Thèse de Gaia Comaschi, Université de Lille, 2019

Université de Lille
Laboratoire Paul Painlevé
Ecole Doctorale SPI 072

Thèse de Doctorat
Mathématiques

Espaces de modules des représentations Pfaffiennes de cubiques de dimension 3

Thèse

présentée pour l'obtention du

Doctorat de l'Université de Lille

par

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Soutenue publiquement le 12 juillet devant le jury composé de:

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Moduli Spaces of Pfaffian Representations of Cubic Threefolds

Gaia COMASCHI

PhD Thesis supervised by Dimitri Markouchevitch

Date of defense: 12 July 2019

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Thèse préparée au Laboratoire Paul Painlevé (UMR CNRS 8524), Université de Lille, Cité Scientifique 59 655 Villeneuve d'Ascq CEDEX, grâce à un financement par la Région Hauts-de-France et le Labex CEMPI (ANR-11-LABX-0007-01).

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Remerciements

J'aimerais tout d'abord remercier mon directeur de thèse, Dimitri Markushevich, pour m'avoir amenée sur la piste d'un thème de recherche tellement fascinant et qui m'a permis de découvrir des branches de la géométrie algébrique qui m'ont beaucoup passionnée pendant ces années .

Je remercie également les rapporteurs, Daniele Faenzi et Laurent Manivel, pour avoir dédié du temps à la lecture de ma thèse et pour avoir, avec leurs remarques et questions, donné vie à des points de réflexion très stimulants.

Mes remerciements sont aussi adresses aux autres membres du jury, pour avoir accepté d'assister à la présentation de mon travail.

A titre plus personnel mes plus grands remerciements vont à ma famille, pour son inconditionnel support et pour m'avoir encouragée même dans les moments les plus difficiles.

Merci à tous mes amis : ceux qui sont proches et avec lesquels je partage mon quotidien et ceux qui sont plus loin mais toujours à mes cotés ; ceux que j'ai eu la chance de rencontrer dans les dernières années et qui ont enrichi ma vie Lilloise et ceux qui m'accompagnent depuis longtemps mais dont l'amitié se montre de plus en plus solide. Votre estime constante c'est mon plus grand soutien.

Merci pour tout.

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Résumé

Dimitri Markushevich et Alexander Tikhomirov ont décrit en 1999 une famille de fibrés vectoriels de rang 2 à cohomologie "naturelle", appelés instantons, sur une hypersurface cubique lisse X de $\mathbb{P}^4$. L'espace de modules $\mathcal{M}_X^{in}$ de ces fibrés instantons s'identifie à un ouvert d'un tore complexe $J(X)$ de dimension 5, la jacobienne intermédiaire de X. Stéphane Druel a donné en 2000 une description complète du bord de $\mathcal{M}_X^{in}$ dans la compactification de Gieseker-Maruyama $\mathcal{M}_X(2;0,2,0)$ paramétrant les faisceaux semistables de rang 2. Il se trouve que $\mathcal{M}_X(2;0,2,0)$ n'est pas $J(X)$, mais un éclatement de $J(X)$. La question se pose si $J(X)$, la compactification naturelle de $\mathcal{M}_X^{in}$ dans la classe des variétés, s'interprète aussi comme une compactification dans la classe des espaces de modules d'objets quelconques liés à X. Cela sert de motivation pour la recherche d'autres compactifications de $\mathcal{M}_X^{in}$ qui soient des espaces de modules. Une autre motivation pour ce problème est de trouver une compactification $\widetilde{M}(X)$ de $\mathcal{M}_X^{in}$ dont la relation à $\mathcal{M}_X(2;0,2,0)$ soit similaire à celle, établie par Jun Li, pour les instantons sur des surfaces algébriques, entre les compactifications de Donaldson-Uhlenbeck et de Gieseker-Maruyama, dont la seconde est un éclatement de la première. Une troisième motivation est d'obtenir une compactification plus maniable que $\mathcal{M}_X(2;0,2,0)$ au cas où X acquiert des singularités, car dans ce cas le bord de $\mathcal{M}_X(2;0,2,0)$ devient intraitable. Enfin, en faisant varier les cubiques X dans la famille des sections hyperplanes d'une cubique Y de $\mathbb{P}^5$, les espaces $\mathcal{M}_X^{in}$ se collent en une variété portant une 2-forme holomorphe symplectique, d'où l'intérêt de la recherche d'un espace de modules compactifié susceptible d'être holomorphiquement symplectique.

Dans la thèse on remplace les fibrés instantons par leurs résolutions localement libres antisymétriques sur $\mathbb{P}^4$, qui ne sont autres que les représentations des cubiques comme les pfaffiens des matrices antisymétriques de taille 6 de formes linéaires sur $\mathbb{P}^4$. Les espaces compactifiés $\widetilde{M}(X)$ se situent dans le lieu des matrices GIT-semistables quotienté par le groupe $SL(6)$. La (semi)stabilité des matrices antisymétriques de formes linéaires sur $\mathbb{P}^4$ se réduit à celle des systèmes $\mathbb{P}^4$ ("hyperwebs") dans l'espace $\mathbb{P}^{14}$ des formes alternés, ou encore à celle des quintuplets de matrices antisymétriques complexes de taille 6. Ce problème est étudié par une méthode similaire à celle de C.T.C.Wall, qu'il a développée pour les systèmes linéaires de quadriques. Des critères de (semi)stabilité dans le cas anti-symétrique sont obtenus, ainsi que la classification géométrique des hyperwebs (semi)stables en fonction de l'intersection de $\mathbb{P}^4$ avec la grassmannienne $\mathrm{Gr}(2,6)$ plongée dans $\mathbb{P}^{14}$ selon Plücker.

L'espace des déformations de la résolution antisymétrique d'un faisceau dans le bord $\mathcal{B}_{X_0}$ de $\mathcal{M}_{X_0}(2;0,2,0)$ est étudié. En notant $\mathcal{B}, \mathcal{M}^{in}$ $\mathcal{M}^{GM}$, $\widetilde{M}$, la réunion des espaces $\mathcal{B}_X$, $\mathcal{M}_X(2;0,2,0)$, $\mathcal{M}_X^{in}$, $\widetilde{M}(X)$ respectivement, pour X parcourant une famille complète des déformations de X_0, on montre que $\mathcal{B}$ est formée de deux diviseurs $\mathcal{B}'$, $\mathcal{B}''$ et que $\mathcal{M}^{GM}$ est, au point générique de $\mathcal{B}'$, un éclatement d'une sous-variété lisse dans $\widetilde{M}$. Conjecturalement le même résultat est valable pour $\mathcal{B}''$. On peut donc considérer $\widetilde{M}$ comme une sorte de compactification de Donaldson-Uhlenbeck de $\mathcal{M}^{in}$.

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Abstract

In 1999, Dimitri Markushevich and Alexander Tikhomirov described a family of rank 2 vector bundles with "natural" cohomology on a smooth cubic hypersurface X of $\mathbb{P}^4$, which they called instantons. The moduli space $\mathcal{M}_X^{in}$ of instanton bundles is isomorphic to an open subset of the intermediate Jacobian $J(X)$ of X, a 5 dimensional complex torus. In 2000, Druel gave a complete description of the boundary of $\mathcal{M}_X^{in}$ in the Gieseker-Maruyama compactification $\mathcal{M}_X(2;0,2,0)$ parametrizing semistable sheaves of rank 2. It turns out that $\mathcal{M}_X(2;0,2,0)$ is not isomorphic to $J(X)$ but to the blowup of $J(X)$ along a smooth surface. A natural question arises whether $J(X)$, the simplest compactification of $\mathcal{M}_X^{in}$ in the class of varieties, can also be interpreted as a compactification in the class of moduli spaces of objects related to X. This question motivates the search for alternative compactifications of $\mathcal{M}_X^{in}$ that are moduli spaces. Another motivation for this problem is to find a compactification $\widetilde{M}(X)$ of $\mathcal{M}_X^{in}$ whose relation with $\mathcal{M}_X(2;0,2,0)$ is similar to the one existing between the Donaldson-Uhlenbeck and the Gieseker-Maruyama compactifications of moduli spaces of instantons on algebraic surfaces. Jun Li proved that the latter is a blowup of the former. Yet another motivation is to obtain a compactification that is easier to handle on singular cubics, since in the singular case the boundary of $\mathcal{M}_X(2;0,2,0)$ is intractable. Moreover, when the cubic X varies in the family of hyperplane sections of a cubic fourfold $Y \subset \mathbb{P}^5$, the spaces $\mathcal{M}_X^{in}$ glue together into a manifold carrying a holomorphic symplectic 2-form. It is thus interesting to look for compactified moduli spaces that might be holomorphically symplectic. In the thesis we replace instanton bundles by their locally free skew-symmetric resolutions in $\mathbb{P}^4$. These are just the representations of cubics as the Pfaffians of 6×6 skew-symmetric matrices of linear forms. The compact moduli space $\widetilde{M}(X)$ is contained in the GIT quotient of the locus of semistable matrices for the action of $SL(6)$. The (semi)stability of skew-symmetric matrices of linear forms on $\mathbb{P}^4$ reduces to the (semi)stability of 4 dimensional linear systems (hyperwebs) in the space $\mathbb{P}^{14}$ of skew-symmetric bilinear forms on $\mathbb{C}^6$ or else, to the (semi)stability of 5-tuples of complex skew-symmetric matrices of size 6. This problem is studied by a method similar to that applied by C.T.C.Wall to linear systems of quadrics. We obtain (semi)stability criteria in the skew-symmetric case and present a classification of semistable hyperwebs in terms of the possible intersections of $\mathbb{P}^4$ with the Grassmannian $\mathrm{Gr}(2,6)$, embedded in $\mathbb{P}^{14}$ via the Plücker embedding. The space of deformations of a sheaf in the boundary $\mathcal{B}_{X_0}$ of $\mathcal{M}_{X_0}(2;0,2,0)$ is studied. Denoting by $\mathcal{B}$, $\mathcal{M}^{in}$, $\mathcal{M}^{GM}$, $\widetilde{M}$ the union of the spaces $\mathcal{B}_X, \mathcal{M}_X^{in}$, $\mathcal{M}_X(2;0,2,0)$, $\widetilde{M}(X)$ respectively, for X varying in a complete family of deformations of X_0, we prove that $\mathcal{B}$ is the union of two divisors $\mathcal{B}'$, $\mathcal{B}''$ and that at the generic point of $\mathcal{B}'$, $\mathcal{M}^{GM}$ is the blowup of $\widetilde{M}$ along a smooth subvariety. We conjecture that the same holds for $\mathcal{B}''$. This allows us to consider $\widetilde{M}$ as a sort of Donaldson-Uhlenbeck compactification of $\mathcal{M}^{in}$.

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Introduction

Instanton bundles originally appeared in the context of Yang-Mills gauge theory. In their seminal work [ADHM], Atiyah, Drinfeld, Hitchin and Manin established a correspondence between the self-dual solutions of the Yang-Mills equations on the four-sphere S^4 of topological charge $n \geq 1$ and the stable rank 2 vector bundles $\mathcal{F}$ on $\mathbb{P}^3$ with Chern classes $c_1(\mathcal{F}) = 0$, $c_2(\mathcal{F}) = n$ and such that $H^1(\mathbb{P}^3, \mathcal{F}(-2))= 0$. Such a bundle $\mathcal{F}$ is referred to as an *instanton*. The vanishing condition on H^1 guarantees that $\mathcal{F}$ can be represented as the cohomology of a very simple complex with free terms, called monad. Here and throughout the thesis, when speaking about sheaves on a projective space, we will call *free* the sheaves that are direct sums of powers of the tautological sheaf $\mathcal{O}(1)$.

Thereafter the investigation of instantons led to several generalizations, including their definition on other three-dimensional Fano varieties. Though there is no ADHM correspondence for the generalizations of instantons to Fano varieties, different from $\mathbb{P}^3$, they share common features with instantons on $\mathbb{P}^3$: they have the maximum of vanishing in cohomology authorized by obvious constraints (Riemann-Roch, Kodaira vanishing and Serre duality), and they can be represented as the cohomology of a very simple complex with free terms. For a Fano threefold X with canonical class divisible by two, a natural generalization of the vanishing condition in the definition of an instanton is $H^1\big(X, \mathcal{F}\big(\frac{1}{2}K_X\big)\big) = 0$.

On a smooth cubic hypersurface X in $\mathbb{P}^4$, the first constructions of instantons were studied by Markushevich-Tikhomirov [MT1]. In this case we have $K_X = \mathcal{O}_X(-2)$, thus the instantons on X are defined as stable rank 2 vector bundles $\mathcal{F}$ with $c_1(\mathcal{F}) = 0$ and $H^1(X, \mathcal{F}(-1)) = 0$. The lowest charge for which instantons exist is $c_2 = 2$. In loc. cit., the authors proved that the moduli space $\mathcal{M}_X^{in}$ of instantons of charge 2 is an étale cover of an open subset of the intermediate Jacobian $J(X)$. Further results on $\mathcal{M}_X^{in}$ were obtained by Iliev-Markushevich [IM] and Beauville [B1]. This work attracted attention to relations between different types of objects related to a cubic threefold X: curves on X, sheaves on X and the intermediate Jacobian $J(X)$. Later Druel [Dr] described the "natural" compactification of $\mathcal{M}_X^{in}$: the Gieseker-Maruyama moduli space $\mathcal{M}_X(2; 0, 2, 0)$ of semistable sheaves of rank 2 and Chern classes $c_1 = 0,\ c_2 = 2,\ c_3 = 0$. He proved that $\mathcal{M}_X(2; 0, 2, 0)$ is smooth of dimension 5 and that furthermore it is isomorphic to $J(X)$ blown up along $F(X)$, the Fano surface parameterizing lines on X.

The growing interest in the moduli space $\mathcal{M}_X^{in}$ is also motivated by its link with the theory of irreducible holomorphic symplectic (IHS) manifolds. It follows indeed by the results obtained by Markushevich-Tikhomirov [MT2] and Kuznetsov-Markushevich [KM], that on a smooth cubic fourfold Y, the moduli space $\mathcal{M}_Y$ of sheaves of the form $i_*\mathcal{F}$, where $[\mathcal{F}] \in \mathcal{M}_{X_h}^{in}$, $X_h = Y \cap h$ runs over the non-singular hyperplane sections of Y and i stands for the natural embedding $X_h \hookrightarrow Y$, is holomorphically symplectic. Moreover, denoting by $U \subset \mathbb{P}^{5^*}$ the open subset parameterizing smooth hyperplane sections X_h of Y, we obtain the natural map $\pi : \mathcal{M}_Y \to U$, whose fiber $\mathcal{M}_{X_h}$ over every point $h \in U$ is a Lagrangian subvariety of $\mathcal{M}_Y$, so that π is a Lagrangian fibration.

A similar phenomenon was observed by Donagi-Markman in [DM]. They proved that the relative intermediate Jacobian $\mathcal{J}(\mathfrak{X}_U/U) = \{J(X_h)\}_{h\in U}$ has a symplectic structure and that also in this case the natural map $\mathcal{J}(\mathfrak{X}_U/U) \to U$ is a Lagrangian fibration. A compactification of $\mathcal{J}(\mathfrak{X}_U/U)$ was constructed by Laza-Saccà-Voisin in [LSV]. This is

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done by extending its construction to the hyperplanes $h \notin \mathcal{U}$ and the resulting manifold $\mathcal{J}(\mathfrak{X}/\mathbb{P}^{5^*})$ is an IHS. However, neither $J(X_h),\ h \in U$, nor the fibers over $\mathbb{P}^{5^*} \setminus U$ "added" in the Laza-Saccà-Voisin compactification have acquired an interpretation as moduli spaces of some objects related to X_h .
As a consequence of these results we ask the following questions:

- Is it possible to find, on a smooth cubic threefold X, a compactification of $\mathcal{M}_X^{in}$ isomorphic to the intermediate Jacobian $J(X)$?
- Is it possible to find a compactification of $\mathcal{M}_X^{in}$ whose construction can be readily extended to singular cubics (or at least to cubics acquiring singularities of the types presented by the varieties of the form $X_h = Y \cap h$)?

These questions are the motivation for a search for alternative compactifications of $\mathcal{M}_X^{in}$.
The Gieseker-Maruyama moduli space, on a smooth cubic threefold X, is birational but not isomorphic to $J(X)$. Moreover its construction on singular threefolds reveals to be troublesome: its boundary is too difficult to treat. We therefore look for a new moduli space associated to cubic threefolds.
The central idea of our construction is to replace instantons by their free skew-symmetric resolution on $\mathbb{P}^4$. To begin with we consider an instanton $\mathcal{F}$ on a smooth cubic X. The twisted bundle $\mathcal{E} := \mathcal{F}(1)$ is a skew-symmetric *Ulrich* bundle; this means that it admits a minimal free resolution in $\mathbb{P}^4$ of the form:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^4}{}^{\oplus 6} \longrightarrow \mathcal{E} \longrightarrow 0,$$

where M is a 6×6 skew-symmetric matrix whose entries are linear forms on $\mathbb{P}^4$ and such that X is defined by the equation $\mathrm{Pf}\,(M) = 0$.
On a smooth cubic hypersurface of $\mathbb{P}^4$ defined by an equation $F = 0$, instanton bundles are then associated to *Pfaffian representations* of F, that is, representations of the form $F = \mathrm{Pf}\,(M)$, where M is a skew-symmetric matrix of size 6 whose entries are linear forms. So the moduli space $\mathfrak{P}$ of free resolutions of instantons, that we construct, is nothing else than the *moduli space of Pfaffian representations of cubic threefolds*. Such a moduli space can be obtained by means of Geometric Invariant Theory (GIT). Considering indeed $\mathcal{P}$, the 74-dimensional projective space of 6×6 skew-matrices of linear forms, we see that the group $GL(6, \mathbb{C})$ acts on $\mathcal{P}$ by conjugation; therefore $\mathfrak{P}$ is the compact projective moduli space of dimension 39 defined as the GIT quotient:

$$\mathfrak{P} := \mathcal{P}^{ss} /\!\!/ SL(6, \mathbb{C})$$

where $\mathcal{P}^{ss}$ is the open set of semistable matrices.
A further reason justifying our attention to the space $\mathfrak{P}$ is the following. Denote by $\mathcal{P}^{in}$ the open set of matrices $M \in \mathcal{P}$ such that the equation $\mathrm{Pf}\,(M) = 0$ individuates an element of $|\mathcal{O}_{\mathbb{P}^4}(3)|$ lying in $\mathcal{U}$, the open set of smooth cubics. Define $\mathfrak{M}^{in}$ as the moduli space of torsion sheaves on $\mathbb{P}^4$ with supports on smooth cubics $X \in \mathcal{U}$ and whose restrictions to X are instanton bundles. We have a commutative diagram:

$$\begin{array}{ccc} \mathcal{P}^{in} & \xrightarrow{\tau} & \mathfrak{M}^{in} \\ & \searrow^{\mathrm{Pf}} & \downarrow \rho \\ & & \mathcal{U} \end{array},$$

where $M \xmapsto{\tau} \mathrm{coker}\,(M) \otimes \mathcal{O}_{\mathbb{P}^4}(-1) \xmapsto{\rho} X = \{\mathrm{Pf}\,(M) = 0\} = \mathrm{Supp}\,(\mathrm{coker}\,(M))$. This induces the diagram:

$$\begin{array}{ccc} \mathfrak{P} & \overset{\overline{\tau}}{\dashrightarrow} & \mathfrak{M} \\ & \searrow^{\overline{\mathrm{Pf}}} & \downarrow \rho \\ & & |\mathcal{O}_{\mathbb{P}^4}(3)| \end{array}, \quad (1)$$

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where $\mathfrak{M}$ is the irreducible component of the moduli space of sheaves on $\mathbb{P}^4$ containing $\mathfrak{M}^{in}$. The morphsim $\overline{\tau}$ is birational and the rational map $\overline{\mathrm{Pf}}$ is generically a fibration with five-dimensional fibers. For a cubic $X \in \mathcal{U}$, the fiber $\widetilde{M}(X) := \overline{\mathrm{Pf}}^{-1}(X)$ is birational to the intermediate Jacobian $J(X)$ and provides a compactification of $\mathcal{M}_X^{in}$ obtained by a new method, different from the "standard" Gieseker-Maruyama one.
The space $\mathfrak{P}$, together with the morphisms $\overline{\mathrm{Pf}}$ and $\overline{\tau}$ are the main objects of study of the entire thesis.
One of the first thing that we observe is that on a smooth X when we compute locally free resolutions of sheaves in the boundary of $\mathcal{M}_X(2;0,2,0)$, skew-symmetric linear complexes still occur. But unlike the case of instantons these linear complexes are now determined by elements of $\Pi \subset \mathcal{P}$, the subvariety parametrizing matrices $M \in \mathcal{P}$ such that $\mathrm{Pf}(M) = 0$.
We pass then to the study of the morphism Pf. This map was known to be surjective onto the open set $\mathcal{U}$, here we prove that it is indeed surjective onto the entire space $|\mathcal{O}_{\mathbb{P}^4}(3)|$. In other words we show that every cubic threefold admits a Pfaffian representation, a result that has been known before only for general cubics.
The actual description of $\mathfrak{P}$ starts with the description of the locus $\mathcal{P}^{ss}$. (Semi)-stability of points in $\mathcal{P}$ can be rephrased in terms of (semi)stability of four-dimensional linear systems (*hyperwebs*) in the space $\mathbb{P}^{14}$ of skew-symmetric forms on a 6-dimensional complex vector space. This issue is dealt with by a method similar to the one adopted by Wall [Wall] for linear systems of quadrics. We show in particular that every point $M \in \mathcal{P}$ having $\mathrm{Pf}(M) \neq 0$, hence every Pfaffian representation of a cubic, is semistable and that Pfaffian representations of smooth cubics are stable.
The converse is not true, namely $\Pi \cap \mathcal{P}^{ss} \neq \varnothing$. We investigate then linear spaces $\mathbb{P}^4$ corresponding to points in Π, namely 4-dimensional linear subspaces of the Pfaffian hypersurface Pf of $\mathbb{P}^{14}$. We give a complete description of those 4-planes that are strictly stable and as a remarkable corollary of this result we get that the only strictly stable points in Π are those appearing in free resolutions of non-instanton sheaves in $\mathcal{M}_X(2;0,2,0)$ for a smooth cubic threefold X. Moreover for linear spaces $\mathbb{P}^4 \subset \mathrm{Pf}$, we classify all the irreducible components of their intersections with the Grassmannian $Gr(2,6)$ (realized as a subvariety of $\mathbb{P}^{14}$ by Plücker's embedding); using this classification we are able to produce several examples of unstable hyperwebs. Our hope is that this classification will be helpful in perspective of a more detailed description of $\mathcal{P}^{ss}$.
Finally we analyze the behavior of the birational map $\overline{\tau}$ at generic points of the components of the boundary. We are thus focused on the description of $\overline{\tau}$ in neighborhoods of points in $\mathfrak{P}$ corresponding to stable orbits contained in Π. This is done by studying the space of deformations of a sheaf in the boundary $\mathcal{B}_{X_0} := \mathcal{M}_{X_0}(2;0,2,0) \setminus \mathcal{M}_{X_0}^{in}$. Denoting by $\mathcal{B}$, $\mathcal{M}^{GM}$, $\widetilde{M}$, the union of the spaces $\mathcal{B}_X$, $\mathcal{M}_X(2;0,2,0)$, $\widetilde{M}(X)$ respectively, for X varying in a complete family of deformations of X_0, we prove that $\mathcal{B}$ is the union of two divisors $\mathcal{B}$' and $\mathcal{B}$". Furthermore at general points of $\mathcal{B}$' $\mathcal{M}^{GM}$ is the blowup of $\widetilde{M}$ along a smooth subvariety; we conjecture that the same holds for $\mathcal{B}$". The relation between $\mathfrak{P}$ and $\mathfrak{M}$ looks then similar to the one existing between the Donaldson-Uhlenbeck and the Gieseker-Maruyama moduli spaces of instantons on algebraic surfaces, where the latter is obtained from the former by a blowup. This allows us to consider $\mathfrak{P}$ as a sort of Donaldson-Uhlenbeck compactification of $\mathfrak{M}^{in}$.

Contents and main results

Now we will describe the contents of the thesis by chapters.

Chapter 1 The first chapter presents the general setting and the principal objects of study of the entire thesis. After recalling some basic notions from sheaf theory we introduce instanton bundles on a smooth cubic threefold X and their moduli space $\mathcal{M}_X^{in}$. We recollect then some of the main results obtained by Markushevich-Tikhomirov and Druel. In [MT1], the authors studied the Abel-Jacobi map $AJ_{\mathcal{H}} : \mathcal{H} \to J(X)$, where $\mathcal{H}$ is the component of the Hilbert scheme $Hilb_X^{5n}$ whose generic point is a normal elliptic

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quintic in X and $J(X)$ is the intermediate Jacobian of X, a 5-dimensional principally polarised abelian variety. They proved that $AJ_{\mathcal{H}}$ factors as:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\phi} & \mathcal{M}_X^{in} \\ & \searrow^{AJ_{\mathcal{H}}} & \downarrow \psi \\ & & J(X) \end{array}$$

The morphism ϕ maps a point $[\mathcal{C}]$ to the vector bundle $\mathcal{E}(-1)$, $\mathcal{E}$ being the rank 2 locally free sheaf obtained from the curve $\mathcal{C}$ by Serre's construction. The map ψ is defined by the second Chern class with values in the Chow group of 1-cycles modulo rational equivalence. Moreover ψ is an étale morphism of degree 1 and determines an isomorphism of $\mathcal{M}_X^{in}$ onto an open subset of $J(X)$.
The standard Gieseker-Maruyama compactification of $\mathcal{M}_X^{in}$, namely the moduli space $\mathcal{M}_X(2;0,2,0)$ of semistable sheaves of rank 2 and Chern classes $c_1 = 0,\ c_2 = 2,\ c_3 = 0$, was studied in [Dr]. In *loc.cit*, Druel proved that $\mathcal{M}_X(2;0,2,0)$ is isomorphic to the intermediate Jacobian $J(X)$ blown up along $F(X)$, the Fano surface of lines on X. Furthermore he gave a complete description of the boundary $\mathcal{B}_X := \mathcal{M}_X(2;0,2,0) \setminus \mathcal{M}_X^{in}$, showing that $\mathcal{B}_X$ is given by the union of two divisors ${\mathcal{B}_X}'$ and ${\mathcal{B}_X}''$. Sheaves belonging to ${\mathcal{B}_X}'$ are parametrized by smooth conics in X and sheaves in ${\mathcal{B}_X}''$ are parametrized by couples of (possibly coincident) lines in X.
In the subsequent sections of the chapter we introduce the main subject of our research: the moduli space $\mathfrak{P}$ of Pfaffian representations of 3-dimensional cubics. Considering a smooth cubic threefold X and looking at minimal free resolutions in $\mathbb{P}^4$ of instanton bundles on X, we describe how instantons correspond to Pfaffian representations of X. Using Druel's description of the boundary of $\mathcal{M}_X(2;0,2,0)$ we are able to compute free resolutions (in $\mathbb{P}^4$) of sheaves in $\mathcal{B}_X$. We show that though these resolutions are not linear they contain a skew-symmetric linear subcomplex of $\mathcal{O}_{\mathbb{P}^4}$-modules which is of the same shape as for instantons. More precisely every sheaf $\mathcal{F} \in \mathcal{B}_X$ fits in a short exact sequence:

$$0 \longrightarrow \mathcal{F} \longrightarrow {\mathcal{O}_X}^{\oplus 2} \longrightarrow \mathcal{G} \longrightarrow 0$$

where $\mathcal{G}$ is a one-dimensional sheaf supported either on a conic or on a couple of lines and whose minimal free resolution in $\mathbb{P}^4$ is the skew-symmetric linear complex:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} \xrightarrow{\gamma} \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 6} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \to \mathcal{G} \to 0. \qquad (2)$$

Here β is 6×6 skew-symmetric matrix such that $\mathrm{Pf}\,(\beta) = 0$, namely $\beta \in \Pi$ and $\gamma = ker(\beta)$ is such that $\alpha = \gamma^T$.
The chapter ends with an appendix on Geometric Invariant Theory.

Chapter 2 Chapter 2 is devoted to the proof of the surjectivity of the morphism Pf (and consequently of the map $\overline{\mathrm{Pf}}$ too), $\mathrm{Pf} : \mathcal{P} \dashrightarrow |\mathcal{O}_{\mathbb{P}^4}(3)|$. As a cubic X belongs to $Im(\mathrm{Pf})$ if and only it is Pfaffian, the surjectivity of Pf is deduced from the following theorem, one of the main result of the thesis:

Theorem (Theorem 2.2.1)**.** *A cubic threefold $X \subset \mathbb{P}^4$ always admits a Pfaffian representation.*

It's known that a smooth cubic threefold is Pfaffian [B2], we then focus our attention on the singular ones. The strategies that we adopt to prove the existence of a Pfaffian representation of a non-smooth cubic X differ depending on the singularities that it presents. We distinguish the following families of singular threefolds:

- Cubic threefolds that are cones;
- Non-normal cubic threefolds;

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- Normal cubic threefolds that present at most double points.

Proving the existence of Pfaffian representations of cubic threefolds that are cones reduces to the study of Pfaffian representations of lower-dimensional cubic hypersurfaces. But as every cubic hypersurface of dimension two or less is Pfaffian (see [B2] or [Ta] for a constructive proof), we deduce that the same holds for three-dimensional cubics that are cones.
Whenever X is not normal (and then its singular locus is a linear space of dimension at least 2) we provide explicitly a matrix $M_X \in \mathcal{P}$ such that X is defined by the equation $\mathrm{Pf}\,(M_X) = 0$. Proving the existence of a Pfaffian representation of X when X is normal and presents at most double points requires a more elaborate argument. In these cases we prove that X carries a rank 2 skew-symmetric Ulrich sheaf $\mathcal{E}$. Indeed if this is the case, as already mentioned earlier, the minimal free resolution in $\mathbb{P}^4$ of $\mathcal{E}$ is a skew-symmetric linear complex of length one:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} \xrightarrow{\varphi_1} \mathcal{O}_{\mathbb{P}^4}{}^{\oplus 6} \longrightarrow \mathcal{E} \longrightarrow 0.$$

The only non-vanishing differential in the resolution of $\mathcal{E}$, φ_1, provides then a Pfaffian representation of X. We show that the existence of such a sheaf $\mathcal{E}$ is guaranteed whenever there exists a nondegenerate arithmetically Gorenstein (AG) quintic elliptic curve $\mathcal{C}$ on X: $\mathcal{E}$ is indeed obtained from $\mathcal{C}$ by means of *Serre correspondence.*
Proving the following:

Theorem (Theorem 2.3.2)**.** *Let X be a normal cubic threefold that is not a cone. Then there exists a non-degenerate AG elliptic quintic curve $\mathcal{C} \subset X$.*

we then complete the proof of theorem 2.2.1.
The central part of the chapter is thus committed to showing the existence of quintic elliptic curves on normal threefolds. To this aim we adapt two different methods used in [MT1] to show the existence of such curves in the smooth case. The first method is based on a deformation argument. We first prove that a general hyperplane section S of X contains a smooth quintic elliptic curve $\mathcal{C}_0$ disjoint from the singular locus of S. Then we prove that $\mathcal{C}_0$ deforms to a nondegenerate curve $\mathcal{C}$ that is still contained in X_{sm}, the smooth locus of X. This argument works exactly as in the smooth case whenever X has a zero-dimensional singular locus $\mathrm{Sing}\,(X)$. When $\mathrm{Sing}\,(X)$ has dimension 1, a general hyperplane section S of X is a cubic surface with isolated singularities. In this case we consider

$$\phi : \tilde{S} \to S,$$

a minimal resolution of singularities of S (the smooth surface $\tilde{S}$ is a so called *weak Del Pezzo surface*) and we show that there exists a smooth curve $\tilde{\mathcal{C}_0} \subset \tilde{S}$ such that $\mathcal{C}_0 := \phi(\tilde{\mathcal{C}_0}) \subset \tilde{S}$ is a smooth elliptic quintic disjoint from the singular locus of S.
The second method is a constructive one and allows us to obtain directly a nondegenerate AG elliptic quintic $\mathcal{C} \subset X$. To apply it we first need to show the existence of a rational quartic $\Gamma \subset X$. Then we present how, starting from Γ it is possible to construct a cubic scroll Σ containing Γ and such that the curve $\mathcal{C}$, residual to Γ in $\Sigma \cap X$, is an AG elliptic quintic. In the case of a smooth X, this method applies starting from a smooth rational quartic Γ (assuming that X is non-singular, there always exists a smooth rational quartic $\Gamma \subset X$. See [MT1]). Dealing with a singular threefold X the scroll Σ is obtained from a reducible rational quartic Γ, union of a twisted cubic C and of a line l meeting C transversely at a point.
A complete classification of 3-dimensional cubic hypersurfaces was obtained by Segre in [Seg]. With the help of this classification, we analyze in detail the singularities that a normal threefold can present, depending on the components of its singular locus we choose which method to apply.

Chapter 3 In chapter 3 we study the semistable locus $\mathcal{P}^{ss}$. To start with we formulate a criterion for (semi)stability. This is done adapting to the skew-symmetric case

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the criterion proved by Wall [Wall] for (semi)stability of linear systems of symmetric forms. Every point in $\mathcal{P}$ can be interpreted as a 4-dimensional linear space $\mathbb{P}(A) \simeq \mathbb{P}^4$ contained in $\mathbb{P}(\bigwedge^2 W^*)$, the 14-dimensional projective space of skew-symmetric forms on a complex vector space W of dimension 6 (namely as a 4-dimensional linear system of skew-symmetric forms on W). A point in $\mathcal{P}$ belongs to Π if and only if the corresponding 4-plane $\mathbb{P}(A)$ is contained in the Pfaffian hypersurface $\mathrm{Pf} \subset \mathbb{P}(\bigwedge^2 W^*)$ or equivalently if and only if the generic element of the linear system of skew-symmetric forms that it determines has rank at most 4. Using this geometric characterization of points in $\mathcal{P}$, we show that our criterion can be formulated in terms of the orthogonal spaces $(\mathbb{P}^4)^\perp \subset \mathbb{P}(\bigwedge^2 W)$, this helps us characterize stable and semi-stable points. We prove the semistability of Pfaffian representations of cubics (this follows immediately from the formulation of the stability criterion) and the stability of representations of the smooth ones. This is deduced from the following result:

Theorem (Theorem 3.2.3)**.** *A 4-dimensional linear subspace $\mathbb{P}(A)$ of $\mathbb{P}(\bigwedge^2 W^*)$ such that $\mathbb{P}(A) \cap \mathrm{Pf}$ is a smooth cubic, is stable.*

These results can be rephrased as:

Theorem (Theorem 3.4.1)**.** *Every point $M \notin \Pi$ is semistable. Moreover every $M \in \mathcal{P}^{in}$ is strictly stable.*

For $M \in \mathcal{P}$, the condition $\mathrm{Pf}\,(M) \neq 0$ is thus sufficient to conclude that M is semistable. The condition is not necessary; indeed, we can find elements of Π that are even stable. We focus then our attention on points lying in Π, namely linear systems $\mathbb{P}^4$(hyperwebs) of skew-symmetric forms of generic rank 4. We prove that the only ones that are stable are exactly those that correspond to points in Π occurring in complexes of the form (2) and hence in locally free resolutions of non-instanton sheaves.

Theorem (Theorem 3.2.4)**.** *Let $\mathbb{P}(A)$ be a stable 4-dimensional linear system of skew-symmetric forms of generic rank less then or equal to four. Then $\mathbb{P}(A)$ is either $SL(W)$-equivalent to the space generated by*

$$\langle e_1 \wedge e_4 + e_2 \wedge e_5,\ e_1 \wedge e_6 + e_3 \wedge e_5,\ e_2 \wedge e_6 - e_3 \wedge e_4,\ e_1 \wedge e_2,\ e_4 \wedge e_5 \rangle$$

or $SL(W)$-equivalent to the space generated by

$$\langle e_1 \wedge e_4 + e_2 \wedge e_5,\ e_1 \wedge e_6 + e_3 \wedge e_5,\ e_2 \wedge e_6 - e_3 \wedge e_4,\ e_1 - e_5 \wedge e_2 + e_4,\ e_1 - e_5 \wedge e_3 + e_6 \rangle.$$

Furthermore in the first case $\mathbb{P}(A)$ meets the Grassmannian $\mathrm{Gr}(2, W^*)$ along a smooth conic $\mathbb{P}^2 \cap \mathrm{Gr}(2,4)$; in the second case $\mathbb{P}(A)$ intersects the $\mathrm{Gr}(2, W^*)$ along a couple of disjoint lines.

The idea of the proof is the following. We first show that a necessary condition for the stability of $\mathbb{P}(A)$ is that the intersection $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$ has dimension 1. This necessary condition implies that $\mathbb{P}(A)$ contains a plane $\mathbb{P}(B)$ of tensors of constant rank 4. As a consequence $\mathbb{P}(A)$ might be written as $\mathbb{P}(A) = \langle \mathbb{P}(B), \omega_3, \omega_4 \rangle$ with $\omega_3,\ \omega_4$ belonging to $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$. The 2-dimensional linear systems of skew-symmetric forms of constant rank 4 have been classified, up to the action of $PGL(W)$ in [MM]. The authors proved that there exist only four distinct $PGL(W)$-orbits of planes of forms of constant rank 4. We show that if $\mathbb{P}(A)$ is stable, $\mathbb{P}(B)$ can only belong to one of these orbits, so that for a suitable choice of independent vectors $e_1, \ldots, e_6$ on W^*, $\mathbb{P}(B)$ is generated by the tensors:

$$\pi_g = \langle e_1 \wedge e_4 + e_2 \wedge e_5, e_1 \wedge e_6 + e_3 \wedge e_5, e_2 \wedge e_6 - e_3 \wedge e_4 \rangle$$

We study then how to choose a couple of rank 2 tensors $\omega_3,\ \omega_4$ on $\mathrm{Gr}(2, W^*)$ in such a way that the 4-plane $\mathbb{P}(A) := \langle \mathbb{P}(B),\ \omega_3, \omega_4 \rangle$ is stable.
We first show that in order to obtain a linear system $\mathbb{P}(A)$ of generic rank 4, $\omega_3,\ \omega_4$ must belong to a rational normal scroll $S_{(2,2,2)}$ admitting a structure of a conic bundle on $\mathbb{P}(B)$. This will also imply that $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$ necessarily consists either of a conic

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or a pair of, possibly coincident, lines. Finally, we will prove that among the hyperwebs obtained in this way the only ones that are stable are those appearing in the statement of the theorem.
The study of points in Π that are strictly semistable is still in progress. Anyway we get a complete classification of the irreducible components of linear sections $\mathbb{P}(A) \cap Gr(2, W^*)$, for $\mathbb{P}(A) \simeq \mathbb{P}^4$ corresponding to a point in Π.

Theorem (Theorem 3.3.1)**.** *Let $\mathbb{P}(A) \subset \mathbb{P}(\bigwedge^2 W^*)$ be a four-dimensional linear space of skew-symmetric forms of generic rank ≤ 4. Let Y be an irreducible component of $\mathbb{P}^4 \cap \mathrm{Gr}(2, W^*)$. Then one of the following cases is realized:*

- *Y is a linear space $Y \simeq \mathbb{P}^r, 1 \leq r \leq 4$.*
- *Y is a variety of minimal degree contained in a smaller Grassmannian $\mathrm{Gr}(2, k) = \mathrm{Gr}(2, U) \subset \mathrm{Gr}(2, 6) = \mathrm{Gr}(2, W^*)$, where U is a vector subspace of W of dimension $k < 6$, and Y is a linear section of $\mathrm{Gr}(2, k)$ of one of the following types:*
  - *$Y = \mathbb{P}^d \cap \mathrm{Gr}(2, d + 2)$, a rational normal curve of degree d, $2 \leq d \leq 4$.*
  - *$Y = \mathbb{P}^{d+1} \cap \mathrm{Gr}(2, d + 2)$, a surface of degree $d = 2, 3$.*
  - *$Y = \mathbb{P}^4 \cap \mathrm{Gr}(2, 4)$, a three-dimensional quadric hypersurface in $\Delta = \mathbb{P}^4$.*
- *Y is an elliptic quintic curve, the image of $\mathbb{P}^4 \cap \mathrm{Gr}(2, 5)$ under some linear embedding $\mathrm{Gr}(2, 5) \hookrightarrow \mathrm{Gr}(2, W^*)$.*

This classification allows us to provide several example of unstable points and we hope that it will be useful for a more accurate future description of $\mathcal{P}^{ss}$.

Chapter 4 In the last chapter we use all our results to study the local behavior of the rational map $\bar{\tau} : \mathfrak{P} \dashrightarrow \mathfrak{M}$. We start considering a sheaf $\mathcal{F}$ on a smooth cubic threefold X corresponding to a point in $\mathcal{B}_X$, the boundary of the moduli space $\mathcal{M}_X(2; 0, 2, 0)$, and we illustrate some properties of its minimal free resolution. It follows from the discussion held in Chapter 1 that $\mathcal{F}$ always admits a minimal free resolution of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} \xrightarrow{G} \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 6} \xrightarrow{B} \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} \to \mathcal{F} \to 0, \qquad (3)$$

in which $B = (\beta'|\beta)$ is a 6-by-8 matrix obtained by concatenation of a 6×2 matrix of quadratic forms β' with a 6×6 skew-symmetric matrix β of linear forms satisfying $\mathrm{Pf}\,(\beta) = 0$. Given a 5-dimensional complex vector space V, the matrix β defines a point in $\bigwedge^2 W^* \otimes V^*$ belonging to $\mathcal{Z}$, where $\mathcal{Z}$ is defined as the locus of matrices $M \in \bigwedge^2 W^* \otimes V^*$ such that $\mathrm{Pf}\,(M) = 0$. In other words, β defines an hyperweb of skew-symmetric forms of generic rank 4 and we show that its intersection with the Grassmanniann $\mathrm{Gr}(2, W^*)$ is a curve C equal to $\mathrm{Sing}\,(\mathcal{F})$. Moreover, applying Theorem 3.2.4 we see that matrices β obtained in this way individuate all the stable points (with respect to the $SL(W)$-action) lying in $\mathcal{Z}$; this implies in particular that their orbits correspond to points in $\mathfrak{P}$.
Given a matrix β as above, we study, in the first part of the chapter, the behavior of $\mathcal{Z}$, $\bigwedge^2 W^* \otimes V^*$ and $\mathfrak{P}$ at β. As the boundary $\mathcal{B}_X$ of $\mathcal{M}_X(2; 0, 2, 0)$ is given by the union of two divisors $\mathcal{B}'_X$, $\mathcal{B}''_X$, β might belong to two different families of elements in $\mathcal{Z}$. We prove that whenever β is obtained from the minimal free resolution of a sheaf $\mathcal{F}$ corresponding to a point $[\mathcal{F}] \in \mathcal{B}'_X$ (resp. $[\mathcal{F}] \in \mathcal{B}''_X$), it belongs to a 47-dimensional component $\mathcal{Z}'$ (resp. 48-dimensional component $\mathcal{Z}''$) of $\mathcal{Z}$, smooth at β. Therefore, the image of its orbit in $\mathfrak{P}$, is a smooth point of a 11-dimensional subvariety $\mathfrak{B}'$, quotient of $\mathcal{Z}'$, (resp. a 12-dimensional subvariety $\mathfrak{B}''$, quotient of $\mathcal{Z}''$,) of $\mathfrak{P}$. However we will show that $\mathfrak{P}$ is smooth at the orbit of β only when $\beta \in \mathcal{Z}'$; this due to the fact that if $\beta \in \mathcal{Z}''$, its stabilizer does not consist only of $\{\pm Id_6\}$.

In the second part of the chapter we study the behavior of a minimal free resolution $\mathcal{R}^{\bullet}$ of $\mathcal{F}$ of the form (3) under deformation. We consider then a polydisk in $\mathbb{C}^N$ and a sheaf $\mathcal{F}_\Delta$ on $\mathbb{P}^4 \times \Delta$ flat over Δ, such that $\forall\, s \in \Delta$, $\mathcal{F}_s$ (the restriction of $\mathcal{F}_\Delta$ to $\mathbb{P}^4 \times s$)

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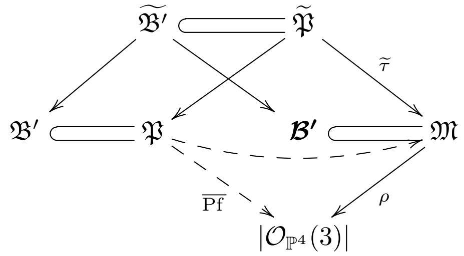
corresponds to a point $[\mathcal{F}_s] \in \mathfrak{M}$ and $\mathcal{F}_0 = \mathcal{F}$. $\mathcal{R}^\bullet$ lifts to a resolution $\mathcal{R}^\bullet_\Delta$ of $\mathcal{F}_\Delta$ of the form:

$$0 \to \mathcal{O}(-3)^{\oplus 2}_{\mathbb{P}^4\times\Delta} \xrightarrow{G(s)} \mathcal{O}_{\mathbb{P}^4\times\Delta}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4\times\Delta}(-2)^{\oplus 6} \xrightarrow{B(s)} \mathcal{O}_{\mathbb{P}^4\times\Delta}(-1)^{\oplus 6} \to 0,$$

$\mathcal{R}^\bullet_\Delta$ is a complex of sheaves on $\mathbb{P}^4 \times \Delta$, its differentials $G(s)$ and $B(s)$ are matrices with entries in $\mathbb{C}\{s\}[X_0, \dots, X_4]$, where $\mathbb{C}\{s\}$ denotes the ring of germs of analytic functions in s at 0. More precisely $B(s)$ presents a block structure $B(s) = (\beta'(s)|\beta(s))$ where $\beta(s)$ is a 6×6 matrix whose entries are elements in $\mathbb{C}\{s\}[X_0, \dots, X_4]$ linear in the variables $X_0, \dots, X_4$. Our aim is to determine when we can deform $\mathcal{F}$ in such a way that $\beta(s)$ is skew-symmetric $\forall\, s \in \Delta$. We prove that such a deformation is always possible whenever the type of the singularity of $\mathcal{F}$ is preserved (Lemma 4.2.1) and whenever $\mathcal{F}$ is a sheaf corresponding to a point in $\mathcal{B}'_X$ and deforming to a sheaf $\mathcal{F}_\Delta$ such that for generic $s \in \Delta$, $\mathcal{F}_s$ is an instanton bundle on a smooth cubic threefold (Lemma 4.2.2). Using this results we are able to describe the local behavior of the diagram (1) in a neighborhood of a generic point $\beta_0 \in \mathcal{Z}'$. We prove the following:

Proposition (Prop. 4.2.4)**.** Let $\beta_0 \in \mathcal{Z}'$ be as above, and consider the diagram (1) in a neighborhood of the orbit $[\beta_0] \in \mathfrak{B}'$, $[\beta_0] = GL(W) \cdot \beta_0$. Then the rational map $\overline{\tau}$ is equivalent to a blowup with center $\mathfrak{B}'$ near $[\beta_0]$. More precisely: let $\widetilde{\mathfrak{P}}$ denote the blowup of $\mathfrak{P}$ with center $\mathfrak{B}'$ and $\widetilde{\mathfrak{B}'}$ its exceptional divisor. Then, in a neighborhood of $[\beta_0]$, (1) can be completed to the diagram



(4)

in which the arrows $\widetilde{\tau}$ and $\widetilde{\tau}|_{\widetilde{\mathfrak{B}'}}$ are isomorphisms.

Here $\boldsymbol{\mathcal{B}'}$ is the divisor of $\mathfrak{M}$ whose generic point is a sheaf $\mathcal{F}$ on a smooth cubic threefold X and corresponding to a point $[\mathcal{F}] \in \mathcal{B}'_X$. Calling now $\boldsymbol{\mathcal{B}''}$ the divisor of $\mathfrak{M}$ whose generic point is a sheaf $\mathcal{F}$ supported on smooth cubic threefolds X and such that $[\mathcal{F}] \in \mathcal{B}''_X$, we conjecture that Proposition 4.2.4 extends literally to $\boldsymbol{\mathcal{B}''}$ and $\mathfrak{B}'' \subset \mathfrak{P}$.

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Chapter 1

Preliminaries

Introduction

In this chapter we present the general framework of the entire thesis together with its main motivations and objectives. We start considering a smooth cubic hypersurface $X \subset \mathbb{P}^4$ and ${\mathcal{M}_X}^{in}$ the moduli space of *instanton bundles*, namely stable vector bundles having rank 2 and Chern classes $c_1 = 0,\ c_2 = 2$. The study of this moduli space relates different types of objects: the sheaves on X, the curves on X and the intermediate Jacobian $J(X)$ of X. Markushevich and Tikhomirov proved indeed in [MT1] that the Abel-Jacobi map $AJ_{\mathcal{H}} : \mathcal{H} \to J(X)$, where $\mathcal{H}$ is the component of the Hilbert scheme Hilb_X^{5n} whose generic point is a normal elliptic quintic in X, factors as:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\phi} & {\mathcal{M}_X}^{in} \\ & {\scriptstyle AJ_{\mathcal{H}}} \searrow & \downarrow {\scriptstyle \psi} \\ & & J(X) \end{array} \qquad (1.1)$$

The morphism ϕ maps a point $[\mathcal{C}]$ to the vector bundle $\mathcal{E}(-1)$, $\mathcal{E}$ being the rank 2 locally free sheaf obtained from the curve $\mathcal{C}$ by Serre's construction. The map ψ is an étale morphism of degree 1 and determines an isomorphism of ${\mathcal{M}_X}^{in}$ onto an open subset of $J(X)$. In [Dr], Druel constructed the "standard" Gieseker-Maruyama compactification of ${\mathcal{M}_X}^{in}$, that is the moduli space $\mathcal{M}_X(2; 0, 2, 0)$ of semistable sheaves of rank 2 and Chern classes $c_1 = 0,\ c_2 = 2,\ c_3 = 0$. He proved that moreover $\mathcal{M}_X(2; 0, 2, 0)$ is isomorphic to the intermediate Jacobian $J(X)$ blown up along $F(X)$, the Fano surface of lines on X. From these facts it is natural to ask whether it is possible to find a compactification of ${\mathcal{M}_X}^{in}$ that is actually isomorphic to $J(X)$. This question leads us to look for a new moduli space associated to cubic threefolds. Our aim is also to obtain a compactification of the moduli of stable bundles that can be readily extended to singular cubics. Indeed in these cases the Gieseker-Maruyama moduli space presents a limit: its boundary becomes too troublesome to handle. The starting point of our construction is the following: for every point $[\mathcal{F}] \in {\mathcal{M}_X}^{in}$, the bundle $\mathcal{F}$ admits a minimal free resolution in $\mathbb{P}^4$ of the form:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 6} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where M is a 6×6 skew-symmetric matrix whose entries are linear forms on $\mathbb{P}^4$ and such that X is defined by the equation $\mathrm{Pf}\,(M) = 0$. Hence instanton bundles are associated to *Pfaffian representations* of the cubic X. Consequently we construct $\mathfrak{P}$, the compact moduli space of Pfaffian representations of cubic threefolds. $\mathfrak{P}$ is obtained by means of Geometric Invariant Theory as follows. We consider $\mathcal{P}$ the 74-dimensional projective space of skew-symmetric matrices of linear forms of size 6; the group $SL(6, \mathbb{C})$ acts on $\mathcal{P}$ by conjugation and the moduli $\mathfrak{P}$ is then defined as the GIT quotient

$$\mathfrak{P} = \mathcal{P}^{ss} /\!\!/ SL(6, \mathbb{C})$$

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where $\mathcal{P}^{ss}$ is the open of semistable matrices.

1.1 Moduli of instanton bundles on cubic threefolds

1.1.1 Generalities and known results

Stable and semistable sheaves

Let X be a projective scheme of dimension n over $\mathbb{C}$. We fix $\mathcal{O}_X(1)$ an ample line bundle on X. (Such an ample line bundle is referred to as a *polarization* of X.) Throughout the rest of the section we fix a coherent sheaf $\mathcal{F}$ on X. We recall that the *Euler characteristic* of $\mathcal{F}$ is defined as $\chi(\mathcal{F}) := \sum_i (-1)^i h^i(X, \mathcal{F})$.

Definition 1.1. The *Hilbert polynomial* $P_{\mathcal{F}}$ of $\mathcal{F}$ is defined by:

$$P_{\mathcal{F}}(m) = \chi(\mathcal{F}(m)).$$

The Hilbert polynomial of $\mathcal{F}$ can be uniquely written in the form:

$$P_{\mathcal{F}}(m) = \sum_{i=0}^{\dim(\mathcal{F})} \alpha_i(\mathcal{F}) \frac{m^i}{i!}$$

where the coefficients $\alpha_0(\mathcal{F}), \dots, \alpha_{\dim(\mathcal{F})}(\mathcal{F})$ are rational numbers. (We recall that the dimension of $\mathcal{F}$ is defined as the dimension of its support).

Definition 1.2. Suppose that $\mathcal{F}$ has dimension $n = \dim(X)$. We call:

$$p_{\mathcal{F}}(m) := \frac{P_{\mathcal{F}}(m)}{\alpha_n(\mathcal{F})}$$

the *reduced Hilbert polynomial of* $\mathcal{F}$.

Definition 1.3. If $\mathcal{F}$ has dimension $n = \dim(X)$, then the *rank* $\mathrm{rk}\,(\mathcal{F})$ of $\mathcal{F}$ is defined by:

$$\mathrm{rk}\,(\mathcal{F}) = \frac{\alpha_n(\mathcal{F})}{\alpha_n(\mathcal{O}_X)}$$

and the *degree* of $\mathcal{F}$ is defined by:

$$\deg(\mathcal{F}) = \alpha_{n-1}(\mathcal{F}) - \mathrm{rk}\,(\mathcal{F})\alpha_{n-1}(\mathcal{O}_X).$$

Remark 1. If X is a smooth projective variety, by Hirzebruch-Riemann-Roch formula we have that the degree of $\mathcal{F}$ reduces to the "usual" definition of degree, namely $\deg(\mathcal{F}) = c_1(\mathcal{F})c_1(\mathcal{O}_X(1))^{n-1}$.

Definition 1.4. If $\mathcal{F}$ has dimension $n = \dim(X)$, its *slope* $\mu(\mathcal{F})$ is defined as the ratio:

$$\mu(\mathcal{F}) = \frac{\deg(\mathcal{F})}{\mathrm{rk}\,(\mathcal{F})}.$$

The Hilbert polynomial and the slope provide two notions of stability for torsion-free coherent sheaves.

Definition 1.5. Suppose that $\mathcal{F}$ has dimension $n = \dim(X)$ and that $\mathcal{F}$ is torsion free. We say that $\mathcal{F}$ is *semistable* (resp. *stable*) if for any proper subsheaf $\mathcal{F}' \subset \mathcal{F}$ we have $p_{\mathcal{F}'} \leq p_{\mathcal{F}}$ (resp. $p_{\mathcal{F}'} < p_{\mathcal{F}}$).

Remember that by $p_{\mathcal{F}'} \leq p_{\mathcal{F}}$ (resp. $p_{\mathcal{F}'} < p_{\mathcal{F}}$), we mean that $p_{\mathcal{F}'}(m) \leq p_{\mathcal{F}}(m)$ (resp. $p_{\mathcal{F}'}(m) < p_{\mathcal{F}}(m)$) for $m >> 0$.

Definition 1.6. Suppose that $\mathcal{F}$ has dimension $n = \dim(X)$ and that $\mathcal{F}$ is torsion free. We say that $\mathcal{F}$ is *μ-semistable* (resp. *μ-stable*) if for any proper subsheaf $\mathcal{F}' \subset \mathcal{F}$ of rank $0 < \mathrm{rk}\,(\mathcal{F}') < \mathrm{rk}\,(\mathcal{F})$, we have $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$ (resp. $\mu(\mathcal{F}') < \mu(\mathcal{F})$).

Remark 2. We can easily prove from the definitions that we have the following chain of implications:

$$\mathcal{F}\ \mu-\text{stable} \implies \mathcal{F}\ \text{stable} \implies \mathcal{F}\ \text{semi-stable} \implies \mathcal{F}\ \mu-\text{semi-stable}.$$

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The intermediate Jacobian and the Abel-Jacobi map

We recall here some facts about cubic threefolds. Throughout the rest of the section X will denote a smooth cubic hypersurfaces in $\mathbb{P}^4$. X is an example of a Fano variety, that is its anticanonical sheaf $\omega_X^{-1} \simeq \mathcal{O}_X(2)$ is ample, that satisfies the following properties:

$$\begin{gathered} h^i(\mathcal{O}_X(k)) = 0, \text{ for } i = 1, 2, \ k \in \mathbb{Z}, \\ h^{i,0} = h^{0,i} = 0 \text{ for } i > 0, \ \ h^{1,2} = h^{2,1} = 5, \end{gathered} \tag{1.2}$$

$$\begin{aligned} \mathrm{Pic}(X) = A_2(X) =& H^2(X, \mathbb{Z}) = \mathbb{Z} \cdot [\mathcal{O}_X(1)], \\ B_1(X) =& H^4(X, \mathbb{Z}) = \mathbb{Z} \cdot l, \\ A_0(X) =& H^6(X, \mathbb{Z}) = \mathbb{Z} \cdot pt. \end{aligned} \tag{1.3}$$

Here A_i and B_i denote the Chow groups of i-dimensional cycles modulo rational and algebraic equivalence, respectively; l is the class of a line and pt is the class of a point.
Because of these equalities, given a coherent sheaf $\mathcal{F}$ on X, we can identify its Chern classes $c_i(\mathcal{F}) \in H^{2i}(X, \mathbb{Z}), \ i = 1, 2, 3$ with integer numbers.
We define the *intermediate Jacobian* of a cubic threefold X by:

$$J(X) = (F^2(H^3(X, \mathbb{C}))^* / \mathrm{im}\,(H_3(X, \mathbb{Z})),$$

where $F^2(H^3(X, \mathbb{C}) = H^{3,0} + H^{2,1}$ and $H_3(X, \mathbb{Z})$ is mapped to $(F^2(H^3(X, \mathbb{C}))^*$ by integration over cycles. From (1.2) we have $h^{3,0} = 0, \ h^{2,1} = 5$, so that the intermediate Jacobian $J(X)$ is a 5 dimensional abelian variety equal to $J(X) = (H^{2,1}(X))^* / \mathrm{im}\,(H_3(X, \mathbb{Z}))$. From the vanishing of $H^{3,0}(X)$ we also deduce, by the Hodge index theorem, that the intersection form on $H_3(X)$ defines a principal polarization on $J(X)$.
Consider now $Hom_1(X)$ the group of 1-cycles homologous to 0. the *Abel-Jacobi* map $AJ : Hom_1(X) \to J(X)$ is defined as follows. Given $Z \in Hom_1(X)$, there exists then an element $Y \in H_3(X, \mathbb{Z})$ such that $Z = \partial Y$. $AJ(Z)$ is given by

$$\left[\omega \mapsto \int_Y \omega\right], \quad \mathrm{mod}\ \mathrm{im}\,(H_3(X, \mathbb{Z})).$$

If $\mathcal{A}$ is a variety parameterizing a family of 1-cycles $\{Z_a\}_{a \in \mathcal{A}}$, whenever we fix a reference point $a_0 \in \mathcal{A}$, we get a set-theoretic map $AJ_{\mathcal{A}} : \mathcal{A} \to J(X), \ b \mapsto AJ(Z_a - Z_{a_0})$. This map is analytical whenever $\mathcal{A}$ is smooth. A particularly interesting case occurs when $\mathcal{A} = F(X)$, the Fano surface of lines on X. In this circumstance the map $AJ_{F(X)} : F(X) \to J(X), \ t \mapsto AJ(l_t - l_s)$, where $s \in F(X)$ is a fixed point, is a closed embedding ([Tyu]). Moreover $F(X)$ is a smooth surface, hence the morphism $AJ_{F(X)} : F(X) \to J(X)$ is analytical. As a consequence $AJ_{F(X)}$ factors through a morphism $\widetilde{AJ}_{F(X)} : \mathrm{Alb}(F(X)) \to J(X)$, where $\mathrm{Alb}(F(X))$ is the Albanese variety of $F(X)$. An accurate description of the Abel-Jacobi map $AJ_{F(X)} : F(X) \to J(X)$ is presented in [CG], where the authors proved that the induced map $\widetilde{AJ}_{F(X)}$ determines an isomorphism:

$$\widetilde{AJ}_{F(X)} : \mathrm{Alb}(F(X)) \xrightarrow{\sim} J(X).$$

The intermediate Jacobian of a smooth cubic threefold can be characterized in another useful way. Indeed, according to [Mu], $J(X)$ is isomorphic to $A_1(0)_X$, the group of algebraic 1-cycles of degree 0 modulo rational equivalence. We see then that for any non negative integer d, denoting by $A_1(d)_X$ the group of 1-cycles of degree d modulo rational equivalence, whenever we fix a class $[Z_0] \in A_1(d)_X$, we induce an isomorphism $A_1(d)_X \xrightarrow{\sim} J(X), \ [Z] \to [Z - Z_0]$ (this is just the Abel- Jacobi map on algebraic cycles of degree d). From now on we will denote by $J^d(X) \simeq A_1(d)_X$ the translate of $J(X)$ parameterizing 1-cycles of degree d on X.

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1.1.2 Instanton bundles on smooth cubic threefolds

Definition 1.7. An *instanton* bundle $\mathcal{F}$ on X is a stable rank 2 vector bundle with Chern classes $c_1(\mathcal{F}) = 0,\ c_2(\mathcal{F}) = 2$.

On a smooth cubic threefold X, as $\text{Pic}(X) = \mathbb{Z}$, a vector bundle $\mathcal{F}$ with $c_1(\mathcal{F}) = 0$ is stable if and only if $h^0(\mathcal{F}) = 0$ (cf. [H1], Lemma 3.1). Consequently, if $\mathcal{F}$ is a stable vector bundle with first Chern class zero, $h^0(\mathcal{F}(i)) = 0,\ \forall\, i \leq 0$. We consider now $\mathcal{M}_X^{in}$, the moduli space of instanton bundles of X. $\mathcal{M}_X^{in}$ is a quasi-projective variety, its Gieseker-Maruyama compactification $\mathcal{M}_X(2;0,2,0)$, parameterizing classes of semi-stable sheaves with $c_1 = 0,\ c_2 = 2,\ c_3 = 0$, has been described by Druel in [Dr]. The author proved the following:

Theorem 1.1.1. *$\mathcal{M}_X(2;0,2,0)$ is a smooth scheme of dimension 5 isomorphic to the blowup of $J(X)$ along the Fano surface.*

This theorem states the existence of a birational map $\mathcal{M}_X(2;0,2,0) \dashrightarrow J(X)$; in the upcoming sections we will give an outline of its construction and of some of its properties. To start with we see that there is a natural map $\psi : {\mathcal{M}_X}^{in} \longrightarrow J(X)$ that can be described as follows. Consider $\mathfrak{c}_2$, the second Chern class with values in $A_1(X)$; given $[\mathcal{F}] \in {\mathcal{M}_X}^{in}$, $\mathfrak{c}_2(\mathcal{F})$ belongs to $A_1(2)_X \simeq J^2(X)$ and $\psi([\mathcal{F}])$ is then defined as $\psi([\mathcal{F}]) = \mathfrak{c}_2(\mathcal{F}) - [Z_0]$, where Z_0 is a fixed element in $A_1(2)_X$. From the results of [MT1] or [IM] we have:

Proposition 1.1.2. *The morphism ψ is a quasi-finite étale morphism of degree 1, thus ψ induces an isomorphism of ${\mathcal{M}_X}^{in}$ onto an open subset of $J(X)$.*

Vector bundles from elliptic quintics

It is possible to give a characterization of the map ψ, or equivalently of the map $\mathfrak{c}_2$, by means of certain curves on X associated to instanton bundles. Before doing this we recollect some known results about instantons. Consider $[\mathcal{F}] \in {\mathcal{M}_X}^{in}$ and denote by $\mathcal{E}$ the twisted bundle $\mathcal{E} := \mathcal{F}(1)$. $\mathcal{E}$ is a rank 2 vector bundle with Chern classes $c_1(\mathcal{E}) = 2,\ c_2(\mathcal{E}) = 5$. We have the following:

Proposition 1.1.3 ([B1], Prop. 1.1)**.** *Let $\mathcal{F}$ be a sheaf on X with $\mathcal{F} \in {\mathcal{M}_X}^{in}$. Then:*

- $h^1(\mathcal{F}(n)) = h^2(\mathcal{F}(n)) = 0,\ \forall\ n \in \mathbb{Z}$;
- $\mathcal{F}(1)$ *is spanned by its global sections.*

Proposition 1.1.4. *Let $\mathcal{F}$ be a sheaf on X with $\mathcal{F} \in {\mathcal{M}_X}^{in}$. Then:*

$$h^0(\mathcal{F}(1)) = 6,\ \ h^i(\mathcal{F}(1)) = 0\ \forall\, i > 0,\ \ h^i(\mathcal{F}(-1)) = 0\ \forall\, i \in \mathbb{Z}.$$

See [MT1] Lemma 5.1.
Take now an element $s \in \mathbb{P}(H^0(X,\mathcal{E})) \simeq \mathbb{P}^5$. Since $\mathcal{E}$ is a rank 2 vector bundle on a smooth 3 dimensional variety, the zero scheme of s is a locally complete intersection curve $\mathcal{C}$ on X. s gives a morphism $\mathcal{O}_X \to \mathcal{E}$ that fits into a short exact sequence:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{\mathcal{C}}(2) \longrightarrow 0; \tag{1.4}$$

where $\mathcal{I}_{\mathcal{C}}$ is the ideal sheaf of $\mathcal{C}$ in X. From this short exact sequence we compute that:

1. $h^0(X,\mathcal{I}_{\mathcal{C}}(1)) = h^0(X,\mathcal{F}) = 0$ and $h^0(X,\mathcal{I}_{\mathcal{C}}) = h^0(X,\mathcal{F}(-1)) = 0$ (from the stability assumption on $\mathcal{F}$). The first equality implies that the curve $\mathcal{C}$ is non degenerate, namely not contained in a hyperplane.

2. $h^1(X,\mathcal{I}_{\mathcal{C}}) = h^1(X,\mathcal{F}(-1)) = 0$ (due to prop. 1.1.3) and $h^2(\mathcal{I}_{\mathcal{C}}) = h^3(\mathcal{O}_X(-2)) = 1$. These equalities imply that $h^0(\mathcal{O}_{\mathcal{C}}) = h^0(\mathcal{O}_X) = 1$ and $h^1(\mathcal{O}_C) = h^2(\mathcal{I}_{\mathcal{C}}) = 1$ hence $\mathcal{C}$ is a curve of (arithmetic) genus 1.

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The curve $\mathcal{C}$ has degree $c_2(\mathcal{E}) = 5$. Note that moreover, we know from proposition 1.1.3 that $\mathcal{E}$ is generated by its global sections. Hence, for s general, the curve $\mathcal{C}$ is smooth. If this is the case we get from 2 that $\mathcal{C}$ is a curve of geometric genus one so that $\omega_{\mathcal{C}} \simeq \mathcal{O}_{\mathcal{C}}$.

Conversely, whenever $\mathcal{C} \subset X$ is a non-degenerate locally complete intersection curve with trivial canonical sheaf $\omega_{\mathcal{C}} \simeq \mathcal{O}_{\mathcal{C}}$ and $h^0(\mathcal{O}_{\mathcal{C}}) = 1$, we have isomorphisms:

$$\operatorname{Ext}^1(\mathcal{I}_{\mathcal{C}}(2), \mathcal{O}_X) \xrightarrow{\sim} \operatorname{Ext}^1(\mathcal{I}_{\mathcal{C}}, \omega_X) \xrightarrow{\sim} \operatorname{Ext}^2(\mathcal{O}_{\mathcal{C}}, \omega_X) \xrightarrow{\sim} H^0(\mathcal{C}, \omega_{\mathcal{C}}) \xrightarrow{\sim} H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}).$$

Therefore, up to isomorphism, there is a unique non-trivial extension:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{\mathcal{C}}(2) \longrightarrow 0. \tag{1.5}$$

Because of the fact that a generator of $H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ is nowhere vanishing, the sheaf $\mathcal{E}$ is locally free. This is due to the following lemma (cf. [MT1] Lemma 2.3)

Lemma 1.1.5. *Let X be a nonsingular variety, $\mathcal{C}$ a locally complete intersection subvariety of X of codimension 2, $\mathcal{L}$ a line bundle on X and:*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I} \otimes \mathcal{L} \longrightarrow 0 \tag{1.6}$$

an extension given by a class $e \in \operatorname{Ext}^1(\mathcal{I}_{\mathcal{C}} \otimes \mathcal{L}, \mathcal{O}_X)$. Then $\mathcal{E}$ is locally free if and only if the image of e in $H^0(\mathcal{C}, \mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{I}_{\mathcal{C}} \otimes \mathcal{L}, \mathcal{O}_X))$ generates the stalk of $\mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{I}_{\mathcal{C}} \otimes \mathcal{L}, \mathcal{O}_X)$ at every point of $\mathcal{C}$.

From (1.5) we compute that $\mathcal{F} := \mathcal{E}(-1)$ is a rank 2 vector bundle with Chern classes $c_1 = 0, 1$ $c_2 = 2$; the non-degeneracy assumption on $\mathcal{C}$ implies the stability of $\mathcal{F}$ since $H^0(X, \mathcal{F}) \simeq H^0(X, \mathcal{I}_{\mathcal{C}}(1)) = 0$.
Summing up we have the following:

Proposition 1.1.6. *[[B1], Prop.1.4] Let $\mathcal{F}$ be a rank 2 vector bundle corresponding to a point $[\mathcal{F}] \in \mathcal{M}_X^{in}$. The scheme of zeros of a global section $s \in H^0(X, \mathcal{F}(1))$ is a non-degenerate locally complete intersection curve $\mathcal{C} \subset X$ with trivial canonical sheaf and $H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \simeq \mathbb{C}$. Conversely, given a curve $\mathcal{C}$ satisfying the aforementioned assumptions, there exists a rank 2 vector bundle $\mathcal{F}$ on X such that $[\mathcal{F}] \in \mathcal{M}_X{}^{in}$ and a section $s \in H^0(X, \mathcal{F}(1))$ whose scheme of zeros is $\mathcal{C}$.*

Remark 3. The construction that we have just described, relating rank 2 sheaves on X to codimension 2 subschemes of X, is the so called *Serre's construction.* We will give a more detailed account of this construction in Chapter 2 where we will also explain how it can be generalized to the case where X is a singular cubic threefold (in this case we are still able to construct rank 2 vector bundles on X, but sometimes they will be obtained from curves that are not necessarily locally complete intersections but that satisfy weaker assumptions).

For an instanton bundle $\mathcal{F}$ on X we thus see that $\mathfrak{c}_2(\mathcal{F}) \in J^2(X)$ satisfies the following equalities:

$$\mathfrak{c}_2(\mathcal{F}) = \mathfrak{c}_2(\mathcal{F}(1)) - h^2 = [\mathcal{C} - h^2],$$

where h^2 is the class of a plane cubic curve isomorphic to $\mathbb{P}^2 \cap X$ and $\mathcal{C}$ is the zero locus of an element in $H^0(X, \mathcal{F}(1))$.

A factorization of the Abel-Jacobi map

Let $\operatorname{Hilb}_X^{5n}$ be the Hilbert scheme of elliptic quintic curves contained in X, we define $\mathcal{H}$ the locally closed subscheme of $\operatorname{Hilb}_X^{5n}$:

$$\mathcal{H} = \left\{ \begin{array}{l} [\mathcal{C}] \in \operatorname{Hilb}_X^{5n} | \ (i) \ \mathcal{C} \text{ is a locally complete intersection of pure dimension 1,} \\ (ii) \ \omega_{\mathcal{C}} \simeq \mathcal{O}_{\mathcal{C}}, \ (iii) \ h^1(\mathcal{I}_{\mathcal{C}}) = h^0(\mathcal{I}_{\mathcal{C}}(1)) = 0, \ (iv) \ h^1(\mathcal{I}_{\mathcal{C}}(2)) = h^2(\mathcal{I}_{\mathcal{C}}(2)) = 0 \end{array} \right\}$$

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Curves $\mathcal{C} \subset X$ corresponding to points in $\mathcal{H}$ are usually referred to as *normal elliptic quintics*. Note that if $\mathcal{C}$ is a normal elliptic quintic, as $h^1(\mathcal{I}_\mathcal{C}) = h^0(\mathcal{I}_\mathcal{C}(1)) = 0$, we get $h^0(\mathcal{O}_\mathcal{C}) = 1$. We see from proposition 1.1.6, that Serre's construction establishes a well defined morphism:

$$\phi : \mathcal{H} \longrightarrow {\mathcal{M}_X}^{in}, \quad [\mathcal{C}] \mapsto \mathcal{F} := \mathcal{E}(-1),$$

where $\mathcal{E}$ is defined by the short exact sequence (1.5) determined by a generator of the vector space $H^0(\mathcal{C}, \omega_\mathcal{C}) \simeq \mathrm{Ext}^1(\mathcal{I}_\mathcal{C}(2), \mathcal{O}_X) \simeq \mathbb{C}$.

Proposition 1.1.7 ([MT1] Prop. 5.4)**.** *The morphism $\phi : \mathcal{H} \to {\mathcal{M}_X}^{in}$ is smooth and projective and all its fibres are 5-dimensional projective spaces. More precisely the fibre of ϕ at $[\mathcal{F}] \in {\mathcal{M}_X}^{in}$ is $\mathbb{P}(H^0(X, \mathcal{F}(1))) \simeq \mathbb{P}^5$.*

Corollary 1.1.8 ([MT1] Prop. 5.5)**.** *$\mathcal{H}$, ${\mathcal{M}_X}^{in}$ are smooth of dimension 10, resp. 5; moreover* Hilb_X^{5n}*, $\mathcal{M}_X(2;0,2,0)$ are smooth at the points of $\mathcal{H}$, resp. ${\mathcal{M}_X}^{in}$.*

We look at the Abel Jacobi map:

$$AJ_\mathcal{H} : \mathcal{H} \longrightarrow J(X)$$

Theorem 1.1.9 ([MT1], Thm. 5.6)**.** *$AJ_\mathcal{H}$ is smooth and every fiber is a disjoint union of 5-dimensional projective spaces. Moreover $AJ_\mathcal{H}$ factors as:*

$$AJ_\mathcal{H} : \mathcal{H} \xrightarrow{\phi} {\mathcal{M}_X}^{in} \xrightarrow{\psi} J(X).$$

We are now going to study how $\mathfrak{c}_2$ behaves on the boundary of $\mathcal{M}_X(2;0,2,0)$.

Boundary of the moduli space

We consider $\mathcal{B}_X := \mathcal{M}_X(2;0,2,0) \setminus {\mathcal{M}_X}^{in}$. In this boundary we find torsion free sheaves that are not locally free. A sheaf $\mathcal{F}$ corresponding to a point $[\mathcal{F}] \in \mathcal{B}_X$ is of one of the following types.

1. Let C be a smooth conic contained in X. We consider the degree one line bundle $\mathcal{O}_C(1pt)$ on C. This line bundle has $h^0(\mathcal{O}_C(1pt)) = 2$ and it is generated by its global sections. Hence the evaluation morphism $H^0(C, \mathcal{O}_C(1pt)) \otimes \mathcal{O}_X \to \mathcal{O}_C(1pt)$ is surjective and its kernel $\mathcal{F}_C$ is a torsion free sheaf on X fitting in a short exact sequence:
$$0 \longrightarrow \mathcal{F}_C \longrightarrow {\mathcal{O}_X}^{\oplus 2} \longrightarrow \mathcal{O}_C(1pt) \longrightarrow 0. \tag{1.7}$$
$\mathcal{F}_C$ is a *stable* torsion free sheaf, from (1.7) we compute that $\mathcal{F}_C$ has rank 2 and that its Chern classes are $c_1 = 0,\ c_2 = 2,\ c_3 = 0$.

2. Let $l_1,\ l_2$ be a couple of (possibly coincident lines) on X. The sheaf $\mathcal{F}_{l_1,l_2} = \mathcal{I}_{l_1} \oplus \mathcal{I}_{l_2}$ is a rank 2 torsion free sheaf with $c_1 = 0,\ c_2 = 2,\ c_3 = 0$ that is clearly *semi-stable* but not stable ($p_{\mathcal{I}_{l_i}} = p_{\mathcal{I}_{l_1} \oplus \mathcal{I}_{l_2}}$).

Sheaves of type (i) are parametrized by smooth conics on X (a 4-dimensional family), hence they give a divisor $\mathcal{B}'_X$ of $\mathcal{M}_X(2;0,2,0)$. Sheaves of type (ii) are parametrized by the symmetric square of $F(X)$ and thus they give an irreducible divisor $\mathcal{B}''_X$ of $\mathcal{M}_X(2;0,2,0)$.

Proposition 1.1.10 ([Dr], Lemma 4.7)**.** *The scheme $\mathcal{B}'_X \subset \mathcal{M}_X(2;0,2,0)$ parameterizing non locally free stable sheaves is a locally closed irreducible subscheme of $\mathcal{M}_X(2;0,2,0)$ of dimension 4. The scheme $\mathcal{B}''_X$ parameterizing non locally free strictly semi-stable sheaves is a locally closed irreducible subscheme of $\mathcal{M}_X(2;0,2,0)$ of dimension 4.*

In [Dr], the authors also prove the smoothness of the entire moduli space $\mathcal{M}_X(2;0,2,0)$.

Theorem 1.1.11 ([Dr], Thm. 4.6)**.** *Let X be a smooth cubic threefold.The moduli space $\mathcal{M}_X(2;0,2,0)$ of semistable sheaves with Chern classes $c_1 = 0,\ c_2 = 2,\ c_3 = 0$ is smooth of dimension 5.*

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We describe the behavior of $\mathfrak{c}_2 : \mathcal{M}_X(2;0,2,0) \to J^2(X)$. When $[\mathcal{F}] \in {\mathcal{M}_X}^{in}$, we saw that in $J^2(X)$, $\mathfrak{c}_2(\mathcal{F}) = [\mathcal{C} - h^2]$, where $\mathcal{C}$ is an elliptic quintic in $\phi^{-1}([\mathcal{F}])$. (as each $\mathcal{C} \in \phi^{-1}([\mathcal{F}])$ is the zero locus of a global section of $\mathcal{F}(1)$, is clear that if $\mathcal{C}$, $\mathcal{C}'$ both belong to $\phi^{-1}([\mathcal{F}])$, $[\mathcal{C}] = [\mathcal{C}']$ in $A_1(X)$). Moreover it follows from proposition 1.1.2 that the restriction of $\mathfrak{c}_2$ to ${\mathcal{M}_X}^{in}$ is an isomorphism onto an open subset of $J^2(X)$. We now look what happens on the boundary.
We have already mentioned that the Abel-Jacobi maps embeds the Fano surface of lines $F(X)$ in $J(X)$. Denote by $F^2(X)$ the variety parameterizing conics contained in X. Given any conic $C \subset X$, the plane $\langle C \rangle$, linear span of C, intersects X in a plane cubic curve of the form $C \cup l$, for a line l. Thus in $A_1(X)$ we have $[C + l] = h^2$. The morphism:

$$F^2(X) \longrightarrow F(X), \quad [C] \mapsto [l]$$

mapping the class of a conic C to the class of the line l, its residual in $\langle C \rangle \cap X$, defines an isomorphism $F^2(X) \xrightarrow{\sim} F(X)$. Consequently the image of $AJ_{F^2(X)} : F^2(X) \to J^2(X)$ is isomorphic to $F(X) \subset J(X)$ through the isomorphism $J^2(X) \xrightarrow{\sim} J(X)$, $Z \mapsto Z - Z_0$ (here Z_0 is a fixed 1-cycle of degree 2).
A point $[\mathcal{F}_C] \in \mathcal{B}'_X$, associated to a conic is mapped by $\mathfrak{c}_2$ to $[C] \in F^2(X) \subset J^2(X)$. Thus $\mathfrak{c}_2$ contracts the divisor $\mathcal{B}'_X$ to the smooth surface $F^2(X)$. Considering now a sheaf $\mathcal{F}_{l_1,l_2} := \mathcal{I}_{l_1} \oplus \mathcal{I}_{l_2} \in \mathcal{B}''_X$, we have that $\mathfrak{c}_2(\mathcal{F}_{l_1,l_2}) = [l_1] + [l_2]$. Hence $\mathfrak{c}_2$ maps $\mathcal{B}''_X$ to the divisor $F(X) + F(X)$ in $J^2(X)$.

Theorem 1.1.12 ([Dr], Thm 4.8)**.** *The morphism $\mathfrak{c}_2 : \mathcal{M}_X(2;0,2,0) \to J^2(X)$ is isomorphic to the blow-up of $J^2(X)$ along the surface $F^2(X)$.*

1.2 Moduli of Pfaffian representations

According to what has been explained so far, we see that we can identify the moduli ${\mathcal{M}_X}^{in}$ with an open subset of the intermediate Jacobian $J(X)$, but its Gieseker-Maruyama compactification is not isomorphic to $J(X)$. We ask then if it is possible to realize $J(X)$ as a compact moduli space of objects associated to X.

Question. Is it possible to obtain a compactification of ${\mathcal{M}_X}^{in}$ isomorphic to $J(X)$?

The Gieseker-Maruyama compactification of the moduli of bundles by means of semistable torsion free sheaves reveals to be troublesome when applied to threefolds acquiring singularities. In these cases the boundary seems too difficult to treat. Our aim is thus to obtain a compactification that can be readily extended to singular cubics.

Question. Is it possible to obtain a compactification of ${\mathcal{M}_X}^{in}$ whose construction can be adapted to singular 3-dimensional cubics?

These questions lead us to construct a new moduli space associated to cubic threefolds, the one parametrizing the skew-symmetric presentation maps for the sheaves that we study. For the instantons, the minimal skew-symmetric resolution is of length one, so the resolution is the same as the presentation map in this case. For the sheavs on the boundary of the compactification, the length of the resolution is 2, and the presentation map is the first step of the resolution.

1.2.1 Skew-symmetric resolutions of sheaves

The starting point of our construction is the following. We consider a smooth cubic hypersurface $X \subset \mathbb{P}^4$ and an instanton $\mathcal{F}$ on X. We saw that the twisted vector bundle $\mathcal{E} := \mathcal{F}(1)$ is obtained from an elliptic curve $\mathcal{C} \subset X$ by means of Serre's construction. Consider now $\mathcal{E}$ as a torsion $\mathcal{O}_{\mathbb{P}^4}$-module. (by an abuse of notation we identify $\mathcal{E}$ with its direct image $i_*\mathcal{E}$, where i is the inclusion $i : X \hookrightarrow \mathbb{P}^4$). It is shown in [B2] that its minimal free resolution in $\mathbb{P}^4$ has the form:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^4}^{\oplus 6} \longrightarrow \mathcal{E} \longrightarrow 0,$$

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where M is a 6×6 skew-symmetric matrix whose entries are linear forms on $\mathbb{P}^4$. Still denoting by M the morphism of vector bundles $\mathcal{O}_{\mathbb{P}^4}^{\oplus 6}(-1) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^4}^{\oplus 6}$, we see that at a generic point $x\in\mathbb{P}^4$, $M(x)$ has rank 6; $M(x)$ has rank 4 whenever x belongs to the cubic hypersurface defined by $\mathrm{Pf}(M)=0$. This means that $\mathcal{E}=\mathrm{coker}(M)$ is supported on $\mathrm{Pf}(M)=0$ and that therefore the matrix M provides a *Pfaffian representation* of X.
Conversely, if we are given M, a skew-symmetric matrix of size 6 whose entries are linear forms and having generic rank 6, we get a short exact sequence of $\mathcal{O}_{\mathbb{P}^4}$-modules:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^4}{}^{\oplus 6} \longrightarrow \mathcal{E} \longrightarrow 0 \tag{1.8}$$

where $\mathcal{E}:=\mathrm{coker}(M)$. The sheaf $\mathcal{E}$ is supported on the cubic X defined by the equation $\mathrm{Pf}(M)=0$; computing the cohomology of $\mathcal{E}$ we deduce that its restriction to the smooth locus of X is a vector bundle. In particular, if X is non-singular, we can easily compute from (1.8) that $\mathcal{F}:=\mathcal{E}(-1)$ is a rank 2 vector bundle on X with $c_1=0,\ c_2=2$ and $h^0(\mathcal{F})=0$, namely an instanton bundle.
We therefore notice that on a smooth cubic threefold X, instanton bundles are related to Pfaffian representations of X. We will now see that 6×6 skew matrices of linear forms also occur in minimal free resolutions (in $\mathbb{P}^4$) of sheaves in the boundary of $\mathcal{M}_X(2;0,2,0)$; but in these cases we will deal with matrices of generic rank 4.

Sheaves on the boundary

Given a smooth cubic X, a sheaf $\mathcal{F}\in\mathcal{B}_X$ fits in a short exact sequence (from now on, whenever $\mathcal{G}$ is a sheaf supported on a subscheme of X, resp. $\mathbb{P}^4$, we will still denote by $\mathcal{G}$ its direct image in X, resp. $\mathbb{P}^4$):

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X{}^{\oplus 2} \longrightarrow \mathcal{G} \longrightarrow 0, \tag{1.9}$$

with $\mathcal{G}\simeq\mathcal{O}_C(1pt)$ when $[\mathcal{F}]\in\mathcal{B}'_X$, $\mathcal{F}=\mathcal{F}_C$, $\mathcal{G}\simeq\mathcal{O}_{l_1}\oplus\mathcal{O}_{l_2}$ when $[\mathcal{F}]\in\mathcal{B}''_X$, $\mathcal{F}=\mathcal{I}_{l_1}\oplus\mathcal{I}_{l_2}$.
In order to obtain a resolution of $\mathcal{F}$ we first build resolutions of the middle and of the right hand terms of (1.9). A resolution of $\mathcal{O}_X{}^{\oplus 2}$ is immediate, calling indeed F the polynomial defining X we have:

$$0 \longrightarrow O_{\mathbb{P}^4}(-3)^{\oplus 2} \xrightarrow{F\cdot id_2} O_{\mathbb{P}^4}{}^{\oplus 2} \longrightarrow \mathcal{O}_X{}^{\oplus 2} \longrightarrow 0. \tag{1.10}$$

We will now describe skew-symmetric resolutions of $\mathcal{G}$.

• $[\mathcal{F}]\in\mathcal{B}'_X$**.** When $[\mathcal{F}]$ belongs to the divisor $\mathcal{B}'_X$, $\mathcal{F}=\mathcal{F}_C$ is a stable sheaf associated to a smooth conic C contained in X. In this case $\mathcal{G}\simeq\mathcal{O}_C(1pt)$. $\mathcal{O}_C(1pt)$ is the dual of a theta characteristic on C, a line bundle associated to a linear symmetric representations of C. We choose coordinates $X_0,\dots,X_4$ on $\mathbb{P}^4$ in such a way that the plane $\langle C\rangle$ has equations $\{X_3=X_4=0\}$ and that the conic is $C:=\{X_3=0,X_4=0,X_1^2-X_0X_2=0\}$. The minimal free resolution of $\mathcal{O}_C(1pt)$ in the plane $\mathbb{P}^2=\langle C\rangle$ is the complex:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \xrightarrow{\begin{pmatrix} X_2 & -X_1\\ -X_1 & X_0\end{pmatrix}} \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \longrightarrow \mathcal{O}_C(1pt) \longrightarrow 0. \tag{1.11}$$

We get thus a resolution of $\mathcal{O}_C(1pt)$ in $\mathbb{P}^4$ (this can be obtained for example starting from the Koszul resolutions of $\mathcal{O}_{\mathbb{P}^2}{}^{\oplus 2}$ and $\mathcal{O}_{\mathbb{P}^2}{}^{\oplus 2}(-1)$) of the form:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} \xrightarrow{\gamma} \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 6} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^4}{}^{\oplus 2} \to \mathcal{O}_C(1pt) \longrightarrow 0; \tag{1.12}$$

where for our choice of coordinates:

$$\alpha=\begin{pmatrix} -X_0 & -X_1 & 0 & -X_3 & 0 & -X_4\\ -X_1 & -X_2 & X_3 & 0 & X_4 & 0\end{pmatrix}, \qquad \gamma=\alpha^T,$$

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$$\beta = \begin{pmatrix} 0 & 0 & 0 & X_4 & 0 & -X_3 \\ 0 & 0 & -X_4 & 0 & X_3 & 0 \\ 0 & X_4 & 0 & 0 & X_2 & -X_1 \\ -X_4 & 0 & 0 & 0 & -X_1 & X_0 \\ 0 & -X_3 & -X_2 & X_1 & 0 & 0 \\ X_3 & 0 & X_1 & -X_0 & 0 & 0 \end{pmatrix}$$

The matrix β is a skew-symmetric matrix of linear forms such that $\mathrm{Pf}(\beta) = 0$ (β has generic rank 4).

• $[\mathcal{F}] \in \mathcal{B}''_X$. If $[\mathcal{F}]$ belongs to $\mathcal{B}''_X$, $\mathcal{F}$ is a strictly semistable sheaf of the form $\mathcal{F} = \mathcal{F}_{l_1,l_2}$, $\mathcal{F} \simeq \mathcal{I}_{l_1} \oplus \mathcal{I}_{l_2}$; where $l_1,\ l_2$ is a couple of lines contained in X. In this case we have $\mathcal{G} = \mathcal{O}_{l_1} \oplus \mathcal{O}_{l_2}$, so that a free resolution of $\mathcal{G}$ can be easily obtained from the Koszul resolutions of each sheaf $\mathcal{O}_{l_i}$. Suppose that the line $l_i,\ i = 1, 2$, is given by the intersection of three hyperplanes $H^i_A,\ H^i_B,\ H^i_C$ defined respectively by linear equations $h^i_A = 0,\ h^i_B = 0,\ h^i_C = 0$. Each sheaf $\mathcal{O}_{l_i}$ admits as minimal resolution in $\mathbb{P}^4$ the Koszul complex:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-3) \xrightarrow{\gamma_i} \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 3} \xrightarrow{\beta_i} \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 3} \xrightarrow{\alpha_i} \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{l_i} \to 0 \qquad (1.13)$$

with:

$$\alpha_i = \begin{pmatrix} h^i_A & h^i_B & h^i_C \end{pmatrix}, \qquad \gamma_i = \alpha_i^T$$

$$\beta_i = \begin{pmatrix} 0 & h^i_C & -h^i_B \\ -h^i_C & 0 & h^i_A \\ h^i_B & -h^i_A & 0 \end{pmatrix}$$

Consequently, we obtain a resolution of the sheaf $\mathcal{G} = \mathcal{O}_{l_1} \oplus \mathcal{O}_{l_2}$:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} \xrightarrow{\gamma} \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 6} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \to \mathcal{O}_{l_1} \oplus \mathcal{O}_{l_2} \to 0 \qquad (1.14)$$

where:

$$\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}, \quad \gamma = \alpha^T.$$

Again, β is a skew-symmetric matrix of size 6 of linear forms with $\mathrm{Pf}(\beta) = 0$.

Remark 4. The linear forms $h^i_A,\ h^i_B, h^i_C$ are a basis for $H^0(\mathbb{P}^4, \mathcal{I}_{l_i/\mathbb{P}^4}(1))$. Hence the linear space $\langle h^1_A,\ h^1_B,\ h^1_C,\ h^2_A,\ h^2_B,\ h^2_C \rangle$ has dimension 5 whenever l_1 and l_2 are disjoint, 4 whenever l_1 and l_2 meet at a point and 3 when the two lines coincide.

Summing up we see that each sheaf $\mathcal{F}$ in the boundary of the moduli space is the kernel of a surjective morphism ${\mathcal{O}_X}^{\oplus 2} \to \mathcal{G}$; the sheaf $\mathcal{G}$ has one dimensional support and admits a minimal resolution in $\mathbb{P}^4$ of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} \xrightarrow{\gamma} \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 6} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \to \mathcal{G} \to 0. \qquad (1.15)$$

Furthermore β is 6×6 skew-symmetric matrix such that $\mathrm{Pf}(\beta) = 0$ and $\gamma = ker(\beta)$ is such that $\alpha = \gamma^T$. The resolution of $\mathcal{G}$ is thus an example of a *skew-symmetric linear complex*. These complexes appearing in the resolutions of sheaves in the boundary of $\mathcal{M}_X(2; 0, 2, 0)$, together with the matrices β determining them will be studied in detail in Chapters 3 and 4. The morphism ${\mathcal{O}_X}^{\oplus 2} \to \mathcal{G}$ lifts to a map of resolutions providing

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the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O}_X^{\oplus 2} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow{\scriptstyle \epsilon} & & \\
 & & & & \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} & \xrightarrow{\mathrm{id}_2} & \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} & & \\
 & & & & {\scriptstyle F\cdot \mathrm{id}_2}\uparrow & & \uparrow{\scriptstyle \alpha} & & \\
 & & & & \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} & \xrightarrow{\beta'} & \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} & & \\
 & & & & \uparrow & & \uparrow{\scriptstyle \beta} & & \\
 & & & & 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 6} & & \\
 & & & & & & \uparrow{\scriptstyle \gamma} & & \\
 & & & & & & \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}
\tag{1.16}
$$

By an easy diagram chase, we deduce a resolution of $\mathcal{F}$ of the form

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} \xrightarrow{G} \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 6} \xrightarrow{B} \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} \to \mathcal{F} \to 0, \tag{1.17}$$

in which $B = (\beta'|\beta)$, the 6-by-8 matrix obtained by concatenation of a 6-by-2 matrix of quadratic forms β' with the matrix β. Further, $G = 0 \oplus \gamma$, that is the image of G is contained in the summand $\mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 6}$, and G coincides with γ as a map $\mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 6}$.

1.2.2 The GIT moduli space of Pfaffian representations

Let W and V be two complex vector spaces of dimension 6 and 5 respectively. We denote by $\mathcal{P}$:

$$\mathcal{P} := \mathbb{P}(V^* \otimes \bigwedge^2 W^*) \simeq \mathbb{P}^{74}$$

the projective space of 6×6 skew-symmetric matrices whose entries are elements in V^*. We consider $\mathcal{P}^{in}$ the open parameterizing matrices $M \in \mathcal{P}$ such that the cubic defined by the equation $\mathrm{Pf}(M) = 0$ is smooth. Let $\mathfrak{M}^{in}$ be the moduli space of torsion sheaves on $\mathbb{P}^4$ with supports on smooth cubic hypersurfaces $X \subset \mathbb{P}^4$, whose restrictions to X are instanton bundles:

$$\mathfrak{M}^{in} := \{[\mathcal{F}] \mid \mathcal{F} \text{ is an instanton on a smooth cubic } X \subset \mathbb{P}^4\};$$

define $\mathfrak{M}$ as the closure of $\mathfrak{M}^{in}$ in the moduli space of sheaves on $\mathbb{P}^4$. We call $\mathcal{U} \subset |\mathcal{O}_{\mathbb{P}^4}(3)| \simeq \mathbb{P}^{34}$ the open subset of $|\mathcal{O}_{\mathbb{P}^4}(3)|$ parameterizing smooth cubics. As for what was discussed in the previous sections, we deduce that we have a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{Q}^{in} & \xrightarrow{\tau} & \mathfrak{M}^{in} \\
 & {\scriptstyle \mathrm{Pf}}\searrow & \downarrow{\scriptstyle \rho} \\
 & & |\mathcal{O}_{\mathbb{P}^4}(3)|
\end{array}
$$

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where $M \overset{\tau}{\longmapsto} \operatorname{coker}(M) \otimes \mathcal{O}_{\mathbb{P}^4}(-1) \overset{\rho}{\longmapsto} X = \{\operatorname{Pf}(M) = 0\} = \operatorname{Supp}(\operatorname{coker}(M))$. According to [B1], the morphism $\tau : \mathcal{P}^{in} \to \mathfrak{M}^{in}$ is a principal bundle with structure group $PGL(6, \mathbb{C})$. The group $GL(6, \mathbb{C})$ acts on the resolutions of instantons as follows. Given $[\mathcal{F}] \in \mathfrak{M}^{in}$ and twisting (1.8) by $\mathcal{O}_{\mathbb{P}^4}(-1)$ we get a resolution of $\mathcal{F}$ of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 6} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} \xrightarrow{N} \mathcal{F} \to 0. \tag{1.18}$$

If it clear that this resolution is uniquely determined by (M, N), where $(N, \mathcal{F}) = \operatorname{coker}(M)$. We then have:

$$g \cdot (M, N) = (gMg^T, Ng^{-1}),\ g \in GL(6, \mathbb{C}).$$

Note also that looking at the morphism $\rho : \mathfrak{M}^{in} \to \mathcal{U}$, the fibre $\rho^{-1}(X)$ over a smooth cubic $X \in \mathcal{U}$ is ${\mathcal{M}_X}^{in}$.
Since our aim is to construct a new moduli space related to instanton bundles on a 3-dimensional cubic X and as we saw that taking free resolutions in $\mathbb{P}^4$, we can associate elements in $\mathcal{P}$ to each point in $\mathcal{M}_X(2; 0, 2, 0)$, we study the moduli space $\mathfrak{P}$ of Pfaffian representations of cubic threefolds. The group $GL(6, \mathbb{C})$ acts on $\mathcal{P}$ by conjugation, therefore the moduli space $\mathfrak{P}$ can be obtained by means of Geometric Invariant Theory (GIT) as the GIT quotient:

$$\mathfrak{P} := \mathcal{P}^{ss} /\!\!/ G$$

where $G = SL(6, \mathbb{C})$ and $\mathcal{P}^{ss}$ is the open parameterizing matrices that are semistable with respect to the action of G. $\mathfrak{P}$ is a compact projective scheme; from the irreducibility of $\mathcal{P}$ we deduce that $\mathfrak{P}$ is irreducible as well. We compute now the dimension of $\mathfrak{P}$. $\mathcal{P}$ is a linear space of dimension 74, hence the open subset $\mathcal{P}^s$ of stable points has dimension 74 too. By the construction of the GIT quotient (see the Appendix at the end of the chapter for details, Thm 1.3.3), there exists an open subset $\mathfrak{P}^s$ of $\mathfrak{P}$ that is a geometric quotient for the action of $SL(6, \mathbb{C})$ on $\mathcal{P}^s$. Since $SL(6, \mathbb{C})$ has dimension 35 we deduce that $\mathfrak{P}^s$ has dimension 39. Therefore $\mathfrak{P}$ is a compact and irreducible scheme having dimension 39 at its general point. This means that $\dim(\mathfrak{P}) = 39$. The morphism $\tau : \mathcal{P}^{in} \to \mathfrak{M}^{in}$ extends to a rational map $\tau : \mathcal{P} \dashrightarrow \mathfrak{M}$, and the latter induces a birational map:

$$\overline{\tau} : \mathfrak{P} \dashrightarrow \mathfrak{M}.$$

The morphism Pf, defined on the open subset parameterizing matrices $M \in \mathcal{P}$ of generic rank six (or equivalently such that $\operatorname{Pf}(M) \neq 0$), determines a rational map $\operatorname{Pf} : \mathcal{P} \dashrightarrow |\mathcal{O}_{\mathbb{P}^4}(3)|$. Since for a matrix $M \in \mathcal{P}$ such that $\operatorname{Pf}(M) \neq 0$, we have $\operatorname{Pf}(M) = \operatorname{Pf}(GL(6, \mathbb{C}) \cdot M)$ we have an induced rational map:

$$\overline{\operatorname{Pf}} : \mathfrak{P} \dashrightarrow |\mathcal{O}_{\mathbb{P}^4}(3)|$$

whose generic fibre is compact and has dimension 5. Summing up, we obtain a commutative diagram:

$$\begin{array}{ccc} \mathfrak{P} & \overset{\overline{\tau}}{\dashrightarrow} & \mathfrak{M} \\ & \overline{\operatorname{Pf}} \searrow & \downarrow \rho \\ & & |\mathcal{O}_{\mathbb{P}^4}(3)|. \end{array} \tag{1.19}$$

Theorem 1.2.1. *1. The map ρ in the diagram (1.19), sending each sheaf to its suport, is a dominant rational map with connected fibers. Its fiber over a smooth cubic 3-fold X is the five-dimensional Druel's moduli space $\mathcal{M}_X(2; 0, 2, 0)$, isomorphic to the blowup of the intermediate Jacobian $J(X)$ with center in a smooth surface.*

2. The map $\overline{\operatorname{Pf}}$ is a surjective morphism.

3. The map $\overline{\tau}$ is birational. Its restriction to the locus of skew-symmetric matrices of linear forms generically of rank 6 is an isomorphism to its image, and its indeterminacy locus is that of matrices of generic rank 4.

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4. *The locus of stable skew-symmetric matrices of linear forms generically of rank 4 is the union of two irreducible components* $\mathfrak{B}'$ *and* $\mathfrak{B}''$*; moreover the map* $\overline{\tau}$ *is the standard blowup over the generic points of* $\mathfrak{B}'$ *transforming it into a divisor* $\boldsymbol{\mathcal{B}}'$*. The generic points of this divisors represent non-locally-free sheaves in* $\mathcal{M}_X(2;0,2,0) \setminus \mathcal{M}_X{}^{in}$ *belonging to* $\mathcal{B}'_X$

Proof. The assertion 1 is the main result of [Dr]. The assertion 2 is proved in Chapter 2, see Theorem 2.2.1. The birationality part of the assertion 3 follows from [B2]: as explained above, when a cubic 3-fold X is smooth, the open part $\mathcal{M}_X{}^{in}$ of $\mathcal{M}_X(2;0,2,0)$ is isomorphic to the quotient $\mathrm{Pf}^{-1}(X)//PGL(6,\mathbb{C})$, and as X runs over the locus of smooth cubics $\mathcal{U}$, this provides a biregular isomorphism $\overline{\mathrm{Pf}}^{-1}(\mathcal{U}) \simeq \mathfrak{M}^{\mathrm{in}}$ via the restriction of $\overline{\tau}$. The fact that it extends as a bijective morphism to the locus of skew-symmetric matrices of linear forms generically of rank 6 follows from the results of Chapter 3 on the description of (semi)stable skew-symmetric matrices of linear forms. We show, in particular, that all the matrices from the generic rank-6 locus are semistable under the action of $SL(6,\mathbb{C})$ (Corollary 3.2.2). The fact that this bijection is a biregular isomorphism follows from the proof of 4.2.4; this is made by comparison of local deformation spaces in the GIT quotient).

The assertion 4 is proved by studying the local deformations of the minimal resolutions of the generic non-locally-free sheaves in the boundary of $\mathfrak{M}$. As we mentioned above, the first step of the resolution for these sheaves, which has the form (1.17), is a 6×8 matrix $B = (\beta'|\beta)$, where β is a skew-symmetric matrix of linear forms of generic rank 4 and β' is a 2×6 matrix of quadratic forms. On the other hand, Chapter 3 provides a criterion for (semi)stability, which allows us to describe the stable locus $\mathcal{P}^s$ and to verify that the matrices β arising in the resolutions of the form (1.17) are all stable under $SL(6,\mathbb{C})$ (Theorem 3.2.4). In Chapter 4, we show that in a deformation of a sheaf of type $\mathcal{B}'_X$ to instantons, β is the limit of Pfaffian representation of the cubics supporting the nearby instantons (Propositions 4.2.3).

On the space of presentation matrices B, we have two regular maps, $B \mapsto \mathcal{F}_B = \mathrm{coker}\,(B) \in \mathfrak{M}$ and $B \mapsto X_B = \mathrm{Supp}\,(\mathcal{F}_B)$. We study the local structure of these maps via the computation of infinitesimal deformations of a randomly chosen element $B_0 = (\beta'_0|\beta_0)$ appearing in a minimal free resolution of a sheaf $\mathcal{F}_0$ on a smooth cubic threefold X_0 and corresponding to a point $[\mathcal{F}_0] \in \mathcal{B}'_{X_0}$. Remark that the general deformation of the resolution of type (1.17) for the sheaf $\mathcal{F}_0$ provides a sheaf of rank 1 over a determinantal sextic in $\mathbb{P}^4$. We restrict ourselves only to deformations that keep the property of the sheaf to be supported on a cubic. We verify that the infinitesimal deformations with this property are unobstructed and are naturally identified with the elements of the normal space to the boundary component $\mathfrak{B}'$, to which belongs the orbit of β_0. The computation uses Macaulay2 [M2] and boils down to a check that the Jacobian of some linear map between spaces of polynomial matrices is of expected rank. □

Remark 5. We conjecture that an equivalent of 1.2.1 (iii) holds also for $\mathfrak{B}''$ and $\boldsymbol{\mathcal{B}}''$, where this latter is the divisor of $\mathfrak{M}$ whose general point is a sheaf in $\mathcal{B}''_X$ for $X \in \mathcal{U}$. However the adaptation of the proofs of 4.2.4 and 4.2.3 in this circumstance presents some technical complications. We hope to solve these issue in the future, it is a work still in progress

We have thus constructed a compactification $\mathfrak{P}$ (resp $\widetilde{M}(X) := \bar{\mathrm{Pf}}^{-1}(X)$), of the moduli space $\mathfrak{M}^{\mathrm{in}}$ (resp. $\mathcal{M}_X^{in}$) of instantons supported on cubics (resp. on a smooth cubic hypersurface X) of $\mathbb{P}^4$, different from Gieseker's. We only deal with stable skew-symmetric matrices of linear forms. Some partial results are also obtained on the behavior of the maps in the diagram (1.19) at strictly semistable points. We are hoping to turn to a systematic study of different strata of the strictly semistable locus in the future.

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1.3 Appendix: Geometric Invariant Theory

In this section we recollect some main features of Geometric Invariant Theory, one of the most efficient way to construct moduli spaces. Using GIT moduli spaces are realized as quotients of varieties, or more generally schemes, with respect to the action of a reductive group G. We will describe here briefly how we construct a GIT quotient for the action of a reductive group on a projective variety. (Our main reference for the section is [Th], see [DG] or [GIT] for an exhaustive dissertation on the subject.)
Let then $X \subset \mathbb{P}^n$ be a projective variety and $\sigma : G \times X \to X$ be the action of a reductive group G on X. We suppose that furthermore G acts on X as $GL(n+1, \mathbb{C})$; this assumption implies that σ lifts to an action $\tilde{\sigma}$ on the affine cone $\tilde{X} \subset \mathbb{A}^{n+1}$, $\tilde{\sigma} : G \times \tilde{X} \to \tilde{X}$. Denote by $\pi : \tilde{X} \to X$ the standard projection. We notice that $\tilde{\sigma}$ satisfies the following:

(i) $\forall\, x \in X, \tilde{x} \in \pi^{-1}(x),\ g \in G$, we have that $g \cdot \tilde{x} \in \pi^{-1}(g \cdot x)$. Moreover, denoting by $l_x \subset \mathbb{A}^{n+1}$ and by $l_{g \cdot x} \subset \mathbb{A}^{n+1}$ the lines corresponding respectively to x and $g \cdot x$, (namely the fibers of the tautological bundle $\mathcal{O}_X(-1)$ at x and $g \cdot x$), we have that the map:

$$\tilde{\sigma}_x(g) : l_x \to l_{g \cdot x}$$

induced by $\tilde{\sigma}$ is a linear isomorphism.

This condition ensures that the origin $0 \in \tilde{X}$ satisfies $G \cdot 0 = 0$.

Remark 6. The situation that we have just described extends to a more general setting, where we have a polarized variety $(X, \mathcal{L})$ and an action $\sigma : G \times X \to X$ of a reductive group G on X. (In this case the variety X will be embedded in a projective space by means of some power $\mathcal{L}^{\otimes n}$ of $\mathcal{L}$). We identify the line bundle $\mathcal{L}$ with its total space, that we still denote by $\mathcal{L}$, and we call $\pi : \mathcal{L} \to X$ the projection. It is possible to lift the action of G to $\mathcal{L}$ whenever $\mathcal{L}$ is *G-linearized.* We recall indeed that we say that the line bundle $\mathcal{L}$ is G-linearized if σ lifts to an action $\overline{\sigma} : G \times \mathcal{L} \to \mathcal{L}$ such that:

- we have a commutative diagram:

$$\begin{array}{ccc} G \times \mathcal{L} & \xrightarrow{\tilde{\sigma}} & \mathcal{L} \\ {\scriptstyle \mathrm{id} \times \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow{\sigma} & X \end{array}$$

- The zero section of $\mathcal{L}$ is G-invariant.

Note that from the definition, if $\mathcal{L}$ is G-linearized, for any $x \in X$ and $g \in G$, the induced morphism on the fibers:

$$\tilde{\sigma}_x(g) : \mathcal{L}_x \to \mathcal{L}_{g \cdot x}$$

is a linear isomorphism.

The GIT quotient
We mentioned at the beginning of the section that we aim to obtain a moduli space as the quotient of an action $\sigma : G \times X \to X$. As furthermore we want our quotient to have "good geometric properties", such as being compact or separated, we can't just consider the orbit space X/G but we will need a new notion of quotient. We describe here how the GIT quotient is defined. Again we consider $X \subset \mathbb{P}^n$, $\sigma : G \times X \to X$ and we suppose that the reductive group G acts through $GL(n+1, \mathbb{C})$. As we have already observed this implies that σ lifts to an action on the affine cone $\tilde{X}$ satisfying (i). Using the language of remark 6, the line bundle $\mathcal{O}_X(1)$ is G-linearized. From the fact that the action is linearized, we see that G acts on each vector space $H^0(\mathcal{O}_X(r))$; we denote then by $H^0(\mathcal{O}_X(r))^G$ the space of G-invariant forms of degree r on X.

Lemma 1.3.1 ([Th], Lemma 3.3)**.** *$\bigoplus_r H^0(\mathcal{O}_X(r))^G$ is finitely generated.*

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$\bigoplus_r H^0(\mathcal{O}_X(r))^G$ is a finitely generated graded $\mathbb{C}$-algebra; we thus define $X /\!\!/ G$ as the projective variety:

$$X /\!\!/ G := Proj(\oplus_r H^0(\mathcal{O}_X(r))^G.$$

Note that by its definition, $X /\!\!/ G$ is endowed with a natural ample line bundle L.

Stable and semi-stable points

Definition 1.8.

- A point $x \in X$ is called *semistable* if there exists an integer $r > 0$ and a section $s \in H^0(\mathcal{O}_X(r))^G$, such that $s(x) \neq 0$.
- A point $x \in X$ is called *stable* if it satisfies the following:
  - $dim\, G = dim\, G \cdot x$ (namely x has finite stabilizer),
  - there exists $r > 0$ and $s \in H^0(\mathcal{O}_X(r))^G$ such that $s(x) \neq 0$ (namely x is semi-stable),
  - the action of G on $X^s := \{y \in X \mid s(x) \neq 0\}$ is closed.

The last condition of the definition of a stable point $x \in X$ consists of requiring the existence of an integer $r > 0$ and a section $s \in H^0(\mathcal{O}_X(r))^G$ that does not vanish at x (this fact guarantees the semistability of x) and that furthermore "separates the orbits " near x. This means that for every point y belonging to the open $X^s := \{y \in X \mid s(x) \neq 0\}$, and for all vectors $v \in T_yX/T_yG \cdot X$, the derivative of s along v is different from zero. We denote by X^{ss} and X^s the open subsets of semi-stable and stable points respectively. It is possible to give a more geometric characterization of stability studying the action of G on $\tilde{X}$.

Theorem 1.3.2 ([Th], Thm. 3.8)**.** *Let x be a point in X and $\tilde{x}$ be a point in $\tilde{X}$ belonging to $\pi^{-1}(x)$.*

- *x is semi-stable if and only if $0 \notin \overline{G \cdot \tilde{x}}$;*
- *x is stable if and only if $G \cdot \tilde{x}$ is closed and $\tilde{x}$ has finite stabilizer.*

The inclusion $\bigoplus_r H^0(\mathcal{O}_X(r))^G \hookrightarrow \bigoplus_r H^0(\mathcal{O}_X(r))$ induces a (surjective) morphism

$$\phi : X^{ss} \to X /\!\!/ G,$$

this morphism is called the *GIT quotient* (sometimes the GIT quotient is simply denoted by $X^{ss} /\!\!/ G$). From the definition we see that $\phi^*(L) = \mathcal{O}_{X^{ss}}(1)$.

The main reason why it is convenient to work with the GIT quotient is that it is a *good quotient.*

Definition 1.9. Let X be an algebraic variety and G a reductive group acting on X. A morphism $\phi : X \to Y$ is a good quotient for the action of G on X if:

- ϕ is G-invariant.
- ϕ is surjective.
- $\mathcal{O}_Y \simeq \phi_*(\mathcal{O}_X^G)$; where $\mathcal{O}_X^G$ is the sheaf that on every open $U \subset X$ is defined by $\mathcal{O}_X^G(U) := \mathcal{O}_X(U)^G$.
- If $W \subset X$ is a G-invariant closed subset of X, $\phi(W)$ is closed in Y.
- If W_1 and W_2 are two disjoint G-invariant closed subset of X, $\phi(W_1)$ and $\phi(W_2)$ are disjoint.
- ϕ is affine.

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If moreover the preimage of each point is a single orbit we say that $\phi : X \to Y$ is a *geometric quotient.*

If $\phi : X \to Y$ is a *good quotient*, it is also a *categorical quotient* namely each G-invariant morphism $\psi : X \to Z$ factors uniquely trough ϕ.

Theorem 1.3.3 (Mumford)**.** *The morphism $\phi : X^{ss} \to X /\!\!/ G$ is a good quotient for the action of G on X^{ss}. Moreover there exists an open subset $Y^s \subset X /\!\!/ G$ such that $\phi^{-1}(Y^s) = X^s$ and $\phi : X^s \to Y^s$ is a geometric quotient for the action of G on X^s.*

The Hilbert-Mumford criterion
The Hilbert-Mumford criterion allows us to detect the stability of a point $x \in X$ just by analyzing its stability with respect to 1-parameter subgroups of G.

Definition 1.10. A one parameter subgroup (1-PS) λ of G is a morphism of algebraic groups:

$$\lambda : \mathbb{G}_m \to G.$$

Theorem 1.3.4 (Hilbert-Mumford criterion)**.**

- *A point $x \in X$ is stable if and only if it is λ-stable for every 1-PS λ of G.*
- *A point $x \in X$ is semistable if and only if it is λ-semistable for every 1-PS λ of G.*

The reason for which it is more advantageous to work with 1 parameter subgroups is that the action of a 1-PS λ of G can always be diagonalized. This means that we can find integers $\lambda_0, \cdots, \lambda_n$ such that $\lambda(t)$ acts as $diag(t^{\lambda_0}, \dots, t^{\lambda_n})$. Taking now $x \in X$ and $\tilde{x} \in \pi^{-1}(x)$, $\tilde{x} = (x_0, \dots x_n)$, we have $\lambda(t) \cdot \tilde{x} = (t^{\lambda_0} x_0, \dots t^{\lambda_n} x_n)$. We define

$$\mu(x, \lambda) := \min_i \{\lambda_i \mid x_i \neq 0\}.$$

It is possible to reformulate the Hilbert-Mumford criterion in terms of the weights $\mu(x, \lambda)$ as follows:

Theorem 1.3.5.

- *x is semistable if and only if $\mu(x, \lambda) \leq 0 \; \forall$ 1-PS λ of G.*
- *x is stable if and only if $\mu(x, \lambda) < 0 \; \forall$ 1-PS λ of G.*

We report here the idea of the proof. In order to analyze the stability of x we study the closure $\overline{\lambda \cdot \tilde{x}}$, that is we study the behavior of $\lambda(t) \cdot \tilde{x}$ for $t \to 0$ and $t \to \infty$. Actually it is enough to restrict to the former as the latter is deduced from λ^{-1}, the 1 parameter subgroup defined by $\lambda^{-1}(t) = \lambda(t^{-1})$. We see then that x fails to be stable if and only if there exists a 1-PS λ such that $\mu(x, \lambda) \geq 0$. Indeed, since $\lambda(t) \cdot \tilde{x} = (t^{\lambda_0} x_0, \dots t^{\lambda_n} x_n)$, if ever $\mu(x, \lambda) \geq 0$, we would have that $\lambda(t) \cdot \tilde{x}$ is bounded for $t \to 0$ implying $\lambda \cdot \tilde{x} \subsetneq \overline{\lambda \cdot \tilde{x}}$. Regarding semistability, we see from the coordinates of $\lambda(t) \cdot \tilde{x}$ that the condition $\mu(x, \lambda) > 0$, is equivalent to requiring that $\lambda(t) \cdot \tilde{x}$ tends to 0 as $t \to 0$, namely $0 \in \overline{\lambda \cdot \tilde{x}}$.

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Chapter 2

Pfaffian representations of cubic threefolds

Introduction

We study Pfaffian representations of a cubic hypersurface X of $\mathbb{P}^4$; namely we try to determine when the polynomial F defining X can be written as the Pfaffian of a 6×6 matrix of skew-symmetric forms. It's known that a smooth cubic threefold is Pfaffian [B2], we will then be concerned with the singular ones. We prove the following:

Theorem (Theorem 2.2.1). *A cubic threefold $X \subset \mathbb{P}^4$ always admits a Pfaffian representation.*

To prove the theorem we will distinguish the following families of singular threefolds:

- Cubic threefolds that are cones;
- Non-normal cubic threefolds;
- Normal cubic threefolds that present at most double points.

As every cubic hypersurface of dimension two or less is Pfaffian, we deduce immediately that the same holds whenever X is a cone. If X is not normal (and then it is singular along a linear space of dimension at least 2), we are able to write explicitly a skew-symmetric matrix of linear forms M_X such that $\mathrm{Pf}\,(M_X) = F$. Things get more interesting when X is singular in codimension at most 2 and presents just double points. Under these assumptions the study of Pfaffian representations relies on the proof of the existence of a rank 2 skew-symmetric Ulrich sheaf $\mathcal{E}$ on X. Indeed if such a sheaf $\mathcal{E}$ on X exists, its minimal free resolution in $\mathbb{P}^4$ is a "linear" complex of length one:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}^{\oplus 6}(-1) \xrightarrow{\varphi_1} \mathcal{O}_{\mathbb{P}^4}^{\oplus 6} \longrightarrow \mathcal{E} \longrightarrow 0;$$

the map φ_1 (the only non-vanishing differential in the resolution of $\mathcal{E}$) provides then a Pfaffian representation of X. Ulrich sheaves on a cubic threefold X can be constructed by means of *Serre correspondence*, starting from nondegenerate arithmetically Gorenstein (AG) quintic elliptic curves on X.
We will prove that these curves always exist on a normal threefold that is not a cone:

Theorem (Theorem 2.3.2). *Let X be a normal cubic threefold that is not a cone. Then there exists a non-degenerate AG elliptic quintic curve $\mathcal{C} \subset X$.*

This result will complete the proof of the existence of Pfaffian representations for all cubic threefolds.
The central part of the chapter is therefore devoted to the proof of the existence of quintic elliptic curves on normal threefolds. This is done by adapting two methods used in [MT1]

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to show the existence of such curves in the smooth case. The first method is based on a deformation argument. We first prove that a general hyperplane section S of X contains a smooth quintic elliptic curve $\mathcal{C}_0$ disjoint from the singular locus of S. Then we prove that $\mathcal{C}_0$ deforms to a nondegenerate curve $\mathcal{C}$ that is still contained in X_{sm}, the smooth locus of X. This arguments works exactly as in the smooth case if $\mathrm{Sing}\,(X)$, the singular locus of X, has dimension zero. When $\dim(\mathrm{Sing}\,(X)) = 1$, a general hyperplane section S of X is a cubic surface with isolated singularities. In this case we will consider

$$\phi : \tilde{S} \to S,$$

a minimal resolution of singularities of S (the surface $\tilde{S}$ is a so called *weak Del Pezzo surface*) and we will prove the existence of a smooth curve $\tilde{\mathcal{C}}_0$ on $\tilde{S}$ such that $\mathcal{C}_0 := \phi(\tilde{\mathcal{C}}_0) \subset \tilde{S}$ is a smooth elliptic quintic disjoint from $\mathrm{Sing}\,(S)$.
The second method is a constructive one and allows us to obtain directly a nondegenerate AG elliptic quintic $\mathcal{C} \subset X$. To apply it we first need to show the existence of a rational quartic (not necessarily irreducible) $\Gamma \subset X$. Then we will present a construction of a cubic scroll Σ containing Γ such that the curve $\mathcal{C}$, residual to Γ in $\Sigma \cap X$, is an AG elliptic quintic.
With the help of Segre classification's of cubic threefolds ([Seg]), we will analyze in detail the singularities that a normal threefold can present; depending on the components of $\mathrm{Sing}\,(X)$ we will choose which method to apply.

2.1 Preliminaries

2.1.1 Rudiments from homological algebra: CM, ACM and AG modules

We give some preliminary algebraic notions that we will use throughout the entire chapter.
Definition 2.1. Let $(A, \mathfrak{m})$ be a noetherian local domain and M an A-module. We say that an A-module M is Cohen-Macaulay (or CM for short) if:

$$\mathrm{depth}\,(M) = \dim(M)$$

Remark 7. We recall that we can give the following cohomological characterization, that will come in use later, of the depth and the dimension of a module M over $(A, \mathfrak{m})$.

- $H^i_{\mathfrak{m}}(M) = 0,\ \forall i < \mathrm{depth}\,(M),\ \forall i > \dim(M)$.
- $H^i_{\mathfrak{m}}(M) \neq 0$ for $i = \mathrm{depth}\,(M)$ and $i = \dim(M)$.

We see then that M is a CM A-module if and only if it has only one non-vanishing local cohomology group, namely $H^i_{\mathfrak{m}}(M),\ i = \dim(M)$.

Definition 2.2. Let A be a noetherian domain and M an A-module. We say that M is Cohen-Macaulay (or CM) if for all maximal ideals $\mathfrak{m} \subset A$, $M_{\mathfrak{m}}$ is a CM $A_{\mathfrak{m}}$-module.

Let now R be the ring of polynomials in $n+1$ variables, $R := \mathbb{C}[X_0, \dots X_n]$; R is a regular ring hence it is CM. Let M be a graded CM R-module. Localizing at $\mathfrak{m}_0 := (X_0, \dots, X_n)$ and applying Auslander-Buchsbaum formula,

$$\mathrm{pd}\,(M_{\mathfrak{m}_0}) + \mathrm{depth}\,(M_{\mathfrak{m}_0}) = \mathrm{depth}\,(R_{\mathfrak{m}_0}),$$

we obtain:

$$\mathrm{pd}\,(M_{\mathfrak{m}_0}) = \dim(R_{\mathfrak{m}_0}) - \dim(M_{\mathfrak{m}_0}).$$

Therefore, M admits a graded minimal free resolution of length $c = \mathrm{codim}\,(M)$:

$$\cdots 0 \longrightarrow F_c \xrightarrow{\varphi_c} F_{c-1} \xrightarrow{\varphi_{c-1}} \cdots \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0, \qquad (2.1)$$

where each term F_i is of the form $F_i = \bigoplus_j R(a_{i,j})$.

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Remark 8. Summing up, we see that if M is a graded CM R-module, then:

- For every maximal ideal $\mathfrak{m} \subset R$, the only non-vanishing local cohomology group is $H^i_{\mathfrak{m}}(M_{\mathfrak{m}}),\ i = \dim(M)$ (this is due to remark 7).
- For every integer $a \in \mathbb{Z}$, $\mathrm{Ext}^i(M, R(a)) = 0$ whenever $i > \mathrm{codim}(M)$ (this is due to the fact that a minimal free resolution of M is of the form (2.1)).

Let now X be a closed subscheme of $\mathbb{P}^n := \mathrm{Proj}(R)$ and denote by $R(X)$ its homogeneous coordinate ring, $R(X) = R/I_X$, (I_X being the saturated ideal of X).

Definition 2.3. A closed subscheme $X \subset \mathbb{P}^n$ is called *arithmetically Cohen-Macaulay* (ACM for short), if its homogeneous coordinate ring $R(X)$ is a Cohen-Macaulay ring.

Remark 9. A projectively embedded curve $C \in \mathbb{P}^n$ is ACM if and only if it is CM and $H^1(\mathcal{I}_C(k)) = 0$ for all $k \in \mathbb{Z}$. This is closely related to the projective normality: C is projectively normal if and only if it is normal and the the restriction maps on global sections $H^0(\mathcal{O}_{\mathbb{P}^n}(k)) \to H^0(\mathcal{O}_C(k))$ are surjective for all k. In particular, if C is smooth, then it is ACM if and only if it is projectively normal.

Definition 2.4. A closed subscheme $X \subset \mathbb{P}^n$ of codimension c is called *arithmetically Gorenstein* (AG for short) if the following holds:

- X is ACM
- The canonical module of $R(X)$, $K_X := \mathrm{Ext}^c_S(R(X), R)(-n-1)$, is isomorphic to $R(X)(a)$ for some $a \in \mathbb{Z}$.

Remark 10.

- The condition that a subscheme $X \subset \mathbb{P}^n$ of codimension c is AG, is equivalent to requiring that $R(X)$ admits a graded minimal free resolution of the form (2.1), such that the term F_c has rank 1.
- If X is AG, from the isomorphism $K_X \simeq R(X)(a)$ we get that the canonical sheaf $\omega_X = \mathscr{E}xt^c_S(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n})(-n-1)$ is isomorphic to $\mathcal{O}_X(a)$, for some $a \in \mathbb{Z}$.

ACM, AG and Ulrich sheaves

Definition 2.5. A coherent sheaf $\mathcal{E}$ on $\mathbb{P}^n$ is an *arithmetically Cohen-Macaulay* (ACM for short) sheaf if $E := \bigoplus_{j\in\mathbb{Z}} H^0(\mathcal{E}(j))$, its module of twisted global sections, is a (graded) Cohen-Macaulay module over $\mathbb{C}[X_0, \dots, X_n]$.

We observe that a coherent sheaf $\mathcal{E}$ on $\mathbb{P}^n$ is ACM if and only if:

- $\mathcal{E}_x$ is a Cohen-Macaulay $\mathcal{O}_{\mathbb{P}^n,x}$ module $\forall x \in \mathbb{P}^n$ (that is, $E_{\mathfrak{m}_x}$ is CM for all maximal ideals $\mathfrak{m}_x$ different from the irrelevant ideal $\mathfrak{m}_0 := (X_0, \dots, X_n)$.
- $H^i(\mathcal{E}(j)) = 0$, for $0 < i < \dim(\mathrm{Supp}(\mathcal{E}))$, for $j \in \mathbb{Z}$. This condition is equivalent to the fact that $E_{\mathfrak{m}_0}$ is a Cohen Macaulay $\mathbb{C}[X_0, \dots, X_n]_{\mathfrak{m}_0}$ module).

If $\mathcal{E}$ is ACM, from a minimal graded resolution of its module of twisted global sections E of the form (2.1), we get a locally free resolution of $\mathcal{E}$

$$\cdots 0 \longrightarrow \bigoplus_j \mathcal{O}_{\mathbb{P}^n}(a_{c,j}) \xrightarrow{\varphi_c} \bigoplus_j \mathcal{O}_{\mathbb{P}^n}(a_{c-1,j}) \xrightarrow{\varphi_{c-1}} \cdots \xrightarrow{\varphi_1} \bigoplus_j \mathcal{O}_{\mathbb{P}^n}(a_{0,j}) \longrightarrow \mathcal{E} \longrightarrow 0 \tag{2.2}$$

of length $c = \mathrm{codim}(\mathrm{Supp}(\mathcal{E}))$. Note that every differential can be represented as a matrix whose entries are homogeneus forms on $\mathbb{P}^n$.

Definition 2.6. A coherent sheaf $\mathcal{E}$ on $\mathbb{P}^n$ is an Ulrich sheaf if:

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- $\mathcal{E}$ is ACM.
- $H^i(\mathcal{E}(-i)) = 0$ for $i \geq 1$ and $H^i(\mathcal{E}(-i-1)) = 0$ for $i \leq \dim(\mathrm{Supp}(\mathcal{E}))$.

Remark 11. Let $\mathcal{E}$ be an ACM sheaf and denote by c the codimension of $\mathrm{Supp}(\mathcal{E})$. Writing a locally free resolution of $\mathcal{E}$ as (2.2), from the definition we get that $\mathcal{E}$ is Ulrich if and only if this resolution is *linear*, that is, if it has the following form:

$$\cdots 0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus r_c}(-c) \xrightarrow{\varphi_c} \mathcal{O}_{\mathbb{P}^n}^{\oplus r_{c-1}}(-c+1) \xrightarrow{\varphi_{c-1}} \cdots \xrightarrow{\varphi_1} \mathcal{O}_{\mathbb{P}^n}^{\oplus r_0} \longrightarrow \mathcal{E} \longrightarrow 0. \tag{2.3}$$

2.1.2 Linear determinantal representations of hypersurfaces and Ulrich sheaves

Let V be a complex vector space of dimension $n+1$, $F \in S^d(V^*)$ a homogeneous form of degree d on $\mathbb{P}(V)$ and r an integer bigger than or equal to one. Denote by $\mathcal{M}_{rd\times rd}$ the vector space of square matrices of size rd. It is a classical problem in algebraic geometry to determine whether some power of F can be written as the determinant of a matrix of linear forms, that is, whether there exists

$$M \in \mathcal{M}_{rd\times rd} \otimes_{\mathbb{C}} V^* \text{ such that } F^r = \det(M).$$

If this is the case, the matrix M is referred to as a *linear determinantal representation* of F. The existence of such a determinantal representation of the polynomial F is equivalent to the existence of a Ulrich sheaf of rank r on X, the hypersurface in $\mathbb{P}(V)$ defined by the equation $\{F = 0\}$.

Proposition 2.1.1. *Let X be a degree d hypersurface in $\mathbb{P}(V) \simeq \mathbb{P}^n$ defined by an equation $F = 0$, $F \in S^d(V^*)$. The following conditions are equivalent:*

- *There exists $M \in \mathcal{M}_{rd\times rd} \otimes_{\mathbb{C}} V^*$ such that $F^r = \det(M)$*
- *There exists an Ulrich sheaf $\mathcal{E}$ supported on X and of rank r as an $\mathcal{O}_X$-module.*

Proof. If F admits a linear determinantal representation $M \in \mathcal{M}_{rd\times rd} \otimes_{\mathbb{C}} V^*$, the matrix M defines a morphism of vector bunldes over $\mathbb{P}^n$ that fits in a short exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus rd}(-1) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^n}^{\oplus rd} \longrightarrow \mathcal{E} \longrightarrow 0 \tag{2.4}$$

so that $\mathcal{E} := \mathrm{coker}\,(M)$. $\mathcal{E}$ is a coherent sheaf supported on X, and as $c_1(\mathcal{E}) = r[dH] = r[X]$, $\mathcal{E}$ has rank r on X. $\mathcal{E}$ is thus a coherent sheaf admitting a minimal free resolution of length $1 = \mathrm{codim}\,(\mathrm{Supp}\,(\mathcal{E}))$, hence it is ACM. Since moreover this resolution is linear, from remark 11 we deduce that $\mathcal{E}$ is Ulrich. Vice versa, let $\mathcal{E} \in \mathrm{Coh}\,(X)$ be a rank r Ulrich sheaf; then $\mathcal{E}$ admits a linear minimal free resolution of length 1. The only nonvanishing differential of this resolution $\mathcal{O}_{\mathbb{P}^n}^{\oplus rd}(-1) \xrightarrow{\varphi_1} \mathcal{O}_{\mathbb{P}^n}^{\oplus rd}$ defines an element $M \in \mathcal{M}_{rd\times rd} \otimes_{\mathbb{C}} V^*$. As $X = \mathrm{Supp}\,(\mathcal{E}) = \{x \in \mathbb{P}^n \mid \mathrm{rk}\,(\phi(x)) \leq rd\}$, $\det(M) = F^r$ so that M is a linear determinantal representation of F. □

We now look for ϵ-symmetric determinantal representations of F, namely we ask if ever we can find an element $M \in \mathcal{M}_{2d\times 2d} \otimes_{\mathbb{C}} V^*$ such that:

- $F^2 = \det(M)$;
- M is ϵ-symmetric that is, $M^T = \epsilon M$, $\epsilon = \pm 1$.

ϵ-symmetric determinantal representations are associated to rank 2 Ulrich sheaves $\mathcal{E}$ on X that, additionally, are *ϵ-symmetric*.

We recall briefly how ϵ-symmetric sheaves are defined. To start with, consider an ACM-sheaf $\mathcal{E}$ on $\mathbb{P}^n$ supported on a hypersurface X and endowed with a morphism of sheaves $\phi : \mathcal{E} \to \mathscr{H}om(\mathcal{E}, \mathcal{O}_X(N))$ for some integer N. Applying the functor $\mathscr{H}om(\,\cdot, \mathcal{O}_X(N))$ we get a morphism $\mathcal{E}^{\vee\vee} \to \mathcal{E}^{\vee}(N)$, composing then with the natural homomorphism $\mathcal{E} \to \mathcal{E}^{\vee\vee}$ we get ϕ^T, the *transpose* of ϕ:

$$\phi^T : \mathcal{E} \to \mathcal{E}^{\vee}(N).$$

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Definition 2.7. An ACM sheaf $\mathcal{E}$ on $\mathbb{P}^n$, endowed with a sheaf morphism $\phi^T : \mathcal{E} \to \mathcal{E}^\vee(N)$, is called ϵ-symmetric if $\phi^T = \epsilon\phi$ ($\epsilon = \pm 1$).

Proposition 2.1.2. *Let X be a degree d hypersurface in $\mathbb{P}(V) \simeq \mathbb{P}^n$ defined by an equation $F = 0$, $F \in S^d(V^*)$. The following conditions are equivalent:*

- *There exists an ϵ-symmetric matrix $M \in \mathcal{M}_{2d\times 2d} \otimes_{\mathbb{C}} V^*$ such that $F^2 = \det(M)$.*
- *There exists a rank 2 ϵ-symmetric Ulrich sheaf $\mathcal{E}$ on X.*

Proof. See [B2], theorem 2.B. □

Remark 12. As for what has been discussed in Chapter 1, sect. 1.2.1, we see that given X a smooth cubic threefold and $\mathcal{F}$ an instanton on X, the twisted bundle $\mathcal{F}(1)$ is a rank 2 skew-symmetric Ulrich bundle on X.

2.2 Pfaffian representations of cubic threefolds

We now study in detail Pfaffian representations of cubic threefolds. Throughout the rest of the chapter we denote by V a 5-dimensional linear space, by $F \in S^3(V^*)$ a homogeneus form of degree 3 and by $X \subset \mathbb{P}(V) \simeq \mathbb{P}^4$ the cubic hypersurface defined by the equation $\{F = 0\}$. We prove the following:
Theorem 2.2.1. *A cubic threefold $X \subset \mathbb{P}^4$ always admits a Pfaffian representation.*

This result is known in the case where X is smooth (see for example [B2]); we will then mainly deal with singular cubic threefolds. The strategy that we adopt in order to prove that X is Pfaffian changes depending on the singularities that X presents; we will indeed distinguish the following situations:

- X is a cone. In this case proving that X is Pfaffian reduces to the study of Pfaffian representations of cubic hypersurfaces of dimension less than or equal to 2.
- X is not normal. In this case we are able to write down explicitely a 6×6 skew-symmetric matrix of linear forms M_X such that $\mathrm{Pf}\,(M_X) = F$.
- X is normal and presents at most double points. Under these hypotheses we will prove the existence of a rank 2 skew-symmetric Ulrich bundle on X. By Proposition 2.1.2, this guarantees that X is Pfaffian.

We now describe how to prove that there exists a rank 2 Ulrich sheaf $\mathcal{E}$ on X endowed with a sheaf isomorphism $\phi : \mathcal{E} \to \mathscr{H}om(\mathcal{E}, \mathcal{O}_X(2))$, such that $\phi^T = -\phi$ (namely a *skew-symmetric Ulrich sheaf*).
As Ulrich sheaves are ACM we start by looking for rank 2 ACM sheaves on X; one possible way to prove the existence of such sheaves is by means of *Serre correspondence.*

2.2.1 Serre correspondence

In Chapter 1, sect. 1.1.2, we introduced Serre's correspondence on a smooth cubic threefold X: a technique that allowed us to obtain a rank 2 Ulrich bundle $\mathcal{E}$ on X (or equivalently the instanton $\mathcal{E}(-1)$), starting from a locally complete intersection elliptic quintic curve $\mathcal{C}$ on X. We are now going to illustrate this technique in greater detail, showing how it applies to cubic threefolds which are not necessarily smooth. More generally, Serre correspondence permits to construct rank 2 ACM sheaves on a variety X, starting from codimension 2 AG subschemes of X. In our specific case a codimension 2 AG subscheme of X is an AG curve $\mathcal{C} \subset X$. Note that moreover, since X is a hypersurface, it is an AG scheme. Its canonical sheaf is $\omega_X \simeq \mathcal{O}_X(-2)$. From now on we will suppose

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that X is integral. To start with, we give a useful characterization of AG curves contained in X.

Proposition 2.2.2. *Let $\mathcal{C} \subset X$ be an AG curve and let $\mathcal{I}_\mathcal{C}$ be the ideal sheaf of $\mathcal{C}$ in X. For every point $x \in \mathcal{C}$, $\operatorname{pd}(\mathcal{I}_{\mathcal{C},x}) = 1$.*

Proof. By Auslander-Buchsbaum formula, we have:

$$\operatorname{pd}(\mathcal{I}_{\mathcal{C},x}) = \operatorname{depth}(\mathcal{O}_{X,x}) - \operatorname{depth}(\mathcal{I}_{\mathcal{C},x}).$$

X is an AG scheme so $\operatorname{depth}(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X,x}) = 3$. In order to determine $\operatorname{depth}(\mathcal{I}_{\mathcal{C},x})$ we compute the local cohomology $H^i_x(\mathcal{I}_{\mathcal{C},x})$. Consider the short exact sequence of sheaves on $\mathbb{P}^4$:

$$0 \longrightarrow \mathcal{I}_{X/\mathbb{P}^4} \longrightarrow \mathcal{I}_{\mathcal{C}/\mathbb{P}^4} \longrightarrow i_*(\mathcal{I}_\mathcal{C}) \longrightarrow 0$$

where $\mathcal{I}_{X/\mathbb{P}^4}$ and $\mathcal{I}_{\mathcal{C}/\mathbb{P}^4}$ denote respectively, the ideal sheaves of X and $\mathcal{C}$ in $\mathbb{P}^4$, and i is the inclusion $i : X \hookrightarrow \mathbb{P}^4$. Localizing at $x \in \mathcal{C}$ we get a short exact sequence of $\mathcal{O}_{\mathbb{P}^4,x}$modules:

$$0 \longrightarrow \mathcal{I}_{X/\mathbb{P}^4,x} \longrightarrow \mathcal{I}_{\mathcal{C}/\mathbb{P}^4,x} \longrightarrow (\mathcal{I}_{\mathcal{C},x}) \longrightarrow 0. \tag{2.5}$$

Since $H^0_x(\mathcal{O}_{\mathbb{P}^4,x}) = 0$, we deduce that $H^0_x(\mathcal{I}_{X/\mathbb{P}^4,x}) = 0$ and $H^0_x(\mathcal{I}_{\mathcal{C}/\mathbb{P}^4,x}) = 0$. Now, $H^i_x(\mathcal{I}_{X/\mathbb{P}^4,x}) \simeq H^{i-1}_x(\mathcal{O}_{X,x}) = 0$, for $i = 1,\dots,3$ (as $\mathcal{O}_{X,x}$ is a local 3-dimensional CM ring). Similarly, $H^1_x(\mathcal{I}_{\mathcal{C}/\mathbb{P}^4,x}) \simeq H^0_x(\mathcal{O}_{\mathcal{C},x}) = 0$. (as $\mathcal{O}_{\mathcal{C},x}$ is a CM local ring of dimension 1). From these computations, we see that, once we take the long exact sequence in local cohomology from (2.5),

$$\cdots \longrightarrow H^i_x(\mathcal{I}_{\mathcal{C}/\mathbb{P}^4,x}) \longrightarrow H^i_x(\mathcal{I}_{\mathcal{C},x}) \longrightarrow H^{i+1}_x(\mathcal{I}_{X/\mathbb{P}^4,x}) \longrightarrow \cdots$$

we get that:

- $H^0_x(\mathcal{I}_{\mathcal{C},x}) = H^1_x(\mathcal{I}_{\mathcal{C},x}) = 0$,
- $H^2_x(\mathcal{I}_{\mathcal{C},x}) \simeq H^2_x(\mathcal{I}_{\mathcal{C}/\mathbb{P}^4,x}) \simeq H^1_x(\mathcal{O}_{\mathcal{C},x}) \neq 0$.

These equalities imply that $\operatorname{depth}(\mathcal{I}_{\mathcal{C},x}) = 2$, hence $\operatorname{pd}(\mathcal{I}_{\mathcal{C},x}) = 1$. □

We will now illustrate how Serre's correspondence works on a (integral) cubic threefold X (for a more general description of Serre correspondence we refer to [HC1]).

We start from an AG curve $\mathcal{C} \subset X$. The canonical sheaf of $\mathcal{C}$ is thus isomorphic to $\mathcal{O}_\mathcal{C}(a)$ for some integer $a \in \mathbb{Z}$. From the isomorphisms

$$\omega_\mathcal{C} \simeq \mathscr{E}xt^2(\mathcal{O}_C, \omega_X) \simeq \mathscr{E}xt^2(\mathcal{O}_C, \mathcal{O}_X(-2)) \simeq \mathscr{E}xt^1(\mathcal{I}_\mathcal{C}, \mathcal{O}_X(-2))$$

we obtain:

$$\mathcal{O}_C \simeq \mathscr{E}xt^1(\mathcal{I}_C, \mathcal{O}_X(-2-a)).$$

As $\mathcal{C}$ is a codimension 2 AG subscheme of X, $\mathscr{E}xt^i(\mathcal{O}_\mathcal{C}, \mathcal{O}_X) \simeq \mathscr{E}xt^{i-1}(\mathcal{I}_\mathcal{C}, \mathcal{O}_X) = 0$, whenever $i \neq 2$; moreover, as $\mathscr{E}xt^2(\mathcal{I}_\mathcal{C}, \mathcal{O}_X)$ is supported on $\mathcal{C}$, $H^i(X, \mathscr{E}xt^1(\mathcal{I}_\mathcal{C}, \mathcal{O}_X)) = 0$ whenever $i \neq 0$ or $i \neq 1$. Therefore, applying the local to global spectral sequence (cf.[God]. II Th.7.3.3),

$$E_2^{p,q} = H^p(X, \mathscr{E}xt^q(\mathcal{O}_\mathcal{C}, \mathcal{O}_X(-2-a))) \Longrightarrow \operatorname{Ext}^{p+q}(\mathcal{O}_\mathcal{C}, \mathcal{O}_X(-2-a))$$

we get isomorphisms:

$$H^0(\mathcal{O}_\mathcal{C}) \simeq H^0(X, \mathscr{E}xt^1(\mathcal{I}_C, \mathcal{O}_X(-2-a))) \simeq \operatorname{Ext}^1(\mathcal{I}_\mathcal{C}, \mathcal{O}_X(-2-a)).$$

The unit element $1 \in \operatorname{Ext}^1(\mathcal{I}_\mathcal{C}, \mathcal{O}_X(-2-a))$ corresponds to a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-2-a) \longrightarrow \mathcal{N} \longrightarrow \mathcal{I}_\mathcal{C} \longrightarrow 0. \tag{2.6}$$

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Proposition 2.2.3. *$\mathcal{N}$ is a rank 2 ACM sheaf.*

Proof. From (2.6) we compute immediately that $\mathcal{N}$ has rank 2 and that the following hold:

- $H^i_x(\mathcal{N}_x) = 0$ for $i = 0, 1$, $\forall\, x \in X$. This is due to the fact that $H^i_x(\mathcal{O}_{X,x}) = 0$, and $H^i_x(\mathcal{I}_{\mathcal{C},x}) = 0$ for $i = 0, 1$.
- $H^1(\mathcal{N}(j)) = 0$, $\forall j \in \mathbb{Z}$. This is due to the fact that $\mathcal{C}$ is AG, hence $H^1_*(\mathcal{I}_\mathcal{C}) = 0$.

To conclude that $\mathcal{N}$ is ACM we still need to check that $H^2_x(\mathcal{N}_x) = 0\ \forall\, x \in X$ (so that $\mathcal{N}_x$ is Cohen-Macaulay $\forall x \in X$) and that $H^2(\mathcal{N}(j)) = 0$, $\forall j \in \mathbb{Z}$. It is proved in [H2] 1.11, that the sheaf $\mathcal{N}$ obtained from an extension corresponding to $1 \in \mathrm{Ext}^1(\mathcal{I}_\mathcal{C}, \mathcal{O}_X(-2-a))$ satisfies:

$$\mathscr{E}xt^1(\mathcal{N}, \mathcal{O}_X) = 0.$$
$$H^1_*(\mathcal{N}^\vee) = 0.$$

This allow us to prove that $\mathcal{N}$ is ACM since:

- Applying local duality, (cf. [Harl], ch. 6) as $\mathcal{O}_{X,x}$ is a local Gorenstein ring, we get:
$$H^2_x(\mathcal{N}_x) \simeq \mathrm{Hom}\,(\mathrm{Ext}^1(\mathcal{N}_x, \mathcal{O}_{X,x}), \omega_{X,x}).$$
This terms are thus both equal to 0 since $\mathrm{Ext}^1(\mathcal{N}_x, \mathcal{O}_{X,x}) \simeq \mathscr{E}xt^1(\mathcal{N}, \mathcal{O}_X)_x$ and $\mathscr{E}xt^1(\mathcal{N}, \mathcal{O}_X) = 0$. This implies that $\mathcal{N}$ is locally Cohen-Macaulay.
- From the previous point $\mathcal{N}$ is locally Cohen-Macaulay. Again, applying local duality we then deduce that $Ext^i_{\mathcal{O}_{X,x}}(\mathcal{N}_x, \mathcal{O}_{X,x}) \simeq Ext^i_{\mathcal{O}_{X,x}}(\mathcal{N}_x, \omega_{X,x}) = 0$, $\forall\, x \in X$, $\forall\, i > 0$, implying that $\mathscr{E}xt^i(\mathcal{N}, \omega_X) = 0$, $\forall\, i > 0$. From the local to global spectral sequence:
$$E_2^{p,q} = H^p(X, \mathscr{E}xt^q(\mathcal{N}, \omega_X)) \Longrightarrow Ext^{p+q}(\mathcal{N}, \omega_X),$$
we obtain an isomorphism $Ext^i(\mathcal{N}, \omega_X) \simeq H^i(X, \mathscr{H}om(\mathcal{N}, \omega_X))$. By Serre's duality we get $Ext^i(\mathcal{N}, \omega_X) \simeq H^{n-i}(\mathcal{N})^*$ and since X is AG with canonical sheaf $\omega_X = \mathcal{O}_X(-2)$, we conclude that $H^i(\mathcal{N}^\vee(-2))$ is dual to $H^{n-i}(\mathcal{N})$ for all $i > 0$. Applying this argument to any twist of $\mathcal{N}$ we get that, for all $j \in \mathbb{Z}$ and for $i = 1$, $H^1(\mathcal{N}^\vee(-j-2))$ is dual to $H^2(\mathcal{N}(j))$. Since $H^1_*(\mathcal{N}^\vee) = 0$ we conclude that $H^2_*(\mathcal{N}) = 0$.

□

2.2.2 Rank 2 Ulrich sheaves from elliptic quintics

Suppose now that $\mathcal{C} \subset X$ is an AG curve having arithmetic genus $p_a(\mathcal{C}) = 1$ (so that $\omega_\mathcal{C} \simeq \mathcal{O}_\mathcal{C}$) and degree 5. Suppose that moreover $\mathcal{C}$ is non-degenerate in $\mathbb{P}^4$ (namely $\mathcal{C}$ spans the entire $\mathbb{P}^4$). Applying Serre's construction we get the short exact sequence (2.6) corresponding to the class of $1 \in H^0(\mathcal{O}_\mathcal{C}) \simeq \mathrm{Ext}^1(\mathcal{I}_\mathcal{C}, \mathcal{O}_X(-2))$ and consequently a rank 2 ACM sheaf $\mathcal{N}$ on X. If now we twist (2.6) by $\mathcal{O}_X(2)$, we get:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_\mathcal{C}(2) \longrightarrow 0 \tag{2.7}$$

where $\mathcal{E} := \mathcal{N}(2)$. From it, since $h^0(\mathcal{I}_\mathcal{C}(2)) = h^0(\mathcal{O}_X(2)) - h^0(\mathcal{O}_\mathcal{C}(2)) = 15 - 10 = 5$, we compute that $h^0(\mathcal{E}) = 6$. Moreover, from the assumption of non-degeneracy of $\mathcal{C}$ we have $h^0(\mathcal{I}_\mathcal{C}(1)) = 0$ so that $h^0(\mathcal{E}(-1)) = 0$.
From these fact and as we are assuming that X is integral, we can conclude that $\mathcal{E}$ is Ulrich, due to:
Proposition 2.2.4. *Let $X \subset \mathbb{P}^n$ be an integral hypersurface of degree d, and $\mathcal{E}$ an ACM sheaf of rank r such that $h^0(\mathcal{E}(-1)) = 0$ and $h^0(\mathcal{E}) = rd$. Then $\mathcal{E}$ is Ulrich.*

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Proof. See [HC2], Lemma 2.2. □

From what we have explained until now, whenever we are able to prove that an integral cubic threefold X carries an AG elliptic quintic $\mathcal{C}$ spanning the entire projective space $\mathbb{P}^4$, we get the existence of a rank 2 Ulrich sheaf $\mathcal{E}$ on X.
Now we want to locate the singular points of $\mathcal{E}$, namely those points $x \in X$ where $\mathcal{E}_x$ (or equivalently $\mathcal{N}_x$) is not a free $\mathcal{O}_{X,x}$-module, as follows. We start by localizing (2.6) at a point $x \in X$, getting in this way a short exact sequence of $\mathcal{O}_{X,x}$ modules:

$$0 \longrightarrow \mathcal{O}_{X,x} \longrightarrow \mathcal{N}_x \longrightarrow \mathcal{I}_{\mathcal{C},x} \longrightarrow 0. \tag{2.8}$$

Note that this short exact sequence corresponds to the class $1_x \in \mathrm{Ext}^1_{\mathcal{O}_{X,x}}(\mathcal{I}_{\mathcal{C},x}, \mathcal{O}_{X,x})$. It's clear that $\forall\, x \notin \mathcal{C}$, $\mathcal{N}_x$ is free. Indeed if $x \notin \mathcal{C}$, we have $\mathcal{I}_{\mathcal{C},x} \simeq \mathcal{O}_{X,x}$, implying that 2.8 splits and that thus $\mathcal{N}_x \simeq \mathcal{O}_{X,x}^{\oplus 2}$. Hence if x is a singular point of $\mathcal{E}$, we must have $x \in \mathcal{C}$. Since by proposition 2.2.2, $\mathcal{I}_{\mathcal{C},x}$ is a $\mathcal{O}_{X,x}$-module of projective dimension 1, we can check if x is a singular point of $\mathcal{E}$ applying the following proposition, proved by Serre (see [OSS], Lemma 5.1.2):

Proposition 2.2.5 ((Serre))**.** *Let A be a Noetherian local ring, $I \subset A$ an ideal admitting a free resolution of length 1:*

$$0 \longrightarrow A^p \longrightarrow A^q \longrightarrow I \longrightarrow 0.$$

Let $e \in \mathrm{Ext}^1(I, A)$ be represented by the extension

$$0 \longrightarrow A \longrightarrow M \longrightarrow I \longrightarrow 0.$$

Then M is a free A-module if and only if e generates the A-module $\mathrm{Ext}^1(I, A)$.

Remark 13. We observe that if ever $\mathcal{E}$ is locally free, then it is a skew-symmetric Ulrich bundle of rank 2. This is due to the fact that if $\mathcal{E}$ is a vector bundle, it is endowed with a "natural" skew-form ϕ, $\phi : \mathcal{E} \to \mathscr{H}om(\mathcal{E}, \mathcal{O}_X(2))$. ϕ is just the natural isomorphism:

$$\phi : \mathcal{E} \xrightarrow{\simeq} \mathcal{E}^\vee \otimes \bigwedge^2 \mathcal{E}$$

that on each fibre $x \in X$ acts as $\phi_x : \mathcal{E}_x \xrightarrow{\simeq} \mathcal{E}_x^\vee \otimes \bigwedge^2 \mathcal{E}_x$, $s \mapsto (t \mapsto s \wedge t)$. Furthermore, whenever $\mathcal{E}$ is locally free, from (2.7) we can compute that $\deg(\mathcal{E}) = 2$, getting, finally, $\phi : \mathcal{E} \xrightarrow{\simeq} \mathcal{E}^\vee \otimes \bigwedge^2 \mathcal{E} \simeq \mathscr{H}om(\mathcal{E}, \bigwedge^2 \mathcal{E}) \simeq \mathscr{H}om(\mathcal{E}, \mathcal{O}_X(2))$. From this remark we understand that the existence of a rank 2 Ulrich bundle on X is sufficient to conclude that X is Pfaffian. In the next section we will show how to obtain, on certain singular cubic threefolds, rank 2 Ulrich sheaves that are, for construction, locally free. Anyway note that conversely, the existence of a Pfaffian representation of a cubic X does not ensure the existence of a Ulrich *bundle* on X; we can indeed have Pfaffian representations corresponding to skew-symmetric Ulrich sheaves that are not necessarily locally free. Anyway in the thesis we won't deal with these cases.

The smooth case

Whenever X is smooth, it was proved in [MT1], that there always exists a smooth quintic elliptic curve not contained in a hyperplane. There are essentially two different ways to prove this. The first is to prove that there exists a smooth elliptic quintic contained in a hyperplane section and then show that it deforms to a nondegenerate one, in the sense that it spans the whole projective space $\mathbb{P}^4$. The second method is "constructive" and produces directly a nondegenerate elliptic quintic. The curve is obtained as the residual one to a rational normal quartic $\Gamma \subset X \cap S$ in $X \cap \Sigma$, where Σ is a cubic scroll. In next sections we will describe in greater detail these techniques showing how we can adapt them to singular cases.

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2.3 Pfaffian representations of normal cubic threefolds

Let X be a normal cubic threefold that does not contain triple points. We show here that X is always Pfaffian and we do this proving the existence of a rank 2 Ulrich bundle on X:

Theorem 2.3.1. *Let X be a normal cubic threefold that does not contain triple points. Then there always exists a rank 2 Ulrich bundle $\mathcal{E}$ on X.*

From this result we deduce immediately the Pfaffian representability of X since, as we saw in remark 13, a rank 2 Ulrich bundle is always skew-symmetric. The first step to show the existence of $\mathcal{E}$ is to show that on X there always exists a non-degenerate AG quintic elliptic curve. We will prove the following:

Theorem 2.3.2. *Let X be a normal cubic threefold that is not a cone. Then there exists a non-degenerate AG elliptic quintic curve $\mathcal{C} \subset X$.*

The Ulrich bundle $\mathcal{E}$ is then obtained from $\mathcal{C}$ by Serre correspondence. Before illustrating the proofs of theorems 2.3.1 and 2.3.2 we present, following Segre's work [Seg], a brief essay on normal cubic threefolds.

2.3.1 Normal cubic threefolds

We recall here some generalities about singular hypersurfaces. Let $X \subset \mathbb{P}^4$ be a cubic hypersurface defined by a homogeneous polynomial F of degree 3; throughout the rest of the section we will always suppose that X does not contain triple points. If X presents a singular point $x \in X$, we might then choose homogeneous coordinates $X_0, \ldots, X_4$ on $\mathbb{P}^4$ in such a way that x is the point $[1:0:0:0:0]$; consequently the polynomial F can be written as:

$$F(X_0, \ldots, X_4) = X_0^2 q_x(X_1, \ldots, X_4) + c_x(X_1, \ldots, X_4).$$

The locus Q_x defined by the equation $q_x(X_1, \ldots X_4) = 0$ is a quadric hypersurface called the *quadric tangent cone* of X at x. It can be characterized as the set of points $Q_x = \{y \in \mathbb{P}^4 \mid \ mult_x(F|_{\overline{xy}}) \geq 3\}$.

Definition 2.8. We say that the point $x \in X$ is a double point of r-th type if Q_x, the quadric tangent cone to X at x is a quadric of rank $5 - r$.

In the upcoming sections we will also refer to a double point $x \in X$ of r-th type, for $r = 3, 2, 1$ as, respectively, a conic node, a binode and a unode.
Take now a hyperplane $H_x \subset \mathbb{P}^4$ not passing through x. Up to a suitable change of coordinates we might assume that H_x has equation $X_0 = 0$. We consider the locus $\mathcal{D}_x^{2,3} \subset H_x$ defined as:

$$\mathcal{D}_x^{2,3} := H_x \cap Q_x \cap X.$$

This is a complete intersection sextic curve and we see that for every point $y \in \mathcal{D}_x$, the polynomial $F|_{\overline{xy}}$ has a root of multiplicity at least 3 in x and a root in y. Since $\deg(F) = 3$, we deduce that the entire line $\overline{xy}$ is contained in X. This means that the union of lines passing through x and contained in X is the cone of vertex x over $\mathcal{D}_x^{2,3}$. Note also that $Q_x \cap H_x$ and $X \cap H_x$ meet at a point $y \in \mathcal{D}_x^{2,3}$ with multiplicity bigger then one if and only if the line $\overline{xy} \subset X$ passes through a singular point of X different from x.
If now we suppose that X is normal, we have that the singular locus $\operatorname{Sing}(X)$, has dimension at most 1. If ever X has non-isolated singularities, $\operatorname{Sing}(X)$ contains then a curve Y. Consequently we give the following definition:

Definition 2.9. We say that a curve $Y \subset \operatorname{Sing}(X)$ is a *double curve of r-th type* if every point in Y is a double point and the generic point of each of its irreducible components is a double point of $(r-1)$-th type.

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Under the additional assumption that X is not a cone (namely that X does not contain triple points), following Segre's classification of three-dimensional cubic hypersurfaces ([Seg]), we conclude that $\mathrm{Sing}\,(X)$ might consist of:

- $N \leq 10$ isolated singular points if $\dim(\mathrm{Sing}\,(X)) = 0$ (including the case when $\mathrm{Sing}\,(X) = \varnothing$).
- The union $Y \cup Z$ of a finite length scheme Z (including the case when $Z = \varnothing$) and a double curve Y, if $(\dim(\mathrm{Sing}\,(X)) = 1$, where Y is one of the following:
  - A line of first or second type.
  - A conic of first or second type (possibly degenerating into a union of two incident lines).
  - The union of three non-coplanar lines of first type meeting at a point.
  - A rational quartic curve of first type (that can possibly degenerate in the union of two conics meeting at a point but not both contained in a hyperplane).

Remark 14. Note that saying that X has multiplicity 2 along the curve $Y \subset X$, means that all the partial derivatives of order 1 of F vanishes along Y. In other words, denoting by $I_Y \subset \mathbb{C}[X_0, \ldots, X_4]$ the homogeneous ideal defining Y, we have that for $i = 0, \ldots, 4$, $\frac{\partial F}{\partial X_i} \in I_Y$. This implies in particular that the polynomial F can be written as $F = \sum_{i=0}^{4} X_i Q_i$ with $Q_i \in I_{Y\,2} := (I_Y \cap \mathbb{C}[X_0, \ldots X_4]_2)$ (the space of homogeneous forms of degree 2 belonging to I_Y). The hypothesis that X has no triple points ensures that $\forall\, y \in Y$, there exists at least one partial derivative of F of order 2 not vanishing at y.

In the upcoming sections we will need to study the behavior of (general) hyperplane sections of X and to this aim it is necessary to understand the kind of singularities that they might present. If ever X has isolated singularities, applying Bertini's theorem we have that for a general hyperplane $H \subset \mathbb{P}^4$, $S := H \cap X$ is smooth. Whenever $\dim(\mathrm{Sing}\,(X)) = 1$, a generic hyperplane section of X is a cubic surface with isolated singularities. More precisely, taking a hyperplane $H \subset \mathbb{P}^4$ corresponding to a point $h \in \mathbb{P}^{4^*}$ that belongs to the open $\mathbb{P}^{4^*} \setminus (X^*)$ (here X^* is the dual variety of X), the intersection $H \cap X$ will be singular along $\mathrm{Sing}\,(X) \cap H$. When X is a cubic threefold in the aforementioned list, following Segre's classification, we have at most a finite number of singular points of X not belonging to the curve Y. This means that for a general hyperplane $H \subset \mathbb{P}^4$, the intersection $S := H \cap X$ satisfies $\mathrm{Sing}\,(S) = Y \cap H$, therefore S is a cubic surface with at most 4 double points. In order to define the type of these singularities it is necessary to describe in greater detail the quadric tangent cones to X at points belonging to 1-dimensional components of $\mathrm{Sing}\,(X)$. This is done writing down a normal form for the polynomial F and with the aid of Segre's work. Our study of the singularities of S relies on Bruce and Walls classification of cubic surfaces presented in [BW].

Cubic threefolds singular along a line

We suppose here that X contains a double line Y supported on the line $\overline{Y} = \nu_1(\mathbb{P}^1)$ where ν_1 is defined as:

$$\begin{aligned} \nu_1 : \mathbb{P}^1 &\longrightarrow \mathbb{P}^4 \\ [t_0 : t_1] &\mapsto [0 : 0 : 0 : t_0 : t_1]. \end{aligned}$$

For an appropriate choice of homogeneous coordinates $X_0, \ldots, X_4$ we can assume that $\overline{Y}$ has equations $\{X_0 = X_1 = X_2 = 0\}$. We call $V := \mathbb{C}\langle X_0, \ldots X_4\rangle$, $V' := \mathbb{C}\langle X_0, X_1, X_2\rangle$, $V'' := \mathbb{C}\langle X_3, X_4\rangle$. We get a decomposition

$$S^3(V) = \bigoplus_{i=0}^{3} S^i(V') \otimes S^{3-i}(V'')$$

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and consequently F can be written as

$$F = \sum_{i=0}^{3} F_i, \quad F_i \in S^i(V') \otimes S^{3-i}(V'')$$

Because of the fact that X has multiplicity 2 along Y, we have $F_0 = 0,\ F_1 = 0$ hence F can be reduced to the following form:

$$F(X_0, \dots X_4) = X_3 q_3(X_0, X_1, X_2) + X_4 q_4(X_0, X_1, X_2) + c(X_0, X_1, X_2).$$

with $\deg(q_3(X_0, X_1, X_2)) = \deg(q_4(X_0, X_1, X_2)) = 2$ and $\deg(c(X_0, X_1, X_2)) = 3$.
The quadric tangent cones to points in Y draw a pencil of quadrics $\mathcal{Q}_Y \subset |\mathcal{O}_{\mathbb{P}^4}(2)|$. The quadric tangent cone to the point $[t_0 : t_1] \in \overline{Y}$ is the element $\mathcal{Q}_{[t_0:t_1]} \in \mathcal{Q}_Y$ defined by:

$$\mathcal{Q}_{[t_0:t_1]} = t_0 q_3(X_0, X_1, X_2) + t_1 q_4(X_0, X_1, X_2). \tag{2.9}$$

We observe that every element of $\mathcal{Q}_Y$ is singular along the line $\overline{Y}$, hence it has rank less than or equal to 3. We recognize here two subcases:

- A general element in $\mathcal{Q}_Y$ has rank 3 (namely the line Y is of *first type*);
- Every quadric in $\mathcal{Q}_Y$ has rank at most 2 (namely the line Y is of *second type*).

Y is a line of first type
If Y is a line of first type, we see from (2.9), that on Y we have 3 binodes $x_1,\ x_2,\ x_3$. (These correspond to the three points where the pencil $\mathcal{Q}_Y$ meets the locus of quadrics of rank less than 3).

Proposition 2.3.3. *Let X be a cubic threefold singular along a line Y of first type. Then a general hyperplane section of X is a cubic surface with one A_1 singularity.*

Proof. We know that for H general, the cubic surface $S := H \cap X$ is singular along $Y \cap H$. Whenever such a hyperplane H meets the line Y in a point $y_0 \neq x_i$ for $i = 0, 1, 2$ (again, all these are "open conditions" on $\mathbb{P}^{4*}$), the resulting intersection S is a cubic surface whose only singular point is y_0. The condition that $Y \cap H = \{y_0\}$ implies that $Y \not\subset H$, so that the quadric tangent cone to S at y_0 is defined by a quadric of rank 3 (singular along y_0), hence y_0 is a conic node. S has thus an A_1 singularity in y_0. □

Y is a line of second type
If the line Y is of second type, a general element $y_0 \in Y$ is a binode; every quadric in the pencil $\mathcal{Q}_Y$ is singular along a plane and is a pair of linear spaces $\mathbb{P}^3$. For a suitable change of coordinates we might suppose that $y_0 = [1 : 0 : 0 : 0 : 0]$, so that F can be written as:

$$F = X_0 q_0(X_3, X_4) + c'(X_1, X_2, X_3, X_4).$$

Consider $H_0 \subset \mathbb{P}^4$ an hyperplane orthogonal to y_0, that we can assume having equation $\{X_0 = 0\}$. All lines passing through y_0 are of the form $\overline{y_0 x}$, where x varies along the sestic curve $\mathcal{D}^{2,3}_{y_0} \subset H_0$:

$$\mathcal{D}^{2,3}_{y_0} = \{X_0 = q_0(X_3, X_4) = c'(X_1, X_2, X_3, X_4) = 0\}.$$

We observe that this curve is not irreducible but is the union of two plane cubic curves. Indeed, denote by S_0 the intersection $X \cap H_0$ and by Q_0 the quadric defined by the equations $\{X_0 = 0,\ q_0(X_3, X_4) = 0\}$. Q_0 is a union of two planes $\Delta_3,\ \Delta_4$, so that $\mathcal{D}^{2,3}_{y_0} = (S_0 \cap \Delta_3) \cup (S_0 \cap \Delta_4)$. This also allows us to notice that X can have at most one other singular point. That's because all singular points of X lie on lines $\overline{y_0 x}$, where x is a point in $\mathcal{D}^{2,3}_{y_0}$ and S_0, Q_0 meet with multiplicity greater than one. As $Q_0 = \Delta_3 \cup \Delta_4$, we deduce that all singular points must project to $S_0 \cap L_{3,4}$ where $L_{3,4} \subset H_0$ is the line $\Delta_3 \cap \Delta_4$. This intersection consist of the point $Y \cap H_0$ and of at most one other point (otherwise we would have X singular along the plane $\langle y_0, L_{3,4} \rangle$).

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Proposition 2.3.4. *Let X be a cubic threefold singular along a line of second type. Then a general hyperplane section of X is a cubic surface with one A_2 singularity.*

Proof. Consider a hyperplane H such that $\mathrm{Sing}\,(X \cap H)$ consists of $Y \cap H = y_0$ and that moreover satisfies the condition $\mathrm{rk}\,(q_0|_H) = 2$ (H is a general hyperplane through y_0). Under generality assumptions, we might also suppose that $H \cap (L_{3,4} \cap S_0) = \varnothing$. The cubic surface $S = X \cap H$ presents just one singular point at y_0, a binode. Following [BW] Lemma 3, in order to determine the nature of the singularity, we have to look at the intersection $H \cap (S_0 \cap L_{3,4})$. Since for our choice of H this intersection is empty, we can conclude that the point y_0 is an A_2 singularity. □

Cubic threefolds singular along a conic

Let be X be a cubic threefold containing a double conic Y supported on $\overline{Y} = \nu_2(\mathbb{P}^1)$ where ν_2 is defined as:

$$\begin{aligned}\nu_1 : \mathbb{P}^1 &\longrightarrow \mathbb{P}^4 \\ [t_0 : t_1] &\mapsto [t_0^2 : t_0t_1 : t_1^2 : 0 : 0].\end{aligned}$$

Choose coordinates on $\mathbb{P}^4$ in such a way that $\overline{Y} := \{X_3 = 0,\ X_4 = 0,\ q(X_0, X_1, X_2) = 0\}$, where $q(X_0, X_1, X_2)$ is the polynomial $X_1^2 - X_0X_2$. We know that if $Y \subset \mathrm{Sing}\,(X)$, we can then write F as $F = \sum_{i=0}^{4} X_iQ_i$ with $Q_i \in (X_3,\ X_4,\ q(X_0, X_1, X_2))_2$. Hence

$$F = \sum_{i,j\in\{3,4\}} l_{ij}(X_0, \ldots X_4)X_iX_j + l_{012}(X_0, \ldots X_4)q(X_0, X_1, X_2) \tag{2.10}$$

where the l_{ij}s and l_{012} are linear forms. From 2.10 we get:

$$F = c(X_3, X_4) + \sum_{i=0}^{2} X_iq_i(X_3, X_4) + l(X_3, X_4)q(X_0, X_1, X_2) + a(X_0, X_1, X_2)$$

where $\deg(l(X_3, X_4)) = 1$, $\deg q_i(X_3, X_4) = \deg(q(X_0, X_1, X_2)) = 2$, $i = 0, \ldots, 2$ and $\deg(c(X_3, X_4)) = \deg(a(X_0, X_1, X_2)) = 3$. Call $\Delta \simeq \mathbb{P}^2$ the plane defined by the equations $\{X_3 = 0, X_4 = 0\}$. As $Y \subset \Delta$, the plane cubic $X \cap \Delta = \{a(X_0, X_1, X_2) = 0\}$ must be singular along Y hence we can conclude that $a(X_0, X_1, X_2)$ must be equal to zero. Finally we obtain that F can be reduced to the following form:

$$F = c(X_3, X_4) + \sum_{i=0}^{2} X_iq_i(X_3, X_4) + l(X_3, X_4)q(X_0, X_1, X_2) \tag{2.11}$$

This time the quadric tangent cones to X at points on Y describe a conic $\mathcal{Q}_Y \subset |\mathcal{O}_{\mathbb{P}^4}(2)|$. The quadric tangent cone at the point $[t_0 : t_1] \in \overline{Y}$ is the element $\mathcal{Q}_{[t_0:t_1]} \in \mathcal{Q}_Y$:

$$\mathcal{Q}_{[t_0:t_1]} = (t_0^2q_0(X_3, X_4) + t_0t_1q_1(X_3, X_4) + t_1^2q_2(X_3, X_4)) - ({t_0}^2X_2 - 2t_0t_1X_1 + {t_1}^2X_0)l(X_3, X_4)$$

Y is a conic of first type

We suppose now that the generic element of $\mathcal{Q}_Y$ is a quadric of rank 3 (namely Y is a double conic of *first type*). If this is so, the conic Y presents two binodes x_1, x_2. These are the two points whose coordinates $[t_0 : t_1]$ are such that the quadric of equation $t_0^2q_0(X_3, X_4) + t_0t_1q_1(X_3, X_4) + t_1^2q_2(X_3, X_4)$ contains the hyperplane $\{l(X_3, X_4) = 0\}$.

Proposition 2.3.5. *Let X be a cubic threefold singular along a conic of first type. Then a general hyperplane section of X is a cubic surface with 2 A_1 singularities.*

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Proof. For $y \neq x_i, i = 1, 2$, the quadric tangent cone at y is a quadric singular along the line $\mathbb{T}_y\overline{Y}$. Consider now a hyperplane $H \subset \mathbb{P}^4$ such that $\mathrm{Sing}\,(X \cap H) = Y \cap H$ consists of two points $y_1,\ y_2$ different from x_1 and x_2. Choosing this hyperplane in such a way that $\mathbb{T}_{y_i}\overline{Y} \not\subset H$, for $i = 1,\ 2$; we get that the hyperplane section $S := X \cap H$ is a cubic surface presenting just two conic nodes $y_1,\ y_2$. S is thus a cubic surface with $2A_1$ singularities. □

Remark 15. The conic Y might also be supported on $\overline{Y} = L_0 \cup L_1$, the union of two lines meeting at a point x_0. We still can choose coordinates on $\mathbb{P}^4$ in a way that F has the form 2.11, but this time the degree two form q can be written as $q(X_0, X_1) = X_0X_1$. We assume that the line L_i has equations $\{X_3 = X_4 = X_i = 0\}$, for $i = 0, 1$. The quadric tangent cones at points in Y define now two pencils of quadrics $\mathcal{Q}_{L_i} \subset |\mathcal{O}_{\mathbb{P}^4}(2)|,\ i = 1, 2$, meeting at a point. Choose an arbitrary point y on one of these lines, say L_0, $y = [0 : t_0 : t_1 : 0 : 0]$. The quadric tangent cone at this point is:

$$\mathcal{Q}_{L_0,[t_0:t_1]} = (t_0q_0(X_3, X_4) + t_1q_2(X_3, X_4)) + l(X_3, X_4)(t_0X_1).$$

Hence a general point on each line is a conic node whether the point x_0 is either a binode or a unode. For a general hyperplane H, $X \cap H$ is again a cubic surface having two A_1 singularities in $H \cap L_i,\ i = 0, 1$.

Y is a conic of second type

Suppose now that Y is a conic of second type, namely a general point of Y is a binode. The condition that Y is a double conic of second type is equivalent to having $l(X_3, X_4)|q_i(X_3, X_4)$ for $i = 0, 1, 2$ in the equation 2.11. The polynomial F can then be written as:

$$F = c(X_3, X_4) + X_3(\sum_{i=0}^{2}(\alpha_i X_4 X_i)) + X_3q(X_0, X_1, X_2)$$

where $c(X_3, X_4)$ is a form of degree 3 such that $X_3 \nmid c(X_3, X_4)$. This time the conic $\mathcal{Q}_Y \subset |\mathcal{O}_{\mathbb{P}^4}(2)|$ parametrizing quadric tangent cones to X at points $[t_0 : t_1] \in Y$ has the form:

$$\mathcal{Q}_{Y[t_0:t_1]} = X_3[t_0^2(\alpha_0X_4 - X_2) + t_0t_1(\alpha_1X_4 - 2X_1) + t_1^2(\alpha_2X_4 - X_0)].$$

Each of these quadric cones decomposes in the union of two spaces. One of them is a hyperplane T fixed (it is the hyperplane of equation $X_3 = 0$), the other varies along a family of hyperplanes over Y, $\mathcal{H}_{Y,[t_0:t_1]} \subset |\mathcal{O}_{\mathbb{P}^4}(1)|$. We denote by $h_{[t_0:t_1]}$ the linear form defining $\mathcal{H}_{Y,[t_0:t_1]}$.

Proposition 2.3.6. *Let X be a cubic threefold singular along a conic of second type. Then a generic hyperplane section of X is a cubic surface with 2 A_2 singularities.*

For a general hyperplane $H \subset \mathbb{P}^4$, the cubic surface $S = X \cap H$ is singular along 2 binodes $\{y_1,\ y_2\} = Y \cap H$. Arguing exactly as for the case of cubic threefolds containing a double line of second type, we can choose H in such a way that at each point y_i, the surface S has an A_2 singularity. Indeed taking any point y having coordinates $[t_0 : t_1]$ on Y and a plane H_y orthogonal to it defined by a linear equation $h(X_0, \ldots, X_4) = 0$, we compute that the intersection $H_y \cap X \cap \mathcal{H}_{Y,[t_0:t_1]} \cap T$ is defined by the equations

$$h(X_0, \ldots, X_4) = h_{[t_0:t_1]}(X_0, \ldots X_4) = X_3 = c(X_3, X_4) = 0$$

and consists then in at most 3 points. Imposing the additional (open) condition that H does not pass through any of these points, we finally have that $S = H \cap X$ presents two A_2 singularities at $y_1,\ y_2$.

Remark 16. Again, the conic Y might also degenerate to the union of two double lines of second type $Y_1,\ Y_2$ meeting at a point x_0. If this is the case, every point $y \in Y$ is a binode except from the point x_0 that is a unode. Anyway arguing exactly as in the smooth case we can conclude that a general hyperplane section of X is still a cubic surface with $2A_2$ singularities.

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Cubic threefolds singular along three non coplanar lines meeting at a point

Suppose that X contains a double curve $Y \subset \mathrm{Sing}\,(X)$ supported on the union of three lines L_0, L_1, L_2 meeting at a point x_0 and not lying in the same plane. Choose linear coordinates in such a way that the lines L_is are given by:

$$L_0 = \{X_3 = X_1 = X_2 = 0\} \quad L_1 = \{X_3 = X_0 = X_2 = 0\} \quad L_2 = \{X_3 = X_0 = X_1 = 0\}.$$

For this choice of coordinates on $\mathbb{P}^4$ the homogeneous ideal defining the curve Y is $(X_3,\ X_0X_1,\ X_0X_2,\ X_1X_2)$ and x_0 is the point $[0:0:0:0:1]$. F belongs to the ideal $(X_3,\ X_0X_1,\ X_0X_2,\ X_1X_2)$ and since F is singular along each line L_i we have

$$F = \sum_{i,j\in\{0,1,2\}} l_{ij}(X_0, \dots X_4)X_iX_j + h(X_0, \dots X_4)X_3^2,$$

where $\deg(l_{ij}(X_0, \dots X_4)) = \deg(h(X_0, \dots X_4)) = 1$ leading to:

$$F = \alpha X_3^3 + X_3^2 l(X_0, X_1, X_2, X_4) + X_3 q(X_0, X_1, X_2) + X_4(\sum_{i=0}^{2} \beta_i \frac{X_0X_1X_2}{X_i})$$

where l is a linear form $l = \sum_{i\neq 3} a_iX_i$ and q is a degree 2 polynomial $q = \sum_{\substack{i,j=0\\ i\neq j}}^{2} b_{ij}X_iX_j$.

The quadric tangent cones at points in Y determine three pencils of quadrics $\mathcal{Q}_{L_i} \subset |\mathcal{O}_{\mathbb{P}^4}(2)|$:

$$\mathcal{Q}_{L_i} = t_0(a_iX_3^2 + X_3(\sum_{\substack{j=0,\\ i\neq j}}^{2} b_{ij}X_j) + X_4(\sum_{\substack{j=0,\\ i\neq j}}^{2} \beta_jX_j)) + t_1(a_4X_3^2 + \sum_{j=0}^{2} \beta_j \frac{X_0X_1X_2}{X_j}),$$

with $[t_0 : t_1] \in \mathbb{P}^1,\ i, j, k \in \{0, 1, 2\}$.
We see thus that a general element of each line is a conic node, the point x_0 is a unode.

Proposition 2.3.7. *Let X be a cubic threefold singular along three non coplanar lines meeting at a point. Then a general hyperplane section of X is a cubic surface with 3 A_1 singularities.*

Proof. Arguing exactly as in the previous cases of curves of first type, we see that for a general hyperplane H, $S = H \cap X$ is a cubic surface with three conic nodes $\{y_0,\ y_1, y_2,\}$ $y_i = H \cap L_i$; hence a cubic surface with 3 A_1 singularities. □

Cubic threefolds singular along a rational normal quartic

We consider X, the secant variety of a rational quartic curve Y. The singular locus of X is the entire curve Y. We express Y as the image of the embedding:

$$\begin{aligned} \nu_4 : \mathbb{P}^1 &\longrightarrow \mathbb{P}^4 \\ [t_0 : t_1] &\mapsto [t_0^4 : t_0^3t_1 : t_0^2t_1^2 : t_0t_1^3 : t_1^4]. \end{aligned}$$

Y is the intersections of the quadrics:

$$q_0 = X_2X_4 - X_3^2,\ q_1 = X_2X_3 - X_1X_4,\ q_2 = 2X_1X_3 - 3X_2^2,$$

$$q_3 = X_1X_2 - X_0X_3,\ q_4 = X_0X_2 - X_1^2.$$

defined by the minors of the matrix:

$$\begin{pmatrix} X_0 & X_1 & X_2 & X_3 \\ X_1 & X_2 & X_3 & X_4 \end{pmatrix}$$

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The polynomial F defining X, secant variety of Y, is the determinant of the matrix:

$$N = \begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_2 & X_3 \\ X_2 & X_3 & X_4 \end{pmatrix}$$

Therefore we get:

$$F = X_0(X_2X_4 - X_3^2) + X_2(X_1X_3 - X_2^2) + X_1(X_3X_2 - X_1X_4).$$

The quadric tangent cones to X at points on Y belong to a rational quartic $\mathcal{Q}_Y$ contained in $|\mathcal{O}_{\mathbb{P}^4}(2)|$. The quadric tangent cone at the point $[t_0 : t_1] \in Y$ is defined by:

$$\mathcal{Q}_{Y,[t_0:t_1]} = t_0^4 q_0 + 2t_0^3 t_1 q_1 + t_0^2 t_1^2 q_1 + 2t_0 t_1^3 q_3 + t_1^4 q_4. \tag{2.12}$$

Proposition 2.3.8. *Let X be the secant cubic threefold. Then a general hyperplane section of X is a cubic surface with 4 A_1 singularities.*

Proof. For a general hyperplane $H \subset \mathbb{P}^4$, the cubic surface $S := H \cap X$ is singular along $Y \cap H$ hence it presents 4 singular points $y_1, \dots y_4$. From (2.12) we see that a general point of Y is a conic node, therefore under generality assumptions we can suppose that S presents 4 A_1 singularities at the points y_i. □

Remark 17. A possible degeneration of the situation we have just studied occurs when the quartic consists of the union of two conics Y_1, Y_2 meeting in a point x_0 but not lying in a same hyperplane. The general point of each conic is a conic node (more precisely every point in Y is a conic node apart from the point x_0 that is a binode). Again, a general hyperplane section is a cubic surface presenting 4 A_1 singularities.

Summing up what we have proved so far is the following:

Proposition 2.3.9. *Let X be a cubic threefold singular along a curve Y of degree d and of r-th type. Then for a general hyperplane $H \subset \mathbb{P}^4$, $S := H \cap X$ is a cubic surface with d A_r singularities.*

The upcoming sections are devoted to the proof of theorem 2.3.2. We will exhibit two different methods to prove the existence of the AG elliptic quintic $\mathcal{C}$ on a normal cubic threefold X. The first method relies on a deformation argument and allows us to prove that on certain normal threefolds there always exists a *smooth* normal elliptic quintic; the second method is constructive and produces directly AG elliptic quintics. Both methods are based on the study of curves on a general hyperplane section of X. From what we have just explained, for a general hyperplane $H \subset \mathbb{P}^4$, the intersection $S := X \cap H$ is a cubic surface with isolated singularities that are at most rational double points (RDPs for short). The study of curves on S will then reduce to the study of curves on a minimal resolution of singularities of S:

$$\phi : \tilde{S} \to S.$$

A cubic surface $S \subset \mathbb{P}^3$ presenting rational double points is a del Pezzo surface of degree 3. A minimal resolution $\tilde{S} \to S$ is a so called *weak Del Pezzo surface* of degree 3.

2.3.2 Singular cubic surfaces and weak Del Pezzo surfaces

We recall in this section some properties of Del Pezzo and weak Del Pezzo surfaces. The main references for the section are [CAG] and [Dem].

Definition 2.10. A Del Pezzo surface S is a nondegenerate irreducible surface of degree d in $\mathbb{P}^d$ that is not a cone and that is not isomorphic to a projection of a surface of degree d in $\mathbb{P}^{d+1}$.

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Theorem 2.3.10. *Let S be a Del Pezzo surface of degree d in $\mathbb{P}^d$. Then all its singularities are rational double points and ω_S^{-1} is an ample invertible sheaf.*

See [CAG] 8.1.3 for a proof.

If S is a singular Del Pezzo surface, a minimal resolution of S, $\phi : \tilde{S} \to S$, is a so called *weak Del Pezzo surface*. Before describing how the resolution ϕ is defined, we give some preliminary notions on weak Del Pezzo surfaces.

Definition 2.11. A weak Del Pezzo surface is a smooth surface $\tilde{S}$ such that the anti-canonical class $-K_{\tilde{S}}$ is nef and big.

A possible way to construct a weak del Pezzo surface $\tilde{S}$, is from a sequence of blowups of N points, $N \leq 8$:

$$X_N := \tilde{S} \xrightarrow{\pi_N} X_{N-1} \xrightarrow{\pi_{N-1}} \dots \xrightarrow{\pi_1} \mathbb{P}^2 := X_0$$

where each morphism $X_i \xrightarrow{\pi_i} X_{i-1}$ is the blowup at a point $p_i \in X_{i-1}$. From now on, whenever $p_i \in E_{i-1},\ E_{i-1} := \pi_i^{-1}(p_{i-1})$, we will write $p_i \succ_1 p_{i-1}$. Roughly speaking, just as for the case of a smooth cubic surface, we are still blowing up an N-tuple of points $p_1, \dots p_N$, but this time we are also allowing configurations with couples of points $p_i,\ p_j$ such that p_i is "infinitely near" p_j. In order to effectively obtain a weak del Pezzo surface, we have constraints, that we are going to illustrate, on the N-tuple $p_1, \dots p_N$.
More generally, consider a surface X_N obtained from an N-tuple of points $p_1, \dots p_N$, by a sequence of blowups:

$$X_N \xrightarrow{\pi_N} X_{N-1} \xrightarrow{\pi_{N-1}} \dots \xrightarrow{\pi_1} \mathbb{P}^2 := X_0$$

with:

$$p_i \in X_{i-1}, \qquad X_i := \mathrm{Bl}_{p_i} X_{i-1}.$$

Set:

$$\pi_{ki} := \pi_{i+1} \circ \dots \circ \pi_k : X_k \to X_i, \qquad \pi_{N0} := \pi : X_N \to \mathbb{P}^2.$$

On each surface X_i, for $i = 1, \dots, N$ we denote by $\mathcal{E}_j$ the total transform of the point $p_j,\ j \leq i$:

$$E_j = \pi_j^{-1}(p_j), \qquad \mathcal{E}_j := \pi_{i,j}{}^*(\ E_j) \quad \text{ for } j = 1, \dots, i.$$

We then define the divisors $\hat{E}_1, \dots \hat{E}_n$ on X_i, as follows. On X_1 we define $\hat{E}_1$ as $\hat{E}_1 := E_1 = \pi_1^{-1}(p_1)$, on X_2 we define $\hat{E}_1$ to be the strict transform of the divisor $\hat{E}_1 \subset X_1$ previously defined and we denote $\hat{E}_2 := E_2 = \pi_2^{-1}(p_2)$; proceeding in this way, at the "i-th step", we will define $\hat{E}_1, \dots \hat{E}_{i-1}$ to be the strict transforms of the divisors $\hat{E}_1, \dots \hat{E}_{i-1}$ on X_{i-1} and we call $\hat{E}_i := E_i = \pi_i^{-1}(p_i)$.
So, from the construction we see that the divisors $\hat{E}_1, \dots \hat{E}_i$ are the irreducible components of $\mathcal{E}_1, \dots, \mathcal{E}_i$. Therefore on X_i we have:

$$\hat{E}_j = \mathcal{E}_j - \epsilon_{j,j+1}\mathcal{E}_{j+1} - \dots - \epsilon_{j,i}E_i \quad j = 1, \dots, i. \tag{2.13}$$

We say that the N-tuple $p_1, \dots p_N$ satisfies the condition (*) if:

- (*) for $i = 1, \dots, N$, if there exists an index $j,\ j \leq i-1$ such that $p_{i+1} \in X_i$ belongs to $\hat{E}_j$, then $\hat{E}_j = \mathcal{E}_j$.

The condition is equivalent to require that in each expression (2.13) we have at most one coefficient $\epsilon_{j,k}$ different from zero.
From now on we will suppose that $N \leq 8$.

Definition 2.12. We say that the N-tuple of points $p_1, \dots, p_N$ is in *almost general position* if:

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- it satisfies condition (*);
- No lines in $\mathbb{P}^2$ passes through four points among the p_is;
- No smooth conics in $\mathbb{P}^2$ passes through seven points among the p_is.

Here, for a curve $C \subset \mathbb{P}^2$, we say that C passes through the point p_i if p_i belongs to the strict transform of C in X_i.

Theorem 2.3.11. *Let X_N be a surface obtained by blowing up an N-tuple of points, $N \leq 8$, in almost general position.Then we have the following:*

- *The anticanonical system $|-K_{X_N}|$ has no fixed component for $N = 8$ and it is base point free for $N < 8$.*
- *The anticanonical divisor $-K_{X_N}$ is big and nef.*

See [Dem] III, Thm 1 for a proof.

From this theorem we see that X_N obtained by blowing up N points in almost general position is a weak Del Pezzo surface. The only weak del Pezzo surfaces that do not arise in this way are a smooth quadric surface $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$, the blowup of the cone over a smooth conic.

Theorem 2.3.12 ([CAG]. Thm 8.1.15)**.** *A weak del Pezzo surface is isomorphic either to a smooth quadric surface either to $\mathbb{F}_2$, the blowup of the cone over a smooth conic, or to the blowup of $N \leq 8$ points in almost general position.*

Suppose now that S is a singular Del Pezzo surface and let $\phi : \tilde{S} \to S$ be a minimal resolution of singularities. We recall that this means that $\tilde{S}$ is a smooth surface, ϕ is a birational morphism that is an isomorphism outside the singular locus of S and moreover ϕ is "minimal" in the sense that it does not factor non-trivially through other resolutions.

Theorem 2.3.13 ([CAG]. Thm 8.1.15)**.** *The minimal resolution of a Del Pezzo surface is a weak Del Pezzo surface.*

Remark 18. We say that an N-tuple of points $p_1, \ldots, p_N$, $N \leq 8$, lies in *general position* if:

- All the points p_i belong to $\mathbb{P}^2$, for $i = 1, \ldots, N$.
- No three points among the p_is are collinear.
- No six points among the p_is lie on a smooth conic.
- No cubic passes through all the points with one of them being a singular point.

Let now X_N be the smooth surface obtained by blowing up these points. It is proved in [Dem], II Thm. I that:

Theorem 2.3.14. *The following condition are equivalent:*

- *The points $p_1, \ldots p_N$ lie in general position.*
- *The anticanonical system $|-K_S|$ is ample.*

Therefore a weak del Pezzo surface is a smooth del Pezzo surface if and only if it is obtained by blowing up $N \leq 8$ points on $\mathbb{P}^2$ in general position. Under this assumptions, when $N = 6$ the surface obtained is a smooth cubic surface.

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The structure of the Picard group

Throughout the rest of the section, we will consider an N-tuple of points $p_1, \dots p_N$, $N \leq 8$, in almost general position and we will denote by X_N the weak Del Pezzo surface obtained blowing them up. We keep the notations adopted so far.
From the construction of X_N, we see that its Picard group $\mathrm{Pic}\,(X_N)$ is a free abelian group of rank $N+1$. Indeed, denote by $e_1, \dots e_N$ the classes of $\mathcal{E}_1, \dots \mathcal{E}_N$ and by l be the class of $\pi^*(L)$ for a line $L \subset \mathbb{P}^2$; we have:

$$\mathrm{Pic}\,(X_N) = \pi^*(\mathrm{Pic}\,(\mathbb{P}^2)) \oplus \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_N = \mathbb{Z}l \oplus \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_N.$$

We compute immediately that the canonical class is:

$$K_{X_N} = -3l + e_1 - \dots + e_N.$$

The intersection product defines a symmetric bilinear form on $\mathrm{Pic}\,(X_N)$. Since we have:

$$l^2 = 1, \quad e_i^2 = -1, \quad e_i \cdot e_j = 0,\ 0 < i \neq j;$$

we see that in the basis $(l, e_1, \dots, e_N)$, the intersection product is represented by the matrix $\mathrm{diag}(1, -1, \dots, -1)$. This endows $\mathrm{Pic}\,(X_N)$ with the structure of a unimodular lattice of signature $(1, N)$.

The E_N lattice

We recall here some basics from lattice theory that we will need in the upcoming sections. Let Λ be a free abelian group of rank $N+1$:

$$\Lambda = \mathbb{Z}e_0 \oplus \dots \oplus \mathbb{Z}e_N$$

endowed with a symmetric bilinear form $(\,\cdot\,) : \Lambda \times \Lambda \to \mathbb{Z}$ defined by the $(N+1) \times (N+1)$ diagonal matrix of $\mathrm{diag}(1, -1, -1, \cdots, -1, -1)$, with respect to the basis $e_0, \cdots e_N$. Λ is a unimodular lattice of signature $(1, N)$. Let K_N be the vector $K_N = -3e_0 + \sum_{i=1}^N e_i$ and denote by E_N the sublattice of Λ defined by:

$$\mathrm{E}_N := (\mathbb{Z}K_N)^{\perp}.$$

We introduce the following subsets of Λ:

$$\mathrm{Exc} := \{v \in \Lambda \mid v^2 = -1,\ v \cdot K_N = -1\}$$
$$\mathcal{R} := \{\alpha \in \mathrm{E}_N \mid \alpha^2 = -2\}$$

Elements belonging to Exc are called *exceptional elements* and elements in $\mathcal{R}$ are called *roots.*

Definition 2.13. A subset $\overline{\beta} \subset \mathcal{R}$, $\overline{\beta} = \{\beta_1, \dots \beta_r\}$ is called a *root basis* if the β_is are linearly independent (over $\mathbb{R}$) and $\beta_i \cdot \beta_j \geq 0$, $\forall\, i \neq j$. A root basis is called *irreducible* if it is not equal to the union of two non-empty subsets $\overline{\beta}'$ and $\overline{\beta}''$ such that $\beta_i \cdot \beta_j = 0$ whenever $\beta_i \in \overline{\beta}'$ and $\beta_j \in \overline{\beta}''$.

Definition 2.14. A subset $\Delta \subset \mathcal{R}$ is called a *simple system* if the elements of Δ are linearly independent over $\mathbb{R}$, they span E_N (namely Δ form a basis for the $\mathbb{R}$ span of $\mathcal{R}$ in $E_N \otimes \mathbb{R}$) and moreover each $\alpha \in \mathcal{R}$ can be written as linear combination of elements in Δ with coefficient all of the same sign (all non-negative or all non-positive).

If $\Delta \subset \mathcal{R}$ is a simple system, a root α in Δ is called a simple root. An example of simple system in E_N is provided by the following roots:

$$\beta_1 = e_0 - e_1 - e_2 - e_3, \quad \beta_i = e_{i-1} - e_i \ \text{ for } i = 2, \cdots .N$$

The bilinear form $(\,\cdot\,)$ restricted to E_N is negative defined and non degenerate, its matrix in the basis $\beta_1, \cdots \beta_N$ is:

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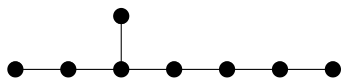
$$C_N = \begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & -2 & 1 \end{pmatrix} \tag{2.14}$$

We have that $C_N = -2I_N + M_N$ where M_N is the incidence matrix of the Coxeter-Dynkin diagram of type E_N.



E_n

Note that whenever we fix a simple system $\Delta = \{\beta_1, \dots \beta_N\}$ we obtain a partition of $\mathcal{R}$, $\mathcal{R} = \mathcal{R}^+ \sqcup \mathcal{R}^-$ where $\mathcal{R}^+$ and $\mathcal{R}^-$ are respectively the sets of non negative and non positive linear combinations of the β_is.

The Weyl group
For every root $\alpha \in R$ we denote by r_α the orthogonal transformation of Λ given by:

$$r_\alpha(v) := v + (v \cdot \alpha)\alpha.$$

r_α is called the *orthogonal reflection relative to* α. We observe that $r_\alpha(\alpha) = -\alpha,\ r_\alpha^2 = id$ and $r_\alpha(v) = 0$ whenever $v \cdot \alpha = 0$. Moreover r_α preserves the intersection form and it fixes K_N. From these facts we deduce that each r_α induces permutations of E_N, Exc and $\mathcal{R}$. The group $W(\mathcal{R})$ (for lattices of type E_N it is sometimes just denoted by $W(\mathrm{E}_N)$) generated by all the reflections r_α for $\alpha \in \mathcal{R}$ is called the *Weyl group.*
We have the following:

Theorem 2.3.15 ([Dem], II Thm. 3)**.**

- *The group $W(\mathcal{R})$ is the group of automorphism of Λ preserving the bilinear form $(\ \cdot\ )$ ad that fix K_N.*
- *$W(\mathcal{R})$ acts transitively on $\mathcal{R}$ and on* Exc *.*

Denote by V the vector space defined as $V := \Lambda \otimes \mathbb{R}$; we extend the form $(\ \cdot\ )$ by linearity to a bilinear form on V. Given any root $\alpha \in \mathcal{R}$, we consider $H_\alpha \subset V$, the hyperplane orthogonal to α. The hyperplanes H_α partition V into finitely many regions. The connected components of $V \setminus \bigcup_{\alpha \in \mathcal{R}} H_\alpha$ are called *open Weyl chambers*, their closure are referred to as *closed Weyl chambers.* If $\Delta \subset \mathcal{R}$ is any simple system, the set $\mathcal{K}_\Delta := \{v \in V \mid v \cdot \alpha > 0\ \forall \alpha \in \Delta\}$ is called the *fundamental Weyl chamber associated to* Δ . Fundamental Weyl chambers associated to simple systems satisfy the following:

Theorem 2.3.16 ([Hum] Thm 1.12)**.**

- *For all vectors $v \in V$ there exist a unique point $u \in \overline{\mathcal{K}}_\Delta$ such that $u = r(v)$, $\exists\ r \in W(\mathcal{R})$.*
- *If $r(v) = u$ for $u,\ v \in \overline{\mathcal{K}}_\Delta$ and $r \in W(\mathrm{E}_N)$, then $v = u$ and r is a product of orthogonal reflections $r_\alpha,\ \alpha \in \Delta$ fixing v (namely reflections r_α with $\alpha \in \Delta$).*

Remark 19. The theorem applies, in a more general contest, to couples $(V,\ \Phi)$ where V is an euclidean real vector space and Φ a so called *root systems* in V. (We refer to [Hum] for further details).

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Negative curves on Weak del Pezzo surfaces

In this section we study *negative curves*, namely irreducible and reduced curves having negative self intersection, on weak Del Pezzo surfaces. As usual we denote by X_N a weak Del Pezzo surface of degree $9-N$ obtained blowing up N points in almost general position. Given any curve irreducible and reduced curve $C \subset X_N$, by adjunction formula we have:

$$K_{X_N} \cdot C + C^2 = 2p_a(C) - 2 \geq -2.$$

The nefness of the anticanonical class $-K_{X_N}$ implies $C^2 \geq -2$, hence if C is a negative curve, we can have either $C^2 = -2$ or $C^2 = -1$.
If ever $C^2 = -1$ we have $0 \leq p_a(C) = 1 + \frac{1}{2}(C^2 + K_{X_N} \cdot C) = 1 + \frac{1}{2}(K_{X_N} \cdot C - 1)$ so that $K_{X_N} \cdot C \geq -1$. $-K_{X_N}$ is nef, hence $-1 \leq K_{X_N} \cdot C \leq 0$; moreover $p_a(C)$ is an integer so that we conclude that $K_{X_N} \cdot C = -1$, $p_a(C) = 0$ and therefore $C \simeq \mathbb{P}^1$. This means that every negative curve having self intersection -1 is a rational curve whose class in $\mathrm{Pic}(X_N)$ lies in Exc , the set of exceptional element. A negative curve C with $C^2 = -1$ is called a $(-1)-$curve.
If C is a negative curve with self intersection equal to -2, so that its class in $\mathrm{Pic}(X_N)$ is a root, we have $0 \leq p_a(C) = \frac{1}{2}(K_{X_N} \cdot C) \leq 0$ and this implies that C is a rational curve satisfying $K_{X_N} \cdot C = 0$. Vice versa, given any irreducible (reduced) curve $C \subset X_N$ such that $C \cdot K_{X_N} = 0$, we have $0 \leq p_a(C) = 1 + \frac{1}{2}(K_{X_N} \cdot C + C^2) = 1 + \frac{1}{2}(C^2)$, leading to $C^2 \geq -2$. We saw in the previous section that the intersection form is nondegenerate and negative defined on $(\mathbb{Z}K_{\tilde{S}})^{\perp}$, thus $C^2 < 0$. Since $p_a(C)$ is an integer, C^2 must be even, consequently we conclude that $C^2 = -2$, $p_a(C) = 0$ and hence $C \simeq \mathbb{P}^1$. This also means that the class of C is a root in $\mathrm{Pic}(X_N)$. An irreducible curve C having self intersection equal to -2 is called a $(-2)-$curve.

Definition 2.15. An *effective root* is a root $\alpha \in \mathrm{Pic}\,(X_N)$ corresponding to the class of an effective divisor. An *irreducible root* is a root $\alpha \in \mathrm{Pic}(X_N)$ corresponding to the class of a $(-2)-$curve. We denote by $\mathcal{R}_e \subset \mathcal{R}$ and by $\mathcal{R}_i \subset \mathcal{R}_e$, respectively, the sets of effective and irreducible roots.

When the points $p_1, \ldots, p_N$ are in general position (namely when X_N is a Del Pezzo surface), $\mathrm{Pic}\,(X_N)$ contains no effective roots, as stated in [Dem], Thm III.1.

Theorem 2.3.17. *[[Dem] Thm III.1 (v)] The following conditions are equivalent on X_N:*

- *X_N is Del Pezzo;*
- *There does not exist an effective element $\alpha \in \mathrm{Pic}\,(X_N)$ such that $\alpha^2 = -2$, $K_{X_N} \cdot \alpha = 0$.*

Thus if X_N is Del Pezzo, the only negative curves on X_N are (-1)-curves. If we ask for the N-tuple $p_1, \ldots, p_N$ to satisfy the weaker condition of being in almost general position, X_N can also contain (-2)-curves so that in $\mathrm{Pic}\,(X_N)$ there might also exist effective roots. (Note that as a negative curve does not move in its linear equivalence class, we can identify it with its class in $\mathrm{Pic}\,(X_N)$). Let now C be any effective divisor with $C^2 = -2$; write C as $\sum_i n_i R_i$ with the R_is effective and irreducible divisors in X_N, and $n_i \geq 0, \forall\, i$. Since

$$0 = C \cdot K_{X_N} = \sum_i n_i (R_i \cdot K_{X_N}), \text{ and } n_i \geq 0\ \forall\ i,$$

by the fact that K_{X_N} is nef, we get that $R_i \cdot K_s = 0,\ \forall\ i$. Therefore, still by adjunction, $\deg(K_{R_i}) = R_i^2 \geq -2$. Now, as the R_is are effective divisors, whenever $i \neq j$ we have that $R_i \cdot R_j \geq 0$. This inequalities together with the condition $C^2 = -2$ implies that $R_i^2 = -2, \forall\ i$. So, summing up, each effective roots $\alpha \in \mathrm{Pic}\,(X_N)$ has a (unique) representative expressed as a linear combination, with non negative coefficients, of irreducible (-2)-curves.

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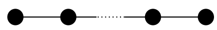
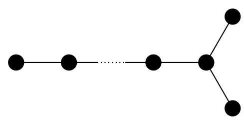
Definition 2.16. A *Dynkin curve* on X_N is a reduced connected curve R such that its irreducible components R_i are (-2)-curves and the intersection matrix $(R_i \cdot R_j)$ is a symmetric integer matrix with:

$$(R_i \cdot R_j) \geq 0 \text{ for } i \neq j \qquad \text{and} \qquad (R_i \cdot R_i) = -2.$$

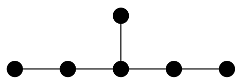
To a Dynkin curve R we can associate a connected graph Γ_R, called Dynkin-Coxeter diagram. To each irreducible component R_i of R corresponds a vertex v_i, weighted by $R_i \cdot R_i$; the vertices v_i and v_j are connected by $R_i \cdot R_j$ edges. A connected graph obtained in this way is of one of the following types:



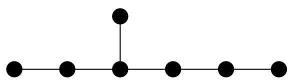
A_n



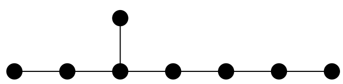
D_n



E_6



E_7



E_8

We will call ADE type of a Dynkin curve $\mathcal{R}$, the ADE type of the corresponding Dynkin-Coxeter diagram.
We conclude the section stating a useful criterion allowing to determine when an exceptional element of $\text{Pic}(X_N)$ correspond to the class of a $(-1)-$curve.

Proposition 2.3.18. *[[CAG]. Lemma 8.2.22] Let X_N be a weak del Pezzo surface of degree $d = 9 - N > 1$ and let D be a divisor class with $D^2 = D \cdot K_{X_N} = -1$. Then $D = E + R$ where R is a non-negative sum of (-2)-curves, and E is a (-1)-curve. Morevoer D is a (-1)-curve if and only if for each (-2)-curve $R_i \subset X_N$, we have $R_i \cdot D \geq 0$.*

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The anticanonical model

We saw that the anticanonical class of X_N is represented in $\mathrm{Pic}\,(X_N)$ by the vector $(3,-1,-1,\dots,-1) = 3l - (\sum_{i_1}^{N} e_i)$. It has self intersection $K^2_{X_N} = 9 - N$.

Definition 2.17. Let X_N be a weak Del Pezzo surface obtained by blowing up an N-tuple of points lying in almost general position. The degree of X_N is the integer $d = 9 - N = K^2_{X_N}$.

Recall that, from theorem 2.3.11, whenever $N < 8$, the anticanonical system is base point free. If furthermore we restrict to $N \le 6$ we have the following:

Theorem 2.3.19. *[[CAG] Thm 8.3.2] Let X_N be a weak del Pezzo surface of degree $d \ge 3$ obtained by blowing up $N := 9 - d$ points in almost general position; let $\mathfrak{R}$ be the union of (-2)-curves on X_N. Then $|-K_{X_N}|$ defines a regular map*

$$\phi_{-K_{X_N}} : X_N \to \mathbb{P}(H^0(-K_{X_N})^*) \simeq \mathbb{P}^d$$

which is an isomorphism outside $\mathfrak{R}$. The image S of this map is a Del Pezzo surface of degree d. The image of each connected component of $\mathfrak{R}$ is a rational double point of S.

The birational morphism $\phi_{-K_{X_N}} : X_N \to S$ is a resolution of the surface S. This map contracts each Dynkin curve $R \subset X_N$ to a RDP p_R of S; p_R is a a singularity of the type of the Dynkin diagram of R. Given now a curve $\tilde{C} \subset X_N$, let $C \subset S$ be its image through ϕ. C passes through the singular point p_R if and only if $\tilde{C}$ intersects a component of the Dynkin curve R contracted to p_R.

Weak del Pezzo surfaces of degree 3

From now on we will always deal with weak del Pezzo surfaces of degree 3. A degree 3 weak Del Pezzo surface $\tilde{S}$ is obtained as the blowup $\pi : \tilde{S} \dashrightarrow \mathbb{P}^2$ of 6 points $p_1, \dots p_6$ in almost general position. Keeping the notations previously adopted we write $\mathrm{Pic}(\tilde{S})$ as the rank 7 free abelian group $\mathrm{Pic}(\tilde{S}) = \mathbb{Z}l \oplus_{i=1}^{6} \mathbb{Z}e_i$ with l the class of $\pi^*(L)$ for a generic line $L \subset \mathbb{P}^2$ and e_i the classes of $\pi^{-1}(p_i)$. The orthogonal to $K_{\tilde{S}} = -3l + \sum_{i=1}^{6} e_i$ is a lattice of type E_6. Following [CAG], Prop. 8.2.7. in E_6 there exists 72 roots and they are all of the following types:

$$\pm\alpha_{ij},\ \pm\alpha_{ijk},\ 2l - e_1 - \cdots - e_6$$

with:

$$\alpha_{ij} = e_i - e_j,\ 1 \le i < j \le 6,\ \alpha_{ijk} = e_0 - e_i - e_j - e_k,\ 1 \le i < j < k \le 6.$$

By theorem 2.3.19, the rational map $\phi_{-K_{\tilde{S}}}$ defined by $-K_{\tilde{S}}$ contracts all Dynkin curves on $\tilde{S}$ to singular points of the cubic surface $S := im(\phi_{-K_{\tilde{S}}})$. All the possible configurations of Dynkin curves on $\tilde{S}$, or equivalently all the possible configurations of RDPs on S are the following (see for example [BW]):

$$A_1,\ A_2,\ A_3,\ A_4,\ A_5,\ 2A_1,\ A_1 + A_2,\ 2A_2,\ A_3 + A_1,\ A_4 + A_1,\ A_5 + A_1$$

$$3A_1,\ 2A_1 + A_2,\ 2A_2 + A_1,\ 3A_2,\ A_3 + 2A_1,\ D_4,\ D_5,\ E_6.$$

2.3.3 Existence of smooth normal elliptic quintics

In this section we use the notions previously introduced to show the existence of smooth elliptic quintics on certain normal cubic threefolds. The method that we adopt to prove the existence of these curves relies on a deformation argument: it consists indeed in first proving that there exists a *smooth* quintic elliptic curve $\mathcal{C}_0$ contained in a hyperplane section of X; then it is shown that $\mathcal{C}_0$ deforms to a nondegenerate curve $\mathcal{C}$ in X. A smooth curve $\mathcal{C}$ obtained in this way is always AG. Indeed, any smooth nondegenerate

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elliptic quintic $\mathcal{C}$ is ACM by Remark 9 and by the projective normality of the smooth curves of genus $g \geq 1$ embedded by complete linear systems of degree $\geq 2g+1$ (see [Mum, Corollary from Theorem 6]); as moreover $\mathcal{O}_{\mathcal{C}} \simeq \omega_{\mathcal{C}}$, $\mathcal{C}$ is subcanonical, hence it is AG. We prove the following:

Theorem 2.3.20. *Let X be a normal cubic threefold that is not a cone and that satisfies one of the following two conditions:*

- *X has isolated singularities;*
- $\mathrm{Sing}\,(X)$ *contains a curve Y that is either a line or a conic of first type or the union of three non-coplanar lines meeting at a point.*

Then there exists a smooth nondegenerate quintic elliptic curve $\mathcal{C} \subset X$.

From Theorem 2.3.20 we will deduce:

Proposition 2.3.21. *Let X be a cubic threefold satisfying one of the hypotheses of 2.3.20. Then there exists a rank 2 skew-symmetric Ulrich bundle $\mathcal{E} \in \mathrm{Coh}\,(X)$.*

Consequently:

Corollary 2.3.22. *Let X be a cubic threefold satisfying one of the hypotheses of 2.3.20. Then X is Pfaffian.*

Illustration of the method in the smooth case

The smooth case was treated in detail in [MT1]. As the proof of the singular case is obtained, in part, by an extension of the method of proof for the smooth case, we start by recalling it. So, first suppose that X is smooth. Applying Bertini's theorem, for a general hyperplane $H \subset \mathbb{P}^4$, the intersection $S := X \cap H$ is a smooth cubic surface. A smooth cubic surface S is a Del Pezzo surface of degree 3, hence it can be realized as the blowup of $\mathbb{P}^2$ in 6 points $p_1, \ldots, p_6$ in *general position*. Denote by $\pi : S \to \mathbb{P}^2$ the corresponding birational morphism and write $\mathrm{Pic}(S) = \mathbb{Z}l \oplus_{i=1}^{6} \mathbb{Z}e_i$ where as usual l is the class of $\pi^*(L)$ for a generic line $L \subset \mathbb{P}^2$ and the e_is are the classes of the exceptional (-1)-curve $E_0, \ldots, E_6$, $E_i = \pi^{-1}(p_i)$.
Choose now 4 points $p_1, \ldots p_4$ among the p_is and consider the linear system

$$|3l - p_1 - p_2 - p_3 - p_4| \subset |\mathcal{O}_{\mathbb{P}^2}(3)|, \qquad |3l - p_1 - p_2 - p_3 - p_4| \simeq \mathbb{P}^5$$

of plane cubics passing through them. For every $C^3 \in |3l - p_1 - p_2 - p_3 - p_4|$ the strict transform $\pi^{-1}(C^3)$ is an elliptic curve (this can be computed on S by adjunction or from $g(\pi^{-1}(C^3)) = g(C^3) = 1$), belonging to the class $(3,-1,-1,-1,-1,0,0) \in \mathrm{Pic}\,(S)$. The linear system $|(3,-1,-1,-1,-1,0,0)|$ is base-point free (this can be deduced from the fact that, by [Hart] Ch. V Thm 4.6, the class $(3,-1,-1,-1,-1)$ is very ample on $\mathrm{Bl}_{p_1,\ldots p_4}\mathbb{P}^2$); thus, by Bertini's theorem, a general element $\mathcal{C}_0 \in |(3,-1,-1,-1,-1,0,0)|$ is smooth. We compute that:

$$\deg(\mathcal{C}_0) = (-K_s) \cdot \mathcal{C}_0 = 5$$

hence $\mathcal{C}_0$ is a smooth elliptic quintic contained in S.

In order to show that $\mathcal{C}_0$ deforms in X to a nondegenerate curve $\mathcal{C}$ we first need to check that Hilb^{5n}_X, the Hilbert scheme of elliptic quintics on X, is smooth at the point $[\mathcal{C}_0]$. Consider now the short exact sequence of sheaves on $\mathcal{C}_0$:

$$0 \longrightarrow \mathcal{N}_{\mathcal{C}_0/S} \longrightarrow \mathcal{N}_{\mathcal{C}_0/X} \longrightarrow \mathcal{N}_{S/X\,|\mathcal{C}_0} \longrightarrow 0. \tag{2.15}$$

$\mathcal{N}_{\mathcal{C}_0/S} \simeq \mathcal{O}_{\mathcal{C}_0}(\mathcal{C}_0)$ is a line bundle of degree $\mathcal{C}_0^2 = 5$, the same holds for $\mathcal{N}_{S/X\,|\mathcal{C}_0} \simeq \mathcal{O}_{\mathcal{C}_0}(1)$. Hence $h^0(\mathcal{N}_{\mathcal{C}_0/S}) = h^0(\mathcal{N}_{S/X\,|\mathcal{C}_0}) = 5$, and $h^1(\mathcal{N}_{\mathcal{C}_0/S}) = h^1(\mathcal{N}_{S/X\,|\mathcal{C}_0}) = 0$. From these

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equalities we get that $h^1(\mathcal{N}_{\mathcal{C}_0/X}) = 0$, $h^0(\mathcal{N}_{\mathcal{C}_0/X}) = 10$ hence Hilb_X^{5n} is smooth of dimension 10 at $[\mathcal{C}_0]$ ([Gro]). The linear system $|\mathcal{C}_0|$ on $S = H \cap X$ has dimension 5; letting the hyperplane H vary in $\mathbb{P}^{4^*}$, we see that the family of elliptic quintics contained in a hyperplane section of X has dimension 9. As Hilb_X^{5n} at $\mathcal{C}_0$ is smooth and of dimension 10, we can finally conclude that $\mathcal{C}_0$ deforms to a nondegenerate smooth elliptic quintic $\mathcal{C}$.

The singular case

We now describe how the method that we have just presented also adapts to certain singular cases. The idea is the following: we look for a smooth elliptic quintic $\mathcal{C}_0$ contained in a hyperplane section $S := X \cap H$ of X and such that $\mathcal{C}_0 \cap \mathrm{Sing}\,(S) = \varnothing$. The fact that $\mathcal{C}_0$ doesn't pass through any of the singular points of S, implies that $\mathcal{C}_0$ is entirely contained in X_{sm}, the smooth locus of X. We thus still get the short exact sequence of line bundles on $\mathcal{C}_0$ 2.15 that allows us to compute again that Hilb_X^{5n} at $\mathcal{C}_0$ is smooth and of dimension 10. Arguing as above, we conclude that $\mathcal{C}_0$ deforms, in X_{sm}, to a nondegenerate smooth elliptic quintic $\mathcal{C}$.
When X is a singular threefold satisfying the hypotheses of theorem 2.3.20, a general hyperplane section S of X is a cubic surface with at most rA_1 singularities, $r \leq 3$ (cf. Prop. 2.3.9). Consider $\tilde{S}$ a degree 3 weak Del Pezzo surface, resolution of S. To start with we exhibit how to construct a linear system on $\tilde{S}$ whose generic element is a smooth curve C such that $C^2 = C \cdot (-K_{\tilde{S}}) = 5$.

Proposition 2.3.23. *Let $\tilde{S}$ be a weak Del Pezzo surface of degree 3, $R \subset \tilde{S}$ a (-2)-curve and $E \subset \tilde{S}$ a (-1)-curve such that $R \cdot E = 1$. Then the linear system $|R - K_{\tilde{S}} + 2E|$ is base point free.*

Proof. The first thing that we show is that $|R - K_{\tilde{S}} + 2E|$ is nef. Given any irreducible effective divisor D on $\tilde{S}$, we can write: $D\cdot(R - K_{\tilde{S}} + 2E) = (D\cdot R) + (D\cdot(-K_{\tilde{S}})) + (2D\cdot E)$. As $-K_{\tilde{S}}$ is nef, we have $D \cdot (-K_{\tilde{S}}) \geq 0$. Since R is a (-2)-curve, $D \cdot R < 0$ if and only if $R = D$. Similarly $E \cdot D < 0$ if and only if $E = D$. Therefore $D \cdot (R - K_{\tilde{S}} + 2E) \geq 0$ whenever $D \neq E$ and $D \neq R$.
Now, if $D = E$ we have:

$$D \cdot (R - K_{\tilde{S}} + 2E) = 1 + 1 - 2 = 0. \tag{2.16}$$

If $D = R$ we get:

$$D \cdot (R - K_{\tilde{S}} + 2E) = -2 + 2 = 0. \tag{2.17}$$

We can thus conclude that $|R - K_{\tilde{S}} + 2E|$ is nef. Given any divisor $D \in |R - K_{\tilde{S}} + 2E|$, we compute that $D \cdot (-K_{\tilde{S}}) = 5$, $D^2 = D \cdot (-K_{\tilde{S}}) = 5$. D is nef, $D^2 > 0$ thus D is big and nef, consequently $D - K_{\tilde{S}}$ is big and nef as well. Applying Kawamata-Viewheg vanishing theorem we have $h^i(\mathcal{O}_{\tilde{S}}(D) = 0$ for $i = 1, 2$. By Riemann-Roch we get

$$\chi(\mathcal{O}_{\tilde{S}}(D)) = h^0(\mathcal{O}_{\tilde{S}}(D)) = \frac{1}{2}(D^2 - K_{\tilde{S}} \cdot D) + 1 = 6.$$

We now prove that $|R - K_{\tilde{S}} + 2E|$ has no fixed component. Indeed, write $|R - K_{\tilde{S}} + 2E|$ as $F + |M|$ where F and $|M|$ are, respectively the fixed and the mobile part of $|R - K_{\tilde{S}} + 2E|$. M is a nef effective divisor satisfying $M^2 \geq 0$ and $\dim|M| = \dim|R - K_{\tilde{S}} + 2E| = 5$, whilst the linear system $|F|$ has dimension 0. By the fact that M nef, we get that $M - K_{\tilde{S}}$ is big and nef, therefore by Kawamata-Viewheg:

$$\chi(\mathcal{O}_{\tilde{S}}(M)) = h^0(\mathcal{O}_{\tilde{S}}(M)) = 6.$$

By Riemann Roch $5 = \frac{1}{2}(M^2 - M \cdot K_{\tilde{S}})$ hence $M^2 = 10 + M \cdot K_{\tilde{S}} = 5 - F \cdot K_{\tilde{S}} \geq 5$. Now, by the nefness of $|R - K_{\tilde{S}} + 2E|$ we get the inequality:

$$5 = (R - K_{\tilde{S}} + 2E)^2 = (R - K_{\tilde{S}} + 2E) \cdot (M + F) \geq (R - K_{\tilde{S}} + 2E) \cdot M;$$

whilst from the nefness of M we deduce:

$$(R - K_{\tilde{S}} + 2E) \cdot M = (M + F) \cdot M = M^2 + M \cdot F \geq M^2.$$

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Therefore

$$M^2 = 5 = (R - K_{\tilde{S}} + 2E)^2 = (R - K_{\tilde{S}} + 2E) \cdot (M + F) =$$
$$= (R - K_{\tilde{S}} + 2E) \cdot F + (M + F) \cdot M = (R - K_{\tilde{S}} + 2E) \cdot F + M \cdot F + M^2.$$

Again, the nefness of $R - K_{\tilde{S}} + 2E$ and M implies $(R - K_{\tilde{S}} + 2E) \cdot F \geq 0,\ M \cdot F \geq 0$ so that a fortiori $(R - K_{\tilde{S}} + 2E) \cdot F = 0,\ M \cdot F = 0$. Since

$$5 = (M + F)^2 = M^2 + 2M \cdot F + F^2 = 5 + F^2$$

we get that $F^2 = 0$, and as $M^2 = 5 = 5 - F \cdot K_{\tilde{S}}$, we deduce $F \cdot K_{\tilde{S}} = 0$. As the intersection form is non-degenerate on $(\mathbb{Z}K_{\tilde{S}})^{\perp}$, we can conclude that $F = 0$, and thus $|R - K_{\tilde{S}} + 2E|$ has no fixed component. By the fact that $|R - K_{\tilde{S}} + 2E|$ has no fixed component, a general element D in it can be written as a sum $D = \sum_i^m D_i$ where all the D_is are numerically equivalent irreducible curves (this follows from [Fr], Ex 5.11). This implies that $\forall\, i \neq j, (D_i - D_j) \cdot D_i = (D_i - D_j) \cdot D_j = 0$ hence $D_i^2 = D_j^2 = D_i \cdot D_j$. Call $a = D_i^2$. We obtain $5 = D^2 = ma + 2(\binom{m}{2}a) = a(m + 2\binom{m}{2})$. This can only occur for $m = 1$ and $a = 5$. Consider now the short exact sequence:

$$0 \to \mathcal{O}_{\tilde{S}} \to \mathcal{O}_{\tilde{S}}(D) \to \mathcal{O}_D(D) \to 0. \tag{2.18}$$

Since $H^1(\mathcal{O}_{\tilde{S}}) = 0$, $H^0(\mathcal{O}_{\tilde{S}}(D)) \to H^0(\mathcal{O}_D(D))$ is surjective so that if ever $|R - K_{\tilde{S}} + 2E|$ has a base point p, we would have that $p \in D$ is a base point of its restriction to D. But since $\mathcal{O}_D(D)$ is a line bundle of degree 5 on D, with $p_a(D) = 1 \geq g(D)$, it can't have base points (see for example [Dem] IV Lemma 1). □

We now show how to construct a smooth elliptic quintic $\mathcal{C}_0$ on a cubic surface with at most rA_1 singularities, $r \leq 3$, and such that $\mathcal{C}_0 \cap \operatorname{Sing}(S) = \varnothing$.

Proposition 2.3.24. *Let S be a cubic surface with $r\,A_1$ singularities, $r \leq 3$. Then there exists a smooth quintic elliptic curve $\mathcal{C}_0 \subset S$ such that $\mathcal{C}_0 \cap \operatorname{Sing}(S) = \varnothing$.*

Proof. We have already illustrated how to obtain elliptic quintics on a smooth cubic surface, hence we consider the cases where $1 \leq r \leq 3$. Let $\tilde{S}$ be a degree 3 weak Del Pezzo surface resolution of S. Denote by $p_1, \cdots p_r$ the singular points of S and by $R_{p_1}, \dots R_{p_r}$ the corresponding $(-2)-$curves on $\tilde{S}$ (since we are assuming that S presents just A_1 singularities, each Dynkin curve on $\tilde{S}$ is a (-2)-curve). We claim the following:

Claim 1. There always exists a (-1)-curve E such that $E \cdot R_{p_1} = 1$ and $E \cdot R_{p_i} = 0$ for $i \neq 1$.

We first show how, assuming the claim, the proposition follows. If such a (-1)-curve E exists, we have that, by proposition 2.3.23, the general element $\tilde{\mathcal{C}_0}$ in $|R_{p_1} - K_{\tilde{S}} + 2E|$ is a smooth curve with $\tilde{\mathcal{C}}_0^2 = 5 = \tilde{\mathcal{C}_0} \cdot (-K_{\tilde{S}})$. We compute that

$$\tilde{\mathcal{C}_0} \cdot R_{p_1} = R_{p_1}^2 + 2E \cdot R_{p_1} = 0,\ \ \tilde{\mathcal{C}_0} \cdot R_{p_i} = R_{p_1} \cdot R_{p_i} + 2E \cdot R_{p_i} = 0 \text{ for } i \neq 1.$$

Therefore the curve $\mathcal{C}_0 := \phi_{-K_{\tilde{S}}}(\tilde{\mathcal{C}_0})$ is a smooth elliptic quintic curve on S disjoint from $\operatorname{Sing}(S)$. We end the proof of the proposition demonstrating the claim. Showing that a (-1)-curve E satisfying the hypotheses of the claim exists is equivalent to showing that on S there always exists a line L passing through p_1 and such that $p_i \notin L$ for $i \neq 1$.
We choose coordinates $X_0, \dots, X_3$ on $\mathbb{P}^3$ in such a way that $p_1 = [1 : 0 : 0 : 0]$. Let $f \in \mathbb{C}[X_0, \dots, X_3]$ be the homogeneous polynomial of degree 3 defining S. For our choice of coordinates f can be written as

$$f(X_0, \cdots, X_3) = X_0 q(X_1, X_2, X_3) + c(X_1, X_2, X_3)$$

where q and c are homogeneous forms of degree, respectively, 2 and 3. Since p_1 is an A_1 singularity, it is a conic node, hence q is a quadric of rank 3. The intersection of the hyperplane $\{X_0 = 0\}$ with the cones having equations $\{q(X_0, X_1, X_2) = 0\}$ and $\{c(X_0, X_1, X_2) = 0\}$ is a zero-dimensional scheme Z of length 6. All the lines contained in

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S and passing through p_1 are of the form $\overline{p_1q}$ for $q \in Z$. A line $\overline{p_1q}, q \in Z$ passes through a singular point p_i different from p_1 if and only if the conic $C_2 := \{X_0 = q(X_1, X_2, X_3) = 0\}$ and the plane cubic $C_3 := \{X_0 = c(X_1, X_2, X_3)\}$ meet at q with multiplicity greater then one. Moreover the point p_i will be an A_{k-1} singularity where k is the multiplicity of the intersecion of C_2 and C_3 at q. Since we are assuming that S has at most r A_1 singularities, C_2 and C_3 meets in $r-1$ points with multiplicity 2 implying that Z consists of $r-1$ double points and of $6-2(r-1)$ simple points. Since $r \leq 3$ there always exists then a point $q \in Z$ where C_2 and C_3 meets transversely so that $\overline{qp_1}$ is a line in S satisfying $\overline{p_1q} \cap \mathrm{Sing}\,(S) = \{p_1\}$.

□

Remark Claim 1 can also be proved by writing down explicitly an exceptional element E of $\mathrm{Pic}(\tilde{S})$ intersecting just one effective root with multiplicity one and orthogonal to all others effective roots, for all possible configuration of (-2)-curves leading to r A_1 singularities. Such an exceptional element E will then correspond to the class of a (-1) curve due to proposition 2.3.18. More precisely it is sufficient to show the existence of E just on one representative $\tilde{S}$ of each $W(\mathrm{E}_6)$ orbit of weak Del Pezzo surfaces of r A_1 type. Indeed let $\tilde{S},\ \tilde{S}'$ be a couple of weak del Pezzo surfaces containing, respectively, r pairwise orthogonal (-2)-curves $\{R_1, \ldots R_r\}$ and $\{R'_1, \cdots R'_r\}$. Suppose the existence of an element $s \in W(\mathrm{E}_6)$ inducing a bijection between $\{R_1, \cdots, R_r\}$ and $\{R'_1, \cdots R'_r\}$. If E is an exceptional element in $\mathrm{Pic}(\tilde{S})$ such that

$$E \cdot R_1 = 1,\ E \cdot R_i = 0 \text{ for } i \neq 1,$$

then $s(E)$ is an exceptional element such that

$$s(E) \cdot s(R_1) = 1,\ s(E) \cdot s(R_i) = 0 \text{ for } i \neq 1,$$

hence the class of a (-1)-curve (still by 2.3.18) meeting just one (-2)-curve.
This is particularly easy for $r = 1$. Indeed in this case, given $\tilde{S}$ and $\tilde{S}'$ each of them containing just a (-2)-curve $R \subset \tilde{S},\ R' \subset \tilde{S}'$, by the transitivity of the action of $W(\mathrm{E}_6)$ on the set of roots, we always get the existence of $s \in W(\mathrm{E}_6)$ such that $s(R) = R'$. Without loss of generality we can thus assume that $\tilde{S}$ is obtained as the blowup:

$$\tilde{S} \simeq \mathrm{Bl}_{\{p_1,\ldots,p_6\}}\mathbb{P}^2, \quad p_i \in \mathbb{P}^2, \quad |h - p_1 - p_2 - p_3| \neq \varnothing$$

of 6 proper points $p_1, \ldots p_6$ on $\mathbb{P}^2$ with just 3 of them, say $p_1,\ p_2,\ p_3$ lying on a line L. The strict transform of L is a $(-2)-$curve R (represented by the vector $(1, -1, -1, -1, 0, 0, 0)$ in $\mathrm{Pic}(\tilde{S})))$, each exceptional divisor $E_i = \pi^{-1}(p_i),\ i = 1, 2, 3$ is a $(-1)-$curve such that $E_i \cdot R = 1$. Hence the generic elements of the classes

$$(4, 0, -2, -2, 0, 0, 0), \quad (4, -2, 0, -2, 0, 0, 0), \quad (4, -2, -2, 0, 0, 0, 0)$$

are mapped by $\phi_{-K_{\tilde{S}}}$ to a smooth elliptic quintic disjoint from $\mathrm{Sing}\,(S)$.

Proof of Theorem 2.3.20. Let X be a cubic threefold satisfying the hypotheses of theorem 2.3.20. Then, by proposition 2.3.24, on a general hyperplane section $S := H \cap X$ there always exists a smooth elliptic quintic $\mathcal{C}_0$ disjoint from $\mathrm{Sing}\,(S)$. The condition $\mathcal{C}_0 \cap \mathrm{Sing}\,(S) = \varnothing$ implies that $\mathcal{C}_0$ is entirely contained in X_{sm}, the smooth locus of X. Applying then to X_{sm} the same argument used in the smooth case we conclude that $\mathcal{C}_0$ deforms in X_{sm} to a nondegenerate smooth elliptic quintic $\mathcal{C} \subset X$. □

Applying theorem 2.3.20 we can prove the following:

Proposition 2.3.25 (Prop. 2.3.21)**.** *Let X be a cubic threefold satisfying the hypotheses of 2.3.20. Then there exists a rank 2 skew-symmetric Ulrich bundle $\mathcal{E} \in \mathrm{Coh}\,(X)$.*

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Proof. Let X be a cubic threefold satisfying the hypotheses of theorem 2.3.20. From what we have just proved, there exists then a smooth quintic elliptic curve $\mathcal{C} \subset X$ that is not contained in any hyperplane section of X. Applying Serre's correspondence, the class $1 \in H^0(\mathcal{O}_\mathcal{C}) \simeq \mathrm{Ext}^1(\mathcal{I}_\mathcal{C}, \mathcal{O}_X(-2))$ corresponds to a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-2) \longrightarrow \mathcal{N} \longrightarrow \mathcal{I}_\mathcal{C} \longrightarrow 0$$

where $\mathcal{N}$ is a rank 2 ACM sheaf. Localizing at a point $x \in \mathcal{C}$ we get:

$$0 \longrightarrow \mathcal{O}_{X,x} \longrightarrow \mathcal{N}_x \longrightarrow \mathcal{I}_{\mathcal{C},x} \longrightarrow 0,$$

corresponding to $1_x \in \mathrm{Ext}^1_{\mathcal{O}_{X,x}}(\mathcal{I}_{\mathcal{C},x}, \mathcal{O}_{X,x})$. Now, since $\mathcal{C}$ is smooth, we have:

$$\mathbb{C} \simeq H^0(\mathcal{O}_\mathcal{C}) \simeq H^0(\mathscr{E}xt^1(\mathcal{I}_\mathcal{C}, \mathcal{O}_X(-2)) \simeq \mathrm{Ext}^1(\mathcal{I}_\mathcal{C}, \mathcal{O}_X(-2)).$$

Hence the class $1 \in H^0(\mathcal{O}_\mathcal{C})$ does not vanishes on any point of $\mathcal{C}$, so that for all points $x \in \mathcal{C}$, 1_x generates $\mathcal{O}_{\mathcal{C},x} \simeq \mathscr{E}xt^1_x(\mathcal{I}_\mathcal{C}, \mathcal{O}_X(-2)) \simeq \mathrm{Ext}^1_{\mathcal{O}_{X,x}}(\mathcal{I}_{\mathcal{C},x}, \mathcal{O}_{X,x})$. These facts, together with propositions 2.2.2 and 2.2.5, implies that $\mathcal{N}_x$ is free. Therefore $\mathcal{N}$ and $\mathcal{E} := \mathcal{N}(2)$ are both vector bundles; by proposition 2.2.4 and remark 13 we can finally conclude that $\mathcal{E}$ is a skew-symmetric Ulrich bundle. ☐

2.3.4 Existence of AG elliptic quintics on normal threefolds

We describe now a constructive method to prove the existence of an *AG* elliptic quintic curve $\mathcal{C} \subset X$. In order to apply this method we need the following couple of geometric objects:

- A rational quartic curve (not necessarily irreducible) $\Gamma \subset X$.
- A cubic scroll $\pi : \Sigma \to \mathbb{P}^1$ containing Γ and such that the class of Γ in $\mathrm{Pic}\,(\Sigma)$ is $\Gamma \sim D + 3F$, where D and F are, respectively, the classes of the directrix and of a fiber $\pi^{-1}(x),\ x \in \mathbb{P}^1$.

Recall that a cubic scroll $\Sigma \to \mathbb{P}^1$ is a rational ruled surface whose Picard group is a free abelian group of rank 2, $\mathrm{Pic}\,(S) = \mathbb{Z}D \oplus \mathbb{Z}F$. The directrix D is a rational (-1)-curve (namely $D^2 = -1$) meeting transversely each fiber, so that the intersection number $D \cdot F$ is equal to one. As two distinct fibers don't meet, F has self intersection equal to zero. From these facts we can compute that, by adjunction formula, the canonical class is $K_\Sigma \sim -2D - 3F$ and that the class of an hyperplane section is $H \sim D + 2F$. Now, if there exists Γ, a rational quartic curve contained in Σ, for a couple of non negative integers $a,\ b$ we can write $\Gamma \sim aD + bF$. Since Γ has genus 0 and degree 4 we must have:

$$\Gamma \cdot H = 3 \text{ and } (\Gamma + K_\Sigma) \cdot \Gamma = -2.$$

Hence a and b must satisfy:

$$(aD+bF)\cdot(D+2F) = a+b = 4, \quad ((a-2)D+(b-3)F)\cdot(aD+bF) = 2b(a-1)-a(a+1) = -2.$$

The only two couple of non negative integers satisfying these equations are $(2,2)$ and $(1,3)$. Suppose now that there exists $\Gamma \subset (X \cap \Sigma)$ a rational quartic such that in $\mathrm{Pic}\,(\Sigma)$ we have $\Gamma \sim D + 3F$. The intersection $X \cap \Sigma$ gives a divisor on Σ whose class in $\mathrm{Pic}\,(\Sigma)$ is $3H = 3D + 6F$; hence the curve $\mathcal{C}$, residual to Γ in $X \cap H$ is an elliptic quintic such that $\mathcal{C} \sim 2D + 3F$. Such elliptic quintic should be necessarily nondegenerate , as:

$$H^0(\Sigma, \mathcal{I}_{\mathcal{C}/\Sigma}(1)) = H^0(\mathcal{O}_\Sigma(H - \mathcal{C})) = H^0(\mathcal{O}_\Sigma(-D - F)) = 0.$$

Theorem 2.3.26. *[[MR]Thm 1.3.11] Let Σ be an ACM surface in $\mathbb{P}^4$ satisfying the condition G_1 (Gorenstein in codimension 1). Let K_Σ be the canonical divisor and let D be an effective divisor linearly equivalent to $mH - K_\Sigma$ for some $m \in \mathbb{Z}$. Then D is an AG curve whose canonical sheaf ω_D is isomorphic to $\mathcal{O}_D(m)$.*

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The condition G_1 consists in requiring that each local ring $\mathcal{O}_{\Sigma,x}$ of dimension less than or equal to 1 is Gorenstein.
We see that a cubic scroll Σ satisfies the hypotheses of the theorem. Indeed it is an ACM surface (see [MR], proof of thm 1.3.15 for a classification of smooth rational ACM surfaces); as moreover Σ is smooth, every local ring is regular hence Gorenstein.
Therefore the curve $\mathcal{C}$ obtained as the residual to Γ in $X \cap \Sigma$ would be an elliptic quintic curve belonging to the class $2D + 3F = -K_\Sigma$; hence an AG curve such that $\mathcal{O}_\mathcal{C} \simeq \omega_\mathcal{C}$.
In the next section we will prove the existence of Γ and Σ as above on a normal cubic threefold X with $\dim(\mathrm{Sing}\,(X)) = 1$. This will lead us to conclude the prove of:

Theorem (2.3.2)**.** *Let X be a normal cubic threefold that is not a cone. Then X contains a non-degenerate AG quintic elliptic curve $\mathcal{C}$.*

We will then show that also this time, the Ulrich sheaves constructed from these curves by mean of Serre's correspondence are locally free.

Proposition 2.3.27. *Let X be a normal cubic threefold that is not a cone. Then there exists a rank 2 skew symmetric Ulrich bundle $\mathcal{E} \in \mathrm{Coh}\,(X)$.*

Illustration of the method in the smooth case

This time again we will first describe how the method works in the smooth case (where there always exists a *smooth* rational quartic) and then we will show how we can adapt it to singular threefolds, where we will need to consider also reducible quartic curves.
Suppose that X is a smooth cubic threefold. It was proved in [MT1] that on X there always exists a smooth rational quartic curve $\Gamma \subset X$.
Starting from a smooth Γ, we can construct a cubic scroll Σ as follows: we take D, a chord of Γ, $D = \overline{p_0 p_1},\ p_i \in \Gamma$. We fix two points $p_2,\ p_2'$ with $p_2 \in \Gamma$, $p_2' \in D$. We consider then all the lines of the form $\overline{pp'}$ where $p \in \Gamma,\ p' \in D$ is a couple of points satisfying the cross-ratio equation:

$$[p_0, p_1, p_2, p] = [p_0, p_1, p_2', p'].$$

The lines $\overline{pp'}$ obtained in this way draw a cubic scroll $\Sigma \subset \mathbb{P}^4$. The directrix of Σ is exactly the line D; the curve Γ intersect D in two points, namely $p_0,\ p_1$ hence $\Gamma \sim D + 3F$ in $\mathrm{Pic}\,(\Sigma)$.

The singular case: construction of the rational quartic Γ

In order to adapt the method that we have just described to the case where X is a singular normal threefold, we will have to consider reducible rational quartics on X. We now explain how to construct such a curve $\Gamma \subset X$; Γ is obtained as the union $\Gamma = L \cup C$ where C is a twisted cubic curve, and L is a line meeting C transversely at a point and not contained in $\langle C \rangle$. Any twisted cubic $C \subset X$ is contained in the hyperplane section $\langle C \rangle \cap X$ and a general hyperplane section of X always contains a twisted cubic. Indeed (as we saw in prop. 2.3.9) a general hyperplane section of X has ADE singularities of the following types:

$$A_1,\ 2A_1,\ 3A_1,\ 4A_1,\ A_2,\ 2A_2,$$

hence it is a cubic surface presenting only rational double points.
A systematic method to obtain twisted cubic curves on a cubic surface S having at most RDPs is presented in [LLSV], where the authors described the Hilbert scheme $\mathrm{Hilb}^{3n+1}(S)$ of curves of degree 3 and arithmetic genus 0 contained in S.
We recall here briefly the construction, exhibited in loc.cit, of these curves.

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Twisted cubics on cubic surfaces with RDPs
The study of curves on a cubic surface S with rational double points, relates to the study of curves on $\tilde{S}$, a weak Del Pezzo surface resolution of singularities of S. Keeping the notations adopted in the previous sections, we call $\mathcal{R} \subset \mathrm{Pic}(\tilde{S})$ the set of roots, $\mathcal{R}_e \subset \mathcal{R}$, $\mathcal{R}_i \subset \mathcal{R}_e$ the sets of, resp., effective and irreducible roots and E_6 lattice $\mathrm{E}_6 := (\mathbb{Z}K_{\tilde{S}})^{\perp}$. Let $R_1, \dots R_m$ the (-2)-curves on $\tilde{S}$ (these are just the irreducible components of the exceptional divisor of $\phi_{-K_{\tilde{S}}}$); again, we identify the curves R_i with their classes in $\mathcal{R}_i \subset \mathrm{Pic}(\tilde{S})$. Define Λ_e as the sublattice of $\mathrm{Pic}(\tilde{S})$ generated by $\mathcal{R}_e$ (since roots are orthogonal to $K_{\tilde{S}}$, Δ_e is a sublattice of E_6).
We consider now $W(\mathcal{R}_e)$ the subgroup of the Weyl group $W(\mathcal{R})$ generated by the reflections r_{R_i}. $W(\mathcal{R}_e)$ acts on $\mathcal{R}$, the orbits contained in $\mathcal{R}_e$ are the connected components of $\mathcal{R}_e$; these corresponds to classes of Dynkin curves on $\tilde{S}$ hence to singular points of S. As the intersection form is nondegenerate on E_6, we get a decomposition:

$$\mathrm{E}_6 \otimes \mathbb{R} = (\Lambda_e \otimes \mathbb{R}) \oplus (\Lambda_e)^{\perp}$$

hence we can write each root $\alpha \in \mathcal{R}$ as:

$$\alpha = \alpha' + \alpha'', \quad \alpha' \in \Lambda_e \otimes \mathbb{R}, \;\; \alpha'' \cdot R_i = 0, \text{ for } i = 1, \dots m.$$

Therefore $\forall\, \beta \in \mathcal{R}_e$, $r_\beta(\alpha) = r_\beta(\alpha')$ so that $W(\mathcal{R}_e) \cdot \alpha = W(\mathcal{R}_e) \cdot \alpha'$. The set $\mathcal{R}_i$ is a simple system for $\Lambda_e \otimes \mathbb{R}$, therefore theorem 2.3.16 applies to the orthogonal projection of any root to $\Lambda_e \otimes \mathbb{R}$.
This implies that given any root $\alpha \in \mathcal{R}$ and denoting by α' its orthogonal projection to $\Lambda_e \otimes \mathbb{R}$, the $W(\mathcal{R}_e)$ orbit of α' intersect each closed fundamental Weyl chamber

$$\overline{\mathcal{K}}_{-\mathcal{R}_i} = \{v \in \Lambda_e \otimes \mathbb{R} \mid v \cdot R_i \leq 0,\ i = 1, \dots m\}, \quad \overline{\mathcal{K}}_{\mathcal{R}_i} = \{v \in \Lambda_e \otimes \mathbb{R} \mid v \cdot R_i \geq 0, i = 1, \dots m\}$$

in exactly one point. We single out in this way two vectors ${\alpha'}^{\pm}$ defined as:

$${\alpha'}^{-} := (W(\mathcal{R}_e) \cdot \alpha') \cap \overline{\mathcal{K}}_{\mathcal{R}_i} \quad {\alpha'}^{+} := (W(\mathcal{R}_e) \cdot \alpha') \cap \overline{\mathcal{K}}_{-\mathcal{R}_i}.$$

By construction we have $\pm{\alpha'}^{\pm} \cdot R_i \leq 0$, $\forall\, i = 1, \dots m$. We define then the vectors $\alpha^{\pm}$ as $\alpha^{\pm} := {\alpha'}^{\pm} + \alpha''$. $\alpha^{\pm}$ belong to $\mathcal{R}$ as $(\alpha^{\pm})^2 = \alpha''^2 + ({\alpha'}^{\pm})^2 = \alpha''^2 + \alpha'^2 = \alpha^2 = -2$. We call α^+ and α^-, respectively, the *maximal and the minimal roots* of the $W(\mathcal{R}_e)$ orbit of α. We can see that $\alpha = \alpha^+ = \alpha^-$ if and only if $\alpha \in {\Lambda_e}^{\perp}$, whether $\alpha^- = -\alpha^+$ if and only if $\alpha \in \mathcal{R}_e$. If now consider a Dynkin curve R_p on $\tilde{S}$ corresponding to a singular point $p \in S$ and an irreducible component R_i of R_p, we see that $(W(\mathcal{R}_e) \cdot R_i) \cap \mathcal{R}_i$ consist of the classes of the components of R_p and ${R_i}^+$ is the class of R_p.
Given $\alpha \in \mathcal{R}$, we consider now the linear system $|\alpha^- - K_{\tilde{S}}|$ and a divisor $D \in |\alpha^- - K_{\tilde{S}}|$. We have $D \cdot (-K_{\tilde{S}}) = -{K_{\tilde{S}}}^2 = 3$ and $D^2 = {\alpha^-}^2 + (-K_{\tilde{S}})^2 = 1$. By Riemann-Roch we get $\chi(\mathcal{O}_{\tilde{S}}(D)) = \frac{1}{2}(D^2 - K_{\tilde{S}} \cdot D) + 1 = 2 + 1 = 3$ and since $2K_{\tilde{S}} - \alpha'$ can not be effective, $h^0(\mathcal{O}_{\tilde{S}}) \geq 3$. For every effective divisor $D \in |\alpha^- - K_{\tilde{S}}|$, its image through $\phi_{-K_{\tilde{S}}}$ is thus a curve on S having degree 3 and arithmetic genus $p_a(D) = 1 + \frac{1}{2}(K_{\tilde{S}} \cdot D + D^2) = 0$, defining a point in $\mathrm{Hilb}^{3n+1}(S)$.
Twisted cubics on S are obtained as the images through $\phi_{-K_{\tilde{S}}}$ of general elements in a linear system $|\alpha^- - K_{\tilde{S}}|$ where α^- is the minimal root of the orbit of an element $\alpha \in \mathcal{R} \setminus \mathcal{R}_e$.

Proposition 2.3.28 ([LLSV] Prop.2.5). *Let $\alpha \in \mathcal{R} \setminus \mathcal{R}_e$, and let α^+ and α^- denote the maximal and the minimal root, respectively, of its orbit.*

(i) The linear system $|\alpha^- - K_{\tilde{S}}|$ does not depend of the choice of α in its $W(\mathcal{R}_e)$ orbit, it has dimension 2 and it is base-point free.

(ii) For every curve $\tilde{C} \in |\alpha^- - K_{\tilde{S}}|$, $C := \phi_{-K_{\tilde{S}}}(\tilde{C})$ is a curve with Hilbert polynomial $P_C(n) = 3n + 1$.

(iii) The image $C = \phi_{-K_{\tilde{S}}}(\tilde{C})$ of a generic curve $\tilde{C} \in |\alpha^- - K_{\tilde{S}}|$ is smooth.

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From (i) we see that whenever $\alpha \in \mathcal{R}$ is not effective, a generic element $\tilde{C} \in |\alpha^- - K_{\tilde{S}}|$ is a smooth rational curve on $\tilde{S}$. Since $\phi_{-K_{\tilde{S}}}$ is an isomorphism outside Sing (S), the curve $C := \phi_{-K_{\tilde{S}}}(\tilde{C})$ can be singular at most along points belonging to Sing (S). If $p \in S$ is a singular point the multiplicity of C at p is given by the integer $(\alpha^- - K_{\tilde{S}}) \cdot R_p$, R_p being the Dynkin curve contracted to p. As we are assuming that α is not effective we must have that $0 \leq \alpha^- \cdot \mathcal{R}_p \leq 1$, namely α'^- (and thus α^-) intersect at most 1 irreducible component of R_p. Let indeed $R_{i_1}, \cdots, R_{i_s}$ be the irreducible components of R_p; by construction, α^- is a root such that $\alpha^- \cdot R_{i_j} \geq 0$ for all $j = 1, \cdots, s$.
If ever α^- meets more than one component of R_p we would have that α^- belongs to the sublattice generated by the R_{i_j}s hence we would get $\alpha^- \in \mathcal{R}_e$ contradicting the assumption $\alpha \in \mathcal{R} \setminus \mathcal{R}_e$.
Therefore $\alpha^- \cdot R_p = 1$ whenever α is not orthogonal to R_p otherwise we get $\alpha^- \cdot R_p = 1$. Therefore whenever α is a non-effective root orthogonal to R_p, the generic element of $|\alpha^- - K_{\tilde{S}}|$ is mapped by $\phi_{-K_{\tilde{S}}}$ to a twisted cubic not passing through p. If $\alpha \cdot R_p \neq 0$ instead, the generic element of $|\alpha^- - K_{\tilde{S}}|$ is mapped to a twisted cubic curve on S having multiplicity 1 at p. Because of the independence of the linear system $|\alpha^- - K_{\tilde{S}}|$ from the choice of the root in its orbit, given a $W(\mathcal{R}_e)$ orbit $B \in \mathcal{R}/W(\mathcal{R}_e)$ we denote by α_B^- the minimal root of the orbit B.
The structure of the Hilbert scheme Hilb $^{3n+1}(S)$ is described by the following theorem:

Theorem 2.3.29 ([?] Thm 2.1)**.** *Let S be a cubic surface with at most rational double points singularities. Then:*

$$\text{Hilb}^{3n+1}(S) \simeq \bigsqcup_{B \in \mathcal{R}/W(\mathcal{R}_e)} |\mathcal{O}_{\tilde{S}}(\alpha_B^- - K_{\tilde{S}})| \simeq (\mathcal{R}/W(\mathcal{R}_e)) \times \mathbb{P}^2.$$

Moreover an orbit of non Cohen-Macaulay or ACM curves depending on whether B contains effective roots or not. The generic curve in a linear system of ACM curves is smooth.

We can finally come back to the construction of rational quartics on the cubic threefold X. Because of the assumptions on X, Sing (X) contains a curve Y, and for a general hyperplane $H \subset \mathbb{P}^4$, $S := X \cap H$ is a cubic surface singular along $Y \cap H$ and that has either $m\ A_1$ singularities, $1 \leq m \leq 4$ or $n\ A_2$ singularities, $n = 1, 2$. Take a root $\alpha \in \text{Pic}(\tilde{S})$, where $\tilde{S}$ is a degree 3 weak del Pezzo surface resolution of S, such that α is not effective; a general element $\tilde{C}$ of $|\alpha^- - K_{\tilde{S}}|$ is then mapped by $\phi_{-K_{\tilde{S}}}$ to a twisted cubic C on S. Consider now a point $p \in C$, under generality assumption we can suppose that $p \notin$ Sing (S) (so that a fortiori $p \notin$ Sing (X)) and that there exists 6 lines $L_1, \dots L_6$ passing through p and entirely contained in X. As we can not have 6 lines on a cubic surface all passing through a smooth point, there always exists $\bar{i} \in \{1, \dots 6\}$ such that $L_{\bar{i}} \not\subset S$ and therefore $L_{\bar{i}} \not\subset \langle S \rangle$. $L_{\bar{i}}$ is thus a line on X meeting C transversely at p and the curve $\Gamma = L_{\bar{i}} \cup C$ is a rational quartic curve entirely contained in X.

The singlar case: construction of the scroll Σ

Let $\Gamma = L \cup C$ be a rational quartic on X, where $L \subset X$ and $C \subset X$ are, respectively, a line and a twisted cubic in X meeting transversely at a point $\bar{p}$. We explain here how to obtain the scroll Σ from Γ. We fix a couple of points $p_0 \in C,\ p_1 \in L,\ p_i \neq \bar{p}$ and we call D the line $D = \overline{p_0 p_1}$. We take two points $p_2,\ p_2'$ with $p_2 \in C$ and $p_2' \in D$. We join with a line all the points $p \in C,\ p' \in D$ satisfying:

$$[\bar{p}, p_1, p_2, p] = [p_0, p_1, p_2', p']. \tag{2.19}$$

The lines obtained in this way form the ruling of a cubic scroll Σ. The scroll Σ has the line D as directrix. Computing the intersections, we see that $L \cap D$ consists just of the point p_0 so that $L \sim F$ in Pic (Σ). $C \sim D + 2F$ intersect D in p_1; the curve Γ is hence a quartic in the class $D + 3F$ metting D in p_0, p_1. In order to see that we are effectively

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drawing a cubic scroll notice that whenever we take a plane $\Delta \subset \mathbb{P}^4$ orthogonal to D, the locus of points $q \in \Delta$ such that:

$$q = \overline{pp'} \cap \Delta,\ p \in C,\ p' \in D \text{ satisfying } (2.19)$$

is simply the image of C under the linear projection from D, hence a conic.
We can now prove theorem 2.3.2.

Proof of theorem 2.3.2. Let X be a normal cubic threefold without triple points. Since we have proved that whenever $\dim(\mathrm{Sing}\,(X)) = 0$, X contains a smooth normal quintic elliptic, we just need to treat the case where $\dim(\mathrm{Sing}\,(X)) = 1$. We consider then Γ a rational quartic and Σ the cubic scroll built from Γ as illustrated above. By construction, the curve $\mathcal{C}$, residual to Γ in $X \cap \Sigma$, belongs to the linear equivalence class $2D + 3F$ in $\mathrm{Pic}(\Sigma)$. Applying theorem 2.3.26, we deduce that $\mathcal{C}$ is an AG curve with $\omega_{\mathcal{C}} \simeq \mathcal{O}_{\mathcal{C}}$ hence an AG quintic curve of arithmetic genus 1. □

We can now prove theorem 2.3.1

Proof of Thm. 2.3.1. Let X be a cubic threefold as in the statement of the proposition. Since we have already showed that there always exists a rank 2 Ulrich bundle on X if ever $\dim(Sing(X)) \leq 0$, we can focus on the case where $\dim(Sing(X)) = 1$. Applying theorem 2.3.2, we get the existence of an AG quintic elliptic curve $\mathcal{C}$ on X contained in $X \cap \Sigma$, where Σ is the cubic scroll constructed from a rational quartic $\Gamma \subset X$ as described in the previous section. $\mathcal{C}$ is an AG curve of (arithmetic) genus 1 and such that $\omega_{\mathcal{C}} \simeq \mathcal{O}_{\mathcal{C}}$. Moreover, from its construction, we know that in $\mathrm{Pic}\,(\Sigma)$, $\mathcal{C} \sim -K_\Sigma \sim 2D+3F$. Consider now the short exact sequence of $\mathcal{O}_\Sigma$-modules:

$$0 \longrightarrow \mathcal{O}_\Sigma(K_\Sigma) \longrightarrow \mathcal{O}_\Sigma \longrightarrow \mathcal{O}_{\mathcal{C}} \longrightarrow 0.$$

Taking the long exact sequence in cohomology we get:

$$0 \longrightarrow H^0(\mathcal{O}_\Sigma(K_\Sigma)) \longrightarrow H^0(\mathcal{O}_\Sigma) \longrightarrow H^0(\mathcal{O}_{\mathcal{C}}) \longrightarrow H^1(\mathcal{O}_\Sigma(K_\Sigma)) \longrightarrow \cdots .$$

$H^0(\mathcal{O}_\Sigma(K_\Sigma)) = H^0(\mathcal{O}_\Sigma(-2D-3F)) = 0$ ($K_\Sigma = -2D-3F$ is not effective). By Serre's duality $h^1(\mathcal{O}_\Sigma(K_\Sigma)) = h^1(\mathcal{O}_\Sigma) = 0$. This implies that $H^0(\mathcal{O}_{\mathcal{C}}) \simeq H^0(\mathcal{O}_\Sigma) \simeq \mathbb{C}$. Therefore the class $1 \in H^0(\mathcal{O}_{\mathcal{C}})$ does not vanish on any point of $\mathcal{C}$ so that 1_x generates $\mathcal{O}_{\mathcal{C},x} \simeq \mathrm{Ext}^1(\mathcal{I}_{\mathcal{C},x}, \mathcal{O}_{X,x})$ forall $x \in \mathcal{C}$. Arguing exactly as in the proof of proposition 2.3.21, we can assert that, by 2.2.2 and 2.2.5, the ACM sheaf $\mathcal{N}$ obtained from $\mathcal{C}$ by Serre's correspondence is locally free. By the non-degeneracy of $\mathcal{C}$ and by remark 13 we can conclude that $\mathcal{E} := \mathcal{N}(2)$ is a skew-symmetric Ulrich bundle. □

As a consequence of theorem 2.3.1 we get:

Corollary 2.3.30. *Let X be a normal cubic threefold that is not a cone. Then X is Pfaffian.*

The last sections of the chapter are devoted to the study of Pfaffian representations of the cubic hypersurfaces we still need to consider, namely:

- Cubic threefolds that are cones over cubic hypersurfaces in $\mathbb{P}^r, r \leq 3$.
- Non-normal cubic threefolds.

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2.4 Non-normal cubic threefolds and cones

Cubic threefolds presenting triple points

Proposition 2.4.1. *Suppose that X is a cone over a cubic hypersurface in $\mathbb{P}^r$, $r \leq 3$. Then X is Pfaffian.*

Proof. If a cubic threefold $X \subset \mathbb{P}(V) \simeq \mathbb{P}^4$ is a cone, there exists then a vector subspace $U < V^*$ of dimension $r+1 \leq 4$ such that $F \in S^3(U)$. X has then multiplicity 3 along the linear space $\mathbb{P}(U^\perp) \subset X$, $\mathbb{P}(U^\perp) \simeq \mathbb{P}^{3-r}$. The polynomial F defines an $(r-1)$-dimensional cubic hypersurface Y and it is clear that the study of Pfaffian representations of X reduces to the study of Pfaffian representations of Y. Since it is well known that a cubic hypersurface of dimension less than or equal to two is always Pfaffian (see for example [B2]), X is Pfaffian too. □

Non-normal cubic threefolds

We still have to analyse the case where X is not normal, namely when $\mathrm{Sing}\,(X)$ has codimension less than 2. We prove the following:
Proposition 2.4.2. *Let X be a non-normal cubic threefold. Then X is Pfaffian.*

Whenever $\mathrm{codim}\,(\mathrm{Sing}\,(X)) \leq 1$, one of the following occurs:

- $\mathrm{codim}\,(\mathrm{Sing}\,(X)) = 1$, hence $\mathrm{Sing}\,(X)$ contains a surface S. From the fact that X contains all the lines generated by points in S we deduce that S must be a plane.
- $\mathrm{codim}\,(\mathrm{Sing}\,(X)) = 0$, hence X is a non-integral cubic threefold, given by the union of a 3-plane H and of a quadric hypersurface Q, $1 \leq \mathrm{rk}\,(Q) \leq 5$.

For each case we will write explicitely a skew matrix of linear forms whose Pfaffian individuates the threefold we are considering.

Cubic threefolds singular along a plane
Suppose that X contains a double plane Δ supported on $\{X_3 = 0, X_4 = 0\}$. Up to an appropriate change of coordinates on $\mathbb{P}^4$, the polynomial F has the form:

$$F(X_0, \ldots X_4) = X_0X_3^2 + X_1X_4^2 + X_2X_3X_4.$$

The skew matrix M_X defined as:

$$M_X = \begin{pmatrix} 0 & X_3 & X_4 & 0 & 0 & X_2 \\ -X_3 & 0 & 0 & 0 & X_4 & 0 \\ -X_4 & 0 & 0 & X_3 & 0 & 0 \\ 0 & 0 & -X_3 & 0 & 0 & X_1 \\ 0 & -X_4 & 0 & 0 & 0 & X_0 \\ -X_2 & 0 & 0 & -X_1 & -X_0 & 0 \end{pmatrix}$$

satisfies $\mathrm{Pf}\,(M_X) = F$ providing then a Pfaffian representation of X.

Non-integral cubic threefolds
Whenever $X := \{F = 0\}$ is not integral, the polynomial F factors as

$$F(X_0, \ldots, X_4) = l(X_0, \ldots X_4)Q(X_0, \ldots X_4),$$

where l is a linear form and Q is a quadratic form of rank less than or equal to five. It is well know that a quadric Q of rank $\mathrm{rk}\,(Q) \leq 5$ is Pfaffian, so that we can find a 4×4 skew matrix of linear forms M_Q such that $\mathrm{Pf}\,(M_Q) = Q(X_0, \ldots X_4)$ (such a matrix can

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be easily written down explicitly, otherwise we refer to [B2]). We construct then M_X, a 6×6 skew-symmetric matrix of linear forms such that $\mathrm{Pf}\,(M_X) = F$ as follows:

$$M_X = \begin{pmatrix} & & & & 0 & 0 \\ & M_Q & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & l \\ 0 & 0 & 0 & 0 & -l & 0 \end{pmatrix}$$

This complete the proof of proposition 2.4.2 and of theorem 2.2.1.

Remark 20. Let $\mathcal{P}$ and $\mathfrak{P} := \mathcal{P}^{ss} /\!\!/ SL(6, \mathbb{C})$ be, respectively, the 74-dimensional projective space of 6×6 skew-symmetric matrices with entries in V^* and the moduli space of Pfaffian representations of cubic threefolds defined in Chapter 1. Consider the rational map $\mathrm{Pf} : \mathcal{P} \dashrightarrow |\mathcal{O}_{\mathbb{P}^4}(3)|,\ M \mapsto \mathrm{Pf}\,(M)$, and the induced rational map $\overline{\mathrm{Pf}} : \mathfrak{P} \dashrightarrow |\mathcal{O}_{\mathbb{P}^4}(3)|$. As an immediate corollary of theorem 2.2.1, we get:

Corollary 2.4.3. *The maps* $\mathrm{Pf} : \mathcal{P} \dashrightarrow |\mathcal{O}_{\mathbb{P}^4}(3)|$ *and* $\overline{\mathrm{Pf}} : \mathfrak{P} \dashrightarrow |\mathcal{O}_{\mathbb{P}^4}(3)|$ *are surjective.*

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Chapter 3

Four-dimensional linear systems of skew-symmetric forms

Introduction

In this chapter we study the four dimensional linear subspaces $\mathbb{P}^4$ of the space $\mathbb{P}(\bigwedge^2 W^*)$ of skew-symmetric forms on a complex vector space W of dimension 6. Linear spaces of such a kind are referred to as 4-dimensional *hyperwebs* of skew-symmetric forms. The first part of the chapter is devoted to the study of their GIT stability. Considering indeed a 4-dimensional linear space $\mathbb{P}(V)$ we see that a hyperweb of skew-symmetric forms defines uniquely a linear embedding $\mathbb{P}(V) \hookrightarrow \mathbb{P}(\bigwedge^2 W^*)$ individuating then a point in the projective space $\mathbb{P}(Hom_{\mathbb{C}}(V, \bigwedge^2 W^*)) \simeq \mathbb{P}(V^* \otimes \bigwedge^2 W^*) \simeq \mathbb{P}^{74}$. The group $SL(W)$ acts on $\mathbb{P}(V^* \otimes \bigwedge^2 W^*)$ so that Geometric Invariant Theory provides a notion of (semi)stability for hyperwebs. Adapting the method used by Wall [Wall], we establish a criterion for (semi)stability. Applying it we can show that whenever a 4-plane $\mathbb{P}(A)$ is not contained in the Pfaffian hypersurface Pf then it is semistable and that it is stable if moreover $\mathbb{P}(A) \cap \mathrm{Pf}$ is smooth.

Theorem (3.2.3)**.** *Let $\mathbb{P}(A)$ be a 4-dimensional linear space of skew-symmetric forms such that the intersection $\mathbb{P}(A) \cap \mathrm{Pf}$ individuates a smooth cubic hypersurface in $\mathbb{P}(A)$. Then $\mathbb{P}(A)$ is stable.*

Afterwards we focus our attention on those linear spaces $\mathbb{P}(A) \subset \mathrm{Pf}$, namely linear spaces whose generic point is a tensor of rank at most 4, aiming to determine the stable ones. We prove the following:

Theorem (**3.2.4**)**.** *Let $\mathbb{P}(A)$ be a stable 4-dimensional linear system of skew-symmetric forms of generic rank less then or equal to four. Then $\mathbb{P}(A)$ is either $SL(W)$-equivalent to the space generated by*

$$\langle e_1 \wedge e_4 + e_2 \wedge e_5,\ e_1 \wedge e_6 + e_3 \wedge e_5,\ e_2 \wedge e_6 - e_3 \wedge e_4,\ e_1 \wedge e_2,\ e_4 \wedge e_5 \rangle$$

or $SL(W)$-equivalent to the space generated by

$$\langle e_1 \wedge e_4 + e_2 \wedge e_5,\ e_1 \wedge e_6 + e_3 \wedge e_5,\ e_2 \wedge e_6 - e_3 \wedge e_4,\ e_1 - e_5 \wedge e_2 + e_4,\ e_1 - e_5 \wedge e_3 + e_6 \rangle.$$

Furthermore in the first case $\mathbb{P}(A)$ meets the Grassmannian $\mathrm{Gr}(2, W^*)$ along a smooth conic isomorphic to $\mathbb{P}^2 \cap \mathrm{Gr}(2, 4)$; in the second case $\mathbb{P}(A)$ intersects the $\mathrm{Gr}(2, W^*)$ along a couple of disjoint lines.

The idea of the proof is the following. We first show that a necessary condition for the stability of $\mathbb{P}(A)$ is that $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$ has dimension 1 (proposition 3.2.6). This necessary condition implies that $\mathbb{P}(A)$ contains a plane $\mathbb{P}(B)$ such that $\forall\, \omega \in \mathbb{P}(B),\ rk(\omega) = 4$

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and that consequently $\mathbb{P}(A)$ might be written $\mathbb{P}(A) = \langle \mathbb{P}(B), \omega_3, \omega_4 \rangle$ with $\omega_3,\ \omega_4$ belonging to $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$. 2-dimensional linear systems of skew-symmetric forms, having constant rank 4 have been classified, up to the action of $PGL(W)$ in [MM]. The authors proved that there exist only four distinct $PGL(W)$-orbits of planes of forms of constant rank 4. We show that if $\mathbb{P}(A)$ is stable, $\mathbb{P}(B)$ can only belong to one of these orbits, so that for a suitable choice of independent vectors $e_1, \dots, e_6$ on W^*, $\mathbb{P}(B)$ is generated by the tensors:

$$\pi_g = \langle e_1 \wedge e_4 + e_2 \wedge e_5, e_1 \wedge e_6 + e_3 \wedge e_5, e_2 \wedge e_6 - e_3 \wedge e_4 \rangle$$

We study then how we can choose a couple of points $\omega_3,\ \omega_4$ on the Grassmannian in order to get a stable 4-plane $\mathbb{P}(A) := \langle \mathbb{P}(B),\ \omega_3, \omega_4 \rangle$.
We first show that in order to get the inclusion $\mathbb{P}(A) \subset \mathrm{Pf}$, $\omega_3,\ \omega_4$ must belong to a rational normal scroll $S_{(2,2,2)}$ admitting the structure of a conic fibration on $\mathbb{P}(B)$. This will also imply that $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$ necessarily consists either of a conic or a pair of, possibly coincident, lines. Finally, we will prove that among the hyperwebs obtained in this way the only ones that are stable are those appearing in the statement of the theorem.
Our classification of stable linear spaces relies on the study of their intersection with the Grassmannian $\mathrm{Gr}(2, W^*)$. Wondering if a similar approach could help us to determine strictly semistable hyperwebs, we investigate varieties obtained as linear sections $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$, $\mathbb{P}(A)$ being a 4 dimensional linear space contained in Pf. This is the main issue we deal with in the second part of the chapter.
We prove the following:

Theorem (3.3.1)**.** *Let $\mathbb{P}(A) \subset \mathbb{P}(\bigwedge^2 W^*)$ be a four-dimensional linear space of skew-symmetric forms of generic rank ≤ 4. Let Y be an irreducible component of $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$. Then one of the following cases is realized:*

- *Y is a linear space $Y \simeq \mathbb{P}^r, 1 \leq r \leq 4$.*
- *Y is a variety of minimal degree,contained in a smaller Grassmannian $\mathrm{Gr}(2, k) = \mathrm{Gr}(2, U) \subset \mathrm{Gr}(2, 6) = \mathrm{Gr}(2, W^*)$, where U is a vector subspace of W of dimension $k < 6$, and Y is a linear section of $\mathrm{Gr}(2, k)$ of one of the following types:*
  - *$Y = \mathbb{P}^d \cap \mathrm{Gr}(2, d+2)$, a rational normal curve of degree d, $2 \leq d \leq 4$.*
  - *$Y = \mathbb{P}^{d+1} \cap \mathrm{Gr}(2, d+2)$, a surface of degree $d = 2, 3$.*
  - *$Y = \mathbb{P}^4 \cap \mathrm{Gr}(2, 4)$, a three-dimensional quadric hypersurface in $\Delta = \mathbb{P}^4$.*
- *Y is an elliptic quintic curve, the image of $\mathbb{P}^4 \cap \mathrm{Gr}(2, 5)$ under some linear embedding $\mathrm{Gr}(2, 5) \hookrightarrow \mathrm{Gr}(2, W^*)$.*

The proof of the theorem reduces to the study of linear sections $\mathbb{P}^r \cap \mathrm{Gr}(2, W^*)$ of the Grassmannian, for $r \leq 4$. More specifically we analyze when Y, an irreducible subvariety of $\mathrm{Gr}(2, W^*)$, spans a linear space $\langle Y \rangle$ contained in Pf and such that $Y = \langle Y \rangle \cap \mathrm{Gr}(2, W^*)$.
The classification provided by theorem 3.3.1 allows us to produce several examples of unstable hyperwebs. In the final part of the chapter we present how these results applies to the study of $\mathfrak{P}$, the moduli space of Pfaffian representation of cubic threefolds.

3.1 Preliminaries

In this section we recollect a few generalities about the Grassmannian, the Pfaffian hypersurface and some of their subvarieties. Let W be a complex vector space of dimension 6 and consider $\bigwedge^2 W^*$, the 15-dimensional vector space of skew-symmetric bilinear forms on W. Each element $\omega \in \bigwedge^2 W^*$ determines a linear morphism $\phi_\omega : W \to W^*$ representable by a 6×6 skew-symmetric matrix M_ω. We define the rank of ω as the rank of M_ω. Note that, as ω is a skew-symmetric bilinear form, its rank is always even, so

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$\forall\, \omega \in \bigwedge^2 W^*$, $\;rk(\omega) = 2k,\; k = 1,2,3$. Throughout the rest of the chapter, for any tensor $\omega \in \bigwedge^2 W^*$, we will denote by U_ω the image of the corresponding linear map ϕ_ω. Consider now $\mathbb{P}(\bigwedge^2 W^*) \simeq \mathbb{P}^{14}$ the projective space of lines in $\bigwedge^2 W^*$. Since the rank of any linear map is invariant under multiplication by a non-zero scalar the notion of rank is well-defined also for elements in $\mathbb{P}(\bigwedge^2 W^*)$.
The locus of all forms having rank less than or equal to 4 individuates an hypersurface, named the Pfaffian hypersurface, $\mathrm{Pf} \subset \mathbb{P}(\bigwedge^2 W^*)$:

$$\mathrm{Pf} := \{\omega \in \mathbb{P}(\bigwedge^2 W^*) \mid rk(\omega) \leq 4\}.$$

Because of the fact that an arbitrary element ω in $\mathbb{P}(\bigwedge^2 W^*)$ is representable by a skew-symmetric matrix, $det\ \omega$ is a square: $|det\ \omega| = \mathrm{Pf}(\omega)^2$. Thus the Pfaffian, being determined by the equation $\mathrm{Pf}(\omega) = 0$, is a cubic hypersurface (as $dim(W^*) = 6$). Consider now $\mathrm{Gr}(2, W^*)$, the Grassmannian of lines in $\mathbb{P}(W^*) \simeq \mathbb{P}^5$. By means of the Plucker's embedding, we can realize $\mathrm{Gr}(2, W^*)$ as a smooth subvariety of $\mathbb{P}(\bigwedge^2 W^*)$ of dimension 8 and degree 14. As any $\omega \in \mathbb{P}(\bigwedge^2 W^*)$ having rank equal to 2, might be written as an *indecomposable* tensor of the form $\omega = v_1 \wedge v_2,\; v_1, v_2 \in W^*$, we see that $\mathrm{Gr}(2, W^*)$ coincides with the locus:

$$\mathrm{Gr}(2, W^*) = \{\omega \in \mathbb{P}(\bigwedge^2 W^*) \mid rk(\omega) = 2\}$$

Consequently, we can identify $\mathrm{Gr}(2, W^*)$ with $\mathrm{Sing}\,(\mathrm{Pf})$ (the set of singular points in Pf). The Pfaffian may also be described as the secant variety of the Grassmannian $\mathrm{Gr}(2, W^*)$:

$$\mathrm{Pf} = \bigcup_{\omega_0, \omega_1 \in \mathrm{Gr}(2, W^*)} \overline{\omega_0 \omega_1};$$

for a point $\omega \in \mathrm{Pf}$, having rank 4, the locus of secants to $\mathrm{Gr}(2, W^*)$ passing through ω is a five dimensional linear space, that we denote by $\mathbb{P}^5_\omega$ (the space $\mathbb{P}^5_\omega$ is isomorphic to $\mathbb{P}(\bigwedge^2(U_\omega)) \simeq \mathbb{P}(\bigwedge^2 \mathbb{C}^4)$). Therefore elements $\omega' \in \mathrm{Pf}$ belonging to $\mathbb{P}^5_\omega$ are exactly those such that $\overline{\omega\omega'} \cap \mathrm{Gr}(2, W^*)$ consists of two (possibly coincident) points.
We end the paragraph stating some properties of the the linear system generated by Plucker's quadrics, $\mathcal{Q} \subset S^2(\bigwedge^2 W), |\mathcal{Q}| \simeq \mathbb{P}(\bigwedge^4 W)$ that will come into use later. More generally, whenever we have a d-dimensional complex vector space V_d, we individuate a linear system $|\mathcal{Q}|$ of quadrics on $\mathbb{P}(\bigwedge^2 V_d)$, $|\mathcal{Q}| \simeq \mathbb{P}(\bigwedge^4 {V_d}^*)$. $|\mathcal{Q}|$ can be spanned by $\binom{d}{4}$ quadrics of rank 6 (the Plucker's quadrics) whose intersection is $\mathrm{Gr}(2, V_d)$. This linear system determines a rational map γ:

$$\gamma : \mathbb{P}(\bigwedge^2 V_d) \dashrightarrow \mathbb{P}(\bigwedge^4 V_d)$$

whose indeterminacy locus is $\mathrm{Gr}(2, V_d)$ (the base locus of $|\mathcal{Q}|$). Whenever $d = 6$ (as it happens for W, the case we will principally be concerned with), we have an isomorphism $\bigwedge^4 W^* \simeq \bigwedge^2 W$ that induces a rational map (that we still denote by γ) $\gamma : \mathbb{P}(\bigwedge^2 W^*) \dashrightarrow \mathbb{P}(\bigwedge^2 W)$ that coincides with the Gauss map of the Pfaffian hypersurface. γ is a Cremona transformation of bidegree 2 mapping the Pfaffian hypersurface to $\mathrm{Gr}(2, W)$ and by means of it we can give a further characterization of the spaces $\mathbb{P}^5_\omega$ associated to a rank 4 tensors $\omega \in \mathrm{Pf}$. Indeed, given any other point ω', we can consider the line $L := \overline{\omega\omega'}$ and the linear system $|\mathcal{Q} - L|$ (i.e. the linear system of quadrics in $\mathcal{Q}$ containing the line L). ω' belongs to $\mathbb{P}^5_\omega$ exactly when the inclusion $|\mathcal{Q} - L| \subset |\mathcal{Q} - \omega|$ is actually an equality; hence when the entire line L is mapped to $\gamma(\omega)$. Therefore we have $\gamma(\mathbb{P}^5_\omega) = \gamma(\omega)$.

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3.1.1 Basics on the geometry of Gr$(2, W^*)$ and Pf

Lines in the Pfaffian hypersurface

Lots among the results that we will prove come from the properties of lines contained in the Pfaffian hypersurface Pf. For a line $L \subset \mathrm{Pf}$, one of the following possibilities occurs:

• $\boldsymbol{L \cap \mathrm{Gr}(2, W^*) = \varnothing}$
A line $L \subset \mathrm{Pf}$ that doesn't meet the Grassmannian (i.e a pencil of skew symmetric forms having constant rank 4) can be of two types (see for ex. [MM]):

- L is a *general line* if it is generated by elements ω_0, ω_1 of the form:

$$\omega_0 = e_1 \wedge e_2 + e_3 \wedge e_4$$
$$\omega_1 = e_1 \wedge e_5 + e_3 \wedge e_6$$

  for an appropriate choice of a basis $e_1, \dots e_6$ of W^*. We notice that $U_{\omega_0} \cap U_{\omega_1}$ has dimension 2, hence, $\mathbb{P}^5_{\omega_0} \cap \mathbb{P}^5_{\omega_1} = \mathbb{P}(\bigwedge^2(U_{\omega_0} \cap U_{\omega_1}))$ consists of just one point ω_2 belonging to $\mathrm{Gr}(2, W^*)$ (in our case $\omega_2 = e_1 \wedge e_3$). ω_2 is the unique point of $\mathrm{Gr}(2, W^*)$ for which we have the inclusion $L \subset \mathbb{T}_{\omega_2}\mathrm{Gr}(2, W^*)$.

- L is a *special line* if there exists a five dimensional subspace W' of W^* such that $L \subset \mathbb{P}(\bigwedge^2 W')$. This means that we can find independent vectors $e_1, \dots, e_5$ in W^* such that L is generated by:

$$\omega_0 = e_1 \wedge e_2 + e_3 \wedge e_4$$
$$\omega_1 = e_1 \wedge e_4 + e_3 \wedge e_5$$

  In this case $\dim(U_{\omega_0} \cap U_{\omega_1}) = 3$ so that $\mathbb{P}^5_{\omega_0} \cap \mathbb{P}^5_{\omega_1} = \mathbb{P}(\bigwedge^2(U_{\omega_0} \cap U_{\omega_1})) \simeq \mathrm{Gr}(2, 3)$ and again there exists a unique point $\omega_2 \in \mathrm{Gr}(2, 3)$ (for our choice of coordinates $\omega_2 = e_1 \wedge e_3$) such that $L \subset \mathbb{T}_{\omega_2}\mathrm{Gr}(2, W^*)$.

• $\boldsymbol{L \cap \mathrm{Gr}(2, W^*) \neq \varnothing}$
Since the Grassmannian is defined by an intersection of quadric hypersurfaces (the Plucker's quadrics), L can either intersect $\mathrm{Gr}(2, W^*)$ in at most two points or be entirely contained in $\mathrm{Gr}(2, W^*)$. More precisely we might have:

- $L \cap \mathrm{Gr}(2, W^*) = \{\omega_0\}$. In this case L can be generated by ω_0 and by a tensor ω_1 of rank 4 such that $\omega_0 \in \mathbb{T}_{\omega_1}\mathrm{Pf}$, but $\omega_0 \notin \mathbb{P}^5_{\omega_1}$. This last condition is equivalent to $\dim(U_{\omega_0} \cap U_{\omega_1}) = 1$. We can thus find independent vectors $e_1, \dots e_5$ such that:

$$\omega_0 = e_4 \wedge e_5$$
$$\omega_1 = e_1 \wedge e_2 + e_3 \wedge e_4$$

  To see this we start choosing coordinates on W^* in such a way that $\omega_0 = e_4 \wedge e_5$. Now, since $\omega_1 \in \mathrm{Pf}$ is a rank 4 tensor, it might be written as $\alpha\omega_1' + \beta\omega_1''$ with ω_1' and ω_1'' in $\mathrm{Gr}(2, W^*)$. Since each tensor in $\mathrm{Gr}(2, W^*)$ is decomposable,we can find 4 independent vectors $\epsilon_1, \dots \epsilon_4$ in W^* such that $\omega_1 = \epsilon_1 \wedge \epsilon_2 + \epsilon_3 \wedge \epsilon_4$. The condition that $L \subset \mathrm{Pf}$ implies that $\dim(\langle e_4,\ e_5\rangle \cap \langle \epsilon_1, \dots \epsilon_4\rangle) = 1$, hence up to the action of $GL(W)$ we can suppose $\omega_1 = e_1 \wedge e_2 + e_3 \wedge e_4$ for independent vectors $e_1,\ e_2,\ e_3$ in W^* such that $\dim\langle e_1, \dots, e_5\rangle = 5$.

- $L \cap \mathrm{Gr}(2, W^*) = \{\omega_0, \omega_1\}$. If L is a secant to the Grassmannian, it can be generated by a rank 4 tensor ω and by a rank 2 tensor ω_0 belonging to $\mathbb{P}^5_\omega$ (this happens if and only if $U_{\omega_0} < U_\omega$) implying in particular that $L \subset \mathbb{P}^5_\omega \simeq \mathbb{P}(\bigwedge^2 U_\omega)$. If the two points ω_0, ω_1 defined by $L \cap \mathrm{Gr}(2, W^*)$ are distinct, then ω correspond to the secant line $\overline{\omega_0, \omega_1}$. Adopting the method used for unisecant lines, we can show that L is

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spanned by two tensors ω_0, ω_1 that, picking a basis $e_1, \dots e_4$ of U_ω, can be written as:

$$\omega_0 = e_1 \wedge e_2, \quad \omega_1 = e_3 \wedge e_4$$

Otherwise, if $\omega_1 = \omega_0$, ω belongs to $\mathbb{T}_{\omega_0}\mathrm{Gr}(2, W^*)$ (so that the entire line is tangent to $\mathrm{Gr}(2, W^*)$ in ω_0); the generators have then the form:

$$\omega_0 = e_1 \wedge e_2, \quad \omega = e_1 \wedge e_3 + e_2 \wedge e_4.$$

- $L \subset \mathrm{Gr}(2, W^*)$. A line L of constant rank 2 can be identified with a point in the Flag variety $F(1, 3, 6)$ (see for example [GH], ch. 5), therefore there always exists a three dimensional subspace W' of W^* such that $L \subset \mathbb{P}(\bigwedge^2 W') \simeq \mathrm{Gr}(2, 3)$. L is generated by a couple of rank two tensors ω_0, ω_1 such that $\omega_i \in \mathrm{Gr}(2, W^*) \cap \mathbb{T}_{\omega_j}\mathrm{Gr}(2, W^*)$, $i, j \in \{1, 2\}$, $i \neq j$. The 3 dimensional space W' coincide with $\langle U_{\omega_0}, U_{\omega_1} \rangle$; we can thus find 3 vectors e_1, e_2, e_3 spanning W' such that we may write:

$$\omega_0 = e_1 \wedge e_2, \quad \omega_1 = e_2 \wedge e_3$$

The group $PGL(W)$ acts on the variety of lines of Pf, each type of line described above corresponds to an orbit of this action.

Linear subspaces of Gr$(2, W^*)$

We remind briefly the structure of linear spaces $\Delta \simeq \mathbb{P}^r$ contained in $\mathrm{Gr}(2, W^*)$. We described lines contained in $\mathrm{Gr}(2, W^*)$ in the previous section, we now treat the case $r \geq 2$.

• $\boldsymbol{\Delta \simeq \mathbb{P}^2}$

A plane Δ contained in $\mathrm{Gr}(2, W^*)$ is of one among the two following types:

- $\Delta \simeq \mathrm{Gr}(2, 3)$: in this case we can find 3 independent vectors in W^*, e_1, e_2, e_3, in such a way that Δ is generated by the tensors:

$$\omega_0 = e_1 \wedge e_2, \quad \omega_1 = e_1 \wedge e_3, \quad \omega_3 = e_2 \wedge e_3.$$

- Δ is of the form $\mathbb{P}(l \wedge W')$, being l a line in W^* and W' a 4 dimensional linear subspace of W^* containing l. Δ correponds then to a point in the Flag variety $F(1, 4, 6)$. Consequently, calling e_1 the generator of l and picking 3 other independent vectors e_2, e_3, e_4 in such a way that $W' = \langle e_1, \dots, e_4 \rangle$, we can write the generators of Δ in the form:

$$\omega_0 = e_1 \wedge e_2, \quad \omega_1 = e_1 \wedge e_3, \quad \omega_3 = e_1 \wedge e_4.$$

• $\boldsymbol{\Delta \simeq \mathbb{P}^3}$

A 3-dimensional linear subspace is always of the form $\mathbb{P}(l \wedge W')$ for a line l in W^* and a 5-dimensional subspace W' of W^* containing it. Choosing then $e_1, \dots e_5$, a basis of W' such that $l = \langle e_1 \rangle$, we can represent the generators of Δ as follows:

$$\omega_0 = e_1 \wedge e_2, \quad \omega_1 = e_1 \wedge e_3, \quad \omega_3 = e_1 \wedge e_4, \quad \omega_3 = e_1 \wedge e_5.$$

Δ corresponds to a point in $F(1, 5, 6)$.

• $\boldsymbol{\Delta \simeq \mathbb{P}^4}$

Similarly to the three dimensional case, a linear subspace of $\mathrm{Gr}(2, W^*)$ of dimension 4, $\Delta \subset \mathrm{Gr}(2, W^*)$ is always of the form $\mathbb{P}(l \wedge W^*)$ for a line l in W^*. Choosing then $e_1, \dots e_6$, a basis of W^* such that $l = \langle e_1 \rangle$, we can represent the generators of Δ as :

$$\omega_0 = e_1 \wedge e_2, \quad \omega_1 = e_1 \wedge e_3, \quad \omega_3 = e_1 \wedge e_4, \quad \omega_3 = e_1 \wedge e_5 \quad \omega_4 = e_1 \wedge e_6.$$

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3.2 Hyperwebs of skew-symmetric forms

Our principal subjects of study are 4-dimensional linear systems of skew-symmetric forms on W. These linear systems are referred to as *hyperwebs* of skew-symmetric forms. A four dimensional hyperweb might equivalently be defined as:

- a 4-dimensional linear subspaces $\mathbb{P}(A)$ of $\mathbb{P}(\bigwedge^2 W^*)$;
- a 6×6 skew-symmetric matrix M_A whose entries are linear forms on $\mathbb{P}(A)$;
- a linear embeddings $\phi : \mathbb{P}(V) \to \mathbb{P}(\bigwedge^2 W^*)$, V being a complex vector space of dimension 5, whose image is $\mathbb{P}(A)$.

Consider a 4-dimensional linear space $\mathbb{P}(A) \subset \mathbb{P}(\bigwedge^2 W^*)$ representing a linear system of skew-symmetric forms; tensors $\omega_0, \ldots, \omega_4$ spanning $\mathbb{P}(A)$ are referred to as generators of the corresponding linear system. Denote by M_A the skew-symmetric matrix of linear forms corresponding to $\mathbb{P}(A)$. The intersection of $\mathbb{P}(A)$ with the Pfaffian hypersurface Pf, $\mathbb{P}(A) \cap \mathrm{Pf}$, individuates those tensors $\omega \in \mathbb{P}(A)$ satisfying $rk(\omega) \leq 4$. The locus $\mathbb{P}(A) \cap \mathrm{Pf}$ is defined by the equation $\mathrm{Pf}\,(M_A) = 0$ and hence it is a cubic hypersurface in $\mathbb{P}(A)$ whenever $\mathbb{P}(A) \not\subset \mathrm{Pf}$.
Tensors $\omega \in \mathbb{P}(A)$ having rank 2 coincide with points belonging to $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$. The locus $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$ is a closed subvariety of $\mathbb{P}(A)$ defined by an intersection of quadrics. Specifically, it is the base locus of $|\mathcal{Q}|_{\mathbb{P}(A)}|$ (the linear system of quadrics in $\mathcal{Q}$, restricted to $\mathbb{P}(A)$), so that:

$$\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*) = \mathbb{P}(A) \cap \Big(\bigcap_{Q \in \mathcal{Q}} Q \Big).$$

The equations defining $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$ can be written down explicitly taking the 4×4 minors of M_A.

3.2.1 GIT stability

Defining hyperwebs we saw that any 4-dimensional linear space of skew-symmetric forms defines a linear embedding $\phi : \mathbb{P}(V) \to \mathbb{P}(\bigwedge^2 W^*)$, where $\mathbb{P}(V) \simeq \mathbb{P}^4$, hence it individuates a point in $\mathbb{P}(V^* \otimes \bigwedge^2 W^*)$. $\mathbb{P}(V^* \otimes \bigwedge^2 W^*)$ is a projective space of dimension 74 and we have a natural action of the group $GL(V) \times GL(W)$ on it .We will focus on the $GL(W)$-action. This is induced by the $GL(W)$ action on the affine cone A of $\mathbb{P}(A)$, so that given $\mathbb{P}(A) \subset \mathbb{P}(\bigwedge^2 W^*)$ generated by $\omega_0, \ldots ,\omega_4$ we have:

$$g \cdot \mathbb{P}(A) = \langle g \cdot \omega_0, \ldots, g \cdot \omega_4 \rangle, \qquad g \in GL(W)$$

(Recall that given $\omega \in \bigwedge^2 W^*, g \in GL(W)$, $g \cdot \omega$ is the skew-symmetric form defined by $g \cdot \omega(u, v) = \omega(g \cdot u, g \cdot v)$, for any couple u, v of vectors in W. Therefore, if M_ω, $M_{g \cdot \omega}$ denote the anti-symmetric matrices representing ω and $g \cdot \omega$ respectively , we have $M_{g \cdot \omega} = g M_\omega\, g^T$.) As $\forall\, \lambda \in \mathbb{C}^*$, $\mathbb{P}(\lambda g \cdot A)$ is clearly equal to $\mathbb{P}(g \cdot A)$, we see that $GL(W)$ acts as the projective general linear group $PGL(W)$. Having an action of $GL(W)$ on $\mathbb{P}(V^* \otimes \bigwedge^2 W^*)$ we get, by geometric invariant theory, a notion of stability for hyperwebs of skew-symmetric forms with respect to this action. By definition of GIT stability, a point $\mathbb{P}(A)$ is stable (resp. semistable) if and only if every point in $\mathbb{C}^* \cdot A \subset V^* \otimes \bigwedge^2 W^*$ (i. e. every point on the affine cone projecting to $\mathbb{P}(A)$) is stable (resp. semistable). Therefore the analysis of the stability of $\mathbb{P}(A)$ is equivalent to analysis of the stability of A. In order to apply invariant theory it is convenient to restrict to the action of $SL(W)$. Indeed in this way we can, by means of Hilbert-Mumford criterion, detect stability just by analyzing the behavior under the action of 1-parameter (1-PS) subgroups of $SL(W)$.

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Hilbert-Mumford criterion. *Let G be a connected reductive complex linear group acting on a projective variety X. A point $x \in X$ is G-(semi)stable if and only if it is stable with repsect to any 1-parameter subsgroup $\lambda : \mathbb{G}_m \to G$ of G.*

Therefore, a point $\mathbb{P}(A)$ fails to be semistable if and only if there exists a 1-PS λ of $SL(W)$ such that $0 = \lim_{t\to\infty} \lambda(t) \cdot A$ (so that $0 \in \overline{\lambda(\mathbb{G}_m) \cdot A}$). Therefore A fails to be stable if and only if there exists a 1-PS λ of $SL(W)$ such that $\lambda(t) \cdot A$ is bounded as $t \to \infty$ (as if this is the case $\overline{(\lambda(\mathbb{G}_m) \cdot A)} \setminus (\lambda(\mathbb{G}_m) \cdot A) \neq \varnothing$). Denote now by $\omega_0, \dots, \omega_4$ the generators of A. Since for $t \in \mathbb{C}^*$, $\lambda(t) \cdot A$ is generated by $\lambda(t) \cdot \omega_i, i = 0, \dots, 4$, we see that A is not λ-(semi)stable if and only if each generator ω_i, $i = 0, \dots, 4$ is not λ(semi)stable. A 1-PS λ of $SL(W)$, $\lambda : \mathbb{C}^* \to SL(W)$, is uniquely determined by a decreasing six-tuple of complex numbers, $\lambda_1 \geq \dots, \geq \lambda_6$, $\sum_{i=1}^6 \lambda_i = 0$, not all equal to zero, such that $\lambda(t) = diag(t^{\lambda_1}, \dots, t^{\lambda_6})$(for this reason we will often denote such a λ with the 6-tuple of weigths of its action, $(\lambda_1, \dots, \lambda_6)$). Adapting the argument used by Wall in [Wall],we prove the following:

Theorem 3.2.1. *Let A be an element of $V^* \otimes \bigwedge^2 W^*$ generated by tensors $\omega_0, \dots, \omega_4$. We denote by M_{ω_k} the skew-symmetric matrices representing the forms ω_k, $M_{\omega_k} = (a^k_{ik})$, $1 \leq i, j \leq 6$, $k = 0, \dots, 4$.*

1. *A is not stable if and only if, for some choice of coordinates on W, there exists an integer $1 \leq s \leq 3$ such that, $a^k_{ij} = 0$ whenever $1 \leq i \leq s$, $i < j \leq 6 - s, 0 \leq k \leq 4$.*
2. *A is not semistable if and only if, for some choice of coordinates on W, there exists an integer $1 \leq s \leq 3$ such that $a^k_{ij} = 0$ whenever $1 \leq i \leq s$, $i < j \leq 7 - s, 0 \leq k \leq 4$.*

Proof. In the first place, notice that given $A \in V^* \otimes \bigwedge^2 W^*$ as in the statement and $\lambda = (\lambda_1, \dots, \lambda_6)$ a 1-PS, $\lambda(t) \cdot A$ is then generated by $\lambda(t) \cdot \omega_k$, tensors represented by the matrices $(M_{\lambda(t)\cdot\omega_k})_{ij} = t^{\lambda_i+\lambda_j} a^k_{ij}$, $k = 0, \dots, 4$.

1. Suppose that A is unstable, that is, non-semistable, and let $\lambda = (\lambda_1, \dots, \lambda_6)$ be a 1-PS for which we have $0 \in \overline{\lambda(\mathbb{G}_m) \cdot A}$. We thus have $0 \in \overline{\lambda(\mathbb{G}_m) \cdot \omega_k}$, $\forall k = 0, \dots, 4$ or, in other words, none among the generators of A is λ-semistable. There exists then an integer $s \in \{1, 2, 3\}$ such that $\lambda_i + \lambda_{7-i} \geq 0$. Indeed, if this was not the case, we would have $\sum_{i=1}^6 \lambda_i < 0$, which is absurd. For such an s we therefore have $\lambda_s + \lambda_{7-s} \geq 0$ and so, from the assumptions on the λ_i, $\lambda_i + \lambda_j \geq 0$, $\forall 1 \leq i \leq s$, $\forall i < j \leq 7 - s$. As we are assuming that A is not λ-semistable, we conclude that $a^k_{i,j} = 0$ whenever $1 \leq i \leq s, i < j \leq 7 - s$. Conversely, if the coordinates a^k_{ij} of A satisfy the hypotheses of the theorem, we are able to construct explicitly a 1-PS of $SL(W)$ refuting the semistability of A. This is the case for $\lambda \in \mathrm{Hom}_{\mathrm{Gr-Alg}}(\mathbb{G}_m, SL(W))$ acting with weights $\lambda_1, \dots, \lambda_6$ defined as follows:

$$\lambda_i = \begin{cases} 6 - s & 1 \leq i \leq s \\ -1 & s + 1 \leq i \leq 7 - s \\ s - 7 & 8 - s \leq i \leq 6 \end{cases}$$

2. Suppose now that A is nonstable. This means that there exists a 1-PS $\lambda = (\lambda_1, \dots, \lambda_6)$ for which $\forall k = 0, \dots, 4$, $\lambda(t) \cdot \omega_k$ is bounded as $t \to \infty$ (implying thus that $\overline{(\lambda(\mathbb{G}_m) \cdot A)} \setminus \lambda(\mathbb{G}_m) \cdot A \neq \varnothing$). Now, if ever for all integers $s \in \{1, 2, 3\}$ we had $\lambda_s + \lambda_{6-s} \leq 0$, we would get $2\sum_{i=1}^5 \lambda_i \leq 0$ hence $\sum_{i=1}^5 \lambda_i \leq 0$. As $\sum_{i=1}^6 \lambda_i = 0$ we should then have $\lambda_6 \geq 0$. Thus for every $1 \leq i \leq 5$, $\lambda_i \geq \lambda_6 \leq 0$ $\forall 1 \leq i \leq 5$, condition that can be satisfied if and only if all the λ_i's are equal to zero, a contradiction. Then there exists an s, $1 \leq s \leq 3$, for which $\lambda_s + \lambda_{6-s} > 0$, and consequently for all i, j with $i \leq s$, $i < j \leq 6 - s$, $\lambda_i + \lambda_j > 0$. As the non-stability requires the boundedness of $\lambda(t) \cdot \omega_k$ as $t \to \infty$ for every $k = 0, \dots, 4$, we must have $a^k_{ij} = 0$ whenever $0 \leq k \leq 4$, $i \leq s$, $i < j \leq 6 - s$.

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For the converse implication, consider $A \in V^* \otimes \bigwedge^2 W^*$ whose affine coordinates a_{ij}^k satisfy the hypotheses of the proposition. Again, we are able to provide a 1-PS λ for which A is not λ-stable. We can consider for example a 1-PS acting with the following weigths:

$$\lambda_i = \begin{cases} 1 & 1 \leq i \leq s \\ 0 & s+1 \leq i \leq 6-s \\ -1 & 6-s+1 \leq i \leq 6 \end{cases}$$

□

Representing an element of $\mathbb{P}(V^* \otimes \bigwedge^2 W^*)$ as a 6×6 skew-symmetric matrix with entries in V^* we see that for an appropriate choice of coordinates on W, a nonstable hyperweb is of one of the following forms:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & * & * & * \\ 0 & * & 0 & * & * & * \\ 0 & * & * & 0 & * & * \\ 0 & * & * & * & 0 & * \\ * & * & * & * & * & 0 \end{pmatrix} \quad (s=1) \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & * & 0 & * & * \\ * & * & * & * & 0 & * \\ * & * & * & * & * & 0 \end{pmatrix} \quad (s=2)$$

$$\begin{pmatrix} 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ * & * & * & 0 & * & * \\ * & * & * & * & 0 & * \\ * & * & * & * & * & 0 \end{pmatrix} \quad (s=3)$$

Similarly, a hyperweb that is not semistable might be represented as a skew-symmetric matrix of one of these types:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & * \\ 0 & * & 0 & * & * & * \\ 0 & * & * & 0 & * & * \\ 0 & * & * & * & 0 & * \\ 0 & * & * & * & * & 0 \end{pmatrix} \quad (s=1) \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & * & 0 & * & * \\ 0 & 0 & * & * & 0 & * \\ * & * & * & * & * & 0 \end{pmatrix} \quad (s=2)$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ * & * & * & * & 0 & * \\ * & * & * & * & * & 0 \end{pmatrix} \quad (s=3)$$

Corollary 3.2.2. *Let $\mathbb{P}(A)$ be a linear space of skew-symmetric forms and denote by M_A the corresponding 6×6 skew-symmetric matrix of linear forms. If $\mathbb{P}(A)$ is not semistable then* $\mathrm{Pf}\,(M_A) = 0$.

Proof. Let $\mathbb{P}(A)$ be an element in $\mathbb{P}(V^* \otimes \bigwedge^2 W^*)$ and M_A, $(M_A)_{ij} = l_{ij}$, $l_{ij} = \sum_{k=0}^4 a_{ij}^k X_k$ the corresponding matrix of linear forms. Denoting by q_{ij} the $(ij)-$th

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entries of the Pfaffian adjugate of M_A, namely $q_{ij} = \mathrm{Pf}(M_A^{i,j})$ is the Pfaffian of the sub-matrix obtained by deleting the ith and jth rows and columns, we have:

$$\mathrm{Pf}(M_A) = \sum_{j=1}^{6} (-1)^j l_{1j} q_{1j}.$$

Suppose now that M_A is not semistable. Applying theorem 3.2.1, we might suppose that, $\exists\ s \in \{1,2,3\}$ such that $a_{ij}^k = 0$ whenever $1 \le i \le s,\ i < j \le 7-s, 0 \le k \le 4$. We can thus easily compute that:

- If $s = 1$, $l_{1j} = 0\ \forall\ 1 < j \le 6$ so that $\mathrm{Pf}(M_A) = 0$.
- If $s = 2$, $\mathrm{Pf}(M_A) = l_{16}q_{16}$; as $l_{2j} = 0,\ \forall\ 2 \le j \le 5$, $q_{16} = 0$, we get $\mathrm{Pf}(M_A) = 0$
- If $s = 3$, $\mathrm{Pf}(M_A) = l_{16}q_{16} - l_{15}q_{15}$; as $l_{2_j} = 0,\ \forall\ 3 \le j \le 4,\ l_{34} = 0$, $q_{15} = q_{16} = 0$, we get $\mathrm{Pf}(M_A) = 0$.

□

Remark 21. The set of all points $A \in V^* \otimes \bigwedge^2 W^*$ that are not semistable (and thus that are such that $\mathbb{P}(A)$ is not semistable), forms a Zariski closed cone $\mathcal{H}_{V^*\otimes\bigwedge^2 W^*}(SL(W))$ in $V^* \otimes \bigwedge^2 W^*$ called the *Hilbert nullcone*. It is proven in [BD] that $\mathcal{H}_{V^*\otimes\bigwedge^2 W^*}(SL(W))$ has exactly 3 irreducible components $\mathcal{H}^s,\ s = 1,2,3$, each one corresponding to the couple of integers $s, 7-s$ determined by theorem 3.2.1, namely:

$$\mathcal{H}^s = \{A \in V^* \otimes \bigwedge^2 W^* \mid\ \exists\ : U' \subset U \subset W,\ \dim(U') = s,\ \dim(U) = 7-s,$$
$$\omega(U', U) = 0,\ \forall \omega \in A\}$$

The stability criterion can also be rephrased in more geometric terms (this will be the characterization of stability that we will mainly use). Indeed we observe that $\mathbb{P}(A) \in V^* \otimes \bigwedge^2 W^*$ fails to be stable (resp. semistable) if there exists an integer $s \in \{1,2,3\}$ and U', U a couple of vector subspaces of W of dimension s and $6-s$ (resp. of dimension s and $7-s$) such that $U' < U$ and $\omega(U', U) = 0,\ \forall\ \omega \in A$. Such a couple of vector spaces satisfies then $\mathbb{P}(U' \wedge U) \subset \mathbb{P}(A)^\perp$. Analyzing each possible value of s we observe that the following situations occur:

- $\boldsymbol{s = 1}$

  For $s = 1$, $\mathbb{P}(A)$ is not stable if and only if there exists a vector $u \in W$ different from 0, for which the forms $\omega(u,\ \cdot) \in W^*, \omega \in A$, satisfy $\bigcap_{\omega\in A} \ker(\omega(u,\ \cdot)) = U$, $\dim(U) \ge 5$. $\mathbb{P}(A)$ is not even semistable when $U = W$ hence when the vector u belongs to $\bigcap_{\omega\in A} \ker(\omega)$. $\mathbb{P}(u \wedge U) \simeq \mathbb{P}^{\dim(U)-1}$ is thus a linear space (of dimension 3 if $\mathbb{P}(A)$ is not stable and of dimension 4 if $\mathbb{P}(A)$ is not semistable) entirely contained in $\mathbb{P}(A)^\perp$.

- $\boldsymbol{s = 2}$

  For $s = 2$, $\mathbb{P}(A)$ is not stable if and only if there exist two independent vectors $u_1,\ u_2$ in W and a vector subspace U of W of dimension greater than or equal to 4, such that $\bigcap_{\omega\in A} \ker(\omega(u_i, \cdot)) = U\ i = 1, 2$

- $\boldsymbol{s = 3}$

  When s is equal to 3, the linear space $\mathbb{P}(A)$ is not stable if and only if there exists a vector subspace $U < W$ of dimension 3 or 4 that is isotropic with the respect to every tensor $\omega \in A$ or equivalently, that satisfies $\mathbb{P}(\bigwedge^2 U) \subset \mathbb{P}(A)^\perp$. More precisely $\mathbb{P}(A)$ is not stable when $\dim(U) = 3$, in this case $\mathbb{P}(\bigwedge^2 U) \simeq \mathrm{Gr}(2, U) \simeq \mathbb{P}^2$. Whenever $\dim(U) = 4$, $\mathbb{P}(A)$ fails to be semistable and $\mathbb{P}(\bigwedge^2 U) \simeq \mathbb{P}^5$. In this circumstance the condition $\mathbb{P}(\bigwedge^2 U) \subset \mathbb{P}(A)^\perp$ yields the inclusion $\mathbb{P}(A) \subset \mathbb{T}_u\mathrm{Gr}(2, W^*)$, being $u := \wedge^2(U^\perp)$.

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Remark 22. Because of the fact that we are working with skew-symmetric forms, if $\mathbb{P}(A)$ is "destabilized" by a couple of spaces $U' < U$ of dimension 2 and 4, then for any hyperplane U'' of U containing U', we have $\mathbb{P}(\bigwedge^2 U'') \subset \mathbb{P}(A)^\perp$. Indeed given any tensor $\omega \in \mathbb{P}(U' \wedge U)^\perp$ and any vector $u \in U,\ u \notin U'$, since $\omega(u,u) = 0$, the 3-dimensional space $U'' := \langle U', u\rangle$ always satisfies $\omega(U'', U'') = 0$. Otherwise we can argue saying that the existence of the aforementioned couple U', U implies that $\mathbb{P}(A)^\perp \cap \mathrm{Gr}(2, U)$ defines an hyperplane section $H \cap \mathrm{Gr}(2, U)$ of $\mathrm{Gr}(2, U)$ by an hyperplane $H \in \mathrm{Gr}(2, U^*)$. Such an hyperplane section is a 4-dimensional quadric of rank 4 that always contains a plane isomorphic to $\mathrm{Gr}(2, 3)$.

Using this "geometric" formulation of our criterion we prove the following.

Theorem 3.2.3. *Let $\mathbb{P}(A)$ be a 4-dimensional linear space of skew-symmetric forms such that the intersection $\mathbb{P}(A) \cap \mathrm{Pf}$ individuates a smooth cubic hypersurface in $\mathbb{P}(A)$. Then $\mathbb{P}(A)$ is stable.*

Proof. Let $\mathbb{P}(A)$ be a linear space satisfying the hypotheses of the theorem. Since by assumption $\mathbb{P}(A) \cap \mathrm{Pf}$ is a cubic hypersurface X in $\mathbb{P}(A)$, $\mathbb{P}(A) \not\subset \mathrm{Pf}$ so that, by corollary 3.2.2, $\mathbb{P}(A)$ is semistable. Assume by contradiction that $\mathbb{P}(A)$ is not stable (hence that it is strictly semistable) and denote then by s, $s \in \{1,\ 2,\ 3\}$, the integer consequently individuated applying the stability criterion 3.2.1. We analyze each possible value of s showing that in any case we can not get a smooth intersection $X = \mathbb{P}(A) \cap \mathrm{Pf}$.

- **s=1 or s=2.** We show that whenever $s = 1$ or $s = 2$, X is reducible. Call $M_A = (l_{ij})$, $1 \le i < j \le 6$, the 6×6 skew-symmetric matrix of linear forms individuated by $\mathbb{P}(A)$ and denote by $q_{ij} := \mathrm{Pf}\,(M_A^{i,j})$ the Pfaffian of the sub-matrix obtained by deleting the ith and jth rows and columns. X is defined by the cubic form $\mathrm{Pf}\,(M_A) = \sum_{j=1}^{6} (-1)^j l_{1j} q_{1j}$. Because of the assumptions we can suppose that $l_{ij} = 0$ for $1 \le s$, $i < j \le 6 - s$. Now, if $s = 1$, we have:

  $$\mathrm{Pf}\,(M_A) = l_{16} q_{16}$$

  hence X is reducible.

  If $s = 2$, we compute that:

  $$\mathrm{Pf}\,(M_A) = l_{16} q_{16} - l_{15} q_{15} = l_{34}(l_{16} l_{25} - l_{15} l_{26})$$

  so that also in this case we get a reducible cubic X.

- **s=3.** In order to treat the case where $s = 3$ we use an approach different from the previous one, that relies on the "geometric formulation" of the stability criterion. To start with we recall that if $X = \mathbb{P}(A) \cap \mathrm{Pf}$ is a smooth threefold, the intersection $\mathbb{P}(A)^\perp \cap \mathrm{Gr}(2, W)$ defines a smooth threefold Y of degree 14. We denote by $F(X)$ the Fano surface of lines on X and by $F(Y)$ the variety parametrizing conics on Y. Under smoothness assumptions we have an isomorphism :

  $$F(X) \simeq F(Y).$$

  If now we suppose that $\mathbb{P}(A)$ is not stable and $s = 3$ is the integer determined by 3.2.1, there exists then a 3 dimensional linear subspace U of W, such that $\mathbb{P}(\bigwedge^2 U) \simeq \mathbb{P}^2$ is contained in $\mathbb{P}(A)^\perp$. As $\mathbb{P}(\bigwedge^2 U) \simeq \mathrm{Gr}(2, U)$, the plane $\mathbb{P}(\bigwedge^2 U)$ is therefore contained in $\mathbb{P}(A)^\perp \cap \mathrm{Gr}(2, W) = Y$. The inclusion $\mathbb{P}(\bigwedge^2 U) \subset Y$, implies that Y contains all the conics in the plane $\mathbb{P}(\bigwedge^2 U)$ and therefore that $F(Y)$ has dimension at least 5. This clearly leads to a contradiction.

□

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3.2.2 Stable hyperwebs of generic rank less than or equal to four

From now on we will work with those 4-dimensional linear subspaces $\mathbb{P}(A)$ of $\mathbb{P}(\bigwedge^2 W^*)$ that moreover are such that $\forall\, \omega \in A,\ rk(\omega) \leq 4$. This condition is equivalent to requiring that $\mathbb{P}(A)$ is contained in Pf, the Pfaffian hypersurface Pf $\subset \mathbb{P}(\bigwedge^2 W^*)$. We refer to these linear systems as hyperwebs of skew-symmetric forms of generic rank less than or equal to 4.
Our first aim is to determine the stable ones. We prove the following:

Theorem 3.2.4. *Let $\mathbb{P}(A)$ be a stable 4-dimensional hyperweb of skew-symmetric forms of generic rank less then or equal to four. Then $\mathbb{P}(A)$ is either $SL(W)$-equivalent to the space generated by*

$$\langle e_1 \wedge e_4 + e_2 \wedge e_5,\ e_1 \wedge e_6 + e_3 \wedge e_5,\ e_2 \wedge e_6 - e_3 \wedge e_4,\ e_1 \wedge e_2,\ e_4 \wedge e_5 \rangle$$

or $SL(W)$-equivalent to the space generated by

$$\langle e_1 \wedge e_4 + e_2 \wedge e_5,\ e_1 \wedge e_6 + e_3 \wedge e_5,\ e_2 \wedge e_6 - e_3 \wedge e_4,\ e_1 - e_5 \wedge e_2 + e_4,\ e_1 - e_5 \wedge e_3 + e_6 \rangle.$$

Furthermore in the first case $\mathbb{P}(A)$ meets the Grassmannian $\mathrm{Gr}(2, W^*)$ along a smooth conic isomorphic to $\mathbb{P}^2 \cap \mathrm{Gr}(2,4)$; in the second case $\mathbb{P}(A)$ intersects the $\mathrm{Gr}(2, W^*)$ along a couple of disjoint lines.

The proof of the theorem relies on the study of the intersection $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$. Given indeed a linear space $\mathbb{P}(A)$ contained in the Pfaffian hypersurface Pf, it is natural to ask whether it intersects the Grassmannian. It turns out that whenever $\dim(\mathbb{P}(A)) = 4$, the answer is always positive and that moreover, $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$ is a variety having dimension at least 1. This is essentially due to the following result, by Manivel-Mezzetti ([MM], cor.11):

Proposition 3.2.5 (Manivel-Mezzetti)**.** *There exists no $\mathbb{P}^3$ of skew-symmetric matrices of order six and constant rank four.*

In [MM] the authors prove that, more precisely, a 3-dimensional hyperweb contained in the Pfaffian hypersurface always meets the Grassmannian along a variety of dimension bigger than or equal to 0.

Dimension of the intersection with the Grassmannian

From proposition 3.2.5, we get that given $\mathbb{P}(A)$, a 4-dimensional linear subspace of Pf, the intersection $\mathrm{Gr}(2, W^*) \cap \mathbb{P}(A)$ is a variety whose dimension is always bigger than or equal to 1. The first step to prove theorem 3.2.4 is to show that if $\mathbb{P}(A)$ is stable, then $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$ must have dimension exactly one.
Proposition 3.2.6. *Let $\mathbb{P}(A)$ be a stable 4-dimensional linear space of skew-symmetric forms of generic rank less than or equal to 4 and let X be the variety defined as the intersection $X := \mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$. Then X is a smooth 1 dimensional variety.*

Proof. To start with we recall that given any rank two tensor $\omega \in \mathrm{Gr}(2, W^*)$ we get the following decomposition of $\mathbb{P}(\bigwedge^2 W^*)$:

$$\mathbb{P}(\bigwedge^2 W^*) = \langle \mathbb{T}_\omega \mathrm{Gr}(2, W^*), \mathbb{P}(\bigwedge^2 \ker(\omega)^*) \rangle.$$

Let indeed $l_\omega \subset \mathbb{P}(W^*)$ be the line corresponding to ω. The tangent space to $\mathrm{Gr}(2, W^*)$ at ω, $\mathbb{T}_\omega \mathrm{Gr}(2, W^*)$, is the 8-dimensional linear subspace of $\mathbb{P}(\bigwedge^2 W^*)$ spanned by tensors corresponding to lines in $\mathbb{P}(W^*)$ meeting l_ω (namely this tangent space can be interpreted as the linear span of the Scubert variety parametrizing lines in $\mathbb{P}(W^*)$ intersecting ω). Consider now the linear space $\mathbb{P}(\ker(\omega)^*) \subset \mathbb{P}(W^*)$; this is a 3 dimensional linear space orthogonal to l_ω. Lines in this 3-plane define the Grassmannian $\mathrm{Gr}(2, \ker(\omega)^*) \simeq \mathrm{Gr}(2,4)$,

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a 4 dimensional variety spanning a 5 plane $\mathbb{P}(\bigwedge^2 \ker(\omega)^*)$ disjoint from $\mathbb{T}_\omega \mathrm{Gr}(2,W^*)$. Consequently we obtain the aforementioned decomposition. Let now $\mathbb{P}(A)$ be a linear space satisfying the hypotheses of the proposition and suppose by contradiction the existence of a point $\omega \in X$ such that $\dim \mathbb{T}_x X = r \geq 2$. Consider now the linear projection π_ω from $\mathbb{T}_\omega \mathrm{Gr}(2,W^*)$ to $\mathbb{P}(\bigwedge^2 \ker(\omega)^*)$:

$$\pi_\omega : \mathbb{P}(\bigwedge^2 W^*) \dashrightarrow \mathbb{P}(\bigwedge^2 \ker(\omega)^*).$$

We analyse the behaviour of $\mathbb{P}(A)$ under this projection. Because of the assumptions on ω, $\dim(\mathbb{T}_\omega X) \geq 2$ so that $\pi(\mathbb{P}(A))$, the image of $\mathbb{P}(A)$ under π_ω is a linear space of dimension $s \leq 1$. (Here we use the convention $\dim(\pi(\mathbb{P}(A))) = -1$ if ever $\pi(\mathbb{P}(A)) = \varnothing$). Since moreover we are supposing that $\mathbb{P}(A) \subset \mathrm{Pf}$, we must have $\mathbb{P}(A) \subset Q_\omega$, condition that is fulfilled if and only if $\pi_\omega(\mathbb{P}(A)) \subset \mathrm{Gr}(2,\ker(\omega)^*)$. This last inclusion implies that $\mathbb{P}(A)^\perp \cap \mathrm{Gr}(2,\ker(\omega))$ is a quadric hypersurface of rank $4-s$ of the linear space $\mathbb{P}(A)^\perp \cap \mathbb{P}(\bigwedge^2 \ker(\omega)) = (\pi_\omega(\mathbb{P}(A)))^\perp \cap \mathbb{P}(\bigwedge^2 \ker(\omega)) \simeq \mathbb{P}^{4-s}$. As such a linear section of the Grassmannian $\mathrm{Gr}(2,\ker(\omega)) \simeq \mathrm{Gr}(2,4)$ always contains a plane isomorphic to $\mathrm{Gr}(2,3)$, there exists $U < \ker(\omega)$ a linear space of dimension 3 such that $\mathbb{P}(\bigwedge^2 U) \subset \mathbb{P}(A)^\perp$. We can therefore conclude that $\mathbb{P}(A)$ can not be stable. □

Destabilizing planes of constant rank 4

The previous proposition ensures that whenever $\mathbb{P}(A)$ is stable, we can always find a plane $\mathbb{P}(B) \subset \mathbb{P}(A)$, such that $rk(\omega) = 4,\ \forall\, \omega \in \mathbb{P}(B)$. 2-dimensional linear spaces $\mathbb{P}(B)$ of $\mathbb{P}(\bigwedge^2 W^*)$ having constant rank 4 have been classified, up to the action of $PGL(W)$, in [MM].
The authors proved that such a $\mathbb{P}(B)$ belongs to the $PGL(W)$-orbit of one of the following planes

$$\begin{aligned} \pi_g &= \langle e_1 \wedge e_4 + e_2 \wedge e_5, e_1 \wedge e_6 + e_3 \wedge e_5, e_2 \wedge e_6 - e_3 \wedge e_4 \rangle \\ \pi_t &= \langle e_1 \wedge e_3 + e_2 \wedge e_4, e_1 \wedge e_4 + e_2 \wedge e_5, e_1 \wedge e_5 + e_2 \wedge e_6 \rangle \\ \pi_p &= \langle e_1 \wedge e_4 + e_2 \wedge e_3, e_1 \wedge e_5 + e_3 \wedge e_4, e_1 \wedge e_6 + e_2 \wedge e_4 \rangle \\ \pi_5 &= \langle e_1 \wedge e_4 + e_2 \wedge e_3, e_1 \wedge e_5 + e_2 \wedge e_4, e_2 \wedge e_5 + e_3 \wedge e_4 \rangle \end{aligned} \tag{3.1}$$

where $e_1, \ldots, e_6$ denotes a basis of W^*.
Proposition 3.2.7. *Let $\mathbb{P}(A) \subset \mathrm{Pf}$ be a 4-dimensional linear space of skew-symmetric forms of generic rank 4 and $\mathbb{P}(B) \subset \mathbb{P}(A)$ be a plane such that $rk(\omega) = 4,\ \forall\, \omega \in B$. If $\mathbb{P}(B)$ is $PGL(W)$-equivalent either to π_t, π_p, π_5, then $\mathbb{P}(A)$ can't be stable.*

Proof. We consider $\gamma : \mathbb{P}(\bigwedge^2 W^*) \dashrightarrow \mathbb{P}(\bigwedge^2 W)$, the Gauss map of the Pfaffian hypersurface and we look at $\gamma|_{\mathbb{P}(B)}$. Since $\mathbb{P}(B)$ has constant rank 4, $\mathbb{P}(B) \cap \mathrm{Gr}(2,W^*) = \varnothing$, so that on $\mathbb{P}(B)$, the map γ is everywhere defined (as $\mathrm{Gr}(2,W^*)$ is the indeterminacy locus of γ). The condition $\mathbb{P}(A) \subset \mathrm{Pf}$ implies that $\forall\, \omega \in \mathbb{P}(A) \setminus \mathrm{Gr}(2,W^*)$, $\mathbb{P}(A) \subset \mathbb{T}_\omega \mathrm{Pf}$, hence $\gamma(\mathbb{P}(B)) \subset \mathbb{P}(A)^\perp$ and consequently $\langle \gamma(\mathbb{P}(B)) \rangle \subset \mathbb{P}(A)^\perp$. By means of the classification (3.1), we can compute directly $\gamma(\mathbb{P}(B))$ (a detailed description of these images can be found in [MM]) and see that whenever $\mathbb{P}(B)$ is $PGL(W)$-equivalent to π_p, π_t or π_5 we can always find a linear subspace $\langle \gamma(\mathbb{P}(B) \rangle \subset \mathbb{P}(A)^\perp$ that prevents the stability of $\mathbb{P}(A)$.

- $\mathbb{P}(B) \in PGL(W) \cdot \pi_t$. A linear space of this kind is always contained in a tangent space $\mathbb{T}_\omega \mathrm{Gr}(2,W^*)$ to $\mathrm{Gr}(2,W^*)$ in a point $\omega \in \mathrm{Gr}(2,W^*)$ (for the choice of coordinates in (3.1), $\pi_t \subset \mathbb{T}_{(e_1 \wedge e_3)} \mathrm{Gr}(2,W^*)$.) In this case $\gamma|_{\mathbb{P}(B)}$ is defined by the complete linear system $|\mathcal{O}_{\mathbb{P}(B)}(2)|$ and $\gamma(\mathbb{P}(B))$ is a Veronese surface contained in $\mathrm{Gr}(2,\ker(\omega)) \subset \mathbb{P}(\bigwedge^2 \ker(\omega)) \simeq \mathbb{P}^5$. Thus $\mathbb{P}(\bigwedge^2 \ker(\omega)) \subset \mathbb{P}(A)^\perp$ and consequently $\mathbb{P}(A)$ can't even be semistable.

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- $\mathbb{P}(B) \in PGL(W) \cdot \pi_p$. In this case the plane $\mathbb{P}(B)$ contains a pencil of special lines (namely lines of tensors of constant rank 4 entirely contained in $\mathbb{P}(\bigwedge W')$, for a 5-dimensional subspace W' of W^*). For any couple L_1, L_2 of generators of this pencyl, there exists a couple of points $\omega_i \in \mathrm{Gr}(2, W^*)$ such that $L_i \subset \mathbb{T}_{\omega_i}\mathrm{Gr}(2, W^*)$, $i = 1, 2$. Denote by $U_{12} < W$ the 3-dimensional space $\ker(\omega_1) \cap \ker(\omega_2)$ (For $\mathbb{P}(B) = \pi_p$ as in (3.1) we can take for example the lines $L_1 = \overline{e_1 \wedge e_4 + e_2 \wedge e_3, e_1 \wedge e_5 + e_3 \wedge e_4}$, $L_2 = \overline{e_1 \wedge e_4 + e_2 \wedge e_3, e_1 \wedge e_6 + e_2 \wedge e_4}$, so that $\omega_1 = e_1 \wedge e_3$, $\omega_2 = e_1 \wedge e_2$ and $U_{12} = \langle e_1, e_2, e_3 \rangle^\perp$). The map $\gamma|_{\mathbb{P}(B)}$ is still a Veronese embedding and the span of Veronese surface $\gamma(\mathbb{P}(B))$ contains the plane $\mathbb{P}(\bigwedge^2 U_{12}) \simeq \mathrm{Gr}(2, 3)$.
- $\mathbb{P}(B) \in PGL(W) \cdot \pi_5$. In this case there exists a vector $v \in W$ such that we have an inclusion $\mathbb{P}(B) \subset \mathbb{P}(\bigwedge^2 (v)^\perp)$; moreover this time $\langle \gamma(\mathbb{P}(B)) \rangle$ is equal to $\mathbb{P}(v \wedge W) \simeq \mathbb{P}^4$, the Schubert variety of 2-dimensional subspaces of W containing v (for the choice of coordinates in (3.1), v will be defined by the intersection of the hyperplanes $e_1, \ldots, e_5$). In this situation the entire space $\mathbb{P}(A)$ must thus be contained in $\mathbb{P}(\bigwedge^2 v^\perp)$; therefore it can't even be semistable.

□

From what we have just proved, we see that a stable $\mathbb{P}(A) \subset \mathrm{Pf}$ can only contain 2-planes of constant rank 4 that are $PGL(W)$-equivalent to π_g. Planes belonging to the orbit $PGL(W) \cdot \pi_g$ have a convenient characterization that we will use in the proof of theorem 3.2.4. Indeed, following [MM], we have that given any $\mathbb{P}(B) \in PGL(W) \cdot \pi_g$, it is possible to find a couple C, D, of 3-dimensional disjoint subspaces of W^*, and a (unique) linear isomorphism $u : C \to D$ such that every tensor $\omega \in \mathbb{P}(B)$ can be written in the form:

$$\omega = x \wedge u(y) - y \wedge u(x), \quad \text{for } x, y \in C.$$

This induces an isomorphism:

$$\begin{aligned} \mathrm{Gr}(2, C) &\longrightarrow \mathbb{P}(B) \\ \langle x, y \rangle &\mapsto x \wedge u(y) - y \wedge u(x). \end{aligned}$$

(Choosing coordinates so that $\mathbb{P}(B)$ is written as in 3.1, we have $C = \langle e_1, e_2, e_3 \rangle$, $D = \langle e_4, e_5, e_6 \rangle$, $u(e_1) = e_5$, $u(e_2) = -e_4$, $u(e_3) = -e_6$.) The restriction of the Gauss map to $\mathbb{P}(B)$:

$$\gamma|_{\mathbb{P}(B)} : \mathbb{P}(B) \longrightarrow \mathbb{P}(\bigwedge^2 W)$$

is defined by the complete linear system $|\mathcal{O}_{\mathbb{P}(B)}(2)|$. The Veronese surface $S_B = \gamma(\mathbb{P}(B))$ is the the variety of lines of the form $\overline{vu^T(v)}$ where:

$$u^T : D^* \to C^*, \;\; D^* \simeq C^\perp, \; C^* \simeq D^\perp$$

is the transpose of u.

We now determine which linear subspaces of W destabilize the plane $\mathbb{P}(B)$. Throughout the rest of the section we fix a basis $e_1, \ldots, e_6$ of W^* in such a way that $\mathbb{P}(B)$ is written as $\mathbb{P}(B) = \pi_g = \langle \omega_0, \omega_1, \omega_2 \rangle$ with the generators ω_i, $i = 0, 1, 2$ of the form appearing in (3.1). To start with, we prove the following:

Proposition 3.2.8. *There is no couple u, U with $u \in W$, $U < W$, $\dim(U) = 5$ such that $\mathbb{P}(u \wedge U) \subset \pi_g^\perp$.*

Proof. Denote by $v_1, \ldots, v_6$ a basis of W dual to $e_1, \ldots, e_6$. As we have already remarked in section 3.2.1, the existence of a couple u, U as in the statement of the proposition is equivalent to the existence of a vector $u \in W$ such that $\bigcap_{\omega \in \pi_g} \ker(\omega(u, \cdot))$ has dimension at least 5. We prove that this can never occur. Indeed, take an arbitrary vector

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$u \in W,\ u = \sum_{i=1}^{6} \alpha_i v_i$. In the basis $e_1, \dots, e_6$, the linear forms in $\omega_0(u, \cdot),\ \omega_1(u, \cdot)$ can be written as:

$$\omega_0(u, \cdot) = \alpha_1 e_4 + \alpha_2 e_5 - \alpha_4 e_1 - \alpha_5 e_2, \quad \omega_1(u, \cdot) = \alpha_1 e_6 + \alpha_3 e_5 - \alpha_6 e_1 - \alpha_5 e_3$$

$$\omega_2(u, \cdot) = \alpha_2 e_6 + \alpha_4 e_3 - \alpha_6 e_2 - \alpha_3 e_4$$

$\bigcap_{i=1}^{3} \ker(\omega_i(u, \cdot))$ has dimension greater than or equal to five if and only if the linear subspace of W^* generated by $\omega_0(u, \cdot), \omega_1(u, \cdot), \omega_2(u, \cdot)$ has dimension at most one or equivalently, if and only if the matrix

$$\begin{pmatrix} -\alpha_4 & -\alpha_5 & 0 & \alpha_1 & \alpha_2 & 0 \\ -\alpha_6 & 0 & -\alpha_5 & 0 & \alpha_3 & \alpha_1 \\ 0 & -\alpha_6 & +\alpha_4 & -\alpha_3 & 0 & \alpha_2 \end{pmatrix}$$

has rank 1. But we can compute directly that this happens if and only if all the α_is vanish.

□

Because of remark (22), a 4-dimensional hyperweb $\mathbb{P}(A) \subset \mathrm{Pf}$ containing a plane π_g is thus not stable if and only if there exists a 3-dimensional space U isotropic with respect to any tensor $\omega \in \mathbb{P}(A)$ and consequently with respect to any tensor in π_g. Every 3-dimensional subspace of W isotropic with respect to any form in π_g can be characterized as follows:

Lemma 3.2.9. *Fix a basis $v_1, \dots, v_6$ on W dual to $e_1, \dots, e_6$. Every 3-dimensional subspace U of W such that $\pi_g \subset \mathbb{P}(U)^\perp$ might be written as:*

$$U = \langle \alpha_3 v_3 + \alpha_6 v_6, \alpha_2 v_2 + \alpha_4 v_4, \alpha_1 v_1 + \alpha_5 v_5 \rangle$$

with $([\alpha_3 : \alpha_6], [\alpha_2 : \alpha_4], [\alpha_1 : \alpha_5]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, satisfying:

$$\alpha_2\alpha_5 - \alpha_4\alpha_1 = 0, \tag{3.2}$$

$$\alpha_3\alpha_5 - \alpha_6\alpha_1 = 0, \tag{3.3}$$

$$\alpha_6\alpha_2 + \alpha_3\alpha_4 = 0. \tag{3.4}$$

Proof. Denote by $\omega_0, \omega_1, \omega_2$ the 3 generators of π_g appearing in (3.1) and let $U < W$ be a 3-dimensional linear space isotropic to $\omega_i,\ i = 0, 1, 2$. The first thing that we can deduce is that every such linear space U is necessarily spanned by 3 independent vectors in W, u_1, u_2, u_3 with $u_i \in \ker(\omega_i)$. This is due to the fact that given a rank 4 tensor ω, if U is a 3-dimensional linear space isotropic with respect to it, then $U \cap \ker(\omega) \neq 0$. To see this we can argue as follows. Take T, a 4-dimensional linear subspace of W disjoint from $\ker(\omega)$. Write then $W = \ker(\omega) \oplus T$ and denote by p_T the linear projection $p_T : W \to T$. If $\omega \in (\bigwedge^2 U)^\perp$, then $\omega|_T \in (\bigwedge^2 (p_T(U))^\perp \subset \bigwedge^2 T$. If ever $U \cap \ker(\omega) = 0$, we would have $\dim(p_T(U)) = 3$ hence $\mathbb{P}(\bigwedge^2 p_T(U))^\perp \subset \mathrm{Gr}(2, T^*)$, leading to a contradiction as $rk(\omega|_T) = 4$. Therefore $U \cap \ker(\omega_i) \neq 0\ \forall \omega_i,\ i = 1, 2, 3$; as whenever $i \neq j$, $\ker(\omega_i) \cap \ker(\omega_j) = 0$, we conclude that we can find a 3-uple of vectors in W, $u_0,\ u_1,\ u_2$ with $u_i, \in \ker(\omega_i)$ spanning the space U. For our choice of coordinates, we have $\ker(\omega_0) = \langle v_3, v_6 \rangle$, $\ker(\omega_1) = \langle v_2, v_4 \rangle$, $\ker(\omega_2) = \langle v_1, v_5 \rangle$, hence 3-dimensional spaces isotropic with respect to any tensor in $\mathbb{P}(B)$ belongs to the family of linear spaces parametrized by $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of the form:

$$U = \langle \alpha_3 v_3 + \alpha_6 v_6, \alpha_2 v_2 + \alpha_4 v_4, \alpha_1 v_1 + \alpha_5 v_5 \rangle$$

with $([\alpha_3 : \alpha_6], [\alpha_2 : \alpha_4], [\alpha_1 : \alpha_5]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Imposing the conditions $\omega_i|_U = 0$, we get the 3 equations (2),(3), (4). To see this embed $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^{11}$ by means of the linear systems of divisors of type $(1, 1, 0),\ (1, 0, 1),\ (0, 1, 1)$. This map is just the morphism

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$$\mathbb{P}(\ker(\omega_0)) \times \mathbb{P}(\ker(\omega_1)) \times \mathbb{P}(\ker(\omega_2)) \longrightarrow \mathbb{P}(\bigwedge^2 W)$$
$$(u_1, u_2, u_3) \mapsto u_1 \wedge u_2 + u_1 \wedge u_3 + u_2 \wedge u_3$$

The conditions $\omega_i \in \mathbb{P}(\bigwedge^2 U)^{\perp}, i = 0, 1, 2$ defines 3 hyperplane sections of $\mathbb{P}^{11}$ that restricted to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ gives the equations (quadratic in the α_is) appearing in the statement. □

Proof of theorem 3.2.4

Proof of theorem 3.2.4. Let $\mathbb{P}(A)$ be a 4-dimensional linear space of skew-symmetric forms satisfying the hypotheses of the theorem. By proposition 3.2.6 $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$ must have dimension one, hence a general 2-dimensional subspace $\mathbb{P}(B)$ of $\mathbb{P}(A)$, is a plane having constant rank 4. Consequently $\mathbb{P}(A)$ might be written as

$$\mathbb{P}(A) = \langle \mathbb{P}(B), \omega_3, \omega_4 \rangle$$

for a couple of points ω_3, ω_4 belonging to $\mathrm{Gr}(2, W^*)$ and $\mathbb{P}(B)$ $PGL(W)$-equivalent to π_g. This last assertion is due to 3.2.7, as we are assuming the stability of $\mathbb{P}(A)$. We need then to detect when 2 points $\omega_3,\ \omega_4$ in the Grassmannian $\mathrm{Gr}(2, W^*)$ are such that $\mathbb{P}(A) := \langle \mathbb{P}(B), \omega_3, \omega_4 \rangle$ is a stable subspace of Pf. We start by studying when for such a couple the space $\mathbb{P}(A)$ is effectively contained in Pf. More generally, since Pf is a cubic hypersurface, we have that for an arbitrary $\omega \in \mathrm{Pf}$, the locus of points $\omega' \in \mathbb{P}(\bigwedge^2 W^*)$ such that $\overline{\omega'\omega} \subset \mathrm{Pf}$ is determined by the intersection:

$$\mathbb{T}_\omega \mathrm{Pf} \cap Q_\omega \cap \mathrm{Pf},$$

where Q_ω is a quadric cone in $\mathbb{P}(\bigwedge^2 W^*)$ with vertex ω. Therefore, given any linear space $\Delta \subset \mathrm{Pf}$, in order to have an inclusion $\langle \Delta, \omega \rangle \subset \mathrm{Pf}$, it is necessary to have $\Delta \subset \mathbb{T}_\omega \mathrm{Pf}$ and consequently $\langle \Delta, \omega \rangle \subset \mathbb{T}_\omega \mathrm{Pf}$. Hence a necessary condition to have $\mathbb{P}(A) \subset \mathrm{Pf}$ is that $\forall\, \omega \in \mathbb{P}(B)$, $\mathbb{P}(A) \subset \mathbb{T}_\omega \mathrm{Pf}$. What we have just explained can be rephrased by means of the morphism:

$$\gamma|_{\mathbb{P}(B)} : \mathbb{P}(B) \to \mathbb{P}(\bigwedge^2 W),$$

restriction of the Gauss map to $\mathbb{P}(B)$, saying that the inclusion $\mathbb{P}(A) \subset \mathrm{Pf}$ implies that $\forall\, \omega \in \mathbb{P}(B)$, $\mathbb{P}(A)$ is contained in the hyperplane corresponding to $\gamma(\omega)$. As we have already seen, $\gamma|_{\mathbb{P}(B)}$ is a degree 2 Veronese embedding; denote by S_B the Veronese surface $\gamma(\mathbb{P}(B))$, by $T \subset \mathbb{P}(\bigwedge^2 W)$, $T \simeq \mathbb{P}^5$ its linear span and by Λ, $\Lambda \subset \mathbb{P}(\bigwedge^2 W^*)$ the 8-plane orthogonal to T. From the discussion presented above, the hypothesis $\mathbb{P}(A) \subset \mathrm{Pf}$ implies that $\mathbb{P}(A) \subset H,\ \forall\, H \in S_B$ leading to $\mathbb{P}(A) \subset \Lambda$. This last inclusion occurs if and only if $\omega_3,\ \omega_4$ are rank 2 tensors both belonging to Λ. We determine then $\mathrm{Gr}(2, W^*) \cap \Lambda$. Recall that since $\mathbb{P}(B)$ belongs to $PGL(W) \cdot \pi_g$, we can find disjoint subspaces $C,\ D$, of W^* of dimension 3 and an isomorhism $u : C \xrightarrow{\sim} D$ (uniquely determined), such that every $\omega \in \mathbb{P}(B)$ has the form:

$$\omega = x \wedge u(y) - y \wedge u(x), \quad \exists\, x, y \in C.$$

This defines an isomorphim ρ:

$$\rho : \mathbb{P}(B) \xrightarrow{\sim} \mathrm{Gr}(2, C)$$
$$x \wedge u(y) - y \wedge u(x) \mapsto x \wedge y$$

Let now $E \simeq \mathbb{P}^1$ be a line and consider the morphism:

$$\psi' : \mathrm{Gr}(2, C) \times E \longrightarrow \mathrm{Gr}(2, W^*)$$
$$(x \wedge y, [t_0 : t_1]) \mapsto (t_0(x) + t_1(u(x))) \wedge (t_0(y) + t_1(u(y))),$$

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and consequently the map:

$$\psi : \mathbb{P}(B) \times E \longrightarrow \mathrm{Gr}(2, W^*)$$
$$(\omega, [t_0 : t_1]) \mapsto \psi'((\rho(\omega), [t_0 : t_1]))$$

Denote by Y the variety $Im(\psi) = Im(\psi')$.
We see from the definition of the maps ψ and ψ' that Y is a rational normal scroll $S_{(2,2,2)}$. This a degree 6 variety of dimension 3, non-degenerate in Λ (hence a minimal variety of dimension 3 in $\Lambda \simeq \mathbb{P}^8$). Y has the structure of a conic fibration over $\mathbb{P}(B)$,

$$Y \xrightarrow{\pi} \mathbb{P}(B), \quad \mathfrak{C}_\omega := \pi^{-1}(\omega) = \psi(\{\omega\} \times E).$$

Note that if $\rho(\omega) = x \wedge y$, the conic $\mathfrak{C}_\omega$ is the locus:

$$\mathfrak{C}_\omega = \pi^{-1}(\omega) = t_0^2(x \wedge y) + t_0 t_1(x \wedge u(y) + u(x) \wedge y) + t_1^2(u(x) \wedge u(y)).$$

Y can also be interpreted as a family of planes over E. Consider the three conics $\mathfrak{C}_{\omega_i} \subset \Lambda$, $i = 0, 1, 2$. The $\mathfrak{C}_{\omega_i}$s are 3 conics lying in 3 disjoint planes, moreover given any point $\omega \in \mathfrak{C}_i$, $\mathbb{T}_\omega \mathrm{Gr}(2, W^*) \cap \mathfrak{C}_j$, $i \neq j$ consists of just one point. This means that once we have fixed an isomorphism $\phi_0 : E \to \mathfrak{C}_{\omega_0}$, we uniquely determine isomorphisms $\phi_i : E \to \mathfrak{C}_{\omega_i}$, $i = 1, 2$ sending a point $p \in E$ to $\mathbb{T}_{\phi_0(p)}\mathrm{Gr}(2, W^*) \cap \mathfrak{C}_{\omega_i}$. $Y \simeq S_{(2,2,2)}$ is thus the rational scroll obtained as:

$$Y = \bigcup_{p \in E} \overline{\phi_0(p), \phi_1(p), \phi_2(p)}, \qquad \overline{\phi_0(p), \phi_1(p), \phi_2(p)} = \psi(\mathbb{P}(B) \times \{p\}).$$

As every point in the Veronese surface $S_B = \gamma(\mathbb{P}(B))$ is of the form $v \wedge u^T(v)$, where $u^T : D^* \to C^*$ is the transpose of u, we see that by construction we have $Y \subset \mathrm{Gr}(2, W^*) \cap \Lambda$. In order to verify that Y is effectively equal to $\mathrm{Gr}(2, W^*) \cap \Lambda$, we first observe that, as $Y \simeq S_{(2,2,2)}$, Y is the base locus of a linear system of quadrics on Λ having dimension $h^0(\mathcal{I}_Y(2)) - 1 = 14$.
Consider now $\mathcal{Q} \simeq \bigwedge^4 W$ the linear system of Plücker's quadrics on $\mathbb{P}(\bigwedge^2 W^*)$. The variety $\Lambda \cap \mathrm{Gr}(2, W^*)$ is the base locus of $|\mathcal{Q}|_\Lambda|$. The inclusion $Y \subset \mathrm{Gr}(2, W^*) \cap \Lambda$ implies that $|\mathcal{Q}|_\Lambda| \subset \mathbb{P}(H^0(\mathcal{I}_Y(2)))$. In order to compute the dimension of $|\mathcal{Q}|_\Lambda|$, we start by taking a 9-tuple of points $\omega_i = 0, \ldots, 8$ spanning Λ; choosing linear coordinates $X_0, \ldots, X_8$, points in $\Lambda \cap \mathrm{Gr}(2, W^*)$ are defined by:

$$\sum_{i=0}^{8} X_i \omega_i \in \Lambda \cap \mathrm{Gr}(2, W^*) \iff \wedge^2(\sum_{i=0}^{8} X_i \omega_i) = 0.$$

$\wedge^2(\sum_{i=0}^{8} X_i \omega_i)$ gives an element in $H^0(\mathcal{O}_\Lambda(2)) \otimes \bigwedge^4 W^* \simeq H^0(\mathcal{O}_\Lambda(2)) \otimes \bigwedge^2 W$ that can thus be written as $\sum_{1 \leq i < j \leq 6} Q_{ij}(v_i \wedge v_j)$, with $Q_{ij} \in H^0(\mathcal{O}_\Lambda(2))$ and with $v_1, \ldots, v_6$ vectors forming a basis of W. $|\mathcal{Q}|_\Lambda|$ is the linear system generated by the quadrics Q_{ij}. Writing $\mathbb{P}(B)$ in the form appearing in (3.1), we can write down explicitly generators of Λ and compute that $\dim(|\mathcal{Q}|_\Lambda|) = 14$. Hence $|\mathcal{Q}|_\Lambda| = \mathbb{P}(H^0(\mathcal{I}_Y(2)))$ from which we deduce that $Y = \Lambda \cap \mathrm{Gr}(2, W^*)$.

We show now that if ω_3, ω_4 are points on Y such that the 4-plane $\mathbb{P}(A) = \langle \mathbb{P}(B),\ \omega_3,\ \omega_4 \rangle$ is a stable 4-plane entirely contained in Pf, we might then suppose that ω_3, ω_4 belong either to a conic $\mathfrak{C}_\omega = \pi^{-1}(\omega)$ for a point $\omega \in \mathbb{P}(B)$, or to a line $\overline{\omega_3 \omega_4}$ entirely contained in Y.

From the construction of Y, we have that each of its points belongs to a unique fiber of $Y \xrightarrow{\pi} \mathbb{P}(B)$. Now, if ω_3, ω_4 lie on the same fiber of π, there exists then a point $\omega \in \mathbb{P}(B)$ such that $\omega_i \in \mathfrak{C}_\omega$, $i = 3, 4$.

Suppose now that ω_3, ω_4 don't belong to the same fiber of π. We claim that we might assume that these two points span a line $\overline{\omega_3 \omega_4}$ contained in Y (more precisely $\overline{\omega_3 \omega_4}$ will be the image through ψ of $L \times \{p\}$, for a line $L \subset \mathbb{P}(B)$ and a point $p \in E \simeq \mathbb{P}^1$.) Without

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loss of generality we can assume that $\omega_3 \in \pi^{-1}(\omega_0)$ and $\omega_4 \in \pi^{-1}(\omega_1)$. We study now the conditions imposed on ω_3, ω_4 by the hypothesis $\mathbb{P}(A) \subset \mathrm{Pf}$. By the assumptions on ω_3, ω_4, $\mathbb{P}(A) \subset \mathrm{Pf}$ holds if and only if $\langle \omega_2, \omega_3, \omega_4 \rangle \subset \mathrm{Pf}$. As we have already remarked at the beginning of the proof, the plane $\langle \omega_2, \omega_3, \omega_4 \rangle$ is contained in Pf if and only if $\overline{\omega_3\omega_4} \subset \mathrm{Pf} \cap Q_{\omega_2} \cap \mathbb{T}_{\omega_2}\mathrm{Pf}$, (where Q_{ω_2} is a quadric cone with vertex in ω_2) and this occurs if and only if $\omega_4 \in \mathbb{T}_{\omega_3} Q_{\omega_2}$. The conic $\mathfrak{C}_{\omega_1}$ intersects the hyperplane $\mathbb{T}_{\omega_3} Q_{\omega_2}$ in two (possibly coincident) points (these are thus the only two points ω on $\mathfrak{C}_{\omega_1}$ leading to an inclusion $\langle \mathbb{P}(B), \omega_3, \omega \rangle \subset \mathrm{Pf}$). We look now at the line $\overline{\omega_0\omega_3}$: this is a secant to $\mathrm{Gr}(2, W^*)$ intersecting $\mathrm{Gr}(2, W^*)$ in ω_3 and in another point ω_3' ($\omega_3 = \omega_3'$ whenever $\omega_0 \in \mathbb{T}_{\omega_3} Gr(2, W^*)$). The two points ω_4, ω_4' on $\mathfrak{C}_{\omega_1}$ defined by $\omega_4 := \mathbb{T}_{\omega_3}\mathrm{Gr}(2, W^*) \cap \mathfrak{C}_{\omega_1}$ and $\omega_4' := \mathbb{T}_{\omega_3'}\mathrm{Gr}(2, W^*) \cap \mathfrak{C}_{\omega_1}$ clearly satisfy $\langle \mathbb{P}(B), \omega_3, \omega_4 \rangle \subset \mathrm{Pf}$, and $\langle \mathbb{P}(B), \omega_3, \omega_4' \rangle = \langle \mathbb{P}(B), \omega_3', \omega_4' \rangle \subset \mathrm{Pf}$. Hence up to replacing ω_3 with ω_3', we might suppose that the generators ω_3, ω_4 lie on a line contained in Y. Note that we can moreover suppose that $\omega_0 \notin \mathbb{T}_{\omega_3}\mathrm{Gr}(2, W^*)$, otherwise we would get $\omega_4 = \omega_4'$ hence the space $\mathbb{P}(A)$ would intersect the Grassmannian along the double line $\overline{\omega_3\omega_4}$. This is not possible by proposition 3.2.6, since we are assuming the stability of $\mathbb{P}(A)$.

We choose now independent vectors $e_1, \ldots, e_6$ in W^*, so that $\mathbb{P}(B)$, being $PGL(W)$ equivalent to π_g, might be written in the form appearing in 3.1, namely $\mathbb{P}(B) = \langle \omega_0, \omega_1, \omega_2 \rangle$ with:

$$\omega_0 = e_1 \wedge e_4 + e_2 \wedge e_5 \quad \omega_1 = e_1 \wedge e_6 + e_3 \wedge e_5, \quad \omega_2 = e_2 \wedge e_6 - e_3 \wedge e_4. \tag{3.5}$$

From what we have showed so far if $\mathbb{P}(A) := \langle \mathbb{P}(B), \omega_3, \omega_4 \rangle$ is stable, up to the action of $GL(W)$, the only possibilities that might occur are:

i ω_3, ω_4 belong to the conic $\mathfrak{C}_{\omega_0}$. Up to an appropriate change of coordinates, $\mathbb{P}(A) = \langle \omega_0, \ldots, \omega_4 \rangle$ with:

$$\omega_3 = e_1 \wedge e_2, \quad \omega_4 = e_4 \wedge e_5.$$

ii $\omega_3 \in \mathfrak{C}_{\omega_0}$, $\omega_1 = (\mathfrak{C}_{\omega_1} \cap \mathbb{T}_{\omega_3}\mathrm{Gr}(2, W^*))$ and $\omega_0 \notin \mathbb{T}_{\omega_3}\mathrm{Gr}(2, W^*)$.
For independent vectors $e_1, \ldots, e_6$, $\mathbb{P}(A) = \langle \omega_0, \ldots,, \omega_4 \rangle$ with:

$$\omega_3 = (e_1 - e_5) \wedge (e_2 + e_4) \quad \omega_4 = (e_1 - e_5) \wedge (e_3 + e_6).$$

Proof of stability. We still need to prove that the orbits listed above are indeed stable. We start considering a point $\omega_3 \in Y$, without loss of generalities we assume that $\omega_3 \in \mathfrak{C}_{\omega_0}$. ω_3 is thus of the form

$$\omega_3 = \psi(\{\omega_0\} \times [t_0 : t_1]) = (t_0 e_1 + t_1 e_5) \wedge (t_0 e_4 - t_1 e_2), \quad [t_0 : t_1] \in E.$$

We look for 3-dimensional linear subspaces U of W that are isotropic with respect to ω_3 and to every tensor in $\mathbb{P}(B)$. By proposition 3.2.9, we know that such a linear space U, satisfying the condition $\mathbb{P}(\bigwedge^2 U) \subset \mathbb{P}(B)^\perp$, necessarily admits a representation in the form:

$$U = \langle \alpha_3 v_3 + \alpha_6 v_6, \alpha_2 v_2 + \alpha_4 v_4, \alpha_1 v_1 + \alpha_5 v_5 \rangle,$$

where the coefficients $([\alpha_3 : \alpha_6], [\alpha_2 : \alpha_4], [\alpha_1 : \alpha_5])$ vary in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $v_1, \ldots, v_6$ is a basis of W dual to $e_1, \ldots, e_6$. Embedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^{11}$ by means of the linear systems of divisors of type $(1,1,0), (1,0,1), (0,1,1)$ (as described in the proof of proposition 3.2.9)), we see that each condition $\omega_i|_U$ defines a hyperplane section $H_i \cap \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $H_i \subset \mathbb{P}^{11}$. Spaces U isotropic to each ω_i, $i = 0, \ldots, 3$ correspond to points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \bigcap_{i=0}^3 H_i$, namely to solutions of the quadratic equations:

$$\begin{aligned}
\alpha_2\alpha_5 - \alpha_4\alpha_1 &= 0, \\
\alpha_3\alpha_5 - \alpha_6\alpha_1 &= 0, \\
\alpha_6\alpha_2 + \alpha_3\alpha_4 &= 0 \\
t_0^2\alpha_1\alpha_2 + t_0t_1(\alpha_2\alpha_5 - \alpha_1\alpha_4) + t_1^2(\alpha_4\alpha_5) &= 0.
\end{aligned}$$

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This system admits solutions if and only if $t_0 = 0$ or $t_1 = 0$. In these cases ω_3 is one of the two points in $\mathfrak{C}_{\omega_0}$ whose tangent lines $\mathbb{T}_{\omega_3}\mathfrak{C}_{\omega_0}$ passes through ω_0 or equivalently, ω_3 is one of the two points in $\mathfrak{C}_{\omega_0}$ such that $\omega_0 \in \mathbb{T}_{\omega_3}\mathrm{Gr}(2, W^*)$. Now if ever $\mathbb{P}(A)$ realizes (i) there always exists a tensor $\omega \in \mathfrak{C}_{\omega_0}$ such that $\omega_0 \notin \mathbb{T}_{\omega}\mathrm{Gr}(2, W^*)$, so there are no 3 dimensional spaces $U < W$ isotropic with respect to every point in $\langle \mathbb{P}(B), \omega \rangle$, implying the stability of $\mathbb{P}(A)$. If $\mathbb{P}(A)$ realizes (ii) instead, by the assumptions on ω_3 and ω_4, $\omega_0 \notin \mathbb{T}_{\omega_3}\mathrm{Gr}(2, W^*)$ (otherwise we would get a double line contained in $\mathrm{Gr}(2, W^*) \cap \mathbb{P}(A)$) hence there are no 3 dimensional spaces U isotropic to every tensor in $\langle \mathbb{P}(B), \omega_3 \rangle$ implying, once again the stability of $\mathbb{P}(A)$. □

3.3 Classification of the irreducible components of $\mathbb{P}(A) \cap \mathbf{Gr}(2, W^*)$

Let $\mathbb{P}(A)$ be a 4 dimensional linear system of skew-symmetric forms having generic rank less than or equal to four. Denote by X the intersection $X := \mathbb{P}(A) \cap Gr(2, W^*)$. The starting point of our classification of stable hyperwebs was proposition 3.2.6, where we proved that if ever $\dim(X) \geq 2$, the linear span of X is then a subspace of $\mathbb{P}(A)$ that prevents the entire $\mathbb{P}(A)$ from being stable. Wondering if a similar strategy might works also to determine strictly semistable hyperwebs, we study then the varieties obtained as linear sections $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$ of the Grassmannian. We prove the following:

Theorem 3.3.1. *Let $\mathbb{P}(A) \subset \mathbb{P}(\bigwedge^2 W^*)$ be a four-dimensional linear space of skew-symmetric forms of generic rank ≤ 4. Let Y be an irreducible component of $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$. Then one of the following cases is realized:*

- *Y is a linear space $Y \simeq \mathbb{P}^r, 1 \leq r \leq 4$.*
- *Y is a variety of minimal degree,contained in a smaller Grassmannian $\mathrm{Gr}(2, k) = \mathrm{Gr}(2, U) \subset \mathrm{Gr}(2, 6) = \mathrm{Gr}(2, W^*)$, where U is a vector subspace of W of dimension $k < 6$, and Y is a linear section of $\mathrm{Gr}(2, k)$ of one of the following types:*
  - *$Y = \mathbb{P}^d \cap \mathrm{Gr}(2, d+2)$, a rational normal curve of degree d, $2 \leq d \leq 4$.*
  - *$Y = \mathbb{P}^{d+1} \cap \mathrm{Gr}(2, d+2)$, a surface of degree $d = 2, 3$.*
  - *$Y = \mathbb{P}^4 \cap \mathrm{Gr}(2, 4)$, a three-dimensional quadric hypersurface in $\Delta = \mathbb{P}^4$.*
- *Y is an elliptic quintic curve, the image of $\mathbb{P}^4 \cap \mathrm{Gr}(2, 5)$ under some linear embedding $\mathrm{Gr}(2, 5) \hookrightarrow \mathrm{Gr}(2, W^*)$.*

The proof of theorem reduces to the study of linear sections $\mathbb{P}^r \cap \mathrm{Gr}(2, W^*)$ for $r \leq 4$. Consider indeed $\mathbb{P}(A)$, $X = \mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$ as above and $\langle X \rangle$ the linear span of X, that is the smallest linear space containing X. $\langle X \rangle$ is a hyperweb of dimension at most 4, still contained in Pf; it might also be characterized as the smallest linear subspace of $\mathbb{P}(A)$ satisfying $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*) = \langle X \rangle \cap \mathrm{Gr}(2, W^*)$. Now, given Y an irreducible component of X we look, just as we did for X, at the linear span $\langle Y \rangle$ of Y. Again, we have that $\langle Y \rangle$ is still a linear space of dimension less than or equal to 4 contained in Pf; moreover Y is an irreducible and non-degenerate component of $\langle Y \rangle \cap \mathrm{Gr}(2, W^*)$. This means that up to replacing $\mathbb{P}(A)$ with $\langle Y \rangle$, we can reduce the study of irreducible components of X to the study of non-degenerate irreducible components of $\Delta \cap \mathrm{Gr}(2, W^*)$, Δ being an r-dimensional, $r \leq 4$, hyperweb of skew-symmetric forms of generic rank 4.

Remark 23. From now we will always denote by Δ an hyperweb of generic rank 4 and by r, $r \leq 4$, its dimension. X will denote $\Delta \cap \mathrm{Gr}(2, W^*)$ and Y a non-degenerate and irreducible component of X not consisting of a linear space. (the fact that $\langle Y \rangle = \Delta$, $Y \neq \Delta$ clearly implies that the same holds for X). We also introduce the following notations, that we will adopt throughout the rest of the chapter: given any variety $Z \subset \mathrm{Gr}(2, W^*)$,

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we define $U_Z \subset W^*$ as the smallest vector subspace of W^* such that $Z \subset \mathbb{P}(\bigwedge^2 U_Z)$ and we indicate by u_Z its dimension. For an arbitrary point on a Grassmannian of lines, $\omega \in \mathrm{Gr}(2, n+1)$, l_ω connotes the corresponding line in $\mathbb{P}^n$.

One of the tools that we will adopt for our classification, is the study of the image of X (and of its irreducible components) under linear projection from one of its points. In next section we present some general results about the behavior of this rational maps.

Behavior under projection from a point

Pick a point $\omega_0 \in X$ and $H \simeq \mathbb{P}^{r-1}$, an hyperplane in Δ not passing through ω_0 and consider the linear projection from ω_0 to H, $\pi_0 : X \setminus \{\omega_0\} \dashrightarrow H$. We look at $\overline{X} = \pi_0(X)$, (by an abuse of notation, for an arbitrary variety Y passing through ω_0, we will always denote by $\pi_0(Y)$ the closed variety $\pi_0(Y) := \overline{\pi_0(Y \setminus \{\omega_0\})}$), the image of X through π_0. Recall that a point $\omega \in H$ belongs to $\pi_0(X \setminus \{\omega_0\})$ if and only if there exists $\omega' \in X$, such that $\omega = \overline{\omega_0\omega'} \cap H$. This means that if $\omega \in \overline{X}$, $\overline{\omega_0\omega}$ intersect X in at least two (possibly coincident) point. But since:

$$\overline{\omega_0\omega} \cap X = \overline{\omega_0\omega} \cap (\Delta \cap \mathrm{Gr}(2, W^*)) = \overline{\omega_0\omega} \cap \mathrm{Gr}(2, W^*),$$

whenever $\omega \in \overline{X}$, the line $\overline{\omega_0\omega}$ meet then the Grassmannian in at least two (possibly coincident) points; hence it is either a line contained in $\mathrm{Gr}(2, W^*)$ either a secant to $\mathrm{Gr}(2, W^*)$. As for what was discussed in section 3.1 concerning lines contained in Pf, we see that for a point $\omega \in \overline{X}$, we have the following possibilities:

- If ω has rank 2, then $\omega \in X \cap H$. $\overline{\omega\omega_0} \cap X$ consists of:
  - $\overline{\omega\omega_0}$ if $\omega \in X \cap \mathbb{T}_{\omega_0}X$
  - $\{\omega_0, \omega\}$ otherwise.
- If ω has rank 4, then $\overline{\omega_0\omega}$ must be a secant to $\mathrm{Gr}(2, W^*)$, implying that $\omega_0 \in \mathbb{P}^5_\omega$. In this case $\overline{\omega\omega_0} \cap X$ consists of two points $\{\omega_1, \omega_0\}$ with $\omega_0 = \omega_1$ whenever $\omega \in \mathbb{T}_{\omega_0}X$.

From the above discussion we observe that whenever $H \not\subset \mathbb{T}_{\omega_0}X$, for a general point $\omega \in \overline{X}$, its preimage $\pi_0^{-1}(\omega) = \overline{\omega_0\omega} \cap (X \setminus \{\omega_0\})$ consists exactly of one point.
Take now ω_0 on Y, such that ω_0 doesn't not belong to any other component of X (points fulfilling this requirement vary then in an open subvariety of Y). The assumption that the only irreducible component passing through ω_0 is Y, implies that we have an equality:

$$\overline{Y} \cap \mathrm{Gr}(2, W^*) = Y \cap H.$$

Indeed, if this was not the case, every line $\overline{\omega_0\omega}$ with $\omega \in (\overline{Y} \cap X) \setminus (Y \cap H)$, would intersect the Grassmannian in at least 3 points and consequently be contained in X (but not in Y), contradicting the assumption. Therefore, a general point in $\overline{Y}$ corresponds to a secant to $\mathrm{Gr}(2, W^*)$ meeting $\mathrm{Gr}(2, W^*)$ in two (possibly coincident) points lying on Y. Consequently, we can state the following:

Proposition 3.3.2. *The linear projection from a general point $\omega_0 \in Y$:*

$$\pi_0 : Y \setminus \{\omega_0\} \dashrightarrow \overline{Y},$$

is a birational morphism.

Proof. Take $\omega_0 \in Y$ a point not belonging to any other component of X, condition that implies that $\mathbb{T}_{\omega_0}X = \mathbb{T}_{\omega_0}Y$. In the first place we notice that for such a point, $\mathbb{T}_{\omega_0}Y \simeq \Delta$ if and only if X, and thus Y, is a cone with vertex ω_0 over $X \cap H$. Recall indeed that X is the base locus of $|\mathcal{Q}|_\Delta|$, so that given $Q_0, \ldots Q_s$ a basis for this linear system, we can write X as:

$$X = \bigcap_{i=0}^{s} Q_i.$$

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Therefore:

$$\mathbb{T}_{\omega_0} Y = \mathbb{T}_{\omega_0} X = (\bigcap_{i=0}^{s} \mathbb{T}_{\omega_0} q_i);$$

implying that $\mathbb{T}_{\omega_0} Y \simeq \Delta$ if and only if $\omega_0 \in \mathrm{Sing}\,(Q_i),\ \forall i = 0, \ldots s$; but this happens if and only if X is a cone with vertex ω_0. From this observation, as we are excluding the case where Y is a linear space, for $\omega_0 \in Y$ general, Y is not a cone with vertex ω_0. For such a ω_0, $H \cap \mathbb{T}_{\omega_0} Y$ is thus a linear space of dimension at most $r-2$, $\overline{Y} = \pi_0(Y)$ has dimension equal to the dimension of Y and $\overline{Y} \not\subseteq \mathbb{T}_{\omega_0} Y$. π_0 maps then $Y \setminus \{\omega_0\}$ birationally to the open set $\overline{Y} \setminus (\mathbb{T}_{\omega_0} Y \cap H)$. □

By means of linear projections, we can also describe $|\mathcal{Q} - H|$ and $|\mathcal{Q} - \Delta|$, namely the linear systems of those quadrics in $\mathcal{Q}$ containing H and Δ, respectively. Take a hyperplane $H \subset \Delta$ and again, a point $\omega_0 \in Y \setminus (Y \cap H)$ such that Y is the only irreducible component of Y passing through it.

Proposition 3.3.3. *We have an equality* $|\mathcal{Q} - H| = |\mathcal{Q} - \Delta|$.

Proof. It's clear that $|\mathcal{Q} - \Delta|$ is a subspace of $|\mathcal{Q} - H|$. We pick now a point $\omega_0 \in Y$ not lying in H and we look at π_0, the linear projection from ω_0 to H. The reverse inclusion holds if and only if $\forall\ Q \in |\mathcal{Q} - H|,\ H < \mathbb{T}_{\omega_0} Q$. Choose now r points $\omega_1, \ldots \omega_r$ on $\overline{Y} \setminus (Y \cap H)$ spanning the hyperplane H (notice that this is alway possible since from $\langle Y \rangle = \Delta$, we have that $\langle \overline{Y} \rangle = H$). The condition $\omega_i \in \overline{Y},\ i = 1, \ldots r$, implies that the point ω_0 belongs then to the intersection:

$$\omega_0 \in \bigcap_{i=1}^{r} \mathbb{P}^5_{\omega_i}.$$

Since $\forall\ i = 1, \ldots, r$, ω_0 belongs to $\mathbb{P}^5_{\omega_i}$, for every quadric $Q \in |\mathcal{Q} - \omega_i|$, the line $\overline{\omega_0 \omega_i}$ is contained in Q. This implies that choosing arbitrarily a generator ω_i of H among $\omega_1, \ldots, \omega_r$, we have $\omega_i \in \mathbb{T}_{\omega_0} Q$, $\forall\ Q \in |\mathcal{Q} - \omega_i|$. Hence whenever $i = 1, \ldots, r$, $\omega_i \in \mathbb{T}_{\omega_0} Q$, $\forall\ Q \in |\mathcal{Q} - H|$ and as H is spanned by the ω_is, we get $H < \mathbb{T}_{\omega_0} Q, \forall\ Q \in |\mathcal{Q} - H|$. □

Corollary 3.3.4. *Let $H \subset \Delta$ be an hyperplane and let $Z \subset H$ be the intersection $Y \cap H$. We have equality $U_Z = U_Y$.*

Proof. We clearly have an inclusion $U_Z \subset U_Y$. Arguing as above, we can now choose r points in $\omega_1, \ldots \omega_r$ on $\overline{Y} \setminus Z$ in general position; for each of these points we have $\omega_0 \in \mathbb{P}^5_{\omega_i}$. As $\mathbb{P}^5_{\omega_i} \subset \mathbb{P}(\bigwedge^2 U_Z)$ and $\omega_0 \in \mathbb{P}(\bigwedge^2 U_Z)$ by 3.3.3, we can conclude that $\Delta \subset \mathbb{P}(\bigwedge^2 U_Z)$. □

Keeping the notations adopted in the previous propositions, we now look at $\overline{X} = \pi_0(X)$. This is a variety defined by an intersection of quadric hypersurfaces in H. To see this, we first look at the line l_{ω_0}. Applying 3.3.4 we see that this line is contained in $\mathbb{P}(U_Z)$. Denote by $\Omega_{l_{\omega_0}}$ the Schubert variety of 3-planes in $\mathbb{P}(U_Z)$ containing the line l_{ω_0}. $\Omega_{l_{\omega_0}}$ is a subvariety of $\mathrm{Gr}(4, U_Z)$ and we see that every point $[\Lambda] \in \Omega_{l_{\omega_0}}$ corresponds to a 3 plane $\Lambda \subset \mathbb{P}(U_Z)$ that can be written as:

$$\Lambda = \langle l_{\omega_0},\ l_{\omega} \rangle,$$

for a line l_ω contained in a $(u_Z - 3)$-plane orthogonal to l_{ω_0}. From this we deduce that we have an isomorphism:

$$\Omega_{l_{\omega_0}} \xrightarrow{\sim} \mathrm{Gr}(2, u_Z - 2)$$

and hence that $\Omega_{l_{\omega_0}}$ has dimension $2(u_Z - 2)$. Consider now the rational map:

$$\gamma|_H : H \dashrightarrow \mathbb{P}(\bigwedge^4 W^*)$$

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restriction of the Gauss map γ to H. This is the map defined by $|\mathcal{Q}|_H|$; its image $\gamma|_H(H)$ spans a linear space T of dimension equal to the dimension of $|\mathcal{Q}|_H|$ and moreover, by the assumptions on X, Y and ω_0, we get that this space T is contained in $\mathbb{P}(\bigwedge^4 U_Z)$. An element $[\Lambda] \in T \cap \Omega_{l_{\omega_0}} = T \cap \langle \Omega_{l_{\omega_0}} \rangle \cap \mathrm{Gr}(4, W^*) = \gamma|_H(H) \cap \Omega_{l_{\omega_0}}$ corresponds thus to a 3-plane Λ containing l_{ω_0} and that satisfies the additional condition:

$$\Lambda = \langle l_\omega,\ l_{\omega'} \rangle, \qquad \exists\ \omega,\ \omega' \in Z.$$

From the discussion held at the beginning of section 3.2.2 we then deduce that pulling back by $\gamma|_H$ the linear equation (on T) individuating $T \cap \langle \Omega_{l_{\omega_0}} \rangle$, we get quadrics on H whose intersection is $\overline{X}$.

Next sections are devoted to the proof of 3.3.1, one of the main result of the chapter, we describe here briefly the idea. Suppose that Y is an n-dimensional irreducible variety of degree d, spanning an r-dimensional linear space $\Delta \simeq \mathbb{P}^r$ (therefore we have inequality $d \geq 1 + (r - n)$). Since Y is clearly non-degenerate in Δ, Y meets every hyperplane $H \subset \Delta$ properly (i.e every hyperplane section $H \cap Y$ has dimension $n - 1$), hence we have that:

$$deg(Y) = deg(Y)deg(H) = \sum_{Z \in Y \cap H} m_Z(Y, H)deg(Z), \tag{3.6}$$

where the sum is taken over the irreducible components Z of $Y \cap H$ and $m_Z(Y, H)$ denotes the intersection multiplicity of Y and H along Z. By the non-degeneracy of Y, we get the non degeneracy of $Y \cap H$, H being a general hyperplane. If moreover we assume that $n \geq 2$, a general hyperplane section $Z := Y \cap H$ is irreducible too. Such a hyperplane $H \simeq \mathbb{P}^{r-1}$ is thus an hyperweb (still having generic rank 4), spanned by $Z = Y \cap H$, an $n - 1$ dimensional irreducible subvariety of $\mathrm{Gr}(2, W^*)$ for which we have equality:

$$deg(Y) = m_Z(Y, H)deg(Z).$$

Y is an irreducible component of $\Delta \cap \mathrm{Gr}(2, W^*)$, thus $m_Z(Y, H) \leq m_Z(\mathrm{Gr}(2, W^*), H)$; this means that whenever $\mathrm{Gr}(2, W^*)$ meets the linear space H along Z with multiplicity one, we have exactly $deg(Y) = deg(Z)$. Our classification starts from the case where Y is an irreducible curve; we will then apply the aforementioned considerations to study, by the aid of formula 3.6, the higher dimensional cases.

3.3.1 One dimensional components

Throughout the section we will be supposing that Y is an irreducible, non degenerate 1-dimensional component of $\Delta \cap \mathrm{Gr}(2, W^*) \simeq \mathbb{P}^r \cap \mathrm{Gr}(2, 6),\ r \leq 4$.
We will prove the following:

Theorem 3.3.5. *Let $\Delta \simeq \mathbb{P}^r$, $r \leq 4$ be an r-dimensional hyperweb of generic rank 4 intersecting $\mathrm{Gr}(2, W^*)$ along a closed variety $X := \Delta \cap \mathrm{Gr}(2, W^*)$ containing a 1-dimensional, non-degenerate irreducible component Y. Then Δ is either:*

- *The r dimensional linear span of Y, a rational normal curve of degree r isomorphic to $\mathbb{P}^r \cap \mathrm{Gr}(2, r + 2)$, $1 \leq r \leq 4$. Additionally, if this is the case, we have $Y = \Delta \cap \mathrm{Gr}(2, W^*)$.*
- *The 4-dimensional linear span of Y, an elliptic quintic curve isomorphic to $\mathbb{P}^4 \cap \mathrm{Gr}(2, 5)$. Additionally, if this is the case we have $Y = \Delta \cap \mathrm{Gr}(2, W^*)$.*

Remark 24. Since by the discussion presented in section 3.1.1, we see immediately that the theorem is verified for $r = 1$ (namely when Y is a line), from now on we will suppose that Y spans an r-plane of dimension $r \geq 2$.

Before demonstrating theorem 3.3.5 we exhibit some generalities about curves contained in Grassmannians of lines.

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The corresponding ruled surface

We can observe that whenever we have an irreducible curve $Y \subset \mathrm{Gr}(2, U_Y)$ we can define a surface S_Y,

$$S_Y := \bigcup_{\omega \in Y} l_\omega$$

where $l_\omega \subset \mathbb{P}(U_Y)$ denotes the line corresponding to the point ω. S_Y is thus a ruled surface in $\mathbb{P}(U_Y)$ whose generators are the lines l_ω, $\omega \in Y$. Consider now the incidence correspondence $\Sigma \subset \mathrm{Gr}(2, U_Y) \times \mathbb{P}(U_Y)$

$$\Sigma := \{(\omega, p) \in \mathrm{Gr}(2, U_Y) \times \mathbb{P}(U_Y) \mid p \in l_\omega\},$$

endowed with the 2 projections $\pi_1 : \Sigma \to \mathrm{Gr}(2, U_Y)$, $\pi_2 : \Sigma \to \mathbb{P}(U_Y)$. Define π_2' as the restriction $\pi_2' := \pi_2|_{\pi_1^{-1}(Y)}$. Suppose now that Y is a curve as in the statement of 3.3.5.

Proposition 3.3.6. *The map π_2' is generically bijective and thus establishes a birational morphism between S_Y and $\pi_1^{-1}(Y) \simeq Y \times \mathbb{P}^1$.*

Proof. Given $p \in S_Y$, we see that the fiber $\pi_2'^{-1}(p)$ consists of all the lines in $\mathbb{P}(U_Y)$ parametrized by points in Y and passing through p. $\pi_2'^{-1}(p)$ is hence isomorphic to $(Y \cap \Omega_p) \times \{p\}$, being $\Omega_p \subset \mathrm{Gr}(2, U_Y)$, $\Omega_p \simeq \mathbb{P}^{u_Y - 2}$ the Schubert variety parameterizing lines containing p. Denote by Z the scheme $Z := Y \cap \Omega_p$. Z must have dimension zero, otherwise we would have $\Delta \subset \Omega_p \subset \mathrm{Gr}(2, U_Y)$. Note that since $Z \subset \Omega_p$, its linear span $\langle Z \rangle$ is contained in Ω_p as well, therefore $\langle Z \rangle \subset \Delta \cap \mathrm{Gr}(2, U_Y)$. If ever Z consits of more then one point, we would get the existence of a linear subspace $\langle Z \rangle$ of $\Delta \cap \mathrm{Gr}(2, U_Y)$ having dimension greater than 0 and parametrizing lines passing through p. Therefore Z would belong to a component of $\Delta \cap Gr(2, U_Y)$ different from Y. □

We can use the previous proposition to compute the degree of the surface S_Y.

Lemma 3.3.7. *The degree of the surface S_Y is equal to the degree of the curve Y*

Proof. Consider a generic codimension 2 plane Λ in $\mathbb{P}(U_Y)$, defined by the intersection of 2 hyperplanes H_1 and H_2, $H_i \in \mathbb{P}({U_Y}^*)$, $i = 1, 2$. From the generality assumption on Λ, we might suppose that $\forall p \in \Lambda \cap S_Y$, $\pi_2'^{-1}(p)$ consists of just one point. Then points in $S_Y \cap \Lambda$, corresponds bijectively to points in the intersection of Y with the hyperplane $H_1 \wedge H_2 \in \mathrm{Gr}(2, {U_Y}^*)$. □

Proof of theorem 3.3.5

Let Δ be a r-dimensional hyperweb of generic rank 4 such that $X := \Delta \cap \mathrm{Gr}(2, W^*)$ contains a one dimensional non-degenerate irreducible component Y. As the curve Y is irreducible and non-degenerate in a r-plane we have $deg(Y) \geq r$ and equality holds if and only if Y is a rational normal curve of degree r. Studying the behavior of hyperplane sections of Y and applying formula (3.6) we will try to determine all the possible value of $deg(Y)$. By the fact that Y spans the entire space Δ, it is possible to find $r+1$ points on Y, $\omega_0, \ldots, \omega_r$, lying in general position. Such a $r+1$-tuple of points might also be chosen in such a way that no other component of X passes through any of them. Consider now r points among the ω_is, say $\omega_1, \ldots \omega_r$, denote by H the hyperplane that they generate and by Z the intersection $H \cap Y$.
Because of the assumptions we made on the ω_is, we see that H is a $r-1$-plane, spanned by r points $\omega_1, \ldots, \omega_r$, $\omega_i \in \mathrm{Gr}(2, W^*)$ and such that $H \cap \mathrm{Gr}(2, W^*) = H \cap X$ contains no irreducible components of dimension greater then 0 passing through any of the ω_is.

Proposition 3.3.8. *Let $H = \langle \omega_1, \ldots \omega_r \rangle$, $r \leq 4$, $Z = Y \cap H$ as above. Then we have the following possibilities:*

1. *Either $Z = \{\omega_1 \ldots \omega_r\}$ and furthermore Z is isomorphic to $\mathbb{P}^{r-1} \cap \mathrm{Gr}(2, r+2)$.*

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2. *Either $r = 4$ and Z is a zero-dimensional subscheme of length 5 isomorphic to* $\mathbb{P}^3 \cap \mathrm{Gr}(2,5)$.

Proof. We prove the proposition analyzing each possible value of r. For each case we describe the corresponding configuration of lines $l_{\omega_i} \subset \mathbb{P}(W^*)$.

r=2 Given $Z = \{\omega_1, \omega_2\},\ \omega_i \in \mathrm{Gr}(2, W^*)$, if $l_{\omega_1} \cap l_{\omega_2} \neq \varnothing$, $\overline{\omega_1\omega_2}$ would be entirely contained in the Grassmannian,contradicting our assumptions. Thus $l_{\omega_1},\ l_{\omega_2}$ are disjoint and consequently span a 3 dimensional linear space $\mathbb{P}(U_Z) \simeq \mathbb{P}^3$. $\overline{\omega_1\omega_2}$ is then a line in $\mathbb{P}(\bigwedge^2 U_Z)$ that will meet $\mathrm{Gr}(2, U_Z) \simeq \mathrm{Gr}(2,4)$ exactly along Z.

r=3 Consider $Z = \{\omega_1, \omega_2, \omega_3\},\ \omega_i \in \mathrm{Gr}(2, W^*)$. Arguing as in the previous point we must have that the corresponding lines $l_{\omega_i} \subset \mathbb{P}(W^*),\ i = 1,2,3$ are pairwise disjoint. The condition $\langle Z \rangle \subset \mathrm{Pf}$ imposes that $(l_{\omega_i} \cap \langle l_{\omega_j}, l_{\omega_k} \rangle) \neq \varnothing$; this intersections must thus consist of a point. Otherwise if ever $l_{\omega_i} \subset \langle l_{\omega_j}, l_{\omega_k} \rangle \simeq \mathbb{P}^3$ we would have $Z \simeq (\mathbb{P}^2 \cap \mathrm{Gr}(2,4))$ and thus we would have a conic passing through the ω_is.

r=4 Given now $Z = \{\omega_1, \ldots \omega_4\}$, we can apply the same arguments exposed in the previous points to see that we must have $l_{\omega_i} \cap l_{\omega_j} = \varnothing,\ dim(l_{\omega_i} \cap \langle l_{\omega_j}, l\omega_k \rangle) = 0, \forall i,j,k \in \{1,2,3,4\},\ i \neq j \neq k$ (here we are adopting the convention $\dim \varnothing = -1$). These requirements ensure that $dim(U_Z) \geq 5$, so that Y is the irreducible component of a variety X isomorphic either to $\mathbb{P}^4 \cap \mathrm{Gr}(2,5)$ or $\mathbb{P}^4 \cap \mathrm{Gr}(2,6)$.

From these facts, we see that a zero-dimensional scheme Z satisfying the hypotheses of the proposition might be of two "types' that will denote by (i) and (ii), respectively.

(i) If Z satisfies (i), then $length(Z) = r$, $dim(U_Z) = r + 2$ and Z consists of a r-tuple of points on $\mathrm{Gr}(2, U_Z)$ in general position. This circumstance occurs when Z corresponds to a r-tuple of pairwise disjoint lines $l_{\omega_1}, \ldots, l_{\omega_r}$, spanning a $r+1$-plane $\mathbb{P}(V_{r+2}) \simeq \mathbb{P}^{r+1}$ and such that:

$$\forall i,j,k \in \{1, \ldots, r\},\ i \neq j \neq k,\ \ l_{\omega_i} \cap \langle l_{\omega_j}, l_{\omega_k} \rangle = \{p_{ijk}\}$$

for a point $p_{ijk} \in \mathbb{P}(V_{r+2})$. More generally, whenever we have such a configuration of lines in a projective space $\mathbb{P}(V_{r+2}) \simeq \mathbb{P}^{r+1}, r \geq 2$, the corresponding tensors $\omega_1, \ldots, \omega_r$ defines a zero-dimensional subscheme Z of $\mathrm{Gr}(2, V_{r+2})$ having length r and such that $\langle Z \rangle \simeq \mathbb{P}^{r-1}$. Furthermore, still denoting by $\mathcal{Q} \simeq \bigwedge^4 {V_{r+2}}^*$ the vector space spanned by Plucker's quadrics on $\mathbb{P}(\bigwedge^2 V_{r+2})$, we can easily compute (for example by induction on r), that $|\mathcal{Q}|_{\langle Z \rangle}| \simeq \mathbb{P}^{\binom{r}{2}-1}$. But since on $\mathbb{P}^{r-1}$, the space of quadrics passing through r points (in general position) has dimension $\binom{r}{2} - 1$ we have an equality $|\mathcal{Q}|_{\langle Z \rangle}| = \mathbb{P}(H^0(\mathcal{I}_Z(2)))$ and consequently $\langle Z \rangle \cap \mathrm{Gr}(2, V_{r+2}) = Z$.

(ii) In Z satisfies (ii) instead, we have a 3-plane intersecting $\mathrm{Gr}(2, U_Z) \simeq \mathrm{Gr}(2,5)$ along a zero dimensional scheme. As $\mathrm{Gr}(2, U_Z)$ is a 6-dimensional subvariety of $\mathbb{P}(\bigwedge^2 U_Z) \simeq \mathbb{P}^9$ of degree 5, a zero-dimensional linear section isomorphic to $\mathbb{P}^3 \cap \mathrm{Gr}(2,5)$ must have length 5. Moreover, for such a linear section $H \cap \mathrm{Gr}(2, U_Z)$, the restriction $|\mathcal{Q}| \to |\mathcal{Q}|_H|$ is injective (hence bijective). As a result, we have that $|\mathcal{Q}|_H| \simeq \mathbb{P}^4 \simeq \mathbb{P}(H^0(\mathcal{I}_Z(2))$; this isomorphism leads to the conclusion that again, $H \cap \mathrm{Gr}(2, U_Z) \simeq Z$.

□

Suppose now that Z is a zero-dimensional scheme of type (i), namely Z consists of a r-tuple of distinct points $\omega_1, \ldots \omega_r,\ \omega_i\ \in \mathrm{Gr}(2, W^*)$ lying in general position and $Z \simeq (\mathbb{P}^{r-1} \cap \mathrm{Gr}(2, r+2)),\ 2 \leq r \leq 4$. Is then always possible to find $\omega_D \in \mathrm{Gr}(2, U_Z)$ such that $\omega_i \in \mathbb{T}_{\omega_D}\mathrm{Gr}(2, U_Z),\ \forall i = 1, \ldots r$. The result is trivial for $r = 2$ (it's enough to consider ω_D corresponding to a line $l_{\omega_D} = \overline{uv},\ u \in l_{\omega_1}, v \in l_{\omega_2}$). For higher values of r we can prove that more in general we have:

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Proposition 3.3.9. *Consider a r-tuple, $r \geq 3$, of pairwise disjoint lines $l_{\omega_1}, \dots l_{\omega_r}$ spanning a $r+1$-plane $\mathbb{P}(V_{r+2}) \simeq \mathbb{P}^{r+1}$ and such that $\forall i, j, k \in \{1, \dots, r\}$, $i \neq j \neq k$, $l_{\omega_i} \cap \langle l_{\omega_j}, l_{\omega_k} \rangle \neq \varnothing$. Then there always exists a line $l_{\omega_D} \subset \mathbb{P}(V_{r+2})$ meeting each l_{ω_i}, $1 \leq i \leq r$ exactly in one point.*

Proof. We argue by induction on r. If $r = 3$, consider 3 lines in $\mathbb{P}^5$, l_{ω_1}, l_{ω_2}, l_{ω_3} satisfying the hypotheses of the proposition. Denoting by $\mathbb{P}(V_4) \simeq \mathbb{P}^3$ the 3-plane generated by l_{ω_1} and l_{ω_2}, we might then write:

$$l_{\omega_3} = \overline{uv}, \; u \in \mathbb{P}(V_4), \; v \notin \mathbb{P}(V_4).$$

Now, as clearly l_{ω_D} must be contained in $\mathbb{P}(V_4)$, it should correspond to a point $\omega_D \in \mathrm{Gr}(2, V_4)$ belonging to $(\mathbb{T}_{\omega_1}\mathrm{Gr}(2, V_4) \cap \mathbb{T}_{\omega_2}\mathrm{Gr}(2, V_4) \cap \Omega_v)$, $\Omega_v \subset \mathrm{Gr}(2, V_4)$ being the Schubert variety of lines in $\mathbb{P}(V_4)$ passing through v. But as $\Omega_v \simeq \mathbb{P}^2$ this intersection consists of just one point.
If $r > 3$, and we are given lines $l_{\omega_1}, \dots l_{\omega_r}$ as in the statement of the proposition, denoting by $\mathbb{P}(V_{r+1})$ the r plane spanned by $l_{\omega_1}, \dots, l_{\omega_{r-1}}$, we see that again we can write:

$$l_{\omega_r} = \overline{uv}, \quad u \in \mathbb{P}(V_{r+1}), \; v \notin \mathbb{P}(V_{r+1}).$$

By inductive hypothesis there exists l_{ω_D} meeting l_{ω_i} in one point $\forall i \; 1 \leq i \leq r-1$. Call $\mathbb{P}(V_{ij}) \simeq \mathbb{P}^3$ the 3-planes spanned by $l_{\omega_i}, l_{\omega_j}$, $1 \leq i, j \leq r-1$.These 3-planes correspond to $\binom{r-1}{2}$ points in $\mathrm{Gr}(4, V_{r+1})$ belonging to $\Omega_{l_{\omega_D}} \simeq \mathrm{Gr}(2, r-1)$, the variety parameterizing 3-planes in $\mathbb{P}(V_{r+1})$ containing l_{ω_D}; note that moreover they intersect exactly along l_{ω_D}. As l_{ω_r} must meet each $\mathbb{P}(V_{ij}), \forall i, j \; 1 \leq i, j \leq r-1$ we deduce that the point u belongs to l_{ω_D}.

□

Corollary 3.3.10. *There always exists $\omega_D \in \mathrm{Gr}(2, W^*)$ such that $Z \subset \mathbb{T}_{\omega_D}\mathrm{Gr}(2, W^*)$.*

Proof of Theorem 3.3.5. . Let Y be a curve satisfying the hypotheses of theorem 3.3.5. Since Y is irreducible and non-degenerate in $\Delta \simeq \mathbb{P}^r$, $deg(Y) \geq r$ and equality holds if and only if Y is a rational normal curve of degree r. Consider a $r+1$ tuple of points $\omega_0, \dots \omega_r$ on Y spanning Δ and, as usual, suppose that no other component of X passes trough any of the ω_is. Pick r among these points, say $\omega_1, \dots \omega_r$ and let H be the hyperplane in Δ spanned by them. We look then at $\pi_0 : \mathbb{P}^d \setminus \{\omega_0\} \dashrightarrow H$, the linear projection from the point ω_0. $\overline{Y} := \pi_0(Y)$ is a non-degenerate curve in H such that:

- $g(Y) = g(\overline{Y})$. This is due to proposition 3.3.2.
- $deg(Y) = deg(\overline{Y}) + 1$, as we are supposing that Y is the only component of X passing through ω_0.

Under the previous assumptions we look at $Z := H \cap Y$. Such a Z must be a scheme of the type described in proposition 3.3.8, leading us to distinguish the two following instances.

- Suppose that Z satisfies (i), hence $Z = \{\omega_1, \dots \omega_r\}$, $Z \simeq \mathbb{P}^{r-1} \cap \mathrm{Gr}(2, r+2)$. In this case the curve Y must be a rational normal curve of degree r satisfying $Y = \Delta \cap \mathrm{Gr}(2, W^*)$, $Y \simeq \mathbb{P}^r \cap \mathrm{Gr}(2, r+2)$. Indeed, from 3.3.8, we have $H \cap \mathrm{Gr}(2, W^*) = Z$ and so $m_{\omega_i}(Y, H) = m_{\omega_i}(\mathrm{Gr}(2, W^*), H) = 1$, $\forall \; i = 1, \dots, r$. Therefore, by formula (3.6), we get $deg(Y) = r$, allowing us to conclude that Y is a rational normal curve of degree r.

  In order to see that Y is actually equal to $\Delta \cap \mathrm{Gr}(2, W^*)$, we look at $|\mathcal{Q}|_\Delta|$. The base locus of this linear system is X, as $X := \Delta \cap \mathrm{Gr}(2, W^*)$. Since Y is contained in X, we have an inclusion:

$$|\mathcal{Q}|_\Delta| \subset \mathbb{P}(H^0(\mathcal{I}_Y(2)) \simeq \mathbb{P}^{\binom{r}{2}-1},$$

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(the last isomoprhism comes from the fact that Y is a rational normal curve of degree r). By proposition 3.3.3, $\dim(|\mathcal{Q}|_H|) = \dim(|\mathcal{Q}|_\Delta|)$ so that, by our hypotheses on Z, $|\mathcal{Q}|_\Delta|$ has dimension $\binom{r}{2} - 1$ too. But then this latter linear system of quadrics is actually equal to $\mathbb{P}(H^0(\mathcal{I}_Y(2))$ hence its base locus is exactly the curve Y. Finally, since $dim(U_Z) = r+2$, applying 3.3.4 we have equality $U_Z = U_Y$ hence $Y \simeq \mathbb{P}^r \cap \mathrm{Gr}(2, r+2)$.

For the moment we have just proved that if ever the intersection of a r-dimensional hyperweb $\Delta \subset \mathrm{Pf}$ with the Grassmannian $\mathrm{Gr}(2, W^*)$ has a non-degenerate component Y of dimension 1, then, unless if Y is a subvariety of $\mathrm{Gr}(2,5)$ spanning a 4-plane, Y is a rational normal curve of degree r contained in $\mathrm{Gr}(2, U_Y) \simeq \mathrm{Gr}(2, r+2)$ and such that $Y = \Delta \cap \mathrm{Gr}(2, U_Y) = \Delta \cap \mathrm{Gr}(2, W^*)$.

We now prove that such a curve Y exists, describing explicitly how to construct it. We start by r distinct points on $\mathrm{Gr}(2, W^*)$ lying in general position, $\omega_1, \dots \omega_r$ corresponding to a configuration of r lines $l_{\omega_1}, \dots l_{\omega_r}$ in $\mathbb{P}(W^*)$ satisfying the hypotheses of proposition 3.3.9. Call Z the zero dimensional scheme $Z := \{\omega_1, \dots, \omega_r\}$, we thus have:

$$U_Z \simeq \mathbb{C}^{r+2}, \quad \langle l_{\omega_1}, \dots l_{\omega_r} \rangle = \mathbb{P}(U_Z).$$

From what we have proved until now, to show the existence of Y, it's enough to exhibit a point $\omega_0 \in \mathrm{Gr}(2, U_Z)$, $\omega_0 \notin H$ such that $\langle \omega_0, H \rangle \cap \mathrm{Gr}(2, U_Z)$ contains a non-degenerate irreducible curve Y passing through the ω_is, $i = 1, \dots, r$ and such that $Y \cap H = Z$.

Applying proposition 3.3.10, we have the existence of a line $l_D \subset \mathbb{P}(U_Z)$ such that $\forall\, i = 1, \dots, r$, $l_{\omega_i} \cap l_{\omega_D}$ consists of exactly one point. Note that the fact that the lines l_{ω_i} and l_{ω_D} intersect at a point is equivalent to having $\omega_i \in \mathbb{T}_{\omega_D}\mathrm{Gr}(2, W^*)$ for all $i = 1, \dots, r$ implying thus an inclusion $Z \subset \mathbb{T}_{\omega_D}\mathrm{Gr}(2, W^*)$.

A general point $\omega_0 \in \mathbb{T}_{\omega_D}\mathrm{Gr}(2, U_Z)$ will then fulfill the above-mentioned requirements. We keep the usual notations, $\Delta := \langle \omega_0, H \rangle$, $X := \Delta \cap \mathrm{Gr}(2, W^*)$ and consider the linear projection

$$\pi_0 : \Delta \dashrightarrow H.$$

We can describe $\overline{X}$, the image of X under this projection and study the equation defining it. We saw that these are quadratic equations that can be described by means of $\gamma|_H : H \dashrightarrow \mathbb{P}(\bigwedge^4 U_Z)$, rational map defined by $|\mathcal{Q}|_H|$. Denote by $T \subset \mathbb{P}(\bigwedge^4 U_Z) \simeq \mathbb{P}(\bigwedge^4 \mathbb{C}^{r+2})$ the linear span of $\gamma|_H(H)$. T is a $\binom{r}{2} - 1$ plane, and it's the linear span of $\Omega_{l_{\omega_D}} \subset \mathrm{Gr}(4, U_Z)$, the Schubert variety parameterizing 3-planes in $\mathbb{P}(U_Z)$ containing the line l_{ω_D}. The variety $\Omega_{l_{\omega_D}}$ satisfies:

$$\Omega_{l_{\omega_D}} \simeq \mathrm{Gr}(r-2, h^0(\mathcal{I}_{l_{\omega_D}}(1))) \simeq \mathrm{Gr}(r-2, r),$$

therefore it has dimension $2(r-2)$. Consider now $\Omega_{\langle l_{\omega_0}, l_{\omega_D} \rangle}$ the subvariety of $\Omega_{l_{\omega_0}}$ parameterizing 3-planes in $\mathbb{P}(U_Z)$ containing the entire plane $\langle l_{\omega_0}, l_{\omega_D} \rangle$. We have:

$$\Omega_{\langle l_{\omega_0}, l_{\omega_D} \rangle} \simeq \mathrm{Gr}(r-2, h^0(\mathcal{I}_{\langle l_{\omega_0}, l_{\omega_D} \rangle}(1))) \simeq \mathbb{P}^{r-2}.$$

Thus $\Omega_{\langle l_{\omega_0}, l_{\omega_D} \rangle}$ is cut by $\binom{r-1}{2}$ hyperplanes in T. Pulling these back by $\gamma|_H$, we obtain a linear system of quadrics in $\mathbb{P}^{r-1}$ spanned by $\binom{r-1}{2}$ quadrics. $\overline{X}$ is then the base locus of this linear system, namely a rational curve of degree $r-1$ passing through the points $\omega_1, \dots \omega_r$. X has thus dimension 1 and contains an irreducible non-degenerate curve Y that is projected birationally into $\overline{X}$. Hence Y is a rational normal curve of degree r and applying the same argumentations presented in the first part of the proof, we get that X=Y and $Y \simeq \mathbb{P}^r \cap \mathrm{Gr}(2, r+2)$.

- We now analyze the case where Y spans a 4-plane $\Delta \simeq \mathbb{P}^4$ and $Z := Y \cap H$ is a scheme of type (ii), namely a zero-dimensional scheme of length 5 isomorphic to $\mathbb{P}^3 \cap \mathrm{Gr}(2,5)$. Note that under these assumptions we have (by 3.3.4) $U_Y = U_Z \simeq \mathbb{C}^5$;

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hence Δ is a 4-plane in $\mathbb{P}(\bigwedge^2 U_Y)$. Irreducible curves in $\mathrm{Gr}(2, U_Y) \simeq \mathrm{Gr}(2,5)$ spanning a 4 dimensional linear space, corresponds to general linear sections of the Grassmannian. Indeed, as $\mathrm{Gr}(2, U_Y)$ is a 6-dimensional smooth subvariety of $\mathbb{P}(\bigwedge^2 U_Y) \simeq \mathbb{P}^9$, a (general) 4-plane Δ intersect $\mathrm{Gr}(2, U_Y)$ along an irreducible curve Y. Moreover, since $\deg(\mathrm{Gr}(2, U_Y)) = 5$, we can compute immediately that the curve Y has degree 5. Since $Y := \mathrm{Gr}(2.U_Y) \cap \mathbb{P}^4$, its genus can be computed by adjunction. We are in $\mathbb{P}^9$, hence a linear section $\mathrm{Gr}(2, U_Y) \cap \mathbb{P}^4$ is realized as $\mathrm{Gr}(2, U_Y) \cap (\bigcap_{i=1}^5 H_i)$ for five hyperplanes H_is in $\mathbb{P}^9$, $i = 1, \dots 5$. The canonical class of $\mathrm{Gr}(2, U_Y)$ is $K_{\mathrm{Gr}(2,U_Y)} = -5H$, where H denote the hyperplane class. By adjunction formula we get that ω_Y has degree 0 and that thus Y has genus 1.

Also in this case we can get a description of the curve Y by means of $\overline{Y}$ its image through the linear projection from a point on Y not lying in H. Recall that H can be generated by any 4-tuple of points in Z, say $\omega_1, \dots, \omega_4$. We saw in 3.3.8, that $|\mathcal{Q}|_H|$ is a 4-dimensional linear system of quadrics equal to $\mathbb{P}(H^0(\mathcal{I}_Z(2))$. $|\mathcal{Q}|_H|$ individuates the rational map:

$$\gamma|_H : H \dashrightarrow \mathbb{P}(\bigwedge^4 U_Y) \simeq \mathbb{P}({U_Y}^*) \simeq \mathbb{P}^4.$$

Whenever we take a point $\omega_0 \in \mathbb{P}(\bigwedge^2 U_Y)$, $\omega_0 \notin H$, we individuate $\Omega_{l_{\omega_0}} \simeq \mathbb{P}^2$, the Scubert variety of 3-planes in $\mathbb{P}(U_Y) \simeq \mathbb{P}^4$, containing the line l_{ω_0}. Pulling back by γ two hyperplanes defining $\Omega_{l_{\omega_0}}$, we get two quadrics in H whose intersection is $\overline{Y}$. Being a complete intersection of two quadrics in $\mathbb{P}^3$, $\overline{Y}$ is a curve of genus 1 and degree 4, so that $\Delta \cap \mathrm{Gr}(2, U_Y)$ is effectively an elliptic quintic curve. To conclude, we observe that again, applying proposition 3.3.3 we get isomorphisms $|\mathcal{Q}|_H| \simeq |\mathcal{Q}|_\Delta| \simeq \mathbb{P}^4$. Y is an elliptic quintic curve, so that $h^0(\mathcal{I}_Y(2)) = 5$, leading indeed to an equality $|\mathcal{Q}|_\Delta| \simeq \mathbb{P}(H^0(\mathcal{I}_Y(2)))$ that, once again, allows us to conclude that $Y \simeq \mathbb{P}^4 \cap Gr(2,5)$.

□

Construction of the corresponding ruled surfaces

Given Y a curve satisfying the hypotheses of theorem 3.3.5, we describe now in greater detail the associated ruled surface S_Y. We already saw, in lemma 3.3.7 that S_Y is a surface having degree equal to the degree of Y. From theorem 3.3.5 we know that Y satisfies $Y = \Delta \cap \mathrm{Gr}(2, U_Y) = \langle Y \rangle \cap \mathrm{Gr}(2, U_Y)$,so that so that the map ${\pi_2}' : \pi_1^{-1}(Y) \to S_Y$ is everywhere bijective. Moreover we saw that for Y rational, or in the case $Y \simeq \mathbb{P}^4 \cap \mathrm{Gr}(2,5)$ for $\Delta \subset \mathbb{P}(\bigwedge^2 U_Y)$ generic, Y is smooth. In these cases S_Y is then a geometrically ruled surface of degree $\deg(Y)$ and irregularity $g(Y)$ over a smooth curve $\mathcal{C} \subset \mathbb{P}(U_Y)$ of genus $g(\mathcal{C}) = g(Y)$.

Remark 25. In the classical language,(see for example Hedge [Hed]), the irregularity of S_Y is referred to as the genus of S_Y. The genus of a ruled surface $S_Y \subset \mathbb{P}^n$ is defined as the genus of a generic hyperplane section. This notion, that can thus be extended also to singular surfaces, is well defined since a generic hyperplane section intersect transversally every generator of the surface S_Y, so that whenever we take two general hyperplane sections, they are in $(1,1)$ correspondence hence birational. Note that this also implies that the genus of S_Y coincide with the genus of the associated curve $Y \subset \mathrm{Gr}(2, n+1)$.

- $Y \simeq \mathbb{P}^r \cap \mathbf{Gr}(2, r+2)$.

  Whenever Y is a rational normal curve of degree r, S_Y is a rational normal scroll of degree r in $\mathbb{P}(U_Y) \simeq \mathbb{P}^{r+1}$. In the proof of theorem 3.3.5, we saw that it is always possible to find a line l_{ω_D} such that $Y \subset \mathbb{T}_{\omega_D}\mathrm{Gr}(2, W^*)$. This line will be the directrix of the scroll S_Y. The projective space $\mathbb{P}(U_Y) \simeq \mathbb{P}^{r+1}$ can then be written as:

$$\mathbb{P}(U_Y) = \langle l_{\omega_D}, \Gamma \rangle, \quad \Gamma \simeq \mathbb{P}^{r-1}, \ \Gamma \cap l_{\omega_D} = \varnothing.$$

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If now we take $r+1$ points $\omega_0, \dots \omega_r$ on Y, we can express each of these points as $\omega_i = u_i \wedge v_i$ with $u_i \in l_{\omega_D}$ and $v_i \in \Gamma$. This $r+1$ tuple of points defines uniquely a degree $r-1$ Veronese embedding ν_{r-1}:

$$\nu_{r-1} : l_{\omega_D} \longrightarrow \Gamma,\ u_i \mapsto v_i,\ i = 0, \dots r.$$

The surface S_Y is therefore the join of the morphism ν_{r-1}, namely:

$$S_Y = \bigcup_{u \in l_{\omega_D}} \overline{u\, \nu_{r-1}(u)}.$$

Notice that vice versa, whenever in a projective space $\mathbb{P}^{r+1}$ we have a line l_{ω_D} and a Veronese embedding $\nu_{r-1} : l_{\omega_D} \to \Gamma$, with $\Gamma \simeq \mathbb{P}^{r-1},\ \Gamma \cap l_{\omega_D} = \varnothing$, the join of this morphism is clearly a rational normal scroll of degree r. This ruled surface corresponds to the rational normal curve on the Grassmannian obtained from the morphism:

$$\tilde{\phi} : l_{\omega_D} \to \mathrm{Gr}(2, r+2),\ u \mapsto (u \wedge \nu_{r-1}(u)).$$

- $Y \simeq \mathbb{P}^4 \cap \mathbf{Gr}(2,5)$.

  In this case the ruled surface S_Y that we obtain tracing the curve Y on the Grassmannian, is a quintic elliptic scroll in $\mathbb{P}^4$. Every such a scroll can be realized as the "translation scroll" associated to a couple (E, P), being E an elliptic quintic curve and $P \in E$ a 2-torsion point (seeing E as a complex torus we can take as a 2-torsion point a half period); that is, S_Y is obtained joining by a line all the couple of points on E differing by P. In other words:

$$S_Y = \bigcup_{x \in E} \overline{x, x+P}.$$

  This is a ruled surface $S_Y \to F$ over the elliptic curve $F := E/\langle P \rangle$. For simplicity, from now on we suppose that F, E and S_Y are smooth. Under these assumption $S_Y \to F$ is a geometrically ruled surface over F, isomorphic to $\mathbb{P}(\mathcal{E}) \to F$, being $\mathcal{E}$ a vector bundle over F of rank 2 and degree 1. We recall here briefly how to describe a quintic elliptic scroll as a translation scroll (see [Cil-Hul] for further details).

  Starting from $S_Y \to F$ an elliptic quintic scroll over a curve F, we have (see Atiyah [At]), that $S_Y \to F$ is isomorphic to the fibration:

$$\pi : Sym^2 F \to F,\ \{x, y\} \mapsto x + y.$$

  Consider now the map $\rho : F \times F \to Sym^2 F,\ (x, y) \mapsto \{x, y\}$, and denote by $F_i \subset F \times F,\ i = 1, 2, 3$, the curve:

$$F_i = \{(x, x + P_i) \in F \times F \mid x \in F\}$$

  being $P_i,\ i = 1, 2, 3$ the 2-torsion points on F different from the origin. Choose one of these curves, say F_1, and denote by $E,\ E \subset Sym^2 F$ its image through ρ. The map $\rho : F_1 \to E$ sends the points (P_2, P_3) and (P_3, P_2) to the 2-torsion point $P \in E,\ P := \{P_2, P_3\},\ 2P \sim 0$. The quotient $E/\langle P \rangle$, obtained identifying couples of points on E differing by P, is isomorphic to F, so that we get an unbranched degree 2 covering $E \to F$. The curve E is a 2-section of the fibration $\pi : S_Y \to F$, namely $\forall\ y \in F$, the fiber $\pi^{-1}(y) \simeq \mathbb{P}^1$ meets the curve E in 2 points $e_1,\ e_2$, $e_i = e_j + P,\ i \neq j$, so that we can write:

$$\pi^{-1}(y) = \overline{e_1 e_2} = \overline{e_1, e_1 + P},$$

  (more concretely $e_1,\ e_2$ might be written in the form $e_1 = \{x, x + P_1\}$, $e_2 = \{x + P_2, x + P_3\},\ x \in F$ being a point such that $2x + P_1 \sim y$). The curve

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E is an elliptic quintic curve isomorphic to $Y \subset \mathrm{Gr}(2,5)$, the curve on the Grassmannian corresponding to S_Y.

Vice versa, starting from a couple (E, P), E being a smooth quintic elliptic curve in $\mathbb{P}^4$ and $P \in E$ a 2-torsion point (note that the translation $x \mapsto x + P$ defines a fixed-point free involution on E), we consider the quotient $q : E \to E/P$. Define F as $F := E/P$, and S_Y the translation scroll obtained from the couple (E, P). S_Y is then a ruled surface over F, $\pi : S_Y \to F$, the fiber over a point $y \in F$ is the line generated by the two points in $q^{-1}(y)$. Thus S_Y is the union:

$$S_Y = \bigcup_{y \in F} \overline{e_1 e_2} \quad \text{with } e_i = e_j + P,\ q(e_1) = q(e_2) = y.$$

Again, we see from the construction that S_Y is represented by a curve Y on the Grassmannian $\mathrm{Gr}(2,5)$ isomorphic to E, therefore it's a ruled surface of degree 5 and irregularity 1, namely a quintic elliptic scroll.

3.3.2 Higher dimensional components

Once we have accomplished the classification of one-dimensional components of X, we can proceed with the study of higher dimensional cases. Let then Y be an n-dimensional non-degenerate irreducible component of X, $n = 2, 3$. As now we are considering Y of dimension greater than 1, for a general hyperplane $H \subset \Delta$, the hyperplane section $H \cap Y$ is still irreducible and non-degenerate, we denote it by Z. Z is then an irreducible, non-degenerate in H, $n-1$ dimensional component of $H \cap \mathrm{Gr}(2, W^*) = \langle Z \rangle \cap \mathrm{Gr}(2, W^*)$. Therefore, by means of theorem 3.3.5, and formula 3.6 we can prove:

Proposition 3.3.11. *Let $\Delta \simeq \mathbb{P}^r$, $r = 3, 4$ be a hyperweb of generic rank 4, and suppose that $X = \Delta \cap \mathrm{Gr}(2, W^*)$ contains a 2 dimensional irreducible component Y. Then Y is:*

- *either a quadric surface (smooth for Δ general) isomorphic to $\mathbb{P}^3 \cap \mathrm{Gr}(2,4)$.*
- *either a rational normal ruled surface of degree 3 isomorphic to $\mathbb{P}^4 \cap \mathrm{Gr}(2,5)$. More precisely Y can be a cone over a twisted cubic or (for Δ general) a smooth cubic scroll.*

Proof. Let Y be a surface satisfying the hypotheses of the proposition and consider $Z = H \cap Y$ an irreducible and non-degenerate hyperplane section. We distinguish the 2 possible value of $r := \dim(\Delta)$:

- **r=3**. In this case Y is an hypersurface in $\mathbb{P}^3$. Applying theorem 3.3.5, we see that Z must be a conic and that moreover $Z = H \cap \mathrm{Gr}(2, U_Z)$. By formula 3.6 and propostion 3.3.4 we deduce that $\Delta \subset \mathbb{P}(\bigwedge^2, U_Y) \simeq \mathbb{P}(\bigwedge^2 \mathbb{C}^4)$. $\mathrm{Gr}(2, U_Y)$ is isomorhic to $\mathrm{Gr}(2,4)$ hence $\Delta \cap \mathrm{Gr}(2, U_Y)$ is a quadric surface. As we are supposing that Y is not a linear space (or equivalently that it spans the entire Δ), $\Delta \cap \mathrm{Gr}(2, U_Y)$ must be irreducible hence it must coincide with Y. We observe that Y is smooth for Δ general, ($\mathrm{rk}\,(Y) = 4$ whenever the pencyl of hyperplanes defining Δ meets $\mathrm{Gr}(2, U_Y^*)$ in 2 distinct points) and $\mathrm{rk}\,(Y) = 3$ whenever this pencyl is tangent to $\mathrm{Gr}(2, U_Y^*)$.
- **r=4**. When the surface Y spans a 4-dimensional linear space, by theorem 3.3.5, Z must be a twisted cubic isomorphic to $\mathbb{P}^3 \cap \mathrm{Gr}(2,5)$. Consequently, we deduce that $\dim(U_Y) = 5$ (so that $Y \subset \mathrm{Gr}(2,5)$), $\deg(Y) = 3$ and $|\mathcal{Q}|_\Delta| \simeq \mathbb{P}^2$. Y, being an irreducible and non-degenerate surface of degree 3 in $\mathbb{P}^4$, is therefore a surface of minimal degree. But it is well known that non-degenerate irreducible surfaces of minimal degree are rational normal ruled surface (see, for example, Harris [Harr]). Thus
$$Y \simeq S_{(a_0,a_1)}, \quad 0 \le a_0 \le a_1,\ a_0 + a_1 = 3.$$
We see there are only 2 possibilities that might occur.

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Either $Y \simeq S_{(0,3)}$, namely is a cone over a twisted cubic, or $Y \simeq S_{(1,2)}$, that is, Y is a cubic scroll. A rational normal ruled surface having degree 3 is the base locus of a 2-dimensional linear system of quadrics (this can be seen, for example, from its determinantal representation), so that $|\mathcal{Q}|_\Delta| \simeq \mathbb{P}(H^0(\mathcal{I}_Y(2))$ from which we conclude that Y is exactly $\Delta \cap \mathrm{Gr}(2, U_Y)$. Choosing 4 points $\omega_1, \dots \omega_4$ on Z, we know by 3.3.10 that there exists a unique point $\omega_D \in \mathrm{Gr}(2, U_Y)$ such that $\omega_i \in \mathbb{T}_{\omega_D}\mathrm{Gr}(2, U_Y)$. Δ can thus be generated by the ω_is, $i = 1, \dots 4$ and by a fifth point $\omega_0 \notin \langle Z \rangle$ belonging to $\mathrm{Gr}(2, U_Y) \cap \mathbb{T}_{\omega_D}\mathrm{Gr}(2, U_Y)$. The surface $Y = \Delta \cap \mathrm{Gr}(2, U_Y)$ is a cubic scroll $Y \simeq S_{(1,2)}$ for $\omega_0 \neq \omega_D$ and degenerates to the cone over Z whenever $\omega_0 = \omega_D$.

□

For what concerns 3-dimensional components $Y \subset X$, the only case that we have to consider (as we are excluding the possibility that $Y \simeq \mathbb{P}^3$) is when Y is an hypersurface in $\Delta \simeq \mathbb{P}^4$. Reasoning as above we can prove:

Proposition 3.3.12. *Let Δ be a four-dimensional hyperweb of generic rank 4 such that $X := \Delta \cap \mathrm{Gr}(2, W^*)$ has dimension 3. Then X is a quadric hypersurface in Δ of rank 4 or 5 and moreover $X \simeq \mathbb{P}^4 \cap \mathrm{Gr}(2, 4)$.*

Proof. Consider a variety X as in the statement of the proposition. Applying 3.3.11, we see that taking a generic hyperplane $H \subset \Delta$, the surface $Z := H \cap X$ is an irreducible quadric surface (and so necesserily non-degenerate) in H that is isomorphic to $\mathbb{P}^3 \cap \mathrm{Gr}(2, 4)$. Consequently $\dim(U_X) = 4$ and X is an hyperplane section, in $\mathbb{P}(\bigwedge^2 U_X^*) \simeq \mathbb{P}^5$ of $\mathrm{Gr}(2, U_X)$. $\mathrm{Gr}(2, U_X)$ is a smooth quadric hypersurface in $\mathbb{P}(\bigwedge^2 U_X)$, hence an hyperplane section migth have rank 5 or 4 (this last instance occurs exactly when we intersect $\mathrm{Gr}(2, U_X)$ with an hyperplane in $\mathrm{Gr}(2, U_X^*)$.) □

The results obtained up to this moment (where we studied irreducible subvarieties of $\mathrm{Gr}(2, W^*)$ not consisting in linear spaces), together with the description of linear subspaces of $\mathrm{Gr}(2, W^*)$ complete the proof of theorem 3.3.1

3.4 Applications to the study of $\mathcal{P}^{ss}$

We are now going to relate the results that we have obtained to the study of the moduli space $\mathfrak{P}$ of Pfaffian representation of cubic threefolds.
Recall that $\mathfrak{P}$ was defined as the GIT quotient $\mathcal{P}^{ss} /\!\!/ SL(6, \mathbb{C})$, where $\mathcal{P} \simeq \mathbb{P}(V^* \otimes \bigwedge^2 W^*)$ is the space of 6×6 skew-symmetric matrices whose entries are linear forms on $\mathbb{P}(V) \simeq \mathbb{P}^4$ and $\mathcal{P}^{ss}$ is the semistable locus. Keeping the notations introduced in Chapter 1, we call Π the locus of matrices $M \in \mathcal{P}$ such that $\mathrm{Pf}(M) = 0$ (namely matrices corresponding to hyperwebs of generic rank ≤ 4) and $\mathcal{P}^{in}$ the locus of matrices $M \in \mathcal{P}$ such that the equation $\mathrm{Pf}(M) = 0$ defines a smooth cubic threefold. $\mathcal{P}^s$ denotes the stable locus. Reformulating the results we got at the beginning of section 3.2.1 we have the following:

Theorem 3.4.1. *let M be a point in $\mathcal{P}$. If $M \notin \Pi$ then $M \in \mathcal{P}^{ss}$. If moreover $M \in \mathcal{P}^{in}$ then $M \in \mathcal{P}^s$.*

Proof. If a point M does not belong to Π, $\mathrm{Pf}(M) \neq 0$ so that M is semistable by corollary 3.2.2. The fact that M does not lie in Π implies that M individuates a linear space $\mathbb{P}(A) \subset \mathbb{P}(\bigwedge^2 W^*)$ not contained in Pf. The equation $\mathrm{Pf}(M) = 0$ defines then the cubic hypersurface $\mathbb{P}(A) \cap \mathrm{Pf}$. Assuming $M \in \mathcal{P}^{in}$, the intersection $\mathbb{P}(A) \cap \mathrm{Pf}$ is a smooth cubic threefold so that M is stable by theorem 3.2.3. □

In other words we see that every Pfaffian representation of a cubic threefold is semistable and that moreover every Pfaffian representation of a smooth cubic threefold is strictly stable.

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Remark 26. Applying theorem 3.2.4, we can see that points in $\Pi \cap \mathcal{P}^s$ are exactly those occurring in locally free resolutions of sheaves in $\mathcal{B}_X$, the boundary of the Gieseker-Maruyama moduli space $\mathcal{M}_X(2;0,2,0)$ on a smooth cubic threefold X. We will give a detailed geometric description of $\Pi \cap \mathcal{P}^s$ in chapter 4.

Examples

We present here some explicit examples of points in $\mathcal{P}$ and we analyze their stability.

We start by giving examples of striclty semistable Pfaffian representation of cubics. Let then $M \in \mathcal{P}$ such that $\mathrm{Pf}(M) \neq 0$ and belonging to $\mathcal{P}^{ss} \setminus \mathcal{P}^s$. Since we are assuming that M is not stable, we apply theorem 3.2.1 and we denote by s, $s \in \{1, \ 2, \ 3\}$, the integer consequently individuated. We saw in proof of theorem 3.2.3 that if $s = 1$ or $s = 2$, then the cubic hypersurface $\mathrm{Pf}(M) = 0$ is reducible. If M is a strictly semistable Pfaffian representation of an irreducible cubic we thus must have $s = 3$. Example of representations of this kind are provided by cubic threefolds X admitting a *linear determinantal representation.* This means that the cubic form F defining X might be written as the determinant of a 3×3 matrix N of linear forms. Every linear determinantal representation N of the cubic X determines a Pfaffian representation: it is indeed enough to consider the matrix $M \in \mathcal{P}$ defined by:

$$M = \left(\begin{array}{c|c} 0 & N \\ \hline -N & 0 \end{array}\right) \tag{3.7}$$

From (3.7) we check directly that $M_{ij} = 0$ whenever $i = 1,2,3, \ j = 1,2,3$ so that M is strictly semistable. Examples of cubic threefolds admitting a linear determinantal representation are the following:

- X **has 6 nodes**. A generic cubic threefold X defined by:

  $$X := \{\det(N) = 0\}, \qquad N = (l_{ij})_{1 \leq i,j \leq 3}$$

  where N is a 3×3 matrix of linear forms l_{ij}, presents 6 nodes lying in general position. Conversely every cubic threefold presenting 6 nodes in general positions admits a linear determinantal representation. This is proved by Segre in [Seg], a modern reformulation can be found in [H-T]

- X **is the secant threefold** We consider X, the secant variety of a rational normal quartic curve Γ. Γ can be defined as the locus of points in $\mathbb{P}^4$ where the matrix of linear forms:

  $$\begin{pmatrix} X_0 & X_1 & X_2 & X_3 \\ X_1 & X_2 & X_3 & X_4 \end{pmatrix} \tag{3.8}$$

  has rank 1. From 3.8 we can determine a linear determinantal representation N of X:

  $$N = \begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_2 & X_3 \\ X_2 & X_3 & X_4 \end{pmatrix} \tag{3.9}$$

We look now to points belonging to Π. Using the classification of the irreducible components of 4-dimensional linear sections of $\mathrm{Gr}(2, W^*)$ we provide some explicit examples of non-stable hyperwebs $\mathbb{P}(A) \subset \mathrm{Pf}$. More specifically we show that whenever

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$X := \mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$ contains a component spanning the entire space $\mathbb{P}(A)$ (and from our classification is thus the only component of $\mathbb{P}(A) \cap \mathrm{Gr}(2, W^*)$, $\mathbb{P}(A)$ can't be semistable.

- $X \simeq \mathbb{P}^4$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & X_0 \\ 0 & 0 & 0 & 0 & 0 & X_1 \\ 0 & 0 & 0 & 0 & 0 & X_2 \\ 0 & 0 & 0 & 0 & 0 & X_3 \\ 0 & 0 & 0 & 0 & 0 & X_4 \\ -X_0 & -X_1 & -X_2 & -X_3 & -X_4 & 0 \end{pmatrix}$$

- X is a smooth 3-dimensional quadric

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_0 & X_1 & X_2 \\ 0 & 0 & -X_0 & 0 & X_3 & X_4 \\ 0 & 0 & -X_1 & -X_3 & 0 & X_0 \\ 0 & 0 & -X_2 & -X_4 & -X_0 & 0 \end{pmatrix}$$

- $X \simeq S_{(0,1,1)}$ is a 3-dimensional quadric of rank 4:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_1 & X_2 \\ 0 & 0 & 0 & 0 & X_3 & X_4 \\ 0 & 0 & -X_1 & -X_3 & 0 & X_0 \\ 0 & 0 & -X_2 & -X_4 & -X_0 & 0 \end{pmatrix}$$

- $X \simeq S_{(1,2)}$ is a cubic scroll:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & X_0 + 2X_4 & -X_0 + X_4 \\ 0 & 0 & 0 & 0 & X_1 + 2X_4 & X_1 + X_4 \\ 0 & 0 & 0 & 0 & X_2 & X_4 \\ 0 & 0 & 0 & 0 & 0 & X_3 + X_4 \\ -X_0 - 2X_4 & -X_1 - 2X_4 & -X_2 & 0 & 0 & 0 \\ X_0 - X_4 & -X_1 - X_4 & -X_4 & -X_3 - X_4 & 0 & 0 \end{pmatrix}$$

- $X \simeq S_{(0,3)}$ is the cone over a twisted cubic:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & X_0 & -X_0 \\ 0 & 0 & 0 & 0 & X_1 & X_1 \\ 0 & 0 & 0 & 0 & X_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & X_3 \\ -X_0 & -X_1 & -X_2 & 0 & 0 & X_4 \\ X_0 & -X_1 & 0 & -X_3 & -X_4 & 0 \end{pmatrix}$$

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- X is a rational quartic curve:

$$\begin{pmatrix} 0 & 0 & 0 & -X_3-2X_4 & X_3-X_4 & 0 \\ 0 & 0 & 0 & -X_2-2X_4 & X_2-X_4 & 0 \\ 0 & 0 & 0 & X_1-2X_4 & -X_4 & 0 \\ X_3+2X_4 & X_2+2X_4 & 2X_4-X_1 & 0 & 0 & 2X_4 \\ X_4-X_3 & X_4-X_2 & X_4 & 0 & 0 & X_0 \\ 0 & 0 & 0 & -2X_4 & -X_0 & 0 \end{pmatrix}$$

- X is a quintic elliptic curve:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X_0-X_3+X_4 & 0 & -X_3+X_4 & X_2+X_4 \\ 0 & X_3-X_0-X_4 & 0 & -X_4 & -X_4 & X_3 \\ 0 & 0 & X_4 & 0 & X_1 & X_2 \\ 0 & X_3-X_4 & X_4 & -X_1 & 0 & X_3+X_4 \\ 0 & -X_2-X_4 & -X_3 & -X_2 & -X_3-X_4 & 0 \end{pmatrix}$$

In these examples, except for the cases of the smooth quadric hypersurface and of the elliptic quintic, X is a rational normal scroll. When X is a rational normal scroll, there exists a point $\omega \in \mathrm{Gr}(2, W^*)$ such that X and so the entire linear space $\mathbb{P}(A) = \langle X \rangle$, are contained in $\mathbb{T}_\omega(\mathrm{Gr}(2, W^*)$. For our choice of coordinates this is the point $e_5 \wedge e_6$. As we have already observed several times, this implies that $\mathbb{P}(\bigwedge^2 \ker(\omega)) \simeq \mathbb{P}^5$ is contained in $\mathbb{P}(A)^\perp$, so that $\ker(\omega)$ is isotropic with respect to every tensor in $\mathbb{P}(A)$. In these cases the hyperwebs $\mathbb{P}(A) = \langle X \rangle$ belong to the same component $\mathcal{H}^3$ of the Hilbert nullcone described in remark 21.

3.5 Appendix: The polarization index

From what we have proved in the chapter we see that for a 4-dimensional hyperweb $\mathbb{P}(A)$, the inclusion $\mathbb{P}(A) \subset \mathrm{Pf}$ is a necessary condition for the non-semistability of $\mathbb{P}(A)$ (this is corollary 3.2.2) but not a sufficient one. Theorem 3.2.4 provides indeed stable hyperwebs of generic rank 4. The fact that the vanishing of the Pfaffian does not prevent a hyperweb from being stable is due to the value of the *polarization index* of the action of $SL(W)$ on $V^* \otimes \bigwedge^2 W^*$. We recall here briefly how this index is defined (We refer to [BD], [Po] for further details). To start with, notice that whenever we have a linear map $\phi : V^* \to U$, from V^* to a complex vector space U of dimension p, $p \leq \dim(V^*)$, tensoring by $\bigwedge^2 W^*$, we obtain a linear map, that we still denote by ϕ,

$$\phi : V^* \otimes \bigwedge^2 W^* \to U \otimes \bigwedge^2 W^*.$$

These vectors spaces have a $SL(W)$-module structure, where $SL(W)$ acts trivially on V^* and U. The map $\phi : V^* \otimes \bigwedge^2 W^* \to U \otimes \bigwedge^2 W^*$ is thus an $SL(W)$-equivariant morphism. Given now any $SL(W)$ invariant $f \in \mathbb{C}[U \otimes \bigwedge^2 W^*]^{SL(W)}$, we can pull it back by ϕ and get $f \circ \phi \in \mathbb{C}[V^* \otimes \bigwedge^2 W^*]^{SL(W)}$ an $SL(W)$-invariant on $V^* \otimes \bigwedge^2 W^*$. The functions obtained in this way, as ϕ varies in $\mathrm{Hom}_{\mathbb{C}}(V^*, U)$ are called *polarizations of f*. Consider now $U = \mathbb{C}$. In this case $\mathbb{C}[U \otimes \bigwedge^2 W^*]^{SL(W)} = \mathbb{C}[\bigwedge^2 W^*]^{SL(W)}$ is a finitely generated $\mathbb{C}$-algebra, generated by the Pfaffian. Denote by $pol_5\mathbb{C}[\bigwedge^2 W^*]^{SL(W)}$ the subalgebra of $\mathbb{C}[V^* \otimes \bigwedge^2 W^*]^{SL(W)}$, whose generators are the polarizations of elements of $\mathbb{C}[\bigwedge^2 W^*]^{SL(W)}$; as this latter is generated by the Pfaffian, $pol_5\mathbb{C}[\bigwedge^2 W^*]^{SL(W)}$ is finitely generated as well and its generators are the polarizations of the Pfaffian.
The inclusions $pol_5\mathbb{C}[\bigwedge^2 W^*]^{SL(W)} \xrightarrow{\iota_p} \mathbb{C}[V^* \otimes \bigwedge^2 W^*]^{SL(W)}$, and $\mathbb{C}[V^* \otimes \bigwedge^2 W^*]^{SL(W)} \xrightarrow{\iota_G} \mathbb{C}[V^* \otimes \bigwedge^2 W^*]$ induce dominant morphisms of affine varieties

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$V^* \otimes \bigwedge^2 W^* \xrightarrow{\pi_p} (V^* \otimes \bigwedge^2 W^*) /\!\!/_p SL(W) := Spec(\ pol_5\mathbb{C}[V^* \otimes \bigwedge^2 W^*]^{SL(W)})$ and $(V^* \otimes \bigwedge^2 W^*) \xrightarrow{\pi_G} V^* \otimes \bigwedge^2 W^* /\!\!/ SL(W) := Spec\ \mathbb{C}[V^* \otimes \bigwedge^2 W^*]^{SL(W)}$, consequently a commutative diagram:

$$\begin{array}{ccc} V^* \otimes \bigwedge^2 W^* & \xrightarrow{\qquad\qquad \pi_G \qquad\qquad} & (V^* \otimes \bigwedge^2 W^*) /\!\!/ SL(W) \\ & {\scriptstyle \pi_p} \searrow \qquad \swarrow & \\ & (V^* \otimes \bigwedge^2 W^*) /\!\!/_p SL(W) & \end{array}$$

Denote the Hilbert nullcone by $\mathcal{H}_{V^* \otimes \bigwedge^2 W^*}(SL(W)) := \pi_G^{-1}(\pi_G(0))$ (recall that this is the cone of unstable points in $V^* \otimes \bigwedge^2 W^*$, namely the closed set of points on which every element of $\mathbb{C}[V^* \otimes \bigwedge^2 W^*]^G$ vanishes) and define $\mathcal{P}_{V^* \otimes \bigwedge^2 W^*}(SL(W)) := \pi_p^{-1}(\pi_p(0))$ (i.e the space of points on which every polarization of the Pfaffian vanishes). The points of $\mathcal{P}_{V^* \otimes \bigwedge^2 W^*}(SL(W))$ thus represent hyperwebs of generic rank less than or equal to 4. Note that, calling $\mathcal{H}_{\bigwedge^2 W^*}(SL(W))$ the nullcone of the action of $SL(W)$ on $\bigwedge^2 W^*$, all the elements of an hyperweb in $\mathcal{P}_{V^* \otimes \bigwedge^2 W^*}(SL(W))$ belong to $\mathcal{H}_{\bigwedge^2 W^*}(SL(W))$.

It's clear that we have an inclusion $\mathcal{H}_{V^* \otimes \bigwedge^2 W^*}(SL(W)) \subseteq \mathcal{P}_{V^* \otimes \bigwedge^2 W^*}(SL(W))$, more precisely it's a *strict* inclusion; this is due to the fact that the polarization index of $\bigwedge^2 W^*$ is strictly smaller then 5. Recall that the polarization index of $\bigwedge^2 W^*$ iz defined by:

$$\text{pol ind } (\bigwedge^2 W^*) := \sup\, n,$$

where the supremum is taken over all positive integers n such that we have an equality $\mathcal{H}_{\mathbb{C}^n \otimes \bigwedge^2 W^*}(SL(W)) = \mathcal{P}_{\mathbb{C}^n \otimes \bigwedge^2 W^*}(SL(W))$. We have in particular the following (Lemma 3.6 in [Po]) : for an integer n , pol ind$(\bigwedge^2 W^*) \leq n$, if and only if for every linear subspace L of $\bigwedge^2 W^*$, with $\dim L \leq n$ and $L \subset \mathcal{H}_{\bigwedge^2 W^*}(SL(W))$, there exists a one-parameter subgroup λ of $SL(W)$ such that every element of L is not λ semistable. In [BD], theorem 4.4, the authors assert that pol ind$(\bigwedge^2 W^*) = 2$, proving at first that pol ind$(\bigwedge^2 W^*) \leq 2$ and then constructing a 3-dimensional subspace $L = \langle A_1, A_2, A_3 \rangle$ of $\bigwedge^2 W^*$ with $L \subset \mathcal{H}_{\bigwedge^2 W^*}(SL(W))$ but such that the triple $(A_1, A_2, A_3) \notin \mathcal{H}_{\mathbb{C}^3 \otimes \bigwedge^2 W^*}(SL(W))$. So, since pol ind$(\bigwedge^2 W^*) < 5$ the inclusion $\mathcal{H}_{V^* \otimes \bigwedge^2 W^*}(SL(W)) \subset \mathcal{P}_{V^* \otimes \bigwedge^2 W^*}(SL(W))$ is strict.

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Chapter 4

Boundary of the Moduli Space

Introduction

In this chapter we illustrate some properties of sheaves belonging to the boundary of the moduli space $\mathcal{M}_X(2;0,2,0)$, X being a smooth cubic threefold, and of their free resolutions. We saw in Chapter 1 that a sheaf $\mathcal{F}$ of this kind always admits a minimal free resolution (as a $\mathcal{O}_{\mathbb{P}^4}$-module) of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} \xrightarrow{G} \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 6} \xrightarrow{B} \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} \to \mathcal{F} \to 0, \qquad (4.1)$$

in which $B = (\beta'|\beta)$ is a 6-by-8 matrix obtained by concatenation of a 6×2 matrix of quadratic forms β' with a 6×6 skew-symmetric matrix β of linear forms satisfying $\mathrm{Pf}(\beta) = 0$. Given a couple of vector spaces $W,\ V$ of dimension 6 and 5 respectively, the matrix β defines a point in $\bigwedge^2 W^* \otimes V^*$ belonging to $\mathcal{Z}$, where $\mathcal{Z}$ is defined as the locus of matrices $M \in \bigwedge^2 W^* \otimes V^*$ such that $\mathrm{Pf}(M) = 0$.
Moreover it follows from the results obtained in Chapter 3 that matrices β obtained in this way individuate all the stable points (with respect to the $SL(W)$-action) lying in $\mathcal{Z}$. Because of this fact the orbit of β defines a point in $\mathfrak{P} := \mathbb{P}(\bigwedge^2 W^* \otimes V^*)^{ss} /\!\!/ SL(W)$, the moduli space of Pfaffian representations of cubic threefolds.
In the first part of the chapter we study the behavior of $\mathcal{Z}$, $\bigwedge^2 W^* \otimes V^*$ and $\mathfrak{P}$ at β. As the boundary $\mathcal{B}_X$ of $\mathcal{M}_X(2;0,2,0)$ is given by the union of two divisors $\mathcal{B}'_X$, $\mathcal{B}''_X$, matrices β as above belong to two different families of elements in $\mathcal{Z}$. We prove that whenever β is obtained from the minimal free resolution of a sheaf $\mathcal{F}$ corresponding to a point $[\mathcal{F}] \in \mathcal{B}'_X$ (resp. $[\mathcal{F}] \in \mathcal{B}''_X$), it belongs to a 47-dimensional component $\mathcal{Z}'$ (resp. 48-dimensional component $\mathcal{Z}''$) of $\mathcal{Z}$, smooth at β. Therefore, the image of its orbit in $\mathfrak{P}$, is a smooth point of a 11-dimensional subvariety $\mathfrak{B}'$, quotient of $\mathcal{Z}'$, (resp. a 12-dimensional subvariety $\mathfrak{B}''$, quotient of $\mathcal{Z}''$) of $\mathfrak{P}$. However we will show that $\mathfrak{P}$ is smooth at the orbit of β only when $\beta \in \mathcal{Z}'$; this due to the fact that if $\beta \in \mathcal{Z}''$, its stabilizer does not consist only of $\{\pm Id_6\}$.

In the second part of the chapter we study how a minimal free resolution $\mathcal{R}^\bullet$ of $\mathcal{F}$ of the form 4.1 behaves under deformation. We consider then a sheaf $\mathcal{F}_\Delta$ on $\mathbb{P}^4 \times \Delta$ flat over Δ, Δ being a polydisk in $\mathbb{C}^N$, such that $\forall\, s \in \Delta$, $\mathcal{F}_s$ (the restriction of $\mathcal{F}_\Delta$ to $\mathbb{P}^4 \times s$) belongs to $\mathfrak{M}$ and $\mathcal{F}_0 = \mathcal{F}$. $\mathcal{R}^\bullet$ lifts to a resolution $\mathcal{R}^\bullet_\Delta$ of $\mathcal{F}_\Delta$ of the form:

$$0 \to \mathcal{O}(-3)^{\oplus 2}_{\mathbb{P}^4 \times \Delta} \xrightarrow{G(s)} \mathcal{O}_{\mathbb{P}^4 \times \Delta}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4 \times \Delta}(-2)^{\oplus 6} \xrightarrow{B(s)} \mathcal{O}_{\mathbb{P}^4 \times \Delta}(-1)^{\oplus 6} \to 0, \quad (4.2)$$

$\mathcal{R}^\bullet_\Delta$ is a complex of sheaves on $\mathbb{P}^4 \times \Delta$, its differentials $G(s)$ and $B(s)$ are matrices with entries in $\mathbb{C}\{s\}[X_0, \dots, X_4]$, where $\mathbb{C}\{s\}$ denotes the ring of germs of analytic functions in s at 0. More precisely $B(s) = (\beta'(s)|\beta(s))$ where $\beta(s)$ is a 6×6 matrix whose entries are elements in $\mathbb{C}\{s\}[X_0, \dots, X_4]$ linear in the variables $X_0, \dots, X_4$. Our goal is to establish when we can deform $\mathcal{F}$ in such a way that $\beta(s)$ is skew-symmetric $\forall\, s \in \Delta$. We prove that such a deformation is always possible whenever the type of the singularity of $\mathcal{F}$ is

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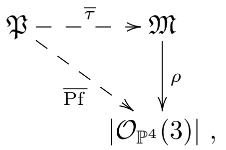
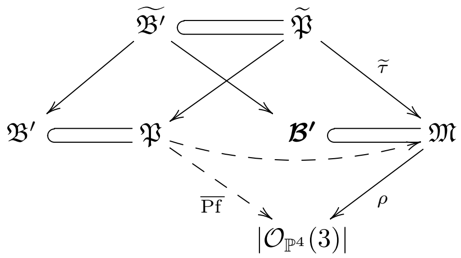
preserved (Lemma 4.2.1) and whenever $\mathcal{F}$ is a sheaf corresponding to a point in $\mathcal{B}'_X$ and deforming to a sheaf $\mathcal{F}_\Delta$ such that for generic $s \in \Delta$, $\mathcal{F}_s$ is an instanton on a smooth cubic threefold (Lemma 4.2.2). Using this results we are able to describe the local behavior of the diagram:



(4.22)

in a neighborhood of the orbit of a generic point $\beta_0 \in \mathcal{Z}'$. We prove the following:

Proposition 4.0.1. *Let $\beta_0 \in \mathcal{Z}'$ be as above, and consider the diagram (4.22) in a neighborhood of the orbit $[\beta_0] \in \mathfrak{B}'$, $[\beta_0] = GL(W) \cdot \beta_0$. Then the rational map $\overline{\tau}$ is equivalent to a blowup with center $\mathfrak{B}'$ near $[\beta_0]$. More precisely: let $\widetilde{\mathfrak{P}}$ denote the blowup of $\mathfrak{P}$ with center $\mathfrak{B}'$ and $\widetilde{\mathfrak{B}'}$ its exceptional divisor. Then, in a neighborhood of $[\beta_0]$, (4.22) can be completed to the diagram*



(4.3)

in which the arrows $\widetilde{\tau}$ and $\widetilde{\tau}|_{\widetilde{\mathfrak{B}'}}$ are isomorphisms.

Here $\boldsymbol{\mathcal{B}'}$ is the divisor of $\mathfrak{M}$ whose generic point is a sheaf $\mathcal{F}$ on a smooth cubic threefold X and corresponding to a point $[\mathcal{F}] \in \mathcal{B}'_X$. Calling now $\boldsymbol{\mathcal{B}''}$ the divisor of $\mathfrak{M}$ whose generic point is a sheaf $\mathcal{F}$ supported on smooth cubic threefolds X and such that $[\mathcal{F}] \in \mathcal{B}''_X$, we conjecture that Proposition 4.2.4 extends literally to $\boldsymbol{\mathcal{B}''}$ and $\mathfrak{B}'' \subset \mathfrak{P}$.

4.1 Skew-symmetric resolutions

Throughout the rest of the section we will work on a smooth cubic threefold X. Consider $\mathcal{M}_X(2;0,2,0)$ the Gieseker-Maruyama moduli space of instanton bundles on X. As for what have been illustrated in Chapter 1, we know that two "families" of sheaves appear in the boundary $\mathcal{B}_X$ of $\mathcal{M}_X(2;0,2,0)$: sheaves associated to conics contained in X and direct sums of ideal sheaves of lines; these two families individuate two divisors $\mathcal{B}'_X$ and $\mathcal{B}''_X$ of $\mathcal{M}_X(2;0,2,0)$. We also recall that given any point $[\mathcal{F}] \in \mathcal{B}_X$, the sheaf $\mathcal{F}$ fits in a short exact sequence of $\mathcal{O}_X-$modules:

$$0 \to \mathcal{F} \to \mathcal{O}_X^2 \to \mathcal{G} \to 0; \tag{4.4}$$

$\mathcal{G}$ being a one dimensional sheaf of the form:

- $\mathcal{G} \simeq \mathcal{O}_\mathcal{C}(1pt)$ for $\mathcal{C}$ a smooth conic (whenever $[\mathcal{F}] \in \mathcal{B}'_X$);
- $\mathcal{O}_{l_1} \oplus \mathcal{O}_{l_2}$ for l_1, l_2 a couple of (possibly incident or even coincident) lines (whenever $[\mathcal{F}] \in \mathcal{B}_X$”).

Consider now $\mathcal{M}_{\mathbb{P}^4}(2m+2)$, the moduli space of sheaves in $\mathbb{P}^4$ having Hilbert polynomial $2m+2$. A sheaf $\mathcal{G}$ as above is a torsion sheaf on $\mathbb{P}^4$ whose support Supp $(\mathcal{G})$ is contained in X and that determines a point $[\mathcal{G}] \in \mathcal{M}_{\mathbb{P}^4}(2m+2)$. Furthermore we saw that such a sheaf $\mathcal{G}$ admits a minimal free resolution $\mathcal{L}^\bullet$ in $\mathbb{P}^4$ of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-3)^2 \xrightarrow{\gamma} \mathcal{O}_{\mathbb{P}^4}(-2)^6 \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^4}(-1)^6 \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^4}^2, \tag{4.5}$$

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in which β is a 6×6 skew-symmetric matrix of linear forms of generic rank 4 and $\gamma = \alpha^T$. We will now show that the 1-dimensional sheaves $\mathcal{G}$ as above are *skew-symmetric ACM sheaves.* Recall that the skew-symmetry of $\mathcal{G}$ means that there exists an isomorphism $k : \mathcal{G} \to \mathcal{G}^\vee(N)$ (for some integer N), that satisfies $k^T = -k$. Here the dual $\mathcal{G}^\vee$ is understood as $\mathscr{H}om_{\mathcal{O}_C}(\ \cdot, \mathcal{O}_C)$, where C is the support of the sheaf. In general this property is too weak for $\mathcal{G}$ to have a skew-symmetric minimal resolution on $\mathbb{P}^4$. A better characterization of the skew-symmetry is given in terms of the derived category of $\mathcal{O}_{\mathbb{P}^4}$-modules. For a d-dimensional sheaf the derived dual is $\mathscr{H}\mathbf{om}(\mathcal{G}, \omega_{\mathbb{P}^4}[d](d+1))$. The derived dual is quasi-isomorphic to a sheaf if and only if $\mathscr{E}xt^i(\mathcal{G}, \omega_{\mathbb{P}^4}) = 0$ for $i \neq 4-d$, and then $\mathcal{G}^\vee = \mathscr{E}xt^{4-d}(\mathcal{G}, \omega_{\mathbb{P}^4})(d+1)$. The vanishing condition for the exts is satisfied when $\mathcal{G}$ is an ACM d-dimensional sheaf, so that the sheaf duality functor $\mathcal{G} \mapsto \mathcal{G}^\vee := \mathscr{E}xt^{4-d}(\mathcal{G}, \omega_{\mathbb{P}^4})(d+1)$ is well-behaved when considered on the category of d-dimensional ACM sheaves on $\mathbb{P}^4$; in particular, it is an involution. We work in this section with ACM sheaves $\mathcal{G}$ of codimension 3; the skew symmetry of such a sheaf means that there is an isomorphism of coherent sheaves $k = \mathcal{G} \xrightarrow{\sim} \mathscr{E}xt^3(\mathcal{G}, \omega_{\mathbb{P}^4})(N)$.

Lemma 4.1.1. *Any sheaf $\mathcal{G}$ on $\mathbb{P}^4$ with minimal free resolution of the form* (4.5) *is a 1-dimensional ACM sheaf with Hilbert polynomial $m \mapsto 2m+2$. If, moreover, the maps of the resolution satisfy the equalities $\beta^T = -\beta$ and $\gamma = \alpha^T$, then $\mathcal{G}$ is skew-symmetric.*

Proof. Suppose that $\mathcal{G}$ admits a minimal free resolution $\mathcal{L}^\bullet$ of the form (4.5). The alternating sum of Hilbert polynomials of the terms of the resolution gives the Hilbert polynoimal of $\mathcal{G}$, equal to $m \mapsto 2m+2$; this implies that $\mathcal{G}$ is a sheaf of dimension 1. Therefore $\mathcal{G}$ is ACM if and only if it is locally Cohen-Macaulay, namely if and only if $\forall\, x \in \operatorname{Supp}(\mathcal{G})$, $\operatorname{depth}(\mathcal{G}_x) = \dim(\mathcal{G}_x)$. As $\mathcal{L}^\bullet$ has length 3, for every $x \in \operatorname{Supp}(\mathcal{G})$, $pd(\mathcal{G}_x) \leq 3$. Applying Auslander-Buchsbaum formula, we get

$$\operatorname{depth}(\mathcal{G}_x) \geq \dim(\mathcal{O}_{\mathbb{P}^4 x}) - 3 = 1 = \dim(\mathcal{G}_x),$$

hence $\operatorname{depth}(\mathcal{G}_x) = \dim(\mathcal{G}_x)$. Thus $\mathcal{G}$ is ACM.
Let's now show that $\mathcal{G}$ is skew-symmetric if the minimal resolution (4.5) satisfies $\beta^T = -\beta$ and $\gamma = \alpha^T$. Under these assumptions, the resolution $\mathcal{L}^\bullet$ is a self-dual complex, up to a twist and a shift by -3: $\mathcal{L}^\bullet \simeq (\mathcal{L}^\bullet)^\vee(-3)[-3]$. In our particular case, the following commutative diagram defines an isomorphism of complexes $\phi : \mathcal{L}^\bullet \xrightarrow{\sim} (\mathcal{L}^\bullet)^\vee(-3)[-3]$:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} & \xrightarrow{\gamma} & \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 6} & \xrightarrow{\beta} & \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} & \xrightarrow{\alpha} & \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} & \longrightarrow 0 \\
& \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle -Id} & & \downarrow{\scriptstyle -Id} & \\
0 \longrightarrow & \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} & \xrightarrow{\gamma} & \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 6} & \xrightarrow{-\beta} & \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} & \xrightarrow{\alpha} & \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} & \longrightarrow 0
\end{array} \tag{4.6}$$

ϕ induces an isomorphism ϕ_0 on the 0-th cohomology of the complexes:

$$\phi_0 : \mathcal{G} \xrightarrow{\sim} \mathscr{E}xt^3(\mathcal{G}, \mathcal{O}_{\mathbb{P}^4}(-3)).$$

Applying the functor $\mathscr{H}om(\ \cdot\ , \mathcal{O}_{\mathbb{P}^4}(-3))$ to the diagram 4.6 we get $\phi_0^T = -\phi_0$ $\square$

Remark 27. From the short exact sequence 4.4 we observe that each sheaf $\mathcal{F}$ such that $[\mathcal{F}] \in \mathcal{B}_X$ is obtained as the *left mutation through* $\mathcal{O}_X$ of the complex $\mathcal{G}[-1]$. This means that $\mathcal{F}$ is the cone of the evaluation map $Ext^\bullet(\mathcal{O}_X, \mathcal{G}[-1]) \otimes \mathcal{O}_X \to \mathcal{G}[-1]$ and therefore fits in a distinguished triangle:

$$Ext^\bullet(\mathcal{O}_X, \mathcal{G}[-1]) \otimes \mathcal{O}_X \to \mathcal{G}[-1] \to \mathcal{F} \tag{4.7}$$

(from which we get 4.4). By abuse of language from now on we will simply say that $\mathcal{F}$ is obtained by mutation of the sheaf $\mathcal{G}$.

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In Chapter 1, section 1.2.1 we showed how to obtain, starting from the complex 4.5, a minimal free resolution of $\mathcal{F}$. Such a free resolution is a complex of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} \xrightarrow{G} \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 6} \xrightarrow{B} \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} \to \mathcal{F} \to 0, \tag{4.8}$$

in which $B = (\beta'|\beta)$ is a 6-by-8 matrix obtained by concatenation of a 6-by-2 matrix of quadratic forms β' with the matrix β and $G = 0 \oplus \gamma$. We recall that, more precisely, we have the following:

• $[\mathcal{F}] \in \boldsymbol{B'_X}$**.** When $[\mathcal{F}]$ belongs to the divisor $\mathcal{B}'_X$, $\mathcal{F} = \mathcal{F}_C$ is a stable sheaf associated to a smooth conic C. Choosing coordinates in such a way that C has equations $\{X_3 = 0, X_4 = 0, X_1^2 - X_0X_2 = 0\}$, we have:

$$\alpha = \begin{pmatrix} -X_0 & -X_1 & 0 & -X_3 & 0 & -X_4 \\ -X_1 & -X_2 & X_3 & 0 & X_4 & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0 & 0 & 0 & X_4 & 0 & -X_3 \\ 0 & 0 & -X_4 & 0 & X_3 & 0 \\ 0 & X_4 & 0 & 0 & X_2 & -X_1 \\ -X_4 & 0 & 0 & 0 & -X_1 & X_0 \\ 0 & -X_3 & -X_2 & X_1 & 0 & 0 \\ X_3 & 0 & X_1 & -X_0 & 0 & 0 \end{pmatrix} \tag{4.9}$$

• $[\mathcal{F}] \in \boldsymbol{\mathcal{B}''_X}$**.** A *general* sheaf $\mathcal{F}$ corresponding to a point $[\mathcal{F}] \in \mathcal{B}''_X$, is the direct sum of the ideal sheaves of two distinct lines l_1 and l_2 contained in X. Choosing coordinates in such a way that l_1 is defined by the equations $\{X_0 = 0,\ X_1 = 0,\ X_2 = 0\}$ and l_2 is defined by $\{X_2 = 0,\ X_3 = 0,\ X_4 = 0\}$ we have:

$$\alpha = \begin{pmatrix} X_0 & X_1 & X_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_2 & X_3 & X_4 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0 & X_2 & -X_1 & 0 & 0 & 0 \\ -X_2 & 0 & X_0 & 0 & 0 & 0 \\ X_1 & -X_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_4 & -X_3 \\ 0 & 0 & 0 & -X_4 & 0 & X_2 \\ 0 & 0 & 0 & X_3 & -X_2 & 0 \end{pmatrix} \tag{4.10}$$

Keeping the notations adopted in the previous chapters, we consider $W,\ V$ two complex vector spaces of dimension 6 and 5 respectively, we denote by $\mathcal{P}$ the projective space $\mathbb{P}(\bigwedge^2 W^* \otimes V^*)$ and by $\mathfrak{P}$ the GIT quotient $\mathcal{P}^{ss} /\!\!/ SL(W)$. We call $\mathcal{Z}$ (resp. Π) the locus of matrices $M \in \bigwedge^2 W^* \otimes V^*$ (resp. $M \in \mathcal{P}$) such that $\mathrm{Pf}(M) = 0$ (in other words $\mathcal{Z}$ is the affine cone over Π). The matrices β appearing in the resolution 4.8 of $\mathcal{F}$ (or equivalently in the resolution 4.5 of $\mathcal{G}$) individuate points in $\bigwedge^2 W^* \otimes V^*$. Moreover, we proved in chapter 3, Theorem 3.2.4 that these are exactly the only stable points in the locus $\mathcal{Z}$ (recall that stability is equally defined on $\bigwedge^2 W^* \otimes V^*$ and on $\mathcal{P}$). This fact implies that their orbit define points in $\mathfrak{P}$. We are now going to study the behavior of $\bigwedge^2 W^* \otimes V^*, \mathcal{Z}$ and $\mathfrak{P}$ at β.

Remark 28. As the study of the orbit of β is made with the help of Macaulay 2 [M2] it is now convenient, unlike Chapter 3, to work on the affine space $\bigwedge^2 W^* \otimes V^*$.

4.1.1 Local study of $\mathfrak{P}$

We start to treat the case where β is associated to a sheaf $\mathcal{F} = \mathcal{F}_C$ corresponding to a point in $\mathcal{B}'_X$. Consider the diagram

$$\begin{array}{ccc} \mathcal{Z}'^{47} & \hookrightarrow & \bigwedge^2 W^* \otimes V^* \\ \downarrow & & \downarrow \\ \mathfrak{B}'^{11} & \hookrightarrow & \mathfrak{P}^{39} \end{array} \tag{4.11}$$

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in which the superscripts denote the dimension of each variety. $\mathcal{Z}'$ is the component of the locus of matrices of generic rank 4 of the same form as β (namely arising from skew-symmetric resolutions of duals of theta characteristics on smooth conics). The vertical arrows in 4.12 are the quotient maps by the action of $GL(6) = GL(W)$. We made them dashed, because the quotient maps are rational ones: they are defined only on the semistable loci. The variety $\mathfrak{B}'$, was introduced in Chapter 1, stating theorem 1.2.1; the dimension of $\mathfrak{P}$, equal to 39, was determined there; the dimensions of $\mathcal{Z}', \mathfrak{B}'$ follow from the following lemma.

Lemma 4.1.2. *Let $B = (\beta'|\beta)$ be the matrix appearing in the minimal free resolution* (4.8) *of $\mathcal{F}_C$ with β of the form* (4.9)*. Then, in a neighborhood of β, the vertical arrows in the diagram* (4.12) *are principal bundles for the group $GL(W)/\{\pm Id_6\}$, the local dimensions of the four varieties at β are as indicated on the diagram, and the four varieties are smooth at β.*

Proof. The first computation we do is the verification that the stabilizer in $GL(W)$ of the special matrix β in the form (4.9) is $\{\pm Id_6\}$, that is the same as for a generic point in $\bigwedge^2 W^* \otimes V^*$.This implies that β is a stable point of the $SL(W)$ action (as we already knew by Theorem 3.2.4)) and that the vertical maps of the diagram are principal bundles for the group $GL(W)/\{\pm Id_6\}$ in a neighborhood of β.
Next we verify that the tangent space to $\mathcal{Z}'$ at β is of dimension 47. This follows from a linear algebra computation: the vanishing of the 35 coefficients of $\mathrm{Pf}(\beta)$ for $\beta \in \bigwedge^2 W^* \otimes V^*$ is a system of 35 cubic equations in 75 variables, which are coordinates on $\bigwedge^2 W^* \otimes V^*$; we ask Macaulay2 to compute the rank of the Jacobian matrix at β, of size 35×75, and it turns out that the rank is 28. Hence $\dim T_\beta \mathcal{Z} = 75 - 28 = 47$. We conclude that $\dim_\beta \mathcal{Z}' \leq 47$, and the equality holds if and only if $\mathcal{Z}'$ is smooth at β. To prove the smoothness of $\mathcal{Z}'$, it is enough to show that $\dim_\beta \mathcal{Z}' \geq 47$ by producing a 47-dimensional family of distinct elements in a neighborhood of β. A 36 dimensional family "arises" from the $GL(W)$-action as follows. The fact that, by stability, the stabilizer of β is finite, implies that the differential of the "orbit map" $\theta_\beta : GL(W) \to \bigwedge^2 W^* \otimes V^*,\ M \mapsto M \cdot \beta$, computed at the identity, will be an injective linear map:

$$d_{Id}\theta_\beta : \mathfrak{gl}(W) \to T_\beta \bigwedge^2 W^* \otimes V^*.$$

We then obtain a 36-dimensional family of independent tangent directions and consequently a 36-dimensional family of deformations of β in $\mathcal{Z}'$, of the form

$$exp(tv) \cdot \beta,\ v \in \mathfrak{gl}(W).$$

We now analyze the $GL(V)$ action. We denote by $GL(V)_\beta$ the stabilizer of β; using M2 we can check that the dimension of its Lie algebra is 0 and thus that $GL(V)_\beta$ is zero-dimensional (once again this means that the differential of the "orbit map" $\tau_\beta : GL(V) \to \bigwedge^2 W^* \otimes V^*$, $N \mapsto N \cdot \beta$ at the identity is injective, providing a 25 dimensional family of deformations of β inside $\mathcal{Z}'$). Still using Macaulay, we can compute that $\langle d_{Id}\theta_\beta, d_{Id}\tau_\beta \rangle$ is a vector subspace of $T_\beta \bigwedge^2 W^* \otimes V^*$ of dimension 47. This means that $\mathcal{Z}'$ is smooth at β and that in this point it's isomorphic to the $GL(W) \times GL(V)$ orbit. This implies that $\mathfrak{B}'$ is smooth of dimension 11 at (the orbit of) β □

We pass now to the case where $\beta \in \mathcal{Z}$ is a matrix occurring in the minimal free resolution of a sheaf $\mathcal{F}_{l_1,l_2} = \mathcal{I}_{l_1} \oplus \mathcal{I}_{l_2}$, l_1 and l_2 being two disjoint lines contained in X. In this circumstance we have a diagram:

$$\begin{array}{ccc} \mathcal{Z}''^{48} & \hookrightarrow & \bigwedge^2 W^* \otimes V^* \\ \downarrow & & \downarrow \\ \mathfrak{B}''^{12} & \hookrightarrow & \mathfrak{P}^{39} \end{array} \tag{4.12}$$

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where $\mathcal{Z}''$ is the component of the locus of matrices of generic rank 4 having the same form of β (that is, arising from skew-symmetric resolutions of direct sum of structure sheaves of disjoint lines). We have the following:

Lemma 4.1.3. *Let $B = (\beta'|\beta)$ be the matrix appearing in the minimal free resolution (4.8) of $\mathcal{F}_{l_1,l_2}$ with β of the form (4.10). Then the local dimensions of the four varieties at β are as indicated on the diagram, and $\mathcal{Z}''$,$\mathfrak{B}''$ are smooth at β.*

Proof. The proof of the smoothness of $\mathcal{Z}''$ and $\mathfrak{B}''$ at β is almost identical to the case of sheaves associated to conics. We start indeed writing β in the form (4.10); we know by Theorem 3.2.4 that β is stable and that thus its stabilizer (with respect to the $GL(W)$ action) is finite. However we can see immediately that it can't consists just of $\{\pm Id_6\}$. Indeed from the block structure of β its stabilizer contains at least four elements: $\{\pm Id_3, \pm Id_3\}$. With the aid of Macaulay we compute that these are indeed all the elements in the stabilizer. Next we calculate that the rank of the Jacobian of Pf at β is 27, implying that the tangent space to $\mathcal{Z}''$ at β is of dimension $75 - 27 = 48$. Again we provide a 48 dimensional family of deformations of β proving smoothness of $\mathcal{Z}''$, and consequently of $\mathfrak{B}''$ at β. Applying the same argument of the proof of lemma 4.1.2, we try to determine then the dimension of the $GL(W) \times GL(V)$ orbit of β. Still denoting by $\theta_\beta : GL(W) \to \bigwedge^2 W^* \otimes V^*$ and $\tau_\beta : GL(V) \to \bigwedge^2 W^* \otimes V^*$ the orbit maps, we use Macaulay to compute that $\langle d_{Id}\theta_\beta, d_{Id}\tau_\beta\rangle$ is a vector subspace of $T_\beta \bigwedge^2 W^* \otimes V^*$ of dimension 48. This means that $\mathcal{Z}''$ is smooth at β and that in this point it's isomorphic to the $GL(W)\times GL(V)$ orbit. This also implies that $\mathfrak{B}''$ is smooth, of dimension 12 at β. Note that this time, from the fact the stabilizer in $GL(W)$ of β is different from $\{\pm Id_6\}$, we don't have the smoothness of $\mathfrak{P}$ at β. Indeed in this circumstance, the quotient map by the action of $GL(W)$ is not a principal bundle near $\mathcal{Z}''$, but a quotient of a principal bundle by an involution, the fiber over the points of $\mathcal{Z}''$ being $GL(W)/\{\pm Id_3, \pm Id_3\}$. □

4.2 Deformations of the resolutions

Let X_0 be a smooth cubic threefold, $\mathcal{F}_0$ a sheaf on X_0 corresponding to a point in $\mathcal{B}_{X_0}$, and $\mathcal{R}_0^\bullet$ a minimal free resolution of $\mathcal{F}_0$ of the form (4.8). Denote by $B_0 = (\beta_0'|\beta_0)$ and G_0 the differentials of the complex $\mathcal{R}_0^\bullet$. Our aim is to study how this complex behaves under deformation of $\mathcal{F}_0$. More precisely we consider the presentation map $B_0 = (\beta_0'|\beta_0)$ in $\mathcal{R}_0^\bullet$; B_0 has a skew-symmetric 6×6 block β_0, (recall that β_0 arises as the middle map of a minimal free resolution $\mathcal{L}_0^\bullet$ of a skew-symmetric 1 dimensional ACM sheaf $\mathcal{G}_0$). We want to check the stability of the skew-symmetry of the right 6×6 block under deformations. Indeed every deformation of $\mathcal{F}_0$ lifts to a deformation $\mathcal{R}^\bullet$ of the complex $\mathcal{R}_0^\bullet$. Calling G and B the differentials of the complex $\mathcal{R}^\bullet$, we see that G and B are deformations of the morphisms G_0 and B_0.

We explain now better what we mean when we talk about deformations of morphisms of vector bundles. Let R be the ring of polynomials $\mathbb{C}[X_0, \ldots X_4]$ and R_d the vector space of homogeneous polynomials of degree d. Call $\mathrm{Mat}_{m,n}(R)$ (resp. $\mathrm{Mat}_{m,n}(R_d)$) the ring of matrices of size $m \times n$ with entries in R (resp. with entries in R_d). A morphism of vector bundles of the form:

$$\mathcal{O}_{\mathbb{P}^4}^{\oplus n}(k) \to \mathcal{O}_{\mathbb{P}^4}^{\oplus m}(k + d)$$

is individuated by a matrix $M_0 \in \mathrm{Mat}_{m,n}(R_d)$. Let now $\mathbb{C}\{s\}$ be the ring of analytic functions in s at 0, $\tilde{R} = \mathbb{C}\{s\}\otimes_{\mathbb{C}} R$ and $\tilde{R}_d = \mathbb{C}\{s\}\otimes_{\mathbb{C}} R_d$. A deformation $M = M(s)$ of M_0 (or equivalently of the morphism that it defines) is an element belonging to $\mathrm{Mat}_{m,n}(\tilde{R}_d)$ (in other words we have to think that we are tracing an analytic arc $s \mapsto M_s := M(s)$ passing through M_0 at $s = 0$.) . One can then represents M in the form:

$$M(s) = \sum_{i \geq 0} M_i s^i \ \in Mat_{m,n}(\tilde{R}_d).$$

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The morphism B in $\mathcal{R}^\bullet$ is thus of the form $B = (\beta'|\beta)$ with $\beta \in \mathrm{Mat}_{6,6}(\tilde{R}_1)$. Our goal is to determine when it is possible to lift $\mathcal{R}_0^\bullet$ to $\mathcal{R}^\bullet$ in such a way that the map β is actually contained $Alt_{6,6}(\tilde{R}_1)$, the space parameterizing skew-symmetric matrices of size 6 with entries in $\tilde{R}_1$ (namely we want $\beta_s := \beta(s)$ to be skew-symmetric $\forall\, s$). This is straightforward when the singularity of the sheaf is preserved:

Proposition 4.2.1. *Let Δ be a polydisk in $\mathbb{C}^N$ and $\mathcal{F}$ a sheaf on $\mathbb{P}^4 \times \Delta$, flat over Δ; denote by $\mathcal{F}_s$ the restriction of $\mathcal{F}$ to $\mathbb{P}^4 \times s$, $s \in \Delta$. Assume that for each $s \in \Delta$, $\mathcal{F}_s$ is a sheaf supported on a smooth cubic 3-fold X_s corresponding to a point $[\mathcal{F}_s] \in \mathcal{B}_{X_s}$ that furthermore can be obtained as the mutation of a skew-symmetric 1-dimensional ACM sheaf $\mathcal{G}_s$ with smooth support (that is, in case $[\mathcal{F}_s] \in \mathcal{B}''_{X_s}$, the support should be the union of two disjoint lines). Then, possibly after shrinking Δ, the following assertions hold:*
1. Any skew-symmetric minimal free resolution $\mathcal{L}_0^\bullet$ of $\mathcal{G}_0$ extends to a complex of analytic sheaves $\mathcal{L}^\bullet$ on $\mathbb{P}^4 \times \Delta$ of the form

$$0 \to \mathcal{O}_{\mathbb{P}^4\times\Delta}(-3)^2 \xrightarrow{\gamma} \mathcal{O}_{\mathbb{P}^4\times\Delta}(-2)^6 \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^4\times\Delta}(-1)^6 \xrightarrow{\alpha} \mathcal{O}^2_{\mathbb{P}^4\times\Delta} \to 0, \tag{4.13}$$

whose maps satisfy the skew symmetry conditions $\beta = -\beta^T$, $\gamma = \alpha^T$, and such that for each $s \in \Delta$, the restriction $\mathcal{L}_s^\bullet$ of $\mathcal{L}^\bullet$ to $\mathbb{P}^4 \times s$ is a skew-symmetric minimal free resolution of $\mathcal{G}_s$.
2. The minimal free resolution $\mathcal{R}_0^\bullet$ of $\mathcal{F}_0$ of the form (4.8) with presentation map $B_0 = (\beta_0'|\beta_0)$, where β_0 is the first syzygy map of the resolution $\mathcal{L}_0^\bullet$, extends to a complex of analytic sheaves $\mathcal{R}^\bullet$ of the form

$$0 \to \mathcal{O}(-3)^{\oplus 2}_{\mathbb{P}^4\times\Delta} \xrightarrow{G} \mathcal{O}_{\mathbb{P}^4\times\Delta}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4\times\Delta}(-2)^{\oplus 6} \xrightarrow{B} \mathcal{O}_{\mathbb{P}^4\times\Delta}(-1)^{\oplus 6} \to 0, \tag{4.14}$$

such that $B = (\beta'|\beta)$ with β the skew-symmetric first syzygy map in $\mathcal{L}^\bullet$ and for each $s \in \Delta$ the restriction $\mathcal{R}_s^\bullet$ of $\mathcal{R}^\bullet$ to $\mathbb{P}^4 \times s$ is a minimal free resolution of $\mathcal{F}_s$.

Proof. Associated to the analytic family of sheaves $\mathcal{F}_s$ is $\mathcal{X}$, the analytic family of their supports; so the cubics X_s are given by a cubic polynomial equation $F(s) = 0$ whose coefficients are analytic functions in $s \in \Delta$. Similarly the family $\mathcal{C}$, the family of the curves $C_s = \mathrm{Supp}\,\mathcal{G}_s$, is analytic, for it is the locus where $\mathcal{F}$ is non-locally-free. From our hypotheses it follows that $\mathcal{C}$ is a family of conics (whenever $[\mathcal{F}_s] \in \mathcal{B}'_{X_s}$ $\forall s \in \Delta$) or of pairs of disjoint lines (whenever $[\mathcal{F}_s] \in \mathcal{B}''_{X_s}$ $\forall s \in \Delta$). Suppose that the central fiber $\mathcal{F}_0$ corresponds to a point $[\mathcal{F}_0] \in \mathcal{B}'_{X_0}$. In the previous section we presented an explicit construction (4.9) of the resolution $\mathcal{L}_0^\bullet$, starting from the equations of the conic C_0 in the form $X_1^2 - X_0X_2 = X_3 = X_4 = 0$. It is easy to see that given an analytic family of conics C_s extending the given conic C_0, one can find linear forms $\ell_i = \ell_i(s)(X_0, \dots, X_4)$ in X_j with coefficients, depending analytically on s, such that the conic C_s is given by the equations $\ell_1^2 - \ell_0\ell_2 = \ell_3 = \ell_4 = 0$. Then the wanted skew-symmetric resolution of $\mathcal{G}_s$ is obtained by replacing X_i by ℓ_i in the formulas defining the maps in $\mathcal{L}_0^\bullet$.
Now the sheaf $\mathcal{F}$, viewed as a sheaf on the family of cubic threefolds $\mathcal{X} \to \Delta$, embeds in its reflexive hull $\mathcal{F}^{\vee\vee}$, and for $s = 0$, we have $\mathcal{F}_0^{\vee\vee} \simeq \mathcal{O}_{X_0}^{\oplus 2}$ and $\mathcal{F}_0^{\vee\vee}/\mathcal{F}_0 \simeq \mathcal{G}_0$ is the cokernel of this inclusion. This implies that $\mathcal{F}^{\vee\vee} \simeq \mathcal{O}_{\mathcal{X}}^{\oplus 2}$ in a neighborhood of the fiber X_0, hence we obtain the exact triple

$$0 \to \mathcal{F} \to \mathcal{O}_{\mathcal{X}}^{\oplus 2} \to \mathcal{G} \to 0.$$

The surjection of this exact triple lifts to a morphism from the natural resolution of $O_{\mathcal{X}}^{\oplus 2}$ to the resolution $\mathcal{L}^\bullet$ of $\mathcal{G}$. This gives rise to a diagram of the form (1.16) over $\mathbb{P}^4 \times \Delta$. By an easy diagram chase we deduce a resolution of the form $\mathcal{R}^\bullet$ with the wanted skew-symmetry property for the right 6×6 bloc of the presentation map B.
The argument for the case of a pair of disjoint lines is completely similar. $\square$

This argument does not work when the singularity of the sheaf is smoothed by deformation, that is when a non-locally-free sheaf from the boundary of the moduli space deforms to an instanton. In this case we provide another proof.

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Proposition 4.2.2. *Let Δ be a polydisk in $\mathbb{C}^N$ and $\mathcal{F}$ a sheaf on $\mathbb{P}^4 \times \Delta$, flat over Δ. Denote by $\mathcal{F}_s$ the restriction of $\mathcal{F}$ to $\mathbb{P}^4 \times s$, $s \in \Delta$, and assume that for each $s \in \Delta$, $\mathcal{F}_s$ is a sheaf supported on a smooth cubic 3-fold X_s. Suppose that for $s = 0$, $\mathcal{F}_0$ is a sheaf corresponding to a point $[\mathcal{F}_0] \in \mathcal{B}_{X_0}$, mutation of a skew-symmetric 1-dimensional ACM sheaf $\mathcal{G}_0$ with smooth support (that is, in case $[\mathcal{F}_0] \in \mathcal{B}''_{X_0}$, the support should be the union of two disjoint lines), and that for generic $s \in \Delta$, $\mathcal{F}_s$ is an instanton on X_s. Let $\mathcal{R}_0^\bullet$ be a minimal free resolution of $\mathcal{F}_0$ of the form* (4.8)*, with its two maps denoted B_0, G_0 such that the block β_0 of the presentation map $B_0 = (\beta_0'|\beta_0)$ is skew-symmetric and $G_0 = \left(\begin{smallmatrix}0\\ \gamma_0\end{smallmatrix}\right)$ with zero block of size 2×2 and γ_0 of size 6×2. Then, possibly after shrinking Δ (for both $[\mathcal{F}_0] \in \mathcal{B}'_{X_0}$ or $[\mathcal{F}_0] \in \mathcal{B}''_{X_0}$) and after pulling back to a double covering of Δ (only whenever $[\mathcal{F}_0] \in \mathcal{B}''_{X_0}$), $\mathcal{R}_0^\bullet$ extends to a resolution of the form* (4.14) *over $\mathbb{P}^4 \times \Delta$ with the same property of the presentation map B: its right block β of size 6×6 is skew-symmetric.*

Proof. Starting with a minimal free resolution $\mathcal{R}_0^\bullet$, we can lift it to a minimal free resolution $\mathcal{R}^\bullet = (R_2 \xrightarrow{G} R_1 \xrightarrow{B} R_0)$ of the form (4.14) over $\mathbb{P}^4 \times \Delta$. The minimality condition implies, in particular, that the map between summands of the terms of the resolution of the same degree is zero, so $G = \left(\begin{smallmatrix}0\\ \gamma\end{smallmatrix}\right)$ and $B = (\beta'|\beta)$, where β', β, γ are respectively some extensions of $\beta_0', \beta_0, \gamma_0$ over Δ. We want to show that β can be made skew-symmetric. As in the previous lemma, we observe that the supports of the sheaves $\mathcal{F}_s$ describe an analytic family $\mathcal{X}$ of smooth cubic threefolds X_s over Δ. This time we have that for generic $s \in \Delta$, $\mathcal{F}_s = \mathcal{F}|_{X_s}$ is a locally free sheaf and that the locus of points $s \in \Delta$ for which $\mathcal{F}_s$ is non-locally-free is a divisor Ξ in Δ, containing 0. For $s \notin \Xi$, the restriction $\mathcal{R}_s^\bullet$ is not minimal, for we know by [B2] that a minimal resolution is of length 1 and can be given by a 6×6 skew-symmetric matrix M_s of linear forms on $\mathbb{P}^4$. A minimal free resolution, say $\mathcal{K}_s^\bullet$, embeds as a direct summand in a non-minimal one, so there is a morphism of complexes giving this embedding:

$$\begin{array}{llcccccccc}
\mathcal{K}_s^\bullet : & & 0 \longrightarrow & \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 6} & \xrightarrow{M_s} & \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} & \xrightarrow{p} & \mathcal{F}_s & \longrightarrow 0 \\
 & & & \downarrow & & \downarrow & & \downarrow \mathrm{id} & \\
\mathcal{R}_s^\bullet : & 0 \longrightarrow R_2 & \xrightarrow{G_s} & R_1 & \xrightarrow{B_s} & R_0 & \xrightarrow{q} & \mathcal{F}_s & \longrightarrow 0
\end{array} \tag{4.15}$$

This implies, in particular, that the 6×6 block β_s of B_s is generically of rank 6 for every $s \notin \Xi$ and that the support X_s of $\mathcal{F}_s$ is the zero locus of the determinant of β_s, which is a cubic form squared. In this argument we have no control of the dependence of M_s on s. So we explain how one can skew-symmetrize β_s by a change analytic in s.
We have shown that for $s \notin \Xi$, we have the following resolution for $\mathcal{F}_s$:

$$\mathcal{K}_s{}^\bullet : \qquad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 6} \xrightarrow{\beta_s} \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} \xrightarrow{p} \mathcal{F}_s \longrightarrow 0.$$

The sheaf $\mathcal{F}_s$ is skew-symmetric, that is there is an isomorphism $k_s : \mathcal{F}_s \to \mathcal{F}_s^\vee$, such that $k_s^\vee : \mathcal{F}_s = \mathcal{F}_s^{\vee\vee} \to \mathcal{F}_s^\vee$ equals $-k_s$ under the canonical identification of $\mathcal{F}_s^{\vee\vee}$ with $\mathcal{F}_s$. Here the duality functor is $\mathscr{H}om_{\mathcal{O}_{X_s}}(\,\cdot, \mathcal{O}_{X_s})$. As in [B2], we lift both isomorphisms to the resolutions and obtain the diagram

$$\begin{array}{lccccccc}
\mathcal{K}^\bullet : & 0 \longrightarrow & K_1 & \xrightarrow{\beta_s} & K_0 & \xrightarrow{p} & \mathcal{F} & \longrightarrow 0 \\
 & & a_0^\vee \Big\downarrow\Big\downarrow a_1 & & a_1^\vee \Big\downarrow\Big\downarrow a_0 & & k_s^\vee \Big\downarrow\Big\downarrow k_s = -k_s^\vee & \\
\mathcal{K}^{\vee\bullet} : & 0 \longrightarrow & K_0^\vee & \xrightarrow{\beta_s^\vee} & K_1^\vee & \xrightarrow{q} & \mathcal{F}_s^\vee & \longrightarrow 0
\end{array} \tag{4.16}$$

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with exact rows, in which the squares with both right hand side (resp. left hand side) vertical arrows are all commutative; $\mathcal{K}^{\vee\bullet}$ denotes the complex $\mathscr{H}om_{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{K}^\bullet, \mathcal{O}_{\mathbb{P}^4}(-3))[1]$. The commutativity and the relation $k_s + k_s^\vee = 0$ imply that $0 = (k_s + k_s^\vee)p = q(a_0 + a_1^\vee)$. The map $a_0 + a_1^\vee : \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6}$ is given by a constant matrix, and the fact that $\ker q$ has no global sections immediately implies that the matrix is zero, hence $a_0 = -a_1^\vee$. Now we define $M_s = a_0\beta_s$. We have $M_s^\vee = \beta_s^\vee a_0^\vee = -\beta_s^\vee a_1 = -a_0\beta_s = -M_s$, thus $M_s : K_1 \to K_1^\vee$ is a skew-symmetric presentation of $\mathcal{F}_s$. The formula for M_s is analytic in s.
In the above procedure for skew-symmetrization of β, we assumed that $s \notin \Xi$. In order to see that it extends to $s \in \Xi$, we have to produce a construction in which a_0 extends on Ξ as an invertible matrix. Let $\delta = 0$ be a local equation of Ξ near 0, and let $\nu = \min\{n \in \mathbb{Z} \mid a_0\delta^n, a_1\delta^n \in \mathrm{Mat}_{6,6}(\mathcal{O}_{\Delta,0})\}$. Then replacing a_i by $a_i\delta^\nu$, we obtain the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^4_\Delta}(-2)^{\oplus 6} & \xrightarrow{\beta} & \mathcal{O}_{\mathbb{P}^4_\Delta}(-1)^{\oplus 6} \\ a_0^\vee \Big\Downarrow a_1 & & a_1^\vee \Big\Downarrow a_0 \\ \mathcal{O}_{\mathbb{P}^4_\Delta}(-2)^{\oplus 6} & \xrightarrow{\beta^\vee} & \mathcal{O}_{\mathbb{P}^4_\Delta}(-1)^{\oplus 6} \end{array} \tag{4.17}$$

in which all the maps are regular analytic on the whole of Δ, and moreover, at least one of the maps a_0, a_1 is not vanishing on Ξ identically. By continuity, the relation $a_0 = -a_1^\vee$ that we have proved for $s \notin \Xi$ holds everywhere in Δ. If we assume that a_{0s} is a nonzero degenerate matrix at some point $s \in \Xi$, then β_s has a diagonal block structure with blocs of size r and $6 - r$, where $r = \mathrm{rk}\, a_{0s}$, the blokc of size r being skew-symmetric. The existence of a block decomposition of β_0 is detected by the jump of the stabilizer in $GL(W) = GL(6)$:

- Whenever $[\mathcal{F}_0] \in \mathcal{B}'_{X_0}$, the stabilizer is $\pm Id_6$, (see Lemma 4.1.2) so a decomposition in blocks is impossible. This proves the invertibility of a_0 in a neighborhood of $s = 0$, which implies the possibility of the global skew-symmetrization of the map β.

- Whenever $[\mathcal{F}_0] \in \mathcal{B}''_{X_0}$, there is a decomposition of β_0 in two blocks of size 3. This does not prevent β_0 from being stable, as we proved in Theorem 3.2.4, but the stabilizer jumps and is of order 4: $\{\pm Id_3, \pm Id_3\}$ (see Lemma 4.1.3). The fact that the stabilizer is of order 4 and not bigger implies that the decomposition in blocks is unique. Assume that in our resolution $\mathcal{R}^\bullet$, the map β is a direct sum of two skew-symmetric blocks of size 3. We can always obtain such a resolution by applying the construction from Section 1.2.1 to a direct sum of two skew-symmetric resolutions of two varying families of lines $l_1(s), l_2(s)$, $s \in \Xi$, and afterwards deforming the resolution over Ξ to an outer direction.

  Let us now work over the analytic local ring $\mathfrak{O}$ of meromorphic functions on Δ without pole on Ξ, with local parameter δ. By base change in $GL(6, \mathfrak{O})$, we can reduce a_0 to a diagonal form $\mathrm{diag}(\delta^{\nu_1}, \ldots, \delta^{\nu_6})$, $\nu_1 \leq \ldots \leq \nu_6$; moreover up to an homothety base change on K_0, we case suppose that $\nu_1 = \nu_2 = \nu_3 = 0 < \nu_4 \leq \nu_5 \leq \nu_6$. Then on Ξ we have $\beta = \alpha' \oplus \alpha''$, the direct sum of two blocks of size 3, with α' skew-symmetric. Let us write down β, $a := a_0$, $M = a_0\beta$ in blocks of size 3 and expand them in powers of δ, $M = \sum_i M^{(i)}\delta^i$, $a_0 = \sum_i a_0^{(i)}\delta^i$, $\beta = \sum_i \beta^{(i)}\delta^i$. The terms of order 0 are:

$$M^{(0)} = \left(\begin{array}{c|c} \alpha' & 0 \\ \hline 0 & 0 \end{array}\right), a_0^{(0)} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right), \beta^{(0)} = \left(\begin{array}{c|c} \alpha' & 0 \\ \hline 0 & \alpha'' \end{array}\right).$$

  Then taking into account the skew-symmetry $M^T = -M$ and looking for the lowest order terms, we find that the expansion the upper-right block of β starts in order

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$\nu_4 + 1$ in δ, with the initial term

$$\beta_{12}^{(\nu_4+1)} = -(a_{22}^{(\nu_4)}\beta_{21}^{(1)})^T,$$

and the expansion of the bloc M_{22} then has its lower order term in degree ν_4. The skew symmetry relation for M the implies:

$$a_{22}^{(\nu_4)}\alpha'' = -(a_{22}^{(\nu_4)}\alpha'')^T.$$

This relation is incompatible with the explicit expression we have for α'' unless $\nu_4 = \nu_5 = \nu_6 = \nu$ and $a_{22} = \delta^\nu \mathrm{id}_3$. Thus M acquires the form

$$M = \left(\begin{array}{c|c} \alpha' & \delta^{\nu+1}\epsilon \\ \hline -\delta^{\nu+1}\epsilon & \delta^\nu\alpha'' \end{array}\right).$$

If ν is even, by a base change $\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \delta^{\nu/2} \end{array}\right)$ we reduce M the diagonal form $\tilde{M}$

$$\tilde{M} = \left(\begin{array}{c|c} \alpha' & \delta\tilde{\epsilon} \\ \hline -\delta\tilde{\epsilon} & \alpha'' \end{array}\right),$$

and if ν is odd, we can lift to a double covering $t^2 = \delta$, which brings us to the case of even ν.

□

In the next section we apply proposition 4.2.2 to study the deformations of a sheaf $\mathcal{F}_0$ such that $[\mathcal{F}_0] \in \mathcal{B}'_{X_0}$. Doing so we are able to describe the behavior of the rational map $\bar{\tau} : \mathfrak{P} \dashrightarrow \mathfrak{M}$ in a neighborhood of the point $[\beta_0] \in \mathfrak{P}$, corresponding to the 6×6 skew-symmetric matrix β_0 appearing in a minimal free resolution of $\mathcal{F}_0$ of the form (4.8).

4.2.1 Sheaves associated to conics

We consider a smooth cubic threeofold X_0 defined by a cubic polynomial F_0 and a sheaf $\mathcal{F}_{C_0}$ on X_0 corresponding to a point $[\mathcal{F}_{C_0}] \in \mathcal{B}'_{X_0}$. $\mathcal{F}_{C_0}$ is obtained as the mutation of $\mathcal{O}_{C_0}(1pt)$, the dual of a theta characteristic on a smooth conic C_0 contained in X_0. Such sheaves fill the 38-dimensional exceptional divisor $\boldsymbol{\mathcal{B}'}$ in the 39-dimensional moduli space $\mathfrak{M}$ of torsion sheaves on $\mathbb{P}^4$ compactifying the open set $\mathfrak{M}^{in}$ of instantons supported on cubic threefolds. Furthermore $\mathcal{F}_{C_0}$ is a flat limit of the instanton bundles $\mathcal{F}$ that are cokernels of Pfaffian representations of X_0:

$$i_*\mathcal{F} = \operatorname{coker}\left(\mathcal{O}(-2)^{\oplus 6} \xrightarrow{M} \mathcal{O}(-1)^{\oplus 6}\right). \tag{4.18}$$

From now on, as we have already done in the previous sections, we denote by $\mathcal{F}, \mathcal{F}_{C_0}$ sheaves on X_0 and we omit i_*, where i is the embedding $X_0 \hookrightarrow \mathbb{P}^4$, when speaking of them as sheaves on $\mathbb{P}^4$. We want to understand what happens to the Pfaffian representations (4.18) as $\mathcal{F}$ degenerates into $\mathcal{F}_{C_0}$. To this end, we will look at the local deformations of a minimal free resolution of $\mathcal{F}_{C_0}$.

Example. We can construct an explicit example of such data by choosing:

$$C_0 = \{X_1^2 - X_0X_2 = X_3 = X_4 = 0\},\ F = X_1(X_1^2 - X_0X_2) + X_3(X_0^2 + X_3^2) + X_4(X_2^2 + X_4^2),$$

The morphisms G_0, B_0 appearing in a complex of type 4.8 are then: $G_0 = 0 \oplus \alpha_0^T$, $B_0 = (\beta'_0|\beta_0)$ with α_0, β_0 of the form 4.9 and β'_0 given by:

$$\beta' = \begin{pmatrix} X_1X_2 & -X_1^2 \\ -X_1^2 & X_0X_1 \\ 0 & X_0^2 + X_3^2 \\ -X_0^2 - X_3^2 & 0 \\ 0 & X_2^2 + X_4^2 \\ -X_2^2 - X_4^2 & 0 \end{pmatrix}. \tag{4.19}$$

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As usual we consider a minimal free resolution of $\mathcal{F}_{C_0}$ of the form (4.8) and we denote by B_0 and G_0 its differentials. In the previous section we saw that a flat deformation of $\mathcal{F}_{C_0}$ as a sheaf on $\mathbb{P}^4$ can be obtained by a deformation B, $s \mapsto B(s) := B_s$, of the sheaf homomorphism B_0 in such a way that the Hilbert polynomial $P_{\operatorname{coker} B_s}$ remains constant (or equivalently in such a way that there also exists a deformation G of G_0 preserving the exactness: $\operatorname{im} G = \ker B$).

Remark 29. Writing explicitly a resolution of type (4.8) of $\mathcal{F}_{C_0}$ with differentials given by the matrices (4.9) and (4.19), we can compute, using Macaulay2, that we have:

$$\dim \operatorname{Ext}^1_{\mathbb{P}^4}(\mathcal{F}_{C_0}, \mathcal{F}_{C_0}) = 132\ , \quad \dim \operatorname{Ext}^2_{\mathbb{P}^4}(\mathcal{F}_{C_0}, \mathcal{F}_{C_0}) = 23;$$

so the deformation space (and the moduli space) of sheaves on $\mathbb{P}^4$ is of local dimension $\geq 109 = 132 - 23$ at the point $[\mathcal{F}_{C_0}]$. The dimension of the space $\mathfrak{M}$ is 39 and the sheaves of the form $\mathcal{F}_{C_0}$, which one obtains in varying the pair $C_0 \subset X_0$ of a conic contained in a cubic threefold, fill a divisorial 38-dimensional subfamily $\boldsymbol{\mathcal{B}'}$. This dimension count shows that a *generic* deformation $\mathfrak{F}$ of $\mathcal{F}_{C_0}$ cannot be supported on a cubic. By the invariance of the Hilbert polynomial of a sheaf under flat deformations and because the leading coefficient of the Hilbert polynomial is $\frac{1}{3!}$ times the degree of the support of $\mathfrak{F}$ times the rank of the sheaf as a module over it support, we conclude that the only possibility is that $\mathfrak{F}$ is supported on an irreducible sextic hypersurface. Thus the 38-dimensional locus $\mathfrak{B}'$ is contained in a component of the moduli space of sheaves, say $\mathfrak{N}$, of dimension ≥ 109, whose generic point represents a rank-1 sheaf $\mathfrak{F}$ on an irreducible sextic hypersurface. We will restrict ourselves to the study of only those deformations which preserve the degree of the support and the rank.

We described in proposition 4.2.1 the deformations of the resolutions inside the 38-dimensional locus $\boldsymbol{\mathcal{B}'}$, namely the deformations preserving the type of singularity of $\mathcal{F}_{C_0}$. We showed that, starting from a presentation of $\mathcal{F}_{C_0}$ by a matrix $B_0 = (\beta'_0|\beta_0)$ as in (4.8), these are given by the deformations of the map B_0 that lift to deformations of the whole diagram (1.16).
Deformations of $\mathcal{F}_{C_0}$ into a locally free sheaf on a smooth cubic were studied in proposition 4.2.2: again, starting from a presentation of $\mathcal{F}_{C_0}$ by a matrix $B_0 = (\beta'_0|\beta_0)$ as in (4.8), these are obtained deforming β_0 inside the skew-symmetric matrices of linear forms (we proved that this is always possible whenever $[\mathcal{F}_{C_0}] \in \mathcal{B}'_{X_0}$). (Note that from [Dr], Lemma 4.2, we have $\operatorname{Ext}^2_{X_0}(\mathcal{F}_{C_0}, \mathcal{F}_{C_0}) = 0$, implying the unobstructedness of infinitesimal deformations of $\mathcal{F}_{C_0}$, no matter whether the support X_0 of the sheaf changes under deformation or is fixed). But not any such deformation lifts to a deformation of B_0 with flat cokernel. A small deformation β of β_0 lifts to a flat deformation B of B_0 if and only if there exists a 6-by-2 matrix of quadratic forms $\beta' \in \operatorname{Mat}_{6,2}(\tilde{R}_2)$ depending on s, such that the Hilbert polynomial of coker $(\beta'_s|\beta_s)$ is constant. In particular, if the cubic Pfaffian $\operatorname{Pf}(\beta_s)$ vanishes only at $s = 0$, then $\mathcal{F}_s = \operatorname{coker} \beta_s$ already has the right Hilbert polynomial for $s \neq 0$ and adding supplementary columns to the presentation matrix should not change it, that is:

$$\operatorname{Pf}(\beta_s) \neq 0 \ \Rightarrow\ \operatorname{coker}(\beta_s) = \operatorname{coker}(\beta'_s|\beta_s) \ \Leftrightarrow\ \operatorname{rk}(\beta'_s|\beta_s)|_{X_s} \leq 4, \qquad (4.20)$$

where $X_s = \operatorname{Supp} \mathcal{F}_s$ is the cubic defined by $\operatorname{Pf}(\beta_s)$ for $s \neq 0$. (Remark that the family X_s defines a deformation of $\operatorname{Supp}(\mathcal{F}_{C_0}) = X_0$, and that $\forall\ s \neq 0$, β_s is of rank 4 on X_s)
We can characterize this condition by explicit equations. The kernel of a skew-symmetric 6-by-6 matrix M of rank 4 is generated by the columns of its Pfaffian-adjugate matrix $M^\vee$ whose (i,j)-th element is $m^\vee_{ij} = (-1)^{i+j+1+\theta(i-j)}\operatorname{Pf}(M_{ij})$, where θ denotes the Heaviside step function, M_{ij} the submatrix of M obtained by deleting the lines and columns with numbers i, j, and the Pfaffian of an odd-dimensional matrix is set to be zero. We can also write $m^\vee_{ij} = \partial\operatorname{Pf}(M)/\partial m_{ij}$ for $i < j$. The Pfaffian-adjugate matrix of a rank-4 skew-symmetric matrix of size 6 is automatically skew-symmetric and is of rank 2.

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We thus see that the condition in (4.20) is equivalent to saying that the two-dimensional space of relations between the columns (or rows) of β'_s is generated by the columns of $\beta_s^\vee$, so that we can rewrite (4.20) as follows:

$$\mathrm{Pf}\,(\beta_s) \neq 0 \implies \beta_s^\vee \cdot \beta'_s|_{X_s} = 0.$$

The equations resulting from this condition are nonlinear, so even if we start by a deformation β linear in s, the respective β'_s may be a power series in s. Remark that the tangent vector of the deformation $\partial_s\beta(0)$ may not be arbitrary matrix, for $\mathrm{Pf}\,(\partial_s\beta(0))$ should be proportional to F_0, the polynomial defining X_0. Summing up, we obtain the following proposition

Proposition 4.2.3. *Let $s \mapsto B(s) = B_s$ be an analytic arc B as above with $B(0) = B_0$, the matrix from the resolution (4.8) of the sheaf $\mathcal{F}_0 = \mathcal{F}_{C_0}$.*

1. *The family of sheaves $s \mapsto \mathcal{F}_s := \operatorname{coker} B_s$ is a flat deformation of sheaves supported on cubic hypersurfaces in $\mathbb{P}^4$ in the neighborhood of $s = 0$ if and only if*

$$\exists\ F(s) \in \tilde{R}_3\ \exists\ \tilde{M}(s) \in \mathrm{Mat}_{6,8}\big(\tilde{R}\big)\ , \text{such that } \beta(s)^\vee \cdot B(s) = F(s) \cdot \tilde{M}(s). \quad (4.21)$$

This condition implies, in particular, that

$$\mathrm{Pf}\,(\beta(s)) \sim F(s) \text{ in } \tilde{R},$$

and that the family of supports $X_s = \operatorname{Supp}\mathcal{F}_s$ is an analytic family of 3-dimensional cubics with equations $F(s) = 0$.

2. *Assume the condition (4.21) verified. Then the deformation of sheaves $s \mapsto \mathcal{F}_s := \operatorname{coker} B_s$ remains in the exceptional locus $\boldsymbol{\mathcal{B}'}$ for small s if and only if $\mathrm{Pf}\,(\beta(s)) \equiv 0$. If the Pfaffian $\mathrm{Pf}\,(\beta(s))$ is not identically zero, then $\mathcal{F}_s$ is an instanton bundle on the smooth cubic*

$$X_s = \{F(s) = 0\} = \{\mathrm{Pf}\,(\beta_s) = 0\}$$

for small $s \neq 0$.

3. *For any $n > 0$, if $B^{(n)} = B_0 + \ldots + B_n s^n$ satisfies the condition (4.21) modulo s^{n+1}, then there exists B_{n+1} such that $B^{(n+1)} := B^{(n)} + B_{n+1}s^{n+1}$ satisfies (4.21) modulo s^{n+2}. In particular, the first order infinitesimal deformations of $\mathcal{F}_0$ are given by the matrices $B_1 = (\beta'_1|\beta_1)$ for which (4.21) is verified modulo s^2, and every such infinitesimal deformation lifts to an analytic deformation, given by a matrix series $B(s)$ with coefficient of s to the power 1 equal to B_1.*

This proposition answers the question, how non-locally-free sheaves in $\mathcal{B}'_{X_0}$ deform to instantons.

Example (continued). For the example considered above, we may choose as an admissible first order deformation $B(s) = B + s \cdot \Delta B$, where we denote $B(s) = \left(\tilde{\beta}'(s)|\tilde{\beta}(s)^\vee\right)$ and set

$$\Delta B = \left(\begin{array}{cc|cccccc} -X_1X_2 & X_0X_2 & 0 & -X_1 & -X_0 & 0 & 0 & 0 \\ X_0X_2 & 0 & X_1 & 0 & 0 & 0 & 0 & -X_2 \\ 0 & 0 & X_0 & 0 & 0 & X_3 & -X_2 & 0 \\ 0 & X_2X_3 & 0 & 0 & -X_3 & 0 & 0 & 0 \\ -X_0X_4 & 0 & 0 & 0 & X_2 & 0 & 0 & X_4 \\ 0 & 0 & 0 & X_2 & 0 & 0 & -X_4 & 0 \end{array}\right).$$

Then we compute:

$$\mathrm{Pf}\,(\tilde{\beta}(s)) = -Fs + X_2^2X_4s^2 - X_1X_3X_4s^3, \quad \tilde{\beta}(s)^\vee \cdot \tilde{\beta}'(s) \equiv 0 \mod (F, s^2).$$

The computation of $B(s)$ can be continued iteratively on degree of s.

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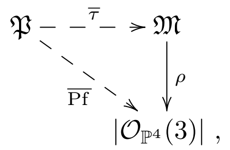
Description of the map $\bar{\tau}$ at a generic point of $\mathcal{B}'$

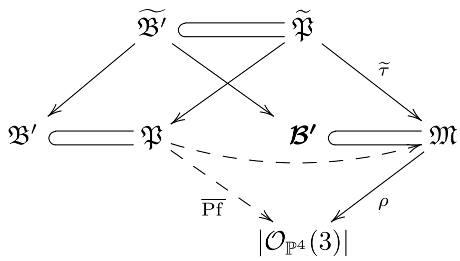
We consider the commutative diagram introduced in Chapter 1:



(4.22)

We will now show how our previous results apply to the study of the rational map $\bar{\tau} : \mathfrak{P} \dashrightarrow \mathfrak{M}$. We still consider a sheaf $\mathcal{F}_0 := \mathcal{F}_{C_0}$ on a smooth cubic threefold X_0 defined by the equation $\{F_0 = 0\}$, corresponding to a point $[\mathcal{F}_{C_0}] \in \mathcal{B}'_{X_0}$. Taking a minimal free resolution of the form (4.8), $\mathcal{F}_{C_0}$ can be described as the cokernel of a map $B_0 = (\beta'_0|\beta_0)$ with β_0 belonging to $\mathcal{Z}$, the locus of matrices in $\bigwedge^2 W^* \otimes V^*$ having Pfaffian equal to zero. The matrix β_0 is the first syzygy of the minimal free resolution of $\mathcal{O}_{C_0}(1pt)$ and for an appropriate choice of coordinates it can be written in the form (4.9) In Lemma 4.1.2 we proved that β_0 belongs to a 47-dimensional variety $\mathcal{Z}' \subset \mathcal{Z}$ and that its image $[\beta_0]$ in $\mathfrak{P}$ is a smooth point of the 11-dimensional variety $\mathfrak{B}' \subset \mathfrak{P}$, quotient of $\mathcal{Z}'$. We now study the behavior of $\bar{\tau}$ in a neighborhood of $[\beta_0]$.

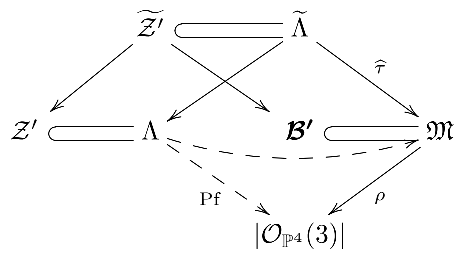
Proposition 4.2.4. *Let $\beta_0 \in \mathcal{Z}'$ be as above, and consider the diagram* (4.22) *in a neighborhood of the orbit $[\beta_0] \in \mathfrak{B}'$, $[\beta_0] = GL(W) \cdot \beta_0$. Then the rational map $\bar{\tau}$ is equivalent to a blowup with center $\mathfrak{B}'$ near $[\beta_0]$. More precisely: let $\widetilde{\mathfrak{P}}$ denote the blowup of $\mathfrak{P}$ with center $\mathfrak{B}'$ and $\widetilde{\mathfrak{B}'}$ its exceptional divisor. Then, in a neighborhood of $[\beta_0]$,* (4.22) *can be completed to the diagram*



(4.23)

in which the arrows $\widetilde{\tau}$ and $\widetilde{\tau}|_{\widetilde{\mathfrak{B}'}}$ are isomorphisms.

Proof. It follows from the Lemma 4.1.2 that to identify $\bar{\tau}$ as a blowup, we can work with $\mathcal{Z}'$ in place of $\mathfrak{B}'$. That is, we can show that there exists a diagram



(4.24)

in which Λ denotes $\bigwedge^2 W^* \otimes V^*$, $\widetilde{\Lambda}$ is the blowup of Λ along $\mathcal{Z}'$, $\widetilde{\mathcal{Z}}'$ is the exceptional divisor. Note that by Lemma 4.1.2, and $\widehat{\tau}$, $\widehat{\tau}|_{\widetilde{\mathcal{Z}}'}$ are principal $GL(W)/\{\pm Id_6\}$-bundles in a neighborhood of β_0. To start with, let us remind that according to the lemma, the rank of the Jacobian matrix of the 35 equations expressing the vanishing of the Pfaffian $\mathrm{Pf}(\beta)$ of a matrix $\beta \in \Lambda$ is equal to 28 at β_0. Since the partial derivative $\partial \mathrm{Pf}(\beta)/\partial b_{ij}$ is the element $b_{ij}^{\vee}$ of the Pfaffian-adjugate matrix $\beta^{\vee}$, and as $\beta_0^{\vee}$ vanishes on the conic C_0, all the 28 cubics in the image of the differential $\mathrm{Pf}_*(\beta_0)$ of Pf at β_0 vanish on C_0. But the whole space of cubics vanishing on C_0 is 28-dimensional. Thus Pf_* identifies the fiber

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of the normal bundle $\mathcal{N}_{\mathcal{Z}'/\Lambda}$ at β_0 with the 28-dimensional vector space of cubic forms vanishing on C_0, and the fibers of the blowdown map $\sigma : \widetilde{\Lambda} \to \Lambda$ are naturally identified with $|\mathcal{I}_{C_0/\mathbb{P}^4}(3)| \simeq \mathbb{P}^{27}$.

Now we will prove that $\widehat{\tau}$ is regular at the points of $\widetilde{\mathcal{Z}}'$ in a neighborhood of $\sigma^{-1}(\beta_0)$. Let $\mathcal{F}_0 = \mathcal{F}_{C_0}$ be a generic sheaf corresponding to a point $[\mathcal{F}_{C_0}] \in \mathcal{B}'_{X_0}$, for a smooth cubic X_0, B_0 its presentation map with block structure $B_0 = (\beta'_0|\beta_0)$ such that β_0 is a 6×6 stable skew-symmetric matrix of generic rank 4.

Let U be a small neighborhood of $[\mathcal{F}_0]$ in $\mathfrak{M}$, a biholomorphic image of a 39-dimensional complex polydisk; we will denote the central point $[\mathcal{F}_0]$ of U just 0. Then there is a sheaf $\mathcal{F}$ over $\mathbb{P}^4_U = \mathbb{P}^4 \times U$, a flat analytic family of sheaves $\mathcal{F}_s$, $s \in U$ which is a locally universal analytic deformation of $\mathcal{F}_0$. Starting with the presentation B_0 for $\mathcal{F}_0$, by Proposition 4.2.2, we can construct a length-2 free resolution $\mathcal{R}^\bullet$ of $\mathcal{F}$, possibly after shrinking U, such that its presentation matrix B, for all $s \in U$, has the same block structure $B = (\beta'|\beta)$ as B_0 with β skew-symmetric of generic rank 4. This provides an analytic map $\varphi : U \to \Lambda$, $s \mapsto \beta_s$. This map is obviously biholomorphic to its image when restricted to the instanton locus, as the inverse can be given by $\beta_s \mapsto [\operatorname{coker} \beta_s]$. This formula fails to determine the image of β_s when $\beta_s \in \mathcal{Z}'$.

Let us denote $\tilde{\varphi}$ the meromorphic map $\sigma^{-1} \circ \varphi : U \to \widetilde{\Lambda}$. There is natural way to extend $\tilde{\varphi}$ to U. Indeed, we saw, that whenever $\beta_s \in \mathcal{Z}'$, the differential $\operatorname{Pf}_*(\beta_s)$ sends the normal space $\mathcal{N}_{\mathcal{Z}'/\Lambda,\ \beta_s}$ isomorphically to $H^0(\mathcal{I}_{C_s,/\mathbb{P}^4}(3))$, where C_s is the conic of singularities of $\mathcal{F}_s$. On the other hand, the fiber $\sigma^{-1}(\beta_s)$ of the blowdown map is naturally identified with the projectivization of the normal space:

$$\sigma^{-1}(\beta_s) = \mathbb{P}(\mathcal{N}_{\mathcal{Z}'/\Lambda,\ \beta_s}) = |\mathcal{I}_{C_s/\mathbb{P}^4}(3)| \simeq \mathbb{P}^{27}.$$

We define $\tilde{\varphi}(s)$ to be the point of $\sigma^{-1}(\beta_s)$ corresponding to the cubic X_s, the support of $\mathcal{F}_s$. It is obviously continuous, which follows from the continuity of the support map and Proposition 4.2.3, according to which the derivative of the cubic F_s defining the support X_s of $\mathcal{F}_s$ is proportional to F_s when $\varphi(s) \in \mathcal{Z}'$. The continuity of a meromorphic map between smooth complex manifolds implies its holomorphy, so $\tilde{\varphi}$ is a holomorphic map, bimeromorphic on its image.

Next we want to find a lower-dimensional polydisc, on which $\tilde{\varphi}$ is injective and which is transversal to the action of $GL(W) \times GL(V)$, in order to construct a tubular neighborhood of an open part of $\widetilde{\mathcal{Z}}'$ carrying a family of sheaves from $\mathfrak{M}$.

As the support map ρ and its restriction to $\boldsymbol{\mathcal{B}'}$ are smooth at $0 = [\mathcal{F}_0]$, there is a smaller polydisc $\Delta \subset U$, 28-dimensional and transversal to $\boldsymbol{\mathcal{B}'}$, such that its image in $\mathbb{P}^{34}$ is an open subset of $\mathbb{P}^{27} = |\mathcal{I}_{C_0/\mathbb{P}^4}(3)|$, where C_0 denotes, as before, the conic of singularities of $\mathcal{F}_0$. We can specify its choice as follows: choose a small 27-dimensional polydisc $\Delta' \in |\mathcal{I}_{C_0/\mathbb{P}^4}(3)|$ centered at X_0, then set Δ'' to be a cross-section of ρ over Δ', contained in U and passing through $[\mathcal{F}_0]$, and Δ a germ of a 28-dimensional manifold in $\rho^{-1}(\mathbb{P}^{27})$, transversal to $\boldsymbol{\mathcal{B}'}$, and fibered in small disks with centers on Δ', each disk being contained in a fiber of ρ. With such a choice, the image of the tangent space $T_0\Delta$ by the differential of the composed map $\operatorname{Pf} \circ \varphi$ will be the whole 28-dimensional vector space of cubics passing through C_0.

The image $\tilde{\varphi}(\Delta')$ (resp. $\tilde{\varphi}(\Delta)$) is then a normal slice to the action of $GL(W) \times GL(V)$ on $\widetilde{\mathcal{Z}}'$ (resp. $\widetilde{\Lambda}$), and $\tilde{\varphi}|_\Delta : \Delta \to \tilde{\varphi}(\Delta)$ is biholomorphic. Let $\psi : \tilde{\varphi}(\Delta) \to \Delta$ be its inverse. Consider the restriction $\mathcal{R}^\bullet_{\tilde{\varphi}(\Delta)} := \mathcal{R}^\bullet \times_\psi \tilde{\varphi}(\Delta)$ of the family of resolutions of the sheaves $\mathcal{F}_s$. The images of $\tilde{\varphi}(\Delta)$ under the action of $GL(W) \times GL(V)$ sweep out a tubular neighborhood D of an open part of $\widetilde{\mathcal{Z}}'$, and we can extend $\mathcal{R}^\bullet_{\tilde{\varphi}(\Delta)}$, via this action, to a flat analytic family $\widetilde{\mathcal{R}}^\bullet_D$ of resolutions of sheaves belonging to $\mathfrak{M}$. The latter gives rise to an analytic family of sheaves $\mathcal{F}_D$ sur $\mathbb{P}^4_D$, whose classifying map $D \to \mathfrak{M}$ is regular and coincides with $\widehat{\tau}$ on the instanton locus, that is on the complement of $\widetilde{\mathcal{Z}}'$. Hence it provides a regular extension of $\widehat{\tau}$ to D. The above construction can be applied to any initial datum $\mathcal{F}_0$ represented by a point of the fiber $\sigma^{-1}(\beta_0)$, so we can get a covering of an open neighborhood of $\sigma^{-1}(\beta_0)$ in $\widetilde{\Lambda}$ by open sets D, to which $\widehat{\tau}$ extends regularly. This proves the regularity of $\widehat{\tau}$ at generic point of $\widetilde{\Lambda}$.

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□

Conjectural description of the map $\bar{\tau}$ at a generic point of $\boldsymbol{\mathcal{B}''}$

Denote by $\boldsymbol{\mathcal{B}''} \subset \mathfrak{M}$ the locus of sheaves supported on a cubic threefold and presenting singularities along a couple of lines (namely we consider the divisor of $\mathfrak{M}$ whose generic point is a sheaf of the form $\mathcal{F}_{l_1,l_2}$ corresponding to a point in $\mathcal{B}''_X$, $X = \operatorname{Supp}(\mathcal{F}_{l_1,l_2})$). We conjecture that Proposition 4.2.4 extends literally to $\boldsymbol{\mathcal{B}''} \subset \mathfrak{M}$ and $\mathfrak{B}'' \subset \mathfrak{P}$. We believe it can be proved along the lines of the proof in the case of hyperwebs belonging to $\mathcal{Z}''$, but there are some technical complications. First, there is no locally universal family of sheaves on a small polydisk in $\mathfrak{M}$ through a general point of $\mathcal{Z}''$, for this is a strictly semistable point, and instead of a polydisk Δ in $\mathfrak{M}$ we should use a Luna slice in a Quot scheme in the framework of the GIT construction of $\mathfrak{M}$. Second, we have a weaker version of skew-symmetrization property (Proposition 4.2.2) for resolutions of sheaves in $\boldsymbol{\mathcal{B}''}$, involving a double covering base change, because of which we cannot affirm, as it happens for hyperwebs appearing resolutions of sheaves in $\boldsymbol{\mathcal{B}'}$, that the differential of the Pfaffian map sends isomorphically the normal space to $\mathcal{Z}'$ onto the space of cubics containing the rank-2 locus of the hyperweb. We hope to resolve these issues in the future, it is a work in progress.

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