Université de Lille

Laboratoire Paul Painlevé

École Doctorale Sciences Pour l'Ingénieur

THÈSE

Pour obtenir le grade de docteur délivré par

L'Université de Lille

Spécialité doctorale : Mathématiques

présentée et soutenue publiquement par

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le 5 Novembre 2019

Analyse stochastique et inférence statistique des solutions d'équations stochastiques dirigées par des bruits fractionnaires gaussiens et non gaussiens

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Remerciements

J'ai le plaisir de commencer ce manuscrit par adresser quelques remerciements. Ma première pensée est destinée à mon directeur de thèse Ciprian Tudor, qui a accepté il y'a trois ans d'encadrer ma thèse. Je commence par lui exprimer ma gratitude la plus profonde pour sa confiance, son optimisme et son encadrement tout au long de ce travail sans qui, il n'aurait jamais abouti. Il m'a initié au calcul de Malliavin et au monde de la recherche académique en me proposant des problématiques riches et stimulantes. S'il a toujours su être disponible et patient pour répondre à mes questions, me conseiller et m'aider, il a également été capable de me recadrer et me motiver quand il était nécessaire. Je n'oublie pas de citer ses nombreuses qualités humaines dont j'ai toujours été très admirative. Je retiendrai en particulier beaucoup de bienveillance, de compréhension, de générosité et de sympathie. Travailler sous sa direction durant ces trois années a été une expérience formidable dont je me souviendrai toujours avec beaucoup de nostalgie. Je lui adresse encore une fois mes remerciements les plus sincères.

J'aimerais également remercier Ivan Nourdin et Francesco Russo qui ont accepté de rapporter ma thèse. C'est un grand honneur qu'ils me font et je suis fière de pouvoir les compter parmi les membres de mon jury. Je tiens également à exprimer ma reconnaissance à Antoine Ayache, Anne Estrade et Eva Löcherbach qui ont accepté de faire partie de mon jury.

Je remercie vivement Soledad Torres qui m'a accueilli durant mes deux visites à l'université de Valparaiso au Chili. C'était un grand plaisir pour moi d'avoir pu collaborer avec elle et son étudiant Hector que je remercie également pour les discussions mathématiques que nous avons pu partager à Valparaiso et à Lille.

Je tiens à remercier aussi toutes les personnes qui m'ont soutenu directement ou indirectement pendant mon doctorat au laboratoire Paul Painlevé. Merci à l'ensemble des membres du laboratoire de maths pour leur gentillesse, le soutien logistique ainsi que pour la très bonne ambiance que j'ai toujours trouvé aux bâtiments M2 et M3.

Je pense aux doctorants (certains devenus docteurs et pour la plupart des bons amis) que j'ai rencontré à Lille et ailleurs. Je les remercie d'abord pour les bons moments passés ensemble et de l'intérêt que certains ont porté à mon travail. Je pense en particulier à François, Mariagiulia, Tania, Alexandre, Mira, Obayda, Joanna, Hector Fadil, Ahmed, Aya, Min, Jacques, Octave, Yassine et Mohammed. Un mot en particulier pour Mira avec qui j'ai eu le plaisir de travailler durant sa visite à Lille. J'espère que nos collaborations seront nombreuses dans le futur. Au delà des échanges mathématiques stimulants et passionnants, le bureau douze a été le témoin de récits d'éxpériences, de nombreux "tu vas y arriver" qui dissipent de nombreux moments de doutes, et de chaleureux et riches échanges sur différents sujets de la vie ! Tout ceci a contribué à rompre quelque peu la solitude de l'expérience de la thèse et à en faire une expérience humaine exceptionnelle.

J'ai ensuite une pensée pour ma grande soeur de thèse Marwa. Je la remercie pour tous les beaux moments passés à Lille et en Tunisie au sein de sa petite famille, que je remercie également pour leur gentillesse et leur hospitalité. Merci également à Tania pour son amitié précieuse, son écoute sans frontières et sa patience. Je garde un excellent souvenir de nos voyages au Chili, à Lille et à Paris. J'ai une pensée également pour tous les moments partagés avec mes amis : Ghizlane et Alaa. Je tenais également à remercier Mehdi et Issa pour leur soutien les derniers mois de la thèse.

Je tiens ensuite à témoigner ma reconnaissance à ma famille, qui ont toujours cru en moi et qui m'ont soutenu dans ce projet de thèse et tout au long de ces nombreuses années d'étude.

Je tiens à exprimer à ma chère Amima Nora toute la tendresse et l'affection que je lui porte et la remercie d'avoir toujours su, à différentes étapes de ma vie, être une force et un grand appui moral. Je la remercie d'avoir géré mes états d'âmes et mes baisses de motivation récurrentes avec tendresse et optimisme constants.

Merci à ma mère pour son amour, sa morale, d'être pour moi un exemple de détermination et de patience.

Je garde le mot de la fin pour mon père. Son soutien indéfectible et ses encouragements sont pour moi les piliers fondateurs de la femme que je suis devenue et de ce que j'ai pu réaliser. Je le remercie de son soutien sans faille, de sa bienveillance inconditionnelle, de m'avoir transmis la soif d'apprendre, de m'avoir encouragé à entreprendre, et d'avoir toujours veillé à m'enseigner l'intégrité et la persévérance. Pour tout ce qui est dit ici et surtout pour tout le reste, Merci !

À *mon père Said,*
À *la mémoire de mon grand-père El Khadir.*

Résumé. Cette thèse est consacrée à l'étude des solutions d'équations différentielles stochastiques dirigées par des bruits fractionnaires gaussiens et non gaussiens. Les bruits fractionnaires considérés sont modélisés par les processus d'Hermite qui forment une famille de processus stochastiques autosimilaires, à accroissements stationnaires et qui sont représentés par des intégrales stochastiques multiples de Wiener-Itô. Dans un premier travail, nous étudions la solution de l'équation stochastique de la chaleur linéaire dirigée par un champ d'Hermite. Nous établissons les différentes propriétés de la solution mild et analysons en particulier sa distribution en probabilité dans le cas non gaussien. La deuxième partie de cette thèse concerne le comportement asymptotique des solutions d'équations stochastiques lorsque l'exposant de Hurst H qui caractérise le bruit fractionnaire converge vers ses valeurs limites. Nous étudions en particulier le comportement en loi de la solution de l'équation de la chaleur stochastique dirigée par un champ d'Hermite et le processus d'Ornstein-Uhlenbeck type Hermite qui est la solution de l'équation de Langevin dirigée par un processus d'Hermite. Dans la dernière partie de ce travail, nous analysons le comportement asymptotique en loi des variations généralisées de la solution de l'équation stochastique des ondes dirigée par un bruit gaussien fractionnaire. Ces résultats ont permis de construire des estimateurs consistants pour l'indice d'autosimilarité H.

Mots clefs : Analyse stochastique, calcul de Malliavin, intégrales stochastiques multiples de Wiener-Ito, théorèmes limites, processus d'Hermite, équation de la chaleur, équation des ondes.

Abstract This doctoral thesis is devoted to the study of the solutions of stochastic differential equations driven by additive Gaussian and non-Gaussian noises. As a non-Gaussian driving noise, we use the Hermite processes. These processes form a family of self-similar stochastic processes with stationary increments and long memory and they can be expressed as multiple Wiener-Itô integrals. The class of Hermite processes includes the well-known fractional Brownian motion which is the only Gaussian Hermite process, and the Rosenblatt process. In a first chapter, we consider the solution to the linear stochastic heat equation driven by a multiparameter Hermite process of any order and with Hurst multi-index H. We study the existence and establish various properties of its mild solution. We discuss also its probability distribution in the non-Gaussian case. The second part deals with the asymptotic behavior in distribution of solutions to stochastic equations when the Hurst parameter converges to the boundary of its interval of definition. We focus on the case of the Hermite Ornstein-Uhlenbeck process, which is the solution of the Langevin equation driven by the Hermite process, and on the case of the solution to the stochastic heat equation with additive Hermite noise. These results show that the obtained limits cover a large class of probability distributions, from Gaussian laws to distribution of random variables in a Wiener chaos of higher order. In the last chapter, we consider the stochastic wave equation driven by an additive Gaussian noise which behaves as a fractional Brownian motion in time and as a Wiener process in space. We show that the sequence of generalized variations satisfies a Central Limit Theorem and we estimate the rate of convergence via the Stein-Malliavin calculus. The results are applied to construct several consistent estimators of the Hurst index.

Keywords : Stochastic analysis, Malliavin calculus, multiple stochastic integrals, limit theorems, Hermite process, stochastic heat equation, stochastic wave equation.

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Introduction et présentation des résultats obtenus

Cette thèse de doctorat d'université, élaborée sous la direction du Professeur Ciprian Tudor, au sein du laboratoire Paul Painlevé de Lille, est consacrée à l'analyse stochastique et à l'inférence statistique des solutions d'équations différentielles stochastiques.

Durant ces dernières années, l'étude des équations aux dérivées partielles stochastiques (EDPS dans la suite), a connu des avancées majeures et est devenue un axe de recherche important de l'analyse stochastique. Depuis le travail pionnier de Walsh en 1986 [108], qui a résolu l'équation des ondes stochastique dirigée par un bruit blanc gaussien, plusieurs auteurs ont travaillé à élargir la classe des bruits, permettant ainsi de tenir compte des situations physiques plus complexes.

Dans ce manuscrit, nous considérons principalement comme bruit aléatoire un processus d'Hermite. Les processus d'Hermite forment une classe de processus stochastiques auto-similaires, à accroissements stationnaires et à mémoire longue, vivant dans un chaos de Wiener. Ce sont des candidats potentiels pour de nombreuses applications pratiques, et ont été intensivement étudiés dans les dernières décennies. La classe des processus d'Hermite contient le mouvement brownien fractionnaire et le processus de Rosenblatt. Nous nous focalisons essentiellement sur deux modèles typiques d'EDPS : l'équation de la chaleur et l'équation des ondes avec un bruit additif. L'outil central pour étudier les solutions de ces équations est le calcul de Malliavin. Connu aussi sous le nom de calcul des variations stochastiques, c'est un calcul différentiel en dimension infinie introduit par Paul Malliavin en 1976 et qui a servi originellement à l'étude de la régularité des solutions d'équations différentielles stochastiques. Le cadre d'application du calcul de Malliavin a connu un développement important dans les décennies qui ont suivi. Un trait majeur de ce calcul est qu'il offre la possibilité d'intégrer par rapport à des processus gaussiens qui ne sont pas des semimartingales comme le mouvement brownien fractionnaire. Ceci a permis de résoudre et d'étudier les équations stochastiques fractionnaires, qu'elles soient différentielles ou aux dérivées partielles. En outre, cela a naturellement conduit à construire et étudier des estimateurs dans ces modèles.

Récemment, le calcul de Malliavin a connu de nouveaux champs d'applications : les théorèmes limites et l'inférence statistique. Ceci est dû principalement à la découverte du théorème du quatrième moment [76], qui permet de caractériser la normalité asymptotique d'une suite d'intégrales multiples à l'aide du calcul de Malliavin. Ce lien entre le calcul de Malliavin et la convergence en loi constitue un outil principal pour les preuves dans ce travail.

Ce manuscrit est divisé en trois parties. Dans une première partie, nous étudions la solution de l'équation de la chaleur stochastique dirigée par un bruit fractionnaire non gaussien que l'on modélise par la version multidimensionnelle des processus d'Hermite. Dans une seconde partie, nous nous intéressons au comportement asymptotique des solutions d'équations à la fois différentielles et aux dérivées partielles fractionnaires lorsque le paramètre fractionnaire converge vers ses valeurs limites. La troisième et dernière partie de cette thèse traite une problématique d'inférence statistique relative à l'équation des ondes dirigée par un bruit fractionnaire gaussien.

Ces parties sont composées des quatre articles suivants :

- ▷ M. Slaoui and C. A. Tudor (2017) : On the linear stochastic heat equation with Hermite noise, *Infinite Dimensional Analysis, Quantum Probability and Related Topics.*
- ▷ M. Slaoui and C. A. Tudor (2018) : Limit behavior of the Rosenblatt Ornstein-Uhlenbeck process with respect to the Hurst index. *Theory of Probability and Mathematical Statistics, 1(98), 173-187.*
- ▷ M. Slaoui and C. A. Tudor (2019) : Behavior with respect to the Hurst index of the Wiener Hermite integrals and application to SPDEs, *Journal of Mathematical Analysis and Applications.*
- ▷ R. Shevchenkon, M. Slaoui and C.A. Tudor (2019) : Generalized k-variations and Hurst parameter estimation for the fractional wave equation via Malliavin calculus, *Journal of Statistical Planning and Inference.*

Dans cette introduction, nous commençons par donner une brève description des principaux objets considérés. Nous inclurons également un résumé des articles qui constituent cette thèse en les situant d'abord dans leur contexte mathématique, puis en explicitant leur contenu et en donnant les idées générales des preuves. Dans la suite de cette partie introductive, nous présenterons en détail les résultats obtenus au cours de la préparation de la thèse. Ces résultats seront présentés sous forme d'articles tels qu'ils ont été soumis ou publiés dans des revues internationales avec comité de lecture.

0.1 Préliminaires

0.1.1 Éléments de Calcul de Malliavin

Puisque nos travaux sont tous fortement basés sur le calcul de Malliavin, nous commencerons par rappeler ses notions fondamentales. Nous renvoyons aux livres [66] et [74] qui font références sur le sujet pour une présentation plus détaillée.

Considérons un espace de probabilité $(\Omega, \mathfrak{F}, \mathbb{P})$ et un espace de Hilbert séparable $\mathcal{H}$ muni d'un produit scalaire qu'on note par $\langle ., . \rangle_{\mathcal{H}}$. Nous notons $\mathcal{H}^{\otimes q}$ (respectivement $\mathcal{H}^{\odot q}$) la q ème puissance tensorielle (respectivement symétrique) de $\mathcal{H}$ munie de la norme $\frac{1}{\sqrt{q!}} \| \cdot \|_{\mathcal{H}^{\otimes q}}$.

▶ Processus isonormal gaussien

Définition 1 (Processus isonormal)**.** *Un processus stochastique W défini par $W = \{W(h), h \in \mathcal{H}\}$ est un processus gaussien isonormal gaussien si c'est une famille de variables aléatoires gaussiennes centrées tel que :*

$$\mathbb{E}\left(W(h)W(g)\right) = \langle h, g\rangle_{\mathcal{H}} \quad \forall h, g \in \mathcal{H}.$$

▶ Opérateur de dérivation

Soit $C_p^\infty(\mathbb{R}^n)$ l'ensemble des fonctions infiniment dérivables $f : \mathbb{R}^n \to \mathbb{R}$ et dont les dérivées sont à croissance polynomiale. Notons $\mathcal{S}$ la classe des variables aléatoires qui s'écrivent sous la forme :

$$F = f\left(W(h_1), \ldots, W(h_n)\right), \tag{1}$$

avec $f \in C_p^\infty(\mathbb{R}^n), h_1, \ldots, h_n \in \mathcal{H}$ et $n \geqslant 1$.

Définition 2 (Opérateur de dérivation)**.** *L'opérateur de dérivation au sens de Malliavin ou dérivée de Malliavin d'une variable aléatoire qui s'exprime sous la forme donnée par (1) est la variable aléatoire à valeur dans $\mathcal{H}$ définie par :*

$$DF = \sum_{k=1}^{n} \frac{\partial^k f}{\partial x^k}\left(W(h_1), \ldots, W(h_n)\right) h_k. \tag{2}$$

▶ Opérateur de divergence

Soient $p, q \in \mathbb{N}^*$. On note $D^{p,q}$ la fermeture de $\mathcal{S}$ par rapport à la norme $\|.\|_{D^{p,q}}$ définie par :

$$\|F\|_{D^{p,q}} = \left(\mathbb{E}(|F|^q) + \sum_{i=1}^{p} \mathbb{E}\left(\|D^i F\|^q_{\mathcal{H}^{\otimes i}}\right)\right)^{\frac{1}{q}}.$$

Définition 3 (Opérateur de divergence)**.** *L'opérateur de divergence est défini comme adjoint de l'opérateur de dérivation sur $L^2(\Omega, \mathcal{H})$ à valeurs dans $L^2(\Omega)$ tel que :*

- ▷ *Le domaine de δ, noté Domδ, est l'ensemble des variables aléatoires $u \in L^2(\Omega, \mathcal{H}) \in$ tel que :*

  $$\mathbb{E}\left(< DF, u >_{\mathcal{H}}\right) \leqslant c_u \sqrt{\mathbb{E}(F^2)}, \quad \forall F \in D^{1,2},$$

  où c_u est une constante qui dépend de u.

- ▷ *Si $u \in$Domδ alors $\delta(u)$ est la variable aléatoire dans $L^2(\Omega)$ caractérisée par cette relation de dualité :*

  $$\mathbb{E}(F\delta(u)) = \mathbb{E}\left(< DF, u >_{\mathcal{H}}\right), \quad \forall F \in D^{1,2}.$$

▶ Polynômes d'Hermite et Chaos de Wiener

Définition 4 (Polynômes d'Hermite)**.** *Les polynômes d'Hermite d'ordre $q \in \mathbb{N}$, sont définis comme suit :*

$$H_q(x) = (-1)^q \exp\left(\frac{x^2}{2}\right) \frac{\mathrm{d}^q}{\mathrm{d}x^q}\left(\exp\left(-\frac{x^2}{2}\right)\right), \ x \in \mathbb{R}. \tag{3}$$

On a en particulier : $H_0(x) = 1, H_1(x) = x, H_2(x) = x^2 - 1, H_3(x) = x^3 - 3x, \ldots.$

Ces polynômes apparaissent dans le développement en série de Taylor de la fonction $F(x,t) = e^{tx - \frac{t^2}{2}}$ de cette manière : $H_q(x) = \frac{\partial^q F(x,t)}{\partial t^q} \mid_{t=0}$.

Ces polynômes vérifient les propriétés suivantes :

1. $H_q'(x) = qH_{q-1}(x)$ et $H_{q+1} = xH_q(x) - qH_{q-1}(x)$,
2. la famille $\left(\frac{1}{\sqrt{q}} H_q(x)\right)_{q \in \mathbb{N}}$ est une base orthonormale de $L^2\left(\mathbb{R}, \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx\right)$,
3. soit (U, V) un vecteur gaussien tel que $U, V \sim \mathcal{N}(0,1)$, alors on a pour tout $p, q \in \mathbb{N}$,

$$\mathbb{E}\left[H_q(U)H_p(V)\right] = \begin{cases} q!\mathbb{E}\left[UV\right]^q & \text{si} \quad p = q, \\ 0 & \quad\quad \text{sinon.} \end{cases} \tag{4}$$

Définissons à présent le q ème chaos de Wiener.

Définition 5 (Chaos de Wiener)**.** *Soient $q \geq 0$ un entier, et W un processus gaussien isonormal sur H. Le q ème chaos de Wiener $\mathcal{H}_q$ associé à W est la fermeture dans $L^2(\Omega)$ du sous-espace vectoriel engendré par $\{H_q(W(h)); h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$.*

Notons que pour $q = 0$, $\mathcal{H}_0$ représente l'espace des fonctions constantes et que pour $q = 1, \mathcal{H}_1$ est l'espace gaussien défini par $\mathcal{H}_1 = \{W(h), h \in \mathcal{H}\}$. De plus l'orthogonalité des polynômes d'Hermite pour $p \neq q$ donnée par (4), implique l'orthogonalité des chaos de Wiener $\mathcal{H}_p$ et $\mathcal{H}_q$.

0.1.2 Intégrales stochastiques multiples

Nous introduisons dans cette partie les intégrales multiples et nous donnons leur principales propriétés. Considérons un processus gaussien isonormal W défini sur un espace de Hilbert $\mathcal{H}$.

Soit $q \geqslant 1$ et introduisons l'application linéaire I_q définie par :

$$I_q(f^{\otimes q}) = H_q(W(f)), \quad \text{pour} \quad f \in \mathcal{H} \quad \text{tel que} \quad \|f\|_{\mathcal{H}} = 1.$$

On peut montrer que l'application I_q se prolonge en une isométrie linéaire entre l'espace de Hilbert $\mathcal{H}^{\odot q}$ muni de la norme $\frac{1}{\sqrt{q!}}\|\cdot\|_{\mathcal{H}^{\otimes q}}$ et le chaos de Wiener $\mathcal{H}_q$ d'ordre q, muni de la norme de $L^2(\Omega)$.

Définition 6. *[Intégrales multiples d'ordre q] Pour tout $f \in \mathcal{H}^{\odot q}$, $I_q(f)$ est l'intégrale multiple d'ordre q associée au processus isonormal gaussien W.*

Soient $p, q \geqslant 1$, $f \in \mathcal{H}^{\otimes q}$ et $g \in \mathcal{H}^{\otimes p}$. La relation qui lie polynômes d'Hermite et intégrales multiples est à l'origine de la formule d'isométrie suivante :

$$\mathbb{E}\Big(I_p(f)I_q(g)\Big) = \begin{cases} q!\langle \tilde{f}, \tilde{g}\rangle_{\mathcal{H}^{\otimes q}} & \text{if } p = q \\ 0 & \text{sinon.} \end{cases} \tag{5}$$

où $\tilde{f}$ désigne la fonction symétrisée de f définie par $\tilde{f}(x_1, \ldots, x_q) = \frac{1}{q!}\sum_{\sigma\in\Sigma_q} f\Big(x_{\sigma(1)}, \ldots, x_{\sigma(q)}\Big)$ avec Σ_q l'ensemble des permutations de $\{1, \ldots, q\}$.

On a alors en particulier $\mathbb{E}\left(I_q(f)^2\right) = q!\|f\|^2_{\mathcal{H}^{\otimes q}}$.

Les moments d'une intégrale multiple d'ordre q vérifient cette propriété d'hypercontractivité :

Proposition 1 (Propriété d'hypercontractivité). *Si $F = I_k(f)$ avec $f \in \mathcal{H}^{\otimes k}$ alors on a :*

$$\mathbb{E}\left(|F|^p\right) \leq (p-1)^{\frac{k}{2}}\left(\mathbb{E}\left(F^2\right)\right)^{\frac{p}{2}}, \tag{6}$$

pour tout $p \geq 2$.

Nous aurons besoin à plusieurs reprises dans les chapitres qui suivent de multiplier deux intégrales multiples. La formule de multiplication dit que le produit de deux intégrales multiples s'écrit comme une somme finie d'intégrales multiples. Afin de l'énoncer, commençons par rappeler la notion de contractions.

Contractions : Soit $(e_k)_{k\geqslant 1}$ une base orthonormale de $\mathcal{H}$. la contraction d'ordre $r = 1, \ldots, p\wedge q$ de $f \in \mathcal{H}^{\otimes p}$ et $g \in \mathcal{H}^{\otimes q}$ notée par $f \otimes_r g$ est l'élément de $\mathcal{H}^{\otimes(p+q-2r)}$, défini comme suit :

$$(f\otimes_r g) \quad = \sum_{j_1,\ldots,j_p=1}^{\infty} < f, e_{k_1}\otimes e_{k_2}\otimes\ldots e_{k_r} >_{\mathcal{H}^{\otimes r}} \otimes < g, e_{k_1}\otimes e_{k_2}\otimes\ldots e_{k_r} >_{\mathcal{H}^{\otimes r}} .$$

Proposition 2 (Formule de multiplication). *Soient $p, q \geqslant 1$, $f \in \mathcal{H}^{\otimes q}$ et $g \in \mathcal{H}^{\otimes p}$. On a :*

$$I_p(f)I_q(g) \quad = \quad \sum_{r=0}^{p\wedge q} r!\binom{p}{r}\binom{q}{r} I_{p+q-2r}\left(f\tilde{\otimes}_r g\right),$$

où $f\tilde{\otimes}_r g$ est la symétrisée de $f\otimes_r g$.

Proposition 3 (Décomposition en chaos de Wiener). *Toute variable aléatoire $F \in L^2(\Omega)$,de carré intégrable et qui est mesurable par rapport a la tribu engendrée par W, admet la décomposition chaotique suivante :*

$$F = \mathbb{E}(F) + \sum_{q=1}^{\infty} I_q(f_q).$$

De plus cette décomposition est unique.

Loi des intégrales multiples

Caractériser les lois des variables aléatoires qui s'expriment sous formes d'intégrales multiples est une problématique récurrente qui apparaît dans nos travaux.

Les variables aléatoires qui s'écrivent sous la forme d'une intégrale multiple d'ordre 1 étant gaussiennes, il est bien connu que leur loi est caractérisée par la donnée des deux premiers moments (espérance et variance).

La loi des intégrales multiples d'ordre 2 est liée à la notion de cumulants que nous introduisons ci dessous.

Définition 7 (Cumulants)**.** *Le cumulant d'ordre m d'une variable aléatoire $F \in L^m(\Omega)$ pour tout $m \geq 1$, est donnée par :*

$$k_m(F) := (-i)^n \frac{\partial^n}{\partial t^n} \ln \mathbb{E}(e^{itF})|_{t=0}.$$

Les moments et cumulants d'une variable aléatoire sont liés par la formule suivante :

$$k_m(F) = \sum_{\sigma=(a_1,..,a_r)\in\mathcal{P}([n])} (-1)^{r-1}(r-1)!\mathbb{E}\left(F^{|a_1|}\right)\ldots\mathbb{E}\left(F^{|a_r|}\right), \tag{7}$$

où pour $d \geqslant 2$ et $a = (a_1, \ldots, a_d) \in \mathbb{N}^d$ on note $|a| = \sum_{i=1}^d a_j$ et où $\mathcal{P}(b)$ désigne l'ensemble des partitions de b.

Notons que si $\mathbb{E}(F) = 0$, nous aurons : $k_1(F) = 0, k_2(F) = \mathbb{E}F^2, k_3(F) = \mathbb{E}F^3$ et $k_4(F) = \mathbb{E}F^4 - 3(\mathbb{E}F^2)^2$.

Rappelons ces résultats fondamentaux démontrés dans la proposition 2.7.13 dans [65] qui caractérisent la loi des intégrales multiples d'ordre 2 :

- ▷ Si F est un variable aléatoire appartenant au second chaos de Wiener ($F = I_2(f)$, avec $f \in L^2(\mathbb{R}^2)$), alors sa loi est entièrement caractérisée par ses cumulants, ou de manière équivalente par ses moments. Ainsi, si F et G sont deux variables aléatoires qui appartiennent au second chaos de Wiener, alors on a l'équivalence suivante :

$$F \overset{loi}{=} G \iff k_m(F) = k_m(G) \quad \forall m \geqslant 1.$$

- ▷ Si $G = I_2(f)$ avec $f \in L^2(\mathbb{R}^2)$ symétrique, son cumulant d'ordre m s'exprime par :

$$k_m(G) = 2^{m-1}(m-1)! \int_{\mathbb{R}^m} du_1 \ldots du_m f(u_1, u_2) f(u_2, u_3) \ldots f(u_{m-1}, u_m) f(u_m, u_1).$$

Soulignons le fait que cette caractérisation par les moments (ou cumulants) des intégrales multiples d'ordre $q = 1$ et $q = 2$ n'est plus valable lorsque $q \geqslant 3$. Nous renvoyons pour plus de détails à [91], la section 2.7.4 de [65] et le chapitre VI de [46].

0.1.3 Théorèmes centraux limites pour les intégrales multiples

Le théorème du quatrième moment, connu aussi sous le nom *Fourth Moment Theorem*, joue un rôle clé dans nos travaux. Il stipule que pour qu'une suite normalisée d'intégrales multiples converge en loi vers une gaussienne centrée réduite, il faut et il suffit que son moment quatrième tende vers 3, (ou de manière équivalente que le cumulant d'ordre 4 converge vers 0). Ce théorème, originellement dû à Nualart et Peccati [76], a été étendu dans [75] et [69].

Théorème 1. *Soit $n \geq 2$. Considérons une suite de variables aléatoires qui appartient au n ème chaos de Wiener $\{F_k = I_n(f_k), k \geq 1\}$ avec $f_k \in \mathcal{H}^{\odot n}$ pour tout $k \geq 1$ et telle que :*

$$\mathbb{E}\left[F_k^2\right] = n!\|f_k\|^2_{\mathcal{H}^{\otimes n}} \to_{k\to\infty} \sigma^2.$$

Nous avons équivalence entre les points suivants :

1. *quand* $k \to \infty$*, la suite* $(F_k)_{k\geq 0}$ *converge en loi vers* $\mathcal{N}(0, \sigma^2)$*,*
2. $\lim\limits_{k\to\infty} \mathbb{E}\left[F_k^4\right] = 3\sigma^4 \quad (\Leftrightarrow \lim\limits_{k\to\infty} k_4(F) = 0)$*,*
3. *pour tout* $1 \leq l \leq n-1$*, on a* $\lim\limits_{k\to\infty} \|f_k \otimes_l f_k\|_{\mathcal{H}^{\otimes 2(n-l)}} = 0$*,*
4. $\lim\limits_{k\to\infty} \|DF_k\|^2_{\mathcal{H}} \to n\sigma^2$ *dans* $L^2(\Omega)$*.*

Peccati et Tudor dans [77] ont obtenu une version multidimensionnelle de ce théorème que nous citons également :

Théorème 2. *Soient* $d \geqslant 2$ *et* $q_d, \ldots, q_1 \geqslant 1$ *des entiers fixés. Considérons le vecteur*

$$F_n = (F_{1,n}, \ldots, F_{d,n}) = \left(I_{q_1}(f_{1,n}), \ldots, I_{q_d}(f_{d,n})\right),$$

avec $f_{i,n} \in \mathcal{H}^{\odot q_i}$*. Soit* C *une matrice symétrique réelle définie positive et supposons que*

$$\lim_{n\to\infty} \mathbb{E}\left(F_{i,n}F_{j,n}\right) = C(i,j) \text{pour chaque } i,\, j \in \{1, \ldots, d\}. \tag{8}$$

Alors, lorsque n *tend vers* $+\infty$*, les deux conditions suivantes sont équivalentes :*

1. F_n *converge en loi vers* $\mathcal{N}_d(0, C)$*,*
2. *pour tout* $1 \leqslant i \leqslant d$ $F_{i,n}$ *converge en loi vers* $\mathcal{N}(0, C(i,i))$*.*

0.1.4 Processus fractionnaires : Définitions et propriétés

Entrons au cœur de cette thèse en définissant la perturbation stochastique ou bruit, qui apparait dans les équations différentielles stochastiques que nous considérons dans nos travaux. Ce bruit est modélisé par des processus stochastiques fractionnaires que nous présentons dans cette partie. Ces processus fractionnaires appartiennent à la classe plus générale des processus auto similaires à accroissements stationnaires.

Commençons par préciser quelques définitions liées à cette classe de processus auto similaires.

Processus autosimilaires L'étude des processus autosimilaires, c'est à dire invariants en loi par changement d'échelle, présente un intérêt à la fois théorique et pratique. En effet, ils ont connu un véritable essor avec le développement de nombreux modèles stochastiques, dont ceux traités dans ce manuscrit, et ont démontré être des candidats puissants à modéliser de nombreux phénomènes réels. Parmi les nombreux domaines d'application, on peut citer l'hydrologie, les mathématiques financières, le trafic internet et le traitement d'image.

La propriété d'auto similarité se traduit de manière probabiliste comme suit :

Définition 8. *[Auto-similarité] Un processus stochastique* $\{X(t)\}_{t\in\mathbb{R}^+}$ *est auto similaire d'indice* $H > 0$ *si, pour toute constante* $c > 0$*, les deux processus* $\{X(ct)\}_{t\in\mathbb{R}^+}$ *et* $\{c^H X(t)\}_{t\in\mathbb{R}^+}$ *ont les mêmes lois finies dimensionnelles.*

Une classe importante de processus auto-similaires est celle des processus à accroissements stationnaires.

Définition 9. *[Stationarité des accroissements] Un processus stochastique* $\{X(t)\}_{t\in\mathbb{R}^+}$ *a des accroissements stationnaires si pour tout réel* $h > 0$ *, la loi de* $\{X(t+h) - X(h)\}_{t\in\mathbb{R}^+}$ *ne dépend pas de* h *, i-e. :*

$$\{X(t+h) - X(h)\}_{t\in\mathbb{R}^+} \overset{(d)}{=} \{X(t) - X(0)\}_{t\in\mathbb{R}^+},$$

où $\overset{(d)}{=}$ *est l'égalité au sens des lois fini-dimensionnelles.*

Nous utiliserons l'abréviation asas lorsque nous parlerons d'un processus auto-similaire à accroissements stationnaires et nous noterons H-asas lorsque nous voudrons préciser que celui-ci est d'ordre H.

Définition 10 (Mémoire longue)**.** *Définissons pour tout entier $n \geq 1$, la fonction d'auto covariance des accroissements d'un processus X asas, donnée par :*
$r(n) = \mathbb{E}(X_1 - X_0)(X_{n+1} - X_n)$. *On dit que ce processus X*

- ▷ *est à mémoire longue si :*
$$\sum_{n \geqslant 0} |r_n| = \infty,$$
- ▷ *est à mémoire courte lorsque :*
$$\sum_{n \geqslant 0} |r_n| < \infty.$$

Remarque 0.1.1. *On peut vérifier sans trop de difficulté que les propriétés d'auto similarité et de stationnarité des accroissements, impliquent que les processus $(X_t)_{t\in\mathbb{R}^+}$ H-asas qui vérifient $\mathbb{E}(X_1^2) < \infty$*

- ▷ *partagent la même fonction de covariance égale à :*
$$\frac{\mathbb{E}(X_1^2)}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad \forall s, t \geqslant 0,$$
- ▷ *sont à mémoire longue lorsque $H > \frac{1}{2}$ et à mémoire courte lorsque $H \leqslant \frac{1}{2}$. En effet, on a lorsque $n \to +\infty$ et $H \neq \frac{1}{2}$,*
$$r(n) = \mathbb{E}\left(X_1(X_{n+1} - X_n)\right) \sim_{n\to\infty} H(2H-1)n^{2H}\mathbb{E}\left(X_1\right)^2$$

Le représentant le plus connu et le plus étudié des processus asas est le mouvement brownien fractionnaire (mbf). C'est en particulier l'unique processus (à constante multiplicative près) asas gaussien.

Le mouvement brownien fractionnaire Rappelons brièvement la définition et les principales propriétés du mbf. Il a été introduit par Kolmogorov en 1940 dans [50], puis rendu célèbre en 1968 par Mandelbrot et Van Ness dans [57] en l'introduisant dans des modèles financiers.

Définition 11. *Soit $H \in (0, 1]$. Le mouvement brownien fractionnaire $(B^H)_{t\geqslant 0}$ est défini comme un processus centré continu et gaussien, qui a pour fonction de covariance :*

$$R^H(t,s) := \mathbb{E}(B_t^H B_s^H) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), s, t \geqslant 0. \tag{9}$$

Le paramètre H est appelé exposant de Hurst ou paramètre fractionnaire et gouverne les propriétés fondamentales du mbf. Listons ces propriétés :

- ▷ le mbf est H-asas,
- ▷ ses trajectoires sont p.s höldériennes pour tout $\delta \in (0, H)$,
- ▷ La valeur de H caractérise la dépendance de ses accroissements : ils sont positivement corrélés lorsque $H > \frac{1}{2}$ et négativement corrélés lorsque$H < \frac{1}{2}$. De plus, le mbf présente un phénomène de mémoire longue lorsque $H > \frac{1}{2}$.
- ▷ lorsque $H = 1$, le mbf est égal p.s à tZ avec $Z \sim \mathcal{N}(0, 1)$, tandis que lorsque $H = \frac{1}{2}$, on a $R^{\frac{1}{2}}(t, s) = t \wedge s$ et le mbf correspond au mouvement brownien standard,
- ▷ lorsque $H > \frac{1}{2}$, on a $R^H(t, s) = H(2H-1)\int_0^t\int_0^s |u-v|^{2H-2}dudv$,

▷ lorsque $H \neq \frac{1}{2}$, le mbf n'est ni un processus de Markov ni une semi martingale par rapport à sa filtration naturelle.

Il existe de nombreuses représentations du mouvement brownien fractionnaire sur un compact de $\mathbb{R}^+$ ou sur $\mathbb{R}^+$ entier. Donnons ici la représentation dite en moyenne mobile : Soit $(W(t))_{t\in\mathbb{R}}$ le processus de Wiener indexé sur $\mathbb{R}$, la représentation en moyenne mobile du mbf obtenu par Mandelbrot et Van Ness est donnée par :

$$B_t^H = \frac{1}{c_H}\int_{\mathbb{R}}\left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right) dW(s).$$

Le lecteur souhaitant une description plus complète du mbf pourra se référer par exemple aux livres [66] et [74].

Processus d'Hermite Présentons maintenant une autre famille de processus asas, cette fois ci non gaussiens introduits originellement par Taqqu dans [95] et [96].

Soient $q \in \mathbb{N}$ et $H \in (\frac{1}{2}, 1)$. Considérons $\{\xi_n\}$ une suite de variables aléatoires, gaussiennes, centrées et de variance unitaire, telle que :

$$\mathbb{E}(\xi_0\xi_n) = n^{\frac{2H-2}{q}} L(n),$$

où $L : (0,\infty) \to (0,\infty)$ est une fonction variant lentement à l'infini au sens où

$$\forall m > 0, \frac{L(n.m)}{L(n)} \to_{n\to\infty} 1.$$

Soit g une fonction mesurable sur $\mathbb{R}$ telle que $\mathbb{E}(g(\xi_0)) = 0, \mathbb{E}(g(\xi_0)^2) < +\infty$ de rang d'Hermite q , c'est à dire :

$$g(x) = \sum_{l=q}^{+\infty} c_l H_l(x) \quad \text{où} \quad c_l = \frac{1}{l!}\mathbb{E}\left(g(\xi_0)H_l(\xi_0)\right)$$

où H_l le l-ème polynôme d'Hermite.

Le théorème de la limite non centrale démontré dans [95] et [96] dit que le processus

$$\frac{1}{n^H}\sum_{i=1}^{[nt]} g(\xi_j)$$

converge au sens des lois fini-dimensionnelles vers un processus stochastique asas qui appartient au q ème chaos de Wiener, appelé processus d'Hermite.

Définition 12. *[Processus d'Hermite] Le processus d'Hermite $(Z_H^q(t))_{t\in\mathbb{R}^+}$ est défini par :*

$$Z_H^q(t) \quad = \quad c(H,q)\int_{\mathbb{R}^q}\int_0^t \left(\prod_{j=1}^q (s-y_j)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} ds\right) dW(y_1)\dots dW(y_q) \tag{10}$$

où $x_+ = \max(x, 0)$ et $(W(y)_{y\in\mathbb{R}}$ le processus de Wiener.

La constante $c(H,q)$ est choisie de telle sorte que $\mathbf{E}\left(Z_H^q(t)\right)^2 = t^{2H}$:

$$c(H,q) = \left(\frac{H(2H-1)}{q!\beta\left(\frac{1}{2} - \frac{1-H}{q}, \frac{2-2H}{q}\right)^q}\right)^{\frac{1}{2}}.$$

Notons que l'intégrale $\int_0^t$ est interprétée comme $-\int_t^0$ lorsque $t < 0$.

Énonçons quelques propriétés fondamentales des processus d'Hermite :

Propriétés des processus d'Hermite

- $\triangleright$ $(Z_H^q(t))_{t\in\mathbb{R}^+}$ est H-asas.
- $\triangleright$ La covariance des processus d'Hermite est alors la même pour tout $q \geqslant 1$:

$$\mathbb{E}\left(Z_H^q(t)Z_H^q(s)\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad \forall s,t \geqslant 0.$$

- $\triangleright$ Lorsque $q = 1$, le processus d'Hermite n'est rien d'autre que le mbf d'indice de Hurst $H \in (\frac{1}{2}, 1)$. C'est en particulier l'unique processus d'Hermite gaussien.
- $\triangleright$ Comme $H > \frac{1}{2}$, $(Z_H^q(t))_{t\in\mathbb{R}^+}$ est à mémoire longue.
- $\triangleright$ Les processus d'Hermite admettent des trajectoires p.s höldériennes pour tout $\delta \in (0, H)$,
- $\triangleright$ Les moments des processus d'Hermite sont finis et pour tout $p \geqslant 1$, on a $\mathbb{E}|Z_H^q(t)|^p = \mathbb{E}|Z_H^q(1)|^p|t|^{2H}$.
- $\triangleright$ Les processus d'Hermite ne sont plus gaussiens pour $q \geqslant 2$.

Les processus d'Hermite peuvent être considérés comme une généralisation non gaussienne du mbf, au sens où ils partagent avec le mbf ses propriétés fondamentales mais ne sont pas gaussiens pour $q \geqslant 2$. D'un point de vue pratique, ceci fait des processus d'Hermite des candidats pertinents pour des modèles où l'hypothèse gaussienne n'est pas réaliste, voir [95] et [109].

Le deuxième processus d'Hermite le plus étudié après le mbf est le processus de Rosenblatt, obtenu pour $q = 2$.

Définition 13 (Processus de Rosenblatt)**.** *Soit $H \in (\frac{1}{2}, 1)$. Le processus de Rosenblatt est le processus d'Hermite non gaussien d'ordre 2 défini par :*

$$Z_H^2(t) = c(H,2)\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^t \left((s-y_1)_+^{\frac{H}{2}-1}(s-y_2)_+^{\frac{H}{2}-1}ds\right)dW(y_1)dW(y_2). \tag{11}$$

Nous renvoyons vers les travaux de Taqqu [97] et de Tudor [101] pour un exposé plus détaillé sur l'analyse stochastique du processus de Rosenblatt et aux contributions [1, 2, 82] pour l'étude des propriétés de ce processus.

Champs d'Hermite Nous introduirons dans nos travaux la version multidimensionnelle des processus d'Hermite d'ordre $q \geqslant 1$, appelés champs d'Hermite.

Avant de les définir et d'énoncer leurs principales propriétés, précisons quelques conventions de notations lorsqu'on se place dans le contexte multidimensionnel.

Soit $d \in \mathbb{N}^*$ et considérons les vecteurs : $\mathbf{a} = (a_1, a_2, \dots, a_d)$, $\mathbf{b} = (b_1, .., b_d)$, $\boldsymbol{\alpha} = (\alpha_1, .., \alpha_d)$ dans $\mathbb{R}^d$. Nous adopterons les notations suivantes dans la suite de ce manuscrit :

$$\mathbf{ab} = \prod_{j=1}^d a_j b_j, \quad |\mathbf{a}-\mathbf{b}|^{\alpha} = \prod_{i=1}^d |a_i - b_i|^{\alpha_i}$$

$$\mathbf{a}/\mathbf{b} = (a_1/b_1, a_2/b_2, \dots, a_d/b_d), \quad [\mathbf{a},\mathbf{b}] = \prod_{i=1}^d [a_i, b_i], \quad (\mathbf{a},\mathbf{b}) = \prod_{i=1}^d (a_i, b_i),$$

$$\sum_{\mathbf{i}\in[\mathbf{0},\mathbf{N}]} a_{\mathbf{i}} = \sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}\dots\sum_{i_d=0}^{N_d} a_{i_1,i_2,\dots,i_d}, \quad \mathbf{a}^{\mathbf{b}} = \prod_{i=1}^d a_i^{b_i}, \ \mathbf{a} < \mathbf{b} \text{ avec } a_1 < b_1, a_2 < b_2, \dots, a_d < b_d$$

Définition 14. *Soient $q \geq 1$ et $\mathbf{H} = (H_1, H_2, \dots, H_d) \in (\frac{1}{2}, 1)^d$. Les* champs d'Hermite d'ordre q, *s'expriment comme une intégrale multiple d'ordre q par rapport au drap brownien $(W(\mathbf{y}), \mathbf{y} \in \mathbb{R}^d)$:*

$$\begin{aligned}
Z^q_{\mathbf{H}}(\mathbf{t}) &= c(\mathbf{H}, q) \int_{\mathbb{R}^{d \cdot q}} \int_0^{t_1} \dots \int_0^{t_d} \left(\prod_{j=1}^{q} (s_1 - y_{1,j})_+^{-\left(\frac{1}{2} + \frac{1-H_1}{q}\right)} \dots (s_d - y_{d,j})_+^{-\left(\frac{1}{2} + \frac{1-H_d}{q}\right)} \right) \\
&\quad ds_d \dots ds_1 \quad dW(y_{1,1}, \dots, y_{d,1}) \dots dW(y_{1,q}, \dots, y_{d,q}) \\
&= c(\mathbf{H}, q) \int_{\mathbb{R}^{d \cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^{q} (\mathbf{s} - \mathbf{y}_j)_+^{-\left(\frac{1}{2} + \frac{1-\mathbf{H}}{q}\right)} d\mathbf{s} \quad dW(\mathbf{y}_1) \dots dW(\mathbf{y}_q) \qquad (12)
\end{aligned}$$

où $x_+ = \max(x, 0)$ et $\mathbf{t} = (t_1, \dots, t_d) \in \mathbb{R}^d_+$.

Pour le cas $q = 1$, on retrouve le drap brownien fractionnaire d'indice de Hurst $\mathbf{H} = (H_1, H_2, \dots, H_d) \in (\frac{1}{2}, 1)^d$ qui est l'unique champ d'Hermite gaussien.

Propriétés des champs d'Hermite

- ▷ Les champs d'Hermite sont autosimilaires d'indice $(H_1, \dots, H_d)$ au sens où le champ aléatoire $\left(\hat{Z}^q_{\mathbf{H}}(\mathbf{t})\right)_{\mathbf{t} \in (\mathbb{R}+)^d}$ défini par

$$\hat{Z}^q_{\mathbf{H}}(\mathbf{t}) = \mathbf{h}^{\mathbf{H}} \hat{Z}^q_{\mathbf{H}} \left(\frac{\mathbf{t}}{\mathbf{h}}\right) = h_1^{H_1} \dots h_d^{H_d} \hat{Z}^q_{\mathbf{H}} \left(\frac{t_1}{h_1}, \dots, \frac{t_d}{h_d}\right)$$

  admet les même lois fini-dimensionnelles que $Z^q_{\mathbf{H}}$.

- ▷ Les champs d'Hermite ont des accroissements stationnaires :
  pour tout $\mathbf{h} \in (\mathbb{R}^+)^d$

$$(\Delta Z^q_{\mathbf{H}}([0, \mathbf{t}]), \mathbf{t} \in \mathbb{R}^d) \overset{(d)}{=} (\Delta Z^q_{\mathbf{H}}([\mathbf{h}, \mathbf{h} + \mathbf{t}]), \mathbf{t} \in \mathbb{R}^d),$$

  Rappelons que les accroissements d'un champ sur un rectangle $[\mathbf{s}, \mathbf{t}] \subset \mathbb{R}^d$ avec $\mathbf{s} = (s_1, \dots, s_d), \mathbf{t} = (t_1, \dots, t_d)$ tel que $\mathbf{s} \leq \mathbf{t}$ sont donnés par :

$$\Delta X([\mathbf{s}, \mathbf{t}]) = \sum_{r \in \{0,1\}^d} (-1)^{d - \sum_{i=1}^d r_i} X_{\mathbf{s} + \mathbf{r} \cdot (\mathbf{t} - \mathbf{s})}. \qquad (13)$$

  Exemples :
  - ▷ $d = 1 \Rightarrow \Delta X([\mathbf{s}, \mathbf{t}]) = X_t - X_s$.
  - ▷ $d = 2 \Rightarrow \Delta X([\mathbf{s}, \mathbf{t}]) = X_{t_1, t_2} - X_{t_1, s_2} - X_{s_1, t_2} + X_{s_1, s_2}$ associé à un rectangle.

- ▷ Pour tout $q \geq 1$, la covariance de $Z^q_{\mathbf{H}}$ est donnée par :

$$\mathbb{E}\left[Z^q_{\mathbf{H}}(\mathbf{t}) Z^q_{\mathbf{H}}(\mathbf{s})\right] = \prod_{i=1}^{d} \left(\frac{1}{2} \left(t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i} \right) \right) \text{ pour tout } \mathbf{s}, \mathbf{t} \in (\mathbb{R}^+)^d.$$

- ▷ On a pour tout $p \geq 2$ et $0 \leq \mathbf{s} \leq \mathbf{t}$,

$$\mathbb{E}\left|\Delta Z^q_{\mathbf{H}}([\mathbf{s}, \mathbf{t}])\right|^p = \mathbb{E}\left|Z_{\mathbf{1}}\right|^p \left(|t_1 - s_1| \cdots |t_d - s_d|\right)^{p\mathbf{H}} = |\mathbf{t} - \mathbf{s}|^{p\mathbf{H}},$$

  ce qui permet, par application du théorème de Kolmogorov-Centsov, de montrer que le champ $Z^q_{\mathbf{H}}$ a des trajectoires p.s höldériennes pour tout $\boldsymbol{\delta} = (\delta_1, \dots, \delta_d) \in (0, \mathbf{H})$.

Nous renvoyons au chapitre 4 de [103] et [26] pour les preuves détaillées de ces points.

Intégrer contre les champs d'Hermite Afin de résoudre et exprimer les solutions des équations stochastiques dirigées par un champ d'Hermite $(Z^q_{\mathbf{H}}(\mathbf{t}))_{t\in\mathbb{R}^d}$, rappelons ici le cadre permettant d'intégrer contre celui ci. La construction de cette intégrale est dû à De la Cerda et Tudor dans [26], en étendant la méthode de construction de l'intégrale contre le processus d'Hermite (correspondant au cas $d = 1$) faite dans [53].

Soient $\mathbf{u} = (u_1, \ldots, u_d)$ et $\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{R}^d$. Considérons l'espace de Hilbert suivant :

$$\mathcal{H}_{\mathbf{H}} := \mathcal{H}_{\mathbf{H}}(\mathbb{R}^d) = \left\{f : \mathbb{R}^d \to \mathbb{R} : \|f\|^2_{\mathcal{H}_{\mathbf{H}}} < +\infty\right\} \tag{14}$$

avec

$$\begin{aligned}\|f\|^2_{\mathcal{H}_{\mathbf{H}}} &:= \mathbf{H(2H-1)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u})f(\mathbf{v})|\mathbf{u}-\mathbf{v}|^{\mathbf{2H-2}} d\mathbf{u}d\mathbf{v}.\\ &= \mathbf{H(2H-1)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u_1,\ldots,u_d)f(v_1,\ldots,v_d)\prod_{j=1}^d |u_j - v^j|^{2H_j-2} d\mathbf{u}d\mathbf{v}.\end{aligned}$$

L'intégrale par rapport au champ d'Hermite $\int_{\mathbb{R}^d} f(\mathbf{s})dZ^q_{\mathbf{H}}(\mathbf{s})$ est définie pour tout $f \in \mathcal{H}_{\mathbf{H}}(\mathbb{R}^d)$ et s'exprime sous la forme d'une intégrale multiple d'ordre q par rapport au drap brownien $(W(\mathbf{y}))_{\mathbf{y}\in\mathbb{R}^d}$:

$$\int_{\mathbb{R}^d} f(\mathbf{s})dZ^q_{\mathbf{H}}(\mathbf{s}) = I_q(Jf) \tag{15}$$

où $Jf \in L^2(\mathbb{R}^{d.q})$ est donné par :

$$Jf(\mathbf{y}_1, ..., \mathbf{y}_q) = c(\mathbf{H}, q)\int_{\mathbb{R}^d} d\mathbf{u}f(\mathbf{u})(\mathbf{u}-\mathbf{y}_1)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)}\ldots(\mathbf{u}-\mathbf{y}_q)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)}. \tag{16}$$

C'est une isométrie entre les espaces $\mathcal{H}_{\mathbf{H}}$ et $L^2(\mathbb{R}^d)$ donnée par :

$$\mathbb{E}\left(\int_{\mathbb{R}^d} f(\mathbf{s})dZ^q_{\mathbf{H}}(\mathbf{s})\int_{\mathbb{R}^d} g(\mathbf{s})dZ^q_{\mathbf{H}}(\mathbf{s})\right) = \langle f, g\rangle_{\mathcal{H}_{\mathbf{H}}} \tag{17}$$

où

$$\langle f, g\rangle_{\mathcal{H}_{\mathbf{H}}} = \mathbf{H(2H-1)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u})g(\mathbf{v})|\mathbf{u}-\mathbf{v}|^{\mathbf{2H-2}} d\mathbf{u}d\mathbf{v}. \tag{18}$$

Soulignons quelque points importants :

- ▷ L'espace $\mathcal{H}_{\mathbf{H}}$ n'est rien d'autre que l'espace de Hilbert associé au drap brownien fractionnaire.
- ▷ Les éléments de $\mathcal{H}_{\mathbf{H}}$ peuvent ne pas être des fonctions mais des distributions ; il est des fois plus judicieux de se placer sur le sous espace de $\mathcal{H}_{\mathbf{H}}$ des fonctions mesurables $f : \mathbb{R}^d \to \mathbb{R}$ noté $|\mathcal{H}_{\mathbf{H}}|$ muni de la norme :

$$\|f\|^2_{|\mathcal{H}_{\mathbf{H}}|} := \mathbf{H}(\mathbf{2H}-\mathbf{1})\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} d\mathbf{u}d\mathbf{v}|f(\mathbf{u})|\cdot|f(\mathbf{v})||\mathbf{u}-\mathbf{v}|^{\mathbf{2H-2}} < +\infty$$

0.2 Partie 1 : Solutions d'EDPS avec bruit fractionnaire non gaussiens.

Les solutions des EDPS type onde et chaleur dirigées par des bruits additifs forment une classe de processus autosimilaires qui a suscité un grand intérêt ces dernières décennies. Afin de positionner notre étude dans la littérature existante, nous donnons dans cette partie une présentation très succincte du contexte général. Nous discutons quelques résultats connus lorsque le bruit se comporte comme un mbf en temps et ou en espace. Nous renvoyons au chapitre 2 de [103] et aux synthèses faites dans [8], [102] pour un exposé plus détaillé. Dans ce manuscrit, les EDPS considérées sont toutes linéaires et le paramètre fractionnaire H est toujours fixé dans l'intervalle $(\frac{1}{2}, 1)$. Notons toutefois, qu'il existe des travaux où les EDPS sont non linéaires ou semi linéaires et où H est inférieur à $\frac{1}{2}$; ils ont été considérés en utilisant une approche spectrale en temps et en espace, voir [7].

Considérons l'équation suivante forcée par le bruit additif W :

$$Lu(t,\mathbf{x}) = \Delta(t,\mathbf{x}) + \dot{W}(t,\mathbf{x}), \quad t \in [0,T], \quad \mathbf{x} \in \mathbb{R}^d \tag{19}$$

avec des conditions initiales nulles. Ici $\dot{W}(t,\mathbf{x})$ est la dérivée formelle de W égale à $\frac{\partial W}{\partial t \partial \mathbf{x}}$, Δ est le Laplacian sur $\mathbb{R}^d$ et L est un opérateur linéaire défini sur $\mathbb{R}^+ \times \mathbb{R}^d$. La solution du problème qu'on considère est de type mild et est définie comme suit :

Définition 15. *[Solution mild] La solution mild de (19) est un champ aléatoire $u = \{u(t,\mathbf{x}); t \geq 0, \mathbf{x} \in \mathbb{R}^d\}$ à carré intégrable défini par :*

$$u(t,\mathbf{x}) = \int_0^t \int_{\mathbb{R}^d} G(t-s, \mathbf{x}-\mathbf{y}) W(ds, d\mathbf{y}), \quad t \geq 0, \mathbf{x} \in \mathbb{R}^d \tag{20}$$

où G est la solution de l'équation homogène $Lu(t,x) - \Delta(t,x) = 0$ et tel que pour tout $T > 0$, $\sup_{t\in[0,T],\mathbf{x}\in\mathbb{R}^d} \mathbf{E}\,|u(t,\mathbf{x})|^2 < \infty$.

- ▷ L'équation de la chaleur linéaire est définie lorsque $L = \frac{\partial}{\partial t}$, et on a :

$$G(t,\mathbf{x}) = \begin{cases} (2\pi t)^{-d/2} \exp\left(-\frac{|\mathbf{x}|^2}{2t}\right) & \text{if } t > 0, \mathbf{x} \in \mathbb{R}^d, \\ 0 & \text{if } t \leq 0, \mathbf{x} \in \mathbb{R}^{\mathbf{d}}. \end{cases} \tag{21}$$

- ▷ L'équation des ondes est définie lorsque $L = \frac{\partial^2}{\partial t^2}$, et la solution fondamentale est définie via sa transformée de Fourier donnée par

$$\mathcal{F}G(t,\cdot)(\xi) = \frac{\sin(t\|\xi\|)}{\|\xi\|}$$

pour tout $\xi \in \mathbb{R}^d$, $t > 0$. Lorsque $d = 1$, on a en particulier $G(t,x) = \frac{1}{2}\mathbb{1}_{\{|x|<t\}}$ où $t > 0$ et $x \in \mathbb{R}$.

Le premier problème qui se pose est de montrer que la solution existe ou de manière équivalente que l'intégrale (20) est bien définie. L'idée principale repose sur la construction de l'intégrale par rapport à W qui vérifie une isométrie linéaire entre l'espace de Hilbert $\mathcal{H}$ associé au bruit W et l'espace de Hilbert $L^2(\Omega)$. Par conséquent la solution existe si et seulement si $G(t-.,\mathbf{x}-.)1_{[0,t]}(.)$ appartient à $\mathcal{H}$. La structure de l'espace de Hilbert associé au bruit W est alors d'une importance considérable pour étudier les propriétés du champ aléatoire défini par (20).

Le bruit blanc espace-temps (qui se comporte comme un mouvement brownien en temps et comme un drap brownien en espace) a été étudiée par Walsh dans [108]. L'espace de Hilbert qui lui est associé est $L^2(\Omega)$ et la solution mild existe si et seulement si :

$$\int_{\mathbb{R}^d} G(t-s, \mathbf{x}-\mathbf{y})^2 ds d\mathbf{y} < +\infty,$$

ce qui est équivalent à la condition $d = 1$ pour l'équation de la chaleur et des ondes.

Afin de contourner cette restriction et obtenir une solution en dimension supérieure, Dalang [35] a introduit un bruit plus général qui admet une structure de corrélation par rapport à la variable espace. Ce bruit coloré noté W, est défini comme un champ gaussien centré qui a pour covariance :

$$\mathbb{E}(W(t,A)W(s,B) = t \wedge s \int_A \int_B f(\mathbf{x}-\mathbf{y})d\mathbf{x}d\mathbf{y}, \quad \forall A, B \in \mathcal{B}_b(\mathbb{R}^d)$$

où f est la transformée de Fourrier d'une mesure tempérée μ sur $\mathbb{R}^d$ c'est à dire $f(x) = \int_{\mathbb{R}^d} e^{-ix.\xi}\mu(d\xi)$. Les équations des ondes et de la chaleur admettent des solutions mild si et seulement si :

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)\mu(d\xi) < \infty.$$

Le développement de l'intégration contre le mbf a permis naturellement de considérer un bruit qui se comporte comme un mbf en temps et/ou en espace. Donnons l'exemple du bruit fractionnaire gaussien introduit dans [10], communément appelé *fractional-colored noise* et noté W^H. Il se comporte comme un mbf d'exposant de Hurst $H > \frac{1}{2}$ en temps et a une structure de corrélation en espace. On définit $W^H = \{W^H(t,A); t \in [0,T], A \in B_b(\mathbb{R}^d)\}$ pour $H \in \left(\frac{1}{2}, 1\right)$ comme un champ aléatoire gaussien centré qui admet la fonction de covariance suivante :

$$\begin{aligned}
\mathbb{E}\left(W^H(t,A)W^H(s,B)\right) &= R_H(t,s)\int_A\int_B f(x-y)dxdy, \quad \forall A, B \in \mathcal{B}_b(\mathbb{R}^d) \\
&:= \langle 1_{[0,t]\times A}, 1_{[0,s]\times B}\rangle_{\mathcal{H}}.
\end{aligned}$$

Soit $\mathcal{E}$ l'espace des fonctions élémentaires de la forme $1_{[0,t]\times A}$ où $(t,A) \in \left([0,T], \mathcal{B}(\mathbb{R}^d)\right)$. L'espace de Hilbert $\mathcal{H}_H$ associé au bruit W est défini comme la fermeture de $\mathcal{E}$ par rapport au produit scalaire $\langle .,. \rangle_{\mathcal{H}}$. On peut montrer que l'application $1_{[0,t]\times A} \to W(t,A)$ est une isométrie entre l'espace $\mathcal{E}$ et l'espace gaussien associé à W^H. Comme $\mathcal{H}$ est défini comme la fermeture de $\mathcal{E}$, cette isométrie peut être étendue à $\mathcal{H}$, ce qui donne l'intégrale stochastique contre le bruit W^H, notée par $\int_0^T \int_{\mathbb{R}^d} f(t,x)W(dt,dx)$, pour $f \in \mathcal{H}$.

De plus, on a pour $\psi, \phi \in \mathcal{H}_H$,

$$\begin{aligned}
\langle \psi, \phi \rangle_{\mathcal{H}} &= H(2H-1)\int_0^T\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(u,x)\phi(v,y)|u-v|^{2H-2}f(x-y)dxdydudv \\
&= H(2H-1)(2\pi)^{-d}\int_0^T\int_0^T\int_{\mathbb{R}^d} \mathcal{F}\psi(u,.)(\xi)\mathcal{F}\phi(v,.)(\xi)|u-v|^{2H-2}dudv\mu(d\xi)
\end{aligned}$$

où la dernière égalité est obtenue par application du théorème de Parseval.

L'équation de la chaleur dirigée par ce bruit a été étudiée par Balan et Tudor dans [10]. Ils montrent que la solution mild existe si et seulement si :

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^{2H} \mu(d\xi) < \infty. \tag{22}$$

L'équation des ondes a été plus tard étudié dans [11] . La condition d'existence est donnée par :

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^{H+\frac{1}{2}} \mu(d\xi) < \infty. \tag{23}$$

Donnons deux exemples de noyaux f auxquels nous ferons plus tard référence :

- La covariance du champ fractionnaire

$$f(x) = \prod_{i=1}^{d} (H_i(2H_i - 1)|x_i|^{2H_i-2} \quad \text{où on a} \quad \mu(d\xi) = \prod_{i=1}^{d} c_{H_i}|\xi|^{1-2H_i}.$$

  Notons que dans ce cas le bruit gaussien considéré se comporte comme un mbf en temps et comme un champ fractionnaire en espace.
  Pour l'équation des ondes,la condition (23) devient $d < \sum_{i=1}^{d}(2H_i - 1) + 2H + 1$ et pour l'équation de la chaleur la condition (22) devient $d < 4H + \sum_{i=1}^{d}(2H_i - 1)$.

- Le noyau de Riesz d'ordre α défini par

$$f(x) = \gamma_{\alpha,d}|x|^{-d+\alpha} \quad \text{avec} \quad 0 < \alpha < d, \quad \text{où on a} \quad \mu(d\xi) = |\xi|^{-\alpha}d\xi.$$

  La condition (22) devient $d < 4H + \alpha$ et le processus $(u(t,x))_{t\in[0,T]}$ pour x fixé, est gaussien auto similaire d'indice $H - \frac{d-\alpha}{2}$; tandis que la condition (23) devient $d < \alpha + 2H + 1$ et le processus $(u(t,x))_{t\in[0,T]}$ pour x fixé est un processus gaussien auto similaire d'indice $H + 1 - \frac{d-\alpha}{2}$.

Des auteurs ont par la suite considéré des bruits fractionnaires non gaussiens, citons par exemple les travaux [17], [26], [31], [34], [33], [100]. Le bruit non gaussien considéré dans ce travail est le champ d'Hermite, qui apparaît comme une extension non gaussienne du drap brownien fractionnaire étudié dans [11]. L'équation des ondes a été traité dans [26] et nous avons considéré dans le chapitre 1 l'équation de la chaleur dirigée par ce bruit.

0.2.1 Résumé du chapitre 1 : Autour de l'équation de la chaleur dirigée par un bruit de type Hermite.

Le Chapitre 1 de cette thèse est constitué de la publication [88], en collaboration avec C.A. Tudor.

Nous considérons l'équation de la chaleur dirigée par un champ type Hermite d'ordre q et de dimension $d+1$. Plus précisément soit le problème suivant :

$$\begin{cases} \frac{\partial u}{\partial t}(t,\mathbf{x}) &= \Delta u(t,\mathbf{x}) + \dot{Z}^q_{\mathbf{H}}(t,\mathbf{x}), \quad t > 0, \mathbf{x} \in \mathbb{R}^d \\ u(0,\mathbf{x}) &= 0, \quad \mathbf{x} \in \mathbb{R}^d \end{cases} \tag{24}$$

où $Z^q_{\mathbf{H}} = \{Z^q_{\mathbf{H}}(t,\mathbf{x}); t \geq 0, \mathbf{x} \in \mathbb{R}^d\}$ avec $\mathbf{H} = (H, H_1, \ldots, H_d) \in (1/2,1)^{(d+1)}$ est un champ d'Hermite de dimension $(d+1)$ qui a pour covariance :

$$\mathbb{E}\{Z^q_{\mathbf{H}}(s,\mathbf{x})Z^q_{\mathbf{H}}(t,\mathbf{y})\} = R_H(t,s)R_{\mathbf{H}_0}(\mathbf{x},\mathbf{y})$$

où,

$$R_H(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad R_{\mathbf{H}_0}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{d} R_{H_j}(x_j,y_j)$$

avec $\mathbf{H}_0 = (H_1,\ldots,H_d)$, s,t$\in \mathbb{R}$ et $\mathbf{x} = (x_1,..,x_d), \mathbf{y} = (y_1,..,y_d) \in \mathbb{R}^d$. De manière équivalente, la covariance du processus $\dot{Z}^q_{\mathbf{H}} = \frac{\partial^2 Z^q_{\mathbf{H}}}{\partial t \partial x}$ est donné par :

$$\mathbb{E}\left\{\dot{Z}^q_{\mathbf{H}}(s,\mathbf{x})\dot{Z}^q_{\mathbf{H}}(t,\mathbf{y})\right\} = H(2H-1)|t-s|^{2H-2}\prod_{i=1}^{d}(H_i(2H_i-1)\cdot|x_i-y_i|^{2H_i-2}).$$

Le bruit que nous considérons se comporte comme un processus d'Hermite par rapport au temps et comme un champ d'Hermite de dimension d par rapport à la variable espace.

La solution mild du problème $u = \{u(t,\mathbf{x}); t \geq 0, \mathbf{x} \in \mathbb{R}^d\}$ s'exprime sous la forme d'une intégrale par rapport au champ d'Hermite $Z^q_{\mathbf{H}}$:

$$u(t,\mathbf{x}) = \int_0^t \int_{\mathbb{R}^d} G(t-s, \mathbf{x}-\mathbf{y}) Z^q_{\mathbf{H}}(ds, d\mathbf{y}), \quad t \geq 0, \mathbf{x} \in \mathbb{R}^d. \tag{25}$$

Comme nous l'avons rappelé précédemment, l'intégrale contre le champ d'Hermite préserve la même structure de covariance que l'intégrale contre le drap brownien fractionnaire ($q = 1$). Fort de cette propriété, l'existence et la régularité de la solution découlent directement des résultats connus dans le cadre gaussien [11].

Proposition 4. *L'équation de la chaleur stochastique (24) admet une unique solution type mild $(u(t,\mathbf{x}))_{t\geq 0, \mathbf{x}\in\mathbb{R}^d}$ si seulement si :*

$$d < 4H + \sum_{i=1}^{d} (2H_i - 1). \tag{26}$$

Dans ce cas, on a $\sup_{t,\mathbf{x}} \mathbb{E}\left(u(t,\mathbf{x})^2\right) < \infty$.

De plus, il existe des constantes $0 < c_1 < c_2$ et $0 < c_3 < c_4$ tel que pour tout $\mathbf{x} \in \mathbb{R}^d$ et $0 \leq s \leq t$, on a :

$$c_1 |t-s|^{2H - \frac{d-\alpha}{2}} \leq \mathbb{E}\left|u(t,\mathbf{x}) - u(s,\mathbf{x})\right|^2 \leq c_2 |t-s|^{2H-\frac{d-\alpha}{2}}, \tag{27}$$

et pour tout $M > 0$, $t > 0$ et $\mathbf{x}, \mathbf{y} \in [-M, M]^d$, on a :

$$c_3 |\mathbf{x}-\mathbf{y}|^{2\beta} \left(\frac{1}{\log|\mathbf{x}-\mathbf{y}|}\right)^{\rho} \leq \mathbb{E}\left|u(t,\mathbf{x}) - u(t,\mathbf{y})\right|^2 \leq c_4 |\mathbf{x}-\mathbf{y}|^{2\beta} \left(\frac{1}{\log|\mathbf{x}-\mathbf{y}|}\right)^{\rho} \tag{28}$$

avec $\beta = \min(1, 2H - \frac{d-\alpha}{2})$ and $\rho = \rho(\beta) = \mathbf{1}_{\{1\}}(\beta)$.

Nous montrons également que le champ stochastique $(u(t,\mathbf{x}))_{t\geq 0,\mathbf{x}\in\mathbb{R}^d}$ est stationnaire par rapport à la variable espace $\mathbf{x} \in \mathbb{R}^d$ et nous établissons son indice d'auto similarité par rapport à la variable t.

Ces résultats s'appuient sur la représentation spectrale de la solution donnée par la proposition suivante :

Proposition 5. *Pour tout $\mathbf{x} \in \mathbb{R}^d$, le processus $(u(t,\mathbf{x}))_{t\geq 0}$ satisfait*

$$u(t,\mathbf{x}) \overset{(d)}{=} C_{0,q} \int_{\mathbb{R}^{d+1}} d\widehat{W}(u_1, \mathbf{z}_1) ... \int_{\mathbb{R}^{d+1}} d\widehat{W}(u_q, \mathbf{z}_q) \left(e^{it(u_1+...+u_q)} - e^{\frac{1}{2}t|\mathbf{z}_1+...+\mathbf{z}_q|^2}\right)$$

$$\frac{1}{i(u_1+...+u_q) + \frac{1}{2}|\mathbf{z}_1 + ... + \mathbf{z}_q|^2} |\mathbf{z}_1|^{-\frac{1}{2}+\frac{1-\mathbf{H}_0}{q}} ... |\mathbf{z}_q|^{-\frac{1}{2}+\frac{1-\mathbf{H}_0}{q}} |u_1|^{-\frac{1}{2}+\frac{1-H}{q}} ... |u_q|^{-\frac{1}{2}+\frac{1-H}{q}}$$

avec $C_{0,q}$ une constante explicitée.

Proposition 6. *Pour tout $x \in \mathbb{R}^d$, le processus $(u(t,\mathbf{x}))_{t\in[0,T]}$ est auto similaire d'indice*

$$\gamma = H + \frac{(H_1 + ... + H_d) - d}{2}. \tag{29}$$

On retrouve dans le cas $d = 1$, l'indice d'auto similarité $H - \frac{1-H_1}{2}$ obtenu dans le cas d'un bruit gaussien fBm-Riesz en temps-espace avec $\alpha = 2H_1 - 1$ (voir [103]).

Le résultat fondamental de ce travail est un théorème de décomposition que nous décrivons dans ce qui suit.

Pour tout $t \geq 0$, $\mathbf{x} \in \mathbb{R}^d$ on peut exprimer la solution (25) sous la forme :

$$u(t, \mathbf{x}) = U(t, \mathbf{x}) - Y(t, \mathbf{x})$$

avec

$$U(t, \mathbf{x}) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(G((t-u)_+, \mathbf{s} - \mathbf{y}) - G((-u)_+, \mathbf{s} - \mathbf{y}) \right) Z^q_{\mathbf{H}}(du, d\mathbf{y}) \tag{30}$$

et

$$Y(t, \mathbf{x}) = \int_{-\infty}^{0} \int_{\mathbb{R}^d} \left(G(t-u, \mathbf{s} - \mathbf{y}) - G(-u, \mathbf{s} - \mathbf{y}) \right) Z^q_{\mathbf{H}}(du, d\mathbf{y}). \tag{31}$$

Cette décomposition a été initialement introduite dans [61] puis étudiée dans différents travaux [39], [105], [43] et [62] afin d'établir diverses propriétés analytiques de la solution.

Le processus U est appelé dans la littérature *the pinned string process*. Dans le cas gaussien $q = 1$ qui correspond au cas de l'équation de la chaleur dirigée par un bruit qui se comporte comme champ fractionnaire de dimension $d + 1$, il a été prouvé dans [105] que la solution s'écrit comme la somme d'un mouvement brownien fractionnaire et d'un processus aux trajectoires régulières. Plus précisément à variable spatiale fixée, U est un mbf (modulo une constante) d'indice de Hurst $H + \frac{H_1 + .. + H_d - d}{2}$ tandis que le processus Y admet p.s des trajectoires continument dérivables sur tout intervalle $[a, b] \subset [0, \infty)$.

Le théorème suivant propose d'étudier les processus U et Y lorsqu'on sort du cadre gaussien pour $q \geqslant 2$:

Théorème 3. *1. Pour tout $\mathbf{x} \in \mathbb{R}^d$, le processus $(U(t, \mathbf{x}))_{t \geq 0}$ défini par (30) est auto similaire d'indice γ (29) et admet des accroissements stationnaires.*

2. Pour tout $\mathbf{x} \in \mathbb{R}^d$ et pour tout entier $k \geq 1$, le processus $(Y(t, \mathbf{x}))_{t \geq 0}$ admet p.s des trajectoires continument dérivables d'ordre k sur tout intervalle $[a, b] \subset [0, \infty)$.

Nous savons que les propriétés d'auto similarité et de stationnarité des accroissements sont caractéristiques du mbf pour les processus gaussiens. Le point 1. de ce théorème implique par conséquent que lorsque $q = 1$, le processus U est un mbf et nous retrouvons le résultat prouvé dans [105]. Lorsqu'on sort du cadre gaussien pour les processus d'Hermite d'ordre $q \geqslant 2$, cette caractérisation n'est plus vraie, U est il toujours un processus d'Hermite comme pour le cas $q = 1$? Afin de donner des éléments de réponse à cette question, nous nous restreignons au cas $q = 2$ où $(U(t, x))_{t \geqslant 0}$ étant un élément du second chaos de Wiener, sa loi est déterminée par ses cumulants. Nous calculons les cumulants des lois fini-dimensionnelles du processus $(U(t, x))_{t \geqslant 0}$ et démontrons qu'ils ne peuvent pas coïncider avec ceux du processus de Rosenblatt.

Une application de ce théorème est le calcul des variations d'ordre α de la solution donnée dans la proposition suivante :

Proposition 7. *Considérons le processus $u = \{u(t, \mathbf{x}); t \geq 0, \mathbf{x} \in \mathbb{R}^d\}$ donnée par 25). Pour tout $x \in \mathbb{R}^d$ fixé, le processus $(u(t, \mathbf{x}))_{t \geq 0}$ admet des α-variations égales à $\mathbb{E} \left| U(1, \mathbf{x}) \right|^{\alpha} t$ pour tout $\alpha > 1$.*

En se restreignant au cas $d = 1$, le dernier résultat de ce chapitre suggère que la solution mild du problème (24) $(u(t, x), t \geq 0, x \in \mathbb{R})$ est aussi une solution faible, qui est définie à partir de la la formulation variationnelle suivante :

$$\int_0^T \int_{\mathbb{R}} u(t, x) \left(\frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2} \right) dx dt = - \int_0^T \int_{\mathbb{R}} \varphi(s, y) Z^q_H(ds, dy)$$

pour tout $T > 0$ et pour toute fonction test $\varphi \in C^{\infty}([0,\infty) \times \mathbb{R})$ avec support compact dans $\mathbb{R}$ et tel que $\varphi(T,x) = 0$ pour tout $x \in \mathbb{R}$.

0.3 Partie 2 : Comportement asymptotique des solutions d'équations stochastiques par rapport au paramètre de Hurst.

0.3.1 Rappels et résultats préliminaires

Durant les dernières années, plusieurs auteurs se sont intéressés au problème du comportement asymptotique des processus fractionnaires lorsque l'exposant de Hurst tend vers ses valeurs limites. Ceci a donné lieu à une série de publications à laquelle les articles qui constituent les chapitres 2 et 3 viennent s'ajouter.

Donnons un bref aperçu de l'état de l'art autour de cette problématique :

- Dans [106], Veillette et Taqqu démontrent que lorsque $H \to 1$, le processus de Rosenblatt $\left(Z^H(t)\right)_{t\geqslant 0}$ converge dans l'espace des fonctions $C([0,T])$ vers le processus $\left(\frac{t}{\sqrt{2}}(Z^2-1)\right)_{t\geqslant 0}$ où Z^2-1 est la variable du chi-deux centrée, tandis qu'il converge vers un mouvement brownien lorsque $H \to \frac{1}{2}$.
- Dans [6], Bai et Taqqu s'intéressent au processus de Rosenblatt généralisé Z^{H_1,H_2} (voir [54] pour une définition), qui est obtenu en remplaçant l'exposant de Hurst H dans le processus de Rosenblatt par deux exposants H_1 et H_2 qui appartiennent au domaine $\Delta = \{0 < H_1 < 1, 0 < H_2 < 1, H_1 + H_2 > 1\}$. La convergence est établie pour les différentes valeurs limites de Δ.
- Dans [12], Bell et Nualart généralisent les résultats précédents en étudiant le processus d'Hermite généralisé. Une conséquence de leur résultat est que le processus d'Hermite $(Z_H^q(t))_{t\geqslant 0}$ converge dans $C([0,T])$ lorsque $H \to \frac{1}{2}$ vers le mouvement brownien.
- Dans [3], Arraya et Tudor étudient quant à eux le comportement asymptotique des champs d'Hermite de dimension d, $\left(Z_{\mathbf{H}}^{q,d}(\mathbf{t})\right)_{\mathbf{t}\geq 0}$, lorsque l'exposant multidimensionnel $\mathbf{H}$ converge vers différents vecteurs limites.

Cette seconde partie de ma thèse traite le comportement asymptotique des solutions d'équations stochastiques fractionnaires par rapport au paramètre de Hurst H.

Nous allons considérer les deux équations stochastiques suivantes, dont nous définissons la solution associée :

- ▷ L'équation de la chaleur stochastique dirigée par un champ d'Hermite dont la solution mild a été définie et étudiée au chapitre 1.
- ▷ L'équation de Langevin stochastique dirigée par un processus d'Hermite d'ordre $q \geq 1$, introduite et étudiée dans [53].

  Rappelons brièvement le contexte. Considérons l'équation différentielle stochastique de Langevin dirigée par un processus d'Hermite Z_H^q où $H \in (\frac{1}{2}, 1)$ donnée par :

$$X_t = \xi - \lambda \int_0^t X_s ds + \sigma Z_H^q(t), t \geq 1 \tag{32}$$

  où $\lambda, \sigma > 0$ et où la condition initiale ξ est une variable aléatoire dans $L^2(\Omega)$.

Cette équation admet une unique solution, appelée le processus d'Ornstein Uhlenbeck type Hermite, qui s'écrit :

$$Y^{H,q}(t) = e^{-\lambda t}\left(\xi + \sigma\int_0^t e^{\lambda u} dZ_H^q(u)\right), \qquad t \geq 0. \tag{33}$$

Notons que l'intégrale $\int_0^t e^{\lambda u} dZ^q(u)$ peut être définie soit comme une intégrale multiple contre le processus d'Hermite ou au sens de Riemann-Stieltjes. Le processus $Y^{H,q}(t)$ a la même fonction de covariance que le processus d'Ornstein Uhlenbeck fractionnaire (voir [19]).

Lorsque la condition initiale est donnée par :

$$\xi = \sigma\int_{-\infty}^0 e^{\lambda u} dZ^H(u) \tag{34}$$

alors l'unique solution de (32), que l'on note $(X^{H,q}(t))_{t\geq 0}$ s'écrit :

$$X^{H,q}(t) = \sigma\int_{-\infty}^t e^{-\lambda(t-u)} dZ_H^q(u), \qquad t \geq 0. \tag{35}$$

Le processus $\left(X^{H,q}(t)\right)_{t\geq 0}$ est stationnaire, nous y ferons référence sous l'appellation processus d'Ornstein Uhlenbeck type Hermite stationnaire. Lorsque $q = 2$, nous parlerons de processus d'Ornstein Uhlenbeck type Rosenblatt et de processus d'Ornstein Uhlenbeck type Rosenblatt stationnaire.

0.3.2 Résumé du chapitre 2 : Comportement asymptotique du processus de Ornstein-Uhlenbeck type Rosenblatt par rapport à l'indice de Hurst H

Le chapitre 2 est constitué de la publication [89], en collaboration avec C.A. Tudor.

Nous étudions la convergence en loi, lorsque $H \to \frac{1}{2}$ et lorsque $H \to 1$, de l'intégrale $\int_{\mathbb{R}} f(u)dZ^H(u)$, où Z^H est le processus de Rosenblatt avec un indice de Hurst $H \in \left(\frac{1}{2}, 1\right)$ et $f \in \mathcal{H}_H$ une fonction déterministe.

Énonçons les deux propositions principales de ce chapitre :

Proposition 8. *Soit $f : \mathbb{R} \to \mathbb{R}$ tel que pour $\varepsilon \in (0, \frac{1}{2})$. On a :*

$$\|f\|^2_{\left|\mathcal{H}_{\frac{1}{2}+\varepsilon}\right|} = \int_{\mathbb{R}}\int_{\mathbb{R}} dudv|f(u)||f(v)||u-v|^{2\varepsilon-1} < \infty \text{ et } f \in L^1(\mathbb{R}). \tag{36}$$

Alors,

$$\int_{\mathbb{R}} f(u)dZ^H(u) \xrightarrow[H\to 1]{(d)} \frac{1}{\sqrt{2}}\left(\int_{\mathbb{R}} f(u)du\right)(Z^2 - 1) \tag{37}$$

avec $Z \sim N(0,1)$ et où $\xrightarrow{(d)}$ désigne la convergence en distribution.

Proposition 9. *Soit $f \in \mathcal{H}_H$. Supposons que*

$$\sigma_f^2 = \lim_{H\to\frac{1}{2}} \|f\|^2_{\mathcal{H}_H} = \lim_{H\to\frac{1}{2}} H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)f(v)|u-v|^{2H-2}dudv \tag{38}$$

existe et est bien définie, et que

$$(2H-1)^2 \int_{\mathbb{R}^4} du_1...du_4 f(u_1)...f(u_4)|u_1-u_2|^{H-1}|u_2-u_3|^{H-1}|u_3-u_4|^{H-1}|u_4-u_1|^{H-1} \xrightarrow[H\to\frac{1}{2}]{} 0. \tag{39}$$

Alors,

$$\int_{\mathbb{R}} f(u)dZ^H(u) \xrightarrow[H\to\frac{1}{2}]{(d)} N(0,\sigma_f^2).$$

Le processus de Rosenblatt ainsi que l'intégrale de Wiener par rapport à celui ci étant des éléments du second chaos de Wiener, les preuves de ces deux propositions reposent sur l'analyse des cumulants et l'application du théorème du quatrième moment. En effet, remarquons que la condition (39) correspond à la condition $k_4\left(\int_{\mathbb{R}} f(u)dZ^H(u)\right) \to 0$ qui vient de l'application du théorème du quatrième moment.

Lorsque $H \to \frac{1}{2}$, nous montrons que pour des noyaux f particuliers, la condition (38) est automatiquement satisfaite.

Rappelons qu'une suite $(f_n)_{n\geq 1}$ d'éléments de $L^1(\mathbb{R}^d)$ est une approximation de l'unité lorsque $n\to\infty$ si les trois conditions suivantes sont vérifiées :

- ▷ $f_n(t) \geq 0$ pour tout $t\in\mathbb{R}$,
- ▷ pour tout $\delta>0$, $\int_{|t|\leq\delta} f_n(t)dt \xrightarrow[n\to\infty]{} 1$,
- ▷ pour tout $\delta>0$, $\int_{|t|>\delta} f_n(t)dt \xrightarrow[n\to\infty]{} 0$.

De plus si $(f_n)_{n\geq 1}$ est une approximation de l'unité, $p\in[1,\infty)$ et $f\in L^p(\mathbb{R})$ alors la convolution $f * f_n$ converge dans $L^p(\mathbb{R})$ vers f lorsque $n\to\infty$.

En remarquant que $2H(2H-1)1_{[0,u]}(v)v^{2H-2}$ est une approximation de l'unité lorsque $H\to\frac{1}{2}$, nous obtenons le corolaire suivant :

Corollaire 1. *Soit $f\in\mathcal{H}_H\cap L^2(\mathbb{R})$ avec $\operatorname{supp}(f)\subset[0,\infty)$ et supposons que (39) soit vérifiée. Alors*

$$\int_{\mathbb{R}} f(u)dZ^H(u) \xrightarrow[H\to\frac{1}{2}]{(d)} N(0,\int_{\mathbb{R}} f^2(u)du).$$

Sachant que lorsque $H\to\frac{1}{2}$, Z^H converge en loi vers le mouvement brownien, soulignons le fait que ce résultat en est une extension naturelle au sens où on obtient que $\int_{\mathbb{R}} f(u)dZ^H(u)$ converge en loi vers $\int_{\mathbb{R}} f(u)dW(u)$, sous les conditions du corolaire (1) sur f .

Comme exemples d'applications des résultats précédents, nous établissons le comportement asymptotique du processus d'Ornstein Uhlenbeck type Rosenblatt stationnaire et non stationnaire. Nous listons ci dessous les résultats obtenus :

- — Comportement asymptotique par rapport à H du processus d'Ornstein Uhlenbeck non stationnaire $\left(Y^{H,2}(t)\right)_{t\in[0,T]}$:
  - — Lorsque $H\to 1$, $(Y^{H,2}(t))_{t\in[0,T]}$ converge en loi sur l'espace des fonctions continues $C[0,T]$, vers le processus $(Y(t))_{t\in[0,T]}$ donné par :

    $$Y(t) = e^{-\lambda t}\xi + \sigma\frac{1}{\sqrt{2}}\left(\int_0^t e^{-\lambda(t-u)}du\right)(Z^2-1) = e^{-\lambda t}\xi + \frac{\sigma}{\sqrt{2}\lambda}(1-e^{-\lambda t})(Z^2-1)$$

    avec $Z\sim N(0,1)$.
  - — Lorsque $H\to\frac{1}{2}$, le processus $(Y^{H,2}(t))_{t\in[0,T]}$ converge en loi sur l'espace des fonctions continues $C[0,T]$, vers le processus d'Ornstein-Uhlenbeck $(Y_0(t))_{t\in[0,T]}$, qui est la solution de l'équation de Langevin dirigée par le processus de Wiener.

— Comportement asymptotique par rapport à H du processus d'Ornstein Uhlenbeck stationnaire $(X^{H,2}(t))_{t\in[0,T]}$:

— Lorsque $H \to 1$, le processus $(X^{H,2}(t))_{t\in[0,T]}$ converge en loi, sur l'espace des fonctions continues $C[0,T]$, vers le processus $(X(t))_{t\in[0,T]}$, défini pour tout $t \in [0,T]$, par :

$$X(t) = \sigma \left(\int_{-\infty}^{t} e^{-\lambda(t-u)} du \right) (Z^2 - 1) = \frac{\sigma}{\lambda}(Z^2 - 1).$$

— Lorsque $H \to \frac{1}{2}$, le processus $(X^{H,2}(t))_{t\in[0,T]}$converge en loi vers le processus d'Ornstein-Uhlenbeck stationnaire $(X_0(t))_{t\in[0,T]}$, qui est la solution de l'équation de Langevin dirigée par le processus de Wiener avec la condition initiale (34) .

Afin de vérifier les conditions d'intégrabilité des propositions 8 et 9, nous avons utilisé le théorème *Power counting theorem* tiré de l'article [98]. L'énoncé de ce résultat ainsi que les calculs détaillés figurent dans le chapitre 2 de ce manuscrit.

0.3.3 Résumé du chapitre 3 : Comportement asymptotique de l'intégrale de Wiener par rapport au processus d'Hermite et applications aux équations stochastiques

Le chapitre 3 est constitué de la publication [90], en collaboration avec C.A. Tudor. Les résultats obtenus dans ce chapitre constituent une extension du travail engagé au chapitre 2. Notre principale motivation est d'étudier le comportement asymptotique de la solution de l'équation de la chaleur stochastique dirigée par un champ d'Hermite et qui a fait l'objet du chapitre 1.

Nous savons que la solution mild du problème (24) est un champ aléatoire $u = \{u(t,\mathbf{x}); t \geq 0, \mathbf{x} \in \mathbb{R}^d\}$ qui s'exprime comme une intégrale par rapport au champ d'Hermite $Z^q_{\mathbf{H}}$ et appartient au q ème chaos de Wiener.

Par analogie avec la démarche adoptée au chapitre 2, nous commençons par analyser le comportement asymptotique de l'intégrale de Wiener-Hermite $\int_{\mathbb{R}^d} f(\mathbf{s})dZ^{q,d}_{\mathbf{H}}(\mathbf{s})$ $\forall q \geq 1$ où $f \in \mathcal{H}_{\mathbf{H}}$ lorsque les composantes H_i tendent vers 1 et/ou $\frac{1}{2}$. Soulignons que l' approche utilisée précédemment dans le chapitre 2, qui est basée sur l'analyse des cumulants ne peut plus être appliquée lorsqu'on travaille dans un chaos de Wiener d'ordre $q \geq 3$

Sous des conditions d'intégrabilités imposées à f que nous détaillerons par la suite, énonçons de manière sommaire les deux comportements asymptotiques obtenus :

1. Lorsqu'il existe au moins une des composantes H_i de $\mathbf{H}$ qui converge vers 1 et aucune vers $\frac{1}{2}$:
   $\hookrightarrow$ La limite en loi de $\int_{\mathbb{R}^d} f(\mathbf{s})dZ^{q,d}_{\mathbf{H}}(\mathbf{s})$ fait apparaitre une variable aléatoire non normale, associée à la distribution d'Hermite.
2. Lorsqu'au moins une des composantes H_i converge vers $\frac{1}{2}$ et les autres sont fixées dans l'intervalle $(\frac{1}{2}, 1)$ ou convergent vers 1 :
   $\hookrightarrow$ La limite en loi de $\int_{\mathbb{R}^d} f(\mathbf{s})dZ^{q,d}_{\mathbf{H}}(\mathbf{s})$ est une variable aléatoire normale centrée qui admet une variance explicite.

Afin d'exposer plus précisément ce travail, nous avons besoin d'introduire les notations suivantes : soient $1 \leq k \leq d$ et $\{j_1,..,j_k\} \subset \{1,..,d\}$, notons

$$A_k = \{j_1,..,j_k\}, \quad \mathbf{H}_{A_k} = (H_{j_1},...,H_{j_k}) \in \left(\frac{1}{2},1\right)^k, \quad \langle \mathbf{t} \rangle_{A_k} = t^{(j_1)}....t^{(j_k)} \text{ si } \mathbf{t} = (t^{(1)},..,t^{(d)}). \tag{40}$$

Introduisons également les espaces suivants qui imposent les conditions nécessaires de régularité sur f lorsqu'on étudie le cas où il existe au moins une composante de $\mathbf{H}$ qui tend vers 1 :

— Pour $1 \leq k < d$, nous introduisons l'espace $\mathcal{H}_{\overline{A}_k}$ des fonctions mesurables $f : \mathbb{R}^d \to \mathbb{R}$ $\mathcal{H}_{\overline{A}_k}$ tel que :

$$\begin{aligned} & \|f\|_{\mathcal{H}_{\overline{A}_k}} := & (41) \\ & \sum_{j=1}^{k} \int_{\mathbb{R}^j} d\mathbf{u}_{A_j} \left| \int_{\mathbb{R}^{d-j}} d\mathbf{v}_{\overline{A}_j} \int_{\mathbb{R}^{d-j}} d\mathbf{w}_{\overline{A}_j} |f(\mathbf{u}_{A_j}, \mathbf{v}_{\overline{A}_j})| \cdot |f(\mathbf{u}_{A_j}, \mathbf{w}_{\overline{A}_j})| |\mathbf{v}_{\overline{A}_j} - \mathbf{w}_{\overline{A}_j}|^{2\mathbf{H}_{\overline{A}_j} - 2} \right|^{\frac{1}{2}} \\ = & \sum_{j=1}^{k} \int_{\mathbb{R}^j} d\mathbf{u}_{A_j} \|f(\mathbf{u}_{A_j}, \cdot)\|_{\mathcal{H}_{\mathbf{H}_{\overline{A}_j}}} < \infty & (42) \end{aligned}$$

muni de la norme $\| \cdot \|_{\mathcal{H}_{\mathbf{H}_{\overline{A}_j}}}$ défini par (18).

— Pour $k = d$, l'espace $\mathcal{H}_{\overline{A}_k} = \mathcal{H}_{\overline{A}_d}$ est celui des fonctions mesurables $f : \mathbb{R}^d \to \mathbb{R}$ tel que :

$$\begin{aligned} & \|f\|_{\mathcal{H}_{\overline{A}_k}} := \|f\|_{L^1(\mathbb{R}^d)} \\ & + \sum_{j=1}^{d-1} \int_{\mathbb{R}^j} d\mathbf{u}_{A_j} \left| \int_{\mathbb{R}^{d-j}} d\mathbf{v}_{\overline{A}_j} \int_{\mathbb{R}^{d-j}} d\mathbf{w}_{\overline{A}_j} |f(\mathbf{u}_{A_j}, \mathbf{v}_{\overline{A}_j})| \cdot |f(\mathbf{u}_{A_j}, \mathbf{w}_{\overline{A}_j})| |\mathbf{v}_{\overline{A}_j} - \mathbf{w}_{\overline{A}_j}|^{2\mathbf{H}_{\overline{A}_j} - 2} \right|^{\frac{1}{2}} \\ := & \|f\|_{L^1(\mathbb{R}^d)} + \|f\|_{\mathcal{H}_{\overline{A}_{d-1}}} < \infty. & (43) \end{aligned}$$

Nous montrons alors les deux résultats suivants :

Proposition 10. *Soit A_k défini par (40) et supposons que $f \in \mathcal{H}_{\overline{A}_k} \cap |\mathcal{H}_{\mathbf{H}}|$.*

— *Si $1 \leq k < d$,*

$$\mathbf{H}_{A_k} \to (1, .., 1) \in \mathbb{R}^k \text{ et } \mathbf{H}_{\overline{A}_k} \in \left(\frac{1}{2}, 1\right)^{d-k} \text{ sont fixés .}$$

Alors la famille des variables aléatoires $\left(X^{\mathbf{H}}, \mathbf{H} \in \left(\frac{1}{2}, 1\right)^d\right)$ définie par :

$$X^{\mathbf{H}} := \int_{\mathbb{R}^d} f(\mathbf{u}) dZ_{\mathbf{H}}^{q,d}(\mathbf{u}) \tag{44}$$

converge en distribution vers la variable aléatoire

$$X := \int_{\mathbb{R}^d} f(u^{(1)}, .., u^{(d)}) dZ_{\overline{A}_k}^{q,d-k}(\mathbf{u}_{\overline{A}_k}) d\mathbf{u}_{A_k} = \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^{d-k}} f(\mathbf{u}_{A_k}, \mathbf{u}_{\overline{A}_{A_k}}) dZ_{\overline{A}_k}^{q,d-k}(\mathbf{u}_{\overline{A}_k}) \right) d\mathbf{u}_{A_k}. \tag{45}$$

— *Si $k = d$ et*

$$\mathbf{H} \to (1, .., 1) \in \mathbb{R}^d,$$

alors la famille de variables aléatoires $\left(X^{\mathbf{H}}, \mathbf{H} \in \left(\frac{1}{2}, 1\right)^d\right)$ donnée par (44) converge en distribution vers

$$\int_{\mathbb{R}^d} f(u^{(1)}, .., u^{(d)}) d\mathbf{u} \frac{1}{\sqrt{q!}} H_q(Z)$$

avec $Z \sim N(0,1)$ où H_q le polynôme d'Hermite d'ordre q (3).

Afin d'établir ce résultat nous établissons la convergence de la fonction caractéristique de $X^{\mathbf{H}}$ vers celle de X. L'idée est d'approcher $X^{\mathbf{H}}$ par une suite de variables aléatoires $X^{n,\mathbf{H}}$ qui s'écrivent comme une combinaison linéaire du champ d'Hermite $Z^{q,d}_{\mathbf{H}}$, puis d'utiliser les résultats obtenus dans [3], qui donnent le comportement asymptotique par rapport à $\mathbf{H}$ de ce dernier. Précisément $X^{n,\mathbf{H}}$ est définie comme suit :

$$X^{n,\mathbf{H}} = \int_{\mathbb{R}^d} f_n(\mathbf{u}) dZ^{q,d}_{\mathbf{H}}(\mathbf{u}) = \sum_{j=1}^{n} a_l(\Delta Z^{q,d}_{\mathbf{H}})((\mathbf{t}_l, \mathbf{t}_{l+1}])$$

où $f_n(\mathbf{u}) = \sum_{l=1}^n a_l 1_{(\mathbf{t}_l,\mathbf{t}_{l+1}]}(\mathbf{u})$ et $\Delta Z^{q,d}_{\mathbf{H}}$ est défini à partir de (13). La preuve se base principalement sur le fait que $X^{n,\mathbf{H}}$ converge dans $L^2(\Omega)$ vers $X^{\mathbf{H}}$ si f_n converges vers f dans $|\mathcal{H}_{\mathbf{H}}|$ par théorème non centrale limite et isométrie de l'intégrale de Wiener Hermite (voir section 3 dans [26]).

Proposition 11. *Soit A_k défini par (40) et $B_p = \{l_1, .., l_p\} \subset \{1, .., d\}$ avec $0 \leq p \leq d, 1 \leq k \leq d, p+k \leq d$ et $A_k \cap B_p = \emptyset$ (si $p = 0$ alors $B_p = \emptyset$.). Soit $f \in |\mathcal{H}_{\mathbf{H}}|$.*

Supposons que cette limite existe et est bien définie

$$\lim_{\mathbf{H}_{A_k} \to (\frac{1}{2},\dots,\frac{1}{2}) \in \mathbb{R}^k} \mathbf{H}(2\mathbf{H}-\mathbf{1}) \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}) f(\mathbf{v}) |\mathbf{u}-\mathbf{v}|^{2\mathbf{H}-2} d\mathbf{u} d\mathbf{v} := \sigma^2_{f,\mathbf{H}_{\overline{A}_k}} \tag{46}$$

et que

$$\begin{aligned} &\sup_{\mathbf{H}_{A_k} \in [\frac{1}{2},1]^k} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} d\mathbf{u} d\mathbf{v} d\mathbf{u}' d\mathbf{v}' f(\mathbf{u}) f(\mathbf{u}') f(\mathbf{v}) f(\mathbf{v}') \\ &\times |\mathbf{u}-\mathbf{v}|^{\frac{2(\mathbf{H}-1)\mathbf{r}}{q}} |\mathbf{u}'-\mathbf{v}'|^{\frac{2(\mathbf{H}-1)\mathbf{r}}{q}} |\mathbf{u}-\mathbf{u}'|^{\frac{2(\mathbf{H}-1)(\mathbf{q}-\mathbf{r})}{q}} |\mathbf{v}-\mathbf{v}'|^{\frac{2(\mathbf{H}-1)(\mathbf{q}-\mathbf{r})}{q}} < \infty. \end{aligned} \tag{47}$$

Lorsque

$$\mathbf{H}_{A_k} \to \left(\frac{1}{2}, ..., \frac{1}{2}\right) \in \mathbb{R}^k, \mathbf{H}_{B_p} \to (1, .., 1) \in \mathbb{R}^p \text{ et } \mathbf{H}_{\overline{A}_k \cup \overline{B}_p} \in \left(\frac{1}{2}, 1\right)^{d-k-p} \text{ fixés,}$$

alors $\int_{\mathbb{R}^d} f(\mathbf{u}) dZ^{q,d}_{\mathbf{H}}(\mathbf{u})$ converge en distribution vers la variable aléatoire de loi $N(0, \sigma^2_{f,\mathbf{H}_{\overline{A}_k}})$.

Remarque 0.3.1. *Lorsque $q = 2$ et $d = 1$, nous retrouvons les résultats obtenus dans le chapitre 2 de ce manuscrit. Tandis que lorsque $f = 1$, nous obtenons le comportement asymptotique du champ d'Hermite [3].*

Énonçons à présent les théorèmes qui donnent la limite en loi de la solution de l'équation de la chaleur stochastique dirigée par un champ d'Hermite, définie par (25). Supposons d'abord que la condition (26) qui assure l'existence de la solution mild, est vérifiée.

Théorème 4. *Fixons $T > 0$ et $\mathbf{x} \in \mathbb{R}^d$, on a alors*

1. *Lorsque*

$$(H_0, \mathbf{H}_{A_k}) \to (1, .., 1) \in \mathbb{R}^{k+1} \text{ et } H_j, j \in \overline{A}_k \text{ sont fixés ,}$$

alors $(u_{H_0,\mathbf{H}}(t, \mathbf{x}), t \in [0, T])$ converge en loi dans l'espace $C[0, T]$ vers le processus $(u(t, \mathbf{x}), t \in [0, T])$ défini par :

$$u(t, \mathbf{x}) = \int_0^t du \int_{\mathbb{R}^k} d\mathbf{y}_{A_k} \int_{\mathbb{R}^{d-k}} dZ^{q,d-k}_{\mathbf{H}_{\overline{A}_k}}(\mathbf{y}_{\overline{A}_k}) G(t-u, \mathbf{x}-\mathbf{y}). \tag{48}$$

2. *Lorsque* $\mathbf{H}_{A_k} \to (1,..,1) \in \mathbb{R}^k$ *et* $H_0, H_j, j \in \overline{A}_k$ *sont fixés, alors* $(u_{H_0,\mathbf{H}}(t,\mathbf{x}), t \in [0,T])$ *converge en loi dans* $C[0,T]$ *vers le processus* $(u(t,\mathbf{x}), t \in [0,T])$ *défini par :*

$$u(t,\mathbf{x}) = \int_{\mathbb{R}^k} d\mathbf{y}_{A_k} \int_0^t \int_{\mathbb{R}^{d-k}} dZ^{q,d+1-k}_{H_0,\mathbf{H}_{\overline{A}_k}}(u, \mathbf{y}_{\overline{A}_k}) G(t-u, \mathbf{x}-\mathbf{y}).$$

3. *Lorsque* $(H_0, \mathbf{H}) \to (1,...,1) \in \mathbb{R}^{d+1}$, *alors* $(u_{H_0,\mathbf{H}}(t,\mathbf{x}), t \in [0,T])$ *converge en loi dans* $C[0,T]$ *vers* $(u(t,\mathbf{x}), t \in [0,T])$ *défini par :*

$$u(t,\mathbf{x}) = \left(\int_0^t \int_{\mathbb{R}^d} G(t-u, \mathbf{x}-\mathbf{y}) d\mathbf{y} du\right) \frac{1}{\sqrt{q!}} H_q(Z).$$

Théorème 5. 1. *Lorsque*

$$(H_0, \mathbf{H}_{A_k}) \to \left(\frac{1}{2}, ..., \frac{1}{2}\right) \in \mathbb{R}^{k+1} \tag{49}$$

et

$$d < 1 + \frac{k}{2} + \sum_{a \in \overline{A}_k} H_a, \tag{50}$$

alors le processus $(u_{H_0,\mathbf{H}}(t,\mathbf{x}), t \in [0,T])$ *donné par (25) converge en loi dans* $C[0,T]$ *vers le processus* $(u(t,\mathbf{x}), t \in [0,T])$ *où* u *est la solution mild de l'équation de la chaleur linéaire suivante :*

$$\begin{cases} \frac{\partial u}{\partial t}(t,\mathbf{x}) & = & \Delta u(t,\mathbf{x}) + \dot{W}^{q,d+1}_{H_0,\mathbf{H}}(t,\mathbf{x}), \quad t > 0, \mathbf{x} \in \mathbb{R}^d \\ u(0,\mathbf{x}) & = & 0, \quad \mathbf{x} \in \mathbb{R}^d \end{cases} \tag{51}$$

où le bruit noté $\left(W_{H_0,\mathbf{H}}(t, A_1 \times A_2), t \in [0,T], A_1 \in \mathcal{B}_b(\mathbb{R}^k), A_2 \in \mathcal{B}_b(\mathbb{R}^{d-k})\right)$ *associé à l'équation de la chaleur, est un champ gaussien qui a pour covariance :*

$$\begin{aligned} & \mathbb{E}\left[W_{H_0,\mathbf{H}}(t, A_1 \times A_2) W_{H_0,\mathbf{H}}(s, B_1 \times B_2)\right] \\ = \quad & (t \wedge s) \lambda_k(A_1 \cap B_1) \int_{A_2 \cap B_2} \mathbf{H}_{\overline{A}_k}(2\mathbf{H}_{\overline{A}_k} - \mathbf{1}) |\mathbf{y}_{\overline{A}_k} - \mathbf{z}_{\overline{A}_k}|^{2\mathbf{H}_{\overline{A}_k} - 2} d\mathbf{y}_{\overline{A}_k} d\mathbf{z}_{\overline{A}_k}. \end{aligned}$$

2. *Lorsque* $\mathbf{H}_{A_k} \to \left(\frac{1}{2}, ..., \frac{1}{2}\right) \in \mathbb{R}^k$, $\mathbf{H}_{B_p} \to (1,..,1) \in \mathbb{R}^p$ *et*

$$d < 2H + \frac{k}{2} + \sum_{a \in \overline{A}_k} H_a, \tag{52}$$

alors le processus $(u_{H_0,\mathbf{H}}(t,\mathbf{x}), t \in [0,T])$ *défini par (25) converge en loi dans* $C[0,T]$ *vers le processus* $(u(t,\mathbf{x}), t \in [0,T])$ *où* u *est la solution mild de l'équation de la chaleur (51) avec le bruit additif gaussien admettant la covariance suivante :*

$$\begin{aligned} & \mathbf{E}\left[W_{H_0,\mathbf{H}}(t, A_1 \times A_2) W_{H_0,\mathbf{H}}(s, B_1 \times B_2)\right] \\ = \quad & R_{H_0}(t,s) \lambda_k(A_1 \cap B_1) \int_{A_2 \cap B_2} \mathbf{H}_{\overline{A}_k}(2\mathbf{H}_{\overline{A}_k} - \mathbf{1}) |\mathbf{y}_{\overline{A}_k} - \mathbf{z}_{\overline{A}_k}|^{2\mathbf{H}_{\overline{A}_k} - 2} d\mathbf{y}_{\overline{A}_k} d\mathbf{z}_{\overline{A}_k}. \end{aligned}$$

3. *Lorsque* $(H_0, \mathbf{H}) \to \left(\frac{1}{2}, ..., \frac{1}{2}\right) \in \mathbb{R}^{d+1}$ *et* $d = 1$, *alors la limite du processus* $(u_{H_0,\mathbf{H}}, t \in [0,T])$ *dans* $C[0,T]$ *est la solution de l'équation de la chaleur (51) dirigée par un bruit blanc espace-temps.*

Pour avoir la convergence en loi dans l'espace des fonctions continues $C([0,T])$, il suffit d'avoir la convergence fini-dimensionnelle de la solution mild et de montrer que la solution est tendue.

La convergence des lois fini-dimensionnelles de la solution mild est établie dans les deux théorèmes par application des propositions 10 et 11. Soient $\lambda_j \in \mathbb{R}, t_j \geq 0$ pour $j = 1, .., N$, nous avons par linéarité de l'intégrale de Wiener-Hermite :

$$\sum_{j=1}^{N} \lambda_j u_{H_0,\mathbf{H}}(t_j, \mathbf{x}) = \int_0^{\infty} \int_{\mathbb{R}^d} \left(\sum_{j=1}^{N} \lambda_j 1_{(0,t_j)}(u) G(t_j - u, \mathbf{x} - \mathbf{y}) \right) dZ^{q,d+1}_{H_0,\mathbf{H}}(u, \mathbf{y}).$$

Nous montrons que l'intégrant $\sum_{j=1}^{N} \lambda_j 1_{(0,t_j)}(u) G(t_j - u, \mathbf{x} - \mathbf{y})$ vérifie les hypothèses de ces propositions en utilisant le théorème *Power Counting*. Cette partie constitue la partie la plus technique des preuves et nous renvoyons au chapitre 3 pour plus de détails sur les calculs effectués.

Sachant que $u_{H_0,\mathbf{H}}(t, \mathbf{x})$ s'écrit comme une intégrale multiple d'ordre $(q+1)$, la caractérisation de la tension est donnée en combinant la relation (27) et la propriété d'hypercontractivité des intégrales multiples. On obtient alors pour tout $p \geq 2$

$$\mathbb{E} \left| u_{H_0,\mathbf{H}}(t, \mathbf{x}) - u_{H_0,\mathbf{H}}(s, \mathbf{x}) \right|^{2p} \leq C |t - s|^{\gamma p} \tag{53}$$

et on conclut en utilisant le critère Billingsley (voir [15, Theorem 12.3] ou [16]).

Remarque 0.3.2. *Il est particulièrement intéressant de remarquer que le processus limite est toujours égal en loi à la solution de l'équation de la chaleur stochastique dirigée par la limite du bruit.*

Dans une deuxième partie de ce travail, nous avons généralisé les résultats obtenus au chapitre 2 en établissant la limite en loi d'un processus d'Ornstein Uhlenbeck type Hermite, solution de l'équation de Langevin dirigée par un processus d'Hermite d'ordre $q \geq 1$ quelconque.

Les résultats obtenus dans ce chapitre montrent que ces modèles offrent une large classe processus appartenant à des chaos de Wiener d'ordre $q \geqslant 1$ et ayant des lois différentes.

0.4 Partie 3 : Inférence statistique des EDPS avec bruit fractionnaire.

0.4.1 Rappels de résultats préliminaires

Dans la dernière partie de cette thèse, nous nous intéressons à une problématique d'inférence statistique relative à une EDPS fractionnaire. Depuis plusieurs décennies une littérature très vaste s'est constituée autour du sujet de l'inférence statistique des paramètres apparaissant dans une EDPS. Nous renvoyons à [24], [52] ou [51] pour un aperçu général sur le sujet. Le développement du calcul stochastique par rapport aux processus fractionnaire tel le mbf et la résolution de certaines équations dirigées par ce processus, ont conduit naturellement à l'étude des estimateurs dans des modèles avec des bruits fractionnaires. Nous citons entre autres les travaux [4], [44], [49], [86], [104] pour des contributions dans ce domaine. Dans le contexte des EDPS fractionnaires, c'est le paramètre de dérive qui a suscité le plus d'intérêt alors que peu de travaux ont considéré l'estimation du paramètre fractionnaire H. Soulignons toutefois que l'estimation de ce paramètre est un problème central lorsqu'on étudie les processus auto similaires. Il y a tout un intérêt à estimer ce paramètre du fait qu'il caractérise les propriétés mathématiques de ces modèles et par conséquent décrit le comportement du système physique qu'ils modélisent. Il existe dans la littérature diverses méthodes pour estimer l'exposant de Hurst H appliquées au processus auto similaires d'indice H telles : méthode R/S, méthode log périodogramme, estimateur du maximum de vraisemblance (estimateur de Wittle), la décomposition en ondelettes ou l'approche basée sur les variations discrètes. C'est cette dernière méthode que nous allons développer dans le cadre de l'équation des ondes fractionnaire.

Dans ce travail, les propriétés des estimateurs de H sont données à partir de l'étude du comportement asymptotique des variations généralisées et sur la base d'observations discrètes de la solution en espace et à t fixé. Avant de décrire ce travail, précisons quelques outils utilisés.

Soit X et Y deux variables aléatoires à valeurs dans $\mathbb{R}^d$. La distance entre la loi de X notée $\mathcal{L}(X)$ et la loi de Y notée $\mathcal{L}(Y)$ est donnée par :

$$d(\mathcal{L}(X),\mathcal{L}(Y)) = \sup_{h\in\mathcal{A}} |\mathbb{E}(h(X)) - \mathbb{E}(h(Y))|$$

où $\mathcal{A}$ est une classe de fonctions tests appropriée.

Spécifions les distances auxquelles nous ferons référence par la suite :

- ▷ Lorsque $\mathcal{A} = \{h : \mathbb{R}^d \to \mathbb{R}; h(z_1,\dots,z_d) = 1_{(-\infty,z_1]}\cdots 1_{(-\infty,z_1]}, z_1,\dots,z_d \in \mathbb{R}\}$, on obtient la distance de Kolmogorov :

$$d_K(\mathcal{L}(X),\mathcal{L}(Y)) = \sup_{z\in\mathbb{R}} |P(X \leqslant z) - P(Y \leqslant z)|.$$

- ▷ Lorsque $\mathcal{A} = \{h : \mathbb{R}^d \to \mathbb{R}; \|h\|_{Lip} \leqslant 1\}$ où $\|h\|_{Lip} = \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|h(x)-h(y)|}{\|x-y\|_{\mathbb{R}^d}}$, on obtient la distance de Wasserstein :

$$d_W(\mathcal{L}(X),\mathcal{L}(Y)) = \sup_{h\in\mathcal{A}} |\mathbb{E}(h(X)) - \mathbb{E}(h(Y))|\,. \tag{54}$$

- ▷ Lorsque $\mathcal{A} = \{h : \mathbb{R}^d \to \mathbb{R}; h(z_1,\dots,z_d) = 1_B(z_1,\dots,z_d), B \in \mathcal{B}(\mathbb{R}^d)\}$, on obtient la distance en variation totale :

$$d_{TV}(\mathcal{L}(X),\mathcal{L}(Y)) = \sup_{B\in\mathcal{B}(\mathbb{R}^d)} |P(X\in B) - P(Y \in B)|\,. \tag{55}$$

0.4.2 Résumé du chapitre 4 : Les variations généralisées et estimation du paramètre de Hurst de la solution de l'EDPS des ondes dirigée par un bruit fractionnaire

Le chapitre 4 est constitué de la publication [87], en collaboration avec R. Shevchenko et C.A. Tudor . Le cadre précis de ce travail est l'équation stochastique des ondes soumise à un bruit gaussien additif W^H :

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) &= \Delta u(t,x) + \dot{W}^H(t,x),\ t\geq 0,\ x\in\mathbb{R}^d,\ d\geq 1,\\ u(0,x) &= 0, \quad x\in\mathbb{R}^d,\\ \frac{\partial u}{\partial t}(0,x) &= 0, \quad x\in\mathbb{R}^d, \end{cases}$$

où Δ est le Laplacien de $\mathbb{R}^d$, $d \geqslant 1$. Le bruit considéré, à savoir $W_H = \{W_t^H(A); t\in[0,T], A\in B_b(\mathbb{R}^d)\}$ pour $H \in \left(\frac{1}{2},1\right)$, est défini comme le champ aléatoire gaussien centré, défini sur un espace de probabilité filtré complet $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geqslant 0},\mathbb{P})$ et qui admet la fonction de covariance suivante :

$$\mathbb{E}\left(W_t^H(A)W_s^H(B)\right) = R_H(t,s)\lambda(A\cap B), \forall A,B\in\mathcal{B}_b(\mathbb{R}^d), \tag{56}$$

où R_H est la covariance du mouvement brownien fractionnaire, et $B_b(\mathbb{R}^d)$ la classe des boréliens de $\mathbb{R}^d$. La solution mild du système (56) est définie comme un champ aléatoire gaussien centré et s'exprime par une intégrale de Wiener par rapport au processus gaussien W^H :

$$u(t,\mathbf{x}) = \int_0^t \int_{\mathbb{R}^d} G_1(t-s,\mathbf{x}-\mathbf{y}) W^H(\mathrm{d}s,\mathrm{d}\mathbf{y}), \quad t\geq 0, \mathbf{x}\in\mathbb{R}^d, \tag{57}$$

où G_1 est la solution fondamentale de l'équation des ondes stochastique. Dans le cas d'étude $d = 1$, on a pour $t \geq 0$ et $x \in \mathbb{R}$, $G_1(t,x) = \frac{1}{2}\mathbb{1}_{\{|x|<t\}}$.

Ce problème a été initialement étudié dans [11] , puis a fait l'objet de plusieurs contributions et extensions. Citons, entre autres, [25], [42], [48], [85].

La motivation principale de ce travail vient de [48] où les auteurs ont construit un estimateur de H à partir de l'étude des variations quadratiques de la solution (57) par rapport à la variable espace. Plus précisément, ils établissent un théorème centrale limite pour les variations quadratiques renormalisées et construisent un estimateur asymptotiquement gaussien pour $H \in (\frac{3}{4}, 1)$. Les résultats que nous détaillons dans le chapitre 4 ont permis d'éviter le changement de régime qui apparaît lorsque $H > \frac{3}{4}$ dans le comportement asymptotique de l'estimateur de H, et de construire ainsi des estimateurs asymptotiquement gaussiens pour tout $H \in (\frac{1}{2}, 1)$ ce qui est fort utile en estimation statistique, notamment pour construire des intervalles de confiance.

L'approche adoptée est de considérer des variations définies sur un filtre, c'est à dire de remplacer les accroissements simples par des accroissements d'ordres supérieurs. Cette démarche a été initiée par Istas et Lang dans [45] puis utilisée dans de nombreux travaux comme [28] pour le mbf, [21] pour le processus de Rosenblatt et [22] pour les processus d'Hermite.

Précisons à ce stade quelques définitions. Un filtre $\alpha = (a_0, a_1, .., a_l)$ d'ordre $p \geqslant 1$ et de longueur $l+1$ désigne un vecteur dont les composantes réelles vérifient :

$$\sum_{q=0}^{l} \alpha_q q^r = 0, \text{ pour } \ r = 0, \dots p-1 \text{ et } \sum_{q=0}^{l} \alpha_q q^p \neq 0.$$

Nous définissons U^α le processus filtré associé à la solution par :

$$U^\alpha\left(\frac{i}{N}\right) = \sum_{r=0}^{l} a_r u\left(t, \frac{i-r}{N}\right) \text{ pour } i = l, .., N. \tag{58}$$

On retrouve pour un filtre de longueur 1 et d'ordre 1 avec $\alpha = (1, -1)$ les accroissements de la solution : $u(t, x_{i+1}) - u(t, x_i)$ et pour le filtre de longueur 2 et d'ordre 2 avec $\alpha = (1, 2, -1)$ les différences secondes : $u\left(t, \frac{i}{N}\right) - 2u\left(t, \frac{i-1}{N}\right) + u\left(t, \frac{i-2}{N}\right)$. Nous verrons qu'il suffit de considérer ce dernier filtre pour contourner la restriction en $\frac{3}{4}$.

Nous définissons à présent les k-variations de la solution par :

$$V_N(k,\alpha) = \frac{1}{N-l} \sum_{i=l}^{N} \left[\frac{\left|U^\alpha\left(\frac{i}{N}\right)\right|^k}{\mathbb{E}\left|U^\alpha\left(\frac{i}{N}\right)\right|^k} - 1 \right] \tag{59}$$

Soit $G_N(k,\alpha) = \sqrt{N-l}V_N(k,\alpha)$. Le théorème central de ce chapitre est le suivant :

Théorème 6. *Soit un filtre α d'ordre $p \geq 1$ et de longueur $l+1 \geq 1$.*

- *Quand $p > H + \frac{1}{4}$, la suite $(G_N(k,\alpha))_{N\geq 1}$ converge en loi , quand $N \to \infty$, vers la variable normale $N(0, \sigma^2)$.*
- *Quand $p = 1$, $H = 3/4$, la suite $\left(\frac{1}{\sqrt{\log(N-l)}} G_N(k,\alpha)\right)_{N\geq 1}$ converge en loi vers $N(0,\, c^2)$.*

De plus, les constantes σ^2, c^2 ont été explicités.

La preuve de ce résultat se base sur la décomposition chaotique de $V_N(k,\alpha)$ en une somme infinie d'intégrales multiples d'ordre 2 à l'infini, par rapport au processus gaussien $(u(t,x))_{x\in\mathbb{R}}$, comme suit :

$$V_N(k,\alpha) = \sum_{q\geq 1} I_{2q}(f_{N,2q})$$

avec

$$f_{N,2q} = \frac{c_{2q}^k}{(2q)!} \frac{1}{N-l} \sum_{i=l}^{N} \frac{C_{i,\alpha}^{\otimes 2q}}{(\pi_H^{\alpha,N}(0))^q}. \tag{60}$$

La norme dans L^2 de chaque terme de la décomposition est analysée en utilisant la propriété d'orthogonalité des intégrales multiples. La condition $p > H + \frac{1}{4}$ découle alors de la condition de convergence du terme dominant de $(2q)!\|f_{N,2q}\|^2_{\mathcal{H}^{\otimes 2q}}$ renormalisé par $(N-l)$. On conclut en utilisant le théorème 6.3.1 dans [66] et en démontrant les points suivants pour $p > H + 1/4$ et $g_{N,2q} = \sqrt{N-l} f_{N,2q}$:

1. $(2q)!\|g_{N,2q}\|^2_{\mathcal{H}^{\otimes 2q}} \to_{N\to\infty} \sigma^2_{2q}$ et $\sigma^2 := \sum_{q\geq 1} \sigma^2_{2q} < \infty$,
2. pour tout $q \geq 1$ et $r = 1, .., 2q-1$, $\|g_{N,2q} \otimes_r g_{N,2q}\|_{\mathcal{H}^{\otimes 4q-2r}} \to_{N\to\infty} 0$,
3. $\lim_{M\to\infty} \sup_{N\geq 1} \sum_{q\geq M+1} (2q)!\|g_{N,2q}\|^2_{\mathcal{H}^{\otimes 2q}} = 0$,

et pour $p = 1$, $H = 3/4$,

1. $\frac{1}{\log(N-l)}(2q)!\|g_{N,2q}\|^2_{\mathcal{H}^{\otimes 2q}} \to_{N\to\infty} 1_{\{q=1\}} c^2$,
2. pour tout $q \geq 1$ et $r = 1, .., 2q-1$, $\frac{1}{\log(N-l)}\|g_{N,2q} \otimes_r g_{N,2q}\|_{\mathcal{H}^{\otimes 4q-2r}} \to_{N\to\infty} 0$,
3. $\lim_{M\to\infty} \sup_{N\geq 1} \sum_{q\geq M+1} \frac{1}{\log(N-l)}(2q)!\|g_{N,2q}\|^2_{\mathcal{H}^{\otimes 2q}} = 0$.

On retrouve le résultat prouvé dans [48] pour $p = 1$: la convergence en loi n'est valable que pour $H \in (\frac{3}{4}, 1)$ et qu'elle est valable pour tout $H \in (0,1)$ dès que $p \geqslant 2$.

Nous donnons pour k pair une borne pour la distance de Wassserstein entre la loi des k-variations normalisées $G_N(k,\alpha)$ et celle de la gaussienne multivariée, et qui donne lieu à ce résultat :

Théorème 7. *Soit k pair.*

— *Soit $Z \sim N(0,\sigma^2)$. Lorsque $p \geq 2$, il existe une constante $C > 0$ tel que :*

$$d_W(G_N(k,\alpha), Z) \leq C \frac{1}{\sqrt{N}}.$$

— *Soit $Z \sim N(0,c^2)$. Lorsque $p = 1$ et $H < 3/4$, il existe une constante $C > 0$ tel que :*

$$d_W(G_N(k,\alpha), Z) \leq C \begin{cases} \frac{1}{\sqrt{N}} & si\ H \in (\frac{1}{2}, \frac{5}{8}), \\ \frac{\log(N)^{3/2}}{\sqrt{N}} & si\ H = \frac{5}{8}, \\ N^{4H-3} & si\ H \in (\frac{5}{8}, \frac{3}{4}). \end{cases}$$

La preuve de ce théorème se base sur la méthode de Stein-Malliavin, principalement sur le corollaire 3.6 dans [68] que nous rappelons ci dessous.

Corollaire 2. *Soient $d \geq 2$ et $1 \leq q_1 \leq \cdots \leq q_d$. Considérons le vecteur $F := (F_1, \ldots F_d) = (I_{q_1}(f_1) \ldots I_{q_d}(f_d))$ avec $f_i \in \mathcal{H}^{\odot q_i}$ pour tout $i = 1, \ldots, d$. Soit $Z \sim N_d(0, C)$ avec C une matrice définie positive. Alors,*

$$d_W(F, Z) \leq c \sqrt{\sum_{1\leq i,\, j\leq d} \mathbb{E}\left[\left(C_{ij} - \frac{1}{q_j}\langle DF_i, DF_j\rangle_{\mathcal{H}}\right)^2\right]},$$

où c est une constante universelle strictement positive.

Remarque 0.4.1. *On retrouve, pour $p = 1$ et $k = 2$, les bornes obtenues dans [48] et qui correspondent aussi à celles des variations quadratiques du mouvement brownien fractionnaire. Soulignons le fait que pour $k = 2$, il est possible d'obtenir des bornes optimales pour la vitesse en variation totale en se basant sur le critère démontré dans [67].*

Dans le cas de non normalité $p = 1$ et $H > \frac{3}{4}$, nous montrons que les variations quadratiques après renormalisation convergent en loi vers une variable aléatoire qui vit dans le second chaos de Wiener et dont on a explicité les cumulants. Cette variable est plus complexe que le processus de Rosenblatt.

En s'inspirant du travail de Coeurjolly [28], nous proposons trois estimateurs de H, prouvons qu'ils sont fortement consistants et établissons leur vitesse de convergence. Décrivons brièvement la démarche adoptée : soit $S_N(k,\alpha)$ le k ème moment absolu empirique des variations discrètes de la solution défini par :

$$S_N(k,\alpha) = \frac{1}{N-l}\sum_{i=l}^{N-1}\left|U^{\alpha}\left(\frac{i}{N}\right)\right|^{k}. \tag{61}$$

La normalité de U^{α} implique :

$$\mathbb{E}\left(S_N(k,\,\alpha)\right) = \left(\pi_H^{\alpha,\,N}(0)\right)^{\frac{k}{2}} E_k.$$

En estimant $\mathbb{E}[S_N(k,\,\alpha)]$ par $S_N(k,\,\alpha)$, un premier estimateur est obtenu en résolvant l'équation suivante :

$$S_N(k,\,\alpha)^{\frac{2}{k}} - E_k^{\frac{2}{k}}\pi_x^{\alpha,\,N}(0) = 0$$

avec

$$\pi_x^{\alpha,\,N}(0) = \frac{t}{2N^{2x}}\Phi_{x,\,\alpha}(0) - \frac{c_x}{N^{2x+1}}\Phi_{x+\frac{1}{2},\,\alpha}(0).$$

En supposant que la taille de l'échantillon N est très grande, (ce qui est toujours le cas en pratique), l'inversibilité $g(x) := \pi_x^{\alpha,\,N}(0)$ est assurée sans trop de difficulté en suivant le raisonnement effectué dans [28]. On obtient alors cette première classe d'estimateur :

$$\widehat{H}_{k,\,N} := \left(\pi_{\cdot}^{\alpha,\,N}(0)\right)^{-1}\left(\left(\frac{S_N(k,\,\alpha)}{E_k}\right)^{\frac{2}{k}}\right). \tag{62}$$

Pour des motivations pratiques, nous définissons une seconde classe d'estimateur en n'inversant que la partie dominante, asymptotiquement parlant de g, donné par $\frac{t}{2N^{2x}}\Phi_{x,\,\alpha}(0) =: \bar{g}(x)$. Nous obtenons alors :

$$\bar{H}_{k,\,N} := \bar{g}^{-1}\left(\left(\frac{S_N(k,\,\alpha)}{E_k}\right)^{\frac{2}{k}}\right). \tag{63}$$

En utilisant les résultats obtenus précédemment, nous démontrons que ces deux estimateurs sont fortement consistants et admettent des vitesses de convergence égales à : $\sqrt{N}\log(N)$ pour $H < p - \frac{1}{4}$, $\sqrt{N}\sqrt{\log(N)}$ pour $(H = \frac{3}{4},\, p = 1)$, et $N^{2-2H}\log(N)$ pour $(H > \frac{3}{4},\, \alpha = (1,\,-1),\, k = 2)$.

Enfin, en supposant que la variable temps $t > 1$ est inconnue, on construit un troisième estimateur noté $\tilde{H}_N$, défini par :

$$\tilde{H}_N := \frac{1}{k}\log_2\left(\frac{S_N(k,\,a^{(2)})}{S_N(k,\,a^{(1)})}\right)$$

où $(a_i^{(1)})_{i\in\{0,\ldots,p\}}$ et $(a_i^{(2)})_{i\in\{0,\ldots,2p\}}$ sont deux filtres tels que $a^{(2)}$ est obtenu en dilatant deux fois la suite $a^{(1)}$ (i-e., $a_{2k}^{(2)} := a_k^{(1)}$ pour $k \in \{0,\ldots,p\}$ et 0 sinon). Nous démontrons que cet estimateur est fortement consistant pour tout $H \geq \frac{1}{2}$. De plus, pour $H < p - \frac{1}{4}$ il est asymptotiquement normal et de vitesse de convergence égale à $\sqrt{N}$. Ce dernier résultat est établi grâce à un TCL multivarié établi pour $k = 2$.

Première partie

Solutions d'ESPS avec bruit fractionnaire non gaussien

Chapitre 1

On the linear stochastic heat equation with Hermite noise

1.1 Introduction

In this work we analyze the stochastic heat equation driven by a Hermite process. The Hermite processes are self-similar stochastic processes with stationary increments. The class of Hermite processes includes the fractional Brownian motion (fBm in the sequel), which is the only Gaussian Hermite process, and the Rosenblatt process.

Since the pioneering work of Walsh [108], probabilists have considering the problem of solving stochastic partial differential equations (SPDEs) in general (and the heat equation in particular) driven by noises that are more general than the time-space white noise, possibly with a correlation structure in time and/or in space. An important particular example is the case when the noise behaves as a fractional Brownian motion in time and/or in space (see e.g. [103] and the references therein), which appearead as a natural consequence of the stochastic calculus with respect to fBm.

Recently, several authors enlarged the class of noises by considering non-Gaussian stochastic processes, such as the Hermite processes (see, among others, [17], [26], [31], [34], [33], [100]). We will focus here on the heat equation with additive Hermite noise (i.e. the noise is a multiparameter Hermite process) and we study the existence and various properties of its mild solution. The solution can be expressed as a Wiener integral with respect to the multiparameter Hermite process. This integral (called in the sequel *Wiener-Hermite integral*) has been introduced in [26]. It has many similarities with the Wiener integral with respect to the fractional Brownian motion ; in particular it preserves the standard Wiener isometry. This allows to get several properties of the solution (existence, covariance, Hölder regularity etc) without much cost, by using the known results from the Gaussian case. On the other hand, other properties of the solution, especially those related to its probability distribution, are more complex because the solution is non-Gaussian, being an element of the qth Wiener chaos. Our purpose is to understand various properties of the solution of the heat equation with non-Gaussian Hermite noise, such as existence, regularity, spectral representation, probability distribution, variation etc. One of our main results is a decomposition theorem for the solution, which extends results in [43], [105]. Recall that in the Gaussian case, the solution to the heat equation driven by an additive time-space fractional Brownian motion can be written, with respect to its time variable, as the sum of a fBm and a smooth Gaussian process. Therefore, many properties of the solution can be deduced from those of the fBm (see [39], [105]). In the non-Gaussian case, our decomposition theorem says that the solution of the heat equation with Hermite noise can be written as the sum of a self-similar process with stationary increments (sssi in the sequel) in the qth Wiener chaos (this is usually called the pinned string process) and a smooth process. In the case $q = 1$, this sssi process from the decomposition is the fBm, which is the only Gausssian sssi process. In the chaos of order strictly higher than 1, one cannot conclude that this pinned string process is a Hermite process. Some properties of this pinned string process are discussed, with a focus on the case $q = 2$, and these properties suggest that the process is not necessarily a Hermite process. As an application of our decomposition theorem, we deduce the α-variation of the solution. Finally we discuss the relation between the mild and weak solution to the heat equation driven by Hermite noise.

We organized our work as follows. Section 2 contains some preliminaries on the multiparameter Hermite processes and the Wiener integral with respect to them. In Section 3 we include our main results on the properties of the solution to the linear stochastioc heat equation with Hermite noise. Section 4 in the appendix which contains the basic tools from the stochastic analysis on Wiener space.

1.2 The Hermite sheet and the Wiener integral with respect to the Hermite sheet

In this preliminary section we recall some basic definitions and properties related to the multiparameter Hermite processes and the Wiener integral with respect to it. We also refer to the monographs [103] and [81] for a more detailed exposition.

1.2.1 The Hermite sheet

We start by introducing multidimensional notation used in the paper. Assume $d \in \mathbb{N}\setminus\{0\}$ and consider the d-dimensional vectors $\mathbf{a} = (a_1, a_2, \ldots, a_d)$, $\mathbf{b} = (b_1, .., b_d)$, $\boldsymbol{\alpha} = (\alpha_1, .., \alpha_d)$ in $\mathbb{R}^d$. We use the following convention :

$$\mathbf{ab} = \prod_{j=1}^{d} a_j b_j, \quad |\mathbf{a} - \mathbf{b}|^{\alpha} = \prod_{i=1}^{d} |a_i - b_i|^{\alpha_i}$$

$$\mathbf{a}/\mathbf{b} = (a_1/b_1, a_2/b_2, \ldots, a_d/b_d), \quad [\mathbf{a}, \mathbf{b}] = \prod_{i=1}^{d} [a_i, b_i], \quad (\mathbf{a}, \mathbf{b}) = \prod_{i=1}^{d} (a_i, b_i),$$

$$\sum_{\mathbf{i} \in [\mathbf{0}, \mathbf{N}]} a_{\mathbf{i}} = \sum_{i_1=0}^{N_1} \sum_{i_2=0}^{N_2} \cdots \sum_{i_d=0}^{N_d} a_{i_1, i_2, \ldots, i_d}, \quad \mathbf{a}^{\mathbf{b}} = \prod_{i=1}^{d} a_i^{b_i}, \ \mathbf{a} < \mathbf{b} \text{ iff } a_1 < b_1, a_2 < b_2, \ldots, a_d < b_d$$

(analogously for the other inequalities).

Let $q \geq 1$ be an integer and consider the Hurst multi-index $\mathbf{H} = (H_1, H_2, \ldots, H_d) \in (\frac{1}{2}, 1)^d$. The *Hermite sheet of order* q is given by

$$\begin{aligned} Z^q_{\mathbf{H}}(\mathbf{t}) &= c(\mathbf{H}, q) \int_{\mathbb{R}^{d \cdot q}} \int_0^{t_1} \cdots \int_0^{t_d} \left(\prod_{j=1}^{q} (s_1 - y_{1,j})_+^{-\left(\frac{1}{2} + \frac{1-H_1}{q}\right)} \cdots (s_d - y_{d,j})_+^{-\left(\frac{1}{2} + \frac{1-H_d}{q}\right)} \right) \\ & \quad ds_d \ldots ds_1 \quad dW(y_{1,1}, \ldots, y_{d,1}) \ldots dW(y_{1,q}, \ldots, y_{d,q}) \\ &= c(\mathbf{H}, q) \int_{\mathbb{R}^{d \cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^{q} (\mathbf{s} - \mathbf{y}_j)_+^{-\left(\frac{1}{2} + \frac{1-\mathbf{H}}{q}\right)} d\mathbf{s} \quad dW(\mathbf{y}_1) \ldots dW(\mathbf{y}_q) \end{aligned} \tag{1.1}$$

where $x_+ = \max(x, 0)$ and $\mathbf{t} \geq 0$. The above stochastic integral is a multiple stochastic integral with respect to the Wiener sheet $(W(\mathbf{y}), \mathbf{y} \in \mathbb{R}^d)$, (see the Appendix). The constant $c(\mathbf{H}, q)$ ensures that $\mathbf{E}\left(Z^q_{\mathbf{H}}(\mathbf{t})\right)^2 = \mathbf{t}^{2\mathbf{H}}$ for every $\mathbf{t} \geq 0$. As pointed out before, when $q = 1$, (1.1) is the fractional Brownian sheet with Hurst multi-index $\mathbf{H} = (H_1, H_2, \ldots, H_d) \in (\frac{1}{2}, 1)^d$. For $q \geq 2$ the process $Z^q_{\mathbf{H}}(\mathbf{t})$ is not Gaussian and for $q = 2$ we denominate it as the *Rosenblatt sheet*.

The Hermite sheet is a $(H_1, .., H_d)$- self-similar stochastic process. That means that for any $\mathbf{h} = (h_1, \ldots, h_d) > 0$ the stochastic process $\left(\hat{Z}^q_{\mathbf{H}}(\mathbf{t})\right)_{\mathbf{t} \in (\mathbb{R}_+)^d}$ given by

$$\hat{Z}^q_{\mathbf{H}}(\mathbf{t}) = \mathbf{h}^{\mathbf{H}} \hat{Z}^q_{\mathbf{H}}\left(\frac{\mathbf{t}}{\mathbf{h}}\right) = h_1^{H_1} ... h_d^{H_d} \hat{Z}^q_{\mathbf{H}}\left(\frac{t_1}{h_1}, \ldots, \frac{t_d}{h_d}\right)$$

has the same finite dimensional distributions as the process $Z^q_{\mathbf{H}}$.

The Hermite sheet also has stationnary increments. Let us recall that the increment of a d-parameter process X on a rectangle $[\mathbf{s}, \mathbf{t}] \subset \mathbb{R}^d$, $\mathbf{s} = (s_1, \ldots, s_d), \mathbf{t} = (t_1, \ldots, t_d)$, with $\mathbf{s} \leq \mathbf{t}$ (denoted by $\Delta X([\mathbf{s}, \mathbf{t}])$) is given by

$$\Delta X([\mathbf{s}, \mathbf{t}]) = \sum_{r \in \{0,1\}^d} (-1)^{d - \sum_{i=1}^d r_i} X_{\mathbf{s} + \mathbf{r} \cdot (\mathbf{t} - \mathbf{s})}. \tag{1.2}$$

When $d = 1$ one obtains $\Delta X([\mathbf{s}, \mathbf{t}]) = X_t - X_s$ while for $d = 2$ one gets $\Delta X([\mathbf{s}, \mathbf{t}]) = X_{t_1, t_2} - X_{t_1, s_2} - X_{s_1, t_2} + X_{s_1, s_2}$, the rectangular increment.

The fact that the process $(Z^q_{\mathbf{H}}(\mathbf{t}), \mathbf{t} \in \mathbb{R}^d)$ has stationary increments means that for every $\mathbf{h} > 0, \mathbf{h} \in \mathbb{R}^d$ the stochastic processes $(\Delta Z^q_{\mathbf{H}}([0, \mathbf{t}]), \mathbf{t} \in \mathbb{R}^d)$ and $(\Delta Z^q_{\mathbf{H}}([\mathbf{h}, \mathbf{h} + \mathbf{t}]), \mathbf{t} \in \mathbb{R}^d)$ have the same finite dimensional distributions.

Moreover, its covariance is the same for every $q \geq 1$ and it coincides with the covariance of the d-parameter fractional Brownian motion, i.e.

$$\mathbf{E}\left[Z^q_{\mathbf{H}}(\mathbf{t}) Z^q_{\mathbf{H}}(\mathbf{s})\right] = \prod_{i=1}^d \left(\frac{1}{2}\left(t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i}\right)\right) \text{ for every } \mathbf{s}, \mathbf{t} > 0.$$

The paths of $Z^q_{\mathbf{H}}$ are Hölder continuous of order $\boldsymbol{\delta} = (\delta_1, .., \delta_d)$ for every $\boldsymbol{\delta} \in (0, \mathbf{H})$, as a consequence of the following relation (see [26] for its proof) which holds for every $p \geq 2$ and $0 \leq \mathbf{s} \leq \mathbf{t}$,

$$\mathbf{E}\left|\Delta Z^q_{\mathbf{H}}([\mathbf{s}, \mathbf{t}])\right|^p = \mathbf{E}\left|Z_{\mathbf{1}}\right|^p \left(|t_1 - s_1| \cdots |t_d - s_d|\right)^{p\mathbf{H}} = |\mathbf{t} - \mathbf{s}|^{p\mathbf{H}}.$$

and of the Kolmogorov continuity criterium.

1.2.2 Wiener-Hermite integrals

Consider the Hilbert space $\mathcal{H} := \mathcal{H}(\mathbb{R}^d)$ given by

$$\mathcal{H} = \left\{f : \mathbb{R}^d \to \mathbb{R} : \|f\|^2_{\mathcal{H}} < +\infty\right\} \tag{1.3}$$

where

$$\|f\|^2_{\mathcal{H}} := \mathbf{H}(\mathbf{2H} - \mathbf{1}) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(\mathbf{u}) f(\mathbf{v}) |\mathbf{u} - \mathbf{v}|^{\mathbf{2H} - \mathbf{2}} d\mathbf{u} d\mathbf{v}.$$

This space is actually the canonical Hilbert space associated to the d- dimensional fractional Brownian sheet.

For $f \in \mathcal{H}(\mathbb{R}^d)$, it is possible to construct Wiener integrals with respect to the Hermite sheet (see [26], see also [53] for the one-parameter case). This object, called in the sequel as *Wiener-Hermite integral*, will be denoted by

$$\int_{\mathbb{R}^d} f(\mathbf{s}) dZ^q_{\mathbf{H}}(\mathbf{s}).$$

It is well-defined for $f \in \mathcal{H}(\mathbb{R}^d)$ and it is the element of the qth Wiener chaos given by

$$\int_{\mathbb{R}^d} f(\mathbf{s}) dZ^q_{\mathbf{H}}(\mathbf{s}) = I_q(Jf) \tag{1.4}$$

where I_q denotes the multiple integral of order q with respect to the d-parametric standard Brownian field $(B(\mathbf{y}))_{\mathbf{y}\in\mathbb{R}^d}$ and $Jf \in L^2(\mathbb{R}^{d.q})$ is given by

$$Jf(\mathbf{y}_1, ..., \mathbf{y}_q) = c(\mathbf{H}, q) \int_{\mathbb{R}^d} d\mathbf{u} f(\mathbf{u})(\mathbf{u}-\mathbf{y}_1)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \dots (\mathbf{u}-\mathbf{y}_q)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)}. \tag{1.5}$$

The Wiener-Hermite integral satisfies the following isometry

$$\mathbf{E}\left(\int_{\mathbb{R}^d} f(\mathbf{s}) dZ^q_{\mathbf{H}}(\mathbf{s}) \int_{\mathbb{R}^d} g(\mathbf{s}) dZ^q_{\mathbf{H}}(\mathbf{s})\right) = \langle f, g\rangle_{\mathcal{H}(\mathbb{R}^d)} \tag{1.6}$$

where

$$\langle f, g\rangle_{\mathcal{H}(\mathbb{R}^d)} = \mathbf{H(2H-1)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}) g(\mathbf{v}) |\mathbf{u}-\mathbf{v}|^{\mathbf{2H-2}} d\mathbf{u} d\mathbf{v}.$$

Notice that the isometry formula (1.6) holds for any $q \geq 1$. Consequently, for every $q \geq 1$ the variance of the Wiener-Hermite integral coincides with the variance of the Wiener integral with respect to the fractional Brownian motion. On the other hand, the law of the object (1.4) is much more complicated when $q \geq 2$.

1.2.3 Spectral representation of the Wiener integral

For certain proofs, it will be more convenient to work with multiple integrals of Fourier type. They constitute a variant of the standard multiple Wiener-Itô integrals, presented in the Appendix. Let us present their definition, basic properties and recall the connection between the two reprensentations.

Let $\widehat{W}$ be a complex-valued Gaussian random spectral measure that satisfies $\mathbf{E}\widehat{W}(A) = 0, \mathbf{E}\left[\widehat{W}(A)\overline{\widehat{W}(B)}\right] = \lambda(A\cap B), \widehat{W}(A) = \overline{\widehat{W}(-A)}$ and $\widehat{W}(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \widehat{W}(A_i)$, for disjoint Borel sets that have finite Lebesgue measure, denoted by λ. The real and imaginary parts of $\widehat{W}(A)$ are independent Gaussian random variables with mean 0 and variance $\frac{\lambda(A)}{2}$.
Let $\overline{L}^2 := L^2\left((\mathbb{R}^d)^q; \mathbb{C}\right)$ be the set of complex-valued functions on $(\mathbb{R}^d)^q$ such that

$$g(-\mathbf{x}_1, ..., -\mathbf{x}_q) = \overline{g(\mathbf{x}_1, ..., \mathbf{x}_q)}$$

for every $\mathbf{x}_i \in \mathbb{R}^d$, $i = 1, ..q$ and

$$\|g\|^2_{\overline{L}^2} = \int_{\mathbb{R}^{d.q}} |g(\mathbf{x}_1, ..., \mathbf{x}_q)|^2 d\mathbf{x}_1 ... d\mathbf{x}_q < \infty.$$

For $g \in L^2\left((\mathbb{R}^d)^q; \mathbb{C}\right)$, the multiple stochastic integral $\widehat{I}_q(g)$ with respect to the Gaussian measure $\widehat{W}$ is defined via an isometry between $L^2\left((\mathbb{R}^d)^q; \mathbb{C}\right)$ and $L^2(\Omega)$, i.e.

$$\mathbf{E}\left[\widehat{I}_p(f)\widehat{I}_q(g)\right] = \left\{\begin{array}{lll} q! < \widetilde{f}, \widetilde{g} >_{\overline{L}^2} & ,\text{if} & q = p. \\ 0 & ,\text{if} & q \neq p. \end{array}\right\}. \tag{1.7}$$

The construction of $\widehat{I}_q(g)$ (that we will call in the sequel *multiple stochastic integral of spectral type* or *of Fourier type*) is similar to the construction of the usual multiple Wiener-Itô integrals presented in the Appendix). We refer to [56] for more details.

We have the following equality in law (see [94], Lemma 6.1), which will be used several times in the sequel : if $f \in L^2((\mathbb{R}^d)^q)$, then

$$I_q(f) \overset{(d)}{=} (2\pi)^{-\frac{q}{2}} \widehat{I}_q(\widehat{f}) \tag{1.8}$$

where $\overset{(d)}{=}$ means equality in distribution and for every $\boldsymbol{\lambda}_1, ..., \boldsymbol{\lambda}_n \in \mathbb{R}^d$, $\widehat{f}$ denotes the Fourier transform of f :

$$\widehat{f}(\boldsymbol{\lambda}_1, , , , , \boldsymbol{\lambda}_n) = \int_{\mathbb{R}^{d.n}} f(\mathbf{y}_1, ..., \mathbf{y}_n) e^{i \sum_{j=1}^n \boldsymbol{\lambda}_j \cdot \mathbf{y}_j} d\mathbf{y}_1 \dots d\mathbf{y}_n$$

We also need the following property of the spectral multiple stochastic integrals (see Proposition 4.2 in [36]) : if $f \in L^2\left(\mathbb{R}^d; \mathbb{C}\right)$ such that $|f(\boldsymbol{\lambda})|^2 = 1$ for every $\lambda \in \mathbb{R}^d$ then

$$\widehat{I}_q(g \cdot f^{\otimes q}) = \widehat{I}_q(g) \tag{1.9}$$

for every $g \in L^2\left((\mathbb{R}^d)^q; \mathbb{C}\right)$.

In the next lemma we give the spectral representation of the Wiener-Hermite integral (1.4).

Lemma 1. *Assume $f \in \mathcal{H}(\mathbb{R}^{d \cdot q})$ and let Jf be defined by (1.5). Then for every $\mathbf{z}_1, .., \mathbf{z}_q \in \mathbb{R}^d$*

$$\widehat{Jf}(\mathbf{z}_1, \dots, \mathbf{z}_q) = c(\mathbf{H}, q) C(\mathbf{z}) \Gamma\left(\frac{1}{2} - \frac{1 - \mathbf{H}}{q}\right)^q \widehat{f}(\mathbf{z}_1 + ... + \mathbf{z}_q) |\mathbf{z}_1|^{-\frac{1}{2} + \frac{1-\mathbf{H}}{q}} \dots |\mathbf{z}_q|^{-\frac{1}{2} + \frac{1-\mathbf{H}}{q}} \tag{1.10}$$

where Γ is the Gamma function and $C(\mathbf{z}) = \prod_{j=1}^q C(\mathbf{z_j})$ with $C(\mathbf{z_j}) = \mathbf{e}^{-\mathbf{i}\frac{\pi}{2}(\frac{1}{2} - \frac{1-\mathbf{H}}{\mathbf{q}})}$ if $\mathbf{z_j} > \mathbf{0}$, $C(-\mathbf{z_j}) = \overline{\mathbf{C}(\mathbf{z_j})}$.

Moreover

$$\int_{\mathbb{R}^d} f(s) dZ^q_{\mathbf{H}}(\mathbf{s}) \overset{(d)}{=} C_{0,q} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} d\widehat{W}(\mathbf{z}_1) \dots d\widehat{W}(\mathbf{z}_q) \widehat{f}(\mathbf{z}_1 + \dots + \mathbf{z}_q) |\mathbf{z}_1|^{-\frac{1}{2} + \frac{1-\mathbf{H}}{q}} \dots |\mathbf{z}_q|^{-\frac{1}{2} + \frac{1-\mathbf{H}}{q}} \tag{1.11}$$

with

$$C_{0,q} = (2\pi)^{\frac{-q}{2}} c(\mathbf{H}, q) \Gamma\left(\frac{1}{2} - \frac{1 - \mathbf{H}}{q}\right)^q. \tag{1.12}$$

Proof : Relation (1.10) has been showed in Lemma 2.2 in [100] (see also Lemma 6.2 in [94]). The equality in distribution (1.11) follows from (1.8), the fact that $C(\mathbf{z}) \widehat{W}(d\mathbf{z}) \overset{(d)}{=} \widehat{W}(d\mathbf{z})$ (see (1.9)) and the expression of the Wiener integral with respect to the Hermite process (1.4). Indeed,

$$\int_{\mathbb{R}^d} f(s) dZ^q_{\mathbf{H}}(\mathbf{s}) = I_q(Jf) = (2\pi)^{-\frac{q}{2}} \widehat{I}_q(\widehat{Jf})$$

and, by (1.10) and (1.9), this is equal to the right hand side of (1.11). ∎

1.3 The heat equation with Hermite noise

Consider the linear stochastic heat equation driven by a Hermite sheet $Z^q_{\mathbf{H}}$ with Hurst multi-index $\mathbf{H} \in (1/2, 1)^{(d+1)}$. That is

$$\begin{cases} \frac{\partial u}{\partial t}(t, \mathbf{x}) = \Delta u(t, \mathbf{x}) + \dot{Z}^q_{\mathbf{H}}(t, \mathbf{x}), & t > 0, \mathbf{x} \in \mathbb{R}^d \\ u(0, \mathbf{x}) = 0, & \mathbf{x} \in \mathbb{R}^d \end{cases} \tag{1.13}$$

Here Δ is the Laplacian on $\mathbb{R}^d$ and $Z^q_{\mathbf{H}} = \{Z^q_{\mathbf{H}}(t,\mathbf{x}); t \geq 0, \mathbf{x} \in \mathbb{R}^d\}$ is the $(d+1)$-parametric Hermite sheet whose covariance is given by

$$\mathbf{E}\left\{Z^q_{\mathbf{H}}(s,\mathbf{x})Z^q_{\mathbf{H}}(t,\mathbf{y})\right\} = R_H(t,s)R_{\mathbf{H}_0}(\mathbf{x},\mathbf{y})$$

if $\mathbf{H} = (H, H_1, \ldots, H_d)$. We denoted by $\mathbf{H}_0 = (H_1, \ldots, H_d)$ and

$$R_H(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad R_{\mathbf{H}_0}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{d} R_{H_j}(x_j, y_j)$$

if $s,t \in \mathbb{R}$ and $\mathbf{x} = (x_1,..,x_d), \mathbf{y} = (y_1,..,y_d) \in \mathbb{R}^d$. Above $\dot{Z}^q_{\mathbf{H}}$ stands for the formal derivative of $Z^q_{\mathbf{H}}$. This means that the noise behaves as a one-parameter Hermite process with respect to the time variable t and as a d-parameter Hermite process in space. Equivalently we can write

$$\mathbf{E}\left\{\dot{Z}^q_{\mathbf{H}}(s,\mathbf{x})\dot{Z}^q_{\mathbf{H}}(t,\mathbf{y})\right\} = H(2H-1)|t-s|^{2H-2}\prod_{i=1}^{d}(H_i(2H_i-1)\cdot|x_i-y_i|^{2H_i-2}).$$

The *mild* solution to (1.13) is a square-integrable process $u = \{u(t,\mathbf{x}); t \geq 0, \mathbf{x} \in \mathbb{R}^d\}$ defined by :

$$u(t,\mathbf{x}) = \int_0^t \int_{\mathbb{R}^d} G(t-s,\mathbf{x}-\mathbf{y})Z^q_{\mathbf{H}}(ds,d\mathbf{y}), \quad t \geq 0, \mathbf{x} \in \mathbb{R}^d. \tag{1.14}$$

The above integral is a Wiener integral with respect to the Hermite sheet, as introduced in Section 2 and $G(t,\mathbf{x})$ is the Green function that satisfies $\frac{\partial u}{\partial t} - \Delta u = 0$, i.e.

$$G(t,\mathbf{x}) = \begin{cases} (2\pi t)^{-d/2}\exp\left(-\frac{|\mathbf{x}|^2}{2t}\right) & \text{if } t > 0, \mathbf{x} \in \mathbb{R}^d, \\ 0 & \text{if } t \leq 0, x \in \mathbb{R}^d. \end{cases} \tag{1.15}$$

Remark 1. *Even if the noise $Z^q_{\mathbf{H}}(s,\mathbf{y})$ is defined for $s,\mathbf{y} \geq 0$, the solution is considered with $\mathbf{x}$ on the whole $\mathbb{R}^d$ via the expression (4.27) and the definition of the Wiener-Hermite integral.*

1.3.1 Existence and Hölder continuity

The existence and some basic properties of the mild solution (4.27) to the heat equation with Hermite noise (1.13) can be easily obtained from the isometry (1.6) and the results known in the Gaussian case. Let us list these facts in the following result.

Proposition 12. *The stochastic heat equation (1.13) admits an unique mild solution $(u(t,\mathbf{x}))_{t\geq 0, \mathbf{x}\in\mathbb{R}^d}$ if and only if*

$$d < 4H + \sum_{i=1}^{d}(2H_i - 1). \tag{1.16}$$

In this case, $\sup_{t,\mathbf{x}} \mathbf{E}\left(u(t,\mathbf{x})^2\right) < \infty$.

Moreover there exist $0 < c_1 < c_2$ and $0 < c_3 < c_4$ such that for every $\mathbf{x} \in \mathbb{R}^d$ and $0 \leq s \leq t$

$$c_1|t-s|^{2H-\frac{d-\alpha}{2}} \leq \mathbf{E}\left|u(t,\mathbf{x}) - u(s,\mathbf{x})\right|^2 \leq c_2|t-s|^{2H-\frac{d-\alpha}{2}} \tag{1.17}$$

and for any $M > 0$, $t > 0$ and $\mathbf{x},\mathbf{y} \in [-M,M]^d$

$$c_3|\mathbf{x}-\mathbf{y}|^{2\beta}\left(\frac{1}{\log|\mathbf{x}-\mathbf{y}|}\right)^{\rho} \leq \mathbf{E}\left|u(t,\mathbf{x})-u(t,\mathbf{y})\right|^2 \leq c_4|\mathbf{x}-\mathbf{y}|^{2\beta}\left(\frac{1}{\log|\mathbf{x}-\mathbf{y}|}\right)^{\rho} \tag{1.18}$$

where $\beta = \min(1, 2H - \frac{d-\alpha}{2})$ *and* $\rho = \rho(\beta) = \mathbf{1}_{\{1\}}(\beta)$.

Proof : In order to check the existence of the mild solution to (1.13), it suffices to check that the Wiener integral in the right-hand side of (4.27) is well-defined. From the construction of the Wiener integral, if $\alpha_H = H(2H-1)$,

$$\begin{aligned}\mathbf{E}\left(u(t,\mathbf{x})^2\right) &= \|G(t-\cdot,\mathbf{x}-*)\mathbf{1}_{(0,t)}(\cdot)\|^2_{\mathcal{H}(\mathbb{R}^{d+1})}\\ &= \alpha_H\int_0^t du\int_0^t dv|u-v|^{2H-2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} d\mathbf{y}d\mathbf{z}G(t-u,\mathbf{x}-\mathbf{y})G(t-v,\mathbf{x}-\mathbf{z})f(\mathbf{y}-\mathbf{z})\end{aligned}$$

where G is defined by (1.15) and

$$f(\mathbf{y}-\mathbf{z}) = \prod_{i=1}^d H_i(2H_i-1)|y_i-z_i|^{2H_i-2} = \mathbf{H}_0(2\mathbf{H}_0-1)|\mathbf{y}-\mathbf{z}|^{2\mathbf{H_0}-2}.$$

By using the Fourier transform of G with respect to the space variable and the Parseval identity, we can also write

$$\begin{aligned}\mathbf{E}\left(u(t,\mathbf{x})^2\right) &= \alpha_H(2\pi)^{-d}\int_0^t du\int_0^t dv|u-v|^{2H-2}\int_{\mathbb{R}^d}\mathcal{F}G(t-u,\cdot)(\boldsymbol{\xi})\mathcal{F}G(t-v,\cdot)(\boldsymbol{\xi})\mu(d\boldsymbol{\xi})\\ &= \alpha_H(2\pi)^{-d}\int_0^t du\int_0^t dv|u-v|^{2H-2}\int_{\mathbb{R}^d} e^{-\frac{1}{2}(t-u)|\boldsymbol{\xi}|^2}e^{-\frac{1}{2}(t-v)|\boldsymbol{\xi}|^2}\mu(d\boldsymbol{\xi}) \qquad (1.19)\end{aligned}$$

where μ is the measure on $\mathbb{R}^d$ whose Fourier transform is f. Actually (see [11])

$$\mu(\boldsymbol{d\xi}) = \mathbf{H}_0(2\mathbf{H}_0-1)\left(\prod_{i=1}^d \xi_i^{2H_i-1}\right)d\xi_1...d\xi_d$$

if $\boldsymbol{\xi} = (\xi_1,..,\xi_d)$. Therefore, it follows from [11] that the quantity (1.19) is finite if and only if

$$d - \sum_{i=1}^d(2H_i-1) < 4H.$$

Similarly, $\mathbf{E}\left|u(t,\mathbf{x})-u(s,\mathbf{x})\right|^2$ has the same value for every $q \geq 1$, the known results on the solution in the Gaussian case $q=1$ can be applied to (4.27). In particular, from [103] we obtain (1.17) and from Theorem 4 in [105], we obtain (1.18). ■

Remark 2. *Let us point out that the condition (1.16) can be also written as* $d < 2H + \sum_{i=1}^d H_i$*. If the parameters* $H, H_i, i = 1,..,d$ *are close to 1, then the solution exists in any dimension, since (1.16) is satisfied. When the Hurst parameters are close to one-half, then (1.16) reduces to* $d < 1+\frac{d}{2}$ *or* $d < 2$ *which means that the solution exists only in spatial dimension* $d = 1$*. So, more regular is the noise, in higher dimension the solution exists.*

1.3.2 Spectral representation and self-similarity

Let us give the expression of the solution (4.27) as a multiple integral in the spectral domain. By $\overset{(d)}{\equiv}$ we will denote the equivalence in the sense of finite dimensional distributions.

Proposition 13. *For every $\mathbf{x} \in \mathbb{R}^d$, the process $(u(t,\mathbf{x}))_{t\geq 0}$ satisfies*

$$u(t,\mathbf{x}) \overset{(d)}{\equiv} C_{0,q} \int_{\mathbb{R}^{d+1}} d\widehat{W}(u_1,\mathbf{z}_1)... \int_{\mathbb{R}^{d+1}} d\widehat{W}(u_q,\mathbf{z}_q) \left(e^{it(u_1+...+u_q)} - e^{\frac{1}{2}t|\mathbf{z}_1+...+\mathbf{z}_q|^2}\right)$$

$$\frac{1}{i(u_1+...+u_q)+\frac{1}{2}|\mathbf{z}_1+...+\mathbf{z}_q|^2} |\mathbf{z}_1|^{-\frac{1}{2}+\frac{1-\mathbf{H_0}}{q}} ... |\mathbf{z}_q|^{-\frac{1}{2}+\frac{1-\mathbf{H_0}}{q}} |u_1|^{-\frac{1}{2}+\frac{1-H}{q}} ... |u_q|^{-\frac{1}{2}+\frac{1-H}{q}}$$

with $C_{0,q}$ from (1.12).

Proof : The Fourier transform of the function $g_1(\mathbf{y}) = G(t-u,\mathbf{x}-\mathbf{y})\mathbf{1}_{(t>u>0)}$ is $\widehat{g}_1(\boldsymbol{\xi}) = e^{i\mathbf{x}\boldsymbol{\xi}} e^{-\frac{1}{2}(t-u)|\boldsymbol{\xi}|^2} 1_{(t>u>0)}$ while the Fourier transform of

$$g_2(u) = e^{-\frac{1}{2}(t-u)|\boldsymbol{\xi}|^2} \mathbf{1}_{(t>u>0)}$$

is (see [105])

$$\widehat{g}_2(u) = \left(e^{itu} - e^{-\frac{1}{2}t|\boldsymbol{\xi}|^2}\right) \frac{1}{iu+\frac{1}{2}|\boldsymbol{\xi}|^2}.$$

From Lemma 6, we have for every $\mathbf{x} \in \mathbb{R}^d$

$$\begin{aligned} u(t,\mathbf{x}) \overset{(d)}{\equiv}\ & C_{0,q} \int_{\mathbb{R}}\int_{\mathbb{R}^d} d\widehat{W}(u_1,\mathbf{z}_1)... \int_{\mathbb{R}}\int_{\mathbb{R}^d} d\widehat{W}(u_q,\mathbf{z}_q) \left(e^{it(u_1+...+u_q)} - e^{\frac{1}{2}t|\mathbf{z}_1+...+\mathbf{z}_q|^2}\right) e^{i\mathbf{x}(\mathbf{z_1}+...\mathbf{z_q})} \\ & \frac{1}{i(u_1+...+u_q)+\frac{1}{2}|\mathbf{z}_1+...+\mathbf{z}_q|^2} |\mathbf{z}_1|^{-\frac{1}{2}+\frac{1-\mathbf{H_0}}{q}} ... |\mathbf{z}_q|^{-\frac{1}{2}+\frac{1-\mathbf{H_0}}{q}} \\ & |u_1|^{-\frac{1}{2}+\frac{1-H}{q}} ... |u_q|^{-\frac{1}{2}+\frac{1-H}{q}} \end{aligned}$$

where $\overset{(d)}{\equiv}$ means the equivalence of finite dimensional distributions. The conclusion follows from (1.9) since

$$d\widehat{W}(u_1,\mathbf{z}_1)...d\widehat{W}(u_q,\mathbf{z}_q) e^{i\mathbf{x}(\mathbf{z_1}+...\mathbf{z_q})} \overset{(d)}{=} d\widehat{W}(u_1,\mathbf{z}_1)...d\widehat{W}(u_q,\mathbf{z}_q).$$

■

It is clear from Proposition 13 that the process u is stationary with respect to the spatial variable $\mathbf{x} \in \mathbb{R}^d$. We show below that it is self-similar in time.

Proposition 14. *For every $x \in \mathbb{R}^d$, the process $(u(t,\mathbf{x}))_{t\in[0,T]}$ is self-similar of order*

$$\gamma = H + \frac{(H_1+...+H_d)-d}{2}. \tag{1.20}$$

Proof : For every $c > 0$, by Proposition 13

$$\begin{aligned} u(ct,\mathbf{x}) \overset{(d)}{\equiv}\ & C_{0,q} \int_{\mathbb{R}}\int_{\mathbb{R}^d} d\widehat{W}(u_1,\mathbf{z}_1)... \int_{\mathbb{R}}\int_{\mathbb{R}^d} d\widehat{W}(u_q,\mathbf{z}_q) \left(e^{ict(u_1+...+u_q)} - e^{-\frac{1}{2}ct(|\mathbf{z}_1+...+\ \mathbf{z}_q|^2}\right) \\ & \frac{1}{i(u_1+...+u_q)+\frac{1}{2}|\mathbf{z}_1+...+\mathbf{z}_q|^2} |\mathbf{z}_1|^{-\frac{1}{2}+\frac{1-\mathbf{H_0}}{q}} ... |\mathbf{z}_q|^{-\frac{1}{2}+\frac{1-\mathbf{H_0}}{q}} \\ & |u_1|^{-\frac{1}{2}+\frac{1-H}{q}} ... |u_q|^{-\frac{1}{2}+\frac{1-H}{q}}. \end{aligned}$$

Recall that $\mathbf{H}_0 = (H_1, .., H_d)$. With the change of variables $\sqrt{c} z_i = \tilde{z}_i$,

$$\begin{aligned}
u(ct,\mathbf{x}) &\overset{(d)}{\equiv} c^{1+\frac{q}{2}\left[\frac{dq}{2}-\frac{d-(H_1+...+H_d)}{q}\right]} C_{0,q} \int_{\mathbb{R}^{d+1}} d\widehat{W}(u_1, \frac{\mathbf{z}_1}{\sqrt{c}}) ... \int_{\mathbb{R}^{d+1}} d\widehat{W}(u_q, \frac{\mathbf{z}_q}{\sqrt{c}}) \\
&\left(e^{ict(u_1+...+u_q)} - e^{-\frac{1}{2}t(|\mathbf{z}_1+...+|\mathbf{z}_q|^2}\right) \\
&\frac{1}{ic(u_1+...+u_q)+\frac{1}{2}|\mathbf{z}_1+...+\mathbf{z}_q|^2} |\mathbf{z}_1|^{-\frac{1}{2}+\frac{1-\mathbf{H}_0}{q}} ... |\mathbf{z}_q|^{-\frac{1}{2}+\frac{1-\mathbf{H}_0}{q}} \\
&|u_1|^{-\frac{1}{2}+\frac{1-H}{q}} ... |u_q|^{-\frac{1}{2}+\frac{1-H}{q}} \\
&\overset{(d)}{\equiv} c^{1-\frac{d-(H_1+...+H_d)}{2}} C_{0,q} \int_{\mathbb{R}^{d+1}} d\widehat{W}(u_1, \mathbf{z}_1) ... \int_{\mathbb{R}^{d+1}} d\widehat{W}(u_q, \mathbf{z}_q) \\
&\left(e^{ict(u_1+...+u_q)} - e^{-\frac{1}{2}t(|\mathbf{z}_1|+...+|\mathbf{z}_q|^2}\right) \\
&\frac{1}{ic(u_1+...+u_q)+\frac{1}{2}|\mathbf{z}_1+...+\mathbf{z}_q|^2} |\mathbf{z}_1|^{-\frac{1}{2}+\frac{1-\mathbf{H}_0}{q}} ... |\mathbf{z}_q|^{-\frac{1}{2}+\frac{1-\mathbf{H}_0}{q}} \\
&|u_1|^{-\frac{1}{2}+\frac{1-H}{q}} ... |u_q|^{-\frac{1}{2}+\frac{1-H}{q}}
\end{aligned}$$

where we used the scaling of the noise $\widehat{W}$ with respect to $\mathbf{z}_i$. Now, with $cu_i = \tilde{u}_i$ and by using the scaling of the noise with respect to the first variable,

$$\begin{aligned}
u(ct,\mathbf{x}) &\overset{(d)}{\equiv} c^{1-\frac{d-(H_1+...+H_d)}{2}} c^{q(\frac{1}{2}-\frac{1-H}{q})} C_{0,q} \int_{\mathbb{R}^{d+1}} d\widehat{W}(\frac{u_1}{c}, \mathbf{z}_1) ... \int_{\mathbb{R}^{d+1}} d\widehat{W}(\frac{u_q}{c}, \mathbf{z}_q) \\
&\left(e^{it(u_1+...+u_q)} - e^{-\frac{1}{2}t(|\mathbf{z}_1+...+|\mathbf{z}_q|^2}\right) \\
&\frac{1}{i(u_1+...+u_q)+\frac{1}{2}|\mathbf{z}_1+...+\mathbf{z}_q|^2} |\mathbf{z}_1|^{-\frac{1}{2}+\frac{1-\mathbf{H}_0}{q}} ... |\mathbf{z}_q|^{-\frac{1}{2}+\frac{1-\mathbf{H}_0}{q}} \\
&|u_1|^{-\frac{1}{2}+\frac{1-H}{q}} ... |u_1|^{-\frac{1}{2}+\frac{1-H}{q}} \\
&\overset{(d)}{\equiv} c^{1-\frac{d-(H_1+...+H_d)}{2}-(1-H)} C_{0,q} \int_{\mathbb{R}^{d}} d\widehat{W}(\frac{u_1}{c}, \mathbf{z}_1) ... \int_{\mathbb{R}^{d}} d\widehat{W}(\frac{u_q}{c}, \mathbf{z}_q) \\
&\left(e^{it(u_1+...+u_q)} - e^{-\frac{1}{2}t(|\mathbf{z}_1+...+|\mathbf{z}_q|^2}\right) \\
&\frac{1}{i(u_1+...+u_q)+\frac{1}{2}|\mathbf{z}_1+...+\mathbf{z}_q|^2} |\mathbf{z}_1|^{-\frac{1}{2}+\frac{1-\mathbf{H}_0}{q}} ... |\mathbf{z}_q|^{-\frac{1}{2}+\frac{1-\mathbf{H}_0}{q}} \\
&|u_1|^{-\frac{1}{2}+\frac{1-H}{q}} ... |u_1|^{-\frac{1}{2}+\frac{1-H}{q}} \\
&\overset{(d)}{\equiv} c^{\gamma} u(t,\mathbf{x})
\end{aligned}$$

with γ given by (1.20). ■

Remark 3. — *In dimension $d = 1$, we retrieve the well known self-similarity index $H - \frac{1-\alpha}{4}$ with $\alpha = 2H_1 - 1$ (see [103]).*

— *From the main result in [70], we can see that for every $t_i \geq 0, \mathbf{x}_i \in \mathbb{R}^d$, with $i = 1, .., N$ and $N \geq 1$ the random vectors $(u(t_1, \mathbf{x}_1), ..., u(t_N, \mathbf{x}_N))$ admits a joint density. Indeed, the main result in [70] says that a random vector whose components are multiple integrals in a Wiener chaos of order $q \geq 1$ admits a density if and only if the components are not proportional, which is obviously the case here.*

1.3.3 A decomposition theorem

For every $t \geq 0$, $\mathbf{x} \in \mathbb{R}^d$ we can express the solution $u(t, \mathbf{x})$ (4.27) as

$$u(t,\mathbf{x}) = U(t,\mathbf{x}) - Y(t,\mathbf{x})$$

where

$$U(t,\mathbf{x}) = \int_{\mathbb{R}}\int_{\mathbb{R}^d} \left(G((t-u)_+, \mathbf{s}-\mathbf{y}) - G((-u)_+, \mathbf{s}-\mathbf{y})\right) Z^q_{\mathbf{H}}(du, d\mathbf{y}) \tag{1.21}$$

and

$$Y(t,\mathbf{x}) = \int_{-\infty}^{0}\int_{\mathbb{R}^d} \left(G(t-u, \mathbf{s}-\mathbf{y}) - G(-u, \mathbf{s}-\mathbf{y})\right) Z^q_{\mathbf{H}}(du, d\mathbf{y}) \tag{1.22}$$

This decomposition has been first given in [61] and then used by many authors, (see [39], [105], [43], [62]). The process U is usually called the *pinned string process.*

If $q=1$, i.e. in the case of the heat-equation with time-space fractional Gaussian noise, the process U is (with respect to the time variable), modulo a constant, a fractional Brownian motion with index $\gamma = H - \frac{d-\alpha}{4} = H + \frac{H_1+..+H_d-d}{2}$, where $\alpha = \sum_{i=1}^d (2H_i - 1)$, while the process Y is a smooth process in time (is continuously differentiable of order any order k with respect to t on any interval $[a,b]\subset[0,\infty)$, see [105] or [43]. That means that the solution to the heat equation can be written as the sum of a fBm and a smooth process. Consequently, many properties for the solution can be obtained via this decomposition, see again [105], [43] or [39]. We will try to analyze the processes U and Y defined before when the noise is not Gaussian anymore.

For the general order $q\geq 1$ we have the following result.

Theorem 1. *1. For every $\mathbf{x}\in\mathbb{R}^d$ the process $(U(t,\mathbf{x}))_{t\geq 0}$ defined by (1.21) is self-similar of order γ (1.20) and it has stationary increments.*

2. For every $\mathbf{x}\in\mathbb{R}^d$ and any integer $k\geq 1$, the process $(Y(t,\mathbf{x}))_{t\geq 0}$ is almost surely continuously differentiable of order k on any time interval $[a,b]\subset[0,\infty)$.

Proof : Since the Fourier transform of the function $g_1(\mathbf{y}) = G((t-u)_+, \mathbf{x}-\mathbf{y})$ is $\widehat{g}_1(\boldsymbol{\xi}) = e^{i\mathbf{x}\boldsymbol{\xi}}e^{-\frac{1}{2}(t-u)|\boldsymbol{\xi}|^2}1_{(t>u)}$ and the Fourier transform of

$$g_2(u) = e^{-\frac{1}{2}(t-u)|\boldsymbol{\xi}|^2}1_{(t>u)} - e^{-\frac{1}{2}(-u)|\boldsymbol{\xi}|^2}1_{(-u>0)}$$

is

$$\widehat{g}_2(u) = \left(e^{itu}-1\right)\frac{1}{iu+\frac{1}{2}|\boldsymbol{\xi}|^2}$$

we have, as in Proposition 13, (also by using the property (1.9))

$$\begin{aligned} U(t,\mathbf{x}) \overset{(d)}{\equiv}\; & C_{0,q}\int_{\mathbb{R}^{d+1}} d\widehat{W}(u_1,\mathbf{z}_1)...\int_{\mathbb{R}^{d+1}} d\widehat{W}(u_q,\mathbf{z}_q)\left(e^{it(u_1+...+u_q)}-1\right) \\ & \frac{1}{i(u_1+...+u_q)+\frac{1}{2}|\mathbf{z}_1+...+\mathbf{z}_q|^2}|\mathbf{z}_1|^{-\frac{1}{2}+\frac{1-\mathbf{H_0}}{q}}...|\mathbf{z}_q|^{-\frac{1}{2}+\frac{1-\mathbf{H_0}}{q}} \\ & |u_1|^{-\frac{1}{2}+\frac{1-H}{q}}...|u_1|^{-\frac{1}{2}+\frac{1-H}{q}} \end{aligned}$$

To prove point 1., we repeat the arguments in the proof of Proposition 14 to obtain the self-similarity. Concerning the stationarity of the increments of $t\to U(t,\mathbf{x})$, we can write

$$\begin{aligned}
U(t+h,\mathbf{x})-U(h,\mathbf{x}) &\overset{(d)}{\equiv} C_{0,q}\int_{\mathbb{R}^{d+1}} d\widehat{W}(u_1,\mathbf{z}_1)...\int_{\mathbb{R}^{d+1}} d\widehat{W}(u_q,\mathbf{z}_q)\\
&\left(e^{i(t+h)(u_1+...+u_q)}-e^{ih(u_1+...+u_q)}\right)\\
&\frac{1}{i(u_1+...+u_q)+\frac{1}{2}|\mathbf{z}_1+...+\mathbf{z}_q|^2}|\mathbf{z}_1|^{-\frac{1}{2}+\frac{1-\mathbf{H_0}}{q}}...|\mathbf{z}_q|^{-\frac{1}{2}+\frac{1-\mathbf{H_0}}{q}}\\
&|u_1|^{-\frac{1}{2}+\frac{1-H}{q}}...|u_1|^{-\frac{1}{2}+\frac{1-H}{q}}\\
&\overset{(d)}{\equiv} C_{0,q}\int_{\mathbb{R}^{d+1}} d\widehat{W}(u_1,\mathbf{z}_1)...\int_{\mathbb{R}^{d+1}} d\widehat{W}(u_q,\mathbf{z}_q)e^{ih(u_1+...+u_q)}\left(e^{it(u_1+...+u_q)}-1\right)\\
&\frac{1}{i(u_1+...+u_q)+\frac{1}{2}|\mathbf{z}_1+...+\mathbf{z}_q|^2}|\mathbf{z}_1|^{-\frac{1}{2}+\frac{1-\mathbf{H_0}}{q}}...|\mathbf{z}_q|^{-\frac{1}{2}+\frac{1-\mathbf{H_0}}{q}}\\
&|u_1|^{-\frac{1}{2}+\frac{1-H}{q}}...|u_1|^{-\frac{1}{2}+\frac{1-H}{q}}\\
&\overset{(d)}{\equiv} U(t,\mathbf{x})
\end{aligned}$$

where we used again (1.9).

The regularity of the process Y can be obtained as in the proof of Theorem 2 of [105] or of Proposition 3.1 in [39]. The distributional derivative of $Y(t,x)$ with respect to t can be written as

$$Y'(t)=\int_{-\infty}^{0}\int_{\mathbb{R}^d} G'(t-u,\mathbf{x}-\mathbf{y})\,W^H(du,d\mathbf{y}),$$

where $G':=\partial G/\partial t$. This is true because for any test function φ vanishing at $\pm\infty$, by using the Fubini theorem, we can interchange the multiple stochastic integral and the integral dt (see Theorem 2.1 in [80]) to get

$$\int_{\mathbb{R}} Y'(t)\varphi(t)dt=-\int_{\mathbb{R}} Y(t)\varphi'(t)dt.$$

On the other hand, we know from [105] (using the fact that the isometry of the Wiener-Hermite integral is the same as in fBm case)

$$\mathbf{E}\big(|Y'(t)-Y'(s)|^2\big)\leq C|t-s|^2.$$

This shows, from the Kolmogorov criterium that $(Y'(t,\mathbf{x}),t\geq 0)$ is a continuous stochastic process, so $t\to Y(t,\mathbf{x})\in C[a,b]$. The same argument can be iterated in order to obtain the differentiable of any order $k\geq 1$. ■

Remark 4. *In the case $q=1$, point 1. in the above theorem allows to conclude that $(U(t,\mathbf{x}))_{t\geq 0}$ is, modulo a constant, a fractional Brownian motion with Hurst index γ (1.20). For $q\geq 2$, the fact $t\to U(t,\mathbf{x})$ is self-similar and has stationary increments does not imply that it is a Hermite process. The class of self-similar processes with stationary increments is strictly much larger that the class of the Hermite processes (see [5], [54]). See also the discussion in Remark 4.*

1.3.4 The case $q=2$: Cumulants

If $q=2$ we can have a better understanding of the law of the stochastic processes u and U given by (4.27) and (1.21) respectively, because they belong to the second Wiener chaos and then their distribution is completely determined by their moments, or equivalently, by their cumulants (see e.g. [66], Theorem 2.7.13).

Recall $G = I_2(f)$ is a multiple integral of order 2 with respect to a Wiener sheet $(B(y))_{y\in\mathbb{R}^d}$, then its cumulants can be computed as follows

$$k_m(G) = 2^{m-1}(m-1)!\langle f \otimes_1^{(m-1)} f, f\rangle_{L^2((\mathbb{R}^d)^2)}$$

where $f \otimes_1^{(m-1)} f$ is defined as $f\otimes_1^{(1)} = f$ and $f \otimes_1^{(p)} f = \left(f \otimes_1^{(p-1)} f\right) \otimes_1 f$ if $p \geq 2$ (see (4.42) for the definition of the contraction). From the above formula we get

$$k_m(G) = 2^{m-1}(m-1)! \int_{(\mathbb{R}^d)^m} f(\mathbf{u}_1, \mathbf{u}_2) f(\mathbf{u}_2, \mathbf{u}_3) \dots f(\mathbf{u}_{m-1}, \mathbf{u}_m) f(\mathbf{u}_m, \mathbf{u}_1) d\mathbf{u}_1 \dots d\mathbf{u}_m. \quad (1.23)$$

In the spectral setup, since G has the same law as $(2\pi)^{-1}\widehat{I}(\widehat{f})$ and since for any two symmetric functions $f, g \in L^2\left((\mathbb{R}^d)^2; \mathbb{C}\right)$ (see e.g. [27], Proposition B.1)

$$(\widehat{f} \otimes_1 \widehat{g})(\mathbf{x}_1, \mathbf{x}_2) = \int_{\mathbb{R}^d} \widehat{f}(\mathbf{x}_1, \mathbf{x}) \widehat{g}(\mathbf{x}_2, -\mathbf{x}) d\mathbf{x}$$

we have

$$k_m(G) = (2\pi)^{-m} 2^{m-1}(m-1)! \int_{(\mathbb{R}^d)^m} \widehat{f}(\mathbf{x}_1, -\mathbf{x}_2) \widehat{f}(\mathbf{x}_2, -\mathbf{x}_3) \dots . \widehat{f}(\mathbf{x}_{m-1}, -\mathbf{x}_m) \widehat{f}(\mathbf{x}_m, -\mathbf{x}_1) d\mathbf{x}_1 \dots d\mathbf{x}_m. \quad (1.24)$$

We will use these formulas in the case of the Wiener integral with respect to the Rosenblatt sheet.

Proposition 15. *The cumulants of a Wiener integral with respect to the Rosenblatt sheet $Z^2_{\mathbf{H}}(\mathbf{t})$ are*

$$\begin{aligned} k_m\left(\int_{\mathbb{R}^d} f(\mathbf{s}) dZ^2_{\mathbf{H}}(\mathbf{s})\right) &= c_{1,m} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} f(\mathbf{y}_1) \dots f(\mathbf{y}_m) |\mathbf{y}_1 - \mathbf{y}_2|^{\mathbf{H}-1} \dots |\mathbf{y}_m - \mathbf{y}_1|^{\mathbf{H}-1} d\mathbf{y}_1 \dots d\mathbf{y}_m \\ &= c_{2,m} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \widehat{f}(\mathbf{x}_1 - \mathbf{x}_2) \widehat{f}(\mathbf{x}_2 - \mathbf{x}_3) \dots \widehat{f}(\mathbf{x}_m - \mathbf{x}_1) \\ &\quad |\mathbf{x}_1|^{-\mathbf{H}} \dots |\mathbf{x}_m|^{-\mathbf{H}} d\mathbf{x}_1 \dots d\mathbf{x}_m. \end{aligned}$$

with $c_{1,m} = 2^{m-1}(m-1)! c(\mathbf{H}, 2)^m \beta(\frac{\mathbf{H}}{2}, \mathbf{H}-1)^m$ and $c_{2,m} = 2^{m-1}(m-1)!(2\pi)^{-m} c(\mathbf{H}, 2)^m \Gamma(\frac{\mathbf{H}}{2})^{2m} e^{-\frac{im\pi\mathbf{H}}{2}}$, where β is the Beta function $\beta(p,q) = \int_0^1 z^{p-1}(1-z)^{q-1} dz, p, q > 0$ and $\beta(\mathbf{p}, \mathbf{q}) = \prod_{i=1}^d \beta(p_i, q_i)$ if $\mathbf{p} = (p_1, .., p_d) \geq 0$ and $\mathbf{q} = (q_1, .., q_d) \geq 0$.

Proof : Put $c_m := (m-1)! 2^{m-1}$ for $m \geq 1$ integer. We can write

$$\int_{\mathbb{R}^d} f(s) dZ^2_H(s) \quad = \quad I_2(Jf)$$

with (see (1.5))

$$Jf(\mathbf{y_1}, \mathbf{y_2}) = c(\mathbf{H}, 2) \int_{\mathbb{R}^d} f(\mathbf{u}) (\mathbf{u} - \mathbf{y_1})_+^{\frac{H}{2}-1} (\mathbf{u} - \mathbf{y_2})_+^{\frac{\mathbf{H}}{2}-\mathbf{1}} d\mathbf{u}.$$

Using (4.31), we have

$$\begin{aligned}
k_m\left(I_2(Jf)\right) &= c_m \int_{(\mathbb{R}^d)^m} d\mathbf{u}_1 \dots d\mathbf{u}_m Jf\left(\mathbf{u}_1, \mathbf{u}_2\right) Jf\left(\mathbf{u}_2, \mathbf{u}_3\right) \dots . Jf\left(\mathbf{u}_{m-1}, \mathbf{u}_m\right) Jf\left(\mathbf{u}_m, \mathbf{u}_1\right) \\
&= c_m \int_{(\mathbb{R}^d)^m} d\mathbf{u}_1 \dots d\mathbf{u}_m c(\mathbf{H},2) \int_{\mathbb{R}^d} f(\mathbf{y}_1)(\mathbf{y}_1-\mathbf{u}_1)_+^{\frac{\mathbf{H}}{2}-1}(\mathbf{y}_1-\mathbf{u}_2)_+^{\frac{\mathbf{H}}{2}-1} d\mathbf{y}_1 \\
&\quad c(\mathbf{H},2) \int_{\mathbb{R}^d} f(\mathbf{y}_2)(\mathbf{y}_2-\mathbf{u}_2)_+^{\frac{\mathbf{H}}{2}-1}(\mathbf{y}_2-\mathbf{u}_3)_+^{\frac{\mathbf{H}}{2}-1} d\mathbf{y}_2 \\
&\quad \times \dots \times c(\mathbf{H},2) \int_{\mathbb{R}^d} f(\mathbf{y}_m)(\mathbf{y}_m-\mathbf{u}_m)_+^{\frac{\mathbf{H}}{2}-1}(\mathbf{y}_m-\mathbf{u}_1)_+^{\frac{\mathbf{H}}{2}-1} d\mathbf{y}_m \\
&= c_m c(\mathbf{H},2)^m \int_{(\mathbb{R}^d)^m} d\mathbf{y}_1 \dots d\mathbf{y}_m f(\mathbf{y}_1) \dots \times f(\mathbf{y}_m) \\
&\quad \int_{\mathbb{R}^d} d\mathbf{u}_1 (\mathbf{y}_1-\mathbf{u}_1)_+^{\frac{\mathbf{H}}{2}-1}(\mathbf{y}_m-\mathbf{u}_1)_+^{\frac{\mathbf{H}}{2}-1} \\
&\quad \times \int_{\mathbb{R}^d} d\mathbf{u}_m (\mathbf{y}_m-\mathbf{u}_m)_+^{\frac{H}{2}-1}(\mathbf{y}_{m-1}-\mathbf{u}_m)_+^{\frac{H}{2}-1}.
\end{aligned}$$

Recall the following identity (see [103], Lemma 3.1)

$$\int_{\mathbb{R}^d} (\mathbf{u}-\mathbf{y})_+^{\mathbf{a}-1}(\mathbf{v}-\mathbf{y})_+^{\mathbf{a}-1} d\mathbf{y} = \beta(\mathbf{a}, 1-2\mathbf{a})|\mathbf{u}-\mathbf{v}|^{2\mathbf{a}-1},$$

with $\beta(\mathbf{a}, 1-2\mathbf{a}) = \prod_{i=1}^d \beta(a_i, 2a_i-1)$ if $\mathbf{a} = (a_1, ..a_d)$. We get

$$\begin{aligned}
k_m(I_2(Jf)) &= c_m c(\mathbf{H},2)^m \beta(\frac{\mathbf{H}}{2}, \mathbf{H}-1)^m \int_{(\mathbb{R}^d)^m} d\mathbf{y}_1 \dots d\mathbf{y}_m f(\mathbf{y}_1) \dots f(\mathbf{y}_m) \\
&\quad |\mathbf{y}_1-\mathbf{y}_2|^{H-1} ... |\mathbf{y}_1-\mathbf{y}_m|^{H-1} \\
&= c_{1,m} \int_{(\mathbb{R}^d)^m} d\mathbf{y}_1 \dots d\mathbf{y}_m f(\mathbf{y}_1) \dots f(\mathbf{y}_m) |\mathbf{y}_1-\mathbf{y}_2|^{H-1} \dots |\mathbf{y}_1-\mathbf{y}_m|^{H-1}
\end{aligned}$$

with $c_{1,m} = c_m c(\mathbf{H},2)^m \beta(\frac{\mathbf{H}}{2}, \mathbf{H}-1)^m$.

The second part of the conclusion can be obtained analogously via Lemma 6 and (2.9). From Lemma 6, we have :

$$\widehat{Jf}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = c(\mathbf{H},2)\widehat{f}(\boldsymbol{\xi}_1+\boldsymbol{\xi}_2)\Gamma(\frac{\mathbf{H}}{2})^2 e^{-\frac{i\pi\mathbf{H}}{2}} |\boldsymbol{\xi}_1|^{-\frac{\mathbf{H}}{2}} |\boldsymbol{\xi}_2|^{-\frac{\mathbf{H}}{2}}$$

Using (2.9), we can write

$$\begin{aligned}
k_m(G) &= c_m (2\pi)^{-m} \int_{(\mathbb{R}^d)^m} \widehat{Jf}(\mathbf{x_1}, -\mathbf{x_2})\widehat{Jf}(\mathbf{x_2}, -\mathbf{x_3}) ... \widehat{Jf}(\mathbf{x_{m-1}}, -\mathbf{x_m})\widehat{Jf}(\mathbf{x_m}, -\mathbf{x_1}) d\mathbf{x_1} ... d\mathbf{x_m} \\
&= c_m (2\pi)^{-m} \int_{(\mathbb{R}^d)^m} \left(c(\mathbf{H},2) e^{-\frac{i\pi\mathbf{H}}{2}} \Gamma\left(\frac{\mathbf{H}}{2}\right)^2 \right)^m \widehat{f}(\mathbf{x_1}-\mathbf{x_2})\widehat{f}(\mathbf{x_2}-\mathbf{x_3}) \dots \\
&\quad \widehat{f}(\mathbf{x_{m-1}}-\mathbf{x_m})\widehat{f}(\mathbf{x_m}-\mathbf{x_1}) |\mathbf{x_1}|^{-\frac{\mathbf{H}}{2}} |\mathbf{x_2}|^{-\frac{\mathbf{H}}{2}} ... |\mathbf{x_m}|^{-\frac{\mathbf{H}}{2}} |\mathbf{x_m}|^{-\frac{\mathbf{H}}{2}} |\mathbf{x_1}|^{-\frac{\mathbf{H}}{2}} d\mathbf{x_1} ... d\mathbf{x_m} \\
&= c_{2,m} \int_{(\mathbb{R}^d)^m} \widehat{f}(\mathbf{x_1}-\mathbf{x_2})\widehat{f}(\mathbf{x_2}-\mathbf{x_3}) ... \widehat{f}(\mathbf{x_{m-1}}-\mathbf{x_m})\widehat{f}(\mathbf{x_m}-\mathbf{x_1}) \\
&\quad |\mathbf{x_1}|^{-\mathbf{H}} |\mathbf{x_2}|^{-\mathbf{H}} ... |\mathbf{x_{m-1}}|^{-\mathbf{H}} |\mathbf{x_m}|^{-\mathbf{H}} d\mathbf{x_1} ... d\mathbf{x_m}
\end{aligned}$$

with $c_{2,m} = c_m (2\pi)^{-m} c(\mathbf{H},2)^m e^{-\frac{im\pi\mathbf{H}}{2}} \Gamma\left(\frac{\mathbf{H}}{2}\right)^{2m}$. ■

Using the same procedure, we can compute the cumulants of finite dimensional distributions of the process $(u(t,x), t \geq 0)$ with $x \in \mathbb{R}$ fixed.

Proposition 16. *For every $N \geq 1$ and for every $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$, $t_1, \ldots, t_N \geq 0$, $\mathbf{x} \in \mathbb{R}^d$, the cumulants of the random variable*

$$V = \lambda_1 u(t_1, \mathbf{x}) + \ldots + \lambda_N u(t_N, \mathbf{x})$$

are

$$\begin{aligned} k_m(V) &= c_{1,m} \sum_{j_1,\ldots,j_m=1}^{N} \lambda_{j_1} \ldots \lambda_{j_m} \int_0^{t_{j_1}} du_1 \ldots \int_0^{t_{j_m}} du_m |u_1 - u_2|^{H-1} \ldots .. |u_m - u_1|^{H-1} \\ &\quad \int_{(\mathbb{R}^d)^m} d\mathbf{y}_1 \ldots d\mathbf{y}_m |\mathbf{y}_1 - \mathbf{y}_2|^{\mathbf{H}_0 - 1} \ldots |\mathbf{y}_m - \mathbf{y}_1|^{\mathbf{H}_0 - 1} G(t_{j_1} - u_1, \mathbf{x} - \mathbf{y}_1) \ldots G(t_{j_m} - u_m, \mathbf{x} - \mathbf{y}_m) \end{aligned}$$

or

$$\begin{aligned} k_m(V) &= c_{2,m} \sum_{j_1,\ldots,j_m=1}^{N} \lambda_{j_1} \ldots \lambda_{j_m} \int_{\mathbb{R}^m} du_1 \ldots du_m \int_{(\mathbb{R}^d)^m} d\mathbf{y}_1 \ldots d\mathbf{y}_m |u_1 ... u_m|^{-H} |\mathbf{y}_1 ... \mathbf{y}_m|^{-\mathbf{H}_0} \\ &\quad \left(e^{it_{j_1}(u_1 - u_2)} - 1\right) \frac{1}{i(u_1 - u_2) + \frac{1}{2}|\mathbf{y}_1 - \mathbf{y}_2|^2} \cdots \left(e^{it_{j_m}(u_m - u_1)} - 1\right) \frac{1}{i(u_m - u_1) + \frac{1}{2}|\mathbf{y}_m - \mathbf{y}_1|^2} \end{aligned}$$

where the constants $c_{1,m}, c_{2,m}$ are as in Proposition 15.

Proof : We can write

$$V = I_2\left(\sum_{j=1}^{N} \lambda_j F_{t_j,\mathbf{x}}\right)$$

with $F_{t,\mathbf{x}} = JG(t - \cdot, \mathbf{x} - *)$ where J is the transform (1.5). We apply formula (4.31) as in Proposition 15 and we obtain

$$\begin{aligned} k_m(V) &= c_{2,m} \int_{\mathbb{R}^m} du_1 \ldots, du_m \int_{(\mathbb{R}^d)^m} d\mathbf{y}_1 \ldots d\mathbf{y}_m \\ &\quad \left(|u_1 - u_2|^{H-1} \ldots |u_m - u_1|^{H-1} |\mathbf{y}_1 - \mathbf{y}_2|^{H_0 - 1} \ldots |\mathbf{y}_m - \mathbf{y}_1|^{H_0 - 1}\right) \\ &\quad \sum_{j_1=1}^{d} \lambda_{j_1} G\left(t_{j_1} - u_1, \mathbf{x} - \mathbf{y}_1\right) \mathbf{1}_{[0,t_{j_1}]}(u_1) \ldots \sum_{j_m=1}^{d} \lambda_{j_m} G\left(t_{j_m} - u_m, \mathbf{x} - \mathbf{y}_m\right) \mathbf{1}_{[0,t_{j_m}]}(u_m) \\ &= c_{2,m} \sum_{j_1,\ldots,j_m=1}^{d} \lambda_{j_1} \ldots \lambda_{j_m} \int_0^{t_{j_1}} du_1 \ldots \int_0^{t_{j_m}} du_m |u_1 - u_2|^{H-1} \ldots |u_m - u_1|^{H-1} \\ &\quad \int_{(\mathbb{R}^d)^m} d\mathbf{y}_1 \ldots d\mathbf{y}_m |\mathbf{y}_1 - \mathbf{y}_2|^{\mathbf{H_0} - 1} \ldots |\mathbf{y}_m - \mathbf{y}_1|^{H_0 - 1} G(t_{j_1} - u_1, \mathbf{x} - \mathbf{y}_1) \ldots G(t_{j_m} - u_m, \mathbf{x} - \mathbf{y}_m). \end{aligned}$$

For the second part of the proof, recall that the Fourier transform of $g_1(y) = G(t_j - u, \mathbf{x} - \mathbf{y}) \mathbf{1}_{[0,t_j]}$ is $\widehat{g}_1(\boldsymbol{\xi}) = e^{i\mathbf{x}\boldsymbol{\xi}} e^{-\frac{1}{2}(t_j - u)|\boldsymbol{\xi}|^2} \mathbf{1}_{[0,t_j]}$ and the Fourier transform of $g_2(u) = e^{i\mathbf{x}\boldsymbol{\xi}} e^{-\frac{1}{2}(t_j - u)|\boldsymbol{\xi}|^2} \mathbf{1}_{[0,t_j]}$ is

$\widehat{g}_2(u) = \frac{e^{i\mathbf{x}\boldsymbol{\xi}}}{iu+\frac{1}{2}|\boldsymbol{\xi}|^2}\left(e^{it_j u}-1\right)$. Using Proposition 4, we get

$$\begin{aligned}
k_m(V) &= c_{2,m}\int_{\mathbb{R}^m} du_1\dots,du_m\int_{(\mathbb{R}^d)^m} d\mathbf{y}_1\dots d\mathbf{y}_m|u_1....u_m|^{-H}|\mathbf{y}_1...\mathbf{y}_m|^{-\mathbf{H}_0}\\
&\left(\sum_{j_1=1}^{m}\lambda_{j_1}\frac{e^{ix(\mathbf{y}_1-\mathbf{y}_2)}}{i(u_1-u_2)+\frac{1}{2}|\mathbf{y}_1-\mathbf{y}_2|^2}\left(e^{it_{j_1}(u_1-u_2)}-1\right)\right)\\
&\dots\left(\sum_{j_m=1}^{m}\lambda_{j_m}\frac{e^{ix(\mathbf{y}_m-\mathbf{y}_1)}}{i(u_m-u_1)+\frac{1}{2}|\mathbf{y}_m-\mathbf{y}_1|^2}\left(e^{it_{j_m}((u_m-u_1)}-1\right)\right)\\
&= c_{2,m}\sum_{j_1,\dots,j_m=1}^{N}\lambda_{j_1}\dots\lambda_{j_m}\int_{\mathbb{R}^m} du_1\dots,du_m\int_{(\mathbb{R}^d)^m} d\mathbf{y}_1\dots d\mathbf{y}_m|u_1...u_m|^{-H}|\mathbf{y}_1...\mathbf{y}_m|^{-\mathbf{H}_0}\\
&\left(e^{it_{j_1}(u_1-u_2)}-1\right)\frac{1}{i(u_1-u_2)+\frac{1}{2}|\mathbf{y}_1-\mathbf{y}_2|^2}\cdots\left(e^{it_{j_m}(u_m-u_1)}-1\right)\frac{1}{i(u_m-u_1)+\frac{1}{2}|\mathbf{y}_m-\mathbf{y}_1|^2}.
\end{aligned}$$

■

Let us comment on the link between the law of (1.21) and the law of the Hermite process if $q=2$.

Remark 5. *Assume $d=1$ and let $\gamma = H+\frac{H_1-1}{2}$. Suppose that the random $u(t,\mathbf{x})$ has, modulo a constant, the same law as a Rosenblatt random variable $Z_\gamma(t)$. Recall that the cumulants of $Z_\gamma(t)$ are*

$$2^{m-1}(m-1)c(\gamma,2)^m\int_{\mathbb{R}^m} du_1...du_m|u_1...u_m|^{-\gamma}\frac{e^{it(u_1-u_2)}-1}{i(u_1-u_2)}\dots\frac{e^{it(u_m-u_1)}-1}{i(u_m-u_1)}. \tag{1.25}$$

From (1.25), with the change of variables $y_i=\sqrt{|u_i|}\tilde{y}_i$, $i=1,..,q$, we have

$$k_m(U(t,x)) = c_{2,m}\int_{\mathbb{R}^m} du_1...du_m|u_1...u_m|^{-\gamma}\frac{e^{it(u_1-u_2)}-1}{i(u_1-u_2)}\dots\frac{e^{it(u_m-u_1)}-1}{i(u_m-u_1)}g_m(u_1,..,u_m) \tag{1.26}$$

where

$$g_m(u_1,..,u_m) = \int_{\mathbb{R}^m} dy_1..dy_m|y_1...y_m|^{-H_1}\frac{1}{1+\frac{1}{2i}}\frac{|y_1\sqrt{|u_1|}-y_2\sqrt{|u_2|}|^2}{u_1-u_2}\dots\frac{1}{1+\frac{1}{2i}}\frac{|y_m\sqrt{|u_m|}-y_1\sqrt{|u_1|}|^2}{u_m-u_1}.$$

By comparing (1.25) and (1.26), asumming that U has the same law as $Z_\gamma(t)$, it would follow that there exists a constant C such that

$$\int_{\mathbb{R}^m} du_1...du_m f_m(u_1,..,u_m) = C^m\int_{\mathbb{R}^m} du_1...du_m f_m(u_1,..,u_m)g_m(u_1,..,u_m)$$

for every $m\geq 2$ where we used the notation

$$f_m(u_1,..,u_m) = |u_1...u_m|^{-\gamma}\frac{e^{it(u_1-u_2)}-1}{i(u_1-u_2)}\dots\frac{e^{it(u_m-u_1)}-1}{i(u_m-u_1)}.$$

A numerical analysis of the above integrals for $m=2,3$ (as done in [5]) should certify that this is not possible and that $(U(t,x))_{t\geq 0}$ is not a Rosenblatt process.

1.3.5 An application : the α-variation

As an application of the decomposition obtained in Section 1.3.3, we compute the α-variation of the solution. Given a continuous stochastic process $(X_t)_{t\geq 0}$, we will say that it admits an α-variation ($\alpha > 0$) if the sequence

$$V_t^{n,\alpha}(X) = \sum_{i=0}^{n-1} \left| X_{t_{i+1}} - X_{t_i} \right|^{\alpha}$$

converges in probability as $n \to \infty$, for every $t \geq 0$, where $t_i = \frac{it}{n}$ for $i = 0, ..., n-1$. Using the decomposition $u = U - Y$ obtained in Section 1.3.3, we can deduce the α -variation of the solution.

Proposition 17. *Consider the process $u = \{u(t, \mathbf{x}); t \geq 0, \mathbf{x} \in \mathbb{R}^d\}$ given by (4.27). For every fixed $x \in \mathbb{R}^d$, the process $(u(t, \mathbf{x}))_{t\geq 0}$ admits an α-variation equal to $\mathbf{E} \left| U(1, \mathbf{x}) \right|^{\alpha} t$ for every $\alpha > 1$.*

Proof : Fix $t > 0$ and let $t_i = \frac{it}{n}$, $i = 0, .., n-1$. By Minkovski inequality,

$$\begin{aligned}
&\left(\sum_{i=0}^{n-1} |U(t_{i+1}, \mathbf{x}) - U(t_i, \mathbf{x})|^{\alpha} \right)^{\frac{1}{\alpha}} - \left(\sum_{i=0}^{n-1} |Y(t_{i+1}, \mathbf{x}) - Y(t_i, \mathbf{x})|^{\alpha} \right)^{\frac{1}{\alpha}} \\
&\leq \left(\sum_{i=0}^{n-1} |u(t_{i+1}, \mathbf{x}) - u(t_i, \mathbf{x})|^{\alpha} \right)^{\frac{1}{\alpha}} \\
&\leq \left(\sum_{i=0}^{n-1} |U(t_{i+1}, \mathbf{x}) - U(t_i, \mathbf{x})|^{\alpha} \right)^{\frac{1}{\alpha}} + \left(\sum_{i=0}^{n-1} |Y(t_{i+1}, \mathbf{x}) - Y(t_i, \mathbf{x})|^{\alpha} \right)^{\frac{1}{\alpha}}. \qquad (1.27)
\end{aligned}$$

By Theorem 1, since $\alpha > 1$

$$\sum_{i=0}^{n-1} |Y(t_{i+1}, \mathbf{x}) - Y(t_i, \mathbf{x})|^{\alpha} \quad \leq \quad \sup_i |Y(t_{i+1}, \mathbf{x}) - Y(t_i, \mathbf{x})|^{\alpha-1} \sum_{i=0}^{n-1} |Y(t_{i+1}, \mathbf{x}) - Y(t_i, \mathbf{x})|$$

so $V_t^{n,\alpha}(Y)$ converges to zero in probability as $n \to \infty$. On the other hand, from the stationarity of the increments and the self-similarity of $U(t, \mathbf{x})$, the sequence

$$\sum_{i=0}^{n-1} |U(t_{i+1}, \mathbf{x}) - U(t_i, \mathbf{x})|^{\alpha}$$

has the same law as

$$\frac{t}{n} \sum_{i=0}^{n-1} |U(j+1, \mathbf{x}) - U(j, \mathbf{x})|^{\alpha}$$

which converges almost surely to $t\mathbf{E} \left| U(1, \mathbf{x}) \right|^{\alpha}$ due to the ergodic theorem for stationary sequences (see e.g. Theorem 3.7 in [107]). The conclusion follows from (1.27). ■

Remark 6. *From the above result, we can easily prove, as in the proof of Theorem 5.1.1 in [74], that $(u(t, \mathbf{x}))_{t\geq 0}$ is not a semimartingale if $\alpha \neq 2$.*

1.3.6 Relation with the weak solution

As a final remark, let us discuss the relation between the mild solution (4.27) and the weak solution of the heat equation with Hermite noise. Let us recall the weak formulation of the solution to the heat equation, which is motivated by the integration by parts. We restrict to the case $d = 1$.

A stochastic process $(u(t,x), t \geq 0, x \in \mathbb{R})$ is a weak solution to (1.13) if

$$\int_0^T \int_{\mathbb{R}} u(t,x) \left(\frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2} \right) dxdt = - \int_0^T \int_{\mathbb{R}} \varphi(s,y) Z_H^q(ds,dy)$$

for every $T > 0$ and for every test function $\varphi \in C^\infty([0,\infty) \times \mathbb{R})$ with compact support in $\mathbb{R}$ such that $\varphi(T,x) = 0$ for every $x \in \mathbb{R}$.

Proposition 18. *A process u is a mild solution to (1.13) if and only if it is a weak solution to (1.13).*

Proof : Suppose u is a mild solution given by (4.27) and consider a test function $\varphi \in C^\infty([0,\infty) \times \mathbb{R})$ with compact support in $\mathbb{R}$ such that $\varphi(T,x) = 0$ for every $x \in \mathbb{R}$. Then, by using Fubini's theorem for multiple stochastic integrals (see Theorem 2.1 in [80]),

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} u(t,x) \left(\frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2} \right) dxdt \\ = &\int_0^T \int_{\mathbb{R}} \varphi(s,y) Z_H^q(ds,dy) \int_s^T dt \int_{\mathbb{R}} dx G(t-s, x-y) \left(\frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2} \right) \end{aligned}$$

and it is known that (see [37], pages 50-51)

$$\int_s^T dt \int_{\mathbb{R}} dx G(t-s, x-y) \left(\frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2} \right) = -\varphi(s,y).$$

The proof of the converse direction is also standard (see [38] or [37]). ■

1.4 Appendix : Multiple Wiener-Itô Integrals

Here, we shall only recall some elementary facts ; our main reference is [74]. Consider $\mathcal{H}$ a real separable infinite-dimensional Hilbert space with its associated inner product $\langle .,. \rangle_{\mathcal{H}}$, and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, which is a centered Gaussian family of random variables such that $\mathbf{E}(B(\varphi)B(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{H}}$, for every $\varphi, \psi \in \mathcal{H}$. Denote by I_q the qth multiple stochastic integral with respect to B. This I_q is actually an isometry between the Hilbert space $\mathcal{H}^{\odot q}$ (symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{q!}} \| \cdot \|_{\mathcal{H}^{\otimes q}}$ and the Wiener chaos of order q, which is defined as the closed linear span of the random variables $H_q(B(\varphi))$ where $\varphi \in \mathcal{H}, \|\varphi\|_{\mathcal{H}} = 1$ and H_q is the Hermite polynomial of degree $q \geq 1$ defined by

$$H_q(x) = (-1)^q \exp\left(\frac{x^2}{2} \right) \frac{\mathrm{d}^q}{\mathrm{d}x^q} \left(\exp\left(-\frac{x^2}{2} \right) \right), \; x \in \mathbb{R}. \tag{1.28}$$

The isometry of multiple integrals can be written as : for $p, \; q \geq 1$, $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$,

$$\mathbf{E}\Big(I_p(f) I_q(g) \Big) = \begin{cases} q! \langle \tilde{f}, \tilde{g} \rangle_{\mathcal{H}^{\otimes q}} & \text{if } p = q \\ 0 & \text{otherwie.} \end{cases} \tag{1.29}$$

Foe every $f \in \mathcal{H}^{\otimes q}$ not necessarily symmetricIt we have

$$I_q(f) = I_q(\tilde{f}),$$

where $\tilde{f}$ denotes the canonical symmetrization of f and it is defined by

$$\tilde{f}(x_1, \ldots, x_q) = \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} f(x_{\sigma(1)}, \ldots, x_{\sigma(q)}),$$

in which the sum runs over all permutations σ of $\{1,\ldots,q\}$.

In the particular case when $\mathcal{H}=L^2(T,\mathcal{B}(T),\mu)$, the rth contraction $f\otimes_r g$ is the element of $\mathcal{H}^{\otimes(p+q-2r)}$, which is defined by

$$\begin{aligned}&(f\otimes_r g)(s_1,\ldots,s_{p-r},t_1,\ldots,t_{q-r})\\ &=\int_{T^r}\mathrm{d}u_1\ldots\mathrm{d}u_r f(s_1,\ldots,s_{p-r},u_1,\ldots,u_r)g(t_1,\ldots,t_{q-r},u_1,\ldots,u_r),\end{aligned}\tag{1.30}$$

for every $f\in L^2(T^p)$, $g\in L^2(T^q)$ and $r=1,\ldots,p\wedge q$.

In our work the isonomal process B will be the d-parameter Wiener process $(W(\mathbf{x}),\mathbf{x}\in\mathbb{R}^d)$ and its associed Hilbert space is $\mathcal{H}=L^2(\mathbb{R}^d)$.

Deuxième partie

Comportement asymptotique des solutions d'équations stochastiques fractionnaires par rapport au paramètre de Hurst

Chapitre 2

Limit behavior of the Rosenblatt Ornstein-Uhlenbeck process with respect to the Hurst index

2.1 Introduction

The Rosenblatt process is a stochastic process in the second Wiener chaos, i.e. it can be expressed as a multiple integral of order two with respect to the Wiener process. It is a non-Gaussian self-similar process with stationary increments that exhibits long memory. The Rosenblatt process belongs to the class of so-called Hermite processes, which are self-similar processes with stationary increments in the qth Wiener chaos ($q \geq 1$) (the Rosenblatt process is obtained for $q = 2$ while for $q = 1$ we have the fractional Brownian motion, which is the only Gaussian Hermite process). The Rosenblatt process has been widely studied in the last decades, see e.g. the monographs [81] and [103] and the references therein.

There are several recent research works that investigate the asymptotic behavior in distribution of some fractional processes (see [12], [6], [3], [106]) with respect to the Hurst parameter. In particular, in the case of the Rosenblatt process $(Z^H(t))_{t\geq 0}$ with self-similarity index $H \in (\frac{1}{2}, 1)$, it has been shown in [106] that Z^H converges weakly, as $H \to \frac{1}{2}$, in the space of continuous functions $C[0,T]$ (for every $T > 0$), to a Brownian motion while if $H \to 1$, it tends weakly to the stochastic process $(t\frac{1}{\sqrt{2}}(Z^2 - 1))_{t\geq 0}$, $Z^2 - 1$ being a so-called *centered chi-square random variable*. The case of the generalized Rosenblatt process has been considered in [6] while the case of the Rosenblatt sheet can be found in [3]. Hermite processes of higher order have been considered in [12], [3].

The purpose of this work is to investigate the asymptotic behavior in distribution, with respect to the Hurst parameter, of the Wiener integral with respect to the Rosenblatt process (or the Wiener-Rosenblatt integral). The Wiener-Rosenblatt integral $\int_{\mathbb{R}} f(u)dZ^H(u)$ has been introduced in [53], for a sufficiently regular deterministic function f. In a first part, we give the asymptotic behavior in distribution, as $H \to \frac{1}{2}$ and as $H \to 1$, of the random variable $\int_{\mathbb{R}} f(u)dZ^H(u)$, by assuming suitable integrability condition on f. We will then focus on the asymptotic behavior with respect to H of the Rosenblatt Ornstein-Uhlenbeck process (ROU for short) which constitutes the unique solution of the Langevin equation driven by the Rosenblatt process. The ROU process can be expressed in the form of a Wiener-Rosenblatt integral with a particular kernel f. In order to check that this kernel verifies the integrability conditions needed in order to apply our general result, we need to use some technical results, in particular the so-called power counting theorem from [98]. We will treat separately the cases of the non-stationary ROU (whose initial value does not depend on H) and of the stationary ROU (with initial value depending on the Hurst parameter). We will prove that the (stationary) ROU converges weakly, as $H \to \frac{1}{2}$, to the (stationary) Gaussian Ornstein-Uhlenbeck process (solution of the Langevin equation driven by the Brownian motion) while as $H \to 1$, the ROU process converges weakly to a chi-square random variable multiplied by a deterministic process. Since we deal with processes in the second Wiener chaos, our proofs rely on the analysis of the asymptotic behavior of the cumulants of the random variables concerned (recall that the distribution of the elements of the second Wiener is completely determined by their cumulants, see [40] or [66]).

We organized our paper as follows. In Section 2 we recall some basic definitions for the Rosenblatt process and the Wiener-Rosenblatt integral and we also state a general result for the limit behavior in law of the Wiener-Rosenblatt integral as the Hurst index converges to its extreme values. In Section 3 we treat the particular case of the Rosenblatt Ornstein-Uhlenbeck process.

2.2 The Rosenblatt process and the Wiener-Rosenblatt integral

Below, in the first part, we present the definition and the basic properties of the Rosenblatt process and of the Wiener integral with respect to the Rosenblatt process. In the second part, we give a general result concerning the convergence in distribution with respect to the Hurst parameter of the Wiener-Rosenblatt integral. This general result will be applied in the next section in order to obtain the limit behavior of the Ornstein-Uhlenbeck process with Rosenblatt noise.

2.2.1 Definition and basic properties

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(B(y))_{y\in\mathbb{R}}$ be a Wiener process on Ω. We will denote by $(Z^H(t))_{t\geq 0}$ the Rosenblatt process with self-similarity index $H \in \left(\frac{1}{2}, 1\right)$. It is defined on Ω as a multiple stochastic integral of order 2 with respect to the Wiener process B via

$$Z^H(t) = c(H,2) \int_{\mathbb{R}}\int_{\mathbb{R}} \left(\int_0^t (u-y_1)_+^{\frac{H}{2}-1}(u-y_2)_+^{\frac{H}{2}-1} du \right) dB(y_1)dB(y_2) = I_2(L_t^H), \quad t \geq 0 \quad (2.1)$$

where I_2 denotes the multiple stochastic integral of order two with respect to B (see the Appendix) and we denoted by L_t^H the kernel of the Rosenblatt process given by, for every $y_1, y_2 \in \mathbb{R}$

$$L_t^H(y_1, y_2) = c(H,2) \int_0^t (u-y_1)_+^{\frac{H}{2}-1}(u-y_2)_+^{\frac{H}{2}-1} du. \quad (2.2)$$

We denoted $x_+ = \max(x, 0)$. It is well-known (see e.g. [103]) that the kernel L_t^H belongs to $L^2(\mathbb{R}^2)$ for every $t \geq 0$ when $H > \frac{1}{2}$, which implies that the multiple integral of order two in (2.1) is well-defined. The strictly positive constant $c(H,2)$ is chosen such that $\mathbf{E}(Z^H(1))^2 = 1$. Actually (see e.g. [103], Proposition 3.1)

$$c(H,2)^2 = \frac{H(2H-1)}{2\beta\left(\frac{H}{2}, 1-H\right)^2}, \quad (2.3)$$

where β is Beta function $\beta(p,q) = \int_0^1 z^{p-1}(1-z)^{q-1}dz, p, q > 0$.

The Rosenblatt process is a H-self-similar process with stationary increments. It exhibits long-range dependence and its sample paths are Hölder continuous of order δ for any $\delta \in (0, H)$. This process has been intensively studied in the last decades, see e.g. the monographs [81] and [103] and the references therein.

The Wiener integral with respect to the Rosenblatt process (or the *Wiener-Rosenblatt integral*) has been constructed in [53]. Let $f \in \mathcal{H}_H$ where $\mathcal{H}_H = \{f : \mathbb{R} \to \mathbb{R} : \|f\|_{\mathcal{H}_H} < \infty\}$ with

$$\|f\|_{\mathcal{H}_H}^2 := H(2H-1) \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)f(v)|u-v|^{2H-2} du dv.$$

Then the Wiener integral of f with respect to Z^H is given by

$$\int_{\mathbb{R}} f(u) dZ^H(u) = I_2(J_H f) \tag{2.4}$$

where the kernel $J_H f$ has the expression, for every $y_1, y_2 \in \mathbb{R}$,

$$(J_H f)(y_1, y_2) = c(H,2) \int_{\mathbb{R}} f(u)(u-y_1)_+^{\frac{H}{2}-1}(u-y_2)_+^{\frac{H}{2}-1} du. \tag{2.5}$$

The Wiener-Rosenblatt integral constitutes an isometry between $\mathcal{H}_H$ and $L^2(\Omega)$ since, for every $f, g \in \mathcal{H}_H$

$$\mathbf{E}\left(\int_{\mathbb{R}} f(u) dZ^H(u) \int_{\mathbb{R}} g(u) dZ^H(u)\right) = \langle f, g\rangle_{\mathcal{H}_H} := H(2H-1) \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)g(v)|u-v|^{2H-2} du dv. \tag{2.6}$$

The space $\mathcal{H}_H$ is not complete and may contains distributions. A subspace of functions included in $\mathcal{H}_H$ is the space $|\mathcal{H}_H|$ of measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\|f\|_{|\mathcal{H}_H|}^2 := \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)f(v)||u-v|^{2H-2} du dv < \infty. \tag{2.7}$$

2.2.2 Asymptotic behavior of the Wiener-Rosenblatt integral

Our purpose is to study the asymptotic behavior, as $H \to \frac{1}{2}$ and $H \to 1$, of the Wiener integral with respect ot the Rosenblatt process

$$\int_{\mathbb{R}} f(u) dZ^H(u)$$

where $(Z^H(t))_{t \geq 0}$ is a Rosenblatt process with self-similarity order $H \in (\frac{1}{2}, 1)$ and $f \in \mathcal{H}_H$.

The proof of the asymptotic behavior of the Wiener-Rosenblatt integral is based on the analysis of its cumulants. Since the random variable

$$\int_{\mathbb{R}} f(u) dZ^H(u)$$

belongs to the second Wiener chaos, see (2.4), its law is completely determined by its cumulants (or equivalently, by its moments). That is, if F, G are elements of the second Wiener chaos then F and G have the same law if and only if they have the same cumulants. Moreover, the convergence of the cumulants implies the convergence in distribution when we deal with sequences in the Wiener chaos of order two. Let us denote by $k_m(F)$, $m \geq 1$ the mth cumulant of a random variable F. It is defined as

$$k_m(F) = (-i)^n \frac{\partial^n}{\partial t^n} \ln \mathbf{E}(e^{itF})|_{t=0}.$$

We have the following link between the moments and the cumulants of F : for every $m \geq 1$,

$$k_m(F) = \sum_{\sigma=(a_1,..,a_r)\in\mathcal{P}(\{1,..,n\})} (-1)^{r-1}(r-1)! \mathbf{E}X^{|a_1|} \ldots \mathbf{E}X^{|a_r|} \tag{2.8}$$

if $F \in L^m(\Omega)$, where $\mathcal{P}(b)$ is the set of all partitions of b. In particular, for centered random variables F, we have $k_1(F) = \mathbf{E}F, k_2(F) = \mathbf{E}F^2, k_3(F) = \mathbf{E}F^3, k_4 = \mathbf{E}F^4 - (\mathbf{E}F^2)^2$.

In the particular situation when $G = I_2(f)$ is a multiple integral of order 2 with respect to a Wiener process $(B(y))_{y\in\mathbb{R}}$, then its cumulants can be computed as (see e.g. [65], Proposition 7.2 or [103])

$$k_m(G) = 2^{m-1}(m-1)! \int_{\mathbb{R}^m} du_1 \dots du_m f(u_1,u_2)f(u_2,u_3)\dots f(u_{m-1},u_m)f(u_m,u_1). \tag{2.9}$$

From the formula (2.9), we can obtain the following expression of the cumulants of the Wiener-Rosenblatt integral (see e.g. [88]).

Proposition 19. *We have, for $m \geq 2$*

$$\begin{aligned} &k_m\left(\int_{\mathbb{R}} f(u)dZ^H(u)\right) \qquad (2.10)\\ &= c_{1,m}\int_{\mathbb{R}}\dots\int_{\mathbb{R}} du_1...du_m f(u_1)...f(u_m)|u_1-u_2|^{H-1}|u_2-u_3|^{H-1}...|u_{m-1}-u_m|^{H-1}|u_m-u_1|^{H-1}\end{aligned}$$

with

$$c_{1,m} = 2^{\frac{m}{2}-1}(m-1)!(H(2H-1))^{\frac{m}{2}} \tag{2.11}$$

and $k_1\left(\int_{\mathbb{R}} f(u)dZ^H(u)\right) = 0$.

We will treat separately the behavior of the Wiener-Rosenblatt integral as H is near $\frac{1}{2}$ and near 1. The limiting process will be different in these two cases.

Convergence when $H \to 1$

We have the following result.

Proposition 20. *Consider $f : \mathbb{R} \to \mathbb{R}$ such that for some $\varepsilon \in (0, \frac{1}{2})$*

$$\|f\|^2_{\left|\mathcal{H}_{\frac{1}{2}+\varepsilon}\right|} = \int_{\mathbb{R}}\int_{\mathbb{R}} dudv|f(u)||f(v)||u-v|^{2\varepsilon-1} < \infty \text{ and } f \in L^1(\mathbb{R}). \tag{2.12}$$

Then

$$\int_{\mathbb{R}} f(u)dZ^H(u) \xrightarrow[H\to 1]{(d)} \frac{1}{\sqrt{2}}\left(\int_{\mathbb{R}} f(u)du\right)(Z^2-1) \tag{2.13}$$

with $Z \sim N(0,1)$.

Proof : First notice that condition (2.12) implies that $f \in |\mathcal{H}_H|$ for $H \geq \frac{1}{2}+\varepsilon$. Indeed, by using the bound $\sup_{H\in[\frac{1}{2}+\varepsilon,1]} |x|^{2H-2} \leq 1 \vee |x|^{2\varepsilon-1}$, we get

$$\begin{aligned} \|f\|^2_{|\mathcal{H}_H|} &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} dudv|f(u)f(v)|(1\vee|u-v|^{2\varepsilon-1})\\ &\leq \left(\int_{\mathbb{R}}|f(u)|du\right)^2 + \int_{\mathbb{R}}\int_{\mathbb{R}} dudv|f(u)||f(v)||u-v|^{2\varepsilon-1} < \infty.\end{aligned}$$

Consider the random variable

$$G := I_2\left(\frac{1}{\sqrt{2}}\left(\int_{\mathbb{R}} f(u)du\right)1^{\otimes 2}_{[0,1]}\right)$$

which has the same law as the right-hand side of (2.13). We have

$$k_m(G) = 2^{m-1}(m-1)! \left(\frac{1}{\sqrt{2}} \left(\int_{\mathbb{R}} f(u)du \right) \right)^m = 2^{\frac{m}{2}-1}(m-1)! \left(\int_{\mathbb{R}} f(u)du \right)^m .$$

On the other hand, since by (2.11)

$$c_{1,m} \xrightarrow[H\to 1]{} 2^{\frac{m}{2}-1}(m-1)!$$

the conclusion will follow if we show that

$$\int_{\mathbb{R}} ... \int_{\mathbb{R}} du_1...du_m f(u_1)...f(u_m)|u_1-u_2|^{H-1}|u_2-u_3|^{H-1}...|u_{m-1}-u_m|^{H-1}|u_m-u_1|^{H-1}$$

converges, as $H \to 1$, to

$$\left(\int_{\mathbb{R}} f(u)du \right)^m .$$

Assume that the integrability condition (2.12) is satisfied. Consider the function g_H defined on $\mathbb{R}^m$ except the diagonals with values in $\mathbb{R}$, given by

$$g_H(u_1,..,u_m) = f(u_1)...f(u_m)|u_1-u_2|^{H-1}|u_2-u_3|^{H-1}...|u_{m-1}-u_m|^{H-1}|u_m-u_1|^{H-1}.$$

Clearly $g_H(u_1,..,u_m)$ converges to $f(u_1)...f(u_m)$ for almost every $u_1,..,u_m \in \mathbb{R}$. Also, using again the bound $\sup_{H\in[\frac{1}{2}+\varepsilon,1]} |x|^{H-1} \leq 1 \vee |x|^{-\frac{1}{2}+\varepsilon}$ for every $x \in \mathbb{R}$, we have

$$\begin{aligned}
\sup_{H\in[\frac{1}{2}+\varepsilon,1]} |g_H(u_1,..,u_m)| &\leq |f(u_1)...f(u_m)| \left(1 \vee (|u_1-u_2|^{-\frac{1}{2}+\varepsilon}....|u_m-u_1|^{-\frac{1}{2}+\varepsilon}) \right) \\
&\leq |f(u_1)...f(u_m)| + |f(u_1)...f(u_m)|.....|u_1-u_2|^{-\frac{1}{2}+\varepsilon}....|u_m-u_1|^{-\frac{1}{2}+\varepsilon}.
\end{aligned}$$

In order to apply the dominated convergence theorem, we need to show that the function

$$|f(u_1)...f(u_m)| + |f(u_1)...f(u_m)|.....|u_1-u_2|^{-\frac{1}{2}+\varepsilon}....|u_m-u_1|^{-\frac{1}{2}+\varepsilon}$$

is integrable over $\mathbb{R}^m$. Since $f \in L^1(\mathbb{R})$, the first summand above is integrable over $\mathbb{R}^m$. On the other hand, $\int_{\mathbb{R}} ... \int_{\mathbb{R}} du_1..du_m|f(u_1)...f(u_m)|.....|u_1-u_2|^{-\frac{1}{2}+\varepsilon}....|u_m-u_1|^{-\frac{1}{2}+\varepsilon}$ represents, modulo a constant, the mth cumulant of the random variable $I_2(J_{\frac{1}{2}+\varepsilon}|f|)$, where $J_{\frac{1}{2}+\varepsilon}$ is given by (2.5). The fact that $\|f\|^2_{\left|\mathcal{H}_{\frac{1}{2}+\varepsilon}\right|} < \infty$ (see (2.10)) by (2.12) and the hypercontractivity property of multiple integrals (2.38) imply that the all the moments of $I_2(J_{\frac{1}{2}+\varepsilon}|f|)$ are finite and consequently, via (2.8), all the cumulants of $I_2(J_{\frac{1}{2}+\varepsilon}|f|)$ are finite. This implies the conclusion by using the dominated convergence theorem. ■

Remark 7. *For example, when f is bounded with compact support, then condition (2.12) is satisfied (in particular if $f = 1_{[0,t]}$ with $t > 0$ fixed). A detailed explanation can be found in the proof of Proposition 22 below.*

Convergence when $H \to \frac{1}{2}$

Concerning the behavior of the Rosenblatt-Wiener integral when H approaches one half, we have the following result.

Proposition 21. *Assume $f \in \mathcal{H}_H$. Also assume that*

$$\sigma_f^2 = \lim_{H\to\frac{1}{2}} \|f\|^2_{\mathcal{H}_H} = \lim_{H\to\frac{1}{2}} H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)f(v)|u-v|^{2H-2}dudv \tag{2.14}$$

exists and it is finite and

$$(2H-1)^2\int_{\mathbb{R}^4} du_1...du_4 f(u_1)...f(u_4)|u_1-u_2|^{H-1}|u_2-u_3|^{H-1}|u_3-u_4|^{H-1}|u_4-u_1|^{H-1} \xrightarrow[H\to\frac{1}{2}]{} 0. \tag{2.15}$$

Then

$$\int_{\mathbb{R}} f(u)dZ^H(u) \xrightarrow[H\to\frac{1}{2}]{(d)} N(0,\sigma_f^2).$$

Proof : We need to analyze the behavior of the cumulants of $\int_{\mathbb{R}} f(u)dZ^H(u)$ as H converges to $\frac{1}{2}$. Recall that $k_1\left(\int_{\mathbb{R}} f(u)dZ^H(u)\right) = \mathbf{E}\left(\int_{\mathbb{R}} f(u)dZ^H(u)\right) = 0$ and

$$k_2\left(\int_{\mathbb{R}} f(u)dZ^H(u)\right) = H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}} du_1du_2 f(u_1)f(u_2)|u_1-u_2|^{2H-2} = \|f\|^2_{\mathcal{H}_H}.$$

By (2.14), we have $k_2\left(\int_{\mathbb{R}} f(u)dZ^H(u)\right) \xrightarrow[H\to\frac{1}{2}]{} \sigma_f^2$. On the other hand, due to condition (2.15),

$$k_4\left(\int_{\mathbb{R}} f(u)dZ^H(u)\right) \xrightarrow[H\to\frac{1}{2}]{} 0.$$

Since $\int_{\mathbb{R}} f(u)dZ^H(u)$ belongs to the second Wiener chaos, by the Fourth Moment Theorem (see [76], see also Theorem 2 in the Appendix) we conclude that $\int_{\mathbb{R}} f(u)dZ^H(u)$ converges in distribution as $H\to\frac{1}{2}$ to a Gaussian random variable with mean zero (the limit of its first cumulant) and variance σ_f^2 (the limit of its second cumulant). ■

We will see below that for certain kernels f the condition (2.14) is automatically satisfied. Recall that a sequence of functions $(f_n)_{n\geq 1}$ is an approximation of the identity as $n\to\infty$ if

- $f_n(t)\geq 0$ for every $t\in\mathbb{R}$.
- For every $\delta>0$, $\int_{|t|\leq\delta} f_n(t)dt \xrightarrow[n\to\infty]{} 1$.
- For every $\delta>0$, $\int_{|t|>\delta} f_n(t)dt \xrightarrow[n\to\infty]{} 0$.

Moreover, if $(f_n)_{n\geq 1}$ is an approximation of the identity, then for every $f\in L^p(\mathbb{R})$ with $p\in[1,\infty)$, the convolution $f*f_n$ converges in $L^p(\mathbb{R})$ to f.

Corollary 1. *Assume $f\in\mathcal{H}_H\cap L^2(\mathbb{R})$ with $\mathrm{supp}(f)\subset[0,\infty)$ and (2.15) holds. Then*

$$\int_{\mathbb{R}} f(u)dZ^H(u) \xrightarrow[H\to\frac{1}{2}]{(d)} N(0,\int_{\mathbb{R}} f^2(u)du).$$

Proof : By Proposition 20, it suffices to check that the limit (2.14) exists and it is equal to $\int_{\mathbb{R}} f^2(u)du$. We have

$$\begin{aligned}\|f\|^2_{\mathcal{H}_H} &= H(2H-1)\int_0^\infty\int_0^\infty dudv f(u)f(v)|u-v|^{2H-2}\\ &= 2H(2H-1)\int_0^\infty du f(u)\int_0^u dv f(u-v)v^{2H-2}.\end{aligned}$$

Notice that the function $2H(2H-1)1_{[0,u]}(v)v^{2H-2}$ constitutes an approximation of the identity as $H\to\frac{1}{2}$. Therefore,

$$\|f\|^2_{\mathcal{H}_H} \xrightarrow[H\to\frac{1}{2}]{} \int_{\mathbb{R}} f^2(u)du.$$

By condition (2.15) and Theorem 2, the conclusion follows. ■

Remark 8. — *The above result shows that, when $H \to \frac{1}{2}$, the Wiener-Rosenblatt integral $\int_{\mathbb{R}} f(u)dZ^H(u)$ converges in distribution to $\int_{\mathbb{R}} f(u)dW(u)$, where W is a Wiener process. This is a natural extension of the results in [3] or [106].*

— *The condition* $\mathrm{supp}\,(f) \subset [0,\infty)$ *in Corollary 1 cannot be omitted. If for example* $\mathrm{supp}\,(f)$ *is* $\mathbb{R}$*, the above proof does not work, since* $2H(2H-1)1_{[0,\infty)}(v)v^{2H-2}$ *is not an approximation of the unity.*

2.3 Asymptotic behavior of the Rosenblatt Ornstein-Uhlenbeck process

The Rosenblatt Ornstein-Uhlenbeck (ROU) process is defined as the unique solution of the Langevin equation

$$X_t = \xi - \lambda \int_0^t X_s ds + \sigma Z^H(t), \quad t \geq 0 \tag{2.16}$$

where $\lambda, \sigma > 0$ and the initial condition ξ is a random variable in $L^2(\Omega)$. The case when the noise in (2.16) is the fractional Brownian motion has been considered in [19].

The unique solution to (2.16) can be expressed as

$$Y^H(t) = e^{-\lambda t}\left(\xi + \sigma \int_0^t e^{\lambda u} dZ^H(u)\right) \tag{2.17}$$

where the stochastic integral with respect to Z^H can be understood both in the Wiener or Riemann-Stieltjes sense.

The stationary Rosenblatt Ornstein-Uhlenbeck process is obtained by taking the initial condition $\xi = \sigma \int_{-\infty}^0 e^{-\lambda u} dZ^H(u)$ in (2.16). Then, the stationary ROU, which will be denoted in the sequel by $(X^H(t))_{t\geq 0}$, can be expressed as, for every $t \geq 0$,

$$X^H(t) = \sigma \int_{-\infty}^t e^{-\lambda(t-u)} dZ^H(u) \tag{2.18}$$

The process $(X^H(t))_{t\geq 0}$ is a stationary Gaussian process, H-self-similar with stationary increments. Moreover, it exhibits long-range dependence since $H > \frac{1}{2}$, see [19] or [53].

In this paragraph, our purpose is to analyze the asymptotic behavior, as $H \to 1$ and as $H \to \frac{1}{2}$, in the sense of the weak convergence, of the processes (2.17) and (2.18). The analysis of the limit behavior at the extreme critical values of the Hurst exponent is different for X^H and Y^H due to the fact that the the initial values depends on H in the case of X^H.

2.3.1 Padded sets and the power counting theorem

We need to recall some notation and results from [98] which are needed in order to check the integrability assumption from Proposition 21.

Consider a set $T = \{M_1, .., M_m\}$ of linear functions on $\mathbb{R}^m$. The power counting theorem (see Theorem 1.1 and Corollary 1.1 in [98]) gives sufficient conditions for the integral

$$I = \int_{\mathbb{R}} ... \int_{\mathbb{R}} du_1 ... du_m f_1(M_1(u_1, .., u_m)) .... f_m(M_m(u_1, .., u_m)) \tag{2.19}$$

to be finite, where $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, .., m$ are such that $|f_i|$ is bounded above on (a_i, b_i) $(0 < a_i < b_i < \infty)$ and

$$|f_i(y)| \leq c_i |y|^{\alpha_i} \text{ if } |y_i| < a_i \text{ and } |f_i(y)| \leq c_i |y|^{\beta_i} \text{ if } |y| > b_i.$$

For a subset $W \subset T$ we denote by $s_T(W) = span(W) \cap T$. A subset W of T is said to be *padded* if $s_T(W) = W$ and any functional $M \in W$ also belongs to $s_T(W \setminus \{M\})$. Denote by span (W) the linear span generated by W and by $r(W)$ the number of linearly independent elements of W.

Then Theorem 1.1 in [98] says that the integral I (2.19) is finite if

$$d_0(W) = r(W) + \sum_{s_T(W)} \alpha_i > 0 \tag{2.20}$$

for any subset W of T with $s_T(W) = W$ and

$$d_\infty(W) = r(T) - r(W) + \sum_{T \setminus s_T(W)} \beta_i < 0 \tag{2.21}$$

for any proper subset W of T with $s_T(W) = W$, including the empty set. If $\alpha_i > -1$ then it suffices to check (2.20) for any padded subset $W \subset T$. Also, it suffices to verify (2.21) only for padded subsets of T if $\beta_i \geq -1$.

The condition (2.20) implies the integrability at the origin while (2.21) gives the integrability of I at infinity.

There is a similar result if one starts with a set T of affine functionals instead of linear functionals.

2.3.2 The (non-stationary) ROU process

We first treat the case of the process (2.17) with initial condition not depending on the Hurst index. In the sequel, we fix $T > 0$ arbitrary chosen.

Convergence when $H \to 1$

Proposition 22. *Assume that the initial condition ξ does not depend on H. Then the process $(Y^H(t))_{t\in[0,T]}$ converges weakly, in the space of continuous functions $C[0,T]$, to the stochastic process $(Y(t))_{t\in[0,T]}$ given by*

$$Y(t) = e^{-\lambda t}\xi + \sigma\frac{1}{\sqrt{2}}\left(\int_0^t e^{-\lambda(t-u)}du\right)(Z^2-1) = e^{-\lambda t}\xi + \frac{\sigma}{\sqrt{2}\lambda}(1-e^{-\lambda t})(Z^2-1) \tag{2.22}$$

with $Z \sim N(0,1)$.

Proof : We start by checking the convergence of the finite dimensional distribution of Y^H of those of Y. Take $\alpha_1, ..., \alpha_d \in \mathbb{R}$ and $t_1, ..., t_d \in [0,T]$. We will prove that $\sum_{j=1}^d \alpha_j Y^H(t_j)$ converges in distribution, as $H \to 1$ to the random variable $\sum_{j=1}^d \alpha_j Y(t_j)$.

We have, by the linearity of the Wiener-Rosenblatt integral,

$$\sum_{j=1}^d \alpha_j Y^H(t_j) = \int_{\mathbb{R}}\left(\sum_{j=1}^d \alpha_j 1_{[0,t_j]}(u)e^{-\lambda(t_j-u)}\right)dZ^H(u) = \int_{\mathbb{R}} f(u)dZ^H(u)$$

with

$$f(u) = \sum_{j=1}^d \alpha_j 1_{[0,t_j]}(u)e^{-\lambda(t_j-u)} \tag{2.23}$$

In order to apply Proposition 20, we need to show that condition (2.12) holds true. Clearly f belongs to $L^1(\mathbb{R})$. Concerning the first part of (2.12), we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} dudv|f(u)f(v)||u-v|^{2\varepsilon-1} \leq \sum_{j,k=1}^{d} |\alpha_j\alpha_k| \int_0^{t_j} du \int_0^{t_k} dve^{-\lambda(t_j-u)}e^{-\lambda(t_k-v)}|u-v|^{2\varepsilon-1}$$
$$\leq \sum_{j,k=1}^{d} |\alpha_j\alpha_k| \int_0^{T} du \int_0^{T} dve^{-\lambda(t_j-u)}e^{-\lambda(t_k-v)}|u-v|^{2\varepsilon-1} \leq \sum_{j,k=1}^{d} |\alpha_j\alpha_k| \int_0^{T} du \int_0^{T} dv|u-v|^{2\varepsilon-1}$$
$$= \frac{1}{\varepsilon(\varepsilon+1)}T^{2\varepsilon+1}\sum_{j,k=1}^{d} |\alpha_j\alpha_k| < \infty.$$

Concerning the tightness, notice that for every $0 \leq s < t \leq T$ we have, since Y^H is a solution to (2.16)

$$\mathbf{E}|Y^H(t) - Y^H(s)|^2 \leq C|t-s|$$

and therefore, for every $p \geq 1$,

$$\mathbf{E}|Y^H(t) - Y^H(s)|^{2p} \leq C_p(\mathbf{E}|Y^H(t) - Y^H(s)|^2)^p \leq c|t-s|^p. \tag{2.24}$$

The tightness is obtained from (2.24) and e.g. Lemma 2.2 in [84]. ■

Note that the limit process $(Y(t))_{t\in[0,T]}$ given by (2.22) is a second chaos stochastic process. Therefore, its finite dimensional distributions are characterized by the cumulants, i.e. for every $\alpha_1, .., \alpha_d \in \mathbb{R}$ and $t_1, .., t_d \in [0,T]$,

$$k_m\left(\sum_{j=1}^{d}\alpha_jY(t_j)\right) = k_m\left(\frac{\sigma}{\sqrt{2\lambda}}1^{\otimes 2}_{[0,1]}\sum_{j=1}^{d}\alpha_j(1-e^{-\lambda t_j})\right) = 2^{\frac{m}{2}-1}(m-1)!\frac{\sigma^m}{\lambda^m}\left(\sum_{j=1}^{d}\alpha_j(1-e^{-\lambda t_j})\right)^m,$$

for $m \geq 2$ and

$$k_1\left(\sum_{j=1}^{d}\alpha_jY(t_j)\right) = \mathbf{E}\left(\sum_{j=1}^{d}\alpha_jY(t_j)\right) = \mathbf{E}\left(\xi\right).\sum_{j=1}^{d}\alpha_je^{-\lambda t_j}.$$

Convergence when $H \to \frac{1}{2}$

The (standard) Ornstein-Uhlenbeck process (denoted Y_0 in the sequel) is given by (2.17) with Z^H replaced by a Wiener process W. Thus it can be written as

$$Y_0(t) = e^{-\lambda t}\left(\xi + \sigma\int_0^t e^{\lambda u}dW(u)\right), \text{ for every } t \geq 0. \tag{2.25}$$

Consequently, $(Y_0(t))_{t\geq 0}$ is a Gaussian process with mean $\mathbf{E}Y_0(t) = e^{-\lambda t}\mathbf{E}\xi$ for any $t \geq 0$ and covariance function

$$Cov(Y_0(t), Y_0(s)) = \frac{\sigma^2}{2\lambda}\left(e^{-\lambda|t-s|} - e^{-\lambda(t+s)}\right)$$

for every $s, t \geq 0$.

The Ornstein-Uhlenbeck process will appear as limit of the ROU process as $H \to \frac{1}{2}$.

Proposition 23. *As $H \to \frac{1}{2}$, the process $Y^H(t)$ converges weakly to the Ornstein-Uhlenbeck process $(Y_0(t))_{t\in[0,T]}$.*

Proof : Consider $\lambda_1, .., \lambda_d \in \mathbb{R}$ and $t_1, .., t_d \in [0, T]$. We will apply Corollary 1. Clearly, as in the proof of Proposition 22, the function $f(u) = \sum_{j=1}^d \alpha_j 1_{[0,t_j]}(u) e^{-\lambda(t_j - u)}$ from (2.23) belongs to $\mathcal{H}_H \cap L^2(\mathbb{R})$.

We need to show that condition (2.15) is satisfied. We have

$$\begin{aligned}
&\int_{\mathbb{R}^4} du_1 ... du_4 f(u_1) ... f(u_4) |u_1 - u_2|^{H-1} |u_2 - u_3|^{H-1} |u_3 - u_4|^{H-1} |u_4 - u_1|^{H-1} \\
\leq &\sum_{j_1, .., j_4 = 1}^{d} |\alpha_{j_1} .... \alpha_{j_4}| \int_0^T ... \int_0^T du_1 .. du_4 |u_1 - u_2|^{H-1} |u_2 - u_3|^{H-1} |u_3 - u_4|^{H-1} |u_4 - u_1|^{H-1}.
\end{aligned}$$

Actually, the function

$$H \to \int_0^T ... \int_0^T du_1 .. du_4 |u_1 - u_2|^{H-1} |u_2 - u_3|^{H-1} |u_3 - u_4|^{H-1} |u_4 - u_1|^{H-1}$$

is finite and continuous on the set $(\frac{1}{4}, 1]$. This follows from Lemma 3.3 in [6] but also by applying the power counting theorem with $(\alpha_1, .., \alpha_4) = (H - 1, .., H - 1)$. Therefore

$$\sup_{H \in [\frac{1}{2}, 1]} \int_{\mathbb{R}} ... \int_{\mathbb{R}} du_1 ... du_4 f(u_1) ... f(u_4) |u_1 - u_2|^{H-1} |u_2 - u_3|^{H-1} |u_3 - u_4|^{H-1} |u_4 - u_1|^{H-1} < \infty$$

and implies (2.15) and consequently, it gives the convergence of finite dimensional distributions of Y^H as $\to \frac{1}{2}$. The tighness follows (2.24). ■

2.3.3 Asymptotic behavior of the stationary Rosenblatt Ornstein-Uhlenbeck process

Now, we analyze the asymptotic behavior of the process (2.18). The main idea is the same as in the previous section but we need to pay attention to the fact that the kernel of the stationary ROU process has the whole real line as support .

Convergence when $H \to 1$

Proposition 24. *The stationary Rosenblatt Ornstein-Uhlenbeck process converges weakly, in the space of continuous functions $C[0, T]$, to the stochastic process $(X(t))_{t \in [0,T]}$ defined by, for every $t \in [0, T]$,*

$$X(t) = \sigma \left(\int_{-\infty}^{t} e^{-\lambda(t-u)} du \right) (Z^2 - 1) = \frac{\sigma}{\lambda} (Z^2 - 1). \tag{2.26}$$

Proof : We prove the convergence of the finite dimensional distributions of X^H to those of X when $H \to \frac{1}{2}$. Consider $\alpha_1, .., \alpha_d \in \mathbb{R}$ and $t_1, .., t_d \in [0, T]$. We can assume that $d \geq 2$ (because the case $d = 1$ has been treated just above) and the t_i are distincts (otherwise, we can reindex them). We prove that

$$\sum_{j=1}^{d} \alpha_j X^H(t_j) \xrightarrow[H \to 1]{(d)} \sum_{j=1}^{d} \alpha_j X(t_j).$$

Notice that, by linearity

$$\sum_{j=1}^{d} \alpha_j X^H(t_j) = \int_{\mathbb{R}} g(u) dZ^H(u)$$

with

$$g(u) = \sigma \sum_{j=1}^{d} \alpha_j e^{-\lambda(t_j-u)} 1_{(-\infty,t_j)}(u). \tag{2.27}$$

We need to show (2.12). First, notice that g belongs to $L^1(\mathbb{R})$ because

$$\int_{\mathbb{R}} |g(u)|du \leq \sigma \sum_{j=1}^{d} |\alpha_j| \int_{-\infty}^{t_j} e^{-\lambda(t_j-u)}du = \sigma \sum_{j=1}^{d} |\alpha_j| \int_0^\infty e^{-\lambda u}du = \frac{\sigma}{\lambda} \sum_{j=1}^{d} |\alpha_j < \infty. \tag{2.28}$$

Concerning the first part of (2.12), we have, with g from (2.27),

$$\int_{\mathbb{R}}\int_{\mathbb{R}} dudv|g(u)g(v)||u-v|^{2\varepsilon-1} \leq \sigma^2 \sum_{j,k=1}^{d} |\alpha_j\alpha_k| \int_0^\infty du \int_0^\infty dve^{-\lambda(u+v)}|u-v|^{2\varepsilon-1}$$

$$= 2\sigma^2 \sum_{j,k=1}^{d} |\alpha_j\alpha_k| \int_0^\infty due^{-\lambda u} \int_0^u e^{-\lambda v}| - v|^{2\varepsilon-1}dv = 2\sigma^2 \sum_{j,k=1}^{d} |\alpha_j\alpha_k| \int_0^\infty due^{-2\lambda u} \int_0^u dve^{\lambda v}v^{2\varepsilon-1}$$

$$= \frac{1}{\lambda}\int_0^\infty dve^{-\lambda v}v^{2\varepsilon-1} = \sigma^2 \sum_{j,k=1}^{d} |\alpha_j\alpha_k|\lambda^{2\varepsilon-1}\Gamma(2\varepsilon) < \infty$$

where Γ denotes the gamma function. $\Gamma(a) = \int_0^\infty z^{a-1}e^{-z}dz$ for $a > 0$.

Again the tightness is obtained since X^H obviously satisfies (2.24). ■

Convergence when $H \to \frac{1}{2}$

We will denote by $(X_0(t))_{t\geq 0}$ the stationary Ornstein-Uhlenbeck process. Recall that the stationary Ornstein-Uhlenbeck process is obtained from (2.25) by taking the initial condition $\xi = \sigma \int_{-\infty}^0 e^{-\lambda u}dW(u)$ where $(W(u))_{u\in\mathbb{R}}$ is a Wiener process on the whole real line. Thus

$$X_0(t) = \sigma \int_{-\infty}^t e^{-\lambda(t-u)}dW(u) \tag{2.29}$$

with $\sigma, \lambda > 0$. The process $(X_0(t))_{t\geq 0}$ is a centered Gaussian process, with stationary increments, with covariance function

$$\mathbf{E}X_0(t)X_0(s) = \frac{\sigma^2}{2\lambda}e^{-\lambda|t-s|} \tag{2.30}$$

for every $s, t \geq 0$. It follows from (2.30) that X_0 is a stationary Gaussian process with stationary increments.

Proposition 25. *The process $(X^H(t))_{t\in[0,T]}$ converges weakly, in the space of continuous functions $C[0,T]$, to the stationary Ornstein-Uhlenbeck process $(X_0(t))_{t\in[0,T]}$.*

Proof : Consider $\alpha_1, .., \alpha_d \in \mathbb{R}$ and $t_1, .., t_d \in [0,T]$. In order to show that the random variable $\sum_{j=1}^d \alpha_j X^H(t_j)$ converges in distribution to $\sum_{j=1}^d \alpha_j X_0(t_j)$ as $H \to \frac{1}{2}$, we use Proposition 21. In order to apply this result, we first notice that the function g given by (2.27) is in $\mathcal{H}_H \cap L^2(\mathbb{R})$: the fact that it belongs to $L^2(\mathbb{R})$ is a consequence of (2.28) while the computation below will show that it also belongs to $\mathcal{H}_H$. We also need to verify (2.14) and (2.15) in Proposition 21. Let us start by checking (2.14). We have to prove that

$$\mathbf{E}\left(\sum_{j=1}^{d} \alpha_j X^H(t_j)\right)^2 \xrightarrow[H\to\frac{1}{2}]{} \mathbf{E}\left(\sum_{j=1}^{d} \alpha_j X_0(t_j)\right)^2. \tag{2.31}$$

First, notice that by (2.30),

$$\begin{aligned}\mathbf{E}\left(\sum_{j=1}^{d}\alpha_j X_0(t_j)\right)^2 &= \sum_{j,k=1}^{d}\alpha_j\alpha_k \mathbf{E}X_0(t_j)X_0(t_k)\\ &= \frac{\sigma^2}{2\lambda}\sum_{j,k=1}^{d}\alpha_j\alpha_k e^{-\lambda|t_j-t_k|}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbf{E}\left(\sum_{j=1}^{d}\alpha_j X^H(t_j)\right)^2 &= \sigma^2 H(2H-1)\sum_{j,k=1}^{d}\alpha_j\alpha_k\int_{-\infty}^{t_j}du\int_{-\infty}^{t_k}dve^{-\lambda(t_j-u)}e^{-\lambda(t_k-v)}|u-v|^{2H-2}\\ &= \sigma^2 H(2H-1)\sum_{j,k=1}^{d}\alpha_j\alpha_k\int_{0}^{\infty}du\int_{0}^{\infty}dve^{-\lambda u}e^{-\lambda v}|u-v-(t_j-t_k)|^{2H-2}.\end{aligned}\tag{2.32}$$

We need to compute the integral

$$I=\int_0^\infty du\int_0^\infty dve^{-\lambda u}e^{-\lambda v}|u-v-K|^{2H-2}\tag{2.33}$$

and we can assume by symmetry that $K\geq 0$. We have

$$\begin{aligned}I &= H(2H-1)\int_0^\infty dve^{-\lambda v}\int_{v+k}^\infty due^{-\lambda u}(u-v-k)^{2H-2}\\ &\quad +H(2H-1)\int_0^\infty dve^{-\lambda v}\int_0^K due^{-\lambda u}(v+K-u)^{2H-2}\\ &:= I_1+I_2.\end{aligned}$$

Let us regard the summand I_1. By the change of variables $\tilde{u}=u-(v+K)$,

$$\begin{aligned}I_1 &= H(2H-1)\int_0^\infty dve^{-\lambda v}\int_0^\infty due^{-\lambda(u+v+k)}u^{2H-2}\\ &= H(2H-1)e^{-\lambda K}\int_0^\infty dve^{-2\lambda v}\lambda^{1-2H}\int_0^\infty dze^{-z}z^{2H-2}\\ &= H(2H-1)e^{-\lambda K}\lambda^{1-2H}\frac{1}{2\lambda}\int_0^\infty dze^{-z}z^{2H-2}.\end{aligned}$$

By integrating by parts

$$H(2H-1)\int_0^\infty dze^{-z}z^{2H-2}\xrightarrow[H\to\frac{1}{2}]{}1$$

so

$$I_1\xrightarrow[H\to\frac{1}{2}]{}\frac{e^{-\lambda K}}{4\lambda}.\tag{2.34}$$

Concerning the summand I_2, with $\tilde{u} = v + K - u$,

$$\begin{aligned} I_2 &= H(2H-1)\int_0^\infty dve^{-\lambda v}\int_0^{v+K} due^{-\lambda(v+K-u)}u^{2H-2} \\ &= H(2H-1)e^{-\lambda K}\int_0^\infty due^{\lambda u}u^{2H-2}\int_{(u-K)\vee 0}^\infty dve^{-2\lambda v} \\ &= H(2H-1)e^{-\lambda K}\int_0^K due^{\lambda u}u^{2H-2}\int_0^\infty dve^{-2\lambda v} \\ &\quad +H(2H-1)e^{-\lambda K}\int_K^\infty due^{\lambda u}u^{2H-2}\int_{u-K}^\infty dve^{-2\lambda v} \\ &= H(2H-1)e^{-\lambda K}\frac{1}{2\lambda}\int_0^K due^{\lambda u}u^{2H-2} + H(2H-1)e^{\lambda K}\frac{1}{2\lambda}\int_K^\infty due^{-\lambda u}u^{2H-2}. \end{aligned}$$

By using the integration by parts, we obtain

$$I_2 = \frac{H}{2\lambda}e^{-\lambda K}\left[e^{\lambda K}K^{2H-1} - \lambda\int_0^K e^{\lambda u}u^{2H-1}du\right] + \frac{H}{2\lambda}e^{\lambda K}\left[-e^{-\lambda K}K^{2H-1} + \lambda\int_K^\infty e^{-\lambda u}u^{2H-1}du\right].$$

Since

$$\lambda\int_0^K e^{\lambda u}u^{2H-1}du \xrightarrow[H\to\frac{1}{2}]{} e^{\lambda K} - 1 \text{ and } \lambda\int_K^\infty e^{-\lambda u}u^{2H-1}du \xrightarrow[H\to\frac{1}{2}]{} e^{-\lambda K}$$

we get

$$I_2 \xrightarrow[H\to\frac{1}{2}]{} \frac{e^{-\lambda K}}{4\lambda}. \tag{2.35}$$

From (2.34) and (2.35), the integral I from (2.33) verifies

$$I \xrightarrow[H\to\frac{1}{2}]{} \frac{e^{-\lambda K}}{2\lambda}. \tag{2.36}$$

Relation (2.36), together with (2.32), will imply that (2.31), i.e. the assumption (2.14) is verified.

Let us now check the assumption (2.15). With g from (2.27),

$$\begin{aligned} &\int_{\mathbb{R}}...\int_{\mathbb{R}} du_1...du_4|g(u_1)...g(u_m)||u_1-u_2|^{H-1}....|u_4-u_1|^{-H-1} \\ \leq\ & \sum_{j_1,j_2,..,j_4=1}^d |\alpha_{j_1}...\alpha_{j_4}|\int_{-\infty}^{t_{j_1}} du_1....\int_{-\infty}^{t_{j_4}} du_m e^{-\lambda(t_{j_1}-u_1)}....e^{-\lambda(t_{j_4}-u_4)}|u_1-u_2|^{H-1}....|u_4-u_1|^{-H-1} \\ =\ & \sum_{j_1,j_2,..,j_4=1}^d |\alpha_{j_1}...\alpha_{j_4}|\int_0^\infty du_1....\int_0^\infty du_4 e^{-\lambda(u_1+..+u_4)} \\ & \times|u_1-u_2-(t_{j_1}-t_{j_2})|^{H-1}...|u_4-u_1-(t_{j_4}-t_{j_1})|^{H-1} \\ \leq\ & e^{\frac{\lambda}{2}(|t_{j_1}-t_{j_2}|+...+|t_{j_4}-t_{j_1}|)}\sum_{j_1,j_2,..,j_4=1}^d |\alpha_{j_1}...\alpha_{j_4}|\int_0^\infty du_1...\int_0^\infty du_4 \\ & e^{-\frac{\lambda}{2}(|u_1-u_2-(t_{j_1}-t_{j_2})|+...+|u_4-u_1-(t_{j_4}-t_{j_1})|)} \\ & \times\left(1\vee|u_1-u_2-(t_{j_1}-t_{j_2})|^{H-1}\right)....\left(1\vee|u_4-u_1-(t_{j_4}-t_{j_1})|^{H-1}\right) \end{aligned}$$

Now, we will consider the set T' of affine functionals on $\mathbb{R}^4$ given by

$$T' = \{u_1-u_2-(t_{j_1}-t_{j_2}), ..., u_4-u_1-(t_{j_4}-t_{j_1})\}.$$

As before, T' is the only paddet subset of T'.

We apply the power counting theorem with

$$(\alpha_1, .., \alpha_4) = (H-1, .., H-1) \text{ and } (\beta_1, .., \beta_4) = (-\gamma, ..., -\gamma)$$

with $\gamma \in (1-\frac{1}{4}, 1) = (\frac{3}{4}, 1)$. We have

$$d_0(T') = 4 - 1 + 4(H-1) > 0 \text{ if } H > \frac{1}{4}$$

and

$$d_\infty(\emptyset) = 4 - 1 - 4\gamma < 0 \text{ if } \gamma > 1 - \frac{1}{4} = \frac{3}{4}.$$

Therefore, the function

$$H \to \int_{\mathbb{R}} ... \int_{\mathbb{R}} du_1 ... du_4 |g(u_1)...g(u_m)||u_1 - u_2|^{H-1} ....|u_4 - u_1|^{H-1}$$

is finite and continuous on the set $D = \{H \in (0,1], H > \frac{1}{4}\}$ which implies that condition (2.15) is satisfied. The conclusion follows from Proposition 21. ■

2.4 Appendix : Multiple stochastic integrals and the Fourth Moment Theorem

Here, we shall only recall some elementary facts; our main reference is [74]. Consider $\mathcal{H}$ a real separable infinite-dimensional Hilbert space with its associated inner product $\langle ., . \rangle_{\mathcal{H}}$, and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, which is a centered Gaussian family of random variables such that $\mathbf{E}(B(\varphi)B(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{H}}$, for every $\varphi, \psi \in \mathcal{H}$. Denote by I_q the qth multiple stochastic integral with respect to B. This I_q is actually an isometry between the Hilbert space $\mathcal{H}^{\odot q}$ (symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{q!}} \| \cdot \|_{\mathcal{H}^{\otimes q}}$ and the Wiener chaos of order q, which is defined as the closed linear span of the random variables $H_q(B(\varphi))$ where $\varphi \in \mathcal{H}$, $\|\varphi\|_{\mathcal{H}} = 1$ and H_q is the Hermite polynomial of degree $q \geq 1$ defined by :

$$H_q(x) = (-1)^q \exp\left(\frac{x^2}{2}\right) \frac{\mathrm{d}^q}{\mathrm{d}x^q} \left(\exp\left(-\frac{x^2}{2}\right)\right), \; x \in \mathbb{R}. \tag{2.37}$$

We recall the hypercontractivity property of multiple stochastic integrals. If $Y = I_2(f)$, with $f \in \mathcal{H}^{\otimes 2}$, then (see relation (2.7.2) in [66])

$$\mathbf{E}\, |Y|^q \leq (q-1)^q E|Y^2|^{\frac{q}{2}} \tag{2.38}$$

for every $q > 2$. We will use the following famous result initially proven in [76] that characterizes the convergence in distribution of a sequence of multiple integrals toward the Gaussian law.

Theorem 2. *Fix $n \geq 2$ and let $(F_k, k \geq 1)$, $F_k = I_n(f_k)$ (with $f_k \in \mathcal{H}^{\odot n}$ for every $k \geq 1$), be a sequence of square-integrable random variables in the nth Wiener chaos such that $\mathbf{E}\left[F_k^2\right] \to 1$ as $k \to \infty$. The following are equivalent :*

1. *the sequence $(F_k)_{k \geq 0}$ converges in distribution to the normal law $\mathcal{N}(0,1)$;*
2. *$\mathbf{E}\left[F_k^4\right] = 3$ as $k \to \infty$;*
3. *for all $1 \leq l \leq n-1$, it holds that $\lim_{k\to\infty} \|f_k \otimes_l f_k\|_{\mathcal{H}^{\otimes 2(n-l)}} = 0$;*

Other equivalent condition can be stated in term of the Malliavin derivatives of F_k, see [66].

Chapitre 3

Behavior with respect to the Hurst index of the Wiener Hermite integrals and application to SPDEs

3.1 Introduction

The Hermite processes are self-similar processes with long-memory and stationary increments. These properties made them good models for many applications. The Hermite processes constitute a non-Gaussian extension of the fractional Brownian motion. Their Hurst parameter, which is contained in the interval $\left(\frac{1}{2}, 1\right)$, characterizes the main properties of this process. The reader may consult the monographs [81] or [103] for a complete exposition on Hermite processes.

Our work deals with stochastic partial differential equations (SPDEs) driven by the Hermite process. Starting with the seminal work [108], many researchers explored the possibility of solving SPDEs with general noises more general than the standard space-time white noise. In our work, such a stochastic perturbation is chosen to be the Hermite noise. Recently, various types of stochastic integral and stochastic equations driven by Hermite noises have been considered by many authors. We refer, among others, to [7], [31], [33], [34], [64], [100], [17], [41], [88], [89]. Our purpose is to analyze the asymptotic behavior in distribution of the solution to the stochastic heat equation with additive Hermite noise, when the Hurst parameter (which is also the self-similarity index of the Hermite process) converges to the extreme values of its interval of definition, i.e when it tends to one and to one half. Our work continues a recent line of research that concerns the limit behavior in distribution with respect to the Hurst parameter of Hermite and related fractional-type stochastic processes. In particular, the papers [12] and [6] deal with the asymptotic behavior of the generalized Rosenblatt process, the work [3] studies the multiparamter Hermite processes while the paper [89] investigates the Ornstein-Uhlenbeck process with Hermite noise of order $q = 2$.

The solution to the heat equation with Hermite noise in $\mathbb{R}^d$ is a $(d+1)$- parameter random field depending on a Hurst index $\mathbf{H} \in \left(\frac{1}{2}, 1\right)^{d+1}$. We prove that the solution converges in distribution to a Gaussian limit when at least one of the components of $\mathbf{H}$ converges to $\frac{1}{2}$ and to a random variable in a Wiener chaos of higher order when at least one of the components of $\mathbf{H}$ tends to 1 (and none of them converges to $\frac{1}{2}$). Moreover, the limit always coincides in distribution with the solution to the stochastic heat equation driven by the limit of the Hermite noise. The results show that these models offer a large flexibilitily, covering a large class of probability distributions, from Gaussian laws to distribution of random variables in Wiener chaos of higher order.

For the proofs we use various techniques, such as the Malliavin calculus and the Fourth Moment Theorem for the normal convergence, the properties of the Wiener integrals with respect to the Hermite process and the so-called power counting theorem. Since the solution to the Hermite-driven heat equation can be expressed as a Wiener integral with respect to a Hermite sheet, we start our analysis by some more general results, i.e by studying the behavior with respect to the Hurst index of such Wiener integrals. This allows to consider other examples, in particular the Hermite Ornstein-Uhlenbeck process.

We organized our paper as follows. Section 2 contains some preliminaries. We introduce the multidimensional Hermite processes and the Wiener integral with respect to them. We also recall some known results concerning the asymptotic behavior of the Hermite sheet. In Section 3, we state general results on the asymptotic behavior of the Wiener-Hermite integrals with respect to the Hurst parameter. We will give two applications of the main results obtained. In Section 4 we analyse the asymptotic behavior of the mild solution of the stochastic heat equation with Hermite noise and finally Section 5 contains the case of the Hermite Ornstein -Uhlenbeck process. The Appendix (Section 6) contains the basic elements of the stochastic analysis on Wiener spaces needed in the paper.

3.2 Preliminaries

In this preliminary section we will introduce the Hermite sheet and the Wiener integral with respect to this multiparameter process. We also recall the main findings from [3] concerning the behavior of the Hermite sheet with respect to its Hurst multi-index. We start with some multidimensional notation, that we will use throughout our work.

3.2.1 Notation

For $d \in \mathbb{N}\setminus\{0\}$ we will work with multi-parametric processes indexed by elements of $\mathbb{R}^d$. We shall use bold notation for multi-indexed quantities, i.e., $\mathbf{a} = (a_1, a_2, \ldots, a_d)$, $\mathbf{b} = (b_1, b_2, .., b_d)$, $\alpha = (\alpha_1, .., \alpha_d)$, $\mathbf{ab} = \prod_{i=1}^d a_i b_i$, $|\mathbf{a}-\mathbf{b}|^{\alpha} = \prod_{i=1}^d |a_1 - b_1|^{\alpha_i}$, $\mathbf{a}/\mathbf{b} = (a_1/b_1, a_2/b_2, \ldots, a_d/b_d)$, $[\mathbf{a}, \mathbf{b}] = \prod_{i=1}^d [a_i, b_i]$, $(\mathbf{a}, \mathbf{b}) = \prod_{i=1}^d (a_i, b_i)$, $\sum_{\mathbf{i}=0}^{\mathbf{N}} a_{\mathbf{i}} = \sum_{i_1=0}^{N_1} \sum_{i_2=0}^{N_2} \ldots \sum_{i_d=0}^{N_d} a_{i_1, i_2, \ldots, i_d}$ if $\mathbf{N} = (N_1, .., N_d)$, $\mathbf{a}^{\mathbf{b}} = \prod_{i=1}^d a_i^{b_i}$, and $\mathbf{a} < \mathbf{b}$ iff $a_1 < b_1, a_2 < b_2, \ldots, a_d < b_d$ (analogously for the other inequalities).

We write $\mathbf{a} - \mathbf{1}$ to indicate the product $\prod_{i=1}^d (a_i - 1)$. By β we denote the Beta function $\beta(p,q) = \int_0^1 z^{p-1}(1-z)^{q-1} dz, p, q > 0$ and we use the notation

$$\beta(\mathbf{a}, \mathbf{b}) = \prod_{i=1}^d \beta\left(a^{(i)}, b^{(i)}\right)$$

if $\mathbf{a} = (a^{(1)}, .., a^{(d)})$ and $\mathbf{b} = (b^{(1)}, .., b^{(d)})$.

Let us recall that the increment of a d-parameter process X on a rectangle $[\mathbf{s}, \mathbf{t}] \subset \mathbb{R}^d$, $\mathbf{s} = (s_1, \ldots, s_d), \mathbf{t} = (t_1, \ldots, t_d)$, with $\mathbf{s} \leq \mathbf{t}$ (denoted by $\Delta X([\mathbf{s}, \mathbf{t}])$) is given by

$$\Delta X([\mathbf{s}, \mathbf{t}]) = \sum_{r \in \{0,1\}^d} (-1)^{d - \sum_{i=1}^d r_i} X_{\mathbf{s}+\mathbf{r}\cdot(\mathbf{t}-\mathbf{s})}. \tag{3.1}$$

When $d = 1$ one obtains $\Delta X([\mathbf{s}, \mathbf{t}]) = X_t - X_s$ while for $d = 2$ one gets $\Delta X([\mathbf{s}, \mathbf{t}]) = X_{t_1, t_2} - X_{t_1, s_2} - X_{s_1, t_2} + X_{s_1, s_2}$.

3.2.2 Hermite processes and Wiener-Hermite integrals

We recall the definition and the basic properties of multiparameter Hermite processes. For a more complete presentation, we refer to [26], [81] or [103]. Let $q \geq 1$ integer and the Hurst multi-index $\mathbf{H} = (H_1, H_2, \ldots, H_d) \in (\frac{1}{2}, 1)^d$. The *Hermite sheet of order q and with self-similarity index* $\mathbf{H}$, denoted $(Z^{q,d}_{\mathbf{H}}(\mathbf{t}), \mathbf{t} \in \mathbb{R}^d_+)$ in the sequel, is given by

$$\begin{aligned} Z^{q,d}_{\mathbf{H}}(\mathbf{t}) &= c(\mathbf{H}, q) \int_{\mathbb{R}^{d\cdot q}} \int_0^{t^{(1)}} \ldots \int_0^{t^{(d)}} \left(\prod_{j=1}^{q} (s_1 - y_{1,j})_+^{-\left(\frac{1}{2} + \frac{1-H_1}{q}\right)} \ldots (s_d - y_{d,j})_+^{-\left(\frac{1}{2} + \frac{1-H_d}{q}\right)} \right) \\ &\quad ds_d \ldots ds_1 \quad dW(y_{1,1}, \ldots, y_{d,1}) \ldots dW(y_{1,q}, \ldots, y_{d,q}) \\ &= c(\mathbf{H}, q) \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^{q} (\mathbf{s} - \mathbf{y}_j)_+^{-\left(\frac{1}{2} + \frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}}\right)} d\mathbf{s} \quad dW(\mathbf{y}_1) \ldots dW(\mathbf{y}_q) \end{aligned} \tag{3.2}$$

for every $\mathbf{t} = (t^1, ..., t^d) \in \mathbb{R}^d_+$, where $x_+ = \max(x, 0)$. The above stochastic integral is a multiple stochastic integral with respect to the Wiener sheet $(W(\mathbf{y}), \mathbf{y} \in \mathbb{R}^d)$, see Section 3.6.1. The constant $c(\mathbf{H}, q)$ ensures that $\mathbf{E}\left(Z^{q}_{\mathbf{H}}(\mathbf{t})\right)^2 = \mathbf{t}^{\mathbf{2H}}$ for every $\mathbf{t} \in \mathbb{R}^d_+$. As pointed out before, when $q = 1$, (3.2) is the fractional Brownian sheet with Hurst multi-index $\mathbf{H} = (H_1, H_2, \ldots, H_d) \in (\frac{1}{2}, 1)^d$. For $q \geq 2$ the process $Z^{q,d}_{\mathbf{H}}$ is not Gaussian and for $q = 2$ we denominate it as the *Rosenblatt sheet*.

The Hermite sheet is a $\mathbf{H}$-self-similar stochastic process and it has stationary increments. Its paths are Hölder continuous of order $\boldsymbol{\delta} < \mathbf{H}$, see [81] or [103]. Its covariance is the same for every $q \geq 1$ and it coincides with the covariance of the d-parameter fractional Brownian motion, i.e.

$$\mathbf{E} Z^{q,d}_{\mathbf{H}}(\mathbf{t}) Z^{q,d}_{\mathbf{H}}(\mathbf{s}) = \prod_{j=1}^{d} \left(\frac{1}{2} \left(t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i} \right) \right) =: R_{\mathbf{H}}(\mathbf{t}, \mathbf{s}), \qquad t_i, s_i \geq 0. \tag{3.3}$$

We will denote by $|\mathcal{H}_{\mathbf{H}}|$ the space of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that

$$\|f\|^2_{|\mathcal{H}_{\mathbf{H}}|} < \infty$$

where

$$\begin{aligned} \|f\|^2_{|\mathcal{H}_{\mathbf{H}}|} &:= \mathbf{H}(\mathbf{2H} - \mathbf{1}) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} d\mathbf{u} d\mathbf{v} |f(\mathbf{u})| \cdot |f(\mathbf{v})| |\mathbf{u} - \mathbf{v}|^{\mathbf{2H}-2} \\ &= \mathbf{H}(\mathbf{2H} - \mathbf{1}) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} du^{(1)} ... du^{(d)} dv^{(1)} ... dv^{(d)} \\ &\qquad \times f(u^{(1)}, .., u^{(d)}) f(v^{(1)}, .., v^{(d)}) \prod_{j=1}^{d} |u^{(j)} - v^{(j)}|^{2H_j - 2} \end{aligned} \tag{3.4}$$

where $\mathbf{u} = (u^{(1)}, .., u^{(d)}), \mathbf{v} = (v^{(1)}, .., v^{(d)}) \in \mathbb{R}^d$.

Notice that the space $|\mathcal{H}_{\mathbf{H}}|$ satisfies the following inclusion (see Remark 3 in [26])

$$L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \subset L^{\frac{1}{\mathbf{H}}}(\mathbb{R}^d) \subset |\mathcal{H}_{\mathbf{H}}|. \tag{3.5}$$

The Wiener integral with respect to the Hermite sheet $Z^{q,d}_{\mathbf{H}}$ has been defined in [26] (following the idea of [53] in the one-parmeter case). In particular, it is well-defined for measurable integrands $f \in |\mathcal{H}_{\mathbf{H}}|$ via the formula

$$\int_{\mathbb{R}^d} f(\mathbf{s}) dZ^{q,d}_{\mathbf{H}}(\mathbf{s}) = \int_{\mathbb{R}^{d.q}} (Jf)(\mathbf{y}_1, ..., \mathbf{y}_q) dW(\mathbf{y}_1) ... dW(\mathbf{y}_q) \tag{3.6}$$

where $\left(W(\mathbf{y}), \mathbf{y} \in \mathbb{R}^d\right)$ is a d-parameter Wiener process and

$$(Jf)(\mathbf{y}_1, ..., \mathbf{y}_q) = c(\mathbf{H}, q) \int_{\mathbb{R}^d} d\mathbf{u} f(\mathbf{u})(\mathbf{u} - \mathbf{y}_1)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \dots (\mathbf{u} - \mathbf{y}_q)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \tag{3.7}$$

with $c(\mathbf{H}, q)$ from (3.2). The stochastic integral $\int_{\mathbb{R}^{d.q}} (Jf)(\mathbf{y}_1, ..., \mathbf{y}_q) dW(\mathbf{y}_1)...dW(\mathbf{y}_q)$ is a multiple Wiener-Itô integral with respect to the Wiener sheet W.

We have the isometry formula, for $f, g \in |\mathcal{H}_{\mathbf{H}}|$

$$\begin{aligned} \mathbf{E}\left(\int_{\mathbb{R}^d} f(\mathbf{s}) dZ_{\mathbf{H}}^{q,d}(\mathbf{s}) \int_{\mathbb{R}^d} g(\mathbf{s}) dZ_{\mathbf{H}}^{q,d}(\mathbf{s})\right) &= \mathbf{H}(2\mathbf{H}-\mathbf{1}) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} d\mathbf{u} d\mathbf{v} f(\mathbf{u}) g(\mathbf{v}) |\mathbf{u}-\mathbf{v}|^{2\mathbf{H}-2} \\ &:= \langle f, g \rangle_{\mathcal{H}_{\mathbf{H}}}. \end{aligned} \tag{3.8}$$

By $\|f\|^2_{\mathcal{H}_{\mathbf{H}}}$ we denote $\langle f, f \rangle_{\mathcal{H}_{\mathbf{H}}}$.

3.2.3 Behavior of the Hermite sheet with respect to the Hurst parameter

In a first step, we analyze the convergence of the integral $\int_{\mathbb{R}^d} f(\mathbf{s}) dZ_{\mathbf{H}}^{q,d}(\mathbf{s})$ when the Hurst indices H_i goes to 1 and/or $\frac{1}{2}$.

Let us introduce the following notation : if $\{j_1, .., j_k\} \subset \{1, .., d\}$ with $1 \le k \le d$ we will denote

$$A_k = \{j_1, .., j_k\}, \quad \mathbf{H}_{A_k} = (H_{j_1}, ..., H_{j_k}) \in \left(\frac{1}{2}, 1\right)^k, \quad \langle \mathbf{t} \rangle_{A_k} = t^{(j_1)} .... t^{(j_k)} \text{ if } \mathbf{t} = (t^{(1)}, .., t^{(d)}). \tag{3.9}$$

We will separate our study into following two situations :

1. At least one parameter converges to 1 and none to $\frac{1}{2}$. Then the limit will be a non-Gaussian random variable related to the Hermite distribution.
2. At least one parameter H_i converges to $\frac{1}{2}$ and the other indices are fixed in $(\frac{1}{2}, 1)$ or converges to 1, i.e. if A_k is as above, $B_p = \{l_1, .., l_p\} \subset \{1, .., d\}$ with $p + k \le d$ and $A_k \cap B_p = \emptyset$, we assume $\mathbf{H}_{A_k} \to (\frac{1}{2}, ..., \frac{1}{2}) \in \mathbb{R}^k$ and $\mathbf{H}_{B_p} \to (1, .., 1) \in \mathbb{R}^p$. In this case we will see that the limit of $\int_{\mathbb{R}^d} f(\mathbf{s}) dZ_{\mathbf{H}}^{q,d}(\mathbf{s})$ is a centered Gaussian random variable with an explicit variance.

We start by recalling the main result in [3] concerning the asymptotic behavior of the Hermite sheet.

Theorem 3. *Let* $\left(Z_{\mathbf{H}}^{q,d}(\mathbf{t})\right)_{\mathbf{t} \ge 0}$ *be given by (3.2) and let* A_k, B_p *be as in (3.9). Fix* $T > 0$.

1. *Assume* $\mathbf{H}_{A_k} \to (1, .., 1) \in \mathbb{R}^k$. *Assume that the parameters* $H_j, j \in \overline{A}_k$ *are fixed. Then the process* $Z_{\mathbf{H}}^{q,d}$ *converges weakly in* $C([0, T]^d)$ *to the d-parameter stochastic process* $(X_{\mathbf{t}})_{\mathbf{t} \ge 0}$ *defined by*
$$X_{\mathbf{t}} = \langle \mathbf{t} \rangle_{A_k} Z_{\mathbf{H}_{\overline{A}_k}}^{q, d-k}(\mathbf{t}_{\overline{A}_k}) \tag{3.10}$$
where $\left(Z_{\mathbf{H}_{\overline{A}_k}}^{q,d-k}(\mathbf{t}_{\overline{A}_k})\right)_{\mathbf{t}_{\overline{A}_k} \in \mathbb{R}_+^{d-k}}$ *is a* $(d-k)$*-parameter Hermite process of order q with Hurst index* $\mathbf{H}_{\overline{A}_k} \in \left(\frac{1}{2}, 1\right)^{d-k}$.
2. *Assume* $(H_1, .., H_d) \to (1, .., 1) \in \mathbb{R}^d$. *Then the process* $Z_{\mathbf{H}}^{q,d}$ *converges weakly in* $C([0, T]^d)$ *to the d-parameter stochastic process* $(X_{\mathbf{t}})_{\mathbf{t} \ge 0}$ *defined by*
$$X_{\mathbf{t}} = \langle \mathbf{t} \rangle_d \frac{1}{\sqrt{q!}} H_q(Z) \tag{3.11}$$
where $Z \sim N(0, 1)$ *and* H_q *is the qth Hermite polynomial (see (3.64)).*

3. *Assume* $\mathbf{H}_{A_k} \to \left(\frac{1}{2}, ..., \frac{1}{2}\right) \in \mathbb{R}^k$. *Assume that the parameters* $H_j, j \in \overline{A}_k$ *are fixed. Then the process* $Z^{q,d}_{\mathbf{H}}$ *converges weakly in* $C([0,T]^d)$ *to a* d*-parameter centered Gaussian process* $(X(\mathbf{t}))_{\mathbf{t}\geq 0}$ *with covariance*

$$\mathbf{E}X_{\mathbf{t}}X_{\mathbf{s}} = \left(\prod_{a\in A_k}\left(t^{(a)} \wedge s^{(a)}\right)\right)\left(\prod_{b\in \overline{A}_k} R_{H_b}(t^{(b)}, s^{(b)})\right) \tag{3.12}$$

with R_{H_b} *defined in* (3.3).

4. *Assume* $\mathbf{H}_{A_k} \to \left(\frac{1}{2}, .., \frac{1}{2}\right) \in \mathbb{R}^k$ *and* $\mathbf{H}_{B_p} \to (1, .., 1) \in \mathbb{R}^p$. *Assume that the* H_j *with* $j \in \{1, 2, .., d\} \setminus (A_k \cup B_p)$ *are fixed. Then the process* $Z^{q,d}_{\mathbf{H}}$ *converges weakly in* $C([0,T]^d)$ *to a* d*-parameter Gaussian process* $(X(\mathbf{t}))_{\mathbf{t}\geq 0}$ *with covariance*

$$\mathbf{E}X_{\mathbf{t}}X_{\mathbf{s}} = \left(\prod_{a\in A_k}(t^{(a)} \wedge s^{(a)})\right)\left(\prod_{b\in B_p} t^{(b)}s^{(b)}\right)\left(\prod_{c\in \overline{A_k\cup B_p}} R_{H_c}(t^{(c)}, s^{(c)})\right). \tag{3.13}$$

We will use the above result in order to get the limit behavior with respect to the Hurst parameter of the Hermite Wiener integral.

3.3 Convergence of the Wiener-Hermite integrals with respect to the Hurst parameter

Let us start the analysis of the behavior of the Wiener-Hermite integral (3.6) when the components of the self-similarity index $\mathbf{H}$ tends to their extreme values. As mentioned above, we will separate our study into two cases : at least one component of $\mathbf{H}$ converges to 1 (and no component tends to $\frac{1}{2}$) and at least one component of $\mathbf{H}$ converges to one-half.

3.3.1 Convergence around 1

We need to introduce new spaces for the deterministic integrand in (3.6). Working on these spaces will ensure the convergence of the Hermite-Wiener integral.

Let A_k be as in (3.9) and assume $1 \leq k < d$. We introduce the space $\mathcal{H}_{\overline{A}_k}$ of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that

$$\|f\|_{\mathcal{H}_{\overline{A}_k}} := \tag{3.14}$$

$$\sum_{j=1}^{k}\int_{\mathbb{R}^j} d\mathbf{u}_{A_j}\left|\int_{\mathbb{R}^{d-j}} d\mathbf{v}_{\overline{A}_j}\int_{\mathbb{R}^{d-j}} d\mathbf{w}_{\overline{A}_j}|f(\mathbf{u}_{A_j}, \mathbf{v}_{\overline{A}_j})| \cdot |f(\mathbf{u}_{A_j}, \mathbf{w}_{\overline{A}_j})||\mathbf{v}_{\overline{A}_j} - \mathbf{w}_{\overline{A}_j}|^{2\mathbf{H}_{\overline{A}_j}-2}\right|^{\frac{1}{2}}$$

$$= \sum_{j=1}^{k}\int_{\mathbb{R}^j} d\mathbf{u}_{A_j}\|f(\mathbf{u}_{A_j}, \cdot)\|_{\mathcal{H}_{\mathbf{H}_{\overline{A}_j}}} < \infty \tag{3.15}$$

with the norm $\|\cdot\|_{\mathcal{H}_{\mathbf{H}_{\overline{A}_j}}}$ defined in (3.4). Notice that for $f \in \mathcal{H}_{\overline{A}_k}$, the integral

$$\int_{\mathbb{R}^k} d\mathbf{u}_{A_k}\int_{\mathbb{R}^{d-k}} dZ^{q,d}_{\mathbf{H}}(u_{\overline{A}_k})f(\mathbf{u}) \tag{3.16}$$

is well-defined in $L^1(\Omega)$. Indeed,

$$
\begin{aligned}
& \mathbf{E}\left|\int_{\mathbb{R}^k} d\mathbf{u}_{A_k} \int_{\mathbb{R}^{d-k}} dZ^{q,d}_{\mathbf{H}}(\mathbf{u}_{\overline{A}_k}) f(\mathbf{u})\right| \\
\leq \;& \int_{\mathbb{R}^k} d\mathbf{u}_{A_k} \mathbf{E}\left|\int_{\mathbb{R}^{d-k}} dZ^{q,d}_{\mathbf{H}}(\mathbf{u}_{\overline{A}_k}) f(\mathbf{u})\right| \leq \int_{\mathbb{R}^k} d\mathbf{u}_{A_k} \left(\mathbf{E}\left|\int_{\mathbb{R}^{d-k}} dZ^{q,d}_{\mathbf{H}}(\mathbf{u}_{\overline{A}_k}) f(\mathbf{u})\right|^2\right)^{\frac{1}{2}} \\
= \;& \left(\mathbf{H}_{\overline{A}_k}(2\mathbf{H}_{\overline{A}_k} - \mathbf{1})\right)^{\frac{1}{2}} \int_{\mathbb{R}^k} d\mathbf{u}_{A_k} \\
& \left|\int_{\mathbb{R}^{d-k}} d\mathbf{v}_{\overline{A}_k} \int_{\mathbb{R}^{d-k}} d\mathbf{w}_{\overline{A}_k} |f(\mathbf{u}_{A_k}, \mathbf{v}_{\overline{A}_k})| \cdot |f(\mathbf{u}_{A_k}, \mathbf{w}_{\overline{A}_k})| |\mathbf{v}_{\overline{A}_j} - \mathbf{w}_{\overline{A}_j}|^{2\mathbf{H}_{\overline{A}_j}-2}\right|^{\frac{1}{2}} \\
\leq \;& \left(\mathbf{H}_{\overline{A}_k}(2\mathbf{H}_{\overline{A}_k} - \mathbf{1})\right)^{\frac{1}{2}} \sum_{j=1}^{k} \int_{\mathbb{R}^j} d\mathbf{u}_{A_j} \\
& \left|\int_{\mathbb{R}^{d-j}} d\mathbf{v}_{\overline{A}_j} \int_{\mathbb{R}^{d-j}} d\mathbf{w}_{\overline{A}_j} |f(\mathbf{u}_{A_j}, \mathbf{v}_{\overline{A}_j})| \cdot |f(\mathbf{u}_{A_j}, \mathbf{w}_{\overline{A}_j})| |\mathbf{v}_{\overline{A}_j} - \mathbf{w}_{\overline{A}_j}|^{2\mathbf{H}_{\overline{A}_j}-2}\right|^{\frac{1}{2}} \\
= \;& \left(\mathbf{H}_{\overline{A}_k}(2\mathbf{H}_{\overline{A}_k} - \mathbf{1})\right)^{\frac{1}{2}} \|f\|_{\mathcal{H}_{\overline{A}_k}} < \infty.
\end{aligned}
$$

If $k = d$, we define $\mathcal{H}_{\overline{A}_k} = \mathcal{H}_{\overline{A}_d}$ to be the set of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that

$$
\begin{aligned}
& \|f\|_{\mathcal{H}_{\overline{A}_k}} := \|f\|_{L^1(\mathbb{R}^d)} \\
& + \sum_{j=1}^{d-1} \int_{\mathbb{R}^j} d\mathbf{u}_{A_j} \left|\int_{\mathbb{R}^{d-j}} d\mathbf{v}_{\overline{A}_j} \int_{\mathbb{R}^{d-j}} d\mathbf{w}_{\overline{A}_j} |f(\mathbf{u}_{A_j}, \mathbf{v}_{\overline{A}_j})| \cdot |f(\mathbf{u}_{A_j}, \mathbf{w}_{\overline{A}_j})| |\mathbf{v}_{\overline{A}_j} - \mathbf{w}_{\overline{A}_j}|^{2\mathbf{H}_{\overline{A}_j}-2}\right|^{\frac{1}{2}} \\
:= \;& \|f\|_{L^1(\mathbb{R}^d)} + \|f\|_{\mathcal{H}_{\overline{A}_{d-1}}} < \infty.
\end{aligned} \tag{3.17}
$$

Remark 9. *Notice that the order of integration in (3.16) is important. That is, the integral*

$$\int_{\mathbb{R}^{d-k}} dZ^{q,d}_{\mathbf{H}}(u_{\overline{A}_k}) \int_{\mathbb{R}^k} d\mathbf{u}_{A_k} f(\mathbf{u})$$

is not necesarily well-defined for $f \in \mathcal{H}_{\overline{A}_k}$.

We have the following non-central limit theorem.

Proposition 26. *Let A_k be as in (3.9) and assume $f \in \mathcal{H}_{\overline{A}_k} \cap |\mathcal{H}_{\mathbf{H}}|$.*

— *Assume $1 \leq k < d$ and*

$$\mathbf{H}_{A_k} \to (1, .., 1) \in \mathbb{R}^k \text{ and } \mathbf{H}_{\overline{A}_k} \in \left(\frac{1}{2}, 1\right)^{d-k} \text{ is fixed.}$$

Then the family of random variables $\left(X^{\mathbf{H}}, \mathbf{H} \in \left(\frac{1}{2}, 1\right)^d\right)$

$$X^{\mathbf{H}} := \int_{\mathbb{R}^d} f(\mathbf{u}) dZ^{q,d}_{\mathbf{H}}(\mathbf{u}) \tag{3.18}$$

converges in distribution to the random variable

$$X := \int_{\mathbb{R}^d} f(u^{(1)}, .., u^{(d)}) dZ^{q,d-k}_{\overline{A}_k}(\mathbf{u}_{\overline{A}_k}) d\mathbf{u}_{A_k} = \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^{d-k}} f(\mathbf{u}_{A_k}, \mathbf{u}_{\overline{A}_{A_k}}) dZ^{q,d-k}_{\overline{A}_k}(\mathbf{u}_{\overline{A}_k})\right) d\mathbf{u}_{A_k}. \tag{3.19}$$

— *Assume $k = d$ and*

$$\mathbf{H} \to (1, .., 1) \in \mathbb{R}^d.$$

Then the limit in distribution of the family $\left(X^{\mathbf{H}}, \mathbf{H} \in \left(\frac{1}{2}, 1\right)^d\right)$ given by (3.18) is

$$\int_{\mathbb{R}^d} f(u^{(1)}, .., u^{(d)}) d\mathbf{u} \frac{1}{\sqrt{q!}} H_q(Z)$$

with $Z \sim N(0,1)$ and H_q the Hermite polinomial of degree q (3.64).

Proof : We will check the convergence of the characteristic function of $X^{\mathbf{H}}$. That is, we will show that for every $\alpha \in \mathbb{R}$,

$$\mathbf{E}e^{i\alpha X^{\mathbf{H}}} \to_{\mathbf{H}_{A_k} \to (1,..,1) \in \mathbb{R}^k} \mathbf{E}e^{i\alpha X}.$$

The idea is to approximate first X by a sequence of random variables that can be written in terms of the linear combinaisons of $Z^{q,d}_{\mathbf{H}}$ and to use the result in Theorem 3. Consider a sequence of step functions

$$f_n(\mathbf{u}) = \sum_{l=1}^{n} a_l 1_{(\mathbf{t}_l, \mathbf{t}_{l+1}]}(\mathbf{u}) = \sum_{l=1}^{n} a_l 1_{(t_l^{(1)}, t_{l+1}^{(1)}]}(u^{(1)}) .... 1_{(t_l^{(d)}, t_{l+1}^{(d)}]}(u^{(d)})$$

(where we used again the notation $\mathbf{u} = (u^{(1)}, .., u^{(d)})$ and $\mathbf{t}_l = (t_l^{(1)}, .., t_l^{(d)})$ for $l = 1, .., n$) such that

$$\|f_n - f\|_{\mathcal{H}_{\overline{A}_k}} \to_{n\to\infty} 0 \text{ and } \|f_n - f\|_{|\mathcal{H}_{\mathbf{H}}|} \to_{n\to\infty} 0. \tag{3.20}$$

The choice of such a sequence $(f_n)_{n\geq 1}$ is possible because for any positive function $f \in \mathcal{H}_{\overline{A}_k} \cap |\mathcal{H}_{\mathbf{H}}|$, there exists an increasing sequence of step functions in $f_n \in \mathcal{H}_{\overline{A}_k} \cap |\mathcal{H}_{\mathbf{H}}|$ which converges poinwise to f and satisfies $|f_n - f| \leq |f|$, and by dominated convergence theorem, it converges in $\mathcal{H}_{\overline{A}_k}$ and in $|\mathcal{H}_{\mathbf{H}}|$. Then, we use the fact that a general function can be decomposed into its positive and negative parts.

Consider the Hermite Wiener integral of f_n with respect to the Hermite sheet

$$X^{n,\mathbf{H}} = \int_{\mathbb{R}^d} f_n(\mathbf{u}) dZ^{q,d}_{\mathbf{H}}(\mathbf{u}) = \sum_{j=1}^{n} a_l (\Delta Z^{q,d}_{\mathbf{H}})((\mathbf{t}_l, \mathbf{t}_{l+1}])$$

with $\Delta Z^{q,d}_{\mathbf{H}}$ given by (3.1). Then we know from [26], Section 3 that $X^{n,\mathbf{H}}$ converges in $L^2(\Omega)$ to $X^{\mathbf{H}}$ if f_n converges to f in $|\mathcal{H}_{\mathbf{H}}|$ due to the isometry of the Hermite Wiener integral (3.8). So we have

$$X^{n,\mathbf{H}} \to_{n\to\infty} X^{\mathbf{H}} := \int_{\mathbb{R}^d} f(\mathbf{s}) dZ^{q,d}_{\mathbf{H}}(\mathbf{s}) \text{ in } L^2(\Omega).$$

Consequently, we can write

$$\lim_{\mathbf{H}_{A_k} \to (1,..,1) \in \mathbb{R}^k} \mathbf{E}e^{i\alpha X^{\mathbf{H}}} = \lim_{\mathbf{H}_{A_k} \to (1,..,1) \in \mathbb{R}^k} \lim_{n\to\infty} \mathbf{E}e^{i\alpha X^{n,\mathbf{H}}}. \tag{3.21}$$

Now, we aim at exchanging the two limits above. Recall that if $f_j, j \geq 1$ is a sequence of functions on $D \subset \mathbb{R}$ converging uniformly to f on D and if a is a limit point for D, then $\lim_{j\to\infty} \lim_{x\to a} f_j(x) = \lim_{x\to a} f(x)$ provided that $\lim_{x\to a} f(x), \lim_{x\to a} f_j(x)$ exist. Therefore it suffices to show that $\mathbf{E}e^{i\alpha X^{n,H}}$ converges uniformly with respect to $\mathbf{H}_{A_k}$ to $\mathbf{E}e^{i\alpha X^H}$.

By the mean value theorem

$$\left|\mathbf{E}e^{i\alpha X^{n,\mathbf{H}}} - \mathbf{E}e^{i\alpha X^{\mathbf{H}}}\right| \leq |\alpha| \mathbf{E}\left|X^{n,\mathbf{H}} - X^{\mathbf{H}}\right| \leq |\alpha| \left(\mathbf{E}\left|X^{n,\mathbf{H}} - X^{\mathbf{H}}\right|^2\right)^{\frac{1}{2}}.$$

Thus, in order to invert the limits in (3.21), it suffices to show that for some $\varepsilon > 0$

$$\sup_{\mathbf{H}_{A_k} \in [\frac{1}{2}+\varepsilon,1]^k} \mathbf{E}\left|X^{n,\mathbf{H}} - X^{\mathbf{H}}\right|^2 \to_{n\to\infty} 0$$

that is proved in Lemma 2 below. The relation (3.21) becomes

$$\lim_{\mathbf{H}_{A_k} \to (1,..,1) \in \mathbb{R}^k} \mathbf{E} e^{i\alpha X^{\mathbf{H}}} = \lim_{n\to\infty} \lim_{\mathbf{H}_{A_k} \to (1,..,1) \in \mathbb{R}^k} \mathbf{E} e^{i\alpha X^{n,\mathbf{H}}}. \tag{3.22}$$

Assume $k < d$. Since, from Theorem 3 $Z^{q,d}_{\mathbf{H}}$ converges weakly to the process $(U_{\mathbf{t}})_{\mathbf{t}\geq 0}$ given by

$$U_{\mathbf{t}} = \langle \mathbf{t} \rangle_{A_k} Z^{q,d-k}_{\mathbf{H}_{\overline{A}_k}}(\mathbf{t}_{\overline{A}_k})$$

it follows from (3.22) that

$$\begin{aligned}
\lim_{\mathbf{H}_{A_k} \to (1,..,1) \in \mathbb{R}^k} \mathbf{E} e^{i\alpha X^{\mathbf{H}}} &= \lim_{n\to\infty} \lim_{\mathbf{H}_{A_k} \to (1,..,1) \in \mathbb{R}^k} \mathbf{E} e^{i\alpha \sum_{l=1}^n a_l (\Delta Z^{q,d}_{\mathbf{H}})((\mathbf{t}_l, \mathbf{t}_{l+1}])} \\
&= \lim_{n\to\infty} \mathbf{E} e^{i\alpha \sum_{l=1}^n a_l (\Delta U)((\mathbf{t}_l, \mathbf{t}_{l+1}])}.
\end{aligned} \tag{3.23}$$

At this point we need to study the convergence as $n \to \infty$ of the sequence

$$X^n := \sum_{l=1}^n a_l (\Delta U)((\mathbf{t}_l, \mathbf{t}_{l+1}]) \tag{3.24}$$

as $n \to \infty$. If $A_k = \{j_1, .., j_k\}$, let us use the notation

$$(\mathbf{t}_l, \mathbf{t}_{l+1}]_{A_k} = (t_l^{(j_1)}, t_{l+1}^{(j_1)}] \times .... \times (t_l^{(j_k)}, t_{l+1}^{(j_k)}].$$

Then it is not difficult to see that

$$(\Delta U)((\mathbf{t}_l, \mathbf{t}_{l+1}]) = (\Delta \langle \mathbf{t} \rangle_{A_k})(\mathbf{t}_l, \mathbf{t}_{l+1}]_{A_k} (\Delta Z^{q,d-k}_{\overline{A}_k})(\mathbf{t}_l, \mathbf{t}_{l+1}]_{\overline{A}_k}.$$

and therefore the sequence (3.24) can be expressed as follows

$$\begin{aligned}
X^n &= \sum_{l=1}^n a_l (\Delta U)((\mathbf{t}_l, \mathbf{t}_{l+1}]] = \sum_{l=1}^n a_l (\Delta \langle \mathbf{t} \rangle_{A_k})(\mathbf{t}_l, \mathbf{t}_{l+1}]_{A_k} (\Delta Z^{q,d-k}_{\mathbf{H}_{\overline{A}_k}})(\mathbf{t}_l, \mathbf{t}_{l+1}]_{\overline{A}_k} \\
&= \int_{\mathbb{R}^d} f_n(u^{(1)}, .., u^{(d)}) d\mathbf{u}_{A_k} dZ^{q,d-k}_{\mathbf{H}_{\overline{A}_k}}(\mathbf{u}_{\overline{A}_k}).
\end{aligned}$$

Now, we show that

$$X^n \to_{n\to\infty} X \text{ in } L^1(\Omega) \tag{3.25}$$

where the random variable X is given by (3.19). We have

$$\begin{aligned}
\mathbf{E}|X^n - X| &= \mathbf{E}\left|\int_{\mathbb{R}^k} d\mathbf{u}_{A_k} \int_{\mathbb{R}^{d-k}} dZ_{\mathbf{H}}^{q,d}(\mathbf{u}_{\overline{A}_k})(f_n(\mathbf{u}_{A_k}, \mathbf{u}_{\overline{A}_k}) - f(\mathbf{u}_{A_k}, \mathbf{u}_{\overline{A}_k}))\right| \\
&\leq \int_{\mathbb{R}^k} d\mathbf{u}_{A_k} \mathbf{E}\left|\int_{\mathbb{R}^{d-k}} dZ_{\mathbf{H}}^{q,d}(\mathbf{u}_{\overline{A}_k})(f_n(\mathbf{u}_{A_k}, \mathbf{u}_{\overline{A}_k}) - f(\mathbf{u}_{A_k}, \mathbf{u}_{\overline{A}_k}))\right| \\
&\leq \int_{\mathbb{R}^k} d\mathbf{u}_{A_k} \left(\mathbf{E}\left|\int_{\mathbb{R}^{d-k}} dZ_{\mathbf{H}}^{q,d}(\mathbf{u}_{\overline{A}_k})(f_n(\mathbf{u}_{A_k}, \mathbf{u}_{\overline{A}_k}) - f(\mathbf{u}_{A_k}, \mathbf{u}_{\overline{A}_k}))\right|^2\right)^{\frac{1}{2}} \\
&= \left(\mathbf{H}_{\overline{A}_k}(2\mathbf{H}_{\overline{A}_k} - \mathbf{1})\right)^{\frac{1}{2}} \int_{\mathbb{R}^k} d\mathbf{u}_{A_k} \left|\int_{\mathbb{R}^{d-k}} \int_{\mathbb{R}^{d-k}} d\mathbf{v}_{\overline{A}_k} d\mathbf{w}_{\overline{A}_k} |\mathbf{v}_{\overline{A}_k} - \mathbf{w}_{\overline{A}_k}|^{2\mathbf{H}_{\overline{A}_k}-2}\right. \\
&\quad \times \left.\left(f_n(\mathbf{u}_{A_k}, \mathbf{v}_{\overline{A}_k}) - f(\mathbf{u}_{A_k}, \mathbf{v}_{\overline{A}_k})\right)\left(f_n(\mathbf{u}_{A_k}, \mathbf{w}_{\overline{A}_k}) - f(\mathbf{u}_{A_k}, \mathbf{w}_{\overline{A}_k})\right)\right|^{\frac{1}{2}} \\
&\leq \left(\mathbf{H}_{\overline{A}_k}(2\mathbf{H}_{\overline{A}_k} - \mathbf{1})\right)^{\frac{1}{2}} \|f_n - f\|_{\mathcal{H}_{\overline{A}_k}} \to_{n\to\infty} 0
\end{aligned}$$

where the last convergence comes from (3.20). We obtain from (3.23) and (3.25)

$$\lim_{\mathbf{H}_{A_k}\to(1,..,1)\in\mathbb{R}^k} \mathbf{E}e^{i\alpha X^{\mathbf{H}}} = \lim_{n\to\infty} \mathbf{E}e^{i\alpha X^n} = \mathbf{E}e^{i\alpha X}$$

and the proof is complete for $1 \leq k < d$.

If $k = d$, the proof is similar. We know that the process $Z_{\mathbf{H}}^{q,d}$ converges weakly in $C[0,T]$ to the process

$$\langle \mathbf{t} \rangle_d \frac{1}{\sqrt{q!}} H_q(Z).$$

Using the same lines as above, we get

$$\lim_{\mathbf{H}\to(1,..,1)\in\mathbb{R}^d} \mathbf{E}e^{i\alpha X^{\mathbf{H}}} = \lim_{n\to\infty} \mathbf{E}e^{i\alpha X^n}$$

and in this case the sequence (3.24) becomes

$$X^n = \sum_{i=1}^{n} (\Delta\langle \mathbf{t} \rangle_d)[\mathbf{t}_l, \mathbf{t}_{l+1}] \frac{1}{\sqrt{q!}} H_q(Z) = \int_{\mathbb{R}} f_n(\mathbf{u}) d\mathbf{u} \frac{1}{\sqrt{q!}} H_q(Z)$$

Clearly, by (3.20)

$$\mathbf{E}|X^n - \int_{\mathbb{R}^d} f(\mathbf{u}) d\mathbf{u} \frac{1}{\sqrt{q!}} H_q(Z)| \leq \left(\int_{\mathbb{R}} |f_n(\mathbf{u}) - f(\mathbf{u})| d\mathbf{u}\right) \frac{1}{\sqrt{q!}} H_q(Z) \to_{n\to\infty} 0.$$

using the definition of the norm in $\mathcal{H}_{\overline{A}_k}$ for $k = d$. Then

$$\lim_{n\to\infty} \mathbf{E}e^{i\alpha X^n} = \mathbf{E}e^{i\alpha(\int_{\mathbb{R}^d} f(\mathbf{u}) d\mathbf{u}) \frac{1}{\sqrt{q!}} H_q(Z)}.$$

■

The below lemma has been needed in the proof of Proposition 26.

Lemma 2. *Let A_k be as in (3.9) with $1 \leq k \leq d$. Assume $f \in \mathcal{H}_{\overline{A}_k} \cap |\mathcal{H}_{\mathbf{H}}|$ and consider a sequence $(f_n)_{n\geq 1}$ of step functions on $\mathbb{R}^d$ such that (3.20) holds true. Let*

$$X^{n,\mathbf{H}} = \sum_{l=1}^{n} a_l (\Delta Z_{\mathbf{H}}^{q,d})((\mathbf{t}_l, \mathbf{t}_{l+1}]).$$

Then for every $\varepsilon > 0$ small enough

$$\sup_{\mathbf{H}_{A_k}\in[\frac{1}{2}+\varepsilon,1]^k} \mathbf{E}\left|X^{n,\mathbf{H}} - X^{\mathbf{H}}\right|^2 \to_{n\to\infty} 0.$$

Proof : From the isometry property (3.8) and from (3.20) we have for every $\mathbf{H} \in (\frac{1}{2}, 1)^d$,

$$\mathbf{E}\left|X^{n,\mathbf{H}} - X^{\mathbf{H}}\right|^2 \to 0. \tag{3.26}$$

Let us show that the above convergence is uniform with respect to $\mathbf{H}_{A_k} \in [\frac{1}{2}+\varepsilon, 1]^k$. By (3.8),

$$\begin{aligned}\mathbf{E}\left|X^{n,\mathbf{H}} - X^{\mathbf{H}}\right|^2 &= \mathbf{H}(2\mathbf{H}-\mathbf{1})\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_n(\mathbf{u})f_n(\mathbf{v})|\mathbf{u}-\mathbf{v}|^{2\mathbf{H}-2}d\mathbf{u}d\mathbf{v}\\ &\quad -2\mathbf{H}(2\mathbf{H}-\mathbf{1})\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_n(\mathbf{u})f(\mathbf{v})|\mathbf{u}-\mathbf{v}|^{2\mathbf{H}-2}d\mathbf{u}d\mathbf{v}\\ &\quad +\mathbf{H}(2\mathbf{H}-\mathbf{1})\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u})f(\mathbf{v})|\mathbf{u}-\mathbf{v}|^{2\mathbf{H}-2}d\mathbf{u}d\mathbf{v}\\ &:= G(\mathbf{H}_{A_k})\end{aligned} \tag{3.27}$$

with the function G considered on the interval $[\frac{1}{2}+\varepsilon, 1]^k$. Assume $k < d$. Let $\mathbf{1}(A_k) = (1, .., 1) \in \mathbb{R}^k$. Then from (3.27)

$$\begin{aligned}& G(\mathbf{1}(A_k))\\ = & \ \mathbf{H}_{\overline{A}_k}(2\mathbf{H}_{\overline{A}_k}-\mathbf{1})\int_{\mathbb{R}^k} d\mathbf{u}_{A_k}\int_{\mathbb{R}^k} d\mathbf{v}_{A_k}\int_{\mathbb{R}^{d-k}} d\mathbf{u}_{\overline{A}_k}\int_{\mathbb{R}^{d-k}} d\mathbf{v}_{\overline{A}_k} f_n(\mathbf{u}_{A_k}, \mathbf{u}_{\overline{A}_k})f_n(\mathbf{v}_{A_k}, \mathbf{v}_{\overline{A}_k})\times\\ & \times|\mathbf{u}_{\overline{A}_k}-\mathbf{v}_{\overline{A}_k}|^{2\mathbf{H}_{\overline{A}_k}-2}\\ & -2\mathbf{H}_{\overline{A}_k}(2\mathbf{H}_{\overline{A}_k}-\mathbf{1})\int_{\mathbb{R}^k} d\mathbf{u}_{A_k}\int_{\mathbb{R}^k} d\mathbf{v}_{A_k}\int_{\mathbb{R}^{d-k}} d\mathbf{u}_{\overline{A}_k}\int_{\mathbb{R}^{d-k}} d\mathbf{v}_{\overline{A}_k} f_n(\mathbf{u}_{A_k}, \mathbf{u}_{\overline{A}_k})f(\mathbf{v}_{H_{A_k}}, \mathbf{v}_{\overline{A}_k})\times\\ & \times|\mathbf{u}_{\overline{A}_k}-\mathbf{v}_{\overline{A}_k}|^{2\mathbf{H}_{\overline{A}_k}-2}\\ & +\mathbf{H}_{\overline{A}_k}(2\mathbf{H}_{\overline{A}_k}-\mathbf{1})\int_{\mathbb{R}^k} d\mathbf{u}_{A_k}\int_{\mathbb{R}^k} d\mathbf{v}_{A_k}\int_{\mathbb{R}^{d-k}} d\mathbf{u}_{\overline{A}_k}\int_{\mathbb{R}^{d-k}} d\mathbf{v}_{\overline{A}_k} f(\mathbf{u}_{A_k}, \mathbf{u}_{\overline{A}_k})f(\mathbf{v}_{A_k}, \mathbf{v}_{\overline{A}_k})\times\\ & \times|\mathbf{u}_{\overline{A}_k}-\mathbf{v}_{\overline{A}_k}|^{2\mathbf{H}_{\overline{A}_k}-2}\end{aligned}$$

and this can be written

$$\begin{aligned}& G(\mathbf{1}(A_k))\\ = & \int_{\mathbb{R}^k} d\mathbf{u}_{A_k}\int_{\mathbb{R}^k} d\mathbf{v}_{A_k}\langle(f_n-f)(\mathbf{u}_{A_k},\cdot),(f_n-f)(\mathbf{v}_{A_k},\cdot)\rangle_{\mathcal{H}_{H_{\overline{A}_k}}}\\ \le & \int_{\mathbb{R}^k} d\mathbf{u}_{A_k}\int_{\mathbb{R}^k} d\mathbf{v}_{A_k}\|(f_n-f)(\mathbf{u}_{A_k},\cdot)\|_{\mathcal{H}_{H_{\overline{A}_k}}}\|(f_n-f)(\mathbf{v}_{A_k},\cdot)\|_{\mathcal{H}_{H_{\overline{A}_k}}}\\ = & \left(\int_{\mathbb{R}^k} d\mathbf{u}_{A_k}\|(f_n-f)(\mathbf{u}_{A_k},\cdot)\|_{\mathcal{H}_{H_{\overline{A}_k}}}\right)^2\\ = & \ \mathbf{H}_{\overline{A}_k}(2\mathbf{H}_{\overline{A}_k}-\mathbf{1})\left[\int_{\mathbb{R}^k} d\mathbf{u}_{A_k}\left|\int_{\mathbb{R}^{d-k}} d\mathbf{v}_{\overline{A}_k}\int_{\mathbb{R}^{d-k}} d\mathbf{w}_{\overline{A}_k}|f(\mathbf{u}_{A_k},\mathbf{v}_{\overline{A}_k})|\cdot|f(\mathbf{u}_{A_k},\mathbf{w}_{\overline{A}_k})||\mathbf{v}_{\overline{A}_k}-\mathbf{w}_{\overline{A}_k}|^{2\mathbf{H}_{\overline{A}_k}-2}\right|^{\frac{1}{2}}\right]^2\\ \le & \ \mathbf{H}_{\overline{A}_k}(2\mathbf{H}_{\overline{A}_k}-\mathbf{1})\|f_n-f\|^2{}_{\mathcal{H}_{\overline{A}_k}}\end{aligned} \tag{3.28}$$

where we used the definition (3.15).

Now, the function G is continuous on $[\frac{1}{2}+\varepsilon, 1]^k$ so there exists $\mathbf{H_0} = (H_{0,1}, .., H_{0,k}) \in [\frac{1}{2}+\varepsilon, 1]^k$ such that

$$\sup_{\mathbf{H}_{A_k}\in[\frac{1}{2}+\varepsilon,1]^k} G(\mathbf{H}_{A_k}) = G(\mathbf{H}_0).$$

If $\mathbf{H}_0 = \mathbf{1}(A_k)$, then the conclusion follows from (3.28) and the assumption (3.20). If $\mathbf{H}_0$ has the form

$$\mathbf{H}_0 = (1, .., 1, H_{0,j+1}, ..., H_{0,k})$$

with $j < k$ then a similar calculation to (3.28) shows that

$$\begin{aligned} & G(\mathbf{H}_0) \\ \leq\ & \mathbf{H}_{\overline{A}_j}(2\mathbf{H}_{\overline{A}_j} - \mathbf{1}) \left[\int_{\mathbb{R}^j} d\mathbf{u}_{A_j} \left| \int_{\mathbb{R}^{d-j}} d\mathbf{v}_{\overline{A}_j} \int_{\mathbb{R}^{d-j}} d\mathbf{w}_{\overline{A}_j} |f(\mathbf{u}_{A_j}, \mathbf{v}_{\overline{A}_j})| \cdot |f(\mathbf{u}_{A_j}, \mathbf{w}_{\overline{A}_j})| |\mathbf{v}_{\overline{A}_j} - \mathbf{w}_{\overline{A}_j}|^{2\mathbf{H}_{\overline{A}_j} - 2} \right|^{\frac{1}{2}} \right]^2 \\ \leq\ & \mathbf{H}_{\overline{A}_j}(2\mathbf{H}_{\overline{A}_j} - \mathbf{1}) \|f_n - f\|^2_{\mathcal{H}_{\overline{A}_k}} \end{aligned} \tag{3.29}$$

and again $G(\mathbf{H}_0) \to 0$ as $\mathbf{H}_{A_k} \to (1, .., 1) \in \mathbb{R}^k$ from (3.20).

Otherwise, if all $H_{0,i}, i = 1, .., k$ are in $[\frac{1}{2} + \varepsilon, 1)$, then the conclusion follows from (3.26).

If $k = d$, the conclusion follows in the same way. Let G be given by (3.27) and let $\mathbf{H_0} = (H_{0,1}, .., H_{0,k}) \in [\frac{1}{2} + \varepsilon, 1]^d$ such that

$$\sup_{\mathbf{H} \in [\frac{1}{2} + \varepsilon, 1]^g} G(\mathbf{H}) = G(\mathbf{H}_0).$$

If $\mathbf{H}_0 = (1, .., 1) \in \mathbb{R}^d$, notice that in this case $G(\mathbf{1}_d) = G(1, ..., 1) = \|f_n - f\|^2_{L^1(\mathbb{R}^d)} \to_{n\to\infty} 0$. If $\mathbf{H}_0$ has the form

$$\mathbf{H}_0 = (1, .., 1, H_{0,j+1}, ..., H_{0,d})$$

with $j < d$ then $G(\mathbf{H}_0)$ satisfies (3.29) and consequently it converges to zero from the assumption (3.20). Il all components of $\mathbf{H}_0$ are strictly contained in the interval $\left(\frac{1}{2}, 1\right)$, then we conclude by (3.26). ■

3.3.2 Convergence around $\frac{1}{2}$

In this section, we will study the convergence in distribution of the Hermite Wiener integral (3.18) when at least one Hurst index converges to one half. Actually, we will assume (recall notation (3.9) from the previous section)

$$\mathbf{H}_{A_k} \to \left(\frac{1}{2}, ..., \frac{1}{2}\right) \in \mathbb{R}^k$$

and

$$\mathbf{H}_{B_p} \to (1, .., 1) \in \mathbb{R}^p$$

with $1 \leq k \leq d, 0 \leq p \leq d$ and $p + k \leq d$. Note that $k \geq 1$ means that at least one Hurst parameter converges to $\frac{1}{2}$ while $p \geq 0$ means that some Hurst parameters (possibly zero) converges to 1.

We have the following result.

Proposition 27. *Assume A_k is as in (3.9) and $B_p = \{l_1, .., l_p\} \subset \{1, .., d\}$ with $0 \leq p \leq d, 1 \leq k \leq d, p + k \leq d$ and $A_k \cap B_p = \emptyset$ (if $p = 0$ then $B_p = \emptyset$.). Let $f \in |\mathcal{H}_{\mathbf{H}}|$. Assume that the following limit exists*

$$\lim_{\mathbf{H}_{A_k} \to (\frac{1}{2}, ..., \frac{1}{2}) \in \mathbb{R}^k} \mathbf{H}(2\mathbf{H} - \mathbf{1}) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(\mathbf{u}) f(\mathbf{v}) |\mathbf{u} - \mathbf{v}|^{2\mathbf{H} - 2} d\mathbf{u} d\mathbf{v} := \sigma^2_{f, \mathbf{H}_{\overline{A}_k}} \tag{3.30}$$

and that

$$\begin{aligned} & \sup_{\mathbf{H}_{A_k} \in [\frac{1}{2}, 1]^k} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} d\mathbf{u} d\mathbf{v} d\mathbf{u}' d\mathbf{v}' f(\mathbf{u}) f(\mathbf{u}') f(\mathbf{v}) f(\mathbf{v}') \\ & \times |\mathbf{u} - \mathbf{v}|^{\frac{2(\mathbf{H}-1)\mathbf{r}}{q}} |\mathbf{u}' - \mathbf{v}'|^{\frac{2(\mathbf{H}-1)\mathbf{r}}{q}} |\mathbf{u} - \mathbf{u}'|^{\frac{2(\mathbf{H}-1)(\mathbf{q}-\mathbf{r})}{q}} |\mathbf{v} - \mathbf{v}'|^{\frac{2(\mathbf{H}-1)(\mathbf{q}-\mathbf{r})}{q}} < \infty. \end{aligned} \tag{3.31}$$

If

$$\mathbf{H}_{A_k} \to \left(\frac{1}{2}, ..., \frac{1}{2}\right) \in \mathbb{R}^k, \mathbf{H}_{B_p} \to (1, .., 1) \in \mathbb{R}^p \textit{ and } \mathbf{H}_{\overline{A}_k \cup \overline{B}_p} \in \left(\frac{1}{2}, 1\right)^{d-k-p} \textit{ is fixed}$$

then the Hermite Wiener integral $\int_{\mathbb{R}^d} f(\mathbf{u}) dZ_{\mathbf{H}}^{q,d}(\mathbf{u})$ *converges in distribution to the Gaussian law* $N(0, \sigma^2_{f,\mathbf{H}_{\overline{A}_k}})$.

Proof : Recall that by (3.6), $\int_{\mathbb{R}^d} f(\mathbf{u}) dZ_{\mathbf{H}}^{q,d}(\mathbf{u}) = I_q(Jf)$ with the operator J defined in (3.7). We can apply the Fourth Moment Theorem to study the normal convergence of (3.18).

First notice that by assumption (3.30), we have

$$\mathbf{E}\left(\int_{\mathbb{R}^d} f(\mathbf{u}) dZ_{\mathbf{H}}^{q,d}(\mathbf{u})\right)^2 = \mathbf{H}(2\mathbf{H}-\mathbf{1}) \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u})f(\mathbf{v})|\mathbf{u}-\mathbf{v}|^{2\mathbf{H}-2} d\mathbf{u}d\mathbf{v}$$

converges to $\sigma^2_{f,\mathbf{H}_{\overline{A}_k}}$. Therefore, in order to apply the Fourth Moment Theorem (see Theorem 6 in the Appendix), it suffices to show that

$$\|Jf \otimes_r Jf\|_{L^2(\mathbb{R}^{d(2q-2r)})} \to 0$$

for every $r = 1, ..., q-1$.

Now, as in the proof of Theorem 3 in [3] (based on relation (13) in this reference)

$$\begin{aligned}
(Jf \otimes_r Jf)(\mathbf{y}_1, .., \mathbf{y}_{2q-2r}) &= \int_{(\mathbb{R}^d)^r} Jf(\mathbf{u}_1, .., \mathbf{u}_r, \mathbf{y}_1, .., \mathbf{y}_{q-r}) Jf(\mathbf{u}_1, .., \mathbf{u}_r, \mathbf{y}_{q-r-1}, .., \mathbf{y}_{2q-2r}) d\mathbf{u}_1 ... d\mathbf{u}_r \\
&= c(\mathbf{H}, q)^2 \int_{(\mathbb{R}^d)^r} d\mathbf{u}_1 ... d\mathbf{u}_r \\
&\quad \int_{\mathbb{R}^d} f(\mathbf{u}) \left(\prod_{j=1}^{q-r} (\mathbf{u}-\mathbf{y}_j)_+^{-(\frac{1}{2}+\frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}})}\right) \left(\prod_{j=1}^{r} (\mathbf{u}-\mathbf{u}_j)_+^{-(\frac{1}{2}+\frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}})}\right) d\mathbf{u} \\
&\quad \times \int_{\mathbb{R}^d} f(\mathbf{v}) \left(\prod_{j=q-r+1}^{2q-2r} (\mathbf{v}-\mathbf{y}_j)_+^{-(\frac{1}{2}+\frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}})}\right) \left(\prod_{j=1}^{r} (\mathbf{v}-\mathbf{u}_j)_+^{-(\frac{1}{2}+\frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}})}\right) d\mathbf{v} \\
&= c(\mathbf{H}, q)^2 \beta\left(\frac{1}{2} - \frac{1-\mathbf{H}}{q}, \frac{2-2\mathbf{H}}{q}\right)^r \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} d\mathbf{u}d\mathbf{v} f(\mathbf{u})f(\mathbf{v})|\mathbf{u}-\mathbf{v}|^{\frac{2(\boldsymbol{H}-\mathbf{1})\mathbf{r}}{q}} \\
&\quad \left(\prod_{j=1}^{q-r} (\mathbf{u}-\mathbf{y}_j)_+^{-(\frac{1}{2}+\frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}})}\right) \left(\prod_{j=q-r+1}^{2q-2r} (\mathbf{v}-\mathbf{y}_j)_+^{-(\frac{1}{2}+\frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}})}\right)
\end{aligned}$$

by using the Fubini theorem and again relation (13) in [3], this leads to

$$\begin{aligned}
&\|Jf \otimes_r Jf\|^2_{L^2(\mathbb{R}^{d(2q-2r)})} \\
&= c(\mathbf{H}, q)^4 \beta\left(\frac{1}{2} - \frac{1-\mathbf{H}}{q}, \frac{2-2\mathbf{H}}{q}\right)^{2r} \beta\left(\frac{1}{2} - \frac{1-\mathbf{H}}{q}, \frac{2-2\mathbf{H}}{q}\right)^{2q-2r} \\
&\quad \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} d\mathbf{u}d\mathbf{v}d\mathbf{u}'d\mathbf{v}'(\mathbf{u})f(\mathbf{u}')f(\mathbf{v})f(\mathbf{v}') \\
&\quad \times |\mathbf{u}-\mathbf{v}|^{\frac{2(\mathbf{H}-\mathbf{1})\mathbf{r}}{q}} |\mathbf{u}'-\mathbf{v}'|^{\frac{2(\mathbf{H}-\mathbf{1})\mathbf{r}}{q}} |\mathbf{u}-\mathbf{u}'|^{\frac{2(\mathbf{H}-\mathbf{1})(\mathbf{q}-\mathbf{r})}{q}} |\mathbf{v}-\mathbf{v}'|^{\frac{2(\mathbf{H}-\mathbf{1})(\mathbf{q}-\mathbf{r})}{q}} \\
&= \frac{1}{q!^2} (\mathbf{H}(2\mathbf{H}-\mathbf{1}))^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} d\mathbf{u}d\mathbf{v}d\mathbf{u}'d\mathbf{v}' f(\mathbf{u})f(\mathbf{u}')f(\mathbf{v})f(\mathbf{v}') \\
&\quad \times |\mathbf{u}-\mathbf{v}|^{\frac{2(\mathbf{H}-\mathbf{1})\mathbf{r}}{q}} |\mathbf{u}'-\mathbf{v}'|^{\frac{2(\mathbf{H}-\mathbf{1})\mathbf{r}}{q}} |\mathbf{u}-\mathbf{u}'|^{\frac{2(\mathbf{H}-\mathbf{1})(\mathbf{q}-\mathbf{r})}{q}} |\mathbf{v}-\mathbf{v}'|^{\frac{2(\mathbf{H}-\mathbf{1})(\mathbf{q}-\mathbf{r})}{q}}.
\end{aligned}$$

The last quantity converges to zero under assumption (3.31). ■

Notice that $q=2$ and $d=1$ we retrieve the results in [89]. For $f=1$, the results in this section reduces to those in Theorem 3 from [3].

3.4 Applications to the stochastic heat equation with Hermite noise

We will apply the main results in the previous section to some particular cases. First, we look to the solution to the heat equation driven by an Hermite noise. That is, we consider the following linear stochastic heat equation driven by an additive Hermite sheet with $d+1$ parameters

$$\begin{cases} \frac{\partial u}{\partial t}(t,\mathbf{x}) &= \Delta u(t,\mathbf{x}) + \dot{Z}^{q,d+1}_{H_0,\mathbf{H}}(t,\mathbf{x}), \quad t\geq 0, \mathbf{x}\in\mathbb{R}^d \\ u(0,\mathbf{x}) &= 0, \quad \mathbf{x}\in\mathbb{R}^d \end{cases} \tag{3.32}$$

We denoted by Δ the Laplacian on $\mathbb{R}^d$ and $Z^{q,d}_{H_0,\mathbf{H}} = \{Z^{q,d+1}_{H_0,\mathbf{H}}(t,\mathbf{x}); t\geq 0, \mathbf{x}\in\mathbb{R}^d\}$ denotes the $(d+1)$-parameter Hermite sheet whose covariance is given by

$$\mathbf{E}\left(Z^{q,d+1}_{H_0,\mathbf{H}}(s,\mathbf{x})Z^{q,d+1}_{H_0,\mathbf{H}}(t,\mathbf{y})\right) = R_{H_0}(t,s)R_{\mathbf{H}}(\mathbf{x},\mathbf{y})$$

if $(H_0,\mathbf{H}) = (H_0,H_1,\dots,H_d)\in\left(\frac{1}{2},1\right)^{d+1}$. We denoted by $\mathbf{H}=(H_1,\dots,H_d)$ and

$$R_H(t,s) = \frac{1}{2}(|t|^{2H}+|s|^{2H}-|t-s|^{2H}), \quad R_{\mathbf{H}}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^d R_{H_j}(x_j,y_j)$$

if $s,t\in\mathbb{R}$ and $\mathbf{x}=(x_1,..,x_d), \mathbf{y}=(y_1,..,y_d)\in\mathbb{R}^d$.

The solution to (3.32) is understood in the mild sense. That is, the *mild* solution to (3.32) is a square-integrable process $u=\{u(t,\mathbf{x}); t\geq 0, \mathbf{x}\in\mathbb{R}^d\}$ defined by :

$$u_{H_0,\mathbf{H}}(t,\mathbf{x}) = \int_0^t\int_{\mathbb{R}^d} G(t-s,\mathbf{x}-\mathbf{y})Z^{q,d+1}_{H_0,\mathbf{H}}(ds,d\mathbf{y}), \quad t\geq 0, \mathbf{x}\in\mathbb{R}^d \tag{3.33}$$

living in the space of jointly measurables random fields $\left(X(t,\mathbf{x}), t\geq 0, \mathbf{x}\in\mathbb{R}^d\right)$ such that for every $T>0$, $\sup_{t\in[0,T],\mathbf{x}\in\mathbb{R}^d}\mathbf{E}\,|X(t,\mathbf{x})|^2<\infty$.

The above integral is a Wiener integral with respect to the Hermite sheet, as introduced in Section 2 and $G(t,\mathbf{x})$ is the Green function (or the fundamental solution) that satisfies $\frac{\partial u}{\partial t}-\Delta u=0$, i.e.

$$G(t,\mathbf{x}) = \begin{cases} (2\pi t)^{-d/2}\exp\left(-\frac{|\mathbf{x}|^2}{2t}\right) & \text{if } t>0, \mathbf{x}\in\mathbb{R}^d, \\ 0 & \text{if } t\leq 0, x\in\mathbb{R}^d. \end{cases} \tag{3.34}$$

The stochastic heat equation (3.32) admits a unique mild solution $(u_{H_0,\mathbf{H}}(t,\mathbf{x}))_{t\geq 0,\mathbf{x}\in\mathbb{R}^d}$ if and only if (see [88])

$$d < 4H_0 + \sum_{i=1}^d (2H_i-1) := \gamma. \tag{3.35}$$

In this case, for every $T>0$, $\sup_{t\in[0,T],\mathbf{x}\in\mathbb{R}^d}\mathbf{E}\left(u(t,\mathbf{x})^2\right)<\infty$.

We will use the following Parseval-type formula (see Lemma A1 in [11]) : for every $f,g\in L^2(a,b)$ and for every $0<\alpha<1$

$$\int_a^b \int_a^b dudv f(u)g(v)|u-v|^{-(1-\alpha)} = q_\alpha \int_{\mathbb{R}} |\tau|^{-\alpha} \mathcal{F}_{a,b}f(\tau)\overline{\mathcal{F}_{a,b}g(\tau)} \tag{3.36}$$

where $(\mathcal{F}_{a,b}f)(\xi) = \int_a^b f(y)e^{-i\xi y}dy$ (we use the notation $\mathcal{F}f = \mathcal{F}_{-\infty,\infty}f$) and

$$q_\alpha = (2^{1-\alpha}\pi^{1/2})^{-1}\frac{\Gamma(\alpha/2)}{\Gamma((1-\alpha)/2)}. \tag{3.37}$$

We recall that the Fourier transform of the function $\mathbf{y} \in \mathbb{R}^d \to G(u, \mathbf{y})$ is $\mathcal{F}G(u, \cdot)(\xi) = e^{-\frac{1}{2}u|\xi|^2}$.

3.4.1 Limit behavior of the solution when the Hurst index tends to 1

The expression "Hurst index tends to 1"means that at least one component of the Hurst multi-index tends to 1. We will apply Proposition 26 to obtain the asymptotic behavior of the solution (3.33) when at least one of the Hurst parameters $H_0, H_1, .., H_d$ converges to 1 and the other parameters are fixed.

Theorem 4. *Assume (3.35) and let A_k be as in (3.9). Fix $T > 0$ and $\mathbf{x} \in \mathbb{R}^d$. Then*

1. *If*

$$(H_0, \mathbf{H}_{A_k}) \to (1, .., 1) \in \mathbb{R}^{k+1} \text{ and } H_j, j \in \overline{A}_k \text{ are fixed}$$

then the stochastic process $(u_{H_0,\mathbf{H}}(t, \mathbf{x}), t \in [0, T])$ converges weakly in $C[0, T]$ to the process $(u(t, \mathbf{x}), t \in [0, T])$ defined by

$$u(t, \mathbf{x}) = \int_0^t du \int_{\mathbb{R}^k} d\mathbf{y}_{A_k} \int_{\mathbb{R}^{d-k}} dZ^{q,d-k}_{\mathbf{H}_{\overline{A}_k}}(\mathbf{y}_{\overline{A}_k})G(t-u, \mathbf{x}-\mathbf{y}). \tag{3.38}$$

2. *If $\mathbf{H}_{A_k} \to (1, .., 1) \in \mathbb{R}^k$ and $H_0, H_j, j \in \overline{A}_k$ are fixed, then $(u_{H_0,\mathbf{H}}(t, \mathbf{x}), t \in [0, T])$ converges weakly in $C[0, T]$ to the stochastic process $(u(t, \mathbf{x}), t \in [0, T])$*

$$u(t, \mathbf{x}) = \int_{\mathbb{R}^k} d\mathbf{y}_{A_k} \int_0^t \int_{\mathbb{R}^{d-k}} dZ^{q,d+1-k}_{H_0,\mathbf{H}_{\overline{A}_k}}(u, \mathbf{y}_{\overline{A}_k})G(t-u, \mathbf{x}-\mathbf{y}).$$

3. *If $(H_0, \mathbf{H}) \to (1, ..., 1) \in \mathbb{R}^{d+1}$, then the weak limit of $(u_{H_0,\mathbf{H}}(t, \mathbf{x}), t \in [0, T])$ in $C[0, T]$ is $(u(t, \mathbf{x}), t \in [0, T])$ with*

$$u(t, \mathbf{x}) = \left(\int_0^t \int_{\mathbb{R}^d} G(t-u, \mathbf{x}-\mathbf{y})d\mathbf{y}du\right)\frac{1}{\sqrt{q!}}H_q(Z).$$

Remark 10. *As usual, by the weak convergence of the family $(u_{H_0,\mathbf{H}}(t, \mathbf{x}), t \in [0, T])$ to $(u(t, \mathbf{x}), t \in [0, T])$ in $C[0, T]$ for fixed $\mathbf{x} \in \mathbb{R}^d$ we mean the weak convergence of the family of distributions of $u_{H_0,\mathbf{H}}(\cdot, \mathbf{x})$ to the law of $u(\cdot, \mathbf{x})$ in $(C[0, T], \mathcal{B}(C[0, T]))$.*

Proof : Consider the function F defined on $\mathbb{R}_+ \times \mathbb{R}$ given by

$$F : (u, \mathbf{y}) \to 1_{(0,t)}(u)(2\pi(t-u))^{-\frac{d}{2}}e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{2(t-u)}}. \tag{3.39}$$

We first show the convergence of finite dimensional distributions Consider the case 1. Let us show that this function belongs to $|\mathcal{H}_{H_0,\mathbf{H}}| \cap \mathcal{H}_{\overline{A}_k}$, with these two spaces defined by (3.4) and (3.15) respectively. We know from [11] that, under (3.35), the function F (3.39) belongs to the space $|\mathcal{H}_{H_0,\mathbf{H}}|$.

Let us check that this function belongs to the space $\mathcal{H}_{\overline{A}_k}$. Writting

$$F(u,\mathbf{y}) = F(u,\mathbf{y}_{A_k},\mathbf{y}_{\overline{A}_k}) = (2\pi u)^{-\frac{d}{2}} e^{-\frac{|\mathbf{x}-\mathbf{y}_{A_k}|^2}{2u}} e^{-\frac{|\mathbf{x}-\mathbf{y}_{\overline{A}_k}|^2}{2u}}$$

we have by the definition of the norm in $\mathcal{H}_{\overline{A}}$ (see (3.15)),

$$\begin{aligned}
\|F\|_{\mathcal{H}_{\overline{A}_k}} &= \sum_{j=1}^{k} \int_0^t du \int_{\mathbb{R}^j} d\mathbf{y}_{A_j} \Bigg| \int_{\mathbb{R}^{d-j}} \int_{\mathbb{R}^{d-j}} d\mathbf{y}_{\overline{A}_j} d\mathbf{z}_{\overline{A}_j} \\
&\quad \times (2\pi u)^{-\frac{d}{2}} e^{-\frac{|\mathbf{y}_{A_j}|^2}{2u}} e^{-\frac{|\mathbf{y}_{\overline{A}_j}|^2}{2u}} (2\pi u)^{-\frac{d}{2}} e^{-\frac{|\mathbf{y}_{A_j}|^2}{2u}} e^{-\frac{|\mathbf{z}_{\overline{A}_j}|^2}{2u}} |\mathbf{y}_{\overline{A}_j} - \mathbf{z}_{\overline{A}_j}|^{2H_{\overline{A}_j}-2} \Bigg|^{\frac{1}{2}} \\
&= \sum_{j=1}^{k} \int_0^t du \int_{\mathbb{R}^j} d\mathbf{y}_{A_j} (2\pi u)^{-\frac{j}{2}} e^{-\frac{|\mathbf{y}_{A_j}|^2}{2u}} \\
&\quad \times \Bigg| \int_{\mathbb{R}^{d-j}} \int_{\mathbb{R}^{d-j}} d\mathbf{y}_{\overline{A}_j} d\mathbf{z}_{\overline{A}_j} (2\pi u)^{-\frac{d-j}{2}} e^{-\frac{|\mathbf{y}_{\overline{A}_j}|^2}{2u}} (2\pi u)^{-\frac{d-j}{2}} e^{-\frac{|\mathbf{z}_{\overline{A}_j}|^2}{2u}} |\mathbf{y}_{\overline{A}_j} - \mathbf{z}_{\overline{A}_j}|^{2H_{\overline{A}_j}-2} \Bigg|^{\frac{1}{2}}.
\end{aligned}$$

By using Parseval's identity (3.36)

$$\int_{\mathbb{R}^{d-j}} \int_{\mathbb{R}^{d-j}} d\mathbf{y}_{\overline{A}_j} d\mathbf{z}_{\overline{A}_j} (2\pi u)^{-\frac{d-j}{2}} e^{-\frac{|\mathbf{y}_{\overline{A}_j}|^2}{2u}} (2\pi u)^{-\frac{d-j}{2}} e^{-\frac{|\mathbf{z}_{\overline{A}_j}|^2}{2u}} |\mathbf{y}_{\overline{A}_j} - \mathbf{z}_{\overline{A}_j}|^{2H_{\overline{A}_j}-2} = C_j \int_{\mathbb{R}^{d-j}} d\xi e^{-u|\xi|^2} |\xi|^{1-2\mathbf{H}_{\overline{A}_j}}$$

so with $C_j, C > 0$

$$\begin{aligned}
\|F\|_{\mathcal{H}_{\overline{A}_k}} &= \sum_{j=1}^{k} C_j \int_0^t du \left| \int_{\mathbb{R}^{d-j}} d\xi e^{-u|\xi|^2} |\xi|^{1-2\mathbf{H}_{\overline{A}_j}} \right|^{\frac{1}{2}} \\
&= C \int_0^t u^{-\frac{d-j}{4} + \frac{1}{4}\sum_{a\in\overline{A}_j}(2H_a-1)} du
\end{aligned}$$

and the last integral is finite if for every $j = 1, .., k$

$$1 - \frac{d-j}{4} + \frac{1}{4} \sum_{a\in\overline{A}_j} (2H_a - 1) > 0 \text{ or } d < 4 + j + \sum_{a\in\overline{A}_j} (2H_a - 1). \tag{3.40}$$

The last bound is true due to (3.35), so the function F given by (3.39) belongs to $|\mathcal{H}_{H_0,\mathbf{H}}| \cap \mathcal{H}_{\overline{A}_k}$.

Take $\lambda_j \in \mathbb{R}, t_j \geq 0$ for $j = 1, .., N$ and denote by

$$Y_N(\mathbf{x}) = \sum_{j=1}^{N} \lambda_j u_{H_0,\mathbf{H}}(t_j, \mathbf{x}) = \int_0^\infty \int_{\mathbb{R}^d} \left(\sum_{j=1}^{N} \lambda_j 1_{(0,t_j)}(u) G(t_j - u, \mathbf{x} - \mathbf{y}) \right) dZ^{q,d+1}_{H_0,\mathbf{H}}(u, \mathbf{y}). \tag{3.41}$$

From the above computations, the integrand $\sum_{j=1}^{N} \lambda_j 1_{(0,t_j)}(u) G(t_j - u, \mathbf{x} - \mathbf{y})$ in (3.41) belongs to $|\mathcal{H}_{H_0,\mathbf{H}}| \cap \mathcal{H}_{\overline{A}}$. Therefore, by Proposition 26, the sequence $Y_N(x)$ (3.41) converges, as $(H_0, \mathbf{H}_{A_k}) \to (1, .., 1) \in \mathbb{R}^{k+1}$ to

$$\sum_{j=1}^{N} \lambda_j \int_0^{t_j} du \int_{\mathbb{R}^k} d\mathbf{y}_{A_k} \int_{\mathbb{R}^{d-k}} dZ^{q,d-k}_{\mathbf{H}_{\overline{A}_k}}(\mathbf{y}_{\overline{A}_k}) G(t_j - u, \mathbf{x} - \mathbf{y}) = \sum_{j=1}^{N} \lambda_j u(t_j, \mathbf{x})$$

with u defined in (3.38). This gives the convergence of the finite dimensional distribution of $(u_{H_0,\mathbf{H}}(t, \mathbf{x}), t \in [0, T])$ to the finite dimensional distributions of $(u(t, \mathbf{x}), t \in [0, T])$.

For the case 2., we have similarly

$$\begin{aligned}\|F\|_{\mathcal{H}_{\overline{A}_k}} &= \sum_{j=1}^{k} C_j \left| \int_0^t \int_0^t dudv|u-v|^{2H_0-2} \int_{\mathbb{R}^{d-j}} d\xi e^{-\frac{1}{2}(u+v)|\xi|^2} |\xi|^{1-2\mathbf{H}_{\overline{A}_j}} \right|^{\frac{1}{2}} \\ &= C \left| \int_0^t \int_0^t dudv|u-v|^{2H_0-2}(u+v)^{-\frac{d-j}{2}+\frac{1}{2}\sum_{a\in\overline{A}_j}(2H_a-1)} \right|^{\frac{1}{2}}\end{aligned}$$

and the above integral is finite under (3.35). For the case 3., we notice in addition that the function F given by (3.39) belongs to $L^1(\mathbb{R}^{d+1})$.

Concerning the tightness, we recall that (see [103]), for every $s,t \in [0,T], \mathbf{x} \in \mathbb{R}^d$,

$$\mathbf{E}\left|u_{H_0,\mathbf{H}}(t,\mathbf{x}) - u_{H_0,\mathbf{H}}(s,\mathbf{x})\right|^2 \leq C|t-s|^{\gamma}$$

with $\gamma > 0$ from (3.35) and C is a constant not depending on $s,t,\mathbf{x}$. Since $u_{H_0,\mathbf{H}}(t,\mathbf{x})$ is an element of the $(q+1)$th Wiener chaos, we use the hypercontractivity property for multiple stochastic integrals to get for every $p \geq 2$

$$\mathbf{E}\left|u_{H_0,\mathbf{H}}(t,\mathbf{x}) - u_{H_0,\mathbf{H}}(s,\mathbf{x})\right|^{2p} \leq C|t-s|^{\gamma p} \tag{3.42}$$

and the tightness follows from (3.42) and the Billingsley criterion (see [15, Theorem 12.3] or [16]). ■

Remark 11. *Notice that when $(H_0, \mathbf{H}_{A_k}) \to (1,..,1) \in \mathbb{R}^{k+1}$, the condition (3.35) "converges" to (3.40).*

3.4.2 Limit behavior when the Hurst index tends to $\frac{1}{2}$

Fix $T > 0$. When at least one of the components of Hurst multi-index goes to one-half, we have a central limit theorem.

Theorem 5. *1. Assume*

$$(H_0, \mathbf{H}_{A_k}) \to \left(\frac{1}{2}, ..., \frac{1}{2}\right) \in \mathbb{R}^{k+1} \tag{3.43}$$

and

$$d < 1 + \frac{k}{2} + \sum_{a\in\overline{A}_k} H_a. \tag{3.44}$$

Then the process $(u_{H_0,\mathbf{H}}(t,\mathbf{x}), t\in[0,T])$ given by (3.33) converges weakly in $C[0,T]$ to the process $(u(t,\mathbf{x}), t \in [0,T])$ where u is the mild solution to the heat equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,\mathbf{x}) &= \Delta u(t,\mathbf{x}) + \dot{W}^{q,d+1}_{H_0,\mathbf{H}}(t,\mathbf{x}), \quad t>0, \mathbf{x}\in\mathbb{R}^d \\ u(0,\mathbf{x}) &= 0, \quad \mathbf{x}\in\mathbb{R}^d \end{cases} \tag{3.45}$$

where $\left(W_{H_0,\mathbf{H}}(t, A_1\times A_2), t\in[0,T], A_1\in\mathcal{B}_b(\mathbb{R}^k), A_2\in\mathcal{B}_b(\mathbb{R}^{d-k})\right)$ is a Gaussian field with covariance

$$\begin{aligned}&\mathbf{E}\left[W_{H_0,\mathbf{H}}(t, A_1\times A_2) W_{H_0,\mathbf{H}}(s, B_1\times B_2)\right] \\ &= (t\wedge s)\lambda_k(A_1\cap B_1)\int_{A_2\cap B_2} \mathbf{H}_{\overline{A}_k}(2\mathbf{H}_{\overline{A}_k}-\mathbf{1})|\mathbf{y}_{\overline{A}_k} - \mathbf{z}_{\overline{A}_k}|^{2\mathbf{H}_{\overline{A}_k}-2} d\mathbf{y}_{\overline{A}_k} d\mathbf{z}_{\overline{A}_k}.\end{aligned}$$

We denoted by λ_k the Lebesque measure on $\mathbb{R}^k$.

2. *If* $\mathbf{H}_{A_k} \to \left(\frac{1}{2}, ..., \frac{1}{2}\right) \in \mathbb{R}^k$, $\mathbf{H}_{B_p} \to (1, .., 1) \in \mathbb{R}^p$ *and*

$$d < 2H + \frac{k}{2} + \sum_{a \in \overline{A}_k} H_a. \tag{3.46}$$

then the process $(u_{H_0,\mathbf{H}}(t, \mathbf{x}), t \in [0, T])$ *given by (3.33) converges weakly in* $C[0, T]$ *to the process* $(u(t, \mathbf{x}), t \in [0, T])$ *where* u *is the mild solution to the heat equation (3.45) where the Gaussian noise has the following covariance*

$$\begin{aligned}
&\mathbf{E}\left[W_{H_0,\mathbf{H}}(t, A_1 \times A_2) W_{H_0,\mathbf{H}}(s, B_1 \times B_2)\right] \\
= \quad & R_{H_0}(t, s) \lambda_k(A_1 \cap B_1) \int_{A_2 \cap B_2} \mathbf{H}_{\overline{A}_k}(2\mathbf{H}_{\overline{A}_k} - \mathbf{1}) |\mathbf{y}_{\overline{A}_k} - \mathbf{z}_{\overline{A}_k}|^{2\mathbf{H}_{\overline{A}_k} - 2} d\mathbf{y}_{\overline{A}_k} d\mathbf{z}_{\overline{A}_k}.
\end{aligned}$$

3. *If* $(H_0, \mathbf{H}) \to \left(\frac{1}{2}, ..., \frac{1}{2}\right) \in \mathbb{R}^{d+1}$ *and* $d = 1$*, then the weak limit of* $(u_{H_0,\mathbf{H}}, t \in [0, T])$ *in* $C[0, T]$ *is the solution to the heat equation (3.45) driven by a space-time white noise.*

Remark 12. *The conditions (3.44), (3.46) and* $d = 1$ *are the "limits" of (3.35) in the cases 1., 2. and 3. respectively.*

Proof : We will prove that the finite dimensional distributions of $(u_{H_0,\mathbf{H}}(t, \mathbf{x}), t \in [0, T])$ converge to those of $(u(t, \mathbf{x}), t \in [0, T])$ which satisfies (3.45). In order to apply Proposition 27, we need to check conditions (3.30) and (3.31).

Checking condition (3.30). Consider the case 1., i.e. assume (3.43) and (3.44).

Take $\lambda_j \in \mathbb{R}, t_j \geq 0$ for $j = 1, .., N$ and denote by

$$Y_N(\mathbf{x}) = \sum_{j=1}^{N} \lambda_j u_{H_0,\mathbf{H}}(t_j, \mathbf{x}) = \int_0^\infty \int_{\mathbb{R}^d} \left(\sum_{j=1}^{N} \lambda_j 1_{(0,t_j)}(u) G(t_j - u, \mathbf{x} - \mathbf{y}) \right) dZ^{q,d+1}_{H_0,\mathbf{H}}(u, \mathbf{y}).$$

We first check condition (3.30) for $Y_N(\mathbf{x})$. Let us calculate $\mathbf{E}\left(Y_N(\mathbf{x})^2\right)$. By using the isometry (3.8),

$$\begin{aligned}
\mathbf{E}\left(Y_N(\mathbf{x})\right)^2 \quad = \quad & \sum_{j,k=1}^{N} \lambda_j \lambda_k H_0(2H_0 - 1) \mathbf{H}(2\mathbf{H} - \mathbf{1}) \\
& \times \int_0^{t_j} du \int_0^{t_k} dv |u - v|^{2H_0 - 2} \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz G(t_j - u, \mathbf{x} - \mathbf{y}) G(t_k - v, \mathbf{x} - \mathbf{z}) |\mathbf{y} - \mathbf{z}|^{2\mathbf{H} - 2}.
\end{aligned}$$

Notice that, if $\mathbf{x} = (x^{(1)}, .., x^{(d)}), \mathbf{y} = (y^{(1)}, .., y^{(d)}), \mathbf{z} = (z^{(1)}, .., z^{(d)})$ we have

$$G(t - u, \mathbf{x} - \mathbf{y}) = 1_{(0,t)}(u) \prod_{a=1}^{d} (2\pi(t - u))^{-\frac{d}{2}} e^{-\frac{|x^{(a)} - y^{(a)}|^2}{2(t-u)}}$$

and so

$$\begin{aligned}
& \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz G(t_j - u, \mathbf{x} - \mathbf{y}) G(t_k - v, \mathbf{x} - \mathbf{z}) |\mathbf{y} - \mathbf{z}|^{2\mathbf{H} - 2} \\
= \quad & \prod_{a=1}^{d} \int_{\mathbb{R}} \int_{\mathbb{R}} dy^{(a)} dz^{(a)} (2\pi(t_j - u))^{-\frac{1}{2}} (2\pi(t_k - v))^{-\frac{1}{2}} e^{-\frac{|x^{(a)} - y^{(a)}|^2}{2(t_j - u)}} e^{-\frac{|x^{(a)} - z^{(a)}|^2}{2(t_k - v)}} |y^{(a)} - z^{(a)}|^{2H_a - 2}.
\end{aligned}$$

We will apply the Parseval identity (3.36) with

$$\alpha = 2H_a - 1 \text{ for every } a = 1, .., d.$$

We get, for every $a = 1, .., d$,

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}} dy^{(a)}dz^{(a)}(2\pi(t_j-u))^{-\frac{1}{2}}(2\pi(t_k-v))^{-\frac{1}{2}}e^{-\frac{|x^{(a)}-y^{(a)}|^2}{2(t_j-u)}}e^{-\frac{|x^{(a)}-z^{(a)}|^2}{2(t_k-v)}}|y^{(a)}-z^{(a)}|^{2H_a-2} \\ = \;& q_{2H_a-1}\int_{\mathbb{R}} d\tau|\tau|^{1-2H_a}e^{-\frac{1}{2}(t_j-u)|\tau|^2}e^{-\frac{1}{2}(t_k-v)|\tau|^2}. \end{aligned}$$

Now, by the change of variables $\tilde{\tau} = (t_j + t_k - 2u)^{\frac{1}{2}}\tau$,

$$\begin{aligned} &\int_{\mathbb{R}} d\tau|\tau|^{1-2H_a}e^{-\frac{1}{2}(t_j-u)|\tau|^2}e^{-\frac{1}{2}(t_k-v)|\tau|^2} \\ = \;& (t_j+t_k-u-v)^{-\frac{1}{2}+\frac{2H_a-1}{2}}\int_{\mathbb{R}} d\tau|\tau|^{1-2H_a}e^{-\frac{1}{2}|\tau|^2} = (t_j+t_k-u-v)^{H_a-1}\int_{\mathbb{R}} d\tau|\tau|^{1-2H_a}e^{-\frac{1}{2}|\tau|^2}. \end{aligned}$$

Thus

$$\begin{aligned} &\mathbf{E}\left(Y_N(\mathbf{x})\right)^2 \\ = \;& \sum_{j,k=1}^{N}\lambda_j\lambda_k H_0(2H_0-1)\mathbf{H}(2\mathbf{H}-\mathbf{1})q_{2\mathbf{H}-\mathbf{1}} \\ &\times\int_0^{t_j}du\int_0^{t_k}dv|u-v|^{2H_0-2}(t_j+t_k-u-v)^{H_1+\ldots+H_d-d}\prod_{a=1}^{d}\int_{\mathbb{R}} d\tau|\tau|^{1-2H_a}e^{-\frac{1}{2}|\tau|^2} \end{aligned} \tag{3.47}$$

where q_{2H_a-1} is defined in (3.37) and

$$q_{2\mathbf{H}-\mathbf{1}} = \prod_{a=1}^{d} q_{2H_a-1}.$$

Notice that for every $H \in (\frac{1}{2}, 1)$, we have

$$H(2H-1)\Gamma(H-\frac{1}{2}) = H(2H-1)\frac{\Gamma(H+\frac{1}{2})}{H-\frac{1}{2}} \to_{H\to\frac{1}{2}} 2\Gamma(1) = 2$$

and then

$$H(2H-1)q_{2H-1} \to_{H\to\frac{1}{2}} (2\pi)^{-1}. \tag{3.48}$$

Relation (3.48) implies

$$\mathbf{H}(2\mathbf{H}-\mathbf{1})q_{2\mathbf{H}-\mathbf{1}} \to_{(H_0,\mathbf{H}_{A_k})\to(\frac{1}{2},\ldots,\frac{1}{2})\in\mathbb{R}^{k+1}} (2\pi)^{-k}q_{2\mathbf{H}_{\overline{A_k}}-\mathbf{1}}. \tag{3.49}$$

Let

$$\gamma := H_1 + \ldots + H_d - d. \tag{3.50}$$

We have, by integrating by parts

$$\begin{aligned}
& H_0(2H_0-1)\int_0^t\int_0^s dudv|u-v|^{2H_0-2}(t+s-u-v)^{-\gamma} \\
= \ & H_0(2H_0-1)\int_0^s\int_0^s dudv|u-v|^{2H_0-2}(t+s-u-v)^{-\gamma} \\
& +H_0(2H_0-1)\int_s^t\int_0^s dudv|u-v|^{2H_0-2}(t+s-u-v)^{-\gamma} \\
= \ & H_0(2H_0-1)2\int_0^s\int_0^u dudv|u-v|^{2H_0-2}(t+s-u-v)^{-\gamma} \\
& +H_0(2H_0-1)\int_s^t\int_0^s dudv|u-v|^{2H_0-2}(t+s-u-v)^{-\gamma} \\
= \ & 2H_0\int_0^s duu^{2H_0-1}(t+s-u)^{-\gamma} \qquad (3.51)\\
& -H_0\int_s^t duu^{2H_0-1}\left((t+s-u)^{-\gamma}-(u-s)^{2H_0-1}(t-u)^{-\gamma}\right) \\
& +2H_0\gamma\int_0^s du\int_0^u dv(u-v)^{2H_0-1}(t+s-u-v)^{-\gamma-1} \\
& +H_0\gamma\int_s^t du\int_0^u dv(u-v)^{2H_0-1}(t+s-u-v)^{-\gamma-1} \\
= \ & 2H_0\int_0^s duu^{2H_0-1}(t+s-u)^{-\gamma} \\
& +H_0\int_s^t duu^{2H_0-1}\left((t+s-u)^{-\gamma}-(u-s)^{2H_0-1}(t-u)^{-\gamma}\right) \\
& +H_0\gamma\int_0^t du\int_0^u dv|u-v|^{2H_0-1}(t+s-u-v)^{-\gamma-1} \qquad (3.52)
\end{aligned}$$

Assuming (3.43), from (3.50)

$$\gamma\to-\left(d-\frac{k}{2}-\sum_{a\in\overline{A}_k}H_a\right):=\gamma_0$$

and, by taking the limit as $\gamma\to\gamma_0$ and $H_0\to\frac{1}{2}$ in (3.52), we get

$$\begin{aligned}
& H_0(2H_0-1)\int_0^t\int_0^s dudv|u-v|^{2H_0-2}(t+s-u-v)^{-\gamma} \\
\to \ & \int_0^s du(t+s-u)^{-\gamma_0} \\
& +\frac{1}{2}\int_s^t du\left((t+s-u)^{-\gamma_0}-(t-u)^{-\gamma_0}\right) \\
& +\frac{1}{2}\gamma_0\int_0^t du\int_0^u dv(t+s-u-v)^{-\gamma_0-1} \\
= \ & \frac{1}{2}\frac{1}{(-\gamma_0+1)}\left((t+s)^{-\gamma_0+1}-|t-s|^{-\gamma_0+1}\right). \qquad (3.53)
\end{aligned}$$

Consequently, as the limit (3.43) holds true, by plugging (3.49) and (3.53) into (3.47), we obtain

$$\begin{aligned}
\mathbf{E}Y_N(\mathbf{x})^2 &\to \frac{1}{2}\frac{1}{-\gamma_0+1}(2\pi)^{-k}\sum_{j,k=1}^{N}\lambda_j\lambda_k\left((t_j+t_k)^{-\gamma_0+1}-|t_j-t_k|^{-\gamma_0+1}\right)q_{2\mathbf{H}_{\overline{A}_k}-\mathbf{1}}\\
&\times\prod_{a\in A_k}\int_{\mathbb{R}}d\tau e^{-\frac{1}{2}|\tau|^2}\prod_{a\in\overline{A}_k}\int_{\mathbb{R}}d\tau|\tau|^{1-2H_a}e^{-\frac{1}{2}|\tau|^2}\\
&= \frac{1}{2}\frac{1}{-\gamma_0+1}(2\pi)^{-k}\sum_{j,k=1}^{N}\lambda_j\lambda_k\left((t_j+t_k)^{-\gamma_0+1}-|t_j-t_k|^{-\gamma_0+1}\right)\\
&\times q_{2\mathbf{H}_{\overline{A}_k}-\mathbf{1}}(\sqrt{2\pi})^k\prod_{a\in\overline{A}_k}\int_{\mathbb{R}}d\tau|\tau|^{1-2H_a}e^{-\frac{1}{2}|\tau|^2}\\
&= \frac{1}{2}\frac{1}{-\gamma_0+1}(2\pi)^{-\frac{k}{2}}\sum_{j,k=1}^{N}\lambda_j\lambda_k\left(t_j+t_k)^{-\gamma_0+1}-|t_j-t_k|^{-\gamma_0+1}\right)\\
&\times q_{2\mathbf{H}_{\overline{A}_k}-\mathbf{1}}\prod_{a\in\overline{A}_k}\int_{\mathbb{R}}d\tau|\tau|^{1-2H_a}e^{-\frac{1}{2}|\tau|^2}.
\end{aligned}$$

On the other hand, if u is the solution to (3.45), then

$$\begin{aligned}
\mathbf{E}\left(\sum_{j=1}^{N}\lambda_j u(t_j,\mathbf{x})\right)^2 &= \sum_{j,k=1}^{N}\lambda_j\lambda_k\int_0^{t_j\wedge t_k}du\int_{\mathbb{R}^k}d\mathbf{y}_{A_k}\int_{\mathbb{R}^{d-k}}\int_{\mathbb{R}^{d-k}}d\mathbf{y}_{\overline{A}_k}d\mathbf{z}_{\overline{A}_k}\\
&\times(2\pi(t_j-u)^{-\frac{d}{2}}e^{-\frac{|\mathbf{y}_{A_k}|^2}{2(t_j-u)}}e^{-\frac{|\mathbf{y}_{\overline{A}_k}|^2}{2(t_j-u)}}(2\pi(t_k-u)^{-\frac{d}{2}}e^{-\frac{|\mathbf{y}_{A_k}|^2}{2(t_k-u)}}e^{-\frac{|\mathbf{z}_{\overline{A}_k}|^2}{2(t_k-u)}}\\
&= \sum_{j,k=1}^{N}\lambda_j\lambda_k\int_0^{t_j\wedge t_k}du(2\pi)^{-k}\int_{\mathbb{R}^k}d\xi e^{-(t_j+t_k-2u)|\xi|^2}q_{2\mathbf{H}_{\overline{A}_k}-\mathbf{1}}\int_{\mathbb{R}^{d-k}}d\tau e^{-(t_j+t_k-2u)|\tau|^2}|\tau|^{\frac{1}{2}\sum_{a\in\overline{A}_k}(2H_a-1)}\\
&= \sum_{j,k=1}^{N}\lambda_j\lambda_k\int_0^{t_j\wedge t_k}du(t_j+t_k-2u)^{-\gamma_0}(2\pi)^{-k}\int_{\mathbb{R}^k}d\xi e^{-|\xi|^2}q_{2\mathbf{H}_{\overline{A}_k}-\mathbf{1}}\prod_{a\in\overline{A}_k}\int_{\mathbb{R}}d\tau e^{-|\tau|^2}|\tau|^{1-2H_a}\\
&= \frac{1}{2}\frac{1}{-\gamma_0+1}(2\pi)^{-\frac{k}{2}}\sum_{j,k=1}^{N}\lambda_j\lambda_k\left((t_j+t_k)^{-\gamma_0+1}-|t_j-t_k|^{-\gamma_0+1}\right)q_{2\mathbf{H}_{\overline{A}_k}-\mathbf{1}}\prod_{a\in\overline{A}_k}\int_{\mathbb{R}}d\tau e^{-|\tau|^2}|\tau|^{1-2H_a}.
\end{aligned}$$

The point 2. follows similarly. Let us discuss point 3. Assume $H_0, H_1, .., H_d$ converge all to $\frac{1}{2}$. Notice that in this case condition (3.35) implies $d<2$ so $d=1$! Then, from (3.50)

$$\gamma\to\frac{d}{2}=\frac{1}{2}.$$

Therefore, from (3.52), as $H_0, H_1, .., H_d\to\frac{1}{2}$

$$\begin{aligned}
&H_0(2H_0-1)\int_0^t\int_0^s dudv|u-v|^{2H_0-2}(t+s-u-v)^{-\gamma}\\
&\to\frac{1}{2}\int_0^t du\left((t-u)^{-\frac{1}{2}}-(t+s-u)^{-\frac{1}{2}}\right)+\frac{1}{2}\times\frac{1}{2}\int_0^t du\int_0^s dv(t+s-u-v)^{-\frac{3}{2}}\\
&= \left((t+s)^{\frac{1}{2}}-|t-s|^{\frac{1}{2}}\right). \qquad (3.54)
\end{aligned}$$

and we obtain, by combining (3.54) and (3.47), by taking the limit (3.43)

$$\begin{aligned}
\mathbf{E}Y_N(\mathbf{x})^2 &\to (2\pi)^{-1}\sum_{j,k=1}^{N}\lambda_j\lambda_k\left((t_j+t_k)^{\frac{1}{2}}-|t_j-t_k|^{\frac{1}{2}}\right)\int_{\mathbb{R}}d\tau e^{-\frac{1}{2}|\tau|^2}\\
&= \sum_{j,k=1}^{N}\lambda_j\lambda_k\left((t_j+t_k)^{\frac{1}{2}}-|t_j-t_k|^{\frac{1}{2}}\right)\sqrt{2\pi}\\
&= (2\pi)^{-\frac{1}{2}}\sum_{j,k=1}^{N}\lambda_j\lambda_k\left((t_j+t_k)^{\frac{1}{2}}-|t_j-t_k|^{\frac{1}{2}}\right)
\end{aligned}$$

which coincides with the $\mathbf{E}\left(\sum_{j=1}^{N}\lambda_j u(t_j,\mathbf{x})\right)^2$ where u is the solution of the heat equation (3.45) driven by a space-time white noise (see [92] or [103]).

Checking condition (3.31). In order to check condition (3.31), we need to show in the case 1. (the other situations are similar) that for every $t_1,t_2,t_3,t_4\in[0,T]$,

$$\begin{aligned}
I: &= \sup_{(H_0,\mathbf{H}_{A_k})\in[\frac{1}{2},1]^{k+1}}\int_0^{t_1}du_1...\int_0^{t_4}du_4|u_1-u_2|^{-\alpha_0}|u_2-u_3|^{-\alpha_0}|u_3-u_4|^{-\beta_0}|u_4-u_1|^{-\beta_0}\\
&\times\int_{\mathbb{R}^d}d\mathbf{y}_1...\int_{\mathbb{R}^d}d\mathbf{y}_4\frac{1}{(2\pi(t_1-u_1))^{\frac{d}{2}}}e^{-\frac{|\mathbf{x}-\mathbf{y}_1|^2}{2(t_1-u_1)}}\frac{1}{(2\pi(t_2-u_2))^{\frac{d}{2}}}e^{-\frac{|\mathbf{x}-\mathbf{y}_2|^2}{2(t_2-u_2)}}\\
&\times\frac{1}{(2\pi(t_3-u_3))^{\frac{d}{2}}}e^{-\frac{|\mathbf{x}-\mathbf{y}_3|^2}{2(t_3-u_3)}}\frac{1}{(2\pi(t_4-u_4))^{\frac{d}{2}}}e^{-\frac{|\mathbf{x}-\mathbf{y}_4|^2}{2(t_4-u_4)}}\\
&|\mathbf{y}_1-\mathbf{y}_2|^{-\alpha}|\mathbf{y}_2-\mathbf{y}_3|^{-\alpha}|\mathbf{y}_3-\mathbf{y}_4|^{-\beta}|\mathbf{y}_4-\mathbf{y}_1|^{-\beta}<\infty
\end{aligned}$$

with

$$\alpha=\frac{2(1-\mathbf{H})r}{q},\quad \beta=\frac{2(1-\mathbf{H})(q-r)}{q},\quad \alpha_0=\frac{2(1-H_0)r}{q},\quad \beta_0=\frac{2(1-H_0)(q-r)}{q}$$

for every $r=1,..,q-1$. After the change of variables $t_i-u_i=\tilde{u}_i,\tilde{\mathbf{y}}=\mathbf{x}-\mathbf{y}$, we will have to show that

$$\begin{aligned}
I &= \sup_{(H_0,\mathbf{H}_{A_k})\in[\frac{1}{2},1]^{k+1}}\int_0^{t_1}du_1...\int_0^{t_4}du_4\\
&|u_1-u_2-(t_1-t_2)|^{-\alpha_0}|u_2-u_3-(t_2-t_3)|^{-\alpha_0}|u_3-u_4-(t_3-t_4)|^{-\beta_0}|u_4-u_1-(t_4-t_1)|^{-\beta_0}\\
&\int_{\mathbb{R}^d}d\mathbf{y}_1...\int_{\mathbb{R}^d}d\mathbf{y}_4\frac{1}{(2\pi u_1)^{\frac{d}{2}}}e^{-\frac{-|\mathbf{y}_1|^2}{2u_1}}\frac{1}{(2\pi u_2)^{\frac{d}{2}}}e^{-\frac{-|\mathbf{y}_2|^2}{2u_2}}\frac{1}{(2\pi u_3)^{\frac{d}{2}}}e^{-\frac{-|\mathbf{y}_3|^2}{2u_3}}\frac{1}{(2\pi u_4)^{\frac{d}{2}}}e^{-\frac{-|\mathbf{y}_4|^2}{2u_4}}\\
&|\mathbf{y}_1-\mathbf{y}_2|^{-\alpha}|\mathbf{y}_2-\mathbf{y}_3|^{-\alpha}|\mathbf{y}_3-\mathbf{y}_4|^{-\beta}|\mathbf{y}_4-\mathbf{y}_1|^{-\beta}<\infty.
\end{aligned}$$

Next, we write for the integrals $d\mathbf{y}_i$

$$\begin{aligned}
&\int_{\mathbb{R}^d}d\mathbf{y}_1...\int_{\mathbb{R}^d}d\mathbf{y}_4...\\
&= \prod_{j\in A_k}\int_{\mathbb{R}}dy_1^{(j)}...\int_{\mathbb{R}}dy_4^{(j)}\frac{1}{\sqrt{2\pi u_1}}e^{-\frac{|y_1^{(j)}|^2}{2u_1}}...\frac{1}{\sqrt{2\pi u_4}}e^{-\frac{|y_4^{(j)}|^2}{2u_4}}\\
&\times\prod_{j\in\overline{A}_k}\int_{\mathbb{R}}dy_1^{(j)}...\int_{\mathbb{R}}dy_4^{(j)}\frac{1}{\sqrt{2\pi u_1}}e^{-\frac{|y_1^{(j)}|^2}{2u_1}}...\frac{1}{\sqrt{2\pi u_4}}e^{-\frac{|y_4^{(j)}|^2}{2u_4}}.
\end{aligned}$$

We will separate the integral $dy_1^{(j)}$, for every $j=1,..,d$, as follows

$$\int_{\mathbb{R}}dy_1^{(j)}=\int_{|y_1|^{(j)}>\sqrt{2T}}dy_1^{(j)}+\int_{|y_1|^{(j)}\leqslant\sqrt{2T}}dy_1^{(j)}$$

and similarly for the integrals $dy_2^{(j)}, dy_3^{(j)}, dy_4^{(j)}$. We use the fact that on the set

$$y^2 > 2T > 2u$$

the function

$$u \to \frac{1}{\sqrt{u}} e^{-\frac{y^2}{2u}} \text{ is increasing}$$

and we majorize

$$\frac{1}{\sqrt{u}} e^{-\frac{y^2}{2u}} \text{ by } \frac{1}{\sqrt{T}} e^{-\frac{y^2}{2T}}$$

On the other hand, on the set

$$y^2 \leqslant 2T$$

we majorize

$$\frac{1}{\sqrt{u}} e^{-\frac{y^2}{2u}} \text{ by a constant.}$$

In this way, the quantity I can be bounded by

$$\begin{aligned}
I \quad \leq \quad & C \sup_{(H_0, \mathbf{H}_{A_k}) \in [\frac{1}{2},1]^{k+1}} \int_0^{t_1} du_1 ... \int_0^{t_4} du_4 \\
& |u_1 - u_2 - (t_1 - t_2)|^{-\alpha_0} |u_2 - u_3 - (t_2 - t_3)|^{-\alpha_0} |u_3 - u_4 - (t_3 - t_4)|^{-\beta_0} |u_4 - u_1 - (t_4 - t_1)|^{-\beta_0} \\
& \prod_{j \in A_k} \int_{\mathbb{R}} dy_1^{(j)} ... \int_{\mathbb{R}} dy_4^{(j)} \left(\frac{1}{\sqrt{2\pi T}} e^{-\frac{|y_1^{(j)}|^2}{2T}} 1_{|y_1^{(j)}| > \sqrt{2T}} + 1_{|y_1|^{(j)} \leqslant \sqrt{2T}} \right) \\
& \times ... \left(\frac{1}{\sqrt{2\pi T}} e^{-\frac{|y_4^{(j)}|^2}{2T}} 1_{|y_4^{(j)}| > \sqrt{2T}} + 1_{|y_4|^{(j)} \leqslant \sqrt{2T}} \right) \\
& \times |y_1^{(j)} - y_2^{(j)}|^{-\alpha_j} |y_2^{(j)} - y_3^{(j)}|^{-\alpha_j} |y_3^{(j)} - y_4^{(j)}|^{-\beta_j} |y_4^{(j)} - y_1^{(j)}|^{-\beta_j} \\
& \times R
\end{aligned}$$

with $\alpha_j = \frac{2(1-H_j)r}{q}, \beta_j = \frac{2(1-H_j)(q-r)}{q}$ for every $j = 1, .., d$ and

$$\begin{aligned}
R \quad = \quad & \prod_{j \in \overline{A}_k} \int_{\mathbb{R}} dy_1^{(j)} ... \int_{\mathbb{R}} dy_4^{(j)} \left(\frac{1}{\sqrt{2\pi T}} e^{-\frac{|y_1^{(j)}|^2}{2T}} 1_{|y_1^{(j)}| \geqslant \sqrt{2T}} + 1_{|y_1|^{(j)} \leqslant \sqrt{2T}} \right) \\
& \times ... \left(\frac{1}{\sqrt{2\pi T}} e^{-\frac{|y_4^{(j)}|^2}{2T}} 1_{|y_4^{(j)}| \geqslant \sqrt{2T}} + 1_{|y_4|^{(j)} \leqslant \sqrt{2T}} \right) \\
& \times |y_1^{(j)} - y_2^{(j)}|^{-\alpha_j} |y_2^{(j)} - y_3^{(j)}|^{-\alpha_j} |y_3^{(j)} - y_4^{(j)}|^{-\beta_j} |y_4^{(j)} - y_1^{(j)}|^{-\beta_j}.
\end{aligned}$$

Consequently, we can write

$$\begin{aligned}
I \quad \leq \quad & C \sup_{H_0 \in [\frac{1}{2},1]} \int_0^{t_1} du_1 ... \int_0^{t_4} du_4 \\
& |u_1 - u_2 - (t_1 - t_2)|^{-\alpha_0} |u_2 - u_3 - (t_2 - t_3)|^{-\alpha_0} |u_3 - u_4 - (t_3 - t_4)|^{-\beta_0} |u_4 - u_1 - (t_4 - t_1)|^{-\beta_0} \\
& \sup_{\mathbf{H}_{A_k} \in [\frac{1}{2},1]^k} \prod_{j \in A_k} \int_{\mathbb{R}} dy_1^{(j)} ... \int_{\mathbb{R}} dy_4^{(j)} \left(\frac{1}{\sqrt{2\pi T}} e^{-\frac{|y_1^{(j)}|^2}{2T}} 1_{|y_1^{(j)}| \geqslant \sqrt{2T}} + 1_{|y_1^{(j)}| \leqslant \sqrt{2T}} \right) \\
& ..... \left(\frac{1}{\sqrt{2\pi T}} e^{-\frac{|y_4^{(j)}|^2}{2T}} 1_{|y_4^{(j)}| \geqslant \sqrt{2T}} + 1_{|y_4^{(j)}| \leqslant \sqrt{2T}} \right) \\
& |y_1^{(j)} - y_2^{(j)}|^{-\alpha_j} |y_2^{(j)} - y_3^{(j)}|^{-\alpha_j} |y_3^{(j)} - y_4^{(j)}|^{-\beta_j} |y_4^{(j)} - y_1^{(j)}|^{-\beta_j} \\
& \times R.
\end{aligned}$$

Note that R does not depend on $H_0, \mathbf{H}_{A_k}$ and

$$\begin{aligned}
&\sup_{H_0\in[\frac{1}{2},1]} \int_0^{t_1} du_1 ... \int_0^{t_4} du_4 \\
&|u_1-u_2-(t_1-t_2)|^{-\alpha_0}|u_2-u_3-(t_2-t_3)|^{-\alpha_0}|u_3-u_4-(t_3-t_4)|^{-\beta_0}|u_4-u_1-(t_4-t_1)|^{-\beta_0} \\
\leq\ & \int_0^T du_1 ... \int_0^T du_4 |u_1-u_2|^{-\alpha_0}|u_2-u_3|^{-\alpha_0}|u_3-u_4|^{-\beta_0}|u_4-u_1|^{-\beta_0}
\end{aligned}$$

which is finite by Lemma 3.3 in [6] since

$$2\alpha+2\beta+4 = 2(2H-2)+4 = 4H > 1.$$

Therefore, in order to conclude, it remains to show that

$$\begin{aligned}
&\sup_{\mathbf{H}_{A_k}\in[\frac{1}{2},1]^k} \prod_{j\in A_k} \int_{\mathbb{R}} dy_1^{(j)} ... \int_{\mathbb{R}} dy_4^{(j)} \\
&\left(\frac{1}{\sqrt{2\pi T}} e^{-\frac{|y_1^{(j)}|^2}{2T}} 1_{|y_1^{(j)}|\geqslant\sqrt{2T}} + 1_{|y_1^{(j)}|\leqslant\sqrt{2T}}\right) ... \left(\frac{1}{\sqrt{2\pi T}} e^{-\frac{|y_4^{(j)}|^2}{2T}} 1_{|y_4^{(j)}|\geqslant\sqrt{2T}} + 1_{|y_4^{(j)}|\leqslant\sqrt{2T}}\right) \\
&\times|y_1^{(j)}-y_2^{(j)}|^{-\alpha_j}|y_2^{(j)}-y_3^{(j)}|^{-\alpha_j}|y_3^{(j)}-y_4^{(j)}|^{-\beta_j}|y_4^{(j)}-y_1^{(j)}|^{-\beta_j} < \infty.
\end{aligned}$$

Assume for simplicity $A_k = \{1, 2, .., k\}$. To check that the above quantity is finite, it suffices to prove that

$$\begin{aligned}
&\sup_{H\in[\frac{1}{2},1]} \int_{\mathbb{R}} dy_1 .... \int_{\mathbb{R}} dy_4 \left(e^{-\frac{|y_1|^2}{2T}} 1_{|y_1|\geqslant\sqrt{2T}} + 1_{|y_1|\leqslant\sqrt{2T}}\right) .... \left(e^{-\frac{|y_4|^2}{2T}} 1_{|y_4|\geqslant\sqrt{2T}} + 1_{|y_4|\leqslant\sqrt{2T}}\right) \\
&\times|y_1-y_2|^{-\alpha}|y_2-y_3|^{-\alpha}|y_3-y_4|^{-\beta}|y_4-y_1|^{-\beta} < \infty.
\end{aligned}$$

Using $\prod_{i=1}^4 (A_i+B_i) = A_1A_2A_3A_4 + A_1B_2B_3B_4 + ... + B_1B_2B_3B_4$, the last integrals can be expressed as a sum of several terms, involving integrals on the sets $|y_i| \geqslant \sqrt{2T}$ and $|y_i| \leqslant \sqrt{2T}$.

Let us start with the first summand, namely

$$\begin{aligned}
T_1 \ :=\ & \sup_{H\in[\frac{1}{2},1]} \int_{\mathbb{R}} dy_1 .... \int_{\mathbb{R}} dy_4 e^{-\frac{y_1^2}{2T}} 1_{|y_1|\geqslant\sqrt{2T}} e^{-\frac{y_2^2}{2T}} 1_{|y_2|\geqslant\sqrt{2T}} e^{-\frac{y_3^2}{2T}} 1_{|y_3|\geqslant\sqrt{2T}} e^{-\frac{y_4^2}{2T}} 1_{|y_4|\geqslant\sqrt{2T}} \\
&\times|y_1-y_2|^{-\alpha}|y_2-y_3|^{-\alpha}|y_3-y_4|^{-\beta}|y_4-y_1|^{-\beta}.
\end{aligned}$$

Since $|y_1-y_2|^2 \leq 2(y_1^2+y_2^2)$ we have

$$|y_1-y_2|^2+|y_2-y_3|^2+|y_3-y_4|^2+|y_4-y_1|^2 \leq 4(y_1^2+y_2^2+y_3^2+y_4^2) \tag{3.55}$$

so

$$e^{-\frac{y_1^2+y_2^2+y_3^2+y_4^2}{2T}} \leq e^{-\frac{1}{8T}(|y_1-y_2|^2+|y_2-y_3|^2+|y_3-y_4|^2+|y_4-y_1|^2)}.$$

Hence, T_1 can be bounded as follows

$$\begin{aligned}
T_1 \ \leq\ & \sup_{H\in[\frac{1}{2},1]} \int_{\mathbb{R}} dy_1 .... \int_{\mathbb{R}} dy_4 e^{-\frac{1}{8T}(|y_1-y_2|^2+|y_2-y_3|^2+|y_3-y_4|^2+|y_4-y_1|^2)} \\
&|y_1-y_2|^{-\alpha}|y_2-y_3|^{-\alpha}|y_3-y_4|^{-\beta}|y_4-y_1|^{-\beta}.
\end{aligned}$$

We apply the power counting theorem, see the Appendix. Consider the set of affine functionals

$$T' = \{y_1 - y_2, y_2 - y_3, y_3 - y_4, y_4 - y_1\}.$$

The only padded subset of T' is T' itself. We apply the power counting theorem with

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(-\frac{2(1-H)r}{q}, -\frac{2(1-H)r}{q}, -\frac{2(1-H)(q-r)}{q}, -\frac{2(1-H)(q-r)}{q}\right)$$

and

$$(\beta_1, \beta_2, \beta_3, \beta_4) = (-\gamma, -\gamma, -\gamma, -\gamma)$$

with $\gamma > 0$ arbitrarly large. We have (d_0 and d_∞ are given by (3.69) and (3.70) respectively)

$$d_0(T') = r(T') + \sum_{i=1}^{4} \alpha_i = 3 + 2(2H-2) = 4H - 1 > 0 \text{ for } H > \frac{1}{4}$$

and

$$d_\infty(\emptyset) = 4 - 1 - 4\gamma < 0 \text{ if } \gamma > \frac{3}{4}.$$

Therefore T_1 is finite. Let us regard the last summand, i.e.

$$\begin{aligned} T_2 \quad := & \sup_{H\in[\frac{1}{2},1]} \int_{\mathbb{R}} dy_1 .... \int_{\mathbb{R}} dy_4 1_{|y_1|\leqslant\sqrt{2T}} ... 1_{|y_4|\leqslant\sqrt{2T}} \\ & \times |y_1 - y_2|^{-\alpha} |y_2 - y_3|^{-\alpha} |y_3 - y_4|^{-\beta} |y_4 - y_1|^{-\beta} < \infty. \end{aligned}$$

This is clearly finite by Lemma 3.3 in [6] since

$$2\alpha + 2\beta + 4 = 4H - 4 + 4 = 4H > 1$$

when $H > \frac{1}{4}$.

The other summands can be handled by combining the arguments used for the two terms above. For instance, consider

$$\begin{aligned} T_3 \quad := & \sup_{H\in[\frac{1}{2},1]} \int_{\mathbb{R}} dy_1 .... \int_{\mathbb{R}} dy_4 e^{-\frac{y_1^2}{2T}} 1_{|y_1|\geqslant\sqrt{2T}} 1_{|y_2|\leqslant\sqrt{2T}} 1_{|y_3|\leqslant\sqrt{2T}} 1_{|y_4|\leqslant\sqrt{2T}} \\ & \times |y_1 - y_2|^{-\alpha} |y_2 - y_3|^{-\alpha} |y_3 - y_4|^{-\beta} |y_4 - y_1|^{-\beta}. \end{aligned}$$

We use the bound (which follows from (3.55)

$$y_1^2 \geq \frac{1}{4}(|y_1 - y_2|^2 + |y_2 - y_3|^2 + |y_3 - y_4|^2 + |y_4 - y_1|^2) - (y_2^2 + y_3^2 + y_4^2)$$

and then

$$\begin{aligned} & e^{-\frac{y_1^2}{2T}} \leq e^{-\frac{1}{8T}(|y_1-y_2|^2+|y_2-y_3|^2+|y_3-y_4|^2+|y_4-y_1|^2)} e^{\frac{y_2^2+y_3^2+y_4^2}{2T}} \\ \leq \quad & Ce^{-\frac{1}{8T}(|y_1-y_2|^2+|y_2-y_3|^2+|y_3-y_4|^2+|y_4-y_1|^2)}. \end{aligned}$$

The term T_3 is thus bounded by

$$\begin{aligned} T_3 \quad \leq \quad & C \sup_{H\in[\frac{1}{2},1]} \int_{\mathbb{R}} dy_1 .... \int_{\mathbb{R}} dy_4 e^{-\frac{1}{8T}(|y_1-y_2|^2+|y_2-y_3|^2+|y_3-y_4|^2+|y_4-y_1|^2)} \\ & \times |y_1 - y_2|^{-\alpha} |y_2 - y_3|^{-\alpha} |y_3 - y_4|^{-\beta} |y_4 - y_1|^{-\beta} \end{aligned}$$

and we follow the proof for the first term. ■

Remark 13. *Notice that the limit process in Theorem coincides in distribution with a bifractional Brownian motion with Hurst parameters* $H = \frac{1}{2}, K = -\gamma_0 + 1 = d - \frac{k}{2} - \sum_{a \in \overline{A}_k} H_a$ *(in the case i.),* $H = \frac{1}{2}, K = d - \frac{k}{2} - \sum_{a \in \overline{A}_k} H_a + (2H - 1)$ *(in the case ii.) and* $H = K = \frac{1}{2}$ *(in the case iii.) We refer to [43], [103], [105] for the definition of the bifractional Brownian motion and for the link between this process and the solution to the heat equation.*

3.5 Applications to Hermite Ornstein-Uhlenbeck process

Let $Z^{q,1} := Z^q$ be a (one-parameter) Hermite process defined by (3.2). The Hermite Ornstein Uhlenbeck process has been introduced in [53]. It is defined as the solution of Langevin equation driven by Hermite noise.

$$X_t = \xi - \lambda \int_0^t X_s ds + \sigma Z_H^q(t), t \geq 1 \tag{3.56}$$

where $\lambda, \sigma > 0$ and the initial condition ξ is a random variable in $L^2(\Omega)$. The unique solution of (3.56) is given by

$$Y^H(t) = e^{-\lambda t} \left(\xi + \sigma \int_0^t e^{\lambda u} dZ_H^q(u) \right), \qquad t \geq 0 \tag{3.57}$$

where the integral $\int_0^t e^{\lambda u} dZ^q(u)$ exists in the Riemann-Stieljes sense.

In particular, by taking the initial condition $\xi = \sigma \int_{-\infty}^0 e^{\lambda u} dZ^H(u)$ in (3.57). The unique solution to (3.56), denoted in the sequel by $(X^H(t))_{t \geq 0}$, can be expressed as

$$X^H(t) = \sigma \int_{-\infty}^t e^{-\lambda(t-u)} dZ_H^q(u), \qquad t \geq 0 \tag{3.58}$$

and the stochastic integral in (3.58) can be also understood in the Wiener sense. The process $\left(X^H(t)\right)_{t \geq 0}$ is a stationary process, H-self similar process with stationary increments.

In [89] the authors have established the asymptotic behavior with respect to H of the Rosenblatt Ornstein Uhlenbeck process which is the solution of (3.56) driven by the Rosenblatt process, i.e. $q = 2$. The proof was based on the analysis of the cumulants, but it is well-known that this method does not work for a Wiener chaos of order $q \geq 3$. In this section, we will study the behavior as $H \to 1$ and as $H \to \frac{1}{2}$ of the processes $\left(X^H(t)\right)_{t \in [0,T]}$ and $\left(Y^H(t)\right)_{t \in [0,T]}$ when $q > 2$. The results obtained give a complete picture for the asymptotic behavior of the Hermite Ornstein Uhlenbeck of any order $q \geq 1$.

3.5.1 Asymptotic behavior of the non stationary Hermite Ornstein-Uhlenbeck

Assume that the initial condition ξ does not depend on H.

Proposition 28. *1 Assume* $H \to 1$*. Then the process* $\left(Y^H(t)\right)_{t \in [0,T]}$ *converges weakly, in the space of the continuous functions* $C[0,T]$ *to the process* $(Y(t))_{t \in [0,T]}$ *given by*

$$Y(t) = e^{-\lambda t} \xi + \sigma \left(1 - e^{-\lambda t}\right) \frac{H_q(Z)}{\sqrt{q!}} \tag{3.59}$$

with $Z \sim \mathcal{N}(0,1)$

2 Assume $H \to \frac{1}{2}$, the process $\left(Y^H(t)\right)_{t\in[0,T]}$ converges weakly, in the space of the continuous functions $C[0,T]$ as $H \to \frac{1}{2}$ to the standard Ornstein Uhlenbeck process $(Y_0(t))_{t\in[0,T]}$ given by

$$Y_0(t) = e^{-\lambda}\left(\xi + \sigma\int_0^t e^{\lambda u}dW(u)\right) \tag{3.60}$$

that is a Gaussian process with mean $\mathbf{E}Y_0(t) = e^{-\lambda t}\mathbf{E}\xi$ for any $t \geq 0$ and covariance function

$$Cov(Y_0(t), Y_0(s)) = \frac{\sigma^2}{2\lambda}\left(e^{-\lambda|t-s|} - e^{-\lambda(t+s)}\right)$$

for every $s, t \geq 0$.

Proof : Consider $\alpha_1, \dots, \alpha_N \in \mathbb{R}$ and $t_1, \dots, t_N \in [0,T]$. We will study the convergence of the finite dimensional distributions of Y^H.

$$\begin{aligned}
Y_N &= \sum_{i=1}^N \alpha_i Y^H(t_i) = \sum_{i=1}^N e^{-\lambda t_i}\xi + \int_{\mathbb{R}}\sum_{i=1}^N \alpha_i 1_{[0,t_i]}(u)e^{-\lambda(t_i-u)}dZ_H^q(u)\\
&= \sum_{i=1}^N e^{-\lambda t_i}\xi + \int_{\mathbb{R}} f(u)dZ_H^q(u)
\end{aligned}$$

with $f(u) = \sum_{i=1}^N \alpha_i 1_{[0,t_i]}(u)e^{-\lambda(t_i-u)}$.

Notice that in this case the space $\mathcal{H}_{A_k}$ given by (3.15) coincides with $L^1(\mathbb{R})$. Since it is clear that f belongs to $|\mathcal{H}_H| \cap L^1(\mathbb{R})$ (see [89]), we get immediatly by Proposition 26 the convergence as $H \to 1$ of $\int_{\mathbb{R}} f(u)dZ_H^q(u)$ to $\left(\int_{\mathbb{R}} f(u)du\right)\frac{H_q(Z)}{\sqrt{q!}}$.

In order to prove the convergence when $H \to \frac{1}{2}$, we will apply Proposition 27. Using the same arguments as for the proof of Proposition 5 in [89], we get

$$\begin{aligned}
&\lim_{H\to\frac{1}{2}} H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)f(v)|u-v|^{2H-2}dudv = \int_{\mathbb{R}}(f(u))^2du\\
&= \sum_{i=1}^N\sum_{j=1}^N \alpha_i\alpha_j\int_0^{t_i\wedge t_j} e^{-\lambda(t_i+t_j-2u)}du = \sum_{i=1}^N\sum_{j=1}^N \alpha_i\alpha_j\frac{\sigma^2}{2\lambda}\left(e^{-\lambda|t_i-t_j|} - e^{-\lambda(t_i+t_j)}\right)
\end{aligned}$$

which coincides with the variance of $\sum_{j=1}^N \alpha_j Y_0(t_j)$. The proof is completed by showing that (3.31) is satisfied. We have

$$\begin{aligned}
&\int_{\mathbb{R}^4} du_1...du_4 f(u_1)...f(u_4)|u_1-u_2|^{H-1}|u_2-u_3|^{H-1}|u_3-u_4|^{H-1}|u_4-u_1|^{H-1}\\
\leq &\sum_{j_1,..,j_4=1}^d |\alpha_{j_1}....\alpha_{j_4}|\int_0^T ... \int_0^T du_1..du_4\\
&\times|u_1-u_2|^{\frac{2(H-1)r}{q}}|u_2-u_3|^{\frac{2(H-1)r}{q}}|u_3-u_4|^{\frac{2(H-1)(q-r)}{q}}|u_4-u_1|^{\frac{2(H-1)(q-r)}{q}}
\end{aligned}$$

is finite and continuous in H on the set $(\frac{1}{4}, 1]$. This follows from Lemma 3.3 in [6] or by applying the power counting theorem with $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\frac{2(H-1)r}{q}, \frac{2(H-1)r}{q}, \frac{2(H-1)(q-r)}{q}, \frac{2(H-1)(q-r)}{q}\right)$. We recall (see [89]) that for $p \geq 1$,

$$\mathbf{E}|Y^H(t) - Y^H(s)|^{2p} \leq C_p(\mathbf{E}|Y^H(t) - Y^H(s)|^2)^p \leq c|t-s|^p. \tag{3.61}$$

The tighness follows from (3.61) and Bilingsley criterium (see [16]). ■

3.5.2 Asymptotic behavior of the stationary Hermite Ornstein-Uhlenbeck

Now we will study the asymptotic behavior of (3.58). The diffrence to the non-stationary case is that the function f from the last proof has support of infinite Lebesque measure an we need to use an argument based on the power counting theorem when H tends to one half. The proof of this results is similar in spirit to the proofs of Proposition 6 and Proposition 7 in [89].

Proposition 29. *1 Assume $H \to 1$. Then the process $\left(X^H(t)\right)_{t\in[0,T]}$ converges weakly, in the space of the continuous functions $C[0,T]$ to the process $(X(t))_{t\in[0,T]}$ defined by*

$$X(t) = \frac{\sigma}{\lambda}\frac{H_q(Z)}{\sqrt{q!}} \tag{3.62}$$

with $Z \sim \mathcal{N}(0,1)$

2 Assume $H \to \frac{1}{2}$, the process $\left(X^H(t)\right)_{t\in[0,T]}$ converges weakly, in the space of the continuous functions $C[0,T]$ as $H \to \frac{1}{2}$ to the stationary Ornstein Uhlenbeck process $(X_0(t))_{t\in[0,T]}$ given by

$$X_0(t) = \sigma \int_{-\infty}^{t} e^{-\lambda(t-u)} dW(u) \tag{3.63}$$

which is a stationary centered Gaussian process with covariance function

$$Cov(X_0(t), X_0(s)) = \frac{\sigma^2}{2\lambda} e^{-\lambda|t-s|}$$

for every $s, t \geq 0$.

Proof : Consider $\alpha_1, \dots, \alpha_N \in \mathbb{R}$ and $t_1, \dots, t_N \in [0,T]$. We will study the convergence of the finite dimensional distributions of Y^H.

$$\begin{aligned}
\sum_{i=1}^{N} \alpha_i X^H(t_i) &= \int_{\mathbb{R}} \sum_{i=1}^{N} \sigma\alpha_i 1_{[-\infty,t_i]}(u) e^{-\lambda(t_i-u)} dZ_H^q(u) \\
&= \int_{\mathbb{R}} g(u) dZ_H^q(u)
\end{aligned}$$

with $g(u) = \sum_{i=1}^{N} \alpha_i 1_{[-\infty,t_i]}(u) e^{-\lambda(t_i-u)}$.

The computations in proofs of Proposition 6 and Proposition 7 in [89] show that g belongs to $|\mathcal{H}_H| \cap L^1(\mathbb{R})$, we get immediatly by Proposition 26 that the random variable $\sum_{i=1}^{N} \alpha_i X^H(t_i)$ converges to$\sum_{i=1}^{N} \alpha_i X(t_i)$ as $H \to 1$.

When $H \to \frac{1}{2}$, the proof with slight changes, follows along the same lines as the proof of Proposition 7 in [89]. We have

$$\mathbf{E}\left(\sum_{j=1}^{d} \alpha_j X^H(t_j)\right)^2 \xrightarrow[H\to\frac{1}{2}]{} \mathbf{E}\left(\sum_{j=1}^{d} \alpha_j X_0(t_j)\right)^2.$$

It remains to prove that the condition (3.31) holds true. We have

$$\begin{aligned}
&\int_{\mathbb{R}^4} du_1...du_4 g(u_1)...g(u_4)|u_1-u_2|^{\frac{2(H-1)r}{q}}|u_2-u_3|^{\frac{2(H-1)r}{q}}\times\\
&\times|u_3-u_4|^{\frac{2(H-1)(q-r)}{q}}|u_4-u_1|^{\frac{2(H-1)(q-r)}{q}}\\
\leq\ & \sum_{j_1,j_2,..,j_4=1}^{d}|\alpha_{j_1}...\alpha_{j_4}|\int_{-\infty}^{t_{j_1}}du_1....\int_{-\infty}^{t_{j_4}}du_m e^{-\lambda(t_{j_1}-u_1)}....e^{-\lambda(t_{j_4}-u_4)}\\
&|u_1-u_2|^{\frac{2(H-1)r}{q}}|u_2-u_3|^{\frac{2(H-1)r}{q}}|u_3-u_4|^{\frac{2(H-1)(q-r)}{q}}|u_4-u_1|^{\frac{2(H-1)(q-r)}{q}}\\
=\ & \sum_{j_1,j_2,..,j_4=1}^{d}|\alpha_{j_1}...\alpha_{j_4}|\int_0^{\infty}du_1....\int_0^{\infty}du_4 e^{-\lambda(u_1+..+u_4)}\\
&\times|u_1-u_2-(t_{j_1}-t_{j_2})|^{\frac{2(H-1)r}{q}}|u_2-u_3-(t_{j_1}-t_{j_2})|^{\frac{2(H-1)r}{q}}\\
&|u_3-u_4-(t_{j_3}-t_{j_4})|^{\frac{2(H-1)(q-r)}{q}}|u_4-u_1-(t_{j_4}-t_{j_1})|^{\frac{2(H-1)(q-r)}{q}}\\
\leq\ & e^{\frac{\lambda}{2}(|t_{j_1}-t_{j_2}|+...+|t_{j_4}-t_{j_1}|)}\sum_{j_1,j_2,..,j_4=1}^{d}|\alpha_{j_1}...\alpha_{j_4}|\int_0^{\infty}du_1...\int_0^{\infty}du_4\\
&e^{-\frac{\lambda}{2}(|u_1-u_2-(t_{j_1}-t_{j_2})|+...+|u_4-u_1-(t_{j_4}-t_{j_1})|)}\\
&\times\left(1\vee|u_1-u_2-(t_{j_1}-t_{j_2})|^{\frac{2(H-1)r}{q}}\right)\left(1\vee|u_2-u_3-(t_{j_1}-t_{j_2})|^{\frac{2(H-1)r}{q}}\right)\\
&\left(1\vee|u_3-u_4-(t_{j_3}-t_{j_4})|^{\frac{2(H-1)(q-r)}{q}}\right)\left(1\vee|u_4-u_1-(t_{j_4}-t_{j_1})|^{\frac{2(H-1)(q-r)}{q}}\right)
\end{aligned}$$

We apply the power counting theorem on the set T' defined by

$$T'=\{u_1-u_2-(t_{j_1}-t_{j_2}),...,u_4-u_1-(t_{j_4}-t_{j_1})\}$$

with $(\alpha_1,..,\alpha_4)=\left(\frac{2(H-1)r}{q},\frac{2(H-1)r}{q},\frac{2(H-1)(q-r)}{q},\frac{2(H-1)(q-r)}{q}\right)$ and $(\beta_1,..,\beta_4)=(-\gamma,...,-\gamma)$ with $\gamma\in(\frac{3}{4},1]$. Since T' is the only paddet subset of T', we have

$$d_0(T')=4-1+\frac{4(H-1)(q-r)}{q}+\frac{4(H-1)(q-r)}{q}=4H-1>0 \text{ if } H>\frac{1}{4}$$

and

$$d_\infty(\emptyset)=4-1-4\gamma<0 \text{ if } \gamma>1-\frac{1}{4}=\frac{3}{4}.$$

Therefore, the function

$$\begin{aligned}
&H\to\int_{\mathbb{R}}...\int_{\mathbb{R}}du_1...du_4|g(u_1)...g(u_m)||u_1-u_2|^{\frac{2(H-1)r}{q}}|u_2-u_3|^{\frac{2(H-1)r}{q}}\times\\
&\times|u_3-u_4|^{\frac{2(H-1)(q-r)}{q}}|u_4-u_1|^{\frac{2(H-1)(q-r)}{q}}
\end{aligned}$$

is finite and continuous on the set $D=\{H\in(0,1],H>\frac{1}{4}\}$.

The conclusion follows from Proposition 27. Again the tighness is obtained by (3.61). ■

3.6 Appendix

The basic tools from the analysis on Wiener space and the power counting theorem proven in [98] are presented in this appendix.

3.6.1 Multiple stochastic integrals and the Fourth Moment Theorem

Here, we shall only recall some elementary facts ; our main reference is [74]. Consider $\mathcal{H}$ a real separable infinite-dimensional Hilbert space with its associated inner product $\langle .,. \rangle_{\mathcal{H}}$, and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, which is a centered Gaussian family of random variables such that $\mathbf{E}\left(B(\varphi)B(\psi)\right) = \langle \varphi, \psi \rangle_{\mathcal{H}}$, for every $\varphi, \psi \in \mathcal{H}$. Denote by I_q the qth multiple stochastic integral with respect to B. This I_q is actually an isometry between the Hilbert space $\mathcal{H}^{\odot q}$ (symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{q!}}\|\cdot\|_{\mathcal{H}^{\otimes q}}$ and the Wiener chaos of order q, which is defined as the closed linear span of the random variables $H_q(B(\varphi))$ where $\varphi \in \mathcal{H}$, $\|\varphi\|_{\mathcal{H}} = 1$ and H_q is the Hermite polynomial of degree $q \geq 1$ defined by :

$$H_q(x) = (-1)^q \exp\left(\frac{x^2}{2}\right) \frac{\mathrm{d}^q}{\mathrm{d}x^q}\left(\exp\left(-\frac{x^2}{2}\right)\right), \; x \in \mathbb{R}. \tag{3.64}$$

The isometry of multiple integrals can be written as : for $p,\ q \geq 1$, $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$,

$$\mathbf{E}\Big(I_p(f)I_q(g)\Big) = \begin{cases} q!\langle \tilde{f}, \tilde{g} \rangle_{\mathcal{H}^{\otimes q}} & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases} \tag{3.65}$$

It also holds that :

$$I_q(f) = I_q(\tilde{f}),$$

where $\tilde{f}$ denotes the canonical symmetrization of f and it is defined by :

$$\tilde{f}(x_1, \ldots, x_q) = \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} f(x_{\sigma(1)}, \ldots, x_{\sigma(q)}),$$

in which the sum runs over all permutations σ of $\{1, \ldots, q\}$.

In the particular case when $\mathcal{H} = L^2(T, \mathcal{B}(T), \mu)$, the rth contraction $f \otimes_r g$ is the element of $\mathcal{H}^{\otimes(p+q-2r)}$, which is defined by :

$$\begin{aligned} &(f \otimes_r g)(s_1, \ldots, s_{p-r}, t_1, \ldots, t_{q-r}) \\ &= \int_{T^r} \mathrm{d}u_1 \ldots \mathrm{d}u_r f(s_1, \ldots, s_{p-r}, u_1, \ldots, u_r) g(t_1, \ldots, t_{q-r}, u_1, \ldots, u_r), \end{aligned} \tag{3.66}$$

for every $f \in L^2([0,T]^p)$, $g \in L^2([0,T]^q)$ and $r = 1, \ldots, p \wedge q$.

An important property of finite sums of multiple integrals is the hypercontractivity. Namely, if $F = \sum_{k=0}^n I_k(f_k)$ with $f_k \in \mathcal{H}^{\otimes k}$ then

$$\mathbf{E}|F|^p \leq C_p \left(\mathbf{E}F^2\right)^{\frac{p}{2}}. \tag{3.67}$$

for every $p \geq 2$.

We will use the following famous result initially proven in [76] that characterizes the convergence in distribution of a sequence of multiple integrals torward the Gaussian law.

Theorem 6. *Fix $n \geq 2$ and let $(F_k, k \geq 1)$, $F_k = I_n(f_k)$ (with $f_k \in \mathcal{H}^{\odot n}$ for every $k \geq 1$), be a sequence of square-integrable random variables in the nth Wiener chaos such that $\mathbf{E}\left[F_k^2\right] \to 1$ as $k \to \infty$. The following are equivalent :*

1. *the sequence $(F_k)_{k\geq 0}$ converges in distribution to the normal law $\mathcal{N}(0,1)$;*
2. *$\mathbf{E}\left[F_k^4\right] = 3$ as $k \to \infty$;*
3. *for all $1 \leq l \leq n-1$, it holds that $\lim_{k\to\infty} \|f_k \otimes_l f_k\|_{\mathcal{H}^{\otimes 2(n-l)}} = 0$;*

Another equivalent condition can be stated in term of the Malliavin derivatives of F_k, see [66].

3.6.2 Power counting theorem

We need to recall some notation and results from [98] which are needed in order to check the integrability assumption from Proposition 27.

Consider a set $T = \{M_1, .., M_m\}$ of linear functions on $\mathbb{R}^m$. The power counting theorem (see Theorem 1.1 and Corollary 1.1 in [98]) gives sufficient conditions for the integral

$$I = \int_{\mathbb{R}} ... \int_{\mathbb{R}} du_1 ... du_m f_1(M_1(u_1, .., u_m)) .... f_m(M_m(u_1, .., u_m)) \tag{3.68}$$

to be finite, where $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, .., m$ are such that $|f_i|$ is bounded above on (a_i, b_i) $(0 < a_i < b_i < \infty)$ and

$$|f_i(y)| \leq c_i |y|^{\alpha_i} \text{ if } |y_i| < a_i \text{ and } |f_i(y)| \leq c_i |y|^{\beta_i} \text{ if } |y| > b_i.$$

For a subset $W \subset T$ we denote by $s_T(W) = span(W) \cap T$. A subset W of T is said to be *padded* if $s_T(W) = W$ and any functional $M \in W$ also belongs to $s_T(W \setminus \{M\})$. Denote by span (W) the linear span generated by W and by $r(W)$ the number of linearly independent elements of W.

Then Theorem 1.1 in [98] says that the integral I (3.68) is finite if

$$d_0(W) = r(W) + \sum_{s_T(W)} \alpha_i > 0 \tag{3.69}$$

for any subset W of T with $s_T(W) = W$ and

$$d_\infty(W) = r(T) - r(W) + \sum_{T \setminus s_T(W)} \beta_i < 0 \tag{3.70}$$

for any proper subset W of T with $s_T(W) = W$, including the empty set. If $\alpha_i > -1$ then it suffices to check (3.69) for any padded subset $W \subset T$. Also, it suffices to verify (3.70) only for padded subsets of T if $\beta_i \geq -1$.

The condition (3.69) implies the integrability at the origin while (3.70) gives the integrability of I at infinity.

There is a similar result if one starts with a set T of affine functionals instead of linear functionals.

Troisième partie

Inférence statistique des EDPS avec bruit fractionnaire

Chapitre 4

Generalized k-variations and Hurst parameter estimation for the fractional wave equation via Malliavin calculus

4.1 Introduction

For several decades the statistical inference in stochastic (partial) differential equations (S(P)DE in the sequel) constitutes an intensive research direction in probability theory and mathematical statistics. Traditionally, the disturbance term in such stochastic models is a standard space-time white noise, i.e. a Gaussian field that behaves as a Brownian motion in time and in space. We refer, among many others, to the monographs or surveys [24], [52] or [51]. A part of the scientific literature on statistical inference for SPDEs concerns the parameter estimation for such equations based on the observation of the solution at discrete points in time and/or in space, which also constitutes the main goal of our work. Among the first contributions to these topics, we refer to [60] and [58] for the maximum likelihood and least square estimators for parabolic or elliptic-type SPDEs respectively, driven by a space-time white noise. The study in [58] has been then extended in [13] by adding a time-varying volatility in the noise term and by using power variation techniques to estimate various parameters of the model. Other recent works on parameter estimates for discretely sampled SPDEs, some of them using generalized variations, are [23], [20] [14] or [83].

Nowadays, a particular case of wide interest is represented by the S(P)DEs driven by fractional Brownian motion (fBm) and related processes, due to the vast area of application of such stochastic models. Many recent works concern the estimation of the drift parameter for stochastic (partial) differential equations driven by fractional Brownian motion (we refer, among many others, to [4], [44], [49], [86], [104]), while fewer works deal with the estimation of the Hurst parameter in such stochastic equations ([48], [99])

In this paper, we consider the one-dimensional stochastic wave equation driven by an additive Gaussian noise which behaves as a fractional Brownian motion in time and as a Wiener process in space (we call it *fractional-white noise*). Our purpose is to construct and analyze estimators for the Hurst parameter of the solution to this SPDE based on the observation of the solution at a fixed time and at a discrete number of points in space. The wave equation with fractional noise in time and/or in space has been studied in several works, such as [11], [25], [42], [48], [85] etc. We will use a standard method to construct estimators for the Hurst parameter, which is based on the k-variations of the observed process. The method has been recently employed in [48] for the case of quadratic variations, i.e. $k = 2$. As for the fBm, it was shown that the standard quadratic variation estimator is not asymptotically normal when the Hurst index becomes bigger than $\frac{3}{4}$ and this is inconvenient for statistical applications. In order to avoid this restriction and to get an estimator which is asymptotically Gaussian for every $H \in \left[\frac{1}{2}, 1\right)$, we will use the generalized k-variations, which means that the usual increment of the process is replaced by a higher order increment. The idea comes from the reference [45] and since then it has been used by many authors (see e.g. [28] or [21]). More precisely, if $(u(t, x), t \geq 0, x \in \mathbb{R})$ denotes the solution to the wave equation with fractional-white noise, we define the (centered) generalized k-variation statistics ($k \geq 1$ integer) as

$$V_N(k, \alpha) = \frac{1}{N-l} \sum_{i=l}^{N} \left[\frac{\left|U^\alpha\left(\frac{i}{N}\right)\right|^k}{\mathbf{E}\left|U^\alpha\left(\frac{i}{N}\right)\right|^k} - 1 \right], \tag{4.1}$$

where $U^\alpha\left(\frac{i}{N}\right)$ represents the spatial increment of the solution u at $\frac{i}{N}$ along a *filter* α of power (order) $p \geq 1$ and length $l + 1 \geq 1$ (see the next section for the precise definition).

By using chaos expansion and recent developments in the Stein-Malliavin calculus we show that the sequence $V_N(k, \alpha)$ satisfies a central limit theorem (CLT) as $N \to \infty$ (in the spirit of [18]) whenever $p > H + \frac{1}{4}$ and in this way the restriction $H < \frac{3}{4}$ can be avoided by choosing a filter of order $p \geq 2$, i.e. by replacing, for example, the usual increment by a rectangular or a higher order increment. We will obtain the rate of convergence under the Wasserstein distance for this convergence in law and we also prove a multidimensional CLT. So we generalize the findings in [48] to filters of any power $p \geq 1$ and to k-variations of any order $k \geq 1$ and in addition we show that in the special case $p = 1$ and $H > \frac{3}{4}$ a non-Gaussian limit theorem occurs with limit distribution related to the Rosenblatt distribution (but having a more complex structure).

These theoretical results are then applied to the estimation of Hurst index of the solution of the fractional-white wave equation. Based on the behavior of the sequence $V_N(k, \alpha)$ we prove that the associated k-variation estimators for H are consistent and asymptotically normal. Moreover, we provide a numerical analysis of the estimators when $k = 2$ by analyzing their performance on various filters and for several values of the Hurst parameter and confirming via simulation the theoretical results.

We organized the paper as follows. Section 2 contains some preliminaries. In this part we present the basic facts concerning the solution to the fractional-white wave equation, we introduce the filters and the increment of the solution along filters. In Section 3, we prove a CLT for the sequence $V_N(k, \alpha)$ for any integer $k \geq 1$, and we obtain the rate of convergence when k is even by using the Stein-Malliavin theory. In Section 4 we show a non-central limit theorem in the case $k = 2, H > \frac{3}{4}$ and for filters of order $p = 1$. Section 5 concerns the estimation of the Hurst parameter of the solution to the fractional-white wave equation. We included theoretical results related to the behavior of the k-variations estimators for the Hurst index as well as simulations and numerical analysis for the performance of the estimators. Section 6 (the Appendix) contains the basic tools from Malliavin calculus needed in the paper and the proofs of some technical results.

4.2 Preliminaries

In this chapter we introduce the fractional-white wave equation and its solution and present the basic definitions and the notation concerning the filters used in our work.

4.2.1 The solution to the wave equation with fractional-colored noise

The object of our study will be the solution to the stochastic wave equation. Though we will only be concerned with the one-dimensional case in the sequel, let us recall the necessary definitions and results for the general case in order to provide some context for this choice. The equation for a general number of dimensions $d \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ is defined as follows :

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) &= \Delta u(t,x) + \dot{W}^H(t,x),\ t \geq 0,\ x \in \mathbb{R}^d,\ d \geq 1, \\ u(0,x) &= 0, \quad x \in \mathbb{R}^d, \\ \frac{\partial u}{\partial t}(0,x) &= 0, \quad x \in \mathbb{R}^d, \end{cases} \tag{4.2}$$

where Δ is the Laplacian on $\mathbb{R}^d$, and W^H is a fractional-white Gaussian noise which is defined as a real valued centered Gaussian field $W^H = \{W_t^H(A); t \geq 0, A \in B_b(\mathbb{R}^d)\}$, over a given complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$, with covariance function given by

$$\mathbf{E}\left(W_t^H(A) W_s^H(B)\right) = R_H(t,s)\lambda(A \cap B), \forall A, B \in \mathcal{B}_b(\mathbb{R}^d), \tag{4.3}$$

where λ denotes the Lebesgue measure and R_H is the covariance of the fractional Brownian motion

$$R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s,t \geq 0.$$

We denoted by $B_b(\mathbb{R}^d)$ the class of bounded Borel subsets of $\mathbb{R}^d$ and we will assume throughout this work $H \in \left[\frac{1}{2}, 1\right)$.

The solution to the equation (4.2) is understood in the mild sense, that is, it is defined as a square-integrable centered field $u = \left(u(t,x);\ t \geq 0, x \in \mathbb{R}^d\right)$ given by

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G_1(t-s, x-y) W^H(\mathrm{d}s, \mathrm{d}y), \quad t \geq 0, x \in \mathbb{R}^d, \tag{4.4}$$

where the integral in (4.4) is a Wiener integral with respect to the Gaussian process W^H and G_1 is the fundamental solution of the wave equation $\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$, which can be defined via its Fourier transform

$$\mathcal{F}G_1(t,\cdot)(\xi) = \frac{\sin(t\|\xi\|)}{\|\xi\|}$$

for any $\xi \in \mathbb{R}^d$, $t > 0$. In particular, for $d = 1$ we have

$$G_1(t,x) = \frac{1}{2}\mathbb{1}_{\{|x|<t\}}, \quad t > 0, \quad x \in \mathbb{R}. \tag{4.5}$$

It is known (see e.g. [11]) that the solution (4.4) is well-defined if and only if

$$d < 2H + 1$$

and it is self-similar in time and stationary in space. So for $H = \frac{1}{2}$ the solution is only well-defined in one dimension while for $H \in \left(\frac{1}{2}, 1\right)$ both $d = 1$ and $d = 2$ are possible. Other properties of the solution can be found in [11], [25] or [103].

Now let us fix $d = 1$ for the rest of this paper. It was shown in [47] for $H = \frac{1}{2}$ and in [48] for $H > \frac{1}{2}$ that the spatial covariance of the solution can be expressed as follows

$$\begin{aligned} \mathbf{E}\left(u(t,x)u(t,y)\right) &= \frac{1}{2}\left(c_H|y-x|^{2H+1} - t\frac{|y-x|^{2H}}{2} + \frac{t^{2H+1}}{2H+1}\right)1_{\{|y-x|<t\}} \\ &\quad + \frac{(2t-|y-x|)^{2H+1}}{8(2H+1)}1_{\{t \leq |y-x| < 2t\}} \end{aligned} \tag{4.6}$$

with $c_H = \frac{4H-1}{4(2H+1)}$. The above formula (4.6) is the key ingredient to analyze of the correlation of the increments of the solution in space and implicitly the behavior of the generalized variations. In order to make this analysis possible, we will consider the situation when the second summand in the right-hand side of (4.6) vanishes. To this end, we assume that $t > T$ whenever $x, y \in [0, T]$. For simplicity and without loss of generality, we will consider $T = 1$. When $t > 1$ and $x, y \in [0, 1]$, this expression reduces to

$$\mathbf{E}\left(u(t,x)u(t,y)\right) \quad = \quad \frac{1}{2}\left(c_H|y-x|^{2H+1} - t\frac{|y-x|^{2H}}{2} + \frac{t^{2H+1}}{2H+1}\right) \text{ for } x,\, y \in [0,\, 1]. \quad (4.7)$$

We will fix for the rest of the work $t > 1$ and we will associate to the process $(u(t,x), x \in [0,1])$ its canonical Hilbert space $\mathcal{H}$ which is defined as the closure of the linear space generated by the indicator functions $\{1_{[0,x]}, x \in [0,1]\}$ with respect to the inner product

$$\langle 1_{[0,x]}, 1_{[0,y]}\rangle_{\mathcal{H}} \quad = \quad \mathbf{E}\left(u(t,x)u(t,y)\right).$$

We will denote by I_q the multiple stochastic integral of order $q \geq 1$ with respect to the Gaussian process $(u(t,x), x \in [0,1])$ and by D the Malliavin derivative with respect to this process. We refer to the Appendix for the basic elements of the Malliavin calculus.

We will also use in Section 4.4 multiple stochastic integrals with respect to the fractional-white noise W^H with covariance (4.3). We use the notation I_q^W to indicate the multiple integral of order $q \geq 1$ with respect to W^H.

4.2.2 Filters

In this section we will define the filters and the increments of the solution to (4.2) along filters. We start with several definitions and some notation needed along this paper.

Definition 1. *Given $l, p \in \mathbb{N}^*$, a vector $\alpha = (\alpha_0, ..., \alpha_l)$ is called a filter of length $l+1$ and order (or power) $p \geq 1$ if*

$$\begin{cases} \sum_{q=0}^{l} \alpha_q q^r & = 0, \quad 0 \leq r \leq p-1, \\ \sum_{q=0}^{l} \alpha_q q^p & \neq 0 \end{cases}$$

with the convention $0^0 = 1$.

For instance, $\alpha = (1, -1)$ is a filter of length 2 and of order $p = 1$ while $\alpha = (1, -2, 1)$ is a filter of length 3 and of power $p = 2$.

For a filter $\alpha = (a_0, a_1, .., a_l)$ of length $l + 1 \geq 1$ and of order $p \geq 1$ we define the space-filtered process (or the spatial increment of the process u along the filter α)

$$U^\alpha\left(\frac{i}{N}\right) = \sum_{r=0}^{l} a_r u\left(t, \frac{i-r}{N}\right) \text{ for } i = l, .., N. \quad (4.8)$$

In the case of the filter $\alpha = (1, -1)$, $U^\alpha\left(\frac{i}{N}\right) = u\left(t, \frac{i}{N}\right) - u\left(t, \frac{i-1}{N}\right)$ is the usual spatial increment of the solution while for $\alpha = (1, -2, 1)$ we have $U^\alpha\left(\frac{i}{N}\right) = u\left(t, \frac{i}{N}\right) - 2u\left(t, \frac{i-1}{N}\right) + u\left(t, \frac{i-2}{N}\right)$ which represents the rectangular spatial increment.
We denote for $j \geq 1$

$$\pi_H^{\alpha,N}(j) := \mathbf{E}\left[U^\alpha\left(\frac{i}{N}\right) U^\alpha\left(\frac{i+j}{N}\right)\right].$$

From the covariance formula (4.6) we can write

$$\begin{aligned} \pi_H^{\alpha,N}(j) &= \sum_{r_1,r_2=0}^{l} a_{r_1}a_{r_2}\mathbf{E}\left[u\left(t,\frac{i-r_1}{N}\right)u\left(t,\frac{i+j-r_2}{N}\right)\right] \\ &= k_1\frac{1}{N^{2H}}\Phi_{H,\alpha}(j)+k_2\frac{1}{N^{2H+1}}\Phi_{H+\frac{1}{2},\alpha}(j), \end{aligned} \tag{4.9}$$

with

$$\Phi_{H,\alpha}(j)=\sum_{r_1,r_2=0}^{l} a_{r_1}a_{r_2}|j+r_1-r_2|^{2H}, \qquad j\geq 0,$$

and $k_1=-\frac{t}{4}$ and $k_2=\frac{c_H}{2}=\frac{4H-1}{8(2H+1)}$. We write for further use

$$c_1(H)=\frac{-t}{4}\sum_{q,r=0}^{l}\alpha_q\alpha_r|q-r|^{2H}, \text{ and } c_2(H)=\frac{c_H}{2}\sum_{q,r=0}^{l}\alpha_q\alpha_r|q-r|^{2H+1}. \tag{4.10}$$

In particular, from (4.9) we obtain

$$\begin{aligned} \pi_H^{\alpha,N}(0) &= \mathbf{E}\left[U^\alpha\left(\frac{i}{N}\right)\right]^2=k_1\frac{1}{N^{2H}}\Phi_{H,\alpha}(0)+k_2\frac{1}{N^{2H+1}}\Phi_{H+\frac{1}{2},\alpha}(0) \\ &= c_1(H)\frac{1}{N^{2H}}+c_2(H)\frac{1}{N^{2H+1}}. \end{aligned}$$

We will need the below technical lemma to establish the asymptotical equivalent of $\Phi_{H,\alpha}$ and similar expressions. The proof of the lemma is based on a Taylor expansion, see [28] or [45].

Lemma 3. *Let $l_1,l_2,p_1,p_2\in\mathbb{N}^*$, $H\in\mathbb{R}^+\backslash\mathbb{N}$ and $\alpha^{(1)}$, $\alpha^{(2)}$ be filters of lengths l_1+1, l_2+2 and of orders p_1,p_2 respectively. Then*

$$\sum_{q=0}^{l_1}\sum_{r=0}^{l_2}\alpha_q^{(1)}\alpha_r^{(2)}|q-r+k|^{2H} \underset{k\to\infty}{\sim} \kappa_H k^{2H-2p},$$

with $\kappa_H=\sum_{q=0}^{l_1}\sum_{r=0}^{l_2}\alpha_q^{(1)}\alpha_r^{(2)}\frac{2H(2H-1)\ldots(2H-2p+1)}{2p!}(q-r)^{2p}$, where $p=\min(p_1,\,p_2)$.

In the sequel, we write $a_k\sim_{k\to\infty}b_k$ to indicate that the sequences a_k,b_k have the same behavior as $k\to\infty$.

4.3 Central limit theorems for the spatial k-variations

In this section we focus on the asymptotic behavior in distribution of the k-variation in space of the solution to the fractional-white wave equation, defined via a filter of power $p\geq 1$. In the first step we show the k-variation satisfies a CLT when $p>H+\frac{1}{4}$. Next, by taking k to be an even integer, we derive a Berry-Esséen type bound for this convergence in distribution via the Stein-Malliavin calculus. Restricting ourselves in addition to $k=2$, we prove a multidimensional CLT, which is needed for the estimation of the Hurst parameter.

4.3.1 Central limit theorem

Fix $t>1$ and $l,p\in\mathbb{N}^*$. Let α be a filter of length $l+1$ and of power p as in Definition 1. Let u be given by (4.4). Recall the definition of the centered spatial k-variations (4.1) of the process $(u(t,x),x\in\mathbb{R})$ for $k\in\mathbb{N}^*$ from the introduction with $U^\alpha\left(\frac{i}{N}\right)$ given by (4.8). We will show that the sequence (4.1) satisfies a CLT using a criterion based on Malliavin calculus.

Chaos expansion

The first step is to derive a Wiener chaos expansion of the k-variation sequence $V_N(k,\alpha)$. Noticing that the filtered process U^α, as a linear combination of centered Gaussian random variables, is a centered Gaussian process, we get

$$\mathbf{E}\left(U^\alpha\left(\frac{i}{N}\right)^k\right) = E_k \mathbf{E}\left(U^\alpha\left(\frac{i}{N}\right)^2\right)^{\frac{k}{2}}, \tag{4.11}$$

where E_k denotes the k-th absolute moment of a standard Gaussian variable given by $E_k = \frac{2^{\frac{k}{2}}\Gamma(\frac{k+1}{2})}{\Gamma(\frac{1}{2})}$. We introduce the variable

$$Z^\alpha\left(\frac{i}{N}\right) = \frac{U^\alpha\left(\frac{i}{N}\right)}{(\pi_H^{\alpha,N}(0))^{1/2}}. \tag{4.12}$$

It is clear that $Z^\alpha\left(\frac{i}{N}\right)$ is a standard Gaussian variable and $\mathrm{Corr}\left(Z^\alpha\left(\frac{i}{N}\right), Z^\alpha\left(\frac{j}{N}\right)\right) = \mathrm{Corr}\left(U^\alpha\left(\frac{i}{N}\right), U^\alpha\left(\frac{j}{N}\right)\right)$, where Corr denotes the correlation coefficient. Using (4.11) and (4.12) we can write V_N as follows :

$$V_N(k,\alpha) \quad = \quad \frac{1}{N-l}\sum_{i=l}^{N}\left[\frac{|U^\alpha(\frac{i}{N})|^k}{\mathbf{E}|U^\alpha(\frac{i}{N})|^k} - 1\right] = \frac{1}{N-l}\sum_{i=l}^{N}\left[\frac{|Z^\alpha(\frac{i}{N})|^k}{E_k} - 1\right].$$

Recall the expansion of the development in Hermite polynomials of the function $H^k(t) = \frac{|t|^k}{E_k} - 1$ given in Lemma 2 of [28] :

$$H^k(t) = \sum_{j=1}^{\infty} c_j^k H_j(t),$$

where $c_{2j+1}^k = 0$ for $j \geqslant 0$, $c_{2j}^k = \frac{1}{(2j)!}\prod_{i=0}^{j-1}(k-2i)$ for $j \geqslant 1$ and $H_j(t)$ denotes the j-th Hermite polynomial defined by

$$H_j(t) = \sum_{a=0}^{[\frac{j}{2}]}(-1)^a \frac{a!}{(j-2a)!a!} 2^{-a} t^{j-2a}.$$

Observing that for

$$C_{i,\,\alpha} := \sum_{q=0}^{l} \alpha_q \mathbf{1}_{[0,\frac{i-q}{N}]}$$

we have from (4.9) that $\left\|\frac{C_{i,\,\alpha}}{(\pi_H^{\alpha,N}(0))^{1/2}}\right\|_{\mathcal{H}} = 1$ we can express $Z^\alpha\left(\frac{i}{N}\right)$ as an integral with respect to the process $(u(t,x), x \in \mathbb{R})$ since the increment $u(t,y) - u(t,x)$ can be expressed as $I_1(\mathbb{1}_{[x,y]})$ (recall that I_1 represents the multiple integral of order 1 with respect to the Gaussian process $(u(t,x), x \in [0,1])$) for every $x < y$:

$$Z^\alpha\left(\frac{i}{N}\right) \quad = \quad I_1\left(\frac{C_{i,\,\alpha}}{(\pi_H^{\alpha,N}(0))^{1/2}}\right).$$

Since we have $H_q(I_1(h)) = \frac{1}{q!}I_q(h^{\otimes q})$ for $\|h\|_{\mathcal{H}} = 1$ we get

$$\begin{aligned} V_N(k,\alpha) &= \frac{1}{N-l}\sum_{i=l}^{N} H^k\left(Z^\alpha\left(\frac{i}{N}\right)\right) = \frac{1}{N-l}\sum_{q\geqslant 1} c_{2q}^k \sum_{i=l}^{N} H_{2q}\left(Z^\alpha\left(\frac{i}{N}\right)\right) \\ &= \frac{1}{N-l}\sum_{q\geqslant 1} c_{2q}^k \sum_{i=l}^{N} H_{2q}\left(I_1\left(\frac{C_{i,\alpha}}{(\pi_H^{\alpha,N}(0))^{1/2}}\right)\right) \\ &= \frac{1}{N-l}\sum_{q\geqslant 1} \frac{c_{2q}^k}{(2q)!} \sum_{i=l}^{N} I_{2q}\left(\left(\frac{C_{i,\alpha}}{(\pi_H^{\alpha,N}(0))^{1/2}}\right)^{\otimes 2q}\right). \end{aligned}$$

Hence, we obtain the following chaotic expansion of the k-variation statistics

$$V_N(k,\alpha) = \frac{1}{N-l}\sum_{i=l}^{N}\sum_{q=1}^{\infty} \frac{c_{2q}^k}{(2q)!} I_{2q}\left(\frac{C_{i,\alpha}^{\otimes 2q}}{(\pi_H^{\alpha,N}(0))^{q}}\right) = \sum_{q\geq 1} I_{2q}(f_{N,2q}), \tag{4.13}$$

with

$$f_{N,2q} = \frac{c_{2q}^k}{(2q)!}\frac{1}{N-l}\sum_{i=l}^{N} \frac{C_{i,\alpha}^{\otimes 2q}}{(\pi_H^{\alpha,N}(0))^{q}}. \tag{4.14}$$

In particular,

$$\mathbf{E}[V_N(k,\alpha)]^2 = \sum_{q\geq 1}(2q)!\|f_{N,2q}\|^2_{\mathcal{H}^{\otimes 2q}}. \tag{4.15}$$

Relation (4.13) shows that the random variable $V_N(k,\alpha)$ admits an infinite chaos expansion, which contains the chaoses of all orders from $q=2$ to infinity. We will study the behavior of each chaos component of $V_N(k,\alpha)$. Let us start by analyzing the asymptotic behavior of the mean square of each kernel $f_{N,2q}$ that appears in the chaos expansion of $V_N(k,\alpha)$.

Lemma 4. *For $N, q \geq 1$, let $f_{N,2q}$ be given by (4.14). Then*

$$(N-l)(2q)!\|f_{N,2q}\|^2_{\mathcal{H}^{\otimes 2q}} \to_{N\to\infty} \frac{(c_{2q}^k)^2}{(2q)!}\sum_{v\in\mathbb{Z}}(\varphi_{H,\alpha}(v))^{2q} := \sigma_{2q}^2,$$

for $H < p - \frac{1}{4q}$ (i.e. $H < 1 - \frac{1}{4q}$ for $p=1$ and $H \in \left[\frac{1}{2}, 1\right)$ for $p \geq 2$), where we use the notation

$$\varphi_{H,\alpha}(v) = \frac{\Phi_{H,\alpha}(v)}{\Phi_{H,\alpha}(0)}. \tag{4.16}$$

Moreover, $\sigma^2 := \sum_{q\geq 1}\sigma_{2q}^2 < \infty$. For $p=q=1$, $H=3/4$, we have

$$\frac{N-l}{\log(N-l)} 2!\|f_{N,2}\|^2_{\mathcal{H}^{\otimes 2}} \to_{N\to\infty} c^2 := \frac{(c_2^k)^2}{2}\lim_{N\to\infty}\log N \sum_{|v|\leq N}(\rho_{H,\alpha}(v))^2 < \infty. \tag{4.17}$$

Proof : See the Appendix. ■

Asymptotic normality for the renormalized k-variation

We will consider the renormalized k-variation sequence

$$G_N(k,\alpha) = \sqrt{N-l}\,V_N(k,\alpha). \tag{4.18}$$

From the above Lemma 4 and (4.15) it follows that

$$\mathbf{E}\left[G_N(k,\alpha)\right]^2 \to_{N\to\infty} \sigma^2,$$

with σ^2 given in the statement of Lemma 4. We will show that the sequence (4.18) satisfies a central limit theorem.

Theorem 7. *Let $l, p \in \mathbb{N}^*$. For a filter α of order p and of length $l+1$, with $p > H+\frac{1}{4}$, let $G_N(k,\alpha)$ be given by (4.18). Then the sequence $(G_N(k,\alpha))_{N\geq 1}$ converges in distribution, as $N \to \infty$, to the Gaussian law $N(0,\sigma^2)$. Moreover, for $p=1$, $H=3/4$, the sequence $\left(\frac{1}{\sqrt{\log(N-l)}} G_N(k,\alpha)\right)_{N\geq 1}$ converges in distribution to $N(0, c^2)$. The constants σ^2, c^2 are those appearing in Lemma 4.*

Proof : Notice that from (4.13), we can write

$$G_N(k,\alpha) = \sum_{q\geq 1} I_{2q}(g_{N,2q}) \text{ with } g_{N,2q} = \sqrt{N-l} f_{N,2q} \tag{4.19}$$

with $f_{N,2q}$ given by (4.14). Our main tool to prove the asymptotic normality of (4.19) is Theorem 6.3.1 from [66]. According to it, for $p > H + 1/4$ it suffices to show that

1. $(2q)! \|g_{N,2q}\|^2_{\mathcal{H}^{\otimes 2q}} \to_{N\to\infty} \sigma^2_{2q}$ and $\sigma^2 := \sum_{q\geq 1} \sigma^2_{2q} < \infty$,
2. for every $q \geq 1$ and $r = 1,..,2q-1$, $\|g_{N,2q} \otimes_r g_{N,2q}\|_{\mathcal{H}^{\otimes 4q-2r}} \to_{N\to\infty} 0$,
3. $\lim_{M\to\infty} \sup_{N\geq 1} \sum_{q\geq M+1} (2q)! \|g_{N,2q}\|^2_{\mathcal{H}^{\otimes 2q}} = 0$

and for $p=1$, $H=3/4$,

1. $\frac{1}{\log(N-l)} (2q)! \|g_{N,2q}\|^2_{\mathcal{H}^{\otimes 2q}} \to_{N\to\infty} 1_{\{q=1\}} c^2$,
2. for every $q \geq 1$ and $r = 1,..,2q-1$, $\frac{1}{\log(N-l)} \|g_{N,2q} \otimes_r g_{N,2q}\|_{\mathcal{H}^{\otimes 4q-2r}} \to_{N\to\infty} 0$,
3. $\lim_{M\to\infty} \sup_{N\geq 1} \sum_{q\geq M+1} \frac{1}{\log(N-l)} (2q)! \|g_{N,2q}\|^2_{\mathcal{H}^{\otimes 2q}} = 0$.

Point 1 in both cases follows from Lemma 4. Let us check point 2. By definition of contraction (see (4.42)), we have for $q \geq 1$ and $r = 1,..,2q-1$

$$g_{N,2q} \otimes_r g_{N,2q} = \frac{1}{N-l} \frac{(c^k_{2q})^2}{(2q)!} \sum_{i,j=l}^{N} \frac{\langle C_{i,\alpha}, C_{j,\alpha}\rangle^r_{\mathcal{H}}}{\pi^{\alpha,N}_H(0)^{2q}} C^{\otimes 2q-r}_{i,\alpha} \otimes C^{\otimes 2q-r}_{j,\alpha}$$

and

$$\begin{aligned}
&\|g_{N,2q} \otimes_r g_{N,2q}\|^2_{\mathcal{H}^{\otimes 4q-2r}} \\
&= \left(\frac{(c^k_{2q})^2}{(2q)!}\right)^2 \frac{1}{(N-l)^2} \sum_{i_1,i_2,i_3,i_4=l}^{N} \frac{\langle C_{i_1,\alpha}, C_{i_2,\alpha}\rangle^{2q-r}_{\mathcal{H}} \langle C_{i_2,\alpha}, C_{i_3,\alpha}\rangle^r_{\mathcal{H}} \langle C_{i_3,\alpha}, C_{i_4,\alpha}\rangle^{2q-r}_{\mathcal{H}} \langle C_{i_4,\alpha}, C_{i_1,\alpha}\rangle^r_{\mathcal{H}}}{\pi^{\alpha,N}_H(0)^{2q}} \\
&= \left(\frac{(c^k_{2q})^2}{(2q)!}\right)^2 \frac{1}{(N-l)^2} \sum_{i_1,i_2,i_3,i_4=l}^{N} \rho^{\alpha,N}_H(i_1-i_2)^{2q-r} \rho^{\alpha,N}_H(i_2-i_3)^r \rho^{\alpha,N}_H(i_3-i_4)^{2q-r} \rho^{\alpha,N}_H(i_4-i_1)^r
\end{aligned}$$

with $\rho^{\alpha,N}_H$ given by (4.45). We use the fact that

$$\begin{aligned}
\sum_{i_1,i_2,i_3,i_4=l}^{N} & \rho^{\alpha,N}_H(i_1-i_2)^{2q-r} \rho^{\alpha,N}_H(i_2-i_3)^r \rho^{\alpha,N}_H(i_3-i_4)^{2q-r} \rho^{\alpha,N}_H(i_4-i_1)^r \\
&\leq \sum_{n,m=l}^{N} \left(\left(\rho^{\alpha,N}_H 1_{\{|\cdot|\leq N-l\}}\right)^{2q-r} * \left(\rho^{\alpha,N}_H 1_{\{|\cdot|\leq N-l\}}\right)^r \right)^2 (n-m),
\end{aligned}$$

(where $*$ denotes convolution on $\mathbb{Z}$) and we obtain

$$
\begin{aligned}
& \|g_{N,2q} \otimes_r g_{N,2q}\|^2_{\mathcal{H}^{\otimes 4q-2r}} \\
\leq \quad & C\frac{1}{N-l}\sum_{v=l}^{N}\left(\left(\rho_H^{\alpha,N}1_{\{|\cdot|\leq N-l\}}\right)^{2q-r} * \left(\rho_H^{\alpha,N}1_{\{|\cdot|\leq N-l\}}\right)^{r}\right)^2(v) \\
\leq \quad & C\frac{1}{N-l}\left\|\left(\rho_H^{\alpha,N}1_{\{|\cdot|\leq N-l\}}\right)^{2q-r}\right\|^2_{l^{4/3}(\mathbb{Z})}\left\|\left(\rho_H^{\alpha,N}1_{\{|\cdot|\leq N-l\}}\right)^{r}\right\|^2_{l^{4/3}(\mathbb{Z})} \\
= \quad & C\frac{1}{N-l}\left(\sum_{|v|\leq N-l}\left(\rho_H^{\alpha,N}(v)\right)^{(2q-r)\frac{4}{3}}\right)^{3/2}\left(\sum_{|v|\leq N-l}\left(\rho_H^{\alpha,N}(v)\right)^{r\frac{4}{3}}\right)^{3/2}
\end{aligned}
$$

by virtue of the Young's inequality as in [48]. Note that for v large enough we have by virtue of (4.49)

$$
b_{N,H}(v)1_{\{|v|\leq N-l\}} \leq C\frac{1}{N}v^{2H+1-2p}1_{\{|v|\leq N-l\}} \leq Cv^{2H-2p} \leq C\varphi_H(v)1_{\{|v|\leq N-l\}},
$$

and since all the powers involved above are positive, this allows us to replace $\rho_H^{\alpha,N}$ with φ_H. Thus, for large N the norm $\|g_{N,2q} \otimes_r g_{N,2q}\|^2_{\mathcal{H}^{\otimes 4q-2r}}$ is bounded by

$$
C\frac{1}{N-l}\left(\sum_{|v|\leq N-l}|v|^{(2H-2p)(2q-r)\frac{4}{3}}\right)^{3/2}\left(\sum_{|v|\leq N-l}|v|^{(2H-2p)r\frac{4}{3}}\right)^{3/2}.
$$

For $p \geq 2$ all these series converge. For $p = 1$ and $H \leq \frac{3}{4}$ the only cases in which some of the series do not converge are $r = 2q-1$ and $r = 1$. However, the observation

$$
\frac{1}{N-l}\sum_{|v|\leq N-l}|v|^{(2H-2)\frac{4}{3}}\sum_{|v|\leq N-l}|v|^{(2H-2)\frac{4}{3}} \leq CN^{-1}N^{\frac{8}{3}H-\frac{5}{3}}N^{\frac{8}{3}H-\frac{5}{3}} \to 0
$$

ensures that even in those cases the term $\|g_{N,2q} \otimes_r g_{N,2q}\|^2_{\mathcal{H}^{\otimes 4q-2r}}$ converges to zero.

Concerning point 3, fix $M \geq 1$ and recall that from (4.46)

$$
(2q)!\|g_{N,2q}\|^2_{\mathcal{H}^{\otimes 2q}} = \frac{(c^k_{2q})^2}{(2q)!}\sum_{v\in\mathbb{Z}}\left(\rho_H^{\alpha,N}(v)\right)^{2q}1_{\{|v|\leq N-l\}}\frac{N-|v|-l}{N-l},
$$

and therefore, since $|\rho_H^{\alpha,N}(v)| \leq 1$ for $|v|$ large enough,

$$
\begin{aligned}
\sup_{N\geq 1}\sum_{q\geq M+1}(2q)!\|g_{N,2q}\|^2_{\mathcal{H}^{\otimes 2q}} \quad \leq \quad & \sup_{N\geq 1}\sum_{q\geq M+1}\frac{(c^k_{2q})^2}{(2q)!}\sum_{v\in\mathbb{Z}}\left(\rho_H^{\alpha,N}(v)\right)^2 1_{\{|v|\leq N-l\}}\frac{N-|v|-l}{N-l} \\
\leq \quad & C\sum_{q\geq M+1}\frac{(c^k_{2q})^2}{(2q)!}\sum_{v\in\mathbb{Z}}\varphi_{H,\alpha}(v)^2 \\
& +C\sup_{N\geq 1}\sum_{q\geq M+1}\frac{(c^k_{2q})^2}{(2q)!}\sum_{v\in\mathbb{Z}}b_{N,H}(v)^2 1_{\{|v|\leq N-l\}}\frac{N-|v|-l}{N-l}.
\end{aligned}
$$

From (4.49)

$$
b_{N,H}(v)^2 \leq C\frac{1}{N^2} \text{ if } p \geq 2,
$$

and

$$
\sum_{|v|\leq N-l}b_{N,H}(v)^2\frac{N-|v|-l}{N-l} \leq C\frac{1}{N^2}\sum_{|v|\leq N-l}v^{(2H-1)2} \leq CN^{4H-3} \text{ if } p = 1, H < \frac{3}{4}.
$$

Consequently,

$$\sup_{N\geq 1}\sum_{q\geq M+1}(2q)!\|g_{N,2q}\|^2_{\mathcal{H}^{\otimes 2q}} \leq C\sum_{q\geq M+1}\frac{(c^k_{2q})^2}{(2q)!}\sum_{v\in\mathbb{Z}}\varphi_{H,\alpha}(v)^2,$$

and this tends to zero as $M\to\infty$, due to the convergence of the series $\sum_{q\geq 1}\frac{(c^k_{2q})^2}{(2q)!}$.

For $\frac{1}{\sqrt{\log(N-l)}}G_N(k,\,\alpha)$ there is nothing to show since the case $q=1$ does not contribute to the limit. ■

4.3.2 Rate of convergence for even power variations

In this section we will further quantify the CLT proved above by deriving a rate of convergence in Wasserstein distance for even power variations.

Let $k\geq 2$ be an even integer. Consider the sequence $G_N(k,\,\alpha)$ defined by (4.18). From (4.13), since the coefficients c^k_{2j} vanish if $2j>k$, we get

$$G_N(k,\alpha)=\frac{1}{\sqrt{N-l}}\sum_{i=l}^{N}\sum_{q=1}^{\frac{k}{2}}\frac{c^k_{2q}}{(2q)!}I_{2q}\left(\frac{C^{\otimes 2q}_{i,\alpha}}{(\pi^{\alpha,N}_H(0))^q}\right). \tag{4.20}$$

Denote for every $q=1,2,..,\frac{k}{2}$ the $2q$-th chaos component of $G_N(k,\alpha)$ by

$$G^{(2q)}_N(k,\alpha)=I_{2q}(g_{N,2q}), \tag{4.21}$$

with $g_{N,2q}$ from (4.19). Let us consider the $\frac{k}{2}$-dimensional random vector

$$\mathbf{G}_N(k,\,\alpha):=\left(G^{(2)}_N(k,\alpha),G^{(4)}_N(k,\alpha),....,G^{(k)}_N(k,\alpha)\right).$$

Notice that for every $q_1,q_2=1,..,\frac{k}{2}$ with $q_1\neq q_2$

$$\mathbf{E}G^{(2q_1)}_N(k,\alpha)G^{(2q_2)}_N(k,\alpha)=0,$$

while for $q_1=q_2=q$

$$\mathbf{E}\left[G^{(2q)}_N(k,\alpha)\right]^2=\frac{(c^k_{2q})^2}{(2q)!}\sum_{v\in\mathbb{Z}}\rho^{\alpha,\,N}_H(v)^{2q}1_{\{|v|\leq N-l\}}\left(1-\frac{|v|}{N-l}\right).$$

Let us introduce the matrix $C=(C_{q_1,q_2})_{q_1,q_2=1,..,\frac{k}{2}}$ with components $C_{q_1,q_2}=0$ if $q_1\neq q_2$ and

$$C_{q,q}=\frac{(c^k_{2q})^2}{(2q)!}\sum_{v\in\mathbb{Z}}\varphi_{H,\alpha}(v)^{2q}. \tag{4.22}$$

The objective in this section is to calculate the rate of convergence of $V_N(k,\,\alpha)$ in the CLT proved in Section 3.1. In order to obtain this rate in terms of the Wasserstein distance we will use Corollary 3.6 from [68] to show that the vector $\mathbf{G}_N(k,\,\alpha)$ converges to a normal distribution with the covariance matrix C and determine its convergence rate. This will provide corresponding results for the k-variation statistics $V_N(k,\alpha)$. For the sake of completeness we cite this corollary here.

Corollary 2. *Fix $d\geq 2$ and $1\leq q_1\leq\cdots\leq q_d$. Consider a vector $F:=(F_1,\dots F_d)=(I_{q_1}(f_1)\dots I_{q_d}(f_d))$ with $f_i\in\mathcal{H}^{\odot q_i}$ for any $i=1,\dots,d$. Let $Z\sim N_d(0,C)$ with C positive definite. Then*

$$d_W(F,\,Z)\leq c\sqrt{\sum_{1\leq i,\,j\leq d}E\left[\left(C_{ij}-\frac{1}{q_j}\langle DF_i,\,DF_j\rangle_{\mathcal{H}}\right)^2\right]}$$

for some constant strictly positive c.

(In the one-dimensional case for a standard normal Z this result is also true and can be found in [66]. For $k=2$ the norming condition is satisfied, and the corollary is applicable.)

In order to apply the corollary for $F_i = G_N^{(2i)}, i = 1, .., \frac{k}{2}$, we will write each summand as

$$\begin{aligned} &\mathbf{E}\left[\left(C_{ij} - \frac{1}{q_j}\langle DF_i, DF_j\rangle_{\mathcal{H}}\right)^2\right] \\ \leq\ & 2\left(C_{ij} - \frac{1}{q_j}\mathbf{E}[\langle DF_i, DF_j\rangle_{\mathcal{H}}]\right)^2 + 2\mathbf{E}\left[\left(\frac{1}{q_j}\mathbf{E}[\langle DF_i, DF_j\rangle_{\mathcal{H}}] - \frac{1}{q_j}\langle DF_i, DF_j\rangle_{\mathcal{H}}\right)^2\right] \end{aligned} \tag{4.23}$$

and conduct separate calculations for both parts. We start with a lemma for the deterministic part.

Lemma 5. *Let $G_N^{(2q)}, C_{q,q}$ be given by (4.21), (4.22) respectively and assume $p \geq 2$. For N large enough and for every $q = 1, .., \frac{k}{2}$ we have for every $H \in \left[\frac{1}{2}, 1\right)$,*

$$\left|\mathbf{E}\left[G_N^{(2q)}(k,\alpha)^2\right] - C_{q,q}\right| \leq C\frac{1}{N}.$$

For $p=1$ we have for $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$

$$\left|\mathbf{E}\left[G_N^{(2q)}(k,\alpha)^2\right] - C_{q,q}\right| \leq CN^{4H-3},$$

and for $p=1$, $H = \frac{1}{2}$,

$$\left|\mathbf{E}\left[G_N^{(2q)}(k,\alpha)^2\right] - C_{q,q}\right| \leq C\frac{\log N}{N}.$$

Proof : See the Appendix. ■

The following proposition provides a bound for the random part in (4.23).

Proposition 30. *Let G_N be given by (4.18). For $q_1, q_2 \in \{1, \ldots, \frac{k}{2}\}$, $p \geq 2$ and $H \in \left[\frac{1}{2}, 1\right)$,*

$$\operatorname{Var}(\langle DG_N^{(2q_1)}(k,\alpha), DG_N^{(2q_2)}(k,\alpha)\rangle_{\mathcal{H}}) \leq C\frac{1}{N}.$$

For $p=1$ and $H < 3/4$

$$\operatorname{Var}(\langle DG_N^{(2q_1)}(k,\alpha), DG_N^{(2q_2)}(k,\alpha)\rangle_{\mathcal{H}}) \leq C\begin{cases} \frac{1}{N} & \text{if } H \in [\frac{1}{2}, \frac{5}{8})\ , \\ \frac{\log(N)^3}{N} & \text{if } H = \frac{5}{8}, \\ N^{8H-6} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}). \end{cases}$$

Proof : See the Appendix. ■

Before stating and proving the main result of this chapter, let us briefly recall the definition of the Wasserstein distance. The Wasserstein distance between the laws of two $\mathbb{R}^d$-valued random variables F and G is defined as

$$d_W(F,G) = \sup_{h\in\mathcal{A}} |\mathbf{E}h(F) - \mathbf{E}h(G)|\,, \tag{4.24}$$

where $\mathcal{A}$ is the class of Lipschitz continuous functions $h : \mathbb{R}^d \to \mathbb{R}$ such that $\|h\|_{Lip} \leq 1$, where

$$\|h\|_{Lip} = \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|h(x) - h(y)|}{\|x-y\|_{\mathbb{R}^d}}.$$

Let us now state and prove the main result of this section.

Theorem 8. *Let σ^2, c^2 be constants as in Lemma 4. Let $p \geq 2$ and consider the sequence (4.20). Let $Z \sim N(0, \sigma^2)$. Then there exists a constant C such that*

$$d_W(G_N(k, \alpha), Z) \leq C \frac{1}{\sqrt{N}}.$$

For $p = 1$ and $H < 3/4$ let $Z \sim N(0, c^2)$. Then there exists a constant C such that

$$d_W(G_N(k, \alpha), Z) \leq C \begin{cases} \frac{1}{\sqrt{N}} & \text{if } H \in [\frac{1}{2}, \frac{5}{8}) \ , \\ \frac{\log(N)^{3/2}}{\sqrt{N}} & \text{if } H = \frac{5}{8}, \\ N^{4H-3} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}). \end{cases}$$

Proof : Consider the function $f : \mathbb{R}^{\frac{k}{2}} \to \mathbb{R}$, $f(x) = \frac{2}{k}(x_1 + ... + x_{\frac{k}{2}})$. Note that f is a Lipschitz continuous function with $\|f\| \leq 1$. From Lemma 5 and Proposition 30 it is easy to see that by Corollary 2

$$d_W\left(\frac{k}{2}\mathbf{G}_N(k, \alpha), \frac{k}{2}\mathbf{Z}\right) = d_W\left(\frac{k}{2}(G_N^{(2)}(k, \alpha), ..., G_N^{(k)}(k, \alpha)), \frac{k}{2}\mathbf{Z}\right) \leq C \frac{1}{\sqrt{N}},$$

where $\mathbf{Z} \sim N(0, C)$ if $p \geq 2$ and

$$d_W(\frac{k}{2}\mathbf{G}_N(k, \alpha), \frac{k}{2}\mathbf{Z}) \leq C \begin{cases} \frac{1}{\sqrt{N}} & \text{if } H \in [\frac{1}{2}, \frac{5}{8}) \ , \\ \frac{\log(N)^{3/2}}{\sqrt{N}} & \text{if } H = \frac{5}{8}, \\ N^{4H-3} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}). \end{cases}$$

if $p = 1$ and $H < 3/4$. Now,

$$\begin{aligned} d_W(G_N(k, \alpha), Z) &= \sup_{\|g\|_{Lip} \leq 1} |\mathbf{E}g(G_N(k, \alpha)) - \mathbf{E}g(Z)| \\ &= \sup_{\|g\|_{Lip} \leq 1} \left|\mathbf{E}(g \circ f)(\frac{k}{2}\mathbf{G}_N(k,\, \alpha)) - \mathbf{E}(g \circ f)(\frac{k}{2}Z)\right| \\ &\leq \sup_{\|h\|_{Lip} \leq 1} \left|\mathbf{E}h(\frac{k}{2}\mathbf{G}_N(k,\, \alpha)) - \mathbf{E}h(\mathbf{Z})\right| \\ &= d_W(\frac{k}{2}\mathbf{G}_N(k,\, \alpha), \mathbf{Z}). \end{aligned}$$

■

Remark 14. *For $p = 1$ and $k = 2$ we retrieve the bounds obtained in [48] (and in [47] for $H = \frac{1}{2}$), which also coincide with the speed of convergence for the quadratic variations of the fBm (see [66]) under the Wasserstein distance. For $k = 2$, it is also possible to get optimal rates under the total variation distance based on the criteria in [67].*

4.3.3 Multivariate central limit theorem

In this part we restrict ourselves to the case of quadratic variations (i.e. $k = 2$) and we derive a multidimensional CLT. This result will be needed in Section 4.5 which deals with the estimation of the Hurst parameter of the solution to (4.2).

To establish multidimensional convergence, we will use Theorem 6.2.3 in [77]. Let us recall its statement.

Theorem 9. *Let $d \geqslant 2$ and $q_1, \dots, q_d \geqslant 1$ be some fixed integers. Consider vectors*

$$F_n = (F_{1,n}, \dots, F_{d,n}) = (I_{q_1}(f_{1,n}), \dots, I_{q_d}(f_{d,n}))$$

with $f_{i,n} \in \mathcal{H}^{\odot q_i}$. Let C be a real-valued symmetric non negative definite matrix and let $N \sim \mathcal{N}_d(0, C)$. Assume that

$$\lim_{n\to\infty} \mathbf{E}\left(F_{i,n}F_{j,n}\right) = C(i,j) \text{ for } i,\, j \in \{1, \dots, d\}. \tag{4.25}$$

Then, as n tends to ∞, the following two conditions are equivalent :

- *F_n converges in law to N,*
- *for every $1 \leqslant i \leqslant d$ $F_{i,n}$ converges in law to $\mathcal{N}(0, C(i,i))$.*

We now state and prove the multivariate CLT for the renormalized sequence (4.1) with $k = 2$.

Theorem 10. *Let $P \geqslant 1$ be an integer and $\alpha^1, \dots, \alpha^P$ be filters of orders $p_1, \dots p_P$ and lengths $l_1 + 1, \dots, l_P + 1$ respectively, where $l_i, p_i \in \mathbb{N}^*, i = 1, .., P$. Let $V_N(2, \alpha)$ be given by (4.1). If $p_1, \dots, p_P > H + \frac{1}{4}$, we have*

$$(\sqrt{N}V_N(2, \alpha^1), \dots, \sqrt{N}V_N(2, \alpha^P)) \to \mathcal{N}\left(0, \Theta\right),$$

where $(\Theta_{i,j})_{i,j=1\dots,P}$ denotes a $P \times P$ matrix with entries given by

$$\Theta_{n,\,m} = \frac{t^2}{8c_1(H)^2} \sum_{k=l}^{\infty} \left(\sum_{q_1=0}^{l_1} \sum_{q_2=0}^{l_2} \alpha^n_{q_1}\alpha^m_{q_2}|k + q_1 - q_2|^{2H} \right)^2. \tag{4.26}$$

Proof : By (4.13) with $k = 2$, with $c_1(H), c_2(H)$ from (2),

$$\begin{aligned}
\mathbf{E}\left(V_N\left(k, \alpha^n\right) V_N\left(k, \alpha^m\right)\right) &= \frac{N^{4H+2}}{(N-l)^2\left(c_1(H)N + c_2(H)\right)^2} \sum_{i,j=l}^{N} \mathbf{E}\left(I_2\left({C_{i,\alpha^n}}^{\otimes 2}\right) I_2\left({C_{j,\alpha^m}}^{\otimes 2}\right)\right) \\
&= \frac{2N^{4H+2}}{(N-l)^2\left(c_1(H)N + c_2(H)\right)^2} \sum_{i,j=l}^{N} \langle C_{i,\alpha^n}, C_{j,\alpha^m}\rangle^2_{\mathcal{H}}.
\end{aligned}$$

By (4.9), we have for $i, j = l, .., N$

$$\begin{aligned}
\langle C_{i,\alpha^n}, C_{j,\alpha^m}\rangle &= \mathbf{E}\left(U^{\alpha^n}\left(\frac{i}{N}\right) U^{\alpha^m}\left(\frac{j}{N}\right)\right) \\
&= \sum_{q_1=0}^{l_1} \sum_{q_2=0}^{l_2} \alpha^n_{q_1}\alpha^m_{q_2} \mathbf{E}\left(u\left(t, \frac{i-q_1}{N}\right) u\left(t, \frac{j-q_2}{N}\right)\right) \\
&= \sum_{q_1=0}^{l_1} \sum_{q_2=0}^{l_2} \alpha^n_{q_1}\alpha^m_{q_2} \left(\frac{N^{-2H-1}}{2} c_H |j - i + q_1 - q_2|^{2H+1} - \frac{tN^{-2H}}{4} |j - i + q_1 - q_2|^{2H}\right).
\end{aligned}$$

Plugging this into the covariance expression and using similar computations as in [48], we get

$$\begin{aligned}
&\mathbf{E}\left(V_N\left(k, \alpha^n\right) V_N\left(k, \alpha^m\right)\right) \sim_{N\to\infty} \frac{2N^{4H+3}}{(N-l)^2\left(c_1(H)N + c_2(H)\right)^2} \\
&\sum_{k=l}^{N} \left(\frac{N^{-2H-1}}{2} c_H \sum_{q_1=0}^{l_1} \sum_{q_2=0}^{l_2} \alpha^n_{q_1}\alpha^m_{q_2}|k + q_1 - q_2|^{2H+1} - \frac{tN^{-2H}}{4} \sum_{q_1=0}^{l_1} \sum_{q_2=0}^{l_2} \alpha^n_{q_1}\alpha^m_{q_2}|k + q_1 - q_2|^{2H} \right)^2 \\
&= P_1 + P_2 + P_3.
\end{aligned}$$

Using Lemma 3, we get with $p := \min(p_n,\, p_m)$

$$P_1 \sim_{N\to\infty} \frac{c_1(H)}{N}\sum_{v=l}^{N} v^{4H-4p}, \quad P_2 \sim_{N\to\infty} \tfrac{c_2(H)}{N^2}\textstyle\sum_{v=l}^{N} v^{4H-4p+1}, \quad P_3 \sim_{N\to\infty} \frac{c_3(H)}{N^3}\sum_{v=l}^{N} v^{4H-4p+2}.$$

This shows that P_1 is the dominant term and it converges for $H < p + \frac{1}{4}$, while the other terms are negligible. We thus obtain the claimed limit :

$$\mathbf{E}\left(\sqrt{N}V_N\left(k, \alpha^n\right)\sqrt{N}V_N\left(k, \alpha^m\right)\right) \underset{N\to\infty}{\to} \Theta_{n,\, m},$$

where $\Theta_{n,\, m}$ are given by (4.26). The second part of the equivalence in Theorem 9 was proved as a particular case of the CLT for higher powers, and thus the statement of the proposition follows. ■

4.4 Noncentral limit theorem

The asymptotic normality obtained in the previous section holds for any filter of order $p \geq 2$ or for any filter of order $p = 1$ and $H \leq \frac{3}{4}$. It remains to understand what happens in the case $p = 1$ and $H > \frac{3}{4}$. We consider in this section the filter $\alpha = (1, -1)$ (which has order $p = 1$) and we will show that, after a proper normalization, the quadratic variation associated to this filter converges in distribution to a non-Gaussian limit. Let us start by estimating the mean square of the quadratic variation.

Lemma 6. *Let $V_N(2, (1, -1))$ be given by (4.1). If $v_N := \mathbf{E}[V_N(2,\, (1,\, -1))^2]$ and $H > \frac{3}{4}$ we have*

$$N^{4-4H} v_N \to_{N\to\infty} \frac{4K_0}{k_1^2},$$

where K_0 will be given in the proof (see (4.52)) and k_1 from (4.9).

Proof : See the Appendix. ■

Recall that the solution to the wave equation with fractional-white noise can be written as

$$u(t, x) = \int_0^t \int_{\mathbb{R}} G_1(t-s, x-y) W^H(\mathrm{d}s, \mathrm{d}y). \tag{4.27}$$

Let $x_i = \frac{i}{N}$, $i = 0, 1, .., N$ be a partition of the unit interval $[0, 1]$. Denote

$$g_{t,i}(s,\, x) = G_1(t-s, x_{i+1} - x) - G_1(t-s, x_i - x)$$

for $i = 0, 1, .., N-1$ and for $t \geq 0, x \in \mathbb{R}$, with G_1 given by (4.5). We can write

$$u(t, x_{i+1}) - u(t, x_i) = I_1^W(g_{t,i}),$$

where I_1^W represents the multiple integral of order 1 with respect to the fractional-white Gaussian noise W^H. Then we have

$$V_N(2, (1, -1)) := V_N = \frac{1}{N}\sum_{i=1}^{N} \frac{I_2^W(g_{t,i}^{\otimes 2})}{\mathbf{E}\left(u(t, x_{i+1}) - u(t, x_i)\right)^2},$$

and so

$$F_N := \frac{V_N}{\sqrt{v_N}} = I_2(f_N) \text{ with } f_N(x_1, x_2) = \frac{1}{\sqrt{N v_N}} \frac{N^{2H+\frac{1}{2}}}{k_1 N + k_2} \sum_{i=1}^{N} g_{t,i}^{\otimes 2}(x_1, x_2). \tag{4.28}$$

Since in this part we will use the multiple stochastic integrals with respect to the Gaussian noise W^H with covariance (4.3), let us recall some facts about them. Designate by ξ the set of linear combinations of the simple functions $\mathbb{1}_{\{[0,t]\times A\}}$, $t \in [0,T]$, $A \in \mathcal{B}_b(\mathbb{R}^d)$. The canonical Hilbert space $\mathcal{H}_W$ associated to the field W^H, when $H > \frac{1}{2}$, is defined as the closure of the linear space generated by ξ with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_W}$ which is expressed by

$$\langle \mathbb{1}_{\{[0,t]\times A\}}, \mathbb{1}_{\{[0,s]\times B\}} \rangle_{\mathcal{H}_W} := \mathbf{E}(W_t^H(A) W_s^H(B)) = \alpha_H \lambda(A\cap B) \int_0^t \int_0^s |\, u - v \,|^{2H-2} \mathrm{d}u\mathrm{d}v.$$

The scalar product in $\mathcal{H}_W$ is given by

$$\langle f, g \rangle_{\mathcal{H}_W} = \mathbf{E}(W^H(f) W^H(g)) = \alpha_H \int_0^T \int_0^T \int_{\mathbb{R}^d} f(u,x) g(v,x) |\, u - v \,|^{2H-2} \mathrm{d}x\mathrm{d}u\mathrm{d}v.$$

for every $f, g \in \mathcal{H}_W$ such that $\int_0^T \int_0^T \int_{\mathbb{R}^d} |f(u,x)g(v,x)||\, u - v\,|^{2H-2} \mathrm{d}x\mathrm{d}u\mathrm{d}v < \infty$.

It is possible to represent the Wiener integral with respect to W^H as an integral with respect to a white noise field with space-time white noise W via a transfer formula given by

$$\int_0^T \int_{\mathbb{R}} f(s,y) dW^H(s,y) = \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathbb{1}_{\{[0,t]\}}(u) f(u,x) (u-s)_+^{H-\frac{3}{2}} du \right) dW(s,y) \tag{4.29}$$

(see [103] for details).

We will analyze the asymptotic behavior of the sequence F_N defined by (4.28). Since F_N belongs to the second Wiener chaos, its law is completely determined by its cumulants (or equivalently, by its moments). That is, if F, G are elements of the second Wiener chaos then F and G have the same law if and only if they have the same cumulants. Moreover, the convergence of the cumulants to cumulants of an element of the second Wiener chaos implies the convergence in distribution. Let us denote by $k_m(F)$, $m \geq 1$, the mth cumulant of a random variable F. Recall that it is defined as

$$k_m(F) = (-i)^n \frac{\partial^n}{\partial t^n} \ln \mathbf{E}(e^{itF})|_{t=0}.$$

We have the following link between the moments and the cumulants of F : for every $m \geq 1$,

$$k_m(F) = \sum_{\sigma=(a_1,..,a_r)\in\mathcal{P}(\{1,..,n\})} (-1)^{r-1}(r-1)! \mathbf{E}X^{|a_1|} \dots \mathbf{E}X^{|a_r|} \tag{4.30}$$

if $F \in L^m(\Omega)$, where $\mathcal{P}(b)$ is the set of all partitions of b. In particular, for centered random variables F we have $k_1(F) = \mathbf{E}F, k_2(F) = \mathbf{E}F^2, k_3(F) = \mathbf{E}F^3, k_4 = \mathbf{E}F^4 - (\mathbf{E}F^2)^2$. In the particular situation when $G = I_2(f)$ its cumulants can be computed as (see e.g. [65], Proposition 7.2 or [103])

$$k_m(G) = 2^{m-1}(m-1)! \int_{\mathbb{R}^m} du_1 \dots du_m f(u_1,u_2) f(u_2,u_3) \dots f(u_{m-1},u_m) f(u_m,u_1). \tag{4.31}$$

Based on the above formula (4.31), we obtain the limit in distribution of (4.28).

Theorem 11. *Let F_N be given by (4.28) with $H > \frac{3}{4}$. Then the sequence $(F_N)_{N\geq 1}$ converges in distribution to a random variable F whose law is determined by the cumulants explicitly defined in the proof (see (4.33) and (4.34)).*

Proof : Note first that by the transfer formula (4.29) $W^H(g_{t,i})$ has a representation as $W(\tilde{g}_{t,i})$ for some (explicitly known) function $\tilde{g}_{t,i}$, where W is a two-dimensional Gaussian noise. Therefore,

$k_1(F_N) = 0, k_2(F_N) = 1$, the above formula for cumulants (4.31) can be applied and we obtain for $m \geq 3$

$$\begin{aligned} k_m(F_N) &= 2^{m-1}(m-1)! \left(\frac{1}{\sqrt{N v_N}} \frac{N^{2H+\frac{1}{2}}}{k_1 N + k_2} \right)^m \\ &\int_{\mathbb{R}^m} \left(\sum_{j_1=1}^{N} \tilde{g}_{t,j_1}^{\otimes 2}(x_1, x_2) \right) \left(\sum_{j_2=1}^{N} \tilde{g}_{t,j_2}^{\otimes 2}(x_2, x_3) \right) \dots \left(\sum_{j_m=1}^{N} \tilde{g}_{t,j_m}^{\otimes 2}(x_m, x_1) \right) dx_1 \dots dx_m \\ &= 2^{m-1}(m-1)! \left(\frac{1}{\sqrt{N v_N}} \frac{N^{2H+\frac{1}{2}}}{k_1 N + k_2} \right)^m \\ &\sum_{j_1,..,j_m=1}^{N} \left(\int_{\mathbb{R}} \tilde{g}_{t,j_1}(x) \tilde{g}_{t,j_2}(x) dx \right) \left(\int_{\mathbb{R}} \tilde{g}_{t,j_2}(x) \tilde{g}_{t,j_3}(x) dx \right) \dots \left(\int_{\mathbb{R}} \tilde{g}_{t,j_m}(x) \tilde{g}_{t,j_1}(x) dx \right). \end{aligned}$$

We use the isometry formula for multiple integrals (4.41) with respect to W as well as the transfer formula in order to get

$$\begin{aligned} \int_{\mathbb{R}} \tilde{g}_{t,j_1}(x) \tilde{g}_{t,j_2}(x) dx &= \mathbf{E}\left(u(t, x_{i+1}) - u(t, x_i)\right)\left(u(t, x_{j+1}) - u(t, x_j)\right) \\ &= k_1 \Phi_H \left(\frac{i-j}{N} \right) + k_2 \Phi_{H+\frac{1}{2}} \left(\frac{i-j}{N} \right), \end{aligned}$$

$$\Phi_H(k) = \frac{1}{2} \left(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H} \right), \quad k \in \mathbb{R}, \tag{4.32}$$

and we obtain

$$\begin{aligned} & k_m(F_N) \\ = \; & 2^{m-1}(m-1)! \left(\frac{1}{\sqrt{N v_N}} \frac{N^{2H+\frac{1}{2}}}{k_1 N + k_2} \right)^m \\ & \sum_{j_1,..,j_m=1}^{N} \left[k_1 \Phi_H \left(\frac{j_1 - j_2}{N} \right) + k_2 \Phi_{H+\frac{1}{2}} \left(\frac{j_1 - j_2}{N} \right) \right] \dots \\ & \times \left[k_1 \Phi_H \left(\frac{j_m - j_1}{N} \right) + k_2 \Phi_{H+\frac{1}{2}} \left(\frac{j_m - j_1}{N} \right) \right]. \end{aligned}$$

By Lemma 6,

$$\begin{aligned} & k_m(F_N) \\ \sim_{N \to \infty} \; & 2^{m-1}(m-1)!(4K_0)^{-\frac{m}{2}} N^m \\ & \sum_{j_1,..,j_m=1}^{N} \left[k_1 \Phi_H \left(\frac{j_1 - j_2}{N} \right) + k_2 \Phi_{H+\frac{1}{2}} \left(\frac{j_1 - j_2}{N} \right) \right] \dots \\ & \times \left[k_1 \Phi_H \left(\frac{j_m - j_1}{N} \right) + k_2 \Phi_{H+\frac{1}{2}} \left(\frac{j_m - j_1}{N} \right) \right]. \end{aligned}$$

By writing

$$\Phi_H \left(\frac{i-j}{N} \right) = H(2H-1) \int_{\frac{i}{N}}^{\frac{i+1}{N}} \int_{\frac{j}{N}}^{\frac{j+1}{N}} |u-v|^{2H-2} du dv,$$

and similarly

$$\Phi_{H+\frac{1}{2}} \left(\frac{i-j}{N} \right) = H(2H+1) \int_{\frac{i}{N}}^{\frac{i+1}{N}} \int_{\frac{j}{N}}^{\frac{j+1}{N}} |u-v|^{2H-1} du dv,$$

we get, for any $m \geq 3$,

$$\begin{aligned}
& k_m(F_N) \\
\sim_{N\to\infty} \quad & 2^{m-1}(m-1)!(4K_0)^{-\frac{m}{2}} N^m \sum_{j_1,..,j_m=1}^{N} \\
& \left(\int_0^1\int_0^1 dudv \left[k_1 H(2H-1)N^{-2H}|u-v+j_1-j_2|^{2H-2}\right.\right. \\
& \left.\left.+k_2 H(2H+1)N^{-2H-1}|u-v+j_1-j_2|^{2H-1}\right]\right) \\
& \dots \\
& \dots \\
& \left(\int_0^1\int_0^1 dudv \left[k_1 H(2H-1)N^{-2H}|u-v+j_m-j_1|^{2H-2}\right.\right. \\
& \left.\left.+k_2 H(2H+1)N^{-2H-1}|u-v+j_m-j_1|^{2H-1}\right]\right).
\end{aligned}$$

Next, we write

$$N^{-2H}|u-v+j_m-j_1|^{2H-2} = N^{-2}\left|\frac{j_1-j_2}{N}\right|^{2H-2}\left|1+\frac{u-v}{j_1-j_2}\right|^{2H-2},$$

and

$$N^{-2H-1}|u-v+j_1-j_2|^{2H-1} = N^{-2}\left|\frac{j_1-j_2}{N}\right|^{2H-1}\left|1+\frac{u-v}{j_1-j_2}\right|^{2H-1},$$

and obtain

$$\begin{aligned}
k_m(F_N) \quad \sim_{N\to\infty} \quad & 2^{m-1}(m-1)!(4K_0)^{-\frac{m}{2}} N^{-m} \sum_{j_1,..,j_m=1}^{N} \\
& \int_0^1\int_0^1 dudv \left[k_1 H(2H-1)\left|\frac{j_1-j_2}{N}\right|^{2H-2}\left|1+\frac{u-v}{j_1-j_2}\right|^{2H-2}\right. \\
& \left.+k_2 H(2H+1)\left|\frac{j_1-j_2}{N}\right|^{2H-1}\left|1+\frac{u-v}{j_1-j_2}\right|^{2H-1}\right] \\
& \dots \\
& \dots \\
& \int_0^1\int_0^1 dudv \left[k_1 H(2H-1)\left|\frac{j_m-j_1}{N}\right|^{2H-2}\left|1+\frac{u-v}{j_m-j_1}\right|^{2H-2}\right. \\
& \left.+k_2 H(2H+1)\left|\frac{j_m-j_1}{N}\right|^{2H-1}\left|1+\frac{u-v}{j_m-j_1}\right|^{2H-1}\right].
\end{aligned}$$

We claim that

$$\begin{aligned}
k_m(F_N) \quad \sim_{N\to\infty} \quad & 2^{m-1}(m-1)!(4K_0)^{-\frac{m}{2}} N^{-m} \sum_{j_1,..,j_m=1}^{N} \\
& \left[k_1 H(2H-1)\left|\frac{j_1-j_2}{N}\right|^{2H-2} + k_2 H(2H+1)\left|\frac{j_1-j_2}{N}\right|^{2H-1}\right] \\
& \dots \\
& \dots \\
& \left[k_1 H(2H-1)\left|\frac{j_m-j_1}{N}\right|^{2H-2} + k_2 H(2H+1)\left|\frac{j_m-j_1}{N}\right|^{2H-1}\right].
\end{aligned}$$

This follows by a standard procedure (see [65] or [103]) from the Taylor expansion in the vicinity of $x = 0$ of the functions

$$1-(1+x)^{2H-2} \text{ and } 1-(1+x)^{2H-1}$$

and by the dominated convergence theorem. Therefore, for $m \geq 3$

$$\begin{aligned} k_m(F_N) \quad \to_{N\to\infty} \quad & 2^{m-1}(m-1)!(4K_0)^{-\frac{m}{2}} \int_{[0,1]^m} dx_1...dx_m \\ & \left(k_1H(2H-1)|x_1-x_2|^{2H-2}+k_2H(2H+1)|x_1-x_2|^{2H-1}\right) \\ & \dots \\ & \left(k_1H(2H-1)|x_m-x_2|^{2H-2}+k_2H(2H+1)|x_m-x_1|^{2H-1}\right). \end{aligned}$$

Notice that the above integral is finite by Lemma 3.3 in [6]. Also, clearly

$$k_1(F_N) = 0 \text{ and } k_2(F_N) = 1.$$

Since F_N belongs to the second Wiener chaos, the convergence of cumulants determines the convergence of F_N in law to a random variable F with cumulants given by

$$\begin{aligned} k_m(F) \quad = \quad & 2^{m-1}(m-1)!(4K_0)^{-\frac{m}{2}} \int_{[0,1]^m} dx_1...dx_m \\ & \left(k_1H(2H-1)|x_1-x_2|^{2H-2}+k_2H(2H+1)|x_1-x_2|^{2H-1}\right) \\ & \dots \\ & \left(k_1H(2H-1)|x_m-x_1|^{2H-2}+k_2H(2H+1)|x_m-x_1|^{2H-1}\right) \end{aligned} \tag{4.33}$$

for $m \geq 3$ and

$$k_1(F) = 0 \text{ and } k_2(F) = 1. \tag{4.34}$$

The existence of such a limit is ensured by the Fréchet-Shohat theorem : It follows from the convergence of cumulants that also all the moments of F_N converge to some real numbers M_m, $m \in \mathbb{N}$, as N tends to infinity. Moreover, by hypercontractivity (4.44) the mth absolute moments of F_N are bounded by $(m-1)^m$. Therefore, also the limits of the moment sequences will be bounded by $(m-1)^m$, which means that the growth condition $\overline{\lim}_{m\to\infty}(\frac{1}{m!}|M_m|)^{1/m} < \infty$ is satisfied, thus yielding the existence of a limiting distribution with the cumulants obtained above. ■

Notice that the limit law with cumulants (4.33) and (4.34) is related to the Rosenblatt distribution but is more complex. For instance, if the constant k_2 vanished in (4.33), then we would have obtained a Rosenblatt distribution in the limit.

4.5 Estimation of the Hurst parameter H

We will apply the theoretical results from Section 4.3 in order to construct and analyze several estimators for the Hurst index of the mild solution (4.4) to the wave equation (4.2). It is worth to emphasize that the estimators are based on the observations of the process u at a fixed time and at discrete points in space.

We will define two kinds of estimators for the Hurst parameter. For the first kind we will consider the observation time t of our equation to be known, and the estimators obtained will be asymptotically normal with the rate of convergence of order $\sqrt{N}\log(N)$ for $H < p - \frac{1}{4}$. In the second case we develop an estimator for H if the time $t > 1$ is not known. This estimator will also be asymptotically normal, but with a slower rate of convergence, namely $\sqrt{N}$. Both kinds of estimators are strongly consistent.

4.5.1 Estimators for known t

We follow the standard procedure from [28] or [21] to construct our estimators. Let us first define the k-th empirical absolute moment of discrete varations of the mild solution $u(t,x)$ for fixed time $t>1$ and a filter α as follows :

$$S_N(k,\alpha) \quad = \quad \frac{1}{N-l}\sum_{i=l}^{N-1}\left|U^{\alpha}\left(\frac{i}{N}\right)\right|^k, \tag{4.35}$$

with $U^{\alpha}\left(\frac{i}{N}\right)$ defined in (4.8). Since $U^{\alpha}\left(\frac{i}{N}\right)$ is Gaussian, we have $\mathbf{E}\left[\left|U^{\alpha}\left(\frac{i}{N}\right)\right|^k\right] = \left(\pi_H^{\alpha,\,N}(0)\right)^{\frac{k}{2}} E_k$, where E_k denotes the k-th absolute moment of a standard Gaussian random variable, and therefore we obtain

$$\mathbf{E}[S_N(k,\,\alpha)] = \left(\pi_H^{\alpha,\,N}(0)\right)^{\frac{k}{2}} E_k.$$

Thus, for a given k, by replacing $\mathbf{E}[S_N(k,\,\alpha)]$ by $S_N(k,\,\alpha)$ we obtain an estimator for H that is a pointwise solution to the equation

$$S_N(k,\,\alpha)^{\frac{2}{k}} - E_k^{\frac{2}{k}}\pi_x^{\alpha,\,N}(0) = 0$$

with respect to x. Recall that (see (4.9))

$$\pi_x^{\alpha,\,N}(0) = \frac{t}{2N^{2x}}\Phi_{x,\,\alpha}(0) - \frac{c_x}{N^{2x+1}}\Phi_{x+\frac{1}{2},\,\alpha}(0),$$

and we denote

$$c_1(x) := \Phi_{x,\,\alpha}(0) = -\frac{1}{2}\sum_{q,\,r=0}^{l}\alpha_q\alpha_r|q-r|^{2x}, \qquad c_2(x) = c_x\Phi_{x+\frac{1}{2},\,\alpha}(0).$$

Note that for large N the function $g(x) := \pi_x^{\alpha,\,N}(0)$ is invertible. In order to see this we consider the derivative

$$g'(x) = \frac{t}{2}\left(\frac{c_1'(x)}{N^{2x}} - \frac{2\log(N)c_1(x)}{N^{2x}}\right) - \left(\frac{c_2'(x)}{N^{2x+1}} - \frac{2\log(N)c_2(x)}{N^{2x+1}}\right).$$

As shown in [28], the first expression inside the parentheses becomes negative for large N, and since it is the asymptotically dominating term, also the whole function will become negative for N large enough. Therefore, for such N the function g is strictly decreasing and we can define estimators by inverting it :

$$\widehat{H}_{k,\,N} := \left(\pi_{\cdot}^{\alpha,\,N}(0)\right)^{-1}\left(\left(\frac{S_N(k,\,\alpha)}{E_k}\right)^{\frac{2}{k}}\right). \tag{4.36}$$

Another estimator can be obtained by inverting only the dominant part of the empirical absolute moment. Notice that asymptotically $\pi_x^{\alpha,\,N}(0)$ is equal to $\frac{t}{2N^{2x}}\Phi_{x,\,\alpha}(0) =: \bar{g}(x)$, which is easier to invert than its exact counterpart. This motivates the definition of another class of estimators,

$$\bar{H}_{k,\,N} := \bar{g}^{-1}\left(\left(\frac{S_N(k,\,\alpha)}{E_k}\right)^{\frac{2}{k}}\right). \tag{4.37}$$

We show that the two estimators constructed above are consistent and we give their limit behavior in distribution.

Proposition 31. *The estimators $\widehat{H}_{N,k}$ and $\bar{H}_{N,k}$ given by (4.36) and (4.37) of the Hurst parameter $H \geq \frac{1}{2}$ are strongly consistent. Moreover, with $v_N^{(k)} := \mathbf{E}[V_N(k,\alpha)^2]$, for $H \leq p - \frac{1}{4}$ we have*

$$\frac{k\log(N)}{\sqrt{v_N^{(k)}}}\left(H - \widehat{H}_{N,k}\right) \xrightarrow{d} \mathcal{N}(0,1),$$

and for $H > \frac{3}{4}$, $\alpha = (1,-1)$, $k = 2$

$$\frac{2\log(N)}{\sqrt{v_N^{(2)}}}\left(H - \widehat{H}_{N,2}\right) \xrightarrow{d}_{N\to\infty} F,$$

where F is the random variable from Proposition 11. The same statements hold for $\bar{H}$.

Proof : Since for every $k \geq 2$,

$$v_N^k = \begin{cases} O(1/N) & \text{if } H < p - \frac{1}{4}\ , \\ O\left(\frac{\log(N)}{N}\right) & \text{if } H = \frac{3}{4},\ p = 1, \\ O\left(\frac{1}{N^{2-2H}}\right) & \text{if } H > \frac{3}{4},\ \alpha = (1,-1),\ k = 2, \end{cases} \tag{4.38}$$

the almost sure convergence to zero of V_N follows by hypercontractivity with a Borel-Cantelli argument, see e.g. [99]. Due to the fact that the functions g and $\bar{g}$ are asymptotically equal we obtain the asymptotic equality of $\bar{H}_{N,k}$ and $\widehat{H}_{N,k}$ and thus also strong consistency of $\bar{H}_{N,k}$.

For the asymptotic behaviour we can refer to the calculations from [28] and obtain

$$V_N(k,\alpha) := V_N = k\log(N)(H - \widehat{H}_{N,k})(1 + o(1)),$$

which means that by Slutsky's Lemma we will get

$$\frac{k\log(N)}{\sqrt{v_N^{(k)}}}\left(H - \widehat{H}_{N,k}\right) \xrightarrow{d} \mathcal{N}(0,1)$$

for $H \leq p - \frac{1}{4}$. For $H < p - \frac{1}{4}$ this implies in particular that

$$k\log(N)\sqrt{N}\left(H - \widehat{H}_{N,k}\right) \xrightarrow{d} \mathcal{N}(0,\sigma^2),$$

with σ^2 defined in Lemma 4. For $H > \frac{3}{4}$, $\alpha = (1,-1)$, $k = 2$ the relation yields

$$\frac{2\log(N)}{\sqrt{v_N}}\left(H - \widehat{H}_{N,2}\right) \xrightarrow{d} F$$

for F given above. The same results follow for $\bar{H}$ due to its asymptotic equality to $\widehat{H}_{N,k}$. ■

Remark 15. *Note that this result provides the following speeds of convergence (see 4.38) : $\sqrt{N}\log(N)$ for $H < p - \frac{1}{4}$, $\sqrt{N}\sqrt{\log(N)}$ for $H = \frac{3}{4}$, $p = 1$ and $N^{2-2H}\log(N)$ for $H > \frac{3}{4}$, $\alpha = (1,-1)$, $k = 2$.*

4.5.2 An estimator for unknown t

Assume that the time $t > 1$ at which the solution (4.4) is observed is not known. Similarly to [45], if two sequences $(a_i^{(1)})_{i\in\{0,\ldots,p\}}$ and $(a_i^{(2)})_{i\in\{0,\ldots,2p\}}$ are considered, where $a^{(2)}$ is obtained by "thinning" the sequence $a^{(1)}$ (i.e., $a_{2k}^{(2)} := a_k^{(1)}$ for $k \in \{0,\ldots,p\}$ and zero otherwise), then it follows that

$$\Phi_{H,a^{(2)}}(0) = 2^{2H}\Phi_{H,a^{(1)}}(0) \text{ and } \Phi_{H+\frac{1}{2},a^{(2)}}(0) = 2^{2H+1}\Phi_{H+\frac{1}{2},a^{(1)}}(0),$$

which implies that for large N we have approximately $\pi_H^{a^{(2)},\,N}(0) \sim 2^{2H}\pi_H^{a^{(1)},\,N}(0)$. This, in turn, can be transferred to S_N :

$$\mathbf{E}[S_N(k,\,a^{(2)})] = \left(\pi_H^{a^{(2)},\,N}(0)\right)^{\frac{k}{2}} E_k \sim 2^{Hk}\left(\pi_H^{a^{(1)},\,N}(0)\right)^{\frac{k}{2}} E_k = 2^{Hk}\mathbf{E}[S_N(k,\,a^{(1)})].$$

This motivates another estimator for H defined by

$$\tilde{H}_N := \frac{1}{k}\log_2\left(\frac{S_N(k,\,a^{(2)})}{S_N(k,\,a^{(1)})}\right). \tag{4.39}$$

Its limit behavior is given below.

Proposition 32. *The estimator $\tilde{H}_N$ (4.39) is strongly consistent for all $H \geq \frac{1}{2}$. Moreover for $H < p - \frac{1}{4}$, we have*

$$\sqrt{N}(\tilde{H}_N - H) \xrightarrow{d} N(0,\,\sigma^2)$$

with $\sigma > 0$.

Proof : It follows from the fact that $V_N \to_{N\to\infty} 0$ almost surely that S_N converges almost surely to its expectation. Thus, strong consistency is clear by construction of $\tilde{H}$. The multivariate convergence statement yields asymptotic normality by the delta method, similarly to [28]. ■

Remark 16. *In our work we assume that the solution to the stochastic wave equation is observed at discrete points in space and at a fixed time (which can be known or unknown), and we estimate the Hurst parameter of the model by using spatial variations. A related problem of interest would be to estimate the Hurst parameter by assuming the the solution is observed at discrete times at a fixed point in space. To this end, a careful analysis of the correlation of the solution is time is needed. This idea has been used in several papers (e.g. [13], [83], [99]) for the estimations of the drift or of the volatility parameter for SPDEs. A more complex approach is to construct estimators based on the space-time grid observation of the process, but in this case the analysis of the behavior of the generalized variations is more technical and it is still an open problem (even in the case $H = \frac{1}{2}$ or for the heat equation).*

4.5.3 Numerical computations and simulation experiments

In this section we conduct simulations of the solution process and compare numerical performances of different estimators introduced in the previous section. More specifically, we are going to analyse the behaviour of $\bar{H}_{2,\,N}$ for filters $(1,\,-1)$ as well as $(1,\,-2,\,1)$, that of its exact counterpart $\hat{H}_{2,\,N}$ for the second filter as well as that of $\tilde{H}_N$ for different values of H. For $N = 1000$ and $t = 3$ we get the following results for MSE computed from 100 iterations :

	$H = 0.51$	$H = 0.7$	$H = 0.95$
$\bar{H}_{2,\,N}(1,\,-1)$	$1.02\cdot 10^{-5}$	$1.61\cdot 10^{-5}$	0.001
$\bar{H}_{2,\,N}(1,\,-2,\,1)$	$1.2\cdot 10^{-5}$	$9.626\cdot 10^{-6}$	$1.98\cdot 10^{-6}$
$\hat{H}_{2,\,N}(1,\,-2,\,1)$	$1.2\cdot 10^{-5}$	$9.634\cdot 10^{-6}$	$1.99\cdot 10^{-6}$
$\tilde{H}_N$	0.002	0.001	0.001

The estimator $\tilde{H}_N$ performs the worst. This can be explained heuristically by the fact that it contains two sources of error instead of one, this being the practical trade-off in the case where time t is not available. Another interesting observation is that the exact estimator $\tilde{H}$ is not performing better than the estimator $\bar{H}$ which uses the inverse of an approximation of the actual function, which justifies the use of the simpler version in applications.

True value H	Mean $\bar{H}_{2,N}(1, -1)$	Mean $\bar{H}_{2,N}(1, -2, 1)$	Mean $\hat{H}_{2,N}(1, -2, 1)$	Mean$\widetilde{H}_N$
0.51	0.5107118	0.5138851	0.5110081	0.5110082
0.55	0.5499827	0.5362797	0.549677	0.549678
0.60	0.5997487	0.6007376	0.5999698	0.5999722
0.65	0.6498786	0.6510065	0.6502865	0.6502909
0.70	0.7005558	0.6925	0.7003125	0.7003196
0.75	0.7500486	0.7482407	0.7499587	0.74997
0.80	0.8005769	0.7966326	0.7998019	0.7998186
0.85	0.8512704	0.8517664	0.8500505	0.8500754
0.90	0.9042009	0.8927607	0.8997257	0.8997638
0.95	0.9587621	0.9540507	0.9498974	0.9499602
0.99	1.01826	0.9959974	0.9898137	0.9899168

Tableau 4.1 – Mean of the estimators for 100 simulations

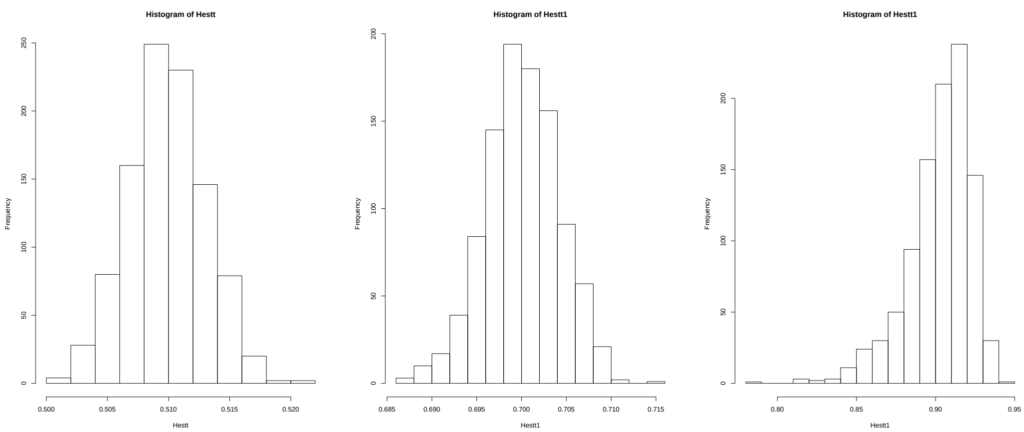

Figure 4.1 – Histograms for $H = 0.51$, 0.7 and 0.9 respectively.

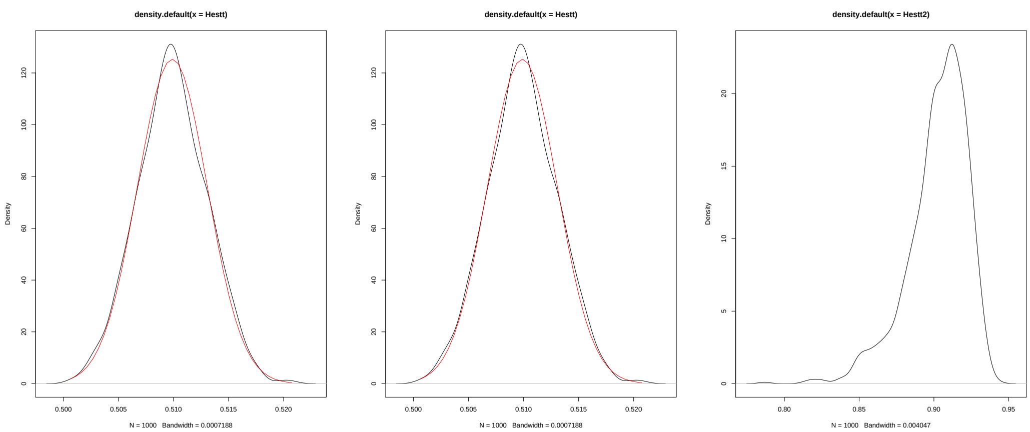

Figure 4.2 – Normal fits of empirical densities for $H = 0.51$, 0.7, density plot for 0.9.

The Figures 4.1 and 4.2 show the change in the limiting distribution, while the boxplots in Figure 4.3 illustrate the changes in the speed of convergence indicated in the discussion for $\bar{H}_{2,N}(1,-1)$ and provide a comparison to the rates of convergence for the other three estimators.

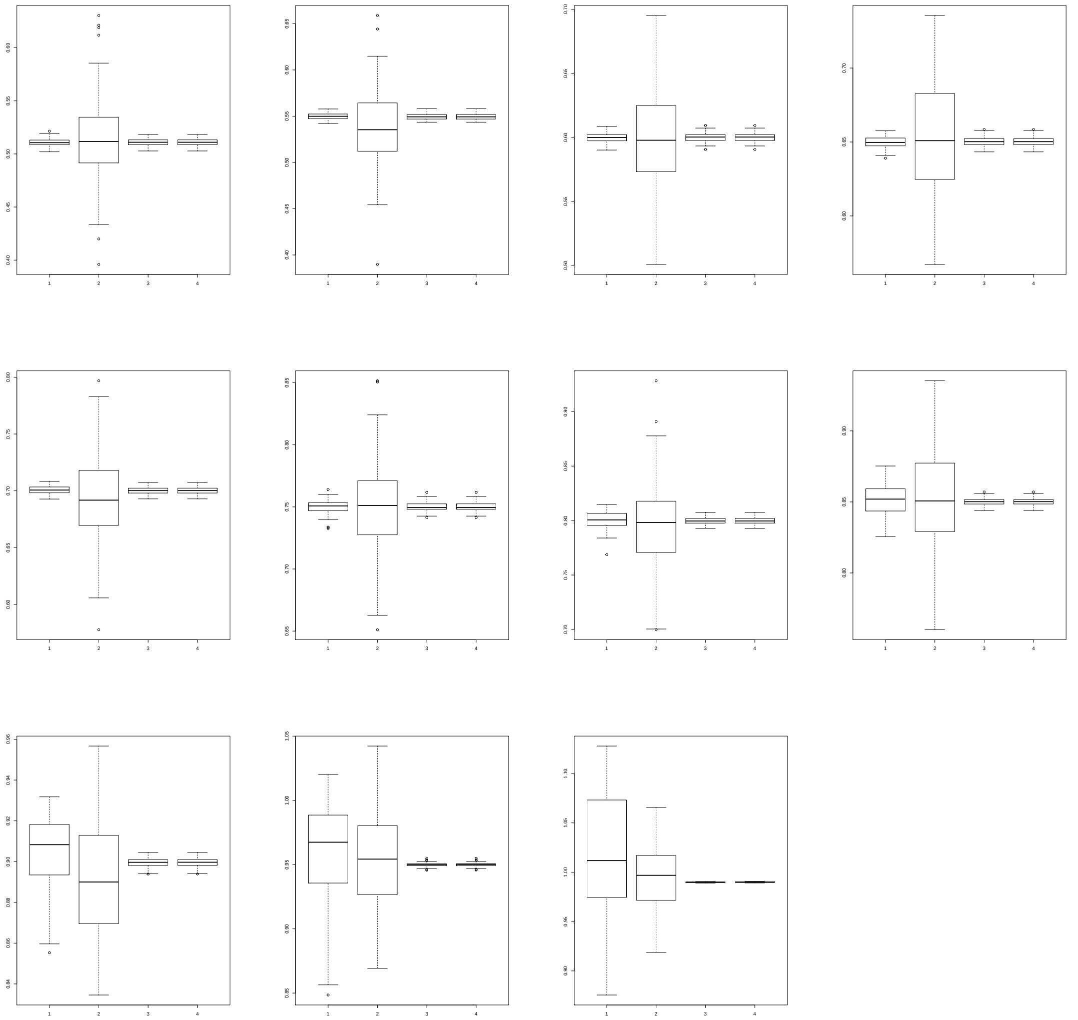

FIGURE 4.3 – Boxplots of $\bar{H}_{2,N}(1,-1)$, $\bar{H}_{2,N}(1,-2,1)$, $\hat{H}_{2,N}(1,-2,1)$, $\tilde{H}_N$ for the values of H listed above.

4.6 Appendix

4.6.1 The basics of Malliavin calculus

The basic tools from the analysis on Wiener space are presented in this section. We will focus on some elementary facts about multiple stochastic integrals. We refer to [74] for a complete review on the topic.

Consider $\mathcal{H}$, a real separable infinite-dimensional Hilbert space with its associated inner product $\langle .,. \rangle_{\mathcal{H}}$, and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, which is a centered Gaussian family of random variables such that $\mathbf{E}\left(B(\varphi)B(\psi)\right) = \langle \varphi, \psi \rangle_{\mathcal{H}}$ for every $\varphi, \psi \in \mathcal{H}$. Denote by I_q the qth multiple stochastic integral with respect to B, which is an isometry between the Hilbert space $\mathcal{H}^{\odot q}$ (symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{q!}}\|\cdot\|_{\mathcal{H}^{\otimes q}}$ and the Wiener chaos of order q, which is defined as the closed linear span of the random variables $H_q(B(\varphi))$ where $\varphi \in \mathcal{H}$, $\|\varphi\|_{\mathcal{H}} = 1$ and H_q is the Hermite polynomial of degree $q \geq 1$ defined by :

$$H_q(x) = (-1)^q \exp\left(\frac{x^2}{2}\right) \frac{\mathrm{d}^q}{\mathrm{d}x^q}\left(\exp\left(-\frac{x^2}{2}\right)\right), \ x \in \mathbb{R}. \tag{4.40}$$

The isometry of multiple integrals can be written as follows : for $p, \ q \geq 1$, $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$

$$\mathbf{E}\Big(I_p(f)I_q(g)\Big) = \begin{cases} q!\langle \tilde{f}, \tilde{g} \rangle_{\mathcal{H}^{\otimes q}} & \text{if } p = q, \\ 0 & \text{otherwise.} \end{cases} \tag{4.41}$$

It also holds that :

$$I_q(f) = I_q\big(\tilde{f}\big),$$

where $\tilde{f}$ denotes the canonical symmetrization of f and is defined by

$$\tilde{f}(x_1, \ldots, x_q) = \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} f(x_{\sigma(1)}, \ldots, x_{\sigma(q)}),$$

where the sum runs over all permutations σ of $\{1, \ldots, q\}$.

We have the following product formula : if $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$, then

$$I_p(f)I_q(g) \quad = \quad \sum_{r=0}^{p\wedge q} r! \binom{p}{r}\binom{q}{r} I_{p+q-2r}\left(f\tilde{\otimes}_r g\right)$$

We need to recall the formula of contraction of elements of tensor products of Hilbert spaces. Consider $\{e_k, k \geq 1\}$ an orthonormal basis of $\mathcal{H}$ and let $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$. For $r = 1, \ldots, p \wedge q$, the rth contraction $f \otimes_r g$ is an element of $\mathcal{H}^{\otimes(p+q-2r)}$, which is defined by

$$(f \otimes_r g) \quad = \quad \sum_{j_1,\ldots,j_p=1}^{\infty} < f, e_{k_1} \otimes e_{k_2} \otimes \ldots e_{k_r} >_{\mathcal{H}^{\otimes r}} \otimes < g, e_{k_1} \otimes e_{k_2} \otimes \ldots e_{k_r} >_{\mathcal{H}^{\otimes r}} . \tag{4.42}$$

In the particular case when $\mathcal{H} = L^2(T)$, the rth contraction $f \otimes_r g$ is the element of $\mathcal{H}^{\otimes(p+q-2r)}$ which is defined by

$$\begin{aligned} &(f \otimes_r g)(s_1, \ldots, s_{p-r}, t_1, \ldots, t_{q-r}) \\ &= \textstyle\int_{T^r} \mathrm{d}u_1 \ldots \mathrm{d}u_r f(s_1, \ldots, s_{p-r}, u_1, \ldots, u_r) g(t_1, \ldots, t_{q-r}, u_1, \ldots, u_r) \end{aligned} \tag{4.43}$$

for every $f \in L^2(T^p)$, $g \in L^2(T^q)$ and $r = 1, \ldots, p \wedge q$. An important property of finite sums of multiple integrals is the hypercontractivity. Namely, if $F = \sum_{k=0}^{n} I_k(f_k)$ with $f_k \in \mathcal{H}^{\otimes k}$ then

$$\mathbf{E}|F|^p \leq C_p \left(\mathbf{E}F^2\right)^{\frac{p}{2}} \tag{4.44}$$

for every $p \geq 2$.

We denote by D the Malliavin derivative operator that acts on cylindrical random variables of the form $F = g(B(\varphi_1), \ldots, B(\varphi_n))$, where $n \geq 1$, $g : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with compact support and $\varphi_i \in \mathcal{H}$. This derivative is an element of $L^2(\Omega, \mathcal{H})$ and it is defined as

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(B(\varphi_1), \ldots, B(\varphi_n))\varphi_i.$$

4.6.2 Proofs of some technical results

Proof of Lemma 4

From (4.14), we get

$$\begin{aligned}(2q)!\|f_{N,2q}\|^2_{\mathcal{H}^{\otimes 2q}} &= \frac{(c^k_{2q})^2}{(2q)!}\frac{1}{(N-l)^2}\sum_{i,j=l}^{N}\frac{\langle C_{i,\alpha}, C_{j,\alpha}\rangle^{2q}_{\mathcal{H}}}{(\pi^{\alpha,N}_H(0))^{2q}} \\ &= \frac{1}{(N-l)^2}\frac{(c^k_{2q})^2}{(2q)!}\sum_{i,j=l}^{N}\left(\rho^{\alpha,N}_H(j-i)\right)^{2q},\end{aligned}$$

where we used the notation

$$\rho^{\alpha,N}_H(v) = \frac{\pi^{\alpha,N}_H(v)}{\pi^{\alpha,N}_H(0)} \text{ for } v \in \mathbb{Z}. \tag{4.45}$$

Next, we write

$$\frac{1}{N-l}\sum_{i,j=l}^{N}\left(\rho^{\alpha,N}_H(j-i)\right)^{2q} = \sum_{v\in\mathbb{Z}}\left(\rho^{\alpha,N}_H(v)\right)^{2q} 1_{\{|v|\leq N-l\}}\frac{N-|v|-l}{N-l},$$

and thus

$$(N-l)(2q)!\|f_{N,2q}\|^2_{\mathcal{H}^{\otimes 2q}} = \frac{(c^k_{2q})^2}{(2q)!}\sum_{v\in\mathbb{Z}}\left(\rho^{\alpha,N}_H(v)\right)^{2q} 1_{\{|v|\leq N-l\}}\frac{N-|v|-l}{N-l}. \tag{4.46}$$

Using the expression

$$\rho^{\alpha,N}_H(v) = \frac{k_1\Phi_{H,\alpha}(v)N^{-2H} + k_2\Phi_{H+\frac{1}{2},\alpha}(v)N^{-2H-1}}{k_1\Phi_{H,\alpha}(0)N^{-2H} + k_2\Phi_{H+\frac{1}{2},\alpha}(0)N^{-2H-1}} = \frac{\Phi_{H,\alpha}(v) + a_N(v)}{\Phi_{H,\alpha}(0) + a_N(0)}$$

with

$$a_N(v) = \frac{k_2}{k_1 N}\Phi_{H+\frac{1}{2},\alpha}(v) \tag{4.47}$$

we can write, with $\varphi_{H,\alpha}$ and $\rho^{\alpha,N}_H$ given by (4.16) and (4.45) respectively,

$$b_{N,H}(v) := \rho^{\alpha,N}_H(v) - \varphi_{H,\,\alpha}(v) \tag{4.48}$$

and remark that due to Lemma 3 for v large enough

$$|b_{N,H}(v)| \underset{|v|\to\infty}{\sim} \left|a_N(v)\frac{1}{\Phi_{H,\,\alpha}(0) + a_N(0)}\right| \leq C\frac{1}{N}v^{2H+1-2p}, \tag{4.49}$$

where $C>0$ does not depend on N, v. With this notation we can write

$$\begin{aligned}
(N-l)(2q)!\|f_{N,2q}\|^2_{\mathcal{H}^{\otimes 2q}} &= \frac{(c^k_{2q})^2}{(2q)!}\sum_{v\in\mathbb{Z}}(\varphi_{H,\alpha}(v)+b_{N,H}(v))^{2q}\,1_{\{|v|\le N-l\}}\frac{N-|v|-l}{N-l}\\
&= \frac{(c^k_{2q})^2}{(2q)!}\sum_{v\in\mathbb{Z}}\sum_{m=0}^{2q}\binom{2q}{m}\varphi_{H,\alpha}(v)^m(b_{N,H}(v))^{2q-m}1_{\{|v|\le N-l\}}\frac{N-|v|-l}{N-l}\\
&= \frac{(c^k_{2q})^2}{(2q)!}\sum_{v\in\mathbb{Z}}\varphi_{H,\alpha}(v)^{2q}1_{\{|v|\le N-l\}}\frac{N-|v|-l}{N-l}+r_{N,q,1},
\end{aligned}$$

with

$$r_{N,q,1}=\frac{(c^k_{2q})^2}{(2q)!}\sum_{v\in\mathbb{Z}}\sum_{m=0}^{2q-1}\binom{2q}{m}\varphi_{H,\alpha}(v)^m(b_{N,H}(v))^{2q-m}1_{\{|v|\le N-l\}}\frac{N-|v|-l}{N-l}. \tag{4.50}$$

Clearly, by the dominated convergence theorem

$$\frac{(c^k_{2q})^2}{(2q)!}\sum_{v\in\mathbb{Z}}\varphi_{H,\alpha}(v)^{2q}1_{\{|v|\le N-l\}}\frac{N-|v|-l}{N-l}\to_{N\to\infty}\sigma^2_{2q},$$

which by Lemma 3 is finite if $p=1,\ H<1-\frac{1}{4q}$, and for all $H\in[1/2,1)$ in the other cases. For $q=p=1,\ H=3/4$,

$$\frac{1}{\log(N-l)}\sum_{v\in\mathbb{Z}}\varphi_{H,\alpha}(v)^2 1_{\{|v|\le N-l\}}\frac{N-|v|-l}{N-l}$$

converges to a positive constant and thus (4.17) is obtained.

In order to conclude it remains to show that the rest term $r_{N,q,1}$ (4.50) converges to 0 as $N\to\infty$, for every $q\ge 1$. From (4.50), using the bound (4.49) and Lemma 3, we have the estimate

$$|r_{N,q,1}| \le C\sum_{m=0}^{2q-1}\binom{2q}{m}\frac{1}{N^{2q-m}}\sum_{1\le v\le N-l}|v|^{(2H-2p)m}|v|^{(2H+1-2p)(2q-m)}:=\sum_{m=0}^{2q-1}r_{N,q,1,m},$$

and for each $m=0,..,2q-1$,

$$r_{N,q,1,m}\le\frac{C}{N^{2q-m}}\sum_{1\le v\le N-l}|v|^{(2H-2p)2q+2q-m}.$$

When the series $\sum_{v\in\mathbb{Z}}|v|^{(2H-2p)2q+2q-m}$ converges we get

$$r_{N,q,1,m}\le C\frac{1}{N^{2q-m}}\le C\frac{1}{N}\to_{N\to\infty}0,$$

and when the series diverges,

$$r_{N,q,1,m}\le C\frac{1}{N^{2q-m}}N^{(2H-2p)2q+2q-m+1}\le CN^{(2H-2p)2q+1}\to_{N\to\infty}0$$

(up to an additional $\log N$ factor appearing whenever the exponent in the sum adds up to minus one) if $p=1, H\in(\frac12,1-\frac{1}{4q})$ or $p\ge 2$ and $H\in[\frac12,1)$. If $p=1$ and $H=\frac12$ we obtain for $m\ne 1$

$$r_{N,q,1,m}\le CN^{-2q+1}\to_{N\to\infty}0$$

and for $m=1$

$$r_{N,q,1,m}\le CN^{-2q+1}\log N\to_{N\to\infty}0.$$

If $p=q=1$, $H=3/4$, the quantity

$$\frac{1}{\log(N-l)} r_{N,q,1}$$

will also converge to zero which can be seen using again (4.49) and Lemma 3.

The fact that the series $\sigma^2 = \sum_{q\geq 1} \sigma^2_{2q} < \infty$ for $H < p - \frac{1}{4q}$ follows from the study of the k-variations of the fractional Brownian motion, see [28] or [66]. ■

Proof of Lemma 5

As in the proof of Lemma 4, we have

$$\mathbf{E}\left[G_N^{(2q)}(k,\alpha)^2\right] = \frac{(c_{2q}^k)^2}{(2q)!} \sum_{v\in\mathbb{Z}} \left(\rho_H^{\alpha,N}(v)\right)^{2q} 1_{\{|v|\leq N-l\}} \frac{N-|v|-l}{N-l}. \tag{4.51}$$

Recall the representation $\rho_H^{\alpha,N}(v) = \varphi_H(v) + b_{N,H}(v)$ introduced in Lemma 4. We obtain by the Newton's formula

$$\begin{aligned}
\mathbf{E}\left[G_N^{(2q)}(k,\alpha)^2\right] &= \frac{(c_{2q}^k)^2}{(2q)!} \sum_{m=0}^{2q} \binom{2q}{m} \sum_{v\in\mathbb{Z}} \varphi_{H,\alpha}(v)^m b_{N,H}(v)^{2q-m} 1_{\{|v|\leq N-l\}} \frac{N-|v|-l}{N-l} \\
&= \frac{(c_{2q}^k)^2}{(2q)!} \sum_{v\in\mathbb{Z}} \varphi_{H,\alpha}(v)^{2q} 1_{\{|v|\leq N-l\}} \frac{N-|v|-l}{N-l} \\
&\quad + r_{N,q,1},
\end{aligned}$$

where we sepatated the summand with $m=2q$ above and we used the notation (4.50). Consequently,

$$\begin{aligned}
\mathbf{E}\left[G_N^{(2q)}(k,\alpha)^2\right] &= \frac{(c_{2q}^k)^2}{(2q)!} \sum_{v\in\mathbb{Z}} \varphi_{H,\alpha}(v)^{2q} - \frac{(c_{2q}^k)^2}{(2q)!} \sum_{|v|\geq N-l+1} \varphi_{H,\alpha}(v)^{2q} \\
&\quad + \frac{(c_{2q}^k)^2}{(2q)!} \sum_{|v|\leq N-l} \varphi_{H,\alpha}(v)^{2q} \left(\frac{N-|v|-l}{N-l} - 1\right) + r_{N,q,1} \\
&= \frac{(c_{2q}^k)^2}{(2q)!} \sum_{v\in\mathbb{Z}} \varphi_{H,\alpha}(v)^{2q} + r_{N,q,3} + r_{N,q,2} + r_{N,q,1}
\end{aligned}$$

with

$$r_{N,q,2} = \frac{(c_{2q}^k)^2}{(2q)!} \sum_{|v|\leq N-l} \varphi_{H,\alpha}(v)^{2q} \left(\frac{N-|v|-l}{N-l} - 1\right),$$

$$r_{N,q,3} = -\frac{(c_{2q}^k)^2}{(2q)!} \sum_{|v|\geq N-l+1} \varphi_{H,\alpha}(v)^{2q}.$$

The asymptotics for $r_{N,q,1}$ has been studied in Lemma 4 :

$$|r_{N,q,1}| \leq C \begin{cases} \frac{1}{N} & \text{if } p\geq 2 \text{ or } p=1, q\geq 2, H\in\left[\frac{1}{2}, 1-\frac{1}{2q}\right), \\ N^{4H-3} & \text{if } p=1, q=1, H\in\left(\frac{1}{2}, \frac{3}{4}\right), \\ \frac{\log N}{N} & \text{if } p=1, q=1, H=\frac{1}{2}. \end{cases}$$

For $r_{N,q,2}$ we calculate

$$|r_{N,q,2}| \leq C \left| \sum_{|v|\leq N-l} \varphi_{H,\alpha}(v)^{2q} \left(\frac{|v|}{N-l}\right) \right| \leq C\frac{1}{N} \sum_{|v|\leq N-l} |v|^{2q(2H-2p)+1}.$$

Note that the above series is convergent for $p \geq 2$ or for $p = 1$ and $q \geq 2$ if $H < 1 - \frac{1}{2q}$ (which is satisfied for $H < \frac{3}{4}$). In these cases, we will find the estimate

$$|r_{N,q,2}| \leq C\frac{1}{N}.$$

For $p = 1$ and $q = 1$, the sequence $\sum_{1\leq v\leq N-l} |v|^{(2H-2p)2q+1} = \sum_{1\leq |v|\leq N-l} |v|^{4H-3}$ behaves as N^{4H-2} and we get

$$|r_{N,q,2}| \leq CN^{4H-3},$$

so here we obtain the bounds

$$|r_{N,\,q,2}| \leq C \begin{cases} \frac{1}{N} & \text{if } p \geq 2 \text{ or } p = 1, q \geq 2, H \in \left[\frac{1}{2},\, 1 - \frac{1}{2q}\right), \\ N^{4H-3} & \text{if } p = 1, q = 1, H \in \left(\frac{1}{2},\, \frac{3}{4}\right), \\ \frac{\log N}{N} & \text{if } p = 1, q = 1, H = \frac{1}{2}. \end{cases}$$

Finally, for $r_{N,\,q,3}$ the same bounds can be established. An application of Lemma 3 yields

$$|r_{N,q,3}| \leq C \sum_{|v|\geq N-l} \varphi_{H,\alpha}(v)^{2q} \leq CN^{(2H-2p)2q+1},$$

and consequently,

$$|r_{N,\,q,3}| \leq C \begin{cases} \frac{1}{N} & \text{if } p \geq 2 \text{ or } p = 1, q \geq 2, H < 1 - \frac{1}{2q} \\ N^{4H-3} & \text{if } p = 1, q = 1, H < \frac{3}{4}. \end{cases}$$

Since $\frac{1}{N} < N^{4H-3}$ for H between $\frac{1}{2}$ and $\frac{3}{4}$, the result for $p = 1$, $q \geq 2$ follows. ■

Proof of Proposition 30

We can explicitly compute the Malliavin derivatives in the statement. For $q \in \{1, \ldots, \frac{k}{2}\}$

$$D.G_N^{(2q)}(k,\alpha) = \frac{1}{\sqrt{N-l}} \sum_{i=l}^{N} \frac{c_{2q}^k}{(2q-1)!} I_{2q-1}\left(\frac{C_{i,\alpha}^{\otimes(2q-1)}}{(\pi_H^{\alpha,N}(0))^q}\right) C_{i,\alpha}(\cdot).$$

Assume without loss of generality $q_1 \leq q_2$. We have

$$\begin{aligned} &\langle DG_N^{(2q_1)}(k,\alpha),\, DG_N^{(2q_2)}(k,\alpha)\rangle_{\mathcal{H}} \\ =\ & \frac{1}{(N-l)(\pi_H^{\alpha,N}(0))^{q_1+q_2}} \frac{c_{2q_1}^k}{(2q_1-1)!} \frac{c_{2q_2}^k}{(2q_2-1)!} \sum_{i,\,j=l}^{N} I_{2q_1-1}\left(C_{i,\alpha}^{\otimes(2q_1-1)}\right) I_{2q_2-1}\left(C_{j,\alpha}^{\otimes(2q_2-1)}\right) \langle C_{i,\,\alpha},\, C_{j,\,\alpha}\rangle_{\mathcal{H}} \\ =\ & \frac{1}{(N-l)(\pi_H^{\alpha,N}(0))^{q_1+q_2}} \frac{c_{2q_1}^k}{(2q_1-1)!} \frac{c_{2q_2}^k}{(2q_2-1)!} \\ & \times \sum_{i,\,j=l}^{N} \left(\sum_{r=0}^{2q_1-1} r! \binom{2q_1-1}{r} \binom{2q_2-1}{r} I_{2q_1+2q_2-2-2r}\left(C_{i,\alpha}^{\otimes(2q_1-1)} \otimes_r C_{j,\alpha}^{\otimes(2q_2-1)}\right)\right) \langle C_{i,\,\alpha},\, C_{j,\,\alpha}\rangle_{\mathcal{H}}, \end{aligned}$$

and $\mathbf{E}[\langle DG_N^{(2q_1)}(k,\alpha),\, DG_N^{(2q_2)}(k,\alpha)\rangle_{\mathcal{H}}]$ is the term containing I_0. It follows that

$$\begin{aligned} &\langle DG_N^{(2q_1)}(k,\alpha),\, DG_N^{(2q_2)}(k,\alpha)\rangle_{\mathcal{H}} - \mathbf{E}[\langle DG_N^{(2q_1)}(k,\alpha),\, DG_N^{(2q_2)}(k,\alpha)\rangle_{\mathcal{H}}] \\ =\ & \frac{1}{(N-l)(\pi_H^{\alpha,N}(0))^{q_1+q_2}} \frac{c_{2q_1}^k}{(2q_1-1)!} \frac{c_{2q_2}^k}{(2q_2-1)!} \\ & \times \sum_{i,\,j=l}^{N} \left(\sum_{r=0}^{2q_1-1-w} r! \binom{2q_1-1}{r} \binom{2q_2-1}{r} I_{2q_1+2q_2-2-2r}\left(C_{i,\alpha}^{\otimes(2q_1-1)} \otimes_r C_{j,\alpha}^{\otimes(2q_2-1)}\right)\right) \langle C_{i,\,\alpha},\, C_{j,\,\alpha}\rangle_{\mathcal{H}}, \end{aligned}$$

where $w = 1$ if $l_1 \neq l_2$ and $w = 2$ otherwise.

Due to the fact that products of integrals of different orders have zero expectation we obtain

$$\begin{aligned}
P \quad &:= \quad \mathbf{E}[(\langle DG_N^{(2q_1)}(k,\alpha),\, DG_N^{(2q_2)}(k,\alpha)\rangle_{\mathcal{H}} - \mathbf{E}[\langle DG_N^{(2q_1)}(k,\alpha),\, DG_N^{(2q_2)}(k,\alpha)\rangle_{\mathcal{H}}]])^2] \\
&\sim_{N\to\infty} \quad \frac{C}{(N-l)^2(\pi_H^{\alpha,N}(0))^{2(q_1+q_2)}} \\
&\quad \times \sum_{r=0}^{2q_1-w} \mathbf{E}\left[\left(\sum_{i,\,j=l}^{N} r!\binom{2q_1-1}{r}\binom{2q_2-1}{r} I_{2q_1+2q_2-2-2r}(C_{i,\,\alpha}^{\otimes(2q_1-1)} \otimes_r C_{j,\,\alpha}^{\otimes(2q_2-1)})\langle C_{i,\,\alpha},\, C_{j,\,\alpha}\rangle_{\mathcal{H}}\right)^2\right] \\
&= \quad \frac{C}{(N-l)^2(\pi_H^{\alpha,N}(0))^{2(q_1+q_2)}} \sum_{r=0}^{2q_1-w} \sum_{i,j,k,m=l}^{N} r!^2\binom{2q_1-1}{r}^2\binom{2q_2-1}{r}^2 \langle C_{i,\,\alpha},\, C_{j,\,\alpha}\rangle_{\mathcal{H}}\langle C_{k,\,\alpha},\, C_{m,\,\alpha}\rangle_{\mathcal{H}} \\
&\quad \times \langle C_{i,\,\alpha}^{\otimes(2q_1-1)} \widetilde{\otimes_r} C_{j,\,\alpha}^{\otimes(2q_2-1)},\, C_{k,\,\alpha}^{\otimes(2q_1-1)} \widetilde{\otimes_r} C_{m,\,\alpha}^{\otimes(2q_2-1)}\rangle_{\mathcal{H}^{\otimes(2q_1+2q_2-2-2r)}} =: \sum_{r=0}^{2q_1-w} P_r.
\end{aligned}$$

We can compute the contractions involved and get via (4.42)

$$C_{i,\,\alpha}^{\otimes(2q_1-1)} \otimes_r C_{j,\,\alpha}^{\otimes(2q_2-1)} = C_{i,\,\alpha}^{\otimes(2q_1-r-1)} \otimes C_{j,\,\alpha}^{\otimes(2q_2-r-1)} \left(\langle C_{i,\,\alpha},\, C_{j,\,\alpha}\rangle_{\mathcal{H}}\right)^r.$$

Consequently, one can write for $r \geq 0$

$$\begin{aligned}
&\left|\langle C_{i,\,\alpha}^{\otimes(2q_1-1)} \widetilde{\otimes_r} C_{j,\,\alpha}^{\otimes(2q_2-1)},\, C_{k,\,\alpha}^{\otimes(2q_1-1)} \widetilde{\otimes_r} C_{m,\,\alpha}^{\otimes(2q_2-1)}\rangle_{\mathcal{H}^{\otimes(2q_1+2q_2-2-2r)}}\right| \\
\lesssim \quad & |(\langle C_{i,\,\alpha},\, C_{j,\,\alpha}\rangle_{\mathcal{H}}\langle C_{k,\,\alpha},\, C_{m,\,\alpha}\rangle_{\mathcal{H}})^r| \\
\times & \max_{a=0,\dots,l_1-r-1}\left|\langle C_{i,\,\alpha},\, C_{k,\,\alpha}\rangle_{\mathcal{H}}^{l_1-r-1-a}\langle C_{j,\,\alpha},\, C_{m,\,\alpha}\rangle_{\mathcal{H}}^{l_2-r-1-a}\langle C_{i,\,\alpha},\, C_{m,\,\alpha}\rangle_{\mathcal{H}}^{a}\langle C_{j,\,\alpha},\, C_{k,\,\alpha}\rangle_{\mathcal{H}}^{a}\right|
\end{aligned}$$

due to symmetrisation : the maximum is taken over all outcomes of different permutations of the first and second component of the inner product, the number a signifying the number of C_i in the first component that are appearing in the same places as C_m in the second component in a given permutation.

In total, we obtain for a fixed $r \in \{0, \dots, 2q_1 - w\}$

$$\begin{aligned}
|P_r| \quad &\leq \quad C\frac{1}{(N-l)^2(\pi_H^{\alpha,N}(0))^{2(q_1+q_2)}} \sum_{i,j,k,m=l}^{N} \left|\langle C_{i,\,\alpha},\, C_{j,\,\alpha}\rangle_{\mathcal{H}}^{r+1}\langle C_{k,\,\alpha},\, C_{m,\,\alpha}\rangle_{\mathcal{H}}^{r+1}\right| \\
&\quad \times \max_{a=0,\dots,l_1-r-1}\left|\langle C_{i,\,\alpha},\, C_{k,\,\alpha}\rangle_{\mathcal{H}}^{l_1-r-1-a}\langle C_{j,\,\alpha},\, C_{m,\,\alpha}\rangle_{\mathcal{H}}^{l_2-r-1-a}\langle C_{i,\,\alpha},\, C_{m,\,\alpha}\rangle_{\mathcal{H}}^{a}\langle C_{j,\,\alpha},\, C_{k,\,\alpha}\rangle_{\mathcal{H}}^{a}\right| \\
&= \quad C\frac{1}{(N-l)^2} \sum_{i,j,k,m=l}^{N} \left|\rho_H^{\alpha,\,N}(i-j)^{r+1}\rho_H^{\alpha,\,N}(k-m)^{r+1}\right| \\
&\quad \times \max_{a=0,\dots,l_1-r-1}\left|\rho_H^{\alpha,\,N}(i-k)^{l_1-r-1-a}\rho_H^{\alpha,\,N}(j-m)^{l_2-r-1-a}\rho_H^{\alpha,\,N}(i-m)^{a}\rho_H^{\alpha,\,N}(j-k)^{a}\right|,
\end{aligned}$$

with $\rho_H^{\alpha,\,N}$ defined in (4.45). Due to boundedness of $\rho_H^{\alpha,\,N}$ we can without loss of generality reduce the number of factors. In particular,

$$\begin{aligned}
&\max_{a=0,\dots,l_1-r-1}\left|\rho_H^{\alpha,\,N}(i-k)^{l_1-r-1-a}\rho_H^{\alpha,\,N}(j-m)^{l_2-r-1-a}\rho_H^{\alpha,\,N}(i-m)^{a}\rho_H^{\alpha,\,N}(j-k)^{a}\right| \\
\leq \quad & C|\rho_H^{\alpha,\,N}(i-k)\rho_H^{\alpha,\,N}(j-m)|,
\end{aligned}$$

since either the factor $|\rho_H^{\alpha,\,N}(i-k)\rho_H^{\alpha,\,N}(j-m)|$ or $|\rho_H^{\alpha,\,N}(i-m)\rho_H^{\alpha,\,N}(j-k)|$ is contained in the product and for symmetry reasons there is no need to distinguish between these cases. Using this inequality and bounding the first two factors in the same way we arrive at a bound

$$\begin{aligned}
|P_r| &\leq C\frac{1}{(N-l)^2}\sum_{i,j,k,m=l}^{N}|\rho_H^{\alpha,N}(i-j)\rho_H^{\alpha,N}(k-m)\rho_H^{\alpha,N}(i-k)\rho_H^{\alpha,N}(j-m)| \\
&\leq C\frac{1}{(N-l)^2}N\left(\sum_{v=1}^{N}|\rho_H^{\alpha,N}(v)|^{4/3}\right)^3,
\end{aligned}$$

where the last step follows via Young's inequality as in [48]. The representation $\rho_H^{\alpha,N}(v)=\varphi_H(v)+b_{N,H}(v)$ together with the fact that for $|v|\leq N$ we have $b_{N,H}(v)\leq C\varphi_H(v)$ for some constant C allows us to replace $\rho_H^{\alpha,N}$ with $\varphi_H(v)$ in the last bound, since the powers involved are positive. Finally, by Lemma 3

$$\sum_{v=1}^{N}|\varphi_H(v)|^{4/3}=\begin{cases}\mathrm{O}(1) & \text{if } H\in(0,\frac{5}{8})\ ,\\ \mathrm{O}(\log(N)) & \text{if } H=\frac{5}{8},\\ \mathrm{O}(N^{\frac{8H}{3}-\frac{5}{3}}) & \text{if } H\in(\frac{5}{8},1)\end{cases}$$

for $p=1$ and $\sum_{v=1}^{N}|\varphi_H(v)|^{4/3}=\mathrm{O}(1)$ for $p>1$, and thus the result follows. ∎

Proof of Lemma 6

As in Lemma 2 in [48], we have by (4.9)

$$\begin{aligned}
v_N &= \frac{2N^{4H}}{(k_1N+k_2)^2}\sum_{i,j=0}^{N-1}\left[\mathbf{E}\left(\left(u(t,\frac{i+1}{N})-u(t,\frac{i}{N})\right)\left(u(t,\frac{j+1}{N})-u(t,\frac{j}{N})\right)\right)\right]^2 \\
&= \frac{2N^{4H}}{(k_1N+k_2)^2}\sum_{i,j=1}^{N}\left[k_1\frac{\Phi_H(i-j)}{N^{2H}}+k_2\frac{\Phi_{H+\frac{1}{2}}(i-j)}{N^{2H+1}}\right]^2 \\
&= \frac{4N^{4H}}{(k_1N+k_2)^2}\sum_{j=1}^{N}\sum_{i=j+1}^{N-1}\left[k_1\frac{\Phi_H(i-j)}{N^{2H}}+k_2\frac{\Phi_{H+\frac{1}{2}}(i-j)}{N^{2H+1}}\right]^2 \\
&\quad+\frac{2N^{4H}}{(k_1N+k_2)^2}\sum_{i=1}^{N}\left[k_1\frac{1}{N^{2H}}+k_2\frac{1}{N^{2H+1}}\right]^2 \\
&= \frac{4N^{4H}}{(k_1N+k_2)^2}\sum_{l=1}^{N}\left[k_1\frac{\Phi_H(l)}{N^{2H}}+k_2\frac{\Phi_{H+\frac{1}{2}}(l)}{N^{2H+1}}\right]^2(N-l) \\
&\quad+\frac{2N^{4H}}{(k_1N+k_2)^2}\sum_{i=1}^{N}\left[k_1\frac{1}{N^{2H}}+k_2\frac{1}{N^{2H+1}}\right]^2.
\end{aligned}$$

The last summand satisfies

$$\frac{2N^{4H}}{(k_1N+k_2)^2}\sum_{i=1}^{N}\left[k_1\frac{1}{N^{2H}}+k_2\frac{1}{N^{2H+1}}\right]^2\leq C$$

for N large enough while the first summand converges to infinity, see below. Using the asymptotic behavior of Φ_H and $\Phi_{H+\frac{1}{2}}$, namely

$$\Phi_H(l)=H(2H-1)l^{2H-2}+o(l^{2H-2})$$

and

$$\Phi_{H+\frac{1}{2}}(l)=H(2H+1)l^{2H-1}+o(l^{2H-1})$$

for l large, we obtain

$$\begin{aligned} v_N \quad &\sim_{N\to\infty} \quad \frac{4}{k_1^2} N^{4H-2} \sum_{l=1}^{N} \left[k_1 H(2H-1) \frac{l^{2H-2}}{N^{2H}} + k_2 H(2H+1) \frac{l^{2H-1}}{N^{2H+1}} \right]^2 (N-l) \\ &= \quad \frac{4}{k_1^2} N^{4H-4} \frac{1}{N} \sum_{l=1}^{N} \left[k_1 H(2H-1) \left(\frac{l}{N}\right)^{2H-2} + k_2 H(2H+1) \left(\frac{l}{N}\right)^{2H-1} \right]^2 \left(\frac{N-l}{N}\right) \end{aligned}$$

and therefore

$$N^{4-4H} \frac{k_1^2}{4K_0} v_N \to_{N\to\infty} 1$$

with

$$\begin{aligned} K_0 &= \int_0^1 \left(k_1 H(2H-1) x^{2H-2} + k_2 H(2H+1) x^{2H-1} \right)^2 (1-x) dx \\ &= k_1^2 \frac{H^2(2H-1)}{2(4H-3)} + 2k_1 k_2 \frac{H^2(2H+1)}{2(4H-1)} + k_2^2 \frac{H(2H+1)^2}{4(4H-1)}. \end{aligned} \tag{4.52}$$

■

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