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Topologie des lissages de singularités non-isolées de surfaces complexes.

Topology of smoothings of non-isolated singularities of complex surfaces.

Soutenue le 17 Juin 2019 devant le jury composé de :

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«- J'ai retrouvé dans cette bibliothèque baroque un texte que je connaissais depuis tout petit mais dont je ne savais plus s'il existait vraiment. À force d'oublier, je m'invente parfois une mémoire. Ce texte s'appelle Les Trois Métamorphoses. Ça commence comme ça : "Je vous dirai trois métamorphoses de l'esprit : comment l'esprit devient chameau, et le chameau, lion, et le lion enfant pour finir."

- Caracole, je t'ai posé une question ! Réponds-moi !

- Qu'est-ce qui est lourd ? demande l'esprit qui respecte et obéit, que je puisse, en héros, en bon hordier, porter les plus lourdes charges. Ainsi parle le chameau. Je te fais la version courte, note bien ! Et solidement harnaché, il marche vers son désert et là il devient lion. Devant lui se dresse le dragon des normes millénaires et sur chacune de ses écailles brillent en lettres d'or ces valeurs et ces mots : "Tu dois." Mais le lion dit "Je veux !" - sauf qu'il ne sait pas encore ce qu'il peut bien vouloir, il n'a fait que se chercher un dernier maître pour le contredire, que se rendre libre pour un devenir qu'il est encore incapable d'incarner. Alors survient la troisième métamorphose : le lion devient enfant. Innocence et oubli, premier mobile, roue qui roule d'elle-même, recommencement et jeu et l'enfant dit "Je crée". Ou plutôt, il ne dit plus rien : il joue, il crée. Il a trouvé son Oui, il a gagné son monde.

- Comment on affronte la neuvième forme du vent ? Je m'en fiche de ton histoire ! Réponds à ma question !

- Ces trois métamorphoses peuvent être les étapes d'une vie, d'un amour, d'une quête – mais tout aussi bien coexister en toi en ce moment même, à différentes vitesses et proportions, en couches fondues. La neuvième forme tue à coup sûr le chameau. Elle blesse à mort le lion. Mais l'enfant que tu sauras peut-être devenir pourrait lui survivre. Penses-y quand ils seront tous morts et que tu resteras debout, seul sur l'alpage avec le ciel nu devant toi. Pense à moi ce jour-là et rappelle-toi de ce moment que nous vivons ici même, rappelle-toi de cette phrase que je prononce à haute voix, de chaque mot qui la compose. Tu m'écoutes, Sov ?

- Oui.

- Rappelle-toi que l'oubli est la seule force vraiment active. Pas la mémoire : l'oubli !»

Alain Damasio, *La horde du contrevent.*

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Résumé : Cette thèse s'intéresse à la topologie des lissages des singularités non-isolées de sufaces complexes. La question est celle de la description de la topologie de la variété, appelée **fibre de Milnor**, qui survient lors de ce procédé de lissage. Devant la difficulté de décrire la totalité de cette topologie, beaucoup de recherches se sont concentrées sur le **bord** de la fibre de Milnor. Dans le cas des singularités isolées, il est connu depuis les travaux de Mumford (1961), que ce bord est une variété graphée, isomorphe au bord de la singularité.

Différents résultats (Michel & Pichon 2003, 2014, Némethi & Szilárd 2012) ont par la suite prouvé que dans le cas des singularités réduites non-isolées de surfaces, si l'espace total du lissage est lui-même lisse, le bord de la fibre de Milnor est encore une variété graphée. Fernández de Bobadilla & Menegon-Neto (2014) ont quant à eux élargi le contexte, considérant le cas d'une surface non réduite dans un espace total à singularité isolée. Dans ce travail, on poursuit l'extension de ce résultat à un plus large contexte, autorisant l'espace total du lissage à présenter des singularités non-isolées, tout en imposant à la surface d'être réduite. Notre preuve s'inspire de celle de Némethi et Szilard, permettant comme chez eux de produire une méthode pour le calcul du bord de la fibre de Milnor. Ceci rend praticable le calcul effectif d'une grande quantité d'exemples, représentant un progrès dans la quête de la compréhension des variétés pouvant apparaître comme bords de fibres de Milnor.

Nous appliquons en particulier la méthode aux singularités Newton-non-dégénérées définies sur des germes toriques tridimensionnels quelconques. Nous généralisons de cette manière un théorème de Oka (1986), en exprimant le bord de la fibre de Milnor en termes du polyèdre de Newton de la singularité.

Abstract: This thesis is dedicated to the study of the topology of smoothings of non-isolated singularities of complex surfaces. The question is to describe the topology of the manifold, called **Milnor fiber**, which appears during this process of smoothing. Considering the great difficulty of a description of the whole of this topology, many researches have focused on the study of the **boundary** of the Milnor fiber. In the case of isolated singularities, it is known since the work of Mumford (1961) that this boundary is a graph manifold, isomorphic to the link of the singularity.

Different results (Michel & Pichon 2003, 2014, Némethi & Szilárd 2012) have then proved that, in the case of reduced non-isolated singularities of surfaces, the boundary of the Milnor fiber is again a graph manifold, while restraining to the case of a smooth total space of smoothing. Fernández de Bobadilla & Menegon-Neto (2014) have widened the context, considering non-reduced surfaces, and allowing the total space to have an isolated singularity. In this work, we pursue the extension of this result to a larger context, allowing the total space to present non-isolated singularities, while restraining ourselves to the study of reduced surface singularities. Our proof is inspired by the one of Némethi and Szilard, and allows us furthermore to provide a method for the computation of the boundary of the Milnor fiber. This makes possible the actual computation of a large number of examples, representing a step forward in the quest for the comprehension of the manifolds that can actually appear as boundaries of Milnor fibers.

We apply in particular the method to Newton non-degenerate singularities defined on 3-dimensional toric germs. This is a generalization of a theorem of Oka (1986), expressing the boundary of the Milnor fiber in terms of the Newton polyhedron of the singularity.

Contents

Chapter 1

Introduction

Version française

L'étude des fibres de Milnor des fonctions holomorphes, débutée dans la seconde moitié du 20ème siècle, a donné lieu a une riche interaction entre l'algèbre et la topologie. Esquissons-en ici les principaux aspects.

John Milnor a proposé dans [37, 1956] les premiers exemples de sphères exotiques, c'est-à-dire de variétés lisses homéomorphes mais non difféomorphes à la sphère de dimension n, dans le cas $n = 7$. De telles variétés présentaient un intérêt important pour les topologues. Milnor et Kervaire ont poursuivi leur étude dans [38, 1959], [27, 1963].

À la même époque, un premier lien entre topologie des variétés et lissité analytique complexe fut exposé par Mumford dans [40, 1961], où il prouva qu'une surface complexe normale qui est une variété topologique est lisse. En explorant la possibilité d'une généralisation d'un tel résultat aux dimensions supérieures, Brieskorn, dans [7, 1963], montra que ce principe était alors mis en défaut, exhibant de nombreux exemples de singularités qui sont des variétés topologiques, explicitement, toutes les singularités de la forme

$$V = \{z_1^2 + \cdots + z_k^2 - z_0^3 = 0\} \subset \mathbb{C}^{k+1},\ k \textit{ impair}.$$

Le **link** $V \cap \mathbb{S}_\varepsilon$ de telles singularités est toujours une sphère nouée, et dans certains cas c'est une sphère exotique, comme il a été démontré par Hirzebruch dans le cas $k = 5$, voir [9, p 46-48] et [24]. Voir aussi [39, Chapter 9], et [55, Section 1].

Cette fabrique de potentielles sphères exotiques fascina Milnor, et le poussa à étudier plus avant la topologie des singularités d'hypersurfaces, cette représentation *via* des équations constituant un moyen d'avoir prise sur des objets par ailleurs très abstraits. Cela l'a finalement mené à l'écriture de son célèbre livre [39] de 1968, destiné à l'étude des singularités isolées d'hypersurfaces de $\mathbb{C}^n$, $V(f) = \{f = 0\}$. Dans ce livre il présenta deux fibrations équivalentes, faisant intervenir respectivement les niveaux de $f/|f|$ sur les sphères centrées aux points critiques de f et les niveaux de f dans $\mathbb{C}^n$. Dans le cas où la singularité est isolée, le bord de la fermeture de la fibre de chacune de ces fibrations est difféomorphe au link de la singularité. Son étude en est ainsi réduite à l'étude du bord de la fibre.

La première de ces deux fibrations est connue de nos jours sous le nom de **fibration de Milnor**, et la fermeture de sa fibre est appelée **fibre de Milnor** F de la fonction f.

La seconde a été étendue à des contextes plus généraux par Lê, voir [29], et est connue sous le nom de **fibration de Milnor-Lê**. Cependant, elle peut faire apparaître des fibres génériques singulières, dues au lieu singulier de l'espace ambiant. Hamm, dans [22, 1971], a proposé un contexte dans lequel la fibration de Milnor-Lê est en fait un **lissage** de $V(f)$, c'est-à-dire un moyen de mettre $V(f)$ dans une famille **plate** de surfaces, telle que la surface générique est lisse. Explicitement, si $(X, 0)$ est un germe d'espace analytique complexe équidimensionnel, et si f est une fonction holomorphe quelconque sur $(X, 0)$ telle que $V(f) \supset \text{Sing}(X)$, alors la fonction f est un lissage de la singularité $(V(f), 0)$.

Cette vision de la fibration a ouvert de nouveaux territoires pour l'exploration de la relation entre l'algèbre et la topologie, puisqu'elle permet de définir une fibration dans différents espaces ambiants, gardant à l'esprit qu'il est très intéressant de réaliser une variété par des équations, ou même simplement de savoir qu'une variété donnée peut être réalisée algébriquement. Pour plus de détails concernant les fibrations de Milnor, on est invité à consulter les survols [64] de Teissier et [55] de Seade.

Cette thèse est dédiée à l'étude des lissages de surfaces complexes analytiques. On part d'un germe de surface complexe (V, p), et, quand un lissage $f : (X, 0) \to (\mathbb{C}, 0)$ de cette singularité existe, on s'interroge sur la topologie de la fibre générique de ce lissage.

Une singularité de surface peut admettre différents lissages, ou n'en admettre aucun. Remarquons cependant qu'une singularité isolée n'admet au plus qu'un nombre fini de lissages différents. L'étude de la topologie des fibres des lissages d'une singularité de surface complexe donnée est très difficile, même dans le cas des singularités isolées, et il n'y a que quelques cas pour lesquels une description de la fibre dans son entier est connue. C'est le cas par exemple pour les singularités Kleinéennes A, D, E, où la fibre de Milnor est unique et difféomorphe à la résolution minimale de la singularité (voir Brieskorn [8]), ainsi que pour les singularités de surfaces toriques normales, avec une description par chirurgie (voir Lisca, [30], et Némethi & Popescu-Pampu [45]), ou encore pour les singularités sandwich (De Jong & Van Straten, [26]). S'agissant des singularités non-isolées, le seul cas connu à ce jour est celui des singularités d'hypersurfaces de la forme $\{f(x, y) + z \cdot g(x, y) = 0\}$, voir Sigurðsson, [59].

D'un autre côté, l'étude du **bord** de la fibre de Milnor est depuis quelques décennies l'objet de recherches très actives.

Comme on l'a dit, le bord de la fibre de Milnor d'un lissage de singularité isolée est unique, et difféomorphe au link de la singularité. Avec les mots d'aujourd'hui, Mumford a prouvé dans [40] que le link d'une singularité isolée de surface complexe est une **variété graphée**, c'est-à-dire une variété descriptible *via* un graphe dont les sommets représentent des fibrations en $\mathbb{S}^1$ au-dessus de surfaces compactes. C'est Walhausen, dans [65, 66, 1967], qui a plus tard introduit ce vocabulaire et a commencé à étudier cette classe de variétés en elle-même. De plus, grâce au travail de Grauert ([21, 1962]), on sait précisément quelles variétés graphées apparaissent comme links de singularités isolées de surfaces complexes. Cependant, ce résultat est tempéré par le fait qu'on ne sait pas, par exemple, quelles variétés apparaissent comme links de singularités d'hypersurfaces de $\mathbb{C}^3$.

Tout de même, on rêverait d'aoir en main un résultat analogue pour les fibres de Milnor de singularités non-isolées de surfaces. Les premiers pas dans la compréhension de la topologie de ces variétés ont été accomplis par Randell dans [54, 1977], puis Siersma dans [57, 1991], qui ont calculé l'homologie du bord ∂F de la fibre de Milnor dans certains cas, et on caractérisé les cas dans lesquels celui-ci est une sphère d'homologie rationelle.

S'agissant de la topologie totale de cette variété, une série de résultats ont visé à prouver que le bord des fibres de Milnor associées à des singularités non-isolées sont, encore, des variétés graphées. Une première preuve de ce résultat fut esquissée en 2003 par Michel et Pichon dans [32], puis corrigée plus tard dans [33] en 2004, pour le cas où l'espace total de lissage est lui-même lisse et où la fonction f est réduite. En 2016, elles publièrent les détails de leur preuve dans [34]. En collaboration avec Weber, elles proposèrent des graphes de plombage explicites pour plusieurs classes de singularités : les **singularités de surfaces d'Hirzebruch** en 2007 [36], et les **suspensions** ($f = g(x, y) + z^n$) en 2009 dans [35]. Fernández de Bobadilla & Menegon-Neto ont de leur côté prouvé ce fait en 2014, dans [16], dans le contexte d'un espace total de lissage admettant une singularité isolée, pour une fonction de la forme $f \cdot \overline{g}$, où f et g sont holomorphes. Mais aucune de ces approches n'était constructive. A contrario, Néméthi et Szilárd ont donné une preuve constructive du fait que ∂F est une variété graphée pour le cas d'une fonction holomorphe réduite $f\colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ dans leur livre [46] de 2012. Explicitement, ils proposèrent un algorithme de calcul du bord de la fibre de Milnor de la fonction f.

Cependant, en général, l'espace total du lissage d'une singularité non-isolée de surface n'a aucune raison d'être lisse, ou même de n'avoir qu'une singularité isolée. Par exemple, dès que la singularité générique (V, q) de V le long d'une composante irréductible de son lieu singulier ne peut pas être plongée holomorphiquement dans $\mathbb{C}^3$, cela signifie que l'espace total d'un lissage sera nécessairement à singularité non-isolée. Il est donc essentiel de généraliser ce résultat concernant la topologie des bords de fibres de Milnor de singularités non-isolées à des espaces plus généraux.

Dans ce travail, nous étendons la stratégie proposée par Néméthi & Szilárd et nous prouvons :

Théorème 1. *Soit $(X, 0)$ un germe d'espace analytique complexe de dimension 3, et $f\colon (X, 0) \to (\mathbb{C}, 0)$ un germe de fonction holomorphe sur $(X, 0)$, tel que $V(f) \supset \mathrm{Sing}(X)$. Alors le bord de la fibre de Milnor de f est une variété orientée de dimension 3, représentable par un graphe de plombage orientable.*

Nous démontrons ce théorème de manière constructive, en adaptant à notre contexte général la preuve de Néméthi & Szilárd. Comme dans leur livre, notre preuve donne lieu à un **algorithme pour le calcul** de ∂F. Elle peut de ce fait représenter un pas en avant dans la quête de l'obtention d'une caractérisation des variétés graphées qui apparaissent comme bords de fibres de Milnor. La stratégie de la preuve et les principales différences sont détaillées au début du Chapitre 4. Insistons simplement ici sur le fait que le lieu singulier de $(X, 0)$ implique une série de complications de la preuve, imposant en particulier de faire intervenir plus de données que dans [46] pour mener à bien le calcul, ainsi que l'introduction de la notion de **compagnon**, qui est faite pour obtenir l'analogue de la notion d'ICIS pour des germes qui ne sont pas des intersections complètes. On a aussi abordé les questions d'orientation d'une manière plus globale, que nous espérons être plus naturelle. Un point commun important avec la situation de [46] est l'utilisation d'un germe $(\mathscr{S}_k, 0)$ de variété réelle analytique de dimension 4 à singularité non-isolée, ayant pour link la variété ∂F, réalisant de fait cette variété comme le link d'une singularité isolée. Le fait que cette variété soit analytique réelle, et non complexe, impose l'usage de décorations $\ominus$ pour certaines arêtes du graphe, ce qui n'arrive jamais dans le cas des singularités isolées de sufaces complexes.

L'autre résultat de ce travail est l'extension de l'algorithme d'Oka (voir [49, 1986]), qui calcule le link d'une singularité isolée Newton-non dégénérée de surface complexe dans $\mathbb{C}^3$. En utilisant l'algorithme général et la théorie de la **géométrie torique**, on produit une méthode de calcul simple du bord de la fibre de Milnor d'une fonction Newton-non dégénérée définie sur un germe de variété torique normale de dimension 3. Ceci répond positivement à la question ouverte posée dans [46, 24.4.20], et ouvre la voie au calcul d'un grand nombre d'exxemples. En conséquence :

Théorème 2. *Soit $(X, 0)$ le germe en son sommet de variété torique normale de dimension 3 définie par un cône de dimension 3 dans un réseau de poids. Soit $f\colon (X, 0) \to (\mathbb{C}, 0)$ une fonction Newton-non-dégénérée dont le lieu des zéros contient $\mathrm{Sing}(X)$. Alors le bord de la fibre de Milnor de f est une variété graphée déterminée par le polyèdre de Newton local de f.*

De plus, ceci nous rapproche de l'extension de ce qui est fait dans [6, 2007], où les auteurs font le chemin inverse, reconstituant un possible polyèdre de Newton de fonction $f : \mathbb{C}^3 \to \mathbb{C}$ ayant une variété graphée donnée comme bord de fibre de Milnor, sous l'hypothèse que celui-ci soit une sphère d'homologie rationelle.

Dans les Chapitres 2 et 3, on introduit les principaux outils et notations requis pour la preuve du Théorème 1 et l'énoncé de la méthode de calcul.

Le Chapitre 4 présente la preuve du Théorème 1.

Le Chapitre 5 expose la manière dont l'algorithme principal peut être appliqué au calcul du bord de la fibre de Milnor d'une fonction Newton-non dégénérée définie sur un germe de variété torique normale de dimension 3, prouvant le Théorème 2.

English version

The study of Milnor fibers of complex-analytic functions, which began in the second half of the 20th century, gave rise to a rich interaction between algebra and topology. Let us describe briefly this interaction.

John Milnor provided in [37, 1956] the first examples of exotic spheres, that is, smooth manifolds homeomorphic but not diffeomorphic to the n-sphere, in the case $n = 7$. Such manifolds were of great interest for topologists. Milnor and Kervaire continued their study in [38, 1959], [27, 1963].

Meanwhile, a first relation between the topology of varieties and complex analytic regularity was discovered by Mumford, in [40, 1961], where he established that a 2-dimensional normal complex surface which is a topological manifold is nonsingular. Exploring the possibility of extending such results to higher dimensions, Brieskorn, in [7, 1963], proved that it is no longer the case, producing many examples of singularities which are topological manifolds, namely all singularities of the form

$$V = \{z_1^2 + \cdots + z_k^2 - z_0^3 = 0\} \subset \mathbb{C}^{k+1},\ k\ odd.$$

The **link** $V \cap \mathbb{S}_\varepsilon$ of such singularities is always a knotted sphere, and in some cases it is an exotic sphere, as was shown by Hirzebruch in the case $k = 5$, see [9, p 46-48] and [24]. See also [39, Chapter 9], and [55, Section 1].

This potential fabric of exotic spheres fascinated Milnor, and inspired him to study further the topology of hypersurface singularities, as this presentation by equations represents a possible way to have a grasp on otherwise very abstract objects. This eventually led him to write his famous 1968 book [39], aimed at the study of isolated singularities of hypersurfaces of $\mathbb{C}^n$, $V(f) = \{f = 0\}$. In this book he introduced two equivalent fibrations, using respectively the levels of $f/|f|$ on spheres centered at the critical points of f and the levels of f in $\mathbb{C}^n$. The point is that, in the case where the singularity is isolated, the boundary of the closure of a fiber is diffeomorphic to the link $\partial V(f)$ of the singularity . Its study is therefore reduced to the study of the boundary of the closure of the fiber.

The first of the two fibrations introduced by Milnor is known nowadays as the **Milnor fibration**, and the closure of its fiber is called the **Milnor fiber** F of the function f.

The second one has been extended to more general contexts by Lê, see [29], and is known as the **Milnor-Lê fibration**. However, it may produce singular generic fibers, due to the singularities of the ambient space. Hamm, in [22, 1971], provided a setting in which the Milnor-Lê fibration is actually a **smoothing** of $V(f)$, that is, a way to put $V(f)$ in a **flat** family of surfaces, where the generic surface is smooth. Namely, if $(X, 0)$ is a germ of equidimensional complex analytic space, and f is any holomorphic function on $(X, 0)$ such that $V(f) \supset \mathrm{Sing}(X)$, then the function f provides a smoothing of the singularity $(V(f), 0)$.

This vision of the fibration opened new areas of exploration for the interplay between algebra and topology, as it allows the definition of a fibration in different ambient spaces, keeping in mind the idea that it is very interesting to realize a manifold *via* equations, or even to know that a given manifold is realizable algebraically. For more details about Milnor fibrations, one may consult the surveys [64] of Teissier and [55] of Seade.

In this work, by Milnor fibration we mean the Milnor-Lê fibration, and we also call its fiber the Milnor fiber.

This thesis is dedicated to the study of smoothings of complex analytic surface singularities. We start with some germ of complex surface (V, p), and, when there exists a smoothing $f: (X, 0) \to (\mathbb{C}, 0)$, wonder about the nature of the generic fiber of this smoothing.

A singularity of surface may admit different smoothings, or even none. Note however that an isolated singularity admits only a finite number of possible smoothings, if any. The study of the topology of the fibers of the smoothings of a given singularity of complex surface is very hard, even for isolated ones, and there is only a few types of singularities where a description of the full fiber is known. It is the case for the Kleinean sigularities A, D, E, where the Milnor fiber is unique and diffeomorphic to the minimal resolution (see Brieskorn, [8]), as well as for the singularities of normal toric surfaces, with a description by surgery (see Lisca, [30] and Némethi & Popescu-Pampu, [45]), and for sandwich singularities (De Jong & Van Straten, [26]). As for nonisolated singularities, the only known case is that of hypersurface singularities of the form $\{f(x, y) + z \cdot g(x, y) = 0\}$, see Sigurðsson, [59].

On the other hand, the study of the **boundary** of the Milnor fiber has been a very active area of resarch in the last decades.

As we already pointed out, the boundary of the fiber of a smoothing of an isolated singularity is unique, and equal to the link of the singularity. In today's words, Mumford, in [40], proved that the link of any isolated singularity of complex surface is a **graph manifold**, that is, a manifold describable using a decorated graph whose vertices represent fibrations in $\mathbb{S}^1$ over compact surfaces. It is Waldhausen, in [65, 66, 1967], that later introduced this vocabulary and began studying this class of varieties in itself. Furthermore, since the work of Grauert ([21, 1962]), one knows exactly which graph manifolds appear as links of isolated singularities of complex surfaces. However, this strong point is tempered by the fact that one still does not know, for example, which of these manifolds appear as links of singularities of **hypersurfaces** of $\mathbb{C}^3$.

Still, one would dream of having an analogous result for boundaries of Milnor fibers associated to non isolated singularities. The first steps towards the comprehension of the topology of these manifolds were made by Randell [54, 1977], then Siersma in [57, 1991], [58, 2000], who computed the homology of the boundary ∂F of the Milnor fiber in certain cases, and characterized the cases in which ∂F is a rational homology sphere.

Concerning the general topology of this manifold, a series of results were aimed at proving that the boundary of the Milnor fibers associated to a non isolated singularity are, again, graph manifolds. A first proof of this result was sketched in 2003 by Michel and Pichon ([32]) and corrected later in [33], in 2004, for the case where the total space of the smoothing is smooth and the function f is reduced. In 2016, they published the details of their proof in [34]. In collaboration with Weber, they provided explicit plumbing graphs for several classes of singularities: **Hirzebruch surface singularities** in 2007 ([35]), and the so-called **suspensions** ($f = g(x, y) + z^n$) in 2009 ([36]). Fernández de Bobadilla and Menegon Neto, in 2014 ([16]), proved it in the context of smoothings of non-isolated and not necessarily reduced singularities whose total space has an isolated singularity, for a function of the form $f \cdot \overline{g}$, with f and g holomorphic. But none of these approaches was constructive. By contrast, Némethi and Szilárd gave a constructive proof for the case of reduced holomorphic functions $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ in their 2012 book [40]. Namely, they provided an algorithm to compute the boundary of the Milnor fiber of the function f.

However, in general, the total space of a smoothing of a non-isolated singularity of surface has no reason to be smooth, or even with isolated singularity. For example, as soon as the generic singularity (V, q) along an irreducible component of the singular locus of V can not be holomorphically embedded in $\mathbb{C}^3$, this means that it can not be seen as a hypersurface of a variety with isolated singularity. It is therefore essential to generalize this result concerning the topology of the boundary of the Milnor fiber to more general spaces.

In this work, we extend the strategy developed in [46] by Némethi and Szilárd and we prove the following:

Theorem 1. *Let $(X, 0)$ be a germ of 3-dimensional complex analytic variety, and $f : (X, 0) \to (\mathbb{C}, 0)$ a germ of holomorphic function on $(X, 0)$, such that $V(f) \supset Sing(X)$. Then the boundary of the Milnor fiber of f is an oriented 3-manifold, which can be represented by an orientable plumbing graph.*

We prove this theorem in a constructive way, by adapting to our more general context the method of Némethi and Szilárd. As was the case in their book, our proof gives rise to an **algorithm for the computation** of ∂F. It may therefore represent a step in the obtention of a characterization of the graph manifolds which appear as boundaries of Milnor fibers. The strategy of proof and the main differences are detailed at the beginning of Chapter 4. Let us simply insist here on the fact that the singular locus of the ambient germ of variety $(X, 0)$ induces a series of complications in the proof, implying in particular the need for more data than in [46] in order to perform the computations, as well as the introduction of the notion of companion, which is made to obtain the analogue of ICIS on germs which may not be complete intersections. We also treated the questions of the orientations in a more global way, which we hope to be more natural. An important common point with the situation of [46], and the main idea of their proof, is the introduction of a real-analytic germ $(\mathscr{S}_k, 0)$ of dimension 4 with isolated singularity having the same boundary as the Milnor fiber, therefore actually realizing this manifold as the link of a singularity. The fact that this variety is real-analytic, and not complex-analytic, imposes the use of $\ominus$ decorations for some edges of the graph, which never happens in the case of isolated complex surface singularities.

The other result of this work is the extension of Oka's algorithm (see [49, 1986]), which computes the link of a non-degenerate isolated singularity of hypersurface of $\mathbb{C}^3$. Using the general algorithm and the theory of **toric geometry**, we provide a simple method for the computation of the Milnor fiber of a Newton-nondegenerate function defined on a normal 3-dimensional toric variety. This answers the open question asked in [46, 24.4.20], and opens the way for the computation of a great number of examples. As a consequence:

Theorem 2. *Let $(X, 0)$ be the germ at its vertex of a 3-dimensional toric variety defined by a 3-dimensional cone in a weight lattice. Let $f : (X, 0) \to (\mathbb{C}, 0)$ be a Newton-nondegenerate function whose zero locus contains $\mathrm{Sing}(X)$. Then the boundary of its Milnor fiber is a graph manifold determined by the local Newton polyhedron of f.*

Furthermore, this gets us closer to the extension of what is done in [6, 2007], where the authors do the opposite work, retrieving a possible Newton polyhedron of a function $f : \mathbb{C}^3 \to \mathbb{C}$ having a given graph manifold as boundary of Milnor fiber, under the hypothesis that this manifold is a rational homology sphere.

In Chapters 2 and 3, we introduce the main tools and notations needed for the proof of theorem 1 and the statement of the method of computation.

Chapter 4 presents the proof of Theorem 1.

Chapter 5 exposes how the main algorithm can be adapted to the computation of the boundary of the Milnor fiber of a non degenerate function defined on a 3-dimensional toric variety, proving Theorem 2.

Chapter 2

Basic tools

In all this work, the spaces considered are real or complex analytic. In the sequel, $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$.

2.1 Smooth and singular points

For details about the notions presented in this Section, one can consult [56, Volume I, Chapter II].

Let us start by defining the central notion of singular point of a variety, through the use of the maximal ideal at this point.

Definition 2.1.1. *The* ***maximal ideal*** *at a point x in a $\mathbb{K}$-analytic space X is the ideal of the local ring $\mathcal{O}_{X,x}$ consisting of the germs of analytic functions at x, cancelling at x. It is denoted by m_x. The quotient ${}^{m_x}\!/_{(m_x)^2}$ is called the* ***Zariski cotangent space*** *of X at x.*

Definition 2.1.2. *An analytic space X is said to be* ***reduced*** *at x if the local ring $\mathcal{O}_{X,x}$ has no nilpotent element. An analytic space is called reduced if it is reduced at any of its points.*

Definition 2.1.3. *If X is an analytic space, its dimension is defined as*

$$\dim(X) = \min_{x \in X} \dim\ \mathcal{O}_{X,x}$$

where $\dim\ \mathcal{O}_{X,x}$ denotes the ***Krull dimension*** *of $\mathcal{O}_{X,x}$, i.e. the maximal length of chains of $\mathbb{K}$-prime ideals in it.*
If all the local rings of X have the same Krull dimension, then X is called ***equidimensional****.*

Definition 2.1.4. *A reduced equidimensional space is called a* ***variety****.*

In the sequel, we will simply say "variety", instead of "analytic variety", unless we need to specify the field.

Definition 2.1.5. *If X is a variety, let us denote by*

$$\mathrm{Sm}(X) := \{x \in X, \dim {}^{m_x}\!/\!{}_{(m_x)^2} = \dim\ X\}$$

the ***smooth locus*** *of X. Its complement, denoted by $\mathrm{Sing}(X)$, is called the* ***singular locus*** *of X. A variety is called smooth if it coincides with its smooth locus.*

Proposition 2.1.6. *If X is a $\mathbb{K}$-analytic variety, its singular set is a nowhere dense closed $\mathbb{K}$-analytic subspace.*

Remark 2.1.7. *The singular set of X is not necessarily equidimensional. Indeed, X may for example have at the same time isolated and non-isolated singular points.*

Proposition 2.1.8. *For any variety X, the number $n = \min_{x\in X} \dim {}^{m_x}\!/\!{}_{(m_x)^2}$ is equal to $\dim\ X$.*
The smooth set $\mathrm{Sm}(X)$ of a variety X is exactly the set of points $x \in X$ where $\mathcal{O}_{X,x}$ is isomorphic to $\mathbb{K}\{z_1, \dots, z_n\}$.

2.2 Modifications and resolutions of singularities

Definition 2.2.1. *Let X be an irreducible variety. A* ***divisor*** *D on X is a collection of pairwise different irreducible closed codimension 1 subvarieties $D_1, \cdots, D_n$ with assigned multiplicities $k_1, \cdots, k_n \in \mathbb{Z}$. We write*

$$D = k_1 D_1 + \cdots + k_n D_n.$$

If all k_i are nonnegative, the divisor D is called ***effective****.*
The union of the D_i's such that $k_i \neq 0$ is called the ***support*** *$\mathrm{Supp}(D)$ of D. The divisor D is called* **reduced** *if all the multiplicities k_i are equal to 1.*

Definition 2.2.2. *A divisor E in a smooth complex variety X is said to be a* ***normal crossings divisor*** *(NCD) iff for all $x \in X$, there exists a local analytic coordinate system $(z_1, \dots, z_n)$ of X based at x such that around this point E admits a local equation of the form $z_1^{a_1} \cdots z_n^{a_n} = 0$, with $(a_1, \dots, a_n) \in \mathbb{Z}^n$*

Definition 2.2.3. *A divisor in a smooth variety is said to be a* ***simple normal crossings divisor*** *(SNCD) iff it is a NCD and all its irreducible components are smooth.*

Let us introduce a weaker *ad hoc* notion, which is the one we will actually need in this work.

Definition 2.2.4. *A reduced divisor D in a variety X is said to be a **SNCD at** $\mathscr{C} \subset X$ if:*

- *Every point $p \in \mathscr{C}$ is a smooth point of X,*

  and

- *For every $p \in \mathscr{C}$, there exists a local analytic coordinate system $(z_1, \ldots, z_n)$ of X based at p such that around p, D admits an equation of the form $z_1^{\epsilon_1} \ldots z_n^{\epsilon_n} = 0$, with $(\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n$.*

Definition 2.2.5. *A **modification** of a variety X is a variety $\tilde{X}$, along with a **proper** morphism $r \colon \tilde{X} \to X$ which is **bimeromorphic**: there exists a dense open subset U of X such that $r \colon r^{-1}(U) \to U$ is an analytic isomorphism, and $r^{-1}(U)$ is dense in $\tilde{X}$.*

Remark 2.2.6. *In particular, by properness, every modification is **surjective**.*

Definition 2.2.7. *A modification $r : \tilde{X} \to X$ of a variety X is called a **resolution** of X, or resolution of the singularities of X, if $\tilde{X}$ is smooth.*

Definition 2.2.8. *Let $r : \tilde{X} \to X$ be a modification of a variety. The **critical locus** of r is the subset of $\tilde{X}$ formed by the points $y \in \tilde{X}$ such that r is **not** a local isomorphism at y. We will also call this set the **exceptional locus** of the modification.*
*The **critical image** or **discriminant locus** $\Delta(r)$ of r is the subset of X which is the image of the critical locus.*

Both sets are closed nowhere dense analytic subsets of their ambient varieties.

Definition 2.2.9. *If $r : \tilde{X} \to X$ is a modification of X, and $\mathscr{S}$ is a closed analytic subset of X which is not contained in the critical image $\Delta(r)$ of r, the **strict transform** of $\mathscr{S}$ by r is the set*

$$\widetilde{\mathscr{S}} := \overline{r^{-1}(\mathscr{S} \setminus \Delta(r))} \subset \tilde{X}.$$

*We call **total transform** of S the set $r^{-1}(\mathscr{S})$, and **exceptional part** of $r^{-1}(\mathscr{S})$ the set $r^{-1}(\mathscr{S} \cap \Delta(r))$.*

Under these conditions, $r_{\mathscr{S}} := r_{|\widetilde{\mathscr{S}}} : \widetilde{\mathscr{S}} \to \mathscr{S}$ is also a modification, this time of $\mathscr{S}$.

Definition 2.2.10. *A **divisorial modification** is a modification of a space whose critical locus is of pure codimension 1. A **divisorial resolution** is a resolution which is a divisorial modification.*

For fluidity purposes, let us state here a definition that involves the definition of a **normal variety**, introduced in section 2.5.

Definition 2.2.11. *Let $X, \tilde{X}$ be normal 3-dimensional varieties, and $r : \tilde{X} \to X$ a divisorial modification of X. Let h be a holomorphic function on X, and r^*h its pullback by r. Then the irreducible components of the support of $div(r^*h)$ can be separated in two types:*

1. *Components whose images by r are* ***points***.
2. *The rest of the components, with multiplicities, constitute the* ***mixed transform*** *of $div(h)$. These components are, again, of two types:*
   (a) *Those which get contracted by r on curves ;*
   (b) *Those whose images by r are* ***surfaces***. *The subdivisor of $div(r^*h)$ supported by their union is called the* ***strict transform*** *of $div(h)$.*

Definition 2.2.12. *Let H^{n-1} be a hypersurface in a smooth variety X of dimension n. A* ***good embedded resolution of the singularities*** *of H is a divisorial modification $\Pi : Y \to X$ such that :*

- *Y is smooth ;*
- *the strict transform $\tilde{H}$ of H is smooth ;*
- *the total transform of H is a SNCD.*

Theorem 2.2.13. *Every hypersurface of an algebraic or analytic variety over a field of characteristic zero admits a good embedded resolution.*

For this theorem, one can consult the recent book [1] of Aroca, Hironaka and Vicente, in the complex analytic case. See also the article [3] of Bierstone and Milman as well as the books [12] of Cutkosky and [28] of Kollár for the case of algebraic varieties of characteristic zero.

2.3 About deformations and smoothings

In this section, following Fischer ([17]), we present the definitions and some properties of the notions of **deformation** and **smoothing** of a germ of analytic variety. Those definitions are built in order to properly define the legal way to "bend" a space, obtaining another space that will be close enough to the initial one. Our main theorem can be formulated in this framework.
First, we need to summon the algebraic notion of **flatness**. Let us join Fischer when he cites Mumford ([41]): "The concept of flatness is a riddle that comes out of algebra, which technically is the answer to many prayers."

Definition 2.3.1. *Let R be a ring, and M an R-module. The module M is said to be R-**flat**, or a **flat** R**-module**, if it satisfies one of the three following equivalent conditions:*

1. *For every exact sequence of R-modules*

$$\cdots \to N_{i-1} \to N_i \to N_{i+1} \to \dots$$

the induced sequence

$$\cdots \to N_{i-1} \otimes_R M \to N_i \otimes_R M \to N_{i+1} \otimes_R M \to \dots$$

is again exact.

2. *For every short exact sequence of R-modules*

$$0 \to N \to N' \to N'' \to 0$$

the induced sequence

$$0 \to N \otimes_R M \to N' \otimes_R M \to N'' \otimes_R M \to 0$$

is again exact.

3. *For every injective morphism of R-modules*

$$N' \to N$$

the induced morphism

$$N' \otimes_R M \to N \otimes_R M$$

is again injective.

This allows us to define the notion of flatness for a morphism:

Definition 2.3.2. *An holomorphic map $f\colon X \to Y$ between complex analytic varieties is said to be **flat** at $x \in X$ if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module.*

The following proposition, which is a particular case of the more general corollary of the section 3.16 of [17], will play a key role in this work:

Proposition 2.3.3. *Let (X, x) be a germ of complex analytic variety. Then any $f \in m_x$ such that f does not divide zero, seen as a germ of morphism $(X, x) \to (\mathbb{C}, 0)$, is a flat morphism.*

Definition 2.3.4. *A **deformation** of a germ of complex analytic variety (S, s) is a germ of flat morphism $f\colon (X, x) \to (Y, y)$, where (X, x) and (Y, y) are germs of complex analytic spaces, together with an isomorphism between (S, s) and the special fiber $(f^{-1}(y), x)$.*

Definition 2.3.5. *Morphisms as in proposition 2.3.3 are called **1-parameter deformations** of (S, s).*

Definition 2.3.6. *A 1-parameter deformation is called a **smoothing** whenever its generic fiber $f^{-1}(t), t \neq 0$ is smooth. A singularity is callled **smoothable** if it admits a smoothing.*

A smoothing does not always exist, even for isolated singularities of surfaces. This results for instance from a theorem of Steenbrink ([60]) about the Milnor numbers of smoothings of Gorenstein normal surface singularities (see [51], Theorem 4.18). However, in the case of isolated complete intersection singularities, it does. In this work the hypersurfaces considered will be smoothable, due to the way they are defined.

2.4 Normal rings

Definition 2.4.1. *Let (X, p) be a germ of $\mathbb{K}$-analytic variety. We will call **unit at** p any analytic function $f \in \mathcal{O}_{X,p}$ on this germ which is not zero at p.*

This denomination is due to the fact that such functions are **units** of the local function ring at that point.

Definition 2.4.2. *Let R be a commutative ring. Let S be the multiplicative system constituted of elements of R that do not divide 0. The **total quotient ring** of R is defined as $S^{-1}R$, denoted Q_R.*

*If R is an integral domain, $Q_R = (R \setminus \{0\})^{-1}R$ is called the **field of fractions** of R.*

*The **integral closure** of a ring R is the ring of all elements of Q_R that are integral over R.*

Definition 2.4.3. *A commutative ring R is called **normal** if it is integrally closed in its total quotient ring Q_R, i.e. if any element of Q_R that is integral over R is in fact an element of R.*

Remark 2.4.4.

1. *If $R \simeq \mathbb{K}[X_1, \cdots, X_n]$ or $\mathbb{K}\{X_1, \cdots, X_n\}$, then R is normal.*

2. *If $R \simeq \mathbb{K}[X_1, \cdots, X_n]/I$ is an integral domain, the integral closure A of R is a finitely generated $\mathbb{K}$-algebra and is normal.*

3. *If R is not an integral domain, then R is not normal.*

2.5 Normalization of varieties

In the sequel, $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.5.1. *A $\mathbb{K}$-analytic variety X is called **normal** if, for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is normal.*

Definition 2.5.2. *The **normal locus** of a variety X is the set of its points x such that $\mathcal{O}_{X,x}$ is normal. Its complement is called the **non-normal locus** of X.*

In most contexts, one can associate canonically a normal variety to a given variety, through a process called **normalization**.

2.5.1 Normalization of an affine algebraic variety

Let us start with a definition of normalization that allows us to deal with $\mathbb{R}$-algebraic varieties.

Definition 2.5.3. *(See [4, p.75]). Let $V = Specm(R)$ be an irreducible $\mathbb{K}$-algebraic variety, where $R = \mathbb{K}[X_1, \cdots, X_n]/I$. Denote A the integral closure of R. Then A is isomorphic to some $\mathbb{K}[X_1, \cdots, X_p]/I'$, and the surjective morphism*

$$N\colon X^N := Specm(A) \to V$$

*induced by the inclusion $R \hookrightarrow A$ is called the **normalization** of X.*

Remark 2.5.4. *The normalization of an irreducible algebraic variety is unique up to isomorphism of algebraic varieties.*

Definition 2.5.5. *If V is an equidimensional $\mathbb{K}$-algebraic variety, denote $V_1, \cdots, V_k$ the irreducible components of V. Then define the normalization V^N of V as*

$$V^N := V_1^N \sqcup \cdots \sqcup V_k^N.$$

Lemma 2.5.6. *The normalization V^N of an algebraic variety V is **normal**, and the morphism $N\colon V^N \to V$ is **finite** and is an isomorphism above the normal locus of V.*

Definition 2.5.7. *Let $V = Specm(A) \subset \mathbb{R}^n$ and $W = Specm(B) \subset \mathbb{R}^p$ be two affine $\mathbb{K}$-algebraic varieties, where $A = \mathbb{K}[X_1, \cdots, X_n]/I$, $B = \mathbb{K}[Y_1, \cdots, Y_p]/J$.*

A map

$$\varphi\colon \begin{array}{l} V \dashrightarrow W \\ (x_1, \cdots, x_n) \mapsto (\varphi_1(x_1, \cdots, x_n), \cdots, \varphi_p(x_1, \cdots, x_n)) \end{array}$$

*where each φ_i is a rational function in the x_j's, is called a **rational map** from V to W.*

The dashed arrow stands for the fact that such a map may have a locus of indeterminacy.

Definition 2.5.8. *Let $V = Specm(A) \subset \mathbb{R}^n$ and $W = Specm(B) \subset \mathbb{R}^p$ be two affine **equidimensional** $\mathbb{K}$-algebraic varieties, where $A = \mathbb{K}[X_1, \cdots, X_n]/I$, $B = \mathbb{K}[Y_1, \cdots, Y_p]/J$.*

*A rational map $\varphi\colon V \dashrightarrow$ is called **birational** if there exists a rational map $\psi\colon W \dashrightarrow V$ and dense open subsets $U_1 \subset V$ and $U_2 \subset W$ such that $\psi \circ \varphi_{|U_1} = Id$ and $\varphi \circ \psi_{|U_2} = Id$. In these conditons, we call ψ **an inverse** of ϕ.*

The following is a direct consequence of the definition.

Lemma 2.5.9. *The map $\varphi^*\colon Q_B \to Q_A$ induced by a birational map $\varphi\colon Specm(A) \dashrightarrow Specm(B)$ is an isomorphism of field extensions of $\mathbb{K}$.*

Finally, we have

Proposition 2.5.10. *Let $V = Specm(A)$ and $W = Specm(B)$ be two affine* ***equidimensional*** *$\mathbb{K}$-algebraic varieties, and $\varphi\colon V \dashrightarrow W$ be a birational map. Then there is a unique birational map $\varphi^N\colon V^N \dashrightarrow W^N$ such that the following diagram commutes:*

$$\begin{array}{ccc} V^N & \overset{\varphi^N}{\dashrightarrow} & W^N \\ {\scriptstyle N_V}\downarrow & \circlearrowleft & \downarrow{\scriptstyle N_W} \\ V & \underset{\varphi}{\dashrightarrow} & W \end{array}$$

Furthermore, let ψ be an inverse of φ. If $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_p$ are respectively generators of A and B such that for any i, $\varphi^*(Y_i)$ is integral over A and for any j, $\psi^*(X_j)$ is integral over B, then φ^N is an isomorphism of algebraic varieties.

2.5.2 Normalization of complex varieties

Much more can be said about the normalization if we restrict ourselves to complex varieties. For fluidity purposes, we will define the notion of normalization in the language of **germs** of analytic spaces, as well as in the one of global analytic spaces. The definitions provided in the two different frameworks are of course compatible. One can find more details in Fischer ([17]), chapter 2, or in de Jong & Pfister ([25]) for the case of germs.

Lemma 2.5.11. *A complex analytic variety X is normal at any of its smooth points. The non-normal locus $NN(X)$ of a variety X is a closed nowhere dense analytic subset. In particular, the set of normal points of a variety X is a dense open subset of X.*

Lemma 2.5.12. *If X is a normal complex analytic variety, its singular set $Sing(X)$ is of codimension at least 2.*

Definition 2.5.13. *Let X be a reduced complex-analytic variety, and denote by $A(X)$ its non-normal locus. We call a* ***normalization*** *of X any* ***finite****, surjective analytic map*

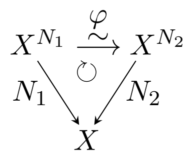
$$N\colon X^N \to X$$

such that

- *X^N is a normal analytic variety.*
- *$N^{-1}(X \setminus A(X))$ is dense in X^N.*
- *The restriction $N'\colon N^{-1}(X \setminus A(X)) \to X \setminus A(X)$ is an analytic isomorphism.*

This definition of normalization is of course compatible with the one given in the algebraic setting.

Proposition 2.5.14. *The normalization of an analytic variety is unique up to unique isomorphism: given two normalizations $N_1 \colon X^{N_1} \to X$ and $N_2 \colon X^{N_2} \to X$ of a variety, there exists a unique analytic isomorphism $\varphi \colon X^{N_1} \xrightarrow{\sim} X^{N_2}$ such that the following diagram commutes.*



Hence, one speaks about ***the*** *normalization of a variety.*

The next result makes explicit one of the main characteristics of the normalization morphism, which is, at a point of X, to "separate" the local analytic irreducible components.

Lemma 2.5.15. *The cardinal of the fiber of the normalization morphism above a point $p \in X$ is equal to the number of local irreducible components of X at p.*

Lemma 2.5.16. *Let X be a reduced analytic variety, K a* ***closed*** *subset of X, and $N \colon \overline{X} \to X$ the normalization of X. Then the restriction $N_K \colon N^{-1}(X \setminus K) \to X \setminus K$ is the normalization of $X \setminus K$.*

Normalization of a germ of analytic variety

Definition 2.5.17. *A germ of analytic variety (X, x) is said to be* ***normal*** *if the local ring $\mathcal{O}_{X,x}$ is normal.*

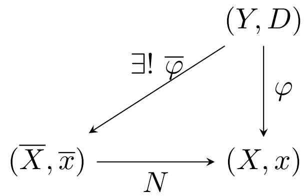
Definition 2.5.18. *A* ***multi-germ*** *of analytic spaces $(\overline{X}, \overline{x})$ is a finite disjoint union $(\overline{X}, \overline{x}) = (X_1, x_1) \bigsqcup \cdots \bigsqcup (X_k, x_k)$ of germs of analytic spaces. By definition the ring $\mathcal{O}_{\overline{X},\overline{x}}$ is equal to $\mathcal{O}_{X_1,x_1} \oplus \cdots \oplus \mathcal{O}_{X_k,x_k}$. We call k the* ***number of components*** *of the multi-germ. Once again, a multi-germ $(\overline{X}, \overline{x})$ is called* ***normal*** *if and only if each $\mathcal{O}_{X_i,x_i}$ is normal.*

Definition 2.5.19. *Let (X, x) be a reduced analytic germ of variety, and denote by $(A(X), x)$ (maybe empty) the germ of its non-normal locus. The* ***normalization*** *of (X, x) is a normal multigerm $(\overline{X}, \overline{x})$, together with a* ***finite****, surjective analytic map $N \colon (\overline{X}, \overline{x}) \to (X, x)$ such that N is an analytic isomorphism outside of $(A(X), x)$.*

Remark 2.5.20. *As in remark 2.5.14, any two such multi-germs are uniquely isomorphic above (X, x), hence we will call any of them* ***the*** *normalization of the germ (X, x).*

Proposition 2.5.21. *Let (X, x) be a germ of analytic variety. Its normalization $(\overline{X}, \overline{x}) \xrightarrow{N} (X, x)$ verifies the following universal property of minimality :*

Let $(Y, D) \xrightarrow{\varphi} (X, x)$ be a ***normal*** *modification of the germ X. Then there is a unique analytic morphism $\overline{\varphi} \colon (Y, D) \to (\overline{X}, \overline{x})$ such that $N \circ \overline{\varphi} = \varphi$.*



Remark 2.5.22.

1. *This universal property is in fact a characterization of the normalization.*

2. *Note that* $\overline{\varphi}$ *is a modification of* $(\overline{X}, \overline{x})$

As a consequence of lemma 2.5.15, we get:

Lemma 2.5.23. *The number of components of the normalization of a germ is equal to the number of irreducible components of this germ. In fact, if* $(X_1, x), \cdots, (X_k, x)$ *are the irreducible components of* (X, x)*, then the normalization of* (X, x) *is the disjoint union of the normalizations of the* (X_i, x)*'s.*

2.6 About graph manifolds and plumbing graphs

The purpose of this section is to make clear the notion of **graph manifolds**, and the correspondence between them and **plumbing graphs**. Note that, in the literature, there are different definitions for the following objects, depending on the level of generality the author wants to attain, and what they use these objects for. Our choices are aimed to the description of the boundaries of Milnor fibers of smoothings of non-isolated complex surface singularities.

2.6.1 Circle bundles over orientable surfaces

We introduce here the Euler number of an $\mathbb{S}^1$-bundle above an orientable surface, that allows one to entirely characterize such a bundle. One can refer to the book [19], especially Example 4.6.5, for what follows.

Definition 2.6.1. *A* ***smooth locally trivial fiber bundle*** (M, p, B, F) *consists of:*

1. *A smooth manifold* M*, called* ***total*** *space.*

2. *A smooth manifold* B*, called* ***base*** *space.*

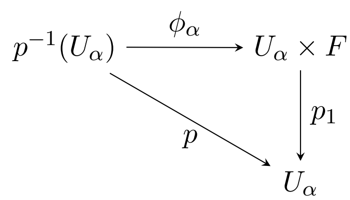
3. *A smooth surjective map* $p\colon M \to B$*, called the* ***projection map****.*

4. *A smooth manifold* F*, called the* ***fiber****.*

The map p is asked to satisfy the following constraint: there exists a collection $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$, called a ***trivialization cover****, where the collection $\{(U_\alpha)\}_{\alpha\in A}$ is an open cover of B, and for each $\alpha \in A$, a diffeomorphism*

$$\phi_\alpha \colon p^{-1}(U_\alpha) \to U_\alpha \times F$$

such that the following diagram is commutative:



$$\forall\ (x, f) \in U_\alpha \times F, p \circ \phi_\alpha^{-1}(x, f) = x.$$

In the diagram, p_1 denotes the projection on the first factor.

Remark 2.6.2. *One may want to insist on the diffeomorphism type of the fiber, in which case a fiber bundle of fiber F is called an* F**-bundle**.

Definition 2.6.3. *A fiber bundle is said to be* ***orientable*** *if there exists a continuous choice of orientations of the fibers.*

Remark 2.6.4. *This does not mean that the global space is orientable. For example, any space of the form $B \times F$, where B is not orientable, and F is orientable, is a trivial orientable F-bundle over B, even if the space $B \times F$ is not orientable. The condition of orientability simply means that one can pick orientations of the fibers, and that these orientations will not be subject to any monodromy phenomenon.*

In all this work, by fiber bundle we mean **smooth** fiber bundle, i.e. so that the total space is smooth.
Let $M \xrightarrow{p} B$ be an orientable $\mathbb{S}^1$-bundle with oriented total space M over an orientable closed smooth surface B. Set an orientation of the base and one of the fibers so that, taken together, they give the orientation of M. Let C be a simple closed curve in B, splitting B in two closed surfaces B_1 and B_2 with C as their common boundary. Denote

$$M_1 := p^{-1}(B_1), M_2 := p^{-1}(B_2), \text{ and } T := M_1 \cap M_2$$

The torus T is the common boundary of M_1 and M_2. It can be oriented either as the boundary of M_1 or as the one of M_2. The two choices determine opposite orientations of T.

Now, we have the following:

Lemma 2.6.5. *Every orientable $\mathbb{S}^1$-bundle over a connected compact surface with non-empty boundary is trivial.*

Using lemma 2.6.5, each M_i can be written as a trivial fibration. In particular they admit global sections. Let Σ_i be such a section in each M_i. Each Σ_i inherits the orientation of B, and this orients also their boundaries in T.

Definition 2.6.6. *In the setting of the previous paragraph, the* ***Euler number*** *of the fibration* $M \xrightarrow{p} B$ *is the intersection number* $\partial\Sigma_1 \cdot \partial\Sigma_2$ *in* $H_1(T, \mathbb{Z})$*, where* T *is oriented as the boundary of* M_1*.*

Remark 2.6.7.

1. *The definition of the Euler number is the same if one reverses the roles of* M_1 *and* M_2*, i.e. if one computes* $\partial\Sigma_2 \cdot \partial\Sigma_1$ *in* $H_1(T, \mathbb{Z})$*, orienting here* T *as the boundary of* M_2*.*

2. *The convention chosen is not arbitrary: if the* $\mathbb{S}^1$*-fibration over an orientable smooth closed surface considered is made of the unit circles in the tangent fibered space, one will find the* ***Euler-Poincaré characteristic*** *of the base, which gives its name to the Euler number.*

3. *The Euler number is in fact the obstruction to the existence of a global section of the fibration. Such a section exists if and only if this number is zero.*

4. *In particular, one can see that reversing the orientation of* M *changes the sign of the Euler number, but, if one keeps the orientation of* M*, changing that of the fibers or of the base will not change the Euler number.*

There is a fundamental correspondence between Euler numbers and self-intersection numbers, explicited in the following:

Proposition 2.6.8. *If* B *is a smooth oriented closed surface embedded in a smooth oriented* 4*-manifold* M*, the Euler number of the normal* $\mathbb{S}^1$*-bundle to* B*, whose total space is oriented as boundary of a tubular neighbourhood of* B*, is equal to the self-intersection of* B *inside* M*.*

Example 2.6.9. *Consider the Hopf fibration on* $\mathbb{S}^3$*, built in the following way: see* $\mathbb{S}^3$ *as the unit sphere around the origin in* $\mathbb{C}^2$*. Every complex line through the origin will intersect* $\mathbb{S}^3$ *along a circle. Those circles are disjoint. Furthermore, the set of all those lines, being* $\mathbb{CP}^1$*, can be identified with* $\mathbb{S}^2$*. This provides* $\mathbb{S}^3$ *as an* $\mathbb{S}^1$*-fibration over* $\mathbb{S}^2$*. The Euler number of this fibration is* -1*.*

Now, one can consider the blowing-up $(X, E) \xrightarrow{\pi} (\mathbb{C}^2, 0)$ *of the origin in* $\mathbb{C}^2$*. The exceptional divisor* E *is a* $\mathbb{CP}^1$*. What we just saw implies that the normal* $\mathbb{S}^1$*-bundle over* E *in* X *is exactly the Hopf fibration, of Euler number* -1*. On the other hand,* -1 *is also the self-intersection of* E *in* X*.*

This correspondence plays a key role in the computation of boundaries of isolated singularities of complex surfaces, or even, in our case, of isolated singularities of real-analytic 4-dimensional varieties, using a more general statement that allows one to compute the boundary of a neighbourhood of a configuration of surfaces in a 4-manifold (see theorem 2.7.9).

We can now state the following fundamental proposition about classification of $\mathbb{S}^1$-bundles over surfaces:

Proposition 2.6.10. *Given a smooth closed orientable surface B of genus g and $e \in \mathbb{Z}$, there exists, up to orientation-preserving isomorphism of fiber bundles, a unique oriented $\mathbb{S}^1$-bundle M over B with e as its Euler number.*

2.6.2 Graphs, decorations and coverings

In this subsection, we recall mostly the material exposed in the first section of [44], also recalled in [46, Chapter 5].

Definition 2.6.11. *A **graph** Γ is the data of two finite sets $\mathscr{V}(\Gamma)$, and $\mathscr{E}(\Gamma)$, together with a map end: $\mathscr{E}(\Gamma) \to \{$subsets of $\mathscr{V}(\Gamma)$ with at most two elements$\}$.*

*The set $\mathscr{V}(\Gamma)$ is called the set of **vertices** of Γ, $\mathscr{E}(\Gamma)$ is called the set of **edges** of Γ, and if $e \in \mathscr{E}(\Gamma)$ and $end(e) = \{v_1, v_2\}$, v_1 and v_2 are called the **end-vertices** of e.*

Remark 2.6.12. *Note that we do not talk about first or second end-point, in order to work in the setting of **undirected** graphs.*

Definition 2.6.13. *An edge e such that $end(e)$ has only one element v is called a **loop** based at v.*

Definition 2.6.14. *A **topological realization** of a graph Γ is a 1-dimensional CW-complex $|\Gamma|$ whose 0-cells and 1-cells correspond respectively to the vertices and edges of Γ, with the appropriate combinatorics.*

If $|\Gamma|$ is a topological realization of Γ, denote c_Γ the rank of $H_1(|\Gamma|, \mathbb{Z})$, following [44, 1.2].

Remark 2.6.15. *Every graph admits a topological realization, which is unique up to homeomorphism. Therefore its first Betti number c_Γ is well-defined.*

Definition 2.6.16. *A connected graph Γ such that $c_\Gamma = 0$ is called a **tree**.*

Definition 2.6.17. *A **covering tree** for a connected graph Γ is a subgraph Γ' of Γ which is a tree and such that $\mathscr{V}(\Gamma') = \mathscr{V}(\Gamma)$.*

Remark 2.6.18. *A covering tree always exists and is not unique as soon as Γ is not a tree.*

Definition 2.6.19. *A **morphism of graphs** $p \colon \Gamma_1 \to \Gamma_2$ consists of two maps*

$$p_{\mathscr{V}} \colon \mathscr{V}(\Gamma_1) \to \mathscr{V}(\Gamma_2)$$

and

$$p_{\mathscr{E}} \colon \mathscr{E}(\Gamma_1) \to \mathscr{E}(\Gamma_2)$$

such that, if v_1, v_2 *are the end-vertices of* $e \in \mathscr{E}(\Gamma_1)$*, then* $p_{\mathscr{V}}(v_1)$ *and* $p_{\mathscr{V}}(v_2)$ *are the end-vertices of* $p_{\mathscr{E}}(e)$*.*

Such a map is called an ***isomorphism of graphs*** *if both* $p_{\mathscr{V}}$ *and* $p_{\mathscr{E}}$ *are bijective.*

Definition 2.6.20. *A* ***decorated graph*** *is a graph* Γ *with additional data, both on* $\mathscr{V}(\Gamma)$ *and on* $\mathscr{E}(\Gamma)$*. In this work the decorations will be of different natures.*

Let Γ *be a graph, and* $\overset{\star}{\Gamma}$ *be a decoration of* Γ*. We say that* Γ *is* ***the graph associated to*** $\overset{\star}{\Gamma}$*.*

Example 2.6.21. *A very classical decoration, that will be at least implicitly present in every graph of this work, makes a distinction between two types of vertices. Write the set of vertices of* Γ *as a disjoint union*

$$\mathscr{V}(\Gamma) = \mathscr{N}(\Gamma) \sqcup \mathscr{A}(\Gamma).$$

The set $\mathscr{N}(\Gamma)$ *is called the set of* ***nodes*** *of* Γ*, while vertices of* $\mathscr{A}(\Gamma)$ *are called* ***arrowheads****. Nodes will be represented by dots* $\bullet$ *while arrowheads are represented by arrowheads* $\rightarrow$*.*

In this work, every vertex is a node, unless stated otherwise.

Definition 2.6.22. *Let* $\overset{\star}{\Gamma}_1$*,* $\overset{\star}{\Gamma}_2$ *be two decorated graphs, with associated graphs* Γ_1 *and* Γ_2*.*

The decorated graphs $\overset{\star}{\Gamma}_1$ *and* $\overset{\star}{\Gamma}_2$ *are said to be* ***isomorphic*** *if there is an isomorphism between* Γ_1 *and* Γ_2 *that preserves the decorations.*

Definition 2.6.23. *(See [44, 1.3].) We say that* $\mathbb{Z}$ ***acts on a graph*** Γ *if there are group actions*

$$a_{\mathscr{V}} \colon \mathbb{Z} \times \mathscr{V}(\Gamma) \to \mathscr{V}(\Gamma)$$

and

$$a_{\mathscr{E}} \colon \mathbb{Z} \times \mathscr{E}(\Gamma) \to \mathscr{E}(\Gamma)$$

such that if v_1 *and* v_2 *are the end-vertices of* $e \in \mathscr{E}(\Gamma)$*, then* $a_{\mathscr{V}}(1, v_1)$ *and* $a_{\mathscr{V}}(1, v_2)$ *are the end-vertices of* $a_{\mathscr{E}}(1, e)$*.*

If Γ_1 *and* Γ_2 *are both endowed with* $\mathbb{Z}$*-actions, a morphism* $p \colon \Gamma_1 \to \Gamma_2$ *is called* ***equivariant*** *if the maps* $p_{\mathscr{V}}$ *and* $p_{\mathscr{E}}$ *are both equivariant with respect to the actions of* $\mathbb{Z}$*, i.e. if they commute with the action of* $\mathbb{Z}$*. If, in addition,* p *is an isomorphism, such a morphism is called an* ***equivariant isomorphism*** *of graphs.*

Definition 2.6.24. *Let* F *be a finite set. A* ***cyclic order*** *on* F *is an automorphism of* F *that generates a transitive group of automorphisms. That is, as a permutation it is a cycle.*

Example 2.6.25. *A $\mathbb{Z}$-action on a graph Γ induces a cyclic order on each orbit, made either of edges or vertices, under this action. Reciprocally, the data of cyclic orders on sets partitioning $\mathscr{V}(\Gamma)$ and $\mathscr{E}(\Gamma)$, with the appropriate compatibility axioms, gives rise to a $\mathbb{Z}$-action on Γ.*

Definition 2.6.26. *Let Γ be a graph, endowed with the trivial action of $\mathbb{Z}$. A* ***$\mathbb{Z}$-covering****, or* ***cyclic covering****, of Γ, is a graph G such that $\mathbb{Z}$ acts on G, together with an equivariant map $p\colon G \to \Gamma$, such that the restriction of the action of $\mathbb{Z}$ to any set of the type $p^{-1}(v)$ or $p^{-1}(e)$ is transitive.*

Notation 2.6.27. *In this setting, if $v \in \mathscr{V}(\Gamma)$ and $e \in \mathscr{E}(\Gamma)$, then the elements of $p^{-1}(v)$ are called the* ***vertices associated to*** *v, while the elements of $p^{-1}(e)$ are called the* ***edges associated to*** *e.*

Remark 2.6.28. *Let G be a cyclic covering of a graph Γ. For any $v \in \mathscr{V}(\Gamma)$ or $e \in \mathscr{E}(\Gamma)$, denote n_v, respectively n_e, the cardinal of $p^{-1}(v)$, respectively $p^{-1}(e)$. Then if $v_1, v_2 \in \mathscr{V}(\Gamma)$ are the end-vertices of $e \in \mathscr{E}(\Gamma)$, there is $d_e \in \mathbb{N}^*$ such that*

$$n_e = d_e \cdot lcm(n_{v_1}, n_{v_2}).$$

Definition 2.6.29. *A* ***covering data*** *for a graph Γ is a system of positive integers $(\mathbf{n_v}, \mathbf{n_e}) = \big\{\{n_v\}_{v\in\mathscr{V}(\Gamma)}, \{n_e\}_{e\in\mathscr{E}(\Gamma)}\big\}$ such that for all $e \in \mathscr{E}(\Gamma)$ with end-vertices v_1, v_2, the number n_e is a multiple of $lcm(n_{v_1}, n_{v_2})$.*

Definition 2.6.30. *Two cyclic coverings G_1, G_2 of a graph Γ, with maps $p_i\colon G_i \to \Gamma$, are called* ***equivalent*** *if there is an equivariant isomorphism $q\colon G_1 \to G_2$ such that $p_1 = p_2 \circ q$.*

The set of equivalence classes of cyclic coverings of Γ with covering data $(\mathbf{n_v}, \mathbf{n_e})$ is denoted $\mathscr{G}(\Gamma, (\mathbf{n_v}, \mathbf{n_e}))$.

Theorem 2.6.31. *[44] The set $\mathscr{G}(\Gamma, (\mathbf{n_v}, \mathbf{n_e}))$ has an abelian group structure, and it is independent of the numbers $\mathbf{n_e}$ such that $(\mathbf{n_v}, \mathbf{n_e})$ is a covering data for Γ.*

See [44, 1.11] for a description of the identity element of this group.

The following results expose cases where the covering data is enough to determine the isomorphism type of the graph G covering Γ.

Theorem 2.6.32. *[44, Theorem 1.19] If Γ is a tree, then for any covering data $(\mathbf{n_v}, \mathbf{n_e})$, $\mathscr{G}(\Gamma, (\mathbf{n_v}, \mathbf{n_e})) = \{0\}$.*

Theorem 2.6.33. *[44, Theorem 1.20] Let Γ be a graph, and denote $\Gamma_{\neq 1}$ the subgraph of Γ obtained from Γ by deleting every $v \in \mathscr{V}(\Gamma)$ such that $n_v = 1$, and every edge having one of those vertices as an endpoint. If $\Gamma_{\neq 1}$ is a disjoint union of trees, then for any covering data $(\mathbf{n_v}, \mathbf{n_e})$, $\mathscr{G}(\Gamma, (\mathbf{n_v}, \mathbf{n_e})) = \{0\}$.*

The next result exposes a case where the covering data may not be enough for the determination of the isomorphism type of the covering graph, but where the missing data is simple to describe. This may be useful in special cases of the method of computation presented in Chapter 4, although there are no known examples yet.

Definition 2.6.34. *A graph is* ***cyclic*** *if* $\mathscr{V}(\Gamma) = \{v_1, \cdots, v_k\}$ *and* $\mathscr{E}(\Gamma) = \{e_1, \cdots, e_k\}$*, such that the end points of* e_i *are* v_i, v_{i+1}*, where* v_{k+1} *denotes* v_1*. In other words, a graph is cyclic if its topological realization is homeomorphic to a circle.*

It may be shown that any graph Γ with $c_\Gamma = 1$ has a unique cyclic subgraph.

Proposition 2.6.35. *[44, Corollary 1.22] Let* Γ *be a graph such that* $c_\Gamma = 1$*, and let* Γ' *be the unique cyclic subgraph of* Γ*. Let* $(\mathbf{n_v}, \mathbf{n_e})$ *be a covering data for* Γ*, and denote* $d := gcd\{n_v, v \in \mathscr{V}(\Gamma')\}$*. Then*

$$\mathscr{G}(\Gamma, (\mathbf{n_v}, \mathbf{n_e})) = \mathbb{Z}_d.$$

In particular, if $d = 1$, the covering data determines the covering graph uniquely.

2.6.3 Graph manifolds

Let us explain here how, in this work, a decorated graph encodes an oriented 3-manifold. See [47] for more details. The manifolds described here are also called **plumbed manifolds**, explored for the first time in themselves as a special class of 3-manifolds by Waldhausen in [65], after having been introduced implicitly in the work [40] of Mumford.

Definition 2.6.36. *Let* Γ *be a graph with decorations of the following type:*

- *Each edge is decorated by a* $\oplus$ *or* $\ominus$ *symbol. We may omit the* $\oplus$ *decoration when representing such graphs.*
- *Each vertex* v *is decorated by a self-intersection* $e_v \in \mathbb{Z}$ *and a genus* $[g_v]$*,* $g_v \in \mathbb{N}$*. We may omit the genus decoration if it is* 0*.*

Such a graph is called an ***orientable plumbing graph****.*

Define the oriented manifold M_Γ *associated to* Γ *in the following way: for each vertex* v *of* Γ *decorated by* $([g_v], e_v)$*, let* M_v *be an oriented* 3*-manifold which is a fibration in* $\mathbb{S}^1$ *of Euler number* e_v *over a closed smooth surface* B_v *of genus* g_v*. Pick an orientation of the base and the fibers so that, taken together, they give the orientation of* M_v*.*

Now, let λ_v *be the number of times the vertex* v *appears as endpoint of an edge. Remove from* B_v *disjoint open disks* $(D_i)_{1 \leqslant i \leqslant \lambda_v}$*, consequently removing as many open solid tori from* M_v*. Each* ∂D_i *is oriented as boundary component of* $B_v \setminus \bigsqcup D_i$*. Denote by* M_v^b *the resulting circle bundle with boundary. Denote* $\partial M_v^b = \bigsqcup T_i$ *a disjoint union of tori.*

For every edge between the vertices v and v', glue the manifolds M_v^b and $M_{v'}^b$ in the following way: pick boundary components $T = \partial D \times \mathbb{S}^1$ in M_v^b, $T' = \partial D' \times \mathbb{S}^1$ in $M_{v'}^b$, and glue T and T' according to the matrix $\epsilon \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, ϵ being the sign on the edge.

Finally, if the graph has several connected components, the resulting manifold is the connected sum of the manifolds corresponding to the different connected components.

The initial graph Γ is what we call a ***plumbing graph*** *representing the final manifold M_Γ obtained by this construction, and the oriented manifold M_Γ is called the* **manifold plumbed according to the graph** Γ*, or the* ***graph manifold*** *associated to Γ.*

Remark 2.6.37. *Let us insist on the fact that a loop based at v in the graph Γ imposes one to remove two disjoint open disks form the base B_v.*

Note that, in the litterature, one may accept negative genera, corresponding to nonorientable surfaces. For the denomination "orientable plumbing graphs", we follow [47, 3.2(i)].

2.6.4 The plumbing calculus

In this section we introduce a version of Neumann's **plumbing calculus**. We follow entirely the notations of [46, Section 4.2], consistently with [47]. Note that we do not allow moves **[R2], [R4]** of [47], as they produce pieces whose bases are non-orientable surfaces.

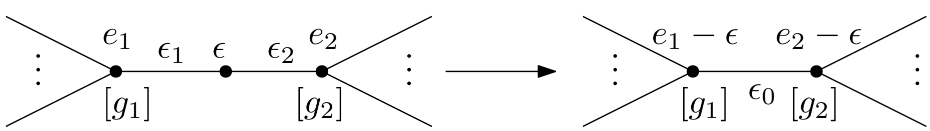
Proposition 2.6.38. *Two manifolds M_{Γ_1} and M_{Γ_2} associated to the plumbing graphs Γ_1 and Γ_2 are orientation-preserving diffeomorphic if and only if one may obtain one graph from the other by applying the following operations or their inverses.*

[R0](a) Reverse the signs on all edges other than loops adjacent to any fixed vertex.

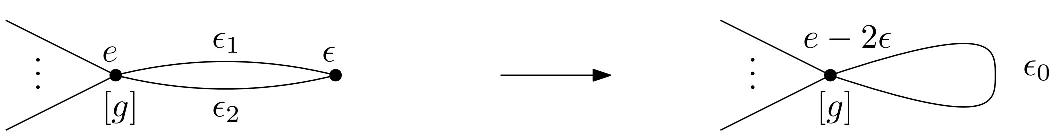
[R1] (blowing down) Here, $\epsilon = \pm 1$, and $\epsilon_0, \epsilon_1, \epsilon_2$ verify $\epsilon_0 = -\epsilon\epsilon_1\epsilon_2$.

a)

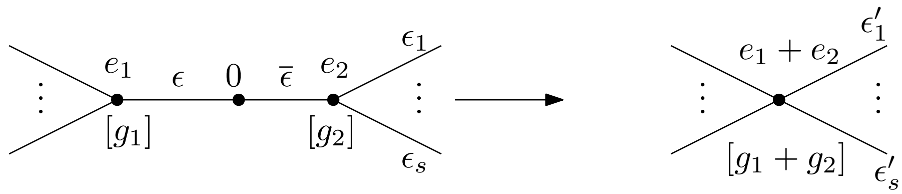
b)



c)

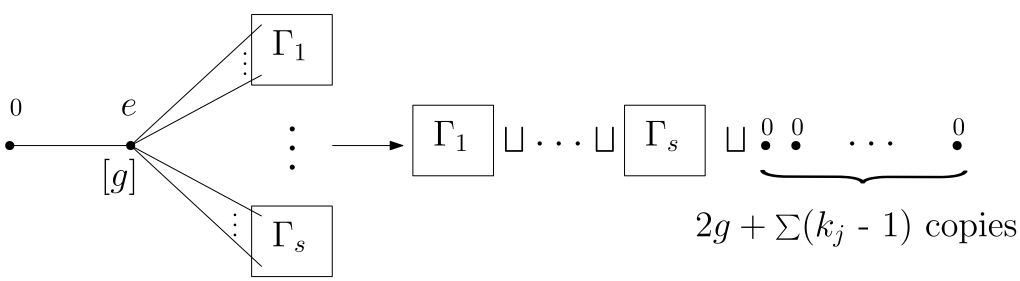


[R3] (0-chain absorption) Here, $\epsilon_i' = \epsilon_i$ if the edge in question is a loop, and $\epsilon_i' = -\epsilon\bar{\epsilon}\epsilon_i$ otherwise.

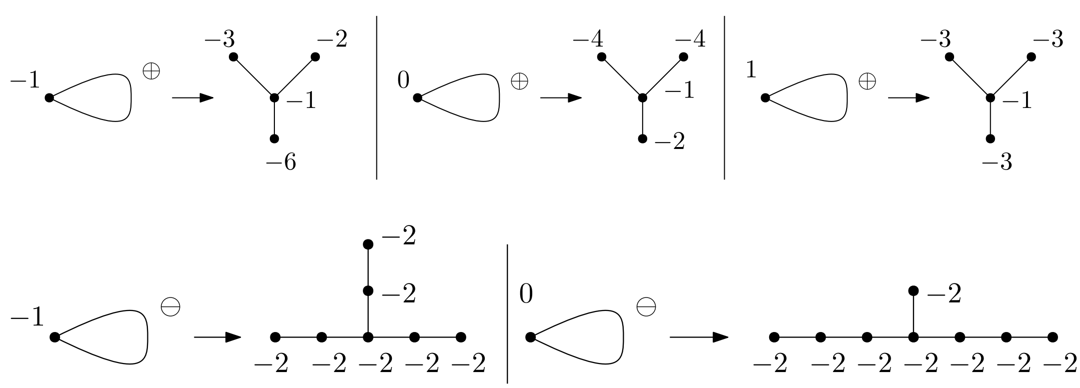


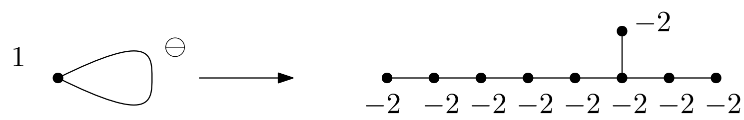
[R5] (oriented handle absorption)

[R6] (Splitting) Here, each Γ_j is a connected graph, and is connected to the vertex e by k_j edges.



[R7] (Seifert graph exchange) According to the original graph, one of the six following modifications:





Remark 2.6.39. *In [46], the authors exclude moves* **[R6],[R7]**, *as operating these moves changes some invariants they are studying, related to monodromy phenomena. See Paragraph 4.2.3 in [46].*

Remark 2.6.40. *Observe that the inverse of move* ***[R6]*** *does not preserve the planarity of graphs. Indeed the three graphs of figure 2.1 are all equivalent, but the first one is not planar, as it contains the complete graph K_5 with five vertices. The two others are planar, one is connected, the other is not.*
However, note that all moves of the plumbing calculus, in the direction in which they are presented, preserve planarity. Furthermore, the method presented in [47] for the computation of the normal form of a graph preserves planarity.
Finally, an example of a non-planar graph in normal form is given by the graph K_5 with any decoration. The fact that the operations leading to the normal form of a graph preserve planarity implies that this graph is not equivalent to a planar graph.

2.7 Boundary of a tubular neighbourhood

The goal of this section is to provide a proper definition of a **tubular neigbourhood** of a **simple configuration of compact real-analytic surfaces** in a 4-dimensional real-analytic manifold, and to express the correspondence between the curve configuration and the topology of the boundary of its tubular neighbourhood.

2.7.1 Plumbing graph associated to a simple configuration of surfaces

Let us start with a general definition that will simply be a matter of vocabulary, valid in a very general setting:

Definition 2.7.1. (Dual graph of a collection.)
Let $E = \bigcup_{finite} E_i$ be a finite covering of a set E by sets $(E_i)_{i \in I}$, satisfying the following two conditions:

1. *For any $i \neq j$, $E_i \cap E_j$ is finite.*
2. *For any $i \neq j \neq k \neq i$, $E_i \cap E_j \cap E_k = \emptyset$.*

The dual graph $\Gamma(E)$ of the covering is constructed in the following way:

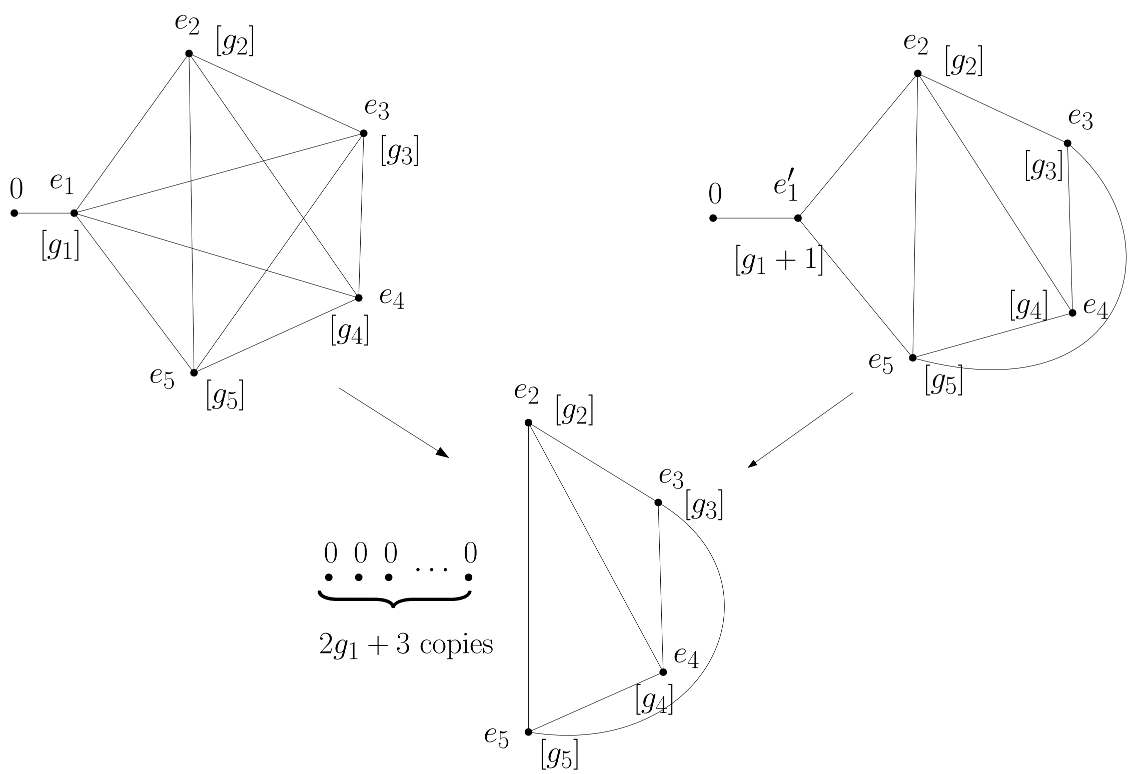


Figure 2.1: Three equivalent plumbing graphs.

- *It has $(v_{E_i})_{i\in I}$ as set of vertices.*
- *The edges having v_{E_i} and v_{E_j} as end-points correspond bijectively to the intersection points of E_i and E_j.*

Definition 2.7.2. (Simple configuration of surfaces, and its plumbing dual graphs.) *Let $\widetilde{\mathscr{S}}$ be a 4-dimensional oriented real-analytic manifold. A* ***simple configuration of compact real-analytic surfaces in $\widetilde{\mathscr{S}}$*** *is a subset $E \subset \widetilde{\mathscr{S}}$ such that:*

1. *$E = \bigcup_{finite} E_i$, such that each E_i is an oriented closed smooth real-analytic surface.*
2. *For all $i \neq j \neq k \neq i$, the intersection $E_i \cap E_j \cap E_k$ is empty.*
3. *For all $i \neq j$, the intersection $E_i \cap E_j$ is either empty or transverse. In particular, it is a finite union of points.*

In this setting, one defines a ***plumbing dual graph*** *$\Gamma_{\widetilde{\mathscr{S}}}(E)$ of E in $\widetilde{\mathscr{S}}$ by decorating its dual graph in the following way:*

1. *Decorate each vertex v_{E_i} by the self-intersection e_i of E_i in $\widetilde{\mathscr{S}}$ and by the genus $[g_i]$ of the surface E_i.*

2. *Decorate each edge of $\Gamma(E)$ corresponding to the intersection point p of $E_i \cap E_j$, by $\oplus$ if the orientation of E_i followed by the orientation of E_j is equal to the orientation of $\widetilde{\mathscr{S}}$ at p, and by $\ominus$ otherwise.*

Remark 2.7.3. *For details about the history of the notion of dual graph, one may consult the article [52].*

2.7.2 Computation of self-intersections

In our context, one will compute the desired self-intersections by using the multiplicities of some special function:

Definition 2.7.4. *Let $E = \bigcup_{finite} E_i$ be a simple configuration of compact oriented real-analytic surfaces in a 4-dimensional real-analytic manifold $\widetilde{\mathscr{S}}$. A real-analytic function $g \colon \widetilde{\mathscr{S}} \to \mathbb{C}$ is called **adapted** to E if*

1. *$E_{tot} := g^{-1}(0)$ is a simple configuration of oriented, not necessarily compact, real-analytic surfaces, such that $E_{tot} \supset E$.*

2. (a) *For any component E_i of E_{tot}, $\forall\, p \in E_i \setminus \bigcup_{j \neq i} E_j$, there is a neighbourhood U_p of p in $\widetilde{\mathscr{S}}$ and complex coordinates (x_p, y_p) on U_p such that $U_p \cap E_i = \{x_p = 0\}$ and $n_i \in \mathbb{N}^*$ such that*
$$g = x_p^{n_i} \cdot \varphi$$
where $\varphi \colon U_p \to \mathbb{C}$ is a unit at p.

   (b) *For any components E_i of E, E_j of E_{tot}, $\forall\, P \in E_i \cap E_j$, there is a neighbourhood U_p of p in $\widetilde{\mathscr{S}}$ and complex coordinates (x_p, y_p) on U_p such that $U_p \cap E_i = \{x_p = 0\}$, $U_p \cap E_k = \{y_p = 0\}$, and $n_i, n_k \in \mathbb{N}^*$ such that*
$$g = x_p^{n_i} y_p^{n_j} \cdot \varphi$$
where $\varphi \colon U_p \to \mathbb{C}$ is a unit at p.

Definition 2.7.5. *In this setting, the integer n_i of point 2a of Definition 2.7.4, independent of the point $p \in E_i \setminus \bigcup_{j \neq i} E_j$, is called **the multiplicity of** g **on** E_i, denoted $m_{E_i}(g)$.*

Remark 2.7.6. *In general, there is no reason to believe that an adapted function will exist. However, in our setting, we will have access to such a function. See Sections 4.11, 4.12.*

Lemma 2.7.7. (Computing self-intersections.)
Let $\widetilde{\mathscr{S}}, E, g, E_{tot}$ be as in Definition 2.7.4. Let $E^{(1)}$ be an irreducible component of E. Then the self-intersection $e^{(1)}$ of the surface $E^{(1)}$ in $\widetilde{\mathscr{S}}$ verifies the following condition: let $p_1, \cdots, p_n$ be the intersection points of $E^{(1)}$ with other components of E_{tot}, $p_j \in E_j \cap E^{(1)}$, where the same component may appear several times. Then

$$n^{(1)} \cdot e^{(1)} = \sum_{i=1}^{n} \epsilon_i \cdot n_i$$

where $\epsilon_i \in \{-1, +1\}$ refers to the sign associated to the intersection p_i in the following sense: if $p \in E_i \cap E_j$, associate $+1$ to p if and only if the combination of the orientations of E_i and E_j at p provides the ambient orientation of $\widetilde{\mathscr{S}}$.

Proof. The proof follows the standard argument in the holomorphic category. The difference of the two members of the equation is the intersection number of $E^{(1)}$ with the cycle defined by $g = 0$. This cycle is homologous with that defined by a nearby level of g, which does not meet $E^{(1)}$ any more. The intersection number being invariant by homology, one gets the desired result.

In order to make this argument rigorous, one has to work in convenient tubular neighborhoods of E and to look at the cycles defined by the levels of g in the homology of the tube relative to the boundary. □

2.7.3 Correspondence between neighbourhoods and graphs

Definition 2.7.8. *(See [15].) Let E be a simple configuration of compact orientable real-analytic surfaces in an oriented 4-dimensional real-analytic manifold $\overline{\mathscr{S}}$. In this context, we call* ***rug function*** *any real-analytic proper function $\rho\colon \overline{\mathscr{S}} \to \mathbb{R}_+$ such that $\rho^{-1}(0) = E$.*

Theorem 2.7.9. *Let E be a simple configuration of compact orientable real-analytic surfaces in an oriented 4-dimensional real analytic manifold $\overline{\mathscr{S}}$, that admits a rug function ρ as in Definition 2.7.8. Then, for $\varepsilon > 0$ small enough, the boundary of the oriented 4-manifold $\{\rho \leqslant \varepsilon\}$ is orientation-preserving homeomorphic to the graph manifold associated to the graph $\Gamma_{\widetilde{\mathscr{S}}}(E)$.*

Definition 2.7.10. *In this setting, the manifold $\{\rho \leqslant \varepsilon\}$ is called a* ***tubular neighbourhood*** *of E.*

About the proof of Theorem 2.7.9. This theorem can be seen as an extension of what is done in [40], in the case of a configuration of complex analytic curves in a smooth complex surface.

In our case, observe first that we can extend the definition of rug functions to semi-analytic functions ρ, and still have a unique homeomorphism type for the boundary of the neighbourhood $\{\rho \leqslant \varepsilon\}$ of E for $\varepsilon > 0$ small enough, following the proof of [15, Proposition

3.5]. Now, one can build by hand a semi-analytic neighbourhood whose boundary is homeomorphic to the manifold $\Gamma_{\widetilde{\mathscr{F}}}(E)$.

This is done by building a rug function for each irreducible component E_i of E, providing a tubular neighbourhood T_i of each E_i whose boundary is an $\mathbb{S}^1$-bundle of Euler class e_i over E_i. One then plumbs those bundles using appropriate normalizations of the rug functions, building a semi-analytic neighbourhood of E which is homeomorphic to the desired graph manifold. □

Remark 2.7.11. *Note that the decorations on the edges of the graph $\Gamma_{\widetilde{\mathscr{F}}}(E)$ depend on the orientations of the surfaces E_i. However, if the surfaces E_i are only orientable, the different possible plumbing dual graphs still encode the same graph manifold, see move* **[R0]** *of the plumbing calculus.*

Chapter 3

Elements of toric geometry

What follows is mainly based on the book [18] of Fulton. To go further, one can consult the books [48] of Oda, [10] of Cox, Little, Schenk, the introductory article [13] of Danilov, as well as the introductory courses [5] of Brasselet and [43] of Musţată.

3.1 Lattices and cones

Definition 3.1.1. *A n-dimensional **lattice** N is a free group of rank n, that is, a group isomorphic to $(\mathbb{Z}^n, +)$. Define the **integral length** of an element $u \in N$ as*

$$l(u) = max\{n \in \mathbb{N}, \exists\ v \in N \setminus \{0\} \text{ such that } u = n \cdot v\}.$$

*An element $u \in N$ is called **primitive** if $l(u) = 1$.*

Denote by $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ the $\mathbb{R}$-vector space associated to N.

Remark 3.1.2. *Note that $l(0) = 0$.*

Once N is identified with $\mathbb{Z}^n$, a vector $u \in N$ is primitive iff its coordinates are coprime as a whole.

Definition 3.1.3. *For $(S, \cdot)$ sub-semigroup of $(\mathbb{R}, \cdot)$, and $v_1, \ldots, v_k \in N$, denote*

$$\langle v_1, \ldots, v_k \rangle_S := \{r_1 \cdot v_1 + \cdots + r_k \cdot v_k, r_1, \ldots, r_k \in S\} \subseteq N_{\mathbb{R}}.$$

*A **rational convex polyhedral cone** σ in $N_{\mathbb{R}}$ is a set of the form $\langle v_1, \ldots, v_k \rangle_{\mathbb{R}_+}$, for some $v_1, \ldots, v_k \in N$, said to be **generating** σ. Let us define the **dimension** of the cone σ as the dimension of $\langle v_1, \ldots, v_k \rangle_{\mathbb{R}}$, the subvector space of $N_{\mathbb{R}}$ generated by $(v_1, \ldots, v_k)$. We will also refer to the **codimension** of σ, defined as the codimension of that same space in $N_{\mathbb{R}}$.*

*Such a cone will be called **strongly convex** if it does not contain any vector subspace of $N_{\mathbb{R}}$.*

In the sequel, all the cones will be rational, convex and polyhedral, so we will omit these terms.

Definition 3.1.4. *A d-dimensional cone σ will be called **simplicial** if it can be generated by d vectors. Furthermore, in this case there is a unique set of d primitive vectors $v_1, \ldots, v_d \in N$, called its **primitive generators**, such that $\sigma = \langle v_1, \ldots, v_d \rangle_{\mathbb{R}_+}$.*

*With these notations, the simplicial cone σ is said to be **regular** if the family $(v_1, \ldots, v_d)$ generates the semigroup $(\sigma \cap N, +)$. All other cones are called **singular**.*

Definition 3.1.5. *For $v_1, \ldots, v_d \in N$, let us define $det(v_1, \ldots, v_d) := l(v_1 \wedge \cdots \wedge v_d) \in \mathbb{N}$, in the lattice $\bigwedge^d N$.*

If N is identified to $\mathbb{Z}^n$, then $det(v_1, \ldots, v_d)$ is equal to the gcd of the $d \times d$ minors of the matrix $(v_1, \ldots, v_d)$.

Lemma 3.1.6. Criterion for regularity.

Let σ be a simplicial cone, and $(v_1, \ldots, v_d)$ its set of primitive generators. Then

$$\sigma \text{ is regular if and only if } det(v_1, \ldots, v_d) = 1.$$

In other words, $\sigma \subset N_{\mathbb{R}}$ is regular if and only if it is generated by a sub family of a basis of the $\mathbb{Z}$-module N.

Definition 3.1.7. *We will usually denote by $M := N^{\checkmark} = Hom(N, \mathbb{Z}) \simeq \mathbb{Z}^n$ the **dual lattice** of N.*

*For σ a convex cone of dimension d in $N_{\mathbb{R}}$ of dimension n, let us define its **dual cone***

$$\sigma^{\checkmark} := \{m \in M_{\mathbb{R}}, \forall\ n \in \sigma, \langle m, n \rangle \geqslant 0\} \subset M_{\mathbb{R}}$$

*and its **orthogonal***

$$\sigma^{\perp} := \{m \in M_{\mathbb{R}}, \forall\ n \in \sigma, \langle m, n \rangle = 0\} \subset \sigma^{\checkmark}.$$

Remark 3.1.8. *The cone $\sigma^{\checkmark}$ is also a rational convex polyhedral cone, while $\sigma^{\perp}$ is a vector subspace of $N_{\mathbb{R}}$ of dimension $n - d$. It is the maximal linear subspace of the cone $\sigma^{\checkmark}$.*

Let us now present some properties of the duality for cones:

Lemma 3.1.9. *Let σ, τ be cones in a lattice.*

1. $(\sigma^{\checkmark})^{\checkmark} = \sigma$.

2. $\tau \subset \sigma \Longleftrightarrow \sigma^{\checkmark} \subset \tau^{\checkmark}$.

3. $\sigma^{\checkmark}$ *is simplicial iff* σ *is simplicial.*

4. *Furthermore, if* σ *is of maximal dimension,* $\sigma^{\checkmark}$ *is regular iff* σ *is regular.*

5. σ *is of maximal dimension iff* $\sigma^{\checkmark}$ *is strongly convex. More generally, the codimension of* σ *is equal to the dimension of the biggest vector space contained in* $\sigma^{\checkmark}$. *This vector space is* $\sigma^{\perp}$.

Definition 3.1.10. *Let* σ *be a cone. A* ***face*** *of* σ *is any cone of the form*

$$\sigma \cap m^{\perp} = \{n \in \sigma, \langle m, n\rangle = 0\}, \text{ for some } m \in \sigma^{\checkmark}.$$

The fact that τ *is a face of* σ *will be denoted* $\tau \preceq \sigma$. *Any face of* σ *different from* σ *is called a* ***proper face****, denoted* $\tau \prec \sigma$.

Define the ***relative interior*** *of a cone* σ *to be*

$$\mathring{\sigma} := \sigma \setminus \left(\bigcup_{\tau \prec \sigma} \tau \right).$$

Remark 3.1.11. *Any face of a strongly convex cone is strongly convex.*

Remark 3.1.12. *There is a* $1-1$ *correspondence between faces of* σ *and faces of* $\sigma^{\checkmark}$ *of complementary dimension, that reverses inclusion, given by:* $\tau \preceq \sigma \leftrightarrow \tau^{\perp} \cap \sigma^{\checkmark} \preceq \sigma^{\checkmark}$.

Note also that if $\tau = \sigma \cap m^{\perp}$, *then* $m \in \tau^{\perp}$.

Lemma 3.1.13. *Let* σ *be a strongly convex cone of dimension* $d \leqslant n$ *in* N, *and* $\tau_1, \ldots, \tau_k$ *its faces of codimension* 1. *Then* $\sigma^{\checkmark}$ *can be written as:*

$$\sigma^{\checkmark} = \sigma^{\perp} \oplus \langle v_1, \ldots, v_k\rangle_{\mathbb{R}_+}$$

where $v_i \in (\tau_i^{\perp} \setminus \sigma^{\perp}) \cap \sigma^{\checkmark}$. *In other words, the* v_i*'s are elements of* $\sigma^{\checkmark}$ *such that*

$$\tau_i = v_i^{\perp} \cap \sigma.$$

Proof. It is clear that $\sigma^{\checkmark}$ can be written as $\sigma^{\checkmark} = \sigma^{\perp} \oplus \mu$, where μ is a strongly convex cone of dimension d. Let us write $\mu = \langle w_1, \ldots, w_l\rangle_{\mathbb{R}_+}$, and recall that $dim(\sigma^{\perp}) = n - d$.

Now, any face of dimension $n-d+1$ of $\sigma^{\checkmark}$ is of the form $\sigma^{\perp} \oplus \langle w_i\rangle_{\mathbb{R}_+}$, and corresponds (by remark 3.1.12) to a face of dimension $d-1$ of σ. This shows that $k = l$.

Using the explicit correspondence gives us that (up to renumbering), $\tau_i^{\perp} \cap \sigma^{\checkmark} = \sigma^{\perp} \oplus \langle w_i\rangle_{\mathbb{R}_+}$, which shows that $w_i \in (\tau_i^{\perp} \setminus \sigma^{\perp}) \cap \sigma^{\checkmark}$. □

Remark 3.1.14. *Furthermore, if M' is a d-dimensional sublattice of M such that $M = M' \oplus \big(\sigma^{\perp} \cap M\big)$, then all the v_i's can be chosen in M'. This means that given a complementary sublattice M' of $\sigma^{\perp} \cap M$ in M, one gets a decomposition*

$$\sigma^{\vee} \cap M = \Big(\sigma^{\perp} \cap M\Big) \oplus \left(\sigma^{\vee} \cap M'\right).$$

This decomposition can in fact be seen more intrinsically.

Definition 3.1.15. *Denote N_σ the sublattice of N generated by the elements of $\sigma \cap N$, and $M_\sigma := N_\sigma^{\vee}$. Then seeing σ as a cone in N_σ, we define its dual $\sigma^{\vee}_{M_\sigma}$ in M_σ.*

Proposition 3.1.16. *Given any complementary sublattice M' of $\sigma^{\perp} \cap M$ in M, there is an isomorphism of semigroups*

$$\left(\sigma^{\vee} \cap M', +\right) \simeq \left(\sigma^{\vee}_{M_\sigma} \cap M_\sigma, +\right).$$

We get as a direct consequence the isomorphism of semigroups

$$\left(\sigma^{\vee} \cap M, +\right) \simeq \sigma^{\perp} \oplus \left(\sigma^{\vee}_{M_\sigma} \cap M_\sigma, +\right).$$

3.2 The affine toric variety defined by a cone

Prologue 3.2.1. In the sequel we will consider affine varieties obtained as spectra of subrings of $A_M := \mathbb{C}[M]$. As a vector space, A_M has a basis formed by elements of the form $\chi^m, m \in M$, with the multiplication determined by the addition in $M : \chi^m \cdot \chi^{m'} = \chi^{m+m'}$.
Let us introduce now the **torus** associated to the lattice N, denoted $\mathcal{T}_N$. It is the maximal ideal spectrum of A_M. It can be seen more concretely as the set of morphisms of groups from $(M, +)$ to $(\mathbb{C}^*, \cdot)$. It is non canonically isomorphic as a group to $((\mathbb{C}^*)^n, \cdot)$, which is an **algebraic torus**. This denomination comes from the fact that $(\mathbb{C}^*)^n$ retracts by deformation to the n-dimensional torus $\left(\mathbb{S}^1\right)^n$. The torus acts on itself by multiplication. If $\phi, \psi \in \mathcal{T}_N$, then the action is defined by:

$$\forall\, m \in M, (\phi \cdot \psi)(m) = \phi(m) \cdot \psi(m).$$

Definition 3.2.2. *Let $\sigma \subset N_{\mathbb{R}}$ be a cone. Denote by*

$$(S_\sigma, +) := (\sigma^{\vee} \cap M, +)$$

*the **semigroup** associated to this cone, and by*

$$A_\sigma := \mathbb{C}[S_\sigma]$$

the algebra generated by this semigroup.

*The **affine toric variety associated to** σ is*

$$X_\sigma := Spec(\mathbb{C}[\sigma^{\vee} \cap M]) = Spec(A_\sigma).$$

Remark 3.2.3. *We see that $dim(X_\sigma) = dim(\sigma^\checkmark)$, hence, as soon as σ is **strongly** convex, by lemma 3.1.9, $dim(X_\sigma) = dim(M) = n$. In the sequel, on the N side, the only cones we will consider will be strongly convex.*

Remark 3.2.4. *A family $m_1, \cdots, m_k \in M$ generating the semigroup $\sigma^\checkmark \cap M$ provides an embedding $X_\sigma \hookrightarrow Spec\,(\mathbb{C}[\chi^{m_1}, \cdots, \chi^{m_k}]) = \mathbb{C}^k{}_{x_1,\cdots,x_k}$.*

Definition 3.2.5. *Given a d-dimensional strongly convex cone σ in $N_\mathbb{R}$, define the **intrinsic** d**-dimensional variety***

$$X_\sigma(N_\sigma) := Spec(\mathbb{C}[\sigma^\checkmark_{M_\sigma} \cap M_\sigma]).$$

Using proposition 3.1.16, we get:

Proposition 3.2.6. *There is a non-canonical isomorphism of affine analytic varieties*

$$X_\sigma \simeq (\mathbb{C}^*)^{n-d} \times X_\sigma(N_\sigma).$$

Proposition 3.2.7. *Closed points of X_σ (maximal ideals of A_σ), which form a dense subset of X_σ, correspond to semigroup morphisms $(S_\sigma, +) \to (\mathbb{C}, \cdot)$.*
This set contains the torus $\mathcal{T}_N$ defined in 3.2.1, corresponding to the semigroup morphisms $(S_\sigma, +) \to (\mathbb{C}^*, \cdot)$.
Furthermore, the torus $\mathcal{T}_N$ is dense in X_σ.

Proof. By strong convexity of σ, $\sigma^\checkmark$ is n-dimensional, hence contains a basis of M. So a semigroup morphism $\sigma^\checkmark \cap M \to \mathbb{C}^*$ is determined by its values on the elements of this basis. In this way we can identify these points with $(\mathbb{C}^*)^n$. Now any semigroup morphism $S_\sigma \cap M \to \mathbb{C}$ can be written as a limit of morphisms $S_\sigma \cap M \to \mathbb{C}^*$, hence the density. ☐

Proposition 3.2.8. *The action of the group $\mathcal{T}_N$ on itself extends to a continuous action on the whole variety X_σ, making $\mathcal{T}_N$ the unique n-dimensional orbit.*

On closed points, the action is defined as on $\mathcal{T}_N$, by multiplication: if $\phi \in X_\sigma$ and $\psi \in \mathcal{T}_N$, then $\phi \cdot \psi \in X_\sigma$ is defined by

$$\forall\ m \in M, (\phi \cdot \psi)(m) = \phi(m) \cdot \psi(m).$$

Proposition 3.2.9. *If $\tau \subset \sigma$, then the inclusion $A_\sigma \subset A_\tau$ gives rise to a canonical birational morphism of algebraic varieties $X_\tau \to X_\sigma$. This morphism is an injection if and only if τ is a face of σ.*

More precisely:

Proposition 3.2.10. *Let τ be a proper face of σ, and $u \in \sigma^\checkmark \cap M$, u primitive, such that $\tau = u^\perp \cap \sigma$. Then:*

1. $S_\tau = S_\sigma + u \cdot \mathbb{Z}_{\leqslant 0}$.

2. X_τ *embeds in* X_σ *as a principal open subset.*

Proof.

1. The inclusion $S_\tau \supset S_\sigma + u \cdot \mathbb{Z}_{\leqslant 0}$ is immediate, because $\sigma^{\checkmark} \supset \tau^{\checkmark}$, and $u \in \tau^{\perp}$.

   Now, let $w \in \tau^{\checkmark} \cap M$. We want to prove the following:

$$\exists\, p \in \mathbb{N}, \forall\, m \in \sigma, \langle w + p \cdot u, m \rangle \geqslant 0.$$

   Let us write $\sigma = \langle m_1, \dots, m_k \rangle_{\mathbb{R}_+}$. For $p \in \mathbb{N}$, and $\sigma \ni m = \sum_{i=1}^{k} a_i m_i$,

$$a_i \in \mathbb{R}_+, \langle w + p \cdot u, m \rangle = \sum_{i=1}^{k} a_i(\langle w, m_i \rangle + p \cdot \langle u, m_i \rangle).$$

   Now, $\forall\ i, \langle u, m_i \rangle \geqslant 0$ because $u \in \sigma^{\checkmark}$, and as soon as $m_i \notin \tau, \langle u, m_i \rangle > 0$. If $m_i \in \tau$, then $\langle w, m_i \rangle \geqslant 0$ so we have nothing to do. Hence we just have to choose $p \geqslant \max_{m_i \notin \tau} \left(-\frac{\langle w, m_i \rangle}{\langle u, m_i \rangle} \right)$

2. It follows from the first assertion that $A_\tau = (A_\sigma)_{\chi^u}$, i.e.

$$X_\tau \cong D(\chi^u) := \{q \in Specm(A_\sigma), (\chi^u) \nsubseteq q\}.$$

□

3.3 The toric variety defined by a fan

Definition 3.3.1. *A* ***fan*** $\mathcal{F}$ *in* $N_\mathbb{R}$ *is a finite set of strongly convex cones such that:*

1. If $\sigma \in \mathcal{F}$*, any face of* σ *is in* $\mathcal{F}$*.*

2. The intersection of two cones of $\mathcal{F}$ *is a face of each.*

The ***support*** $|\mathcal{F}|$ *of the fan* $\mathcal{F}$ *is the union* $\bigcup_{\sigma \in \mathcal{F}} \sigma$ *of the cones composing it.*

Remark 3.3.2. *A cone* σ*, together with the collection of all its faces, defines a fan. For short, in the sequel, we will simply call it* ***the fan*** σ*.*

Definition 3.3.3. *A fan $\mathcal{F}$ in $N_{\mathbb{R}}$ defines a **toric variety** $X_{\mathcal{F}}$ in the following way: take the disjoint union of the X_σ's, for all σ in $\mathcal{F}$, and, if σ and σ' are cones of $\mathcal{F}$, glue X_σ and $X_{\sigma'}$ along $X_{\sigma\cap\sigma'}$.*

$$X_{\mathcal{F}} := \left(\bigsqcup_{\sigma\in\mathcal{F}} X_\sigma\right) \Big/ \left(X_\sigma \underset{X_{\sigma\cap\sigma'}}{\sim} X_{\sigma'}\right)$$

Proposition 3.3.4. *The actions of the torus $\mathcal{T}_N$ on each X_σ glue together, in agreement with the gluing of the X_σ's, giving rise to a global action of the torus on $X_{\mathcal{F}}$, under which the torus $\mathcal{T}_N$ is the unique n-dimensional orbit. Furthermore $\mathcal{T}_N$ is open and dense in $X_{\mathcal{F}}$.*

This is why we call it a **toric variety**. More generally:

Definition 3.3.5. *A **toric variety** is an algebraic variety X containing a torus $\mathcal{T}$ (i.e. a group isomorphic to $((\mathbb{C}^*)^n, .)$, for some n) as a Zariski dense open subset, for which the action of $\mathcal{T}$ on itself by multiplication extends to the whole variety X.*

Proposition 3.3.6. *A toric variety is **normal** if and only if it can be defined by a fan.*

One of the main advantages of such a description of a variety is that much topological information about the variety is encoded combinatorially:

Proposition 3.3.7. (Decomposition into orbits.)
Let σ be a strongly convex polyhedral cone of dimension d belonging to a fan $\mathcal{F}$ in $N_{\mathbb{R}}$, where N is n-dimensional. Let $\tau_1, \ldots, \tau_k$ be the faces of σ of codimension 1.

Then

$$O_\sigma := X_\sigma \setminus \bigcup X_{\tau_i}$$

is an orbit of $X_{\mathcal{F}}$ under the action of $\mathcal{T}_N$. It is made of all morphisms $(\sigma^{\vee} \cap M, +) \to (\mathbb{C}, +)$ that are different from zero exactly on $\sigma^\perp \cap M$.

In other words, a point of O_σ is a maximal ideal of $\mathbb{C}[\sigma^{\vee} \cap M]$ containing the monomial $\chi^m \in \mathbb{C}[\sigma^{\vee} \cap M]$ if and only if $m \notin \sigma^\perp \cap M$.

Furthermore, the variety O_σ can be seen as

$$O_\sigma = Spec(\mathbb{C}[\sigma^\perp \cap M])$$

providing, non-canonically, the isomorphism

$$O_\sigma \simeq (\mathbb{C}^*)^{n-d}.$$

Remark 3.3.8. *In particular, $O_{\{0\}} = T_N$.*

*Furthermore, if $\mathcal{F} = \sigma$, then O_σ is the unique minimal-dimensional orbit in X_σ. If $dim(\sigma) = n$, it is called the **origin** of X_σ, and denoted 0_{X_σ}.*

With this notation, if σ is of maximal dimension, we get the ***germ of toric variety*** $(X_\sigma, 0_{X_\sigma})$*, or* $(X_\sigma, 0)$*, whose local ring of germs of holomorphic functions is*

$$\mathcal{O}_{X_\sigma,0} = \mathbb{C}\{\sigma^\vee \cap M\}.$$

Proof of proposition 3.3.7. An element of $X_\sigma \setminus \bigcup X_{\tau_i}$ is a semigroup morphism $\phi : S_\sigma \to \mathbb{C}$ that is not a semigroup morphism $(S_{\tau_i}, +) \to (\mathbb{C}, \cdot)$, for any i. Let us give a characterization of such morphisms.
We know that $\sigma^\perp \cap M \subset S_\sigma$, and is a group, hence the morphism ϕ must be everywhere different from 0 here.

On the other hand, if $v_i \in S_\sigma$ such that $\tau_i = v_i^\perp \cap \sigma$, we know by proposition 3.2.10 that $S_{\tau_i} = S_\sigma + v_i \cdot \mathbb{Z}_{\leqslant 0}$, hence if $\phi(v_i) \neq 0$, ϕ can be extended to S_{τ_i}. This means that we must have $\phi(v_i) = 0$ for all such v_i, for all i.

This discussion, combined with the decomposition $\sigma^\vee = \sigma^\perp \oplus \langle v_1, \ldots, v_k\rangle_{\mathbb{R}_+}$ provided by lemma 3.1.13, leads to the following characterization:

A point in $X_\sigma \setminus \bigcup X_{\tau_i}$ is a semigroup morphism $\phi : (\sigma^\vee \cap M, +) \to (\mathbb{C}, \cdot)$ such that:

$$\forall\ m \in \sigma^\vee \cap M, \phi(m) \neq 0 \text{ iff } m \in \sigma^\perp.$$

This presents the set of points of $X_\sigma \setminus \bigcup X_{\tau_i}$ as an orbit under the action of the torus. □

Considering this for any cone of $\mathcal{F}$ will give us a decomposition of $X_\mathcal{F}$ into orbits under the action of the torus, a cone of $\mathcal{F}$ corresponding to an orbit:

$$X_\mathcal{F} = \bigsqcup_{\sigma \in \mathcal{F}} O_\sigma.$$

Proposition 3.3.9. (Closure of an orbit.)

The closure in $X_\mathcal{F}$ of an orbit corresponding to a cone $\tau \in \mathcal{F}$ is made of the union of the orbits corresponding to cones of $\mathcal{F}$ having τ as a face:

$$\overline{O_\tau} = \bigcup_{\tau \preceq \sigma} O_\sigma.$$

Remark 3.3.10. *In particular, the unique closed orbits are the minimal-dimensional ones, and we also get again the fact that $\mathcal{T}_N$ is dense in $X_\mathcal{F}$, the cone $\{0\}$ being a face of every cone.*

Proof of proposition 3.3.9. If $\tau \preceq \sigma$, then $S_\sigma \subset S_\tau$, and $\sigma^\perp \subset \tau^\perp$. An element of O_τ is a morphism $(S_\sigma, +) \to (\mathbb{C}, \cdot)$ that is zero exactly outside of $\tau^\perp \cap M$, in particular it is zero outside of $\sigma^\perp \cap M$. Hence any morphism that is zero outside of $\sigma^\perp \cap M$ can be seen as limit of elements of O_τ. Hence

$$\tau \preceq \sigma \Rightarrow O_\sigma \subset \overline{O_\tau}.$$

To conclude the equality, observe that every X_σ is an open subset of $X_\mathcal{F}$, and that if $\tau \npreceq \sigma$, then $O_\tau \cap X_\sigma = \emptyset$. □

Remark 3.3.11. *Note that this implies that $0_{X_\sigma(N_\sigma)}$ is in the closure of every orbit of $X_\sigma(N_\sigma)$, every cone of the fan associated to σ being a face of σ.*

, This leads to the following:

Remark 3.3.12. *The geometry of $X_\sigma(N_\sigma)$ is reflected in the germ $(X_\sigma(N_\sigma), 0_{X_\sigma(N_\sigma)})$.*

Proposition 3.3.13. *The intrinsic variety $X_\sigma(N_\sigma)$ is smooth if and only if σ is regular.*

Proof. Indeed, $X_\sigma(N_\sigma)$ is regular if and only if $\sigma^{\checkmark}_{M_\sigma} \cap M_\sigma \simeq \mathbb{N}^d$, if and only if $\sigma^{\checkmark}_{M_\sigma}$ is regular, if and only if σ is regular. □

Proposition 3.3.14. *Let $\sigma \in \mathcal{F}$ be a d-dimensional cone in an n-dimensional lattice N, and $p \in O_\sigma \in X_{\mathcal{F}}$. Then there is an isomorphism of analytic germs*

$$(X_{\mathcal{F}}, p) \simeq (X_\sigma(N_\sigma), 0_{X_\sigma(N_\sigma)}) \times \mathbb{C}^{n-d}.$$

Proof. This equality comes from Proposition 3.3.9, implying that the only orbits whose closure contain p are those corresponding to faces of τ. Hence we can identify $X_{\mathcal{F}}$ at p with the variety X_τ at any point of its minimal-dimensional orbit $O_\tau \simeq (C^*)^{n-d}$. □

The previous results imply in particular the following:

Proposition 3.3.15. *Let $\mathcal{F}$ be a fan in $N_{\mathbb{R}}$, and $\sigma \in \mathcal{F}$. Then $X_{\mathcal{F}}$ is smooth along O_σ, i.e. at every point of O_σ, if and only if σ is regular.*

Proposition 3.3.16. *Let $\tau \in \mathcal{F}$. Then the closure $\overline{O_\tau}$ of the orbit associated to τ in $X_{\mathcal{F}}$ is a toric variety, associated to the fan*

$$\overline{\mathcal{F}} = \{\overline{\sigma}, \tau \preceq \sigma\}$$

in $N(\tau)$, where $N(\tau) = {}^{N}\!/_{N_\tau}$, and $\overline{\sigma}$ is the image of σ in $N(\tau)_{\mathbb{R}}$.

Proof. We already know that any point in $\overline{O_\tau}$ is a morphism $(S_\sigma, +) \to (\mathbb{C}, \cdot)$ that is zero outside of $\tau^\perp$, for some σ such that $\tau \preceq \sigma$. It is entirely determined by its restriction to $M(\tau) := \tau^\perp \cap M$. Hence one can see it as a morphism $\tau^\perp \cap \sigma^{\checkmark} \to \mathbb{C}$. Now note that N_τ is canonically dual to M_τ, and that $\left(\sigma^{\checkmark} \cap \tau^\perp\right)^{\checkmark} = \overline{\sigma}$. □

Let us study compactness properties in toric varieties:

Definition 3.3.17. *A cone σ in a fan $\mathcal{F}$ is said to be an **external cone** if $\sigma \subset \partial|\mathcal{F}|$, where $\partial|\mathcal{F}|$ denotes the topological boundary of the subset $|\mathcal{F}|$ in $N_{\mathbb{R}}$. Any other cone of $\mathcal{F}$ is called **internal**.*

Example 3.3.18. *Heuristically, a cone is internal if it is of maximal dimension or "surrounded" by cones of $\mathcal{F}$. In the variety corresponding to the fan represented in figure 3.1, the only internal cones are τ_2, σ_1 and σ_2.*

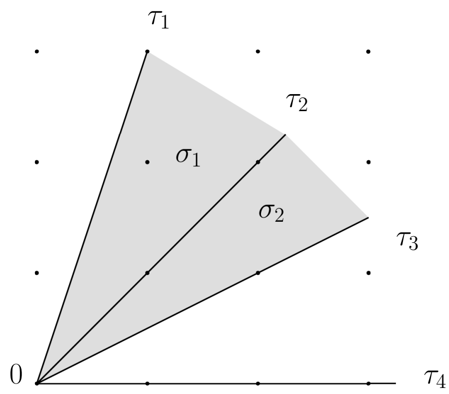


Figure 3.1: A fan in a 2-dimensional lattice.

Proposition 3.3.19. *The closure of the orbit $O_\sigma \subset X_{\mathcal{F}}$ is compact iff σ is an internal cone of $\mathcal{F}$.*

Recalling that a fan is a **finite** union of cones, the previous proposition has the following immediate consequence:

Proposition 3.3.20. *The variety $X_{\mathcal{F}}$ is compact iff the support $|\mathcal{F}|$ of the fan $\mathcal{F}$ to which it is associated is equal to the whole of $N_{\mathbb{R}}$.*

3.4 The modification defined by a refinement

Definition 3.4.1. *A **refinement** of a fan $\mathcal{F}$ in $N_{\mathbb{R}}$ is another fan $\mathcal{F}'$ in $N_{\mathbb{R}}$ such that:*

$$|\mathcal{F}| = |\mathcal{F}'| \text{ and } \forall\, \sigma' \in \mathcal{F}', \exists\, \sigma \in \mathcal{F} \text{ such that } \sigma' \subset \sigma.$$

*A **refinement** of a cone is a refinement of the fan formed by its faces.*

Definition 3.4.2. *Let $\mathcal{F}$ be a fan in N, and $\mathcal{F}'$ a refinement of $\mathcal{F}$.*

*The **toric morphism** $\Pi_{\mathcal{F}',\mathcal{F}} : X_{\mathcal{F}'} \to X_{\mathcal{F}}$ associated to this refinement is obtained by gluing the morphisms given by the inclusions of cones of $\mathcal{F}'$ in the cones of $\mathcal{F}$, defined in Proposition 3.2.9.*

Recall Definition 2.2.5 of a modification.

Proposition 3.4.3. *The morphism $\Pi_{\mathcal{F}',\mathcal{F}}$ is a modification of $X_{\mathcal{F}}$. It has the following combinatorial property: if $\sigma' \in \mathcal{F}'$, let σ be the minimal cone of $\mathcal{F}$ containing σ'. Then $\Pi_{\mathcal{F}',\mathcal{F}}(O_{\sigma'}) \subset O_\sigma$.*

The critical locus $E_{\mathcal{F}',\mathcal{F}}$ of $\Pi_{\mathcal{F}',\mathcal{F}}$ is exactly the union

$$\bigsqcup_{\tau\in\mathcal{F}',\tau\notin\mathcal{F}} O_\tau \in X_{\mathcal{F}'}$$

of orbits of $X_{\mathcal{F}'}$ corresponding to new cones, and the discriminant locus $\Delta(\Pi_{\mathcal{F}',\mathcal{F}})$ is

$$\bigsqcup_{\tau\in\mathcal{F},\tau\notin\mathcal{F}'} O_\tau \in X_{\mathcal{F}},$$

the union of orbits of $X_{\mathcal{F}}$ corresponding to cones that have been subdivided.

3.5 Modification associated to a germ of function

3.5.1 Local Newton polyhedron and the associated modification

Let N be an n-dimensional lattice, $\sigma \subset N_{\mathbb{R}}$ be a strongly convex cone, and $(X_\sigma, 0)$ the germ of affine normal variety associated to it. Let $f \in \mathbb{C}\{\sigma^{\checkmark} \cap M\}$. We want to study the germ of hypersurface $(V(f), 0) \subset (X_\sigma, 0)$, where $V(f)$ is the zero locus of f on X_σ.

In order to do this, let us introduce the Local Newton Polyhedron of a germ of function on a germ of normal toric variety. To define this we will refer to the following

Definition 3.5.1. *Let A, B be two subsets of an abelian semigroup $(S, +)$. Their* ***Minkowski sum*** *$A + B \subset S$ is defined by*

$$A + B := \{a + b, a \in A, b \in B\}.$$

Definition 3.5.2.
- *Let $f = \sum_{m_i\in\sigma^{\checkmark}\cap M} a_i\chi^{m_i} \in \mathbb{C}\{\sigma^{\checkmark} \cap M\}$. The* ***support of*** *f is:*

$$\mathrm{Supp}(f) := \bigcup_{a_i\neq 0} \{m_i\} \subset \sigma^{\checkmark} \cap M$$

- *The* ***Local Newton polyhedron of*** *f* ***at the origin of*** *X_σ is defined as*

$$\mathrm{LNP}(f) := \mathrm{Conv}\left(\mathrm{Supp}(f) + \sigma^{\checkmark}\right)$$

where "+" denotes the Minkowski sum in $\sigma^{\checkmark}$.

This definition of Newton polyhedron for general germs may be found in [61, Definition 5], or [53, Definition 8.7].

Lemma 3.5.3. *The ideal $I(\tau) \subset \mathbb{C}\{\sigma^{\checkmark} \cap M\}$ of functions cancelling on the orbit O_τ of X_σ is made of the functions $f \in \mathbb{C}\{\sigma^{\checkmark} \cap M\}$ such that* $\mathrm{Supp}(f) \cap \tau^{\perp} = \emptyset$.

Definition 3.5.4. *A function $f \in \mathbb{C}\{\sigma^{\checkmark} \cap M\}$ is called **suitable** if $V(f)$ does not contain any $(n-1)$-dimensional orbit of X_σ, or equivalently, if $\mathrm{Supp}(f)$ has points in each $(n-1)$-dimensional face of $\sigma^{\checkmark}$.*

In the sequel, every function considered will be suitable.

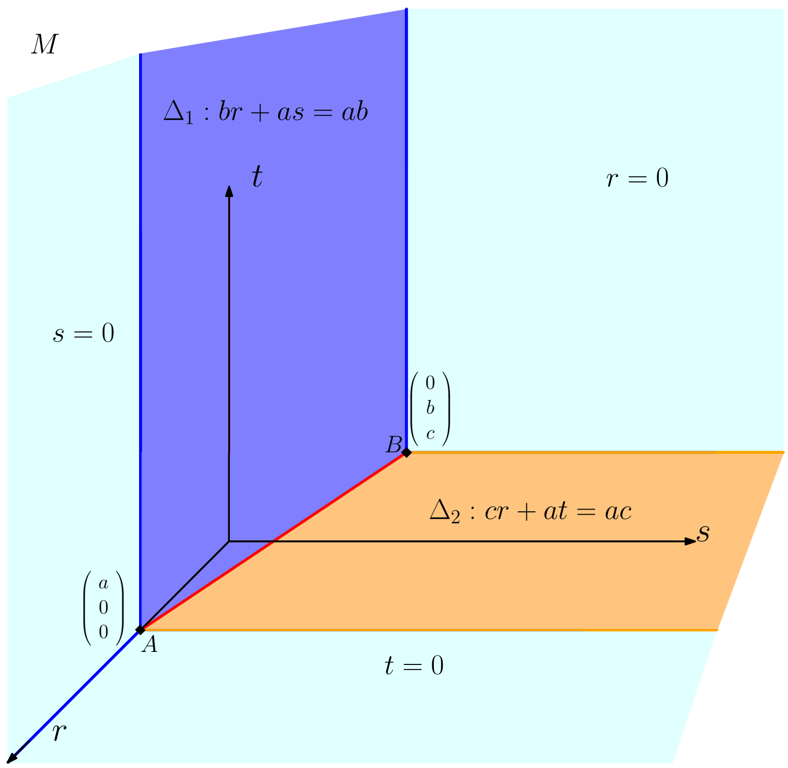


Figure 3.2: The local Newton polyhedron of $x^a - \theta \cdot y^b z^c \in \mathbb{C}[x, y, z]$ in $M = \mathbb{Z}^3_{r,s,t}$.

Example 3.5.5. *Figure 3.2 shows the local Newton polyhedron of the function $h(x, y, z) = x^a - \theta \cdot y^b z^c$ defined on $(\mathbb{C}^3, 0) = Spec\left(\mathbb{C}\{\mathbb{N}^3_{r,s,t}\}\right)$, for $\theta \in \mathbb{C}^*$. Here, to each point corresponding to an element of* $\mathrm{Supp}(f)$*, we added the positive octant, corresponding to $(\mathbb{C}^3, 0)$. We used lighter colors for the faces situated in the planes of coordinates.*

The equations of the faces of codimension 1 will turn out to be important when we will use this object to define a modification of the ambient germ.

Definition 3.5.6. *Let $v \in \sigma$, and $f \in \mathbb{C}\{\sigma^\vee \cap M\}$. Define the* ***height of*** $\mathrm{LNP}(f)$ ***in the direction*** v *to be*

$$h_v(f) := \min_{m \in \mathrm{LNP}(f)} \langle m, v \rangle \in \mathbb{R}_+.$$

Definition 3.5.7. *A* **face** *of* $\mathrm{LNP}(f)$ *is any subset of the form*

$$\Delta_v := \{m \in LNP(f), \langle m, v\rangle = h_v(f)\}$$

for some $v \in \sigma$.

Remark 3.5.8. *Note that $\Delta_0 = \mathrm{LNP}(f)$.*

Definition 3.5.9. *Let σ be a cone in N, and $f \in \mathbb{C}\{\sigma^\vee \cap M\}$. Let Δ be a face of* $\mathrm{LNP}(f)$ *of dimension d. Then the set*

$$\tau_\Delta := \{v \in \sigma, \Delta_v = \Delta\} = \{v \in \sigma / \forall\ m \in \Delta, \langle m, v\rangle = h_v(f)\}$$

is a cone of codimension d contained in σ. The set

$$\mathcal{F}_f := \{\tau_\Delta, \Delta \subset \mathrm{LNP}(f)\}$$

is the ***fan associated to*** f***.***

The morphism $\Pi_{\mathcal{F}_f} \colon X_{\mathcal{F}_f} \to X_\sigma$ coming from the refinement of σ is called the ***modification of*** X_σ ***associated to*** f***.***

The fan $\mathcal{F}_f$ is a refinement of the cone σ, in the sense of Definition 3.4.1.

Example 3.5.10. *Figure 3.3 shows the fan in N associated to the function $x^a - \theta y^b z^c \in \mathbb{C}\{x, y, z\}$. The notation (a, b) stands short for $gcd(a, b)$. This figure shows the fan "seen from the origin", i.e. this drawing is meant to be understood as the cone over what is drawn, with vertex the origin. This is the way we will represent 3-dimensional fans in the following drawings.*

Remark 3.5.11. *It is important to notice that, on any cone belonging to the fan $\mathcal{F}_f$, the height $h_v(f)$ is linear in the argument v.*

This cutting of σ in domains of linearity of the height function is in fact an alternative definition of the fan associated to the function f.

Remark 3.5.12. *The definition of τ_Δ implies that the face Δ is "parallel" to $\tau^\perp$, in the sense that there is a non-unique element $v_\Delta \in M$ such that $\{v_\Delta\} + \tau^\perp$ is the affine subspace of $M_\mathbb{R}$ spanned by Δ.*

Furthermore $\Delta_1 \subset \Delta_2 \Leftrightarrow \tau_{\Delta_2} \preceq \tau_{\Delta_1}$. Hence, again, there is a correspondence between the faces of $\mathrm{LNP}(f)$ *and the orbits of $X_{\mathcal{F}_f}$. This correspondence will not hold anymore as*

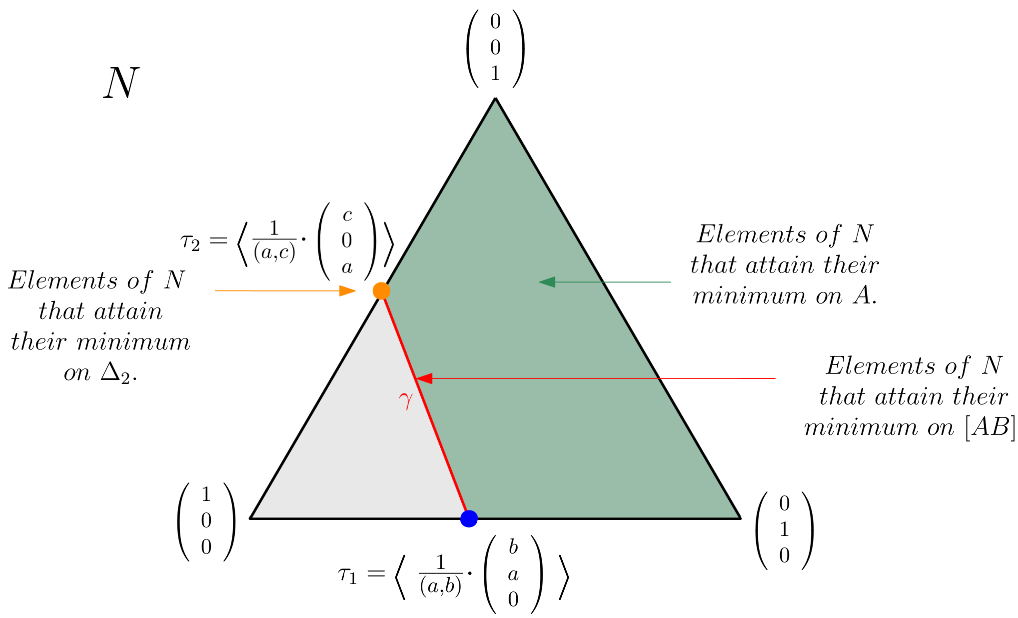


Figure 3.3: The fan associated to $x^a - \theta \cdot y^b z^c \in \mathbb{C}[x, y, z]$.

soon as we will refine $\mathcal{F}_f$, but is still pertinent. For Δ a face of $\mathrm{LNP}(f)$*, we will sometimes denote*

$$O_\Delta := O_{\tau_\Delta} \in X_{\mathcal{F}_f}.$$

It is also clear that

$$\overline{O_\Delta} = \bigsqcup_{\Delta' \subset \Delta} O_{\Delta'}.$$

Now, the modification of X_σ associated to f is adapted to the function f in the following sense:

3.5.2 Behaviour of the strict tranform, Newton-nondegeneracy

Let $\mathcal{F}$ be a refinement of $\mathcal{F}_f$, and $\tau \subset \sigma$ a cone of $\mathcal{F}$. Denote

$$\Pi_{\mathcal{F}} \colon (X_{\mathcal{F}}, E_{\mathcal{F}}) \to (X_\sigma, 0)$$

the modification associated to $\mathcal{F}$, and $\tilde{f} = f \circ \Pi_{\mathcal{F}}$ the pullback of f by this modification. Let

$$\Delta_\tau := \{m \in \mathrm{LNP}(f), \forall\ v \in \tau, \langle m, v\rangle = h_v(f)\}.$$

Note that we may have $\tau \neq \tau'$ and still $\Delta_\tau = \Delta_{\tau'}$. However, the assumption that $\mathcal{F}$ is a refinement of $\mathcal{F}_f$ ensures that this definition makes sense. In fact, if γ is the minimal cone of $\mathcal{F}_f$ containing τ, then $\Delta_\tau = \Delta_\gamma$.

Note also that $dim(\Delta_\tau) \leqslant codim(\tau)$, and that $\tau \preceq \sigma \Rightarrow \Delta_\sigma \subset \Delta_\tau$.

Let us study the intersection of the strict transform of $V(f)$ with the orbit $O_\tau \in X_{\mathcal{F}}$. Pick a basis $v_1, \cdots, v_n$ of the $\mathbb{Z}$-module N, such that

$$\forall\ i \in \{1, \dots, d\}, v_i \in \tau \text{ and } \forall\ i \in \{d+1, \dots, n\}, v_i \in \sigma \setminus \tau.$$

Denote $m_1, \cdots, m_n$ its dual basis in M, that is, $\forall\ i, j, \langle m_i, v_j \rangle = \delta_{i,j}$. This implies in particular that the family $(m_i)_{d+1 \leqslant i \leqslant n}$ is a basis of the $\mathbb{Z}$-module $\tau^\perp$.

Denote M' the sub $\mathbb{Z}$-module of M generated by the family $v_1, \cdots, v_d$. We have

$$M = \left(M \cap \tau^\perp\right) \oplus M'.$$

Now, if $m \in \sigma^\checkmark$, then $m = \sum \langle m, v_i \rangle v_i$, and $\forall\ i, \langle m, v_i \rangle \geqslant 0$, because every v_i is in σ. Furthermore,

$$m \in \Delta_\tau \Leftrightarrow \forall\ 1 \leqslant i \leqslant d, \langle m, v_i \rangle = h_{v_i}(f).$$

Remark 3.5.13. *Note that $\overline{O_\tau}$ is compact if and only if Δ_τ is compact. This comes from the fact that the only faces that are "surrounded" by other faces are the compact ones.*

Furthermore, Proposition 3.4.3 implies

Lemma 3.5.14. *The preimage $E_{\mathcal{F}}$ of the origin of X_σ by the modification $\Pi_{\mathcal{F}}$ is equal to*

$$\bigsqcup_{\tau : \Delta_\tau \ compact} O_\tau .$$

Definition 3.5.15. *For a face Δ of LNP(f), of any dimension, we define f_Δ, f **truncated relatively to the face** Δ, as the function obtained by keeping only the terms of f corresponding to points of this face:*

$$f_\Delta := \sum_{m_i \in \Delta} a_i \chi^{m_i}.$$

Denoting $h_i := h_{v_i}(f)$, this definition, combined with the preceding considerations, provides the factorization

$$f = \chi^{h_1 m_1 + \cdots + h_d m_d} \left(\tilde{f}_\tau + g\right)$$

where

$$\tilde{f}_\tau := \frac{f_{\Delta_\tau}}{\chi^{h_1 m_1 + \cdots + h_d m_d}} \in \mathbb{C}[\tau^\perp \cap M],$$

and every monomial of g is divisible by χ^{m_i}, for some $1 \leqslant i \leqslant d$.

In addition, the orbit O_τ is characterized by:

$$O_\tau = \{x \in X_{\mathcal{F}}, \chi^{m_i}(x) \neq 0 \text{ iff } d+1 \leqslant i \leqslant n\}.$$

This leads us to:

Lemma 3.5.16. *The intersection of the strict transform $\widetilde{V(f)}$ of $V(f)$ with the orbit $O_\tau = Spec\left(\mathbb{C}[\chi^{\pm m_{d+1}}, \cdots, \chi^{\pm m_n}]\right)$ of $X_{\mathcal{F}}$ associated to the cone $\tau \in \mathcal{F}$ is given by*

$$\widetilde{V(f)} \cap O_\tau = \left\{\tilde{f}_\tau = 0\right\}.$$

Furthermore, we can deduce from the previous paragraph:

Lemma 3.5.17. *If $\tau = \langle v \rangle_{\mathbb{R}_+}$ for some primitive vector v in N, then $\tilde{f} := f \circ \Pi_{\mathcal{F}}$ vanishes along $\overline{O_\tau}$, and the multiplicity of $\tilde{f}$ along this hypersurface is*

$$m_{\overline{O_\tau}}(\tilde{f}) = h_v(f).$$

The polynomial $\tilde{f}_\tau$ depends on the initial choice of the basis $(v_i)_{1\leqslant i \leqslant n}$, but it is always, up to a monomial of $\mathbb{C}[M' \cap \sigma^\vee]$, equal to f_{Δ_τ}.

Remark 3.5.18. *A direct consequence is that if $\mathcal{F}$ is a refinement of $\mathcal{F}_f$, and $\tau \in \mathcal{F}$, then*

$$\widetilde{V(f)} \cap O_\tau \neq \emptyset \textit{ iff } \dim \Delta_\tau \geqslant 1.$$

The following elementary lemma establishes then the final link between the regularity of $\widetilde{V(f)} \cap O_\tau$ and the initial function f.

Lemma 3.5.19. *Let $n \geqslant k$, and $\underline{X}^{\underline{m}} \in \mathbb{C}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$. Then the polynomial $f \in \mathbb{C}[X_1^{\pm 1}, \cdots, X_k^{\pm 1}]$ defines a smooth hypersurface of $(\mathbb{C}^*)^k$ if and only if $\underline{X}^{\underline{m}} \cdot f$ defines a smooth hypersurface of $(\mathbb{C}^*)^n$.*

This motivates the following definition, see also [61, Definition 5]:

Definition 3.5.20.

- *A germ of suitable function $f \in \mathbb{C}\{\sigma^\vee \cap M\}$ is said to be **nondegenerate relatively to a compact face** $\Delta \subset \mathrm{LNP}(f)$ if and only if $V(f_\Delta)$ is smooth in $(\mathbb{C}^*)^n$.*
- *A germ of function is said to be **Newton-nondegenerate**, or **NND** if it is nondegenerate relatively to every compact face of its local Newton polyhedron.*

Example 3.5.21. *Any suitable binomial is Newton-nondegenerate.*

The previous paragraph implies:

Proposition 3.5.22. *Let $f \in \mathbb{C}\{\sigma^\vee \cap M\}$ be a NND germ of analytic function on $(X_\sigma, 0)$, and $\mathcal{F}$ a refinement of $\mathcal{F}_f$, such that $X_\mathcal{F}$ is smooth along O_τ and Δ_τ is compact. Then the intersection $\widetilde{V(f)} \cap O_\tau$ is smooth, and at any point p of this intersection, $\left(\widetilde{V(f)}, p\right)$ is a germ of smooth hypersurface of $X_\mathcal{F}$ intersecting O_τ transversally.*

Denote $E := \widetilde{V(f)} \cap E_\mathcal{F}$. Lemma 3.5.14 implies that

$$E = \bigsqcup_{\Delta_\tau \text{ compact}} O_\tau \cap \widetilde{V(f)}.$$

Then

$$\pi := \Pi_{\mathcal{F}_{|\widetilde{V(f)}}} \colon (\widetilde{V(f)}, E) \to (V(f), 0)$$

is a modification of the germ $(V(f), 0)$, and the previous proposition implies

Corollary 3.5.23. *Let f be a germ of NND function on $(X_\sigma, 0)$ and $\mathcal{F}$ a refinement of $\mathcal{F}_f$ such that any cone $\tau \in \mathcal{F}$ such that Δ_τ is compact and $dim(\Delta_\tau) \geqslant 1$ is regular. Then the modification $\pi \colon (\widetilde{V(f)}, E) \to (V(f), 0)$ is a resolution of the germ $(V(f), 0)$.*

Furthermore, lemma 3.5.16 implies the following, that will be heavily used in Chapter 5:

Proposition 3.5.24. *Let f be a Newton-nondegenerate function on X_σ, and $\mathcal{F}$ be a refinement of $\mathcal{F}_f$. Let τ be a cone of $\mathcal{F}$, such that Δ_τ is compact. Then,*

- *If $\operatorname{codim}(\tau) = 1$, denote $l(\Delta_\tau)$ the integral length of this face in M. Then*

$$Card\left(\widetilde{V(f)} \cap O_\tau\right) = l(\Delta_\tau).$$

- *If $\operatorname{codim}(\tau) = 2$, denote $i(\Delta_\tau)$ the number of points of M in the **interior** of Δ_τ. Then*

$$C(\tau) := \widetilde{V(f)} \cap \overline{O_\tau}$$

  is a smooth curve. Furthermore, if $\dim(\Delta_\tau) = 2$, this curve is irreducible and

$$g(C(\tau)) = i(\Delta_\tau).$$

  If $\dim(\Delta_\tau) = 1$, $C(\tau)$ is a disjoint union of $l(\Delta_\tau)$ smooth curves of genus 0.

Remark 3.5.25. *Let us remind that, in both cases, if $\dim(\Delta_\tau) = 0$, then $\widetilde{V(f)} \cap \overline{O_\tau} = \emptyset$.*

Let us conclude this section with a lemma giving a precise meaning to the genericity of Newton-nondegenerate functions.

Lemma 3.5.26. *Let f be a holomorphic function on X_σ. Then in the space of coefficients of all those functions h such that $\mathrm{LNP}(h) = \mathrm{LNP}(f)$, those which are non-degenerate are Zariski-dense.*

3.5.3 Application to some Hirzebruch-Jung singularities

Let us treat as an example the particular case of Hirzebruch-Jung singularities. Such singularities appear in the proof of the main result (see Subsections 4.11.2 and 4.11.3).

Definition 3.5.27. *A **Hirzebruch-Jung singularity** is any multigerm of the form*

$$\left(\left\{x^a - \theta \cdot y^b z^c = 0\right\}, 0\right)^{norm}, \theta \in \mathbb{C}^*.$$

Usually one considers only irreducible Hirzebruch-Jung singularities (i.e when $gcd(a, b, c) = 1$). Therefore the previous definition is a slight extension of the usual notion.

The tools developed earlier can be applied to compute a description and a resolution of such singularities.

Denote $h(x, y, z) := x^a - \theta y^b z^c$, and, as usual, $V(h) := \{h = 0\} \subset \mathbb{C}^3$. Denote by $\overline{V(h)}$ the strict tranform of $V(h)$ by the modification $\Pi_{\mathcal{F}_h}$ of $\mathbb{C}^3$ associated to h. This modification is given by the fan described in figure 3.3. Denote by

$$norm\colon \overline{V(h)} \to V(h)$$

the restriction of $\Pi_{\mathcal{F}_h}$ to the strict transform of $V(h)$.

Lemma 3.5.28. *(See [20]). The morphism norm is a normalization of $V(h)$.*

Example 3.5.29. *Figure 3.5.3 shows in blue and orange the preimages by the normalization of the axes of coordinates contained in $V(h)$. These non-compact curves are represented as arrows. Proposition 3.5.24 implies that $\overline{V(h)}$ intersects $\overline{O_\gamma} = {\Pi_{\mathcal{F}_h}}^{-1}(0)$ in*

$$d := (a, b, c)$$

points $p_1, \cdots, p_d$. At each $p_i \in O_\gamma$, the variety $\overline{V(h)}$ "inherits" the transversal singularity $(X_\gamma(N_\gamma), 0)$ of $X_{\mathcal{F}_h}$ along O_γ.

The morphism of multigerms

$$norm\colon \left(\left(\overline{V(h)}, p_1\right) \bigsqcup, \cdots, \bigsqcup \left(\overline{V(h)}, p_d\right)\right) \to (V(h), 0)$$

is a first step towards a resolution of $(V(h), 0)$. We know that the only singularities of $\overline{V(h)}$ are located at its intersections with orbits along which $X_{\mathcal{F}_h}$ is singular.

The only possible such orbit is O_γ. The variety $X_{\mathcal{F}_h}$ is singular along it iff the cone γ is singular.

There is a canonical way to subdivide a singular 2-dimensional cone in regular cones. See [50, Section 2.2].

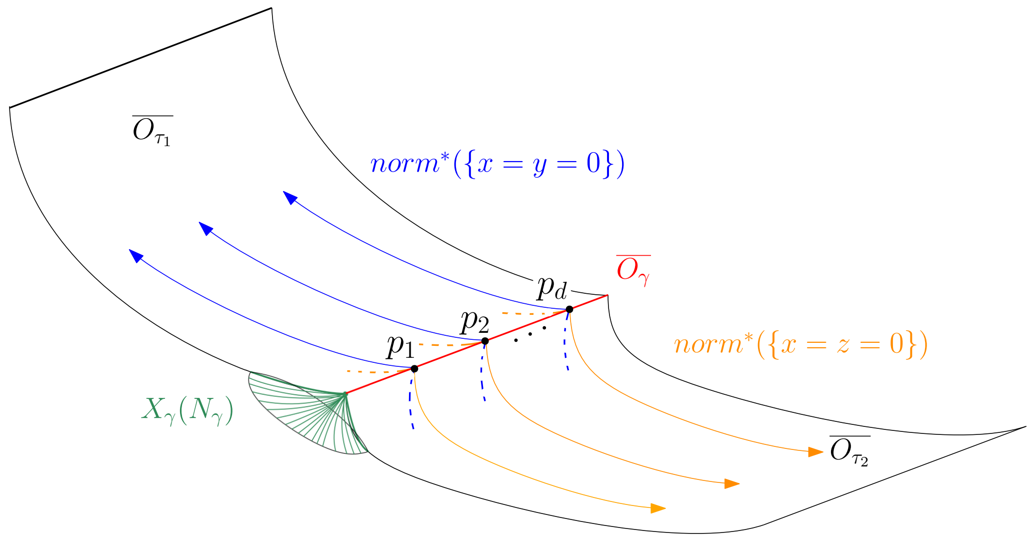


Figure 3.4: Preimages of the axes $\{x = y = 0\}$ and $\{x = z = 0\}$ by the normalization of $V(h)$.

Lemma 3.5.30. *Let $\gamma := \langle u, v\rangle_{\mathbb{R}_+} \subset N_{\mathbb{R}}$ be a 2-dimensional cone. Let $\delta = det(u, v)$. If $\delta \neq 1$, there is a unique $\alpha \in \{1, \cdots, \delta - 1\}$ such that*

$$u_1 := \frac{1}{\delta} \cdot (\alpha u + v) \in N. \tag{3.1}$$

Let

$$\frac{\delta}{\alpha} = k_1 - \cfrac{1}{k_2 - \cfrac{1}{\cdots - \cfrac{1}{k_l}}} \tag{3.2}$$

be the negative continued fraction expansion of $\frac{\delta}{\alpha}$. Set $u =: u_0, u_1, \cdots, u_l, u_{l+1} := v$, such that $\forall\ 2 \leqslant i \leqslant l$,

$$u_i := k_{i-1}u_{i-1} - u_{i-2}. \tag{3.3}$$

Then $\forall\ 0 \leqslant i \leqslant l$, the cone

$$\gamma_i := \langle u_i, u_{i+1}\rangle_{\mathbb{R}_+}$$

is regular.

Proposition 3.5.31. *The collection of 2-dimensional cones obtained by the above process is the minimal regular subdivision of γ in the sense of refinements. It is called the* ***canonical regular subdivision*** *of γ. One says that we subdivided γ regularly.*

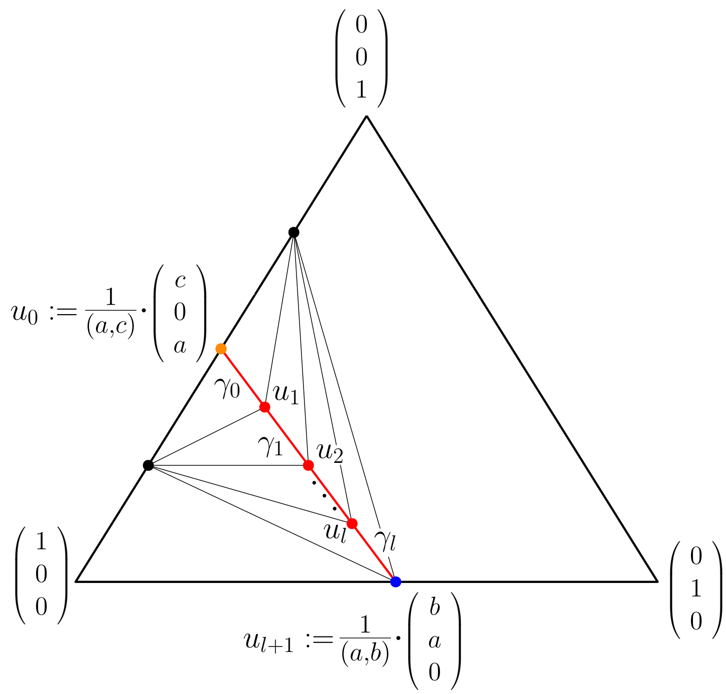


Figure 3.5: A possible $\mathcal{F}$.

Simple computations lead to:

Lemma 3.5.32. *In our case,*

$$\delta = \frac{ad}{(a,c)(a,b)} \tag{3.4}$$

and α *is the unique integer in* $[0, \delta - 1]$ *such that*

$$ad \mid \alpha c(a,b) + b(a,c). \tag{3.5}$$

Now we can use this refinement of γ to build a new fan $\mathcal{F}$, refining $\mathcal{F}_h$, such that $X_{\mathcal{F}_h}$ will be regular along the orbits intersected by the strict tranform $\widetilde{V(h)}$ of $V(h)$ by $\Pi_{\mathcal{F}_h} \circ \Pi_{\mathcal{F},\mathcal{F}_h} = \Pi_{\mathcal{F}}$. Note that $\widetilde{V(h)}$ is also the strict transform of $\overline{V(h)}$ by $\Pi_{\mathcal{F},\mathcal{F}_h}$.

Figure 3.5 shows a possible fan $\mathcal{F}$. The only orbits that are intersected correspond to colored cones, the γ_i's or $\langle u_j \rangle$'s. Proposition 3.5.24 implies that, among them, each orbit of dimension 2 is intersected by a disjoint union of $d = (a,b,c)$ rational curves, and each orbit of dimension 1 is intersected in d points. Along each one of these orbits, the variety $X_{\mathcal{F}}$ is smooth. Hence the strict transform $\widetilde{V(h)}$ of $V(h)$ by $\Pi_{\mathcal{F},\mathcal{F}_h}$ is also smooth. Denote

$$\pi := \Pi_{\mathcal{F},\mathcal{F}_h}{}_{|\widetilde{V(h)}} : \widetilde{V(h)} \to \overline{V(h)}, \text{ and } \Pi := \Pi_{\mathcal{F}}{}_{|\widetilde{V(h)}}.$$

The situation is summarized in the diagram of Figure 3.6.

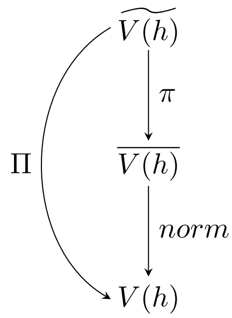


Figure 3.6: The successive toric modifications.

The morphism π is a resolution of $\overline{V(h)}$, while Π is a resolution of $V(h)$. Their common exceptional divisor **is a disjoint union of** (a, b, c) **identical chains of smooth rational curves intersecting transversally**. Furthermore, by compatibility of the complex orientations, we get:

Lemma 3.5.33. *At any intersection point of two such curves, the combination of the complex orientation of each of them gives the ambient complex orientation of* $\widetilde{V(h)}$.

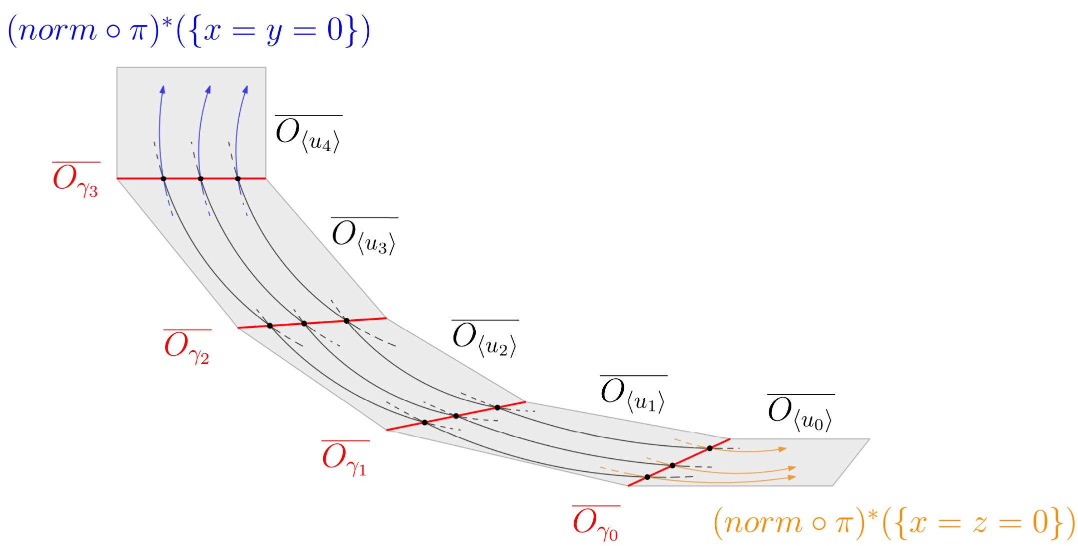


Figure 3.7: The configuration of curves.

Figure 3.7 shows a visualization of this configuration in the case $l = 3, d = 3$. The arrows represent the non-compact curves of intersection of $\widetilde{V(h)}$ with the non-compact orbits corresponding to $\langle u_0 \rangle$ and $\langle u_{l+1} \rangle$.

The self-intersection of each compact curve in $\widetilde{V(h)}$ is given by the number $-k_i$. In order to see this, let us compute the multiplicities of some regular function g on $\widetilde{V(h)}$ such that g vanishes on each of those curves. Such a function is adapted to the exceptional divisor of the resolution in the sense of Definition 2.7.4, allowing the computation of the self-intersections of its irreducible components by the use of Lemma 2.7.7.

Let $n_1, n_2, n_3 \in \mathbb{N}$, and $g(x, y, z) := x^{n_1} y^{n_2} z^{n_3}$. Then the multiplicities of the pullback of g by Π on each curve of one of these chains are indicated on figure 3.8. Each vertex corresponds to one of the curves, with an arrow for the non-compact ones. The multiplicities $\mu_i = h_{u_i}(g)$ are written in parenthesis, and they verify

$$\forall\ 1 \leqslant i \leqslant l, \mu_{i+1} = \mu_{i-1} - k_i \cdot \mu_i \tag{3.6}$$

because the height function for the **monomial** g is linear, and because of the relation (3.3) of lemma 3.5.30. In particular, the linearity of the multiplicity provides the value

$$\mu_1 = \frac{\alpha \mu_0 + \mu_{l+1}}{\delta}. \tag{3.7}$$

In conclusion, starting from the values a, b, c, Equations (3.4), (3.5) provide δ and α, hence the integers k_i *via* Equation (3.2). If $\delta = 1$, the collection of k_i's is empty. Finally, Equation (3.7) provides the value of μ_1, while (3.6) provides the other multiplicities.

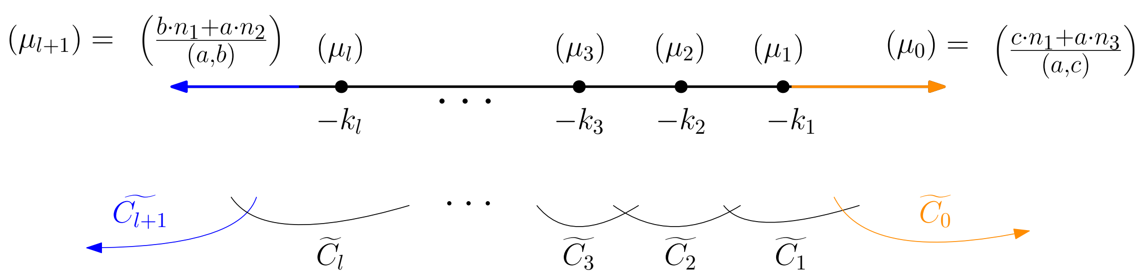


Figure 3.8: The string $Str(a; b, c | n_1; n_2, n_3)$.

Notation 3.5.34. *The bamboo of figure 3.8 is denoted*

$$Str(a; b, c | n_1; n_2, n_3)$$

following to a certain extent the notation introduced in [46, Definition 4.3.10].

3.5.4 Counting points of intersection

In this section, we present the theorem of Bernstein-Koushnirenko-Khovanskii (see [2, 50, 11, 62]), which will be useful for the toric version of the main algorithm.

In the sequel, M is an n-dimensional lattice, and we denote $N := M^{\checkmark}$.

Definition 3.5.35. *Denote $\mathscr{P}(M)$ the set of convex polytopes in $M_{\mathbb{R}}$ with vertices in M. Call **volume** the only function*

$$Vol\colon \mathscr{P}(M) \to \mathbb{R}_+$$

defined by the following rules:

1. *$dim(P) \leqslant n-1 \Rightarrow Vol(P) = 0$.*
2. *Vol is invariant by translation.*
3. *If $m_1, \cdots, m_n$ is a basis of the $\mathbb{Z}$-module M, $Vol\left(Conv(0, m_1, \cdots, m_n)\right) = 1$.*
4. *$Vol(P \cup Q) = Vol(P) + Vol(Q) - Vol(P \cap Q)$ whenever $P \cup Q$ is convex.*

Definition 3.5.36. *If $P \in \mathscr{P}(M)$ and $\lambda \in \mathbb{Z}$, denote*

$$\lambda \cdot P = \{\lambda \cdot a, a \in P\}.$$

Relatively to this multiplication and the Minkowski sum, there exists a unique n-linear form V on $\mathscr{P}(M)$ such that

$$V(P, \cdots, P) = Vol(P).$$

*If $P_1, \cdots, P_n \in \mathscr{P}(M)$, the number $V(P_1, \cdots, P_n)$ is called the **mixed volume** of $P_1, \cdots, P_n$.*

The mixed volume is a non-negative integer. See [18, Section 5.4], and [11, Theorem 7.4.12] for more properties of the mixed volume.

Let $P \in \mathscr{P}(M)$. Adapting the construction of a fan from a local Newton polyhedron executed in subsection 3.5.1, we will talk about the inner normal fan $\mathcal{F}_P$ associated to the polytope P.

In this context, the fan $\mathcal{F}_P$ will always be **complete**, in the sense that its support $|\mathcal{F}_P|$ is always equal to $M_{\mathbb{R}}$. The birational map

$$\Pi_{\mathcal{F}_P, M_{\mathbb{R}}}\colon X_{\mathcal{F}_P} \to X_{M_{\mathbb{R}}}$$

is a **compactification** of the torus $X_{M_{\mathbb{R}}} \simeq (\mathbb{C}^*)^n$.

Remark 3.5.37. *Note that $\mathcal{F}_{\sum P_i}$ is the minimal fan that is a refinement of each $\mathcal{F}_{P_i}$.*

Definition 3.5.38. *If $f \in \mathbb{C}[M]$, denote*

$$P(f) := \mathrm{Conv}\left(\mathrm{Supp}(f)\right)$$

*the **Newton polytope** of f.*

Now, let $f_1, \cdots, f_n \in \mathbb{C}[M]$, denote $P_i = P(f_i)$, and $P = \sum_{i=1}^{n} P_i$. Denote $\widetilde{V(f_i)}$ the strict tranform of $V(f_i)$ by the modification $\Pi_{\mathcal{F}_P, M_{\mathbb{R}}}$.

Theorem 3.5.39. (The Bernstein-Koushnirenko-Khovanskii theorem, [11, Theorem 7.5.4].)
For a generic choice of coefficients for each f_i, with fixed Newton polytopes,

$$\operatorname{Card}\left(\widetilde{V(f_1)} \cap \cdots \cap \widetilde{V(f_n)}\right) = V(P_1, \cdots, P_n)$$

and this intersection is entirely realized in the n-dimensional orbit O_0 of $X_{\mathcal{F}_P}$.

Furthermore the union $\widetilde{V(f_1)} \cup \cdots \cup \widetilde{V(f_n)}$ is a simple normal crossings divisor at $\widetilde{V(f_1)} \cap \cdots \cap \widetilde{V(f_n)}$, in the sense of Definition 2.2.4.

In this theorem, the genericity of the choice of the coefficients means that each function should be Newton-nondegenerate, and that the surfaces $V(f_i)$ should be in relative generic positions.

Remark 3.5.40. *In the continuity of Remark 3.5.37, note that $\mathcal{F}_{f_1 \cdots f_n} = \mathcal{F}_{P_1 + \cdots + P_n}$, and is the **minimal** refinement of the cone $M_{\mathbb{R}}$ which is a refinement of each $\mathcal{F}_{f_i}$.*

For us, Theorem 3.5.39 will be used through the following corollary:

Corollary 3.5.41. *Let M be a 2-dimensional lattice, and let $f \in \mathbb{C}[M]$ be a Newton-nondegenerate function. Denote $P_1 := P(f)$. Then for any $P_2 \in \mathscr{P}(M)$, a generic choice of coefficients for the elements of $P_2 \cap M$ will provide a function g such that $P(g) = P_2$ and, denoting $P = P_1 + P_2$, the compactifications $\widetilde{V(f)}$ and $\widetilde{V(g)}$ of $V(f)$ and $V(g)$ in $X_{\mathcal{F}_P}$ intersect transversally only on O_0, in $V(P_1, P_2)$ points.*

Remark 3.5.42. *To compute 2-dimensional mixed volumes, one can use the identity*

$$V(P_1, P_2) = \frac{Vol(P_1 + P_2) - Vol(P_1) - Vol(P_2)}{2!}$$

which has an analogue in dimension n.

Chapter 4

The general result and the main algorithm

4.1 Introduction and strategy

Let $(X, 0)$ be a germ of complex analytic variety of dimension 3, and let $f\colon (X, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function on $(X, 0)$. The function f defines a germ $(V(f), 0)$ of hypersurface on $(X, 0)$, where $V(f) := \{x \in X, f(x) = 0\}$. Denote by $\mathrm{Sing}(V(f))$ the singular locus of $V(f)$, and by $\mathrm{Sing}(X)$ the one of X. We are about to introduce the well-known Milnor fibration associated to f, after a few preliminary definitions. In the sequel, if $(X, 0)$ is a germ, X will denote a representative of this germ.

Definition 4.1.1. *We say that a real-analytic function $\rho\colon X \to \mathbb{R}_+$ **defines** 0 in X if 0 is isolated in $\rho^{-1}(0)$, i.e. if there is another representative $X' \subset X$ of $(X, 0)$ such that $\rho^{-1}_{|X'}(0) = \{0\}$.*

Theorem 4.1.2. (H. Hamm, [22, Satz 1.6], Lê [29, Theorem 1.1])
Given a real-analytic function ρ defining 0 in X, and $\varepsilon > 0$, denote $X_\varepsilon := X \cap \rho^{-1}([0, \varepsilon))$, and $S_\varepsilon := X \cap \{\rho = \varepsilon\}$. Let $f\colon (X, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function, such that $X \setminus V(f)$ is smooth. Then there exists $\varepsilon_0 > 0$, such that $\forall\ 0 < \varepsilon \leqslant \varepsilon_0, \exists\ \delta_\varepsilon > 0$ such that $\forall\ 0 < \delta \leqslant \delta_\varepsilon$, the following two maps are diffeomorphic smooth fibrations:

- $\frac{f}{|f|}\colon S_\varepsilon \setminus V(f) \to \mathbb{S}^1$
- $f\colon \partial\left(\{|f| = \delta\} \cap X_\varepsilon\right) \to \partial\left(D_\delta\right)$, *where D_δ denotes the closed disc of radius δ around 0 in $\mathbb{C}$.*

Definition 4.1.3. *The first of the two fibrations above is referred to as the **Milnor fibration of** f, and the second one is called the **Milnor-Lê fibration** (see [29, Theorem 1.1]). The closure of the fiber of the Milnor-Lê fibration is called the **Milnor fiber** of the germ of function f.*

Remark 4.1.4. *The Milnor-Lê fibration is also sometimes referred to as the Milnor fibration.*

Using transversality arguments, one may show that the diffeomorphism type of the Milnor fiber does not depend on the chosen representative, so we speak about ***the Milnor fiber of the germ of function*** $f \in \mathcal{O}_{X,0}$

The goal of the present chapter is to generalize:

Theorem 4.1.5. (Michel-Pichon, [32], [33], [34], Némethi-Szilard, [46, 10.2.10])

If $(X, 0) = (\mathbb{C}^3, 0)$*, then the boundary of the Milnor fiber of the reduced function* f *is an oriented* 3*-manifold, which can be represented by an orientable plumbing graph.*

The main theorem of this thesis is the following generalization of Theorem 4.1.5:

Theorem 4.1.6. *Let* $(X, 0)$ *be a germ of* 3*-dimensional complex analytic variety, and* $f : (X, 0) \to (\mathbb{C}, 0)$ *a germ of holomorphic function on* $(X, 0)$*, such that* $V(f) \supset Sing(X)$*. Then the boundary of the Milnor fiber of* f *is an oriented* 3*-manifold, which can be represented by an orientable plumbing graph.*

When $(X, 0)$ has isolated singularity, an extension of this theorem to functions of the type $f\overline{g}$ was already proved by Fernández de Bobadilla and Menegon Neto in [16]. Our theorem extends this result to any ambient germ, with possibly non-isolated singularity. The study of a class of examples of this type is carried out explicitly in Chapter 5, in the case of toric ambient germs.

4.1.7. The strategy. Our strategy of proof generalizes that developed by Némethi and Szilárd in [46], with some complications, due to the singular locus of X, along which the transversal type of X may be arbitrarily complicated. As in [46], our proof provides a method for the computation of the boundary of the Milnor fiber, but the singular locus of X imposes the need for additional considerations.

- As a prelude, in section 4.3, we show that the ambient germ X can be assumed to be normal.

- Then, in section 4.4, with the help of a second function g in general position relatively to f, we will build a new germ of variety $(\mathscr{S}_k, 0)$, which will have the same boundary as the Milnor fiber. This variety will have an isolated singularity at the origin, but will be real-analytic, instead of complex-analytic.

- The rest of the proof, from Sections 4.5 to 4.11, essentially consists in building an *adapted resolution* of $\mathscr{S}_k$, i.e. a proper map from a 4-dimensional manifold to $\mathscr{S}_k$, which will be an analytic isomorphism outside the preimage of the origin. The preimage E of 0 by this resolution will be a simple configuration of smooth surfaces. We obtain a function g which is adapted to E in the sense of Definition 2.7.4, and therefore allows the computation of the self-intersections of the irreducible components of E, using

Lemma 2.7.7. Finally, we also obtain a rug function ρ for E, in the sense of Definition 2.7.8, which defines a tubular neighbourhood of E whose boundary is therefore, by Theorem 2.7.9, a graph manifold described by E.
Finally, in Section 4.12, using the fact that the resolution is an orientation-preserving analytic isomorphism outside the origin, one concludes the proof that the link of $\mathscr{S}_k$ is the same graph manifold as the boundary of the tubular neighbourhood of E. This part is introduced in section 4.5. The resolution is constructed step by step as a composition of several real-analytic modifications, defined and studied from section 4.6 to section 4.11.

- Along the process, one keeps track of the structure of the preimage of the origin by the previous modifications, therefore computing a plumbing graph for the boundary of $\mathscr{S}_k$, hence also for the boundary of the Milnor fiber. This structure will be encoded in a graph that we will modify at each step of the process, according to the data in it. This process will be called the **main algorithm**, because of the possibility in some cases, as for example the case $(X, 0) = (\mathbb{C}^3, 0)$, for one to blindly apply calculation rules to an initial decorated graph in order to obtain a plumbing graph for the boundary of the Milnor fiber of f. The explicit construction of the initial decorated graph is the object of section 4.9.

4.1.1 Main contibutions

As mentioned earlier, this work is an extension of the proof carried out by Némethi and Szilárd in [46] for the case where X is smooth. Before proceeding to our proof, let us stress out the main differences that had to be introduced to carry out the proof in the general case:

- The Definition 4.4.2 of a companion is a new *ad hoc* definition introduced in order to keep the main properties of an Isolated Complete Intersection Singularity (ICIS), which is the notion used in [46]. As mentioned in Lemma 4.4.4, if the ambient germ X is a complete intersection, our definition is a restriction of the one of ICIS. The proof of the fact that a companion always exists required the use of more results of the literature, as explained in proof of Lemma 4.4.8.

- Instead of using an embedded resolution as in 6.1.2 of [46], we introduce the notion of adapted modification, see Definition 4.6.1. This different definition is less restrictive but still good enough to carry out the computation, and allows simpler computations in the toric case studied in Chapter 5.

- In the general case, the computation of the Milnor fiber requires the use of data that do not appear in the decorated graph $\overset{\star}{\Gamma}(\mathscr{C})$ which is the starting point of the algorithm presented in the Chapter 10 of [46]. Those data, related to the singular locus of X, are of two kinds.

- The first is the collection of switches associated to a non-rational curve of $\mathscr{C}$, introduced in Subsection 4.10.2.
- The second is a datum concerning the global arrangement of irreducible components of the variety $\widetilde{\mathscr{S}_k}$ when one follows a cycle in the graph $\overset{\star}{\Gamma}(\mathscr{C})$. This second problem is explained in Subsection 4.10.4.

Both these data are unnecessary in the case where X is smooth, as shown by Némethi and Szilárd. In Chapter 5, we prove that even in the more general case where $(X, 0)$ is a germ of normal toric variety and the function f is Newton-nondegenerate, one does not need these data.

- Our approach of the problems of orientations is quite different from the one developed in [46], because we carefully keep track from the beginning of an orientation for every object appearing in the process of computation. The questions of orientations represent one of the main differences with the case of links of isolated singularities of complex surfaces, and this is a rather delicate point to carry out.

4.2 Deformation theoretical reformulation of our main theorem

The main Theorem 4.1.6 may be reformulated in the language of 1-parameter deformations. We recall here this language.

Definition 4.2.1. *A 1-**parameter deformation** of a reduced germ of complex surface (S, s) is a holomorphic morphism $f\colon (X, x) \to (\mathbb{C}, 0)$, where (X, x) is an equidimensional germ of complex 3-dimensional analytic space, together with an isomorphism between (S, s) and the special fiber $(f^{-1}(0), x)$.*

Definition 4.2.2. *A 1-parameter deformation is called a **smoothing** whenever its generic fibers are smooth in a neighborhood of x in X.*

Now, the hypothesis for $f\colon (X, x) \to (\mathbb{C}, 0)$ to be a smoothing of (S, s) implies that one has the hypotheses of Theorem 4.1.2, and can then talk about the Milnor fiber of f, in the sense, again, of the Milnor-Lê fibration. In this context it is also referred to as the **Milnor fiber of the smoothing**.

Hence, the main theorem can be reformulated as:

Theorem 4.2.3. *The boundary of the Milnor fiber of a smoothing of a reduced singularity of complex surface is a graph manifold which can be represented by an orientable plumbing graph.*

4.3 The ambient germ can be assumed normal

First, let us show that we can suppose the ambient germ of variety $(X, 0)$ to be **normal**:

Lemma 4.3.1. *Let $(X, 0)$ be a germ of complex variety of dimension 3, and let $N\colon (\overline{X}, \overline{x}) \to (X, 0)$ be its normalization. Let f be a germ of analytic function on $(X, 0)$. Then N provides an orientation-preserving isomorphism of complex-analytic manifolds from the Milnor fiber of $\tilde{f}\colon = f \circ N$ defined on the normal germ $(\overline{X}, \overline{x})$ to the one of f.*

Proof. Let $(\overline{X}, \overline{x}) = (X_1, x_1) \bigsqcup \cdots \bigsqcup (X_k, x_k) \xrightarrow{N} (X, 0)$ be the normalization of $(X, 0)$. Then f and r induce analytic functions $f_i = f \circ N,\ r_i = r \circ N$ on each X_i, and r_i defines x_i in X_i. Now, let $\varepsilon > 0,\ \delta_\varepsilon > 0$ be adapted (in the sense of Theorem 4.1.2) to every triple (f_i, r_i, X_i), and to (f, r, X).
In this situation, we now have a disjoint union of Milnor fibers $F_1 \bigsqcup \cdots \bigsqcup F_k$ in $X_1 \bigsqcup \cdots \bigsqcup X_k$, each associated with the analytic function f_i, $F_i = f_i^{-1}(\delta) \cap X_{i_\varepsilon}$, and N provides a diffeomorphism between F, the Milnor fiber of f, and $F_1 \bigsqcup \cdots \bigsqcup F_k$, because

$$F = f^{-1}(\delta) \cap X_\varepsilon \subset \mathrm{Sm}(X),$$

and $\mathrm{Sm}(X)$ is itself contained in the set of normal points of X. □

4.4 The real-analytic variety $\mathscr{S}_k$

From now on, unless stated otherwise, the ambient variety X will be assumed **normal**.

The goal of this section is to introduce a 4-dimensional real analytic variety with isolated singularity $(\mathscr{S}_k, 0)$, which is built in such a way as to have a link diffeomorphic to the boundary of the Milnor fiber of f.

Let us introduce a second function g, that will be used as a computational tool.

Definition 4.4.1. *Let $g\colon (X, 0) \to (\mathbb{C}, 0)$ be a reduced holomorphic function. Together, f and g define a germ of morphism $\Phi := (f, g) : (X, 0) \to (\mathbb{C}^2, 0)$.*

*The **critical locus** of Φ, denoted C_Φ, is the closure of the set of smooth points of X where Φ is not a local submersion.*

*The **discriminant locus** of Φ, denoted by Δ_Φ, is the image of C_Φ by Φ.*

For our purposes, the function g must be in a generic position relatively to f, more precisely we will ask g to be a companion of f in the following sense:

Definition 4.4.2. *We say that the holomorphic function $g\colon (X, 0) \to (\mathbb{C}, 0)$ is a **companion of** f if it satisfies the following conditions:*

1. *The surface $V(g)$ is smooth outside the origin.*

2. *$V(g) \cap Sing(V(f)) = \{0\}$.*

3. *The surfaces $V(g)$ and $V(f)$ intersect transversally outside the origin, along a smooth punctured curve.*

4. *The discriminant locus Δ_Φ is an analytic curve.*

Remark 4.4.3. *This definition is analogue to that of "Isolated Complete Intersection Singularity" (ICIS) described in the work [46] of Némethi and Szilárd, in section 3.1, or in the book [31] of Looijenga. The denomination "companion" is introduced for the first time in this work. It is a little different from the one of ICIS, mainly to adapt it to any ambient normal germ $(X, 0)$ while keeping the fundamental properties of Definition 4.4.2 that are also verified by ICIS.*

Theorem 2.8, point (vi) of [31] implies the following:

Lemma 4.4.4. *Let $(X, 0)$ be a complete intersection singularity*

$$(X, 0) = (V(f_1, \cdots, f_{N-3}), 0) \subset (\mathbb{C}^N, 0)$$

and g a companion of f. Denote F and G holomorphic functions extending respectively f and g to $\mathbb{C}^N$. Then the morphism $\widetilde{\Phi} := (f_1, \cdots, f_{N-3}, F, G)$ is an Isolated Complete Intersection Singularity.

Definition 4.4.5. *(See [31, 2.8]) The diffeomorphism type of $\Phi^{-1}(x, y), (x, y) \notin \Delta_\Phi$ is called the* ***Milnor fiber*** *of Φ. In the same way, the diffeomorphism type of $\widetilde{\Phi}^{-1}(\underline{x}), \underline{x} \notin \Delta_{\widetilde{\Phi}}$ is called the* ***Milnor fiber*** *of $\widetilde{\Phi}$.*

Note that if $(x, y) \notin \Delta_\Phi$, then $(0, \cdots, 0, x, y) \notin \Delta_{\widetilde{\Phi}}$. Therefore, we have the following:

Lemma 4.4.6. *If $(X, 0)$ is a complete intersection germ and g is a companion of f, the Milnor fiber of the morphism Φ is the same as the one of a 1-dimensional ICIS.*

Finally, we arrive to the following:

Proposition 4.4.7. *[31, (5.8)] If $(X, 0)$ is a complete intersection singularity, the Milnor fiber of Φ is* ***connected****.*

Lemma 4.4.8. *Given a holomorphic function $f : (X, 0) \to (\mathbb{C}, 0)$ such that $Sing(V(f)) \supset Sing(X)$, there exists a function $g : (X, 0) \to (\mathbb{C}, 0)$ that is a companion of f.*

Furthermore, given an embedding $(X, 0) \subset (\mathbb{C}^N, 0)$ and a function f on X, there is a Zariski open dense set Ω in the space of affine hyperplanes of $\mathbb{C}^N$ at 0 such that if $\widehat{g} = 0$ belongs to Ω, the restriction to X of $\widehat{g} : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$ will be a companion of f.

Proof. To verify Conditions 1, 2 and 3, it is enough to take a hyperplane which contains neither the tangent cone to $V(f)$ nor that to $\mathrm{Sing}(X)$ (which is contained in the first one). These finite conditions are therefore realized by a dense subset of hyperplanes through 0.

Now, the fact that a generic linear form satisfies Condition 4 follows from the corollary 1 of section 5 of [23] and theorem 2.1 of [29]. □

Remark 4.4.9. *Note that Condition 2 implies in particular that* $V(g) \cap Sing(X) = \{0\}$.

Notation 4.4.10. *In the sequel,* g *will denote a companion of* f*, and* Φ *denotes the morphism* $(f, g)\colon (X, 0) \to (\mathbb{C}^2_{x,y}, 0)$.

Definition 4.4.11. *For any* $k \in 2\mathbb{N}^*$*, we define the real analytic smooth variety*

$$Z_k := \{(x, y) \in \mathbb{C}^2, x = |y|^k\} \subset \mathbb{C}^2.$$

This variety is a real surface which we orient by taking the pullback of the complex orientation of the y*-axis by the projection on the second coordinate :* $\left\{ \begin{array}{ccc} Z_k & \to & \{x = 0\} \\ (|y|^k, y) & \mapsto & y \end{array} \right.$

Remark 4.4.12. *Note that if the integer* k *is odd, the variety* Z_k *is not analytic.*

Lemma 4.4.13. *For* $k \in 2\mathbb{N}^*$ *large enough, there is a neighbourhood* U_k *of the origin in* $\mathbb{C}^2$ *on which:*

$$U_k \cap \Delta_\Phi \setminus V(x) \subset U_k \cap \{|x| > |y|^k\}.$$

The content of this lemma is represented schematically in figure 4.4.

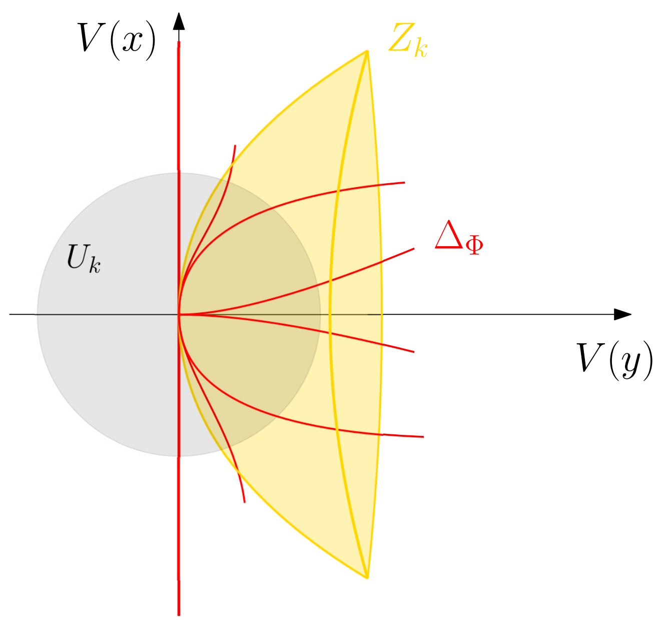


Figure 4.1: Position of Z_k relatively to Δ_Φ.

Definition 4.4.14. *In this setting, the neighbourhood U_k of $0_{\mathbb{C}^2}$ is called* ***small enough for the pair*** (Δ_Φ, k)*, or, if there is no ambiguity, for k.*

Note that a neighbourhood that is small enough for k is also small enough for any even $k' \geqslant k$.

Before beginning the proof of Lemma 4.4.13, let us note:

Remark 4.4.15. *Since $V(f)$ has non-isolated singularities, and because of the properties of transversality asked to f and g, the x-axis is contained in Δ_Φ.*

Proof of Lemma 4.4.13. The discriminant locus Δ_Φ has a finite number of branches $V(x) =: D_1, \cdots, D_n$. For a branch $D \neq D_1$, let the Puiseux expansion of D be given by

$$y = \sum_{j \geqslant j_0} a_j \cdot x^{j/n},\ a_{j_0} \neq 0$$

Then for any $k \in \mathbb{N}$,

$$|y|^k \leqslant |a_{j_0}|^k \cdot |x|^{k \cdot j_0/n} + o(|x|^{k \cdot j_0/n})$$

so if $\frac{k \cdot j_0}{n} > 1$,

$$|y|^k - |x| \geqslant -|x| + |a_{j_0}|^k \cdot |x|^{k \cdot j_0/n} + o(|x|^{k \cdot j_0/n})$$

which is strictly negative on a neighbourhood of the origin. Now it suffices to choose an integer k that is big enough for every branch of Δ_Φ.

Let us conclude this proof by noting that the exponent n/j_0 is the slope of one of the compact faces of the Newton polygon of the analytic function giving Δ_Φ. □

Remark 4.4.16. *One sees from the proof that a suitable value for k is easily derived from Δ_Φ, once one knows the Newton polygon of the corresponding function. Indeed, thanks to the observation made at the end of the previous proof, it is enough for k to be greater than the greatest slope of the compact faces of this Newton polygon.*

Definition 4.4.17. *An even integer k is said to be* ***large enough*** *for $\Phi = (f, g)$ if it is large enough in the sense of Lemma 4.4.13. Note that if k is large enough for Φ, then any even integer $k' \geqslant k$ is also large enough for Φ.*

Denote by k_0 the smallest even number that is large enough for $\Phi = (f, g)$.

In the sequel, we are going to build an explicit representative of the Milnor fiber of f, in order to compute its boundary. To do this, we have to build an appropriate representative of the germ $(X, 0)$, and to take an appropriate value for the level of f.

Definition 4.4.18. *If $k \in 2\mathbb{N}$ is large enough for Φ, a representative X_ε of $(X, 0)$ is said to be* ***small enough for*** *k if $\Phi(X_\varepsilon)$ is small enough for k.*

Condition 1. on ε_0. *We want $\varepsilon_0 > 0$ to be such that X_{ε_0} is small enough for k_0.*

Given any $k \in 2\mathbb{N}^*$ greater than k_0, and $0 < \varepsilon < \varepsilon_0$, the neighbourhood $\Phi(X_\varepsilon)$ of $0_{\mathbb{C}^2}$ will then also be small enough for k.

More conditions will be imposed on ε_0 later.

Definition 4.4.19. *Set $k \geqslant k_0$. Then we define the germ*

$$(\mathscr{S}_k, 0) := \big(\Phi^{-1}(Z_k), 0\big) = \Big(\{f = |g|^k\}, 0\Big) \subset (X, 0).$$

Lemma 4.4.20. *The germ $(\mathscr{S}_k, 0)$ is an isolated singularity of real-analytic 4-variety. Furthermore there exists a representative $\mathscr{S}_k$ of it such that $\mathscr{S}_k \setminus \{0\}$ is orientable.*

Proof. Thanks to Lemma 4.4.13, the variety $\mathscr{S}_k \setminus (V(f) \cap V(g)) \cap X_{\varepsilon_0}$ is a smooth locally trivial fibration over $Z_k \setminus \{0\} \cap \Phi(X_{\varepsilon_0})$, with fiber a smooth complex curve of the form $f^{-1}(x_0) \cap g^{-1}(y_0) \cap \overline{X_{\varepsilon_0}}$.

The smoothness of $\mathscr{S}_k \cap X_\varepsilon$ at $V(f) \cap V(g) \setminus \{0\}$ comes from the fact that on $V(f) \cap V(g) \setminus \{0\}$, because of the conditions of Definition 4.4.2, $V(f)$ and $V(g)$ are smooth surfaces intersecting transversally. Hence if $p \in V(f) \cap V(g) \setminus \{0\}$, there are local coordinates (u, v, w) at p such that $V(f) = \{u = 0\}$ and $V(g) = \{v = 0\}$.

$$\text{Locally, } \begin{cases} f = I_f(u, v, w) \cdot u \\ g = I_g(u, v, w) \cdot v \end{cases}$$

where I_f, I_g are units at p. Hence,

$$\text{at } p,\ \mathscr{S}_k = \{I(u, v, w)u = |v|^k\}$$

where I is a unit at p. This shows that $\mathscr{S}_k$ is smooth along $V(f) \cap V(g) \setminus \{0\}$.

Now, an orientation of $\mathscr{S}_k \setminus (V(f) \cap V(g))$ is built by orienting the fibers using the complex structure, and then taking the product of the orientations of the base and fibers. This provides an orientation of $\mathscr{S}_k$. □

In the sequel, $k \geqslant k_0$ is fixed and $\mathscr{S}_k$ is considered as an **oriented** variety, with the orientation described in the previous proof.

Now we are going to build cautiously actual representatives of $\mathscr{S}_k$ and of the Milnor fiber of f, that will allow us to compare their boundaries.

Recall that for $\varepsilon > 0$, we denote $S_\varepsilon := \rho^{-1}(\varepsilon) \subset X$.

Condition 2. on ε_0. *From now on, in addition to the Condition 1, we ask $\varepsilon_0 > 0$ to be such that $\forall\ 0 < \varepsilon < \varepsilon_0$, $\mathscr{S}_k$ intersects transversally S_ε.*

Definition 4.4.21. *Set $0 < \varepsilon < \varepsilon_0$, and take $k_0 \leqslant k \in 2\mathbb{N}^*$. Then we define the real-analytic 4-dimensional oriented variety*

$$\mathscr{S}_k := \overline{\Phi^{-1}(Z_k) \cap X_\varepsilon} = \overline{\{f = |g|^k\} \cap X_\varepsilon} \subset X$$

Definition 4.4.22. *In these conditions, the* ***oriented*** *manifold*

$$\partial\mathscr{S}_k := \mathscr{S}_k \cap S_\varepsilon,$$

independent of the choice of $0 < \varepsilon < \varepsilon_0$*, is called the* ***link of*** $\mathscr{S}_k$*.*

Before we can state the final result of this section, allowing us to study the link of $\mathscr{S}_k$ instead of the boundary of F, let us impose a final condition on the neighbourhoods of the origins of X and $\mathbb{C}^2$ that we want to consider.

Condition 3. on ε_0**.** *We ask in the sequel the number* $\varepsilon_0 > 0$ *to be as in the fibration Theorem 4.1.2, and to be such that the intersection* $V(f) \cap V(g)$ *is transverse in* $X_{\varepsilon_0} \setminus \{0\}$*, and for all* $0 < \varepsilon < \varepsilon_0$*, the intersection* $(V(f) \cap V(g)) \cap S_\varepsilon$ *is transverse.*

First, let us sum up here the conditions asked to the representative X_{ε_0} of the germ $(X, 0)$.

Definition 4.4.23. *In our context, the representative* X_{ε_0} *will be called a* ***good representative of the germ*** $(X, 0)$ *if* $\varepsilon_0 > 0$ *verifies Conditions 1,2 and 3 on* ε_0 *defined above.*

From now on, fix a good representative X_{ε_0} of $(X, 0)$ and $0 < \varepsilon < \varepsilon_0$. This is enough to build a convenient variety $\mathscr{S}_k$.

Let us now build an appropriate representative of the Milnor fiber of f.

Condition 3 on ε_0 implies that $\exists\ \delta_1, \delta_2 > 0$ such that $\Phi_{|S_\varepsilon}$ induces a smooth locally trivial fibration above $D_{\delta_1} \times D_{\delta_2} \subset \mathbb{C}^2$. Note that it implies that $D_{\delta_1} \times D_{\delta_2} \subset \Phi(X_\varepsilon)$.

Set $\delta > 0$ to be as in Theorem 4.1.2 relatively to ε, and such that $\delta \leqslant \delta_1$ and $\delta^{1/k} \leqslant \delta_2$. The representative of the Milnor fiber that we choose will be

$$F := \overline{\{f = \delta\} \cap X_\varepsilon}$$

with boundary

$$\partial F = f^{-1}(\delta) \cap S_\varepsilon.$$

We can now state and prove the following:

Proposition 4.4.24. *In the setting described in this section, the oriented manifolds* $\partial\mathscr{S}_k$ *and* ∂F *are orientation-preserving diffeomorphic.*

Proof. We are going to build an isotopy from ∂F to $\partial\mathscr{S}_k$ in the smooth locus $Sm(X_\varepsilon)$ of X_ε, using vector fields. This isotopy will be built in three steps, represented schematically in figure 4.2.

In the sequel, we will use the notion of gradient for some functions defined on parts of $Sm(X_\varepsilon)$. Take an embedding $X_\varepsilon \subset \mathbb{C}^N$. Then this gradient will be defined with respect to the restriction of the usual hermitian form of $T_p\mathbb{C}^N$ on $T_pSm(X_\varepsilon)$, for any $p \in Sm(X_\varepsilon)$.

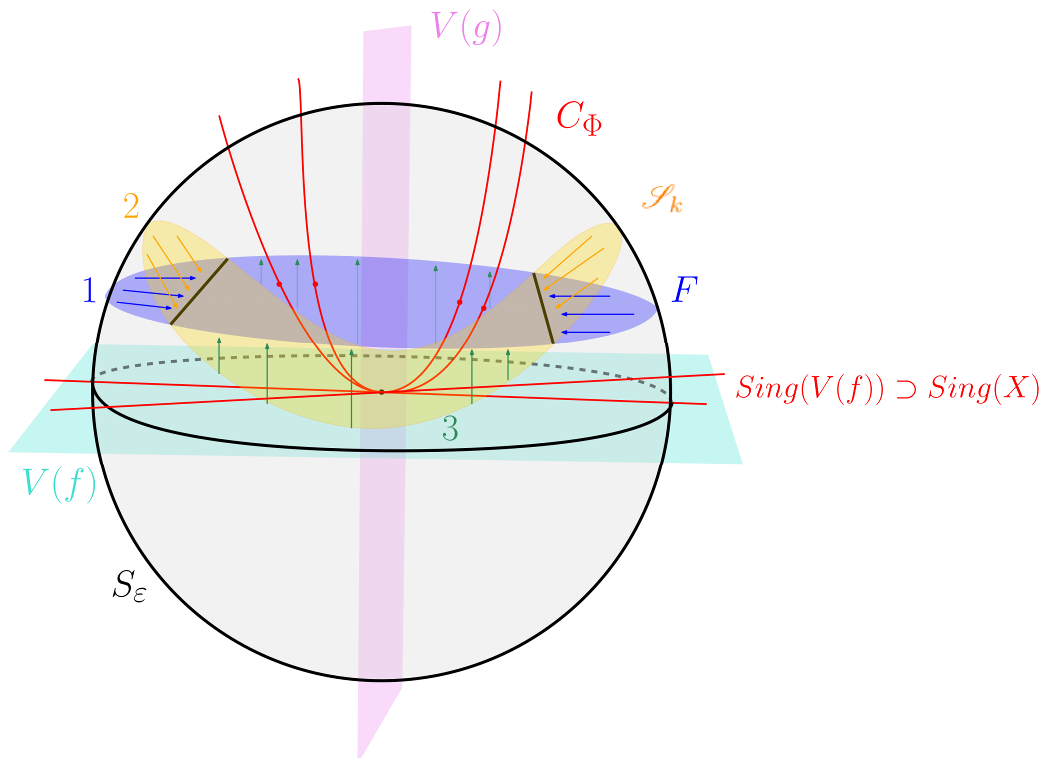


Figure 4.2: The isotopy of boundaries.

First, observe that there is an isotopy from F to the manifold with corners

$$F^{\square} := \{f = \delta\} \cap \{|g| \leqslant \delta^{1/k}\} \cap \overline{X_\varepsilon} \tag{4.1}$$

This isotopy follows from integrating the vector field provided by the gradient of the function

$$|g| \colon F \cap \{|g| \geqslant \delta^{1/k}\} \to \mathbb{R}_+$$

To see that this vector field is everywhere non zero on the compact set $F \cap \{|g| \geqslant \delta^{1/k}\}$, observe that the hypothesis $k \geq k_0$ ensures that $F \cap \{|g| \geqslant \delta^{1/k}\} \cap C_\Phi = \emptyset$, which means that the levels of f and g intersect transversally everywhere on $F \cap \{|g| \geqslant \delta^{1/k}\}$, guaranteeing that the same is true for f and $|g|$.

In the same spirit, there is an isotopy from $\mathscr{S}_k$ to the manifold with corners

$$\mathscr{S}_k^{\square} := \mathscr{S}_k \cap \{|g| \leqslant \delta^{1/k}\} \cap X_\varepsilon \tag{4.2}$$

This one is obtained, again, by integrating the gradient of the function

$$|g| \colon \mathscr{S}_k \cap \{|g| \geqslant \delta^{1/k}\} \to \mathbb{R}_+$$

which is an everywhere non-zero vector field on the compact $\mathscr{S}_k \cap \{|g| \geqslant \delta^{1/k}\}$.

Now, observe that we have the following two decompositions for the boundaries :

$$\partial F^{\square} = \left(\{f=\delta\} \cap \{|g| = \delta^{1/k}\} \cap \overline{X_\varepsilon}\right) \bigcup_{\{f=\delta\}\cap\{|g|=\delta^{1/k}\}\cap S_\varepsilon} \left(\{f=\delta\} \cap \{|g| \leqslant \delta^{1/k}\} \cap S_\varepsilon\right)$$

$$\partial \mathscr{S}_k^{\square} = \left(\{f=\delta\} \cap \{|g| = \delta^{1/k}\} \cap \overline{X_\varepsilon}\right) \bigcup_{\{f=\delta\}\cap\{|g|=\delta^{1/k}\}\cap S_\varepsilon} \left(\{f=|g|^k\} \cap \{|g| \leqslant \delta^{1/k}\} \cap S_\varepsilon\right)$$

We now build an isotopy on S_ε taking the second part of the decomposition of $\partial F^{\square}$ to the one of $\partial \mathscr{S}_k^{\square}$, and preserving $\{f=\delta\} \cap \{|g| = \delta^{1/k}\} \cap S_\varepsilon$. It will, again, be obtained by integrating an everywhere non-zero vector field. This one is built in the following way:

At $p \in \{Im(f) = 0\} \cap \{|g|^k \leqslant Re(f) \leqslant \delta\} \cap S_\varepsilon$, the vector is given by the gradient of the function

$$Re(f)\colon \{g = g(p)\} \cap S_\varepsilon \to \mathbb{R}$$

The fact that this vector field is nowhere zero on the compact $\{Im(f) = 0\} \cap \{|g|^k \leqslant Re(f) \leqslant \delta\} \cap S_\varepsilon$ comes from the condition asked on δ. Indeed δ is chosen so that Φ induces a smooth locally trivial fibration over $D_\delta \times D_{\delta^{1/k}}$, ensuring that at a point of this compact, the levels of f and g intersect transversally, guaranteeing the same for the levels of $Re(f)$ and g.

The combination of these three isotopies provides an isotopy between $\partial \mathscr{S}_k$ and ∂F, hence an orientation-preserving homeomorphism. To conclude this proof, it is enough to invoke the fact that two 3-manifolds that are orientation-preserving homeomorphic are also orientation-preserving diffeomorphic. See e.g. Munkres [42].

Let us note however that if one wants to build a diffeomorphism, it can be obtained using the three isotopies that we composed in this proof, and smoothing the corners at every step. For more details on this idea of smoothing corners, see [14].

□

4.5 The tower of morphisms

In the next sections 4.6 to 4.11, we build step by step a proper morphism

$$\Pi\colon (\widetilde{\mathscr{S}}, E) \to (\mathscr{S}_k, 0)$$

where E is a simple configuration of compact smooth real-analytic oriented surfaces as in Definition 2.7.2 in a oriented real-analytic manifold $\widetilde{\mathscr{S}}$ of dimension 4.

This morphism will be an analytic isomorphism outside E, and will therefore allow us to identify the boundary $\partial \mathscr{S}_k$ with its preimage by Π. We then prove that this preimage can be seen as the boundary of a tubular neighbourhood of E in $\widetilde{\mathscr{S}}$, which is a graph manifold determined by the configuration E, as explained in Theorem 2.7.9.

At every step, the preimage of the origin will be a configuration of real-analytic oriented surfaces, that we will represent by its dual graph, with some decorations. Note that until the penultimate step, some of these surfaces may be singular. However at every step it will be possible to associate a dual graph to the preimage of the origin.

The first level of this morphism will come from a modification r_X of X, respecting certain conditions relatively to f and g. We will consider the strict transform $\widetilde{\mathscr{S}_k}$ of $\mathscr{S}_k$ by r_X. Denote the restriction of r_X to $\widetilde{\mathscr{S}_k}$ by

$$r_{\mathscr{S}}\colon \left(\widetilde{\mathscr{S}_k}, \mathscr{C}\right) \to (\mathscr{S}_k, 0)\,.$$

The second step is the normalization of $\widetilde{\mathscr{S}_k}$

$$N_{\mathscr{S}}\colon \left(\widetilde{\mathscr{S}_k}^N, N_{\mathscr{S}}^*(\mathscr{C})\right) \to \left(\widetilde{\mathscr{S}_k}, \mathscr{C}\right).$$

Then, using local equations for $\widetilde{\mathscr{S}_k}$, we build local morphisms κ_p from complex surfaces to a finite number of open sets covering $(\widetilde{\mathscr{S}_k}, \mathscr{C})$, and use their liftings to the normalizations of the source and the target to build a global morphism

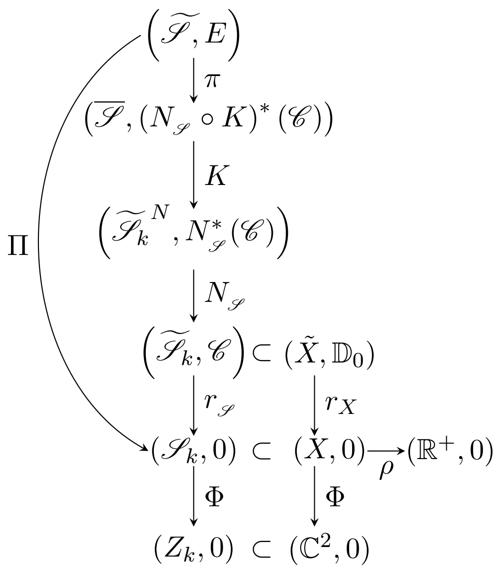
$$K\colon \left(\overline{\mathscr{S}}, (N_{\mathscr{S}} \circ K)^* (\mathscr{C})\right) \to \left(\widetilde{\mathscr{S}_k}^N, N_{\mathscr{S}}^*(\mathscr{C})\right)$$

from a 4-dimensional variety $\overline{\mathscr{S}}$ to $\widetilde{\mathscr{S}_k}^N$, that will be a diffeomorphism outside $N_{\mathscr{S}}^*(\mathscr{C})$.

The variety $\overline{\mathscr{S}}$ will have controlled isolated singularities, namely of Hirzebruch-Jung type. A standard resolution of such singularities is explained in Subsection 3.5.3, and requires only few data to be computed, leading to the morphism of the final step

$$\pi\colon \left(\widetilde{\mathscr{S}}, E\right) \to \left(\overline{\mathscr{S}}, (N_{\mathscr{S}} \circ K)^* (\mathscr{C})\right).$$

The construction of the composed morphism Π is summarized in the following diagram



Algorithmic aspect

We will decorate the different dual graphs, obtained as in Definition 2.7.1,

$$\Gamma(\mathscr{C}), \Gamma((N_{\mathscr{S}} \circ K)^{*}(\mathscr{C})), \Gamma(E)$$

obtaining the decorated graphs

$$\overset{\star}{\Gamma}(\mathscr{C}), \overset{\star}{\Gamma}((N_{\mathscr{S}} \circ K)^{*}(\mathscr{C})), \overset{\star}{\Gamma}(E).$$

The decorations of these graphs will be of different natures.

They are built and decorated in order to get the plumbing graph

$$\overset{\star}{\Gamma}(E) = \Gamma_{\widetilde{\mathscr{S}}}(E)$$

for the boundary of a tubular neighbourhood of E in $\widetilde{\mathscr{S}}$, which is the aim of this study.

It is very important to notice that, at every step, some of the information needed to compute a decorated graph is encoded in the previous one. Namely:

- $\overset{\star}{\Gamma}(E)$ is entirely determined by $\overset{\star}{\Gamma}((N_{\mathscr{S}} \circ K)^{*}(\mathscr{C}_{tot}))$.
- $\overset{\star}{\Gamma}((N_{\mathscr{S}} \circ K)^{*}(\mathscr{C}_{tot}))$ is entirely determined by $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ and some information relative to the singularities of X, namely the collection of switches introduced in Subsection 4.10.2, and the additional covering data mentioned in Subsection 4.10.4. In general, accessing that information may represent a difficult task, for which one should have,

among other things, a good understanding of the structure of the initial modification r_X. However this does not stop us from carrying on the proof of the general result. In some cases, that additional information is unnecessary, as for example in the case where X is smooth (see [46]), or in the toric case studied in Chapter 5.

In conclusion, this relation between the graphs, which in some cases allows to deduce one graph from the previous one, permits us to build a method for the computation of ∂F, whose starting point will be the decorated graph $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ together with the required additional information relative to the singularities of X. The construction and definition of $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ and $\widetilde{\mathscr{S}_k}$ is the object of the next section.

4.6 The variety $\widetilde{\mathscr{S}_k}$

Now let us introduce a complex analytic modification (in the sense of Definition 2.2.5) $r_X : \tilde{X} \to X$ of X, that will verify some conditions relative to f and g:

Definition 4.6.1. (Modification adapted to Φ.)
*A modification $r_X : (\tilde{X}, \mathbb{D}_0) \to (X, 0)$ is said to be **adapted** to the morphism $\Phi = (f, g)$, where g is a companion of f, if the two following conditions are simultaneously satisfied:*

1. *The modification r_X is an isomorphism outside of $(V(f) \setminus V(g)) \cup \{0\}$. Or equivalently, r_X may not be an isomorphism only at $V(f) \setminus V(g)$ or $\{0\}$.*
2. *Denote $\mathbb{D}_f := \overline{r_X^{-1}(V(f) \setminus 0)}$, $\mathbb{D}_g := \overline{r_X^{-1}(V(g) \setminus 0)}$, as well as*

$$\mathscr{C} := \mathbb{D}_f \cap \mathbb{D}_0$$

and

$$\mathscr{C}_{tot} := \mathscr{C} \cup (\mathbb{D}_f \cap \mathbb{D}_g).$$

In order to emphasize that the global configuration of the irreducible components of the curves $\mathscr{C}$ and $\mathscr{C}_{tot}$ is essential for us, we will say that the two curves are **curve configurations**.

With these notations, the second condition imposed on r_X is that the total transform

$$\mathbb{D} := r_X^{-1}(V_{f \cdot g})$$

of $V_{f \cdot g}$ shall be a simple normal crossings divisor at $\mathscr{C}_{tot}$.

Example 4.6.2. *Figure 4.3 shows an example of divisor $\mathbb{D}$. The components of $\mathbb{D}_f$ are represented in blue, those of $\mathbb{D}_g$ in green, and those of $\mathbb{D}_0$ in red. Full lines represent irreducible components of $\mathscr{C}_{tot}$, and the arrow represents a non-compact component of $\mathbb{D}_f \cap \mathbb{D}_g$. The points of intersection of two irreducible components of $\mathscr{C}_{tot}$ are decorated using signs, anticipating the convention introduced in Definition 4.7.6.*

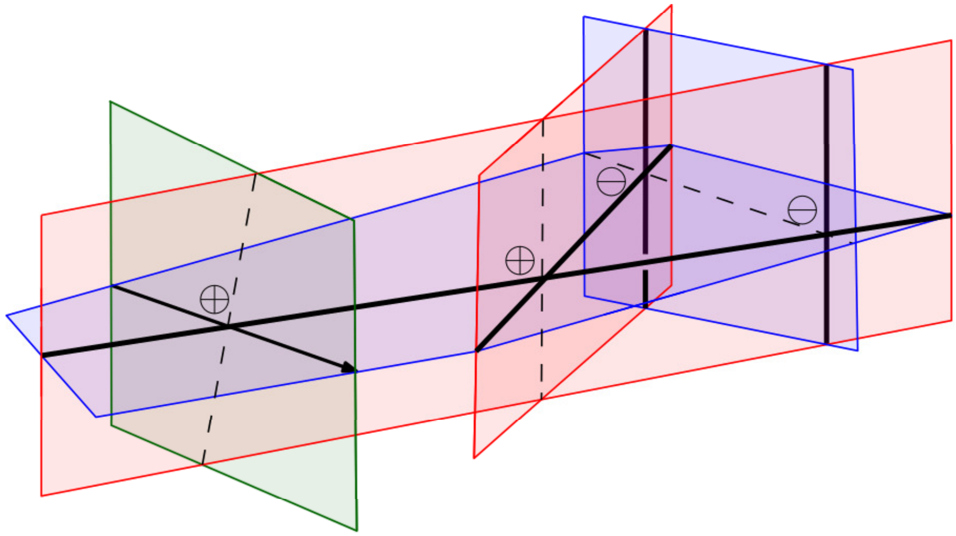


Figure 4.3: The divisor $\mathbb{D}$, and the curve configurations $\mathscr{C}$, $\mathscr{C}_{tot}$.

Proposition 4.6.3. *(See [46, Corollary 7.1.5], and Proposition 4.6.3). If the germ $(X, 0)$ is a complete intersection, the curve configurations $\mathscr{C}$ and $\mathscr{C}_{tot}$ are connected.*

Remark 4.6.4. *Both $\mathscr{C}$ and $\mathscr{C}_{tot}$ can be seen as complex curve configurations in $\tilde{X}$ or as real surfaces configurations in $\mathscr{S}_k$. Note that the configuration $\mathscr{C}$ is compact.*

These configurations will play a key role in the sequel. They are in general complicated to compute, but in the toric case, exposed in Chapter 5, we can express them with only combinatorial considerations.

Definition 4.6.5. *Orient the irreducible components of $\mathscr{C}$ and $\mathscr{C}_{tot}$ by their complex structures in $\tilde{X}$.*

Condition 1 of Definition 4.6.1 implies that we do not want to build an embeddded resolution of the surface $V_{f \cdot g}$. Indeed we do not want to modify it along $V(f) \cap V(g) \setminus \{0\}$, where it is singular, but with simple singularities, i.e. it looks locally like a transverse intersection of smooth surfaces.

The existence of a modification of X adapted to Φ is guaranteed by the following

Lemma 4.6.6. (Resolution Lemma), [63, p.633]

Let X be an irreducible variety over an algebraically closed field of characteristic 0, V a pure codimension one reduced subvariety of X, and U an open subset of the smooth locus of X such that $U \cap V$ has smooth components crossing normally. Then there is a projective morphism $r_X : \tilde{X} \to X$ which satisfies the following conditions:

1. r_X *is a composition of blowing ups of smooth subvarieties.*

2. r_X *induces an isomorphism over* U.

3. $\tilde{X}$ *is smooth.*

4. $r_X^{-1}(V \cup (X \setminus U))$ *is a divisor. Moreover, it has smooth components crossing normally.*

In our context, one can for example take the open subset U to be $X \setminus Sing(V(f))$.

Remark 4.6.7. 1. *Because of the properties of the modification* r_X, $\mathbb{D}_g$ *is exactly the strict transform of* $V(g)$ *by* r_X. *However,* $\mathbb{D}_f$ *is the* ***mixed transform****, in the sense of Definition 2.2.11, of* $V(f)$ *by* r_X. *Indeed it will also contain some components of the exceptional divisor, the surfaces that get contracted by* r_X *on curves in* $(V(f) \setminus V(g)) \cup \{0\}$, *in particular the preimage of the singular locus of* $V(f)$.

2. *Point 1 of Definition 4.6.1 implies that* $\mathbb{D}_f \cap \mathbb{D}_g$ *is exactly the intersection of the strict transforms of* $V(f)$ *and* $V(g)$ *by* r_X.

3. *The Resolution Lemma 4.6.6 implies that in general one can always build a modification* r_X *such that* $\mathbb{D}$ *is globally a SNCD, which will imply Condition 2 on* r_X. *However, what we really need in order to proceed is this weaker condition.*

4. *In the same spirit, unlike what is required in [46, 6.1.2], we allow the morphism* r_X *to modify some curves in* $(V(f) \setminus V(g)) \cup \{0\}$ *along which* $V(f)$ *may be smooth. This point and the previous one will be important for our analysis of Newton nondegenerate surface singularities in Chapter 5.*

5. *We get a partition of the set of irreducible components of the total transform* $\mathbb{D}$ *in three parts: the set of irreducible components of* $\mathbb{D}_f, \mathbb{D}_g$ *or* $\mathbb{D}_0$. *By abuse of notation, we will sometimes refer to an irreducible component of* $\mathbb{D}$ *as "in" (*$\in$*) one of these three surfaces, instead of "subset of" (*$\subset$*).*

Notation 4.6.8. *Denote*

$$f_{\tilde{X}} := f \circ r_X, \ g_{\tilde{X}} := g \circ r_X : \tilde{X} \to \mathbb{C}$$

the pullbacks of f *and* g *to* $\tilde{X}$. *In the sequel, if* D_i *is an irreducible component of* $\mathbb{D}$, *we will denote*

$$m_i := mult_{D_i}(f_{\tilde{X}}), \ n_i := mult_{D_i}(g_{\tilde{X}})$$

the respective multiplicities of $f_{\tilde{X}}, g_{\tilde{X}}$ *along this component.*

Note that if $D_1 \in \mathbb{D}_g$, *then necessarily* $m_1 = 0$ *and* $n_1 = 1$.

The firt step of the construction of the desired morphism Π is the modification of $\mathscr{S}_k$ obtained by restricting r_X to the strict transform

$$(\widetilde{\mathscr{S}_k}, \mathscr{C}) := \left(\overline{r_X^{-1}(\mathscr{S}_k \setminus \{0\})}, \mathscr{C}\right) \subset (\tilde{X}, 0).$$

of $\mathscr{S}_k$ by r_X. It will have non-isolated singularities, being from this point of view more complicated than $(\mathscr{S}_k, 0)$. However we can domesticate these singularities. This is the point of the next section.

4.7 Local equations of $\widetilde{\mathscr{S}_k}$ along $\mathscr{C}_{tot}$, and complex models

Denote $r_{\mathscr{S}} := r_{X|\widetilde{\mathscr{S}_k}} : \widetilde{\mathscr{S}_k} \to \mathscr{S}_k$.

Proposition 4.7.1. *Let $k \in 2\mathbb{N}$ be such that for every component $D_1 \in \mathbb{D}_0$, $kn_1 > m_1$. Then:*

1. *The preimage in $\widetilde{\mathscr{S}_k}$ of the origin in $\mathscr{S}_k$ is*

$$r_{\mathscr{S}}^{-1}(0) = \mathscr{C} = \mathbb{D}_f \cap \mathbb{D}_0. \tag{4.3}$$

2. *The support of the total transform of $V(g) \cap \mathscr{S}_k$ by $r_{\mathscr{S}}$ is equal to $\mathscr{C}_{tot} = (\mathbb{D}_f \cap \mathbb{D}_g) \cup (\mathbb{D}_f \cap \mathbb{D}_0)$. In other words, denoting*

$$g_{\widetilde{\mathscr{S}_k}} := g \circ r_{\mathscr{S}} : \widetilde{\mathscr{S}_k} \to \mathbb{C}$$

the pullback of g to $\widetilde{\mathscr{S}_k}$, we have

$${g_{\widetilde{\mathscr{S}_k}}}^{-1}(0) = \mathscr{C}_{tot}. \tag{4.4}$$

The statement of Proposition 4.7.1 implies that we need new conditions on k in order to proceed to the computation of the boundary of $\mathscr{S}_k$. In the sequel, more will appear. Let us make those conditions explicit.

Condition 1. on k. *The integer k must be even and large enough in the sense of Lemma 4.4.13.*

Condition 2. on k. *The integer k must be such that, for every component $D_1 \in \mathbb{D}_0$, $kn_1 > m_1$.*

Notation 4.7.2. *Let $H \subset \mathbb{C}^3_{u,v,w}$. Denote H^{str} the* ***strict part*** *of H defined as the closure*

$$H^{str} := \overline{H \setminus \{uvw = 0\}}.$$

Proof of Proposition 4.7.1. Let p be a generic point of a component $D_1 \in \mathbb{D}_0$. Then one can take local holomorphic coordinates (u, v, w) on a neigbourhood U_p of p in $\tilde{X}$ such that $D_1 \cap U_p = \{u = 0\} \cap U_p$. Then on U_p,

$$\begin{cases} f_{\tilde{X}} = I_f(u, v, w)u^{m_1} \\ g_{\tilde{X}} = I_g(u, v, w)u^{n_1} \end{cases}$$

where I_f, I_g are units at p, as in Definition 2.4.1.

Then

$$\widetilde{\mathscr{S}_k} \cap U_p = \overline{\{I_f \cdot u^{m_1} = |I_g|^k |u|^{kn_1}\} \setminus V(u)} \cap U_p.$$

Now the condition $kn_1 > m_1$ implies $kn_1 \neq m_1$, hence $\widetilde{\mathscr{S}_k}$ does not contain p.

The same considerations at generic points of $\mathbb{D}_f$ or $\mathbb{D}_g$ will show, even without conditions on k, that $\widetilde{\mathscr{S}_k}$ does not contain them neither.

Consider a point $p \in D_1 \cap D_2$, where $D_1 \in \mathbb{D}_0, D_2 \in \mathbb{D}_g$, and p is on no other component of $\mathbb{D}$. Then, as $\mathbb{D}$ is a simple normal crossings divisor at p, we can set local holomorphic coordinates (u, v, w) on a neighbourhood U_p of p in $\tilde{X}$ such that $D_1 = \{u = 0\}, D_2 = \{v = 0\}$. Then on U_p,

$$\begin{cases} f_{\tilde{X}} = I_f(u, v, w)u^{m_1} \\ g_{\tilde{X}} = I_g(u, v, w)u^{n_1}v^{n_2} \end{cases}$$

where I_f, I_g are units at p.

On U_p, $\widetilde{\mathscr{S}_k}$ is then defined by $\overline{\{f_{\tilde{X}} = |g_{\tilde{X}}|^k\} \setminus V(u)}$, that is,

$$\widetilde{\mathscr{S}_k} \cap U_p = \{I_f \cdot u^{m_1} = |I_g|^k |u|^{kn_1} |v|^{kn_2}\}^{str} \cap U_p.$$

Then the condition $kn_1 > m_1$ shows that $\widetilde{\mathscr{S}_k}$ does not contain p.

The rest of this proposition, stating that $\widetilde{\mathscr{S}_k}$ contains $\mathscr{C}_{tot}$, is proved in the following Subsections 4.7.1 and 4.7.2, where we provide local equations of $\widetilde{\mathscr{S}_k}$ along $\mathscr{C}_{tot}$. □

The reason why we want to keep track of the total transform of $V(g) \cap \mathscr{S}_k$ in $\widetilde{\mathscr{S}_k}$, is because the pullback of g by Π will be the adapted function that we use to compute the self-intersections of the irreducible components of E in $\widetilde{\mathscr{S}}$, as explained in Lemma 2.7.7.

We will now determine local equations of $\widetilde{\mathscr{S}_k}$ around a point of $\mathscr{C}_{tot}$, in order to define and study the morphisms $N_{\mathscr{S}}$ and K mentioned in section 4.5.

Let $p \in \mathscr{C}_{tot}$. Because $\mathbb{D}$ is a normal crossings divisor at $\mathscr{C}_{tot}$, p will be on no more than three irreducible components of $\mathbb{D}$ at the same time. We face two different situations. In each of them we first build local coordinates giving $\widetilde{\mathscr{S}_k}$ a simple equation, and use it to provide a "local algebraic model" of this surface. We then study the lifting of this model to the normalizations of the source and the target. This lifting, globally, provides the morphism K.

Discussion 4.7.3. *In the sequel, we show among other things that there is a finite open covering of* $(\widetilde{\mathscr{S}_k}, \mathscr{C}_{tot})$ *such that, on each one of these open sets,* $\widetilde{\mathscr{S}_k}$ *can be presented as an affine real-algebraic variety. This allows us, using Subsection 2.5.1, to speak about the normalization* $\widetilde{\mathscr{S}_k}^N$ *of* $\widetilde{\mathscr{S}_k}$*, which is a real-analytic variety. Denote by* $N_{\mathscr{S}}: \widetilde{\mathscr{S}_k}^N \to \widetilde{\mathscr{S}_k}$ *the* ***normalization morphism*** *of* $\widetilde{\mathscr{S}_k}$*.*

From now on, if $x \in \mathbb{R}_+, n \in \mathbb{N}^*$, $x^{1/n}$ will denote the unique n-th root of x in $\mathbb{R}_+$.

4.7.1 Generic points of $\mathscr{C}_{tot}$

Take a point $p \in D_1 \cap D_2$, where $D_1 \in \mathbb{D}_0$ or $\mathbb{D}_g$, $D_2 \in \mathbb{D}_f$, with the notations of Definition 4.6.1, and p is on no other component of $\mathbb{D}$. Then, as $\mathbb{D}$ is a simple normal crossings divisor at p, we can set local holomorphic coordinates (u', v', w') on a neighbourhood U'_p of p in $\tilde{X}$ such that $D_1 = \{u' = 0\}, D_2 = \{v' = 0\}$. Then locally:

$$\begin{cases} f_{\tilde{X}} = I_f(u', v', w') u'^{m_1} v'^{m_2} \\ g_{\tilde{X}} = I_g(u', v', w') u'^{n_1} \end{cases}$$

where I_f, I_g are units at p, as in Definition 2.4.1.

Now, choose ϕ, ψ units at p such that $\phi^{n_1} = I_g$ and $\psi^{m_2} = I_f \cdot \phi^{-m_1}$, and build new local coordinates (u, v, w) around p defined by

$$\begin{cases} u = \phi \cdot u' \\ v = \psi \cdot v' \\ w = w' \end{cases}$$

on a neighbourhood $U_p \subset U'_p$ where ϕ and ψ are non-zero.

On U_p, $\widetilde{\mathscr{S}_k}$ is then defined by $\overline{\{f_{\tilde{X}} = |g_{\tilde{X}}|^k\} \setminus V(u)}$, that is,

$$\widetilde{\mathscr{S}_k} \cap U_p = \{u^{m_1} v^{m_2} = |u|^{kn_1}\}^{str} \cap U_p. \tag{4.5}$$

Note that Condition 2 on k ensures that p is in $\widetilde{\mathscr{S}_k}$.

Now, we can provide a birational morphism:

$$\kappa_p \colon \{(x, y, z) \in \mathbb{C}^3, x^{m_1} = y^{m_2}\} \to \{(u, v, w) \in \mathbb{C}^3, u^{m_1} v^{m_2} = |u|^{kn_1}\}^{str}$$

given by

$$\begin{cases} u = x \\ v = y^{-1} |x|^{kn_1/m_2} = y^{-1} |y|^{kn_1/m_1} \\ w = z \end{cases} \qquad \begin{cases} x = u \\ y = v^{-1} |u|^{kn_1/m_2} \\ z = w \end{cases} \tag{4.6}$$

If $kn_1/m_2 \in 2\mathbb{N}^*$, the map κ_p is birational.

We then require

Condition 3. on k. *For every κ_p to be birational, the integer k must be such that, $\forall\ D_1 \in \mathbb{D}_0$ that intersects a component $D_2 \in \mathbb{D}_f$,*

$$\frac{kn_1}{m_2} \in 2\mathbb{N}.$$

In addition to the birationality of κ_p, the identity

$$y^{m_2} = u^{m_1}$$

shows that the coordinates $x, y, z, \overline{x}, \overline{y}$ and $\overline{z}$ of the source are integral on the ring of regular functions of $\widetilde{\mathscr{S}_k} \cap U_p$. Reciprocally, the identity

$$v^{m_1} = y^{-m_1}|y|^{kn_1}$$

shows that, thanks to Condition 2 on k, the coordinates of the target are integral on the ring of functions of the source.

Note that these two identities taken together, with the Condition 2 on k, ensure that κ_p is a homeomorphism.

Setting $U_p^{\mathbb{C}} := \kappa_p^{-1}(U_p)$ a neighbourhood of the origin in $\mathbb{C}^3$ and using Proposition 2.5.10, we get:

Lemma 4.7.4. *The homeomorphism κ_p induces an isomorphism of real analytic varieties*

$$\overline{\kappa_p}\colon \left(\{(x,y,z) \in \mathbb{C}^3, x^{m_1} = y^{m_2}\} \cap U_p^{\mathbb{C}}\right)^{norm} \xrightarrow{\sim} \left(\widetilde{\mathscr{S}_k} \cap U_p\right)^{norm}$$

at the level of normalizations.

Discussion 4.7.5. Orientation of $\widetilde{\mathscr{S}_k}$ and the model near a generic point.

In section 4.4, we provided a description of the orientation of $\mathscr{S}_k$. This orientation is pulled-back by $r_{\mathscr{S}}$ to provide an orientation of $\widetilde{\mathscr{S}_k}$, making $r_{\mathscr{S}}$ orientation-preserving.

Another way to retrieve this orientation on an open U_p as in subsection 4.7.1 is to observe that it is the one that makes $\partial\widetilde{\mathscr{S}_k} := r_{\mathscr{S}}^{-1}(\partial\mathscr{S}_k) \simeq \partial\mathscr{S}_k$ orientation-preserving diffeomorphic to ∂F, where the last equivalence symbol denotes an orientation-preserving diffeomorphism.

The modification r_X induces a biholomorphic morphism over F, and is hence an orientation-preserving diffeomorphism from $\widetilde{F} := r_X^{-1}(F)$ to F, both being oriented by their complex structures. $\widetilde{F}$ is a complex manifold, and

$$\widetilde{F} \cap U_p = \{I(u,v,w)u^{m_1}v^{m_2} = \delta\}$$

for some holomorphic unit I at p. The complex orientation of $\widetilde{F} \cap U_p$ is given by $du \wedge d\overline{u} \wedge dw \wedge d\overline{w}$ (or, equivalently, $dv \wedge d\overline{v} \wedge dw \wedge d\overline{w}$, but for our purposes the first expression is more natural).

Now, the orientation of $\widetilde{\mathscr{S}_k}$ on U_p is the one that is compatible with this orientation of $\widetilde{F}$, that is, the one that is given on the smooth part of $\widetilde{\mathscr{S}_k} \cap U_p$ by $du \wedge d\overline{u} \wedge dw \wedge d\overline{w}$.

One can see now that the morphism κ_p is orientation-preserving, ***if the source is taken with its complex orientation*** $dx \wedge d\overline{x} \wedge dz \wedge d\overline{z}$.

Furthermore, the normalizations of the source and target of κ_p are now canonically oriented by the pullbacks of these orientations, making, again, $\overline{\kappa_p}$ orientation-preserving as soon as the source is oriented by its complex structure.

Finally, remind that in Definition 4.6.5, we oriented $\mathscr{C} \cap U_p = \{u = v = 0\} \cap U_p$ via its complex structure, i.e. by $dw \wedge d\overline{w}$. Pulling back this orientation to its preimage by κ_p gives, again, the complex orientation of the axis $\{x = y = 0\} \subset U_p^{\mathbb{C}}$.

Therefore, the pullback of the orientation of $\mathscr{C}_{tot} \cap U_p$ by $N_{\mathscr{S}} \circ \kappa_p$ ***orients its preimage*** $(N_{\mathscr{S}} \circ \kappa_p)^{-1}(\mathscr{C}_{tot} \cap U_p)$ ***by its complex structure.***

4.7.2 Double points of $\mathscr{C}_{tot}$

Let $p \in D_1 \cap D_2 \cap D_3$, where $D_1 \in \mathbb{D}_0, D_2 \in \mathbb{D}_f, D_3 \in \mathbb{D}$, with the notations of Definition 4.6.1. Set local holomorphic coordinates (u', v', w') on a neighbourhood U'_p of p in $\tilde{X}$ such that $D_1 = \{u' = 0\}, D_2 = \{v' = 0\}, D_3 = \{w' = 0\}$. Then we can write locally:

$$\begin{cases} f_{\tilde{X}} = I_f(u', v', w') u'^{m_1} v'^{m_2} w'^{m_3} \\ g_{\tilde{X}} = I_g(u', v', w') u'^{n_1} w'^{n_3} \end{cases}$$

where I_f, I_g are units at p.

In the same way as in 4.7.1, we obtain local coordinates (u, v, w) on a neighbourhood $U_p \subset U'_p$ such that

$$\widetilde{\mathscr{S}_k} \cap U_p = \{u^{m_1} v^{m_2} w^{m_3} = |u|^{kn_1} |w|^{kn_3}\}^{str} \cap U_p. \tag{4.7}$$

and $\widetilde{\mathscr{S}_k}$ contains p because, again, of Condition 2 on k for D_1.

We will provide, again, a local algebraic model for $\widetilde{\mathscr{S}_k}$ around p, that will depend on the nature of the double point p.

Definition 4.7.6. *(See [46, section 6.2])*

The double point p is said to be ***of type*** $\oplus$ *if $D_3 \in \mathbb{D}_g$ or $\mathbb{D}_0$, and* ***of type*** $\ominus$ *if $D_3 \in \mathbb{D}_f$.*

That is, p is of type $\oplus$ if and only if $n_3 \neq 0$.

Example 4.7.7. *For the sake of clarity, we repeat here in Figure 4.4 an example of divisor $\mathbb{D}$. As in Figure 4.3, the components of $\mathbb{D}_f$ are represented in blue, those of $\mathbb{D}_g$ in green, and those of $\mathbb{D}_0$ in red. Full lines represent irreducible components of $\mathscr{C}_{tot}$, and the arrow represents a non-compact component of $\mathbb{D}_f \cap \mathbb{D}_g$.*

Remark 4.7.8. *We separate the two types of points using signs, anticipating the fact that they will give different signs of edges in the final plumbing graph for ∂F. Némethi and Szilárd use respectively 1 and 2, instead of $\oplus$ and $\ominus$, to designate the two types of double points.*

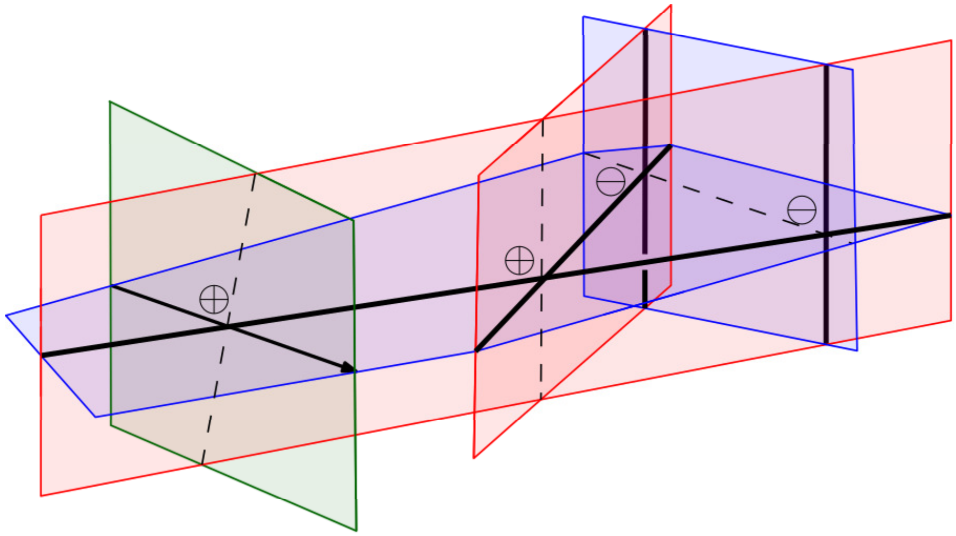


Figure 4.4: The divisor $\mathbb{D}$, and the curve configurations $\mathscr{C}$, $\mathscr{C}_{tot}$.

Points of type $\oplus$.

If p is of type $\oplus$, things go more or less the same way as in the case of generic points. Namely, there is a birational map

$$\kappa_p^{\oplus}\colon \{(x,y,z)\in\mathbb{C}^3, y^{m_2}=x^{m_1}z^{m_3}\}\to\{u^{m_1}v^{m_2}w^{m_3}=|u|^{kn_1}|w|^{kn_3}\}^{str}$$

given by

$$\begin{cases} u=x \\ v=y^{-1}|x|^{kn_1/m_2}|z|^{kn_3/m_2} \\ w=z \end{cases} \qquad \begin{cases} x=u \\ y=v^{-1}|u|^{kn_1/m_2}|w|^{kn_3/m_2} \\ z=w \end{cases} \tag{4.8}$$

Condition 3 on k shows that this map is birational. Furthermore it is a homeomorphism.

Moreover, as in 4.7.1, using $y^{m_2}=u^{m_1}w^{m_3}$ and reciprocally, $v^{m_2}=|x|^{kn_1}x^{-m_1}|z|^{kn_3}z^{-m_3}$, still with $kn_i>m_i$, and denoting $U_p^{\mathbb{C}}:={\kappa_p^{\oplus}}^{-1}(U_p)$ a neighbourhood of the origin in $\mathbb{C}^3$, we have:

Lemma 4.7.9. *The morphism $\kappa_p^{\oplus}$ is a homeomorphism, and it induces an isomorphism of real analytic varieties*

$$\overline{\kappa_p^{\oplus}}\colon \left(\{(x,y,z)\in\mathbb{C}^3, y^{m_2}=x^{m_1}z^{m_3}\}\cap U_p^{\mathbb{C}}\right)^{norm}\xrightarrow{\sim}\left(\widetilde{\mathscr{S}_k}\cap U_p\right)^{norm}$$

at the level of normalizations.

Discussion 4.7.10. Orientation of $\widetilde{\mathscr{S}_k}$ and its model near a point $\oplus$.
An argument analogous to the one developped in discussion 4.7.5, with, this time,

$$\widetilde{F} \cap U_p = \{I(u,v,w)u^{m_1}v^{m_2}w^{m_3} = \delta\}$$

provides again an orientation of $\widetilde{\mathscr{S}_k}$ given by $du \wedge d\overline{u} \wedge dw \wedge d\overline{w}$, and $\overline{\kappa_p^{\oplus}}$ is then orientation-preserving as soon as ***its source is oriented by its complex structure.***

Furthermore, the two irreducible components of $\mathscr{C}_{tot}$ that are visible here, given by $\{u = v = 0\}$ and $\{w = v = 0\}$, are oriented as complex curves, respectively by $dw \wedge d\overline{w}$ and $du \wedge d\overline{u}$. The morphism $\kappa_p^{\oplus}$ then orients their preimages, respectively $\{x = y = 0\}$ and $\{z = y = 0\}$, by their complex orientations $dz \wedge d\overline{z}$ and $dx \wedge d\overline{x}$. Hence, again, the pullback of the orientation of $\mathscr{C}_{tot} \cap U_p$ by $N_{\mathscr{S}} \circ \kappa^{\oplus}$ ***orients its preimage*** *$(N_{\mathscr{S}} \circ \kappa^{\oplus})^{-1}(\mathscr{C}_{tot} \cap U_p)$* ***by its complex structure.***

Points of type $\ominus$.

Now if p is a double point of type $\ominus$, one can say less, but still provide a local algebraic model for $\widetilde{\mathscr{S}_k}$. Let us introduce the following morphism:

$$\kappa_p^{\ominus} \colon \{(x,y,z) \in \mathbb{C}^3, x^{m_1} = y^{m_2}z^{m_3}\} \to \{u^{m_1}v^{m_2}w^{m_3} = |u|^{kn_1}\}^{str}$$

given by

$$\begin{cases} u = x \\ v = y^{-1}|y|^{kn_1/m_1} \\ w = z^{-1}|z|^{kn_1/m_1} \end{cases} \tag{4.9}$$

Condition 2 on k implies that $\kappa_p^{\ominus}$ is regular. Furthermore it is a homeomorphism. Indeed, identifying the arguments and moduli, one can get

$$\begin{cases} x = u \\ y = v^{-1}|v|^{kn_1/(kn_1-m_1)} \\ z = w^{-1}|w|^{kn_1/(kn_1-m_1)} \end{cases}$$

Remark 4.7.11. *However, this morphism is* ***not*** *birational as soon as $\frac{kn_1}{kn_1-m_1}$ is not an even integer.*

For example if $kn_1 > 2m_1$, $\frac{kn_1}{kn_1-m1} = 1 + \frac{m_1}{kn_1-m_1}$ is not an integer.

The morphism $\kappa_p^{\ominus}$, sending no irreducible component of the source to the non-normal locus of the target (see Proposition 2.5.10), lifts to a morphism

$$\overline{\kappa_p^{\ominus}} \colon \left(\{(x,y,z) \in \mathbb{C}^3, x^{m_1} = y^{m_2}z^{m_3}\} \cap U\right)^{norm} \to \left(\widetilde{\mathscr{S}_k} \cap U_p\right)^{norm}$$

at the level of normalizations.

Although $\overline{\kappa_p^{\ominus}}$ is not an isomorphism, we have the following:

Lemma 4.7.12. *Setting* $U_p^{\mathbb{C}} := \kappa_p^{\ominus -1}(U_p)$*, the restriction of* $\overline{\kappa_p^{\ominus}}$ *induces an isomorphism of analytic varieties*

$$\overline{\kappa_p^{\ominus *}}\colon \left(\{(x,y,z)\in\mathbb{C}^3, x^{m_1}=y^{m_2}z^{m_3}\}\cap U_p^{\mathbb{C}}\setminus\{0\}\right)^{norm}\to\left(\widetilde{\mathscr{S}_k}\cap U_p\setminus\{0\}\right)^{norm}$$

at the level of "punctured" normalizations.

Proof. Cover $\left(\widetilde{\mathscr{S}_k}\cap U_p\setminus\{0\}\right)^{norm} = N_{\mathscr{S}}^{-1}\left(\widetilde{\mathscr{S}_k}\cap U_p\setminus\{0\}\right)$ with two sets:

$$\left(\widetilde{\mathscr{S}_k}\cap U_p\setminus\{0\}\right)^{norm} = N_{\mathscr{S}}^{-1}\left(\widetilde{\mathscr{S}_k}\cap U_p\setminus V(v)\right)\bigcup_{N_{\mathscr{S}}^{-1}\left(\widetilde{\mathscr{S}_k}\cap U_p\setminus V(v\cdot w)\right)} N_{\mathscr{S}}^{-1}\left(\widetilde{\mathscr{S}_k}\cap U_p\setminus V(w)\right)$$

We are going to prove that, in restriction to each of these two sets, $\overline{\kappa_p^{\ominus *}}$ induces a diffeomorphism.

In order to do this, consider the birational map

$$\alpha_{p,v}\colon \{(x',y',z')\in\mathbb{C}^3, x'^{m_1}=y'^{m_2}z'^{m_3}\}\setminus V(y')\to\{u^{m_1}v^{m_2}w^{m_3}=|u|^{kn_1}\}^{str}\setminus V(v)$$

given by

$$\begin{cases} u = x' \\ v = y'^{-1} \\ w = z'^{-1}|x'|^{kn_1/m_3}\end{cases}\qquad\qquad \begin{cases} x' = u \\ y' = v^{-1} \\ z' = w^{-1}|u|^{kn_1/m_3}\end{cases}$$

Thanks to Condition 3 on k, this map is birational. Furthermore, with Condition 2 on k for D_1, the identities

$$z'^{m_3} = u^{m_1}v^{m_2}$$

and

$$w^{m_3} = y'^{m_2}|x'|^{kn_1}x'^{-m_1}$$

show that, setting $U'_v := \alpha_{p,v}^{-1}(U_p)$, it induces an isomorphism of real analytic varieties

$$\overline{\alpha_{p,v}}\colon \left(\{(x',y',z')\in\mathbb{C}^3, x'^{m_1}=y'^{m_2}z'^{m_3}\}\cap U'_v\setminus V(y')\right)^{norm}\xrightarrow{\sim} N_{\mathscr{S}}^{-1}\left(\widetilde{\mathscr{S}_k}\cap U_p\setminus V(v)\right)$$

We can use this isomorphism to understand the restriction $\overline{\kappa_{p,v}^{\ominus}}$ of $\overline{\kappa_p^{\ominus}}$ to

$$\left(\{(x,y,z)\in\mathbb{C}^3, x^{m_1}=y^{m_2}z^{m_3}\}\cap U_p^{\mathbb{C}}\setminus V(y)\right)^{norm}.$$

We use the composition with $\alpha_{p,v}^{-1}$ to obtain a morphism:

$$\alpha_{p,v}^{-1}\circ\kappa_{p,v}^{\ominus}\colon \{(x,y,z)\in\mathbb{C}^3, x^{m_1}=y^{m_2}z^{m_3}\}\cap U_p^{\mathbb{C}}\setminus V(y)$$

$$\to \{(x', y', z') \in \mathbb{C}^3, x'^{m_1} = y'^{m_2} z'^{m_3}\} \cap U'_v \setminus V(y')$$

given by

$$\begin{cases} x' = x \\ y' = y|y|^{-kn_1/m_1} \\ z' = z \cdot |z|^{-kn_1/m_1} |x|^{kn_1/m_3} = z|y|^{kn_1 m_2 / m_1 m_3} \end{cases}.$$

The morphism $\alpha_{p,v}^{-1} \circ \kappa_{p,v}^{\ominus}$ is not, in general, birational, but it induces an isomorphism of real-analytic varieties at the level of normalizations. Indeed, consider the morphism

$$N_1 \colon \{(x_1, y_1, z_1) \in \mathbb{C}^3, y_1^{m_2} z_1^{d_3} = 1\} \to \{(x, y, z) \in \mathbb{C}^3, x^{m_1} = y^{m_2} z^{m_3}\} \setminus V(y)$$

given by

$$\begin{cases} x = x_1^{m_3/d_3} z_1^{a_1} \\ y = y_1 \\ z = x_1^{m_1/d_3} z_1^{c_1} \end{cases}$$

where $d_3 = gcd(m_1, m_3)$, and $a_1, c_1 \in \mathbb{N}$ are such that $c_1 m_3 - a_1 m_1 = d_3$.

The morphism N_1 is a normalization of $\{(x, y, z) \in \mathbb{C}^3, x^{m_1} = y^{m_2} z^{m_3}\} \setminus V(y)$. In the same way, build the normalization

$$N'_1 \colon \{(x'_1, y'_1, z'_1) \in \mathbb{C}^3, y_1'^{m_2} z_1'^{d_3} = 1\} \to \{(x', y', z') \in \mathbb{C}^3, x'^{m_1} = y'^{m_2} z'^{m_3}\} \setminus V(y').$$

Set $U_1 := N_1^{-1}(U_p^{\mathbb{C}}), U'_1 := N_1'^{-1}(U'_v)$. The morphism $\alpha_{p,v}^{-1} \circ \kappa_{p,v}^{\ominus}$ then lifts to the normalizations in the following way:

$$\overline{\alpha_{p,v}^{-1} \circ \kappa_{p,1}^{\ominus}} \colon \{(x_1, y_1, z_1) \in \mathbb{C}^3, y_1^{m_2} z_1^{d_3} = 1\} \cap U_1 \to \{(x'_1, y'_1, z'_1) \in \mathbb{C}^3, y_1'^{m_2} z_1'^{d_3} = 1\} \cap U'_1$$

$$\begin{cases} x'_1 = x_1 |y_1|^{-a_1 \frac{kn_1 m_2}{m_1 m_3}} \\ y'_1 = y_1 |y_1|^{-\frac{kn_1}{m_1}} \\ z'_1 = z_1 |y_1|^{\frac{kn_1 m_2}{m_1 d_3}} \end{cases} \qquad \begin{cases} x_1 = x'_1 |y'_1|^{a_1 \frac{kn_1 m_2}{m_3(n_1 - km_1)}} \\ y_1 = y'_1 |y'_1|^{\frac{kn_1}{(m_1 - kn_1)}} \\ z_1 = z'_1 |y'_1|^{-\frac{kn_1 m_2}{d_3(m_1 - kn_1)}} \end{cases}$$

which is, as claimed, an isomorphism of real-analytic varieties, both functions $|y_1|$ and $|y'_1|$ being everywhere different from 0.

Now $\overline{\alpha_{p,v}^{-1} \circ \kappa_{p,v}^{\ominus}}$ and $\overline{\alpha_{p,v}^{-1}}$ are diffeomorphisms, whence $\overline{\kappa_{p,v}^{\ominus}}$ also.

The same strategy will prove that the restriction $\overline{\kappa_{p,w}^{\ominus}}$ of $\overline{\kappa_p^{\ominus}}$ to

$$\left(\{(x, y, z) \in \mathbb{C}^3, x^{m_1} = y^{m_2} z^{m_3}\} \cap U_p^{\mathbb{C}} \setminus V(z)\right)^{norm}$$

is also an isomorphism of real-analytic varieties. In this chart the required explicit normalization is provided by the morphism

$$N_2 \colon \{(x_2, y_2, z_2) \in \mathbb{C}^3, y_2^{d_2} z_2^{m_3} = 1\} \to \{(x, y, z) \in \mathbb{C}^3, x^{m_1} = y^{m_2} z^{m_3}\} \setminus V(z)$$

given by

$$\begin{cases} x = x_2^{\frac{m_2}{d_2}} y_2^{a_2} \\ y = x_2^{\frac{m_1}{d_2}} y_2^{b_2} \\ z = z_2 \end{cases}$$

with $d_2 = gcd(m_1, m_2)$, and $a_2, b_2 \in \mathbb{N}$ such that $b_2 m_2 - a_2 m_1 = d_2$, and

$$\alpha_{p,w} \colon \{(x', y', z') \in \mathbb{C}^3, x'^{m_1} = y'^{m_2} z'^{m_3}\} \setminus V(z') \to \{u^{m_1} v^{m_2} w^{m_3} = |u|^{kn_1}\}^{str} \setminus V(w)$$

is given by

$$\begin{cases} u = x' \\ v = y'^{-1} |x'|^{\frac{kn_1}{m_2}} \\ w = z'^{-1} \end{cases} \qquad \begin{cases} x' = u \\ y' = v^{-1} |u|^{\frac{kn_1}{m_2}} \\ z' = w^{-1} \end{cases}$$

In conclusion, the images by $\overline{\kappa_p^{\ominus *}}$ of $N_{\mathscr{S}}^{-1}\left(\widetilde{\mathscr{S}_k} \cap U_p \cap V(v)\right)$ and of $N_{\mathscr{S}}^{-1}\left(\widetilde{\mathscr{S}_k} \cap U_p \cap V(w)\right)$ being disjoint, $\overline{\kappa_p^{\ominus *}}$ is an isomorphism of real-analytic varieties. □

Discussion 4.7.13. Orientation of $\widetilde{\mathscr{S}_k}$ and its model near a point $\ominus$.

Here again, the orientation of $\widetilde{\mathscr{S}_k}$ is given by $du \wedge d\overline{u} \wedge dw \wedge d\overline{w}$. But in order for $\overline{\kappa_p^{\ominus}}$ to be orientation-preserving, we need to orient its source with ***the opposite of the orientation given by its complex structure****, $-dx \wedge d\overline{x} \wedge dy \wedge d\overline{y}$.*

Furthermore, the two irreducible components of $\mathscr{C}_{tot}$ that are visible here, given by $\{u = v = 0\}$ and $\{u = w = 0\}$, are oriented respectively as complex curves by $dw \wedge d\overline{w}$ and $dv \wedge d\overline{v}$. The morphism $\kappa_p^{\ominus}$ then orients their preimages, respectively $\{x = y = 0\}$ and $\{x = z = 0\}$, by the opposite of their complex orientations, $-dz \wedge d\overline{z}$ and $-dy \wedge d\overline{y}$.

So, here, the pullback of the orientation of $\mathscr{C}_{tot} \cap U_p$ by $N_{\mathscr{S}} \circ \kappa^{\ominus}$ ***orients its preimage*** *$(N_{\mathscr{S}} \circ \kappa^{\ominus})^{-1}(\mathscr{C}_{tot} \cap U_p)$* ***by the opposite of its complex structure.***

4.8 A variety $\overline{\mathscr{S}}$

In the previous section is implicitly provided a description of an analytic variety $\overline{\mathscr{S}}$, as well as of a morphism $K \colon (\overline{\mathscr{S}}, (N_{\mathscr{S}} \circ K)^* (\mathscr{C})) \to (\widetilde{\mathscr{S}_k}^N, N_{\mathscr{S}}^*(\mathscr{C}))$. In this section we make it more explicit. In particular we explain the orientation of $(\overline{\mathscr{S}}, (N_{\mathscr{S}} \circ K)^* (\mathscr{C}))$ and of the surfaces of $(N_{\mathscr{S}} \circ K)^* (\mathscr{C})$.

4.8.1 Definition of $(\overline{\mathscr{S}}, (N_{\mathscr{S}} \circ K)^* (\mathscr{C}))$

Choose a covering of $\mathscr{C} \subset \widetilde{\mathscr{S}_k}$ with connected neighbourhoods of the form $U_p \cap \widetilde{\mathscr{S}_k}$ providing equations for $\widetilde{\mathscr{S}_k}$ as (4.5) and (4.7) in subsections 4.7.1 and 4.7.2. Such a covering may be chosen finite because of the compactness of $\mathscr{C}$.

In the sequel, if $p \in \mathscr{C}$, κ_p denotes κ_p, $\kappa_p^{\ominus}$ or $\kappa_p^{\oplus}$, according to the type of the point p.

Definition 4.8.1. (Local complexification.)

If $U_p \cap \widetilde{\mathscr{S}_k}$ is an open subset in this covering, and if p is a generic point of $\mathscr{C}$ at the intersection of $D_1 \in \mathbb{D}_0$ and $D_2 \in \mathbb{D}_f$, denote

$$\overline{U_p} := (\overline{\kappa_p} \circ N_{\mathscr{S}})^{-1} \left(U_p \cap \widetilde{\mathscr{S}_k} \right) \subset \{(x_p, y_p, z_p) \in \mathbb{C}^3, x_p^{m_1} = y_p^{m_2}\}^{norm}.$$

In other cases, the definition of $\overline{U_p}$ is the same, with the appropriate equation for the right-hand term, according to the type of p.

Then we define a real-analytic variety $\overline{\mathscr{S}}$, called ***local complexification of*** $\widetilde{\mathscr{S}_k}^N$, *by gluing together the open sets $\left(\overline{U_p}\right)_p$ in the following way: if $U_p \cap U_q \neq \emptyset$, glue $\overline{U_p}$ and $\overline{U_q}$ along*

$$(\overline{\kappa_p} \circ N_{\mathscr{S}})^{-1} \left(U_p \cap U_q \cap \widetilde{\mathscr{S}_k} \right) \subset \overline{U_p}$$

and

$$(\overline{\kappa_q} \circ N_{\mathscr{S}})^{-1} \left(U_p \cap U_q \cap \widetilde{\mathscr{S}_k} \right) \subset \overline{U_q}$$

using $\overline{\kappa_p}^{-1} \circ \overline{\kappa_q}$, which is an isomorphism of real-analytic varieties in restriction to $\overline{U_p} \cap \overline{U_q}$.

The latter is indeed an isomorphism: if p is a point of $\mathscr{C}$, and $q \neq p$ a double point of $\mathscr{C}_{tot}$, we may have $p \in U_q$, but never $q \in U_p$, because of the equations of $\widetilde{\mathscr{S}_k}$ on these open sets, see section 4.7. The isomorphism follows then from Lemmas 4.7.4, 4.7.9 and 4.7.12.

Remark 4.8.2. *Note that the open sets $\overline{U_p}$ are not, in general, connected. See Section 4.10 for more details.*

Now, the real-analytic morphism

$$K : (\overline{\mathscr{S}}, (N_{\mathscr{S}} \circ K)^* (\mathscr{C})) \to (\widetilde{\mathscr{S}_k}^N, N_{\mathscr{S}}^* (\mathscr{C}))$$

is defined using the local morphisms κ_p, on the covering of $\overline{\mathscr{S}}$ by the $\overline{U_p}$'s.

Lemmas 4.7.4, 4.7.9 and 4.7.12 imply:

Lemma 4.8.3. *The real-analytic morphism K is an isomorphism of real-analytic varieties outside $N_{\mathscr{S}}^* (\mathscr{C})$.*

4.8.2 Orientation of $\overline{\mathscr{S}}$

Although $\overline{\mathscr{S}}$ is only real-analytic, it admits local complex equations, and an orientation of $\overline{\mathscr{S}}$ can be defined relatively to the local complex orientations, using Discussions 4.7.5, 4.7.10 and 4.7.13:

- On an open $\overline{U_p}$ coming from a generic point of $\mathscr{C}$, $\overline{\mathscr{S}}$ is oriented by its local complex structure, and the preimage of $\mathscr{C}$ also.
- On an open $\overline{U_p}$ coming from a double point $\oplus$ of $\mathscr{C}_{tot}$, $\overline{\mathscr{S}}$ is oriented by its local complex structure, and the components of the preimage of $\mathscr{C}_{tot}$ also.
- On an open $\overline{U_p}$ coming from a double point $\ominus$ of $\mathscr{C}_{tot}$, $\overline{\mathscr{S}}$ is oriented by the opposite of its local complex structure, and the preimage of $\mathscr{C}_{tot}$ is also oriented by the opposite of its complex structure.

These choices of orientation are compatible, because by construction they ensure that every morphism $\overline{\kappa_p}^{-1} \circ \overline{\kappa_q}$ of Definition 4.8.1 is orientation-preserving. Again, by construction, the choices of orientation for the preimages of the irreducible components of $\mathscr{C}_{tot}$ by $N_{\mathscr{S}} \circ K$ are compatible (see Discussions 4.7.5, 4.7.10 and 4.7.13). Furthermore, as can be checked using those same discussions,

Lemma 4.8.4. *The morphism K is orientation-preserving.*

4.9 The decorated graph $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ of configuration of curves

In order to make the description of $N^*_{\mathscr{S}}(\mathscr{C}_{tot}) \subset \overline{\mathscr{S}}$ easier, and in the perspective of the main algorithm, let us introduce the decorated graph $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$.

Definition 4.9.1. *Denote $\Gamma(\mathscr{C}_{tot})$ the dual graph of the configuration of complex curves $\mathscr{C}_{tot}$, as defined in 2.7.1. The decorated graph $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ is obtained from $\Gamma(\mathscr{C}_{tot})$ in the following way:*

- *If C is an irreducible component of $D_1 \cap D_2$, where $D_1 \in \mathbb{D}_f$ and $D_2 \in \mathbb{D}_0 \cup \mathbb{D}_g$, decorate v_C, the vertex corresponding to C, with the triple $(m_1; m_2, n_2)$ (with the Notation 4.6.8), and with its genus $[g]$, in square brackets. If $D_2 \in \mathbb{D}_g$, then the vertex associated to the non-compact curve C is an arrowhead.*
- *Decorate each edge e_p of $\Gamma(\mathscr{C}_{tot})$ corresponding to a double point p of $\mathscr{C}_{tot}$ with a $\oplus$ or a $\ominus$ sign, according to the type of the point p, introduced in Definition 4.7.6.*

Note that Proposition 4.6.3 implies:

Lemma 4.9.2. *If the germ $(X, 0)$ is a complete intersection singularity, the graphs $\Gamma(\mathscr{C}_{tot})$ and $\Gamma(\mathscr{C})$ are connected.*

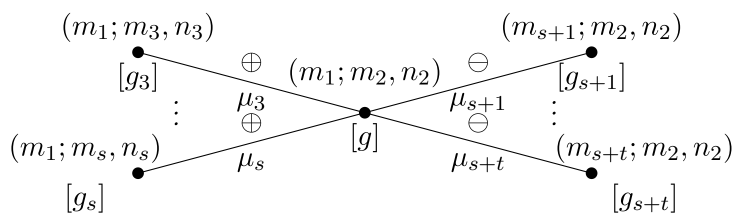
Remark 4.9.3.

1. *If $p, q \in C_1 \cap C_2$, then p and q are of the same type. Hence the different edges between two vertices are all of the same type.*

2. *By point 2 of Remark 4.6.7, if $D_2 \in \mathbb{D}_g$, then $m_1 = n_2 = 1$. Furthermore, there is a unique edge containing this arrowhead vertex, and it is decorated with $\oplus$.*

3. *Because of the hypothesis of simpleness in the point 2 of Definition 4.6.1, the graph $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ contains no loop.*

Definition 4.9.4. *Let v_C be a vertex of $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$. Its* ***star*** *is the subgraph of $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ whose vertices are v_C and its neighbours, with the edges connecting v_C to them. It will therefore have the following form:*



where the decoration μ_i means that the edge is repeated μ_i times.

4.10 Description of $\overset{\star}{\Gamma}\left((N_{\mathscr{S}} \circ K)^* (\mathscr{C}_{tot})\right)$

On a local chart of $\overline{\mathscr{S}}$ of the form $\overline{U_p}$ (see Definition 4.8.1), we know how to recognize the configuration $(N_{\mathscr{S}} \circ K)^* (\mathscr{C})$. However, we also need to know it as a global object. Namely, two questions remain:

1. Given an irreducible curve $C \in \mathscr{C}$, what is the structure of $(N_{\mathscr{S}} \circ K)^* (C)$?

2. Given two irreducible curves $C_1, C_2 \in \mathscr{C}$ such that $C_1 \cap C_2 \neq \emptyset$, what can we say about $(N_{\mathscr{S}} \circ K)^* (C_1) \cap (N_{\mathscr{S}} \circ K)^* (C_2)$?

Let us look here more precisely at the configuration $(N_{\mathscr{S}} \circ K)^* (\mathscr{C}) \subset \overline{\mathscr{S}}$.

We know that in every open set $\overline{U_p}$, $(N_{\mathscr{S}} \circ K)^* (\mathscr{C})$ is a configuration of complex curves.

In the sequel, C is an irreducible component of $\mathscr{C}$, contained in $D_1 \cap D_2$, where $D_1 \in \mathbb{D}_f$ and $D_2 \in \mathbb{D}_g$ or $\mathbb{D}_0$. Denote $p_3, \cdots, p_l$ the double points of $\mathscr{C}$ on C, and

$$d := gcd(m_1, m_2).$$

4.10.1 Cyclic orders on components

In this subsection, we show how one can endow coherently the set of local irreducible components of $\widetilde{\mathscr{S}}_k$ at a generic point of an irreducible component C of $\mathscr{C}$ with a cyclic order. Then we show that the same is true for double points on C, and that the cyclic orders defined are compatible.

Lemma 4.10.1. (Cyclic order at generic points.)
If p is a generic point of the irreducible component C of $D_1 \cap D_2$, $(N_{\mathscr{S}} \circ K)^* (C) \cap \overline{U_p}$ is a disjoint union of d complex curves. Furthermore there is a cyclic order (in the sense of Definition 2.6.24) on them. This order is compatible with the gluings of charts $\overline{U_q}$ coming from generic points $q \in \mathscr{C}_{tot}$.

Namely, if $\overline{\mathscr{S}} \cap \overline{U_p} = \left(\left\{x_p^{m_1} = y_p^{m_2}\right\} \cap U_p^{\mathbb{C}}\right)^{norm}$, then the decomposition in irreducible components

$$\left\{x_p^{m_1} = y_p^{m_2}\right\} = \bigcup_{j=0}^{d-1} \left\{x_p^{m_1/d} = e^{\frac{2ij\pi}{d}} y_p^{m_2/d}\right\},$$

where d denotes $gcd(m_1, m_2)$, provides $\overline{\mathscr{S}} \cap \overline{U_p}$ as a disjoint union

$$\overline{\mathscr{S}} \cap \overline{U_p} = \bigsqcup_{j=0}^{d-1} \left(\left\{x_p^{m_1/d} = e^{\frac{2ij\pi}{d}} y_p^{m_2/d}\right\}\right)^{norm} \cap \overline{U_p}$$

and finally

$$(N_{\mathscr{S}} \circ K)^* (C) \cap \overline{U_p} = \bigsqcup_{j=0}^{d-1} C^{(j)} \cap \overline{U_p}$$

where $C^{(j)}$ is the preimage of the curve $\{x_p = y_p = 0\}$ by the normalization of $\{x_p^{m_1/d} = e^{\frac{2ij\pi}{d}} y_p^{m_2/d}\}$. Each $C^{(j)}$ is a smooth complex curve. The cyclic order on the connected components of $(N_{\mathscr{S}} \circ K)^* (C)$ is then given by the rule:

$C^{(j+1)}$ follows $C^{(j)}$, where the addition is to be understood in $\mathbb{Z}/d\mathbb{Z}$.

or equivalently

$$\left\{x_p^{m_1/d} = e^{\frac{2i(j+1)\pi}{d}} y_p^{m_2/d}\right\} \cap U_p^{\mathbb{C}} \text{ follows } \left\{x_p^{m_1/d} = e^{\frac{2ij\pi}{d}} y_p^{m_2/d}\right\} \cap U_p^{\mathbb{C}}$$

Proof. We want to prove that, given two open subsets $\overline{U_p}$ and $\overline{U_q}$, coming from generic points $p, q \in C$, such that $\overline{U_q} \cap \overline{U_p} \neq \emptyset$, the orderings on the components of $(N_{\mathscr{S}} \circ K)^* (C)$ on $\overline{U_p} \cap \overline{U_q}$ are the same. The point is that this ordering on curves is the same as an ordering on irreducible components.

First, note that we can relate the coordinates (x_p, y_p, z_p) and (x_q, y_q, z_q) by descending to $\widetilde{\mathscr{S}_k}$: take coordinates (u_p, v_p, w_p) on $\widetilde{\mathscr{S}_k} \cap U_p$ and (u_q, v_q, w_q) on $\widetilde{\mathscr{S}_k} \cap U_q$, as in section 4.7.1. Then there are functions λ, μ, ν, nowhere zero on $U_p \cap U_q$ such that, on $U_p \cap U_q$,

$$\begin{cases} u_q = u_p \cdot \lambda(u_p, v_p, w_p) \\ v_q = v_p \cdot \mu(u_p, v_p, w_p) \\ w_q = \nu(u_p, v_p, w_p) \end{cases}$$

which leads, using equations (4.6) of subsection 4.7.1, to the following identities on $U_p^{\mathbb{C}} \cap U_q^{\mathbb{C}}$:

$$\begin{cases} x_q = x_p \cdot \lambda \\ y_q = y_p \cdot \mu^{-1}|\lambda|^{kn_1/m_2} \\ z_q = \nu \end{cases}$$

where λ denotes $\lambda(x_p, y_p^{-1}|x_p|^{kn_1/m_2}, z_p)$, and similarily for μ, ν. The functions $\lambda, \mu^{-1}|\lambda^{kn_1/m_2}|, \nu$ are nowhere zero on $U_p^{\mathbb{C}} \cap U_q^{\mathbb{C}}$.

Whence the identification

$$\left\{x_q^{m_1/d} = e^{\frac{2ij\pi}{d}} y_q^{m_2/d}\right\} \cap U_p^{\mathbb{C}} \cap U_q^{\mathbb{C}} = \left\{x_p^{m_1/d}\lambda^{m_1/d} = e^{2ij\pi/d} y_p^{m_2/d} \mu^{-m_2/d}|\lambda|^{kn_1/d}\right\} \cap U_p^{\mathbb{C}} \cap U_q^{\mathbb{C}}.$$

This implies that there exists $0 \leqslant l \leqslant j-1$ such that

$$e^{2ij\pi/d}\mu^{-m_2/d}|\lambda|^{kn_1/d}\lambda^{-m_1/d} = e^{2il\pi/d}.$$

This allows one to identify the j-th irreducible component, as seen in $U_q^{\mathbb{C}}$, with the l-th component, in $U_p^{\mathbb{C}}$. And replacing j by $j+1$ in the previous computation leads now to the component $l+1$ in $U_q^{\mathbb{C}}$.

In other words, we gave a meaning to the expression "the following irreducible component" at a point of $\{x_p^{m_1} = y_p^{m_2}\} \cap U_p^{\mathbb{C}}$, and hence to the expression "the following curve" in $\overline{\mathscr{S}} \cap \overline{U_p}$. □

Discussion 4.10.2. (Cyclic order at double points.) *Recall the notations introduced at he beginnning of Section 4.10. Let p_i be a double point of $\mathscr{C}_{tot}$ in C, $p_i \in D_i$. Then $(N_{\mathscr{S}} \circ K)^{-1}(p_i)$ is made of $d_i := gcd(m_1, m_2, m_i)$ points $\overline{p_i^1}, \cdots, \overline{p_i^{d_i}}$, each one of them corresponding to an irreducible component of $\kappa_{p_i}^{-1}(\widetilde{\mathscr{S}_k} \cap U_{p_i})$.*

Furthermore, one can define a cyclic order on these irreducible components, in the following way:

***If** p_i **is of type** $\oplus$, then*

$$\kappa_{p_i}^{-1}(\widetilde{\mathscr{S}_k} \cap U_{p_i}) = \bigcup_{j=0}^{d_i-1} \left\{y_i^{m_2/d_i} = e^{2ij\pi/d_i} x_i^{m_1/d_i} z_i^{m_3/d_i}\right\} \cap U_{p_i}^{\mathbb{C}} \tag{4.10}$$

where the coordinates (x_i, y_i, z_i) are defined from local coordinates (u_i, v_i, w_i) at p_i via *Equation 4.8 applied at p_i. This union is the decomposition of $\kappa_{p_i}^{-1}(\widetilde{\mathscr{S}_k} \cap U_{p_i})$ in irreducible components.*

We define a cyclic order on the set of irreducible components by the following rule:

$$\left\{y_i^{m_2/d_i} = e^{2ij\pi/d_i} x_i^{m_1/d_i} z_i^{m_3/d_i}\right\}$$

is followed by

$$\left\{y_i^{m_2/d_i} = e^{2i(j-1)\pi/d_i} x_i^{m_1/d_i} z_i^{m_3/d_i}\right\}.$$

***If** p_i **is of type** $\ominus$, then*

$$\kappa_{p_i}^{-1}(\widetilde{\mathscr{S}_k} \cap U_{p_i}) = \bigcup_{j=0}^{d_i-1} \left\{x_i^{m_1/d_i} = e^{2ij\pi/d_i} y_i^{m_2/d_i} z_i^{m_3/d_i}\right\} \cap U_{p_i}^{\mathbb{C}} \tag{4.11}$$

where the coordinates (x_i, y_i, z_i) are defined from local coordinates (u_i, v_i, w_i) at p_i via *Equation 4.9 applied at p_i.*

Then, again, define a cyclic order by the rule: the component

$$\left\{x_i^{m_1/d_i} = e^{2ij\pi/d_i} y_i^{m_2/d_i} z_i^{m_3/d_i}\right\}$$

is followed by the component

$$\left\{x_i^{m_1/d_i} = e^{2i(j+1)\pi/d_i} y_i^{m_2/d_i} z_i^{m_3/d_i}\right\}.$$

The normalization of complex-analytic varieties having the topological effect of separating local irreducible components (see Lemma 2.5.15), this means that $\overline{\mathscr{S}} \cap \overline{U_{p_i}}$ is made of a disjoint union of d_i normal varieties, with a cyclic order induced by the one on the irreducible components of $\kappa_{p_i}^{-1}(\widetilde{\mathscr{S}_k} \cap U_{p_i})$. Incidentally, this provides also a cyclic order on the set $\left\{\overline{p_i^1}, \cdots, \overline{p_i^{d_i}}\right\}$ of preimages of p_i by $(N_{\mathscr{S}} \circ K)$.

Moreover, the intersection of $(N_{\mathscr{S}} \circ K)^* (C)$ with each of these connected components is an irreducible curve, namely the pullback of $\{x_i = y_i = 0\}$ by the normalization of the irreducible component, with the notations of Equations (4.10) or (4.11). See Subsection 3.5.3 for more details. Recall again the setting of the beginning of Section 4.10. Then:

Lemma 4.10.3. (Compatibility of orders.)

Let $p_i \in C \cap D_i$ be one of the double points of $\mathscr{C}_{tot}$ on C. Let p be a generic point of C such that $\widetilde{U} := \overline{U_p} \cap \overline{U_{p_i}} \neq \emptyset$. Consider the cyclic orders defined above. If an irreducible component of $\kappa_p^{-1}(\widetilde{\mathscr{S}_k} \cap \overline{U})$ is contained in an irreducible component of $\kappa_{p_i}^{-1}(\widetilde{\mathscr{S}_k} \cap \overline{U_{p_i}})$, then the following component of $\kappa_p^{-1}(\widetilde{\mathscr{S}_k} \cap \overline{U})$ is contained in the following irreducible component of $\kappa_{p_i}^{-1}(\widetilde{\mathscr{S}_k} \cap \overline{U_{p_i}})$.

In particular, two curves in $(N_{\mathscr{S}} \circ K)^* (C) \cap \overline{U}$ are in the same branch of $(N_{\mathscr{S}} \circ K)^* (C) \cap \overline{U_{p_i}}$ if and only if their positions in the cyclic order differ by a multiple of d_i.

Proof of Lemma 4.10.3. Denote (u, v, w) and (u_i, v_i, w_i) coordinates on U_p and U_{p_i} giving respectively equations of the form (4.5) and (4.7) for $\widetilde{\mathscr{S}}_k$, as in section 4.7.

Then

$$\overline{\mathscr{S}} \cap \overline{U_p} = \left(\{x^{m_1} = y^{m_2}\} \cap U_p^{\mathbb{C}}\right)^{norm} = \bigsqcup_{j=0}^{d-1} \left(\left\{x^{m_1/d} = e^{2ij\pi/d} y^{m_2/d}\right\} \cap U_p^{\mathbb{C}}\right)^{norm}$$

where

$$\begin{cases} x = u \\ y = v^{-1}|u|^{kn_1/m_2} \\ z = w \end{cases}$$

Then there exist functions λ, μ, ν, nowhere zero on $U_p \cap U_{p_i}$, such that, on $U_p \cap U_{p_i}$,

$$\begin{cases} u = u_i.\lambda(u_i, v_i, w_i) \\ v = v_i.\mu(u_i, v_i, w_i) \\ w = \nu(u_i, v_i, w_i) \end{cases}$$

We face two different situations, according to the type of the double point p_i. Denote

$$\delta_i := \frac{d}{d_i}.$$

<u>Case $\oplus$:</u> If p_i is of type $\oplus$, then

$$\overline{\mathscr{S}} \cap \overline{U_{p_i}} = \left(\{y_i^{m_2} = x_i^{m_1} z_i^{m_3}\} \cap U_{p_i}^{\mathbb{C}}\right)^{norm} =$$

$$\bigsqcup_{j=0}^{d_i-1} \left(\left\{y_i^{m_2/d_i} = e^{2ij\pi/d_i} x_i^{m_1/d_i} z_i^{m_3/d_i}\right\} \cap U_{p_i}^{\mathbb{C}}\right)^{norm} \tag{4.12}$$

where

$$\begin{cases} u_i = x_i \\ v_i = y_i^{-1}|x_i|^{kn_1/m_2}|z_i|^{kn_3/m_2} \\ w_i = z_i \end{cases}$$

which gives, on $U_{p_i}^{\mathbb{C}} \cap U_p^{\mathbb{C}}$,

$$\begin{cases} x = x_i\lambda \\ y = y_i|z_i|^{-kn_3/m_2}\mu^{-1}|\lambda|^{kn1/m_2} \\ z = \nu \end{cases}$$

where λ (or μ, ν) denotes $\lambda(x_i, y_i^{-1}|x_i|^{kn_1/m_2}|z_i|^{kn_3/m_2}, z_i)$. Hence

$$\left\{x^{m_1/d} = e^{2ij\pi/d} y^{m_2/d}\right\} \cap U_p^{\mathbb{C}} \cap U_{p_i}^{\mathbb{C}} =$$

$$\left\{x_i^{m_1/d} = y_i^{m_2/d}\lambda^{-m_1/d}|\lambda|^{kn_1/d}|z_i|^{-kn_3/d}\mu^{-m_2/d}e^{2ij\pi/d}\right\} \cap U_p^{\mathbb{C}} \cap U_{p_i}^{\mathbb{C}}.$$

Elevating to the power δ_i shows that

$$\left\{x^{m_1/d} = e^{2ij\pi/d}y^{m_2/d}\right\} \cap U_p^{\mathbb{C}} \cap U_{p_i}^{\mathbb{C}} \subset$$

$$\left\{y_i^{m_2/d_i} = x_i^{m_1/d_i}\lambda^{m_1/d_i}|\lambda|^{-kn_1/d_i}|z_i|^{kn_3/d_i}\mu^{m_2/d_i}e^{-2ij\pi/d_i}\right\} \cap U_p^{\mathbb{C}} \cap U_{p_i}^{\mathbb{C}}.$$

Now there exists $0 \leqslant l \leqslant d_i - 1$ such that on $U_{p_i}^{\mathbb{C}} \cap U_p^{\mathbb{C}}$,

$$\lambda^{m_1/d_i}|\lambda|^{-kn_1/d_i}|z_i|^{kn_3/d_i}\mu^{m_2/d_i}e^{-2ij\pi/d_i} = e^{2il\pi/d_i}z_i^{m_3/d_i}.$$

The previous inclusion can then be written as

$$\left\{x^{m_1/d} = e^{2ij\pi/d}y^{m_2/d}\right\} \cap U_p^{\mathbb{C}} \cap U_{p_i}^{\mathbb{C}} \subset \left\{y_i^{m_2/d_i} = e^{2il\pi/d_i}x_i^{m_1/d_i}z_i^{m_3/d_i}\right\} \cap U_p^{\mathbb{C}} \cap U_{p_i}^{\mathbb{C}}.$$

Taking the successor of this component, we get indeed

$$\left\{x^{m_1/d} = e^{2i(j+1)\pi/d}y^{m_2/d}\right\} \cap U_p^{\mathbb{C}} \cap U_{p_i}^{\mathbb{C}} \subset \left\{y_i^{m_2/d_i} = e^{2i(l-1)\pi/d_i}x_i^{m_1/d_i}z_i^{m_3/d_i}\right\} \cap U_p^{\mathbb{C}} \cap U_{p_i}^{\mathbb{C}}.$$

Case $\ominus$: If p_i is of type $\ominus$, then

$$\overline{\mathscr{S}} \cap \overline{U_{p_i}} = \left(\{x_i^{m_1} = y_i^{m_2}z_i^{m_3}\} \cap U_{p_i}^{\mathbb{C}}\right)^{norm} =$$

$$\bigsqcup_{j=0}^{d_i-1}\left(\left\{x_i^{m_1/d_i} = e^{2ij\pi/d_i}y_i^{m_2/d_i}z_i^{m_3/d_i}\right\} \cap U_{p_i}^{\mathbb{C}}\right)^{norm} \tag{4.13}$$

where

$$\begin{cases} u_i = x_i \\ v_i = y_i^{-1}|y_i|^{kn_1/m_1} \\ w_i = z_i^{-1}|z_i|^{kn_1/m_1} \end{cases}$$

which gives, on $U_{p_i}^{\mathbb{C}} \cap U_p^{\mathbb{C}}$,

$$\begin{cases} x = x_i\lambda \\ y = y_i|y_i|^{-kn_1/m_1}|x_i|^{kn1/m_2}\mu^{-1}|\lambda|^{kn_1/m_2} \\ z = \nu \end{cases}$$

where λ (or μ, ν) denotes $\lambda(x_i, y_i^{-1}|y_i|^{kn_1/m_1}, z_i^{-1}|z_i|^{kn_1/m_1})$. Hence

$$\left\{x^{m_1/d} = e^{2ij\pi/d}y^{m_2/d}\right\} \cap U_p^{\mathbb{C}} \cap U_{p_i}^{\mathbb{C}} =$$

$$\left\{x_i^{m_1/d} = y_i^{m_2/d}|y_i|^{-kn_1m_2/dm_1}|x_i|^{kn_1/d}\lambda^{-m_1/d}|\lambda|^{kn_1/d}|z_i|^{-kn_3/d}\mu^{-m_2/d}e^{2ij\pi/d}\right\} \cap U_p^{\mathbb{C}} \cap U_{p_i}^{\mathbb{C}}.$$

Elevating to the power δ_i shows that

$$\left\{x^{m_1/d} = e^{2ij\pi/d} y^{m_2/d}\right\} \cap U_p^{\mathbb{C}} \cap U_{p_i}^{\mathbb{C}} \subset$$

$$\left\{x_i^{m_1/d_i} = y_i^{m_2/d_i} |y_i|^{-kn_1m_2/d_im_1} |x_i|^{kn_1/d_i} \lambda^{-m_1/d_i} |\lambda|^{kn_1/d_i} |z_i|^{-kn_3/d_i} \mu^{-m_2/d_i} e^{2ij\pi/d_i}\right\} \cap U_p^{\mathbb{C}} \cap U_{p_i}^{\mathbb{C}}.$$

Now there exists $0 \leqslant l \leqslant d_i - 1$ such that on $U_{p_i}^{\mathbb{C}} \cap U_p^{\mathbb{C}}$,

$$|y_i|^{-kn_1m_2/d_im_1} |x_i|^{kn_1/d_i} \lambda^{-m_1/d_i} |\lambda|^{kn_1/d_i} |z_i|^{-kn_3/d_i} \mu^{-m_2/d_i} e^{2ij\pi/d_i} = e^{2il\pi/d_i} z_i^{m_3/d_i}.$$

The previous inclusion can then be written as

$$\left\{x^{m_1/d} = e^{2ij\pi/d} y^{m_2/d}\right\} \cap U_p^{\mathbb{C}} \cap U_{p_i}^{\mathbb{C}} \subset \left\{x_i^{m_1/d_i} = e^{2il\pi/d_i} y_i^{m_2/d_i} z_i^{m_3/d_i}\right\} \cap U_p^{\mathbb{C}} \cap U_{p_i}^{\mathbb{C}}.$$

Taking the successor of this component, we get indeed

$$\left\{x^{m_1/d} = e^{2i(j+1)\pi/d} y^{m_2/d}\right\} \cap U_p^{\mathbb{C}} \cap U_{p_i}^{\mathbb{C}} \subset \left\{x_i^{m_1/d_i} = e^{2i(l+1)\pi/d_i} y_i^{m_2/d_i} z_i^{m_3/d_i}\right\} \cap U_p^{\mathbb{C}} \cap U_{p_i}^{\mathbb{C}}.$$

□

Remark 4.10.4. *In other words, taking into account a double point p_i of C shows that some of the* a priori *different components of $(N_{\mathscr{S}} \circ K)^* (C)$ are in the same branch, and that this identification is made uniformly with respect to the cyclic order.*

However, other identifications may occur, due to the fundamental group of C. We make this aspect more precise in Subsection 4.10.2.

The following lemma describes a favorable situation in which the data contained in $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ is sufficient to determine the structure of $(N_{\mathscr{S}} \circ K)^* (C)$. To state it we need the following:

Definition 4.10.5. *Let $(L, 0)$ be a branch of* $\mathrm{Sing}(V(f))$. *The* ***transversal type*** *of $(X, V(f))$ along L is the equisingularity type of a transverse section of the couple $(X, V(f))$ along L, where two curves in surfaces (V_1, C_1) and (V_2, C_2) are called equisingular if they admit homeomorphic good embedded resolutions.*

Remark 4.10.6. *This definition makes sense because there exists a representative X of $(X, 0)$ such that for any branch L of* $\mathrm{Sing}(V(f))$, *all transverse sections along L are equisingular.*

Lemma 4.10.7. *Let C be an ireducible curve of $\mathscr{C}$, and denote D the component of $\mathbb{D}_f$ containing C. If D is not contained in the strict transform $\widetilde{V(f)}$ of $V(f)$, denote by L the curve $r_X(D)$. If the transversal type of $(X, V(f))$ along L has a resolution whose exceptional divisor is a union of rational curves, then the curve C is rational.*

The proof of this lemma follows the one of [46, Proposition 7.4.8, a)]. In the case studied by Némethi and Szilárd, it has the following consequence:

Corollary 4.10.8. *Let* $(X, 0) = (\mathbb{C}^3, 0)$*, and* $C \in \mathscr{C}$ *be decorated by a triple of multiplicities* $(m_1; m_2, n_2)$*. Then if* $m_1 \geqslant 2$*, then the curve* C *is rational.*

Indeed, in this case all transversal types of the surface $V(f)$ are germs of plane curves, whose resolutions use only rational curves.

4.10.2 Non-rational curves

However, in general, the curve C will not be rational, whence the need of a little additional information, namely about the identifications of local irreducible components of $\widetilde{\mathscr{S}_k}$ along loops in C. The goal of this subsection is to make this precise.

Let C be an irreducible curve of $\mathscr{C}$. Again, let $p_3, \dots, p_l$ be the double points of $\mathscr{C}$ on C. Let $p \in C \setminus \{p_3, \dots, p_l\}$. Then $\pi_1(C \setminus \{p_3, \dots, p_l\}, p)$ acts on the set of local irreducible components of $\widetilde{\mathscr{S}_k}$ at p in the following way: let $[\gamma] \in \pi_1(C \setminus \{p_3, \dots, p_l\}, p)$. Consider a tubular neighbourhood $U = \bigcup_{i=0}^{j} U_i$ of γ in $\tilde{X}$ such that, on each U_i, we can have local coordinates in $\tilde{X}$ in such a way as to get local equations as in 4.7.1 for $\widetilde{\mathscr{S}_k}$. Then one can follow an irreducible component at p along γ. By consistency of the cyclic order, this action is completely defined by the data of the number δ such that a component is sent on the one which is δ further in the cyclic order. Furthermore,

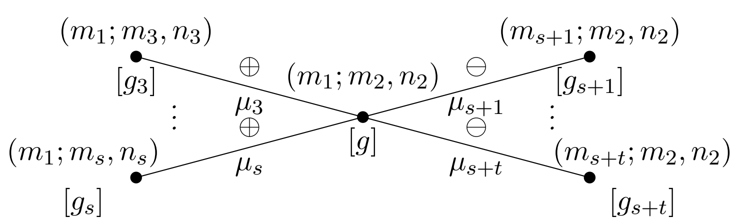
Lemma 4.10.9. *For* $[\gamma]$ *in* $\pi_1(C \setminus \{p_3, \dots, p_l\})$*, the number* δ *is independent of the choice of the base point* p *or the representative* γ*.*

Definition 4.10.10. *The number* δ *defined above is called the* ***switch*** *associated to* γ*.*

Now, $\pi_1(C \setminus \{p_3, \dots, p_l\}, p)$ can be generated by $\alpha_1, \dots, \alpha_{2g}, \gamma_1, \dots, \gamma_l$, where $\alpha_1, \dots, \alpha_{2g}$ generate $\pi_1(C, p)$, and γ_i is a "simple loop" around p_i, i.e. $\gamma_i \neq 1$ in $\pi_1(C \setminus \{p_1, \dots, p_l\}, p)$ but $\gamma_i = 1$ in $\pi_1(C \setminus \{p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_l\}, p)$

Lemmas 4.10.1 and 4.10.3, together with the previous considerations, imply the following:

Theorem 4.10.11. (Structure of $(N_{\mathscr{S}} \circ K)^* (C)$**.)**
Let C *be a curve of* $\mathscr{C}$ *of genus* g*, and let its star be*



Let δ_i be the switch associated to α_i, where $\alpha_1, \ldots, \alpha_{2g}$ generate $\pi_1(C)$. Then $(N_{\mathscr{S}} \circ K)^* (C)$ is the union of

$$n_C := gcd(m_1, \ldots, m_{s+t}, \delta_1, \ldots, \delta_{2g})$$

disjoint connected irreducible oriented surfaces.

The set of connected components of $(N_{\mathscr{S}} \circ K)^* (C)$ is endowed with a cyclic order induced by the one on the local irreducible components of $\kappa_p^{-1}(\widetilde{\mathscr{S}_k} \cap U_p)$ at generic points p of C. Denote $\overline{C_1}, \cdots, \overline{C_{n_C}}$ those surfaces, where the numbering respects the cyclic order.

Furthermore, the common Euler characteristic $\chi\left(\overline{C}\right)$ of each of these surfaces verifies

$$n_C \cdot \chi\left(\overline{C}\right) = \left(2 - 2g - \sum_{i=3}^{s+t} \mu_i\right) \cdot gcd(m_1, m_2) + \sum_{i=3}^{s+t} gcd(m_1, m_2, m_i) \cdot \mu_i. \tag{4.14}$$

Proof of the last identity. This identity comes by recognising $gcd(m_1, m_2)$ as the cardinal of the preimage of a generic point, and $gcd(m_1, m_2, m_i)$ as the cardinal of the preimage of a double point, and applying the Riemann-Hurwitz formula for the map $N_{\mathscr{S}} \circ K$ restricted to the irreducible curve C. □

Remark 4.10.12. *The collection of switches associated to the α_i's is of course not unique, but the number n_C is.*

4.10.3 Preimage of an intersection point

Now, the final information we need to define the graph $\Gamma(N_{\mathscr{S}}^*(\mathscr{C}))$ is the data above the double points of $\mathscr{C}$. Let C, C' be irreducible curves in $\mathscr{C}$, and $p \in C \cap C'$, $p \in D_1 \cap D_2 \cap D_3$. Denote $d := gcd(m_1, m_2, m_3)$. Denote $\overline{p^1}, \cdots, \overline{p^d}$ the ordered preimages of p by $(N_{\mathscr{S}} \circ K)$ (see Discussion 4.10.2). Recall the notations of Theorem 4.10.11. Lemma 4.10.3 implies the following:

Lemma 4.10.13. *for any $\overline{p^i} \in (N_{\mathscr{S}} \circ K)^{-1}(p)$, there are components $\overline{C_j}, \overline{C'_l}$, respectively of $(N_{\mathscr{S}} \circ K)^* (C)$ and $(N_{\mathscr{S}} \circ K)^*(C')$, such that $\overline{p^i} \in \overline{C_j} \cap \overline{C'_l}$.*

Furthermore, if $\overline{p^l} \in \overline{C_i} \cap \overline{C'_j}$, then $\overline{p^{l+1}} \in \overline{C_{i+1}} \cap \overline{C'_{j+1}}$.

4.10.4 The graph $\overset{\star}{\Gamma}\left((N_{\mathscr{S}} \circ K)^* (\mathscr{C}_{tot})\right)$

In this subsection, we sum up what has been developped throughout the previous ones to describe the decorated graph $\overset{\star}{\Gamma}\left((N_{\mathscr{S}} \circ K)^* (\mathscr{C}_{tot})\right)$.

If p is a double point of $\mathscr{C}_{tot}$, $p \in D_1 \cap D_2 \cap D_3$, denote $d_p = gcd(m_1, m_2, m_3)$ the cardinal of $(N_{\mathscr{S}} \circ K)^{-1}(p)$.

Recall the notations of Section 4.9. Theorem 4.10.11 and Lemma 4.10.13 imply:

Lemma 4.10.14. *The graph $\Gamma((N_{\mathscr{S}} \circ K)^* (\mathscr{C}_{tot}))$ is a cyclic covering of $\Gamma(\mathscr{C}_{tot})$ with covering data $(\{n_C\}_{C\in\mathscr{C}_{tot}}, \{d_p\}_{p \text{ double point}})$.*

However, in general, this data does not determine $\Gamma((N_{\mathscr{S}} \circ K)^* (\mathscr{C}_{tot}))$ uniquely. See Theorems 2.6.32, 2.6.33 and Proposition 2.6.35 for cases where this data is enough. In particular, if $(X, 0) = (\mathbb{C}^3, 0)$, $\mathscr{G}(\{n_C\}_{C\in\mathscr{C}_{tot}}, \{d_p\}_{p \text{ double point}}) = 0$, hence the covering data is enough, see [46, Theorem 7.4.16].

If $v \in \mathscr{V}(\Gamma(\mathscr{C}_{tot}))$ is associated to a curve $C \in \mathscr{C}_{tot}$, denote $n_v := n_C$, and denote $\overline{v_1}, \cdots, \overline{v_{n_v}}$ the vertices associated to v. In the same fashion, if $e \in \mathscr{E}(\Gamma(\mathscr{C}_{tot}))$ is associated to a double point $p \in \mathscr{C}_{tot}$, denote $d_e := d_p$, and denote $\overline{e_1}, \cdots, \overline{e_{d_e}}$ the edges associated to e.

To help overcome the ambiguity, note that the structure of cyclic covering implies that if one knows the end-vertices of one of the edges $\overline{e_i}$ for each $e \in \mathscr{E}(\Gamma(\mathscr{C}_{tot}))$, then one knows $\Gamma((N_{\mathscr{S}} \circ K)^* (\mathscr{C}_{tot}))$, up to isomorphism of cyclic coverings. It is therefore enough, for every double point $p \in C \cap C'$ of $\mathscr{C}_{tot}$, to figure out two components $\overline{C_i}$ and $\overline{C'_l}$ intersecting at one of the $\overline{p^j}$'s.

Definition 4.10.15. (Decorations of $\Gamma((N_{\mathscr{S}} \circ K)^* (\mathscr{C}_{tot}))$.)
*Decorate every vertex and every edge of $\Gamma((N_{\mathscr{S}} \circ K)^* (\mathscr{C}_{tot}))$ with the decorations of its image in $\Gamma(\mathscr{C}_{tot})$, **except for the genus decorations**: if $v \in \mathscr{V}(\Gamma(\mathscr{C}_{tot}))$ corresponds to a curve C, for each $\overline{v_i}$ corresponding to v, replace the genus decoration by the Euler characteristic $\chi\left(\overline{C}\right)$ of $\overline{C_i}$ computed in Theorem 4.10.11.*

4.11 The resolution step and a variety $(\widetilde{\mathscr{S}}, E_t)$

4.11.1 A variety $(\widetilde{\mathscr{S}}, E_t)$

The final step of the construction of the morphism Π announced in section 4.5 is the resolution π of the remaining isolated singular points of $\overline{\mathscr{S}}$. For each singular point $\overline{p}$ of $\overline{\mathscr{S}}$, we are going to decribe a resolution $\pi_{\overline{p}}$ of the singularity $(\overline{\mathscr{S}}, \overline{p})$, in the following subsections 4.11.2 and 4.11.3.

Once this process is complete, denote $(\widetilde{\mathscr{S}}, E_t)$ the variety obtained from $(\overline{\mathscr{S}}, (N_{\mathscr{S}} \circ K)^* (\mathscr{C}_{tot}))$ by resolving its singular points, and

$$\pi\colon (\widetilde{\mathscr{S}}, E_t) \to (\overline{\mathscr{S}}, (N_{\mathscr{S}} \circ K)^* (\mathscr{C}_{tot}))$$

the global resolution of $(\overline{\mathscr{S}}, (N_{\mathscr{S}} \circ K)^* (\mathscr{C}_{tot}))$ obtained by gluing the $\pi_{\overline{p}}$'s.

Finally, denote

$$\Pi := r_{\mathscr{S}} \circ N_{\mathscr{S}} \circ K \circ \pi\colon (\widetilde{\mathscr{S}}, E) \to (\mathscr{S}_k, 0)$$

the composed morphism, which is a resolution of the 4-dimensional real analytic singularity $(\mathscr{S}_k, 0)$.

Note that the construction of Π ensures that it is an isomorphism of real-analytic varieties outside the origin.

Let $\overline{p} \in \overline{C} \cap \overline{C'}$, where $\overline{C}, \overline{C'}$ are irreducible real surfaces of $(N_{\mathscr{S}} \circ K)^* (\mathscr{C}_{tot})$, be a potentially singular point of $\overline{\mathscr{S}}$. We face two different situations, according to the type of the point

$$p := (N_{\mathscr{S}} \circ K)(\overline{p}) \in D_1 \cap D_2 \cap D_3,$$

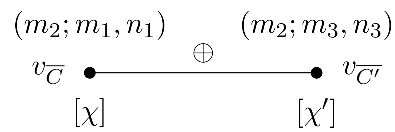
where $D_1 \in \mathbb{D}_0 \cup \mathbb{D}_g, D_2 \in \mathbb{D}_f$ and $D_3 \in \mathbb{D}$. Denote

$$C := (N_{\mathscr{S}} \circ K) \left(\overline{C}\right), C' := (N_{\mathscr{S}} \circ K) \left(\overline{C'}\right).$$

Denote also $\overline{U}$ the connected component of $\overline{U_p}$ containing $\overline{p}$, and $d := (m_1, m_2, m_3)$.

4.11.2 Point of type $\oplus$

If p is of type $\oplus$, then the edge associated to $\overline{p}$ in $\overset{\star}{\Gamma} \left((N_{\mathscr{S}} \circ K)^* (\mathscr{C}_{tot})\right)$ is of the form



and, on $\overline{U}$, the equation of $\overline{\mathscr{S}}$ is of the form

$$\overline{\mathscr{S}} \cap \overline{U} = \left(\left\{y^{m_2/d} = e^{2ij\pi/d} x^{m_1/d} z^{m_3/d}\right\} \cap U_p^{\mathbb{C}}\right)^{norm}.$$

(see equation (4.12) in the proof of Lemma 4.10.3.)

Denote

$$\pi_{\overline{p}} \colon \widetilde{U} \to \overline{\mathscr{S}} \cap \overline{U}$$

the resolution of the surface $\left(\left\{x^{m_1/d} = e^{2ij\pi/d} y^{m_2/d} z^{m_3/d}\right\}\right)^{norm} \cap \overline{U}$ described in Subsection 3.5.3, up to exchanging the roles of the coordinates x and y. In the sequel we refer to this Subsection for the notations.

Note that

$$\overline{C} \cap \overline{U} = (norm^* (\{y = x = 0\})) \cap \overline{U}$$

and

$$\overline{C'} \cap \overline{U} = (norm^* (\{y = z = 0\})) \cap \overline{U}.$$

Denote $\widetilde{C}$, $\widetilde{C'}$ the strict transforms of $\overline{C}, \overline{C'}$ by the morphism π.

Denote

$$g^{\mathbb{C}} := g \circ r_{\mathscr{S}} \circ N_{\mathscr{S}} \circ \kappa_p$$

the pullback of the function g to $\{y^{m_2} = x^{m_1} z^{m_3}\} \cap U_p^{\mathbb{C}}$. Up to a unit,

$$g^{\mathbb{C}} = x^{n_1} z^{n_3}.$$

Under the morphism π, the preimage of

$$(N_{\mathscr{S}} \circ K)^* (\mathscr{C}_{tot}) \cap \overline{U} = norm^* (\{xz = y = 0\}) \cap \overline{U}$$

is a chain of complex curves, namely

$$Str\left(\frac{m_2}{d}; \frac{m_1}{d}, \frac{m_3}{d} | 0; n_1, n_3\right)$$

where the notation Str has been introduced in 3.5.3. The multiplicities on the curves of the string are those of the function

$$\tilde{g} := g \circ \Pi$$

which is the pullback of the function g to $\widetilde{\mathscr{S}} \cap \pi^{-1}(\overline{U})$, and where $\widetilde{C_0} = \widetilde{C'}$, and $\widetilde{C_{l+1}} = \widetilde{C}$. Denote $\widetilde{U} := \pi^{-1}\left(\overline{U}\right)$. Note that the morphism $\pi \colon \widetilde{U} \to \overline{U}$ is an isomorphism outside of $\overline{p}$.

Discussion 4.11.1. Orientation compatibilities, point $\oplus$.
Let us remind here the choices of orientation made in subsection 4.8.2.

The pullbacks of the orientations of the curves $\overline{C} \cap \overline{U}, \overline{C'} \cap \overline{U}$ by the biholomorphism π are again the orientations given by their local complex structure.

*Each new curve $\widetilde{C_i}$ is taken oriented by **its complex structure**.*

The pullback of the orientation of $\overline{\mathscr{S}} \cap \overline{U}$ by the bimeromorphism $\pi \colon \widetilde{U} \to \overline{U}$ is, again, the orientation given by the complex structure. Now, Lemma 3.5.33 implies that at each intersection point $\widetilde{C_i} \cap \widetilde{C_{i+1}}$, the combination of the orientations of the two curves gives the ambient orientation of $\widetilde{U}$.

Remark 4.11.2. *It is useful in practice to notice that, if $m_2 = 1$, the point $\overline{p}$ is already a smooth point of $\overline{\mathscr{S}}$. The morphism π described in Subsection 3.5.3 is in this case an isomorphism.*

In the end, the preimage of $((N_{\mathscr{S}} \circ K)^* (\mathscr{C}_{tot}), \overline{p})$ by π is a chain of complex curves, whose dual decorated graph is the bamboo of figure 4.5, where the $\oplus$ signs refer to the orientation compatibilities. The integers μ_i represent the multiplicities of $\tilde{g}$ on each curve. For the definition of the multiplicity of $\tilde{g}$ on the real surfaces $\widetilde{C}$ and $\widetilde{C'}$, we refer to Definition 2.7.5.

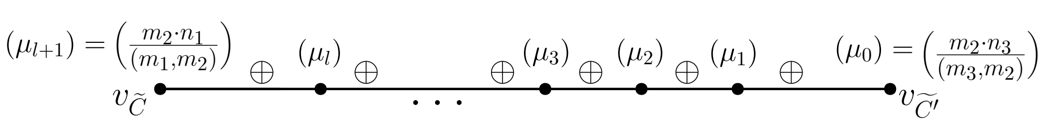
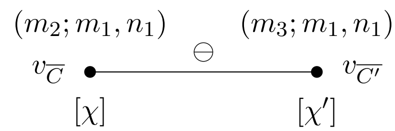


Figure 4.5: The string $Str^{\oplus}\left(\frac{m_2}{d}; \frac{m_1}{d}, \frac{m_3}{d} | 0; n_1, n_3\right)$.

Recall that (a, b) denotes $gcd(a, b)$.

4.11.3 Point of type $\ominus$

If p is of type $\ominus$, then the edge associated to $\overline{p}$ in $\overset{\star}{\Gamma}\left(\left(N_{\mathscr{S}}\circ K\right)^{*}\left(\mathscr{C}_{tot}\right)\right)$ is of the form



and, on $\overline{U}$, the equation of $\overline{\mathscr{S}}$ is of the form

$$\overline{\mathscr{S}}\cap\overline{U}=\left(\left\{x^{m_1/d}=e^{2ij\pi/d}y^{m_2/d}z^{m_3/d}\right\}\cap U_p^{\mathbb{C}}\right)^{norm}.$$

See equation (4.13) in the proof of Lemma 4.10.3.

Denote

$$\pi_{\overline{p}}\colon \widetilde{U}\to\overline{\mathscr{S}}\cap\overline{U_p}$$

the resolution of the surface $\left(\left\{x^{m_1/d}=e^{2ij\pi/d}y^{m_2/d}z^{m_3/d}\right\}\right)^{norm}\cap\overline{U}$ described in Subsection 3.5.3. Again, in the sequel we refer to this section for the notations.

Note that

$$\overline{C}\cap\overline{U}=\left(norm^{*}\left(\{x=z=0\}\right)\right)\cap\overline{U}$$

and

$$\overline{C'}\cap\overline{U}=\left(norm^{*}\left(\{x=y=0\}\right)\right)\cap\overline{U}.$$

Denote $\widetilde{C}$, $\widetilde{C'}$ the strict transforms of $\overline{C},\overline{C'}$ by the morphism π.

Denote

$$g^{\mathbb{C}}:=g\circ r_{\mathscr{S}}\circ N_{\mathscr{S}}\circ\kappa_p$$

the pullback of the function g to $\{x^{m_1}=y^{m_2}z^{m_3}\}\cap U_p^{\mathbb{C}}$. Up to a unit,

$$g^{\mathbb{C}}=x^{n_1}.$$

Under the morphism $\pi_{\overline{p}}$, the preimage of

$$\left(N_{\mathscr{S}}\circ K\right)^{*}\left(\mathscr{C}_{tot}\right)\cap\overline{U}=norm^{*}\left(\{yz=x=0\}\right)\cap\overline{U}$$

is a chain of complex curves, namely

$$Str\left(\frac{m_1}{d};\frac{m_2}{d},\frac{m_3}{d}|n_1;0,0\right)$$

where the notation Str has been introduced in 3.5.3. The multiplicities on the curves of the string are those of the function

$$\tilde{g}:=g\circ\Pi$$

the pullback of the function g to $\widetilde{\mathscr{S}}\cap\pi^{-1}(\overline{U})$, and where $\widetilde{C_0}=\widetilde{C'}$, and $\widetilde{C_{l+1}}=\widetilde{C}$. Note that the morphism $\pi_{\overline{p}}\colon\widetilde{U}\to\overline{U}$ is an isomorphism outside of $\overline{p}$.

Discussion 4.11.3. Orientation compatibilities, point $\ominus$.

Let us remind here the choices of orientation made in subsection 4.8.2.

The pullbacks of the orientations of the curves $\overline{C} \cap \overline{U}, \overline{C'} \cap \overline{U}$ by the biholomorphism $\pi_{\overline{p}}$ are again the opposites of the orientations given by their local complex structure.

*Each new curve $\widetilde{C_i}$ is taken oriented by **the opposite of its complex structure**.*

The pullback of the orientation of $\overline{\mathscr{S}} \cap \overline{U}$ by the bimeromorphism $\pi_{\overline{p}} \colon \widetilde{U} \to \overline{U}$ is, again, the opposite of the orientation given by the complex structure. Now, Lemma 3.5.33 implies that at each intersection point $\widetilde{C_i} \cap \widetilde{C_{i+1}}$, the combination of the orientations of the two curves gives the opposite of the ambient orientation of $\widetilde{U}$.

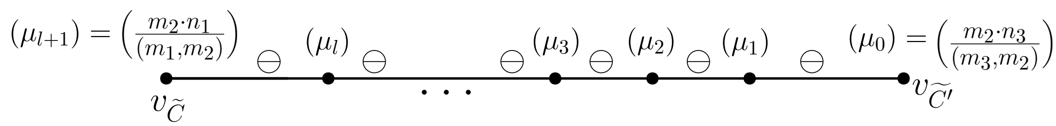


Figure 4.6: The string $Str^{\ominus}\left(\frac{m_1}{d}; \frac{m_2}{d}, \frac{m_3}{d} | n_1; 0, 0\right)$.

Recall that (a, b) denotes $gcd(a, b)$.

In the end, the preimage of $((N_{\mathscr{S}} \circ K)^* (\mathscr{C}_{tot}), \overline{p})$ by $\pi_{\overline{p}}$ is a chain of complex curves, whose dual decorated graph is the bamboo of figure 4.6, where the $\ominus$ signs refer to the orientation compatibilities. The integers μ_i represent the multiplicities of $\tilde{g}$ on each curve. Again, for the definition of the multiplicity of $\tilde{g}$ on the real surfaces $\widetilde{C}$ and $\widetilde{C'}$, see Definition 2.7.5.

4.11.4 The decorated graph $\Gamma^{\mu}(E_{tot})$

The nature of the resolution process described in Subsection 3.5.3, affecting each coordinate axis in only one point, whose preimage is a point, implies:

Lemma 4.11.4. *If $\overline{C}$ is an irreducible surface of $(N_{\mathscr{S}} \circ K)^* (\mathscr{C}_{tot})$, and $\widetilde{C}$ is the strict transform of $\overline{C}$ by π, then*

$$\chi(\widetilde{C}) = \chi(\overline{C}). \tag{4.15}$$

Furthermore, the surfaces added in the resolution process are all of genus 0.

Definition 4.11.5. *Denote $\Gamma^{\mu}(E_{tot})$ the graph obtained from $\overset{\star}{\Gamma}((N_{\mathscr{S}} \circ K)^* (\mathscr{C}_{tot}))$ by introducing the strings described in Subsections 4.11.2 and 4.11.3, where*

- *Vertices coming from arrowheads are still represented as arrowheads, and every vertex is decorated with the corresponding multiplicity of $\tilde{g}$.*

- *The vertex $v_{\widetilde{C}}$ corresponding to the strict transform of the curve $\overline{C}$ by π has genus decoration*

$$[g_{\widetilde{C}}] = [1 - \frac{\chi(\overline{C})}{2}]. \tag{4.16}$$

and the new vertices introduced in the strings have genus decoration 0*.*

The genus decorations of Equation (4.16) are motivated by the Euler characteristic identities of Equation (4.15). For the computation of $\chi\left(\overline{C}\right)$, see Equation (4.14) of Theorem 4.10.11.

4.12 Boundary of the Milnor fiber

Now, one has all the information required to compute the boundary of the Milnor fiber of f. Recall that this one is identified *via* Proposition 4.4.24 to $\partial\mathscr{S}_k = \rho^{-1}_{|\mathscr{S}_k}(\varepsilon)$, for any $\varepsilon > 0$ as in Definition 4.4.23.

Denote $\widetilde{\rho} := \rho \circ \Pi$. The fact that Π is an orientation-preserving real-analytic isomorphism outside 0 allows us to say that $\partial\mathscr{S}_k$ is orientation-preserving homeomorphic to its preimage by Π, which is the boundary ∂T of the tubular neighbourhood

$$T := \{\widetilde{\rho} \leqslant \varepsilon\}$$

of the preimage E of the origin by Π.

Now, $\tilde{g}$ and $\widetilde{\rho}$ are respectively an adapted and a rug function for the configuration E in $\widetilde{\mathscr{S}}$. Theorem 2.7.9 implies that

$$\partial T \simeq M_{\Gamma_{\widetilde{\mathscr{S}}}(E)},$$

where the equivalence symbol denotes an orientation-preserving homeomorphism.

In conclusion, the orientation-preserving diffeomorphism between $\partial\mathscr{S}_k$ and ∂T implies

$$\partial\mathscr{S}_k \simeq M_{\Gamma_{\widetilde{\mathscr{S}}}(E)},$$

where the equivalence symbol denotes an orientation-preserving diffeomorphism.

Definition 4.12.1. *Denote $\overset{\star}{\Gamma}(E)$ the graph obtained from $\Gamma^{\mu}(E_{tot})$ by keeping the genera decorations and replacing the multiplicity decorations of nodal vertices by the self-intersection decorations, using Lemma 2.7.7, then removing the arrowhead vertices.*

The construction of the decorations of $\overset{\star}{\Gamma}(E)$ is made in order to have the equality $\overset{\star}{\Gamma}(E) = \Gamma_{\widetilde{\mathscr{S}}}(E)$.

Finally we get the following generalization of [46, Theorem 10.2.10]:

Theorem 4.12.2. *The boundary of F is a graph manifold, and a possible plumbing graph for ∂F is the decorated graph $\overset{\star}{\Gamma}(E)$, which has only nonnegative genera decorations.*

This implies in particular Theorem 1 of the Introduction, Chapter 1.

Chapter 5

A toric version of the main algorithm

In this chapter, we describe how the main algorithm can be run in the case of a Newton-nondegenerate germ of function on a 3-dimensional germ of toric variety. The algorithm exposed here is a generalization of the algorithm proposed by Oka in [49] to compute the boundary of a nondegenerate isolated singularity of surface in $\mathbb{C}^3$, and answers the following open question: *"Determine ∂F for weighted homogeneous or Newton non-nondegenerate singularities in terms of their Newton diagram."*, asked by Némethi and Szilárd in [46, 24.4.20].

Let N be a 3-dimensional lattice, $\sigma \subset N_{\mathbb{R}}$ be a strongly convex rational cone, and $X := X_\sigma$ be the 3-dimensional toric variety associated to σ. Denote $M := N^{\checkmark}$ the dual lattice of N, and denote 0 the origin of X, its only 0-dimensional orbit. More generally, recall the notations introduced in Chapter 3.

Let $f\colon (X_\sigma, 0) \to (\mathbb{C}, 0)$ be a suitable germ of Newton-non-degenerate function, such that $V(f) \supset \mathrm{Sing}(X_\sigma)$. By Lemma 3.5.3, this last requirement is equivalent to the fact that, if $\tau \prec \sigma$ is a singular cone, $\mathrm{Supp}(f)$ has no point in the face $\tau^\perp \cap \sigma^{\checkmark}$ of $\sigma^{\checkmark}$.

Furthermore, Lemma 3.5.3 implies:

Remark 5.0.1. *Let τ be a regular 2-dimensional face of σ. Then the hypersurface $V(f)$ is singular along O_τ if and only if every element m of $\mathrm{Supp}(f)$ can be written as $m = m_1 + m_2$, where $m_1, m_2 \in \left(\sigma^{\checkmark} \setminus \tau^\perp\right) \cap M$.*

We propose in this chapter a way to obtain the decorated graph $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ that is the starting point of the general algorithm described in Chapter 4 from the Newton polyhedron $LNP(f)$ of f. Furthermore, we show that, in this case, the data of this decorated graph is sufficient to run the algorithm of computation of the boundary of the Milnor fiber of f.

This provides an algorithm taking as an input the cone σ and the Newton polyhedron of f, and computing the boundary of the Milnor fiber of f.

A direct consequence of this construction is the planarity of the plumbing graph, explicited in Proposition 5.3.1.

5.1 A companion for everyone ?

Although there is no chance to describe a function g that will be a companion (in the sense of Definition 4.4.2) of each Newton-nondegenerate function f on X, there is indeed some object that will work for every f.

Definition 5.1.1. *For any sequence $G = (m_1, \cdots, m_k)$ generating the semigroup $\sigma^\vee \cap M$, denote*

$$P(G) := Conv\left(\bigcup_{1\leqslant i\leqslant k} m_i + \sigma^\vee\right).$$

In the sequel, we fix such a family G.

By Remark 3.2.4, the sequence G leads to an embedding $(X, 0) \hookrightarrow \mathbb{C}^k{}_{x_1,\cdots,x_k}$. From this viewpoint, $P(G)$ is in fact the local Newton polyhedron of the restriction to X of a generic linear form of $\mathbb{C}^k{}_{x_1,\cdots,x_k}$.

Now, Lemmas 4.4.8 and 3.5.26 imply:

Lemma 5.1.2. *For any germ of function f on $(X, 0)$, there is a function g that is a companion of f and such that*

$$LNP(g) = P(G).$$

Remark 5.1.3. *One can note that the Newton-nondegeneracy of f is not required for this lemma. However, for what follows, the hypothesis of Newton-nondegeneracy is central.*

5.2 Computing $\mathscr{C}_{tot}$

In this section, we explain how to get access to the decorated graph $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$. We construct a fan $\widetilde{\mathcal{F}}$ refining σ and such that the associated modification

$$\widetilde{r} := \Pi_{\widetilde{\mathcal{F}}} \colon X_{\widetilde{\mathcal{F}}} \to X$$

verifies the conditions of Definition 4.6.1.

5.2.1 First refinement of σ

Denote $\overline{\mathcal{F}} := \mathcal{F}_{f\cdot g}$ the fan associated to the germ $f \cdot g$. Remark 3.5.37 implies that this fan is the minimal refinement of both $\mathcal{F}_f$ and $\mathcal{F}_g$. Denote $\overline{X} = X_{\overline{\mathcal{F}}}$ and

$$\overline{r} := \Pi_{\mathcal{F}_{f\cdot g}} : \overline{X} \to X$$

the modification associated to this refinement.

Note that we have the following commutative diagram:

$$\begin{array}{ccccc} & \Pi_{\overline{\mathcal{F}},\mathcal{F}_f} & \overline{X} & \Pi_{\overline{\mathcal{F}},\mathcal{F}_g} & \\ X_{\mathcal{F}_f} & \swarrow & \downarrow \overline{r} & \searrow & X_{\mathcal{F}_g} \\ & \searrow \Pi_{\mathcal{F}_f} & X & \Pi_{\mathcal{F}_g} \swarrow & \end{array}$$

Denote $\overline{V(f)}, \overline{V(g)}$ the strict transforms of $V(f), V(g)$ by $\overline{r}$. Denote

$$\overline{\mathbb{D}_f} := \overline{\overline{r}^{-1}(V(f) \setminus \{0\})}$$

the mixed transform of $V(f)$ by $\overline{r}$, which is the analogue at this step of the divisor $\mathbb{D}_f$ of Definition 4.6.1.

The fact that the fan $\mathcal{F}_{f \cdot g}$ is the minimal refinement of both $\mathcal{F}_f$ and $\mathcal{F}_g$ implies that it can be computed by "superposing" the two fans $\mathcal{F}_f$ and $\mathcal{F}_g$, as in figure 5.1.

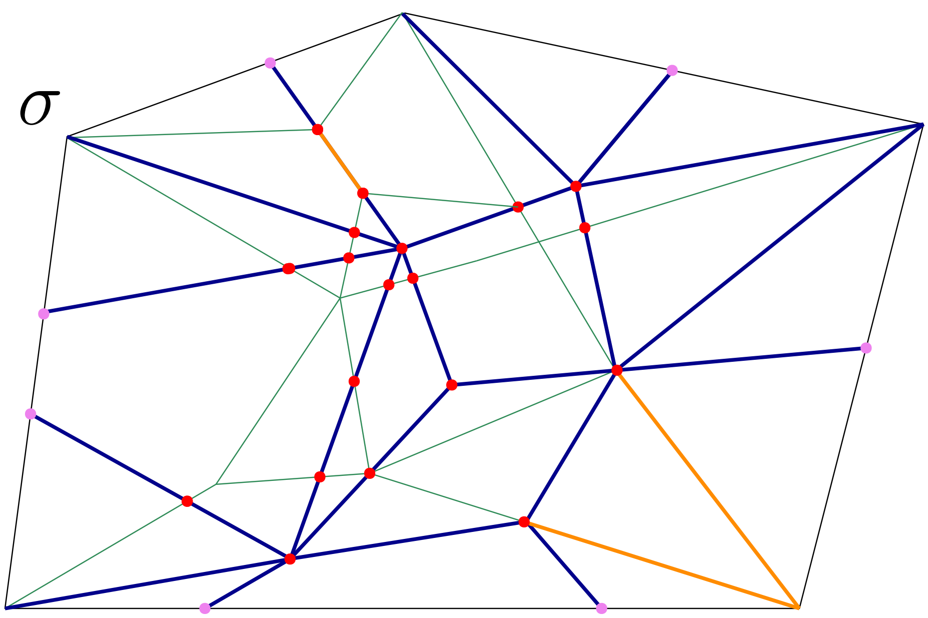


Figure 5.1: The fan $\mathcal{F}_{f \cdot g}$ as a superposition of $\mathcal{F}_f$ and $\mathcal{F}_g$.

Let us explain the colours in this figure:

- In orange are drawn the 2-dimensional cones that are in both $\mathcal{F}_f$ and $\mathcal{F}_g$.
- The fan $\mathcal{F}_g$ is drawn in green and orange.
- The fan $\mathcal{F}_f$ is drawn in blue and orange.

- The red marks correspond to the 2-dimensional compact orbits of $X_{\mathcal{F}_{f\cdot g}}$ that are intersected by $\overline{V(f)}$.
- The violet marks correspond to the non compact 2-dimensional orbits whose closures are irreducible components of $\overline{\mathbb{D}_f}$. They are contained in fact in the exceptional part $\overline{\overline{\mathbb{D}_f} \setminus \overline{V(f)}}$ of this mixed transform.

  Indeed, if γ is a 2-dimensional face of σ, $V(f) \supset O_\gamma$ if and only if $\mathcal{F}_f$ contains a 1-dimensional cone τ in the interior of γ. In these conditions, $\overline{O_\tau}$ is an irreducible component of $\overline{\overline{\mathbb{D}_f} \setminus \overline{V(f)}}$.

Remark 5.2.1. *Note that our construction of g ensures that $\mathcal{F}_g$ has no 1-dimensional cone contained in the interior of a 2-dimensional face of σ.*

5.2.2 Second refinement

Now, refine $\overline{\mathcal{F}}$ in such a way as to subdivide regularly every 2-dimensional cone corresponding to an orbit intersected by $\overline{V(f)}$, as in Lemma 3.5.30. Denote $\widehat{\mathcal{F}}$ the fan obtained at this step, $\widehat{X} := X_{\widehat{\mathcal{F}}}$, and $\widehat{V(f)}, \widehat{V(g)}, \widehat{\mathbb{D}_f}$ respectively the strict transforms of f, g and the mixed transform of $V(f)$ by

$$\widehat{r} := \Pi_{\widehat{\mathcal{F}}} \colon \widehat{X} \to X,$$

the associated modification of X. The following figure 5.2 shows such a possible refinement. The violet 2-dimensional cones correspond to 1-dimensional orbits whose closures are curves of $\mathbb{D}_f \cap \mathbb{D}_0$. The discontinuous ones represent such orbits that are **not** intersected by $\widehat{V(f)}$.

5.2.3 Last refinement

At this point, the last problems stand at the 0-dimensional orbits contained in the closures of orbits corresponding to violet 2-dimensional cones. In $X_{\overline{\mathcal{F}}}$, such points are the orbits 0_{γ_i}, where γ_i varies among the 3-dimensional cones of the fan $\overline{\mathcal{F}}$ that have a face of dimension 1 which is contained in a 2-dimensional face of σ but in no 1-dimensional face.

Hence the purpose of the last refinement is to subdivide regularly those cones γ_i. However, there is no canonical way to refine regularly a cone of dimension 3 or more. For a way to compute such a refinement, see [49], paragraph 3. A refinement of those cones will always start with the canonical refinements of their 2-dimensional faces. Denote $\widetilde{\mathcal{F}}$ this last fan, and $\widetilde{V(f)}, \widetilde{V(g)}, \mathbb{D}_f$ the strict and mixed tranforms. Denote

$$\widetilde{r} := \Pi_{\widetilde{\mathcal{F}}} \colon X_{\widetilde{\mathcal{F}}} \to X$$

and

$$\mathbb{D}_0 := \widetilde{r}^{-1}(0).$$

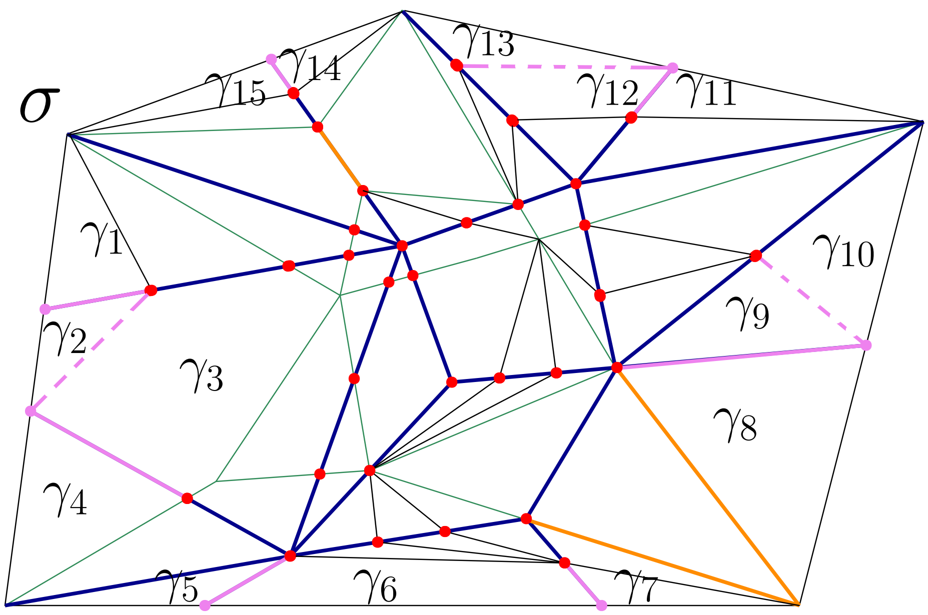


Figure 5.2: A possible $\widehat{\mathcal{F}}$, refinement of $\overline{\mathcal{F}}$.

The point is that, up to choosing generic coefficients for g, the modification $\widetilde{r}$ is **adapted** to the couple (f,g), in the sense of Definition 4.6.1. Denote $\mathbb{D}_{f,ex}$ the union of irreducible components of $\mathbb{D}_f$ that are not in $\widetilde{V(f)}$. This way, we get a decomposition

$$\mathbb{D}_f = \widetilde{V(f)} \cup \mathbb{D}_{f,ex}$$

of the set of irreducible components of $\mathbb{D}_f$.

Figure 5.3 shows a possible fan $\widetilde{\mathcal{F}}$. The set $\mathbb{D}_{f,ex}$ is the union of the closures of the 2-dimensional orbits of $X_{\widetilde{\mathcal{F}}}$ corresponding to 1-dimensional cones of $\widetilde{\mathcal{F}}$ whose minimal containing cone of σ is of dimension 2. Those orbits are again represented by violet marks.

5.2.4 Reading $\mathscr{C}$

Definition 5.2.2. *For $\tau \in \widetilde{\mathcal{F}}$, denote repectively $\Delta_\tau(f)$ and $\Delta_\tau(g)$ the corresponding faces of $LNP(f)$ and $LNP(g)$.*

To make short, if the dimension of those faces is 1 or less, denote

$$l_\tau(f) := l(\Delta_\tau(f)) \text{ and } l_\tau(g) := l(\Delta_\tau(g))$$

with the convention that the length of a 0-dimensional face is 0.

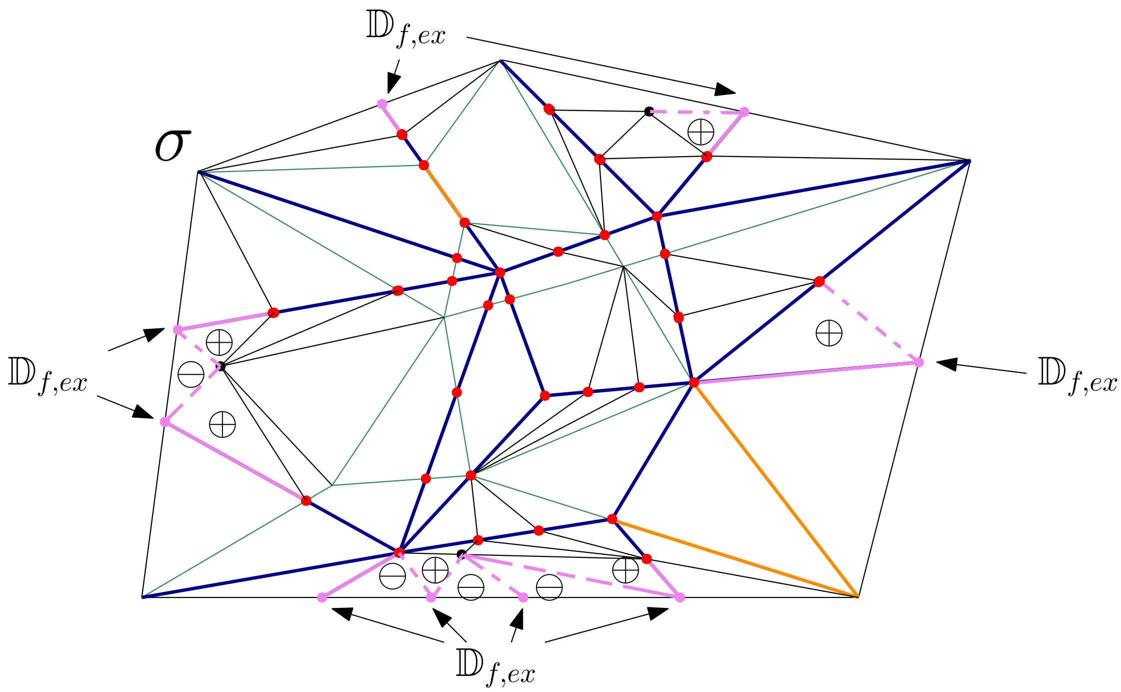


Figure 5.3: A possible $\widetilde{\mathcal{F}}$, refinement of $\widehat{\mathcal{F}}$.

If $dim(\Delta_\tau(f)) \leqslant 2$, denote i_τ the number of interior points of $\Delta_\tau(f)$, with the convention that the number of interior points of a face of dimension 0 or 1 is 0.

*Finally, call τ **pertinent** if and only if $dim(\Delta_\tau(f)) \geqslant 1$ and τ is not contained in a strict face of σ.*

Remark 5.2.3. *Remark 3.5.18 implies that the intersection $\widetilde{V(f)} \cap O_\tau$ is non-empty if and only if τ is pertinent. In figure 5.3, pertinent 1-dimensional cones are represented by red marks, while pertinent 2-dimensional cones are represented by either blue or **continuous** violet segments.*

One can read the configuration $\mathscr{C}$ on the fan $\widetilde{\mathcal{F}}$, together with $LNP(f)$ and $LNP(g)$. First, note that the curves of $\mathscr{C} = \mathbb{D}_f \cap \mathbb{D}_0$ are of two types, according to the decomposition $\mathbb{D}_f = \widetilde{V(f)} \cup \mathbb{D}_{f,ex}$.

Discussion 5.2.4. (Two types of compact curves.)

1. ***The first type is made of curves in $\mathbb{D}_{f,ex} \cap \mathbb{D}_0$.*** *Such curves are of the form $\overline{O_\gamma}$, where γ is a 2-dimensional cone with one 1-dimensional face in the interior of σ, and the other in the interior of a 2-dimensional face of σ. Such cones are represented by violet lines, continuous or not, in figure 5.3.*

Two such curves intersect if and only if the corresponding cones are faces of a same 3-dimensional cone of $\widetilde{\mathcal{F}}$. Our construction ensures that $\tilde{X}$ is smooth along the union of those curves, and that they intersect transversally in exactly one point. Furthermore, the nature of their intersection point can be read on the fan: it is $\oplus$ if and only if they are in the same component of $\mathbb{D}_{f,ex}$.

2. ***The second type is made of curves of** $\widetilde{V(f)} \cap \mathbb{D}_0$, that is, intersection of $\widetilde{V(f)}$ with closures of orbits corresponding to 1-dimensional cones in the interior of σ. Let τ be such a 1-dimensional cone of $\widetilde{\mathcal{F}}$. Recall that $\overline{O_\tau}$ is intersected by $\widetilde{V(f)}$ if and only if $dim(\Delta_\tau(f)) \geqslant 1$.*

   Furthermore, we know the nature of this intersection: by Proposition 3.5.24, given a pertinent cone τ of dimension 1, the intersection $\overline{O_\tau} \cap \widetilde{V(f)}$ is

   - *an irreducible curve C_τ if $dim(\Delta_\tau(f)) = 2$.*
   - *a disjoint union $C_\tau = {C_\tau}^1 \bigsqcup \cdots \bigsqcup {C_\tau}^{l_\tau(f)}$ of $l_\tau(f)$ irreducible rational curves if $dim(\Delta_\tau(f)) = 1$.*

   By construction of $\widetilde{\mathcal{F}}$ and the non-degeneracy of f, these curves are smooth, $\tilde{X}$ is smooth along each of them, and they intersect transversally.

   Furthermore, let τ_1, τ_2 be two 1-dimensional pertinent cones. The possibly disconnected curves C_{τ_1} and C_{τ_2} intersect if and only if τ_1 and τ_2 are faces of the same 2-dimensional pertinent cone γ. In these conditions,

   $$Card(C_{\tau_1} \cap C_{\tau_2}) = l_\gamma(f)$$

   and these intersection points are all of type $\oplus$, because these curves are in the same component of $\mathbb{D}_f$, that is, $\widetilde{V(f)}$.

   Note that, in this situation, if $dim(\tau_1) = 1$, then $l_{\tau_1}(f) = l_\gamma(f)$, and each connected component of C_{τ_1} intersects C_{τ_2} in exactly one point.

Lemma 5.2.5. (Intersection of curves of the first and of the second type.) *Let $C_1 = \overline{O_\gamma}$ be a curve of the first type, and $C_2 = C_\tau$ be of the second type, for some 2-dimensional cones τ, γ. Then*

$$C_1 \cap C_2 \neq \emptyset \Leftrightarrow \tau \prec \gamma \text{ and } \gamma \text{ is pertinent.}$$

In these conditions,

$$Card(C_1 \cap C_2) = l_\gamma(f)$$

and each intersection point is of type $\ominus$.

Discussion 5.2.6. *The last data required for the description of the configuration $\mathscr{C}$ is the decorations of the curves:*

1. *If $C = \overline{O_\gamma}$ is a curve of type 1, denoting τ_1 the 1-dimensional face of γ that is not in the boundary of σ and τ_2 the one in the interior, the decoration of C is the triple*

$$(h_{\tau_1}(f); h_{\tau_2}(f), h_{\tau_2}(g))$$

with genus 0.

2. *If $C \subset C_\tau$ is of the second type, then the decoration of C is the triple*

$$(1; h_\tau(f), h_\tau(g))$$

with genus

$$g(C) = i_\tau,$$

see Proposition 3.5.24.

5.2.5 Non-compact curves

Pick generic coefficients of g so that for any 2-dimensional orbit O_τ of $\tilde{X}$ intersected by $\widetilde{V(f)}$, the truncations $f_{\Delta_\tau(f)}$ and $g_{\Delta_\tau(g)}$ verify the hypothesis of Corollary 3.5.41.

Definition 5.2.7. *Denote $V(\tau)$ the mixed 2-dimensional volume of $\Delta_\tau(f)$ and $\Delta_\tau(g)$.*

With this choice of g, we can access the rest of the configuration $\mathscr{C}_{tot}$. Indeed,

Lemma 5.2.8. (Adding non-compact curves)
Let C_τ be a curve of $\mathscr{C}$ of the second type, possibly a disconnected union of curves. Then

$$Card\left(C_\tau \cap \widetilde{V(g)}\right) = V(\tau),$$

each of these intersection points being an intersection point of C_τ with a curve in $\widetilde{V(f)} \cap \widetilde{V(g)}$.

Furthermore, if $dim(\Delta_\tau(f)) = 1$, $\exists\ k \in \mathbb{N}$ s.t. $V(\tau) = k \cdot l_\tau(f)$, and each connected component of C_τ is intersected in k points.

Each of these points is of type $\oplus$, and, in $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$, the new curves are represented by arrowheads decorated with $(1; 0, 1)$.

Remark 5.2.9. *If $dim(\Delta_\tau(g)) = 0$, then $V(\tau) = 0$.*
If $\Delta_\tau(f)$ and $\Delta_\tau(g)$ are parallel segments, again, $V(\tau) = 0$
If $\Delta_\tau(f)$ and $\Delta_\tau(g)$ are non parallel segments, $V(\tau) = l_\tau(f) \cdot l_\tau(g)$.

Figure 5.4 shows a possible form of the graph $\Gamma(\mathscr{C}_{tot})$, where the decorations $\oplus$ and the triples of multiplicities are omitted. Violet marks represent curves of type 1. We respected to a large extent the form of the fan $\widetilde{\mathcal{F}}$, and the reader can follow the construction of this graph from the fan, with the appropriate values for the lengths and mixed volumes. Every curve without genus decoration is rational, and the indicated genera g_i's may be zero, according to the number of integral points of the 2-dimensional compact faces of $LNP(f)$.

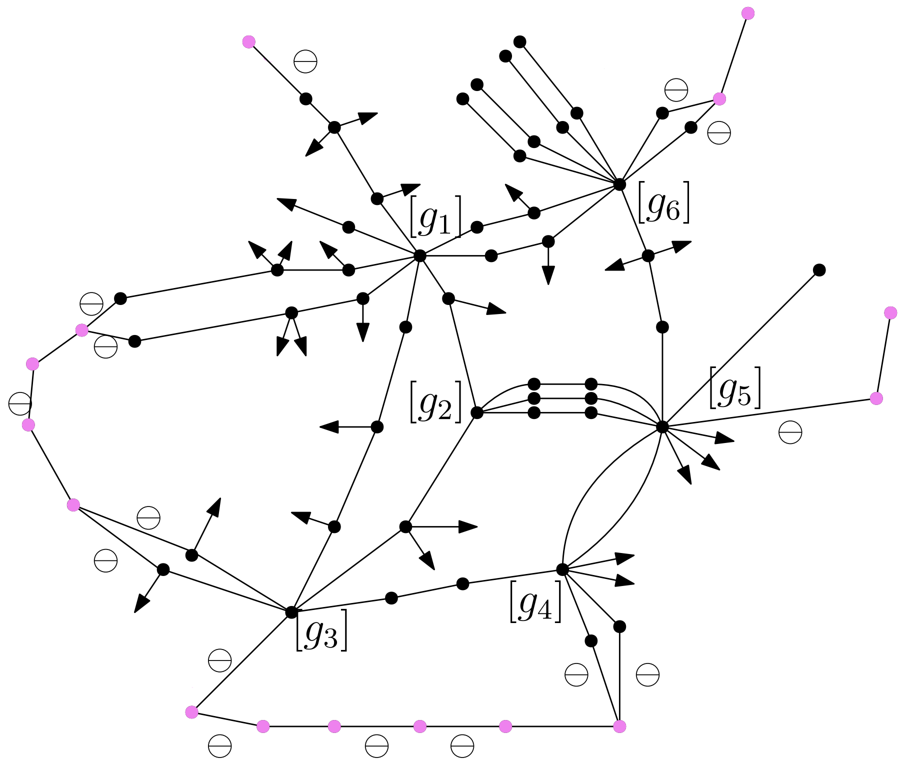


Figure 5.4: A possible graph $\Gamma(\mathscr{C}_{tot})$.

5.3 Sufficiency of $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$

Let us conclude this chapter by two observations that show that the data of the decorated graph $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ computed in the previous sections is sufficient to apply the main algorithm presented in Chapter 4.

5.3.1 Switches

Observe that, if the genus of a compact irreducible component of $\mathscr{C}_{tot}$ is not 0, then the triple of multiplicities of this curve is of the form $(1; m, n)$. Combining this with Theorem 4.10.11, we see that there is only one component in its preimage by the normalization morphism. This makes the data of the switches of Definition 4.10.10 unnecessary.

5.3.2 Covering graph

In the same spirit, observe that, by our construction, the graph obtained from $\overset{\star}{\Gamma}(\mathscr{C})$ by removing all vertices corresponding to curves of type 2 is a disjoint union of trees. Again, each of those curves of type 2 being decorated by a triple of the form $(1; m, n)$, this prevents by Theorem 2.6.33 any ambiguity in the definition of the covering graph corresponding to the process of normalization and complexification of $\widetilde{\mathscr{S}_k}$, see Section 4.10.

Hence, one can proceed to the main algorithm with the sole data of $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$, obtained from $LNP(f)$ and σ.

Furthermore, we can observe the following, which comes from our construction and Remark 2.6.40:

Proposition 5.3.1. *In the toric case, the graph produced by our algorithm is planar. The normal form of this graph in the sense of [47] is also planar.*

5.4 Two examples of computation

In this section we give two examples of computation of plumbing graphs of the boundaries of Milnor fibers of Newton non-degenerate functions on three-dimensional toric germs, performed using the algorithm described before. In the first example the toric germ is smooth, therefore we are in the setting of Némethi and Szilárd's book [46]. Nevertheless, let us recall that the family of Newton non-degenerate functions was not treated in that book. In the second example the toric germ is singular, with 1-dimensional singular locus. Therefore the fact that the boundary of the associated Milnor fiber is a graph manifold is not a consequence of any previous work.

5.4.1 An example in $\mathbb{C}^3$

Consider the function $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ given by

$$f(x, y, z) = x^5 + y^2 + xyz.$$

Figure 5.5 shows the local Newton polyhedron of f, while Figure 5.6 shows the local Newton polyhedron of a generic linear form $g(x, y, z) = \alpha \cdot x + \beta \cdot y + \gamma \cdot z$.

Figure 5.7 shows the fan $\mathcal{F}_{f \cdot g}$, with the conventions introduced in Subsection 5.2.1.

In this example, every blue cone of dimension 2 is already regular. The cone generated by $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$ is refined by introducing the vectors $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$.

Figure 5.8 shows the refinement $\widetilde{\mathcal{F}}$ of $\mathcal{F}_{f \cdot g}$ respecting the conditions asked in Subsection 5.2.3.

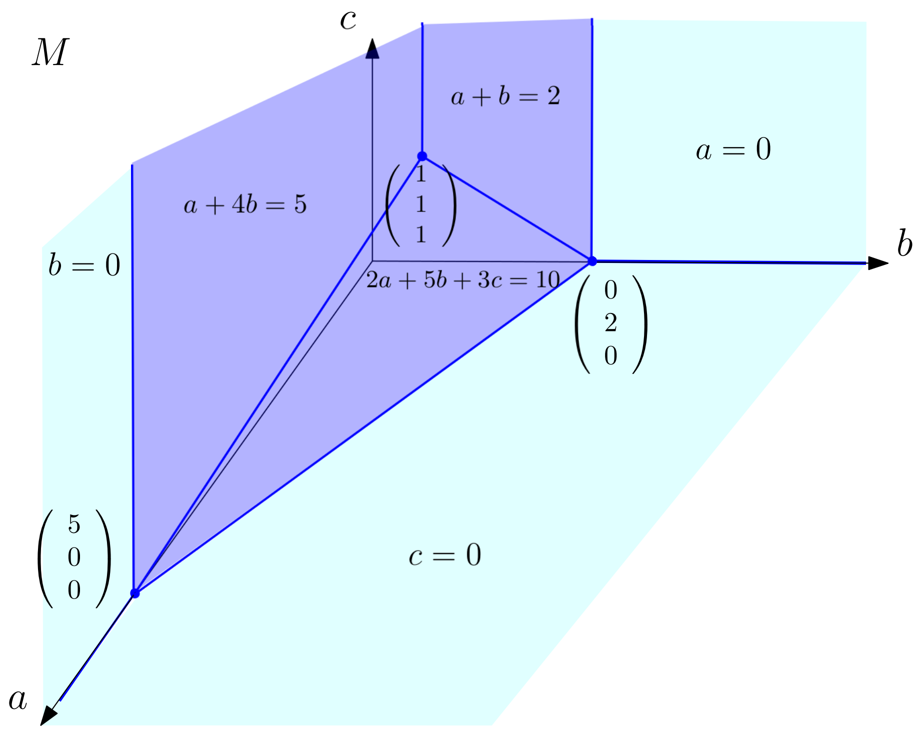


Figure 5.5: $LNP(f)$.

Denote $u = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$. The number of points of intersection of the strict transforms $\widetilde{V(f)}$ and $\widetilde{V(g)}$ in the orbit $O_{\langle u \rangle}$ is equal to the mixed volume of the faces $\Delta_u(f) = \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right]$ and $\Delta_u(g) = \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]$, where $[AB]$ denotes the segment between the points A and B. They are noncolinear segments, both of length 1, so this mixed volume is equal to 1. In the same way, the number of points of intersection of $\widetilde{V(f)}$ and $\widetilde{V(g)}$ in $O_{\langle v \rangle}$ is 1.

The multiplicities of f and g on the closures of the orbits are given either by direct lecture on the respective Newton polyhedra or using the linearity of each multiplicity function in each cone of $\mathcal{F}_{f \cdot g}$. For example, $u = \frac{1}{3}\left(\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right)$, so $m_{\overline{O_u}}(f) = (10+2)/3 = 4$.

The graph $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ is represented in Figure 5.9, with the decorations $\oplus$ omitted. Note

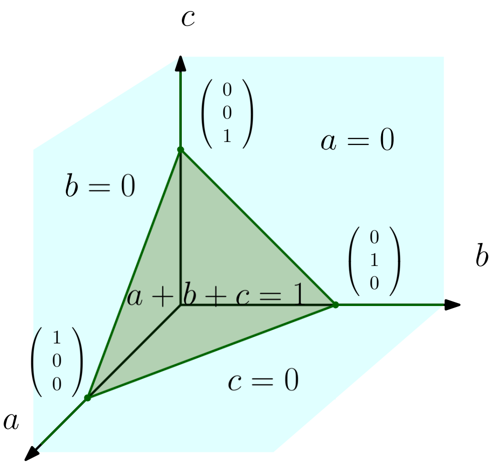
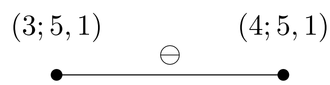


Figure 5.6: $LNP(g)$.

that, in this example, no compact face has interior points, which explains the nullity of all genera, and the fact that there is no multiple edge.

By equation (4.14), each surface of $\overset{\star}{\Gamma}\left((N_{\mathscr{S}} \circ K)^{*}\left(\mathscr{C}_{tot}\right)\right)$ has Euler characteristic 2, which will eventually lead to genus 0, so we omit these decorations. The next step is the computation of the strings inserted in place of the edges. We explain the computation on the edge



which is replaced by the string $Str^{\ominus}(5;3,4|1;0,0)$. Here, with the notations of Subsection 3.5.3, $a=5, b=3, c=4, n_1=1, n_2=n_3=0$, $d=(a,b,c)=1$, and $\delta=\dfrac{ad}{(a,b)(a,c)}=5$. Now, α is the only integer between 0 and 4 such that

$$ad \mid c(a,b)\alpha + b(a,c).$$

In our case, $\alpha = 3$. Then we compute

$$\frac{\delta}{\alpha}=\frac{5}{3}=2-\frac{1}{3}.$$

This gives us $k_1=2$ and $k_2=3$, and $l=2$. Finally, we get $\mu_0=\dfrac{cn_1+an_3}{(a,c)}=4$, $\mu_3=\dfrac{bn_1+an_2}{(a,b)}=3$, and $\mu_1=\dfrac{\alpha\mu_0+\mu_{l+1}}{\delta}=3$. Now μ_2 is computed thanks to the identity $\mu_2\cdot k_2=\mu_1+\mu_3$. We get $\mu_2=2$. Finally the string $Str^{\ominus}(5;3,4|1;0,0)$ is equal to

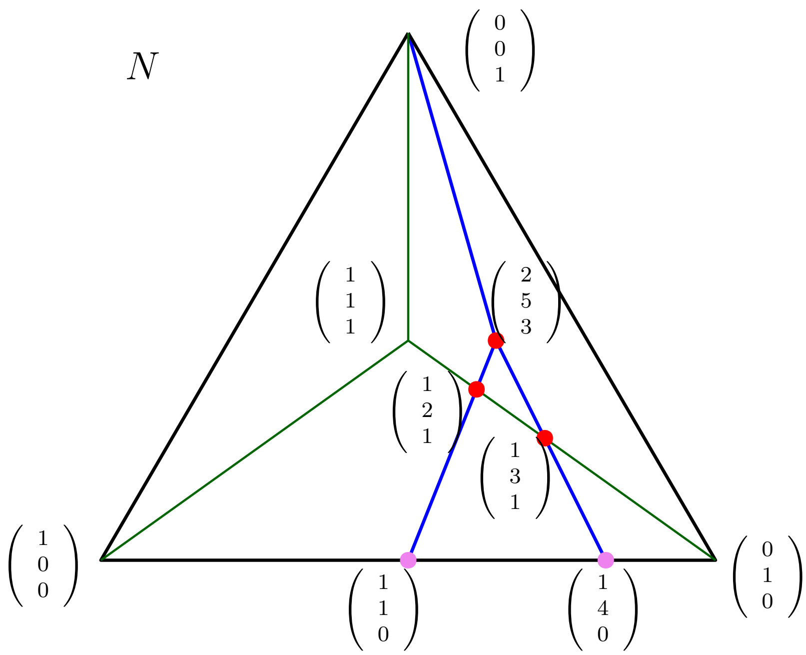
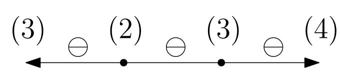


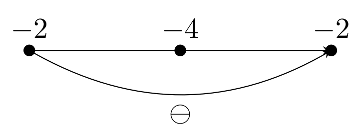
Figure 5.7: The fan $\mathcal{F}_{f \cdot g}$.



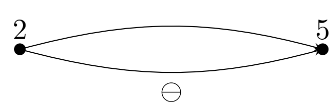
and the graph $\Gamma^{\mu}(E_{tot})$ is the one represented in Figure 5.10, with the decorations ⊕ omitted.

Finally, the plumbing graph $\Gamma_{\widetilde{\mathscr{F}}}(E)$ obtained as a result of the main algorithm is the one represented in Figure 5.11.

Its normal form in the sense of [47] is



Note that, in [46, p. 210], the authors propose the form



These two graphs encode the same graph manifold, and are related by a succession of blowing-downs and blowing-ups.

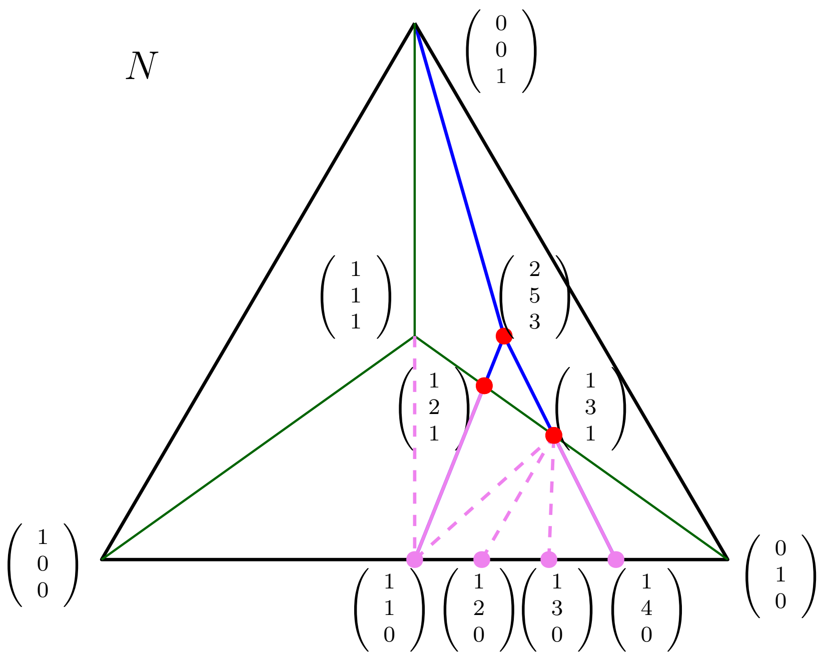


Figure 5.8: The fan $\widetilde{\mathcal{F}}$.

5.4.2 An example with X singular

Denote $M := \mathbb{Z}^3$, and let

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, u_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

be vectors in $\mathbb{R}^3$, and

$$\sigma = \langle u_1, u_2, u_3, u_4 \rangle_{\mathbb{R}_+}.$$

Let $X := X_\sigma$ be the 3-dimensional toric variety corresponding to σ. The cone σ is not simplicial, hence X is singular at the origin. Furthermore, the face $\tau_{1,2} := \langle u_1, u_2 \rangle_{\mathbb{R}_+}$ of σ is singular, hence X is singular along $0_{\tau_{1,2}}$.

The cone $\sigma^\vee$ is generated by the vectors

$$U = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, V = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, W = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, X = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}.$$

The face of $\sigma^\vee$ corresponding to the face $\tau_{1,2}$ of σ is the ray generated by W.

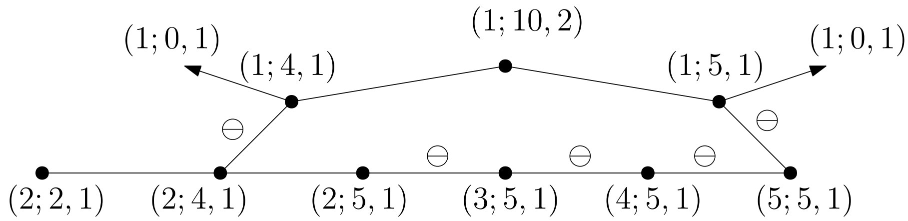


Figure 5.9: The configuration $\mathscr{C}_{tot}$.

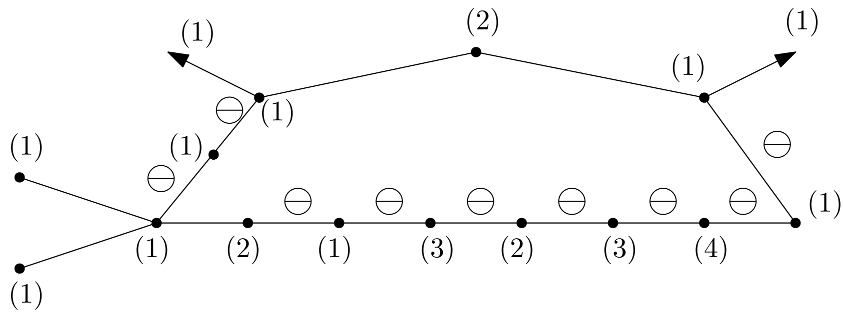


Figure 5.10: The graph $\Gamma^{\mu}(E_{tot})$.

Furthermore, the semigroup $S_\sigma = \sigma^{\checkmark} \cap M$ is generated by U, V, W, X and $Y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

The relations between these vectors provide the description

$$X = Spec\left(\mathbb{C}[U,V,W,X,Y] \big/ (Y^3 - XV, UX - WY^2)\right)$$

Figure 5.12 shows the local Newton polyhedron of the restriction g to X of a generic linear form in $\mathbb{C}^5_{U,V,W,X,Y}$. Full black lines represent the cone σ, while dashed lines represent the axes of coordinates, left for clarity.

Consider the function $f \in \mathbb{C}[\sigma^{\checkmark} \cap M]$ given by

$$f = \chi^{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} + \chi^{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}} + \chi^{\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}} + \chi^{\begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix}}$$

Note that $V(f) \supset \mathrm{Sing}(X)$, since f admits no multiple of W in its support. Figure 5.13 shows the local Newton polyhedron of f, where we kept again the coordinate axes and the cone $\sigma^{\checkmark}$. Blue marks represent the elements of $\mathrm{Supp}(f)$, while black marks represent the other points of M in the compact faces of $LNP(f)$.

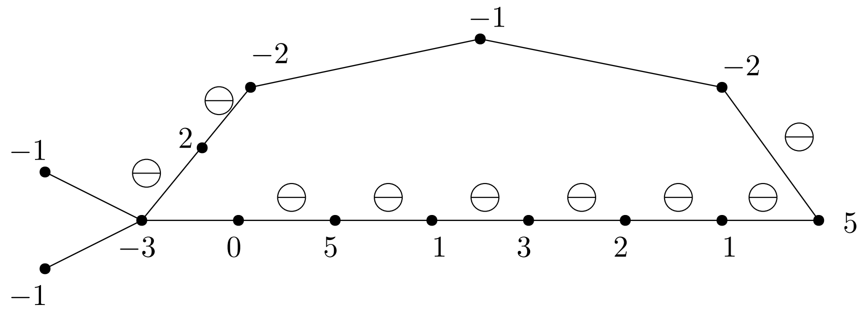


Figure 5.11: The graph $\Gamma_{\widetilde{\mathscr{F}}}(E)$.

The final refinement of $\mathcal{F}_{fg}$ is represented in Figure 5.14, with the codes of colours introduced in the description of the general case, see Subsection 5.2.1. Next to each vector corresponding to a 2-dimensional orbit of $X_{\mathcal{F}_{fg}}$, we indicate the couple of multiplicities of the pullbacks of the functions f, g on the orbit in question. Again, the missing multiplicities are computed using the linearity of each multiplicity function on the cones of $\mathcal{F}_{fg}$. For example,

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)$$

hence $m_{\overline{O}_{\left\langle \left(\begin{smallmatrix} 1 \\ 2 \\ 2 \end{smallmatrix}\right) \right\rangle}}(\tilde{g}) = \frac{1}{2} \cdot 2 = 1.$

Now, set $u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, w = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

The face $\Delta_{\langle u \rangle}(f)$ is a segment of length 2, and the mixed volume $V_{\langle u \rangle}$ of $\Delta_{\langle u \rangle}(f)$ and $\Delta_{\langle u \rangle}(g)$ is equal to 2.

We show an example of computation of the mixed volume by hand in the case of v:

$\Delta_{\langle v \rangle}(f)$ is the triangle $\left[\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}\right]$ which can be translated to the triangle $\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}\right]$.

$\Delta_{\langle v \rangle}(g)$ can be translated into the polygon $\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}\right]$.

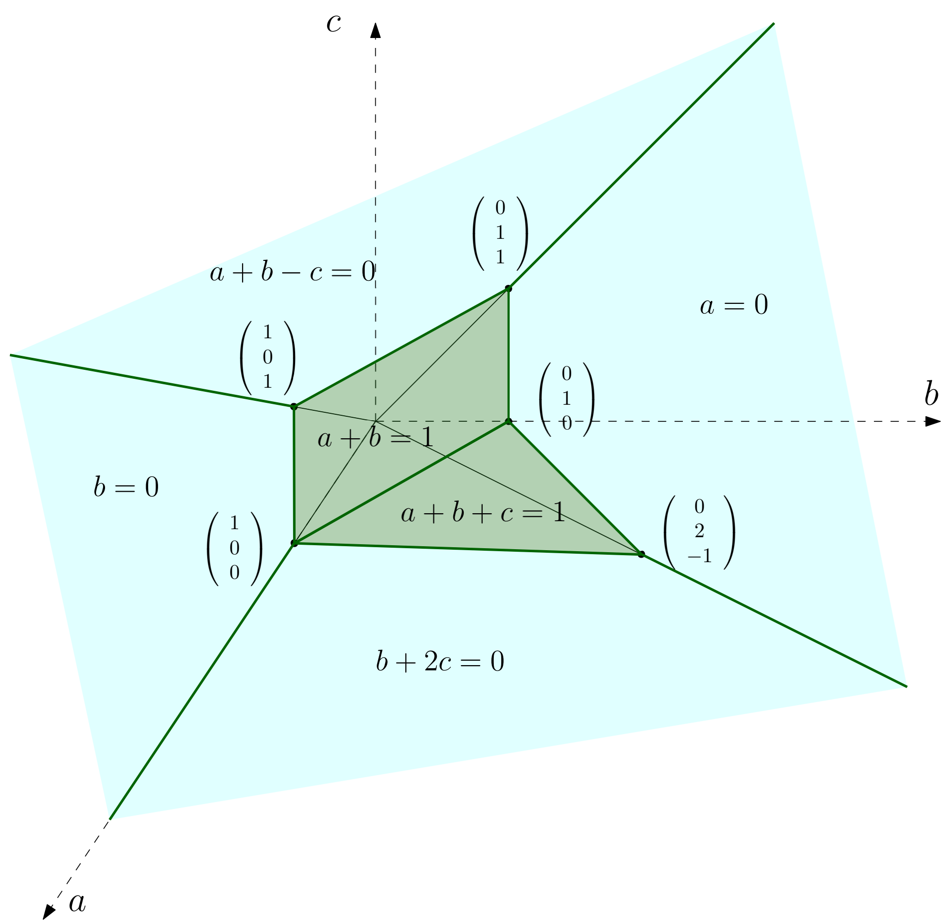


Figure 5.12: $LNP(g)$.

Now, choosing the basis $x = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, y = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ for the plane containing these two polygons, we get the picture of Figure 5.15.

The mixed volume is read on Figure 5.16, reminding that an elementary triangle has volume 1, and that $V(P,Q) = (Vol(P+Q) - Vol(P) - Vol(Q))/2$. We get $V_{\langle v \rangle} = 4$.

Finally, $V_{\langle w \rangle} = 0$ because $\Delta_{\langle w \rangle}(g)$ is a point. Keeping in mind that face $\Delta_{\langle v \rangle}(f)$ has one interior point, we get finally the graph of configuration of Figure 5.17.

This leads to the multiplicity graph $\Gamma^{\mu}(E_{tot})$ represented in Figure 5.18. For the computation of the genera, notice that, by formula 4.14, the genus of a curve where the multiplicities of $\tilde{f}$ are coprime remains unchanged.

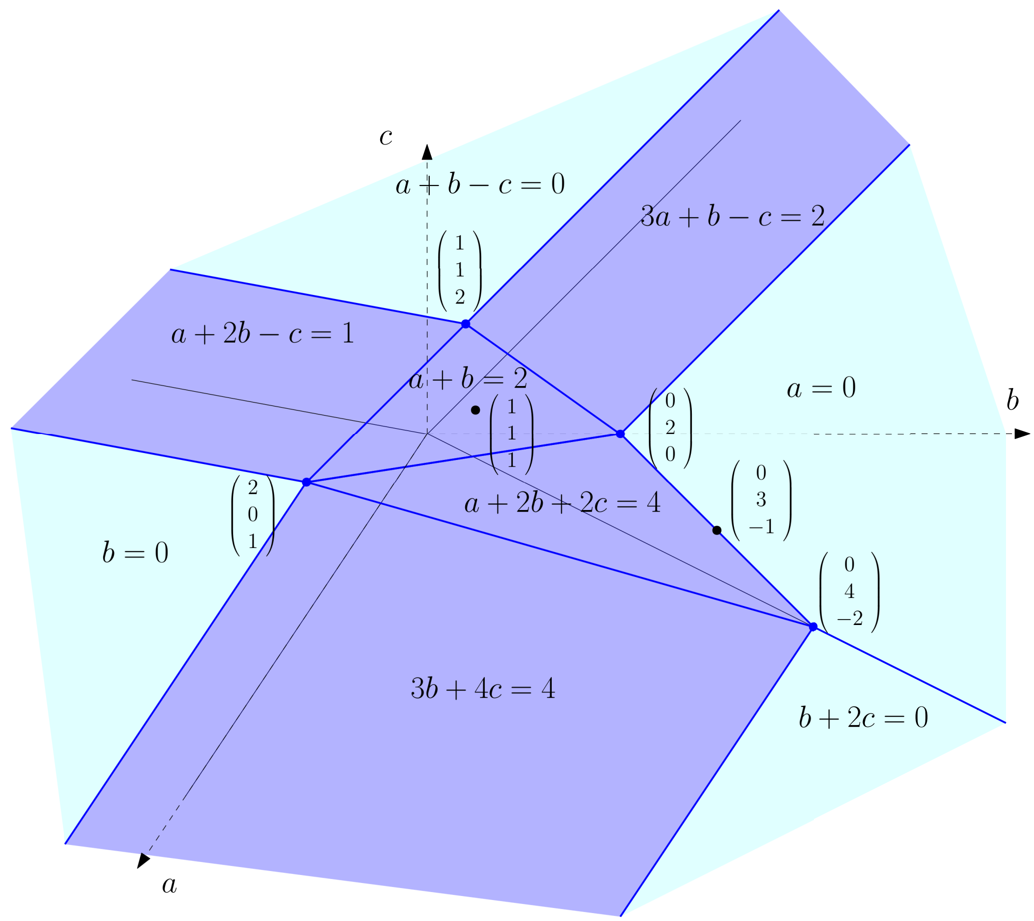


Figure 5.13: $LNP(f)$.

Finally, we get the plumbing graph $\Gamma_{\widetilde{\mathscr{S}}}(E)$ represented in Figure 5.19. It can be reduced with blowing-downs to the graph of figure 5.20. Note that this graph is in normal form in the sense of [47].

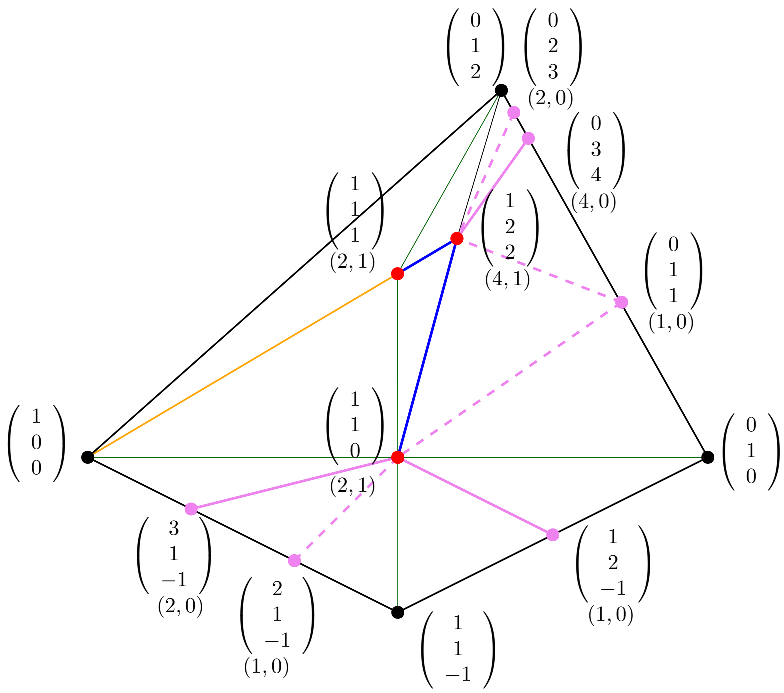


Figure 5.14: Refinement of $\mathcal{F}_{fg}$.

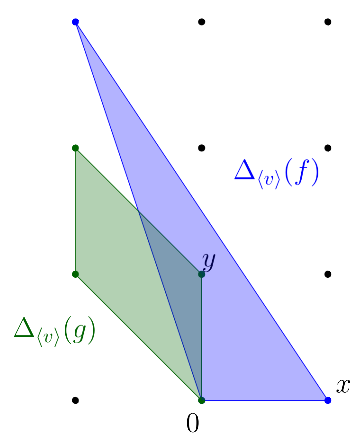


Figure 5.15: Translations of $\Delta_{\langle v\rangle}(f)$ and $\Delta_{\langle v\rangle}(g)$.

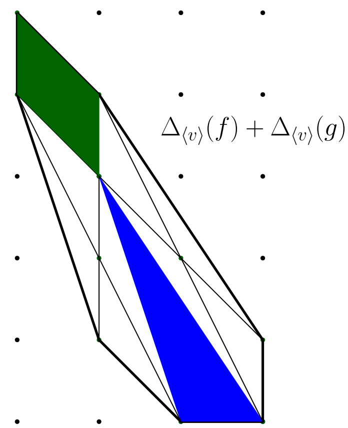


Figure 5.16: The Minkowski sum $\Delta_{\langle v \rangle}(f) + \Delta_{\langle v \rangle}(g)$.

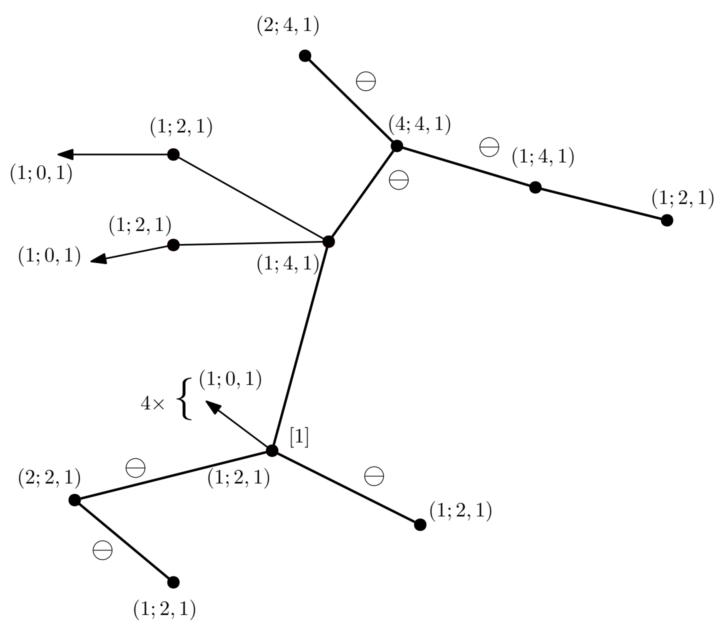


Figure 5.17: The configuration $\mathscr{C}_{tot}$.

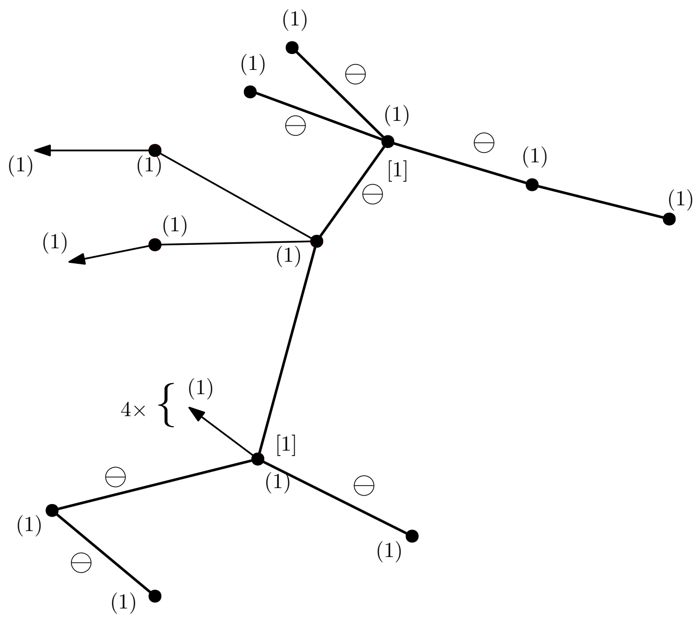


Figure 5.18: The graph $\Gamma^{\mu}(E_{tot})$.

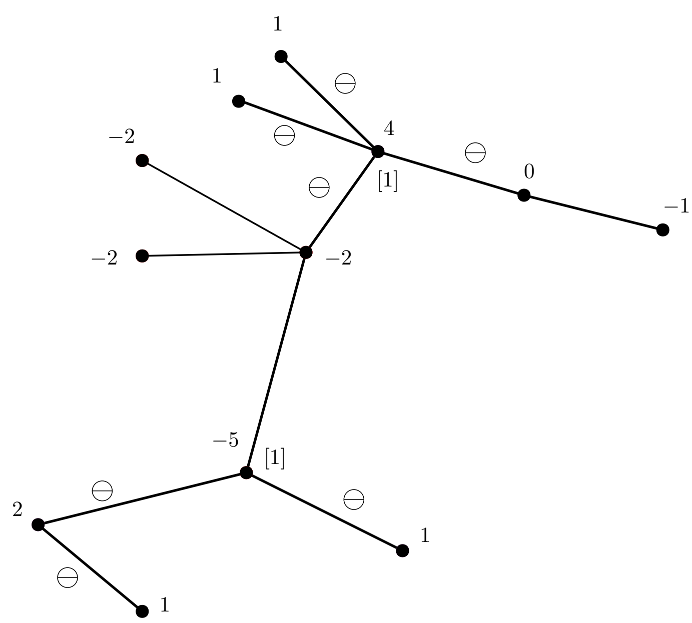


Figure 5.19: The graph $\Gamma_{\widetilde{\mathscr{S}}}(E)$.

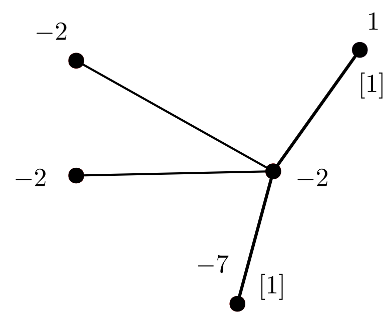


Figure 5.20: The equivalent normal graph.

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Δ_Φ Discriminant locus of Φ, page 75

Δ_v Face of $\mathrm{LNP}(f)$ associated to the vector v, page 59

$\mathbb{D}, \mathbb{D}_f, \mathbb{D}_0, \mathbb{D}_g$, page 85

D_δ Disc of radius δ in $\mathbb{C}$, page 71

$\mathscr{E}(\Gamma)$ Set of edges of the graph Γ, page 35

e_i Self-intersection decoration of a surface in the dual graph, page 43

e_p Edge of $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ corresponding to the double point p, page 99

F Representative of the Milnor fiber, page 80

f_Δ Truncation of f relatively to the face Δ of $\mathrm{LNP}(f)$, page 61

$\mathcal{F}$ A fan, page 52

$|\mathcal{F}|$ Support of the fan $\mathcal{F}$, page 52

$\mathcal{F}_f$ The fan associated to the funciton f, page 59

$\mathcal{F}_P$ Fan associated to the compact polyhedron P, page 69

$f_{\tilde{X}}, g_{\tilde{X}}$ Pullbacks of f and g to $\tilde{X}$, page 87

g A companion of f, page 75

$[g_i]$ Genus decoration of a surface in the dual graph, page 43

$\tilde{g}$ Pullback of the function g to $\widetilde{\mathscr{S}} \cap \pi^{-1}(\overline{U})$, page 111

$\mathscr{G}(\Gamma, (\mathbf{n_v}, \mathbf{n_e}))$ Set of equivalence classes of cyclic coverings of Γ with data $(\mathbf{n_v}, \mathbf{n_e})$, page 37

Γ Topological realization of the graph Γ, page 35

$\Gamma(E)$ Dual graph of the collection E, page 41

$\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ Decorated dual graph of $\mathscr{C}_{tot}$, page 99

$\Gamma_{\widetilde{\mathscr{S}}}(E)$ Plumbing dual graph of a simple configuration of surfaces, page 42

$\overset{\star}{\Gamma}((N_{\mathscr{S}} \circ K)^*(\mathscr{C}_{tot}))$ The decorated graph associated to the configuration $(N_{\mathscr{S}} \circ K)^*(\mathscr{C}_{tot})$, page 108

$\Gamma^\mu(E_{tot})$ Dual graph of E_{tot} decorated with the multiplicities of $\tilde{g}$, page 113

H^{str} Strict part of H, page 88

$h_v(f)$ Height of LNP(f) relatively to the vector v, page 59

$i(\Delta)$ Number of interior points of the face Δ of dimension 2, page 63

K Local complexification morphism of $\widetilde{\mathscr{S}_k}$, page 98

k_0 Smallest large enough integer k for Φ, page 78

κ_p , page 90

$\kappa_p^{\ominus}$, page 94

$\kappa_p^{\oplus}$, page 93

$\overline{\kappa_p}$, page 91

$\overline{\kappa_p^{\ominus}}$, page 95

$\overline{\kappa_p^{\oplus}}$, page 93

LNP(f) The local Newton polyhedron of the germ of function f, page 57

$l(\Delta)$ Integral length of the face Δ of dimension 1, page 63

$l(u)$ Integral length of the element u of a lattice, page 47

M The dual lattice of N, page 48

M_Γ Graph manifold associated to the graph Γ, page 38

M_σ Dual lattice of N_σ, page 50

m_i, n_i Multiplicities of $f_{\tilde{X}}$ and $g_{\tilde{X}}$ along the irreducible component D_i of $\mathbb{D}$, page 87

N Lattice of weights, page 47

$N_\mathbb{R}$ Vector space associated to the lattice N, page 47

N_σ Lattice generated by elements of σ, page 50

n_C Number of connected components of $(N_{\mathscr{S}} \circ K)^{-1}(C)$, page 108

$N_{\mathscr{S}}$ The normalization morphism of $\widetilde{\mathscr{S}_k}$, page 90

$\mathcal{O}_{X,x}$ The local ring of the variety X at the point x, page 23

O_Δ The orbit associated to the face Δ, page 59

O_σ Orbit associated to the cone σ, page 53

Φ The germ of function (f, g), page 75

Π The total modification of $\mathscr{S}_k$, page 109

π Resolution map of $\overline{\mathscr{S}}$, page 109

$\Pi_{\mathcal{F}}$ The modification of X_σ associated to the refinement $\mathcal{F}$ of σ, page 60

$\pi_{\overline{p}}$ resolution of $(\overline{\mathscr{S}}, \overline{p})$, page 110

$\Pi_{\mathcal{F}',\mathcal{F}}$ Modification of $X_{\mathcal{F}}$ associated to the refinement $\mathcal{F}'$, page 56

$\overline{p}$ A preimage of the double point p by $(N_{\mathscr{S}} \circ K)$, page 110

$p_{\mathscr{E}}$ Map of edges, page 35

$p_{\mathscr{V}}$ Map of vertices, page 35

ρ Function defining the origin in X, page 71

r_X Modification of X adapted to Φ, page 85

$r_{\mathscr{S}}$ Restriction of r_X to $\widetilde{\mathscr{S}_k}$, page 88

$\sigma^\perp$ Orthogonal of the cone σ, page 48

$\sigma^\vee$ Dual cone of σ, page 48

$\mathring{\sigma}$ The relative interior of the cone σ, page 49

$\mathrm{Sing}(X)$ Singular locus of X, page 24

$\mathrm{Sm}(X)$ Smooth locus of X, page 24

$\mathscr{S}_k$, page 79

$\widetilde{\mathscr{S}_k}$, page 88

$\widetilde{\mathscr{S}_k}^N$ Normalization of $\widetilde{\mathscr{S}_k}$, page 90

$\overline{\mathscr{S}}$ Local complexification of $\widetilde{\mathscr{S}_k}^N$, page 98

$\widetilde{\mathscr{S}}$ Resolution of $\overline{\mathscr{S}}$, page 109

$\mathrm{Supp}(f)$ Support of the function f, page 57

S_σ The semigroup associated to the cone σ, page 50

S_ε Sphere in X of radius ε centered at the origin, page 71

$\tau \prec \sigma$ τ is a proper face of σ, page 49

$\tau \preceq \sigma$ τ is a face of σ, page 49

τ_Δ Cone associated to the face Δ, page 59

$\mathcal{T}_N$ Torus associated to the lattice N, page 50

$U_p^{\mathbb{C}}$, page 91

$U_p^{\mathbb{C}}$, page 93

$U_p^{\mathbb{C}}$, page 95

$\overline{U_p}$ Neighbourhood of a point in $\overline{\mathscr{S}}$, page 98

$\widetilde{U}$ Neighbourhood of a point in $\widetilde{\mathscr{S}}$, page 111

U_k A small enough neighbourhood for k, page 78

U_p , page 90

$V(f)$ Zero locus of the function f, page 71

$\mathscr{V}(\Gamma)$ Set of vertices of the graph Γ, page 35

$V(P)$ Volume of the polytope P, page 69

$V(P_1, \cdots, P_n)$ Mixed volume of the polytopes $P_1 \cdots, P_n$, page 69

v_C Vertex of $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ corresponding to the curve C, page 99

$X_\sigma(N_\sigma)$ The intrinsic variety associated to the cone σ, page 51

X Ambient 3-dimensional germ of complex analytic variety, page 71

$X_\mathcal{F}$ The toric variety associated to the fan $\mathcal{F}$, page 53

X_σ The affine variety associated to the cone σ, page 50

X_ε Open ball in X of radius ε centered at the origin, page 71

Z_k , page 77

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