicted in Figure 3.1, together with their mirror images $\widetilde{M}_{1}, \ldots, \widetilde{M}_{5}$.


Figure 3.1. Branched moves $\operatorname{BMP}(2,3)$

The moves $\operatorname{BMP}(0,2)$ are branched versions of the move MP $(0,2)$. They are the moves $L_{1}, L_{2}, L_{3}$ depicted in Figure 3.2, Note that these moves are self-mirror (the mirror image of $L_{i}$ is $L_{i}$ ).

By BMP-moves on branched spines of ( $M, \nu$ ), we mean ambient isotopies of branched spines in $M$ together with the moves $\operatorname{BMP}(0,2), \operatorname{BMP}(2,3)$ and their inverses.

The following result provides a calculus for combed 3-manifolds:
Theorem 3.5 ([BP3, Theorem 4.3.2]). Any two branched spines of a combed 3-manifold are related by a finite sequence of BMP-moves.


Figure 3.2. Branched moves $\operatorname{BMP}(0,2)$

### 3.3. Combed 3-manifold via o-graphs

The theory of o-graphs has been introduced and developed in [BP2, BP1]. The o-graphs encode a particular class of standard polyhedra which is sufficient to encode all combed 3-manifolds.

In what follows, we always orient the plane $\mathbb{R}^{2}$ counterclockwise.
3.3.1. o-graphs. A vertex $v$ of an oriented graph $G$ is said to be of crossing type if:

- $v$ is quadrivalent with 2 incoming half-edges and 2 outgoing half-edges;
- the set $E_{v}$ of half-edges incident to $v$ is endowed with a cyclic order;
- there are 2 distinguished half-edges which are not consecutive (with respect to the cyclic order on $E_{v}$ ) and such that one is incoming and the other is outgoing.
A vertex of crossing type of an oriented graph $G$ is positive if the distinguished outgoing half-edge is followed (with respect to the cyclic order on half-edges) by an outgoing half-edge. Otherwise, it is said to be negative. In what follows, we depict a vertex of crossing type by a crossing, the overcrossing strand representing the distinguished half-edges:


Here, the cyclic order of the set of half-edges is given by the counterclockwise orientation of the plane.

Equivalently, a vertex $v$ of a graph $G$ is of crossing type if there is an embedding of a neighborhood of $v$ into the oriented plane "resembling" to a crossing of an oriented curve (i.e., a multiple point which is double and transverse with a distinguished strand).

An o-graph is a non-empty connected oriented graph where all vertices are of crossing type. (This notion of an o-graph corresponds to that of a normal o-graph in [BP2]). An isomorphism between two o-graphs is an isomorphism between their underlying oriented graphs which preserves the crossing types.

Any o-graph can be represented by a planar diagram obtained by immersing generically the o-graph into the oriented plane. (Here generically means that the multiple points of the immersion are double transverse and distinct from the image of the vertices). For example, the diagram

represents an o-graph with 3 vertices and 6 edges.
Two such diagrams represent isomorphic o-graphs if and only if one can be obtained from the other by a finite sequence of isotopies and the following Reidemeister-type moves:




Here, the orientations (not depicted) must agree before and after the moves. For example, the diagrams

and

represent two non-isomorphic o-graphs with 1 vertex and 2 edges.
3.3.2. From o-graphs to branched polyhedra. To each o-graph $\Gamma$ is associated a branched polyhedron $P_{\Gamma}$ defined as follows. Replace each positive vertex $v$ of $\Gamma$ with the following portion of a branched polyhedron :


Replace each negative vertex of $\Gamma$ with the following portion of a branched polyhedron :


Replace each edge of $\Gamma$ with the following portion of a branched polyhedron :


Finally, the branched polyhedron $P_{\Gamma}$ is obtained by gluing together this pieces according to smoothing (i.e., in such a way to respect the orientations).
3.3.3. Combed 3 -manifold associated to o-graphs. To any o-graph $\Gamma$ is associated a combed 3 -manifold ( $M_{\Gamma}, \nu_{\Gamma}$ ) with non-empty boundary. This combed 3 -manifold is defined by

$$
\left(M_{\Gamma}, \nu_{\Gamma}\right)=\left(M_{P_{\Gamma}}, \nu_{P_{\Gamma}}\right)
$$

where $P_{\Gamma}$ is the branched polyhedron associated to $\Gamma$ (see Section 3.3.2) and ( $M_{P_{\Gamma}}, \nu_{P_{\Gamma}}$ ) is the combed 3-manifold associated to $P_{\Gamma}$ (see Section 3.2.3).
3.3.4. Moves on o-graphs. The local transformations on o-graphs depicted in Figures 3.3 and 3.4 turn any o-graph into another o-graph. In these figures, if the orientations of some edges are omitted, then these edges can be oriented arbitrarily but in a same way before and after the move. By sliding moves, we mean moves in Figures 3.3 and 3.4 and their inverses together with isomorphisms of o-graphs.

The following result provides a calculus for combed 3-manifolds with nonempty boundary:

Theorem 3.6 ([日B2, Corollary 4.3.5]). (a) Any combed 3-manifold with non-empty boundary is equivalent to $\left(M_{\Gamma}, \nu_{\Gamma}\right)$ for some o-graph $\Gamma$.


Figure 3.3. Snake move


Figure 3.4. Sliding moves
(b) The combed 3-manifolds associated to two o-graphs are equivalent if and only if the o-graphs are related by a finite sequence of sliding moves.
3.3.5. Closed combed 3-manifolds via closed o-graphs. In this section, we consider a class of o-graph which encodes closed combed 3-manifolds.

By a circuit, we mean an oriented closed immersed plane curve such that all its multiple points are double and transverse.

An o-graph $\Gamma$ is closed if it satisfies the following three conditions:
(i) The number of circuits obtained from $\Gamma$ by removing all its vertices is exactly one.
(ii) The (trivalent) graph obtained from $\Gamma$ by applying the rules of Figure 3.5 is connected.
(iii) The number of circuits obtained from $\Gamma$ by applying the rules of Figure 3.6 is exactly one more than the number of vertices of $\Gamma$.

For example, the following o-graph:



Figure 3.5.




Figure 3.6.
is closed. Indeed, Condition (i) is clear and Conditions (ii) and (iii) are respectively verified by


The combed 3-manifold ( $M_{\Gamma}, \nu_{\Gamma}$ ) associated with a closed o-graph $\Gamma$ has trivial spherical boundary. Indeed, Condition $(i)$ implies that the Euler characteristic of the boundary $\partial M_{\Gamma}$ is 2 . Condition (ii) implies that $\partial M_{\Gamma}$ is connected and so together with $(i)$ implies that $\partial M_{\Gamma}$ is homeomorphic to a 2 -sphere. Finally, Condition (iii) implies that the submanifold of $\partial M_{\Gamma}$ where $\nu_{\Gamma}$ is tangent is a circle. (For details, we refer to [BP2, Section 5.2].)

Consequently, by Section [3.2.5, to any closed o-graph $\Gamma$ is associated the closed combed 3-manifold

$$
\left(\widehat{M}_{\Gamma}, \widehat{\nu}_{\Gamma}\right)=\left(M_{\Gamma}, \nu_{\Gamma}\right) \cup\left(B^{3}, \nu_{\text {triv }}\right) .
$$



Figure 3.7. Pontrjagin move
Note that any sliding move (see Section 3.3.4) transforms a closed o-graph into a closed o-graph. Combining Theorem 3.6 and Lemma 3.3, we obtain the following calculus for closed combed 3-manifolds:

THEOREM 3.7 ([BP2, Theorem 1.4.1]). (a) Any closed combed 3-manifold is equivalent to $\left(\widehat{M}_{\Gamma}, \widehat{\nu}_{\Gamma}\right)$ for some closed o-graph $\Gamma$.
(b) The closed combed 3-manifolds associated to two closed o-graphs are equivalent if and only if the o-graphs are related by a finite sequence of sliding moves.
3.3.6. The Pontrjagin move. The local transformation on o-graphs depicted in Figure 3.7 is called the Pontrjagin move. Note that this move transforms any closed o-graph into another closed o-graph, but their associated closed combed 3 -manifolds may be non-equivalent.

The Pontrjagin move allows to relate all combings on the same underlying closed 3 -manifold (see [BP2, Theorem 6.3.1]). More precisely, let $\Gamma$ and $\Gamma^{\prime}$ be closed o-graphs. Consider their associated closed combed 3-manifolds ( $\widehat{M}_{\Gamma}, \widehat{\nu}_{\Gamma}$ ) and ( $\widehat{M}_{\Gamma^{\prime}}, \widehat{\nu}_{\Gamma^{\prime}}$ ). Then the 3-manifolds $\widehat{M}_{\Gamma}$ and $\widehat{M}_{\Gamma^{\prime}}$ are homeomorphic if and only if $\Gamma$ and $\Gamma^{\prime}$ are related by a finite sequence of sliding moves and Pontrjagin moves.
3.3.7. Summary. We summarize the main results of this chapter as follows:

THEOREM (Non-empty boundary case). There is a one-to-one correspondence between:
(1) Combed 3-manifolds with non-empty boundary up to equivalence;
(2) Branched polyhedra up to BMP-moves;
(3) o-graphs up to sliding moves.

Theorem (Closed case). There is a one-to-one correspondence between:
(1') Closed combed 3-manifolds up to equivalence;
(2') Closed branched polyhedra up to BMP-moves;
(3') Closed o-graphs up to sliding moves.

## CHAPTER 4

## A state sum invariant of combed 3-manifolds

Fix, throughout this chapter, a pivotal fusion $\mathbb{k}$-category $\mathcal{C}$ and a representative set $I$ of simple objects of $\mathcal{C}$. We derive from this data a scalar topological invariant of combed 3 -manifolds.

### 4.1. An invariant of combed 3-manifolds

In this section, we construct a state sum topological invariant of combed 3-manifolds.
4.1.1. The state sum invariant via branched spines. Let $(M, \nu)$ be a combed 3 -manifold (with or without boundary). Pick a branched spine $P$ of $(M, \nu)$, see Sections 3.2.4 and 3.2.6. Recall from Section 3.1.2 the set $\operatorname{Reg}(P)$ of regions of $P$.

A coloring of $P$ is a map $c: \operatorname{Reg}(P) \rightarrow I$. The object $c(s) \in I$ assigned to $s \in \operatorname{Reg}(P)$ is called the $c$-color of $s$. We associate a scalar $|c| \in \mathbb{k}$ to each coloring $c$ of $P$ as follows.

By definition $P$ has at least one vertex and so it has at least one edge (stratified 2-polyhedra have no isolated vertices). By an oriented edge of $P$ we mean an edge of $P$ endowed with an orientation. Each oriented edge $e$ of $P$ yields a cyclic $\mathcal{C}$-set defined as follows. The orientations of $e$ and $M$ determine a positive direction on a small loop in $M$ encircling $e$. The resulting oriented loop determines a cyclic order on the set $P_{e}$ of branches of $P$ at $e$ (see Section 3.1.2). To each branch $\delta \in P_{e}$, we assign the $c$-color of the region of $P$ containing $\delta$ and a sign equal to + if the orientation of $\delta$ induces the one of $e \subset \partial \delta$ (that is, the orientation of $\delta$ is given by the orientation of $e$ followed by a vector at a point of $e$ directed inside $\delta$ ) and equal to - otherwise. In this way, $P_{e}$ becomes a cyclic $\mathcal{C}$-set and we consider its multiplicity module $H_{c}(e)=H\left(P_{e}\right)$. Let

$$
H_{c}=\bigotimes_{e} H_{c}(e)
$$

be the unordered tensor product (see Appendix (A) of the $\mathbb{k}$-modules $H_{c}(e)$ over all oriented edges $e$ of $P$. Since each $\mathbb{k}$-module $H_{c}(e)$ is projective of finite type and there are finitely many oriented edges of $P$, Appendix A yields a canonical $\mathbb{k}$-linear isomorphism

$$
H_{c}^{\star} \simeq \bigotimes_{e} H_{c}(e)^{\star}
$$



Figure 4.1. The colored graph $\Gamma_{x}^{c}$
Next, we associate to each (unoriented) edge e of $P$ a vector

$$
*_{e} \in H_{c}\left(e_{1}\right) \otimes H_{c}\left(e_{2}\right)
$$

where $e_{1}$ and $e_{2}$ are the two opposite oriented edges of $P$ corresponding to $e$. Recall from Section 3.2.2 that there is a branch $b_{e}$ of $P$ at $e$ which induces an orientation on $e$ which is opposite to the orientations induced by the other two branches of $P$ at $e$. We choose notation so that $e_{1}$ is $e$ endowed with the orientation induced by $b_{e}$. By Section 2.3.2, the element $b_{e} \in P_{e_{1}}$ determines a vector

$$
*_{e}=*_{P_{e_{1}}}^{b_{e}} \in H\left(P_{e_{1}}\right) \otimes H\left(P_{e_{1}}^{\mathrm{op}}\right)=H_{c}\left(e_{1}\right) \otimes H_{c}\left(e_{2}\right) .
$$

Set

$$
*_{c}=\otimes_{e} *_{e} \in H_{c}
$$

where $\otimes_{e}$ is the unordered tensor product over all the (unoriented) edges $e$ of $P$.

For a vertex $x$ of $P$, consider the link graph $\Delta_{x} \subset \partial B_{x}$ where $B_{x} \subset M$ is a $P$-cone neighborhood of $x$ (see Section 3.1.5). Here we endow $\partial B_{x}$ with the orientation induced by that of $M$ restricted to $M \backslash \operatorname{Int}\left(B_{x}\right)$. Every edge $a$ of $\Delta_{x}$ lies in a region $r_{a}$ of $P$. We color $a$ with $c\left(r_{a}\right) \in I$ and endow $a$ with the orientation induced by that of $r_{a} \backslash \operatorname{Int}\left(B_{x}\right)$. In this way, $\Delta_{x}$ becomes a $\mathcal{C}$-colored graph in $\partial B_{x}$ denoted by $\Delta_{x}^{c}$. The combing $\nu$ at $x$ determines a connected component of $\partial B_{x} \backslash \Delta_{x}$. (This follows from the definition of the combing $\nu$ at $x$, see Section 3.2.3, and the fact that $B_{x}$ is a $P$-cone neighborhood of $x$.) Pick a point $p$ in this connected component. The image of $\Delta_{x}^{c}$ under the (orientation-preserving) stereographic projection $\partial B_{x} \backslash\{p\} \rightarrow \mathbb{R}^{2}$ with pole $p$ is a $\mathcal{C}$-colored planar graph denoted by $\Gamma_{x}^{c}$. (An example is given in Figure4.1). Section 2.2.4 yields a vector

$$
\mathbb{F}_{\mathcal{C}}\left(\Gamma_{x}^{c}\right) \in H\left(\Gamma_{x}^{c}\right)^{\star}=\operatorname{Hom}_{\mathbb{k}}\left(H\left(\Gamma_{x}^{c}\right), \mathbb{k}\right) .
$$

Note that the cyclic $\mathcal{C}$-set associated with any vertex $v$ of $\Gamma_{x}^{c}$ (see Section 2.2.3) is canonically isomorphic to the cyclic $\mathcal{C}$-set $P_{e}$, where $e=e(v)$ is the edge of $P$ containing $v$ and oriented away from $x$. Therefore, there are canonical isomorphisms

$$
H\left(\Gamma_{x}^{c}\right) \simeq \bigotimes_{e_{x}} H_{c}\left(e_{x}\right) \quad \text { and } \quad H\left(\Gamma_{x}^{c}\right)^{\star} \simeq \bigotimes_{e_{x}} H_{c}\left(e_{x}\right)^{\star}
$$

where $e_{x}$ runs over all edges of $P$ incident to $x$ and oriented away from $x$. (An edge with both endpoints in $x$ appears in each of these tensor product twice with opposite orientations.) The tensor product of the previous isomorphisms over all vertices $x$ of $P$ yields a $\mathbb{k}$-linear isomorphism

$$
\bigotimes_{x} H\left(\Gamma_{x}^{c}\right)^{\star} \simeq \bigotimes_{x} \bigotimes_{e_{x}} H_{c}\left(e_{x}\right)^{\star} \simeq \bigotimes_{e} H_{c}(e)^{\star} \simeq H_{c}^{\star}
$$

where $e$ runs over all oriented edges of $P$. The image under this $\mathbb{k}$-linear isomorphism of the unordered tensor product $\bigotimes_{x} \mathbb{F}_{\mathcal{C}}\left(\Gamma_{x}^{c}\right)$, where $x$ runs over all vertices of $P$, is a vector $V_{c} \in H_{c}^{\star}$. We evaluate $V_{c}$ on $*_{c}$ and set

$$
|c|=V_{c}\left(*_{c}\right) \in \mathbb{k}
$$

Finally, let

$$
\operatorname{dim}_{l}(c)=\prod_{s \in \operatorname{Reg}(P)} \operatorname{dim}_{l}(c(s))
$$

and set

$$
\begin{equation*}
\mathrm{I}_{\mathcal{C}}(M, \nu)=\sum_{c} \operatorname{dim}_{l}(c)|c| \in \mathbb{k}, \tag{4.1}
\end{equation*}
$$

where $c$ runs over all colorings of $P$. Note that the right-hand side of (4.1) is well defined because there are finitely many colorings of $P$ (since both $\operatorname{Reg}(P)$ and $I$ are finite).

Theorem 4.1. The scalar $\mathrm{I}_{\mathcal{C}}(M, \nu)$ is a topological invariant of $(M, \nu)$ independent of the choice of $P$ and $I$.

We will prove Theorem 4.1 in Section 4.2,
4.1.2. Properties. 1. Let $(M, \nu)$ be a combed 3 -manifold with trivial spherical boundary. Consider the closed combed 3-manifold ( $\widehat{M}, \widehat{\nu}$ ) associated to $(M, \nu)$ as in Section 3.2.5. Then

$$
\mathrm{I}_{\mathcal{C}}(\widehat{M}, \widehat{\nu})=\mathrm{I}_{\mathcal{C}}(M, \nu)
$$

This follows from the fact that any branched spine of $(M, \nu)$ is a branched spine of $(\widehat{M}, \widehat{\nu})$.
2. By considering some examples, we prove in Section 5.3 that the invariant $\mathrm{I}_{\mathcal{C}}$ is non-trivial and does depend on the combing: it may distinguish two non-homotopic combings on the same 3 -manifold (see Theorem 55.2).
3. Suppose that $\mathcal{C}$ is spherical (see Section 1.2.8). Then for any combed 3 -manifold $(M, \nu)$, the invariant $\mathrm{I}_{\mathcal{C}}(M, \nu)$ does not depend on $\nu$ and

$$
\mathrm{I}_{\mathcal{C}}(M, \nu)=\operatorname{TV}_{\mathcal{C}}(M)
$$

where $\mathrm{TV}_{\mathcal{C}}$ is the Turaev-Viro invariant of compact oriented 3-manifolds defined using $\mathcal{C}$ (in the formulation of [TVi, Section 13.2.2] denoted by $\|\cdot\|_{\mathcal{C}}$ ).
4. It follows from the definitions that for any combed 3-manifold ( $M, \nu$ ),

$$
\mathrm{I}_{\mathcal{C}}(-M, \nu)=\mathrm{I}_{\mathcal{C}^{\otimes o \mathrm{p}}}(M, \nu)
$$

where $-M$ is $M$ with opposite orientation and $\mathcal{C}^{\otimes \mathrm{op}}=\left(\mathcal{C}, \otimes^{\mathrm{op}}, \mathbb{1}\right)$.
5. The naturality of the invariant $\mathbb{F}_{\mathcal{C}}$ of $\mathcal{C}$-colored graphs (see Section 2.2.6) implies that

$$
\mathrm{I}_{\mathcal{C}^{\prime}}(M, \nu)=\mathrm{I}_{\mathcal{C}}(M, \nu)
$$

for any combed 3-manifold $(M, \nu)$ and any pivotal fusion $\mathbb{k}$-category $\mathcal{C}^{\prime}$ equivalent to $\mathcal{C}$. In particular

$$
\mathrm{I}_{\mathcal{C}^{\mathrm{p}}}(M, \nu)=\mathrm{I}_{\mathcal{C}^{\otimes \mathrm{op}}}(M, \nu)=\mathrm{I}_{\mathcal{C}}(-M, \nu) \quad \text { and } \quad \mathrm{I}_{\mathcal{C}^{\mathrm{rev}}}(M, \nu)=\mathrm{I}_{\mathcal{C}}(M, \nu) \text {, }
$$

since $\mathcal{C}^{\mathrm{op}}=\left(\mathcal{C}^{\mathrm{op}}, \otimes, \mathbb{1}\right)$ is equivalent to $\mathcal{C}^{\otimes \mathrm{op}}$ and $\mathcal{C}^{\mathrm{rev}}=\left(\mathcal{C}^{\mathrm{op}}, \otimes^{\mathrm{op}}, \mathbb{1}\right)$ is equivalent to $\mathcal{C}$, see Section 1.4.5.
4.1.3. Computation via o-graphs. In this section, we provide an algorithm to compute the invariant $\mathrm{I}_{\mathcal{C}}$ of Theorem 4.1 starting from the presentation of combed 3 -manifolds by means of o-graphs (see Section 3.3).

Let $(M, \nu)$ be a combed 3-manifold. Let $\Gamma$ be an o-graph such that $(M, \nu)$ is equivalent to $\left(M_{\Gamma}, \nu_{\Gamma}\right)$ if $\partial M \neq \emptyset$ (see Section 3.3.3) or to $\left(\widehat{M}_{\Gamma}, \widehat{\nu}_{\Gamma}\right)$ if $\partial M=\emptyset$ (see Section 3.3.5).

Denote by $\operatorname{Circ}(\Gamma)$ the set of the circuits obtained from $\Gamma$ by applying the rules of Figure 3.6. A coloring of $\Gamma$ is a map $c: \operatorname{Circ}(\Gamma) \rightarrow I$. The object $c(\gamma) \in I$ assigned to $\gamma \in \operatorname{Circ}(\Gamma)$ is called the $c$-color of $\gamma$. We associate a scalar $|c| \in \mathbb{k}$ to each coloring $c$ of $\Gamma$ as follows.

Each edge $e$ of $\Gamma$ yields a cyclic $\mathcal{C}$-set $\Gamma_{e}$ defined in the following way. Set $\Gamma_{e}=\{1,2,3\}$ with cyclic order $1<2<3<1$. There are four types of edges of $\Gamma$, depending on the nature (distinguished/undistinguished, incoming/outgoing) of the two half-edges forming an edge:


Recall that the rules of Figure 3.6 associate to $e$ three portions of circuits (eventually coinciding) in $\operatorname{Circ}(\Gamma)$. Define $f_{e}: \Gamma_{e} \rightarrow \operatorname{Circ}(\Gamma)$ according to the type of $e$ :


Define $\varepsilon_{e}: \Gamma_{e} \rightarrow\{+,-\}$ by setting $\varepsilon_{e}(1)=\varepsilon_{e}(2)=+$ and $\varepsilon_{e}(3)=-$. Then

$$
\Gamma_{e}=\left(\Gamma_{e}, c \circ f_{e}, \varepsilon_{e}\right)
$$

is a cyclic $\mathcal{C}$-set. Consider the unordered tensor product of the multiplicity modules associated with $\Gamma_{e}$ and $\Gamma_{e}^{\mathrm{op}}$ :

$$
H_{c}(e)=H\left(\Gamma_{e}\right) \otimes H\left(\Gamma_{e}^{\mathrm{op}}\right) .
$$

By Section 2.3.2, the element $3 \in \Gamma_{e}$ determines a vector

$$
*_{e}=*_{\Gamma_{e}}^{3} \in H\left(\Gamma_{e}\right) \otimes H\left(\Gamma_{e}^{\mathrm{op}}\right)=H_{c}(e) .
$$

Set

$$
H_{c}=\otimes_{e} H_{c}(e) \quad \text { and } \quad *_{c}=\otimes_{e} *_{e} \in H_{c},
$$

where $\otimes_{e}$ is the unordered tensor product over all the edges $e$ of $\Gamma$.
Next, we associate to each vertex $x$ of $\Gamma$ a $\mathcal{C}$-colored planar graph $\Gamma_{x}^{c}$ as follows. If the vertex $x$ is positive, we associate:


If the vertex $x$ is negative, we associate:


Here, the middle pictures represent the portions of circuits associated with $x$ together with their $c$-colors $i, j, k, l, m, n \in I$. Section 2.2.4 yields a vector

$$
\mathbb{F}_{\mathcal{C}}\left(\Gamma_{x}^{c}\right) \in H\left(\Gamma_{x}^{c}\right)^{\star}=\operatorname{Hom}_{\mathbb{k}}\left(H\left(\Gamma_{x}^{c}\right), \mathbb{k}\right)
$$

Note that the cyclic $\mathcal{C}$-set associated with any vertex $v$ of $\Gamma_{x}^{c}$ (see Section 2.2.3) is canonically isomorphic to the cyclic $\mathcal{C}$-set $\Gamma_{e}^{\epsilon(e)}$, where $e=e(v)$ is the edge of $\Gamma$ containing $v, \epsilon(e)=\emptyset$ if $e$ is oriented away from $x$, and $\epsilon(e)=$ op if $e$ is oriented towards $x$. Therefore, there are canonical isomorphisms

$$
H\left(\Gamma_{x}^{c}\right) \simeq \bigotimes_{e_{x}} H\left(\Gamma_{e_{x}}^{\epsilon\left(e_{x}\right)}\right) \quad \text { and } \quad H\left(\Gamma_{x}^{c}\right)^{\star} \simeq \bigotimes_{e_{x}} H\left(\Gamma_{e_{x}}^{\epsilon\left(e_{x}\right)}\right)^{\star}
$$

where $e_{x}$ run over all edges of $\Gamma$ incident to $x$. The tensor product of the latter isomorphisms over all vertices $x$ of $\Gamma$ yields a $\mathbb{k}$-linear isomorphism

where $e$ runs over all the edges of $\Gamma$. The image under this isomorphism of the unordered tensor product $\bigotimes_{x} \mathbb{F}_{\mathcal{C}}\left(\Gamma_{x}^{c}\right)$, where $x$ runs over all vertices of $\Gamma$, is a vector $V_{c} \in H_{c}^{\star}$. Recall the vector $*_{c} \in H_{c}$. Set

$$
\operatorname{dim}_{l}(c)=\prod_{\gamma \in \operatorname{Circ}(\Gamma)} \operatorname{dim}_{l}(c(\gamma)) \in \mathbb{k} \quad \text { and } \quad|c|=V_{c}\left(*_{c}\right) \in \mathbb{k} .
$$

Theorem 4.2. We have:

$$
\mathrm{I}_{\mathcal{C}}(M, \nu)=\sum_{c} \operatorname{dim}_{l}(c)|c|
$$

where $c$ runs aver all colorings of $\Gamma$.
We prove Theorem 4.2 in Section 4.2.
In Section 5.2, we apply Theorem 4.2 for a particular pivotal fusion $\mathbb{k}$ category $\mathcal{C}=G_{\text {kk }}^{d}$ (see Theorem 5.1).

### 4.2. Proof of Theorem 4.1 and Theorem 4.2

Let $(M, \nu)$ be a combed 3 -manifold and $P$ be a branched spine of $(M, \nu)$. Denote the right hand side of (4.1) by $\mathrm{I}_{\mathcal{C}}(P)$, that is,

$$
\mathrm{I}_{\mathcal{C}}(P)=\sum_{c} \operatorname{dim}_{l}(c)|c|
$$

where $c$ runs over all colorings of $P$.
In Section 4.2.1, we prove the invariance of $\mathrm{I}_{\mathcal{C}}(P)$ under the application of a BMP $(2,3)$ move to $P$. In Section 4.2.2, we prove the invariance of $\mathrm{I}_{\mathcal{C}}(P)$ under the application of a $\operatorname{BMP}(0,2)$ move to $P$. Finally, we prove Theorem 4.1 in Section 4.2.3 and Theorem 4.2 in Section 4.2.4.
4.2.1. Invariance under $\operatorname{BMP}(\mathbf{2 , 3})$. The application of a $\operatorname{BMP}(2,3)$ move transforms $P$ into another branched spine $P^{\prime}$ of $(M, \nu)$. The move acts locally on $P$ leaving unchanged all the regions of $P$ except those involved in the move. For this reason, we only consider the contribution of the vertices involved in the move to the quantity $\mathrm{I}_{\mathcal{C}}(P)$. We denote by $x, y$ the two vertices of $P$ and by $u, v, z$ the three vertices of $P^{\prime}$ that are involved in the move.

Pick a coloring $c$ of $P$. In what follows, we denote by $a, b, c, d, f, g, h, i, l \in I$ the $c$-colors of the regions of $P$ involved in the moves. The coloring $c$ of $P$ extends to a coloring $c^{\prime}$ of $P^{\prime}$ by adding a color $j \in I$ to the new region created by the move.

We now analyze in detail the contribution given to the state sum by each BMP $(2,3)$ move (see Section 3.2.7).

Invariance under the move $M_{1}$ :
The move $P \xrightarrow{M_{1}} P^{\prime}$ is represented by


By Section 4.1.1, the $\mathcal{C}$-colored planar graphs $\Gamma_{x}^{c}$ and $\Gamma_{y}^{c}$ associated to the vertices $x, y$ of $P$ are


Consequently, the contribution to the state sum of the vertices $x, y$ and of the edge $e$ connecting them is

$$
\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{x}^{c}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{y}^{c}\right)\right)\left(*_{e}\right) \stackrel{(i)}{=} \mathbb{F}_{\mathcal{C}}(
$$



Here, the equality $(i)$ follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ (see Section 2.2.6) and from the graphical representation of evaluations (see Section 2.3.4), and (ii) from Lemma 2.3(c).

Now, the $\mathcal{C}$-colored planar graphs $\Gamma_{u}^{c^{\prime}}, \Gamma_{v}^{c^{\prime}}, \Gamma_{z}^{c^{\prime}}$ associated to the vertices $u, v, z$ of $P^{\prime}$ are


Consequently, the contribution to the state sum of the vertices $u, v, z$, of the three edges $e_{1}, e_{2}, e_{3}$ connecting them, and of the new created region is

$$
\begin{aligned}
& \left.\sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right) \text { (i)} \sum_{j \in I} \operatorname{dim}_{l}(j) \mathbb{F}_{\mathcal{C}}()^{\prime}\right)
\end{aligned}
$$




Here, $(i)$ follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and the graphical representation of evaluations, (ii) and (iii) from Lemma 2.3(c), and (iv) from Lemma 2.3(a).

We deduce that

$$
\begin{gathered}
\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{x}^{c}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{y}^{c}\right)\right)\left(*_{e}\right) \\
=\sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right) .
\end{gathered}
$$

This proves that $\mathrm{I}_{\mathcal{C}}(P)=\mathrm{I}_{\mathcal{C}}\left(P^{\prime}\right)$.
Invariance under the move $\widetilde{M}_{1}$ :
The move $P \xrightarrow{\widetilde{M_{1}}} P^{\prime}$ is represented by


By Section 4.1.1, the $\mathcal{C}$-colored planar graphs $\Gamma_{x}^{c}$ and $\Gamma_{y}^{c}$ associated to the vertices $x, y$ of $P$ are


Consequently, the contribution to the state sum of the vertices $x, y$ and of the edge $e$ connecting them is



Here, the equality $(i)$ follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and from the graphical representation of evaluations, and (ii) from Lemma 2.3(c).

Now, the $\mathcal{C}$-colored planar graphs $\Gamma_{u}^{c^{\prime}}, \Gamma_{v}^{c^{\prime}}, \Gamma_{z}^{c^{\prime}}$ associated to the vertices $u, v, z$ of $P^{\prime}$ are


Consequently, the contribution to the state sum of the vertices $u, v, z$, of the three edges $e_{1}, e_{2}, e_{3}$ connecting them, and of the new created region is

$$
\sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right)
$$

$$
\stackrel{(i i)}{=} \sum_{j \in I} \operatorname{dim}_{l}(j) \mathbb{F}_{\mathcal{C}}(
$$



Here, $(i)$ follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and the graphical representation of evaluations, (ii) Lemma 2.3(c), and (iii) from Lemma 2.3(a).

We deduce that

$$
\begin{gathered}
\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{x}^{c}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{y}^{c}\right)\right)\left(*_{e}\right) \\
=\sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right) .
\end{gathered}
$$

This proves that $\mathrm{I}_{\mathcal{C}}(P)=\mathrm{I}_{\mathcal{C}}\left(P^{\prime}\right)$.
Invariance under the move $M_{2}$ :
The move $P \xrightarrow{M_{2}} P^{\prime}$ is represented by


By Section 4.1.1 the $\mathcal{C}$-colored planar graphs $\Gamma_{x}^{c}$ and $\Gamma_{y}^{c}$ associated to the vertices $x, y$ of $P$ are


Consequently, the contribution to the state sum of the vertices $x, y$ and of the edge $e$ connecting them is


Here, the equality $(i)$ follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and from the graphical representation of evaluations, and (ii) from Lemma 2.3(c).

Now, the $\mathcal{C}$-colored planar graphs $\Gamma_{u}^{c^{\prime}}, \Gamma_{v}^{c^{\prime}}, \Gamma_{z}^{c^{\prime}}$ associated to the vertices $u, v, z$ of $P^{\prime}$ are

Consequently, the contribution to the state sum of the vertices $u, v, z$, of the three edges $e_{1}, e_{2}, e_{3}$ connecting them, and of the new created region is

$$
\begin{aligned}
\sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right) \\
\stackrel{(i)}{=} \sum_{j \in I} \operatorname{dim}_{l}(j) \mathbb{F}_{\mathcal{C}} \\
\stackrel{(i i)}{=} \sum_{j \in I} \operatorname{dim}_{l}(j) \mathbb{F}_{\mathcal{C}}
\end{aligned}
$$

Here, $(i)$ follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and the graphical representation of evaluations, (ii) from Lemma 2.3(c), and (iii) from Lemma 2.3(a).

We deduce that

$$
\begin{gathered}
\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{x}^{c}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{y}^{c}\right)\right)\left(*_{e}\right) \\
=\sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right)
\end{gathered}
$$

This proves that $\mathrm{I}_{\mathcal{C}}(P)=\mathrm{I}_{\mathcal{C}}\left(P^{\prime}\right)$.

Invariance under the move $\widetilde{M}_{2}$ :
The move $P \xrightarrow{\widehat{M_{2}}} P^{\prime}$ is represented by


By Section 4.1.1, the $\mathcal{C}$-colored planar graphs $\Gamma_{x}^{c}$ and $\Gamma_{y}^{c}$ associated to the vertices $x, y$ of $P$ are


Consequently, the contribution to the state sum of the vertices $x, y$ and of the edge $e$ connecting them is



Here, the equality $(i)$ follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and from the graphical representation of evaluations, and (ii) from Lemma 2.3(c).

Now, the $\mathcal{C}$-colored planar graphs $\Gamma_{u}^{c^{\prime}}, \Gamma_{v}^{c^{\prime}}, \Gamma_{z}^{c^{\prime}}$ associated to the vertices $u, v, z$ of $P^{\prime}$ are

$$
\Gamma_{u}^{c^{\prime}}=
$$

Consequently, the contribution to the state sum of the vertices $u, v, z$, of the three edges $e_{1}, e_{2}, e_{3}$ connecting them, and of the new created region is

$$
\begin{aligned}
& \sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right) \\
& \stackrel{(i)}{=} \sum_{j \in I} \operatorname{dim}_{l}(j) \mathbb{F}_{\mathcal{C}} \\
& \stackrel{(i i)}{=} \sum_{j \in I} \operatorname{dim}_{l}(j) \mathbb{F}_{\mathcal{C}}
\end{aligned}
$$

Here, $(i)$ follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and the graphical representation of evaluations, (ii) from Lemma 2.3(c), and (iii) from Lemma 2.3(a).

We deduce that

$$
\begin{gathered}
\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{x}^{c}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{y}^{c}\right)\right)\left(*_{e}\right) \\
=\sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right) .
\end{gathered}
$$

This proves that $\mathrm{I}_{\mathcal{C}}(P)=\mathrm{I}_{\mathcal{C}}\left(P^{\prime}\right)$.
Invariance under the move $M_{3}$ :
The move $P \xrightarrow{M_{3}} P^{\prime}$ is represented by


By Section 4.1.1, the $\mathcal{C}$-colored planar graphs $\Gamma_{x}^{c}$ and $\Gamma_{y}^{c}$ associated to the vertices $x, y$ of $P$ are


Consequently, the contribution to the state sum of the vertices $x, y$ and of the edge $e$ connecting them is

$$
\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{x}^{c}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{y}^{c}\right)\right)\left(*_{e}\right) \stackrel{(i)}{=} \mathbb{F}_{\mathcal{C}}
$$



Here, the equality $(i)$ follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and from the graphical representation of evaluations, and (ii) from Lemma 2.3(c).

Now, the $\mathcal{C}$-colored planar graphs $\Gamma_{u}^{c^{\prime}}, \Gamma_{v}^{c^{\prime}}, \Gamma_{z}^{c^{\prime}}$ associated to the vertices $u, v, z$ of $P^{\prime}$ are


Consequently, the contribution to the state sum of the vertices $u, v, z$, of the three edges $e_{1}, e_{2}, e_{3}$ connecting them, and of the new created region is


Here, ( $i$ ) follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and the graphical representation of evaluations, (ii) from Lemma 2.3(c), and (iii) from Lemma 2.3(a).

We deduce that

$$
\begin{gathered}
\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{x}^{c}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{y}^{c}\right)\right)\left(*_{e}\right) \\
=\sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right)
\end{gathered}
$$

This proves that $\mathrm{I}_{\mathcal{C}}(P)=\mathrm{I}_{\mathcal{C}}\left(P^{\prime}\right)$.

Invariance under the move $\widetilde{M}_{3}$ :
The move $P \xrightarrow{\widetilde{M_{3}}} P^{\prime}$ is represented by


By Section 4.1.1 the $\mathcal{C}$-colored planar graphs $\Gamma_{x}^{c}$ and $\Gamma_{y}^{c}$ associated to the vertices $x, y$ of $P$ are


Consequently, the contribution to the state sum of the vertices $x, y$ and of the edge $e$ connecting them is



Here, the equality $(i)$ follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and from the graphical representation of evaluations, and (ii) from Lemma 2.3(c).

Now, the $\mathcal{C}$-colored planar graphs $\Gamma_{u}^{c^{\prime}}, \Gamma_{v}^{c^{\prime}}, \Gamma_{z}^{c^{\prime}}$ associated to the vertices $u, v, z$ of $P^{\prime}$ are


Consequently, the contribution to the state sum of the vertices $u, v, z$, of the three edges $e_{1}, e_{2}, e_{3}$ connecting them, and of the new created region is

$$
\sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right)
$$



Here, $(i)$ follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and the graphical representation of evaluations, (ii) from Lemma 2.3(c), and (iii) from Lemma 2.3(a).

We deduce that

$$
\begin{gathered}
\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{x}^{c}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{y}^{c}\right)\right)\left(*_{e}\right) \\
=\sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right) .
\end{gathered}
$$

This proves that $\mathrm{I}_{\mathcal{C}}(P)=\mathrm{I}_{\mathcal{C}}\left(P^{\prime}\right)$.
Invariance under the move $M_{4}$ :
The move $P \xrightarrow{M_{4}} P^{\prime}$ is represented by


By Section 4.1.1, the $\mathcal{C}$-colored planar graphs $\Gamma_{x}^{c}$ and $\Gamma_{y}^{c}$ associated to the vertices $x, y$ of $P$ are


Consequently, the contribution to the state sum of the vertices $x, y$ and of the edge $e$ connecting them is



Here, the equality $(i)$ follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and from the graphical representation of evaluations, and (ii) from Lemma 2.3(c).

Now, the $\mathcal{C}$-colored planar graphs $\Gamma_{u}^{c^{\prime}}, \Gamma_{v}^{c^{\prime}}, \Gamma_{z}^{c^{\prime}}$ associated to the vertices $u, v, z$ of $P^{\prime}$ are


Consequently, the contribution to the state sum of the vertices $u, v, z$, of the three edges $e_{1}, e_{2}, e_{3}$ connecting them, and of the new created region is

$$
\begin{aligned}
& \sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right) \\
& \stackrel{(i)}{=} \lim _{j \in I}(j) \mathbb{F}_{\mathcal{C}} \\
& \stackrel{(i i)}{=} \sum_{j \in I} \operatorname{dim}_{l}(j) \mathbb{H}_{\mathcal{C}}
\end{aligned}
$$



Here, ( $i$ ) follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and the graphical representation of evaluations, (ii) from Lemma 2.3(c), and (iii) from Lemma 2.3(a).

We deduce that

$$
\begin{gathered}
\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{x}^{c}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{y}^{c}\right)\right)\left(*_{e}\right) \\
=\sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right)
\end{gathered}
$$

This proves that $\mathrm{I}_{\mathcal{C}}(P)=\mathrm{I}_{\mathcal{C}}\left(P^{\prime}\right)$.
Invariance under the move $\widetilde{M}_{4}$ :
The move $P \xrightarrow{\widetilde{M_{4}}} P^{\prime}$ is represented by


By Section 4.1.1, the $\mathcal{C}$-colored planar graphs $\Gamma_{x}^{c}$ and $\Gamma_{y}^{c}$ associated to the vertices $x, y$ of $P$ are


Consequently, the contribution to the state sum of the vertices $x, y$ and of the edge $e$ connecting them is


Here, the equality $(i)$ follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and from the graphical representation of evaluations, and (ii) from Lemma 2.3(c).

Now, the $\mathcal{C}$-colored planar graphs $\Gamma_{u}^{c^{\prime}}, \Gamma_{v}^{c^{\prime}}, \Gamma_{z}^{c^{\prime}}$ associated to the vertices $u, v, z$ of $P^{\prime}$ are


Consequently, the contribution to the state sum of the vertices $u, v, z$, of the three edges $e_{1}, e_{2}, e_{3}$ connecting them, and of the new created region is

$$
\sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right)
$$



Here, $(i)$ follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and the graphical representation of evaluations, (ii) from Lemma 2.3(c), and (iii) from Lemma 2.3(a).

We deduce that

$$
\begin{gathered}
\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{x}^{c}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{y}^{c}\right)\right)\left(*_{e}\right) \\
=\sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right)
\end{gathered}
$$

This proves that $\mathrm{I}_{\mathcal{C}}(P)=\mathrm{I}_{\mathcal{C}}\left(P^{\prime}\right)$.

Invariance under the move $M_{5}$ :
The move $P \xrightarrow{M_{5}} P^{\prime}$ is represented by


By Section 4.1.1, the $\mathcal{C}$-colored planar graphs $\Gamma_{x}^{c}$ and $\Gamma_{y}^{c}$ associated to the vertices $x, y$ of $P$ are


Consequently, the contribution to the state sum of the vertices $x, y$ and of the edge $e$ connecting them is


Here, the equality $(i)$ follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and from the graphical representation of evaluations, and (ii) from Lemma 2.3(c).

Now, the $\mathcal{C}$-colored planar graphs $\Gamma_{u}^{c^{\prime}}, \Gamma_{v}^{c^{\prime}}, \Gamma_{z}^{c^{\prime}}$ associated to the vertices $u, v, z$ of $P^{\prime}$ are


Consequently, the contribution to the state sum of the vertices $u, v, z$, of the three edges $e_{1}, e_{2}, e_{3}$ connecting them, and of the new created region is

$$
\begin{aligned}
& \sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right) \\
& \stackrel{(i)}{=} \sum_{j \in I} \operatorname{dim}_{l}(j) \mathbb{F}_{\mathcal{C}} \\
& \left.\stackrel{(i i)}{=} \sum_{j \in I} \operatorname{dim}_{l}(j) \mathbb{F}_{\mathcal{C}} \text { (iii)} \mathbb{F}_{\mathcal{C}}()_{c}\right)
\end{aligned}
$$

Here, $(i)$ follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and the graphical representation of evaluations, (ii) from Lemma 2.3(c), and (iii) from Lemma 2.3(a).

We deduce that

$$
\begin{gathered}
\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{x}^{c}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{y}^{c}\right)\right)\left(*_{e}\right) \\
=\sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right)
\end{gathered}
$$

This proves that $\mathrm{I}_{\mathcal{C}}(P)=\mathrm{I}_{\mathcal{C}}\left(P^{\prime}\right)$.

Invariance under the move $\widetilde{M}_{5}$ :
The move $P \xrightarrow{\widetilde{M_{5}}} P^{\prime}$ is represented by


By Section 4.1.1, the $\mathcal{C}$-colored planar graphs $\Gamma_{x}^{c}$ and $\Gamma_{y}^{c}$ associated to the vertices $x, y$ of $P$ are


Consequently, the contribution to the state sum of the vertices $x, y$ and of the edge $e$ connecting them is


Here, the equality $(i)$ follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and from the graphical representation of evaluations, and (ii) from Lemma 2.3(c).

Now, the $\mathcal{C}$-colored planar graphs $\Gamma_{u}^{c^{\prime}}, \Gamma_{v}^{c^{\prime}}, \Gamma_{z}^{c^{\prime}}$ associated to the vertices $u, v, z$ of $P^{\prime}$ are


Consequently, the contribution to the state sum of the vertices $u, v, z$, of the three edges $e_{1}, e_{2}, e_{3}$ connecting them, and of the new created region is

$$
\begin{aligned}
& \sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right) \\
& \stackrel{(i)}{=} \sum_{j \in I} \operatorname{dim}_{l}(j) \mathbb{F}_{\mathcal{C}}(\underbrace{\prime})
\end{aligned}
$$



Here, ( $i$ ) follows from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$ and the graphical representation of evaluations, (ii) from Lemma 2.3(c), and (iii) from Lemma 2.3(a).

We deduce that

$$
\begin{gathered}
\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{x}^{c}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{y}^{c}\right)\right)\left(*_{e}\right) \\
=\sum_{j \in I} \operatorname{dim}_{l}(j)\left(\mathbb{F}_{\mathcal{C}}\left(\Gamma_{u}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{v}^{c^{\prime}}\right) \otimes \mathbb{F}_{\mathcal{C}}\left(\Gamma_{z}^{c^{\prime}}\right)\right)\left(*_{e_{1}} \otimes *_{e_{2}} \otimes *_{e_{3}}\right) .
\end{gathered}
$$

This proves that $\mathrm{I}_{\mathcal{C}}(P)=\mathrm{I}_{\mathcal{C}}\left(P^{\prime}\right)$.
4.2.2. Invariance under $\operatorname{BMP}(\mathbf{0 , 2})$. The application of a $\operatorname{BMP}(0,2)$ move transforms $P$ into another branched spine $P^{\prime}$ of $(M, \nu)$. The move acts locally on $P$ leaving unchanged all the regions of $P$ except those involved in the move. For this reason, we only consider the contribution of the vertices involved in the move to the quantity $\mathrm{I}_{\mathcal{C}}(P)$. We denote by $u, v$ the two vertices of $P^{\prime}$ that are involved in the move.

Pick a coloring $c$ of $P$. In what follows, we denote by $i, k, l, m, n \in I$ the $c$-colors of the regions of $P$ involved in the moves. The coloring $c$ of $P$ extends to a coloring $c^{\prime}$ of $P^{\prime}$ by considering two new colors $j, k^{\prime} \in I$. The small region of $P^{\prime}$ created by the move is colored by $j$. The region $r$ of $P$ whose $c$-color is $k$ splits into to regions $r^{\prime}$ and $r^{\prime \prime}$ of $P^{\prime}$ that we color by $k^{\prime}$ and $k$, respectively.

We now analyze in detail the contribution given to the state sum by every $\operatorname{BMP}(0,2)$ move (see Section 3.2.7).

Invariance under the move $L_{1}$ :
The move $P \xrightarrow{L_{1}} P^{\prime}$ is represented by


By Section 4.1.1, the contribution to the state sum of the edges of $P$ represented in the move $L_{1}$ is


Now, the $\mathcal{C}$-colored planar graphs $\Gamma_{u}^{c^{\prime}}$ and $\Gamma_{v}^{c^{\prime}}$ associated to the vertices $u$ and $v$ of $P^{\prime}$ are


Consequently, the contribution to the state sum of the vertices $u$ and $v$, of all theall the edges involved in the move (i.e., the two edges connecting $u, v$ and
the other four edges represented in move $L_{1}$ ), and of the new created region is



Here, $(i)$ and (iv) follow from Lemma 2.3(c) and the graphical representation of evaluations (see Section 2.3.4), (ii) from Lemma 2.3(a), (iii) from Lemma 2.1 using the fact that $\operatorname{Hom}_{\mathcal{C}}\left(k, k^{\prime}\right)=0$ if $k \neq k^{\prime}$, and (v) from the $\otimes$ multiplicativity of $\mathbb{F}_{\mathcal{C}}$ (see Section 2.2.6). We deduce that the contributions to the state sum before and after the move $L_{1}$ are equal.

This proves that $\mathrm{I}_{\mathcal{C}}(P)=\mathrm{I}_{\mathcal{C}}\left(P^{\prime}\right)$.
Invariance under the move $L_{2}$ :
The move $P \xrightarrow{L_{2}} P^{\prime}$ is represented by


By Section 4.1.1, the contribution to the state sum of the edges of $P$ represented in the move $L_{2}$ is


Now, the $\mathcal{C}$-colored planar graphs $\Gamma_{u}^{c^{\prime}}$ and $\Gamma_{v}^{c^{\prime}}$ associated to the vertices $u$ and $v$ of $P^{\prime}$ are


Consequently, the contribution to the state sum of the vertices $u$ and $v$, of all the edges involved in the move (i.e., the two edges connecting $u, v$ and the other four edges represented in move $L_{2}$ ), and of the new created region is



Here, (i) and (iv) follow from Lemma 2.3(c) and the graphical representation of evaluations, ( $i$ i $)$ from Lemma 2.3(a), (iii) from Lemma 2.1, and $(v)$ from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$. We deduce that the contributions to the state sum before and after the move $L_{2}$ are equal.

This proves that $\mathrm{I}_{\mathcal{C}}(P)=\mathrm{I}_{\mathcal{C}}\left(P^{\prime}\right)$.
Invariance under the move $L_{3}$ :
The move $P \xrightarrow{L_{3}} P^{\prime}$ is represented by


By Section 4.1.1, the contribution to the state sum of the edges of $P$ represented in the move $L_{3}$ is


Now, the $\mathcal{C}$-colored planar graphs $\Gamma_{u}^{c^{\prime}}$ and $\Gamma_{v}^{c^{\prime}}$ associated to the vertices $u$ and $v$ of $P^{\prime}$ are


Consequently, the contribution to the state sum of the vertices $u$ and $v$, of all the edges involved in the move (i.e., the two edges connecting $u, v$ and the other four edges represented in move $L_{3}$ ), and of the new created region is



Here, $(i)$ and $(i v)$ follow from Lemma 2.3(c) and the graphical representation of evaluations, (ii) from Lemma 2.3(a), and (iii) from Lemma 2.1, and (v) from the $\otimes$-multiplicativity of $\mathbb{F}_{\mathcal{C}}$. We deduce that the contributions to the state sum before and after the move $L_{3}$ are equal.

This proves that $\mathrm{I}_{\mathcal{C}}(P)=\mathrm{I}_{\mathcal{C}}\left(P^{\prime}\right)$.
4.2.3. Proof of Theorem 4.1. Let $(M, \nu)$ be a combed 3 -manifold. Let $P$ be a branched spine of $(M, \nu)$. It follows from the definitions that $\mathrm{I}_{\mathcal{C}}(P)$ remains unchanged under the application of an ambient isotopy in $M$. Also $\mathrm{I}_{\mathcal{C}}(P)$ remains unchanged under the application of a $\operatorname{BMP}(2,3)$ move (by Section 4.2.1) or a $\operatorname{BMP}(0,2)$ move (by Section 4.2.2). Therefore, by Theorem 3.5,
the scalar $\mathrm{I}_{\mathcal{C}}(M, \nu)=\mathrm{I}_{\mathcal{C}}(P)$ is well defined, i.e., does not depend on the choice of $P$.

If $I^{\prime}$ is another representative set of simple object of $\mathcal{C}$, then there is a unique bijection $\varphi: I \rightarrow I^{\prime}$ such that the objects $i$ and $\varphi(i)$ are isomorphic for all $i \in I$. Consequently, the naturality of $\mathbb{F}_{\mathcal{C}}$ (see Section 2.2.6) implies that $\mathrm{I}_{\mathcal{C}}(M, \nu)$ does not depend on the choice of the representative set $I$.

Let $(M, \nu)$ and $\left(M^{\prime}, \nu^{\prime}\right)$ be two equivalent combed 3 -manifolds (see Section 3.2.1). There is an orientation-preserving diffeomorphism $\phi: M \rightarrow M^{\prime}$ such that the combings $\nu^{\prime \prime}=\phi_{*} \circ \nu \circ \phi^{-1}$ and $\nu^{\prime}$ are homotopic within the class of combings on $M^{\prime}$. Pick a branched spine $P$ of $(M, \nu)$ and a branched spine $P^{\prime}$ of $\left(M^{\prime}, \nu^{\prime}\right)$. Then $P^{\prime \prime}=\phi(P)$ is a branched spine of ( $M^{\prime}, \nu^{\prime \prime}$ ). Clearly $\mathrm{I}_{\mathcal{C}}(P)=\mathrm{I}_{\mathcal{C}}\left(P^{\prime \prime}\right)$. Also, since $\nu^{\prime \prime}$ and $\nu^{\prime}$ are homotopic within the class of combings on $M^{\prime}$, it follows from the definition of the state sum that $\mathrm{I}_{\mathcal{C}}\left(P^{\prime \prime}\right)=\mathrm{I}_{\mathcal{C}}\left(P^{\prime}\right)$. Consequently,

$$
\mathrm{I}_{\mathcal{C}}(M, \nu)=\mathrm{I}_{\mathcal{C}}(P)=\mathrm{I}_{\mathcal{C}}\left(P^{\prime \prime}\right)=\mathrm{I}_{\mathcal{C}}\left(P^{\prime}\right)=\mathrm{I}_{\mathcal{C}}\left(M^{\prime}, \nu^{\prime}\right)
$$

4.2.4. Proof of Theorem 4.2. It follows from Sections 3.3 .2 and 4.1.3 that there are

- a bijection between the set of vertices of the o-graph $\Gamma$ and the set of vertices of the branched polyhedron $P_{\Gamma}$ associated to $\Gamma$;
- a bijection between the set of edges of $\Gamma$ and the set of edges of $P_{\Gamma}$;
- a bijection between the set $\operatorname{Circ}(\Gamma)$ and the set of regions of $P_{\Gamma}$.

By pre-composing with the latter bijection, any coloring $c$ of $\Gamma$ induces a coloring $\tilde{c}$ of $P_{\Gamma}$. Clearly $\operatorname{dim}_{l}(c)=\operatorname{dim}_{l}(\tilde{c})$. By definition, for any vertex $x$ of $\Gamma$, the associated graph $\Gamma_{x}^{c}$ (see Section 4.1.3) is equal to the graph $\Gamma_{\tilde{x}}^{\tilde{c}}$ associated to the vertex $\tilde{x}$ of $P_{\Gamma}$ corresponding to $x$ (see Section 4.1.1). Also, the contraction vector $*_{c}$ provided by Section 4.1.3 is equal to the contraction vector $*_{\tilde{c}}$ provided by Section 4.1.1. It follows that $|c|=|\tilde{c}|$. Now, the above map $c \mapsto \tilde{c}$ is a bijection between the set of colorings of $\Gamma$ and the set of colorings of $P_{\Gamma}$. Consequently, we deduce that

$$
\sum_{c} \operatorname{dim}_{l}(c)|c|=\sum_{c} \operatorname{dim}_{l}(\tilde{c})|\tilde{c}|=\mathrm{I}_{\mathcal{C}}\left(P_{\Gamma}\right)=\mathrm{I}_{\mathcal{C}}(M, \nu),
$$

where $c$ runs over all colorings of $\Gamma$.

## CHAPTER 5

## A particular case

In this chapter, we consider a pivotal fusion category $G_{\mathrm{k}}^{d}$ associated with a character $d$ of a finite group $G$ and study in detail the invariant $\mathrm{I}_{G_{\mathbf{k}}^{d}}(M, \nu)$ of combed 3 -manifolds defined with this category. In particular, we prove that this invariant is non-trivial and corresponds to the evaluation by the character $d$ on the Euler class of the real vector bundle of rank 2 associated to $\nu$.

Throughout this chapter, $G$ is a finite group and $d$ is a character of $G$ over the (non-zero) commutative ring $\mathbb{k}$, that is, a group homomorphism from $G$ to the multiplicative group $\mathbb{k}^{*}$ of $\mathbb{k}$.

### 5.1. The pivotal fusion category $G_{\mathrm{kk}}^{d}$

In this section, we define a pivotal fusion $\mathbb{k}$-category $G_{\mathbb{k}}^{d}$ as follows. The objects of $G_{\mathrm{kk}}^{d}$ are the elements of $G$. By definition,

$$
\operatorname{End}_{G_{\mathfrak{k}}^{d}}(g)=\mathbb{k} \quad \text { and } \quad \operatorname{Hom}_{G_{k}^{d}}(h, l)=\{0\} \subset \mathbb{k}
$$

for all $g \in G$ and distinct $h, l \in G$. The composition of morphisms in $G_{\mathrm{kk}}^{d}$ is induced by multiplication in $\mathbb{k}$. The identity of an object $g \in G$ is $\operatorname{id}_{g}=1_{\mathbb{k}}$.

The category $G_{\mathbb{k}}^{d}$ is strict monoidal with monoidal product defined by

$$
g \otimes h=g h \quad \text { and } \quad \lambda \otimes \mu=\lambda \mu
$$

for all $g, h \in G$ and all morphisms $\lambda, \mu$ in $\mathcal{C}$ (which are elements of $\mathbb{k}$ ). The unit object of $G_{\mathrm{kk}}^{d}$ is the unit element $1 \in G$.

The monoidal category $G_{\mathrm{k}}^{d}$ is pivotal with pivotal duality

$$
\left\{\left(g^{*}=g^{-1}, \mathrm{ev}_{g}=1_{\mathrm{k}}, \widetilde{\mathrm{ev}}_{g}=d(g)^{-1}\right)\right\}_{g \in G}
$$

More precisely,

$$
\begin{aligned}
& \mathrm{ev}_{g}=1_{\mathbb{k}} \in \mathbb{k}=\operatorname{Hom}_{G_{\mathbb{k}}^{d}}\left(g^{-1} \otimes g, 1\right), \\
& \widetilde{\mathrm{ev}}_{g}=d(g)^{-1} \in \mathbb{k}=\operatorname{Hom}_{G_{\mathbb{k}}^{d}}\left(g \otimes g^{-1}, 1\right) .
\end{aligned}
$$

The corresponding coevaluation morphisms are computed by

$$
\begin{aligned}
& \operatorname{coev}_{g}=1_{\mathbb{k}} \in \mathbb{k}=\operatorname{Hom}_{G_{\mathbf{k}}^{d}}\left(1, g \otimes g^{-1}\right), \\
& \widetilde{\operatorname{coev}_{g}}=d(g) \in \mathbb{k}=\operatorname{Hom}_{G_{\mathbf{k}}^{d}}\left(1, g^{-1} \otimes g\right) .
\end{aligned}
$$

Note that the dual functor of $G_{\mathbb{k}}^{d}$ acts as the inversion on objects and as the identity on morphisms. By definition, the dimensions of an object $g \in G$ are computed by

$$
\operatorname{dim}_{l}(g)=d(g) \in \mathbb{k} \quad \text { and } \quad \operatorname{dim}_{r}(g)=d(g)^{-1} \in \mathbb{k} .
$$

Consequently, $G_{\mathrm{k}}^{d}$ is spherical if and only if $d(g)^{2}=1_{\mathrm{k}}$ for all $g \in G$.
We endow $G_{\mathbb{k}}^{d}$ with a structure of monoidal $\mathbb{k}$-category defined by providing each Hom-set (which is either $\mathbb{k}$ or 0 ) with the left $\mathbb{k}$-module structure given by multiplication. The pivotal $\mathbb{k}$-category $G_{\mathbb{k}}^{d}$ is then fusion with $G$ as a representative set of simple objects.

### 5.2. A direct computation of $\mathrm{I}_{G_{\mathrm{k}}^{d}}$

Let $(M, \nu)$ be a combed 3 -manifold. Let $\Gamma$ be an o-graph such that $(M, \nu)$ is equivalent to $\left(M_{\Gamma}, \nu_{\Gamma}\right)$ if $\partial M \neq \emptyset$ (see Section 3.3.3) or to $\left(\widehat{M}_{\Gamma}, \widehat{\nu}_{\Gamma}\right)$ if $\partial M=\emptyset$ (see Section 3.3.5).

Recall from Section 4.1.3 that a coloring of $\Gamma$ is a map from the set $\operatorname{Circ}(\Gamma)$ of the circuits obtained from $\Gamma$ by applying the rules of Figure 3.6 to the set $G$. We say that a coloring $c$ of $\Gamma$ is admissible if

$$
c\left(f_{e}(3)\right)=c\left(f_{e}(1)\right) c\left(f_{e}(2)\right) \in G
$$

for all edge $e$ of $\Gamma$, where the map $f_{e}: \Gamma_{e}=\{1,2,3\} \rightarrow \operatorname{Circ}(\Gamma)$ is defined in Section 4.1.3.

Let $c$ be an admissible coloring of $\Gamma$. For $\gamma \in \operatorname{Circ}(\Gamma)$, set

$$
d_{c}(\gamma)=d(c(\gamma)) .
$$

We associate to a vertex $x$ of $\Gamma$ a scalar $\kappa_{c}(x)$ defined as follows. If $x$ is positive, then


If $x$ is negative, then


Here, the middle pictures represent the portions of circuits associated with $x$ together with their $c$-colors $i, j, k, l, m, n \in I$.

We associate to an edge $e$ of $\Gamma$ a scalar $\theta_{c}(e)$ defined as follows. Recall from the definition of the map $f_{e}$ that to $e$ is associated three portions of circuits (eventually coinciding) in $\operatorname{Circ}(\Gamma)$. The rightmost portion is a portion of the circuit $f_{e}(3) \in \operatorname{Circ}(\Gamma)$. Set

$$
\theta_{c}(e)=d\left(c\left(f_{e}(3)\right)\right) .
$$

In the next theorem, we compute the invariant $\mathrm{I}_{G_{k}^{d}}(M, \nu)$ of $(M, \nu)$ derived from $G_{\mathrm{k}}^{d}$ using the scalars $d_{c}(\gamma), \kappa_{c}(x)$, and $\theta_{c}(e)$ defined above.

Theorem 5.1. We have:

$$
\mathrm{I}_{G_{\mathbf{k}}^{d}}(M, \nu)=\sum_{c}\left(\prod_{\gamma} d_{c}(\gamma)\right)\left(\prod_{x} \kappa_{c}(x)\right)\left(\prod_{e} \theta_{c}(e)\right),
$$

where $c$ runs over all admissible colorings of $\Gamma, \gamma$ runs over all circuits in $\operatorname{Circ}(\Gamma), x$ runs over all vertices of $\Gamma$, and e runs over all edges of $\Gamma$.

Proof. By Theorem 4.2, we have that

$$
\mathrm{I}_{G_{\mathbf{k}}^{d}}(M, \nu)=\sum_{c} \operatorname{dim}_{l}(c)|c|
$$

where $c$ runs over all colorings of $\Gamma$. Since

$$
\operatorname{dim}_{l}(c)=\prod_{\gamma \in \operatorname{Circ}(\Gamma)} d_{c}(\gamma)
$$

it suffices to prove that if $c$ is a non-admissible coloring of $\Gamma$, then $|c|=0$, and that if $c$ is an admissible coloring of $\Gamma$, then

$$
\begin{equation*}
|c|=\left(\prod_{x} \kappa_{c}(x)\right)\left(\prod_{e} \theta_{c}(e)\right) \tag{5.1}
\end{equation*}
$$

where $x$ runs over all vertices of $\Gamma$ and $e$ runs over all edges of $\Gamma$. Fix a coloring $c$ of $\Gamma$.

Recall from Section 4.1.3 the cyclic $G_{\mathbb{k}}^{d}$-set $\Gamma_{e}=\left(\{1,2,3\}, c \circ f_{e}, \varepsilon_{e}\right)$ associated to an edge $e$ of $\Gamma$. It follows from the definitions that

$$
H\left(\Gamma_{e}\right) \simeq \operatorname{Hom}_{G_{k}^{d}}\left(1, c\left(f_{e}(1)\right) c\left(f_{e}(2)\right) c\left(f_{e}(3)\right)^{-1}\right)
$$

and so

$$
H_{c}(e)=H\left(\Gamma_{e}\right) \otimes H\left(\Gamma_{e}^{\mathrm{op}}\right) \simeq \begin{cases}\mathbb{k} & \text { if } c\left(f_{e}(1)\right) c\left(f_{e}(2)\right)=c\left(f_{e}(3)\right), \\ 0 & \text { if } c\left(f_{e}(1)\right) c\left(f_{e}(2)\right) \neq c\left(f_{e}(3)\right) .\end{cases}
$$

Therefore

$$
H_{c}=\bigotimes_{e} H_{c}(e) \simeq \begin{cases}\mathbb{k} & \text { if } c \text { is admissible }, \\ 0 & \text { otherwise }\end{cases}
$$

Consequently, if $c$ is not admissible, then $|c|=0$.
Assume that $c$ is admissible. Let $x$ be a vertex of $\Gamma$. Recall from Section 4.1.3 the $G_{\mathbb{k}^{d}}^{d}$-colored planar graph $\Gamma_{x}^{c}$ associated to $x$. Then

$$
H\left(\Gamma_{x}^{c}\right)=H\left(\Gamma_{e_{1}}^{\epsilon\left(e_{1}\right)}\right) \otimes H\left(\Gamma_{e_{2}}^{\epsilon\left(e_{2}\right)}\right) \otimes H\left(\Gamma_{e_{3}}^{\epsilon\left(e_{3}\right)}\right) \otimes H\left(\Gamma_{e_{4}}^{\epsilon\left(e_{4}\right)}\right) \simeq \mathbb{k}^{\otimes 4}
$$

where $e_{1}, \ldots, e_{4}$ are the edges incident to $x$. Therefore the contribution of $x$ to $|c|$ is

$$
\mathbb{F}_{G_{\mathfrak{k}}^{d}}\left(\Gamma_{x}^{c}\right)\left(1_{\mathrm{k}} \otimes 1_{\mathbb{k}} \otimes 1_{\mathfrak{k}} \otimes 1_{\mathfrak{k}}\right) \in H\left(\Gamma_{x}^{c}\right)^{\star} \simeq \mathbb{k}
$$

If $x$ is positive, then

where $i, j, k, l, m, n \in I$ are $c$-colors of the circuits associated with $x$ and so


Here, $(i)$ follows from the definition of $\mathbb{F}_{G_{k}^{d}}$, (ii) from the definition of the pivotal duality of $G_{\mathfrak{k}}^{d}$, and (iii) from the definition of $\kappa_{c}(x)$. If $x$ is negative, then

where $i, j, k, l, m, n \in I$ are $c$-colors of the circuits associated with $x$ and so


$$
\stackrel{(i i i)}{=} d(k)^{-1} d(j)^{-1} d(m)^{-1} d(n)^{-1} \stackrel{(i i i)}{=} d(i)^{-1} d(m)^{-1} d(n)^{-1} \stackrel{(i v)}{=} \kappa_{c}(x) .
$$

Here, $(i)$ follows from the definition of $\mathbb{F}_{G_{k}^{d}},(i i)$ from the definition of the pivotal duality of $G_{\mathrm{k}}^{d}$, (iii) from the admissibility of $c$, and $(i v)$ from the definition of $\kappa_{c}(x)$.

Recall from Section 4.1.3 that the contraction vector associated to an edge $e$ of $\Gamma$ is

$$
*_{e}=*_{\Gamma_{e}}^{3} \in H\left(\Gamma_{e}\right) \otimes H\left(\Gamma_{e}^{\mathrm{op}}\right)=H_{c}(e) \simeq \mathbb{k} .
$$

It is computed by

where $g=c\left(f_{e}(1)\right), h=c\left(f_{e}(2)\right)$, and $k=c\left(f_{e}(3)\right)$. Here, ( $i$ ) follows from Lemma 2.2, (ii) from the definition of the pivotal duality of $G_{\mathbb{k}}^{d}$, (iii) from the admissibility of $c$, and (iv) from the definition of $\theta_{c}(e)$.

Consequently, since $|c|$ is the evaluation of $*_{c}=\otimes_{e} *_{e}$, where $e$ runs over all edges of $\Gamma$, by $\bigotimes_{x} \mathbb{F}_{\mathcal{C}}\left(\Gamma_{x}^{c}\right)$, where $x$ runs over all vertices of $\Gamma$, we obtain that the equality (5.1) is satisfied.

### 5.3. Non-triviality of $\mathrm{I}_{G_{\mathrm{k}}^{d}}$

Consider the following two o-graphs:


The o-graphs $\Gamma$ and $\Gamma^{\prime}$ are closed and so encode closed combed 3-manifolds $(M, \nu)$ and ( $M^{\prime}, \nu^{\prime}$ ), respectively (see Section 3.3.5).

Theorem 5.2. (a) The 3-manifolds $M$ and $M^{\prime}$ are homeomorphic.
(b) The combed 3-manifolds $(M, \nu)$ and $\left(M^{\prime}, \nu^{\prime}\right)$ are not equivalent.
(c) We have:

$$
\mathrm{I}_{G_{\mathfrak{k}}^{d}}(M, \nu)=\sum_{g \in G} d(g)^{2} \quad \text { and } \quad \mathrm{I}_{G_{\mathfrak{k}}^{d}}\left(M^{\prime}, \nu^{\prime}\right)=|G| 1_{\mathfrak{k}} .
$$

(d) There are examples of a finite group $G$ and of a character $d$ of $G$ such that $\mathrm{I}_{G_{k}^{d}}(M, \nu) \neq \mathrm{I}_{G_{k}^{d}}\left(M^{\prime}, \nu^{\prime}\right)$.

Note that the parts (a),(b),(d) of Theorem 5.2 implies that the invariant $\mathrm{I}_{\mathcal{C}}$ of Theorem 4.1 is non-trivial and does depend on the combing: it may distinguish two non-homotopic combings on the same 3-manifold.

Proof. Part (a) follows from Section 3.3.6 since $\Gamma^{\prime}$ is obtained from $\Gamma$ by applying a Pontrjagin move.

Let us prove part (c). We compute $\mathrm{I}_{G_{k}^{d}}(M, \nu)$ by using its expression given by Theorem 5.1. Let $c$ be an admissible coloring of $\Gamma$. There are 6 circuits
$\gamma_{1}, \ldots, \gamma_{6}$ obtained from $\Gamma$ by applying the rules of Figure 3.6


Here, the $c$-colors of the circuits are denoted as follows:

$$
\begin{aligned}
i=c\left(\gamma_{1}\right) & \text { red, } & j=c\left(\gamma_{2}\right) & \text { green, } \\
k=c\left(\gamma_{3}\right) & \text { blue, } & m=c\left(\gamma_{4}\right) & \text { black, } \\
n=c\left(\gamma_{5}\right) & \text { grey, } & t=c\left(\gamma_{6}\right) & \text { pink. }
\end{aligned}
$$

The scalars associated to the vertices $x_{1}, \ldots, x_{5}$ of $\Gamma$ are:

$$
\begin{array}{ll}
\kappa_{c}\left(x_{1}\right)=d(j)^{-1} d(m)^{-1} d(i)^{-1}, & \kappa_{c}\left(x_{2}\right)=d(i)^{-1} d(t)^{-1} d(j)^{-1}, \\
\kappa_{c}\left(x_{3}\right)=d(j)^{-3}, & \kappa_{c}\left(x_{4}\right)=d(i)^{-1} d(k)^{-1} d(j)^{-1}, \\
\kappa_{c}\left(x_{5}\right)=d(j)^{-1} d(i)^{-1} d(n)^{-1} . &
\end{array}
$$

The scalars associated to the edges $e_{1}, \ldots, e_{10}$ are:

$$
\begin{array}{llll}
\theta_{c}\left(e_{1}\right)=d(m), & \theta_{c}\left(e_{2}\right)=d(t), & \theta_{c}\left(e_{3}\right)=d(j), & \theta_{c}\left(e_{4}\right)=d(j), \\
\theta_{c}\left(e_{5}\right)=d(j), & \theta_{c}\left(e_{6}\right)=d(j), & \theta_{c}\left(e_{7}\right)=d(k), & \theta_{c}\left(e_{8}\right)=d(i), \\
\theta_{c}\left(e_{9}\right)=d(n), & \theta_{c}\left(e_{10}\right)=d(i) . & &
\end{array}
$$

The scalars associated to the circuits $\gamma_{1}, \ldots, \gamma_{6}$ are:

$$
\begin{array}{lll}
d_{c}\left(\gamma_{1}\right)=d(i), & d_{c}\left(\gamma_{2}\right)=d(j), & d_{c}\left(\gamma_{3}\right)=d(k), \\
d_{c}\left(\gamma_{4}\right)=d(m), & d_{c}\left(\gamma_{5}\right)=d(n), & d_{c}\left(\gamma_{6}\right)=d(t) .
\end{array}
$$

Now, the admissibility of $c$ imposes conditions on the colors, one for each of the 10 edges of $\Gamma$ :

$$
\begin{array}{llll}
e_{1} \rightsquigarrow m=j i, & e_{2} \rightsquigarrow t=i j, & e_{3} \rightsquigarrow j=j i, & e_{4} \rightsquigarrow j=i j, \\
e_{5} \rightsquigarrow j=i j, & e_{6} \rightsquigarrow j=j i, & e_{7} \rightsquigarrow k=i j, & e_{8} \rightsquigarrow i=i^{2}, \\
e_{9} \rightsquigarrow n=j i, & e_{10} \rightsquigarrow i=i^{2}, & &
\end{array}
$$

that is,

$$
i=1 \quad \text { and } \quad j=k=m=n=t .
$$

Therefore

$$
\begin{gathered}
\mathrm{I}_{G_{\mathrm{k}}^{d}}(M, \nu)=\sum_{c} \prod_{q=1}^{6} d_{c}\left(\gamma_{q}\right) \prod_{r=1}^{5} \kappa_{c}\left(x_{r}\right) \prod_{s=1}^{10} \theta_{c}\left(e_{s}\right) \\
=\sum_{i, j, k, m, n, t \in G} \delta_{i, 1} \delta_{j, k} \delta_{j, m} \delta_{j, n} \delta_{j, t} d(i)^{-1} d(j)^{-2} d(k) d(m) d(n) d(t)=\sum_{j \in G} d(j)^{2} .
\end{gathered}
$$

We compute similarly $\mathrm{I}_{G_{\mathrm{k}}^{d}}\left(M^{\prime}, \nu^{\prime}\right)$. Let $c$ be an admissible coloring of $\Gamma^{\prime}$. There are 8 circuits $\gamma_{1}^{\prime}, \ldots, \gamma_{8}^{\prime}$ obtained from $\Gamma^{\prime}$ by applying the rules of Figure 3.6


Here, the $c$-colors of the circuits are denoted as follows:

$$
\begin{array}{rlrlrl}
i & =c\left(\gamma_{1}^{\prime}\right) & \text { red, } & j & =c\left(\gamma_{2}^{\prime}\right) & \\
\text { green, } \\
k= & =c\left(\gamma_{3}^{\prime}\right) & \text { pink, } & m & =c\left(\gamma_{4}^{\prime}\right) & \\
\text { black, } \\
n & =c\left(\gamma_{5}^{\prime}\right) & \text { blue, } & t & =c\left(\gamma_{6}^{\prime}\right) & \\
\text { yellow, } \\
u & =c\left(\gamma_{7}^{\prime}\right) & \text { grey, } & v & =c\left(\gamma_{8}^{\prime}\right) & \\
\text { light blue. }
\end{array}
$$

The scalars associated to the $x_{1}^{\prime}, \ldots, x_{7}^{\prime}$ are:

$$
\begin{array}{ll}
\kappa_{c}\left(x_{1}^{\prime}\right)=d(i)^{-1} d(u)^{-1} d(n)^{-1}, & \kappa_{c}\left(x_{2}^{\prime}\right)=d(i)^{-1} d(j)^{-1} d(m)^{-1}, \\
\kappa_{c}\left(x_{3}^{\prime}\right)=d(m)^{-1} d(j)^{-1} d(k)^{-1}, & \kappa_{c}\left(x_{4}^{\prime}\right)=d(i)^{-1} d(j)^{-2}, \\
\kappa_{c}\left(x_{5}^{\prime}\right)=d(i)^{-1} d(k)^{-1} d(v)^{-1}, & \kappa_{c}\left(x_{6}^{\prime}\right)=d(i)^{-2} d(m)^{-1}, \\
\kappa_{c}\left(x_{7}^{\prime}\right)=d(n)^{-1} d(t)^{-1} d(i)^{-1} . &
\end{array}
$$

The scalars associated to the edges $e_{1}^{\prime}, \ldots, e_{14}^{\prime}$ are:

$$
\begin{array}{llll}
\theta_{c}\left(e_{1}^{\prime}\right)=d(t), & \theta_{c}\left(e_{2}^{\prime}\right)=d(i), & \theta_{c}\left(e_{3}^{\prime}\right)=d(i), & \theta_{c}\left(e_{4}^{\prime}\right)=d(m), \\
\theta_{c}\left(e_{5}^{\prime}\right)=d(v), & \theta_{c}\left(e_{6}^{\prime}\right)=d(i), & \theta_{c}\left(e_{7}^{\prime}\right)=d(k), & \theta_{c}\left(e_{8}^{\prime}\right)=d(j), \\
\theta_{c}\left(e_{9}^{\prime}\right)=d(j), & \theta_{c}\left(e_{10}^{\prime}\right)=d(m), & \theta_{c}\left(e_{11}^{\prime}\right)=d(j), & \theta_{c}\left(e_{12}^{\prime}\right)=d(i), \\
\theta_{c}\left(e_{13}^{\prime}\right)=d(u), & \theta_{c}\left(e_{14}^{\prime}\right)=d(n) . & &
\end{array}
$$

The scalars associated to the circuits $\gamma_{1}^{\prime}, \ldots, \gamma_{8}^{\prime}$ are:

$$
\begin{array}{lll}
d_{c}\left(\gamma_{1}^{\prime}\right)=d(i), & d_{c}\left(\gamma_{2}^{\prime}\right)=d(j), & d_{c}\left(\gamma_{3}^{\prime}\right)=d(k), \\
d_{c}\left(\gamma_{4}^{\prime}\right)=d(m), & d_{c}\left(\gamma_{5}^{\prime}\right)=d(n), & d_{c}\left(\gamma_{6}^{\prime}\right)=d(t), \\
d_{c}\left(\gamma_{7}^{\prime}\right)=d(u), & d_{c}\left(\gamma_{8}^{\prime}\right)=d(v) . &
\end{array}
$$

Now, the admissibility of $c$ imposes conditions on the colors, one for each of the 14 edges of $\Gamma$ :

$$
\begin{aligned}
& e_{1}^{\prime} \rightsquigarrow t=n i, \quad e_{2}^{\prime} \rightsquigarrow i=n i, \quad e_{3}^{\prime} \rightsquigarrow i=j i, \quad e_{4}^{\prime} \rightsquigarrow m=i j, \\
& e_{5}^{\prime} \rightsquigarrow v=i k, \quad e_{6}^{\prime} \rightsquigarrow i=n m, \quad e_{7}^{\prime} \rightsquigarrow k=j k, \quad e_{8}^{\prime} \rightsquigarrow j=i k, \\
& e_{9}^{\prime} \rightsquigarrow j=m k, \quad e_{10}^{\prime} \rightsquigarrow m=i j, \quad e_{11}^{\prime} \rightsquigarrow j=n j, \quad e_{12}^{\prime} \rightsquigarrow i=i n, \\
& e_{13}^{\prime} \rightsquigarrow u=i n, \quad e_{14}^{\prime} \rightsquigarrow n=n^{2},
\end{aligned}
$$

that is,

$$
i=m=t=u, \quad j=n=v=1, \quad \text { and } \quad k=i^{-1} .
$$

Therefore

$$
\begin{gathered}
\quad \mathrm{I}_{G_{\mathbf{k}}^{d}}\left(M^{\prime}, \nu^{\prime}\right)=\sum_{c} \prod_{q=1}^{8} d_{c}\left(\gamma_{q}\right) \prod_{r=1}^{7} \kappa_{c}\left(x_{r}\right) \prod_{s=1}^{14} \theta_{c}\left(e_{s}\right) \\
=\sum_{i, j, k, m, n, t, u, v \in G} \delta_{i, m} \delta_{i, t} \delta_{i, u} \delta_{j, 1} \delta_{n, 1} \delta_{v, 1} \delta_{k, i-1} d(i)^{-2} d(t) d(u) d(v) \\
=\sum_{i \in G} 1_{\mathbb{k}}=|G| 1_{\mathbb{k}} .
\end{gathered}
$$

Let us prove Part (d). Consider the cyclic group $G=\mathbb{Z} / 3 \mathbb{Z}$ and the character $d: \mathbb{Z} / 3 \mathbb{Z} \rightarrow \mathbb{C}^{*}$ defined by $\bar{k} \mapsto \exp (2 i \pi k / 3)$. By Part (b), we have:

$$
\mathrm{I}_{\mathcal{C}}(M, \nu)=d(\overline{0})^{2}+d(\overline{1})^{2}+d(\overline{2})^{2}=1+\exp (4 i \pi / 3)+\exp (2 i \pi / 3)=0
$$

and

$$
\mathrm{I}_{\mathcal{C}}\left(M^{\prime}, \nu^{\prime}\right)=|\mathbb{Z} / 3 \mathbb{Z}| 1_{\mathbb{C}}=3 \neq 0 .
$$

Finally, Part (b) follows from part (d) and the fact that $\mathrm{I}_{G_{\mathrm{k}}^{d}}$ is an invariant of combed 3-manifolds.

### 5.4. An interpretation of $\mathrm{I}_{G_{k}^{d}}$

In this section, we give an interpretation (in terms of classical topological invariants) of the state sum invariant $\mathrm{I}_{G_{k}^{d}}$ defined with the category $G_{\mathrm{k}}^{d}$.

Throughout this section, we fix a closed combed 3-manifold ( $M, \nu$ ).
5.4.1. The Euler class of a combing. The combing $\nu: M \rightarrow T M$ is a nowhere zero vector field on the (closed oriented connected) 3-manifold $M$. Therefore $\nu$ generates a vector sub-bundle $L^{\nu} \rightarrow M$ of rank 1 of the tangent bundle $T M \rightarrow M$. Then the quotient bundle

$$
F^{\nu}=T M / L^{\nu} \rightarrow M
$$

is a vector bundle on $M$ of rank 2. By definition, for any $x \in M$,

$$
F_{x}^{\nu}=T_{x} M / L_{x}^{\nu}=T_{x} M / \mathbb{R} \nu_{x}
$$

The Euler class of $\nu$ is the Euler class of the vector bundle $F^{\nu}$ :

$$
E_{\nu}=e\left(F^{\nu}\right) \in H^{2}(M ; \mathbb{Z})
$$

Recall that this class is defined as follows (e.g., see [Ha, Chapter 4]). Consider a generic section $s$ of $F^{\nu}$. Then $s^{-1}(0)$ is a closed submanifold of $M$ of dimension 3 -2 $=1$. Its homology $\left[s^{-1}(0)\right] \in H_{1}(M, \mathbb{Z})$ does not depend on the choice of $s$. The image of this class under the Poincaré duality isomorphism $H_{1}(M, \mathbb{Z}) \simeq H^{2}(M ; \mathbb{Z})$ is the Euler class $e\left(F^{\nu}\right) \in H^{2}(M ; \mathbb{Z})$ of $F^{\nu}$.
5.4.2. Computation of the Euler class from closed o-graphs. Let $\Gamma$ be a closed o-graph such that $(M, \nu)$ is equivalent to $\left(\widehat{M}_{\Gamma}, \widehat{\nu}_{\Gamma}\right)$ (see Section 3.3.5). In this section, we compute the Euler class $E_{\nu}$ of $\nu$ from $\Gamma$.

The branched polyhedron $P=P_{\Gamma}$ associated to $\Gamma$ (see Section 3.3.2) is a branched spine of $(M, \nu)$. Recall that there is a bijection between the set of edges of $\Gamma$ and the set of edges of $P$. Let $e$ be an edge of $P$. Then $e$ inherits an orientation from its corresponding edge of $\Gamma$. The orientations of $e$ and $M$ determine a positive direction on a small loop in $M$ encircling $e$. The resulting oriented loop determines a cyclic order on the set $\left\{\delta^{e}, \delta_{-}^{e}, \delta_{+}^{e}\right\}$ of branches of $P$ at $e$. We choose notation so that this cyclic order is $\delta^{e}<\delta_{-}^{e}<\delta_{+}^{e}<\delta^{e}$ and the orientation of $e$ coincide with that induced by $\delta_{-}^{e}$ and $\delta_{+}^{e}$. Denote by $\Delta^{e}, \Delta_{-}^{e}, \Delta_{+}^{e}$ the regions of $P$ (eventually coinciding) containing $\delta^{e}, \delta_{-}^{e}, \delta_{+}^{e}$ respectively:


Lemma 5.3 ( $\mathbf{B P 2}$, Lemma 10.1.1]). $H^{2}(P ; \mathbb{Z})$ is the $\mathbb{Z}$-module generated by the regions $\Delta$ of $P$ subject to the relations

$$
\Delta^{e}=\Delta_{+}^{e}+\Delta_{-}^{e},
$$

as e runs over the edges of $P$.

Recall that there is a bijection between the set $V_{\Gamma}$ of vertices of $\Gamma$ and the set of vertices of $P$. Recall that the set $\operatorname{Circ}(\Gamma)$ of circuits obtained from $\Gamma$ by applying the rules of Figure 3.6 is in bijection with the set $\operatorname{Reg}(P)$ of regions of $P$. Associate to any vertex $x$ of $\Gamma$ the following cohomology class:

$$
\lambda(x)=\Delta_{1}^{x}+\Delta_{2}^{x}+\Delta_{3}^{x} \in H^{2}(P ; \mathbb{Z}) .
$$

Here, $\Delta_{1}^{x}, \Delta_{2}^{x}, \Delta_{3}^{x}$ are regions of $P$ corresponding to three portions of circuits in $\operatorname{Circ}(\Gamma)$ induced by $x$ when applying the rule of Figure 3.6. If $x$ is positive, then:


If $x$ is negative, then:


Denote by $E_{\Gamma}$ the set of edges of $\Gamma$ (which is in bijection with the set of edges of $P$ ).

Lemma 5.4. The Euler class $E_{\nu}$ of $\nu$ is the image of

$$
\mu_{\Gamma}=\sum_{\Delta \in \operatorname{Reg}(P)} \Delta-\sum_{x \in V_{\Gamma}} \lambda(x)+\sum_{e \in E_{\Gamma}} \Delta^{e} \in H^{2}(P ; \mathbb{Z})
$$

under the isomorphism $H^{2}(P ; \mathbb{Z}) \simeq H^{2}(M ; \mathbb{Z})$ induced by the inclusion of $P$ in $M$.

Proof. Since $P$ is branched, it may be provided with a $C^{1}$-structure (see Section 3.2.2). The Euler class is the image of a class $\mu_{\Gamma} \in H^{2}(P ; \mathbb{Z})$ under the isomorphism $H^{2}(P ; \mathbb{Z}) \simeq H^{2}(M ; \mathbb{Z})$ induced by the inclusion of $P$ in $M$. The class $\mu_{\Gamma}$ is just the obstruction to the existence of a nowhere-zero tangent vector field on $P$. For each region $\Delta$ of $P$, remove the interior of a disk embedded in $\Delta$. The result is a regular neighborhood $N$ of the vertices and edges of $P$. The boundary of $N$ is the disjoint union of the circles bounding the disks. Following [BP2, Propositin 7.1.1], construct a nowhere-zero tangent vector field near on $N$ using the following rules:


The red points represent the points where the vector field is tangent to the circles bounding the removed disks. For a region $\Delta$ of $P$, let $n_{\Delta}$ be the number of red dots in $\Delta$. By construction, the red points in $\Delta$ split the circle bounding the removed disk in $\Delta$ into segments on which the field points alternatively inside and outside the disk. Thus $n_{\Delta}$ is even. If $n_{\Delta}=0$, then we can extend the vector field on $\Delta$ with a zero of index 1 . If $n_{\Delta}>0$, then we can extend the vector field on $\Delta$ with $\frac{n_{\Delta}}{2}-1$ zeros of index -1 . Consequently,

$$
\begin{equation*}
\mu_{\Gamma}=\sum_{\Delta \in \operatorname{Reg}(P)}\left(1-\frac{n_{\Delta}}{2}\right) \Delta \in H^{2}(P ; \mathbb{Z}) . \tag{5.2}
\end{equation*}
$$

Let $x$ be a vertex of $\Gamma$. Since $x$ is of crossing type (see Section 3.3), it has two incoming half-edges $h_{1}^{\text {in }}, h_{2}^{\text {in }}$ and two outgoing half-edges $h_{1}^{\text {out }}, h_{2}^{\text {out }}$. We choose notation so that the distinguished half-edges are $h_{1}^{\text {in }}$ and $h_{1}^{\text {out }}$. Denote the corresponding edges of $\Gamma$ by $e_{1}^{x, \text { in }}, e_{2}^{x, \text { in }}, e_{1}^{x, \text { out }}, e_{2}^{x, \text { out }}$. Since the edges of $\Gamma$ are oriented, we have:

$$
e_{1}^{x, \text { in }} \neq e_{2}^{x, \text { in }} \quad \text { and } \quad e_{1}^{x, \text { out }} \neq e_{2}^{x, \text { out }}
$$

Denote by $A_{1}^{x}, A_{2}^{x}, A_{3}^{x}, A_{4}^{x}, A_{5}^{x}, A_{6}^{x}$ the regions of $P$ corresponding to the portions of circuits in $\operatorname{Circ}(\Gamma)$ induced by $x$ when applying the rule of Figure 3.6. We choose notation according to the sign of $x$. If $x$ is positive, then:


If $x$ is negative, then:


Since any edge is outgoing from a unique vertex, since there are exactly two half-edges outgoing from a vertex, and since $e_{1}^{x, \text { out }} \neq e_{2}^{x, \text { out }}$, we have:

$$
\sum_{e \in E_{\Gamma}} \Delta^{e}=\sum_{x \in V_{\Gamma}}\left(\Delta^{e_{1}^{x, \text { out }}}+\Delta^{e_{2}^{x, \text { out }}}\right) .
$$

Similarly,

$$
\sum_{e \in E_{\Gamma}} \Delta^{e}=\sum_{x \in V_{\Gamma}}\left(\Delta^{e_{1}^{x, \text { in }}}+\Delta^{e_{2}^{x, \text {, }}}\right) .
$$

Consequently,

$$
2 \sum_{e \in E_{\Gamma}} \Delta^{e}=\sum_{x \in V_{\Gamma}}\left(\Delta^{e_{1}^{x, \text { out }}}+\Delta^{e_{2}^{x, \text { out }}}+\Delta^{e_{1}^{x, \text { in }}}+\Delta^{e_{2}^{x, \text {,n }}}\right)
$$

Now, if $x$ is positive, then

$$
\Delta^{e_{1}^{x, \text { out }}}=A_{3}^{x}, \quad \Delta^{e_{2}^{x, \text { out }}}=A_{5}^{x}, \quad \Delta^{e_{1}^{x, \text { in }}}=A_{1}^{x}, \quad \Delta^{e_{2}^{x, \text { in }}}=A_{3}^{x}
$$

if $x$ is negative, then

$$
\Delta_{1}^{e_{1}^{x, \text { out }}}=A_{1}^{x}, \quad \Delta_{2}^{e_{2}^{x, \text { out }}}=A_{3}^{x}, \quad \Delta^{e_{1}^{x, \text { in }}}=A_{3}^{x}, \quad \Delta_{2}^{e_{2}^{x, \text { in }}}=A_{5}^{x} .
$$

Therefore

$$
2 \sum_{e \in E_{\Gamma}} \Delta^{e}=\sum_{x \in V_{\Gamma}}\left(A_{1}^{x}+2 A_{3}^{x}+A_{5}^{x}\right) .
$$

By definition, for any vertex $x$ of $\Gamma$,

$$
\lambda(x)=A_{1}^{x}+A_{3}^{x}+A_{5}^{x} .
$$

Since $n_{\Delta}$ is the number of red dots in $\Delta$, we have:

$$
\sum_{\Delta \in \operatorname{Reg}(P)} n_{\Delta} \Delta=\sum_{x \in V_{\Gamma}}\left(A_{1}^{x}+A_{5}^{x}\right) .
$$

Consequently,

$$
-2 \sum_{x \in V_{\Gamma}} \lambda(x)+2 \sum_{e \in E_{\Gamma}} \Delta^{e}=-\sum_{x \in V_{\Gamma}}\left(A_{1}^{x}+A_{5}^{x}\right)=-\sum_{\Delta \in \operatorname{Reg}(P)} n_{\Delta} \Delta .
$$

Using the expression (5.2), we conclude that

$$
\mu_{\Gamma}=\sum_{\Delta \in \operatorname{Reg}(P)}\left(1-\frac{n_{\Delta}}{2}\right) \Delta=\sum_{\Delta \in \operatorname{Reg}(P)} \Delta-\sum_{x \in V_{\Gamma}} \lambda(x)+\sum_{e \in E_{\Gamma}} \Delta^{e} .
$$

5.4.3. Interpretation of $\mathrm{I}_{G_{k}^{d}}$. Let $B G$ be the (pointed) classifying space of the group $G$. The character $d: G \rightarrow \mathbb{k}^{*}$ represents an element

$$
[d] \in H^{1}\left(G ; \mathbb{k}^{*}\right) \cong H^{1}\left(B G ; \mathbb{k}^{*}\right)
$$

Pick a point $* \in M$. Denote by $\operatorname{Hom}\left(\pi_{1}(M, *), G\right)$ the set of group homomorphisms from the fundamental group $\pi_{1}(M, *)$ to $G$. Any $f \in \operatorname{Hom}\left(\pi_{1}(M, *), G\right)$ induces a pointed map $\widetilde{f}: M \rightarrow B G$ and so a homomorphism

$$
\widetilde{f}^{*}: H^{1}\left(B G ; \mathbb{k}^{*}\right) \rightarrow H^{1}\left(M ; \mathbb{k}^{*}\right) .
$$

Consider the pairing

$$
\langle\cdot, \cdot\rangle: H^{1}\left(M ; \mathbb{k}^{*}\right) \times H^{2}(M ; \mathbb{Z}) \rightarrow \mathbb{k}
$$

induced by the Poincaré duality isomorphism $H^{2}(M ; \mathbb{Z}) \simeq H_{1}(M ; \mathbb{Z})$ and the evaluation pairing $H^{1}\left(M ; \mathbb{k}^{*}\right) \times H_{1}(M ; \mathbb{Z}) \rightarrow \mathbb{k}$.

Theorem 5.5. We have:

$$
\mathrm{I}_{G_{\mathbf{k}}^{d}}(M, \nu)=\sum_{f \in \operatorname{Hom}\left(\pi_{1}(M, *), G\right)}\left\langle\widetilde{f}^{*}([d]), E_{\nu}\right\rangle,
$$

where $E_{\nu}$ is the Euler class of the combing $\nu$.

Proof. Let $\Gamma$ be a closed o-graph such that $(M, \nu)$ is equivalent to $\left(\widehat{M}_{\Gamma}, \widehat{\nu}_{\Gamma}\right)$ (see Section 3.3.5). Let $P=P_{\Gamma}$ be the branched polyhedron associated to $\Gamma$. The set $M \backslash P$ is homeomorphic to an open 3-ball. We may assume that $* \in M$ is the center of this ball. For any region $\Delta$ of $P$, pick a loop $\gamma_{\Delta}$ in $M$ based in $*$ which is positively transverse to $\Delta$. The fundamental group $\pi_{1}(M, *)$ is generated by the homotopy classes $\left[\gamma_{\Delta}\right]$ with $\Delta \in \operatorname{Reg}(P)$. The only relations are $\left[\gamma_{e}\right]=1$ with $e$ an edge of $P$, where $\gamma_{e}=\gamma_{\Delta_{-}^{e}} \gamma_{\Delta_{+}^{e}}\left(\gamma_{\Delta^{e}}\right)^{-1}$ :


Consequently,

$$
\left.\pi_{1}(M, *)=\left\langle\left[\gamma_{\Delta}\right], \Delta \text { region of } P\right|\left[\gamma_{\Delta^{e}}\right]=\left[\gamma_{\Delta_{-}^{e}}\right]\left[\gamma_{\Delta_{+}^{e}}\right], e \text { edge of } P\right\rangle .
$$

Thus any $f \in \operatorname{Hom}\left(\pi_{1}(M, *), G\right)$ induces an admissible coloring $c_{f}$ of $\Gamma$ defined by

$$
c_{f}(\Delta)=f\left(\left[\gamma_{\Delta}\right]\right) \in G
$$

for all region $\Delta$ of $P$ (through the obvious bijection between $\operatorname{Circ}(\Gamma)$ and $\operatorname{Reg}(P))$. Also the assignment $f \mapsto c_{f}$ is bijective.

Let $f \in \operatorname{Hom}\left(\pi_{1}(M, *), G\right)$ and denote by $c=c_{f}$ its associated admissible coloring of $\Gamma$. By Theorem 5.1, it suffices to prove that

$$
\begin{equation*}
\left\langle\widetilde{f^{*}}([d]), E_{\nu}\right\rangle=\prod_{\Delta \in \operatorname{Reg}(P)} d_{c}(\Delta) \prod_{x \in V_{\Gamma}} \kappa_{c}(x) \prod_{e \in E_{\Gamma}} \theta_{c}(e), \tag{5.3}
\end{equation*}
$$

where $V_{\Gamma}$ is the set of vertices of $\Gamma$ and $E_{\Gamma}$ is the set of edges of $\Gamma$. Recall the presentation of $H^{2}(P ; \mathbb{Z})$ given by Lemma 5.3 and define a group homomorphism

$$
\varphi_{c}: H^{2}(P ; \mathbb{Z}) \rightarrow \mathbb{k}^{*}
$$

by setting $\varphi_{c}(\Delta)=d(c(\Delta))$ for all $\Delta \in \operatorname{Reg}(P)$. By Lemma 5.4, the Euler class $E_{\nu}$ is the image of

$$
\mu_{\Gamma}=\sum_{\Delta \in \operatorname{Reg}(P)} \Delta-\sum_{x \in V_{\Gamma}} \lambda(x)+\sum_{e \in E_{\Gamma}} \Delta^{e} \in H^{2}(P ; \mathbb{Z})
$$

(with the notation of Section 5.4.2) under the isomorphism $H^{2}(P ; \mathbb{Z}) \simeq H^{2}(M ; \mathbb{Z})$ induced by the inclusion of $P$ in $M$. Then, we have:

$$
\left\langle\widetilde{f}^{*}([d]), E_{\nu}\right\rangle=\varphi_{c}\left(\mu_{\Gamma}\right)=\prod_{\Delta \in \operatorname{Reg}(P)} \varphi_{c}(\Delta) \prod_{x \in V_{\Gamma}} \varphi_{c}(\lambda(x))^{-1} \prod_{e \in E_{\Gamma}} \varphi_{c}\left(\Delta^{e}\right)
$$

Now it follows from the definitions that

$$
\varphi_{c}(\Delta)=d_{c}(\Delta), \quad \varphi_{c}(\lambda(x))^{-1}=\kappa_{c}(x), \quad \varphi_{c}\left(\Delta^{e}\right)=\theta_{c}(e)
$$

for all regions $\Delta$ of $P$, all vertices $x$ of $\Gamma$, and all edges $e$ of $\Gamma$. Therefore (5.3) holds.
5.4.4. Remark. Recall from Section 4.1.2 that if the pivotal fusion $\mathbb{k}$ category $G_{\mathrm{kk}}^{d}$ is spherical, then $\mathrm{I}_{G_{k}^{d}}(M, \nu)$ does not depend on $\nu$. This can be recovered from Theorem 5.5 as follows. It is well-known (see for example Tul) that the Euler class $E_{\nu}$ is even, that is, $E_{\nu}=2 D_{\nu}$ with $D_{\nu} \in H^{2}(M ; \mathbb{Z})$. Therefore, for any $f \in \operatorname{Hom}\left(\pi_{1}(M, *), G\right)$,

$$
\left\langle\widetilde{f}^{*}([d]), E_{\nu}\right\rangle=\left\langle\widetilde{f}^{*}\left(\left[d^{2}\right]\right), D_{\nu}\right\rangle .
$$

Consequently, if $G_{\mathrm{k}}^{d}$ is spherical, or equivalently if $d(g)^{2}=1_{\mathrm{k}}$ for all $g \in G$ (see Section (5.1), then $\left\langle\widetilde{f}^{*}([d]), E_{\nu}\right\rangle=1$ and Theorem [5.5 gives that

$$
\mathrm{I}_{G_{\mathfrak{k}}^{d}}(M, \nu)=\left|\operatorname{Hom}\left(\pi_{1}(M, *), G\right)\right| 1_{\mathbb{k}}
$$

In particular, $\mathrm{I}_{G_{k}^{d}}(M, \nu)$ does not depend on $\nu$.

## APPENDIX A

## Unordered tensor products of modules

By a module we mean a left module over the commutative ring $\mathbb{k}$. Given a finite family $E$ of modules, we define the unordered tensor product $\otimes_{M \in E} M$ as follows. Let $n=\# E$ be the number of elements of $E$, and let $\mathcal{S}=\mathcal{S}(E)$ be the set of bijections $\{1, \ldots, n\} \rightarrow E$. For any bijection $\sigma \in \mathcal{S}$, consider the module

$$
E_{\sigma}=\sigma(1) \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} \sigma(n) .
$$

For $\sigma, \mu \in \mathcal{S}$, let $p_{\sigma, \mu}: E_{\sigma} \rightarrow E_{\mu}$ be the $\mathbb{k}$-linear isomorphism induced by the permutations of modules: given any vectors $m_{i} \in \sigma(i)$ with $i=1, \ldots, n$,

$$
p_{\sigma, \mu}\left(m_{1} \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} m_{n}\right)=m_{\sigma^{-1} \mu(1)} \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} m_{\sigma^{-1} \mu(n)}
$$

It follows from the definitions that for arbitrary $\sigma, \mu, \nu \in \mathcal{S}$,

$$
p_{\mu, \nu} p_{\sigma, \mu}=p_{\sigma, \nu}: E_{\sigma} \rightarrow E_{\nu} \quad \text { and } \quad p_{\sigma, \sigma}=\operatorname{id}_{E_{\sigma}} .
$$

The unordered tensor product of the modules $M \in E$ is the projective limit of the system $\left(E_{\sigma}, p_{\sigma, \mu}\right)_{\sigma, \mu \in \mathcal{S}}$ :

$$
\otimes_{M \in E} M=\lim _{\check{m}} E_{\sigma} .
$$

This is a module (over $\mathbb{k}$ ) equipped with an isomorphism $\otimes_{M \in E} M \cong E_{\sigma}$ for each $\sigma \in \mathcal{S}$. The latter isomorphisms are called the cone isomorphisms. They commute with $p_{\sigma, \mu}$ for all $\sigma, \mu \in \mathcal{S}$. If all modules $M \in E$ are projective of finite type, then so is $\otimes_{M \in E} M$ and there is a canonical isomorphism

$$
\left(\otimes_{M \in E} M\right)^{\star} \simeq \otimes_{M \in E} M^{\star} .
$$

The unordered tensor product of an empty set of modules is the ground ring $\mathbb{k}$.
Given a bijection $\varphi: E \rightarrow F$ between two finite families of modules, an arbitrary family $\left\{f_{M}: M \rightarrow \varphi(M)\right\}_{M \in E}$ of $\mathbb{k}$-linear homomorphisms induces a $\mathbb{k}$-linear homomorphism

$$
\otimes_{M \in E} f_{M}: \otimes_{M \in E} M \rightarrow \otimes_{N \in F} N .
$$

It is uniquely determined by the property that for all $\sigma \in \mathcal{S}(E)$, the following diagram commutes:

where the vertical isomorphisms are the cone isomorphisms. If all $f_{M}$ are isomorphisms, then so is $\otimes_{M \in E} f_{M}$.

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## Invariants par somme d'états des 3-variétés peignées

Cette thèse concerne la topologie quantique, une branche des mathématiques née dans les années 1980 suite aux travaux de Jones, Drinfeld et Witten. Un exemple fondamental d'invariant quantique des 3 -variétés est due à Turaev-Viro en 1992. Leur approche, dans sa forme générale due à Barrett et Westbury, utilise une catégorie de fusion sphérique comme ingrédient principal et consiste en une somme d'états sur un squelette de la 3 -variété dont les sommets sont coloriés par les $6 j$-symboles de la catégorie.

Le résultat principal de la thèse est la construction d'un invariant topologique des 3 -variétés peignées (c'est-à-dire des 3 -variétés munies d'un champ de vecteurs jamais nuls) qui généralise celui de Turaev-Viro. Ce nouvel invariant est défini au moyen d'une catégorie de fusion pivotale et consiste en une somme d'états sur un squelette ramifié représentant la 3 -variété peignée.

Lorsque la catégorie de fusion pivotale n'est pas sphérique, l'invariant permet en général de distinguer des champs de vecteurs non homotopes sur une même 3 -variété. Ceci est montré en considérant une catégorie de fusion pivotale associée à un caractère d'un groupe fini. Pour cette catégorie, l'invariant correspond à l'évaluation par le caractère de la classe d'Euler d'un certain fibré vectoriel de rang 2 associé au champ de vecteurs.

## State sum invariants of combed 3-manifolds

This thesis concerns quantum topology, a branch of mathematics born in the 1980s after the work of Jones, Drinfeld and Witten. A fundamental example of a quantum invariant of 3 -manifolds is due to Turaev-Viro in 1992. Their approach, in its general form due to Barrett and Westbury, uses a spherical fusion category as the main ingredient and consists in a state sum on a skeleton of the 3 -manifold whose vertices are colored by the $6 j$-symbols of the category.

The main result of the thesis is the construction of a topological invariant of combed 3-manifolds (that is, of 3-manifolds endowed with a nowhere-zero vector field) which generalizes that of Turaev-Viro. This new invariant is defined by means of a pivotal fusion category and consists in a state sum on a branched skeleton representing the combed 3 -manifold.

When the pivotal fusion category is not spherical, the invariant allows in general to distinguish non homotopic vector fields on the same 3 -manifold. This is proved by considering a pivotal fusion category associated with a character of a finite group. For this category, the invariant corresponds to the evaluation by the character of the Euler class of a certain vector bundle of rank 2 associated to the vector field.

