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Analyse qualitative de plusieurs types de systèmes de maladies infectieuses avec effets de réaction ou de diffusion

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Contents

CHAPTER 1

Introduction

1. Background and objectives

The research of infectious diseases based on mathematical models, especially dynamic models, has a long and diverse history [40]. It may be traced back to D. Bernoulli's mathematical study of smallpox vaccination in 1760 [15]. In 1873-1894, P. D. En'ko established modern mathematical models of infectious diseases [36]. In 1906, in order to understand the repeated epidemics of measles, W. H. Hamer constructed a discrete mathematical model and analyzed the dynamics of the model [49]. Five years later, the Nobel laureate, R. Ross, used a differential dynamical system to study the dynamics of malaria transmission between mosquitoes and people in detail [94]. To study the epidemic regularity of black death in 1665-1666 and plague in 1906, W. O. Kermack and A. G. McKendrick proposed the most influential model—SIR model in 1927 [5, 59]:

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI, \\ \dfrac{dI}{dt} = \beta SI - \gamma I, \\ \dfrac{dR}{dt} = \gamma I, \end{cases} \tag{1.1}$$

where the total population (N) in the affected area is divided into three classes: susceptible (S), infected (I) and recovered (R). Birth and death are ignored, that is, $N(t) = S(t) + I(t) + R(t)$ is constant. The parameters β and γ denote the transmission and recovery rates, respectively. Subsequently, W. O. Kermack and A. G. McKendrick put forward the SIS compartmental model in 1932 [60]:

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI + \gamma I, \\ \dfrac{dI}{dt} = \beta SI - \gamma I. \end{cases} \tag{1.2}$$

On this basis, they proposed a "threshold theory" that distinguishes whether the disease becomes popular. When the basic reproduction number $R_0 := \frac{\beta S_0}{\gamma} < 1$, the disease dies out; when $R_0 > 1$, the disease remains and becomes endemic, where S_0 denotes the number or density of the initial susceptible population.

These two basic dynamic models (SIR and SIS) and the corresponding theories established by W. O. Kermack and A. G. McKendrick lay a foundation for the study of the dynamics of infectious diseases. Since then, mathematical modeling and dynamical analysis of infectious diseases began to flourish, and the representative work is the first book on the mathematical theory of infectious diseases and its applications published by N. T. Bailey in 1957 [9]. Especially in the past 30 years, there has been a rapid progress for the mathematical analysis of the dynamical systems originating from infectious diseases in the world. A large number of papers and books [5, 50, 58] have been devoted to analyze the dynamical properties of the established models with regard to various infectious diseases.

In this thesis, we shall consider three types of dynamical systems extracted from infectious disease research, i.e., (a) reaction-diffusion system, (b) ordinary differential system, and (c) network-based system. The first two categories are deterministic models, and the last one belongs to stochastic models. Systems (a) and (c) have the diffusion effect, while system (b) only has the reaction effect. In what follows, we introduce their research progress in recent years, respectively.

(I) A diffusive influenza system with multiple strains

Influenza, having been a major cause of excessive morbidity and mortality [102], is a serious cytopathogenic, drastic respiratory infectious disease caused by an RNA virus in the Orthomyxoviridae family [35]. Besides, influenza poses a considerable economic burden of society and becomes a problem of public health [113]. Thus, it is imperative to prevent and contain the outbreak of influenza via increasing our understanding of the dynamics of influenza transmission.

Among pharmaceutical interventions, antiviral treatment remains one of the most effective measures to lower disease transmission and reduce the health burden of infections [38]. On the other hand, abundant use of antiviral drugs (such as oseltamivir and zanamivir) is a significant factor in producing resistant strains. Recently, some mathematical models have been used to explore the potential effects of drug resistance on the transmission of influenza [93] and identify effective treatment strategies for resistance management [69, 91]. These studies have provided useful insights into the emergence, spread and control of drug-resistant influenza. However, these models are all ordinary differential models. If the random movement of individuals in space plays a very important role in the dynamics of influenza transmission, it is necessary to consider the influence of spatial diffusion, which is usually characterized by reaction-diffusion equations.

The traveling wave solution (or referred to as traveling wave), which appears to be traveling with constant shape and velocity, is one of the elementary notions in the study of reaction-diffusion equations. The study of traveling wave solutions for nonlinear reaction-diffusion equations began with Fisher's equation. In 1937, R. A. Fisher proposed the following equation [39]:

$$u_t = d\Delta u + ru(1-u),\ t>0,\ x \in \mathbb{R}^n, \tag{1.3}$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, the parameters r and d are both positive, to describe the spatial spread of an advantageous allele and explored its traveling wave solutions. At the same time, A. N. Kolmogorov et al. gave a more general reaction-diffusion equation [61]:

$$u_t = du_{xx} + f(u),\ t>0,\ x \in \mathbb{R}, \tag{1.4}$$

and analyzed its traveling wave solutions. Thereafter, traveling wave solutions of nonlinear reaction-diffusion equations have been extensively and deeply studied, wherein its existence is the most fundamental problem in determining the long-term behavior of other solutions of the systems [54, 96, 122]. In epidemiology, the existence and nonexistence of nontrivial traveling waves indicate whether an infectious disease can persist as a wave front of infection that travels geographically across vast distances. The minimal wave speed for a traveling wave is a key parameter to characterize the speed at which the disease spreads in a spatial domain [33, 96]. Therefore, the study of traveling waves and minimal wave speeds is of great significance to the prevention and control of diseases.

In 2014, T. Zhang and W. Wang established a reaction-diffusion influenza model with treatment [123]:

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_s \dfrac{\partial^2 S}{\partial x^2} - \beta(I_u + \delta I_h)S, \\ \dfrac{\partial I_u}{\partial t} = d_u \dfrac{\partial^2 I_u}{\partial x^2} + (1-\mu)\beta(I_u + \delta I_h)S - k_u I_u, \\ \dfrac{\partial I_h}{\partial t} = d_h \dfrac{\partial^2 I_h}{\partial x^2} + \mu\beta(I_u + \delta I_h)S - k_h I_h, \\ \dfrac{\partial R}{\partial t} = d_r \dfrac{\partial^2 R}{\partial x^2} + k_u I_u + k_h I_h, \end{cases} \tag{1.5}$$

and focused on the existence and nonexistence of traveling wave solutions. As a result of the high mobility of population and the emergence of drug resistance due to treatment, we will further show the effects of demographic factors (recruitment and natural deaths) and drug resistance on the spatial spread of influenza. With the increase of the practical factors, some interesting and novel dynamics will appear, which are different from those in previous studies [122, 123, 124].

(II) Predator-prey type eco-epidemiological systems

Ecology [76] and epidemiology [5], as two different subjects in the field of mathematical biology [79], have received considerable attention in their own right. However, since R. M. Anderson and R. M. May (1978, 1986) showed that invasion of a resident predator-prey (or host-parasite) system by a new strain of parasites could cause destabilization and give rise to limit cycles [4, 77], quite a number of research papers (e.g., see [6, 25, 47, 72, 109] and the references in [72]) have already appeared in this intercrossed direction, linking ecological and epidemiological issues together. This leads to a new field of research popularly known as eco-epidemiology, which is coined by J. Chattopadhyay and O. Arino [25].

In studying eco-epidemiological systems, the interaction between species are primarily assumed to be predator-prey type [4, 6, 25, 47, 72, 77], although competitive type [4, 72] and symbiotic type [72, 109] are also common. In terms of predator-prey type eco-epidemiological systems, disease spread can happen within prey/host population [4, 6, 25, 72, 77], or within predator/host population [4, 72], or between prey and predator populations [47].

From the perspective of modeling, the dynamical behavior of predator-prey system with infection in the prey population is an important research topic. Generally speaking, such eco-epidemiological system with SI or SIS type disease (see [4, 6, 25, 72, 77]) can be defined by the following Kolmogorov-type differential equations

$$\begin{cases} \dfrac{dS}{dt} = S f_1(S, I, P), \\ \dfrac{dI}{dt} = I f_2(S, I, P), \\ \dfrac{dP}{dt} = P f_3(S, I, P), \end{cases} \tag{1.6}$$

in the state space

$$\mathbb{R}^3_+ = \{(S, I, P) \in \mathbb{R}^3 : S \geq 0, I \geq 0, P \geq 0\},$$

where $S(t), I(t)$ and $P(t)$ represent the population densities/numbers of susceptible prey, infected prey, and predator at time t, respectively. The functions $f_k \in C^1(\mathbb{R}^3_+, \mathbb{R}), k = 1, 2, 3$.

Since eco-epidemiological system (1.6) is a combination of a predator-prey model and an epidemiological model, there are two factors that need attention from a large perspective: (i) the choice of predator-prey model, generalized Gause or Leslie-Gower type [72]; (ii) the types of infectious diseases, SI or SIS [5, 79].

On the other hand, Allee effect is an important dynamic phenomenon in conservation biology, which is named after W. C. Allee(1885-1955) [32]. In recent decades, researchers have conducted extensive theoretical studies on Allee effect within different biological contexts, such as biological invasion [110], infectious diseases [51], etc. However, to the best of our knowledge, there is little research involving the impact of Allee effect on eco-epidemiology [17, 56, 97], wherein [17, 56] considered the case that susceptible prey is subject to Allee effect.

Given the above considerations, system (1.6) can be simplified to a less abstract but biologically more intuitive system as follows:

$$\begin{cases} \dfrac{dS}{dt} = Sg(S,I)a(S,\theta) - \psi(S,I) - \phi_1(S,I,P)P, \\ \dfrac{dI}{dt} = \psi(S,I) - \phi_2(S,I,P)P - \mu I, \\ \dfrac{dP}{dt} = \gamma_1\phi_1(S,I,P)P + \gamma_2\phi_2(S,I,P)P - dP, \end{cases} \tag{1.7}$$

where the parameters d and μ represent the death rate (includes an additional disease-induced death) of the predator and infected prey, γ_1 and γ_2 are the conversion rates of the susceptible and infected prey biomass into the predator biomass, respectively; $g(S,I)$ denotes the per capita growth rate of susceptible prey in the absence of Allee effect and predation; $a(S,\theta)$ models an Allee effect that affects susceptible prey; $\psi(S,I)$ is the transmission/incidence function; $\phi_1(S,I,P)$ and $\phi_2(S,I,P)$ are the functional/trophic responses or feeding rates for the predator with respect to susceptible and infected preys, respectively.

In particular, we assume that infected prey has less but positive contribution to the growth of the predator in comparison to susceptible prey, that is, $0 < \gamma_2 < \gamma_1$. Moreover, μ, d, γ_1 and γ_2 lie in the interval $(0,1)$. The form of carrying capacity plays a central rule in the function $g(S,I)$, which usually has two types, namely, explicit and implicit/emergent carrying capacities [99]. In most of the model-based eco-epidemiological studies [6, 17, 25, 47, 56, 97], especially when resources are limited, the growth function with explicit carrying capacity (wherein the competitive abilities for both susceptible and infected preys are the same) is a better choice as it is easy to interpret and comparatively straightforward to estimate from real life observations. However, the experimental study in [13] suggests that disease can change the competitive abilities within prey. For this reason, we first assume that infected prey does not contribute to the reproduction of newborns but competes for resources. By considering different competition coefficients for two possible interactions, based on the growth function $g(S,I) = rS(1-\frac{S+I}{K})$ with explicit carrying capacity, we modify it as $g(S,I) = rS(1-\frac{c_1S+c_2I}{K})$, where c_1 and c_2 $(0 < c_1, c_2 \leq 1)$ represent the weights of intra-class competition in susceptible prey and inter-class competition between susceptible and infected preys, respectively.

Although the mathematical expressions modeling Allee effect are varied [31, 108], most of them can be proven to be topologically equivalent. Here, we adopt the most common form for Allee effect [62], i.e., $a(S,\theta) = S - \theta$, where $-\frac{K}{c_1} \leq \theta \ll \frac{K}{c_1}$. When $\theta > 0$, this phenomenon is called a strong Allee effect [108, 110], and θ is known as Allee threshold. When $\theta \leq 0$, it is said that susceptible prey is affected by a weak Allee effect [110].

The transmission/incidence function is usually either density-dependent (also referred to the law of mass action or the bilinear incidence, i.e., $\psi(S, I) = kSI$) or frequency-dependent (also called the standard incidence, i.e., $\psi(S, I) = \frac{kSI}{S+I}$) [20]. By taking into account more biological details, several different nonlinear transmission functions (e.g., the saturated transmission rate $\frac{\beta SI}{1+\alpha I}$ [23] and nonlinear incidence $\beta S^p I^q$ [70]) were proposed. We think the saturated transmission form is more reasonable since it represents a "crowding effect" or "protection measure" and prevents the unboundedness of the contact rate, so in our work, we use $\psi(S, I) = \frac{\beta IS}{1+\alpha I}$, where $\alpha > 0$, βI measures the infection force of the disease and $\frac{1}{1+\alpha I}$ measures the inhibition effect from the behavioral change of susceptible prey when their number increases or from the crowding effect of infective prey.

Different functional responses have been considered in the context of ecology and eco-epidemiology, see [10, 21, 80]. Mathematically, they can be classified into two main categories: monotonic and non-monotonic. To keep systems more general, the functional response is not specified in our work. Moreover, we assume that the predator is not smart enough to distinguish between susceptible and infected preys. Hence, the algebraic expressions of $\phi_k(S, I, P), k = 1, 2$ used in this work take $\phi_1(S, I, P) = \frac{bS^m}{(S+I)^n+a^n}$ and $\phi_2(S, I, P) = \frac{bI^m}{(S+I)^n+a^n}$, where $n, m \in \mathbb{N}_+$, $n \geq m \geq 1$, b is the searching efficiency constant or the predation rate on the prey and a is the half-saturation constant of the predator.

Our model (1.7) is different from the models used in the previous works [17, 56, 97] due to several aspects:

(D1) We use different competition coefficients within prey that arises due to disease-modified inter-specific competition. This is an extension of the initial idea of M. Sieber et al. [99].

(D2) The incidence function exhibits the feature of transmission saturation.

(D3) The predator not only captures infected prey but also catches susceptible prey based on the work of S. Biswas et al. [17]. Moreover, the consumption of infected prey has less contribution to the predator's growth.

(D4) We use the generalized functional responses of the predator, whose advantage is that our results are not restricted to a particular model but applicable to certain classes of models.

(III) Network-based systems coupling epidemic spread and information diffusion

Complex networks have recently attracted attention in many disciplines, including epidemiology, physics, and social sciences [57]. Whereas the conventional compartmental models of epidemic transmission are based on the assumption of a homogenous and well-mixed population, the application of complex networks explicitly models connectivity between individuals. In particular, the discovery of scale-free networks has shifted the focus of network-based research on disease transmission from small world networks to scale-free networks [11]. One of the most influential results is the pioneering work done by R. Pastor-Satorras and A. Vespignani [86, 87, 88]. After that, more and more scholars concentrate on the study of epidemic models on complex networks [19, 81, 85].

In real-world situations, when an epidemic begins to spread, people generally become increasingly aware of its presence. Their perception of the nature of the epidemic is shaped by the information obtained through a variety of distinct channels: in their social or spatial neighborhood, from the mass media (e.g., TV, radio, newspapers, etc.), through various online social media, and under the influence of various environmental factors. This spreading information then causes some people to change their behaviors — either to protect themselves or to reduce the risk of transmission [37]. As a result, the extent of the spread of the infection can be significantly reduced [43].

This implies that the outbreak of an epidemic can trigger behavioral responses from individuals, conversely, human behavioral changes induced by epidemics could have great influence on the epidemic dynamics and even epidemic network structure. So, it is of great interest to incorporate the change of human behaviors into mathematical models for infectious disease transmission, making an exploration and simulation of epidemic spread and its control.

Recently, several works have addressed the problem from different perspectives [8, 42, 64, 74, 116], for example, risk perception, behavioral changes, competing viral agents, or collective behaviors. For simplicity, we collectively denote all individual and community preventive behaviors against epidemics as *adaptive behaviors*. To examine the interplay between adaptive behaviors and epidemic spread, in Fig. 1, we depict a specific logical loop in the process of epidemic spread. Therefore, in order to accurately model the spread of epidemics in complex

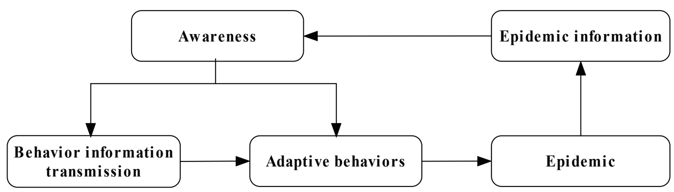


Figure 1. A specific logical loop in the process of epidemic spread.

networks, we should take account of multiple dynamic processes simultaneously, including epidemic spread, behavioral dynamics, information transmission, and the interplay. A general interplay model between human adaptive behaviors and epidemic spread in complex networks can be described by

$$\begin{cases} \dot{X}(t) = F(X(t), C(t)), \\ \dot{Y}(t) = G(Y(t), E(t)), \\ \dot{C}(t) = H(Y(t), E(t)), \end{cases} \tag{1.8}$$

where the variable $X(t) = (x_1^T(t), x_2^T(t), \cdots, x_N^T(t))^T$ with $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in \mathbb{R}^n$ denotes the behavior state variable of all individuals in a behavior information network with size N, which can display collective behaviors under suitable conditions. The variable $C(t) = (c_1(t), c_2(t), \cdots, c_N(t))^T$ with $c_i(t) \in \mathbb{R}$ denotes the coupling weight of each individual in the behavior information network. In the second equality of (1.8), the variable $Y(t) = (y_1(t), y_2(t), \cdots, y_N(t))^T$ with $y_i(t) \in \mathbb{R}$ denotes the infection probability of each individual in an epidemic spreading network with size N. The variable $E(t) = (E_1(t), E_2(t), \cdots, E_N(t))^T$ with $E_i(t) \in \mathbb{R}$ denotes the state error of each individual, which can be defined in different forms. F, G, and H represent three different mappings, respectively. The mapping $F : (\mathbb{R}^{nN}, \mathbb{R}^N) \to \mathbb{R}^{nN}$ controls the dynamical change process of the state variable $X(t)$. The mapping $G : (\mathbb{R}^N, \mathbb{R}^N) \to \mathbb{R}^N$ characterizes the dynamical change process of the infection probability $Y(t)$. The mapping $H : (\mathbb{R}^N, \mathbb{R}^N) \to \mathbb{R}^N$ defines an adaptive update law of the coupling weight $C(t)$.

In system (1.8), human adaptive behaviors $X(t)$ play a role in the epidemic spreading process $Y(t)$ by embedding the behavior state error $E(t)$, and epidemic spreading process $Y(t)$ influences human adaptive behaviors $X(t)$ by changing their coupling weight $C(t)$.

Knowing that the spread of behavioral information is quite different from that of the underlying epidemic, we intend to study the complex interplay between adaptive behaviors and epidemic spread in multiplex networks [103], in which the nodes represent the same entities in all layers but the connection patterns at each layer are different. In view of the authenticity of the network and the accuracy of feedback of nodes (i.e., point-to-point feedback), the topological structure of a multiplex network is set to be quenched. On the other hand, for emerging epidemics, people cannot promptly manufacture effective vaccines or produce targeted drugs. In these circumstances, governments, mass media, or public health authorities typically choose to guide individual behaviors to an optimal state of self-protection to reduce one's susceptibility to infection, so we will design optimized control schemes from the new perspective of behavioral regulation.

From a mathematical point of view, we shall focus on the qualitative structure (or topological structure) of the limit sets (or invariant sets) in these three classes of dynamical systems. The limit sets usually contain at least one of the following three parts: (1) singularities/equilibria, (2) periodic solutions (closed orbits), and (3) singularities and the orbits which tend to these singularities when $t \to -\infty$ or $t \to +\infty$ (e.g., homoclinic and heteroclinic orbits). Supposing that the limit sets of systems are discussed clearly, their qualitative structure can be basically determined. Furthermore, the stability of the limit sets and bifurcation phenomena are also the main content of our concern.

2. Main work of the thesis

This thesis is divided into five chapters, and mainly studies the dynamical behavior of three types of epidemiological mathematical systems, i.e., a diffusive influenza system with multiple strains, predator-prey type eco-epidemiological systems, and two network-based systems coupling epidemic spread and information diffusion. For these epidemiological systems, we focus on their qualitative analysis. The specific work of this thesis is stated as follows.

In Chapter 2, we consider the following diffusive influenza system with multiple strains

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \dfrac{\partial^2 S}{\partial x^2} + \Lambda - \mu S - [\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R]S, \\ \dfrac{\partial I_{SU}}{\partial t} = d_{SU} \dfrac{\partial^2 I_{SU}}{\partial x^2} + (1-f)\beta_S(I_{SU} + \delta I_{ST})S - (k_U + \mu)I_{SU}, \\ \dfrac{\partial I_{ST}}{\partial t} = d_{ST} \dfrac{\partial^2 I_{ST}}{\partial x^2} + f(1-r)\beta_S(I_{SU} + \delta I_{ST})S - (k_T + \mu)I_{ST}, \\ \dfrac{\partial I_R}{\partial t} = d_R \dfrac{\partial^2 I_R}{\partial x^2} + [fr\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R]S - (k_R + \mu)I_R, \\ \dfrac{\partial R}{\partial t} = d_{\widetilde{R}} \dfrac{\partial^2 R}{\partial x^2} + k_U I_{SU} + k_T I_{ST} + k_R I_R - \mu R. \end{cases} \tag{1.9}$$

Our main purpose is to establish the conditions for the existence of the three kinds of traveling waves for system (1.9) starting from the disease-free equilibrium $E^0(S^0, 0, 0, 0, 0)$ (at the initial stage of influenza transmission): semi-traveling waves, strong traveling waves and weak (persistent) traveling waves.

So far, the existence of traveling waves for monotone systems (e.g., competitive or cooperative models) has been well understood. However, system (1.9) is a non-monotone system, implying that some standard methods, such as the monotone iteration and comparison argument [68, 126], are no longer suitable. Though there have been some recent progress in the study of non-monotonic systems [2, 54, 120, 122, 123, 124], the methods in these references (such as singular perturbation argument, geometric method, squeeze method, etc.) seem to be powerless for system (1.9). Several specific reasons are: (i) The non-monotonic systems in the above literatures mostly consist of two equations, while system (1.9) contains five equations; (ii) Unlike the handling techniques in [123], we have no restrictions on the diffusion coefficients of different variables; (iii) To be more realistic, recruitment and natural mortality factors of population are considered, which increase the dynamical complexity of the system.

Due to the complexity of system (1.9), some dynamical problems on this system become very challenging, implying that we need to improve the previous approaches. To do so, we shall further extend the method of upper-lower solutions developed in [2, 122, 123, 124]. Since it is difficult to construct a pair of appropriate upper-lower solutions connecting the disease-free equilibrium E^0 for system (1.9), we first introduce an auxiliary system, the existence of semi-traveling waves of which is easy to prove. The existence of semi-traveling waves of the auxiliary system, together with limit arguments imply the existence of semi-traveling waves of system (1.9). Then we construct an appropriate Lyapunov function and apply persistent theory of dynamical systems in [105] to prove the existence of strong and weak traveling waves of system (1.9), respectively. Finally, the nonexistence of semi-traveling waves for system (1.9) in four cases is obtained by the comparison principle, the negative one-sided and two-sided Laplace transforms, which are introduced by [24, 124]. Furthermore, the interval estimation of the minimal wave speed is given.

In Chapter 3, we study a class of predator-prey type eco-epidemiological systems in $\mathbb{R}^3_+$, given by the following set of nonlinear differential equations:

$$\begin{cases} \dfrac{dS}{dt} = rS(1-\dfrac{c_1S+c_2I}{K})(S-\theta) - \dfrac{\beta I}{1+\alpha I}S - \dfrac{bS^m}{(S+I)^n+a^n}P, \\ \dfrac{dI}{dt} = \dfrac{\beta I}{1+\alpha I}S - \dfrac{bI^m}{(S+I)^n+a^n}P - \mu I, \\ \dfrac{dP}{dt} = \gamma_1\dfrac{bS^m}{(S+I)^n+a^n}P + \gamma_2\dfrac{bI^m}{(S+I)^n+a^n}P - dP, \end{cases} \tag{1.10}$$

with initial conditions

$$S(0)\geq 0,\ I(0)\geq 0,\ P(0)\geq 0. \tag{1.11}$$

Our main objective is to explore the abundant dynamic behavior exhibited by the proposed system and to identify the crucial parameters that ensure specific population behaviors. Firstly, to gain insight into the boundary dynamics of system (1.10), we divide it into three independent subsystems in $\mathbb{R}^2_+$, that is, the epidemiological (S-I subsystem), predator-prey (S-P subsystem) and predator-infected-prey(I-P subsystem) subsystems. Then, by: (i) obtaining a global topological sketches of the dynamics of (1.10) and its three subsystems, with different Allee effects or competitive coefficients; and (ii) comparing the dynamics of (1.10) and its S-P subsystem with monotonic functional response to those with non-monotonic functional response, we conclude that (a) strong Allee effect can create a separatrix curve (or surface), leading to multi-stability; (b) different competitive abilities between prey can greatly change the dynamics of S-I subsystem, and (c) S-P subsystem with non-monotonic functional response has richer dynamical behavior than that with monotonic functional response. Finally, we provide the sufficient conditions for the local and global stability of boundary equilibria of (1.10), derive the criteria for which (1.10) will persist, and identify an interior periodic orbit by applying Poincaré map and bifurcation theory.

In Chapter 4, we present two network-based systems coupling epidemic spread and information diffusion, namely, a concrete interplay system in quenched multiplex networks

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) + c_i(t) \sum_{j=1}^{N} b_{ij} \Gamma [x_j(t) - x_i(t)], \\ \dot{\rho}_i(t) = -\rho_i(t) + \lambda \phi_i(t)[1 - \rho_i(t)] \sum_{j=1}^{N} a_{ij} \rho_j(t), \ i = 1, 2, \cdots, N, \\ \dot{c}_i(t) = \beta \psi(k_i^a)[\theta(k_i^a)\rho_i^l(t) + (1 - \theta(k_i^a))\rho^g(t)] e_i^T(t) e_i(t), \end{cases} \tag{1.12}$$

and an epidemic control system

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) - c_i(t) \sum_{j=1}^{N} l_{ij} \Gamma x_j(t) + u_i(t), \\ u_i(t) = -c_i(t) d_i e_i(t), \ i = 1, 2, \cdots, l, \\ u_i(t) = 0, \ i = l+1, l+2, \cdots, N, \\ \dot{\rho}_i(t) = -\rho_i(t) + \lambda \phi_i(t)[1 - \rho_i(t)] \sum_{j=1}^{N} a_{ij} \rho_j(t), \ i = 1, 2, \cdots, N, \\ \dot{c}_i(t) = \beta \psi(k_i^a)[\theta(k_i^a)\rho_i^l(t) + (1 - \theta(k_i^a))\rho^g(t)] e_i^T(t) e_i(t). \end{cases} \tag{1.13}$$

Our main aim is to investigate the uniform persistence and stability of systems (1.12) and (1.13). Through the next-generation matrix approach, we calculate the basic reproduction number of the epidemic spreading system in (1.12) and (1.13), which is a critical quantity that determines whether epidemics are persistent. With the aid of the theories of nonnegative matrices, we explore the impact of adaptive behaviors on epidemic spread by comparing the epidemic thresholds. Then we utilize the comparison principle, construct appropriate Lyapunov functions and applying the LaSalle's invariance principle to prove the globally asymptotically stability of two network-based systems, including the disease-free and endemic equilibria of the network-based epidemic spreading system and the synchronization manifold of the network-based behavioral information diffusion system. For theoretically unprovable parts, we perform some numerical simulations to supplement the mathematical analysis of systems (1.12) and (1.13).

CHAPTER 2

Traveling waves and estimation of minimal wave speed for a diffusive influenza system with multiple strains

In this chapter, we consider the following diffusive influenza system with multiple strains

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \dfrac{\partial^2 S}{\partial x^2} + \Lambda - \mu S - [\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R]S, \\ \dfrac{\partial I_{SU}}{\partial t} = d_{SU} \dfrac{\partial^2 I_{SU}}{\partial x^2} + (1-f)\beta_S(I_{SU} + \delta I_{ST})S - (k_U + \mu)I_{SU}, \\ \dfrac{\partial I_{ST}}{\partial t} = d_{ST} \dfrac{\partial^2 I_{ST}}{\partial x^2} + f(1-r)\beta_S(I_{SU} + \delta I_{ST})S - (k_T + \mu)I_{ST}, \\ \dfrac{\partial I_R}{\partial t} = d_R \dfrac{\partial^2 I_R}{\partial x^2} + [fr\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R]S - (k_R + \mu)I_R, \\ \dfrac{\partial R}{\partial t} = d_{\widetilde{R}} \dfrac{\partial^2 R}{\partial x^2} + k_U I_{SU} + k_T I_{ST} + k_R I_R - \mu R, \end{cases} \tag{2.1}$$

where $S(x,t), I_{SU}(x,t), I_{ST}(x,t), I_R(x,t)$ and $R(x,t)$ represent the quantities of susceptible, infected with the sensitive strain and untreated, infected with the sensitive strain and treated, infected with the resistant strain, and recovered population at position x and time t, respectively. The parameters d_S, d_{SU}, d_{ST}, d_R and $d_{\widetilde{R}}$ are the diffusion coefficients of the above five subclasses. The constant Λ is the recruitment rate of the population and μ is per-capita natural death rate. Here we assume that each infected individual with the sensitive strain will receive treatment with proportion f, and each individual who received treatment will develop drug resistance with probability r. The parameters β_S and β_R are the transmission coefficients of the untreated and drug-resistant infected individuals. Due to antiviral treatment, the transmission rate by an individual who received treatment will be reduced by the factor δ. Each individual in I_j subclasses can recover with the corresponding rate k_j, $j = SU, ST, R$. All parameters are assumed to be positive.

The corresponding reaction system of (2.1) is described by the following system of ODEs:

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - \mu S - [\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R]S, \\ \dfrac{dI_{SU}}{dt} = (1-f)\beta_S(I_{SU} + \delta I_{ST})S - (k_U + \mu)I_{SU}, \\ \dfrac{dI_{ST}}{dt} = f(1-r)\beta_S(I_{SU} + \delta I_{ST})S - (k_T + \mu)I_{ST}, \\ \dfrac{dI_R}{dt} = [fr\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R]S - (k_R + \mu)I_R, \\ \dfrac{dR}{dt} = k_U I_{SU} + k_T I_{ST} + k_R I_R - \mu R. \end{cases} \tag{2.2}$$

In Section 1, we shall give conditions for the existence of equilibria of system (2.2). Under some conditions, (2.2) has a unique disease-free equilibrium E^0, and under some other conditions, apart from the disease-free equilibrium, there exist a boundary equilibrium $\hat{E}$ and/or an interior equilibrium E^*.

Since the first four equations of (2.1) are independent of the last one, it suffices to consider the following reduced reaction-diffusion system:

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \dfrac{\partial^2 S}{\partial x^2} + \Lambda - \mu S - [\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R]S, \\ \dfrac{\partial I_{SU}}{\partial t} = d_{SU} \dfrac{\partial^2 I_{SU}}{\partial x^2} + (1-f)\beta_S(I_{SU} + \delta I_{ST})S - (k_U + \mu)I_{SU}, \\ \dfrac{\partial I_{ST}}{\partial t} = d_{ST} \dfrac{\partial^2 I_{ST}}{\partial x^2} + f(1-r)\beta_S(I_{SU} + \delta I_{ST})S - (k_T + \mu)I_{ST}, \\ \dfrac{\partial I_R}{\partial t} = d_R \dfrac{\partial^2 I_R}{\partial x^2} + [fr\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R]S - (k_R + \mu)I_R. \end{cases} \tag{2.3}$$

More specifically, we consider a solution of (2.3), $(S(x,t), I_{SU}(x,t), I_{ST}(x,t), I_R(x,t))$, with the following form

$$S(x,t) = S(\xi),\ I_i(x,t) = I_i(\xi),\ \xi = x + ct, \tag{2.4}$$

where $i = SU, ST, R$, and $c > 0$ is the wave speed.

The solution $(S(x,t), I_{SU}(x,t), I_{ST}(x,t), I_R(x,t))$ having the form (2.4) is called a traveling wave solution (or referred to as traveling wave) if $S(\xi)$ and $I_i(\xi),\ i = SU, ST, R$, are defined for all $\xi \in \mathbb{R}$ and are nonnegative functions.

For the convenience of discussions below, we first list several definitions of different kinds of traveling waves as follows [2, 54, 124].

Definition 0.1. *(see [2, 54]). A traveling wave $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi))$ is called a semi-traveling wave connected to the disease-free equilibrium E^0 (for convenience, here we still use the same notation E^0 to represent the equilibrium of system* (2.3)*) if it satisfies the boundary condition*

$$\lim_{\xi\to-\infty}(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi)) = E^0(S^0, 0, 0, 0). \tag{2.5}$$

Definition 0.2. *(see [124]). A traveling wave $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi))$ is strong if it satisfies*

$$\begin{aligned} (S(-\infty), I_{SU}(-\infty), I_{ST}(-\infty), I_R(-\infty)) &= E^0, \\ (S(+\infty), I_{SU}(+\infty), I_{ST}(+\infty), I_R(+\infty)) &= \hat{E}/E^*, \end{aligned} \tag{2.6}$$

where $U(\pm\infty) = \lim_{\xi\to\pm\infty} U(\xi)$.

Definition 0.3. *(see [124]). A traveling wave $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi))$ is weak or persistent if there exist two positive constants M_1 and M_2 such that*

$$\begin{gathered} (S(-\infty), I_{SU}(-\infty), I_{ST}(-\infty), I_R(-\infty)) = E^0, \\ M_1 < \liminf_{\xi\to+\infty} S(\xi),\ \limsup_{\xi\to+\infty} S(\xi) < M_2, \\ M_1 < \liminf_{\xi\to+\infty} I_i(\xi),\ \limsup_{\xi\to+\infty} I_i(\xi) < M_2,\ i = SU, ST, R. \end{gathered} \tag{2.7}$$

The remaining parts of this chapter are organized as follows. A preliminary lemma on the existence of equilibria for the reaction system is given in Section 1. Section 2 is devoted to establish the existence of semi-traveling waves of an auxiliary system. By using the results derived in Section 2, conditions for the existence of three different kinds of traveling waves of the original system are obtained in Section 3. By means of the comparison principle and the negative one-sided and two-sided Laplace transforms, some sufficient conditions for the nonexistence of semi-traveling waves of the original system and an estimation of the minimal wave speed are given in Section 4. Finally, Section 5 concludes this work.

1. Preliminary lemma on the existence of equilibria for the reaction system

In this section, we give a brief discussion about the existence of equilibria of corresponding reaction equations (2.2) of (2.1).

Denote by $N(t)$ the total quantity of the population at time t, namely,

$$N(t) = S(t) + I_{SU}(t) + I_{ST}(t) + I_R(t) + R(t).$$

Note that the total population quantity $N(t)$ satisfies the equation

$$\frac{dN}{dt} = \Lambda - \mu N. \tag{2.8}$$

It is clear that $N(t) = \frac{\Lambda}{\mu}$ is a solution of equation (2.8), and for any initial value $N(t_0) \geq 0$, the general solution of (2.8) is

$$N(t) = \frac{1}{\mu}[\Lambda - (\Lambda - \mu N(t_0)) \exp^{-\mu(t-t_0)}]. \tag{2.9}$$

By the expression (2.9) of the general solution of (2.8), we have $\lim_{t\to+\infty} N(t) = \frac{\Lambda}{\mu}$.

Through the above analysis, we know that the biologically feasible set of reaction system (2.2) is given by

$$\Gamma = \{(S, I_{SU}, I_{ST}, I_R, R)|0 \leq S, I_{SU}, I_{ST}, I_R, R, S + I_{SU} + I_{ST} + I_R + R \leq \frac{\Lambda}{\mu}\}. \tag{2.10}$$

Obviously, the set Γ is positively invariant for system (2.2).

There is a key parameter in epidemiological models, the basic reproduction number, commonly denoted by R_0, defined as the expected number of secondary infections generated by a single infectious individual during the infection period in an entirely susceptible population [5, 50, 119]. When certain control measures (such as immunization, isolation, treatment, etc) are introduced, we use the control reproduction number, denoted by R_C, to determine whether the epidemic can be contained [5]. Similarly, we can use the same approach developed in [106] to calculate the control reproduction number of reaction system (2.2) with treatment terms.

Note that system (2.2) always has a disease-free equilibrium $E^0 = (S^0, 0, 0, 0, 0)$, where $S^0 := \frac{\Lambda}{\mu}$. System (2.2) has three infected variables, namely, I_{SU}, I_{ST} and I_R, linearizing the equations of these three variables at disease-free equilibrium $E^0(S^0, 0, 0, 0, 0)$, the matrices F and V (corresponding to the new infection and remaining transfer terms, respectively) are given by

$$F = \begin{pmatrix} (1-f)\beta_S S^0 & (1-f)\beta_S \delta S^0 & 0 \\ f(1-r)\beta_S S^0 & f(1-r)\beta_S \delta S^0 & 0 \\ fr\beta_S S^0 & fr\beta_S \delta S^0 & \beta_R S^0 \end{pmatrix}, \tag{2.11}$$

and

$$V = \begin{pmatrix} k_U + \mu & 0 & 0 \\ 0 & k_T + \mu & 0 \\ 0 & 0 & k_R + \mu \end{pmatrix}. \tag{2.12}$$

Thus,

$$FV^{-1} = \begin{pmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{pmatrix},$$

where

$$F_{11} = \begin{pmatrix} \frac{(1-f)\beta_S S^0}{k_U+\mu} & \frac{(1-f)\beta_S \delta S^0}{k_T+\mu} \\ \frac{f(1-r)\beta_S S^0}{k_U+\mu} & \frac{f(1-r)\beta_S \delta S^0}{k_T+\mu} \end{pmatrix}, \ F_{21} = \begin{pmatrix} \frac{fr\beta_S S^0}{k_U+\mu} & \frac{fr\beta_S \delta S^0}{k_T+\mu} \end{pmatrix}, \ F_{22} = \frac{\beta_R S^0}{k_R+\mu}.$$

Let

$$R_{SU} = \frac{\beta_S}{k_U+\mu}, \ R_{ST} = \frac{\beta_S \delta}{k_T+\mu}, \ R_R = \frac{\beta_R}{k_R+\mu}, \tag{2.13}$$

then we have

$$R_{SC} = \rho(F_{11}) = \frac{(1-f)\beta_S S^0}{k_U+\mu} + \frac{f(1-r)\beta_S \delta S^0}{k_T+\mu} = S^0[(1-f)R_{SU} + f(1-r)R_{ST}], \tag{2.14}$$

and

$$R_{RC} = \rho(F_{22}) = \frac{\beta_R S^0}{k_R+\mu} = S^0 R_R, \tag{2.15}$$

where $\rho(A)$ is the spectral radius of the nonnegative matrix A.

Thus, the control reproduction number of system (2.2) is given by

$$R_C = \rho(FV^{-1}) = \max\{R_{SC}, R_{RC}\}. \tag{2.16}$$

To study equilibria of system (2.2) and their corresponding conditions of parameters, we present the following lemma.

LEMMA 1.1. *(1) The disease-free equilibrium E^0 always exists;*
(2) If $R_C < 1$, there exists a unique disease-free equilibrium E^0;
(3) If $R_C > 1$, then in addition to the disease-free equilibrium E^0, system (2.2) *has a boundary equilibrium $\hat{E}$ when $R_{RC} > 1$, and an interior (positive) equilibrium E^* when $R_{SC} > 1$ and $R_{RC} < R_{SC}$.*

Proof: The equilibria of system (2.2) are the solutions of the following equations:

$$\begin{cases} \Lambda - \mu S - [\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R]S = 0, \\ (1-f)\beta_S(I_{SU} + \delta I_{ST})S - (k_U + \mu)I_{SU} = 0, \\ f(1-r)\beta_S(I_{SU} + \delta I_{ST})S - (k_T + \mu)I_{ST} = 0, \\ [fr\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R]\, S - (k_R + \mu)I_R = 0, \\ k_U I_{SU} + k_T I_{ST} + k_R I_R - \mu R = 0. \end{cases} \tag{2.17}$$

In order to solve algebraic equations (2.17), we divide it into the following three cases.

Case I: $I_R = 0$. For this case, based on the setting of parameters and the fact that $S > 0$, from the fourth equation of (2.17), we have $I_{SU} = I_{ST} = 0$. Substituting $I_R = I_{SU} = I_{ST} = 0$ into the first equation of (2.17), we can obtain $S = \frac{\Lambda}{\mu} = S^0$.

Case II: $I_R > 0$ and $I_{ST} = 0$. For this case, it follows from the third and fourth equations of (2.17) that $I_{SU} = 0$ and $S = \frac{k_R+\mu}{\beta_R} = \frac{S^0}{R_{RC}} := \hat{S}$. Substituting $I_{SU} = I_{ST} = 0$ and $S = \hat{S}$ into (2.17), we get the following reduced equations

$$\begin{cases} \Lambda - \mu\hat{S} - \beta_R I_R \hat{S} = 0, \\ k_R I_R - \mu R = 0. \end{cases} \tag{2.18}$$

By solving (2.18), we obtain $I_R = \frac{\mu S^0(1-\frac{1}{R_{RC}})}{k_R+\mu} := \hat{I}_R$ and $R = \frac{k_R S^0(1-\frac{1}{R_{RC}})}{k_R+\mu} := \hat{R}$.

Case III: $I_R > 0$ and $I_{ST} > 0$. For this case, we first deal with the second and third equations of (2.17), which can be regarded as equations with unknown quantities I_{SU} and I_{ST}. In view of the fact that $I_{ST} > 0$, by Cramer's Rule, we have

$$\begin{vmatrix} (1-f)\beta_S S - (k_U+\mu) & (1-f)\beta_S \delta S \\ f(1-r)\beta_S S & f(1-r)\beta_S\delta S - (k_T+\mu) \end{vmatrix} = 0.$$

Calculate the above determinant, we can get the value of S as follows

$$S = \frac{(k_U+\mu)(k_T+\mu)}{(k_T+\mu)(1-f)\beta_S + (k_U+\mu)f(1-r)\beta_S\delta} = \frac{S^0}{R_{SC}} := S^*.$$

Substituting $S = S^*$ into (2.17), after some algebraic computations, we can solve the remaining four unknown quantities of equations (2.17) with $S = S^*$ as follows

$$I_{SU} = \frac{\mu(R_{SC}-1)}{\beta_S(1+\delta a)+\beta_R b} := I^*_{SU},\ I_{ST} = aI^*_{SU} := I^*_{ST},$$
$$I_R = bI^*_{SU} := I^*_R,\ R = \frac{k_U I^*_{SU} + k_T I^*_{ST} + k_R I^*_R}{\mu} := R^*,$$

where $a = \frac{f(1-r)(k_U+\mu)}{(1-f)(k_T+\mu)}$ and $b = \frac{fr(k_U+\mu)}{(1-f)(k_R+\mu)(1-\frac{R_{RC}}{R_{SC}})}$.

From the above discussions, it follows that (2.17) has three possible nonnegative solutions. Accordingly, system (2.2) has three possible equilibria $E^0 = (S^0, 0, 0, 0, 0)$, $\hat{E} = (\hat{S}, 0, 0, \hat{I}_R, \hat{R})$ and $E^* = (S^*, I^*_{SU}, I^*_{ST}, I^*_R, R^*)$. Based on the expression of $\hat{I}_R$ in Case II, we know that $\hat{I}_R > 0$ if and only if $R_{RC} > 1$. Under the condition $I^*_{SU} > 0$, from the expression of I^*_R in Case III, we can easily see that $I^*_R > 0$ if and only if $R_{RC} < R_{SC}$, which implies $b > 0$. Return to the expression of I^*_{SU}, we can similarly determine that $I^*_{SU} > 0$ if and only if $R_{SC} > 1$. When $R_C < 1$, i.e., $R_{SC} < 1$ and $R_{RC} < 1$, it follows that $\hat{I}_R < 0$ and $I^*_{SU} < 0$. Thus, when $R_C < 1$, system (2.2) has a unique disease-free equilibrium E^0. When $R_{RC} > 1$, it follows that the boundary equilibrium $\hat{E}$ exists. When $R_{RC} < R_{SC}$ and $R_{SC} > 1$, the interior (positive) equilibrium E^* exists. No matter how R_{SC} and R_{RC} are valued, the disease-free equilibrium E^0 always exists. □

In Table 1, we present a diagram to clearly show the relationship between the existence of equilibria and the values of the parameters R_{SC}, R_{RC} and R_C.

2. Semi-traveling waves for an auxiliary system

In this section, to prove the existence of semi-traveling waves for the original system (2.3) (here, in view of the equivalence between systems (2.1) and (2.3), we also refer to system (2.3) as the original system in the latter study), we first introduce an auxiliary system, the technique of which has been widely used (see [71, 122, 123, 124]). Then, by linearizing the wave equations of the original system (2.3) at disease-free equilibrium E^0, we construct a pair of

Table 1. Existence of equilibria on the values of R_{SC}, R_{RC} and R_C

Parameters \ Equilibria	E^0	$\hat{E}$	E^*
$R_C < 1$	Y	N	N
$R_{SC} > 1 > R_{RC}$	Y	N	Y
$R_{SC} > R_{RC} > 1$	Y	Y	Y
$R_{RC} > 1 > R_{SC}$	Y	Y	N
$R_{RC} > R_{SC} > 1$	Y	Y	N

Remark: Y: Exists; N: Does not exist

upper-lower solutions for the auxiliary system. Finally, we use Schauder's fixed-point theorem to establish the existence of semi-traveling waves for the auxiliary system.

2.1. An auxiliary system. An auxiliary system related to the original system (2.3) can be described by

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \frac{\partial^2 S}{\partial x^2} + \Lambda - \mu S - [\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R]S, \\ \frac{\partial I_{SU}}{\partial t} = d_{SU} \frac{\partial^2 I_{SU}}{\partial x^2} + (1-f)\beta_S(I_{SU} + \delta I_{ST})S - (k_U + \mu)I_{SU} - \Upsilon I_{SU}^2, \\ \frac{\partial I_{ST}}{\partial t} = d_{ST} \frac{\partial^2 I_{ST}}{\partial x^2} + f(1-r)\beta_S(I_{SU} + \delta I_{ST})S - (k_T + \mu)I_{ST} - \Upsilon I_{ST}^2, \\ \frac{\partial I_R}{\partial t} = d_R \frac{\partial^2 I_R}{\partial x^2} + [fr\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R]S - (k_R + \mu)I_R - \Upsilon I_R^2, \end{cases} \tag{2.19}$$

where Υ is a small positive constant.

Substituting the wave profile $S(x,t) = S(\xi), I_i(x,t) = I_i(\xi),\ i = SU, ST, R,\ \xi = x + ct$ into (2.19), and denoting $x + ct$ by ξ, we obtain the corresponding wave equations

$$\begin{cases} cS' = d_S S'' + \Lambda - \mu S - [\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R]S, \\ cI'_{SU} = d_{SU} I''_{SU} + (1-f)\beta_S(I_{SU} + \delta I_{ST})S - (k_U + \mu)I_{SU} - \Upsilon I_{SU}^2, \\ cI'_{ST} = d_{ST} I''_{ST} + f(1-r)\beta_S(I_{SU} + \delta I_{ST})S - (k_T + \mu)I_{ST} - \Upsilon I_{ST}^2, \\ cI'_R = d_R I''_R + [fr\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R]S - (k_R + \mu)I_R - \Upsilon I_R^2. \end{cases} \tag{2.20}$$

The limiting equations of (2.20) when $\Upsilon \to 0$ become the wave equations of the original system (2.3). For the convenience of use, we give their specific form as follows:

$$\begin{cases} cS' = d_S S'' + \Lambda - \mu S - [\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R]S, \\ cI'_{SU} = d_{SU} I''_{SU} + (1-f)\beta_S(I_{SU} + \delta I_{ST})S - (k_U + \mu)I_{SU}, \\ cI'_{ST} = d_{ST} I''_{ST} + f(1-r)\beta_S(I_{SU} + \delta I_{ST})S - (k_T + \mu)I_{ST}, \\ cI'_R = d_R I''_R + [fr\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R]S - (k_R + \mu)I_R. \end{cases} \tag{2.21}$$

2.2. Linearization of the wave system at E^0. Linearizing system (2.21) at the disease-free equilibrium $E^0(S^0, 0, 0, 0)$ and only considering the last three equations of the linearized system, we have

$$\begin{cases} c\varphi'_2 = d_{SU}\varphi''_2 + (1-f)\beta_S(\varphi_2 + \delta\varphi_3)S^0 - (k_U + \mu)\varphi_2, \\ c\varphi'_3 = d_{ST}\varphi''_3 + f(1-r)\beta_S(\varphi_2 + \delta\varphi_3)S^0 - (k_T + \mu)\varphi_3, \\ c\varphi'_4 = d_R\varphi''_4 + [fr\beta_S(\varphi_2 + \delta\varphi_3) + \beta_R\varphi_4]S^0 - (k_R + \mu)\varphi_4, \end{cases} \tag{2.22}$$

where the functions $\varphi_i(\xi),\ i = 2, 3, 4$ correspond to $I_j(\xi),\ j = SU, ST, R$, respectively.

We look for the solutions with the form $(\varphi_2(\xi), \varphi_3(\xi), \varphi_4(\xi)) = e^{\lambda\xi}(\kappa_2, \kappa_3, \kappa_4)$, where $\kappa_i > 0$, $i = 2, 3, 4$ and $\lambda > 0$. Substituting them into equations (2.22), we obtain the following eigenvalue equations

$$\begin{cases} c\lambda\kappa_2 = d_{SU}\lambda^2\kappa_2 + (1-f)\beta_S(\kappa_2 + \delta\kappa_3)S^0 - (k_U + \mu)\kappa_2, \\ c\lambda\kappa_3 = d_{ST}\lambda^2\kappa_3 + f(1-r)\beta_S(\kappa_2 + \delta\kappa_3)S^0 - (k_T + \mu)\kappa_3, \\ c\lambda\kappa_4 = d_R\lambda^2\kappa_4 + [fr\beta_S(\kappa_2 + \delta\kappa_3) + \beta_R\kappa_4]S^0 - (k_R + \mu)\kappa_4. \end{cases} \tag{2.23}$$

Let $\tilde{A} = \mathrm{diag}(d_{SU}, d_{ST}, d_R)$, $\tilde{B} = \mathrm{diag}(c, c, c)$ and $\widetilde{M}(\lambda, c) := \tilde{A}\lambda^2 - \tilde{B}\lambda + F - V$, where the matrices F and V are given by (2.11) and (2.12). Then the eigenvalue equations (2.23) can be rewritten as

$$\widetilde{M}(\lambda, c)\mathcal{K} = 0, \tag{2.24}$$

where $\mathcal{K} = (\kappa_2, \kappa_3, \kappa_4)^T$.

Make the new transformation $A = V^{-1}\tilde{A}$ and $B = V^{-1}\tilde{B}$, we obtain the equivalent form of equation (2.24) as follows

$$M(\lambda, c)\mathcal{K} = \mathcal{K}, \tag{2.25}$$

where $M(\lambda, c) = (-A\lambda^2 + B\lambda + I)^{-1}(V^{-1}F)$.

A direct calculation gives

$$M(\lambda, c) = \begin{pmatrix} \frac{(1-f)\beta_S S^0}{\Theta_2(\lambda,c)} & \frac{(1-f)\beta_S\delta S^0}{\Theta_2(\lambda,c)} & 0 \\ \frac{f(1-r)\beta_S S^0}{\Theta_3(\lambda,c)} & \frac{f(1-r)\beta_S\delta S^0}{\Theta_3(\lambda,c)} & 0 \\ \frac{fr\beta_S S^0}{\Theta_4(\lambda,c)} & \frac{fr\beta_S\delta S^0}{\Theta_4(\lambda,c)} & \frac{\beta_R S^0}{\Theta_4(\lambda,c)} \end{pmatrix}, \tag{2.26}$$

where

$$\Theta_2(\lambda, c) = -d_{SU}\lambda^2 + c\lambda + k_U + \mu,$$
$$\Theta_3(\lambda, c) = -d_{ST}\lambda^2 + c\lambda + k_T + \mu, \Theta_4(\lambda, c) = -d_R\lambda^2 + c\lambda + k_R + \mu.$$

Take $d = \max\{d_{SU}, d_{ST}, d_R\}$, since $\Theta_i(\frac{c}{2d}, c)$ is strictly increasing and nonnegative in $c \in [0, +\infty)$, we can deduce that the matrix $M(\frac{c}{2d}, c)$ is decreasing for $c \in [0, +\infty)$.

Denote by $\rho(M(\lambda, c))$ the principal eigenvalue of the nonnegative matrix $M(\lambda, c)$ for $\lambda \in [0, \frac{c}{2d}]$. Since $\rho(M(\lambda, c))$ is continuous and monotonically increasing with respect to the nonnegative matrix $M(\lambda, c)$, $\rho(M(\frac{c}{2d}, c))$ is strictly decreasing in $c \in [0, +\infty)$. In particular, we have $\rho(M(0, 0)) = \rho(V^{-1}F)$ when $c = 0$ and $\rho(M(\frac{c}{2d}, c)) \to 0$ when $c \to +\infty$.

For the continuation of the analysis, here, we give a brief proof of $\rho(V^{-1}F) = R_C$. By the definition of the control reproduction number R_C in (2.16), we know $R_C = \rho(FV^{-1})$, implying that R_C is the Perron-Frobenius eigenvalue of the matrix FV^{-1}. So there exists a positive eigenvector $P = (p_1, p_2, p_3)$ with $p_i > 0$, $i = 1, 2, 3$ such that $(FV^{-1})P = R_C P$. Then we have $V^{-1}P > 0$ and $(V^{-1}F)(V^{-1}P) = V^{-1}(FV^{-1})P = R_C V^{-1}P$. This implies that R_C is a nonnegative eigenvalue of the matrix $V^{-1}F$ with positive eigenvector $V^{-1}P$. It is easy to see that $V^{-1}F$ is irreducible, that is, $(V^{-1}F + I)^2 > 0$. Using Perron-Frobennius theorem, we get $\rho(V^{-1}F) = R_C$.

Combining with $\rho(M(0, 0)) = \rho(V^{-1}F)$ yields $\rho(M(0, 0)) = R_C$. Consequently, when $R_C > 1$, there exists a unique $c^* > 0$ such that

$$\rho(M(\frac{c}{2d}, c)) \begin{cases} > 1, & c \in [0, c^*); \\ = 1, & c = c^*; \\ < 1, & c \in (c^*, +\infty). \end{cases}$$

Now we fix $c > c^*$, note that $\Theta_i(\lambda, c)(i = 2, 3, 4)$ is strictly increasing in $\lambda \in [0, \frac{c}{2d}]$, then we obtain that $\rho(M(\lambda, c))$ is strictly decreasing and nonnegative in $\lambda \in [0, \frac{c}{2d}]$. In view of the facts $\rho(M(0, c)) = \rho(M(0, 0)) = R_C > 1$ and $\rho(M(\frac{c}{2d}, c)) < 1$, then there exists a $\lambda_c \in (0, \frac{c}{2d})$ such that

$$\rho(M(\lambda, c)) \begin{cases} > 1, & \lambda \in [0, \lambda_c); \\ = 1, & \lambda = \lambda_c; \\ < 1, & \lambda \in (\lambda_c, \frac{c}{2d}]. \end{cases}$$

Based on the above discussion, we have the following lemma.

LEMMA 2.1. *Assume that $R_C = \rho(FV^{-1}) > 1$. Then there exists $c^* > 0$ such that for any $c > c^*$, we can always find $\lambda_c \in (0, \frac{c}{2d})$ and $\mathcal{K}_c = (\kappa_2, \kappa_3, \kappa_4)^T$ with $\kappa_i > 0,\ i = 2, 3, 4$ satisfying $\det \widetilde{M}(\lambda_c, c) = 0$ and $\widetilde{M}(\lambda_c, c)\mathcal{K}_c = 0$.*

Proof: It follows from the above arguments that $\rho(M(\lambda_c, c)) = 1$. By the Perron-Frobenius theorem, we conclude that there is a vector $\mathcal{K}_c \in \mathbb{R}^3$ with positive components such that $M(\lambda_c, c)\mathcal{K}_c = \mathcal{K}_c$. Multiplying the matrix $-A\lambda_c^2 + B\lambda_c + I$ on both sides of the above equality, we have $(A\lambda_c^2 - B\lambda_c + V^{-1}F - I)\mathcal{K}_c = 0$. Multiplying the diagonal matrix V to both sides of the above equality, we obtain $(\tilde{A}\lambda_c^2 - \tilde{B}\lambda_c + F - V)\mathcal{K}_c = \widetilde{M}(\lambda_c, c)\mathcal{K}_c = 0$. □

Let $\mathcal{K}_c = (\kappa_2, \kappa_3, \kappa_4)^T$ as obtained in Lemma 2.1, the following lemma is straightforward.

LEMMA 2.2. *The vector valued function $\varphi(\xi) = (\varphi_2(\xi), \varphi_3(\xi), \varphi_4(\xi))$ with $\varphi_i(\xi) = \kappa_i e^{\lambda_c \xi}$, $i = 2, 3, 4$ satisfies equations (2.22).*

2.3. Construction and properties of upper-lower solutions. In the next subsection, by using the Schauder's fixed-point theorem, we establish the existence of semi-traveling waves of the auxiliary system (2.20). For this, we need to define a pair of upper-lower solutions of system (2.20) as follows.

$$\begin{array}{ll} \bar{S}(\xi) := S^0, & \underline{S}(\xi) := \max\{S^0 - \sigma e^{\alpha\xi}, 0\}, \\ \bar{I}_{SU}(\xi) := \min\{\kappa_2 e^{\lambda_c\xi}, \kappa_2 K^*\}, & \underline{I}_{SU}(\xi) := \max\{\kappa_2 e^{\lambda_c\xi}(1 - Qe^{\varepsilon\xi}), 0\}, \\ \bar{I}_{ST}(\xi) := \min\{\kappa_3 e^{\lambda_c\xi}, \kappa_3 K^*\}, & \underline{I}_{ST}(\xi) := \max\{\kappa_3 e^{\lambda_c\xi}(1 - Qe^{\varepsilon\xi}), 0\}, \\ \bar{I}_R(\xi) := \min\{\kappa_4 e^{\lambda_c\xi}, \kappa_4 K^*\}, & \underline{I}_R(\xi) := \max\{\kappa_4 e^{\lambda_c\xi}(1 - Qe^{\varepsilon\xi}), 0\}, \end{array} \tag{2.27}$$

where the constants $\kappa_2, \kappa_3, \kappa_4$ and λ_c have been determined in Lemma 2.1. The positive constants $K^*, \sigma, \alpha, Q, \varepsilon$ will be determined later.

We next show that such constructed upper and lower solutions satisfy some properties in Lemmas 2.3, 2.4 and 2.5.

LEMMA 2.3. *For $K^* > 1$ large enough, the functions $\bar{I}_{SU}(\xi), \bar{I}_{ST}(\xi)$ and $\bar{I}_R(\xi)$ satisfy the following inequalities*

$$\begin{cases} c\bar{I}'_{SU} \geq d_{SU}\bar{I}''_{SU} + (1-f)\beta_S(\bar{I}_{SU} + \delta\bar{I}_{ST})S^0 - (k_U + \mu)\bar{I}_{SU} - \Upsilon\bar{I}_{SU}^2, \\ c\bar{I}'_{ST} \geq d_{ST}\bar{I}''_{ST} + f(1-r)\beta_S(\bar{I}_{SU} + \delta\bar{I}_{ST})S^0 - (k_T + \mu)\bar{I}_{ST} - \Upsilon\bar{I}_{ST}^2, \\ c\bar{I}'_R \geq d_R\bar{I}''_R + [fr\beta_S(\bar{I}_{SU} + \delta\bar{I}_{ST}) + \beta_R\bar{I}_R]S^0 - (k_R + \mu)\bar{I}_R - \Upsilon\bar{I}_R^2, \end{cases} \tag{2.28}$$

for any $\xi \neq \xi_1 := \frac{\ln K^*}{\lambda_c}$.

Proof: Define the operator

$$L[I_{SU}(\cdot), I_{ST}(\cdot), I_R(\cdot)](\xi) := \begin{pmatrix} d_{SU}I''_{SU} - cI'_{SU} + (1-f)\beta_S(I_{SU} + \delta I_{ST})S^0 - (k_U + \mu)I_{SU} \\ d_{ST}I''_{ST} - cI'_{ST} + f(1-r)\beta_S(I_{SU} + \delta I_{ST})S^0 - (k_T + \mu)I_{ST} \\ d_RI''_R - cI'_R + [fr\beta_S(I_{SU} + \delta I_{ST}) + \beta_RI_R]S^0 - (k_R + \mu)I_R \end{pmatrix},$$

then, the differential inequalities (2.28) can be transformed into the following equivalent operator inequalities

$$L[I_{SU}(\cdot), I_{ST}(\cdot), I_R(\cdot)](\xi) \leq \Upsilon(\bar{I}_{SU}^2, \bar{I}_{ST}^2, \bar{I}_R^2). \tag{2.29}$$

So, as long as we prove operator inequality (2.29), we complete the proof of the lemma. Below, we can prove operator inequalities (2.29) in two cases:

When $\xi < \xi_1$, by (2.27), we have

$$(\bar{I}_{SU}(\xi), \bar{I}_{ST}(\xi), \bar{I}_R(\xi)) = (\kappa_2, \kappa_3, \kappa_4)e^{\lambda_c \xi}.$$

Substituting it into the equations of operator L, yields

$$L[I_{SU}(\cdot), I_{ST}(\cdot), I_R(\cdot)](\xi) = e^{\lambda_c}\widetilde{M}(\lambda_c, c)\mathcal{K}_c = 0.$$

Obviously, operator inequalities (2.29) hold.

When $\xi > \xi_1$, by (2.27), we have $(\bar{I}_{SU}(\xi), \bar{I}_{ST}(\xi), \bar{I}_R(\xi)) = (\kappa_2, \kappa_3, \kappa_4)K^*$. Taking the first inequality of operator inequalities (2.29) as an example, we substitute the upper solutions into it, yielding

$$\begin{aligned} &d_{SU}\bar{I}_{SU}'' - c\bar{I}_{SU}' + (1-f)\beta_S(\bar{I}_{SU} + \delta\bar{I}_{ST})S^0 - (k_U + \mu)\bar{I}_{SU} - \Upsilon\bar{I}_{SU}^2 \\ &= \{[(1-f)\beta_S S^0 - (k_U + \mu)]\kappa_2 + (1-f)\beta_S\delta S^0\kappa_3 - \Upsilon\kappa_2^2 K^*\}K^*. \end{aligned}$$

To ensure that the value of the above equality is smaller or equal to 0, we require

$$K^* > \frac{[(1-f)\beta_S S^0 - (k_U + \mu)]\kappa_2 + (1-f)\beta_S\delta S^0\kappa_3}{\Upsilon\kappa_2^2}.$$

To make the remaining two inequalities of operator inequalities (2.29) also hold, similarly, we can choose

$$K^* > \frac{f(1-r)\beta_S S^0\kappa_2 + [f(1-r)\beta_S\delta S^0 - (k_T + \mu)]\kappa_3}{\Upsilon\kappa_3^2},$$

and

$$K^* > \frac{fr\beta_S S^0(\kappa_2 + \delta\kappa_3) + [\beta_R S^0 - (k_R + \mu)]\kappa_4}{\Upsilon\kappa_4^2}.$$

By selecting $K^* > 1$ satisfying the above three inequalities, we complete the proof of operator inequalities (2.29) when $\xi > \xi_1$. □

LEMMA 2.4. *For* $0 < \alpha < \min\{\frac{c}{d_S}, \lambda_c\}$, $\sigma > \max\{S^0, \frac{[\beta_S(\kappa_2+\delta\kappa_3)+\beta_R\kappa_4]S^0}{(c-d_S\alpha)\alpha+\mu}\}$, *the function* $\underline{S}(\xi)$ *satisfies the following inequality*

$$c\underline{S}' \leq d_S\underline{S}'' + \Lambda - \mu\underline{S} - [\beta_S(\bar{I}_{SU} + \delta\bar{I}_{ST}) + \beta_R\bar{I}_R]\underline{S} \tag{2.30}$$

for any $\xi \neq \xi_2 := \frac{1}{\alpha}\ln\frac{S^0}{\sigma}$.

Proof: If $\xi > \xi_2$, then $\underline{S}(\xi) = 0$. Obviously, the inequality (2.30) holds.

If $\xi < \xi_2$, then $\underline{S}(\xi) = S^0 - \sigma e^{\alpha\xi}$. From the choice of K^* and σ, we knows that $\xi_2 = \frac{1}{\alpha}\ln\frac{S^0}{\sigma} < 0 < \xi_1$, implying $(\bar{I}_{SU}(\xi), \bar{I}_{ST}(\xi), \bar{I}_R(\xi)) = (\kappa_2, \kappa_3, \kappa_4)e^{\lambda_c\xi}$ when $\xi < \xi_2$. Through direct calculations, we have

$$
\begin{aligned}
& d_S\underline{S}'' - c\underline{S}' + \Lambda - \mu\underline{S} - [\beta_S(\bar{I}_{SU} + \delta\bar{I}_{ST}) + \beta_R\bar{I}_R]\underline{S} \\
=& - d_S\sigma\alpha^2 e^{\alpha\xi} + c\sigma\alpha e^{\alpha\xi} + \Lambda - \mu(S^0 - \sigma e^{\alpha\xi}) - [\beta_S(\kappa_2 e^{\lambda_c\xi} + \delta\kappa_3 e^{\lambda_c\xi}) + \beta_R\kappa_4 e^{\lambda_c\xi}](S^0 - \sigma e^{\alpha\xi}) \\
=& \{c\sigma\alpha + \mu\sigma - d_S\sigma\alpha^2 - [\beta_S(\kappa_2 + \delta\kappa_3) + \beta_R\kappa_4](S^0 - \sigma e^{\alpha\xi})e^{(\lambda_c-\alpha)\xi}\}e^{\alpha\xi} \\
\geq& \{(c\alpha + \mu - d_S\alpha^2)\sigma - [\beta_S(\kappa_2 + \delta\kappa_3) + \beta_R\kappa_4]S^0\}e^{\alpha\xi} \\
\geq& 0
\end{aligned}
$$

where we use the fact that $e^{(\lambda_c-\alpha)\xi} < 1$ due to $\alpha < \lambda_c$ and $\xi < 0$, and the conditions that $0 < \alpha < \frac{c}{d_S}$ and $\sigma > \frac{[\beta_S(\kappa_2+\delta\kappa_3)+\beta_R\kappa_4]S^0}{(c-d_S\alpha)\alpha+\mu}$. □

LEMMA 2.5. *Let $\varepsilon > 0$ be small enough with $\varepsilon < \alpha, \varepsilon < \lambda_c$ and $\lambda_c + \varepsilon < \frac{c}{2d}$, then for sufficiently large $Q > 1$, the functions $\underline{I}_{SU}(\xi), \underline{I}_{ST}(\xi)$ and $\underline{I}_R(\xi)$ satisfy the following inequalities*

$$
\begin{cases}
c\underline{I}'_{SU} \leq d_{SU}\underline{I}''_{SU} + (1-f)\beta_S(\underline{I}_{SU} + \delta\underline{I}_{ST})\underline{S} - (k_U + \mu)\underline{I}_{SU} - \Upsilon\underline{I}^2_{SU}, \\
c\underline{I}'_{ST} \leq d_{ST}\underline{I}''_{ST} + f(1-r)\beta_S(\underline{I}_{SU} + \delta\underline{I}_{ST})\underline{S} - (k_T + \mu)\underline{I}_{ST} - \Upsilon\underline{I}^2_{ST}, \\
c\underline{I}'_R \leq d_R\underline{I}''_R + [fr\beta_S(\underline{I}_{SU} + \delta\underline{I}_{ST}) + \beta_R\underline{I}_R]\underline{S} - (k_R + \mu)\underline{I}_R - \Upsilon\underline{I}^2_R,
\end{cases}
\tag{2.31}
$$

for any $\xi \neq \xi_3 := -\frac{\ln Q}{\varepsilon}$.

Proof: Choose $Q > 1$ sufficiently large and ε small enough such that $\xi_3 < \xi_2 < 0$, this implies that $Q > \max\{(\frac{\sigma}{S^0})^{\frac{\varepsilon}{\alpha}}, 1\}$.

When $\xi > \xi_3$, based on the definition of the lower solutions in (2.27), we have $\underline{I}_{SU}(\xi) = \underline{I}_{ST}(\xi) = \underline{I}_R(\xi) = 0$. It is clear that inequalities (2.31) hold.

When $\xi < \xi_3$, by (2.27), we have $(\underline{I}_{SU}(\xi), \underline{I}_{ST}(\xi), \underline{I}_R(\xi)) = (\kappa_2, \kappa_3, \kappa_4)e^{\lambda_c\xi}(1 - Qe^{\varepsilon\xi})$ and $\underline{S}(\xi) = S^0 - \sigma e^{\alpha\xi}$. For the first inequality of (2.31), we can show

$$
\begin{aligned}
& c\underline{I}'_{SU} - d_{SU}\underline{I}''_{SU} - (1-f)\beta_S(\underline{I}_{SU} + \delta\underline{I}_{ST})\underline{S} + (k_U + \mu)\underline{I}_{SU} + \Upsilon\underline{I}^2_{SU} \\
=& c\kappa_2 e^{\lambda_c\xi}[\lambda_c(1 - Qe^{\varepsilon\xi}) - Q\varepsilon e^{\varepsilon\xi}] - d_{SU}\kappa_2 e^{\lambda_c\xi}[\lambda_c^2(1 - Qe^{\varepsilon\xi}) - \lambda_c Q\varepsilon e^{\varepsilon\xi} - (\lambda_c + \varepsilon)Q\varepsilon e^{\varepsilon\xi}] \\
& - (1-f)\beta_S(\kappa_2 + \delta\kappa_3)e^{\lambda_c\xi}(1 - Qe^{\varepsilon\xi})(S^0 - \sigma e^{\alpha\xi}) + (k_U + \mu)\kappa_2 e^{\lambda_c\xi}(1 - Qe^{\varepsilon\xi}) \\
& + \Upsilon[\kappa_2 e^{\lambda_c\xi}(1 - Qe^{\varepsilon\xi})]^2 \\
=& e^{\lambda_c\xi}(1 - Qe^{\varepsilon\xi})\{[-d_{SU}\lambda_c^2 + c\lambda_c - (1-f)\beta_S S^0 + k_U + \mu]\kappa_2 - (1-f)\beta_S\delta S^0\kappa_3\} \\
& + e^{(\lambda_c+\varepsilon)\xi}\{[(2\lambda_c + \varepsilon)d_{SU} - c]Q\varepsilon\kappa_2 + (1-f)\beta_S(\kappa_2 + \delta\kappa_3)\sigma e^{(\alpha-\varepsilon)\xi}(1 - Qe^{\varepsilon\xi}) \\
& + \Upsilon\kappa_2^2 e^{(\lambda_c-\varepsilon)\xi}(1 - Qe^{\varepsilon\xi})^2\} \\
=& e^{(\lambda_c+\varepsilon)\xi}\{[(2\lambda_c + \varepsilon)d_{SU} - c]Q\varepsilon\kappa_2 + (1-f)\beta_S(\kappa_2 + \delta\kappa_3)\sigma e^{(\alpha-\varepsilon)\xi}(1 - Qe^{\varepsilon\xi}) \\
& + \Upsilon\kappa_2^2 e^{(\lambda_c-\varepsilon)\xi}(1 - Qe^{\varepsilon\xi})^2\} \\
\leq& e^{(\lambda_c+\varepsilon)\xi}\{[(2\lambda_c + \varepsilon)d_{SU} - c]Q\varepsilon\kappa_2 + (1-f)\beta_S(\kappa_2 + \delta\kappa_3)\sigma e^{-(\alpha-\varepsilon)\frac{\ln Q}{\varepsilon}} + \Upsilon\kappa_2^2 e^{-(\lambda_c-\varepsilon)\frac{\ln Q}{\varepsilon}}\},
\end{aligned}
$$

where we use the conditions $\varepsilon < \alpha$ and $\varepsilon < \lambda_c$.

In view of the condition that $\varepsilon > 0$ and $\lambda_c + \varepsilon < \frac{c}{2d}$, we have

$$(2\lambda_c + \varepsilon)d_{SU} - c < 2(\lambda_c + \varepsilon)d_{SU} - c < 2(\lambda_c + \varepsilon)d - c < 0.$$

Thus, we can choose sufficiently large $Q > 1$ such that

$$[(2\lambda_c + \varepsilon)d_{SU} - c]Q\varepsilon\kappa_2 + (1-f)\beta_S(\kappa_2 + \delta\kappa_3)\sigma e^{-(\alpha-\varepsilon)\frac{\ln Q}{\varepsilon}} + \Upsilon\kappa_2^2 e^{-(\lambda_c-\varepsilon)\frac{\ln Q}{\varepsilon}} \leq 0,$$

indicating that the first inequality of (2.31) holds. Similarly, we can verify the second and third inequalities of (2.31) also hold. □

2.4. The existence of semi-traveling waves. We look for semi-traveling waves of the auxiliary system (2.20) in the following profile set

$$\begin{aligned}\mathfrak{L} = \{&(S(\cdot), I_{SU}(\cdot), I_{ST}(\cdot), I_R(\cdot)) \in C_\nu(\mathbb{R}, \mathbb{R}^4) : \\ &\underline{S}(\xi) \le S(\xi) \le \bar{S}(\xi), \underline{I}_j(\xi) \le I_j(\xi) \le \bar{I}_j(\xi), j = SU, ST, R, \text{ for all } \xi \in \mathbb{R}\}.\end{aligned} \tag{2.32}$$

Note that $C_\nu(\mathbb{R}, \mathbb{R}^4)$ is a Banach space with the norm $\|\cdot\|$ formulated by

$$\|\Phi\| := |\Phi(\cdot)|_\nu = \max\{\sup_{\xi\in\mathbb{R}} |\varphi_i(\xi)|e^{-\nu|\xi|},\ i = S, SU, ST, R\}, \tag{2.33}$$

where $\Phi(\xi) = (\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi), \varphi_4(\xi)) \in C_\nu(\mathbb{R}, \mathbb{R}^4)$, ν is a positive constant which will be determined later. Obviously, $\mathfrak{L}$ is a bounded nonempty closed convex subset of $C_\nu(\mathbb{R}, \mathbb{R}^4)$.

$C_\nu(\mathbb{R}, \mathbb{R})$ is a Banach space with the sup norm $|\varphi(\cdot)|_\nu := \sup_{\xi\in\mathbb{R}} |\varphi(\xi)|e^{-\nu|\xi|}$, where $\varphi(\xi) \in C_\nu(\mathbb{R}, \mathbb{R})$. Let $T_i : \mathfrak{L} \to C_\nu(\mathbb{R}, \mathbb{R}),\ i = S, SU, ST, R$ be operators defined by

$$T_S(S, I_{SU}, I_{ST}, I_R)(\xi) := \vartheta_S S(\xi) + \Lambda - \mu S(\xi) - [\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi)) + \beta_R I_R(\xi)]S(\xi),$$

$$T_{SU}(S, I_{SU}, I_{ST}, I_R)(\xi) := \vartheta_{SU} I_{SU}(\xi) + (1-f)\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi))S(\xi) - (k_U + \mu)I_{SU}(\xi) - \Upsilon I_{SU}^2(\xi),$$

$$T_{ST}(S, I_{SU}, I_{ST}, I_R)(\xi) := \vartheta_{ST} I_{ST}(\xi) + f(1-r)\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi))S(\xi) - (k_T + \mu)I_{ST}(\xi) - \Upsilon I_{ST}^2(\xi),$$

$$T_R(S, I_{SU}, I_{ST}, I_R)(\xi) := \vartheta_R I_R(\xi) + [fr\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi)) + \beta_R I_R(\xi)]S(\xi) - (k_R + \mu)I_R(\xi) - \Upsilon I_R^2(\xi),$$

where

$$\begin{aligned}&\vartheta_S > \mu + [\beta_S(\kappa_2 + \delta\kappa_3) + \beta_R\kappa_4]K^*,\\ &\vartheta_{SU} > 2\Upsilon\kappa_2 K^* + (k_U + \mu),\\ &\vartheta_{ST} > 2\Upsilon\kappa_3 K^* + (k_T + \mu), \vartheta_R > 2\Upsilon\kappa_4 K^* + (k_R + \mu).\end{aligned}$$

Then the auxiliary system (2.20) can be now rewritten as

$$\begin{cases} -d_S S''(\xi) + cS'(\xi) + \vartheta_S S(\xi) = T_S(S, I_{SU}, I_{ST}, I_R)(\xi),\\ -d_{SU} I''_{SU}(\xi) + cI'_{SU}(\xi) + \vartheta_{SU} I_{SU}(\xi) = T_{SU}(S, I_{SU}, I_{ST}, I_R)(\xi),\\ -d_{ST} I''_{ST}(\xi) + cI'_{ST}(\xi) + \vartheta_{ST} I_{ST}(\xi) = T_{ST}(S, I_{SU}, I_{ST}, I_R)(\xi),\\ -d_R I''_R(\xi) + cI'_R(\xi) + \vartheta_R I_R(\xi) = T_R(S, I_{SU}, I_{ST}, I_R)(\xi). \end{cases} \tag{2.34}$$

Let $\zeta_{i1} < 0 < \zeta_{i2}$ be the roots of the quadratic equation

$$d_i\zeta_i^2 - c\zeta_i - \vartheta_i = 0,$$

then, define the operators $G_i : \mathfrak{L} \to C_\nu(\mathbb{R}, \mathbb{R})$ by

$$\begin{aligned}G_i(S, I_{SU}, I_{ST}, I_R)(\xi) := &\frac{1}{d_i\zeta_i}\\ &\left[\int_{-\infty}^{\xi} e^{\zeta_{i1}(\xi-s)} T_i(S, I_{SU}, I_{ST}, I_R)(s)ds + \int_{\xi}^{+\infty} e^{\zeta_{i2}(\xi-s)} T_i(S, I_{SU}, I_{ST}, I_R)(s)ds\right],\end{aligned}$$

where $\zeta_i = \zeta_{i2} - \zeta_{i1},\ i = S, SU, ST, R$.

Then $G = (G_S, G_{SU}, G_{ST}, G_R) : \mathfrak{L} \to C_\nu(\mathbb{R}, \mathbb{R}^4)$ is a well-defined map, and satisfies

$$\begin{cases} -d_S G''_S(\xi) + cG'_S(\xi) + \vartheta_S G_S(\xi) = T_S(G_S, G_{SU}, G_{ST}, G_R)(\xi),\\ -d_{SU} G''_{SU}(\xi) + cG'_{SU}(\xi) + \vartheta_{SU} G_{SU}(\xi) = T_{SU}(G_S, G_{SU}, G_{ST}, G_R)(\xi),\\ -d_{ST} G''_{ST}(\xi) + cG'_{ST}(\xi) + \vartheta_{ST} G_{ST}(\xi) = T_{ST}(G_S, G_{SU}, G_{ST}, G_R)(\xi),\\ -d_R G''_R(\xi) + cG'_R(\xi) + \vartheta_R G_R(\xi) = T_R(G_S, G_{SU}, G_{ST}, G_R)(\xi), \end{cases} \tag{2.35}$$

for any $(S(\cdot), I_{SU}(\cdot), I_{ST}(\cdot), I_R(\cdot)) \in C_\nu(\mathbb{R}, \mathbb{R}^4)$. Thus, any fixed point of the operator G is a solution of (2.34), which is a traveling wave of the auxiliary system (2.20). On the other hand, a solution of (2.34) is also a fixed point of the operator G.

To apply Schauder's fixed-point theorem, we need to prove that the operators G_S, G_{SU}, G_{ST} and G_R admit the following properties:

LEMMA 2.6. *The operator G maps $\mathfrak{L}$ into $\mathfrak{L}$, i.e., $G(\mathfrak{L}) \subset \mathfrak{L}$.*

Proof: If $(S(\cdot), I_{SU}(\cdot), I_{ST}(\cdot), I_R(\cdot)) \in \mathfrak{L}$, that is,

$$\underline{S}(\xi) \leq S(\xi) \leq \bar{S}(\xi) = S^0,\ \underline{I}_i(\xi) \leq I_i(\xi) \leq \bar{I}_i(\xi),\ i = SU, ST, R$$

for any $\xi \in \mathbb{R}$. Then it suffices to show

$$\underline{S}(\xi) \leq G_S(S, I_{SU}, I_{ST}, I_R)(\xi) \leq \bar{S}(\xi) = S^0,$$

$$\underline{I}_i(\xi) \leq G_i(S, I_{SU}, I_{ST}, I_R)(\xi) \leq \bar{I}_i(\xi),\ i = SU, ST, R.$$

We now prove the first inequality about the operator G_S. First consider the left-hand side of the first inequality.
If $\xi \geq \xi_2$, then $\underline{S}(\xi) = 0$ by (2.27). It follows from the choice of ϑ_S that $T_S(S, I_{SU}, I_{ST}, I_R)(\xi) \geq 0$ for all $\xi \in \mathbb{R}$, which implies that $G_S(S, I_{SU}, I_{ST}, I_R)(\xi) \geq 0 = \underline{S}(\xi)$ when $\xi \geq \xi_2$.
If $\xi < \xi_2$, then by Lemma 2.4, we obtain

$$\begin{aligned}
-d_S\underline{S}''(\xi) + c\underline{S}'(\xi) + \vartheta_S\underline{S}(\xi) &\leq \vartheta_S\underline{S}(\xi) + \Lambda - \mu\underline{S}(\xi) - [\beta_S(\bar{I}_{SU}(\xi) + \delta\bar{I}_{ST}(\xi)) + \beta_R\bar{I}_R(\xi)]\underline{S}(\xi) \\
&= \{\vartheta_S - \mu - [\beta_S(\bar{I}_{SU}(\xi) + \delta\bar{I}_{ST}(\xi)) + \beta_R\bar{I}_R(\xi)]\}\underline{S}(\xi) + \Lambda \\
&\leq \{\vartheta_S - \mu - [\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi)) + \beta_R I_R(\xi)]\}S(\xi) + \Lambda \\
&= T_S(S, I_{SU}, I_{ST}, I_R)(\xi),
\end{aligned}$$

which follows

$$\begin{aligned}
G_S(S, I_{SU}, I_{ST}, I_R)(\xi) =& \frac{1}{d_S\zeta_S}\left[\int_{-\infty}^{\xi} e^{\zeta_{S1}(\xi-s)} + \int_{\xi}^{+\infty} e^{\zeta_{S2}(\xi-s)}\right] T_S(S, I_{SU}, I_{ST}, I_R)(s)ds \\
\geq& \frac{1}{d_S\zeta_S}\left[\int_{-\infty}^{\xi} e^{\zeta_{S1}(\xi-s)} + \int_{\xi}^{+\infty} e^{\zeta_{S2}(\xi-s)}\right] [-d_S\underline{S}''(s) + c\underline{S}'(s) + \vartheta_S\underline{S}(s)]ds \\
=& \frac{1}{d_S\zeta_S}\int_{-\infty}^{\xi} e^{\zeta_{S1}(\xi-s)}[-d_S\underline{S}''(s) + c\underline{S}'(s) + \vartheta_S\underline{S}(s)]ds \\
&+ \frac{1}{d_S\zeta_S}\int_{\xi}^{\xi_2} e^{\zeta_{S2}(\xi-s)}[-d_S\underline{S}''(s) + c\underline{S}'(s) + \vartheta_S\underline{S}(s)]ds \\
&+ \frac{1}{d_S\zeta_S}\int_{\xi_2}^{+\infty} e^{\zeta_{S2}(\xi-s)}[-d_S\underline{S}''(s) + c\underline{S}'(s) + \vartheta_S\underline{S}(s)]ds \\
=& \underline{S}(\xi) + \frac{1}{\zeta_S}e^{\zeta_{S2}(\xi-\xi_2)}[\underline{S}'(\xi_2+0) - \underline{S}'(\xi_2-0)] \\
\geq& \underline{S}(\xi),
\end{aligned}$$

where the final inequality uses the fact that $\underline{S}'(\xi_2+0) = 0$ and $\underline{S}'(\xi_2-0) < 0$.

To summarize, $G_S(S, I_{SU}, I_{ST}, I_R)(\xi) \geq \underline{S}(\xi)$ holds for all $\xi \in \mathbb{R}$.

Next, we prove the right-hand side of the first inequality, i.e., $G_S(S, I_{SU}, I_{ST}, I_R)(\xi) \leq \bar{S}(\xi) = S^0$ for all $\xi \in \mathbb{R}$.

It is easy to verify the validity of the following inequality for all $\xi \in \mathbb{R}$

$$c\bar{S}'(\xi) \geq d_S\bar{S}''(\xi) + \Lambda - \mu\bar{S}(\xi) - [\beta_S(\underline{I}_{SU}(\xi) + \delta\underline{I}_{ST}(\xi)) + \beta_R\underline{I}_R(\xi)]\bar{S}(\xi),$$

it follows that

$$\begin{aligned}-d_S\bar{S}''(\xi) + c\bar{S}'(\xi) + \vartheta_S\bar{S}(\xi) &\geq \vartheta_S\bar{S}(\xi) + \Lambda - \mu\bar{S}(\xi) - [\beta_S(\underline{I}_{SU}(\xi) + \delta\underline{I}_{ST}(\xi)) + \beta_R\underline{I}_R(\xi)]\bar{S}(\xi)\\ &= \{\vartheta_S - \mu - [\beta_S(\underline{I}_{SU}(\xi) + \delta\underline{I}_{ST}(\xi)) + \beta_R\underline{I}_R(\xi)]\}\bar{S}(\xi) + \Lambda\\ &\geq \{\vartheta_S - \mu - [\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi)) + \beta_R I_R(\xi)]\}S(\xi) + \Lambda\\ &= T_S(S, I_{SU}, I_{ST}, I_R)(\xi).\end{aligned}$$

Through the above inequality, we can prove

$$\begin{aligned}G_S(S, I_{SU}, I_{ST}, I_R)(\xi) =&\frac{1}{d_S\zeta_S}\left[\int_{-\infty}^{\xi} e^{\zeta_{S1}(\xi-s)} + \int_{\xi}^{+\infty} e^{\zeta_{S2}(\xi-s)}\right] T_S(S, I_{SU}, I_{ST}, I_R)(s)ds\\ \leq&\frac{1}{d_S\zeta_S}\left[\int_{-\infty}^{\xi} e^{\zeta_{S1}(\xi-s)} + \int_{\xi}^{+\infty} e^{\zeta_{S2}(\xi-s)}\right] [-d_S\bar{S}''(s) + c\bar{S}'(s) + \vartheta_S\bar{S}(s)]ds\\ =&\frac{1}{d_S\zeta_S}\int_{-\infty}^{\xi} e^{\zeta_{S1}(\xi-s)}[-d_S\bar{S}''(s) + c\bar{S}'(s) + \vartheta_S\bar{S}(s)]ds\\ &+ \frac{1}{d_S\zeta_S}\int_{+\infty}^{\xi} e^{\zeta_{S2}(\xi-s)}[-d_S\bar{S}''(s) + c\bar{S}'(s) + \vartheta_S\bar{S}(s)]ds\\ =&\bar{S}(\xi),\end{aligned}$$

for all $\xi \in \mathbb{R}$.

In a similar way, we can also show that the remaining three inequalities about operators $G_i,\ i = SU, ST, R$ hold for any $\xi \in \mathbb{R}$. □

In what follows, we shall apply Schauder's fixed-point theorem to the operator G, which requires the continuity and compactness of G. To achieve the two properties, we need to introduce a topology in $C_\nu(\mathbb{R}, \mathbb{R}^4)$. Let $\nu \in (0, \min\{-\zeta_{i1}, \zeta_{i2},\ i = S, SU, ST, R\})$. Denote

$$B_\nu(\mathbb{R}, \mathbb{R}^4) = \{\Phi(\xi) = (\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi), \varphi_4(\xi)) \in C_\nu(\mathbb{R}, \mathbb{R}^4) : |\Phi(\cdot)|_\nu < +\infty\}$$

with the same norm as that in (2.33), then it is easy to verify that $(B_\nu(\mathbb{R}, \mathbb{R}^4), |\cdot|_\nu)$ is a Banach space.

LEMMA 2.7. *The operator $G = (G_S, G_{SU}, G_{ST}, G_R) : \mathfrak{L} \to \mathfrak{L}$ is continuous with respect to the norm $|\cdot|_\nu$ in $B_\nu(\mathbb{R}, \mathbb{R}^4)$.*

Proof: For $\Phi(\xi) = (\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi), \varphi_4(\xi)), \Psi(\xi) = (\psi_1(\xi), \psi_2(\xi), \psi_3(\xi), \psi_4(\xi)) \in \mathfrak{L}$, by the definition of the operator T_S, we easily get

$$\begin{aligned}|T_S(\Phi)(\xi) - T_S(\Psi)(\xi)|e^{-\nu|\xi|} =&|\{\vartheta_S - \mu - [\beta_S(\varphi_2(\xi) + \delta\varphi_3(\xi)) + \beta_R\varphi_4(\xi)]\}(\varphi_1(\xi) - \psi_1(\xi))\\ &+ \psi_1(\xi)\{\beta_S[(\psi_2(\xi) - \varphi_2(\xi)) + \delta(\psi_3(\xi) - \varphi_3(\xi))]\\ &+ \beta_R(\psi_4(\xi) - \varphi_4(\xi))\}|e^{-\nu|\xi|}\\ \leq& N_S|\Phi(\cdot) - \Psi(\cdot)|_\nu,\end{aligned}$$

where $N_S = \vartheta_S - \mu - [\beta_S(\kappa_2 + \delta\kappa_3) + \beta_R\kappa_4]K^* + [\beta_S(1 + \delta) + \beta_R]S^0 > 0$.

Then, by the definition of the operator G_S, we obtain

$$\begin{aligned}|G_S(\Phi)(\xi)-G_S(\Psi)(\xi)|e^{-\nu|\xi|} \leq & \frac{e^{-\nu|\xi|}}{d_S\zeta_S}\left[\int_{-\infty}^{\xi} e^{\zeta_{S1}(\xi-s)}+\int_{\xi}^{+\infty} e^{\zeta_{S2}(\xi-s)}\right]|T_S(\Phi)(s)-T_S(\Psi)(s)|ds \\ \leq & \frac{N_S e^{-\nu|\xi|}}{d_S\zeta_S}\left[\int_{-\infty}^{\xi} e^{\zeta_{S1}(\xi-s)+\nu|s|}ds+\int_{\xi}^{+\infty} e^{\zeta_{S2}(\xi-s)+\nu|s|}ds\right]|\Phi(\cdot)-\Psi(\cdot)|_\nu \\ = & \frac{N_S}{d_S\zeta_S}\left[\int_{-\infty}^{\xi} e^{\zeta_{S1}(\xi-s)-\nu|\xi|+\nu|s|}ds+\int_{\xi}^{+\infty} e^{\zeta_{S2}(\xi-s)-\nu|\xi|+\nu|s|}ds\right]|\Phi(\cdot)-\Psi(\cdot)|_\nu \\ \leq & \frac{N_S}{d_S\zeta_S}\left[\int_{-\infty}^{\xi} e^{\zeta_{S1}(\xi-s)+\nu|\xi-s|}ds+\int_{\xi}^{+\infty} e^{\zeta_{S2}(\xi-s)+\nu|\xi-s|}ds\right]|\Phi(\cdot)-\Psi(\cdot)|_\nu \\ = & \frac{N_S}{d_S\zeta_S}\frac{\zeta_{S1}-\zeta_{S2}+2\nu}{(\zeta_{S1}+\nu)(\zeta_{S2}-\nu)}|\Phi(\cdot)-\Psi(\cdot)|_\nu.\end{aligned}$$

Hence, $G_S:\mathfrak{L}\to C_\nu(\mathbb{R},\mathbb{R})$ is continuous with respect to the norm $|\cdot|_\nu$ in $B_\nu(\mathbb{R},\mathbb{R})$. Similarly, we can show the remaining operators $G_i:\mathfrak{L}\to C_\nu(\mathbb{R},\mathbb{R}),\ i=SU,ST,R$ are also continuous with respect to the norm $|\cdot|_\nu$ in $B_\nu(\mathbb{R},\mathbb{R})$. This implies that $G:\mathfrak{L}\to\mathfrak{L}$ is continuous with respect to the norm $|\cdot|_\nu$ in $B_\nu(\mathbb{R},\mathbb{R}^4)$. □

LEMMA 2.8. *The operator $G=(G_S,G_{SU},G_{ST},G_R):\mathfrak{L}\to\mathfrak{L}$ is compact with respect to the norm $|\cdot|_\nu$ in $B_\nu(\mathbb{R},\mathbb{R}^4)$.*

Proof: Let $\Phi(\xi)=(\varphi_1(\xi),\varphi_2(\xi),\varphi_3(\xi),\varphi_4(\xi))\in\mathfrak{L}$, obviously, for all $\xi\in\mathbb{R}$, we have

$$|T_S(\Phi)(\xi)|=|\vartheta_S\varphi_1(\xi)+\Lambda-\mu\varphi_1(\xi)-[\beta_S(\varphi_2(\xi)+\delta\varphi_3(\xi))+\beta_R\varphi_4(\xi)]\varphi_1(\xi)|\leq\tilde{N}_S,$$

where $\tilde{N}_S=\Lambda+\{\vartheta_S+\mu+[\beta_S(\kappa_2+\delta\kappa_3)+\beta_R\kappa_4]K^*\}S^0$.

Consequently,

$$\begin{aligned}\left|\frac{d}{d\xi}G_S(\Phi)(\xi)\right| = & \frac{1}{d_S\zeta_S}\left|\left[\zeta_{S1}\int_{-\infty}^{\xi} e^{\zeta_{S1}(\xi-s)}+\zeta_{S2}\int_{\xi}^{+\infty} e^{\zeta_{S2}(\xi-s)}\right]T_S(\Phi)(s)ds\right| \\ \leq & \frac{\tilde{N}_S}{d_S\zeta_S}\left[|\zeta_{S1}|\int_{-\infty}^{\xi} e^{\zeta_{S1}(\xi-s)}ds+\zeta_{S2}\int_{\xi}^{+\infty} e^{\zeta_{S2}(\xi-s)}ds\right] \\ = & \frac{2\tilde{N}_S}{d_S\zeta_S},\end{aligned}$$

which implies $\left|\frac{d}{d\xi}G_S(\Phi)(\cdot)\right|_\nu<\frac{2\tilde{N}_S}{d_S\zeta_S}$.

So, we see that $|\frac{d}{d\xi}G_S(\Phi)(\xi)|_\nu$ is bounded. Using the similar arguments as above, we can show that $|\frac{d}{d\xi}G_i(\Phi)(\xi)|_\nu, i=SU,ST,R$ are also bounded. This means that $G(\mathfrak{L})$ is uniformly bounded and equicontinuous with respect to the norm $|\cdot|_\nu$.

For fixed positive integer n, we define an operator $G^n=(G^n_S,G^n_{SU},G^n_{ST},G^n_R)$ by

$$G^n(\Phi)(\xi)=\begin{cases}G(\Phi)(-n),\xi\in(-\infty,-n],\\ G(\Phi)(\xi),\xi\in[-n,n],\\ G(\Phi)(n),\xi\in[n,+\infty).\end{cases}$$

By Arzelà-Ascoli theorem, $G^n : \mathfrak{L} \to \mathfrak{L}$ is compact with respect to the norm $|\cdot|_\nu$ in $B_\nu(\mathbb{R}, \mathbb{R}^4)$. Since

$$\begin{aligned}|G_S(\Phi)(\xi)| =& \frac{1}{d_S\zeta_S}\left|\left[\int_{-\infty}^{\xi} e^{\zeta_{S1}(\xi-s)} + \int_{\xi}^{+\infty} e^{\zeta_{S2}(\xi-s)}\right] T_S(\Phi)(s)ds\right| \\ \leq& \frac{\tilde{N}_S}{d_S\zeta_S}\left[\int_{-\infty}^{\xi} e^{\zeta_{S1}(\xi-s)}ds + \int_{\xi}^{+\infty} e^{\zeta_{S2}(\xi-s)}ds\right] \\ =& \frac{\tilde{N}_S}{d_S|\zeta_{S1}|\zeta_{S2}},\end{aligned}$$

we have

$$\begin{aligned}|G_S^n(\Phi)(\cdot) - G_S(\Phi)(\cdot)|_\nu &= \sup_{\xi\in R} |G_S^n(\Phi)(\xi) - G_S(\Phi)(\xi)|e^{-\nu|\xi|} \\ &= \sup_{\xi\in(-\infty,-n]\cup[n,+\infty)} |G_S^n(\Phi(\cdot))(\xi) - G_S(\Phi(\cdot))(\xi)|e^{-\nu|\xi|} \\ &\leq \frac{2\tilde{N}_S}{d_S|\zeta_{S1}|\zeta_{S2}}e^{-\nu n}.\end{aligned}$$

When $n \to +\infty$, we have $|G_S^n(\Phi)(\cdot) - G_S(\Phi)(\cdot)|_\nu \to 0$. By similar arguments, we can also show that $|G_i^n(\Phi)(\cdot) - G_i(\Phi)(\cdot)|_\nu \to 0$ when $n \to +\infty$ for $i = SU, ST, R$.

Overall, $|G^n(\Phi)(\cdot) - G(\Phi)(\cdot)|_\nu \to 0$ when $n \to +\infty$. By Proposition 2.12 in [121], we know that G^n converges to G in $\mathfrak{L}$ with respect to the norm $|\cdot|_\nu$. Therefore, the operator $G = (G_S, G_{SU}, G_{ST}, G_R) : \mathfrak{L} \to \mathfrak{L}$ is compact with respect to the norm $|\cdot|_\nu$ in $B_\nu(\mathbb{R}, \mathbb{R}^4)$. □

Now we state our main results of this section as follows.

LEMMA 2.9. *If $R_C > 1$, then there exists $c^* > 0$, defined by Lemma 2.1, such that for any $c > c^*$, the auxiliary system* (2.20) *admits a nonnegative bounded semi-traveling wave* $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi))$ *satisfying the asymptotic boundary condition* (2.5).

Proof: Based on the above discussion, we conclude that there exists a fixed point of the operator G, denoted by $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi)) \in \mathfrak{L}$, by Schauder's fixed-point theorem and Lemmas 2.6, 2.7 and 2.8, which is equivalent to say that $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi))$ is a nonnegative bounded traveling wave of the auxiliary system (2.20).

We can further show that $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi))$ satisfies the asymptotic boundary condition (2.5). It is easy to see that $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi)) \to E^0(S^0, 0, 0, 0)$ when $\xi \to -\infty$ due to the definition of upper-lower solutions in (2.27) and Lemmas 2.3, 2.4 and 2.5. So, the nonnegative bounded traveling wave $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi))$ is a semi-traveling wave satisfying the asymptotic boundary condition (2.5). □

3. Semi-, strong and weak traveling waves for the original system

In this section, we discuss the conditions for the existence of three kinds of traveling waves starting from the disease-free equilibrium E^0: semi-traveling waves, strong traveling waves and weak (persistent) traveling waves.

3.1. Semi-traveling waves.

THEOREM 3.1. *If $R_C > 1$, then there exists $c^* > 0$ (defined by Lemma 2.1) such that for any $c > c^*$, system (2.3) admits a positive semi-traveling wave $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi))$ satisfying the asymptotic boundary condition (2.5) and $S(\xi) < S^0$ for any $\xi \in R$. Furthermore,*

$$\lim_{\xi \to -\infty} I_i(\xi) e^{-\lambda_c \xi} = \kappa_i, \ \lim_{\xi \to -\infty} I_i'(\xi) e^{-\lambda_c \xi} = \lambda_c \kappa_i, \ i = SU, ST, R, \tag{2.36}$$

where $\kappa_{SU} = \kappa_2, \kappa_{ST} = \kappa_3, \kappa_R = \kappa_4$.

Proof: Set $\Upsilon = \Upsilon_n := \frac{1}{n}$. Obviously, the sequence $\{\Upsilon_n\}$ satisfies $0 < \Upsilon_{i+1} < \Upsilon_i < 1$ and $\Upsilon_n \to 0$ as $n \to +\infty$. By Lemma 2.9, there is a nonnegative semi-traveling wave

$$\Phi_n(\xi) = (\varphi_{1n}(\xi), \varphi_{2n}(\xi), \varphi_{3n}(\xi), \varphi_{4n}(\xi)) \in \mathfrak{L}$$

of the auxiliary system (2.20) with $\Upsilon = \Upsilon_n$ satisfying the asymptotic boundary condition

$$(\varphi_{1n}(-\infty), \varphi_{2n}(-\infty), \varphi_{3n}(-\infty), \varphi_{4n}(-\infty)) = E^0(S^0, 0, 0, 0).$$

From the proof of Lemma 2.8, we know that $\{|\varphi_{in}'(\xi)|\}$ are uniformly bounded for $i = 1, 2, 3, 4$ since $\Phi_n(\xi) \in \mathfrak{L}$ is a fixed point of the operator G. In addition, $\{|\varphi_{in}''(\xi)|\}$ and $\{|\varphi_{in}'''(\xi)|\}$ are also uniformly bounded since $\Phi_n(\xi)$ is the solution of the auxiliary system (2.20) with $\Upsilon = \Upsilon_n$. Therefore, $\{\Phi_n(\xi)\}, \{\Phi_n'(\xi)\}, \{\Phi_n''(\xi)\}$ are equicontinuous and uniformly bounded in $\mathbb{R}$. Then Arzelà-Ascoli theorem implies that there exists a subsequence $\{\Upsilon_{n_k}\}$ such that

$$\Phi_{n_k}(\xi) \to \Psi(\xi), \Phi_{n_k}'(\xi) \to \Psi'(\xi), \Phi_{n_k}''(\xi) \to \Psi''(\xi)$$

uniformly in any bounded closed interval when $k \to +\infty$, and pointwise on $\mathbb{R}$, where $\Psi(\xi) = (\psi_1(\xi), \psi_2(\xi), \psi_3(\xi), \psi_4(\xi))$.

Since $\Phi_{n_k}(\xi)$ is the solution of the auxiliary system (2.20), and let $\Upsilon_{n_k} \to 0$, we obtain

$$\begin{cases} c\psi_1' = d_S \psi_1'' + \Lambda - \mu \psi_1 - [\beta_S(\psi_2 + \delta\psi_3) + \beta_R \psi_4]\psi_1, \\ c\psi_2' = d_{SU}\psi_2'' + (1-f)\beta_S(\psi_2 + \delta\psi_3)\psi_1 - (k_U + \mu)\psi_2, \\ c\psi_3' = d_{ST}\psi_3'' + f(1-r)\beta_S(\psi_2 + \delta\psi_3)\psi_1 - (k_T + \mu)\psi_3, \\ c\psi_4' = d_R \psi_4'' + [fr\beta_S(\psi_2 + \delta\psi_3) + \beta_R\psi_4]\psi_1 - (k_R + \mu)\psi_4. \end{cases} \tag{2.37}$$

Therefore, $\Psi(\xi)$ is a nonnegative semi-traveling wave of the original system (2.3) satisfying the asymptotic boundary condition (2.5).

Next, we show that $\Psi(\xi)$ is a positive semi-traveling wave of the original system (2.3), i.e., $\psi_i(\xi) > 0, \ i = S, SU, ST, R$ for any $\xi \in \mathbb{R}$.

Suppose that $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi)) \in \mathfrak{L}$ is a nonnegative semi-traveling wave of the original system (2.3). For fixed $c > c^*$ and $\Upsilon \in (0, 1]$, then ξ_1, ξ_2 and ξ_3 defined in Lemmas 2.3, 2.4 and 2.5, can be chosen such that they do not depend on the choice of Υ. So, by the definition of the profile set $\mathfrak{L}$ in (2.32), we know that there exists a constant $\xi_0 \le \xi_2$ such that $S(\xi) > 0$ for any $\xi < \xi_0$. Now we show that $S(\xi) > 0$ for any $\xi \in \mathbb{R}$. On the contrary, we suppose that there exists ξ^* such that $S(\xi^*) = 0$. Since $S(\xi) \ge 0$ for any $\xi \in \mathbb{R}$, $S(\xi^*)$ is a minimum, implying that $S'(\xi^*) = 0$ and $S''(\xi^*) \ge 0$. Employing the first equation of system (2.21) yields

$$d_S S''(\xi^*) + \Lambda = 0,$$

a contradiction. Thus we have $S(\xi) > 0$ for any $\xi \in \mathbb{R}$.

We then claim that $I_{SU}(\xi) > 0$ for any $\xi \in \mathbb{R}$. In fact, there exists $\widetilde{\xi}_0 \le \xi_3$ such that $I_{SU}(\xi) > 0$ when $\xi < \widetilde{\xi}_0$. If there exists $\widetilde{\xi}^*$ such that $I_{SU}(\widetilde{\xi}^*) = 0$, then there exist constants a_1, a_2 such that $a_1 < \xi_3 < a_2$ and $\widetilde{\xi}^* \in (a_1, a_2)$. It implies that $I_{SU}(\xi)$ achieves its minimum in (a_1, a_2) for any $\xi \in [a_1, a_2]$. From the second equation of system (2.21), we know that $I_{SU}(\xi)$ satisfies

$$-d_{SU} I''_{SU}(\xi) + cI'_{SU}(\xi) + (k_U + \mu) I_{SU}(\xi) = (1-f)\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi))S(\xi) \geq 0,\ \xi \in [a_1, a_2].$$

By the elliptic strong maximum principle (see Theorem 3.3.6 in [45]), it follows that $I_{SU}(\xi) \equiv 0$ for $\xi \in [a_1, a_2]$. On the other hand, by Lemma 2.3, we have $I_{SU}(\xi) > 0$ for $\xi \in [a_1, \xi_3)$, a contradiction. Similarly, by the elliptic strong maximum principle, it can be shown $I_i(\xi) > 0,\ i = ST, R$ for any $\xi \in \mathbb{R}$.

Based on the above arguments, we easily obtain $S(\xi) < S^0$ for any $\xi \in \mathbb{R}$. Otherwise, there exists $\widehat{\xi}^*$ such that

$$d_S S''(\widehat{\xi}^*) = [\beta_S(I_{SU}(\widehat{\xi}^*) + \delta I_{ST}(\widehat{\xi}^*)) + \beta_R I_R(\widehat{\xi}^*)]S(\widehat{\xi}^*) > 0,$$

a contradiction due to $S''(\widehat{\xi}^*) \leq 0$.

Finally, we show that the positive semi-traveling wave $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi))$ of the original system (2.3) satisfies (2.36).

Let $(S_n(\xi), I_{SUn}(\xi), I_{STn}(\xi), I_{Rn}(\xi)) \in \mathfrak{L}$ be a nonnegative semi-traveling wave of the auxiliary system (2.20) with $\Upsilon = \Upsilon_n$ in Lemma 2.9. Let $\kappa_{SU} = \kappa_2, \kappa_{ST} = \kappa_3, \kappa_R = \kappa_4$. Since the selection of $\kappa_i, i = 2, 3, 4$ is independent on Υ, by the definition of upper-lower solutions of the auxiliary system (2.20), we have

$$\kappa_j e^{\lambda_c \xi}(1 - Qe^{\varepsilon\xi}) \leq \underline{I}_{jn}(\xi) \leq I_{jn}(\xi) \leq \bar{I}_{jn}(\xi) \leq \kappa_j e^{\lambda_c \xi}$$

which follows that

$$\lim_{\xi \to -\infty} I_{jn}(\xi) e^{-\lambda_c \xi} = \kappa_j,\ j = SU, ST, R.$$

In addition, note that $(S_n(\xi), I_{SUn}(\xi), I_{STn}(\xi), I_{Rn}(\xi)) \in \mathfrak{L}$ is a fixed point of the operator G_n. Applying L'Höspital rule to the maps $G_{in},\ i = S, SU, ST, R$, it is easy to see that $S'_n(-\infty) = 0$ and $I'_{jn}(-\infty) = 0,\ j = SU, ST, R$. Integrating both sides of the second equation of the auxiliary system (2.20) from $-\infty$ to ξ gives

$$\begin{aligned} d_{SU} I'_{SUn}(\xi) =& cI_{SUn}(\xi) - (1-f)\beta_S \int_{-\infty}^{\xi} (I_{SUn}(s) + \delta I_{STn}(s)) S_n(s) ds \\ &+ (k_U + \mu) \int_{-\infty}^{\xi} I_{SUn}(s) ds + \Upsilon_n \int_{-\infty}^{\xi} I^2_{SUn}(s) ds. \end{aligned}$$

Recall the proven results that $S_n(-\infty) = S^0$, $S'_n(-\infty) = 0$, $I_{jn}(-\infty) = 0$, $I'_{jn}(-\infty) = 0$ and $\lim_{\xi \to -\infty} I_{jn}(\xi) e^{-\lambda_c \xi} = \kappa_j,\ j = SU, ST, R$, then we have

$$\begin{aligned} \lim_{\xi \to -\infty} I'_{SUn}(\xi) e^{-\lambda_c \xi} =& \lim_{\xi \to -\infty} \frac{1}{d_{SU}} \left[ce^{-\lambda_c \xi} I_{SUn}(\xi) - (1-f)\beta_S e^{-\lambda_c \xi} \int_{-\infty}^{\xi} (I_{SUn}(s) + \delta I_{STn}(s)) S_n(s) ds \right] \\ &+ \lim_{\xi \to -\infty} \frac{1}{d_{SU}} \left[(k_U + \mu) e^{-\lambda_c \xi} \int_{-\infty}^{\xi} I_{SUn}(s) ds + \Upsilon_n e^{-\lambda_c \xi} \int_{-\infty}^{\xi} I^2_{SUn}(s) ds \right] \\ =& \frac{c\lambda_c \kappa_{SU} - (1-f)\beta_S(\kappa_{SU} + \delta\kappa_{ST})S^0 + (k_U + \mu)\kappa_{SU}}{d_{SU}\lambda_c}. \end{aligned}$$

By the first equation of the eigenvalue equations (2.23), we know

$$\frac{c\lambda_c \kappa_{SU} - (1-f)\beta_S(\kappa_{SU} + \delta\kappa_{ST})S^0 + (k_U + \mu)\kappa_{SU}}{d_{SU}\lambda_c} = \lambda_c \kappa_{SU}.$$

So, $\lim_{\xi \to -\infty} I'_{SUn}(\xi) e^{-\lambda_c \xi} = \lambda_c \kappa_{SU}$.

Based on the previous discussion in this lemma, we suppose that there exists a subsequence $\{n_k\}$ such that

$$S_{n_k}(\xi) \to S(\xi), I_{jn_k}(\xi) \to I_j(\xi),\ j = SU, ST, R,$$

where $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi))$ is a positive semi-traveling wave of the original system (2.3) satisfying the asymptotic boundary condition (2.5). Applying the limiting arguments, yields

$$\lim_{\xi\to-\infty} I'_{SU}(\xi)e^{-\lambda_c\xi} = \lim_{k\to+\infty}\lim_{\xi\to-\infty} I'_{SUn_k}(\xi)e^{-\lambda_c\xi} = \lambda_c\kappa_{SU}.$$

Similarly, we can also demonstrate $\lim_{\xi\to-\infty} I'_j(\xi)e^{-\lambda_c\xi} = \lambda_c\kappa_j,\ j = ST, R.$ □

Remark 2.1. *In Theorem 3.1, we establish the existence of positive semi-traveling waves connecting the disease-free equilibrium E^0 for the original system* (2.3)*, where the disease-free equilibrium E^0 is the population state before the transmission of influenza. Meanwhile, its presence means that the influenza will spread among the crowd.*

3.2. Strong traveling waves.

Theorem 3.2. *Under the condition of $R_C > 1$, if $R_{SU} \le R_R$ and $R_{ST} \le R_R$ (defined in* (2.13)*), then for any $c > c^*$ ($c^* > 0$ is defined by Lemma 2.1), system* (2.3) *has a strong traveling wave $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi))$ connecting $E^0(S^0, 0, 0, 0)$ and $\hat{E}(\hat{S}, 0, 0, \hat{I}_R)$.*

Proof: From Theorem 3.1, we see that when $R_C > 1$, there exists $c^* > 0$ such that for any $c > c^*$, system (2.3) admits a positive semi-traveling wave $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi))$ satisfying

$$(S(-\infty), I_{SU}(-\infty), I_{ST}(-\infty), I_R(-\infty)) = E^0(S^0, 0, 0, 0)$$

with $S(\xi) < S^0$ for any $\xi \in \mathbb{R}$.

To complete the proof, it is sufficient to show

$$(S(+\infty), I_{SU}(+\infty), I_{ST}(+\infty), I_R(+\infty)) = \hat{E}(\hat{S}, 0, 0, \hat{I}_R).$$

We first claim that $\frac{S'(\xi)}{S(\xi)}$ and $\frac{I'_j(\xi)}{I_j(\xi)},\ j = SU, ST, R$ are bounded for any $\xi \in \mathbb{R}$. To get the result, we rewrite wave equations (2.21) of the original system (2.3) as follows

$$\begin{pmatrix} d_S & 0 & 0 & 0 \\ 0 & d_{SU} & 0 & 0 \\ 0 & 0 & d_{ST} & 0 \\ 0 & 0 & 0 & d_R \end{pmatrix}\begin{pmatrix} S'' \\ I''_{SU} \\ I''_{ST} \\ I''_R \end{pmatrix} - c\begin{pmatrix} S' \\ I'_{SU} \\ I'_{ST} \\ I'_R \end{pmatrix} + \begin{pmatrix} b_{11}(\xi) & 0 & 0 & 0 \\ b_{21}(\xi) & b_{22}(\xi) & 0 & 0 \\ b_{31}(\xi) & 0 & b_{33}(\xi) & 0 \\ b_{41}(\xi) & 0 & 0 & b_{44}(\xi) \end{pmatrix}\begin{pmatrix} S \\ I_{SU} \\ I_{ST} \\ I_R \end{pmatrix} = 0, \tag{2.38}$$

where

$$\begin{aligned} b_{11}(\xi) &= \frac{\Lambda}{S(\xi)} - \mu - [\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi)) + \beta_R I_R(\xi)], \\ b_{21}(\xi) &= (1-f)\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi)),\ b_{22}(\xi) = -(k_U + \mu), \\ b_{31}(\xi) &= f(1-r)\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi)),\ b_{33}(\xi) = -(k_T + \mu), \\ b_{41}(\xi) &= fr\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi)) + \beta_R I_R(\xi),\ b_{44}(\xi) = -(k_R + \mu). \end{aligned}$$

By Lemmas 2.3, 2.4 and 2.5, it is not difficult to see that the functions $b_{k1}(\xi), b_{kk}(\xi),\ k = 1, 2, 3, 4$ are bounded. Moreover, $b_{k1}(\xi) > 0,\ k = 2, 3, 4$ due to the fact $S(\xi) > 0$ and $I_j(\xi) > 0,\ j = SU, ST, R$ for any $\xi \in \mathbb{R}$. We can apply Harnack inequality (see Theorem

1.1 in [27]) for system (2.38), it follows that there exists a constant $D > 0$ such that for any $\xi \in \mathbb{R}$, we have

$$\max_{[s-1,s+1]} S(\xi) \leq D \min_{[s-1,s+1]} S(\xi),$$
$$\max_{[s-1,s+1]} I_j(\xi) \leq D \min_{[s-1,s+1]} I_j(\xi),\ j = SU, ST, R,$$

where D depends only on the coefficients of system (2.38) and the length of interval $[s-1, s+1]$. As a consequence, we can deduce that there exists some constant $D_1 > 0$ such that

$$\left|\frac{S'(\xi)}{S(\xi)}\right| + \sum_{j=SU,ST,R} \left|\frac{I_j'(\xi)}{I_j(\xi)}\right| \leq D_1,\ \xi \in R.$$

Set $V_S(\xi) = S'(\xi), V_j(\xi) = I_j'(\xi),\ j = SU, ST, R$, wave equations (2.21) of the original system (2.3) can be transformed into the following equivalent system

$$\begin{cases} S' = V_S, \\ d_S V_S' = cV_S - \Lambda + \mu S + [\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R]S, \\ I_{SU}' = V_{SU}, \\ d_{SU} V_{SU}' = cV_{SU} - (1-f)\beta_S(I_{SU} + \delta I_{ST})S + (k_U + \mu)I_{SU}, \\ I_{ST}' = V_{ST}, \\ d_{ST} V_{ST}' = cV_{ST} - f(1-r)\beta_S(I_{SU} + \delta I_{ST})S + (k_T + \mu)I_{ST}, \\ I_R' = V_R, \\ d_R V_R' = cV_R - [fr\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R]S + (k_R + \mu)I_R. \end{cases} \tag{2.39}$$

Finally, we complete the proof of the theorem by introducing a Lyapunov function, determining that positive semi-traveling wave $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi))$ of the original system (2.3) converge to the boundary equilibrium $\hat{E}(\hat{S}, 0, 0, \hat{I}_R)$ as $\xi \to +\infty$. Equivalently, it corresponds to the convergence of semi-traveling wave

$$(S(\xi), V_S(\xi), I_{SU}(\xi), V_{SU}(\xi), I_{ST}(\xi), V_{ST}(\xi), I_R(\xi), V_R(\xi))$$

of system (2.39) to $\hat{E}(\hat{S}, 0, 0, 0, 0, 0, \hat{I}_R, 0)$ (for the convenience, we still use the same notation $\hat{E}$ to denote the equilibrium of system (2.39)). To this end, we consider the following Lyapunov function $L(\xi) := L_S(\xi) + L_{SU}(\xi) + L_{ST}(\xi) + L_R(\xi)$, where

$$L_S(\xi) = cS - d_S V_S + \frac{\hat{S} d_S V_S}{S} - c\int_{\hat{S}}^{S} \frac{\hat{S}}{\eta} d\eta,$$

$$L_{SU}(\xi) = cI_{SU} - d_{SU} V_{SU},$$

$$L_{ST}(\xi) = cI_{ST} - d_{ST} V_{ST},$$

$$L_R(\xi) = cI_R - d_R V_R + \frac{\hat{I}_R d_R V_R}{I_R} - c\int_{\hat{I}_R}^{I_R} \frac{\hat{I}_R}{\eta} d\eta.$$

Then, through a simple calculation, the derivative of $L_S(\xi)$ along the traveling wave of system (2.39) satisfies

$$\begin{aligned}\frac{dL_S(\xi)}{d\xi} =& cV_S(\xi) - d_S V_S'(\xi) + \frac{d_S \hat{S}(V_S'(\xi)S(\xi) - V_S(\xi)S'(\xi))}{S^2(\xi)} - c\hat{S}\frac{\hat{S}}{S(\xi)} \\ =& [cV_S(\xi) - d_S V_S'(\xi)]\frac{S(\xi) - \hat{S}}{S(\xi)} - \frac{d_S \hat{S} V_S^2(\xi)}{S^2(\xi)} \\ \leq& [cV_S(\xi) - d_S V_S'(\xi)]\frac{S(\xi) - \hat{S}}{S(\xi)}.\end{aligned}$$

Similarly, we can calculate

$$\frac{dL_{SU}(\xi)}{d\xi} = cV_{SU}(\xi) - d_{SU} V_{SU}'(\xi),$$

$$\frac{dL_{ST}(\xi)}{d\xi} = cV_{ST}(\xi) - d_{ST} V_{ST}'(\xi),$$

and

$$\frac{dL_R(\xi)}{d\xi} \leq [cV_R(\xi) - d_R V_R'(\xi)]\frac{I_R(\xi) - \hat{I}_R}{I_R(\xi)}.$$

Therefore, we have

$$\begin{aligned}\frac{dL(\xi)}{d\xi} =& \frac{L_S(\xi)}{d\xi} + \frac{L_{SU}(\xi)}{d\xi} + \frac{L_{ST}(\xi)}{d\xi} + \frac{L_R(\xi)}{d\xi} \\ \leq& [cV_S(\xi) - d_S V_S'(\xi)]\frac{S(\xi) - \hat{S}}{S(\xi)} + [cV_{SU}(\xi) - d_{SU} V_{SU}'(\xi)] \\ & + [cV_{ST}(\xi) - d_{ST} V_{ST}'(\xi)] + [cV_R(\xi) - d_R V_R'(\xi)]\frac{I_R(\xi) - \hat{I}_R}{I_R(\xi)}.\end{aligned}$$

By system (2.39), we can further get

$$\begin{aligned}\frac{dL(\xi)}{d\xi} \leq& \{\Lambda - \mu S(\xi) - [\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi)) + \beta_R I_R(\xi)]S(\xi)\}\frac{S(\xi) - \hat{S}}{S(\xi)} \\ & + [(1-f)\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi))S(\xi) - (k_U + \mu)I_{SU}(\xi)] \\ & + [f(1-r)\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi))S(\xi) - (k_T + \mu)I_{ST}(\xi)] \\ & + \{[fr\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi)) + \beta_R I_R(\xi)]S(\xi) - (k_R + \mu)I_R(\xi)\}\frac{I_R(\xi) - \hat{I}_R}{I_R(\xi)}.\end{aligned}$$

Together with the following equilibrium conditions $\Lambda = \mu\hat{S} + \beta_R\hat{S}\hat{I}_R$ and $k_R + \mu = \beta_R\hat{S}$, we have

$$\begin{aligned}\frac{dL(\xi)}{d\xi} \leq& \left\{\mu\hat{S} + \beta_R\hat{S}\hat{I}_R - \mu S(\xi) - \left[\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi)) + \beta_R I_R(\xi)\right] S(\xi)\right\} \frac{S(\xi) - \hat{S}}{S(\xi)} \\ &+ \left[(1-f)\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi))S(\xi) - (k_U + \mu)I_{SU}(\xi)\right] \\ &+ \left[f(1-r)\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi))S(\xi) - (k_T + \mu)I_{ST}(\xi)\right] \\ &+ \left\{\left[fr\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi)) + \beta_R I_R(\xi)\right] S(\xi) - \beta_R\hat{S}I_R(\xi)\right\} \frac{I_R(\xi) - \hat{I}_R}{I_R(\xi)} \\ =& -\mu\frac{(S(\xi) - \hat{S})^2}{S(\xi)} + \beta_R\hat{S}\hat{I}_R\left(2 - \frac{\hat{S}}{S(\xi)} - \frac{S(\xi)}{\hat{S}}\right) \\ &+ \beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi))\hat{S} - (k_U + \mu)I_{SU}(\xi) - (k_T + \mu)I_{ST}(\xi) \\ &- \frac{fr\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi))S(\xi)\hat{I}_R}{I_R(\xi)} \\ \leq& \beta_R\hat{S}\hat{I}_R\left(2 - \frac{\hat{S}}{S(\xi)} - \frac{S(\xi)}{\hat{S}}\right) \\ &+ (k_U + \mu)I_{SU}(\xi)\left(\frac{\beta_S\hat{S}}{k_U + \mu} - 1\right) + (k_T + \mu)I_{ST}(\xi)\left(\frac{\beta_S\delta\hat{S}}{k_T + \mu} - 1\right).\end{aligned}$$

Let $\Theta(x) := 1 - x + \ln x$, for $x \in (0, +\infty)$. Using the fact $\hat{S} = \frac{S^0}{R_{RC}} = \frac{1}{R_R}$ and the property that $\Theta(x) \leq 0$ with $\Theta(x) = 0$ if and only if $x = 1$, gives

$$\begin{aligned}\frac{dL(\xi)}{d\xi} \leq& \beta_R\hat{S}\hat{I}_R\left[\Theta\left(\frac{\hat{S}}{S(\xi)}\right) + \Theta\left(\frac{S(\xi)}{\hat{S}}\right)\right] \\ &+ (k_U + \mu)I_{SU}(\xi)\left(\frac{R_{SU}}{R_R} - 1\right) + (k_T + \mu)I_{ST}(\xi)\left(\frac{R_{ST}}{R_R} - 1\right) \\ \leq& (k_U + \mu)I_{SU}(\xi)\left(\frac{R_{SU}}{R_R} - 1\right) + (k_T + \mu)I_{ST}(\xi)\left(\frac{R_{ST}}{R_R} - 1\right).\end{aligned}$$

It is obvious that $\frac{dL(\xi)}{d\xi} \leq 0$ holds for all $\xi \in \mathbb{R}$ when $R_{SU} \leq R_R$ and $R_{ST} \leq R_R$, implying that $L(\xi)$ is decreasing. Furthermore, $\frac{dL(\xi)}{d\xi} = 0$ if and only if

$$(S, V_S, I_{SU}, V_{SU}, I_{ST}, V_{ST}, I_R, V_R) = \hat{E}(\hat{S}, 0, 0, 0, 0, 0, \hat{I}_R, 0).$$

LaSalle's invariance principle [63] implies

$$(S(\xi), V_S(\xi), I_{SU}(\xi), V_{SU}(\xi), I_{ST}(\xi), V_{ST}(\xi), I_R(\xi), V_R(\xi)) \to \hat{E}(\hat{S}, 0, 0, 0, 0, 0, \hat{I}_R, 0)$$

as $\xi \to +\infty$. That is, $(S(+\infty), I_{SU}(+\infty), I_{ST}(+\infty), I_R(+\infty)) = \hat{E}(\hat{S}, 0, 0, \hat{I}_R)$. □

REMARK 2.2. *From $R_{SU} \leq R_R$ and $R_{ST} \leq R_R$, we can easily derive that $R_{SC} < R_{RC}$. So we address the connection problem between two equilibria E^0 and $\hat{E}$ in the later two cases of Table 1 in Section 1. The existence of this strong traveling wave shows that the spread of influenza is successful, which describes the influenza propagation into the susceptible individuals from an initial disease-free equilibrium to the final boundary equilibrium with only resistant-strain.*

3.3. Weak traveling waves. We can easily show that the positive semi-traveling wave $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi))$ of (2.3) in Theorem 3.1 satisfies that $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi)) \in \mathfrak{L}$, which implies

$$\limsup_{\xi\to+\infty} S(\xi) \le S^0,\ \limsup_{\xi\to+\infty} I_i(\xi) \le \kappa_i K^*,\ i = SU, ST, R,$$

where $\kappa_{SU} = \kappa_2, \kappa_{ST} = \kappa_3, \kappa_R = \kappa_4$.

To prove that the positive semi-traveling wave $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi))$ is persistent, we only need to prove

$$\liminf_{\xi\to+\infty} S(\xi) > 0,\ \liminf_{\xi\to+\infty} I_i(\xi) > 0,\ i = SU, ST, R.$$

For this, we will apply the uniform persistence Theorem 4.5 in [105] and restate it as a lemma as follows.

Lemma 3.3. *Let X be locally compact, and let X_2 be compact in X and X_1 be forward invariant under the continuous semiflow Φ on X. Assume that Ω_2, defined by*

$$\Omega_2 = \bigcup_{y\in Y_2} \omega(y),\ Y_2 = \{x \in X_2 : \Phi_t(x) \in X_2, \forall t > 0\},$$

has an acyclic isolated covering $M = \bigcup_{k=1}^m M_k$. If each part M_k of M is a weak repeller for X_1, then X_2 is a uniform strong repeller for X_1.

To use Lemma 3.3, we define

$$\begin{aligned} X_1 =& \{(S, V_S, I_{SU}, V_{SU}, I_{ST}, V_{ST}, I_R, V_R) : (S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi)) \text{ is a positive semi-traveling} \\ & \text{wave of system (2.3) in Theorem 3.1 and } V_S(\xi) = S'(\xi), V_j(\xi) = I_j'(\xi),\ i = SU, ST, R\}, \\ X_2 =& \{(S, V_S, 0, 0, 0, 0, I_R, V_R) : 0 \le S \le S^0,\ |V_S| \le D_1|S|,\ 0 \le I_R \le \kappa_4 K^*,\ |V_R| \le D_1|I_R|\}, \end{aligned}$$

where κ_4 and K^* have been determined in Lemmas 2.1 and 2.3, and D_1 is a positive constant that is just determined in Theorem 3.2.

Lemma 3.4. *If $R_{SC} > 1 > R_{RC}$ (defined in (2.14) and (2.15)), and $c > \max\{c^*, \tilde{c}^*\}$ ($c^* > 0$ is defined by Lemma 2.1, $\tilde{c}^* > \max\{2\sqrt{d_{SU}P_{SU}(0)}, 2\sqrt{d_{ST}P_{ST}(0)}\}$) hold. Let $W^s(E^0)$ denote the stable manifold of system (2.39) at the equilibrium E^0, then we have*

$$W^s(E^0) \cap X_1 = \emptyset,$$

where $E^0 = (S^0, 0, 0, 0, 0, 0, 0, 0)$ and $\emptyset$ denotes the empty set.

Proof: First, we calculate the Jacobian matrix of system (2.39) at E^0 as follows:

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu}{d_S} & \frac{c}{d_S} & \frac{\beta_S S^0}{d_S} & 0 & \frac{\beta_S \delta S^0}{d_S} & 0 & \frac{\beta_R S^0}{d_S} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{k_U+\mu-(1-f)\beta_S S^0}{d_{SU}} & \frac{c}{d_{SU}} & \frac{-(1-f)\beta_S\delta S^0}{d_{SU}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-f(1-r)\beta_S S^0}{d_{ST}} & 0 & \frac{k_T+\mu-f(1-r)\beta_S\delta S^0}{d_{ST}} & \frac{c}{d_{ST}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-fr\beta_S S^0}{d_R} & 0 & \frac{-fr\beta_S\delta S^0}{d_R} & 0 & \frac{k_R+\mu-\beta_R S^0}{d_R} & \frac{c}{d_R} \end{pmatrix}.$$

Let

$$J_{11} = \begin{pmatrix} 0 & 1 \\ \frac{\mu}{d_S} & \frac{c}{d_S} \end{pmatrix}, \; J_{33} = \begin{pmatrix} 0 & 1 \\ \frac{k_R+\mu-\beta_R S^0}{d_R} & \frac{c}{d_R} \end{pmatrix},$$

$$J_{22} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{k_U+\mu-(1-f)\beta_S S^0}{d_{SU}} & \frac{c}{d_{SU}} & \frac{-(1-f)\beta_S \delta S^0}{d_{SU}} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-f(1-r)\beta_S S^0}{d_{ST}} & 0 & \frac{k_T+\mu-f(1-r)\beta_S \delta S^0}{d_{ST}} & \frac{c}{d_{ST}} \end{pmatrix}.$$

Obviously, the characteristic polynomial of J will be determined by the characteristic polynomial of J_{11}, J_{22} and J_{33}. That is to say, the eigenvalues of J consist of the eigenvalues of J_{11}, J_{22} and J_{33}, so we consider the characteristic equations of J_{11}, J_{22} and J_{33} and calculate their eigenvalues, respectively.

Upon a direct computation, one is able to verify that J_{11} and J_{33} have eigenvalues

$$\lambda_{11}^{\pm} = \frac{c \pm \sqrt{c^2 + 4d_S\mu}}{2d_S}, \; \lambda_{33}^{\pm} = \frac{c \pm \sqrt{c^2 + 4d_R(k_R+\mu)(1-R_{RC})}}{2d_R}.$$

J_{11} has one positive eigenvalue λ_{11}^+ and a negative eigenvalue λ_{11}^-, the eigenvector of matrix J corresponding to the negative eigenvalue λ_{11}^- is $(1, \lambda_{11}^-, 0, 0, 0, 0, 0, 0)^T$. When $R_{RC} < 1$, J_{33} has one positive eigenvalue λ_{33}^+ and a negative eigenvalue λ_{33}^-, the corresponding eigenvector to λ_{33}^- is $(0, 0, 0, 0, 0, 0, 1, \lambda_{33}^-)^T$.

In addition, the characteristic equation of J_{22} is

$$H(\lambda) := P_{SU}(\lambda)P_{ST}(\lambda) - \gamma = 0, \tag{2.40}$$

where

$P_{SU}(\lambda) = d_{SU}\lambda^2 - c\lambda + (1-f)\beta_S S^0 - (k_U+\mu)$,
$P_{ST}(\lambda) = d_{ST}\lambda^2 - c\lambda + f(1-r)\beta_S \delta S^0 - (k_T+\mu)$ and
$\gamma = (1-f)\beta_S^2 \delta f(1-r)(S^0)^2$.

It is easy to verify that $R_{SC} > 1$ if and only if $\mathfrak{A}(S^0) < 0$, where $\mathfrak{A}(S^0) = H(0)$. From Lemma 2.1 (e) in [122], it follows that J_{22} has only one negative eigenvalue, denote by λ_{22}^- when $c \geq \tilde{c}^*$ ($\tilde{c}^* > \max\{2\sqrt{d_{SU}P_{SU}(0)}, 2\sqrt{d_{ST}P_{ST}(0)}\}$). Suppose that $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8)$ is the eigenvector of matrix J corresponding to λ_{22}^-, wherein $(\alpha_3, \alpha_4, \alpha_5, \alpha_6)$ is the eigenvector of matrix J_{22} corresponding to λ_{22}^-, then the relationship among the components $\alpha_i, \; i = 3, 4, 5, 6$ can be described by

$$\begin{cases} \lambda_{22}^-\alpha_3 = \alpha_4, \\ P_{SU}(\lambda_{22}^-)\alpha_3 = -G_{23}^0\alpha_5, \\ P_{ST}(\lambda_{22}^-)\alpha_5 = -G_{32}^0\alpha_3, \\ \lambda_{22}^-\alpha_5 = \alpha_6, \end{cases} \tag{2.41}$$

where $G_{23}^0 = (1-f)\beta_S\delta S^0, G_{32}^0 = f(1-r)\beta_S S^0$. It follows from (2.41) that the vector $(\alpha_3, \alpha_4, \alpha_5, \alpha_6)$ has the form $(G_{23}^0, \lambda_{22}^- G_{23}^0, -P_{SU}(\lambda_{22}^-), -\lambda_{22}^- P_{SU}(\lambda_{22}^-))$ or another equivalent form $(G_{32}^0, \lambda_{22}^- G_{32}^0, -P_{ST}(\lambda_{22}^-), -\lambda_{22}^- P_{ST}(\lambda_{22}^-))$. From Lemma 2.1 (e) in [122], we know that $P_{SU}(\lambda_{22}^-) > 0$ when $R_{SC} > 1$ and $c \geq \tilde{c}^*$. So, α_3 and α_5 have the opposite sign.

If λ_{11}^-, λ_{22}^- and λ_{33}^- are not equal to each other, the stable subspace of the linearized system of system (2.39) at E^0 is spanned by $(1, \lambda_{11}^-, 0, 0, 0, 0, 0, 0)$, α and $(0, 0, 0, 0, 0, 0, 1, \lambda_{33}^-)$. In view of $\alpha_3\alpha_5 < 0$, $\lambda_{11}^- < 0$ and $\lambda_{33}^- < 0$, together with the tangency of stable manifold to the stable subspace in stable manifold theorem [89], then we have $W^s(E^0) \cap X_1 = \emptyset$.

If only two of λ_{11}^-, λ_{22}^- and λ_{33}^- are equal or all three are equal, without loss of generality, we suppose that $\lambda_{22}^- = \lambda_{11}^-$ or $\lambda_{22}^- = \lambda_{11}^- = \lambda_{33}^-$. Since λ_{22}^- is a simple eigenvalue of J_{22} and a multiple eigenvalue of J with multiplicity 1 or 2, then the stable subspace of the linearized system of system (2.39) at E^0 is spanned by $(1, \lambda_{11}^-, 0, 0, 0, 0, 0, 0)$, $\tilde{\alpha}$ and $(0, 0, 0, 0, 0, 0, 1, \lambda_{33}^-)$, where the elements $\tilde{\alpha}_3$ and $\tilde{\alpha}_5$ of the eigenvector $\tilde{\alpha}$ satisfy $\tilde{\alpha}_3\tilde{\alpha}_5 < 0$ by (2.41). Similar to the above discussion, we can get the conclusion that $W^s(E^0) \cap X_1 = \emptyset$. □

THEOREM 3.5. *If $R_{SC} > 1 > R_{RC}$ (defined in* (2.14) *and* (2.15)*), and $c > \max\{c^*, \tilde{c}^*\}$ ($c^* > 0$ is defined by Lemma 2.1, $\tilde{c}^*$ is defined by Lemma 3.4) hold. System* (2.3) *admits a weak (or say persistent) traveling wave $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi))$ satisfying the asymptotic boundary condition* (2.5).

Proof: Assume that

$$(S(\xi_i), V_S(\xi_i), I_{SU}(\xi_i), V_{SU}(\xi_i), I_{ST}(\xi_i), V_{ST}(\xi_i), I_R(\xi_i), V_R(\xi_i)) \to (S^*, V_S^*, I_{SU}^*, V_{SU}^*, I_{ST}^*, V_{ST}^*, I_R^*, V_R^*)$$

when $\xi_i \to +\infty$ as $i \to +\infty$.

We first prove several results as follows:

(a) $I_{SU}^* = 0 \Rightarrow V_{SU}^* = I_{ST}^* = V_{ST}^* = 0$;

In view of the fact that $I_{SU}^* = 0$ and $V_{SU} = I_{SU}'$, if $V_{SU}^* \neq 0$, by the Taylor formula, we know there exists ξ^* such that $I_{SU}(\xi^*) < 0$, a contradiction. In addition, we assume that $\lim_{\xi\to+\infty} I_{SU}''(\xi)$ exists, then we can show that $\lim_{\xi\to+\infty} I_{SU}''(\xi) \geq 0$. By selecting a subsequence of ξ_i, denoted by ξ_{i_j}, we directly give the Taylor expansion of $I_{SU}(\xi_{i_j})$ at $\xi_{i_j}^0$:

$$I_{SU}(\xi_{i_j}) = I_{SU}(\xi_{i_j}^0) + I_{SU}'(\xi_{i_j}^0)(\xi_{i_j} - \xi_{i_j}^0) + I_{SU}''(\xi_{i_j}^0)(\xi_{i_j} - \xi_{i_j}^0)^2 + o((\xi_{i_j} - \xi_{i_j}^0)^2). \tag{2.42}$$

Let $\xi_{i_j}^0 \to +\infty$, implying that $I_{SU}(\xi_{i_j}^0) \to 0$ and $I_{SU}'(\xi_{i_j}^0) \to 0$. Combining with $I_{SU}(\xi_{i_j}) \geq 0$, we can get $\lim_{\xi\to+\infty} I_{SU}''(\xi) \geq 0$ from (2.42).

Now go back to the equation of I_{SU} in wave equations (2.21) of the original system (2.3)

$$cI_{SU}'(\xi) = d_{SU}I_{SU}''(\xi) + (1-f)\beta_S(I_{SU}(\xi) + \delta I_{ST}(\xi))S - (k_U + \mu)I_{SU}(\xi). \tag{2.43}$$

Take the limit on both sides of equality (2.43), in order to make the limiting equation still hold, we have $I_{SU}''(+\infty) = 0$ and $I_{ST}(+\infty) = I_{ST}^* = 0$. Similar to the proof of $V_{SU}^* = 0$, we can show $V_{ST}^* = 0$.

Through a similar discussion with (a), we can also prove the following results (b) and (c).

(b) $I_{ST}^* = 0 \Rightarrow V_{ST}^* = I_{SU}^* = V_{SU}^* = 0$;

(c) $I_R^* = 0 \Rightarrow V_R^* = I_{SU}^* = V_{SU}^* = I_{ST}^* = V_{ST}^* = 0$.

By (a), (b) and (c), we can find that $I_{SU}^* = 0 \Leftrightarrow I_{ST}^* = 0$, $I_R^* = 0 \Rightarrow I_{SU}^* = I_{ST}^* = 0$, while $I_{SU}^* = I_{ST}^* = 0$ does not imply $I_R^* = 0$. So we only need to show that X_2 excludes X_1, if we want to prove that $S(\xi), I_{SU}(\xi), I_{ST}(\xi)$ and $I_R(\xi)$ are persistent. Now we study the dynamics of system (2.39) in X_2. Equivalently, we consider the subsystem of system (2.39)

$$\begin{cases} S' = V_S, \\ d_S V_S' = cV_S - \Lambda + \mu S + \beta_R I_R S, \\ I_R' = V_R, \\ d_R V_R' = cV_R - \beta_R I_R S + (k_R + \mu)I_R. \end{cases} \tag{2.44}$$

If $R_{RC} < 1$, system (2.44) has a unique equilibrium $\bar{E}^0(S^0, 0, 0, 0)$. Now we consider the Jacobian matrix of (2.44) at $\bar{E}^0$, which has the form

$$\bar{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\mu}{d_S} & \frac{c}{d_S} & \frac{\beta_R S^0}{d_S} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k_R+\mu-\beta_R S^0}{d_{IR}} & \frac{c}{d_{IR}} \end{pmatrix}.$$

The characteristic equation of $\bar{J}$ is

$$(d_S\lambda^2 - c\lambda - \mu)\left[d_R\lambda^2 - c\lambda + \beta_R S^0 - (k_R + \mu)\right] = 0. \tag{2.45}$$

It is easy to calculate the eigenvalues of $\bar{J}$ as follows:

$$\lambda_S^{\pm} = \frac{c \pm \sqrt{c^2 + 4d_S\mu}}{2d_S}, \ \lambda_R^{\pm} = \frac{c \pm \sqrt{c^2 + 4d_R(k_R + \mu)(1 - R_{RC})}}{2d_R}.$$

When $R_{RC} < 1$ and $c > 0$, the real part of all eigenvalues of $\bar{J}$ is nonzero, by the Hopf bifurcation theorem [75], we know there is no periodic solution around $\bar{E}^0$. Obviously, there is no heteroclinic orbit connecting $\bar{E}^0$ for any $c > 0$ as $R_{RC} < 1$.

Finally, we need to rule out the possibility of a homoclinic connection at $\bar{E}^0$. For the non-degenerate critical point $\bar{E}^0$, if there is a homoclinic orbit l connecting $\bar{E}^0$, then we have $l \subseteq W^U(\bar{E}^0) \cap W^S(\bar{E}^0)$. The eigenvectors of matrix $\bar{J}$ corresponding to eigenvalues $\lambda_i^{\pm}, i = S, R$ are

$$\begin{array}{ll} h_S^+ = (1, \lambda_S^+, 0, 0), & h_R^+ = (\beta_R S^0, \lambda_R^+\beta_R S^0, P_S(\lambda_R^+), \lambda_R^+ P_S(\lambda_R^+)), \\ h_S^- = (1, \lambda_S^-, 0, 0), & h_R^- = (\beta_R S^0, \lambda_R^-\beta_R S^0, P_S(\lambda_R^-), \lambda_R^- P_S(\lambda_R^-)), \end{array} \tag{2.46}$$

where $P_S(\lambda) = d_S\lambda^2 - c\lambda - \mu$.

The unstable subspace of the linearized system of (2.44) at $\bar{E}^0$ is spanned by h_S^+ and h_R^+, and the stable subspace is spanned by h_S^- and h_R^-. If there is a homoclinic connection at $\bar{E}^0$, then $l - \bar{E}^0 \neq \emptyset$, we suppose that there is some point $P^0 \in l - \bar{E}^0$ such that $P^0 \in W^U(\bar{E}^0) \cap W^S(\bar{E}^0)$. Through simple calculations, together with $h_S^+ h_S^- < 0$ and $h_R^+ h_R^- < 0$, we can show $P^0 = \bar{E}^0$, a contradiction. So there do not exist homoclinic orbits in X_2 for system (2.44).

The above discussions imply that the sets Ω_2 and M in Lemma 3.3 are given by $\Omega_2 = E^0 = M$. Obviously, M is an acyclic isolated covering of Ω_2. Applying Lemmas 3.3 and 3.4 completes the proof of the theorem. □

REMARK 2.3. *In Theorem 3.5, we only give the existence of weak traveling waves connecting the disease-free equilibrium since it is difficult to construct a Lyapunov function or a pair of closed upper-lower solutions which converge to the positive equilibrium E^* as $\xi \to +\infty$. Although we can determine the components of the final state of weak traveling waves are positive, it does not mean the final state of the weak traveling waves is E^*. At the end of influenza spread, susceptible individuals and infected individuals with sensitive and resistant strains may coexist at constant levels (E^*) or periodically fluctuating state. However, Theorem 3.5 can still tell us the propagation speed of infection into susceptible individuals. Biologically, it is indeed of public health importance, indicating that if few infectives are introduced into a completely susceptible population, then the infected individuals with sensitive and resistant strains would not vanish at the end of the wavefront.*

4. Nonexistence of semi-traveling waves and estimation of minimal wave speed for the original system

In this section, we show the nonexistence of semi-traveling waves for the original system (2.3) in the following four cases: (I) $R_C < 1$ and $c > 0$; (II) $R_{RC} > 1$, $R_{SC} \neq 1$ and $0 < c < c_1^*$; (III) $R_{SC} > 1$, $R_{RC} \neq 1$ and $0 < c < c_2^*$; and (IV) $R_C > 1$, $R_i \neq 1, i = SC, RC$ and $0 < c < \min\{c_1^*, c_2^*\}$. In addition, we also give the estimation of the minimal wave speed.

4.1. Nonexistence of semi-traveling waves.

4.1.1. *Case I: $R_C < 1$ and $c > 0$.*

Theorem 4.1. *Suppose that $R_C < 1$, then for any $c > 0$, the original system* (2.3) *has no nonnegative bounded semi-traveling waves (nontrivial) satisfying the asymptotic boundary condition* (2.5). *That is, in addition to the trivial semi-traveling wave, the original system* (2.3) *does not admit any traveling wave connecting the disease-free steady state E^0 itself.*

Proof: Suppose that the original system (2.3) admits a nonnegative bounded semi-traveling wave (nontrivial) satisfying the asymptotic boundary condition (2.5). Without loss of generality, we assume that $0 \leq S(\xi) \leq S^0$ and $I(\xi) \geq 0$ for $\xi \in \mathbb{R}$.

Note that the three equations for (I_{SU}, I_{ST}, I_R) in wave equations (2.21) of the original system (2.3) can be transformed into

$$\begin{cases} I_{SU}(\xi) = \int_{-\infty}^{\xi} \frac{k_U+\mu}{\rho_{SU}} e^{\lambda_{SU}^-(\xi-s)} \frac{1}{k_U+\mu} H_{SU}(s)ds + \int_{\xi}^{+\infty} \frac{k_U+\mu}{\rho_{SU}} e^{\lambda_{SU}^+(\xi-s)} \frac{1}{k_U+\mu} H_{SU}(s)ds, \\ I_{ST}(\xi) = \int_{-\infty}^{\xi} \frac{k_T+\mu}{\rho_{ST}} e^{\lambda_{ST}^-(\xi-s)} \frac{1}{k_T+\mu} H_{ST}(s)ds + \int_{\xi}^{+\infty} \frac{k_T+\mu}{\rho_{ST}} e^{\lambda_{ST}^+(\xi-s)} \frac{1}{k_T+\mu} H_{ST}(s)ds, \\ I_R(\xi) = \int_{-\infty}^{\xi} \frac{k_R+\mu}{\rho_R} e^{\lambda_R^-(\xi-s)} \frac{1}{k_R+\mu} H_R(s)ds + \int_{\xi}^{+\infty} \frac{k_R+\mu}{\rho_R} e^{\lambda_R^+(\xi-s)} \frac{1}{k_R+\mu} H_R(s)ds, \end{cases} \tag{2.47}$$

where

$$\lambda_{SU}^{\pm} = \frac{c \pm \sqrt{c^2 + 4d_{SU}(k_U+\mu)}}{2d_{SU}}, \ \lambda_{ST}^{\pm} = \frac{c \pm \sqrt{c^2 + 4d_{ST}(k_T+\mu)}}{2d_{ST}}, \ \lambda_R^{\pm} = \frac{c \pm \sqrt{c^2 + 4d_R(k_R+\mu)}}{2d_R},$$

$$\rho_{SU} = \lambda_{SU}^+ - \lambda_{SU}^-, \ \rho_{ST} = \lambda_{ST}^+ - \lambda_{ST}^-, \ \rho_R = \lambda_R^+ - \lambda_R^-, \ H_{SU}(s) = (1-f)\beta_S(I_{SU}(s) + \delta I_{ST}(s))S(s),$$

$$H_{ST}(s) = f(1-r)\beta_S(I_{SU}(s) + \delta I_{ST}(s))S(s), \ H_R(s) = [fr\beta_S(I_{SU}(s) + \delta I_{ST}(s)) + \beta_R I_R(s)]S(s).$$

According to (2.47) and assumptions, we have

$$\begin{cases} I_{SU}(\xi) \leq & \int_{-\infty}^{\xi} \frac{k_U+\mu}{\rho_{SU}} e^{\lambda_{SU}^-(\xi-s)} \frac{1}{k_U+\mu}(1-f)\beta_S(I_{SU}(s) + \delta I_{ST}(s))S^0 ds \\ & + \int_{\xi}^{+\infty} \frac{k_U+\mu}{\rho_{SU}} e^{\lambda_{SU}^+(\xi-s)} \frac{1}{k_U+\mu}(1-f)\beta_S(I_{SU}(s) + \delta I_{ST}(s))S^0 ds, \\ I_{ST}(\xi) \leq & \int_{-\infty}^{\xi} \frac{k_T+\mu}{\rho_{ST}} e^{\lambda_{ST}^-(\xi-s)} \frac{1}{k_T+\mu} f(1-r)\beta_S(I_{SU}(s) + \delta I_{ST}(s))S^0 ds \\ & + \int_{\xi}^{+\infty} \frac{k_T+\mu}{\rho_{ST}} e^{\lambda_{ST}^+(\xi-s)} \frac{1}{k_T+\mu} f(1-r)\beta_S(I_{SU}(s) + \delta I_{ST}(s))S^0 ds, \\ I_R(\xi) \leq & \int_{-\infty}^{\xi} \frac{k_R+\mu}{\rho_R} e^{\lambda_R^-(\xi-s)} \frac{1}{k_R+\mu}[fr\beta_S(I_{SU}(s) + \delta I_{ST}(s)) + \beta_R I_R(s)]S^0 ds \\ & + \int_{\xi}^{+\infty} \frac{k_R+\mu}{\rho_R} e^{\lambda_R^+(\xi-s)} \frac{1}{k_R+\mu}[fr\beta_S(I_{SU}(s) + \delta I_{ST}(s)) + \beta_R I_R(s)]S^0 ds. \end{cases} \tag{2.48}$$

Making further simplification of inequalities (2.48), yields

$$\begin{cases} I_{SU}(\xi) \leq & (V^{-1}F)_1 \left[\int_{-\infty}^{\xi} \frac{k_U+\mu}{\rho_{SU}} e^{\lambda_{SU}^-(\xi-s)} I(s)ds + \int_{\xi}^{+\infty} \frac{k_U+\mu}{\rho_{SU}} e^{\lambda_{SU}^+(\xi-s)} I(s)ds \right], \\ I_{ST}(\xi) \leq & (V^{-1}F)_2 \left[\int_{-\infty}^{\xi} \frac{k_T+\mu}{\rho_{ST}} e^{\lambda_{ST}^-(\xi-s)} I(s)ds + \int_{\xi}^{+\infty} \frac{k_T+\mu}{\rho_{ST}} e^{\lambda_{ST}^+(\xi-s)} I(s)ds \right], \\ I_R(\xi) \leq & (V^{-1}F)_3 \left[\int_{-\infty}^{\xi} \frac{k_R+\mu}{\rho_R} e^{\lambda_R^-(\xi-s)} I(s)ds + \int_{\xi}^{+\infty} \frac{k_R+\mu}{\rho_R} e^{\lambda_R^+(\xi-s)} I(s)ds \right], \end{cases} \tag{2.49}$$

where $(V^{-1}F)_i,\ i=1,2,3$ denotes the i-th row of the matrix $V^{-1}F$ and

$$V^{-1}F=\begin{pmatrix}\frac{(1-f)\beta_S S^0}{k_U+\mu} & \frac{(1-f)\beta_S\delta S^0}{k_U+\mu} & 0\\ \frac{f(1-r)\beta_S S^0}{k_T+\mu} & \frac{f(1-r)\beta_S\delta S^0}{k_T+\mu} & 0\\ \frac{fr\beta_S S^0}{k_R+\mu} & \frac{fr\beta_S\delta S^0}{k_R+\mu} & \frac{\beta_R S^0}{k_R+\mu}\end{pmatrix},\ I(t)=\begin{pmatrix}I_{SU}(t)\\ I_{ST}(t)\\ I_R(t)\end{pmatrix}.$$

Let $I_j^0:=\sup_{\xi\in\mathbb{R}} I_j(\xi),\ j=SU,ST,R$. Then $I^0:=(I^0_{SU},I^0_{ST},I^0_R)^T\geq \mathbf{0}$ and $I^0\neq\mathbf{0}$, where T represents the transpose of vectors. Furthermore, by (2.49), we have

$$I^0\leq (V^{-1}F)I^0. \tag{2.50}$$

In subsection 2.2, we have proven that $\rho(V^{-1}F)=\rho(FV^{-1})=R_C$. Through the Perron-Frobenius theorem, we see that there exists a vector $P=(p_1,p_2,p_3)^T\in\mathbb{R}^3$ with $p_i>0, i=1,2,3$ such that $(V^{-1}F)P=R_CP$. As I^0 is bounded, we can suppose that there exists a constant $\chi>0$ such that $I^0\leq\chi P$. Iterating inequalities (2.50), we have

$$I^0\leq (V^{-1}F)^nI^0\leq\chi(V^{-1}F)^nP=\chi(R_C)^nP. \tag{2.51}$$

When $R_C<1$, by selecting a sufficiently large n, we get $I^0=0$, which is in contradiction with the assumption. □

REMARK 2.4. *Theorem 4.1 determines whether nonnegative bounded semi-traveling waves connecting the disease-free equilibrium E^0 itself exist in the first case of Table 1 in Section 1. The results show that the control reproduction number R_C is a critical threshold determining whether nonnegative bounded semi-traveling waves exist.*

4.1.2. *Case II: $R_{RC}>1$, $R_{SC}\neq 1$ and $0<c<c_1^*$.*

LEMMA 4.2. *Suppose that $R_C>1$, $R_i\neq 1,\ i=SC,RC$ are satisfied. For any $c>0$, if $(S(\xi),I_{SU}(\xi),I_{ST}(\xi),I_R(\xi))$ is a nonnegative semi-traveling wave of the original system* (2.3) *satisfying the asymptotic boundary condition* (2.5)*, then there exists a positive constant η such that*

$$\sup_{\xi\in\mathbb{R}}\{|S^0-S(\xi)|e^{-\eta\xi}\}<+\infty,\quad \sup_{\xi\in\mathbb{R}}\{|I_j(\xi)|e^{-\eta\xi}\}<+\infty,\ j=SU,ST,R. \tag{2.52}$$

Proof: Since the nonnegative traveling wave $(S(\xi),I_{SU}(\xi),I_{ST}(\xi),I_R(\xi))$ of the original system (2.3) satisfies the boundary condition (2.5), we have

$$(S(\xi),V_S(\xi),I_{SU}(\xi),V_{SU}(\xi),I_{ST}(\xi),V_{ST}(\xi),I_R(\xi),V_R(\xi))\to\bar{E}^0(S^0,0,0,0,0,0,0,0),$$

as $\xi\to-\infty$.

It is easy to calculate the characteristic polynomial of the linearized system of equivalent system (2.39) of wave equations (2.21) at $\bar{E}^0$ as follows

$$P_S(\lambda)H(\lambda)P_R(\lambda)=0, \tag{2.53}$$

where $P_R(\lambda)=d_R\lambda^2-c\lambda+\beta_RS^0-(k_R+\mu)$.

When $R_{RC}\neq 1$, we know that the roots of the polynomials $P_S(\lambda)$ and $P_R(\lambda)$ have no zero real part. Next, we determine whether the polynomial $H(\lambda)=P_{SU}(\lambda)P_{ST}(\lambda)-\gamma$ has a root with zero real part. Since $H(0)=P_{SU}(0)P_{ST}(0)-\gamma=(k_U+\mu)(k_T+\mu)(1-R_{SC})\neq 0$ when $R_{SC}\neq 1$, $\lambda=0$ is not the root of $H(\lambda)=0$. By $H(\lambda)=0$, we get the following quartic polynomial of λ

$$\lambda^4+a_3\lambda^3+a_2\lambda^2+a_1\lambda+a_0=0, \tag{2.54}$$

where

$$a_3 = -\frac{c(d_{SU}+d_{ST})}{d_{SU}d_{ST}}, \qquad a_2 = \frac{c^2}{d_{SU}d_{ST}} + \frac{(1-f)\beta_S S^0-(k_U+\mu)}{d_{SU}} + \frac{f(1-r)\beta_S\delta S^0-(k_T+\mu)}{d_{ST}},$$
$$a_1 = -\frac{c[f(1-r)\beta_S\delta S^0-(k_T+\mu)+(1-f)\beta_S S^0-(k_U+\mu)]}{d_{SU}d_{ST}},$$
$$a_0 = \frac{[(1-f)\beta_S S^0-(k_U+\mu)][f(1-r)\beta_S\delta S^0-(k_T+\mu)]}{d_{SU}d_{ST}} - \frac{(1-f)\beta_S\delta S^0 f(1-r)\beta_S S^0}{d_{SU}d_{ST}}.$$

Suppose that (2.54) has a pure imaginary root, denoted by $\lambda = \beta i, \beta \neq 0$, then we substitute $\lambda = \beta i$ into (2.54), yielding

$$\beta^4 - a_2\beta^2 + a_0 = 0, \; a_1 = a_3\beta^2. \tag{2.55}$$

On account of $\beta^2 > 0$ and $a_3 < 0$, we have $a_1 < 0$, implying

$$f(1-r)\beta_S\delta S^0 - (k_T+\mu) + (1-f)\beta_S S^0 - (k_U+\mu) > 0.$$

Combining the two equalities in (2.55), we obtain

$$a_1^2 + a_0a_3^2 - a_1a_2a_3 = 0. \tag{2.56}$$

Through calculation, we can get the following inequality

$$\begin{aligned}
&a_1^2 + a_0a_3^2 - a_1a_2a_3\\
=&-\frac{c^2}{d_{SU}^3d_{ST}^3}\lfloor(d_{SU}+d_{ST})[f(1-r)\beta_S\delta S^0-(k_T+\mu)+(1-f)\beta_S S^0-(k_U+\mu)]c^2\\
&+\{d_{SU}[f(1-r)\beta_S\delta S^0-(k_T+\mu)]-d_{ST}[(1-f)\beta_S S^0-(k_U+\mu)]\}^2\\
&+(1-f)\beta_S S^0 f(1-r)\beta_S\delta S^0(d_{SU}+d_{ST})^2\rfloor\\
<&0, \text{ for } c>0,
\end{aligned}$$

which is in contradiction with (2.56). Thus, when $R_C > 1$ and $R_i \neq 1, \; i = SC, RC$, the characteristic polynomial (2.53) has no roots with zero real parts for any $c > 0$, implying that the equilibrium $\bar{E}^0$ is hyperbolic. By using stable manifold theorem in [89], we know that there exists a positive constant η such that (2.52) holds. □

THEOREM 4.3. *If $R_{RC} > 1$ and $R_{SC} \neq 1$ are satisfied, then for any $c \in (0, c_1^*)$, the original system* (2.3) *has no nonnegative bounded semi-traveling waves satisfying the asymptotic boundary condition* (2.5)*, where $c_1^* = 2\sqrt{d_R(k_R+\mu)(R_{RC}-1)}$.*

Proof: We first define the two-sided Laplace transform by

$$\mathcal{L}[U(\cdot)](\lambda) := \int_{-\infty}^{+\infty} e^{-\lambda t}U(t)dt, \tag{2.57}$$

for $\lambda \geq 0$.

We can rewrite (2.57) as follows

$$\mathcal{L}[U(\cdot)](\lambda) = \mathcal{L}^-[U(\cdot)](\lambda) + \mathcal{L}^+[U(\cdot)](\lambda), \tag{2.58}$$

where $\mathcal{L}^-[U(\cdot)](\lambda) := \int_{-\infty}^0 e^{-\lambda t}U(t)dt$ is referred to as the negative one-sided Laplace transform (see [124]), $\mathcal{L}^+[U(\cdot)](\lambda) := \int_0^{+\infty} e^{-\lambda t}U(t)dt$.

It follows from (2.58) that the convergence of $\mathcal{L}[U(\cdot)](\lambda)$ is equivalent to that of $\mathcal{L}^-[U(\cdot)](\lambda)$ if $U(t)$ is bounded in $[0, +\infty)$. From the definition of $\mathcal{L}^-[U(\cdot)](\lambda)$, we can find that $\mathcal{L}^-[U(\cdot)](\lambda)$ is increasing in $[0, \lambda^*)$, where $\lambda^* = +\infty$ or $\lambda^* < +\infty$ with $\lim_{\lambda\to\lambda^*} \mathcal{L}[U(\cdot)](\lambda) = +\infty$.

It is easy to verify that the two-sided and negative one-sided Laplace transforms have the following properties:

$$\mathcal{L}[U'(\cdot)](\lambda) = \lambda\mathcal{L}[U(\cdot)](\lambda), \qquad \mathcal{L}[U''(\cdot)](\lambda) = \lambda^2\mathcal{L}[U(\cdot)](\lambda), \tag{2.59}$$

and

$$\begin{aligned} &\mathcal{L}^-[U'(\cdot)](\lambda) = \lambda\mathcal{L}^-[U(\cdot)](\lambda) + U(0), \\ &\mathcal{L}^-[U''(\cdot)](\lambda) = \lambda^2\mathcal{L}^-[U(\cdot)](\lambda) + \lambda U(0) + U'(0). \end{aligned} \tag{2.60}$$

Set

$$J_i(\lambda) := \mathcal{L}[I_i(\cdot)](\lambda),\ i = SU, ST, \tag{2.61}$$

and

$$J_R^-(\lambda) := \mathcal{L}^-[I_R(\cdot)](\lambda), \tag{2.62}$$

for $\lambda \in [0, \lambda_i^*),\ i = SU, ST, R$. By Lemma 4.2, it follows that $\lambda_i^* \geq \eta,\ i = SU, ST, R$.

The latter three equations of wave equations (2.21) of the original system (2.3) can be rewritten as

$$\begin{cases} d_{SU}I_{SU}'' - cI_{SU}' + [(1-f)\beta_S S^0 - (k_U+\mu)]I_{SU} = (1-f)\beta_S(S^0 - S)(I_{SU} + \delta I_{ST}) - (1-f)\beta_S\delta S^0 I_{ST}, \\ d_{ST}I_{ST}'' - cI_{ST}' + [f(1-r)\beta_S S^0 - (k_T+\mu)]I_{ST} = f(1-r)\beta_S(S^0 - S)(I_{SU} + \delta I_{ST}) - f(1-r)\beta_S S^0 I_{SU}, \\ d_R I_R'' - cI_R' + [\beta_R S^0 - (k_R+\mu)]I_R = (S^0 - S)[fr\beta_S(I_{SU} + \delta I_{ST}) + \beta_R I_R] - fr\beta_S S^0(I_{SU} + \delta I_{ST}). \end{cases} \tag{2.63}$$

Define $\upsilon = \min\{P_R(\lambda) : \lambda \geq 0\}$, where $P_R(\lambda) = d_R\lambda^2 - c\lambda + \beta_R S^0 - (k_R + \mu)$. It follows from the condition $0 < c < c_1^* = 2\sqrt{d_R(k_R+\mu)(R_{RC}-1)}$ that $\upsilon > 0$. Now we suppose that there is a nonnegative semi-traveling wave $(S(\xi), I_{SU}(\xi), I_{ST}(\xi), I_R(\xi))$ of the original system (2.3) satisfying the asymptotic boundary condition (2.5). According to the boundary condition (2.5), without loss of generality, we can assume that $S^0 - S(\xi) < \frac{\upsilon}{2}$ for all $\xi < 0$. By the third equation of (2.63), we get

$$\begin{aligned} d_R I_R'' - cI_R' + [\beta_R S^0 - (k_R+\mu)]I_R &= \beta_R(S^0 - S)I_R - fr\beta_S S(I_{SU} + \delta I_{ST}) \\ &\leq \beta_R(S^0 - S)I_R \leq \frac{\upsilon\beta_R}{2}I_R \\ &\leq \frac{\upsilon}{2}I_R. \end{aligned} \tag{2.64}$$

Taking the negative one-sided Laplace transform of the above inequality (2.64) and making use of the properties of $\mathcal{L}^-[\cdot]$ in (2.60), we obtain

$$P_R(\lambda)J_R^-(\lambda) + Q(\lambda) \leq \frac{\upsilon}{2}J_R^-(\lambda), \tag{2.65}$$

where $Q(\lambda) = (d_R\lambda - c)I_R(0) + d_R I_R'(0)$. Therefore, by (2.65), we have

$$\Xi(\lambda) := [P_R(\lambda) - \frac{\upsilon}{2}]J_R^-(\lambda) + Q(\lambda) \leq 0. \tag{2.66}$$

If $\lambda_R^* < +\infty$, we have $\lim_{\lambda\to\lambda_R^{*-}} J_R^-(\lambda) = +\infty$, which implies $\lim_{\lambda\to\lambda_R^{*-}} \Xi(\lambda) = +\infty$, which is in contradiction with (2.66). If $\lambda_R^* = +\infty$, since $J_R^-(\lambda)$ is monotonically increasing, together with the definitions of $P_R(\lambda)$ and $Q(\lambda)$, we have $\lim_{\lambda\to\lambda_R^{*-}} \Xi(\lambda) = +\infty$, which is still in contradiction with (2.66). □

4.1.3. *Case III:* $R_{SC} > 1$, $R_{RC} \neq 1$ *and* $0 < c < c_2^*$.

LEMMA 4.4. *For any* $c \in (0, c_2^*)$, *there is no positive real root* λ^* *for*

$$H(\lambda) = P_{SU}(\lambda)P_{ST}(\lambda) - \gamma = 0$$

such that $P_{SU}(\lambda^*) < 0$ *and* $P_{ST}(\lambda^*) < 0$ *hold, where*

$$\begin{array}{c} c_2^* := \inf_{\lambda>0} \frac{\widetilde{P}_{SU}(\lambda)+\widetilde{P}_{ST}(\lambda)+\sqrt{(\widetilde{P}_{SU}(\lambda)-\widetilde{P}_{ST}(\lambda))^2+4\gamma}}{2\lambda}, \\ \widetilde{P}_{SU}(\lambda) = d_{SU}\lambda^2 + (1-f)\beta_S S^0 - (k_U + \mu), \\ \widetilde{P}_{ST}(\lambda) = d_{ST}\lambda^2 + f(1-r)\beta_S \delta S^0 - (k_T + \mu). \end{array} \tag{2.67}$$

Proof: Suppose that $H(\lambda) = 0$ has a positive real root λ^*, satisfying

$$H(\lambda^*) = P_{SU}(\lambda^*)P_{ST}(\lambda^*) - \gamma = 0, \tag{2.68}$$

and

$$P_{SU}(\lambda^*) < 0, \ \ P_{ST}(\lambda^*) < 0. \tag{2.69}$$

Since $0 < c < c_2^*$, by the definition of c_2^* in (2.67), we have

$$c\lambda^* < \frac{\widetilde{P}_{SU}(\lambda^*) + \widetilde{P}_{ST}(\lambda^*) + \sqrt{(\widetilde{P}_{SU}(\lambda^*) - \widetilde{P}_{ST}(\lambda^*))^2 + 4\gamma}}{2}. \tag{2.70}$$

Then, we obtain

$$0 > P_{SU}(\lambda^*) = \widetilde{P}_{SU}(\lambda^*) - c\lambda^* > \frac{\widetilde{P}_{SU}(\lambda^*) - \widetilde{P}_{ST}(\lambda^*) - \sqrt{(\widetilde{P}_{SU}(\lambda^*) - \widetilde{P}_{ST}(\lambda^*))^2 + 4\gamma}}{2}, \tag{2.71}$$

and

$$0 > P_{ST}(\lambda^*) = \widetilde{P}_{ST}(\lambda^*) - c\lambda^* > \frac{\widetilde{P}_{ST}(\lambda^*) - \widetilde{P}_{SU}(\lambda^*) - \sqrt{(\widetilde{P}_{SU}(\lambda^*) - \widetilde{P}_{ST}(\lambda^*))^2 + 4\gamma}}{2}. \tag{2.72}$$

It follows

$$0 < P_{SU}(\lambda^*)P_{ST}(\lambda^*) < \gamma, \tag{2.73}$$

which is in contradiction with (2.68). □

THEOREM 4.5. *If* $R_{SC} > 1$ *and* $R_{RC} \neq 1$ *are satisfied, then for any* $c \in (0, c_2^*)$, *the original system* (2.3) *has no nonnegative bounded semi-traveling waves satisfying the asymptotic boundary condition* (2.5).

Proof: We prove the theorem by contradiction. For fixed $c \in (0, c_2^*)$, we suppose that there exists a nonnegative bounded semi-traveling wave of the original system (2.3) satisfying the asymptotic boundary condition (2.5).

Based on the definition of the two-sided Laplace transform (see (2.57)) in subsubsection 4.1.2, we take the two-sided Laplace transform of first and second equations of (2.63), yielding

$$\begin{cases} P_{SU}(\lambda)J_{SU}(\lambda) = (1-f)\beta_S G(\lambda) - (1-f)\beta_S \delta S^0 J_{ST}(\lambda), \\ P_{ST}(\lambda)J_{ST}(\lambda) = f(1-r)\beta_S G(\lambda) - f(1-r)\beta_S S^0 J_{SU}(\lambda), \end{cases} \tag{2.74}$$

where $G(\lambda) = \mathcal{L}[g(\cdot)](\lambda)$, $g(t) = (S^0 - S)(I_{SU} + \delta I_{ST})$.

Now, we illustrate $\lambda_i^* < +\infty, i = SU, ST$. By the first equation of (2.74), we obtain

$$H_{SU}(\lambda) := [d_{SU}\lambda^2 - c\lambda - (k_U + \mu)]J_{SU}(\lambda) + (1-f)\beta_S \mathcal{L}[S(\cdot)(I_{SU}(\cdot) + \delta I_{ST}(\cdot))](\lambda) = 0. \tag{2.75}$$

By aid of the two-sided Laplace transform in (2.57), we have

$$J_{SU}(\lambda) > 0, \tag{2.76}$$

and

$$\mathcal{L}[S(\cdot)(I_{SU}(\cdot) + \delta I_{ST}(\cdot))](\lambda) > 0, \tag{2.77}$$

for $\lambda \in [0, \lambda^*_{SU})$.

If $\lambda^*_{SU} = +\infty$, using (2.76) and (2.77), we get $H_{SU}(+\infty) = +\infty$, which is in contradiction with (2.75). So we can conclude that $\lambda^*_{SU} < +\infty$. Similarly, we can also prove that $\lambda^*_{ST} < +\infty$.

Then, we show $\lambda^*_{SU} = \lambda^*_{ST}$. Assume that $\lambda^*_{SU} < \lambda^*_{ST}$, which means $\lim_{\lambda\to\lambda^*_{SU}} J_{SU}(\lambda) = +\infty$ and $\lim_{\lambda\to\lambda^*_{SU}} J_{ST}(\lambda) = J_{ST}(\lambda^*_{SU}) < +\infty$. From Lemma 4.2 and the definition of the two-sided Laplace transform in (2.57), together with the boundedness of semi-traveling waves, we know that $G(\lambda^*_{SU}) < +\infty$, which follows that the second equation of (2.74) does not hold. So, $\lambda^*_{SU} \geq \lambda^*_{ST}$. On the other hand, we suppose that $\lambda^*_{SU} > \lambda^*_{ST}$, by a similar discussion, we have $\lambda^*_{SU} \leq \lambda^*_{ST}$. Based on the above analysis, we can get the conclusion that $\lambda^* := \lambda^*_{SU} = \lambda^*_{ST}$.

Next, let us further consider $P_{SU}(\lambda^*)$ and $P_{ST}(\lambda^*)$. If $P_{SU}(\lambda^*) \geq 0$, we have

$$P_{SU}(\lambda^*)J_{SU}(\lambda^*) + (1-f)\beta_S\delta S^0 J_{ST}(\lambda^*) = +\infty > (1-f)\beta_S G(\lambda^*), \tag{2.78}$$

which is in contradiction with the first equation of (2.74). So, we have $P_{SU}(\lambda^*) < 0$. Similarly, we can also prove $P_{ST}(\lambda^*) < 0$.

Finally, we multiply the first equation by the second one of (2.74), yielding

$$H(\lambda)J_{SU}(\lambda)J_{ST}(\lambda) = f(1-f)(1-r)\beta_S^2 G(\lambda)[G(\lambda) - S^0(J_{SU}(\lambda) + \delta J_{ST}(\lambda))]. \tag{2.79}$$

Consequently, we have

$$H(\lambda^*) = \lim_{\lambda\to\lambda^{*-}} = \frac{f(1-f)(1-r)\beta_S^2 G(\lambda)[G(\lambda) - S^0(J_{SU}(\lambda) + \delta J_{ST}(\lambda))]}{J_{SU}(\lambda)J_{ST}(\lambda)} = 0, \tag{2.80}$$

which is in contradiction with Lemma 4.4. □

4.1.4. *Case IV: $R_C > 1$, $R_i \neq 1, i = SC, RC$ and $0 < c < \min\{c_1^*, c_2^*\}$.* Combined with Lemma 4.2, Theorems 4.3 and 4.5, we can give the following theorem directly.

THEOREM 4.6. *If $R_C > 1$ and $R_i \neq 1, i = SC, RC$ hold, then for any $c \in (0, \min\{c_1^*, c_2^*\})$, the original system (2.3) has no nonnegative bounded semi-traveling waves satisfying the asymptotic boundary condition (2.5).*

4.2. Estimation of the minimal wave speed. Biologically speaking, epidemics can spread for $c \geq c_{min}$ while they can not spread for any $c < c_{min}$, where c_{min} is the minimal wave speed, an important threshold value to determine whether epidemics can spread or not. Theorem 4.6 provides the basis for our estimation of the range of minimal wave speed. Combining with Theorem 3.1, we can conjecture that the minimal wave speed c_{min} of the original system (2.3) satisfies $c_{min} \in [\min\{c_1^*, c_2^*\}, c^*]$. We find that the lower bound of minimal wave speed c_{min} depends on the minimum value of the minimal wave speeds of its two subsystems where $I_R = 0$ or $I_{SU} = I_{ST} = 0$, which seems to be a new phenomenon.

5. Concluding remarks

In this chapter, we investigate a diffusive influenza system (2.1) with multiple strains. By solving algebraic equations, we find all equilibria of the reaction system (2.2) and the corresponding conditions that guarantee their existence (see Table 1). There are three possible equilibria for the reaction system, i.e., two boundary equilibria (the disease-free equilibrium E^0 and the boundary equilibrium $\hat{E}$) and an interior (positive) equilibrium E^*. We introduce three parameters, R_{SC}, R_{RC} and R_C, to determine the region where each equilibrium exists.

As we all know, traveling waves starting from the disease-free equilibrium are of biological significance since we can get a lot of information from them, such as whether epidemics will spread, asymptotic speed of propagation, the final state of the wavefront, etc. By introducing an auxiliary system and using the Schauder's fixed-point theorem, we first establish the existence of positive semi-traveling waves connecting the disease-free equilibrium E^0 in terms of R_C and the critical wave speed c^* for the original system (2.3). On the basis of the existence of semi-traveling waves, we construct an appropriate Lyapunov function and use LaSalle's invariance principle to obtain the existence condition of strong traveling waves connecting the disease-free equilibrium E^0 and boundary equilibrium $\hat{E}$. In addition, persistence theory of dynamical systems is creatively applied to prove the existence of weak (persistent) traveling waves starting from the disease-free equilibrium E^0. In view of these three types of traveling waves, we give some biological interpretations about their analytical results in Remarks 2.1, 2.2 and 2.3, respectively. Biologically, the existence of semi-traveling waves connecting the disease-free equilibrium E^0 indicates that the spread of influenza will occur. The existence of strong traveling waves which connect the disease-free equilibrium E^0 and boundary equilibrium $\hat{E}$ indicates that there is a transition zone moving from the steady state with no infective individuals to the steady state with only drug-resistant infected individuals. In particular, the presence of persistent traveling waves indicates that the infection with sensitive and resistant strains does not disappear at the end of the wavefront.

By using the comparison principle and the negative one-side and two-side Laplace transforms, we also prove the nonexistence of nonnegative bounded semi-traveling waves which connects the disease-free steady state E^0 itself in four cases. Our results are based on the fact that the diffusion coefficients of five subpopulations are extremely different, implying the research on the original system (2.3) is more biologically meaningful. In the text, we do not give the discussion of the existence of semi-traveling waves connecting the disease-free equilibrium E^0 under the condition $c = c^*$. We can show that semi-traveling wave solutions in case of $c > c^*$ converge to semi-traveling waves corresponding to $c = c^*$ by picking a sequence $\{c_n\}$ satisfying $c_n > c^*$ and $c_n \to c^*$ as $n \to +\infty$.

CHAPTER 3

Dynamical behavior for a class of predator-prey type eco-epidemiological systems in $\mathbb{R}^3_+$

In this chapter, we study a class of predator-prey type eco-epidemiological systems in $\mathbb{R}^3_+$ as follows:

$$\begin{cases} \frac{dS}{dt} = rS(1-\frac{c_1S+c_2I}{K})(S-\theta) - \frac{\beta I}{1+\alpha I}S - \frac{bS^m}{(S+I)^n+a^n}P, \\ \frac{dI}{dt} = \frac{\beta I}{1+\alpha I}S - \frac{bI^m}{(S+I)^n+a^n}P - \mu I, \\ \frac{dP}{dt} = \gamma_1\frac{bS^m}{(S+I)^n+a^n}P + \gamma_2\frac{bI^m}{(S+I)^n+a^n}P - dP, \end{cases} \tag{3.1}$$

with initial conditions

$$S(0) \geq 0,\ I(0) \geq 0,\ P(0) \geq 0, \tag{3.2}$$

where $r, b, d, \beta, \mu, \gamma_1, \gamma_2 \in (0,1)$; $0 < c_1, c_2 \leq 1$; $0 < K, \alpha$; $-\frac{K}{c_1} \leq \theta \ll \frac{K}{c_1}$; $n, m \in \mathbb{N}_+$, $1 \leq m \leq n$; $0 < a < \frac{K}{c_1}$; and $\gamma_2 < \gamma_1$.

The rest of the chapter is organized as follows. In Section 1, we provide a brief survey of some of the relevant portions of the Conley index theory. In Section 2, we give some preliminary results associated with the solutions of system (3.1) (also called the full system). In Section 3, to understand the boundary dynamics of the full system, we separately analyze the dynamical behavior of its three subsystems in $\mathbb{R}^2_+$. In Section 4, we carry out a detailed analysis on the complete dynamics of the full system. Our results include the conditions for the local and global asymptotic stability of boundary equilibria, the uniform persistence of the full system, and certain criterion for which there is an interior periodic solution (limit cycle) via the Poincaré map and bifurcation method. Finally, we conclude our findings and provide potential applications in Section 5.

1. Conley index and restricted Conley index

In order to analyze and discuss the qualitative and bifurcation behaviors of system (3.1), in this section, we give a brief introduction to the Conley index theory.

1.1. Conley index. Basic references for this material are [30, 78, 100]. The objects of primary interest in Conley's approach to dynamical systems are isolating neighborhoods and their associated invariant sets.

DEFINITION 1.1. *Let $\varphi : \mathbb{R} \times X \to X$ be a flow on a locally compact topological space. A compact set $N \subset X$ is an isolating neighbourhood if its maximal invariant set is contained strictly in its interior, i.e.,*

$$Inv(N, \varphi) := \{x \in N : \varphi(\mathbb{R}, x) \subset N\} \subset Int(N).$$

If $S = Inv(N, \varphi)$ for some isolating neighbourhood N, then S is called an isolated invariant set.

The Conley index studies isolated invariant set S, the essential tool for this study is an index pair for S, i.e., a compact pair (N, L), whose definition is as follows:

Definition 1.2. *Let S be an isolated invariant set. A pair of compact sets (N, L) where $L \subset N$ is called an index pair for S if:*
(i) $S = Inv(cl(N \setminus L))$ and $N \setminus L$ is a neighborhood of S;
(ii) L is positively invariant in N; that is, given $x \in L$ and $\varphi([0,t],x) \subset N$, then $\varphi([0,t],x) \subset L$;
(iii) L is an exit set for N; that is, given $x \in N$ and $t_l > 0$ such that $\varphi(t_l, x) \notin N$, then there exists $t_0 \in [0, t_l]$ for which $\varphi([0,t_0],x) \subset N$ and $\varphi(t_0, x) \in L$.

It is shown in [30] that given an isolated invariant set S, there exists an index pair. For an isolated invariant set S with index pair (N, L), we give the definition for the (homotopy) Conley index of S, denoted by $h(S)$ [30, 78, 100].

Definition 1.3. *The homotopy Conley index of S is*

$$h(S) = h(S, \varphi) \sim (N/L, [L]).$$

The index has been defined in terms of an isolated invariant set, but it can be extended to an index of isolating neighborhoods as follows. Let N be an isolating neighborhood. The Conley index of N is defined to be

$$h(N) = h(N, \varphi) \sim h(Inv(N, \varphi)).$$

Observe that the Conley index of S has been defined in terms of any index pair. Furthermore, typically an isolated invariant set possesses a multitude of isolating neighborhoods. Therefore one needs the following theorem.

Theorem 1.4. *(The Conley index is well defined). Let (N, L) and (N', L') be index pairs for an isolated invariant set S. Then*

$$(N/L, [L]) \sim (N'/L', [L']).$$

Now, we state the continuation theorem for the Conley index.

Let $\varphi^\lambda : \mathbb{R} \times X \to X, \lambda \in \Lambda$, be a continuously parameterized family of flows, where the parameter space Λ is a compact locally contractible, connected metric space. The parameterized flow corresponding to the family φ^λ is the continuous flow,

$$\begin{aligned} \Phi : \mathbb{R} \times X \times \Lambda &\to X \times \Lambda \\ (t, x, \lambda) &\mapsto \Phi(t, x, \lambda) := (\varphi^\lambda(t, x), \lambda). \end{aligned}$$

Let $N \subset X \times \Lambda$ and $N^\lambda := N \cap (X \times \{\lambda\})$.

Definition 1.5. *Let $\lambda_i \in \Lambda, i = 0, 1$, and let S^i be isolated invariant set for φ^{λ_i}. S^0 and S^1 are related by continuation if there exists an isolating neighborhood $N \subset X \times \Lambda$ of the parameterized flow Φ such that $Inv(N^{\lambda_0}, \varphi^{\lambda_0}) = S^0$ and $Inv(N^{\lambda_1}, \varphi^{\lambda_1}) = S^1$.*

Theorem 1.6. *(Continuation property). Let S^0 and S^1 be isolated invariant sets that are related by continuation. Then,*

$$h(S^0) \sim h(S^1).$$

The most useful result about Conley index is as follows:

THEOREM 1.7. *(Ważewski property). Let N be an isolating neighborhood and assume that $h(Inv(N)) \neq \bar{0}$. Then, $Inv(N) \neq \emptyset$.*

This result provides the simplest example of an existence result which can be obtained via the Conley index. It also demonstrates an important point concerning the way one wishes to view the Conley index.

The following theorem is fundamental to many significant applications of the index theory to date. In particular, its converse is of greatest use.

THEOREM 1.8. *(Summation property). Assume that $S = S_0 \cup S_1$ is an isolated invariant set where S_0 and S_1 are disjoint invariant sets. Then*

$$h(S) = h(S_0) \vee h(S_1).$$

After the empty set, the simplest isolated invariant sets are hyperbolic fixed points. In this case, the following standard result will be used to determine the appropriate Conley index.

PROPOSITION 1.9. *If x_0 is a hyperbolic critical point with unstable manifold $W^u(x_0)$ of dimension n, then $\{x_0\}$ is an isolated invariant set and $h(x_0) = \Sigma^n$, the pointed n-sphere.*

The following result gives an isolated neighbourhood:

PROPOSITION 1.10. *(see [30]). Suppose $\frac{dx}{dt} = f(x)$ is a differential equation on $\mathbb{R}^n$ and let $V(x)$ be a smooth function on $\mathbb{R}^n$. Suppose there is a compact set $K \subset \mathbb{R}^n$ and a constant $\varepsilon > 0$ such that, for $x \in \mathbb{R}^n \setminus K$,*

$$\frac{d}{dt}V(x(t)) \leq -\varepsilon \|f(x(t))\|.$$

Then the set of bounded solutions of the equation is compact (in particular it is isolated, so has an index).

1.2. The restricted Conley index. The restricted Conley index is a simple generalization of the Conley index, which was proposed by A. F. A. Ismail [55]. Most of the content below, please refer to [55].

Let $\mathfrak{X}(\mathbb{R}^n)$ be the set of C^2 vector fields on $\mathbb{R}^n$. Consider the flow $\varphi_f(t,x)$ generated by the solutions of

$$\frac{dx}{dt} = f(x) \tag{3.3}$$

where $f \in \mathfrak{X}(\mathbb{R}^n)$ is a smooth vector field on $\mathbb{R}^n$.

Suppose that I is an affine subspace in $\mathbb{R}^n$ such that I is invariant for the flow φ_f generated by (3.3). Note that this is the case if and only if $x \in I$ implies that $f(x) \in T_x I$ (the tangent space of I at x). We write $f \in \mathfrak{X}_I$ to be the subset of vector fields in $\mathfrak{X}$ that leave I invariant.

LEMMA 1.11. *Suppose $N \subset \mathbb{R}^n$ is an isolating neighbourhood for $f \in \mathfrak{X}_I$ and $N \cap I \neq \emptyset$. Then $N \cap I$ is an isolating neighbourhood for f_I.*

DEFINITION 1.12. *An isolating neighbourhood N is called an isolating block if there are no internal tangencies of the flow to the boundary of N.*

Deviating slightly from the notation we have used in Conley index, we use $h^f(N)$ to denote the (homotopy) Conley index for the flow generated by f for the isolating block N.

DEFINITION 1.13. *For the flow φ_f, we define the Conley index for N restricted to I by*

$$h_I^f(N) = h^{f_I}(N \cap I),$$

if N is an isolating block; and similarly

$$h_I^f(S) = h^{f_I}(S \cap I),$$

if S is an isolated invariant set for the flow.

LEMMA 1.14. *If N is an isolating block for the flow generated by $f \in \mathfrak{X}_I$ and $N \cap I \neq \emptyset$, then the restricted Conley index $h_I(N)$ is well defined.*

THEOREM 1.15. *Suppose that, for some $f \in \mathfrak{X}_I$, $\varphi_f(t, x)$ has hyperbolic equilibria x_1 and x_2 and that S is the set of connections between these equilibria. If $S \cap I$ is a nonempty isolated invariant set of connections, then $f_{I\cap N}$ is a locally gradient-like vector field. Moreover, if*

$$h_I(S) \neq h_I(x_1) \vee h_I(x_2)$$

then the connection is robust to perturbations in $\mathfrak{X}_I$.

2. Preliminary lemmas

In this section, we show some basic results associated with system (3.1) which are necessary for the understanding of subsequent results. Therefore, we have the following lemmas, which ensure the existence, uniqueness, positivity and uniform ultimate boundedness of the solutions of (3.1).

LEMMA 2.1. *Every solution of system* (3.1) *with initial conditions* (3.2) *exists and is bounded in the interval* $[0, +\infty)$ *and* $S(t) \geq 0, I(t) \geq 0, P(t) \geq 0$ *for all* $t \geq 0$.

Proof: Obviously, system (3.1) can be written as the following Kolmogorov-type differential equations

$$\begin{cases} \frac{dS}{dt} = Sf_1(S, I, P), \\ \frac{dI}{dt} = If_2(S, I, P), \\ \frac{dP}{dt} = Pf_3(S, I, P). \end{cases} \tag{3.4}$$

Since the functions f_1, f_2 and f_3 are C^1 and locally Lipschitzian in $\mathbb{R}_+^3$, the local existence and uniqueness of the solution $(S(t), I(t), P(t))$ of system (3.1) with initial conditions (3.2) hold, namely, it exists and is unique in the interval $[0, \xi)$, where $0 < \xi \leq +\infty$ [48]. According to system (3.4) and nonnegative initial conditions (3.2), we have

$$\begin{cases} S(t) = S(0)e^{\int_0^t f_1(S(\tau),I(\tau),P(\tau))d\tau} \geq 0, \\ I(t) = I(0)e^{\int_0^t f_2(S(\tau),I(\tau),P(\tau))d\tau} \geq 0, \\ P(t) = P(0)e^{\int_0^t f_3(S(\tau),I(\tau),P(\tau))d\tau} \geq 0. \end{cases} \tag{3.5}$$

For any solution $(S(t), I(t), P(t))$ of (3.1) which starts in $\mathbb{R}_+^3$, we first prove the component $S(t)$ is bounded. Choose any point $(S, I, P) \in \mathbb{R}_+^3$ such that $S > \frac{K}{c_1}$, since (S, I, P) is positively invariant in $\mathbb{R}_+^3$, we have

$$\frac{dS}{dt}|_{S>\frac{K}{c_1}} \leq rS(1 - \frac{c_1S + c_2I}{K})(S - \theta)|_{S>\frac{K}{c_1}} < 0, \tag{3.6}$$

which implies that $S(t)$ is bounded, without loss of generality, we assume $S(t) < M_s$.

For the component $I(t)$, we have

$$\frac{dI}{dt} = \frac{\beta I}{1+\alpha I}S - \frac{bI^m}{(S+I)^n + a^n}P - \mu I \leq \frac{\beta}{\alpha}M_s - \mu I. \tag{3.7}$$

By applying the theory of differential inequality [16], we obtain

$$I(t) \leq e^{-\mu t}I(0) + \frac{\beta}{\alpha\mu}M_s(1-e^{-\mu t}) \leq \max\{I(0), \frac{\beta}{\alpha\mu}M_s\} := M_i, \tag{3.8}$$

which implies that $I(t)$ is also bounded.

Now we define a time-dependent function by $Z(t) = S(t) + I(t) + P(t)$, then the time-derivatives of $Z(t)$ along the solutions of (3.1) satisfy

$$\begin{aligned}\frac{dZ}{dt} =& rS(1-\frac{c_1S+c_2I}{K})(S-\theta) + (\gamma_1 - 1)\phi_1(S,I,P)P + (\gamma_2-1)\phi_2(S,I,P)P - \mu I - dP\\ \leq& rS(1-\frac{c_1S+c_2I}{K})(S-\theta) - \mu I - dP\\ =& rS(1-\frac{c_1S+c_2I}{K})(S-\theta) + dS + (d-\mu)I - dZ\\ \leq& M_z - dZ,\end{aligned} \tag{3.9}$$

where $M_z = \max_{0\leq S\leq M_s, 0\leq I\leq M_i}\{rS(1-\frac{c_1S+c_2I}{K})(S-\theta) + dS + (d-\mu)I\}$, and we use the condition that $0<\gamma_k<1,\ k=1,2$. Applying the theory of differential inequality [16] yields $Z(t) \leq \frac{M_z}{d}$, indicating that $P(t)$ is also bounded.

In summary, every solution $(S(t), I(t), P(t))$ of (3.1) with initial conditions (3.2) is bounded, so its existence interval is $[0, +\infty)$. □

LEMMA 2.2. *Assume that* $0 < c_2 \leq c_1 \leq 1$ *and* $0 \leq c_2 r\theta^2 \leq 4\mu K$*, then for any* $\epsilon > 0$*, all solutions of system* (3.1) *initiating in* $\mathbb{R}^3_+$ *are uniformly ultimately bounded within the region* $\mathbb{W}_\epsilon = \{(S,I,P) \in \mathbb{R}^3_+ : G \leq K+\epsilon, S+I+P \leq \frac{U}{\eta} + \epsilon\}$*, where* $G = c_1S + c_2I$*,* $0 < \eta \leq \min\{\mu, d\}$ *and* $U = \max_{0\leq S\leq \frac{K}{c_1}, 0\leq G\leq K}\{S[\eta + r(1-\frac{G}{K})(S-\theta)]\}$.

Proof: Choose any point $(S,I,P) \in \mathbb{R}^3_+$ such that $S = \frac{K}{c_1}$, we have $\frac{dS}{dt}|_{S=\frac{K}{c_1}, I=0, P=0} = 0$ and $\frac{dS}{dt}|_{S=\frac{K}{c_1}, I+P>0} < 0$. Together with (3.6) and the fact that system (3.1) has no equilibrium when $S > \frac{K}{c_1}$, we obtain

$$\limsup_{t\to+\infty} S(t) \leq \frac{K}{c_1}. \tag{3.10}$$

Now we define a time-dependent function by $G(t) = c_1S(t) + c_2I(t)$, then we show that $\limsup_{t\to+\infty} G(t) = c_1S(t) + c_2I(t) \leq K$. The time-derivatives of $G(t)$ along the solutions of (3.1) satisfy

$$\begin{aligned}\frac{dG}{dt} =& c_1\left[rS(1-\frac{G}{K})(S-\theta) - \psi(S,I) - \phi_1(S,I,P)P\right] + c_2\left(\psi(S,I) - \phi_2(S,I,P)P - \mu I\right)\\ =& c_1rS(1-\frac{G}{K})(S-\theta) - c_2\mu I - (c_1-c_2)\psi(S,I) - (c_1\phi_1(S,I,P) + c_2\phi_2(S,I,P))P\\ \leq& c_1rS(1-\frac{G}{K})(S-\theta) - c_2\mu I,\end{aligned} \tag{3.11}$$

where we use the condition that $0 < c_2 \leq c_1 \leq 1$. From (3.1), it follows that

$$\begin{cases} \frac{dS}{dt} = rS(1 - \frac{c_1S+c_2I}{K})(S-\theta) - \frac{\beta I}{1+\alpha I}S - \frac{bS^m}{(S+I)^n+a^n}P \leq rS(1-\frac{c_1S+c_2I}{K})(S-\theta) - \frac{\beta I}{1+\alpha I}S, \\ \frac{dI}{dt} = \frac{\beta I}{1+\alpha I}S - \frac{bI^m}{(S+I)^n+a^n}P - \mu I \leq \frac{\beta I}{1+\alpha I}S - \mu I = (\frac{\beta S}{1+\alpha I} - \mu)I, \end{cases} \tag{3.12}$$

which means that the S and I's dynamics of (3.1) can be governed by that of S-I subsystem.

When $\theta \leq 0$, we have $\frac{dG}{dt}|_{G>K} \leq c_1 rS(1-\frac{G}{K})(S-\theta) - c_2\mu I|_{G>K} < 0$, which implies $\limsup_{t\to+\infty} G(t) \leq K$.

When $\theta > 0$, we first assume that $S(0) < \theta$. If $\frac{\beta S}{1+\alpha I} \leq \mu$ for $0 \leq S \leq \frac{K}{c_1}$ and $I \geq 0$, then (3.12) has no interior equilibrium. Moreover, there is no equilibrium on I-axis. By Poincaré-Bendixson theorem [114], we see that any trajectory converges to a boundary equilibrium located on S-axis. Thus, we have $\limsup_{t\to+\infty} I(t) = 0$. Then we have $S(t) \leq \theta$ for all $t > 0$ due to the fact $\frac{dS}{dt}|_{S=\theta} < 0$. Then, the limiting system of (3.12) is $\frac{dS}{dt} = rS(1-\frac{c_1S}{K})(S-\theta)$ with $S(t) \leq \theta$, which indicates $\lim_{t\to+\infty} S(t) = 0$. So, we have $\lim_{t\to+\infty} G(t) = 0$. If $\frac{\beta S}{1+\alpha I} > \mu$ for $0 < S \leq \frac{K}{c_1}$ and $I \geq 0$, then we have

$$\begin{aligned} \frac{dS}{dt} \leq rS(1-\frac{c_1S+c_2I}{K})(S-\theta) - \mu I &= rS\left[(1-\frac{c_1S}{K})(S-\theta) - \frac{rc_2S^2 - \theta rc_2S + \mu K}{rKS}I\right] \\ &\leq rS(1-\frac{c_1S}{K})(S-\theta), \end{aligned}$$

where we use the condition that $0 \leq c_2 r\theta^2 \leq 4\mu K$. Thus, we have $\lim_{t\to+\infty} G(t) = 0$.
Assume $S(0) = \theta$, then we have $S(t) < \theta$ if $I(0) + P(0) > 0$ or $S(t) = \theta$ if $I(0) + P(0) = 0$. Similar to the above argument, we can show $\limsup_{t\to+\infty} G(t) \leq K$.
Assume $S(t) > \theta$ for all $t > 0$, then we have $\frac{dG}{dt}|_{G>K} \leq c_1 rS(1-\frac{G}{K})(S-\theta) - c_2\mu I|_{G>K} < 0$, implying that $\limsup_{t\to+\infty} G(t) \leq K$.

With aid of (3.9), we have

$$\frac{dZ}{dt} \leq rS(1-\frac{G}{K})(S-\theta) - \mu I - dP.$$

Similarly, defining $0 < \eta \leq \min\{\mu, d\}$, for any $\epsilon > 0$, there is a T large enough such that for any $t > T$, we have

$$\frac{dZ}{dt} + \eta Z \leq S\left[\eta + r(1-\frac{G}{K})(S-\theta)\right] + (\eta-\mu)I + (\eta-d)P \leq U_\epsilon, \tag{3.13}$$

where $U_\epsilon = \max_{0\leq S\leq \frac{K}{c_1}+\epsilon, 0\leq G\leq K+\epsilon} \left\{S\left[\eta + r(1-\frac{G}{K})(S-\theta)\right]\right\}$. Applying the theory of differential inequality [16] and letting $\epsilon \to 0$, yields

$$\limsup_{t\to+\infty} Z(t) = \limsup_{t\to+\infty}\{S(t) + I(t) + P(t)\} \leq \frac{U}{\eta}, \tag{3.14}$$

where $U = \max_{0\leq S\leq \frac{K}{c_1}, 0\leq G\leq K} \left\{S\left[\eta + r(1-\frac{G}{K})(S-\theta)\right]\right\}$. Thus, all solutions of system (3.1) are uniformly ultimately bounded for any initial value in $\mathbb{R}^3_+$. □

Remark 3.1. *Due to limited resources, no interacting species grows suddenly or exponentially over a long time interval. The assumption that $c_1 \geq c_2$ implies that intra-class competition in susceptible prey is greater than or equal to inter-class competition between susceptible and infected preys. Compared with susceptible prey, infected one is weaker and less competitive.*

3. Dynamics of three subsystems

To understand the boundary dynamics of the full system well, in this section, we divide system (3.1) into three independent subsystems in $\mathbb{R}_+^2$: the first subsystem (S-I subsystem) is obtained by assuming the absence of the predator, the second subsystem (S-P subsystem) is obtained in the absence of infected prey, and the third subsystem(I-P subsystem) is obtained by assuming the absence of susceptible prey. We first give some lemmas and definitions that will be used in later analysis.

LEMMA 3.1. *(Bifurcation point). Consider a dynamical system in* $\mathbb{R}^n$,

$$\frac{dv}{dt} = \Phi(v, \lambda), \tag{3.15}$$

if v_0 *is a hyperbolic equilibrium of system* (3.15) *for* $\lambda \in (\lambda_0 - \delta, \lambda_0) \cup (\lambda_0, \lambda_0 + \delta)$ *whose Conley index changes at* $\lambda = \lambda_0$, *then* (v_0, λ_0) *is a bifurcation point of nontrivial bounded invariant sets of system* (3.15).

Proof: Based on Conley's index theory [30], the change of Conley index of the equilibrium v_0 has the following consequences for system (3.15): $\{v_0\} \in \mathbb{R}^n$ is an isolated invariant set of (3.15) for all $\lambda \in (\lambda_0 - \delta, \lambda_0) \cup (\lambda_0, \lambda_0 + \delta)$ and let $N_\epsilon \subset \overline{B}_\epsilon(v_0) \subset \mathbb{R}^n$ be an isolating neighborhood of $\{v_0\}$ for (3.15) with $\lambda = \lambda_0 \pm \epsilon, 0 < \epsilon < \delta$. Then there is some $\widetilde{\lambda} \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ such that (3.15) has a global solution whose trajectory is in N_ϵ for all $t \in (-\infty, \infty)$ and touches the boundary of N_ϵ at some t. If there were no such $\widetilde{\lambda} \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, then N_ϵ would define a continuation from $\lambda_0 - \epsilon$ to $\lambda_0 + \epsilon$, and the Conley indices would be the same, contradicting with hypothesis. The union of all such bounded trajectories forms a nontrivial bounded invariant set in $N_\epsilon \subset \overline{B}_\epsilon(v_0)$ for (3.15) with $\widetilde{\lambda} \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$. Since $0 < \epsilon < \delta$ is arbitrary, we can say that (v_0, λ_0) is a bifurcation point of nontrivial bounded invariant sets of system (3.15). □

REMARK 3.2. *In other words, any change of the local Conley index of* $D_v\Phi(v_0, \lambda)$ *at* $\lambda = \lambda_0$ *implies bifurcation of system* (3.15) *at* $\lambda = \lambda_0$.

REMARK 3.3. *Generally speaking, the Conley index remains the same under a change of the parameter* λ, *that is,* $h(v_0, \lambda_1) = h(v_0, \lambda_2)$. *If the Conley index of the equilibrium* v_0 *has changed, then for some intermediate value of the parameter* $\lambda \in (\underline{\lambda}, \overline{\lambda})$, *the neighbourhood* N *has ceased to be isolating. Recall that the violation of the condition of being isolating is equivalent to the existence of a boundary point* $\bar{v} \in \partial N$ *such that the trajectory passing through it is entirely contained in* N. *But the converse statement is not true. If the condition of being isolating is violated for some value of* λ, *Conley's theory ceases to work and cannot give a definite answer as to whether or not the index will change.*

One of the most illustrative extensions of Lemma 3.1 is the bifurcation of the birth of a cycle on a plane when for each value of the parameter $\lambda \in [\underline{\lambda}, \overline{\lambda}]$ there is exactly one equilibrium with complex multipliers $\beta_1(\lambda) \pm i\beta_2(\lambda)$. Next we give a proposition of the Hopf bifurcation in the plane by using Conley index.

PROPOSITION 3.2. *(Hopf bifurcation). Consider a planar system in* $\mathbb{R}^2$

$$\begin{cases} \frac{dx}{dt} = g_1(x, y, \lambda), \\ \frac{dy}{dt} = g_2(x, y, \lambda), \end{cases} \tag{3.16}$$

where g_1 and g_2 are smooth. Suppose that (x_0, y_0) is a hyperbolic equilibrium of system (3.16) for $\lambda \in (\underline{\lambda}, \lambda_0) \cup (\lambda_0, \overline{\lambda})$ whose Conley index changes transversely from Σ^0 at $\lambda \in (\underline{\lambda}, \lambda_0)$ to Σ^2 at $\lambda \in (\lambda_0, \overline{\lambda})$ (wherein the transverse change refers to $\frac{d\beta_1(\lambda)}{d\lambda}|_{\lambda=\lambda_0} \neq 0$), then system (3.16) undergoes a Hopf bifurcation at (x_0, y_0, λ_0). Moreover, if the Conley index of (x_0, y_0) at $\lambda = \lambda_0$ is Σ^0, the Hopf bifurcation is supercritical; if the Conley index of (x_0, y_0) at $\lambda = \lambda_0$ is Σ^2, the Hopf bifurcation is subcritical; if the Conley index of (x_0, y_0) at $\lambda = \lambda_0$ is $\bar{0}$, the Hopf bifurcation is degenerate.

Proof: Obviously, (x_0, y_0, λ_0) is a bifurcation point of system (3.16), see Lemma 3.1. Since the equilibrium (x_0, y_0) is hyperbolic for $\lambda \in (\underline{\lambda}, \lambda_0) \cup (\lambda_0, \overline{\lambda})$ and its Conley index changes from Σ^0 in $\lambda < \lambda_0$ to Σ^2 in $\lambda > \lambda_0$, it follows that the real part of the multipliers, i.e., $\beta_1(\lambda)$, changes sign from minus to plus, resulting in the attracting point turning into a repelling one. Since $\beta_1(\lambda)$ is C^1 function of the parameter λ, the non-hyperbolicity and transversality conditions are satisfied. If the standard ϵ-neighbourhood of the equilibrium is fixed in such a way that it is isolating for both $\lambda = \underline{\lambda}$ and $\lambda = \overline{\lambda}$, then for $\lambda = \lambda_0$ it will cease to be isolating, and this will mean that a closed trajectory entirely lying in this neighborhood passes through some boundary point of the neighborhood. It is clear that this is a limit cycle, since there can be nothing else due to dimensional settings. Thus, the Hopf bifurcation occurs at (x_0, y_0, λ_0). By the Hopf bifurcation theorem in [28], we see that the type of this bifurcation (super- or subcritical) is determined by the stability of the equilibrium (x_0, y_0) at $\lambda = \lambda_0$. If the Conley index of (x_0, y_0) at $\lambda = \lambda_0$ is Σ^0, this implies that (x_0, y_0) at $\lambda = \lambda_0$ is a weak attractor. So, we conclude that there is a supercritical Hopf bifurcation at this Hopf point. Similarly, we can show the Hopf bifurcation is subcritical if the Conley index of (x_0, y_0) at $\lambda = \lambda_0$ is Σ^2. If the Conley index of (x_0, y_0) at $\lambda = \lambda_0$ is $\bar{0}$, this suggests (x_0, y_0) is neither attractive nor repulsive, thus a degenerate Hopf bifurcation takes place at $\lambda = \lambda_0$. □

Different types of solutions can be found for these subsystems, so we introduce the following definitions:

Definition 3.3. *(Heteroclinic orbit). A solution curve $v(t)$ of system (3.15) is called a heteroclinic orbit between equilibria if it connects two equilibria $v_1 \neq v_2$. In this case, $\alpha(v(0)) = v_1$ and $\omega(v(0)) = v_2$, where $\alpha(\cdot)$ and $\omega(\cdot)$ are called α- and ω- limit sets, respectively.*

Definition 3.4. *(Bifurcation point of heteroclinic orbits). Consider the parametrized family of differential equations (3.15) and choose a point $(v_0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}$ such that $\Phi(v_0, \lambda_0) = 0$. This point (v_0, λ_0) is said to be a bifurcation point of heteroclinic orbits of (3.15) if for any open neighborhood $\mathcal{V}$ of $(v_0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}$ there exists a heteroclinic orbit of (3.15) included in $\mathcal{V}$.*

Definition 3.5. *(Heteroclinic cycle). A finite collection Ξ of heteroclinic orbits which connect a finite number of equilibria $\{v_1, v_2, \cdots, v_n\}$ in a cycle for system (3.15) is called a heteroclinic cycle. That is*

$$v_1 \to v_2 \to \cdots \to v_n \to v_1$$

where $v_i \to v_j$ means there exists (at least one) heteroclinic orbit from v_i to v_j for all $i, j \in \mathbb{N}_+$.

In the following, we perform a detailed analysis for the complete dynamics of each subsystem.

3.1. S-I subsystem. The S-I subsystem in the absence of predation in (3.1) is represented as

$$X^1_\zeta : \begin{cases} \frac{dS}{dt} = rS(1-\frac{c_1S+c_2I}{K})(S-\theta) - \frac{\beta I}{1+\alpha I}S, \\ \frac{dI}{dt} = \frac{\beta I}{1+\alpha I}S - \mu I, \end{cases} \tag{3.17}$$

where $\zeta \in \aleph = \Big\{(r,K,\alpha,c_1,c_2,\beta,\mu,\theta) \in \mathbb{R}^3_+ \times (0,1]^4 \times (-\frac{K}{c_1},\frac{K}{c_1}) : c_1 \geq c_2\Big\}$.
System (3.17) or vector field X^1_ζ is defined in the set:

$$\Omega_1 = \left\{(S,I) \in \mathbb{R}^2_+ : 0 \leq c_1S + c_2I \leq K\right\}.$$

In order to reduce the number of parameters and make an adequate description of dynamical behavior of system (3.17), we follow the methodology used in [1], making a change of variables and time rescaling given by the function: $\varphi_1 : \widetilde{\Omega}_1 \times \mathbb{R} \to \Omega_1 \times \mathbb{R}$ such that

$$\varphi_1(s,i,\tau) = \left(\frac{K}{c_1}s, \frac{K}{c_2}i, \frac{c_1}{rK}\frac{\alpha K i + c_2}{\alpha K}\tau\right) = (S,I,t),$$

where $\widetilde{\Omega}_1 = \{(s,i) \in \mathbb{R}^2_+ : 0 \leq s+i \leq 1\}$. We have $\det D\varphi_1(s,i,\tau) > 0$, that is, φ_1 is a diffeomorphism preserving the orientation of the time. In the new coordinates, the vector field $Y^1_{\overline{\zeta}} = \varphi_1 \circ X^1_\zeta$ is topologically equivalent to the vector field X^1_ζ, and its associated differential equations are given by

$$Y^1_{\overline{\zeta}} : \begin{cases} \frac{ds}{d\tau} = [(1-s-i)(s-\Theta)(i+A) - B_1 i]\, s, \\ \frac{di}{d\tau} = B_2\,[s - C(i+A)]\, i, \end{cases} \tag{3.18}$$

where $A = \frac{c_2}{\alpha K}, B_1 = \frac{\beta c_1}{\alpha r K}, B_2 = \frac{\beta c_2}{\alpha r K}, C = \frac{\mu\alpha c_1}{\beta c_2}$ and $\Theta = \frac{\theta c_1}{K}$, with

$$\overline{\zeta} = (\Theta, A, B_1, B_2, C) \in (-1,1) \times \mathbb{R}^4_+.$$

Since φ_1 is a diffeomorphism, system (3.18) has the same qualitative behavior as system (3.17). The equilibria of (3.18) in $\widetilde{\Omega}_1$ are $E^1_0(0,0), E^1_1(1,0)$ which always exist, $E^1_\Theta(\Theta,0)$ whose existence depends on the value of Θ, and the positive equilibria satisfying the equations of the isoclines

$$\begin{cases} (1-s-i)(s-\Theta)(i+A) - B_1 i = 0, \\ s - C(i+A) = 0, \end{cases} \tag{3.19}$$

By the second equation of (3.19), if $AC \geq 1$, we have $i = \frac{s-AC}{C} \leq 0$ in $\widetilde{\Omega}_1$, indicating that system (3.18) has no positive equilibrium. So, in the following we only discuss the case where $AC < 1$.

By solving (3.19), we see that the abscissa of the positive equilibria satisfies the following cubic equation:

$$Q(s) = (C+1)s^3 - [\Theta(C+1) + C(A+1)]\, s^2 + C\,[\Theta(A+1) + B_1]\, s - AB_1C^2 = 0. \tag{3.20}$$

Lemma 3.6. *For system* (3.18) *or vector field* $Y^1_{\overline{\zeta}}$*, if* $AC < 1$*, we have:*

(i) *when* $\Theta < AC$*, system* (3.18) *has a unique positive equilibrium* (s^*, i^*) *where* $s^* \in (AC, \frac{C(A+1)}{C+1})$*;*

(ii) *when* $\Theta = AC$*, if* $\widetilde{\Delta}_1 < 0$ *or* $\widetilde{\Delta}_1 = 0$ *and* $\widetilde{s}^* \leq AC$ *or* $\widetilde{\Delta}_1 > 0$ *and* $\widetilde{s}_+ \leq AC$*, system* (3.18) *has no positive equilibrium;*
if $\widetilde{\Delta}_1 = 0$ *and* $\widetilde{s}^* > AC$*, system* (3.18) *has a unique positive equilibrium* $(\widetilde{s}^*, \widetilde{i}^*)$*;*
if $\widetilde{\Delta}_1 > 0$ *and* $\widetilde{s}_- \leq AC < \widetilde{s}_+$*, system* (3.18) *has a unique positive equilibrium* $(\widetilde{s}_+, \widetilde{i}_+)$*;*

if $\widetilde{\Delta}_1 > 0$ *and* $AC < \widetilde{s}_-$, *system* (3.18) *has two different positive equilibria* $(\widetilde{s}_-, \widetilde{i}_-)$ *and* $(\widetilde{s}_+, \widetilde{i}_+)$; *where*

$$\widetilde{s}^* = \frac{C(A+1)}{2(C+1)}, \quad \widetilde{\Delta}_1 = C^2(A+1)^2 - 4B_1C(C+1), \quad \widetilde{s}_\pm = \frac{C(A+1) \pm \sqrt{\widetilde{\Delta}_1}}{2(C+1)};$$

(iii) *when* $AC < \Theta < \frac{C(A+1)}{C+1}$ *or* $\Theta > \frac{C(A+1)}{C+1}$,
if $\overline{\Delta}_1 < 0$, *system* (3.18) *has no positive equilibrium;*
if $\overline{\Delta}_1 = 0$, *system* (3.18) *has a unique positive equilibrium* $(\overline{s}^*, \overline{i}^*)$;
if $\overline{\Delta}_1 > 0$, *system* (3.18) *has two different positive equilibria* $(\overline{s}_-, \overline{i}_-)$ *and* $(\overline{s}_+, \overline{i}_+)$; *where*

$$\overline{s}^* = \frac{\Theta(C+1) + C(A+1) - \bar{s}_0(C+1)}{2(C+1)} \in \left(\Theta, \frac{C(A+1)}{C+1}\right) \text{ or } \left(\frac{C(A+1)}{C+1}, \Theta\right),$$

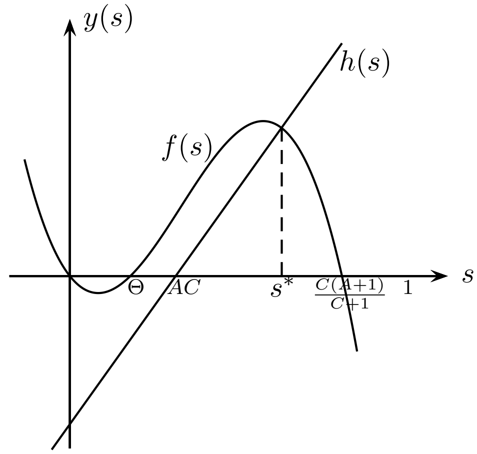
$$\overline{\Delta}_1 = [\Theta(C+1) + C(A+1) - \bar{s}_0(C+1)]^2 + 4(C+1)\frac{AC(\bar{s}_0 - \Theta)[(C+1)\bar{s}_0 - (A+1)C]}{\bar{s}_0 - AC},$$

$$\overline{s}_\pm = \frac{[\Theta(C+1) + C(A+1) - \bar{s}_0(C+1)] \pm \sqrt{\overline{\Delta}_1}}{2(C+1)} \in \left(\Theta, \frac{C(A+1)}{C+1}\right) \text{ or } \left(\frac{C(A+1)}{C+1}, \Theta\right),$$

$\bar{s}_0$ *is a real positive root of* (3.20), *which always exists in the interval* $(0, AC)$;

(iv) *when* $\Theta = \frac{C(A+1)}{C+1}$, *system* (3.18) *has no positive equilibrium.*

Proof: To obtain the number of positive equilibria of system (3.18), we can rewrite (3.20) as $f(s) - h(s) = 0$, where $f(s) = s(s - \Theta)[C(A+1) - (C+1)s]$ and $h(s) = B_1C(s - AC)$. Obviously, $f(s)$ has three roots $0, \Theta$ and $\frac{C(A+1)}{C+1}$, and $h(s)$ intersects the positive s-axis at $s = AC$. Through graph analysis, we easily obtain the results of (i) and (iv).



Next, we give the proof of (ii). When $\Theta = AC$, the curves $f(s)$ and $h(s)$ intersect at $s = AC$. So, $Q(s)$ can be rewritten as $Q(s) = (s - AC)[(C+1)s^2 - C(A+1)s + B_1C]$. This implies that in addition to the root $s = AC$, the remaining roots of $Q(s) = 0$ are determined by the quadratic equation

$$(C+1)s^2 - C(A+1)s + B_1C = 0. \tag{3.21}$$

Let $\widetilde{\Delta}_1 = C^2(A+1)^2 - 4B_1C(C+1)$.

If $\widetilde{\Delta}_1 < 0$, (3.21) has no real root, implying that system (3.18) has no positive equilibrium.

If $\widetilde{\Delta}_1 = 0$, (3.21) has a positive real root $\widetilde{s}^*$ with multiplicity 2. In this case, if $\widetilde{s}^* > AC$ (i.e., $A(2C+1) < 1$), system (3.18) has a unique positive equilibrium $(\widetilde{s}^*, \widetilde{i}^*)$; otherwise, system (3.18) has no positive equilibrium.

If $\widetilde{\Delta}_1 > 0$, (3.21) has two different positive real roots $\widetilde{s}_\pm$. In this case, if $AC < \widetilde{s}_-$, system (3.18) has two different positive equilibria $(\widetilde{s}_-, \widetilde{i}_-)$ and $(\widetilde{s}_+, \widetilde{i}_+)$; if $\widetilde{s}_- \leq AC < \widetilde{s}_+$, system (3.18) has a unique positive equilibrium $(\widetilde{s}_+, \widetilde{i}_+)$; if $\widetilde{s}_+ \leq AC$, system (3.18) has no positive equilibrium.

Finally, we prove the results of (iii). When $AC < \Theta < \frac{C(A+1)}{C+1}$ or $\Theta > \frac{C(A+1)}{C+1}$, we see that (3.20) always has a positive real root lying in interval $(0, AC)$, denoted by $\bar{s}_0$. Dividing the polynomial $Q(s)$ by $s - \bar{s}_0$, yields $Q(s) = (s - \bar{s}_0)Q_1(s)$, where

$$\begin{aligned} Q_1(s) =&(C+1)s^2 - [\Theta(C+1) + C(A+1) - \bar{s}_0(C+1)]\, s + \Theta C(A+1) \\ &+ B_1 C - \bar{s}_0 [\Theta(C+1) + C(A+1) - \bar{s}_0(C+1)] \end{aligned} \tag{3.22}$$

is a factor of $Q(s)$. By the expression (3.20) of $Q(s)$, we have

$$R(s) = \bar{s}_0 \{\Theta C(A+1) + B_1 C - \bar{s}_0 [\Theta(C+1) + C(A+1) - \bar{s}_0(C+1)]\} - AB_1C^2 = 0.$$

Then, we can solve $B_1 = \frac{\bar{s}_0\{\bar{s}_0[\Theta(C+1)+C(A+1)-\bar{s}_0(C+1)]-\Theta C(A+1)\}}{C(\bar{s}_0 - AC)}$. Substituting it into (3.22), we have

$$Q_1(s) = (C+1)s^2 - [\Theta(C+1) + C(A+1) - \bar{s}_0(C+1)]\, s - \frac{AC(\bar{s}_0-\Theta)[(C+1)\bar{s}_0-(A+1)C]}{\bar{s}_0 - AC}. \tag{3.23}$$

In addition to the positive real root $\bar{s}_0$, the number of the remaining roots of (3.20) will be determined by that of the roots of (3.23). To this end, we define

$$\overline{\Delta}_1 = [\Theta(C+1) + C(A+1) - \bar{s}(C+1)]^2 + 4(C+1)\frac{AC(\bar{s}-\Theta)\,[(C+1)\bar{s} - (A+1)C]}{\bar{s} - AC}.$$

Considering the sign of $\overline{\Delta}_1$, we conclude that

(a) if $\overline{\Delta}_1 < 0$, (3.23) has no real root, implying that system (3.18) has no positive equilibrium;

(b) if $\overline{\Delta}_1 = 0$, (3.23) has one positive real root s^* with multiplicity two, indicating that system (3.18) has a unique positive equilibrium $(\overline{s}^*, \overline{i}^*)$;

(c) if $\overline{\Delta}_1 > 0$, (3.23) has two different positive real roots $s_\pm$, meaning that system (3.18) has two different positive equilibria $(\overline{s}_-, \overline{i}_-)$ and $(\overline{s}_+, \overline{i}_+)$. □

To determine the local nature of equilibria, we first give the Jacobian matrix of system (3.18) at any point (s, i) as follows:

$$J_1(s,i) = \begin{pmatrix} J_1(s,i)_{11} & J_1(s,i)_{12} \\ J_1(s,i)_{21} & J_1(s,i)_{22} \end{pmatrix}, \tag{3.24}$$

where

$J_1(s,i)_{11} = (1-s-i)(s-\Theta)(i+A) - B_1 i + s(i+A)(\Theta+1-2s-i)$,

$J_1(s,i)_{12} = s(s-\Theta)(1-A-s-2i) - B_1 s,\ J_1(s,i)_{21} = B_2 i,\ J_1(s,i)_{22} = B_2[s - C(2i+A)]$.

It is clear that the equilibrium (s, i) is an elementary equilibrium, a hyperbolic saddle or a degenerated equilibrium if $\det(J_1(s,i)) \neq 0$, $\det(J_1(s,i)) < 0$ or $\det(J_1(s,i)) = 0$, respectively.

After substituting $E_1^1(1,0)$ into (3.24), we obtain its stability by computing the corresponding eigenvalues: E_1^1 is locally asymptotically stable (node) if $AC > 1$, it is a hyperbolic saddle if $AC < 1$, and it is a stable saddle-node if $AC = 1$. Both the eigenvalues associated with (3.24) at E_1^1 are $\rho_1 = A(\Theta - 1)(< 0)$ and $\rho_2 = B_2(1 - AC)$. As the existence or stability of the equilibria $E_0^1(0,0)$ and $E_\Theta^1(\Theta, 0)$ depends on the value of Θ, we discuss them in the following two cases.

Case: $\Theta > 0$

(I) E_0^1 is always locally asymptotically stable (node), because both the eigenvalues associated with (3.24) at E_0^1 are $\rho_1 = -\Theta A(< 0)$ and $\rho_2 = -AB_2C(< 0)$;

(II) E_Θ^1 is an unstable node if $\Theta > AC$, it is a hyperbolic saddle if $\Theta < AC$, and it is an unstable saddle-node if $\Theta = AC$. Both the eigenvalues associated with (3.24) at E_Θ^1 are $\rho_1 = A\Theta(1-\Theta)(>0)$ and $\rho_2 = B_2(\Theta - AC)$;

Case: $\Theta \leq 0$

(I) E_0^1 is a hyperbolic saddle if $-1 < \Theta < 0$, and it is a stable saddle-node if $\Theta = 0$. Both the eigenvalues associated with (3.24) at E_0^1 are given by

$$\rho_1 = -\Theta A \begin{cases} >0, \text{ if } -1<\Theta<0, \\ =0, \text{ if } \Theta = 0. \end{cases} \quad \text{and } \rho_2 = -AB_2C(<0).$$

Let us consider the local stability of positive equilibria. First, we consider the case where there is a unique positive equilibrium, denoted by $E_-^1(\bar{s},\bar{i})$ ($\bar{i} = \frac{\bar{s}-AC}{C}, \bar{s} > AC$), for system (3.18). The Jacobian matrix at $E_-^1(\bar{s},\bar{i})$ is

$$J_1(E_-^1(\bar{s},\bar{i})) = \begin{pmatrix} J_1(E_-^1(\bar{s},\bar{i}))_{11} & J_1(E_-^1(\bar{s},\bar{i}))_{12} \\ J_1(E_-^1(\bar{s},\bar{i}))_{21} & J_1(E_-^1(\bar{s},\bar{i}))_{22} \end{pmatrix}, \tag{3.25}$$

where

$$J_1(E_-^1(\bar{s},\bar{i}))_{11} = \frac{\bar{s}^2}{C^2}\left[(\Theta + A + 1)C - (2C+1)\bar{s}\right],$$
$$J_1(E_-^1(\bar{s},\bar{i}))_{12} = \frac{\bar{s}}{C}\{(\bar{s}-\Theta)[(A+1)C - (C+2)\bar{s}] - B_1C\},$$
$$J_1(E_-^1(\bar{s},\bar{i}))_{21} = \frac{B_2(\bar{s}-AC)}{C}, \qquad J_1(E_-^1(\bar{s},\bar{i}))_{22} = B_2(AC-\bar{s}).$$

Then, we have

$$\begin{aligned} &\det(J_1(E_-^1(\bar{s},\bar{i}))) \\ =& -\frac{B_2\bar{s}}{C^2}(\bar{s}-AC)\left\{-3(C+1)\bar{s}^2 + 2\left[\Theta(C+1) + (A+1)C\right]\bar{s} - \Theta C(A+1) - B_1C\right\} \\ =&\frac{B_2\bar{s}}{C^2}(\bar{s}-AC)[h'(\bar{s}) - f'(\bar{s})] \geq 0, \end{aligned} \tag{3.26}$$

and the trace is given by

$$\begin{aligned} tr(J_1(E_-^1(\bar{s},\bar{i}))) =& \frac{\bar{s}^2}{C^2}[(\Theta + A + 1)C - (2C+1)\bar{s}] + B_2(AC-\bar{s}) \\ =& \frac{1}{C^2}\{\bar{s}^2[(\Theta + A + 1)C - (2C+1)\bar{s}] - B_2C^2(\bar{s}-AC)\}. \end{aligned} \tag{3.27}$$

If $tr(J_1(E_-^1(\bar{s},\bar{i}))) = 0$, then $B_2 = B_2^* := \frac{\bar{s}^2[(\Theta+A+1)C-(2C+1)\bar{s}]}{C^2(\bar{s}-AC)}$. Since $\bar{s} > AC$ and $B_2 > 0$, we have $AC < \frac{\Theta+1}{2}$. Let $\Delta_1 = (tr(J_1(E_-^1(\bar{s},\bar{i}))))^2 - 4\det(J_1(E_-^1(\bar{s},\bar{i})))$. For system (3.18), the unique positive equilibrium $E_-^1(\bar{s},\bar{i})$ has the following properties:

LEMMA 3.7. *Let $E_-^1(\bar{s},\bar{i})$ be the unique positive equilibrium of* (3.18), *if $0 < AC < \frac{\Theta+1}{2}$, then we have*

(I) If $\det(J_1(E_-^1(\bar{s},\bar{i}))) > 0$, and

(i) If $tr(J_1(E_-^1(\bar{s},\bar{i}))) < 0$, i.e., $B_2 > B_2^*$, then it is an attractor. Furthermore,

(i.1) It is an attractor node if $\Delta_1 > 0$,

(i.2) It is an attractor focus if $\Delta_1 < 0$;

(ii) If $tr(J_1(E_-^1(\bar{s},\bar{i}))) > 0$, i.e., $B_2 < B_2^*$, then it is a repeller. Furthermore,

(ii.1) It is a repeller node if $\Delta_1 > 0$,

(ii.2) It is a repeller focus if $\Delta_1 < 0$;

(iii) If $tr(J_1(E_-^1(\bar{s},\bar{i}))) = 0$, i.e., $B_2 = B_2^*$, then it is a weak focus (center).

(II) If $\det(J_1(E_-^1(\bar{s},\bar{i}))) = 0$, and

(i) If $tr(J_1(E^1_-(\bar{s},\bar{i}))) < 0$, *i.e.,* $B_2 > B_2^*$, *then it is a stable saddle-node;*

(ii) If $tr(J_1(E^1_-(\bar{s},\bar{i}))) > 0$, *i.e.,* $B_2 < B_2^*$, *then it is an unstable saddle-node;*

(iii) If $tr(J_1(E^1_-(\bar{s},\bar{i}))) = 0$, *i.e.,* $B_2 = B_2^*$, *then it is a cusp (i.e., Bogdanov-Takens bifurcation point).*

Let $E^1_{-1}(\bar{s}_1,\bar{i}_1)$ and $E^1_{-2}(\bar{s}_2,\bar{i}_2)$ be two different positive equilibria of (3.18), where $AC < \bar{s}_1 < \bar{s}_2$, $\bar{i}_{1,2} = \frac{\bar{s}_{1,2}-AC}{C}$. Then, we have

$$\det(J_1(E^1_{-1}(\bar{s}_1,\bar{i}_1))) = \frac{B_2\bar{s}_1(\bar{s}_1 - AC)}{C^2}(h'(\bar{s}_1) - f'(\bar{s}_1)) < 0,$$

and

$$\det(J_1(E^1_{-2}(\bar{s}_2,\bar{i}_2))) = \frac{B_2\bar{s}_2(\bar{s}_2 - AC)}{C^2}(h'(\bar{s}_2) - f'(\bar{s}_2)) > 0,$$

and the trace of $J_1(E^1_{-2}(\bar{s}_2,\bar{i}_2)$ is given by

$$tr(J_1(E^1_{-2}(\bar{s}_2,\bar{i}_2)) = \frac{1}{C^2}\left\{\bar{s}_2^2\left[(\Theta + A + 1)C - (2C+1)\bar{s}_2\right] - B_2C^2(\bar{s}_2 - AC)\right\}.$$

Let $\Delta_{12} = (tr(J_1(E^1_{-2}(\bar{s}_2,\bar{i}_2)))^2 - 4\det(J_1(E^1_{-2}(\bar{s}_2,\bar{i}_2))$ and $B_{22}^* = \frac{\bar{s}_2^2[(\Theta+A+1)C-(2C+1)\bar{s}_2]}{C^2(\bar{s}_2-AC)}$. For system (3.18), the positive equilibria $E^1_{-1}(\bar{s}_1,\bar{i}_1)$ and $E^1_{-2}(\bar{s}_2,\bar{i}_2)$ have the following properties:

LEMMA 3.8. *Let* $E^1_{-1}(\bar{s}_1,\bar{i}_1)$ *and* $E^1_{-2}(\bar{s}_2,\bar{i}_2)$ *be the two different positive equilibria of system* (3.18)*, then we have*

(I) The positive equilibrium $E^1_{-1}(\bar{s}_1,\bar{i}_1)$ *is always a saddle;*

(II) For the positive equilibrium $E^1_{-2}(\bar{s}_2,\bar{i}_2)$, *if* $0 < AC < \frac{\Theta+1}{2}$, *and*

(i) If $tr(J_1(E^1_{-2}(\bar{s}_2,\bar{i}_2))) < 0$, *i.e.,* $B_2 > B_{22}^*$, *then it is an attractor. Furthermore,*

(i.1) It is an attractor node if $\Delta_{12} > 0$,

(i.2) It is an attractor focus if $\Delta_{12} < 0$;

(ii) If $tr(J_1(E^1_{-2}(\bar{s}_2,\bar{i}_2))) > 0$, *i.e.,* $B_2 < B_{22}^*$, *then it is a repeller. Furthermore,*

(ii.1) It is a repeller node, if $\Delta_{12} > 0$,

(ii.2) It is a repeller focus if $\Delta_{12} < 0$;

(iii) If $tr(J_1(E^1_{-2}(\bar{s}_2,\bar{i}_2))) = 0$, *i.e.,* $B_2 = B_{22}^*$, *then it is a weak focus (center).*

REMARK 3.4. *According to Lemma 3.1, we see that the points*

$$(E^1_0,\Theta) = (E^1_0, 0),\ (E^1_\Theta,\Theta) = (E^1_\Theta, AC),$$

$$(E^1_1, A) = (E^1_1, \frac{1}{C}),\ (E^1_-, B_2) = (E^1_-, B_2^*),\ (E^1_{-2}, B_2) = (E^1_{-2}, B_{22}^*)$$

are all bifurcation points of system (3.18)*. When the parameter* Θ *cross the critical value 0 from the left, a new boundary equilibrium* E^1_Θ *occurs.*

PROPOSITION 3.9. *Let* $E^1_-(\bar{s},\bar{i})$ *be the unique positive equilibrium of system* (3.18)*. Assume that* $\det(J_1(E^1_-(\bar{s},\bar{i}))) > 0$, $0 < AC < \frac{\Theta+1}{2}$ *and* $B_2 > B_2^*$*. If* $\Theta \le 0$ *and* $B_2C \ge B_1 - \Theta(A+1)$*, then* $E^1_-(\bar{s},\bar{i})$ *is globally asymptotically stable.*

Proof: According to Lemma 3.7, if $0 < AC < \frac{\Theta+1}{2}$, $\det(J_1(E^1_-(\bar{s},\bar{i}))) > 0$ and $B_2 > B_2^*$, we see that the equilibrium $E^1_-(\bar{s},\bar{i})$ is locally asymptotically stable. To examine the global behavior of $E^1_-(\bar{s},\bar{i})$, we use Dulac theorem to exclude the limit cycle. Let $V = \frac{1}{s^2 i}$ (a Dulac function), if $\Theta \le 0$ and $B_2C \ge B_1 - \Theta(A+1)$, then we have

$$\frac{\partial VF_1}{\partial s} + \frac{\partial VF_2}{\partial i} = \frac{1}{s^2 i}\left[(i+A)(\Theta - s^2) - \Theta i^2 - (\Theta A + B_2C - B_1)i\right] \le 0.$$

where

$$F_1 = [(1-s-i)(s-\Theta)(i+A) - B_1 i]s,\ F_2 = B_2[s - C(i+A)]i.$$

So, system (3.18) has no limit cycle in $\mathbb{R}^2_+$, which implies $E^1_-(\bar{s},\bar{i})$ is globally asymptotically stable. □

COROLLARY 3.10. *Let* $E^1_-(\bar{s},\bar{i})$ *be the unique positive equilibrium of system* (3.18)*, suppose that* $\det(J_1(E^1_-(\bar{s},\bar{i}))) > 0$ *and* $0 < AC < \frac{\Theta+1}{2}$*, then system* (3.18) *undergoes a Hopf bifurcation at* $E^1_-(\bar{s},\bar{i})$ *at* $B_2 = B_2^*$*. When* $h(E^1_-(\bar{s},\bar{i}), B_2^*) = \Sigma^0$*, the direction of the Hopf bifurcation is supercritical and the bifurcating periodic solutions are orbitally asymptotically stable; when* $h(E^1_-(\bar{s},\bar{i}), B_2^*) = \Sigma^2$*, the direction of the Hopf bifurcation is subcritical and the bifurcating periodic solutions are unstable.*

Proof: Obviously, the equilibrium $E^1_-(\bar{s},\bar{i})$ is hyperbolic when $B_2 \neq B_2^*$, and its Conley index changes transversely from Σ^0 at $B_2 > B_2^*$ to Σ^2 at $B_2 < B_2^*$. From Proposition 3.2, we conclude that a Hopf bifurcation occurs at $E^1_-(\bar{s},\bar{i})$ for the bifurcation value $B_2 = B_2^*$, whose type depends the Conley index of $E^1_-(\bar{s},\bar{i})$ at $B_2 = B_2^*$. □

REMARK 3.5. *Similarly, if system* (3.18) *has two different positive equilibria* $E^1_{-1}(\bar{s}_1,\bar{i}_1)$ *and* $E^1_{-2}(\bar{s}_2,\bar{i}_2)$*, then a Hopf bifurcation occurs at* $E^1_{-2}(\bar{s}_2,\bar{i}_2)$ *for* $B_2 = B_{22}^*$*. In addition, suppose* $\det(J_1(E^1_-(\bar{s},\bar{i}))) = 0$ *and* $0 < AC < \frac{\Theta+1}{2}$*, system* (3.18) *may undergo a Bogdanov-Takens bifurcation of codimension 2 around* $E^1_-(\bar{s},\bar{i})$ *when* $B_2 = B_2^*$*, refer to [95, 125] for its detailed analysis.*

Finally, we give some results concerning the existence of heteroclinic orbits, a bifurcation point of heteroclinic orbits, and a heteroclinic cycle for system (3.18).

PROPOSITION 3.11. *For system* (3.18)*: (i) If* $\Theta \in (0,1)$*, there exist two heteroclinic orbits* $\gamma^{(1)}(t)$ *and* $\gamma^{(2)}(t)$ *satisfying* $\alpha(\gamma^{(1)}(t)) = \alpha(\gamma^{(2)}(t)) = E^1_\Theta$*,* $\omega(\gamma^{(1)}(t)) = E^1_0$ *and* $\omega(\gamma^{(2)}(t)) = E^1_1$*;*
(ii) If $\Theta \in (-1,0]$*, there is a heteroclinic orbit* $\gamma^{(3)}(t)$ *satisfying* $\alpha(\gamma^{(3)}(t)) = E^1_0$ *and* $\omega(\gamma^{(3)}(t)) = E^1_1$*;*
(iii) The point $(E^1_0,\Theta) = (E^1_0,0)$ *is a bifurcation point of heteroclinic orbits of* (3.18)*.*

Proof: Since three heteroclinic orbits in (i) and (ii) locate on the s-axis (denoted by I_s), we use the restricted Conley index (see subsection 1.2) to complete their proof.

(i) Fix $\Theta \in (0,1)$, the restricted Conley indices of three equilibria are given by

$$h_{I_s}(E^1_0) = \Sigma^0,\ h_{I_s}(E^1_\Theta) = \Sigma^1,\ h_{I_s}(E^1_1) = \Sigma^0.$$

Now, let us consider two intervals $G' = [\frac{\Theta-1}{2}, \frac{\Theta+1}{2}] \in \mathbb{R}$ and $G'' = [\frac{\Theta}{2}, \frac{\Theta+2}{2}] \in \mathbb{R}$. G' can be viewed as the intersection of the invariant subspace I_s and some isolating neighborhood K' including $\{E^1_0, E^1_\Theta\}$ in $\mathbb{R}^2$, so G'' is, i.e., $G'' = I_s \cap K''$ where $\{E^1_\Theta, E^1_1\} \subset K'' \subset \mathbb{R}^2$. Computing the restricted Conley indices of K' and K'', we obtain $h_{I_s}(K') = h_{I_s}(K'') = \bar{0}$. From the summation property of the restricted Conley index, we obtain $h_{I_s}(E^1_0 \cup E^1_\Theta) = h_{I_s}(E^1_0) \vee h_{I_s}(E^1_\Theta) = \Sigma^0 \vee \Sigma^1$ and similarly $h_{I_s}(E^1_\Theta \cup E^1_1) = \Sigma^0 \vee \Sigma^1$.

Since $h_{I_s}(K') = \bar{0} \neq \Sigma^0 \vee \Sigma^1 = h_{I_s}(E^1_0 \cup E^1_\Theta)$, we determine the existence of a heteroclinic connection of system (3.18), denoted by $\gamma^{(1)}(t)$ and lying in the interior of G'. Moreover, the equilibria E^1_0 and E^1_Θ are its $\omega-$ and $\alpha-$ limit sets, respectively. Analogously for G'', we conclude that there exists a heteroclinic orbit, say $\gamma^{(2)}(t)$, included in the interior of G''.

(ii) The proof is similar to that of (i).

(iii) Let $(\mathcal{D}_k)$ be a descending sequence of discs in $\mathbb{R}^2$ centered at E_0^1 such that $\cap\mathcal{D}_k = \{E_0^1\}$. Now, for each $k \in \mathbb{N}_+$ there exists an isolating block $\mathcal{B}_k \subseteq \mathcal{D}_k$ of E_0^1. By Definition 3.4, it is sufficient to show that for every open neighborhood $\mathcal{U}$ of $(E_0^1, 0) \in \mathbb{R}^3$ there exists a heteroclinic orbit of (3.18). Fix an open neighborhood $\mathcal{U}$ of $(E_0^1, 0) \in \mathbb{R}^3$. Since $\mathcal{B}_k$ is compact, we obtain $\mathcal{B}_k \times (-\epsilon, \epsilon) \subset \mathcal{U}$ if $k \in \mathbb{N}_+$ is sufficiently large and $\epsilon > 0$ is small enough. Let $\Theta \in (0, \epsilon)$, then $\mathcal{B}_k \times \{\Theta\}$ is an isolating block for some invariant set containing E_0^1 and E_Θ^1. Since $h_{I_s}(\mathcal{B}_k \times \{\Theta\}) = \bar{0}$, $h_{I_s}(E_0^1 \times \{\Theta\}) = h((E_0^1 \cap I_s) \times \{\Theta\}) = \Sigma^0$ and $h_{I_s}(E_\Theta^1 \times \{\Theta\}) = h((E_\Theta^1 \cap I_s) \times \{\Theta\}) = \Sigma^1$, we obtain the existence of a heteroclinic orbit in $\mathcal{B}_k \times \{\Theta\} \subset \mathcal{U}$. □

REMARK 3.6. *Let* $E_-^1(\bar{s}, \bar{i})$ *be the unique positive equilibrium of* (3.18)*, a heteroclinic cycle (loop)* $\gamma_h = (\Theta, 0) \cup \gamma_{\Theta 1} \cup (1, 0) \cup \gamma_{1\Theta}$ *exists for certain parameter values, wherein* $\gamma_{1\Theta}$ *lies above the s-axis.*

3.2. S-P subsystem. The S-P subsystem in the absence of the disease in (3.1) is presented as

$$X_\pi^2 : \begin{cases} \frac{dS}{dt} = rS(1 - \frac{c_1 S}{K})(S - \theta) - \frac{bS^m}{S^n + a^n} P, \\ \frac{dP}{dt} = \gamma_1 \frac{bS^m}{S^n + a^n} P - dP, \end{cases} \tag{3.28}$$

where $\pi \in \Gamma = \left\{(r, K, a, b, c_1, \gamma_1, d, m, n, \theta) \in \mathbb{R}_+^2 \times (0, \frac{K}{c_1}) \times (0, 1]^4 \times \mathbb{N}_+^2 \times (-\frac{K}{c_1}, \frac{K}{c_1}) : n \geq m \geq 1\right\}$.

Let $\tilde{h}(x) = \frac{bx^m}{x^n + a^n}$. If $n = m$, $\tilde{h}(x)$ is monotonically increasing in $[0, \infty)$. Particularly, the functions corresponding to $n = 1, 2$ are respectively called Holling types II and III [53], and $\tilde{h}(x)$ with $n > 2$ is also biologically significant [92].

If $n > m$, $\tilde{h}(x)$ is non-monotonic. When $n = 2, m = 1$, $\tilde{h}(x) = \frac{bx}{x^2 + a^2}$ is known as Holling type IV functional response (or called a simplified Monod-Haldane function) [101]. Through calculation, we have $\tilde{h}'(x) = b\frac{x^{m-1}[ma^n - (n-m)x^n]}{(x^n + a^n)^2}$, and

$$\tilde{h}''(x) = b\frac{x^{m-2}[(n-m)(n-m+1)x^{2n} + (2m^2 - n^2 - 2mn - 2m + n)a^n x^n + m(m-1)a^{2n}]}{(x^n + a^n)^3}.$$

Thus, the function $\tilde{h}(x)$ has a maximum value for $x = a\sqrt[n]{\frac{m}{n-m}}$ and one inflexion point for $m = 1$, or two for $m > 1$. For fixed n, the properties of $\tilde{h}(x)$ depending on m may have an impact on the dynamics of S-P subsystem and full system.

System (3.28) or vector field X_π^2 is defined in

$$\Omega_2 = \left\{(S, P) \in \mathbb{R}^2 : 0 \leq S \leq \frac{K}{c_1}, P \geq 0\right\}.$$

We construct the diffeomorphism transformation $\varphi_2 : \widetilde{\Omega}_2 \times \mathbb{R} \to \Omega_2 \times \mathbb{R}$ such that

$$\varphi_2(s, p, \tau) = \left(\frac{K}{c_1}s, \frac{r}{b}(\frac{K}{c_1})^{n-m+2}p, \frac{c_1}{rK}\frac{(Ks)^n + (ac_1)^n}{K^n}\tau\right) = (S, P, t)$$

with $\widetilde{\Omega}_2 = \left\{(s, p) \in \mathbb{R}^2 : 0 \leq s \leq 1, p \geq 0\right\}$.

The transformed differential equations are given by

$$\overline{Y}_{\overline{\pi}}^2 : \begin{cases} \frac{ds}{d\tau} = \left[(1 - s)(s - \Theta)(s^n + D) - s^{m-1}p\right] s := f^{(1)}(s, p), \\ \frac{dp}{d\tau} = E\left[s^m - F(s^n + D)\right] p := f^{(2)}(s, p), \end{cases} \tag{3.29}$$

where $D = (\frac{ac_1}{K})^n, E = \frac{\gamma_1 b}{r}(\frac{c_1}{K})^{n-m+1}, F = \frac{d}{\gamma_1 b}(\frac{K}{c_1})^{n-m}$ and $\Theta = \frac{\theta c_1}{K}$, with

$$\overline{\pi} \in \overline{\Gamma} = \left\{(D, E, F, m, n, \Theta) \in (0, 1) \times \mathbb{R}_+^2 \times \mathbb{N}_+^2 \times (-1, 1) : n \geq m \geq 1\right\}.$$

The equilibria of system (3.29) in $\widetilde{\Omega}_2$ are always on the curve $p = \frac{1}{F}s(1-s)(s-\Theta)$ and they are $E_0^2(0,0), E_1^2(1,0)$ which always exist, $E_\Theta^2(\Theta,0)$ depending on the value of Θ, and those whose s-component satisfies $s^m - F(s^n + D) = 0$. The Jacobian matrix or variational matrix of system (3.29) at any point (s,p) is given by

$$J_2(s,p) = \begin{pmatrix} J_2(s,p)_{11} & J_2(s,p)_{12} \\ J_2(s,p)_{21} & J_2(s,p)_{22} \end{pmatrix}, \tag{3.30}$$

where

$$\begin{aligned} J_2(s,p)_{11} =& -(n+3)s^{n+2} + (\Theta+1)(n+2)s^{n+1} - \Theta(n+1)s^n - ms^{m-1}p \\ & -3Ds^2 + 2D(\Theta+1)s - \Theta D, \\ J_2(s,p)_{12} =& -s^m,\ J_2(s,p)_{21} = E(ms^{m-1} - nFs^{n-1})p,\ J_2(s,p)_{22} = E[s^m - F(s^n + D)]. \end{aligned}$$

3.2.1. *Local stability of boundary equilibria.* Substituting $E_1^2(1,0)$ into (3.30), yields that both the eigenvalues associated with (3.30) at E_1^2 are $\rho_1 = (\Theta-1)(D+1)(<0)$ and $\rho_2 = E[1 - F(D+1)]$. So E_1^2 is locally asymptotically stable (node) if $1 - F(D+1) < 0$, it is a hyperbolic saddle if $1 - F(D+1) > 0$, and it is a stable saddle-node if $1 - F(D+1) = 0$. As the existence or stability of $E_0^2(0,0)$ and $E_\Theta^2(\Theta,0)$ depends on the value of Θ, we discuss them in two situations:

Case: $\Theta > 0$

(I) E_0^2 is always locally asymptotically stable (node) for any parameter values, because both the eigenvalues associated with (3.30) at E_0^2 are $\rho_1 = -\Theta D(<0)$ and $\rho_2 = -DEF(<0)$;

(II) E_Θ^2 is an unstable node if $\Theta^m - F(\Theta^n + D) > 0$, it is a hyperbolic saddle if $\Theta^m - F(\Theta^n + D) < 0$, and it is an unstable saddle-node if $\Theta^m - F(\Theta^n + D) = 0$. Both the eigenvalues associated with (3.30) at E_Θ^2 are $\rho_1 = -\Theta(\Theta-1)(\Theta^n + D)(>0)$ and $\rho_2 = E[\Theta^m - F(\Theta^n + D)]$.

Case: $\Theta \leq 0$

(I) E_0^2 is a hyperbolic saddle if $-1 < \Theta < 0$, and it is a stable saddle-node if $\Theta = 0$. Both the eigenvalues associated with (3.30) at E_0^2 are given by

$$\rho_1 = -\Theta D \begin{cases} > 0, & \text{if } -1 < \Theta < 0, \\ = 0, & \text{if } \Theta = 0, \end{cases} \quad \text{and } \rho_2 = -DEF(<0).$$

Remark 3.7. *Similar to the analysis in subsection 3.1, if we assume that system* (3.29) *has a unique positive equilibrium, when* $\Theta > 0$, $\Theta^m - F(\Theta^n + D) < 0$ *and* $1 - F(D+1) > 0$, *a heteroclinic cycle (loop)* $\widetilde{\gamma}_h = (\Theta,0) \cup \widetilde{\gamma}_{\Theta 1} \cup (1,0) \cup \widetilde{\gamma}_{1\Theta}$ *exists, wherein* $\widetilde{\gamma}_{1\Theta}$ *lies above the* s*-axis.*

In the following, we study the existence and properties of positive equilibria of system (3.29) with monotonic functional response ($n = m$) and non-monotonic functional response ($n > m$), respectively.

3.2.2. *Main results with monotonic functional response.* The conditions for the existence of positive equilibria of system (3.29) with monotonic functional response ($n = m$) are established as follows:

Lemma 3.12. *(i) If* $\max\{|\Theta|^n sign(\Theta), 0\} < \frac{DF}{1-F} < 1$, *then system* (3.29) *has a unique positive equilibrium* $E_*^2(\check{s},\check{p}) \in int(\widetilde{\Omega}_2)$, *where* $\check{s} = (\frac{DF}{1-F})^{\frac{1}{n}}$ *and* $\check{p} = \frac{1}{F}\check{s}(1-\check{s})(\check{s}-\Theta)$;
(ii) If $\frac{DF}{1-F} > 1$, *then system* (3.29) *has no positive equilibria in the interior of* $\widetilde{\Omega}_2$.

Proof: The positive equilibria of system (3.29) with monotonic functional response satisfy

$$\begin{cases} (1-s)(s-\Theta)(s^n+D) - s^{n-1}p = 0, \\ (1-F)s^n - DF = 0. \end{cases} \tag{3.31}$$

(i) From the second equation of (3.31), we have that if $0 < F < 1$, the equation $(1-F)s^n - DF = 0$ has a unique positive solution $\check{s}$. Correspondingly, system (3.29) has a unique positive equilibrium $(\check{s}, \check{p})$. Furthermore, since $\check{p} > 0$, we have $\check{s} \in (\Theta, 1)$ $(\Theta > 0)$ or $\check{s} \in (0,1)$ $(\Theta \leq 0)$, yielding $\max\{|\Theta|^n sign(\Theta), 0\} < \frac{DF}{1-F} < 1$.

(ii) Obviously, if $\frac{DF}{1-F} > 1$, system (3.29) has no positive equilibria in the interior of $\widetilde{\Omega}_2$. □

By Lemma 3.12, we see that there exists a unique positive equilibrium $E_*^2(\check{s}, \check{p}) \in int(\widetilde{\Omega}_2)$ for system (3.29) if $\max\{|\Theta|^n sign(\Theta), 0\} < \frac{DF}{1-F} < 1$. Then, we discuss its nature in the following lemmas.

Let $\Delta_2 = (\Theta-1)^2[n(1-F)-2]^2 + 4\Theta$, $\check{s}_\pm = \frac{(\Theta+1)[n(1-F)-2] \pm \sqrt{\Delta_2}}{2[n(1-F)-3]}$ and $\check{s}_0 = \frac{2\Theta}{\Theta+1}$.

LEMMA 3.13. *Assume $\Theta > 0$ and let $E_*^2(\check{s}, \check{p})$ be the unique positive equilibrium for system* (3.29).
(i) If $n(1-F)-3 > 0$, when $\check{s} > \check{s}_-$, E_*^2 is an attractor (stable node or focus); when $\check{s} < \check{s}_-$, E_*^2 is a repeller (unstable node or focus); when $\check{s} = \check{s}_-$, E_*^2 is a center (weak focus).
(ii) If $n(1-F)-3 < 0$, when $\check{s} > \check{s}_+$, E_*^2 is an attractor (stable node or focus); when $\check{s} < \check{s}_+$, E_*^2 is a repeller (unstable node or focus); when $\check{s} = \check{s}_+$, E_*^2 is a center (weak focus).
(iii) If $n(1-F)-3 = 0$, when $\check{s} > \check{s}_0$, E_*^2 is an attractor (stable node or focus); when $\check{s} < \check{s}_0$, E_*^2 is a repeller (unstable node or focus); when $\check{s} = \check{s}_0$, E_*^2 is a center (weak focus).

Proof: By (3.30), we have

$$J_2(E_*^2(\check{s}, \check{p})) = \begin{pmatrix} J_2(E_*^2(\check{s}, \check{p})) & -\check{s}^n \\ nE(1-F)\check{s}^{n-1}\check{p} & 0 \end{pmatrix},$$

where

$$\begin{aligned} J_2(E_*^2(\check{s}, \check{p}))_{11} = &-(n+3)\check{s}^{n+2} + (\Theta+1)(n+2)\check{s}^{n+1} - \Theta(n+1)\check{s}^n - n\check{s}^{n-1}\check{p} \\ &- 3D\check{s}^2 + 2D(\Theta+1)\check{s} - \Theta D. \end{aligned}$$

Thus we have

$$\det(J_2(E_*^2(\check{s}, \check{p}))) = nE(1-F)\check{s}^{2n-1}\check{p} > 0,$$

and

$$tr(J_2(E_*^2(\check{s}, \check{p})) = \frac{D}{1-F}\left\{[n(1-F)-3]\,\check{s}^2 + (\Theta+1)\,[2-n(1-F)]\,\check{s} + \Theta\,[n(1-F)-1]\right\}.$$

Let $\check{h}(\check{s}) = [n(1-F)-3]\,\check{s}^2 + (\Theta+1)\,[2-n(1-F)]\,\check{s} + \Theta\,[n(1-F)-1]$, we have $\check{h}(\Theta) = \Theta(1-\Theta) > 0$ and $\check{h}(1) = \Theta - 1 < 0$. Thus, there exists $\check{s}_* \in (\Theta, 1)$ such that $\check{h}(\check{s}_*) = 0$.

If $n(1-F)-3 > 0$, then we have $\check{s}_* = \check{s}_-$. When $\check{s} > \check{s}_-$, $tr(J_2(E_*^2(\check{s}, \check{p})) < 0$, E_*^2 is an attractor (stable node or focus); when $\check{s} < \check{s}_-$, $tr(J_2(E_*^2(\check{s}, \check{p})) > 0$, E_*^2 is a repeller (unstable node or focus); when $\check{s} = \check{s}_-$, $tr(J_2(E_*^2(\check{s}, \check{p})) = 0$, E_*^2 is a center (weak focus).

If $n(1-F)-3 < 0$, then we have $\check{s}_* = \check{s}_+$. When $\check{s} > \check{s}_+$, $tr(J_2(E_*^2(\check{s}, \check{p})) < 0$, E_*^2 is an attractor (stable node or focus); when $\check{s} < \check{s}_+$, $tr(J_2(E_*^2(\check{s}, \check{p})) > 0$, E_*^2 is a repeller (unstable node or focus); when $\check{s} = \check{s}_+$, $tr(J_2(E_*^2(\check{s}, \check{p})) = 0$, E_*^2 is a center (weak focus).

If $n(1-F)-3=0$, then we have $\check{s}_*=\check{s}_0$. When $\check{s}>\check{s}_0$, $tr(J_2(E_*^2(\check{s},\check{p}))<0$, E_*^2 is an attractor (stable node or focus); when $\check{s}<\check{s}_0$, $tr(J_2(E_*^2(\check{s},\check{p}))>0$, E_*^2 is a repeller (unstable node or focus); when $\check{s}=\check{s}_0$, $tr(J_2(E_*^2(\check{s},\check{p}))=0$, E_*^2 is a center (weak focus). □

Lemma 3.14. *Assume* $\Theta \le 0$ *and let* $E_*^2(\check{s},\check{p})$ *be the unique positive equilibrium for system* (3.29).
(i) If $n(1-F)\ge 2$, *then* E_*^2 *is always an attractor (stable node or focus).*
(ii) If $1<n(1-F)<2$,

(ii.1) when $\Delta_2<0$, E_*^2 *is always an attractor (stable node or focus);*

(ii.2) when $\Delta_2=0$, *if* $\check{s}\neq\check{s}_\diamond$, *then* E_*^2 *is an attractor (stable node or focus); if* $\check{s}=\check{s}_\diamond$, *then* E_*^2 *is a center (weak focus), where* $\check{s}_\diamond=\frac{(\Theta+1)[n(1-F)-2]}{2[n(1-F)-3]}$.*;*

(ii.3) when $\Delta_2>0$, *if* $\check{s}_*=\check{s}_-$ *or* $\check{s}_*=\check{s}_+$, *then* E_*^2 *is a center (weak focus); if* $0<\check{s}<\check{s}_-$ *or* $\check{s}_+<\check{s}<1$, *then* E_*^2 *is an attractor (stable node or focus); if* $\check{s}_-<\check{s}<\check{s}_+$, *then* E_*^2 *is a repeller (unstable node or focus).*
(iii) If $n(1-F)\le 1$, *when* $\check{s}>\check{s}_+$, E_*^2 *is an attractor (stable node or focus); when* $\check{s}<\check{s}_+$, E_*^2 *is a repeller (unstable node or focus); when* $\check{s}=\check{s}_+$, E_*^2 *is a center (weak focus).*

Proof: Similar to the proof of Lemma 3.13, we have $\check{h}(0)=\Theta[n(1-F)-1]$ and $\check{h}(1)=\Theta-1<0$.

If $n(1-F)\ge 2$, then we have $\check{h}(0)\le 0$ and $tr(J_2(E_*^2(\check{s},\check{p}))<0$ holds for any $\check{s}\in(0,1)$. Thus, E_*^2 is always an attractor (stable node or focus).

If $1<n(1-F)<2$, then we have $\check{h}(0)\le 0$. And, we need to determine the sign of Δ_2.

If $\Delta_2<0$, then $tr(J_2(E_*^2(\check{s},\check{p}))<0$ holds for any $\check{s}\in(0,1)$, implying that E_*^2 is always an attractor (stable node or focus).

If $\Delta_2=0$, then we have $\check{s}_*=\check{s}_\diamond$. When $\check{s}\neq\check{s}_\diamond$, $tr(J_2(E_*^2(\check{s},\check{p}))<0$, E_*^2 is an attractor (stable node or focus); when $\check{s}=\check{s}_\diamond$, $tr(J_2(E_*^2(\check{s},\check{p}))=0$, E_*^2 is a center (weak focus).

If $\Delta_2>0$, then we have $\check{s}_*=\check{s}_-$ or $\check{s}_*=\check{s}_+$. When $\check{s}_*=\check{s}_-$ or $\check{s}_*=\check{s}_+$, $tr(J_2(E_*^2(\check{s},\check{p}))=0$, E_*^2 is a center (weak focus); when $0<\check{s}<\check{s}_-$ or $\check{s}_+<\check{s}<1$, $tr(J_2(E_*^2(\check{s},\check{p}))<0$, E_*^2 is an attractor (stable node or focus); when $\check{s}_-<\check{s}<\check{s}_+$, $tr(J_2(E_*^2(\check{s},\check{p}))>0$, E_*^2 is a repeller (unstable node or focus).

If $n(1-F)\le 1$, then we have $\check{h}(0)\ge 0$. Thus, there exists $\check{s}_*\in(0,1)$ such that $h(\check{s}_*)=0$. According to the graph of $h(\check{s})$, we have $\check{s}_*=\check{s}_+$. When $\check{s}>\check{s}_+$, $tr(J_2(E_*^2(\check{s},\check{p}))<0$, E_*^2 is an attractor (stable node or focus); when $\check{s}<\check{s}_+$, $tr(J_2(E_*^2(\check{s},\check{p}))>0$, E_*^2 is a repeller (unstable node or focus); when $\check{s}=\check{s}_+$, $tr(J_2(E_*^2(\check{s},\check{p}))=0$, E_*^2 is a center (weak focus). □

Remark 3.8. *From Lemma 3.13 or (ii.3) and (iii) in Lemma 3.14, we can easily find that system* (3.29) *will undergo a Hopf bifurcation at some parameter values (e.g.,* Θ, n, D *and* F*).*

Proposition 3.15. *Let* $E_*^2(\check{s},\check{p})$ *be the unique positive equilibrium for system* (3.29)*, if* $(\check{s}-s)\left(\frac{s(1-s)(s-\Theta)(s^n+D)}{s^n}-\frac{\check{s}(1-\check{s})(\check{s}-\Theta)}{F}\right)\ge (or \le) 0$ *for all* $s\in(0,\check{s})\cup(\check{s},1)$, *and* $\frac{s(1-s)(s-\Theta)(s^n+D)}{s^n}\neq\frac{\check{s}(1-\check{s})(\check{s}-\Theta)}{F}$ *for* $0<|s-\check{s}|\ll 1$ *hold, then* E_*^2 *is globally asymptotically stable in* $\widetilde{\Omega}_2$.

Proof: System (3.29) can be rewritten as

$$\begin{cases} \frac{ds}{d\tau}=\widetilde{\psi}(s)-\widetilde{\xi}(p)h_1(s), \\ \frac{dp}{d\tau}=\widetilde{\eta}(p)h_2(s), \end{cases} \tag{3.32}$$

where

$$\widetilde{\psi}(s) = s(1-s)(s-\Theta)(s^n + D),\ \widetilde{\xi}(p) = p,$$
$$h_1(s) = s^n,\ h_2(s) = E[(1-F)s^n - DF],\ \widetilde{\eta}(p) = p.$$

By Theorem 3.3 in [117], we see if $(\check{s}-s)(\frac{\widetilde{\psi}(s)}{h_1(s)} - \widetilde{\xi}(\check{p})) \geq$ (or $\leq$)0 for all $s \in (0,\check{s}) \cup (\check{s},1)$, and $\frac{\widetilde{\psi}(s)}{h_1(s)} \not\equiv \widetilde{\xi}(\check{p})$ for $0 < |s-\check{s}| \ll 1$ hold, that is, $(\check{s}-s)(\frac{s(1-s)(s-\Theta)(s^n+D)}{s^n} - \frac{\check{s}(1-\check{s})(\check{s}-\Theta)}{F}) \geq$ (or $\leq$)0 for all $s \in (0,\check{s}) \cup (\check{s},1)$, and $\frac{s(1-s)(s-\Theta)(s^n+D)}{s^n} \not\equiv \frac{\check{s}(1-\check{s})(\check{s}-\Theta)}{F}$ for $0 < |s-\check{s}| \ll 1$ hold, then E^2_* is globally asymptotically stable in the interior of the first quadrant. □

3.2.3. *Main results with non-monotonic functional response.* The results concerning the existence of positive equilibria of system (3.29) with non-monotonic functional response ($n > m$) are given by the following lemma:

LEMMA 3.16. *(i) If* $\max\{|\Theta|^{n-m} sign(\Theta), 0\} < \frac{m}{nF} < 1$ *and* $D = \frac{n-m}{nF}(\frac{m}{nF})^{\frac{m}{n-m}}$*, then system* (3.29) *has a unique positive equilibrium* $E^2_\circ(\hat{s},\hat{p}) \in int(\widetilde{\Omega}_2)$*, where* $\hat{s} = (\frac{m}{nF})^{\frac{1}{n-m}}$ *and* $\hat{p} = \frac{1}{F}\hat{s}(1-\hat{s})(\hat{s}-\Theta)$*;*
(ii) If $\max\left\{0, |\frac{\Theta^m - F\Theta^n}{F}| sign(\Theta), \frac{1-F}{F}\right\} < D < \frac{n-m}{nF}(\frac{m}{nF})^{\frac{m}{n-m}}$*, then system* (3.29) *has two different positive equilibria* $E^2_{\circ 1}(\hat{s}_1,\hat{p}_1)$ *and* $E^2_{\circ 2}(\hat{s}_2,\hat{p}_2)$ *in the interior of* $\widetilde{\Omega}_2$ *whose components satisfy* $\max\{\Theta, 0\} < \hat{s}_1 < \hat{s} < \hat{s}_2 < 1$ *and* $\hat{p}_k = \frac{1}{F}\hat{s}_k(1-\hat{s}_k)(\hat{s}_k - \Theta),\ k = 1,2$*;*
(iii) If $D > \frac{n-m}{nF}(\frac{m}{nF})^{\frac{m}{n-m}}$*, then system* (3.29) *has no positive equilibria in* $int(\widetilde{\Omega}_2)$*.*

Proof: The components of positive equilibria of system (3.29) satisfy:

$$\begin{cases} (1-s)(s-\Theta)(s^n + D) - s^{m-1}p = 0, \\ s^m - F(s^n + D) = 0. \end{cases} \tag{3.33}$$

From (3.33), it follows that s-component is the root in the interval $(\max\{\Theta, 0\}, 1)$ of $s^m - F(s^n + D) = 0$. Let $\tilde{f}(s) = s^m$ and $\tilde{g}(s) = F(s^n + D)$, we have $\frac{d\tilde{f}(s)}{ds} = ms^{m-1}$ and $\frac{d\tilde{g}(s)}{ds} = nFs^{n-1}$. If the curves of $\tilde{f}(s)$ and $\tilde{g}(s)$ intersect in $int(\widetilde{\Omega}_2)$, there must be a point $\hat{s}$ such that $\frac{d\tilde{f}(s)}{ds}|_{s=\hat{s}} = \frac{d\tilde{g}(s)}{ds}|_{s=\hat{s}}$, yielding that $\hat{s} = (\frac{m}{nF})^{\frac{1}{n-m}}$. We discuss the number of positive equilibria of system (3.29) in three cases:

(i) If $\tilde{f}(\hat{s}) - \tilde{g}(\hat{s}) = 0$, then system (3.29) has a unique positive equilibrium $(\hat{s},\hat{p}) \in int(\widetilde{\Omega}_2)$, where $\hat{p} = \frac{1}{F}\hat{s}(1-\hat{s})(\hat{s}-\Theta)$. Substituting $\hat{s} = (\frac{m}{nF})^{\frac{1}{n-m}}$ into $\tilde{f}(\hat{s}) - \tilde{g}(\hat{s}) = 0$, we obtain $D = \frac{n-m}{nF}(\frac{m}{nF})^{\frac{m}{n-m}}$. Furthermore, since $\hat{p} > 0$, $\hat{s}$ should be located in the interval $(\Theta, 1)$ $(\Theta > 0)$ or $(0,1)$ $(\Theta \leq 0)$, thus we have $\max\{|\Theta|^{n-m} sign(\Theta), 0\} < \frac{m}{nF} < 1$.

(ii) If $\tilde{f}(\hat{s}) - \tilde{g}(\hat{s}) > 0$, that is, if $D < \frac{n-m}{nF}(\frac{m}{nF})^{\frac{m}{n-m}}$, then system (3.29) has two different positive equilibria $E^2_{\circ 1}(\hat{s}_1,\hat{p}_1)$ and $E^2_{\circ 2}(\hat{s}_2,\hat{p}_2)$. To ensure that $\hat{p}_k > 0$, we have $\max\{\Theta, 0\} < \hat{s}_1 < \hat{s} < \hat{s}_2 < 1$, meaning $\tilde{f}(\Theta) - \tilde{g}(\Theta) < 0$ $(\Theta > 0)$ or $\tilde{f}(0) - \tilde{g}(0) < 0$ $(\Theta \leq 0)$, and $\tilde{f}(1) - \tilde{g}(1) < 0$. Thus, we obtain $D > \max\left\{0, |\frac{\Theta^m - F\Theta^n}{F}| sign(\Theta), \frac{1-F}{F}\right\}$.

(iii) If $\tilde{f}(\hat{s}) - \tilde{g}(\hat{s}) < 0$, that is, if $D > \frac{n-m}{nF}(\frac{m}{nF})^{\frac{m}{n-m}}$, then system (3.29) has no positive equilibria in $int(\widetilde{\Omega}_2)$. □

REMARK 3.9. *According to Lemma 3.6 and the definition of basic reproduction number in [34], we can calculate the epidemiological basic reproduction number of system* (3.18)*, denoted by* $R^I_0 = \frac{1}{AC}$*. Similarly, by Lemma 3.12, the ecological basic reproduction number of system* (3.29) *with monotonic functional response can be defined by* $R^{P1}_0 = \frac{1-F}{DF}$*. For* *non-monotonic case, we can use* $R_0^{P2} = \frac{1}{F(D+1)}$ *to represent its ecological basic reproduction number, at which a backward bifurcation may occur. In addition to the epidemiological as well as ecological basic reproduction numbers, we find the existence of interior equilibrium in S-I (or S-P) subsystem is closely related to the value of Allee effect.*

Let us consider the local stability of positive equilibria of system (3.29). We first study the case that $\max\left\{0, |\frac{\Theta^m - F\Theta^n}{F}| sign(\Theta), \frac{1-F}{F}\right\} < D < \frac{n-m}{nF}(\frac{m}{nF})^{\frac{m}{n-m}}$.

Lemma 3.17. *Assume that system* (3.29) *has two positive equilibria* $E^2_{\circ 1}(\hat{s}_1, \hat{p}_1)$ *and* $E^2_{\circ 2}(\hat{s}_2, \hat{p}_2)$ *in* $int(\widetilde{\Omega}_2)$ *whose components satisfy* $\max\{\Theta, 0\} < \hat{s}_1 < \hat{s} < \hat{s}_2 < 1$ *and* $\hat{p}_k = \frac{1}{F}\hat{s}_k(1-\hat{s}_k)(\hat{s}_k - \Theta),\ k = 1, 2.$
(i) For the positive equilibria $E^2_{\circ 1}(\hat{s}_1, \hat{p}_1)$*:*
(i1) If $J_2(E^2_{\circ 1}(\hat{s}_1, \hat{p}_1))_{11} < 0$*, then* $E^2_{\circ 1}$ *is an attractor (stable node or focus);*
(i2) If $J_2(E^2_{\circ 1}(\hat{s}_1, \hat{p}_1))_{11} > 0$*, then* $E^2_{\circ 1}$ *is a repeller (unstable node or focus);*
(i3) If $J_2(E^2_{\circ 1}(\hat{s}_1, \hat{p}_1))_{11} = 0$*, then* $E^2_{\circ 1}$ *is a center (weak focus).*
(ii) The positive equilibrium $E^2_{\circ 2}(\hat{s}_2, \hat{p}_2)$ *is always a saddle. Moreover, if* $J_2(E^2_{\circ 2}(\hat{s}_2, \hat{p}_2))_{11} \neq 0$*, it is a hyperbolic saddle; otherwise, it is a non-hyperbolic saddle.*

Proof: (i) By (3.30), we have

$$J_2(E^2_{\circ 1}(\hat{s}_1, \hat{p}_1)) = \begin{pmatrix} J_2(E^2_{\circ 1}(\hat{s}_1, \hat{p}_1))_{11} & -\hat{s}_1^m \\ E(m\hat{s}_1^{m-1} - nF\hat{s}_1^{n-1})\hat{p}_1 & 0 \end{pmatrix},$$

where

$$\begin{aligned} J_2(E^2_{\circ 1}(\hat{s}_1, \hat{p}_1))_{11} = & -(n+3)\hat{s}_1^{n+2} + (\Theta+1)(n+2)\hat{s}_1^{n+1} - \Theta(n+1)\hat{s}_1^n - m\hat{s}_1^{m-1}\hat{p}_1 \\ & - 3D\hat{s}_1^2 + 2D(\Theta+1)\hat{s}_1 - \Theta D. \end{aligned}$$

Suppose that $\hat{s}_1 = \delta_1 \hat{s}$, where $0 < \delta_1 < 1$, then we have

$$\det(J_2(E^2_{\circ 1}(\hat{s}_1, \hat{p}_1))) = mE\hat{s}_1^{2m-1}(1 - \delta_1^{n-m})\hat{p}_1 > 0.$$

Thus, the nature of $E^2_{\circ 1}(\hat{s}_1, \hat{p}_1)$ is dependent of the sign of $J_2(E^2_{\circ 1}(\hat{s}_1, \hat{p}_1))_{11}$.
(ii) The Jacobian matrix of system (3.29) at $E^2_{\circ 2}(\hat{s}_2, \hat{p}_2)$ is:

$$J_2(E^2_{\circ 2}(\hat{s}_2, \hat{p}_2)) = \begin{pmatrix} J_2(E^2_{\circ 2}(\hat{s}_2, \hat{p}_2))_{11} & -\hat{s}_2^m \\ E(m\hat{s}_2^{m-1} - nF\hat{s}_2^{n-1})\hat{p}_2 & 0 \end{pmatrix},$$

where

$$\begin{aligned} J_2(E^2_{\circ 2}(\hat{s}_2, \hat{p}_2))_{11} = & -(n+3)\hat{s}_2^{n+2} + (\Theta+1)(n+2)\hat{s}_2^{n+1} - \Theta(n+1)\hat{s}_2^n - m\hat{s}_2^{m-1}\hat{p}_2 \\ & - 3D\hat{s}_2^2 + 2D(\Theta+1)\hat{s}_2 - \Theta D. \end{aligned}$$

Suppose that $\hat{s}_2 = \delta_2 \hat{s}$, where $\delta_2 > 1$, then we have

$$\det(J_2(E^2_{\circ 2}(\hat{s}_2, \hat{p}_2))) = mE\hat{s}_2^{2m-1}(1 - \delta_2^{n-m})\hat{p}_2 < 0.$$

Thus, $E^2_{\circ 2}(\hat{s}_2, \hat{p}_2)$ is always a saddle. If $tr(J_2(E^2_{\circ 2}(\hat{s}_2, \hat{p}_2))) = J_2(E^2_{\circ 2}(\hat{s}_2, \hat{p}_2))_{11} \neq 0$, it is a hyperbolic saddle; otherwise, it is a non-hyperbolic saddle. □

Remark 3.10. *Based on (i) of Lemma 3.17, if we further assume* $\hat{s}_1 \neq \frac{2(\Theta+1)+\sqrt{(2\Theta-1)^2+3}}{6}$ *and* $D_* = \frac{-(n+3)\hat{s}_1^{n+2} + (\Theta+1)(n+2)\hat{s}_1^{n+1} - \Theta(n+1)\hat{s}_1^n - m\hat{s}_1^{m-1}\hat{p}_1}{3\hat{s}_1^2 - 2(\Theta+1)\hat{s}_1 + \Theta} > 0$*, then system* (3.29) *undergoes a Hopf bifurcation at* D_**, which can be obtained from the change of the Conley index of* $E^2_{\circ 1}(\hat{s}_1, \hat{p}_1)$ *near* D_* *by Proposition 3.2.*

Next, we consider the case that there is a unique positive equilibrium for system (3.29).

LEMMA 3.18. *Assume that* $\max\{|\Theta|^{n-m}sign(\Theta), 0\} < \frac{m}{nF} < 1$ *and* $D = \frac{n-m}{nF}(\frac{m}{nF})^{\frac{m}{n-m}}$, *then the positive equilibria* $E^2_{\circ 1}(\hat{s}_1, \hat{p}_1)$ *and* $E^2_{\circ 2}(\hat{s}_2, \hat{p}_2)$ *collapse, there exists a unique positive equilibrium* $E^2_\circ(\hat{s}, \hat{p}) \in int(\widetilde{\Omega}_2)$.
(i) If $F = F^*$, *then* $E^2_\circ$ *is a cusp (i.e., Bogdanov-Takens bifurcation point);*
(ii) If $F > F^*$, *then* $E^2_\circ$ *is a saddle-node repeller;*
(iii) If $F < F^*$, *then* $E^2_\circ$ *is a saddle-node attractor;*
where $F^* = \frac{m}{n}\left(\frac{6}{2(\Theta+1)+\sqrt{(2\Theta-1)^2+3}}\right)^{n-m}$.

Proof: The Jacobian matrix of system (3.29) at $E^2_\circ(\hat{s}, \hat{p})$ is:

$$J_2(E^2_\circ(\hat{s}, \hat{p})) = \begin{pmatrix} J_2(E^2_\circ(\hat{s}, \hat{p}))_{11} & -\hat{s}^m \\ 0 & 0 \end{pmatrix}, \tag{3.34}$$

where

$$\begin{aligned} J_2(E^2_\circ(\hat{s}, \hat{p}))_{11} = & -(n+3)\hat{s}^{n+2} + (\Theta+1)(n+2)\hat{s}^{n+1} - \Theta(n+1)\hat{s}^n - m\hat{s}^{m-1}\hat{p} \\ & - 3D\hat{s}^2 + 2D(\Theta+1)\hat{s} - \Theta D \end{aligned}$$

Then we have $\det(J_2(E^2_\circ(\hat{s}, \hat{p}))) = 0$, and

$$\begin{aligned} tr(J_2(E^2_\circ(\hat{s}, \hat{p}))) = & \hat{s}^m\left[-(n+3)\hat{s}^{n-m+2} + (\Theta+1)(n+2)\hat{s}^{n-m+1} - \Theta(n+1)\hat{s}^{n-m} - \frac{m}{F}(1-\hat{s})(\hat{s}-\Theta)\right] \\ & - 3D\hat{s}^2 + 2D(\Theta+1)\hat{s} - \Theta D. \end{aligned}$$

Since $\hat{s} = (\frac{m}{nF})^{\frac{1}{n-m}}$, we obtain $tr(J_2(E^2_\circ(\hat{s}, \hat{p}))) = (\frac{m}{nF}\hat{s}^m + D)[-3\hat{s}^2 + 2(\Theta+1)\hat{s} - \Theta]$. Obviously, $tr(J_2(E^2_\circ(\hat{s}, \hat{p}))) = 0$ if and only if $-3\hat{s}^2 + 2(\Theta+1)\hat{s} - \Theta = 0$. Since $\hat{s} \in (\max\{\Theta, 0\}, 1)$, we have $\hat{s} = \frac{2(\Theta+1)+\sqrt{(2\Theta-1)^2+3}}{6}$, equivalently, $F = F^*$. In this case, $E^2_\circ$ is a cusp (i.e., Bogdanov-Takens bifurcation point). In addition, if $F > F^*$ ($tr(J_2(E^2_\circ(\hat{s}, \hat{p}))) > 0$), $E^2_\circ$ is a saddle-node repeller; if $F < F^*$ ($tr(J_2(E^2_\circ(\hat{s}, \hat{p}))) < 0$), $E^2_\circ$ is a saddle-node attractor. □

PROPOSITION 3.19. *Let* $D^* = \frac{n-m}{nF}(\frac{m}{nF})^{\frac{m}{n-m}}$, *consider* D *as a bifurcation parameter, then system* (3.29) *undergoes a saddle-node bifurcation at* $D = D^*$ *if* $\max\{|\Theta|^{n-m}sign(\Theta), 0\} < \frac{m}{nF} < 1$ *and* $F < F^*$. *In addition, system* (3.29) *does not attain any transcritical and pitchfork bifurcation around* $E^2_\circ(\hat{s}, \hat{p})$.

Proof: To show that system (3.29) undergoes a saddle-node bifurcation, we use Sotomayor's theorem [89] by considering D as the bifurcation parameter. The Jacobian matrix, $J_2(E^2_\circ(\hat{s}, \hat{p}))$, of (3.29) at $E^2_\circ(\hat{s}, \hat{p})$ is given by (3.34), one eigenvalue of which is zero. If $F < F^*$, then the other eigenvalue is negative.

Now, let $V = (v_1, v_2)^T$ and $W = (w_1, w_2)^T$ be the eigenvectors of $J_2(E^2_\circ)$ and $J_2^T(E^2_\circ)$ corresponding to zero eigenvalue, respectively. A simple calculation yields $V = (\hat{s}^m, J_2(E^2_\circ)_{11})^T$ and $W = (0, 1)^T$. Therefore, we have

$$W^T f_D((\hat{s}, \hat{p}), D) = -EF\hat{p} < 0 \text{ and } W^T[D^2 f((\hat{s}, \hat{p}), D)(V, V)] = E\hat{s}^m[m(m-1)\hat{s}^{m-2} - n(n-1)F\hat{s}^{n-2}]\hat{p} < 0,$$

where

$$f_D((\hat{s}, \hat{p}), D) := \frac{\partial f((\hat{s}, \hat{p}), D)}{\partial D} = \begin{pmatrix} \hat{s}(1-\hat{s})(\hat{s}-\Theta) \\ -EF\hat{p} \end{pmatrix},$$

and

$$D^2 f((\hat{s},\hat{p}),D)(V,V) = \begin{pmatrix} \hat{s}^{m-1}\left[\hat{s}\frac{\partial J_2(E_*^2(\hat{s},\hat{p}))_{11}}{\partial \hat{s}} + (1-m)J_2(E_*^2(\hat{s},\hat{p}))_{11} - m\hat{s}^m\right] \\ E\hat{s}^m\left[m(m-1)\hat{s}^{m-2} - n(n-1)F\hat{s}^{n-2}\right]\hat{p} \end{pmatrix},$$

which implies that the transversality condition for saddle-node bifurcation is satisfied.

From Sotomayor's theorem, it follows that system (3.29) undergoes a saddle-node bifurcation around $E_\circ^2(\hat{s},\hat{p})$ at $D = D^*$. Hence, we can conclude that when the parameter D passes from right side of $D = D^*$ to the left side, the number of interior equilibria of system (3.29) changes from zero to two. Since $W^T[D^2 f((\hat{s},\hat{p}),D)(V,V)] \neq 0$, system (3.29) does not attain any transcritical and pitchfork bifurcation around $E_\circ^2(\hat{s},\hat{p})$ [89]. □

3.3. I-P subsystem. The I-P subsystem when $S \equiv 0$ in (3.1) is written as

$$X_\varsigma^3 : \begin{cases} \frac{dI}{dt} = -\frac{bI^m}{I^n+a^n}P - \mu I, \\ \frac{dP}{dt} = \gamma_2 \frac{bI^m}{I^n+a^n}P - dP, \end{cases} \tag{3.35}$$

where $\varsigma \in \Pi = \left\{(a,b,\mu,\gamma_2,d,m,n) \in (0,\frac{K}{c_1}) \times (0,1)^4 \times \mathbb{N}_+^2 : n \geq m \geq 1\right\}$.
System (3.35) or vector field X_ς^3 is defined in the set:

$$\Omega_3 = \left\{(I,P) \in \mathbb{R}^2 : 0 \leq I \leq \frac{K}{c_2}, P \geq 0\right\}.$$

We make a change of variables and time rescaling given by the function: $\varphi_3 : \widetilde{\Omega}_3 \times \mathbb{R} \to \Omega_3 \times \mathbb{R}$ such that

$$\varphi_3(i,p,\tau) = (i,p,(i^n + a^n)\tau) = (I,P,t)$$

with $\widetilde{\Omega}_3 = \{(i,p) \in \mathbb{R}^2 : 0 \leq i \leq 1, p \geq 0\}$. Since $\det D\varphi(i,p,\tau) = i^n + a^n > 0$, the new vector field, denoted by Y_ς^3, is topologically equivalent to the vector field X_ς^3, and its associated differential equations are given by

$$Y_\varsigma^3 : \begin{cases} \frac{di}{d\tau} = -b[i^{m-1}p + L(i^n + a^n)]i, \\ \frac{dp}{d\tau} = M[i^m - N(i^n + a^n)]p, \end{cases} \tag{3.36}$$

where $L = \frac{\mu}{b}, M = \gamma_2 b$ and $N = \frac{d}{\gamma_2 b}$, with

$$\overline{\varsigma} \in \overline{\Pi} = \left\{(a,b,L,M,N,m,n) \in (0,\frac{K}{c_1}) \times (0,1) \times \mathbb{R}_+^3 \times \mathbb{N}_+^2 : n \geq m \geq 1\right\}.$$

Obviously, system (3.36) has a unique equilibrium $E_0^3(0,0)$ in $\widetilde{\Omega}_3$ which always exist. The Jacobian matrix of system (3.36) at $E_0^3(0,0)$ is given by:

$$J_3(E_0^3(0,0)) = \begin{pmatrix} -bLa^n & 0 \\ 0 & -MNa^n \end{pmatrix}. \tag{3.37}$$

The eigenvalues of $J_3(E_0^3(0,0))$ are $-bLa^n$ and $-MNa^n$, which are both negative. So $E_0^3(0,0)$ is always locally asymptotically stable for any parameter values. Moreover, there is only an equilibrium in $\widetilde{\Omega}_3$ for system (3.36), thus it is globally asymptotically stable in $\widetilde{\Omega}_3$.

4. Dynamical analysis of the full system

After obtaining a complete dynamics of three subsystems of system (3.1), in this section, we continue to study the dynamical behavior of the full system, including: 1) the boundary equilibria and their stability; 2) the uniform persistence; 3) a heteroclinic network; and finally 4) an interior periodic orbit.

$$X_\upsilon : \begin{cases} \frac{dS}{dt} = rS(1-\frac{c_1S+c_2I}{K})(S-\theta) - \frac{\beta I}{1+\alpha I}S - \frac{bS^m}{(S+I)^n+a^n}P, \\ \frac{dI}{dt} = \frac{\beta I}{1+\alpha I}S - \frac{bI^m}{(S+I)^n+a^n}P - \mu I, \\ \frac{dP}{dt} = \gamma_1 \frac{bS^m}{(S+I)^n+a^n}P + \gamma_2 \frac{bI^m}{(S+I)^n+a^n}P - dP, \end{cases} \tag{3.38}$$

where $\upsilon \in \Upsilon = \{(r,K,\alpha,a,c_1,c_2,b,d,\beta,\mu,\gamma_1,\gamma_2,m,n,\theta) \in \mathbb{R}^3_+ \times (0,\frac{K}{c_1}) \times (0,1]^8 \times \mathbb{N}^2_+ \times (-\frac{K}{c_1},\frac{K}{c_1}) : n \geq m \geq 1\}$. System (3.38) or vector field X_υ is defined in

$$\Omega = \{(S,I,P) \in \mathbb{R}^3_+ : 0 \leq c_1S + c_2I \leq K, P \geq 0\}.$$

To simplify the calculation, we make a change of variables and a time rescaling by the function $\varphi : \widetilde{\Omega} \times \mathbb{R} \to \Omega \times \mathbb{R}$ such that

$$\varphi(s,i,p,\tau) = \left(\frac{K}{c_1}s, \frac{K}{c_2}i, \frac{r}{b}(\frac{K}{c_1})^{n-m+2}p, \frac{c_1}{rK}\frac{\alpha Ki + c_2}{\alpha K}\frac{(Ks+\frac{c_1}{c_2}Ki)^n + (ac_1)^n}{K^n}\tau\right) = (S,I,P,t)$$

with $\widetilde{\Omega} = \{(s,i,p) \in \mathbb{R}^3_+ : 0 \leq s+i \leq 1, p \geq 0\}$. Thus, we have that $\det D\varphi(s,i,p,\tau) > 0$, implying that φ is a diffeomorphism preserving the orientation of the time. The vector field, denoted by $Y_{\overline{\upsilon}}$, in the new coordinates is topologically equivalent to the vector field X_υ, and its associated differential equations are described by the following Kolmogorov polynomial system:

$$Y_{\overline{\upsilon}} : \begin{cases} \frac{ds}{d\tau} = \left\{[(1-s-i)(s-\Theta)(i+A) - B_1 i]\,[(s+\varepsilon i)^n + D] - (i+A)s^{m-1}p\right\} s, \\ \frac{di}{d\tau} = B_2\left\{[s - C(i+A)]\,[(s+\varepsilon i)^n + D] - M_1(i+A)i^{m-1}p\right\} i, \\ \frac{dp}{d\tau} = E(i+A)\left\{s^m + M_2 i^m - F\,[(s+\varepsilon i)^n + D]\right\} p, \end{cases} \tag{3.39}$$

where $A = \frac{c_2}{\alpha K}, B_1 = \frac{\beta c_1}{\alpha r K}, B_2 = \frac{\beta c_2}{\alpha r K}, C = \frac{\mu \alpha c_1}{\beta c_2}, D = (\frac{ac_1}{K})^n, E = \frac{\gamma_1 b}{r}(\frac{c_1}{K})^{n-m+1}, F = \frac{d}{\gamma_1 b}(\frac{K}{c_1})^{n-m}, M_1 = \frac{\alpha r K c_1^{m-1}}{\beta c_2^m}, M_2 = \frac{\gamma_2 c_1^m}{\gamma_1 c_2^m}, \varepsilon = \frac{c_1}{c_2}$ and $\Theta = \frac{\theta c_1}{K}$ with

$$\overline{\upsilon} \in \overline{\Upsilon} = \left\{(D,A,B_1,B_2,C,E,F,M_1,M_2,m,n,\varepsilon,\Theta) \in (0,1) \times \mathbb{R}^8_+ \times \mathbb{N}^2_+ \times [1,+\infty) \times (-1,1) : n \geq m \geq 1\right\}.$$

Since φ is a diffeomorphism, system (3.39) has the same qualitative behavior as system (3.38). In the following, we start with the existence of boundary equilibria and their stability.

4.1. Boundary equilibria and their stability. Through the analysis of three subsystems in Section 3, it is easy to check that system (3.39) has seven possible boundary equilibria. The trivial equilibrium $E_0(0,0,0)$ and axial equilibrium $E_1(1,0,0)$ always exist. The axial equilibrium $E_\Theta(\Theta,0,0)$ exists if $\Theta > 0$. In *si*-plane, system (3.39) may have a planar equilibrium $E_{si}(\bar{s},\bar{i},0)$, or two planar equilibria $E^1_{si}(\bar{s}_1,\bar{i}_1,0)$ and $E^2_{si}(\bar{s}_2,\bar{i}_2,0)$, see Lemma 3.6. In *sp*-plane, system (3.39) may have a planar equilibrium $\check{E}_{sp}(\check{s},0,\check{p})$ (monotonic case, see Lemma 3.12) or $\hat{E}_{sp}(\hat{s},0,\hat{p})$ (non-monotonic case, see Lemma 3.16), or two planar equilibria $E_{sp}^1(\hat{s}_1, 0, \hat{p}_1)$ and $E_{sp}^2(\hat{s}_2, 0, \hat{p}_2)$, see Lemma 3.16. Below we study the local and global stability of these boundary equilibria, respectively.

The Jacobian matrix of system (3.39) at any point (s, i, p) is given by:

$$J(s,i,p) = \begin{pmatrix} J(s,i,p)_{11} & J(s,i,p)_{12} & J(s,i,p)_{13} \\ J(s,i,p)_{21} & J(s,i,p)_{22} & J(s,i,p)_{23} \\ J(s,i,p)_{31} & J(s,i,p)_{32} & J(s,i,p)_{33} \end{pmatrix}, \tag{3.40}$$

where

$$\begin{aligned} J(s,i,p)_{11} =& \{(i+A)(1+\Theta-2s-i)[(s+\varepsilon i)^n + D] + n(s+\varepsilon i)^{n-1}[(1-s-i)(s-\Theta)(i+A) \\ & - B_1 i]\}s + [(1-s-i)(s-\Theta)(i+A) - B_1 i][(s+\varepsilon i)^n + D] - (i+A)s^{m-1}p, \\ J(s,i,p)_{12} =& \{[(s-\Theta)(1-A-s-2i) - B_1][(s+\varepsilon i)^n + D] + \varepsilon n(s+\varepsilon i)^{n-1} \\ & [(1-s-i)(s-\Theta)(i+A) - B_1 i] - s^{m-1}p\}s, \\ J(s,i,p)_{13} =& -(i+A)s^m, \\ J(s,i,p)_{21} =& B_2\{n(s+\varepsilon i)^{n-1}[s - C(i+A)] + [(s+\varepsilon i)^n + D]\}i, \\ J(s,i,p)_{22} =& B_2\{\varepsilon n(s+\varepsilon i)^{n-1}[s - C(i+A)] - C[(s+\varepsilon i)^n + D] - M_1 p[mi + A(m-1)]i^{m-2}\}i \\ & + B_2\{[s - C(i+A)][(s+\varepsilon i)^n + D] - M_1(i+A)i^{m-1}p\}, \\ J(s,i,p)_{23} =& -B_2 M_1 (i+A) i^m, \\ J(s,i,p)_{31} =& E(i+A)[ms^{m-1} - nF(s+\varepsilon i)^{n-1}]p, \\ J(s,i,p)_{32} =& E\{s^m + M_2 i^m - F[(s+\varepsilon i)^n + D] + (i+A)[mM_2 i^{m-1} - \varepsilon nF(s+\varepsilon i)^{n-1}]\}p, \\ J(s,i,p)_{33} =& E(i+A)\{s^m + M_2 i^m - F[(s+\varepsilon i)^n + D]\}. \end{aligned}$$

By (3.40), the variational matrix of system (3.39) at $E_0(0,0,0)$ is

$$J(0,0,0) = \begin{pmatrix} -\Theta AD & 0 & 0 \\ 0 & -AB_2CD & 0 \\ 0 & 0 & -ADEF \end{pmatrix},$$

then its eigenvalues are $-\Theta AD$, $-AB_2CD$ and $-ADEF$. Thus, E_0 is locally asymptotically stable (node) if $\Theta > 0$, an attracting saddle if $\Theta < 0$, and a non-hyperbolic attractor if $\Theta = 0$.

Remark 3.11. *Although the equilibrium E_0 is locally asymptotically stable if $\Theta > 0$, we cannot construct a Lyapunov function or use other methods to show that it is also globally asymptotically stable in $\widetilde{\Omega}$. We guess that when $\Theta > 0$, $E_\Theta(\Theta, 0, 0)$ arises and it is always unstable (see the analysis below). There is a great possibility that a separatrix surface forms in the first octant which intersects with s-axis at E_Θ, and multi-stability appears.*

By (3.40), the variational matrix of system (3.39) at $E_\Theta(\Theta, 0, 0)$ $(\Theta > 0)$ can be written as

$$J(\Theta,0,0) = \begin{pmatrix} A\Theta(1-\Theta)(\Theta^n + D) & -B_1\Theta(\Theta^n + D) & -A\Theta^m \\ 0 & B_2(\Theta - AC)(\Theta^n + D) & 0 \\ 0 & 0 & AE[\Theta^m - F(\Theta^n + D)] \end{pmatrix},$$

then its eigenvalues are $A\Theta(1-\Theta)(\Theta^n + D) > 0$, $B_2(\Theta - AC)(\Theta^n + D)$ and $AE[\Theta^m - F(\Theta^n + D)]$. Thus, E_Θ is always unstable. If $(\Theta - AC)[\Theta^m - F(\Theta^n + D)] < 0$, then it is a repelling saddle; if $\Theta > AC$ and $\Theta^m - F(\Theta^n + D) > 0$, then it is an unstable node; if $\Theta < AC$ and

$\Theta^m - F(\Theta^n + D) < 0$, then it is an attracting saddle, if $(\Theta - AC)[\Theta^m - F(\Theta^n + D)] = 0$, then it is non-hyperbolic.

By (3.40), the variational matrix of system (3.39) at $E_1(1, 0, 0)$ can be computed as

$$J(1,0,0) = \begin{pmatrix} -A(1-\Theta)(D+1) & -[A(1-\Theta)+B_1](D+1) & -A \\ 0 & B_2(1-AC)(D+1) & 0 \\ 0 & 0 & AE[1-F(D+1)] \end{pmatrix},$$

then its eigenvalues are $-A(1-\Theta)(D+1) < 0$, $B_2(1-AC)(D+1)$ and $AE[1-F(D+1)]$. If $AC > 1$ and $1 - F(D+1) < 0$, then E_1 is locally asymptotically stable (node); if $AC < 1$ or $1 - F(D+1) > 0$, then it is a saddle.

THEOREM 4.1. *When $AC > 1$, $DF > 1$ and $\Theta \leq 0$, the equilibrium $E_1(1, 0, 0)$ of system (3.39) is globally asymptotically stable in $\widetilde{\Omega}$.*

Proof: Consider a Lyapunov function $V_1(i,p) = \frac{EM_2}{B_2 M_1} i + p$, and its derivative along the trajectories of (3.39) is

$$\begin{aligned} \frac{dV_1}{d\tau} =& \frac{EM_2}{M_1} i\,[s - C(i+A)]\,[(s+\varepsilon i)^n + D] + E(i+A)\,\{s^m - F\,[(s+\varepsilon i)^n + D]\}\,p \\ \leq& \frac{EM_2}{M_1}(1-AC)\,[(s+\varepsilon i)^n + D]\,i + E(i+A)(1-DF)p. \end{aligned}$$

Thus, when $AC > 1$ and $DF > 1$, we have $\frac{dV_1}{d\tau} \leq 0$. Furthermore, it can be verified that $\{(s,i,p) \in \widetilde{\Omega} : \frac{dV_1}{d\tau} = 0\} = \{i = 0, p = 0\}$. By LaSalle's invariance principle, we have $\lim_{\tau\to+\infty} i(\tau) = 0$ and $\lim_{\tau\to+\infty} p(\tau) = 0$ for all solutions of system (3.39). So, the limiting system of (3.39) is $\frac{ds}{d\tau} = As(1-s)(s-\Theta)(s^n + D)$. Obviously, the equilibrium $s = 1$ of this equation is globally asymptotically stable if $\Theta \leq 0$. By the limiting argument [104], we obtain that $E_1(1,0,0)$ is globally asymptotically stable in $\widetilde{\Omega}$ if $AC > 1$, $DF > 1$ and $\Theta \leq 0$. □

By (3.40), the variational matrix of system (3.39) at $E_{si}(\bar{s}, \bar{i}, 0)$ can be written as

$$J(\bar{s},\bar{i},0) = \begin{pmatrix} J_1(\bar{s},\bar{i})_{11}[(\bar{s}+\varepsilon\bar{i})^n + D] & J_1(\bar{s},\bar{i})_{12}[(\bar{s}+\varepsilon\bar{i})^n + D] & -(\bar{i}+A)\bar{s}^m \\ J_1(\bar{s},\bar{i})_{21}[(\bar{s}+\varepsilon\bar{i})^n + D] & J_1(\bar{s},\bar{i})_{22}[(\bar{s}+\varepsilon\bar{i})^n + D] & -B_2 M_1(\bar{i}+A)\bar{i}^m \\ 0 & 0 & J(\bar{s},\bar{i},0)_{33} \end{pmatrix},$$

where $J(\bar{s},\bar{i},0)_{33} = E(\bar{i}+A)\{\bar{s}^m + M_2\bar{i}^m - F[(\bar{s}+\varepsilon\bar{i})^n + D]\}$.

Clearly, its first two eigenvalues, denoted by $\bar{\rho}_{1,2}$, satisfy $\bar{\rho}_k = [(\bar{s}+\varepsilon\bar{i})^n + D]\rho_k, k = 1,2$, where $\rho_{1,2}$ are two eigenvalues of $J_1(\bar{s},\bar{i})$. So, $E_{si}(\bar{s},\bar{i},0)$ has the same stability properties as $E_*^1(\bar{s},\bar{i})$ in si-plane. The third eigenvalue of $J(\bar{s},\bar{i},0)$ is $\bar{\rho}_3 = J(\bar{s},\bar{i},0)_{33}$. According to (I.i) of Lemma 3.7, if both eigenvalues ρ_1 and ρ_2 have negative real parts and $\bar{\rho}_3 < 0$, then $E_{si}(\bar{s},\bar{i},0)$ is locally asymptotically stable. When $\bar{\rho}_3 > 0$, $E_{si}(\bar{s},\bar{i},0)$ is always a saddle. Assume that $\bar{\rho}_3 < 0$ holds, then system (3.39) undergoes a Hopf bifurcation near $E_{si}(\bar{s},\bar{i},0)$ as the parameter B_2 passes through the critical value B_2^*.

Similarly, combined with Lemma 3.8, we can discuss the local stability of $E_{si}^1(\bar{s}_1,\bar{i}_1,0)$ and $E_{si}^2(\bar{s}_2,\bar{i}_2,0)$ when system (3.39) has two different planar equilibria in si-plane. Next, we investigate its global stability when there is only one planar equilibrium $E_{si}(\bar{s},\bar{i},0)$ in si-plane for system (3.39).

Theorem 4.2. *Assume that $E_{si}(\bar{s},\bar{i},0)$ is a unique planar equilibrium in si-plane for system (3.39), and $0 < AC < \frac{\Theta+1}{2}$, $\det(J_1(E_-^1(\bar{s},\bar{i}))) > 0$ and $B_2 > B_2^*$. If $\Theta \le 0$, $B_2C \ge B_1 - \Theta(A+1)$ and $DF - M_2 > 1$, then E_{si} is globally asymptotically stable in $\widetilde{\Omega}$.*

Proof: We consider a Lyapunov function $V_2(p) = p$, and its derivative along the trajectories of system (3.39) is given by

$$\begin{aligned}\frac{dV_2}{d\tau} =& E(i+A)\left\{s^m + M_2 i^m - F\left[(s+\varepsilon i)^n + D\right]\right\} p \\ \le& E(i+A)(1+M_2-DF)p.\end{aligned}$$

Thus, when $DF - M_2 > 1$, we have $\frac{dV_2}{d\tau} \le 0$. Furthermore, it can be verified that $\{(s,i,p) \in \widetilde{\Omega} | \frac{dV_2}{d\tau} = 0\} = \{p = 0\}$. By LaSalle's invariance principle, we have $\lim_{\tau\to+\infty} p(\tau) = 0$ for all solutions of (3.39). So, the limiting system of (3.39) is

$$\begin{cases} \frac{ds}{d\tau} = \left[(1-s-i)(s-\Theta)(i+A) - B_1 i\right]\left[(s+\varepsilon i)^n + D\right] s, \\ \frac{di}{d\tau} = B_2\left[s - C(i+A)\right]\left[(s+\varepsilon i)^n + D\right] i. \end{cases} \tag{3.41}$$

Making a time rescaling $\tau' = [(s+\varepsilon i)^n + D]\tau$, yields the equations (3.18) of S-I subsystem. By Proposition 3.9, we see that $E_-^1(\bar{s},\bar{i})$ is globally asymptotically stable when $\det(J_1(E_-^1(\bar{s},\bar{i}))) > 0$, $0 < AC < \frac{\Theta+1}{2}$, $B_2 > B_2^*$, $\Theta \le 0$ and $B_2C \ge B_1 - \Theta(A+1)$. By the limiting argument, we have that $E_{si}(\bar{s},\bar{i},0)$ is globally asymptotically stable in $\widetilde{\Omega}$. □

By (3.40), the variational matrix of system (3.39) at $\check{E}_{sp}(\check{s},0,\check{p})$ can be written as

$$J(\check{s},0,\check{p}) = \begin{pmatrix} J_2(\check{s},\check{p})_{11}A & J(\check{s},0,\check{p})_{12} & J_2(\check{s},\check{p})_{12}A \\ 0 & J(\check{s},0,\check{p})_{22} & 0 \\ J_2(\check{s},\check{p})_{21}A & J(\check{s},0,\check{p})_{32} & J_2(\check{s},\check{p})_{22}A \end{pmatrix},$$

where $J(\check{s},0,\check{p})_{12} = \{[(\check{s}-\Theta)(1-A-\check{s}) - B_1](\check{s}^n + D) + \varepsilon n A \check{s}^{n-1}(1-\check{s})(\check{s}-\Theta) - \check{s}^{m-1}\check{p}\}\check{s}$; if $n = 1$, $J(\check{s},0,\check{p})_{22} = B_2[(\check{s}-AC)(\check{s}+D) - AM_1\check{p}]$ and $J(\check{s},0,\check{p})_{32} = AE(M_2 - \varepsilon F)\check{p}$, otherwise, $J(\check{s},0,\check{p})_{22} = B_2(\check{s}-AC)(\check{s}^n + D)$ and $J(\check{s},0,\check{p})_{32} = -\varepsilon n AEF\check{s}^{n-1}\check{p}$.
So, the two eigenvalues of $J(\check{s},0,\check{p})$ in s and p-directions satisfy $\check{\rho}_k = A\rho_k, k = s,p$, where $\rho_{s,p}$ are two eigenvalues of $J_2(\check{s},\check{p})$. According to Lemmas 3.13 and 3.14, we can determine the sign of $\check{\rho}_s$ and $\check{\rho}_p$. Under the conditions that both $\check{\rho}_s$ and $\check{\rho}_p$ have negative real parts, then $\check{E}_{sp}(\check{s},0,\check{p})$ is locally asymptotically stable if the eigenvalue in i-direction satisfies $\check{\rho}_i = J(\check{s},0,\check{p})_{22} < 0$. If $\check{\rho}_i < 0$, system (3.39) undergoes a Hopf bifurcation around $\check{E}_{sp}$ in the condition of occurrence of a Hopf bifurcation near $E_*^2(\check{s},\check{p})$.

Similarly, by combining Lemmas 3.17 and 3.18, we can further deal with the local stability of one planar equilibrium $\hat{E}_{sp}(\hat{s},0,\hat{p})$ or two planar equilibria $E_{sp}^1(\hat{s}_1,0,\hat{p}_1)$ and $E_{sp}^2(\hat{s}_2,0,\hat{p}_2)$ of system (3.39) with non-monotonic functional response. Finally, we give the sufficient conditions for the global asymptotic stability of $\check{E}_{sp}(\check{s},0,\check{p})$.

Theorem 4.3. *Assume that $\check{E}_{sp}(\check{s},0,\check{p})$ is a unique planar equilibrium in sp-plane for system (3.39) with $n = m$, if $(\check{s}-s)\left(\frac{s(1-s)(s-\Theta)(s^n+D)}{s^n} - \frac{\check{s}(1-\check{s})(\check{s}-\Theta)}{F}\right) \ge$ (or $\le$)0 for all $s \in (0,\check{s}) \cup (\check{s},1)$, and $\frac{s(1-s)(s-\Theta)(s^n+D)}{s^n} \not\equiv \frac{\check{s}(1-\check{s})(\check{s}-\Theta)}{F}$ for $0 < |s-\check{s}| \ll 1$ hold, then $\check{E}_{sp}$ is globally asymptotically stable in $\widetilde{\Omega}$.*

Proof: We consider a Lyapunov function $V_3(i) = i$, and its derivative along the trajectories of system (3.39) is given by

$$\begin{aligned}\frac{dV_3}{d\tau} =& B_2\{[s - C(i + A)][(s + \varepsilon i)^n + D] - M_1(i + A)i^{n-1}p\}i \\ \leq & B_2(1 - AC)[(s + \varepsilon i)^n + D]i.\end{aligned}$$

Thus, when $AC > 1$, we have $\frac{dV_3}{d\tau} \leq 0$. Furthermore, it can be verified that $\{(s, i, p) \in \widetilde{\Omega} | \frac{dV_3}{d\tau} = 0\} = \{i = 0\}$. By LaSalle's invariance principle, we have $\lim_{\tau\to+\infty} i(\tau) = 0$ for all solutions of (3.39) if $AC > 1$. So, the limiting system of (3.39) is

$$\begin{cases} \frac{ds}{d\tau} = A[(1 - s)(s - \Theta)(s^n + D) - s^{n-1}p]s, \\ \frac{dp}{d\tau} = AE[(1 - F)s^n - DF]p, \end{cases} \tag{3.42}$$

We make a time rescaling $\tau' = A\tau$, yielding (3.32). By Proposition 3.15, we have that $E^2_*(\check{s}, \check{p})$ is globally asymptotically stable when $(\check{s} - s)(\frac{s(1-s)(s-\Theta)(s^n+D)}{s^n} - \frac{\check{s}(1-\check{s})(\check{s}-\Theta)}{F}) \geq (\text{or } \leq) 0$ for all $s \in (0, \check{s}) \cup (\check{s}, 1)$, and $\frac{s(1-s)(s-\Theta)(s^n+D)}{s^n} \not\equiv \frac{\check{s}(1-\check{s})(\check{s}-\Theta)}{F}$ for $0 < |s - \check{s}| \ll 1$ hold. By the limiting argument, we have that $\breve{E}_{sp}(\check{s}, 0, \check{p})$ is globally asymptotically stable in $\widetilde{\Omega}$. □

4.2. Uniform persistence. There exists a positive equilibrium for system (3.39) if and only if there is a positive solution for the following equations

$$\begin{cases} [(1 - s - i)(s - \Theta)(i + A) - B_1 i] [(s + \varepsilon i)^n + D] - (i + A)s^{m-1}p = 0, \\ [(s - C(i + A)] [(s + \varepsilon i)^n + D] - M_1(i + A)i^{m-1}p = 0, \\ s^m + M_2 i^m - F [(s + \varepsilon i)^n + D] = 0. \end{cases} \tag{3.43}$$

Due to the complexity of the algebraic expression of (3.43), it is not an easy task to find the explicit criteria for the existence and quantity of the interior equilibria in terms of the parameters. We would like to study the uniform persistence of system (3.39), by using the method of average Lyapunov function [44, 52].

THEOREM 4.4. *If* $\Theta < 0$, $AC < 1$, $F(D + 1) < 1$, *then system* (3.39) *is uniformly persistent if one of the following conditions is satisfied:*
(i) $n = m$, $\bar{s}^m + M_2\bar{i}^m > F[(\bar{s} + \varepsilon\bar{i})^m + D]$, $0 < \frac{DF}{1-F} < 1$ *and* $(\check{s} - AC)(\check{s} + D) > AM_1\check{p}$ *(*$m = 1$*) or* $\check{s} > AC$ *(*$m > 1$*);*
(ii) $n > m$, $\bar{s}^m + M_2\bar{i}^m > F[(\bar{s} + \varepsilon\bar{i})^n + D]$, $\frac{m}{nF} < 1$, $D = \frac{n-m}{nF}(\frac{m}{nF})^{\frac{m}{n-m}}$ *and* $(\hat{s} - AC)(\hat{s}^n + D) > AM_1\hat{p}$ *(*$m = 1$*) or* $\hat{s} > AC$ *(*$m > 1$*).*

Proof: We consider the average Lyapunov function with the form

$$V(s, i, p) = s^{\alpha_1} i^{\alpha_2} p^{\alpha_3},$$

where α_k $(k = 1, 2, 3)$ are positive constants to be determined later. In the interior of $\mathbb{R}^3_+$, we define

$$\begin{aligned}\Psi(s, i, p) := \frac{\dot{V}}{V} =& \alpha_1 \left\{[(1 - s - i)(s - \Theta)(i + A) - B_1 i] [(s + \varepsilon i)^n + D] - (i + A)s^{m-1}p\right\} \\ &+ \alpha_2 B_2 \left\{[s - C(i + A)] [(s + \varepsilon i)^n + D] - M_1(i + A)i^{m-1}p\right\} \\ &+ \alpha_3 E(i + A) \{s^m + M_2 i^m - F [(s + \varepsilon i)^n + D]\}.\end{aligned}$$

We have already proved that all solutions of system (3.39) are uniformly ultimately bounded in $\mathbb{R}^3_+$ (see Lemma 2.2). To establish the uniform persistence of the solutions, we have to show that all boundary equilibria are repellers under certain conditions, i.e., the function

$\Psi(s,i,p) > 0$ at each of the boundary equilibria for a suitable choice of $\alpha_k > 0$ $(k = 1, 2, 3)$. We first consider two boundary equilibria that always exist, i.e., $E_0(0,0,0)$ and $E_1(1,0,0)$. Here, we choose $\alpha_1 > -\frac{\alpha_2 B_2 C + \alpha_3 EF}{\Theta} > 0$ $(\Theta < 0)$, then we have

$$\Psi(0,0,0) = -\alpha_1 \Theta AD - \alpha_2 AB_2CD - \alpha_3 ADEF = -AD(\alpha_1\Theta + \alpha_2 B_2 C + \alpha_3 EF) > 0.$$

For $E_1(1,0,0)$, we have

$$\Psi(1,0,0) = \alpha_2 B_2(1+D)(1-AC) + \alpha_3 AE[1 - F(D+1)] > 0,$$

which follows from the assumption that $AC < 1$ and $F(D+1) < 1$.

Since $AC < 1$ and $\Theta < 0$, there is a unique planar equilibrium $E_{si}(\bar{s},\bar{i},0)$ in *si*-plane for system (3.39) (see Lemma 3.6). For $E_{si}(\bar{s},\bar{i},0)$, we have

$$\Psi(\bar{s},\bar{i},0) = \alpha_3 E(\bar{i} + A)\left\{\bar{s}^m + M_2\bar{i}^m - F\left[(\bar{s}+\varepsilon\bar{i})^n + D\right]\right\} > 0,$$

which follows from the assumption that $\bar{s}^m + M_2\bar{i}^m > F[(\bar{s}+\varepsilon\bar{i})^n + D]$ $(n \geq m \geq 1)$.

When system (3.39) has two different planar equilibria in *sp*-plane, one of which will always be a saddle, so we only need to consider the situation that there is a unique planar equilibrium in *sp*-plane, i.e., $\check{E}_{sp}(\check{s},0,\check{p})$ (the monotonic case, see Lemma 3.12) or $\hat{E}_{sp}(\hat{s},0,\hat{p})$ (the non-monotonic case, see Lemma 3.16). If the condition (i) holds, then $\check{E}_{sp}(\check{s},0,\check{p})$ exists and we have

$$\Psi(\check{s},0,\check{p}) = \begin{cases} \alpha_2 B_2[(\check{s}-AC)(\check{s}+D) - AM_1\check{p}] > 0, \ \text{if } m = 1, \\ \alpha_2 B_2(\check{s}-AC)(\check{s}^m + D) > 0, \ \text{if } m > 1. \end{cases}$$

Similarly, if the condition (ii) holds, we can show $\hat{E}_{sp}(\hat{s},0,\hat{p})$ is a repeller. Hence $V(s,i,p)$ is an average Lyapunov function and system (3.39) is uniformly persistent. □

REMARK 3.12. *Biologically, uniform persistence (permanence) is often a better measure of ecological stability, which is the research focus of most ecologists. Uniform persistence means that the minimal densities of all populations are bounded away from zero, thus they can survive for all future time.*

4.3. Robustness of heteroclinic orbits and a heteroclinic network. In subsections 3.1 and 3.2, we have discussed the existence of heteroclinic cycles in *si*-plane and *sp*-plane, respectively (see Remarks 3.6 and 3.7). For the heteroclinic connection (orbit) $\gamma_{\Theta 1}$ whenever it exists, we always have

$$h_{I_s}(\gamma_{\Theta 1}) \neq h_{I_s}(E_\Theta) \vee h_{I_s}(E_1),$$

where h_{I_s} is the Conley index restricted to the invariant set I_s (*s*-axis) of system (3.39).

By Theorem 1.15 in subsection 1.2, it can be shown that the heteroclinic connection $\gamma_{\Theta 1}$ is robust to perturbations of the vector field in I_s. However, with respect to the heteroclinic cycles γ_h in *si*-plane and $\widetilde{\gamma}_h$ in *sp*-plane, we have

$$h_{I_{si}}(\gamma_h) = h_{I_{si}}(E_\Theta) \vee h_{I_{si}}(E_1) \text{ and } h_{I_{sp}}(\widetilde{\gamma}_h) = h_{I_{sp}}(E_\Theta) \vee h_{I_{sp}}(E_1),$$

which cannot ensure the robustness of the heteroclinic cycles, implying that arbitrarily small perturbation of parameters may destroy the heteroclinic connection $\gamma_{1\Theta}$ or $\widetilde{\gamma}_{1\Theta}$.

DEFINITION 4.5. *(see [7]). A heteroclinic network is a connected union of heteroclinic cycles.*

Combining Remarks 3.6 and 3.7, a heteroclinic network consisting of two cycles γ_h and $\widetilde{\gamma}_h$ exists for certain parameter values. The existence of a heteroclinic network is very interesting due to a common heteroclinic connection between the two cycles. On the other hand, it is impossible for either of the cycles to attract all nearby trajectories. Biologically, in the presence of Allee effect, the emergence of either of the cycles is as a result of the outbreak of the disease or predator's successful invasion.

4.4. An interior periodic orbit. We have shown that when system (3.39) has a unique planar equilibrium $E_{si}(\bar{s}, \bar{i}, 0)$ in si-plane, if $\det(J_1(\bar{s}, \bar{i})) > 0$, $0 < AC < \frac{\Theta+1}{2}$ and $h(E_{-}^{1}(\bar{s}, \bar{i}), B_2^*) = \Sigma^0$, then the bifurcating periodic solutions in si-plane are orbitally asymptotically stable (see Corollary 3.10). Moreover, when system (3.39) has two different planar equilibria $E_{si}^1(\bar{s}_1, \bar{i}_1, 0)$ and $E_{si}^2(\bar{s}_2, \bar{i}_2, 0)$ in si-plane, if $0 < AC < \frac{\Theta+1}{2}$ and $h(E_{-2}^{1}(\bar{s}_2, \bar{i}_2), B_{22}^*) = \Sigma^0$, then the bifurcating periodic solutions in si-plane are also orbitally asymptotically stable (see Remark 3.5). Now, we take the first case as an example to show that this limit cycle can be bifurcated into the interior of the positive octant of sip-space.

First, we denote

$$O_1 = \frac{1}{T}\int_0^T (i(\tau) + A)s^m(\tau)d\tau, \qquad O_2 = \frac{1}{T}\int_0^T (i(\tau) + A)i^m(\tau)d\tau,$$

and

$$O_3 = \frac{1}{T}\int_0^T (i(\tau) + A)[(s(\tau) + \varepsilon i(\tau))^n + D]d\tau,$$

for some positive T.

Let $(s(\tau), i(\tau))$ be the unique periodic solution of system (3.18) for a fixed set of parameters A, B_1, B_2, C and Θ, and T be the period of this solution. Then $(s(\tau), i(\tau), 0)$ is a periodic solution of system (3.39) for any choice of D, E, F, M_1, M_2, m and n. Fix D, E, M_1, M_2, m and n, and the remaining parameter F will be treated as a bifurcation parameter and $F^* = \frac{O_1 + M_2 O_2}{O_3}$ will turn out to be the critical value.

Before presenting the main theorem, we give the following three technical lemmas, whose proofs can be found in [22, 75].

Lemma 4.6. *(see [22]). Let $\mathcal{W}$ be an open neighborhood of $0 \in \mathbb{R}^n$ and let I be an open interval about $0 \in \mathbb{R}$. Let $\Phi_\nu : \mathcal{W} \to \mathbb{R}^n$ be such that the map $(\nu, x) \to \Phi_\nu(x)$ is a C^k map ($k \geq 1$) from $I \times \mathcal{W}$ to $\mathbb{R}^n$, and such that $\Phi_\nu(0) = 0$ for all $\nu \in I$. Define L_ν to be the differential map $d\Phi_\nu(0)$ and suppose that all eigenvalues of L_ν lie inside the unit circle of the complex plane for $\nu < 0$. Assume that there is a real, simple eigenvalue $l(\nu)$ of L_ν such that $l(0) = 1$ and $(dl/d\nu)(0) > 0$. Let v_0 be the eigenvector corresponding to $l(0)$. Then there is a C^{k-1} curve $\mathscr{C}$ of fixed points of $\Phi : (\nu, x) \to (\nu, \Phi_\nu(x))$ near $(0,0)$ in $I \times \mathbb{R}^n$ which, together with the points $(\nu, 0)$, are the only fixed points of Φ near $(0,0)$. The curve $\mathscr{C}$ is tangent to v_0 at $(0,0)$ in $I \times \mathbb{R}^n$.*

Lemma 4.7. *(see [75]). The spectrum of the linearization of the Poincaré map union $\{1\}$ is equal to the spectrum of the linearization of the solution map.*

Lemma 4.8. *(see [22]). Let $H(t)$ be a periodic matrix of period T and suppose that the linear system*

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = H(t) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

has Floquet exponents 0 *and* $-\sigma < 0$. *Let* $h_{13}(t), h_{23}(t)$ *and* $h_{33}(t)$ *be functions of period* T *such that the mean value of* $h_{33}(t)$ *is equal to* ν. *Then the linear system*

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \overline{H}(t) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

where

$$\overline{H}(t) = \left(\begin{array}{cc:c} \multicolumn{2}{c:}{\multirow{2}{*}{$H(t)$}} & h_{13}(t) \\ & & h_{23}(t) \\ \hdashline 0 & 0 & h_{33}(t) \end{array}\right)$$

has Floquet exponents 0, $-\sigma < 0$ *and* ν.

THEOREM 4.9. *Let* A, B_1, B_2, C *and* Θ *be given such that* $(s(\tau), i(\tau))$ *is a locally unique periodic orbit with period* T *in the interior of the first quadrant of si-plane. If there exists an* F^* *denoted by* $F^* = \frac{O_1 + M_2 O_2}{O_3}$ *such that as* $|F - F^*| \ll 1$, *there is a periodic orbit in the interior of the first octant arbitrarily close to the si-plane.*

Proof: Denote by $\mathscr{P}$ the orbit corresponding to $(s(\tau), i(\tau), 0)$ where $(s(\tau), i(\tau))$ is the locally unique periodic orbit of system (3.39) in *si*-plane. Let $\mathscr{T}$ be a two dimensional, local, transverse section of $\mathscr{P}$. For each value of F, the Poincaré map $\mathcal{P} : W_0 \to W_1$ exists, where W_0 and W_1 are open subsets of $\mathscr{T}$. Given a periodic orbit, there is a relationship between the linearization about the orbit and the linearization of the Poincaré map about the corresponding fixed point, made precise in the statement of Lemma 4.7. That is, there is an eigenvalue 1, an eigenvalue determined by the stability of the periodic orbit and an eigenvalue determined by the linearization of system (3.39) at the periodic orbit, followed from Lemma 4.8. As assumed, the periodic orbit in *si*-plane is a stable limit cycle, yielding one Floquet multiplier inside the unit circle.

In order to apply the bifurcation theorem it is necessary to show that the remaining eigenvalue crosses the unit circle transversally. This will be accomplished by showing that one of the Floquet exponents passes through zero (transversally) and applying Lemma 4.7.

Let $(s(\tau), i(\tau))$ be the unique, periodic solution of system (3.18). As noted, the Floquet exponents are 0 and $-\sigma < 0$. The coefficient matrix of the linearization about the periodic orbit of a solution of (3.39) can be denoted by $J_1(s, i)$, see (3.24), while for a solution of (3.39), with $p = 0$, it takes the form

$$\left(\begin{array}{cc:c} \multicolumn{2}{c:}{J_1(s,i)} & -(i+A)s^m \\ & & -B_2 M_1 (i+A) i^m \\ \hdashline 0 & 0 & E(i+A)\{s^m + M_2 i^m - F[(s+\varepsilon i)^n + D]\} \end{array}\right).$$

By Lemma 4.8, the Floquet exponents for the linearization of (3.39) about $(s(\tau), i(\tau), 0)$ are 0, $-\sigma < 0$ and $\nu = EO_3(F^* - F)$. As F decreases to F^*, $\frac{d\nu}{dF}|_{F=F^*} = -EO_3 < 0$, so the crossing is transversal. Hence the Poincaré map has one eigenvalue $e^{-\sigma}$ inside the unit circle and one eigenvalue e^{ν} crossing the unit circle transversally as ν increases through 0. Let $\mathscr{P}$ be the orbit corresponding to $(s(\tau), i(\tau), 0)$. Select $p_0 \in \mathscr{P}$ and identify the transverse section $\mathscr{T}$ of $\mathscr{P}$ through p_0 with $\mathbb{R}^2$, identifying p_0 with $0 \in \mathbb{R}^2$. Let Φ_ν denote the Poincaré map associated with $\mathscr{P}$, p_0 and the section $\mathscr{T}$, for (3.39) with $F = F^* - \frac{\nu}{EO_3}$. From the analytic dependence of the vector field defined by (3.39) on its parameters, it follows that solutions are analytic in parameters and initial conditions [29]. So is the Poincaré map. It follows that there is a neighborhood $\mathcal{W}$ of p_0 in $\mathscr{P}$ such that for all ν sufficiently close to 0, say $\nu \in I$, the map Φ_ν, is defined on $\mathcal{W}$. Making our identification with $\mathbb{R}^2$, we see that $(\nu, x) \to \Phi_\nu(x)$ is analytic from $I \times \mathcal{W}$ to $\mathbb{R}^2$, $\Phi_\nu(0) = 0$ for all $\nu \in I$ and $L_\nu = d\Phi_\nu(0)$ has eigenvalues $e^{-\sigma}$, e^{ν}. Applying Lemma 4.6, we obtain an analytic curve $\mathscr{C}$ of fixed points of $\Phi : (\nu, x) \to (\nu, \Phi_\nu(x))$ bifurcating from $(\nu, 0)$ at $(0, 0)$.

For such (ν, x) , we have $x = \Phi_\nu(x)$, so x is a fixed point of the Poincaré map Φ_ν. $\mathscr{C}$ therefore corresponds to a l-parameter family of periodic solutions of (3.39). In addition, $\mathscr{C}$ is tangent to the eigenvector v_0 associated with the eigenvalue 1 of $d\Phi_0$. The direction of v_0 is transverse to the *si*-plane for system (3.39) (the other eigenvector of $d\Phi_0$ lies in this plane). It follows that there is a branch of periodic solutions of (3.39) in the positive octant for $|F - F^*| = |\nu|$ small. □

REMARK 3.13. *The main idea of the proof is to apply Lemma 4.6 to the Poincaré map on a section of the unique periodic orbit in the plane. All of the parameters will be fixed except one, F, which will be adjusted so as to produce a bifurcation of a fixed point of the Poincaré map. There are also a branch of periodic solutions not in the positive octant, but these are without biological interest. Similarly, if S-P subsystem* (3.29) *has a stable limit cycle, then this limit cycle can also be bifurcated into the interior of the positive octant of the sip-space by selecting an appropriate bifurcation parameter.*

5. Concluding remarks

Under some basic assumptions, we study a class of predator-prey type eco-epidemiological systems. We first show the existence, uniqueness, positivity and uniform ultimate boundedness of the solutions for the proposed systems, which imply that (3.1) is biologically well behaved. By assuming in turn one of the three populations is absent, we reduce the 3-dimensional system (3.1) to three 2-dimensional subsystems (3.18), (3.29) and (3.36), and study their respective dynamics. Then we observe that S-I subsystem (3.18) and S-P subsystem (3.29) possess very complicated features due to the nonlinearity growth of susceptible prey including Allee effect and competition. By Lemmas 3.6, 3.12 and 3.16, we find the existence of interior equilibrium in S-I (or S-P) subsystem is closely related to two quantities, i.e., the epidemiological (or ecological) basic reproduction number and the value of Allee effect. For certain parameter values, there are periodic orbits, heteroclinic cycles and multi-stability in both *si*- and *sp*-planes. We use Conley index and restricted Conley index (defined by restricting Conley index to specific invariant subspaces) to detect the bifurcation point (Hopf bifurcation and bifurcation of heteroclinic orbits) and heteroclinic orbits (cycles) and to show the robustness of heteroclinic orbits in S-I subsystem (3.18). We also observe that different intra-class and inter-class competitions between prey influence the existence and stability of the interior equilibrium and the stability of the boundary equilibrium belong to the strong Allee effect for S-I subsystem (3.18). The non-monotonic functional response makes the dynamics of S-P subsystem (3.29) more abundant. For example, under suitable conditions, it has two interior equilibria and undergoes backward bifurcation and saddle-node bifurcation. By comparing strong and week Allee effects in S-I subsystem (3.18) (or S-P subsystem (3.29)), we conclude that the Allee effects can generate or destroy the interior equilibrium of S-I (or S-P) subsystem and have the ability to destabilize the subsystems and even make them prone to extinction. However, the dynamics of I-P subsystem (3.36) is relatively simple, a unique trivial equilibrium is unconditionally globally asymptotically stable. These results provide us an access to investigate the dynamics of the full system.

Under suitable parametric conditions, the full system (3.39) admits seven possible boundary equilibria, i.e., one trivial equilibrium, two axial equilibria and four planar equilibria. By analyzing the corresponding characteristic equations, the local stability of each of boundary equilibria of (3.39) is established, respectively. By constructing appropriate Lyapunov functions or using Bendixson-Dulac criteria, we work out analytically the conditions for the global stability of one axial equilibrium and two planar equilibria. Furthermore, the strong Allee effect can induce a separatrix surface leading to multi-stability and the Hopf bifurcation is observed around two planner equilibria. As it is not an easy task to solve an interior equilibrium where both the prey and the predator species coexist, we turn to study the uniform persistence of (3.39) by applying the average Lyapunov function, which ensures that all populations coexist for longer period and none of them becomes extinct. If prey is subject to the strong Allee effect, it is impossible for (3.39) to persist for any initial value, indicating that the strong Allee effect in prey makes initial conditions being extremely important for the persistence of prey as well as predator. Moreover, it is shown that disease-induced competition can affect the coexistence of all populations and tremendously alter the stability of (3.39). These analyses reveal the impact of Allee effects, the possibility of coexistence and the outcomes of different competitions. Interestingly, we find that a heteroclinic network formed by two heteroclinic cycles in si- and sp-planes emerges under certain conditions of parameter values. Finally, we perform a detailed analysis to show that a stable limit cycle created through the Hopf bifurcation in si-plane may be bifurcated into the interior of the first octant under some restrictions.

These findings may have wide applications in management policy of species conservation. For example, the introduction of disease or predation may act as a biological control to regulate population sizes and save the population from extinction. Due to the usage of generalized models, they can be generalized in obvious ways to food chain systems, and may be applied to the situation where disease is transmitted by migratory bird population, like the West Nile virus, salmonella, and so on.

CHAPTER 4

Mathematical analysis of network-based systems coupling epidemic spread and information diffusion

In this chapter, we investigate two network-based systems coupling epidemic spread and information diffusion, i.e., a concrete interplay system and an epidemic control system. The organization of the chapter is as follows. In Section 1, we list some basic knowledge and modeling approaches of complex networks. In Section 2, we construct a concrete interplay system in quenched multiplex networks using a well-known SIS (susceptible-infected-susceptible) model. Then, we make some preparations, analyze the epidemic threshold, and obtain the conditions of global stability of the concrete interplay system. In Section 3, we formulate an SIS epidemic control system and analyze its stability. In Section 4, some numerical simulations are carried out to complement our theoretical analysis in Sections 2 and 3. Finally, Section 5 concludes this chapter and presents some discussion.

1. Complex networks and network-based approaches for epidemic spread

In mathematical terms, a network can be defined as a graph $G = (V, E)$, where V is the set of nodes (vertices) and E is the set of links (edges) [19]. Two nodes are called neighbors if they are connected with a link. The degree (connectivity) of a node is the total number of its neighbors. The degree distribution of a network, $P(k)$, can be defined as the fraction of nodes having degree k or the probability that a randomly chosen node has degree k. Instead of using a list of nodes and edges, we can represent the network as an adjacency matrix $A = (a_{ij})_{N\times N}$. The matrix element $a(i, j)$ is unit if node i and node j are connected; otherwise, it is zero.

Complex networks are the underlying structures of many complex systems in nature and society [3, 19, 81]. In the context of network theory, a complex network is a graph (network) with nontrivial topological features—features that do not occur in simple networks such as lattices or random graphs but often occur in graphs modeling of real systems. Two well-known and much studied classes of complex networks are small-world networks [112] and scale-free networks [12], whose discovery and definition are canonical case studies in the field.

A network is called a small-world network by analogy with the small-world phenomenon (popularly known as six degrees of separation). In 1998, the first small-world network model (the WS model) [112] was published by D. J. Watts and S. H. Strogatz, which through a single parameter smoothly interpolates between completely regular and purely random networks, see Fig. 1. Subsequently, M. E. Newman and D. J. Watts built the NW small-world network model using an asymptotically exact real-space renormalization group method [82].

A network is named scale-free [11, 12] if its degree distribution follows a power law, at least asymptotically. That is, the fraction $P(k)$ of nodes in the network having k connections to other nodes goes for large values of k as

$$P(k) \sim k^{-\gamma}$$

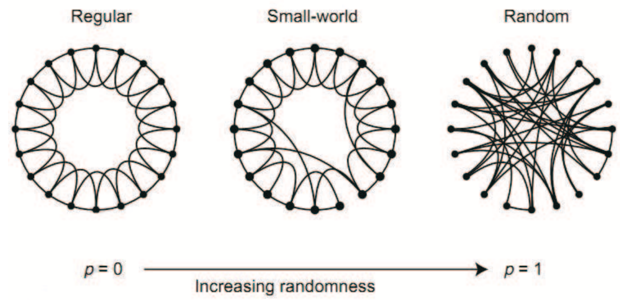


FIGURE 1. Regular, random, and WS small-world networks. Image courtesy of [112].

where the exponent γ is a parameter whose value is typically in the range $2 < \gamma < 3$ (wherein the second moment of $k^{-\gamma}$ is infinite but the first moment is finite), although occasionally it may lie outside these bounds [84]. There are various algorithms to generate a scale-free network. Particularly, in Section 4, we use the Barábasi-Albert algorithm to generate multiple scale-free networks.

In classical compartmental models, it is assumed that the population is homogeneously mixed. However, in reality, individuals have different connectivity patterns. To develop better models, complex networks [3, 19, 57, 81, 85] have been used in mathematical modeling of epidemic spread by relaxing the homogenous mixing assumption. Generally speaking, the most widely used theoretical approaches for epidemic spreading on complex networks in terms of increasing complexity, include the mean-field, the heterogeneous mean-field, the quenched mean-field, dynamical message-passing, link percolation, and pairwise approximation [111].

As far as the mean-field approaches are concerned, the quenched mean-field approach is effective to model the spread of an infectious disease in a heterogenous population [107]. We consider the standard SIS model in a quenched network of size N. Let $\rho_i(t)$ denote the infection probability of node i at time t. Neglecting correlations between infected and susceptible nodes, the evolution equation of node i can be described by

$$\dot{\rho}_i(t) = -\rho_i(t) + \lambda[1 - \rho_i(t)] \sum_{j=1}^{N} a_{ij}\rho_j(t), \ i = 1, 2, \cdots, N, \tag{4.1}$$

where the infection rate $\lambda \in (0, 1]$ denotes the probability that each susceptible node is infected if it is connected to one infected node. a_{ij} is an element of the adjacency matrix, it is assigned with 1 if there is an edge between nodes i and j $(i \neq j)$ or 0 otherwise; if $i = j$, then $a_{ii} = 0$.

2. A concrete interplay system and its stability analysis

2.1. A concrete interplay system. Our interplay system is implemented in multiplex networks (in fact, a duplex network). One layer of this network is the behavior information network and the other is epidemic contact network. Here, let $(a_{ij}), (b_{ij})$ be the adjacency matrices of the epidemic spreading (contact) network and the behavior information network, respectively. We shall denote by $k_i^a = \sum_{j=1}^{N} a_{ij}, k_i^b = \sum_{j=1}^{N} b_{ij}$ the physical connectivity and virtual connectivity of the i-th individual in multiplex networks, respectively. We make the following basic assumptions:

(i) There is weak coupling between individuals in the behavior information network when an epidemic begins to spread.

(ii) Each individual promptly obtain relatively accurate epidemic information through multiple channels and is affected by both the local and global epidemic information. In particular, if an individual has more connected neighbors, he/she will pay more attention to the local epidemic information.

(iii) The individual with more connected neighbors is likely more concerned about epidemic information (including both the local and global epidemic information), so his/her coupling weight in the behavior information network will increase more significantly, implying that the extent of individual response to epidemic information is related to one's physical degrees in the epidemic spreading network. Here, we use the function $\psi(k_i^a)$ to express the response extent of the i-th individual with physical degree k_i^a to epidemic information.

(iv) When the global infection density (measured by $\rho^g(t) = \sum_{i=1}^N \rho_i(t)/N$) or local infection density (measured by $\rho_i^l(t) = \sum_{j=1}^N a_{ij}\rho_j(t)/k_i^a$) becomes higher in the epidemic spreading network, individuals will communicate the information of risk-averse behaviors with their neighbors more frequently to protect themselves in the behavior information network, indicating that the rate of change of the coupling weight of the i-th individual, $\dot{c}_i(t)$, is directly proportional to the global infection density $\rho^g(t)$ and local infection density $\rho_i^l(t)$.

(v) When the collective protective behaviors increase sufficiently, the communication of protective information among individuals will become saturated because they have come to an agreement on optimal protective measures. Thus, the proportional relationship between the rate of change of the coupling weight of the i-th individual, $\dot{c}_i(t)$, and its synchronization error $e_i^T(t)e_i(t)$ always remains valid, where $e_i(t) = x_i(t) - s(t)$ for $i = 1, 2, \cdots, N$.

(vi) Meanwhile, individual adaptive behaviors will in turn suppress the spread of the epidemic, which can reduce one's susceptibility to infection (i.e., the admission rate [83]).

Based on the above assumptions (i)-(vi) and the general interplay model (1.8), we construct a concrete interplay system in quenched multiplex networks:

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) + c_i(t)\sum\limits_{j=1}^N b_{ij}\Gamma[x_j(t) - x_i(t)], \\ \dot{\rho}_i(t) = -\rho_i(t) + \lambda\phi_i(t)[1-\rho_i(t)]\sum\limits_{j=1}^N a_{ij}\rho_j(t),\ i = 1, 2, \cdots, N, \\ \dot{c}_i(t) = \beta\psi(k_i^a)[\theta(k_i^a)\rho_i^l(t) + (1-\theta(k_i^a))\rho^g(t)]e_i^T(t)e_i(t), \end{cases} \tag{4.2}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in \mathbb{R}^n$ is the behavior state variable of the i-th individual at time t, $t \in [0, +\infty)$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously nonlinear function and describes the local dynamics of individuals, $c_i(t) > 0$ denotes the coupling weight or strength of the i-th individual. The matrix $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbb{R}^{n\times n}$ represents the inner-coupling matrix, which is a positive definite diagonal matrix. The matrices $A = (a_{ij})_{N\times N}$ and $B = (b_{ij})_{N\times N}$ represent the adjacency matrices of the epidemic spreading and behavior information networks, respectively. If there are respective connections between node i and node j $(i \neq j)$ in the epidemic spreading and behavior information networks, then $a_{ij} = a_{ji} = 1, b_{ij} = b_{ji} = 1$; otherwise, $a_{ij} = a_{ji} = 0, b_{ij} = b_{ji} = 0$.

In a quenched epidemic network of size N, the variable $\rho_i(t)$ denotes the infection probability of node (individual) i at time t. The transmission rate $\lambda \in (0, 1]$ denotes the probability that an infected node would actually transmit an infection through an edge connected to one susceptible node. The term $\phi_i(t)$ is the admission rate that susceptible node i would actually admit an infection through an edge connected to an infected node. In the absence of adaptive behaviors, it is usually assumed that $\phi_i(t) = 1$ for $i = 1, 2, \cdots, N$. The admission rate $\phi_i(t)$ of node i is dependent on the strength of its adaptive behaviors and contact number (denoted by its physical degree). Intuitively, with fixed adaptive strength, larger contact number means higher risk of infection. Accordingly, in the condition of the same contact number, larger adaptive strength implies lower risk of infection.

Based on the above analysis, we give a specific expression of the admission rate $\phi_i(t)$ for node i as follows:

$$\phi_i(t) = [1 - \alpha(k_i^a)]E_i(t) + \alpha(k_i^a), \ i = 1, 2, \cdots, N, \tag{4.3}$$

where $\alpha(x)$ is an increasing function of x, $0 < \alpha(x) < 1$, and the variable $E_i(t)$ is set as follows:

$$E_i(t) = \frac{e_i^T(t)e_i(t)}{1 + e_i^T(t)e_i(t)} \in [0, 1), \ i = 1, 2, \cdots, N, \tag{4.4}$$

where $e_i(t) = x_i(t) - s(t)$ is the synchronization error of node i, and $s(t)$ is the synchronous state of individual adaptive behaviors in the behavior information network, which can be an equilibrium point, a periodic orbit, or even a chaotic attractor, satisfying $\dot{s}(t) = f(s(t))$.

Individuals uniformly change their behaviors in response to epidemic information. When the adaptive behaviors of all individuals achieve synchronization, i.e., $E_i(t) \to 0$ as $t \to +\infty$ for $i = 1, 2, \cdots, N$, then $x_i(t)$ is the optimal behavioral status, namely, the admission rate $\phi_i(t)$ of the i-th individual achieves the minimum $\alpha(k_i^a)$. The impact of adaptive behaviors on the epidemic can be quantified by the variable $\phi_i(t)$, accordingly, the infection rate λ becomes $\lambda\phi_i(t)$, indicating that the risk of infection from others has fallen to a certain extent.

The parameter $\beta > 0$, $\rho_i^l(t) = \sum_{j=1}^N a_{ij}\rho_j(t)/k_i^a$ is the local infection density in the neighborhood of node i (i.e., the local epidemic information) and $\rho^g(t) = \sum_{i=1}^N \rho_i(t)/N$ is the global infection density in a whole community (i.e., the global epidemic information). The term $\theta(k_i^a)$ represents the extent that the i-th individual is affected by the local epidemic information, accordingly, $1 - \theta(k_i^a)$ is the proportion of the effect of the global information on individual i. By assumption (ii), we know that $\theta(x)$ is an increasing function of x, $0 < \theta(x) < 1$. The term $\psi(k_i^a)$ characterizes the response strength of individual i to epidemic information. Obviously, individuals with more neighbors should have a stronger sense of epidemic information — $\psi(x)$ is an increasing function of x.

The initial condition of system (4.2) can be set as follows: the initial infection probability $\rho_i(0) = \epsilon$, the initial state $x_i(0) = (x_{i1}(0), x_{i2}(0), \cdots, x_{in}(0))^T \in \mathbb{R}^n$, and the initial coupling weight $c_i(0) = \tau$ for $i = 1, 2, \cdots, N$, where $0 < \epsilon \ll 1$ and $0 < \tau \ll 1$.

System (4.2) gives a concrete interplay model describing the interaction between adaptive behaviors and epidemic spread in quenched multiplex networks. Compared to the general interplay model (1.8), we have the behavior state variable $X(t) = (x_1^T(t), x_2^T(t), \cdots, x_N^T(t))^T$ and the variable $Y(t) = (\rho_1(t), \rho_2(t), \cdots, \rho_N(t))^T$. In the behavior information network, as the epidemic spreads, the coupling weight from node i to node j, $w_{ij}^b(t) = c_i(t)b_{ij}$, becomes larger. However, in the epidemic spreading network, $w_{ij}^a(t) = \phi_i(t)a_{ij}$ can be regarded as the contact weight from node i to node j, and it will decrease as the strength of the adaptive behaviors of node i becomes large. Taking into account the heterogeneity of nodes, we have $w_{ij}^a(t) \neq w_{ji}^a(t), w_{ij}^b(t) \neq w_{ji}^b(t)$ if there are respective connections between node i and node j $(i \neq j)$ in the epidemic spreading and behavior information networks, which implies that with the epidemic spread, the epidemic and behavior information networks become directed.

In the light of the adjacency matrix B of the behavior information network, its Laplacian matrix $L = (l_{ij})_{N\times N}$ can be defined as follows:

$$l_{ij} = \begin{cases} -b_{ij}, \ i \neq j, \\ \sum\limits_{\substack{k=1\\k\neq i}}^{N} b_{ik}, \ i = j. \end{cases} \tag{4.5}$$

It is clear that the diagonal elements of the Laplacian matrix L satisfy the following condition:

$$l_{ii} = -\sum_{\substack{j=1\\j\neq i}}^{N} l_{ij} = -\sum_{\substack{j=1\\j\neq i}}^{N} l_{ji} = k_i^b, \ i = 1, 2, \cdots, N, \tag{4.6}$$

where k_i^b denotes the virtual degree of node i in the behavior information network.

It is assumed that B is an irreducible matrix, which implies that the network is connected in the sense of having no isolated clusters. It follows from [115] that zero is the smallest eigenvalue of matrix L with multiplicity 1 and all the other eigenvalues are strictly positive.

The concrete interplay system (4.2) in quenched multiplex networks can now be reformulated in terms of the Laplacian matrix L as

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) - c_i(t) \sum\limits_{j=1}^{N} l_{ij}\Gamma x_j(t), \\ \dot{\rho}_i(t) = -\rho_i(t) + \lambda\phi_i(t)[1-\rho_i(t)] \sum\limits_{j=1}^{N} a_{ij}\rho_j(t), \ i = 1, 2, \cdots, N, \\ \dot{c}_i(t) = \beta\psi(k_i^a)[\theta(k_i^a)\rho_i^l(t) + (1-\theta(k_i^a))\rho^g(t)]e_i^T(t)e_i(t). \end{cases} \tag{4.7}$$

2.2. Stability analysis.

2.2.1. *Preliminaries.* Let $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n\times n}$ be nonnegative matrices, namely, all of their entries are nonnegative. We say $A \geq B$ if $a_{ij} \geq b_{ij}$ for all i and j, and $A > B$ if $A \geq B$ and $A \neq B$.

DEFINITION 2.1. *For $n > 1$, a matrix $A \in \mathbb{R}^{n\times n}$ is reducible if there exists a permutation matrix Q, such that*

$$QAQ^T = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix},$$

where A_1 and A_3 are square matrices. Otherwise, A is irreducible.

In [14], the following properties of nonnegative matrices are given:

- R1. If A is nonnegative, then the spectral radius $\varrho(A)$ of A is one of its eigenvalues, and A has a nonnegative eigenvector corresponding to $\varrho(A)$.
- R2. If A is nonnegative and irreducible, then $\varrho(A)$ is a simple eigenvalue, and A has a positive eigenvector ω corresponding to $\varrho(A)$.

R3. If $0 \leq A \leq B$, then $\varrho(A) \leq \varrho(B)$. Moreover, if $0 \leq A < B$ and $A + B$ is irreducible, then $\varrho(A) < \varrho(B)$.

where $\varrho(A)$ denotes the spectral radius of a matrix A.

2.2.2. *Epidemic threshold.* For the epidemic spreading network in system (4.7), we set $\beta_{ij} = \lambda a_{ij}$, and nonnegative matrices

$$F(\rho) = (\phi_i(t)\beta_{ij}(1 - \rho_i(t)))_{N\times N},\ F_0 = (\phi_i(t)\beta_{ij})_{N\times N}. \tag{4.8}$$

By setting $\rho = (\rho_1, \rho_2, \cdots, \rho_N)^T$, the epidemic spreading system in system (4.7) can be rewritten in a more compact form as

$$\dot{\rho}(t) = F(\rho)\rho - \rho. \tag{4.9}$$

Define

$$R(t) = \varrho(F(\rho)),\ R_0(t) = \varrho(F_0), \tag{4.10}$$

where $R(t)$ is known as the effective reproduction number [5, 18], while $R_0(t)$ is known as the basic reproduction number [34, 106]. We have $R(t) = R_0(t)$ only when the entire population is susceptible.

When the adaptive behaviors of individuals achieve synchronization, i.e., $E_i(t) \to 0$ as $t \to \infty$ for $i = 1, 2, \cdots, N$ by the expression (4.3) of the admission rate $\phi_i(t)$, then the epidemic spreading system in system (4.7) can be rewritten as

$$\dot{\rho}_i(t) = -\rho_i(t) + \lambda\alpha(k_i^a)[1 - \rho_i(t)]\sum_{j=1}^{N} a_{ij}\rho_j(t),\ i = 1, 2, \cdots, N. \tag{4.11}$$

If we set $\beta_{ij} = \lambda a_{ij}$, and nonnegative matrices

$$\bar{F}(\rho) = (\alpha(k_i^a)\beta_{ij}(1 - \rho_i(t)))_{N\times N},\ \bar{F}_0 = (\alpha(k_i^a)\beta_{ij})_{N\times N}. \tag{4.12}$$

By a simple computation, we can then obtain the basic reproduction number of the epidemic spreading system (4.11):

$$\bar{R}_0 = \varrho(\bar{F}_0). \tag{4.13}$$

The classical network-based epidemic spreading system in the absence of human adaptive behaviors can be described by the following differential equation:

$$\dot{\rho}_i(t) = -\rho_i(t) + \lambda[1 - \rho_i(t)]\sum_{j=1}^{N} a_{ij}\rho_j(t),\ i = 1, 2, \cdots, N. \tag{4.14}$$

If we set $\beta_{ij} = \lambda a_{ij}$, and nonnegative matrices

$$\tilde{F}(\rho) = (\beta_{ij}(1 - \rho_i(t)))_{N\times N},\ \tilde{F}_0 = (\beta_{ij})_{N\times N}, \tag{4.15}$$

then we can get the basic reproduction number of the epidemic spreading system (4.14):

$$\tilde{R}_0 = \varrho(\tilde{F}_0). \tag{4.16}$$

Next, we compare the basic reproduction numbers of systems (4.7), (4.11), and (4.14). For arbitrary elements $f_{ij}, \bar{f}_{ij}$, and $\tilde{f}_{ij},\ i, j = 1, 2, \cdots, N$ in $F_0, \bar{F}_0$, and $\tilde{F}_0$, we have $0 \leq \bar{f}_{ij} \leq f_{ij} \leq \tilde{f}_{ij}$, which implies that $0 \leq \bar{F}_0 \leq F_0 \leq \tilde{F}_0$. Then, we can obtain $\varrho(\bar{F}_0) \leq \varrho(F_0) \leq \varrho(\tilde{F}_0)$ by R3 of subsection 2.2.1. Moreover, $F_0 \neq \tilde{F}_0$ and $F_0 + \tilde{F}_0$ is irreducible, so $\varrho(\bar{F}_0) \leq \varrho(F_0) < \varrho(\tilde{F}_0)$, namely, $\bar{R}_0 \leq R_0(t) < \tilde{R}_0$, which implies that individual adaptive behaviors can effectively lower the basic reproduction number of epidemic transmission. Only when the adaptive behaviors of all individuals achieve synchronization, do we have $R_0(t) = \bar{R}_0$.

By the relationship $\lambda_c = \lambda/R_0$ between the epidemic threshold λ_c and the basic reproduction number R_0, we can obtain the respective epidemic thresholds $\lambda_c(t) = \lambda/R_0(t) = \lambda/\varrho(F_0) = 1/\varrho((\phi_i(t)a_{ij})_{N\times N})$, $\bar{\lambda}_c = \lambda/\bar{R}_0 = \lambda/\varrho(\bar{F}_0) = 1/\varrho((\alpha(k_i^a)a_{ij})_{N\times N})$, and $\tilde{\lambda}_c = \lambda/\tilde{R}_0 = \lambda/\varrho(\tilde{F}_0) = 1/\varrho((a_{ij})_{N\times N})$ of systems (4.7), (4.11), and (4.14). Owing to $\varrho(\bar{F}_0) \le \varrho(F_0) < \varrho(\tilde{F}_0)$, we have $\tilde{\lambda}_c < \lambda_c(t) \le \bar{\lambda}_c$.

The ω-limit sets of the epidemic spreading system in system (4.7) are contained in the following bounded region:

$$\Gamma = \{(S_1, \rho_1, \cdots, S_N, \rho_N) \in \mathbb{R}_+^{2N} | 0 \le S_i \le 1, S_i + \rho_i = 1, i = 1, 2, \cdots, N\}. \tag{4.17}$$

It can be verified that Γ is positively invariant with respect to the epidemic spreading system in system (4.7). Let Γ° denote the interior of Γ, and the disease-free equilibrium E_0 of the epidemic spreading system in system (4.7) is on the boundary of Γ.

PROPOSITION 2.2. *Assume that the matrix $A = (a_{ij})$ is irreducible. Then the following results hold:*
(i) If $\tilde{R}_0 \le 1$ (i.e., $\lambda \le \tilde{\lambda}_c < \bar{\lambda}_c$), then E_0 is the unique equilibrium of the epidemic spreading system in system (4.7) and it is globally asymptotically stable in Γ.
(ii) If $\bar{R}_0 > 1$ (i.e., $\tilde{\lambda}_c < \bar{\lambda}_c < \lambda$), then E_0 is unstable and the epidemic spreading system in system (4.7) is uniformly persistent.

Proof: Let $\rho = (\rho_1, \rho_2, \cdots, \rho_N)^T \in \mathbb{R}^N$ and $\rho_0 = (0, 0, \cdots, 0)^T$. The epidemic spreading system in system (4.7) can be rewritten as a vector equation as

$$\dot{\rho}(t) = F(\rho)\rho - \rho, \tag{4.18}$$

where $f_{ij} = \phi_i(t)\beta_{ij}(1 - \rho_i(t))$, $\beta_{ij} = \lambda a_{ij}$.

For $1 \le i \le N$, $0 \le \rho_i \le 1$, we have $0 \le F(\rho) \le F(\rho_0) = F_0 < \tilde{F}_0$, and if $\rho \ne \rho_0$, then $F(\rho) < F(\rho_0)$. Since A is irreducible, we know $F(\rho), F_0$, and $\tilde{F}_0$ are irreducible. Furthermore, $F_0 + \tilde{F}_0$ is irreducible, thus we obtain $\varrho(F(\rho)) \le \varrho(F_0) < \varrho(\tilde{F}_0)$ by R3 of subsection 2.2.1.

If $\tilde{R}_0 = \varrho(\tilde{F}_0) \le 1$, then $\varrho(F(\rho)) < 1$, and $F(\rho)\rho - \rho = 0$ has only the trivial solution $\rho = \rho_0$. Thus E_0 is the only equilibrium of the epidemic spreading system of system (4.7) in Γ if $\tilde{R}_0 \le 1$.

Let $u = (u_1, u_2, \cdots, u_N)$ be a left eigenvector of $\tilde{F}_0$ corresponding to $\varrho(\tilde{F}_0)$, i.e., $u\varrho(\tilde{F}_0) = u\tilde{F}_0$. Since $\tilde{F}_0$ is irreducible, we know $u_i > 0$ for $i = 1, 2, \cdots, N$ by R2 of subsection 2.2.1. Calculating the derivative of the Lyapunov function $L_1(t) = \sum_{i=1}^N u_i\rho_i$ along the solution of system (4.18), we then have

$$\begin{aligned}\frac{dL_1(t)}{dt} &= \sum_{i=1}^N u_i\dot{\rho}_i = u[F(\rho)\rho - \rho] \le u[F_0\rho - \rho] \le u[\tilde{F}_0\rho - \rho] = u\tilde{F}_0\rho - u\rho \\ &= u\varrho(\tilde{F}_0)\rho - u\rho = [\varrho(\tilde{F}_0) - 1]u\rho = (\tilde{R}_0 - 1)u\rho \le 0 \; if \; \tilde{R}_0 \le 1. \end{aligned} \tag{4.19}$$

If $\tilde{R}_0 < 1$, then $\frac{dL_1(t)}{dt} = 0 \Leftrightarrow \rho = \rho_0$. If $\tilde{R}_0 = 1$, then $\frac{dL_1(t)}{dt} = 0$ implies $u[F(\rho)\rho - \rho] = 0$. If $\rho \ne \rho_0$, then $u[F(\rho)\rho - \rho] < u[F_0\rho - \rho] < u[\tilde{F}_0\rho - \rho] = 0$ by inequality (4.19), a contradiction to the equality that $u[F(\rho)\rho - \rho] = 0$. Thus $u[F(\rho)\rho - \rho] = 0$ has only the trivial solution $\rho = \rho_0$. Therefore, $\frac{dL_1(t)}{dt} = 0 \Leftrightarrow \rho = \rho_0$ if $\tilde{R}_0 \le 1$. It can be verified that the only compact invariant subset of the set where $\frac{dL_1(t)}{dt} = 0$ is the singleton $\{E_0\}$. By LaSalle's invariance principle, E_0 is globally asymptotically stable in Γ if $\tilde{R}_0 \leq 1$.

Let $v = (v_1, v_2, \cdots, v_N)$ be a left eigenvector of $\bar{F}_0$ corresponding to $\varrho(\bar{F}_0)$, i.e., $v\varrho(\bar{F}_0) = v\bar{F}_0$. Since $\bar{F}_0$ is irreducible, we know $v_i > 0$ for $i = 1, 2, \cdots, N$ by R2 of subsection 2.2.1. Calculating the derivative of the function $L_2(t) = \sum_{i=1}^{N} v_i\rho_i$ along the solution of system (4.18), we then have

$$\frac{dL_2(t)}{dt} = \sum_{i=1}^{N} v_i\dot{\rho}_i = v[F(\rho)\rho - \rho] \geq v[\bar{F}(\rho)\rho - \rho] \tag{4.20}$$

and

$$\begin{aligned} v[\bar{F}(\rho)\rho - \rho] &\leq v[\bar{F}_0\rho - \rho] = v\bar{F}_0\rho - v\rho = v\varrho(\bar{F}_0)\rho - v\rho \\ &= [\varrho(\bar{F}_0) - 1]v\rho = (\bar{R}_0 - 1)v\rho. \end{aligned} \tag{4.21}$$

If $\bar{R}_0 > 1$ and $\rho \neq \rho_0$, we know that $v\bar{F}_0 - v = (\bar{R}_0 - 1)v > 0$, and thus $v[\bar{F}(\rho)\rho - \rho] > 0$ in a neighborhood of E_0 in Γ° by continuity. By inequality (4.20), we know that $\frac{dL_2(t)}{dt} = v[F(\rho)\rho - \rho] > 0$ in a neighborhood of E_0 in Γ°, which implies that E_0 is unstable. Using a uniform persistence result from [41] and a similar argument as in the proof of Proposition 3.3 in [65], we can show that, when $\bar{R}_0 > 1$, the instability of E_0 implies the uniform persistence of the epidemic spreading system in system (4.7). □

2.2.3. *Global stability.*

Definition 2.3. *The synchronization manifold of the network-based behavioral information diffusion system can be defined as* $S = \{(x_1^T, x_2^T, \cdots, x_N^T)^T \in \mathbb{R}^{nN} : x_i = x_j, i, j = 1, 2, \cdots, N\}$, *where* $x_i = (x_{i1}, x_{i2}, \cdots, x_{in})^T \in \mathbb{R}^n$ *and* x_i^T *represents the transpose of* x_i.

Define the synchronization error variable by $e_i(t) = x_i(t) - s(t),\ i = 1, 2, \cdots, N$, where the synchronous state $s(t)$ can be an equilibrium point, a periodic orbit, or even a chaotic attractor, satisfying $\dot{s}(t) = f(s(t))$, then the error system with respect to the behavioral information diffusion system in system (4.7) can be written as follows:

$$\dot{e}_i(t) = f(x_i(t)) - f(s(t)) - c_i(t)\sum_{j=1}^{N} l_{ij}\Gamma e_j(t),\ i = 1, 2, \cdots, N. \tag{4.22}$$

Denote $e(t) = (e_1^T(t), e_2^T(t), \cdots, e_N^T(t))^T, F(t) = (f^T(x_1(t)) - f^T(s(t)), f^T(x_2(t)) - f^T(s(t)), \cdots, f^T(x_N(t)) - f^T(s(t)))^T, C(t) = \mathrm{diag}(c_1(t), c_2(t), \cdots, c_N(t))$, then we can rewrite (4.22) in a simple compact form

$$\dot{e}(t) = F(t) - (C(t)L \otimes \Gamma)e(t),\ i = 1, 2, \cdots, N, \tag{4.23}$$

where $\otimes$ represents the Kronecker product.

Assumption 2.4. *(see [26, 67]). There exists a positive definite diagonal matrix* P *where* $P = \mathrm{diag}(p_1, p_2, \cdots, p_n)$ *and a constant* ξ, *such that the nonlinear vector-valued continuous function* $f(x(t))$ *satisfies*

$$(x(t) - y(t))^T P[f(x(t)) - f(y(t))] \leq \xi(x(t) - y(t))^T(x(t) - y(t)) \tag{4.24}$$

for all $x(t), y(t) \in \mathbb{R}^n, t \geq 0$.

THEOREM 2.5. *Suppose that the matrices* $A = (a_{ij})$ *and* $B = (b_{ij})$ *are irreducible. If* $\bar{R}_0 > 1$ *(i.e.,* $\tilde{\lambda}_c < \bar{\lambda}_c < \lambda$*), then the epidemic spreading system in system* (4.7) *has a unique endemic equilibrium* E^**, and* E^* *is globally asymptotically stable in* Γ°*. Moreover, the synchronization manifold of the behavioral information diffusion system in system* (4.7) *is also globally asymptotically stable.*

Proof: Since $S_i(t) + \rho_i(t) = 1$ for $i = 1, 2, \cdots, N$, we can rewrite the epidemic spreading system in system (4.7) as follows:

$$\begin{cases} \dot{S}_i(t) = 1 - S_i(t) - \sum\limits_{j=1}^{N} \lambda\phi_i(t)a_{ij}S_i(t)\rho_j(t), \\ \dot{\rho}_i(t) = -\rho_i(t) + \sum\limits_{j=1}^{N} \lambda\phi_i(t)a_{ij}S_i(t)\rho_j(t),\ i = 1, 2, \cdots, N. \end{cases} \tag{4.25}$$

Let $E^* = \{S_1^*, \rho_1^*; \cdots, S_N^*, \rho_N^*\} \in \Gamma^\circ$, where $S_i^*, \rho_i^* > 0$ for $1 \le i \le N$, denote the unique endemic equilibrium of the epidemic spreading system in system (4.7). Set $\bar{a}_{ij} = \lambda\alpha(k_i^a)a_{ij}S_i^*\rho_j^*$, $1 \le i, j \le N$, and utilize the following matrix:

$$\bar{A} = \begin{pmatrix} \Sigma_{j\neq 1}\bar{a}_{1j} & -\bar{a}_{12} & \cdots & -\bar{a}_{1N} \\ -\bar{a}_{21} & \Sigma_{j\neq 2}\bar{a}_{2j} & \cdots & -\bar{a}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{a}_{N1} & -\bar{a}_{N2} & \cdots & \Sigma_{j\neq N}\bar{a}_{Nj} \end{pmatrix}. \tag{4.26}$$

We consider the Lyapunov function candidate $V(t) = V^A(t) + V^B(t)$ with

$$V^A(t) = \sum_{i=1}^{N} z_i V_i^A(t), \tag{4.27}$$

where $z_i > 0$ denotes the cofactor of the i-th diagonal entry of $\bar{A}, 1 \le i \le N$, and

$$V_i^A(t) = S_i(t) - S_i^* \ln S_i(t) + \rho_i(t) - \rho_i^* \ln \rho_i(t), \tag{4.28}$$

and

$$V^B(t) = \frac{1}{2}e^T(t)(I_N \otimes P)e(t) + \sum_{i=1}^{N} \frac{\sigma\delta_i}{2\beta\psi(k_i^a)}[c_i^* - c_i(t)]^2, \tag{4.29}$$

where $\sigma = \lambda_{\min}(P\Gamma) > 0$, $\lambda_{\min}(P\Gamma)$ denotes the minimal eigenvalue of matrix $P\Gamma$, and positive constants c_i^*, δ_i will be decided later.

Next, we show that $V_i^A(t)$ satisfies the assumptions of Theorem 3.1 in [66].

The derivative of $V_i^A(t)$ along the solution of the epidemic spreading system in system (4.7) as follows:

$$\begin{aligned} \frac{dV_i^A(t)}{dt} =& [1 - \frac{S_i^*}{S_i(t)}]\frac{dS_i(t)}{dt} + [1 - \frac{\rho_i^*}{\rho_i(t)}]\frac{d\rho_i(t)}{dt} \\ =& [1 - \frac{S_i^*}{S_i(t)}][1 - S_i(t) - \sum_{j=1}^{N} \lambda\phi_i(t)a_{ij}S_i(t)\rho_j(t)] \\ &+ [1 - \frac{\rho_i^*}{\rho_i(t)}][-\rho_i(t) + \sum_{j=1}^{N} \lambda\phi_i(t)a_{ij}S_i(t)\rho_j(t)]. \end{aligned} \tag{4.30}$$

Using the equilibrium equations

$$1 = S_i^* + \sum_{j=1}^{N} \lambda\alpha(k_i^a)a_{ij}S_i^*\rho_j^* \text{ and } \rho_i^* = \sum_{j=1}^{N} \lambda\alpha(k_i^a)a_{ij}S_i^*\rho_j^*, \tag{4.31}$$

we can obtain

$$\begin{aligned}
\frac{dV_i^A(t)}{dt} =& [1 - \frac{S_i^*}{S_i(t)}][S_i^* + \sum_{j=1}^{N} \lambda\alpha(k_i^a)a_{ij}S_i^*\rho_j^* - S_i(t) - \sum_{j=1}^{N} \lambda\phi_i(t)a_{ij}S_i(t)\rho_j(t)] \\
&+ [1 - \frac{\rho_i^*}{\rho_i(t)}][\sum_{j=1}^{N} \lambda\phi_i(t)a_{ij}S_i(t)\rho_j(t) - \sum_{j=1}^{N} \lambda\alpha(k_i^a)a_{ij}S_i^*\rho_j^*\frac{\rho_i(t)}{\rho_i^*}] \\
=& -\frac{1}{S_i(t)}[S_i(t) - S_i^*]^2 + \sum_{j=1}^{N} \lambda\alpha(k_i^a)a_{ij}S_i^*\rho_j^*[2 - \frac{S_i^*}{S_i(t)} - \frac{\rho_i(t)}{\rho_i^*}] \\
&+ \sum_{j=1}^{N} \lambda\phi_i(t)a_{ij}[S_i^*\rho_j(t) - \frac{S_i(t)\rho_j(t)\rho_i^*}{\rho_i(t)}].
\end{aligned} \tag{4.32}$$

Substituting the expression (4.3) of $\phi_i(t)$ into (4.32), since $[S_i(t) - S_i^*]^2 \geq 0$, we have

$$\begin{aligned}
\frac{dV_i^A(t)}{dt} \leq& \sum_{j=1}^{N} \lambda\alpha(k_i^a)a_{ij}S_i^*\rho_j^*[2 - \frac{S_i^*}{S_i(t)} - \frac{\rho_i(t)}{\rho_i^*}] \\
&+ \sum_{j=1}^{N} \lambda[(1 - \alpha(k_i^a))E_i(t) + \alpha(k_i^a)]a_{ij}[S_i^*\rho_j(t) - \frac{S_i(t)\rho_j(t)\rho_i^*}{\rho_i(t)}] \\
=& \sum_{j=1}^{N} \lambda\alpha(k_i^a)a_{ij}S_i^*\rho_j^*[2 - \frac{S_i^*}{S_i(t)} - \frac{\rho_i(t)}{\rho_i^*} - \frac{S_i(t)\rho_j(t)\rho_i^*}{S_i^*\rho_j^*\rho_i(t)} + \frac{\rho_j(t)}{\rho_j^*}] \\
&+ \sum_{j=1}^{N} \lambda(1 - \alpha(k_i^a))E_i(t)a_{ij}[S_i^*\rho_j(t) - \frac{S_i(t)\rho_j(t)\rho_i^*}{\rho_i(t)}].
\end{aligned} \tag{4.33}$$

Let $\bar{a}_{ij} = \lambda\alpha(k_i^a)a_{ij}S_i^*\rho_j^*$, $G_i(\rho_i) = -\frac{\rho_i}{\rho_i^*} + \ln\frac{\rho_i}{\rho_i^*}$, and $F_{ij}(S_i, \rho_i, \rho_j(\cdot)) = 2 - \frac{S_i^*}{S_i(t)} - \frac{\rho_i(t)}{\rho_i^*} - \frac{S_i(t)\rho_j(t)\rho_i^*}{S_i^*\rho_j^*\rho_i(t)} + \frac{\rho_j(t)}{\rho_j^*}$. Since $0 < 1 - \alpha(k_i^a) \leq 1$, by expression (4.4) of $E_i(t)$ and (4.33), we have

$$\begin{aligned}
\frac{dV_i^A(t)}{dt} \leq& \sum_{j=1}^{N} \bar{a}_{ij}F_{ij}(S_i, \rho_i, \rho_j(\cdot)) + \sum_{j=1}^{N} \lambda E_i(t)a_{ij}S_i^*\rho_j(t) \\
=& \sum_{j=1}^{N} \bar{a}_{ij}F_{ij}(S_i, \rho_i, \rho_j(\cdot)) + \lambda S_i^*E_i(t)\sum_{j=1}^{N} a_{ij}\rho_j(t) \\
\leq& \sum_{j=1}^{N} \bar{a}_{ij}F_{ij}(S_i, \rho_i, \rho_j(\cdot)) + \lambda k_i^a S_i^*E_i(t) \\
\leq& \sum_{j=1}^{N} \bar{a}_{ij}F_{ij}(S_i, \rho_i, \rho_j(\cdot)) + \lambda k_i^a S_i^* e_i^T(t)e_i(t).
\end{aligned} \tag{4.34}$$

Let $\Phi(a) = 1 - a + \ln a$. Then $\Phi(a) \leq 0$ for $a > 0$ and equality holds only at $a = 1$. Furthermore,

$$F_{ij}(S_i, \rho_i, \rho_j(\cdot)) = G_i(\rho_i) - G_j(\rho_j) + \Phi\left(\frac{S_i^*}{S_i}\right) + \Phi\left(\frac{S_i \rho_j(t) \rho_i^*}{S_i^* \rho_j^* \rho_i}\right) \leq G_i(\rho_i) - G_j(\rho_j). \quad (4.35)$$

We can show that $V_i^A, F_{ij}, G_i, \bar{a}_{ij}$ satisfy the assumptions of Theorem 3.1 and Corollary 3.3 in [66]. Therefore, the function $V^A = \sum_{i=1}^N z_i V_i^A$ as defined in Theorem 3.1 in [66] is a Lyapunov function for the epidemic spreading system in system (4.7), namely,

$$\frac{dV^A(t)}{dt} \leq \sum_{i,j=1}^{N} z_i \lambda k_i^a S_i^* e_i^T(t) e_i(t) = \sum_{i=1}^{N} \lambda z_i k_i^a S_i^* e_i^T(t) e_i(t) \quad (4.36)$$

for all $(S_1, \rho_1, \cdots, S_N, \rho_N) \in \Gamma^\circ$.

Let $\Delta = \mathrm{diag}(\delta_1, \delta_2, \cdots, \delta_N)$, then the derivative of $V^B(t)$ along the trajectory of the behavioral information diffusion system in system (4.7) is given by

$$\begin{aligned}\frac{dV^B(t)}{dt} =& e^T(t)(I_N \otimes P)F(t) - e^T(t)(C(t)L \otimes P\Gamma)e(t) - \sum_{i=1}^{N} \frac{\sigma \delta_i}{\beta \psi(k_i)} \dot{c}_i(t)[c_i^* - c_i(t)] \\ =& \sum_{i=1}^{N} e_i^T(t) P[f(x_i(t)) - f(s(t))] - e^T(t)(C(t)L \otimes P\Gamma)e(t) \\ & - \sigma \sum_{i=1}^{N} \delta_i [c_i^* - c_i(t)][\theta(k_i^a)\rho_i^l(t) + (1 - \theta(k_i^a))\rho^g(t)] e_i^T(t) e_i(t). \end{aligned} \quad (4.37)$$

Using the inequality in Assumption 2.4 and equality (4.37), we can get

$$\begin{aligned}\frac{dV^B(t)}{dt} \leq& \xi \sum_{i=1}^{N} e_i^T(t) e_i(t) - e^T(t)(C(t)L \otimes P\Gamma)e(t) \\ & + \sigma \sum_{i=1}^{N} \delta_i c_i(t)[\theta(k_i^a)\rho_i^l(t) + (1 - \theta(k_i^a))\rho^g(t)] e_i^T(t) e_i(t) \\ & - \sigma \sum_{i=1}^{N} \delta_i c_i^* [\theta(k_i^a)\rho_i^l(t) + (1 - \theta(k_i^a))\rho^g(t)] e_i^T(t) e_i(t). \end{aligned} \quad (4.38)$$

By the definition of $\theta(\cdot), \rho_i^l(\cdot)$ and $\rho_i^g(\cdot)$, we see that $0 < \theta(\cdot) < 1$ and $0 \leq \rho_i^l(\cdot), \rho_i^g(\cdot) \leq 1$. So, we obtain $0 \leq \theta(k_i^a)\rho_i^l(t) + (1 - \theta(k_i^a))\rho^g(t) \leq 1$, together with (4.38), yields

$$\begin{aligned}\frac{dV^B(t)}{dt} \leq& \xi \sum_{i=1}^{N} e_i^T(t) e_i(t) - e^T(t)(C(t)L \otimes P\Gamma)e(t) + \sigma \sum_{i=1}^{N} \delta_i c_i(t) e_i^T(t) e_i(t) \\ & - \sigma \sum_{i=1}^{N} \delta_i c_i^* [\theta(k_i^a)\rho_i^l(t) + (1 - \theta(k_i^a))\rho^g(t)] e_i^T(t) e_i(t). \end{aligned} \quad (4.39)$$

Rewrite (4.39) in a compact form as

$$\begin{aligned}\frac{dV^B(t)}{dt} \leq &\xi \sum_{i=1}^{N} e_i^T(t)e_i(t) - e^T(t)(C(t)L \otimes P\Gamma)e(t) + e^T(t)(C(t)\Delta \otimes P\Gamma)e(t) \\ &- \sigma \sum_{i=1}^{N} \delta_i c_i^* [\theta(k_i^a)\rho_i^l(t) + (1-\theta(k_i^a))\rho^g(t)] e_i^T(t)e_i(t) \\ = &\sum_{i=1}^{N} \{\xi - \sigma\delta_i c_i^* [\theta(k_i^a)\rho_i^l(t) + (1-\theta(k_i^a))\rho^g(t)]\} e_i^T(t)e_i(t) \\ &+ e^T(t)[C(t)(\Delta - L) \otimes P\Gamma]e(t). \end{aligned} \tag{4.40}$$

It is clear that $c_i(t) > 0$ when $t > 0$ for all $i = 1, 2, \cdots, N$, namely, the matrices $C(t), P$, and Γ are positive definite. Therefore, by picking appropriate constants δ_i such that $\{\Delta - L\}^s \leq 0$, we then have

$$\begin{aligned}\frac{dV^B(t)}{dt} &\leq \sum_{i=1}^{N} \{\xi - \sigma\delta_i c_i^* [\theta(k_i^a)\rho_i^l(t) + (1-\theta(k_i^a))\rho^g(t)]\} e_i^T(t)e_i(t) \\ &\leq \sum_{i=1}^{N} [\xi - \sigma\delta_i (1-\theta(k_i^a)) c_i^* \rho^g(t)] e_i^T(t)e_i(t). \end{aligned} \tag{4.41}$$

From Proposition 2.2, we know that if $\bar{R}_0 > 1$ (i.e., $\tilde{\lambda}_c < \bar{\lambda}_c < \lambda$), then the epidemic spreading system in system (4.7) is uniformly persistent in Γ°. Combining the continuity and this uniformly persistent property of function $\rho^g(t)$, we can conclude that if $\bar{R}_0 > 1$, then there exists $\rho^* \in (0, 1]$ such that $\lim_{t\to+\infty} \rho^g(t) = \rho^*$, which implies that there exists $\varepsilon \in (0, \rho^*]$ and $t_0 > 0$ such that $\rho^g(t) > \rho^* - \varepsilon \geq 0$ for all $t > t_0$. So when $t > t_0$, we can further obtain

$$\frac{dV^B(t)}{dt} \leq \sum_{i=1}^{N} [\xi - \sigma\delta_i (1-\theta(k_i^a)) c_i^* (\rho^* - \varepsilon)] e_i^T(t)e_i(t). \tag{4.42}$$

Integrating the above discussions, we have

$$\frac{dV(t)}{dt} \leq \sum_{i=1}^{N} [\xi + \lambda z_i k_i^a S_i^* - \sigma\delta_i (1-\theta(k_i^a)) c_i^* (\rho^* - \varepsilon)] e_i^T(t)e_i(t). \tag{4.43}$$

Thus, we can select adequately large constants c_i^* such that $\frac{dV(t)}{dt} \leq 0$, which implies $V(t)$ is non-increasing. It is obvious that the singleton

$$H^* = (S_1^*, S_2^*, \cdots, S_N^*, \rho_1^*, \rho_2^*, \cdots, \rho_N^*, 0, 0, \cdots, 0, c_1^*, c_2^*, \cdots, c_N^*)$$

is the largest invariant set of

$$H = \{(S_1, S_2, \cdots, S_N, \rho_1, \rho_2, \cdots, \rho_N, e_1, e_2, \cdots, e_N, c_1, c_2, \cdots, c_N) | \frac{dV}{dt} = 0\}.$$

By LaSalle's invariance principle, H^* is globally asymptotically stable, that is, for system (4.7), the unique endemic equilibrium E^* of the epidemic spreading system is globally asymptotically stable in Γ°, and the synchronization manifold of the behavioral information diffusion system is also globally asymptotically stable. □

3. Epidemic control system and its stability analysis

3.1. Epidemic control system. At the early stage of an emerging epidemic, it is almost impossible for governments or public health and surveillance systems to respond sufficiently rapidly to develop a vaccine to curb an emerging epidemic by containing it at source. Massive news coverage and fast information flow can generate profound psychological impact on the public, and hence governments or public health authorities could propagate, announce, or release appropriate behavioral information of self-protection (such as staying at home, washing hands frequently, and wearing surgical face masks) to guide individuals at higher risk of infection (e.g., officers, teachers, and doctors, etc.) through various means, aiming at altering their behaviors to achieve an optimal state of self-protection.

Without loss of generality, suppose that the first l nodes are controlled, thus we construct an SIS epidemic control system as follows:

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) - c_i(t) \sum\limits_{j=1}^{N} l_{ij}\Gamma x_j(t) + u_i(t), \\ u_i(t) = -c_i(t) d_i e_i(t),\ i = 1, 2, \cdots, l, \\ u_i(t) = 0,\ i = l+1, l+2, \cdots, N, \\ \dot{\rho}_i(t) = -\rho_i(t) + \lambda\phi_i(t)[1 - \rho_i(t)] \sum\limits_{j=1}^{N} a_{ij}\rho_j(t),\ i = 1, 2, \cdots, N, \\ \dot{c}_i(t) = \beta\psi(k_i^a)[\theta(k_i^a)\rho_i^l(t) + (1 - \theta(k_i^a))\rho^g(t)]e_i^T(t)e_i(t), \end{cases} \tag{4.44}$$

where $d_i = \mathrm{diag}(d_{i1}, d_{i2}, \cdots, d_{in}), i = 1, 2, \cdots, l$, are positive definite feedback gain matrices, $1 \leq l \ll N$, the parameter $\beta > 0$, the local epidemic information $\rho_i^l(t) = \sum_{j=1}^{N} a_{ij}\rho_j(t)/k_i^a$, and the global epidemic information $\rho^g(t) = \sum_{i=1}^{N} \rho_i(t)/N$. The initial condition of system (4.44) can be set as follows: the initial infection probability $\rho_i(0) = \epsilon$, the initial state $x_i(0) = (x_{i1}(0), x_{i2}(0), \cdots, x_{in}(0))^T \in \mathbb{R}^n$, and the initial coupling weight $c_i(0) = \tau$ for $i = 1, 2, \cdots, N$, where $0 < \epsilon \ll 1$ and $0 < \tau \ll 1$.

3.2. Stability analysis.

LEMMA 3.1. *(see [26]). If $G = (g_{ij})_{N\times N}$ is an irreducible matrix with $Rank(G) = N - 1$ and satisfying $g_{ij} = g_{ji} \geq 0$ if $i \neq j$, and $\sum_{j=1}^{N} g_{ij} = 0$ for $i = 1, 2, \cdots, N$. Then, the matrix*

$$\begin{pmatrix} g_{11} - \varepsilon & g_{12} & \cdots & g_{1N} \\ g_{21} & g_{22} & \cdots & g_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N1} & g_{N2} & \cdots & g_{NN} \end{pmatrix}$$

is negative definite for any constant ε.

THEOREM 3.2. *Assume that the matrices $A = (a_{ij})$ and $B = (b_{ij})$ are irreducible. If $\bar{R}_0 > 1$ (i.e., $\tilde{\lambda}_c < \bar{\lambda}_c < \lambda$), then the epidemic spreading system in system* (4.44) *has a unique endemic equilibrium E^*, and E^* is globally asymptotically stable in Γ°. Moreover, the synchronization manifold of the behavioral information diffusion system in system* (4.44) *is also globally asymptotically stable.*

Proof: The error system with respect to the behavioral information diffusion system in system (4.44) can be written as:

$$\dot{e}_i(t) = f(x_i(t)) - f(s(t)) - c_i(t)\sum_{j=1}^{N} l_{ij}\Gamma e_j(t) + u_i(t),\ i = 1, 2, \cdots, N. \tag{4.45}$$

We construct the Lyapunov function $V(t) = V^A(t) + V^B(t)$ with the same $V^A(t)$ as that of Theorem 2.5 and a different $V^B(t)$ as

$$V^B(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)Pe_i(t) + \sum_{i=1}^{N}\frac{\delta_i}{2\beta\psi(k_i^a)}[c_i^* - c_i(t)]^2, \tag{4.46}$$

where positive constants c_i^* and δ_i will be decided later.

Similarly to the analysis in subsection 2.2.3, we can show that the derivative of $V^A(t)$ satisfies

$$\frac{dV^A(t)}{dt} \le \sum_{i,j=1}^{N} z_i\lambda k_i^a S_i^* e_i^T(t)e_i(t) = \sum_{i=1}^{N}\lambda z_i k_i^a S_i^* e_i^T(t)e_i(t) \tag{4.47}$$

for all $(S_1, \rho_1, \cdots, S_N, \rho_N) \in \Gamma^\circ$.

Denote $\tilde{e}_k(t) = (e_{1k}(t), e_{2k}(t), \cdots, e_{Nk}(t))$, $D_k = \mathrm{diag}(d_{1k}, d_{2k}, \cdots, d_{lk}, 0, \cdots, 0)$, $L_k = (l_{ij}^k) \in R^{N\times N}$, and $\Delta_k = \mathrm{diag}(\delta_{1k}, \delta_{2k}, \cdots, \delta_{Nk})$ for $k = 1, 2, \cdots, n$, where $l_{ij}^k = \gamma_k l_{ij}$ and $\delta_{ik} = \delta_i/p_k$. The matrices $-L_k$ are assumed to be irreducible and symmetric, and all of their off-diagonal entries are nonnegative. Then, for the positive semidefinite matrix D_k with nonzero entry $d_{ik} > 0, i \le l$, one can easily derive that $-L_k - D_k < 0$ according to Lemma 3.1.

Then, the derivative of $V^B(t)$ along the trajectory of the behavioral information diffusion system in system (4.44) satisfies

$$\begin{aligned}
\frac{dV^B(t)}{dt} =& \sum_{i=1}^{N} e_i^T(t)P[f(x_i(t)) - f(s(t)) - c_i(t)\sum_{j=1}^{N} l_{ij}\Gamma e_j(t) + u_i(t)] \\
&- \sum_{i=1}^{N}\frac{\delta_i}{\beta\psi(k_i^a)}\dot{c}_i(t)[c_i^* - c_i(t)] \\
\le& \xi\sum_{i=1}^{N} e_i^T(t)e_i(t) - \sum_{i=1}^{N} e_i^T(t)Pc_i(t)\sum_{j=1}^{N} l_{ij}\Gamma e_j(t) + \sum_{i=1}^{l} e_i^T(t)Pu_i(t) \\
&+ \sum_{i=1}^{N}\delta_i c_i(t)[\theta(k_i^a)\rho_i^l(t) + (1-\theta(k_i^a))\rho^g(t)]e_i^T(t)e_i(t) \\
&- \sum_{i=1}^{N}\delta_i c_i^*[\theta(k_i^a)\rho_i^l(t) + (1-\theta(k_i^a))\rho^g(t)]e_i^T(t)e_i(t) \\
\le& \xi\sum_{i=1}^{N} e_i^T(t)e_i(t) - \sum_{i=1}^{N}\sum_{j=1}^{N} c_i(t)l_{ij}e_i^T(t)P\Gamma e_j(t) - \sum_{i=1}^{l} c_i(t)e_i^T(t)Pd_ie_i(t) \\
&+ \sum_{i=1}^{N}\delta_i c_i(t)e_i^T(t)e_i(t) - \sum_{i=1}^{N}\delta_i c_i^*(1-\theta(k_i^a))\rho^g(t)e_i^T(t)e_i(t).
\end{aligned} \tag{4.48}$$

By rewriting some of the summation formulas in (4.48), we further have

$$\begin{aligned}\frac{dV^B(t)}{dt} \leq & \xi\sum_{i=1}^{N} e_i^T(t)e_i(t) - \sum_{k=1}^{n} p_k\tilde{e}_k^T(t)C(t)L_k\tilde{e}_k(t) - \sum_{k=1}^{n} p_k\tilde{e}_k^T(t)C(t)D_k\tilde{e}_k(t) \\ & + \sum_{k=1}^{n} p_k\tilde{e}_k^T(t)C(t)\Delta_k\tilde{e}_k(t) - \sum_{i=1}^{N}\delta_i(1-\theta(k_i^a))c_i^*\rho^g(t)e_i^T(t)e_i(t) \\ = & \sum_{i=1}^{N}[\xi - \delta_i(1-\theta(k_i^a))c_i^*\rho^g(t)]e_i^T(t)e_i(t) \\ & + \sum_{k=1}^{n} p_k\tilde{e}_k^T(t)C(t)(\Delta_k - L_k - D_k)\tilde{e}_k(t). \end{aligned} \tag{4.49}$$

It is clear that $c_i(t) > 0$ when $t > 0$ for all $i = 1, 2, \cdots, N$, namely, the matrix $C(t)$ is positive definite. Since $-L_k - D_k < 0$, one can choose appropriate constants δ_i such that $\{\Delta_k - L_k - D_k\}^s < 0$, then we have

$$\frac{dV^B(t)}{dt} \leq \sum_{i=1}^{N}[\xi - \delta_i(1-\theta(k_i^a))c_i^*\rho^g(t)]e_i^T(t)e_i(t). \tag{4.50}$$

From Proposition 2.2, we can conclude that if $\bar{R}_0 > 1$, then there exists $\rho^* \in (0, 1]$ such that $\lim_{t\to+\infty}\rho^g(t) = \rho^*$, which implies that there exists $\varepsilon \in (0, \rho^*]$ and $t_0 > 0$ such that $\rho^g(t) > \rho^* - \varepsilon \geq 0$ for all $t > t_0$. So when $t > t_0$, we can further obtain

$$\frac{dV^B(t)}{dt} \leq \sum_{i=1}^{N}[\xi - \delta_i(1-\theta(k_i^a))c_i^*(\rho^* - \varepsilon)]e_i^T(t)e_i(t). \tag{4.51}$$

Integrating the above discussions, we have

$$\frac{dV(t)}{dt} \leq \sum_{i=1}^{N}[\xi + \lambda z_i k_i^a S_i^* - \delta_i(1-\theta(k_i^a))c_i^*(\rho^* - \varepsilon)]e_i^T(t)e_i(t). \tag{4.52}$$

Thus, we can select adequately large constants c_i^* such that $\frac{dV(t)}{dt} \leq 0$, which implies $V(t)$ is non-increasing. It is obvious that the singleton H^* is the largest invariant set of H. By LaSalle's invariance principle, H^* is globally asymptotically stable, that is, for system (4.44), the unique endemic equilibrium E^* of the epidemic spreading system is globally asymptotically stable in Γ°, and the synchronization manifold of the behavioral information diffusion system is also globally asymptotically stable. □

4. Numerical simulations

We consider multiplex networks with N nodes, composed of two subnetworks with which we encode epidemic and behavior information propagation. We generate the epidemic spreading and behavior information networks by the Barabási-Albert (BA) preferential attachment algorithm [11]: Starting with m_0 fully connected nodes, at each time step, a new node is added and connected to m existing nodes in the network with the probability $\prod_i = k_i/\sum_j k_j$, which is a linear preferential attachment strategy. Here, we set $N = 100$, $m_0 = 4, m = 3$ in the epidemic spreading network, $m_0 = 3, m = 3$ in the behavior information network.

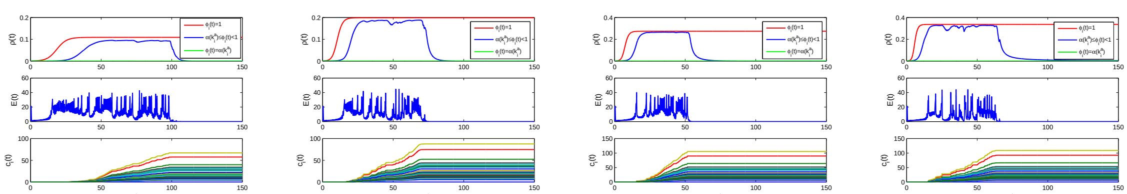
From the point of view of the physical propagation process, it is difficult for us to identify the behavior dynamics of individuals within a community. Without loss of generality, we assume that the local dynamics of the dynamical behavior network are identical and defined as the chaotic Lorenz oscillation. We use a chaotic dynamical system to model the behavioral state of each node as we are primarily looking for synchronisation (consensus of opinion) among nodes in the opinion network. Of course, the choice is arbitrary and other models could be used instead. However, eventual synchronization behaviors are determined by the network architecture and the form of coupling, not the specificities of the underlying dynamical system [90]. Rather than attempting to construct an accurate model of consciousness and opinion of individual agents in the community, we provide a chaotic caricature.

A single Lorenz oscillator, as the desired orbit, can be described by

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix} \begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{pmatrix} + \begin{pmatrix} 0 \\ -x_{i1}x_{i3} \\ x_{i1}x_{i2} \end{pmatrix}, \tag{4.53}$$

which has a chaotic attractor for any initial values.

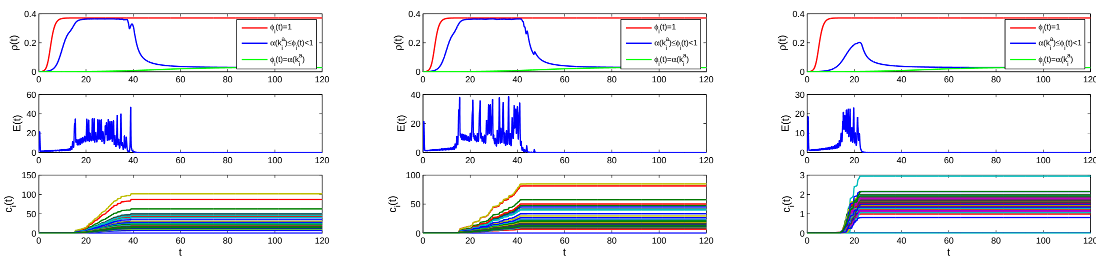
In the following simulations, the inner-coupling matrix is set as $\Gamma = I_3$, the parameter $\beta = 0.001$. The synchronization error is set as $E(t) = \frac{1}{N-1}\sum_{i=2}^{N} \|x_i(t) - x_1(t)\|$. Based on the explanation of the functions $\alpha(x), \theta(x)$ and $\psi(x)$ in Section 2.1, they can be set as $\alpha(x) = \frac{x}{2(1+x)}$, $\theta(x) = \frac{3x}{4(1+x)}$ and $\psi(x) = x$, respectively. For the epidemic control system (4.44), the number of controlling nodes is selected as $l = 20$, the feedback gain matrix is chosen as $d_i = \mathrm{diag}(20000, 30000, 40000)$ for $i = 1, 2, \cdots, 20$. The initial infection probability $\rho_1(0) = \rho_2(0) = 0.01, \rho_i(0) = 0, i \neq 1, 2$, the initial values of the state variables are chosen randomly in $[0, 1]$ with uniform distribution, and the initial coupling weights $c_i(0) = 0.001, i = 1, 2, \cdots, 100$.



(a) (b) (c) (d)

Figure 2. The changes of the infection prevalence $\rho(t)$, synchronization error $E(t) = \frac{1}{N-1}\sum_{i=2}^{N} \|x_i(t) - x_1(t)\|$, and coupling weights $c_i(t)$ when $\tilde{\lambda}_c < \lambda \leq \bar{\lambda}_c$ for system (4.7) with $\tilde{\lambda}_c = 0.1040, \bar{\lambda}_c = 0.2305$. (a) $\lambda = 0.1356, \bar{R}_0 = 0.5884, \tilde{R}_0 = 1.3040$; (b) $\lambda = 0.1672, \bar{R}_0 = 0.7256, \tilde{R}_0 = 1.6080$; (c) $\lambda = 0.1989, \bar{R}_0 = 0.8628, \tilde{R}_0 = 1.9121$; (d) $\lambda = 0.2305, \bar{R}_0 = 1, \tilde{R}_0 = 2.2161$.

From Fig. 2, we can see that the infection prevalence $\rho(t)$ of system (4.7) increases, then oscillates [46, 98], finally converges to zero and the synchronization error $E(t)$ converges to zero when the transmission rate λ satisfies $\tilde{\lambda}_c < \lambda \leq \bar{\lambda}_c$. This implies that although the infection does not persist, the epidemic dynamics successfully induce the synchronization of individual behaviors when $\tilde{\lambda}_c < \lambda \leq \bar{\lambda}_c$. When $\tilde{\lambda}_c < \lambda < \bar{\lambda}_c$, we can find that the convergence speed of $\rho(t)$ and $E(t)$ to zero for system (4.7) increases, which suggests that the epidemic dynamics not only induce the synchronization of individual behaviors, but also enhance the speed of synchronization. However, when $\lambda = \bar{\lambda}_c$, the speed of synchronization has not been accelerated. Compared with system (4.14) without adaptive reactions (i.e., $\phi_i(t) = 1$), we find that the infection prevalence $\rho(t)$ of system (4.7) with adaptive reactions is always lower than that of system (4.14) without adaptive reactions, indicating that human adaptive behaviors not only slow down the spread of the infection and lower the size of its outbreak, but also prevent it from growing into an epidemic [42]. Furthermore, the larger is the transmission rate λ, the greater is the value of the highest peak of $\rho(t)$ of system (4.7), but cannot exceed the value of the highest peak of $\rho(t)$ of system (4.14).



(a) Both the epidemic spreading and behavior information networks have different BA scale-free network topologies for system (4.7).

(b) Both the epidemic spreading and behavior information networks have the same BA scale-free network topology: $m_0 = 4, m = 3$ for system (4.7) .

(c) Both the epidemic spreading and behavior information networks have different BA scale-free network topologies for system (4.44).

FIGURE 3. The changes of the infection prevalence $\rho(t)$, synchronization error $E(t) = \frac{1}{N-1}\sum_{i=2}^{N} \|x_i(t) - x_1(t)\|$, and coupling weights $c_i(t)$ when $\lambda > \bar{\lambda}_c > \tilde{\lambda}_c$ for systems (4.7) and (4.44) with $\tilde{\lambda}_c = 0.1040, \bar{\lambda}_c = 0.2305$. $\lambda = 0.2505$, $\bar{R}_0 = 1.0868$, $\tilde{R}_0 = 2.4084$.

From Fig. 3(a), we can see that the infection prevalence $\rho(t)$ of system (4.7) reaches a peak, then oscillates, finally converges to the positive state and the synchronization error $E(t)$ converges to zero when the transmission rate λ satisfies $\lambda > \bar{\lambda}_c > \tilde{\lambda}_c$, which implies that the infection becomes endemic and the epidemic dynamics successfully induce the synchronization of individual behaviors with a large transmission rate. In Figs. 3(a) and 3(b), the epidemic level of system (4.7) is significantly lower than that of system (4.14), which means that human adaptive behaviors collectively lower the final incidence of the infection. Under the condition that the other parameters remain the same, through the comparison of Figs. 3(a) and 3(b), we find no essential difference by only changing the behavior information network topology. Interestingly, it seems that the structural difference between the epidemic spreading and behavior information networks accelerates the generation of collective synchronization and the fall of the epidemic. Through the comparison of Figs. 3(a) and 3(c), we find that the behavioral regulation not only enhances the convergence speed of $\rho(t)$ towards the endemic equilibrium and the speed of collective synchronization, but also significantly reduces the value of the highest peak of $\rho(t)$ and the steady-state values of the coupling weights $c_i(t)$, which suggests that our epidemic control strategy from the perspective of behavioral control is very valid.

5. Concluding remarks

In this chapter, we first present a concrete interplay system (4.2) in quenched multiplex networks using a well-known SIS model. We focus on the epidemic threshold, the uniform persistence and global stability of system (4.2). Through the theoretical analysis, we discover that the epidemic threshold $\lambda_c(t)$ of the epidemic system in (4.2) is time varying, with an upper bound $\bar{\lambda}_c$ and a lower bound $\tilde{\lambda}_c$. If the transmission rate λ satisfies $\lambda \leq \tilde{\lambda}_c$, then the disease-free equilibrium E_0 is the unique equilibrium of the epidemic spreading system in system (4.2) and it is globally asymptotically stable in the feasible region Γ. If $\lambda > \bar{\lambda}_c$, then the disease-free equilibrium E_0 is unstable, and the epidemic spreading system in system (4.2) is uniformly persistent and has a unique endemic equilibrium E^*, which is globally asymptotically stable in Γ°. Moreover, the synchronization manifold of the behavioral information diffusion system in system (4.2) is also globally asymptotically stable when $\lambda > \bar{\lambda}_c$ [64, 118]. The numerical results show that although the infection does not persist, the epidemic dynamics successfully induce the synchronization of individual behaviors when $\tilde{\lambda}_c \leq \lambda \leq \bar{\lambda}_c$. The individual adaptive behaviors triggered by the emergence of an epidemic can slow down the spread of the infection, lower the final epidemic size, in some cases (e.g., when $\tilde{\lambda}_c < \lambda \leq \bar{\lambda}_c$), can prevent the infection from becoming widespread. That is, the synchronous behaviors of individuals have a stronger impact on the epidemic threshold and prevalence than asynchronous adaptive behaviors. These results provide us with an alternative idea for understanding why some infections do not cause major outbreaks or reach the epidemic threshold in the absence of immunization policy or territory-wide quarantine/isolation measures [42, 73]. Although the transmission rate λ can determine whether the adaptive behaviors of individuals achieve synchronization, it is not the only factor that determines the speed of synchronization. Interestingly, it seems that the difference between the epidemic spreading and behavior information networks accelerates the generation of collective synchronization and the fall of the epidemic, but cannot change the final size of the infection burden.

To further contain the spread of epidemics, we construct an SIS epidemic control system (4.44) from the perspective of behavioral control. By constructing an appropriate Lyapunov function and applying matrix theory and LaSalle's invariance principle, we obtain the similar results to those of system (4.2). Through numerical comparison, we find that behavioral control not only enhances the speed with which the epidemic tends to become stable and the speed of collective synchronization, but also significantly reduces the value of the highest peak of the infection prevalence and the steady-state values of the coupling weights, which indicates that our epidemic control system does provide a reference basis on which to design the practical quasi-optimal control strategies or policies, and to assess their effectiveness.

Furthermore, our results also provide several useful suggestions for responding to the emergence of new epidemics in future. For individuals themselves, if people in affected areas can quickly take effective measures to protect themselves, then the outbreak of epidemics may be effectively controlled; for governments, mass media, and other society organizations, the effective epidemic and behavioral protection information that they provide can have a significant and substantial impact on emerging epidemics, such as decreasing incidence, or even eventually eradicating them.

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Analyse qualitative de plusieurs types de systèmes de maladies infectieuses avec effets de réaction ou de diffusion

Résumé. Cette thèse étudie quelques problèmes qualitatifs pour les systèmes d'équations différentielles modélisant des maladies infectieuses avec des effets de réaction ou de diffusion. Il se compose en trois parties.

Premièrement, nous étudions un système de réaction-diffusion complex décrivant la propagation spatio-temporelle de la grippe avec de multiples souches. Nous établissons des conditions d'existence d'ondes semi-progressives, progressives fortes et faibles (persistantes) à partir de l'équilibre sans maladie. Nous discutons en outre plusieurs situations dans lesquelles les ondes semi-progressives n'existent pas, et donnent une estimation de la vitesse minimale d'onde. Deuxièmement, nous analysons une classe de systèmes éco-épidémiologiques dans lesquels les proies sont sujettes à l'effet Allee et à l'infection. Pour certains sous-systèmes, nous déterminons l'existence du point de bifurcation (bifurcation Hopf et bifurcation d'orbites hétéroclines). Nous montrons que l'effet Allee fort peut créer une courbe séparatrice (ou une surface), conduisant à une stabilité multiple. Nous trouvons que les cycles hétéroclines forment un réseau hétérocline et identifient une orbite périodique intérieure. Enfin, nous donnons une analyse qualitative de deux systèmes différentiels basés sur le réseau couplant la propagation de l'épidémie et la diffusion de l'information: le système d'interaction et le système de contrôle des épidémies. Plus spécifiquement, nous obtenons l'existence de l'équilibre sans maladie, l'équilibre endémique et la variété de synchronisation, ainsi que leur stabilité asymptotique globale.

Mots clés: Systémes de maladies infectieuses, ondes progressives, vitesse minimale d'onde, réseau hétérocline, orbite périodique intérieure, variété de synchronisation, stabilité globale

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Qualitative analysis of some classes of infectious disease systems with reaction or diffusion effects

Abstract. This thesis studies some qualitative problems for systems of differential equations modeling infectious diseases with reaction or diffusion effects. It consists of three parts.

Firstly, we study a complex reaction-diffusion system describing the spatiotemporal spread of influenza with multiple strains. We establish conditions for the existence of semi-, strong and weak (persistent) traveling waves starting from the disease-free equilibrium. We further discuss several situations in which semi-traveling waves do not exist, and give an estimation of minimal wave speed. Secondly, we analyze a class of eco-epidemiological systems where prey is subject to Allee effect and infection. For certain subsystems, we determine the existence of the bifurcation point (Hopf bifurcation and bifurcation of heteroclinic orbits). We show that the strong Allee effect can create a separatrix curve (or surface), leading to multi-stability. We find that the heteroclinic cycles form a heteroclinic network and identify an interior periodic orbit. Finally, we give a qualitative analysis of two network-based differential systems coupling epidemic spread and information diffusion: the interplay system and the epidemic control system. More specifically, we obtain the existence of the disease-free equilibrium, endemic equilibrium and synchronization manifold, and their global asymptotic stability.

Keywords: Infectious disease systems, traveling wave, minimal wave speed, heteroclinic network, interior periodic orbit, synchronization manifold, global stability