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DOCTORAL THESIS

Discipline: MATHEMATICS

Reflective modular forms and Weyl invariant E_8 Jacobi modular forms

Les formes modulaires réflexives et les formes de Jacobi de $\mathrm{W}(\mathrm{E}_8)$-invariantes

submitted by

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Résumé

Cette thèse comprend deux parties indépendantes.

Dans la première partie, nous développons une approche fondée sur la théorie des formes de Jacobi dont l'indice est un réseau pour classifier les formes modulaires réflexives sur des réseaux de niveau arbitraire. Les formes modulaires réflexives ont des applications en géométrie algébrique, en algèbre de Lie et en arithmétique. La classification des formes modulaires réflexives est un problème ouvert et a été étudiée par Borcherds, Gritsenko, Nikulin, Scheithauer et Ma depuis 1998. Dans cette partie, nous établissons de nouvelles conditions nécessaires à l'existence d'une forme modulaire réflexive. Nous prouvons la nonexistence de formes modulaires réflexives et de formes modulaires 2-réflexives sur des réseaux de grand rang. Nous donnons également une classification complète des formes modulaires 2-réflexives sur des réseaux contenant deux plans hyperboliques.

La deuxième partie est consacrée à l'étude des formes de Jacobi de $W(E_8)$-invariantes. Ce type de formes de Jacobi a une signification dans les variétés de Frobenius, la théorie de Gromov-Witten et la théorie des cordes. En 1992, Wirthmüller a prouvé que l'espace des formes de Jacobi pour tout système de racines irréductible excepté E_8 est une algèbre polynomiale. Très peu de choses sont connues dans le cas de E_8. Dans cette partie, nous montrons que l'anneau bigradué des formes de Jacobi $W(E_8)$-invariantes n'est pas une algèbre polynomiale et prouvons que chacune de ces formes de Jacobi peut être exprimée uniquement sous la forme d'un polynôme en neuf formes de Jacobi algébriquement indépendantes introduites par Sakai avec des coefficients méromorphes $\mathrm{SL}_2(\mathbb{Z})$-modulaires. Ce dernier résultat implique que, à indice fixé, l'espace des formes de Jacobi $W(E_8)$-invariantes est un module libre sur l'anneau des formes $\mathrm{SL}_2(\mathbb{Z})$-modulaires et que le nombre de générateurs peut être calculé via une série génératrice. Nous déterminons et construisons tous les générateurs pour des indices petits. Ces résultats étendent un théorème de type de Chevalley au cas du réseau E_8.

Mots clés

réseaux, formes modulaires de Jacobi, formes modulaires pour la représentation de Weil, formes modulaires pour les groupes orthogonaux, produits de Borcherds, formes modulaires réflexives, formes modulaires 2-réflexives, groupes de réflexion, groupe de Weyl, système de racine E_8, théorie invariante.

Abstract

This thesis consists of two independent parts.

In the first part we develop an approach based on the theory of Jacobi forms of lattice index to classify reflective modular forms on lattices of arbitrary level. Reflective modular forms have applications in algebraic geometry, Lie algebra and arithmetic. The classification of reflective modular forms is an open problem and has been investigated by Borcherds, Gritsenko, Nikulin, Scheithauer and Ma since 1998. In this part, we establish new necessary conditions for the existence of a reflective modular form. We prove the nonexistence of reflective modular forms and 2-reflective modular forms on lattices of large rank. We also give a complete classification of 2-reflective modular forms on lattices containing two hyperbolic planes.

The second part is devoted to the study of Weyl invariant E_8 Jacobi forms. This type of Jacobi forms has significance in Frobenius manifolds, Gromov–Witten theory and string theory. In 1992, Wirthmüller proved that the space of Jacobi forms for any irreducible root system not of type E_8 is a polynomial algebra. But very little has been known about the case of E_8. In this paper we show that the bigraded ring of Weyl invariant E_8 Jacobi forms is not a polynomial algebra and prove that every such Jacobi form can be expressed uniquely as a polynomial in nine algebraically independent Jacobi forms introduced by Sakai with coefficients which are meromorphic $\mathrm{SL}_2(\mathbb{Z})$ modular forms. The latter result implies that the space of Weyl invariant E_8 Jacobi forms of fixed index is a free module over the ring of $\mathrm{SL}_2(\mathbb{Z})$ modular forms and that the number of generators can be calculated by a generating series. We determine and construct all generators of small index. These results give a proper extension of the Chevalley type theorem to the case of E_8.

Keywords

lattices, Jacobi modular forms, modular forms for the Weil representation, modular forms on orthogonal groups, Borcherds products, reflective modular forms, 2-reflective modular forms, reflection groups, Weyl group, root system E_8, invariant theory.

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Contents

Notations

$\mathbb{C}$	the set of complex numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Z}$	the set of integers
$\mathbb{N}$	the set of natural numbers i.e. 0, 1, 2, ...
$e(x)$	$\exp(2\pi i x), x \in \mathbb{C}$
$M^\vee$	the dual lattice of the lattice M
$D(M)$	the discriminant group of M
$M(a)$	the rescaling of the lattice M by a
$\mathrm{rank}(L)$	the rank of the lattice L
$\mathrm{div}(x)$	the divisor of $x \in M : (x, M) = \mathrm{div}(x)\mathbb{Z}$
U	the hyperbolic plane: the even unimodular lattice of signature $(1,1)$
$\mathrm{sign}(D)$	the signature (i.e. $r - s$) of the discriminant form D
$\mathrm{O}(M)$	the integral orthogonal group of the lattice M
$\mathrm{O}(D)$	the orthogonal group of the discriminant form D
$\widetilde{\mathrm{O}}^+(M)$	the stable orthogonal group of M acting trivially on $D(M)$
$\mathcal{D}(M)$	the Hermitian symmetric domain of type IV
$\mathcal{D}_v(M)$	the rational quadratic divisor associated to the vector v
R_L	the set of 2-roots of L
$\mathrm{II}_{r,s}(D)$	the genus of lattice
ρ_D	the Weil representation
$H(L)$	the integral Heisenberg group associated to L
$J^{w.h.}_{k,L,t}$	the space of weakly holomorphic Jacobi forms for L
$J^{w}_{k,L,t}$	the space of weak Jacobi forms of weight k and index t for L
$J_{k,L,t}$	the space of holomorphic Jacobi forms for L
$J^{cusp}_{k,L,t}$	the space of Jacobi cusp forms of weight k and index t for L
$\eta(\tau)$	the Dedekind η-function
υ_η	the multiplier system of the Dedekind η-function
$\vartheta(\tau, z)$	the odd Jacobi theta function, see Example 1.4.5
$\Gamma_0(n)$	the Hecke congruence subgroup of level n

$\Delta(\tau)$	the normalized cusp form of weight 12
M_*	the ring of $\mathrm{SL}_2(\mathbb{Z})$ modular forms
$E_4(\tau)$	the Eisenstein series of weight 4
$E_6(\tau)$	the Eisenstein series of weight 6
$H_k(\cdot)$	the weight raising differential operator
$T_-(m)$	the index raising Hecke operator
$\mathrm{Grit}(\phi)$	the additive lifting of ϕ
$\mathrm{Borch}(\psi)$	the Borcherds product of ψ
σ_r	the reflection associated to the vector r
$\mathcal{H}(\lambda, m)$	the Heegner divisor of discriminant (λ, m), see Eq 2.1.2
$\mathcal{H}$	(-2)-Heegner divisor, see Eq 2.1.3
Φ_{12}	the Borcherds modular form of weight 12
$N(R)$	the Niemeier lattice with root system R
$[i]$	the elements of discriminant groups, see section 2.3.1
$\mathrm{II}_{2,2+8n}$	the even unimodular lattice of signature $(2, 2+8n)$
$\mathcal{H}_0, \mathcal{H}_\mu$	see section 2.4
$\mathrm{Sing}(\phi)$	the singular Fourier coefficients of the Jacobi form ϕ
$\mathrm{div}(F)$	the zero divisor of the modular form F
N_8	the Nikulin's lattice, see Remark 2.3.2
Norm_2	the condition Norm_2, see Equation (2.3.2)
A_n	the root lattice (root system) of type A_n
D_n	the root lattice (root system) of type D_n
E_n	the root lattice (root system) of type E_n, $n = 6, 7, 8$
$W(E_8)$	the Weyl group of the root system E_8
$\mathrm{orb}(m)$	the Weyl orbit associated to $m \in E_8$, see Equation 3.2.3
ϑ_{E_8}	the Jacobi theta function for the root lattice E_8
$\sum_{2i}$	the Weyl orbits of norm $2i$, see section 3.4.1
$J_{k,E_8,t}^{\mathrm{w},W(E_8)}$	the space of Weyl invariant E_8 weak Jacobi forms
$J_{k,E_8,t}^{W(E_8)}$	the space of Weyl invariant E_8 holomorphic Jacobi forms
$J_{k,E_8,t}^{\mathrm{cusp},W(E_8)}$	the space of Weyl invariant E_8 Jacobi cusp forms

Introduction

Modular forms are everywhere in mathematics and have even turned up in the study of theoretical physics recently. Jacobi modular forms are an elegant intermediate between different types of modular forms. The arithmetic theory of Jacobi forms was first systematically studied in the 1980s in Eichler and Zagier's monograph, [EZ85]. One example of Jacobi forms is the first Fourier–Jacobi coefficient of a Siegel modular form of genus 2. In the 1990s, Gritsenko generalized this fact and defined Jacobi forms of lattice index as modular forms with respect to a parabolic subgroup of an orthogonal group of signature $(2, n)$ (see [Gri91]). Gritsenko also constructed the additive Jacobi lifting which lifts holomorphic Jacobi forms to holomorphic modular forms on orthogonal groups, which is an advanced version of Maass lift (see [Gri95]). In his celebrated work, Borcherds discovered a multiplicative lifting (Borcherds product) which maps modular forms for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ to meromorphic orthogonal modular forms (see [Bor95, Bor98]). There is an isomorphism between modular forms for the Weil representation and Jacobi forms. One can also construct Borcherds products from Jacobi forms (see [GN98b]). Thus Jacobi forms are a bridge between vector-valued modular forms and orthogonal modular forms. In recent years, Jacobi forms become more and more popular due to their application in algebra and geometry and their connection to string theory. They determine Lorentzian Kac–Moody Lie algebras of Borcherds type (see [GN97, GN98a]) and some Jacobi forms of weight 0 can be viewed as the elliptic genus of Calabi–Yau manifolds (see [BL00, Gri99, Gri99b, Tot00]). Some interesting partition functions in geometry and physics can be expressed in terms of certain Jacobi forms (see [MNV+98, Moh02]).

This Ph.D. thesis investigates two types of modular forms, reflective modular forms and Weyl invariant E_8 Jacobi forms. The theory of Jacobi forms plays a crucial role in our work. We use the Jacobi forms approach to classify reflective modular forms and study the ring of Jacobi forms for the remarkable unimodular lattice E_8.

Part I. Reflective modular forms

Let M be an even integral lattice of signature $(2, n)$ with $n \geq 3$ and $M^\vee$ be its dual lattice. We choose one of the two connected components of the Hermitian symmetric domain of type IV

$$\mathcal{D}(M) = \{[\mathcal{Z}] \in \mathbb{P}(M \otimes \mathbb{C}) : (\mathcal{Z}, \mathcal{Z}) = 0, (\mathcal{Z}, \bar{\mathcal{Z}}) > 0\}^+.$$

Let $\mathrm{O}^+(M)$ denote the subgroup of the orthogonal group $\mathrm{O}(M)$ preserving $\mathcal{D}(M)$. Let $\widetilde{\mathrm{O}}^+(M) < \mathrm{O}^+(M)$ denote the subgroup acting trivially on the discriminant form $D(M)$.

A holomorphic function F on the affine cone $\mathcal{D}(M)^\bullet$ is called a modular form of weight $k \in \mathbb{Z}$ and character χ with respect to a finite index subgroup $\Gamma < \mathrm{O}^+(M)$ if

$$\begin{aligned} F(t\mathcal{Z}) &= t^{-k} F(\mathcal{Z}), \quad \forall t \in \mathbb{C}^*, \\ F(g\mathcal{Z}) &= \chi(g) F(\mathcal{Z}), \quad \forall g \in \Gamma. \end{aligned}$$

The theory of automorphic Borcherds products (see [Bor95, Bor98]) provides a great way to construct modular forms with rational quadratic divisors of the form

$$\mathcal{D}_v(M) = \{[\mathcal{Z}] \in \mathcal{D}(M) : (\mathcal{Z}, v) = 0\}.$$

A non-constant holomorphic modular form with special rational quadratic divisors associated to reflective (resp. 2-reflective) vectors is called reflective (resp. 2-reflective). Here a reflective vector is a primitive vector $r \in M$ with $(r,r) < 0$ such that the reflection

$$\sigma_r : l \mapsto l - \frac{2(l,r)}{(r,r)}r$$

is in $\mathrm{O}^+(M)$. A primitive vector $r \in M$ is called 2-reflective if $(r,r) = -2$. The definition implies that 2-reflective modular forms are a particular class of reflective modular forms. Note that 2-reflective modular forms have the geometric interpretation as automorphic discriminants of moduli of K3 surfaces (see [Nik96]). A lattice is called (2-)reflective if it admits a (2-)reflective modular form.

The notion and examples of reflective modular forms first appeared in the works of Borcherds [Bor98] and Gritsenko-Nikulin [GN98a]. Reflective modular forms are usually Borcherds products of some vector-valued modular forms (see [Bru02, Bru14]). The Igusa form Δ_{10}, namely the first cusp form for the Siegel modular group of genus 2, is the first such modular form (see [GN97]). The Borcherds form Φ_{12} for $\mathrm{II}_{2,26}$ the even unimodular lattice of signature $(2,26)$ is the last such modular form (see [Bor95]).

Reflective modular forms are of great importance. Such modular forms play a crucial role in classifying interesting Lorentzian Kac-Moody algebras, as their denominator identities are usually reflective modular forms (see [GN98a, GN98b, GN02, GN18, Sch04, Sch06]). This type of modular forms also has applications in algebraic geometry, as the existence of a particular reflective modular form determines the Kodaira dimension of the corresponding modular variety (see [GHS07, GH14, Ma18]). In addition, reflective modular forms are beneficial to the research of hyperbolic reflection groups and hyperbolic reflective lattices (see [Bor98, Bor00]), as the existence of a reflective modular form with a Weyl vector of positive norm indicates that the hyperbolic lattice is reflective. It means that the subgroup generated by reflections has finite index in the integral orthogonal group of the lattice. Recently, as joint work with Gritsenko, we use the pull-backs of certain reflective modular forms of singular weight to build infinite families of remarkable Siegel paramodular forms of weights 2 and 3 (see [GW17, GW18b, GW19a]). Besides, the first Fourier-Jacobi coefficients of reflective modular forms give interesting holomorphic theta blocks (see [GSZ18, Gri18]).

The classification of reflective modular forms is an old open problem since 1998 when Gritsenko and Nikulin first conjectured that the number of lattices having reflective modular forms is finite in [GN98a]. This problem has been widely studied by several mathematicians and many classification results have been obtained. Borcherds [Bor00] constructed many interesting reflective modular forms related to extraodinary hyperbolic groups as Borcherds products of nearly holomorphic modular forms on congruence subgroups. Gritsenko and Nikulin [GN02] classified reflective modular forms of signature $(2,3)$ by means of the classification of hyperbolic reflective lattices. Scheithauer classified some special reflective modular forms with norm 0 Weyl vectors. More precisely, based on the theory of vector-valued modular forms, he found a necessary condition for the existence of a reflective form in [Sch06]. Using this condition, the classification of strongly reflective modular forms of singular weight on lattices of squarefree level is almost completed (see [Sch17, Dit18]). From an algebraic geometry approach, Ma derived finiteness of lattices admitting 2-reflective modular forms and reflective modular forms of bounded vanishing order, which proved partly the conjecture of Gritsenko and Nikulin (see [Ma17, Ma18]).

Scheithauer's condition is very hard to use when the lattice is not of squarefree level because in this case the Fourier coefficients of vector-valued Eisenstein series are very complicated and it is difficult to characterize the discriminant form of the lattice. Ma's approach is useless to give the list of reflective lattices because his estimate is rather rough. There is no efficient way to classify reflective modular forms on general lattices. In this thesis we shall classify reflective lattices of arbitrary level.

In order to do this, the theory of Jacobi forms of lattice index (see [EZ85, CG13]) might be very useful. We know from [Bru14] that every reflective modular form on a lattice of type $U \oplus U(m) \oplus L$ is in fact a Borcherds product of a suitable vector-valued modular form. Thus the existence of a reflective modular form is determined by the existence of a certain vector-valued modular form. In view of the isomorphism between vector-valued modular forms and Jacobi forms, we can use Jacobi forms to study reflective modular forms. In some sense, Jacobi forms are more powerful than vector-valued modular forms. We can take the product and tensor product of different Jacobi forms. We can also consider the pull-backs of Jacobi forms from a certain lattice to its sublattices. There are the Hecke type operators to raise the index of Jacobi forms and the differential operators to raise the weight of Jacobi forms. The structure of the space of Jacobi forms for some familiar lattices was known (see [Wir92] for the case of root systems). Besides, we usually focus on the genus of a lattice when we use vector-valued modular forms. But we will see all the faces of a reflective modular form when we work with Jacobi forms, because there are different Jacobi forms on the expansions of an orthogonal modular form at different one-dimensional cusps. For example, the Borcherds modular form Φ_{12} is constructed as the Borcherds product of the inverse of Ramanujan Delta function $\Delta^{-1}(\tau) = q^{-1} + 24 + O(q)$. But in the context of Jacobi forms, there are 24 different constructions of this modular form corresponding to 24 classes of positive-definite even unimodular lattices of rank 24 (see [Gri18]).

The following is our first main theorem of this part, which gives a complete classification of 2-reflective lattices of large rank.

Theorem 0.1 (see Theorem 2.6.8)**.** *Let M be a 2-reflective lattice of signature $(2, n)$ with $n \geq 14$. Then it is isomorphic to $\mathrm{II}_{2,18}$, or $2U \oplus 2E_8(-1) \oplus A_1(-1)$, or $\mathrm{II}_{2,26}$.*

We have mentioned that there is a relation between hyperbolic 2-reflective lattices and 2-reflective modular forms. The full classification of hyperbolic 2-reflective lattices was known due to the work of Nikulin and Vinberg [Nik81, Nik84, Vin07]. Vinberg [Vin72] proved that if $U \oplus L(-1)$ is a hyperbolic 2-reflective lattice then the set of 2-roots of each lattice in the genus of L generates the whole space $L \otimes \mathbb{R}$. In this paper, we prove an analogue of Vinberg's result (see Theorem 2.6.1) and use it to give a complete classification of 2-reflective lattices.

Theorem 0.2 (see Theorem 2.6.9)**.** *There are only three types of 2-reflective lattices containing two hyperbolic planes:*

(a) $\mathrm{II}_{2,26}$*;*

(b) $2U \oplus L(-1)$ *: every lattice in the genus of L has no 2-root. In this case, the corresponding 2-reflective modular form has a Weyl vector of norm zero and has weight $12\beta_0$, where β_0 is the multiplicity of the principal Heegner divisor $\mathcal{H}_0$;*

(c) $2U \oplus L(-1)$ *: every lattice in the genus of L has 2-roots and the 2-roots generate a sublattice* *of the same rank. In this case, L is in the genus of one of the following 50 lattices*

n	L
3	A_1
4	$2A_1$, A_2
5	$3A_1$, $A_1 \oplus A_2$, A_3
6	$4A_1$, $2A_1 \oplus A_2$, $A_1 \oplus A_3$, A_4, D_4, $2A_2$
7	$5A_1$, $2A_1 \oplus A_3$, $A_1 \oplus 2A_2$, $A_1 \oplus A_4$, $A_1 \oplus D_4$, A_5, D_5
8	$6A_1$, $2A_1 \oplus D_4$, $A_1 \oplus A_5$, $A_1 \oplus D_5$, E_6, $3A_2$, $2A_3$, A_6, D_6
9	$7A_1$, $3A_1 \oplus D_4$, $A_1 \oplus D_6$, $A_1 \oplus E_6$, E_7, A_7, D_7
10	$8A_1$, $4A_1 \oplus D_4$, $2A_1 \oplus D_6$, $A_1 \oplus E_7$, E_8, $2D_4$, D_8, N_8
11	$5A_1 \oplus D_4$, $A_1 \oplus 2D_4$, $A_1 \oplus D_8$, $A_1 \oplus E_8$
12	$2A_1 \oplus E_8$
18	$2E_8$
19	$2E_8 \oplus A_1$

Note that $5A_1 \oplus D_4$ and $A_1 \oplus N_8$, $A_1 \oplus 2D_4$ and $3A_1 \oplus D_6$, $A_1 \oplus D_8$ and $2A_1 \oplus E_7$, $2A_1 \oplus E_8$ and D_{10} are in the same genus, respectively. Here, $N_8 \cong D_8^\vee(2)$ is the Nikulin lattice. Moreover, every lattice has a 2-reflective modular form with a positive norm Weyl vector. Thus, every associated Lorentzian lattice $U \oplus L(-1)$ is hyperbolic 2-reflective.

The above (c) characterizes the 2-reflective lattices giving arithmetic hyperbolic 2-reflective groups. By this characterization, we further prove that there are exactly 18 hyperbolic 2-reflective lattices of rank bigger than 5 not associated to 2-reflective modular forms (see Theorem 2.6.17), which gives a negative answer of [Bor98, Problem 16.1]. It now remains to classify 2-reflective lattices of type (b). We conjecture that this type of lattices might be viewed as sublattices of Leech lattice. It seems very difficult to classify such lattices because they correspond to hyperbolic parabolically 2-reflective lattices (see [Bor00, GN18]) whose full classification is unknown.

As a corollary of the above theorems, we figure out the classification of 2-reflective modular forms of singular weight.

Theorem 0.3 (see Theorem 2.6.12)**.** *If $M = 2U \oplus L(-1)$ has a 2-reflective modular form of singular weight, then L is in the genus of $3E_8$ or $4A_1$.*

We now explain the main idea of the proof. Our proof is based on manipulation of Jacobi forms and independent of the work of Nikulin and Vinberg on the classification of hyperbolic 2-reflective lattices. Suppose that M contains two hyperbolic planes i.e. $M = 2U \oplus L(-1)$, and F is a 2-reflective modular form for $\widetilde{\mathrm{O}}^+(M)$. The divisor of F consists of the following two types of reflective divisors

$$\mathcal{H}_0 = \bigcup_{\substack{l \in M, l^2=-2 \\ \mathrm{div}(l)=1}} \mathcal{D}_l(M) \qquad \text{and} \qquad \mathcal{H}_\mu = \bigcup_{\substack{l \in M^\vee, l^2=-\frac{1}{2} \\ l \in \mu + M}} \mathcal{D}_l(M),$$

where μ is an element of order 2 and norm $-1/2$ in the discriminant form of M. The integer $\mathrm{div}(l)$ is the natural number generating the ideal (l, M). The existence of F implies that there exists a weakly holomorphic Jacobi form $\phi_{0,L}$ of weight 0 for the lattice L. The divisor of F determines the singular Fourier coefficients of $\phi_{0,L}$. Here the singular Fourier coefficients are its Fourier coefficients of type $f(n,\ell)$ with negative hyperbolic norm $2n - (\ell,\ell) < 0$. The singular Fourier coefficients of $\phi_{0,L}$ have hyperbolic norm -2 or $-1/2$.

There is a differential operator which maps a Jacobi form of weight k to a Jacobi form of weight $k+2$ (see Lemma 1.4.11). Using this operator, we can construct Jacobi forms of weight 2, 4, 6 which have the same type of singular Fourier coefficients as $\phi_{0,L}$. Taking their combinations to cancel the singular Fourier coefficients, we can construct several holomorphic Jacobi forms of low weights, whose existence yields the nonexistence of 2-reflective modular forms on lattices of large rank due to the singular weight. This proves one part of the first theorem.

Through more refined analysis, we find that all Fourier coefficients in q^{-1} and q^0 terms of $\phi_{0,L}$ are singular except the constant term $f(0,0)$ giving the weight of F. The remarkable thing is that the coefficients in q^n-terms ($n \leq 0$) of any Jacobi form of weight 0 satisfy certain relations (see Lemma 1.4.12)

$$C := \frac{1}{24}\sum_{\ell\in L^\vee} f(0,\ell) - \sum_{n<0}\sum_{\ell\in L^\vee} f(n,\ell)\sigma_1(-n) = \frac{1}{2\operatorname{rank}(L)}\sum_{\ell\in L^\vee} f(0,\ell)(\ell,\ell)$$
$$\sum_{\ell\in L^\vee} f(0,\ell)(\ell,\mathfrak{z})^2 = 2C(\mathfrak{z},\mathfrak{z}), \quad \forall \mathfrak{z}\in L\otimes\mathbb{C}.$$

From the first identity we deduce a formula to express the weight of F in terms of the multiplicities of the irreducible components of the divisor of F. From the second identity we derive that if L has 2-roots then the set of all 2-roots spans the whole space $L\otimes\mathbb{R}$. Moreover, all irreducible root components not of type A_1 have the same Coxeter number (see Theorem 2.6.1). Furthermore, the q^0-term of F also defines a holomorphic Jacobi form as a theta block

$$\eta(\tau)^{f(0,0)}\prod_{\ell>0}\left(\frac{\vartheta(\tau,(\ell,\mathfrak{z}))}{\eta(\tau)}\right)^{f(0,\ell)}.$$

From its holomorphicity we also deduce a necessary condition. The 2-reflective modular forms for lattices listed in statement (c) can be constructed as quasi pull-backs of the Borcherds modular form Φ_{12} (see §2.3.1). For other lattices, the quasi pull-backs of Φ_{12} are not exactly 2-reflective modular forms and usually have additional divisors. But this is not bad. By considering its difference with the assumed 2-reflective modular form, we construct some Jacobi forms whose nonexistence can be proved by the structure of the space of Jacobi forms. Combining these arguments together, the theorems can be proved.

We also use the similar argument to classify reflective modular forms. We prove the following nonexistence results.

Theorem 0.4 (see Theorem 2.5.9). *There is no any reflective lattice of signature $(2,n)$ with $23\leq n\leq 25$.*

Theorem 0.5 (see Theorem 2.7.1). *The lattice $T_n = 2U\oplus 2E_8(-1)\oplus\langle -2n\rangle$ is reflective if and only if $n = 1,\ 2$.*

The lattice T_n is related to the moduli space of K3 surfaces. Looijenga [Loo03] proved that the lattice T_n is 2-reflective if and only if $n = 1$ which answered a question of Nikulin [Nik96]. Our theorem gives an extension of Looijenga's result.

Part II. Weyl invariant E_8 Jacoi forms

For the lattice constructed from the classical root system R, one can define the Jacobi forms which are invariant with respect to the Weyl group $W(R)$. Such Jacobi forms are called $W(R)$-invariant Jacobi forms. In this setting, the classical Jacobi forms due to Eichler and Zagier [EZ85] are actually the $W(A_1)$-invariant Jacobi forms. All $W(R)$-invariant weak Jacobi forms make up a bigraded ring graded by the weight and the index. The problem on the algebraic structure of such bigraded ring was inspired by the work of E. Looijenga [Loo76, Loo80] and K. Saito [Sai90] on the invariants of generalized root systems. This problem is closely related to the theory of Frobenius manifolds (see [Ber00a, Ber00b, Dub96, Dub98, Sat98]). The first solution to this problem was given in 1992 by Wirthmüller [Wir92], which stated that the bigraded ring of $W(R)$-invariant weak Jacobi forms is a polynomial algebra over $\mathbb{C}$ except for the root system E_8. This result can be regarded as the Chevalley type theorem for affine root systems (see [BS06]). But Wirthmüller's solution was totally non-constructive because it did not give the construction of generators like in the case of A_1 considered in the book [EZ85] of Eichler–Zagier. On account of this defect, later Bertola [Ber99] and Satake [Sat98] reconsidered this problem and found explicit constructions of the generators for the root systems A_n, B_n, G_2, D_4 and E_6.

The case $R = E_8$ was not covered by Wirthmüller's theorem and remains completely open for 27 years. In recent years, $W(E_8)$-invariant Jacobi forms appear in various contexts in mathematics and physics and have applications in Gromov–Witten theory and string theory. The original Seiberg-Witten curve for the E-string theory is expressed in terms of $W(E_8)$-invariant Jacobi forms [Sak17a]. The Gromov-Witten partition function $Z_{g;n}(\tau|\mu)$ of genus g and winding number n of the local B_9 model is a $W(E_8)$-invariant quasi-Jacobi form of index n and weight $2g-6+2n$ up to a factor [Moh02, MNV+98]. The $\mathbb{P}_1$-relative Gromov-Witten potentials of the rational elliptic surface are $W(E_8)$-invariant quasi-Jacobi forms numerically and the Gromov-Witten potentials of the Schoen Calabi-Yau threefold (relative to $\mathbb{P}_1$) are $E_8\times E_8$ quasi-bi-Jacobi forms [OP17]. But unfortunately, very little has been known about the ring of $W(E_8)$-invariant Jacobi forms. The purpose of this paper is to study the space of $W(E_8)$-invariant Jacobi forms.

Let $\tau\in\mathbb{H}$ and $\mathfrak{z}\in E_8\otimes\mathbb{C}$. A holomorphic function $\varphi(\tau,\mathfrak{z})$ is called a $W(E_8)$-invariant weak Jacobi form of weight k and index t if it satisfies the following transformations under the action of the Jacobi group which is the semidirect product of $\mathrm{SL}_2(\mathbb{Z})$ with the integral Heisenberg group of E_8

$$\varphi\left(\frac{a\tau+b}{c\tau+d},\frac{\mathfrak{z}}{c\tau+d}\right)=(c\tau+d)^k\exp\left(t\pi i\frac{c(\mathfrak{z},\mathfrak{z})}{c\tau+d}\right)\varphi(\tau,\mathfrak{z}),\quad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}),$$
$$\varphi(\tau,\mathfrak{z}+x\tau+y)=\exp\left(-t\pi i[(x,x)\tau+2(x,\mathfrak{z})]\right)\varphi(\tau,\mathfrak{z}),\quad x,y\in E_8,$$

if it admits a Fourier expansion of the form

$$\varphi(\tau,\mathfrak{z})=\sum_{n\in\mathbb{N}}\sum_{\ell\in E_8}f(n,\ell)e^{2\pi i(n\tau+(\ell,\mathfrak{z}))},$$

and if it is invariant under the Weyl group i.e. $\varphi(\tau,\sigma(\mathfrak{z}))=\varphi(\tau,\mathfrak{z})$ for all $\sigma\in W(E_8)$.

Sakai [Sak17a] constructed nine independent $W(E_8)$-invariant holomorphic Jacobi forms, noted by A_1, A_2, A_3, A_4, A_5, B_2, B_3, B_4, B_6. The Jacobi forms A_m, B_m are of weight 4, 6 and index m respectively. Sakai [Sak17b] also conjectured that the number of generators of $W(E_8)$-invariant Jacobi forms of index m coincides with the number of fundamental representations at level m. In this thesis we give an explicit mathematical description of his conjecture and prove it to be true. We first study the values of $W(E_8)$-invariant Jacobi forms at $q=0$, which are called q^0-terms of Jacobi forms. We prove that the q^0-term of any $W(E_8)$-invariant Jacobi form can be written as a particular polynomial in terms of the Weyl orbits of eight fundamental weights of E_8 (see Lemma 3.3.2). We then deduce our first main theorem of this part, which gives an extension of Wirthmüller's theorem to the case of E_8.

Theorem 0.6 (see Theorem 3.3.1). *Let t be a positive integer. Then the space $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ of $W(E_8)$-invariant weak Jacobi forms of index t is a free module of rank $r(t)$ over the ring M_* of $\mathrm{SL}_2(\mathbb{Z})$ modular forms, where $r(t)$ is given by*

$$\frac{1}{(1-x)(1-x^2)^2(1-x^3)^2(1-x^4)^2(1-x^5)(1-x^6)}=\sum_{t\geq 0}r(t)x^t.$$

Equivalently, we have that

$$J^{\mathrm{w},W(E_8)}_{*,E_8,*}=\bigoplus_{t\geq 0}J^{\mathrm{w},W(E_8)}_{*,E_8,t}\subsetneq\mathbb{C}\left(E_4,E_6\right)\left[A_1,A_2,A_3,A_4,A_5,B_2,B_3,B_4,B_6\right],$$

and that A_1, A_2, A_3, A_4, A_5, B_2, B_3, B_4, B_6 *are algebraically independent over* M_*. *Here* $\mathbb{C}(E_4, E_6)$ *denotes the fractional field of* $\mathbb{C}[E_4, E_6]$.

It is well-known that the space $J^{\mathrm{w},W(E_8)}_{*,E_8,1}$ is generated over M_* by the theta function of the root lattice E_8. In this thesis we further elucidate the structure of $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ for $t = 2, 3, 4$. We next explain the main ideas.

Our main theorem says that the space $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ is a free module over M_* and the number of generators is $r(t)$. It means that we only need to find all generators in order to characterize the structure of $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$. We do this by analyzing the spaces of weak Jacobi forms of fixed index and non-positive weight.

We first construct many basic Jacobi forms of non-positive weight in terms of Jacobi theta functions. The main tool is the weight raising differential operators (see Lemma 3.2.4).

We then determine the dimension of the space of Jacobi forms of fixed index and any given negative weight. To do this, we study the orbits of E_8 vectors of fixed norm under the action of the Weyl group and represent q^0-terms of Jacobi forms as linear combinations of these orbits. By means of the following two crucial facts,

1. From a given Jacobi form of weight k, we can construct a Jacobi form of weight $k + 2j$ for every positive integer j by the differential operators.

2. If one takes $\mathfrak{z} = 0$, then q^0-term of any Jacobi form of negative weight will be zero. In addition, coefficients of q^0-term of any Jacobi form of weight zero satisfy a linear relation (see Lemma 3.2.5).

from a Jacobi form of negative weight with given q^0-term, we can build a certain system of linear equations defined by the coefficients of the orbits of E_8 vectors in q^0-terms of Jacobi forms. By solving the linear system, we can know if the Jacobi form of negative weight with given q^0-term exists.

The following theorem describes the structure of $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ for $t = 2, 3, 4$.

Theorem 0.7 (see Theorems 3.4.7, 3.4.10, 3.4.13)**.**

$$\begin{aligned}
J^{\mathrm{w},W(E_8)}_{*,E_8,2} &= M_*\langle \varphi_{-4,2}, \varphi_{-2,2}, \varphi_{0,2}\rangle,\\
J^{\mathrm{w},W(E_8)}_{*,E_8,3} &= M_*\langle \varphi_{-8,3}, \varphi_{-6,3}, \varphi_{-4,3}, \varphi_{-2,3}, \varphi_{0,3}\rangle,\\
J^{\mathrm{w},W(E_8)}_{*,E_8,4} &= M_*\langle \varphi_{-2k,4},\ 0 \le k \le 8;\ \psi_{-8,4}\rangle,
\end{aligned}$$

where $\varphi_{k,t}$ *is a* $W(E_8)$*-invariant weak Jacobi form of weight* k *and index* t.

Combining the above two theorems together, we prove that the bigraded ring $J^{\mathrm{w},W(E_8)}_{*,E_8,*}$ is in fact not a polynomial algebra over M_* and its structure is rather complicated (see Theorem 3.4.8). It means that the Chevalley type theorem does not hold for E_8.

Furthermore, we apply our results about weak Jacobi forms to determining the structures of the modules of $W(E_8)$-invariant holomorphic and cusp Jacobi forms of index 2, 3, 4.

Theorem 0.8 (see Theorems 3.4.7, 3.4.11, 3.4.14)**.**

$$\begin{aligned}
J^{W(E_8)}_{*,E_8,2} &= M_*\langle A_2, B_2, A_1^2\rangle,\\
J^{W(E_8)}_{*,E_8,3} &= M_*\langle A_3, B_3, A_1A_2, A_1B_2, A_1^3\rangle,
\end{aligned}$$

and $J^{W(E_8)}_{*,E_8,4}$ *is generated over* M_* *by two Jacobi forms of weight* 4*, two Jacobi forms of weight* 6*, three Jacobi forms of weight* 8*, two Jacobi forms of weight* 10 *and one Jacobi form of weight* 12.

Table 1: Number of generators of $J^{\text{cusp},W(E_8)}_{*,E_8,t}$

weight	8	10	12	14	16
$t = 2$	0	0	1	1	1
$t = 3$	0	1	2	1	1
$t = 4$	1	2	3	2	2

Theorem 0.9 (see Theorems 3.4.7, 3.4.12, 3.4.14)**.** *Let* $t = 2$, 3 *or* 4. *The numbers of generators of indicated weight of* $J^{\text{cusp},W(E_8)}_{*,E_8,t}$ *are shown in Table 1.*

We also present two isomorphisms between spaces of weak Jacobi forms for lattices of different types, which give new descriptions of $J^{\text{w},W(E_8)}_{*,E_8,2}$ and $J^{\text{w},W(E_8)}_{*,E_8,3}$ (see §3.4.5). Besides, we develop an approach based on pull-backs of E_8 Jacobi forms into classical Jacobi forms to discuss the possible minimum weight of the generators of $J^{\text{w},W(E_8)}_{*,E_8,t}$ for $t = 5$ and 6 (see §3.4.6 and §3.4.7). As an application, we estimate the dimension of the space of modular forms for the orthogonal group $\mathrm{O}^+(2U \oplus E_8(-1))$ and give it an upper bound using our theory of $W(E_8)$-invariant Jacobi forms. This upper bound almost coincides with the exact dimension obtained by [HU14].

The thesis is principally based on the following three works by the author.

(1) *Reflective modular forms: A Jacobi forms approach.* arXiv:1801.09590, accepted for publication in **Int. Math. Res. Not. IMRN**. (appear in Chapter 2)

(2) *The classification of* 2*-reflective modular forms.* Preprint 2019, 33 pages. (appear in Chapter 2)

(3) *Weyl invariant* E_8 *Jacobi forms.* arXiv: 1801.08462, submitted. (appear in Chapter 3)

Chapter 1

Preliminaries

1.1 Lattices and discriminant forms

In this section we recall some basic results on lattices and discriminant forms. The main references for this material are [Bou60, Ebe02, Nik80, SC98].

1.1.1 Lattices

Let V be a finite-dimensional rational vector space with the nondegenerate symmetric bilinear form $(\cdot,\cdot)$ and the associated quadratic form $Q(x) := (x,x)/2$. A free $\mathbb{Z}$-submodule $M \subseteq V$ is a **lattice** in V if $V = M \otimes_{\mathbb{Z}} \mathbb{Q}$. For every lattice $M \subseteq V$ and every $a \in \mathbb{Z}\backslash\{0\}$, the lattice obtained by **rescaling** M with a is denoted by $M(a)$. It is endowed with the quadratic form $a \cdot Q$ instead of Q. For $x \in M$ and $x \neq 0$, if $Q(x) = 0$ then it is called **isotropic**, otherwise it is called **anisotropic**. If $x \in M$ satisfying $\mathbb{Q}x \cap M = \mathbb{Z}x$, then it is called **primitive**. Let $\mathrm{GL}(V)$ be the group of linear automorphisms of vector space V. The **orthogonal group** of M is defined by

$$\mathrm{O}(M) = \{g \in \mathrm{GL}(V) : Q(g(v)) = Q(v), g(v) \in M, \forall\, v \in M\}.$$

By Sylvester's law of intertia from linear algebra, the quadratic space (V,Q) is isomorphic to a quadratic space $\mathbb{R}^{r+s}$ equipped with the quadratic form defined by

$$q_{r,s}(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2.$$

The pair (r,s) is uniquely determined by (V,Q) and is called the **signature** of V (or M).

A lattice M is called **integral** if $(x,y) \in \mathbb{Z}$ for all $x,y \in M$. It is **even** if (x,x) is even for all $x \in M$. Let $\{e_1,...,e_n\}$ be a basis of M. The symmetric matrix $G = ((e_i,e_j))_{1\leq i,j\leq n}$ is called the **Gram matrix** of M with respect to the basis $\{e_1,...,e_n\}$. Its determinant is independent of the basis and called the **discriminant** of M, denoted by $disc(M)$.

For an integral lattice M, the **dual lattice** of M is the subgroup

$$M^\vee = \{x \in V : (x,y) \in \mathbb{Z}, \forall\, y \in M\}.$$

It is clear that $M^\vee$ is a lattice containing M. Then we can consider the quotient $D(M) := M^\vee/M$, which is called the **discriminant group** of M. The elementary divisors theorem implies that $|M^\vee/M| = |disc(M)|$. If $M^\vee = M$, then it is called **unimodular**.

From now on, we presume that M is an even lattice with quadratic form Q. The **level** of M is the smallest positive integer N such that $NQ(x) \in \mathbb{Z}$ for all $x \in M^\vee$. For any non-zero $x \in M$ the **divisor** of x is the natural number $\mathrm{div}(x)$ defined by $(x,M) = \mathrm{div}(x)\mathbb{Z}$. If M is of level N then $NM^\vee \subseteq M$. An embedding $M_1 \hookrightarrow M_2$ of even lattices is called **primitive** if M_2/M_1 is a free $\mathbb{Z}$-module. A given embedding $M \hookrightarrow M_1$ of even lattices, for which M_1/M is a finite abelian group, is called an even **overlattice** of M.

1.1.2 Discriminant forms

We next introduce the notion of a discriminant form. A **discriminant form** (D, q) is a finite abelian group D together with a nondegenerate quadratic form $q : D \to \mathbb{Q}/\mathbb{Z}$, i.e. a function satisfying the properties

(i) $q(ax) = a^2 q(x)$ for all $a \in \mathbb{Z}$, $x \in D$,

(ii) $(x, y) := q(x + y) - q(x) - q(y)$ is a nondegenerate bilinear form.

Obviously the discriminant group $D(M) := M^\vee/M$ with the induced quadratic form Q is a discriminant form. A subgroup G of $D(M)$ is called **isotropic** if $q(\gamma) = 0$ for any $\gamma \in G$. There is a one-to-one correspondence between even overlattices of M and isotropic subgroups of $D(M)$. On the one hand, if M_1 is an even overlattice of M, then M_1/M is an isotropic subgroup of $D(M)$. On the other hand, if G is an isotropic subgroup of $D(M)$, then the lattice generated by G over M is an even overlattice of M.

From [Nik80], we know that every discriminant form is isomorphic to the discriminant group of an even lattice. Besides, two even lattices M, M_0 have isometric (or isomorphic) discriminant forms if and only if there are even unimodular lattices II, II_0 such that $M \oplus \mathrm{II} \cong M_0 \oplus \mathrm{II}_0$. Any even unimodular lattice of signature (r, s) satisfies that $r - s \in 8\mathbb{Z}$. Therefore, every discriminant form (D, Q) has a well-defined **signature**

$$\operatorname{sign}(D) = \operatorname{sign}(M) \in \mathbb{Z}/8\mathbb{Z},$$

where M is an even lattice such that $D \cong M^\vee/M$.

1.1.3 Genus of lattices

A suitable notion to classify even lattices is that of genus. The **genus** of a lattice M is the set of lattices M' of the same signature as M such that $M \otimes \mathbb{Z}_p \cong M' \otimes \mathbb{Z}_p$ for every prime number p. By [Nik80], two even lattices of the same signature are in the same genus if and only if their discriminant forms are isomorphic. Thus we here use the following equivalent definition of genus. Let M be an even lattice of signature (r, s) with discriminant form D. The **genus** of M, which is denoted by $\mathrm{II}_{r,s}(D)$, is the set of all even lattices of signature (r, s) whose discriminant form is isomorphic to D. For our purpose, we state the following theorems proved in [Nik80, Corollary 1.10.2] and [Nik80, Corollary 1.13.3], which tell us when a given genus is non-empty and when a given genus contains only one lattice up to isomorphism.

Theorem 1.1.1. *Let D be a discriminant form and $r, s \in \mathbb{Z}$. If $r \geq 0$, $s \geq 0$, $r - s = \operatorname{sign}(D) \bmod 8$ and $r + s > l(D)$, then there is an even lattice of signature (r, s) having discriminant form D. Here $l(D)$ is the minimum number of generators of the group D.*

Theorem 1.1.2. *Let D be a discriminant form and $r, s \in \mathbb{Z}$. If $r \geq 1$, $s \geq 1$ and $r + s \geq 2 + l(D)$, then all even lattices of genus $\mathrm{II}_{r,s}(D)$ are isomorphic.*

Let U be a **hyperbolic plane** i.e. $U = \mathbb{Z}e + \mathbb{Z}f$ with $(e, e) = (f, f) = 0$ and $(e, f) = 1$. The lattice U is an even unimodular lattice of signature $(1, 1)$. As a consequence of Theorems 1.1.1 and 1.1.2, we prove the following criterion.

Lemma 1.1.3. *Let M be an even lattice of signature $(n, 2)$ with $n \geq 3$. If the minimum number of generators of $D(M)$ satisfies $n - 2 > l(D(M))$, then there exists a positive definite even lattice L such that $M = 2U \oplus L$.*

Proof. By Theorem 1.1.1, there exists a positive definite even lattice L of rank $n - 2$ whose discriminant form is isomorphic to $D(M)$. By Theorem 1.1.2, the lattice $2U \oplus L$ is isomorphic to M. The proof is completed. □

A discriminant form can decompose into a sum of indecomposable Jordan components. This decomposition is usually not unique. But if two Jordan decompositions can be transformed into each other using certain summation rules, then they describe the same discriminant form. In this way, we can classify even lattices. We next give a brief overview of the possible nontrivial Jordan components following [Sch09, §2] and the references given there.

1. Let $q > 1$ be a power of an odd prime p. The nontrivial p-adic Jordan components of exponent q are $q^{\pm n}$ for $n \geq 1$. The indecomposable components are $q^{\pm 1}$, generated by an element γ with $q\gamma = 0$, $\gamma^2/2 = a/q \mod 1$ where a is an integer with $\left(\frac{2a}{p}\right) = \pm 1$. These components all have level q. The p-excess is given by
$$\text{p-excess}(q^{\pm n}) = n(q-1) + 4k \mod 8$$
where $k = 1$ if q is not a square and the exponent is $-n$ and $k = 0$ otherwise. We define
$$\gamma_p(q^{\pm n}) = e(-\text{p-excess}(q^{\pm n})/8).$$

2. Let $q > 1$ be a power of 2. The nontrivial even 2-adic Jordan components of exponent q are $q_{\mathrm{II}}^{\pm 2n}$ for $n \geq 1$. The indecomposable components are $q_{\mathrm{II}}^{\pm 2}$, generated by two elements γ and δ with $q\gamma = q\delta = 0$, $(\gamma, \delta) = 1/q \mod 1$ and $\gamma^2/2 = \delta^2/2 = 0 \mod 1$ for q_{II}^{+2} and $\gamma^2/2 = \delta^2/2 = 1/q \mod 1$ for q_{II}^{-2}. These components all have level q. The oddity is given by
$$\text{oddity}(q_{\mathrm{II}}^{\pm 2n}) = 4k \mod 8$$
where $k = 1$ if q is not a square and the exponent is $-n$ and $k = 0$ otherwise. We define
$$\gamma_2(q_{\mathrm{II}}^{\pm 2n}) = e(\text{oddity}(q_{\mathrm{II}}^{\pm 2n})/8).$$

3. Let $q > 1$ be a power of 2. The nontrivial odd 2-adic Jordan components of exponent q are $q_t^{\pm n}$ with $n \geq 1$ and $t \in \mathbb{Z}/8\mathbb{Z}$. The indecomposable components are $q_t^{\pm 1}$ where the subscript t satisfies $\left(\frac{t}{2}\right) = \pm 1$, generated by an element γ with $q\gamma = 0$ and $\gamma^2/2 = t/2q \mod 1$. These components all have level $2q$. The oddity is given by
$$\text{oddity}(q_t^{\pm n}) = t + 4k \mod 8$$
where $k = 1$ if q is not a square and the exponent is $-n$ and $k = 0$ otherwise. We define
$$\gamma_2(q_t^{\pm n}) = e(\text{oddity}(q_t^{\pm n})/8).$$

The sum of two Jordan components with the same prime power q is given by multiplying the signs, adding the ranks and, if any components have a subscript t, adding the subscripts t.

The factors γ_p are multiplicative. Let D be a discriminant form, we have the following **oddity formula**
$$\prod \gamma_p(D) = e(\text{sign}(D)/8). \tag{1.1.1}$$

Since the level of every odd 2-adic Jordan component is divisible by 4, the signature of a discriminant form of level N with $4 \nmid N$ is even. Therefore, the rank of an even lattice of level N with $4 \nmid N$ is even. From the oddity formula, we also see that the signature of a lattice of level 2 is divisible by 4.

Table 1.1: Coxeter numbers of irreducible root lattices

L	A_n	D_n	E_6	E_7	E_8
h	$n+1$	$2(n-1)$	12	18	30

1.1.4 Root lattices

At the end of this section, we recall some basic facts on root lattices following [Ebe02]. Let L be an even lattice in $\mathbb{R}^N$. An element $r \in L$ is called a root if $(r,r) = 2$. The set of all roots is denoted by R_L. The lattice L is called a root lattice if L is generated by R_L. Every root lattice can be written as an orthogonal direct sum of the irreducible root lattices of types $A_n(n \geq 1)$, $D_n(n \geq 4)$, E_6, E_7 and E_8. For a root lattice L of rank n the number $h = |R_L|/n$ is called the Coxeter number of L. The Coxeter numbers of irreducible root lattices are listed in Table 1.1.

By [Ebe02, Proposition 1.6], we have the following.

Proposition 1.1.4. *Let $L \subset \mathbb{R}^n$ be an irreducible root lattice. Then for any $x \in \mathbb{R}^n$ we have*

$$\sum_{r \in R_L} (r,x)^2 = 2h(x,x).$$

Let R_L^+ be the set of positive roots of L. The Weyl vector of L is defined as $\rho = \frac{1}{2}\sum_{r\in R_L^+} r$. We know from [Ebe02, Lemma 1.16] that the norm of Weyl vector of an irreducible root lattice is given by $\rho^2 = \frac{1}{12}h(h+1)\,\mathrm{rank}(L)$.

1.2 Modular forms on orthogonal groups

In this section we briefly introduce the theory of modular forms on orthogonal groups, which is one of the main tools in the study of the geometry of the modular varieties. The general reference is [Bor95].

We start with the general settings. Let M be an even integral lattice with a quadratic form of signature $(2,n)$ with $n \geq 3$, and let

$$\mathcal{D}(M) = \{[\mathcal{Z}] \in \mathbb{P}(M \otimes \mathbb{C}) : (\mathcal{Z},\mathcal{Z}) = 0, (\mathcal{Z},\bar{\mathcal{Z}}) > 0\}^+ \tag{1.2.1}$$

be the associated Hermitian symmetric domain of type IV (here + denotes one of its two connected components). The affine cone over $\mathcal{D}(M)$ is

$$\mathcal{D}(M)^\bullet = \{\mathcal{Z} \in M \otimes \mathbb{C} : [\mathcal{Z}] \in \mathcal{D}(M)\}. \tag{1.2.2}$$

Let us denote the index 2 subgroup of the orthogonal group $\mathrm{O}(M)$ preserving $\mathcal{D}(M)$ by $\mathrm{O}^+(M)$. Let $D(M) = M^\vee/M$ be the discriminant group of M. The subgroup of $\mathrm{O}^+(M)$ acting trivially on $D(M)$ is called the **stable orthogonal group** and denoted by $\widetilde{\mathrm{O}}^+(M)$. For any $v \in M \otimes \mathbb{Q}$ satisfying $(v,v) < 0$, the **rational quadratic divisor** associated to v is defined as

$$\mathcal{D}_v(M) = \{[\mathcal{Z}] \in \mathcal{D}(M) : (\mathcal{Z},v) = 0\}. \tag{1.2.3}$$

Definition 1.2.1. Let Γ be a finite index subgroup of $\mathrm{O}^+(M)$ and $k \in \mathbb{Z}$. A modular form of weight k and character $\chi : \Gamma \to \mathbb{C}^*$ with respect to Γ is a holomorphic function $F : \mathcal{D}(M)^\bullet \to \mathbb{C}$ on the affine cone $\mathcal{D}(M)^\bullet$ satisfying

$$F(t\mathcal{Z}) = t^{-k}F(\mathcal{Z}), \quad \forall t \in \mathbb{C}^*,$$
$$F(g\mathcal{Z}) = \chi(g)F(\mathcal{Z}), \quad \forall g \in \Gamma.$$

A modular form is called a cusp form if it vanishes at every cusp (i.e. a boundary component of the Baily-Borel compactification of the modular variety $\Gamma\backslash\mathcal{D}(M)$).

We denote the spaces of modular and cusp forms of weight k and character χ by $M_k(\Gamma,\chi)$ and $S_k(\Gamma,\chi)$ respectively.

If $n<3$, we can also define orthogonal modular forms but we have to add to Definition 1.2.1 the condition that F is holomorphic at boundary. According to Koecher's principle (see [Bai66]) this condition is automatically fulfilled if $n\geq 3$. In this thesis we only consider the case $n\geq 3$. In this case the order of any character χ in Definition 1.2.1 is finite. The quotient

$$\mathcal{F}_M(\Gamma)=\Gamma\backslash\mathcal{D}(M) \tag{1.2.4}$$

is called a *modular variety of orthogonal type* or *orthogonal modular variety*. It is a quasi-projective variety of dimension n. For a recent account of the theory of $\mathcal{F}_M(\Gamma)$ we refer to [GHS13]. We next introduce the Fourier expansion of an orthogonal modular form at 0-dimensional cusps.

A 0-dimensional cusp of $\mathcal{D}(M)$ is defined by a primitive isotropic vector $c\in M$ (up to sign: c and $-c$ define the same cusp). We can show that for any $\mathcal{Z}\in\mathcal{D}(M)^\bullet$ there exists a unique $\alpha\in\mathbb{C}^*$ such that $(\alpha\mathcal{Z},c)=1$. From this, it follows that

$$\mathcal{D}(M)_c=\{\mathcal{Z}\in\mathcal{D}(M)^\bullet:(\mathcal{Z},c)=1\}\cong\mathcal{D}(M).$$

The lattice

$$M_c=c^\perp/c=c_M^\perp/\mathbb{Z}c \tag{1.2.5}$$

is an integral lattice of signature $(1,n-1)$. We fix an element $b\in M^\vee$ such that $(c,b)=1$. A choice of b gives a realisation of the hyperbolic lattice M_c as a sublattice in M

$$M_c\cong M_{c,b}=M\cap c^\perp\cap b^\perp. \tag{1.2.6}$$

This yields a decomposition

$$M\otimes\mathbb{Q}=M_{c,b}\otimes\mathbb{Q}\oplus(\mathbb{Q}b+\mathbb{Q}c).$$

Using the hyperbolic lattice $M_c\otimes\mathbb{R}$ we can define a positive cone

$$C(M_c)=\{X\in M_c\otimes\mathbb{R}:(X,X)>0\}.$$

We choose $C^+(M_c)$, one of the two connected components of $C(M_c)$ and define the corresponding tube domain, which is the complexification of $C^+(M_c)$

$$\mathcal{H}_c(M)=M_c\otimes\mathbb{R}+iC^+(M_c). \tag{1.2.7}$$

Then we have an isomorphism $\mathrm{pr}_c:\mathcal{H}_c(M)\to\mathcal{D}(M)_c\cong\mathcal{D}(M)$ defined by

$$\mathrm{pr}_c:\quad Z\mapsto Z\oplus\left[b-\frac{(Z,Z)+(b,b)}{2}c\right]. \tag{1.2.8}$$

Using the coordinate $Z\in\mathcal{H}_c(M)$ defined by the choice of c and b we can identify an arbitrary orthogonal modular form F of weight k with a modular form $F_{c,b}$ (or simply F_c) on the tube domain $\mathcal{H}_c(M)$:

$$F_{c,b}(Z)=F(\mathrm{pr}_c(Z)). \tag{1.2.9}$$

For every $g\in\mathrm{O}^+(M)$ and $Z\in\mathcal{H}_c(M)$, there exist $J_{c,b}(g,Z)\in\mathbb{C}^*$ and $g\langle Z\rangle\in\mathcal{H}_c(M)$ such that

$$g\,\mathrm{pr}_c(Z)=J_{c,b}(g,Z)\,\mathrm{pr}_c(g\langle Z\rangle). \tag{1.2.10}$$

The above relation defines an action of $\mathrm{O}^+(M)$ on $\mathcal{H}_c(M)$. A modular form in Definition 1.2.1 satisfies

$$F_{c,b}|_k g = \chi(g)F_{c,b}$$

where

$$(F_{c,b}|_k g)(Z) := J_{c,b}(g,Z)^{-k}F_{c,b}(g\langle Z\rangle).$$

Let $F \in M_k(\widetilde{\mathrm{SO}}^+(M))$. Since the *Eichler transvection* (see [Eic52, §3])

$$t(c,a) : v \mapsto v - (a,v)c + (c,v)a - \frac{1}{2}(a,a)(c,v)c \tag{1.2.11}$$

belongs to $\widetilde{\mathrm{SO}}^+(M)$ for all $a \in M_{c,b}$ and $t(c,a)(\mathrm{pr}_c(Z)) = \mathrm{pr}_c(Z+a)$, we have $F_c(Z+a) = F_c(Z)$, which gives the Fourier expansion of F at the cusp c:

$$F_c(Z) = \sum_{l \in M_{c,b}^\vee} f(l)\exp(2\pi i(l,Z)). \tag{1.2.12}$$

By Koecher's principle, the function F_c is holomorphic at the cusp c, which means that

$$f(l) \neq 0 \Longrightarrow l \textit{ belongs to the closure of } C^+(M_c).$$

By [Bor95, corollary 3.3], any modular form on $\mathcal{D}(M)$ either has weight 0 in which case it is constant, or has weight at least $n/2-1$. The minimum possible weight is called the **singular weight**. If a modular form has singular weight then all the Fourier coefficients corresponding to vectors of nonzero norm vanish.

1.3 Modular forms for the Weil representation

In this section we introduce the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ and the vector-valued modular forms. We refer to [Bor98, Bru02] for more details.

1.3.1 Weil representation and vector-valued modular forms

We first recall the *metaplectic group* $\mathrm{Mp}_2(\mathbb{R})$, which is a double cover of $\mathrm{SL}_2(\mathbb{R})$. Let us denote by $\sqrt{z}$ the principal branch of the square root i.e. $\arg(\sqrt{z}) \in (-\pi/2, \pi/2]$. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, let ϕ be a holomorphic function on $\mathbb{H}$ satisfying $\phi(\tau)^2 = c\tau + d$, $\tau \in \mathbb{H}$. The elements of $\mathrm{Mp}_2(\mathbb{Z})$ are pairs $(A, \phi(\tau))$. The product of two elements of $\mathrm{Mp}_2(\mathbb{R})$ is given by

$$(A,\phi_1(\tau))(B,\phi_2(\tau)) = (AB, \phi_1(B\tau)\phi_2(\tau)).$$

The map

$$A \longmapsto \widetilde{A} = \left(A, \sqrt{c\tau+d}\right) \tag{1.3.1}$$

defines a locally isomorphic embedding of $\mathrm{SL}_2(\mathbb{R})$ into $\mathrm{Mp}_2(\mathbb{R})$. Let $\mathrm{Mp}_2(\mathbb{Z})$ be the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$. It is well-known that $\mathrm{Mp}_2(\mathbb{Z})$ is generated by

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right), \qquad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right).$$

We have the relations $S^2 = (ST)^3 = Z$, where

$$Z = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i\right)$$

is the standard generator of the center of $\mathrm{Mp}_2(\mathbb{Z})$.

Let D be a discriminant form with quadratic form $q : D \to \mathbb{Q}/\mathbb{Z}$ and associated bilinear form $(\cdot,\cdot)$. Let $\{\mathbf{e}_\gamma : \gamma \in D\}$ be the basis of the group ring $\mathbb{C}[D]$. The **Weil representation** of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[D]$ is a unitary representation defined by the action of the generators of $\mathrm{Mp}_2(\mathbb{Z})$ as follows:

$$\rho_D(T)\mathbf{e}_\gamma = e(-\gamma^2/2)\mathbf{e}_\gamma, \tag{1.3.2}$$

$$\rho_D(S)\mathbf{e}_\gamma = \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta\in D} \exp((\gamma,\beta))\mathbf{e}_\beta. \tag{1.3.3}$$

We note that

$$\rho_M(Z)\mathbf{e}_\gamma = e(\mathrm{sign}(D)/4)e_{-\gamma}. \tag{1.3.4}$$

Definition 1.3.1. Let $f(\tau) = \sum_{\gamma\in D} f_\gamma(\tau)\mathbf{e}_\gamma$ be a holomorphic function on $\mathbb{H}$ with values in $\mathbb{C}[D]$ and $k \in \frac{1}{2}\mathbb{Z}$. Then f is a nearly holomorphic modular form for ρ_D of weight k if

$$f(A\tau) = \phi(\tau)^{2k}\rho_D(A)f(\tau), \quad \forall (A,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$$

and f is meromorphic at $i\infty$. If f is also holomorphic at $i\infty$, then it is called a holomorphic modular form for ρ_D. If f vanishes at $i\infty$, then it is called a cusp form for ρ_D.

The invariance of F under T implies that the functions $e(q(\gamma)\tau)f_\gamma(\tau)$ are periodic with period 1. Thus f has a Fourier expansion

$$f(\tau) = \sum_{\gamma\in D} \sum_{n\in\mathbb{Z}-q(\gamma)} c_\gamma(n)e^{2\pi i n\tau}\mathbf{e}_\gamma. \tag{1.3.5}$$

Here, the sum

$$\sum_{\gamma\in D} \sum_{\substack{n\in\mathbb{Z}-q(\gamma)\\ n<0}} c_\gamma(n)e^{2\pi i n\tau}\mathbf{e}_\gamma. \tag{1.3.6}$$

is called the **principal part** of f.

The orthogonal group $\mathrm{O}(D)$ acts on $\mathbb{C}[D]$ via

$$\sigma\left(\sum_{\gamma\in D} a_\gamma \mathbf{e}_\gamma\right) = \sum_{\gamma\in D} a_\gamma \mathbf{e}_{\sigma(\gamma)}$$

and this action commutes with that of ρ_D on $\mathbb{C}[D]$ i.e. $\sigma(\rho_D(A)x) = \rho_D(A)(\sigma x)$, for $\sigma \in \mathrm{O}(D)$, $A \in \mathrm{SL}_2(\mathbb{Z})$, $x \in \mathbb{C}[D]$. Thus the modular forms invariant under $\mathrm{O}(D)$ can be well defined: a modular form F is called invariant under $\mathrm{O}(D)$ if $\sigma F = F$ for all $\sigma \in \mathrm{O}(D)$.

1.3.2 Borcherds products

We now introduce the Borcherds product (also called theta lift), which is a multiplicative lift due to Borcherds (see [Bor95, Bor98]) from modular forms for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ to modular forms on orthogonal groups.

Let M be an even lattice of signature $(2,n)$ with $n \geq 3$. Let $c \in M$ be a primitive norm 0 vector i.e. a 0-dimensional cusp. Then there exists a $b \in M^\vee$ satisfying $(c,b) = 1$. Let $\mathrm{div}(c)$ be the divisor of c i.e. the unique positive integer such that $(c,M) = \mathrm{div}(c)\mathbb{Z}$. Then we have $c/\mathrm{div}(c) \in M^\vee$. Let $\zeta \in M$ satisfying $(c,\zeta) = \mathrm{div}(c)$. It can be uniquely represented as

$$\zeta = \zeta_{M_{c,b}} + \mathrm{div}(c)b + Bc \tag{1.3.7}$$

with $\zeta_{M_{c,b}} \in M_{c,b}^\vee$ and $B \in \mathbb{Q}$. By [Bru02, Proposition 2.2], we have the direct sum

$$M = M_{c,b} \oplus \mathbb{Z}\zeta \oplus \mathbb{Z}c. \tag{1.3.8}$$

We define the following sublattice of $M^\vee$

$$M_0' = \{\lambda \in M^\vee : (\lambda, c) \in \operatorname{div}(c)\mathbb{Z}\}. \tag{1.3.9}$$

From [Bru02, page 41], we have an isomorphism

$$\begin{aligned} p : M_0' &\longrightarrow M_{c,b}^\vee, \\ \lambda &\longmapsto \lambda_{M_{c,b}} - \frac{(\lambda, c)}{\operatorname{div}(c)}\zeta_{M_{c,b}}, \end{aligned} \tag{1.3.10}$$

where $\lambda = \lambda_{M_{c,b}} + xc + yb \in M_0'$ with $x \in \mathbb{Q}$, $y \in \operatorname{div}(c)\mathbb{Z}$, $\lambda_{M_{c,b}} \in M_{c,b}$. We can check that $p(M) = M_{c,b}$. Thus p induces a surjective map $M_0'/M \to M_{c,b}^\vee/M_{c,b}$ which will also be denoted by p.

From [Bor98, Theorem 13.3] or [Bru02, Theorem 3.22], we have the following result.

Theorem 1.3.2 (Borcherds 98)**.** *Let M be an even lattice of signature $(2, n)$ with $n \geq 3$. Let f be a nearly holomorphic modular form of weight $k = 1 - n/2$ for the Weil representation ρ_D, where D is the discriminant form of the lattice $M(-1)$. Assume that the Fourier expansion of f is of the form*

$$f(\tau) = \sum_{\gamma \in D} \sum_{n \in \mathbb{Z} - q(\gamma)} c_\gamma(n) e^{2\pi i n\tau} \boldsymbol{e}_\gamma$$

satisfying that $c_\gamma(n)$ are integral if $n < 0$. Then there is a meromorphic function $\Psi : \mathcal{D}(M)^\bullet \to \mathbb{C}$ with the following properties.

1. *The function Ψ is a meromorphic modular form of weight $c_0(0)/2$ for the group*
$$\mathrm{O}(M, f)^+ = \{\sigma \in \mathrm{O}(M)^+ : \sigma f = f\}$$
with some multiplier system χ of finite order. If $c_0(0)$ is even, then χ is a character.
2. *The only zeros or poles of Ψ lie on rational quadratic divisors $\mathcal{D}_\gamma(M)$ where γ is a primitive vector of negative norm in $M^\vee$. The divisor $\mathcal{D}_\gamma(M)$ has order*
$$\sum_{m>0} c_{m\gamma}(m^2\gamma^2/2).$$
3. *For each primitive isotropic vector $c \in M$, an associated vector $b \in M^\vee$ with $(c, b) = 1$ and for each Weyl chamber W of $M_{c,b}$ and f, the restriction Ψ_c has an infinite product expansion converging in a neighbourhood of the cusp corresponding to c that is up to a constant*
$$e((Z, \rho)) \prod_{\substack{\lambda \in M_{c,b}^\vee \\ (\lambda, W) > 0}} \prod_{\substack{\delta \in M_0'/M \\ p(\delta) = \lambda + M_{c,b}}} (1 - e((\delta, b) + (\lambda, Z)))^{c_\delta(\lambda^2/2)},$$
where the projection p is defined in (1.3.10) and ρ is the Weyl vector attached to W and f.

The function Ψ is called the Borcherds product of f.

Bruinier proved the following converse theorem [Bru14, Theorem 1.2].

Theorem 1.3.3. *Let M be an even lattice of signature $(2, n)$ with $n \geq 3$ and Ψ a meromorphic modular form for $\widetilde{\mathrm{O}}^+(M)$ whose divisor is a linear combination of rational quadratic divisors. If $M = U \oplus U(m) \oplus L(-1)$ for some positive integer m and a positive definite even lattice L, then up to a constant factor the function Ψ is the Borcherds product of a modular form for the Weil representation associated to M.*

We remark that Bruinier also proved that the existence of a Borcherds product of non-zero weight whose divisor is supported on certain special divisors and showed that every meromorphic Borcherds product is the quotient of two holomorphic ones (see [Bru17]).

1.3.3 Construction of vector-valued modular forms

In this subsection we introduce many useful properties of discriminant forms of squarefree level and the lifting from scalar-valued modular forms on congruence subgroups to modular forms for the Weil representation following [Sch06, Sch09, Sch15].

Let D be a discriminant form of even signature and N a positive integer such that the level of D divides N. There is an explicit formula to calculate the action of the Weil representation ρ_D on $\mathrm{SL}_2(\mathbb{Z})$ (see [Sch09, Theorem 4.7]). In particular, when $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, the matrix A acts in ρ_D as (see [Sch09, Proposition 4.8])

$$\rho_D(A)\mathbf{e}_\gamma = \left(\frac{a}{|D|}\right) e((a-1)\,\mathrm{oddity}(D)/8) e(-bd\gamma^2/2)\mathbf{e}_{d\gamma}. \tag{1.3.11}$$

From this, we see that the group $\Gamma(N)$ acts trivially in ρ_D. We can check that

$$\chi_D(A) = \left(\frac{a}{|D|}\right) e((a-1)\,\mathrm{oddity}(D)/8) \tag{1.3.12}$$

defines a quadratic Dirichlet character $\chi_D : \Gamma_0(N) \to \mathbb{C}^*$ and for $\gamma \in D$

$$\chi_\gamma(A) = e(-b\gamma^2/2) \tag{1.3.13}$$

defines a character $\chi_\gamma : \Gamma_1(N) \to \mathbb{C}^*$.

Let $F = \sum_{\gamma \in D} F_\gamma \mathbf{e}_\gamma$ be a modular form of weight k for ρ_D. Then we have

$$F_\gamma|_A = \left(\frac{d}{|D|}\right) e((d-1)\,\mathrm{oddity}(D)/8) e(-ab\gamma^2/2) F_{a\gamma}, \quad A \in \Gamma_0(N). \tag{1.3.14}$$

From this, we deduce that F_0 is a modular form on $\Gamma_0(N)$ of weight k with character χ_D and F_γ is a modular form on $\Gamma_1(N)$ of weight k with character χ_γ.

$\boxed{\text{We next assume that the number } N \text{ is squarefree.}}$ In this case the character χ_D reduces to

$$\chi_D(A) = \left(\frac{a}{|D|}\right), \quad A \in \Gamma_0(N). \tag{1.3.15}$$

In his series of works, Scheithauer constructed several liftings from scalar-valued modular forms on congruence subgroups to modular forms for the Weil representation (see for example [Sch15, Theorem 3.1]). For our purpose, we recall the following lifting which may be the simplest one. We refer to [Sch06, Theorem 6.2, Theorem 6.5] for a proof.

Theorem 1.3.4. *Let f be a nearly holomorphic modular form for $\Gamma_0(N)$ (holomorphic except at cusps) of weight $k \in \mathbb{Z}$ and character χ_D. Then*

$$F_{\Gamma_0(N),f,0}(\tau) = \sum_{A \in \Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})} f|_A(\tau)\rho_D(A^{-1})e_0$$

is a nearly holomorphic modular form for ρ_D of weight k which is invariant under $\mathrm{O}(D)$.

For each positive divisor c of N we choose a matrix $A_c = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $d = 1 \mod c$ and $d = 0 \mod c'$ where $c' = N/c$. Let $f_c(\tau) = f|_{A_c}(\tau)$ be the expansion of f at the cusp $1/c$. We can write uniquely

$$f_c(\tau) = g_{c',0}(\tau) + g_{c',1}(\tau) + \ldots + g_{c',c'-1}(\tau)$$

where

$$g_{c',j}(\tau+1) = e(j/c')g_{c',j}(\tau), \quad \forall 0 \le j \le c'.$$

Then we can calculate $F_{\Gamma_0(N),f,0}$ *explicitly as*

$$F(\tau) = \sum_{c|N}\sum_{\mu\in D_{c'}} \xi_c \frac{\sqrt{|D_c|}}{\sqrt{|D|}} c' g_{c',j_{\mu,c'}}(\tau)\boldsymbol{e}_\mu,$$

where

$$D_n = \{\gamma \in D : n\gamma = 0\}, \quad n|N,$$

$$\xi_c = \left(\frac{-c}{|D_{c'}|}\right)\prod_{p|c'}\gamma_p(D),$$

and $j_{\mu,c'}/c' = -\mu^2/2 \mod 1$ *for* $\mu \in D_{c'}$.

The following proposition can be found in [Sch15, Proposition 5.1].

Proposition 1.3.5. *Let D be a discriminant form of squarefree level N. Two elements of D are in the same orbit under $\mathrm{O}(D)$ if and only if they have the same norm and order.*

The next result was proved in [Sch15, Proposition 5.3].

Proposition 1.3.6. *Let D be a discriminant form of squarefree level N and $F = \sum_{\gamma\in D} F_\gamma \boldsymbol{e}_\gamma$ a modular form for ρ_D which is invariant under $\mathrm{O}(D)$. Then the complex vector space generated by the components F_γ, $\gamma \in D$ is generated by the functions $F_0|_A$, $A \in \mathrm{SL}_2(\mathbb{Z})$. In particular $F = 0$ if $F_0 = 0$.*

The following result proved in [Sch15, Corollary 5.5] tells us that the lift $f \mapsto F_{\Gamma_0(N),f,0}$ is in fact surjective.

Proposition 1.3.7. *Let D be a discriminant form of squarefree level N and F a modular form for ρ_D which is invariant under $\mathrm{O}(D)$. Then $F = F_{\Gamma_0(N),f,0}$ for a suitable modular form f on $\Gamma_0(N)$ with character χ_D.*

1.4 Jacobi forms of lattice index

In this section we give a brief overview of the theory of Jacobi forms of lattice index. We refer to [CG13, Gri95] for more details.

1.4.1 Definition of Jacobi forms and basic properties

Let L be an even positive definite lattice with bilinear form $(\cdot,\cdot)$ and $L^\vee$ be its dual lattice. The real **Heisenberg group** associated to L is

$$H(L\otimes\mathbb{R}) = \{[x,y:r] : x,y \in L\otimes\mathbb{R}, r\in\mathbb{R}\}$$

together with the operation

$$[x_1,y_1:r_1]\cdot[x_2,y_2:r_2] = \left[x_1+x_2, y_1+y_2 : r_1+r_2+\frac{1}{2}\left((x_1,y_2)-(x_2,y_1)\right)\right]. \tag{1.4.1}$$

The group $\mathrm{SL}_2(\mathbb{R})$ acts on $H(L\otimes\mathbb{R})$ via

$$A\cdot[x,y:r] = [(x,y)A^{-1}:r] = [dx-cy, ay-bx:r], \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1.4.2}$$

The real **Jacobi group** $\Gamma^J(L\otimes\mathbb{R})$ is the semi-direct product $\mathrm{SL}_2(\mathbb{R})\rtimes H(L\otimes\mathbb{R})$ with the multiplication

$$(A,h_1)\cdot(B,h_2)=(AB,(B^{-1}\cdot h_1)\cdot h_2), \tag{1.4.3}$$

where $A,B\in\mathrm{SL}_2(\mathbb{R})$, $h_1,h_2\in H(L\otimes\mathbb{R})$. For simplicity of notations, we write $\{A\}$, h, $\{A\}h$ instead of $(A,0)$, $(1,h)$, $(A,0)\cdot(1,h)$, respectively. Let $k\in\frac{1}{2}\mathbb{Z}$, $t\in\mathbb{Q}$. The real Jacobi group $\Gamma^J(L\otimes\mathbb{R})$ acts on the space of holomorphic functions on $\mathbb{H}\times(L\otimes\mathbb{C})$ via

$$\left(\varphi|_{k,t}\{A\}\right)(\tau,\mathfrak{z})=(c\tau+d)^{-k}e^{-\pi i t\frac{c(\mathfrak{z},\mathfrak{z})}{c\tau+d}}\varphi\left(\frac{a\tau+b}{c\tau+d},\frac{\mathfrak{z}}{c\tau+d}\right), \tag{1.4.4}$$

$$\left(\varphi|_{k,t}h\right)(\tau,\mathfrak{z})=e^{\pi i t((x,x)\tau+2(x,\mathfrak{z})+(x,y)+2r)}\varphi(\tau,\mathfrak{z}+x\tau+y), \tag{1.4.5}$$

for all $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{R})$ and $h=[x,y:r]\in H(L\otimes\mathbb{R})$. The integral **Jacobi group** which is a subgroup of $\Gamma^J(L\otimes\mathbb{R})$ is defined as $\Gamma^J(L)=\mathrm{SL}_2(\mathbb{Z})\rtimes H(L)$, where

$$H(L)=\left\{[x,y:r]:x,y\in L, r+\frac{1}{2}(x,y)\in\mathbb{Z}\right\}$$

is the integral **Heisenberg group** associated to L. Let $\chi:\Gamma^J(L)\to\mathbb{C}^*$ be a character (or a multiplier system) of finite order. By [CG13], its restriction to $\mathrm{SL}_2(\mathbb{Z})$ is a power υ_η^D of the multiplier system of the Dedekind η-function and we have

$$\chi(\{A\}\cdot[x,y:r])=\upsilon_\eta^D(A)\cdot\chi_{H(L)}([x,y:r]), \tag{1.4.6}$$

and

$$\chi_{H(L)}([x,y:r])=e^{\pi i t((x,x)+(y,y)-(x,y)+2r)}, \tag{1.4.7}$$

where $t\in\mathbb{Q}$ such that $t\cdot s(L)\in\mathbb{Z}$ and $s(L)$ denotes the generator of the integral ideal generated by (x,y) for all x and y in L. We set

$$\nu_{H(L)}([x,y:r])=e^{\pi i((x,x)/2+(y,y)/2-(x,y)/2+r)}. \tag{1.4.8}$$

The Jacobi forms are defined as follows

Definition 1.4.1. A holomorphic function $\varphi:\mathbb{H}\times(L\otimes\mathbb{C})\to\mathbb{C}$ is called a weakly holomorphic Jacobi form of weight $k\in\frac{1}{2}\mathbb{Z}$ and index $t\in\mathbb{Q}_{\geq 0}$ with a character (or a multiplier system) of finite order $\chi:\Gamma^J(L)\to\mathbb{C}^*$ if φ satisfies

$$\varphi|_{k,t}g=\chi(g)\varphi,\quad\forall g\in\Gamma^J(L)$$

and φ admits a Fourier expantion of the form

$$\varphi(\tau,\mathfrak{z})=\sum_{\substack{n\geq n_1, n\equiv\frac{D}{24}\bmod\mathbb{Z}\\ \ell\in\frac{1}{2}L^\vee}}f(n,\ell)e^{2\pi i(n\tau+(\ell,\mathfrak{z}))}, \tag{1.4.9}$$

where n_1 is a certain integer and the number D is given by $\chi|_{\mathrm{SL}_2(\mathbb{Z})}=\upsilon_\eta^D$.

If φ satisfies the condition

$$f(n,\ell)\neq 0\Longrightarrow n\geq 0$$

then it is called a weak Jacobi form. If φ further satisfies the condition

$$f(n,\ell)\neq 0\Longrightarrow 2nt-(\ell,\ell)\geq 0$$

then it is called a holomorphic Jacobi form. If φ further satisfies the stronger condition

$$f(n,\ell) \neq 0 \Longrightarrow 2nt - (\ell,\ell) > 0$$

then it is called a Jacobi cusp form. We denote by $J^{w.h.}_{k,L,t}(\chi)$ the vector space of weakly holomorphic Jacobi forms of weight k and index t and the corresponding spaces of weak Jacobi forms, holomorphic Jacobi forms and Jacobi cusp forms are denoted by $J^{w}_{k,L,t}(\chi)$, $J_{k,L,t}(\chi)$ and $J^{cusp}_{k,L,t}(\chi)$, respectively. If the character is trivial we also write $J_{k,L,t} = J_{k,L,t}(1)$ for short.

Remark 1.4.2. By [CG13, page193], we know that if $J^{w}_{k,L,t}(\chi) \neq \{0\}$ then $t \cdot s(L) \in \mathbb{Z}$ and $\chi|_{H(L)} = \nu^{2t}_{H(L)}$. Let $L(t)$ be the lattice L equipped with bilinear form $t(\cdot,\cdot)$. Assume that $J_{k,L,t}(\chi) \neq \{0\}$. Then the lattice $L(t)$ is integral.

1. If $L(t)$ is an even lattice then

$$J_{k,L,t}(\chi) = J_{k,L(t),1}(\chi).$$

   If this space is non-trivial then $\nu_{H(L)}([x,y:\frac{1}{2}(x,y)]) = 1$, for all $x,y \in L$.

2. If $L(t)$ is an odd lattice then

$$J_{k,L,t}(\chi) = J_{k,L(2t),\frac{1}{2}}(\chi).$$

   In this case, $\nu_{H(L)}([x,y:\frac{1}{2}(x,y)])^2 = 1$, for all $x,y \in L$ and there exist $x,y \in L$ such that $\nu_{H(L)}([x,y:\frac{1}{2}(x,y)]) = -1$.

Therefore, we only need to distinguish the Jacobi forms of index 1 and index $\frac{1}{2}$.

Remark that in the literature Jacobi forms of weight k and index t for the lattice L are also called Jacobi forms of weight k and index $L(t)$.

We next explain the Jacobi group and give another definition of Jacobi forms from the perspective of orthogonal modular forms. Let M be an even lattice of signature $(2, n_0+2)$ containing two hyperbolic planes, i.e. $M = U \oplus U_1 \oplus L(-1)$, where L is a positive definite even lattice of rank $\operatorname{rank}(L) = n_0$ and $L(-1)$ denotes its rescaling by -1. We fix a basis of the hyperbolic plane $U = \mathbb{Z}e \oplus \mathbb{Z}f$: $(e,f) = 1$, $(e,e) = (f,f) = 0$. Similarly $U_1 = \mathbb{Z}e_1 \oplus \mathbb{Z}f_1$. We choose a basis of M of the form $(e, e_1, ..., f_1, f)$ where ... denotes a basis of $L(-1)$. In this basis, the quadratic form associated to the bilinear form on M has the following Gram matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -S & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where S is the Gram matrix of L. Let F be the totally isotropic plane spanned by e and e_1 and let P_F be the parabolic subgroup of $\mathrm{SO}^+(M)$ that preserves F. The subgroup of P_F of elements acting trivially on the sublattice L is isomorphic to $\Gamma^J(L)$ i.e. the integral Jacobi group of L. In fact, it has a subgroup isomorphic to $\mathrm{SL}_2(\mathbb{Z})$. For any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we denote

$$\{A\} = \begin{pmatrix} A^* & 0 & 0 \\ 0 & \mathbf{I}_{n_0} & 0 \\ 0 & 0 & A \end{pmatrix} \in \Gamma^J(L), \quad A^* = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

In addition, it has another subgroup generated by the elements

$$[x,y:r] = \begin{pmatrix} 1 & 0 & y^{tr}S & \frac{1}{2}(x,y)-r & \frac{1}{2}(y,y) \\ 0 & 1 & x^{tr}S & \frac{1}{2}(x,x) & \frac{1}{2}(x,y)+r \\ 0 & 0 & \mathbf{I}_{n_0} & x & y \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $x,y \in L$ and $r \in \frac{1}{2}\mathbb{Z}$ such that $r + \frac{1}{2}(x,y) \in \mathbb{Z}$, which is isomorphic to the Heisenberg group $H(L)$. Let G be a modular form of weight k for $\widetilde{\mathrm{O}}^+(M)$. Following the notations in section 1.2, we choose $c = e$ and $b = f$. We fix a tube realization of the homogeneous domain $\mathcal{D}(M)$ related to the 1-dimensional boundary component determined by the totally isotropic plane $F = \langle e, e_1\rangle$ as

$$\mathcal{H}(L) := \mathcal{H}_e(M) = \{Z = (\tau,\mathfrak{z},\omega) \in \mathbb{H} \times (L\otimes\mathbb{C}) \times \mathbb{H} : (\operatorname{Im} Z, \operatorname{Im} Z) > 0\}$$

where $(\operatorname{Im} Z, \operatorname{Im} Z) = 2\operatorname{Im}\tau \operatorname{Im}\omega - (\operatorname{Im}\mathfrak{z}, \operatorname{Im}\mathfrak{z})$. In this setting, the modular form G_e has a Fourier-Jacobi expansion of the form

$$\begin{aligned} G_e(Z) &= \sum_{m\geq\mathbb{N}} \sum_{\substack{n\in\mathbb{N} \\ 2nm\geq(\ell,\ell)}} f(n,\ell,m) e^{2\pi i(n\tau - (\ell,\mathfrak{z}) + m\omega)} \\ &= \sum_{m\geq\mathbb{N}} G_m(\tau,\mathfrak{z}) e^{2\pi i m\omega}. \end{aligned}$$

Since G_e is invariant under $\Gamma^J(L)$, the Fourier-Jacobi coefficient $G_m(\tau,\mathfrak{z})$ is a holomorphic Jacobi form of weight k and index m for $m \in \mathbb{N}$. Conversely, we can also define holomorphic Jacobi forms in the following way

Definition 1.4.3. Let χ be a character (or a multiplier system) of finite order of $\Gamma^J(L)$, $k \in \frac{1}{2}\mathbb{Z}$ and $t \in \mathbb{Q}_{\geq 0}$. A holomorphic function $\varphi : \mathbb{H} \times (L\otimes\mathbb{C}) \to \mathbb{C}$ is called a holomorphic Jacobi form of weight k and index t with a character (or a multiplier system) χ if the modified function

$$\widetilde{\varphi}(Z) = \varphi(\tau,\mathfrak{z}) e^{2\pi i t\omega}, \quad Z = (\tau,\mathfrak{z},\omega) \in \mathcal{H}(L) = \mathcal{H}_e(M)$$

satisfies the functional equation

$$\widetilde{\varphi}|_k g = \chi(g)\widetilde{\varphi}, \quad \forall g \in \Gamma^J(L)$$

and is holomorphic at "infinity".

From the invariance of Jacobi forms under the Heisenberg group $H(L)$, we can deduce the following lemma.

Lemma 1.4.4. *Let $\varphi \in J^{w.h.}_{k,L,1}(v_\eta^D)$. Then its Fourier coeffcients $f(n,\ell)$ depend only on the class of ℓ in $L^\vee/L$ and the number $2n - (\ell,\ell)$. If φ is a weak Jacobi form then*

$$f(n,\ell) \neq 0 \Longrightarrow 2n - (\ell,\ell) \geq -\min\{(v,v) : v \in \ell + L\}.$$

The number $2n - (\ell,\ell)$ is called the **hyperbolic norm** of Fourier coefficient $f(n,\ell)$. The Fourier coefficients $f(n,\ell)$ with negative hyperbolic norm are called **singular Fourier coefficients**, which play a crucial role in the theory of Borcherds products. By definition, a weakly holomorphic Jacobi form without singular Fourier coefficient is a holomorphic Jacobi form.

We next introduce the odd Jacobi theta-series which is the most important Jacobi form because it can be used to construct many basic Jacobi forms and holomorphic Borcherds products. For more details, we refer to [GN98b, Mum83].

Example 1.4.5. The Jacobi theta-series of characteristic $(\frac{1}{2}, \frac{1}{2})$ is defined by

$$\vartheta(\tau,z) = \sum_{n\in\mathbb{Z}} \left(\frac{-4}{n}\right) q^{\frac{n^2}{8}} \zeta^{\frac{n}{2}} = -q^{\frac{1}{8}}\zeta^{-\frac{1}{2}} \prod_{n\geq 1}(1-q^{n-1}\zeta)(1-q^n\zeta^{-1})(1-q^n)$$

where $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$ and $\zeta = e^{2\pi iz}$, $z \in \mathbb{C}$. The above Jacobi triple product formula reflects the fact that $\vartheta(\tau,z)$ is the Kac–Weyl denominator function of the simplest affine Lie algebra (see [Kac90]). The theta-series ϑ satisfies two functional equations

$$\vartheta(\tau, z + x\tau + y) = (-1)^{x+y} e^{-i\pi(x^2\tau+2xz)}\vartheta(\tau,z), \quad \forall\, x,y \in \mathbb{Z},$$

$$\vartheta\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = \upsilon_\eta^3(A)(c\tau+d)^{\frac{1}{2}} e^{\frac{i\pi cz^2}{c\tau+d}}\vartheta(\tau,z), \quad \forall\, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Using our notations, $A_1 = \langle 2\rangle = \langle \mathbb{Z}, 2x^2\rangle$, we have

$$\vartheta \in J_{\frac{1}{2},A_1,\frac{1}{2}}(\upsilon_\eta^3 \times \upsilon_H)$$

where, for short, $\upsilon_{H(A_1)} = \upsilon_H$ is defined by

$$\upsilon_H([x,y:r]) = (-1)^{x+y+xy+r}, \quad [x,y:r] \in H(A_1).$$

The function ϑ is the simplest example of Jacobi form of half-integral index. We remind

$$\vartheta(\tau,-z) = -\vartheta(\tau,z) \quad \text{and} \quad \mathrm{div}(\vartheta) = \{z = x\tau + y : x,y \in \mathbb{Z}\}.$$

We also mention the Dedekind η-function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \in J_{\frac{1}{2},A_1,0}(\upsilon_\eta).$$

Using ϑ and η one can define the theta block which gives a great way to construct holomorphic Jacobi forms of small weight (see [GSZ18]).

1.4.2 Theta decomposition of Jacobi forms

It is well-known that Jacobi forms can be considered as vector-valued modular forms for the Weil representation. In order to state the result, we first introduce the theta functions for the positive definite even lattice L defined as

$$\Theta_\mu^L(\tau,\mathfrak{z}) = \sum_{\ell\in\mu+L} e^{\pi i((\ell,\ell)\tau + 2(\ell,\mathfrak{z}))}, \quad \mu \in L^\vee/L. \tag{1.4.10}$$

These series converge locally uniformly and are invariant with respect to the action of the integral Heisenberg group. Moreover, the set $\{\Theta_\mu^L : \mu \in L^\vee/L\}$ is linearly independent for fixed τ, hence $\{\Theta_\mu^L : \mu \in L^\vee/L\}$ is free over the ring of holomorphic functions on the upper half plan $\mathbb{H}$. Let $\Theta^L = (\Theta_\mu^L)_{\mu\in L^\vee/L}$. The following transformation formula for Θ^L can be found in [Oda77].

Proposition 1.4.6. *For any matrix* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, *the theta vector* Θ^L *has the following transformation property*

$$\Theta^L\left(\frac{a\tau+b}{c\tau+d}, \frac{\mathfrak{z}}{c\tau+d}\right) = (c\tau+d)^{\frac{\mathrm{rank}(L)}{2}} U(A) e^{\pi it\frac{c(\mathfrak{z},\mathfrak{z})}{c\tau+d}} \Theta^L(\tau,\mathfrak{z}),$$

where $U(A)$ *is a unitary matrix. In particular,*

$$U(T) = diag\left(e^{\pi i(\mu,\mu)}\right)_{\mu\in L^\vee/L},$$

$$U(S) = ([L^\vee : L])^{-\frac{1}{2}}(-i)^{\frac{\mathrm{rank}(L)}{2}} \left(e^{-2\pi i(\mu,\nu)}\right)_{\mu,\nu\in L^\vee/L}.$$

The above proposition means that the function $\Theta^L(\tau,0)$ is a modular form of weight $\frac{1}{2}\operatorname{rank}(L)$ for the Weil representation associated to the discriminant form of the lattice $L(-1)$. Denote the level of L by N. Let $\varphi \in J^{w.h.}_{k,L,1}$ be a weakly holomorphic Jacobi form of weight k and index one for L. Then we have the following representation (see [Gri95, Lemma 2.3])

$$\varphi(\tau,\mathfrak{z}) = \sum_{\mu\in L^\vee/L} \phi_\mu(\tau)\Theta^L_\mu(\tau,\mathfrak{z})$$

where

$$\varphi_\mu(\tau) = \sum_{\substack{r\in\mathbb{Z}\\ \frac{2r}{N}+(\mu,\mu)\in 2\mathbb{Z}}} f\left(\frac{2r+N(\mu,\mu)}{2N},\mu\right)\exp\left(\frac{2\pi i r\tau}{N}\right).$$

Example 1.4.7. Let E be an even positive definite unimodular lattice. The above proposition yields that the theta function

$$\Theta_E(\tau,\mathfrak{z}) = \sum_{\ell\in E} e^{\pi i(\ell,\ell)\tau+2\pi i(\ell,\mathfrak{z})} \in J_{\frac{\operatorname{rank}(E)}{2},E,1} \tag{1.4.11}$$

is a holomorphic Jacobi form of weight $\operatorname{rank}(E)/2$ and index 1 for E.

Let $\Phi(\tau) = (\varphi_\mu(\tau))_{\mu\in L^\vee/L}$. Then $\Phi(\tau)$ is a nearly holomorphic modular form of weight $k - \frac{\operatorname{rank}(L)}{2}$ for the Weil representation associated to the discriminant form $L^\vee/L$. Moreover, when φ is a holomorphic Jacobi form, the function $\Phi(\tau)$ will be a holomorphic modular form. Hence the space $J_{k,L,1}$ is finite dimensional, and $J_{k,L,1}$ is trivial if $k < \frac{\operatorname{rank}(L)}{2}$. The minimum possible weight $k = \frac{\operatorname{rank}(L)}{2}$ is called the **singular weight**. For any holomorphic Jacobi form of singular weight the hyperbolic norms of nonzero Fourier coefficients are always equal to zero. The above facts hold in the case of Jacobi forms of half-integral index or with character. In fact, if φ is a Jacobi form of weight k and index t for L, then $\varphi(\tau,2\mathfrak{z})$ is a Jacobi form of weight k and index $4t$ for L. If φ is a Jacobi form of weight k and index one with a character of order n for L, then the function $\varphi^{\otimes^n} = \varphi\otimes\cdots\otimes\varphi$ is a Jacobi form of weight nk and index one with the trivial character for the lattice $L^n = L\oplus\cdots\oplus L$.

We now give the isomorphism between modular forms for the Weil representation and Jacobi forms.

Theorem 1.4.8. *Let L be an even positive definite lattice with discriminant form $D(L)$. The map*

$$F(\tau) = \sum_{\gamma\in D(L)} F_\gamma(\tau)\boldsymbol{e}_\gamma \longmapsto \sum_{\gamma\in D(L)} F_\gamma(\tau)\Theta^L_\gamma(\tau,\mathfrak{z}) \tag{1.4.12}$$

defines an isomorphism between the vector spaces of nearly holomorphic modular forms for the Weil representation $\rho_{D(L)}$ and weakly holomorphic Jacobi forms of index one for L. It maps a vector-valued modular form of weight k to a Jacobi form of weight $k+\operatorname{rank}(L)/2$. The principal part of F corresponds to the singular Fourier coefficients of the above Jacobi form. The map also induces an isomorphism between the subspaces of holomorphic modular (resp. cusp) forms for the Weil representation $\rho_{D(L)}$ and holomorphic (resp. cusp) Jacobi forms of index one for L.

By means of the above isomorphism and Theorem 1.3.4, we obtain the following lifting.

Proposition 1.4.9. *Under the assumptions of Theorem 1.3.4, if we write*

$$F_{\Gamma_0(N),f,0}(\tau) = \sum_{\gamma\in D(L)} F_{\Gamma_0(N),f,0;\gamma}(\tau)e_\gamma,$$

then the function

$$\mathbb{J}_{\Gamma_0(N),f,0}(\tau,\mathfrak{z}) = \sum_{\gamma\in D(L)} F_{\Gamma_0(p),f,0;\gamma}(\tau)\Theta^L_\gamma(\tau,\mathfrak{z}) \tag{1.4.13}$$

is a weakly holomorphic Jacobi form of weight $k+\frac{1}{2}\operatorname{rank}(L)$ and index 1 for L which is invariant under the integral orthogonal group $\mathrm{O}(L)$. Moreover, this application maps holomorphic modular (resp. cusp) forms to holomorphic (resp. cusp) Jacobi forms.

Example 1.4.10. As an application, we use this lifting to construct the generator of weight -4 of the ring of Weyl invariant D_8 weak Jacobi forms. Fix $D_8 = \{x \in \mathbb{Z}^8 : \sum_{i=1}^{8} x_i = 0 \mod 2\}$. Its level is 2. In this case, the input of the above lifting is the modular forms for the congruence subgroup $\Gamma_0(2)$ with trivial character. We construct this generator as

$$\begin{aligned}\varphi_{-4,D_8} =& \mathbb{J}_{\Gamma_0(2),\theta_{10}^8(\tau)-\frac{1}{2}E_4(\tau),0}/\Delta \\ =& 128 - 16\sum_{i=1}^{8}\zeta_i^{\pm 1} + \frac{1}{2}\sum \zeta_1^{\pm\frac{1}{2}}\zeta_2^{\pm\frac{1}{2}}\cdots\zeta_8^{\pm\frac{1}{2}} + O(q) \in J^{w,\mathrm{O}(D_8)}_{-4,D_8,1},\end{aligned} \tag{1.4.14}$$

where $\theta_{10}(\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{(2n+1)^2}{8}}$ is a theta constant of order two and $\zeta_i = e^{2\pi i z_i}$.

1.4.3 Differential operators

In this subsection we recall the following weight raising differential operator which will be used later. Such technique can also be found in [CK00] for the general case or in [EZ85] for classical Jacobi forms.

Lemma 1.4.11. *Let $\psi(\tau,\mathfrak{z}) = \sum a(n,l)q^n\zeta^l$ be a weakly holomorphic Jacobi form of weight k and index one for L. Then $H_k(\psi)$ is a weakly holomorphic Jacobi form of weight $k+2$ and index one for L, where*

$$H_k(\psi) = H(\psi) + (2k - \operatorname{rank}(L))G_2\psi, \tag{1.4.15}$$

$$H(\psi)(\tau,\mathfrak{z}) = \frac{1}{2}\sum_{n\in\mathbb{Z}}\sum_{l\in L^\vee}[2n-(l,l)]\,a(n,l)q^n\zeta^l, \tag{1.4.16}$$

and $G_2(\tau) = -\frac{1}{24} + \sum_{n\geq 1}\sigma(n)q^n$ is the Eisenstein series of weight 2 on $\mathrm{SL}_2(\mathbb{Z})$.

Proof. Let $\{\alpha_1,\ldots,\alpha_n\}$ be a basis of L and $\{\alpha_1^*,\ldots,\alpha_n^*\}$ be its dual basis. We write $\mathfrak{z} = \sum_{i=1}^n z_i\alpha_i \in L\otimes\mathbb{C}$, $z_i \in \mathbb{C}$. We define $\frac{\partial}{\partial\mathfrak{z}} = \sum_{i=1}^n \alpha_i^*\frac{\partial}{\partial z_i}$. Then we have that

$$\left(\frac{\partial}{\partial\mathfrak{z}},\frac{\partial}{\partial\mathfrak{z}}\right)e^{2\pi i(l,\mathfrak{z})} = -4\pi^2(l,l)e^{2\pi i(l,\mathfrak{z})}$$

and the operator $H(\cdot)$ is equal to the heat operator

$$H = \frac{1}{2\pi i}\frac{\partial}{\partial\tau} + \frac{1}{8\pi^2}\left(\frac{\partial}{\partial\mathfrak{z}},\frac{\partial}{\partial\mathfrak{z}}\right).$$

By formulas (3.5) and (3.7) in [CK00, Lemma 3.3], the transformations of the function $H(\psi)$ with respect to the actions of $\mathrm{SL}_2(\mathbb{Z})$ and the Heisenberg group of L are known:

$$\begin{aligned}H(\psi|_{k,1}A) &= H(\psi)|_{k+2,1}A + \frac{1}{4\pi i}(\operatorname{rank}(L)-2k)\frac{c}{c\tau+d}\psi|_{k,1}A,\\ H(\psi|_{k,1}[x,y:(x,y)/2]) &= H(\psi)|_{k+2,1}[x,y:(x,y)/2],\end{aligned}$$

for $A\in\mathrm{SL}_2(\mathbb{Z})$ and $x,y\in L$. From these transformations, we see that $H(\psi)$ does not transform like a Jacobi form, so we make an automorphic correction by considering the quasi-modular Eisenstein series G_2 of weight 2. By direct calculations, we can show that $H_k(\psi)$ is invariant under $\mathrm{SL}_2(\mathbb{Z})$ and the Heisenberg group. Therefore, it is a weakly holomorphic Jacobi form of weight $k+2$ and index one for L. □

The next lemma gives a quite useful identity related to singular Fourier coefficients of any Jacobi form of weight zero, which plays a key role in this thesis. It was proved in [Gri12, Proposition 2.2]. Its variant in the context of vector-valued modular forms was first proved in [Bor98, Theorem 10.5].

Lemma 1.4.12. *Assume that ϕ is a weakly holomorphic Jacobi form of weight 0 and index 1 for L with the Fourier expansion*

$$\phi(\tau,\mathfrak{z}) = \sum_{n\in\mathbb{Z}}\sum_{\ell\in L^\vee} f(n,\ell)q^n\zeta^\ell.$$

Then we have the following identity

$$C := \frac{1}{24}\sum_{\ell\in L^\vee} f(0,\ell) - \sum_{n<0}\sum_{\ell\in L^\vee} f(n,\ell)\sigma(-n) = \frac{1}{2\,\mathrm{rank}(L)}\sum_{\ell\in L^\vee} f(0,\ell)(\ell,\ell). \tag{1.4.17}$$

Moreover, we have

$$\sum_{\ell\in L^\vee} f(0,\ell)(\ell,\mathfrak{z})^2 = 2C(\mathfrak{z},\mathfrak{z}), \quad \forall \mathfrak{z}\in L\otimes\mathbb{C}. \tag{1.4.18}$$

Proof. From Lemma 1.4.11, it follows that $H_0(\phi)$ is a weakly holomorphic Jacobi form of weight 2. Therefore $H_0(\phi)(\tau,0)$ is a nearly holomorphic modular form of weight 2 for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. By [Bor95, Lemma 9.2], the modular form $H_0(\phi)(\tau,0)$ has zero constant term, which establishes the first desired identity.

In order to prove the second identity, we consider the automorphic correction of ϕ as in [Gri99b, Proposition 1.5]. We define

$$\Psi(\phi)(\tau,\mathfrak{z}) := e^{-4\pi^2 G_2(\tau)(\mathfrak{z},\mathfrak{z})}\phi(\tau,\mathfrak{z}).$$

By direct calculations, we obtain

$$\Psi(\phi)\left(\frac{a\tau+b}{c\tau+d},\frac{\mathfrak{z}}{c\tau+d}\right) = \Psi(\phi)(\tau,\mathfrak{z}), \quad \forall \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z}).$$

Let $t\in\mathbb{R}$ and define $\Phi(\phi)(t) = \Psi(\phi)(\tau,t\mathfrak{z})$. We consider the Taylor expansion of $\Phi(\phi)(t)$ at the point $t=0$

$$\Phi(\phi)(t) = \sum_{n\geq 0} f_n(\tau,\mathfrak{z})t^n.$$

From the relation

$$\frac{d\Phi(\phi)}{dt} = \frac{\partial\Phi(\phi)}{\partial t\mathfrak{z}}\frac{\partial t\mathfrak{z}}{\partial t} = \frac{\partial\Psi(\phi)}{\partial\mathfrak{z}}(\tau,t\mathfrak{z})\frac{\partial t\mathfrak{z}}{\partial t},$$

we derive that the function $f_n(\tau,\mathfrak{z})$ is a nearly holomorphic modular form of weight n on $\mathrm{SL}_2(\mathbb{Z})$ for any $\mathfrak{z}\in L\otimes\mathbb{C}$. Therefore, the modular form f_2 has weight 2 and then its constant term is zero, which yields the second identity. □

Using the above Lemma, we give a simple proof of the following result which was firstly proved by Borcherds in [Bor98, Theorem 11.2].

Corollary 1.4.13. *Let ϕ be a weakly holomorphic Jacobi form of weight 0 and index 1 for L with the Fourier expansion*

$$\phi(\tau,\mathfrak{z}) = \sum_{n\in\mathbb{Z}}\sum_{\ell\in L^\vee} f(n,\ell)q^n\zeta^\ell.$$

Assume that $f(n,\ell)\in\mathbb{Z}$ for all $n\leq 0$ and $\ell\in L^\vee$. Let $(n(L))$ denote the ideal of $\mathbb{Z}$ generated by (x,y), $x,y\in L$. Then we have

$$\frac{n(L)}{24}\sum_{\ell\in L^\vee} f(0,\ell)\in\mathbb{Z}.$$

Proof. By (1.4.18), we have

$$\sum_{\ell\in L^\vee} f(0,\ell)(\ell,x)(\ell,y) = 2C(x,y), \quad \forall x,y\in L.$$

Since $f(0,\ell)=f(0,-\ell)\in\mathbb{Z}$, we get $C(x,y)\in\mathbb{Z}$ for all $x,y\in L$, which yields $n(L)C\in\mathbb{Z}$. We thus complete the proof by (1.4.17). □

Remark 1.4.14. Let Λ be an even positive definite unimodular lattice of rank 24. Assume that the set R_Λ of roots of Λ is not empty. Let $R(\Lambda)$ denote the root lattice generated by R_Λ. The theta-function for the lattice Λ is a holomorphic Jacobi form of weight 12 and index 1 for Λ. Thus, we have

$$\psi_{0,\Lambda}(\tau,\mathfrak{z}) = \frac{\Theta_\Lambda(\tau,\mathfrak{z})}{\Delta(\tau)} = q^{-1} + \sum_{r\in R_\Lambda}\zeta^r + 24 + O(q) \in J^{w.h.}_{0,\Lambda,1}.$$

By Lemma 1.4.12, we prove the identity $\sum_{r\in R_\Lambda}(r,\mathfrak{z})^2 = 2h(\mathfrak{z},\mathfrak{z})$. From this, it follows that the lattice $R(\Lambda)$ has rank 24 and all its irreducible components have the same Coxeter numbers. In this thesis, we shall use the similar idea to classify reflective modular forms.

1.4.4 Additive liftings and Borcherds products

In this subsection we introduce the additive lifting and Borcherds product, which are two main ways to construct orthogonal modular forms from Jacobi forms.

We first recall the following index raising operator introduced in [Gri91, Corollary 1].

Proposition 1.4.15. *Let $\varphi\in J^w_{k,L,t}$. For any positive integer m, we have*

$$\varphi|_{k,t}T_-(m)(\tau,\mathfrak{z}) = m^{-1}\sum_{\substack{ad=m,a>0\\0\le b<d}} a^k\varphi\left(\frac{a\tau+b}{d}, a\mathfrak{z}\right)\in J^w_{k,L,mt}.$$

If $f(n,\ell)$ are the Fourier coefficients of φ, then the Fourier coefficients of $\varphi|_{k,t}T_-(m)(\tau,\mathfrak{z})$ are given by

$$f_m(n,\ell) = \sum_{\substack{a\in\mathbb{N}\\a|(n,\ell,m)}} a^{k-1} f\left(\frac{nm}{a^2},\frac{\ell}{a}\right),$$

here $a\,|\,(n,\ell,m)$ means that $a\,|\,(n,m)$ and $a^{-1}\ell\in L^\vee$.

We now introduce the additive lifting. We refer to [Gri95, Theorem 3.1] for a proof.

Theorem 1.4.16. *Let $\varphi\in J_{k,L,t}$. Then the function*

$$\mathrm{Grit}(\varphi)(Z) = f(0,0)G_k(\tau) + \sum_{m\ge1}\varphi|_{k,t}T_-(m)(\tau,\mathfrak{z})e^{2\pi i m\omega}$$

defines a modular form of weight k with respect to $\widetilde{\mathrm{O}}^+(2U\oplus L(-t))$, where $G_k = 1+O(q)$ is the normalized Eisenstein series of weight k on $\mathrm{SL}_2(\mathbb{Z})$. If φ is a Jacobi cusp form, then $\mathrm{Grit}(\varphi)$ will be a cusp form.

Since $f(0,0)=0$ if $k<4$ or k is odd, the first term of $\mathrm{Grit}(\varphi)$ is well defined. We note that the additive lifting is symmetric i.e. $\mathrm{Grit}(\tau,\mathfrak{z},\omega)=\mathrm{Grit}(\omega,\mathfrak{z},\tau)$. The cusp condition follows from [Ma18, Theorem 3.5]. In fact, one derives the cusp condition from the calculation of the Fourier expansion of $\mathrm{Grit}(\varphi)$ at every 0-dimensional cusp due to Borcherds (see [Bor98, Theorem 14.3] where Borcherds used the language of modular forms for the Weil representation to give an extended version of additive Jacobi lifting).

The cusp condition above means that the additive lifting is a cusp form if it vanishes at the standard cusp (0-dimensional cusp of level 1, namely $c = e$). This does not hold for general modular forms. By [GHS07, Theorem 4.2], when every isotropic subgroup of the discriminant group of $2U \oplus L(-1)$ is cyclic, a modular form G is cuspidal if it vanishes at the standard cusp because in this case the closure of every 1-dimensional cusp contains the standard 0-dimensional cusp.

There are also index raising operators and additive liftings for Jacobi forms with character (see [CG13, GN98b]).

Proposition 1.4.17 (see Proposition 3.1 in [CG13])**.** *Let $\varphi \in J^w_{k,L,t}(v_\eta^D \times \nu)$ be a weak Jacobi forms of weight k and index t for a positive definite even lattice L and it is not identically zero. We assume that k is integral, t is rational and D is an even divisor of 24. If $Q = \frac{24}{D}$ is odd, we also assume that the character of the minimal integral Heisenberg group ν is trivial. Then for any natural m coprime to Q, we have*

$$\varphi|_{k,t}T_-^{(Q)}(m)(\tau,\mathfrak{z}) = m^{-1}\sum_{\substack{ad=m, a>0\\ 0\le b<d}} a^k v_\eta^D(\sigma_a)\varphi\left(\frac{a\tau+bQ}{d}, a\mathfrak{z}\right) \in J^w_{k,L,mt}(v_\eta^{D\cdot x}\times\nu),$$

where $x, y \in \mathbb{Z}$ such that $mx + Qy = 1$, and

$$\sigma_a = \begin{pmatrix} dx + Qdxy & -Qy \\ Qy & a \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Moreover, if we assume $f(n,\ell)$ are the Fourier coefficients of φ, then the Fourier coefficients of $\varphi|_{k,t}T_-^{(Q)}(m)(\tau,\mathfrak{z})$ have the following form

$$f_m(n,\ell) = \sum_{a|(n,\ell,m)} a^{k-1} v_\eta^D(\sigma_a) f\left(\frac{nm}{a^2}, \frac{\ell}{a}\right).$$

Theorem 1.4.18 (see Theorem 3.2 in [CG13])**.** *Let $\varphi \in J_{k,L,t}(v_\eta^D \times \nu)$, k be integral, t be rational and D be an even divisor of 24. If the conductor $Q = 24/D$ is odd, we assume that ν is trivial. Then the function*

$$\mathrm{Grit}(\varphi)(Z) = f(0,0)G_k(\tau) + \sum_{\substack{m\equiv 1 \bmod Q\\ m>0}} \varphi|_{k,t}T_-^{(Q)}(m)(\tau,\mathfrak{z})e^{2\pi i\frac{m}{Q}\omega}$$

is a modular form of weight k with respect to the stable orthogonal group $\widetilde{O}^+(2U \oplus L(-Qt))$ of the even lattice $L(Qt)$ with a character χ of order Q induced by $v_\eta^D \times \nu$ and the relations $\chi([0,0:r]) = e^{2\pi i\frac{r}{Q}}$, $\chi(V) = 1$, where $V : (\tau,\mathfrak{z},\omega) \to (\omega,\mathfrak{z},\tau)$.

The input of original Borcherds lifting is the modular forms for the Weil representation. The constructed modular forms have nice infinite product expansions at the rational 0-dimensional cusps. By means of the isomorphism between modular forms for the Weil representation and Jacobi forms (see 1.4.8), Gritsenko and Nikulin [GN98b] proposed a variant of Borcherds products, which lifts weakly holomorphic Jacobi forms of weight 0 to modular forms on orthogonal groups. In this case, any constructed modular form has a nice product expansion at each rational 1-dimensional cusp and can be expressed as a product of a general theta block with the exponential of additive lifting.

Theorem 1.4.19 (see Theorem 3.1 in [Gri12])**.** *Let*

$$\varphi(\tau,\mathfrak{z}) = \sum_{n\in\mathbb{Z}}\sum_{\ell\in L^\vee} f(n,\ell)q^n\zeta^\ell \in J^{w.h.}_{0,L,1}.$$

Assume that $f(n,\ell) \in \mathbb{Z}$ *for all* $2n - (\ell,\ell) \le 0$. *We set*

$$A = \frac{1}{24}\sum_{\ell \in L^\vee} f(0,\ell), \qquad \vec{B} = \frac{1}{2}\sum_{\ell>0} f(0,\ell)\ell, \qquad C = \frac{1}{2\,\mathrm{rank}(L)}\sum_{\ell\in L^\vee} f(0,\ell)(\ell,\ell).$$

Then the product

$$\mathrm{Borch}(\varphi)(Z) = q^A\zeta^{\vec{B}}\xi^C \prod_{\substack{n,m\in\mathbb{Z},\ell\in L^\vee\\ (n,\ell,m)>0}} (1-q^n\zeta^\ell\xi^m)^{f(nm,\ell)},$$

where $Z = (\tau,\mathfrak{z},\omega) \in \mathcal{H}(L)$, $q = \exp(2\pi i\tau)$, $\zeta^\ell = \exp(2\pi i(\ell,\mathfrak{z}))$, $\xi = \exp(2\pi i\omega)$, *defines a meromorphic modular form of weight* $f(0,0)/2$ *with respect to the stable orthogonal group* $\widetilde{\mathrm{O}}^+(2U\oplus L(-1))$ *with a character* χ *induced by*

$$\chi|_{\mathrm{SL}_2(\mathbb{Z})} = v_\eta^{24A}, \qquad \chi|_{H(L)}([\lambda,\mu;r]) = e^{\pi i C((\lambda,\lambda)+(\mu,\mu)-(\lambda,\mu)+2r)}, \qquad \chi(V) = (-1)^D,$$

where $V : (\tau,\mathfrak{z},\omega) \to (\omega,\mathfrak{z},\tau)$ *and* $D = \sum_{n<0}\sigma_0(-n)f(n,0)$. *The poles and zeros of* $\mathrm{Borch}(\varphi)$ *lie on the rational quadratic divisors* $\mathcal{D}_v(2U\oplus L(-1))$, *where* $v \in 2U\oplus L^\vee(-1)$ *is a primitive vector with* $(v,v)<0$. *The multiplicity of this divisor is given by*

$$\mathrm{mult}\,\mathcal{D}_v(2U\oplus L(-1)) = \sum_{d\in\mathbb{Z},d>0} f(d^2n, d\ell),$$

where $n\in\mathbb{Z}$, $\ell\in L^\vee$ *such that* $(v,v) = 2n-(\ell,\ell)$ *and* $v \equiv \ell \mod 2U\oplus L(-1)$. *Moreover, we have*

$$\mathrm{Borch}(\varphi) = \psi_{L,C}(\tau,\mathfrak{z})\xi^C \exp\left(-\mathrm{Grit}(\varphi)\right),$$

where

$$\psi_{L,C}(\tau,\mathfrak{z}) = \eta(\tau)^{f(0,0)}\prod_{\ell>0}\left(\frac{\vartheta(\tau,(\ell,\mathfrak{z}))}{\eta(\tau)}\right)^{f(0,\ell)}. \tag{1.4.19}$$

Remark 1.4.20. A finite multiset of vectors $\{\ell; m(\ell)\}$ from $L^\vee$ (one takes every vector $m(\ell)$ times) satisfying (1.4.18) is called *vector system* defined in [Bor95, §6]. Thus the finite multiset set $X = \{\ell; f(0,\ell)\}$ from Theorem 1.4.19 is a vector system and we can define its *Weyl chamber* as a connected component of

$$L\otimes\mathbb{R}\backslash\left(\bigcup_{x\in X\backslash\{0\}}\{v\in L\otimes\mathbb{R} : (x,v)=0\}\right).$$

Let W be a fixed Weyl chamber. For $\ell\in L^\vee$ we define an ordering on $L^\vee$ by

$$\ell > 0 \iff \exists\, w\in W \ s.t.\ (\ell,w)>0.$$

The notation $(n,\ell,m)>0$ in Theorem 1.4.19 means that either $m>0$, or $m=0$ and $n>0$, or $m=n=0$ and $\ell<0$.

The vector $(A,\vec{B},C)$ is called the **Weyl vector** of the Borcherds product.

The following result is called the **Eichler criterion** introduced by Eichler [Eic52].

Proposition 1.4.21. *Let* $M = U\oplus U_1\oplus L(-1)$ *be an even lattice containing two hyperbolic planes, where* $U = \mathbb{Z}e+\mathbb{Z}f$ *with* $(e,e)=(f,f)=0$, $(e,f)=1$. *If* u, $v\in M$ *are primitive,* $(u,u)=(v,v)$ *and* $\frac{u}{\mathrm{div}(u)} \equiv \frac{v}{\mathrm{div}(v)} \mod M$, *then there exists* $\sigma\in E_U(L_1)$ *such that* $\sigma(u)=v$, *where*

$$E_U(L_1) = \langle\{t(e,a), t(f,a) : a\in U_1\oplus L(-1)\}\rangle$$

and $t(e,a)$, $t(f,a)$ *are the Eichler transvections (see* (1.2.11)*).*

The Eichler criterion tells that the $\widetilde{\mathrm{O}}^+(2U \oplus L(-1))$-orbit of any primitive vector $v \in 2U \oplus L^\vee(-1)$ is uniquely determined by its norm (v,v) and by the image of v in the discriminant group of $2U \oplus L(-1)$. Therefore, there exists $(0,n,\ell,1,0) \in 2U \oplus L^\vee(-1)$ such that $(v,v) = 2n-(\ell,\ell) < 0$ and $v \equiv \ell \bmod 2U \oplus L(-1)$, which implies

$$\widetilde{\mathrm{O}}^+(2U \oplus L(-1)) \cdot \mathcal{D}_v(2U \oplus L(-1)) = \widetilde{\mathrm{O}}^+(2U \oplus L(-1))\langle \{Z \in \mathcal{H}(L) : n\tau - (\ell, \mathfrak{z}) + \omega = 0\}\rangle.$$

At the end of this section, we discribe the generators of orthogonal groups following [GHS09]. Let $M = U \oplus U_1 \oplus L(-1)$ and $L_1 = U_1 \oplus L(-1)$. Note that $E_U(L_1) \subset \widetilde{\mathrm{SO}}^+(M)$ and $\widetilde{\mathrm{O}}^+(M) = \langle \widetilde{\mathrm{SO}}^+(M), V\rangle$. By [GHS09, Proposition 3.3], we have

$$\begin{aligned} &E_U(L_1) < \langle \Gamma^J(L), V\rangle, \\ &\mathrm{O}^+(M) = \langle E_U(L_1), \mathrm{O}(L_1)\rangle, \\ &\widetilde{\mathrm{O}}^+(M) = \langle E_U(L_1), \widetilde{\mathrm{O}}^+(L_1)\rangle. \end{aligned}$$

For any prime p the p-rank of M, denoted by $\mathrm{rank}_p(M)$, is the maximal rank of the sublattices M' in M such that $disc(M')$ is coprime to p. By [GHS09, Corollary 1.8, Proposition 3.4], if $\mathrm{rank}_2(M) \geq 6$ and $\mathrm{rank}_3(M) \geq 5$, then

$$\widetilde{\mathrm{SO}}^+(M) = E_U(L_1), \qquad \widetilde{\mathrm{O}}^+(M) = \langle \Gamma^J(L), V\rangle,$$

and the group $\widetilde{\mathrm{O}}^+(M)$ has only one non-trivial character, namely det, and $\widetilde{\mathrm{SO}}^+(M)$ has no non-trivial character. Besides, $\mathrm{O}^+(M) = \langle \widetilde{\mathrm{O}}^+(M), \mathrm{O}(L)\rangle$ if the natural homomorphism $\mathrm{O}(L) \to \mathrm{O}(D(L))$ is surjective.

Chapter 2

Reflective modular forms

2.1 Reflective modular forms and 2-reflective modular forms

$\boxed{\text{Let } M \text{ be an even lattice of signature } (2,n) \text{ with } n \geq 3.}$ The **reflection** with respect to the hyperplane defined by an anisotropic vector r is given by

$$\sigma_r(x) = x - \frac{2(r,x)}{(r,r)}r, \quad x \in M. \tag{2.1.1}$$

A primitive vector $l \in M$ of negative norm is called **reflective** (or a **root**) if the reflection σ_r is in $\mathrm{O}^+(M)$. If $v \in M^\vee$ and $(v,v) < 0$, the rational quadratic divisor $\mathcal{D}_v(M) = v^\perp \cap \mathcal{D}(M)$ is called a **reflective divisor** if $\sigma_v \in \mathrm{O}^+(M)$. For $\lambda \in D(M) = M^\vee/M$ and $m \in \mathbb{Q}$ we define

$$\mathcal{H}(\lambda, m) = \bigcup_{\substack{v \in M+\lambda \\ (v,v)=2m}} \mathcal{D}_v(M) \tag{2.1.2}$$

as the **Heegner divisor** of discriminant (λ, m).

Remark that a primitive vector $l \in M$ with $(l,l) = -2d$ is reflective if and only if $\mathrm{div}(l) = 2d$ or d. We set $\lambda = [l/\mathrm{div}(l)] \in D(M)$. Then $\mathcal{D}_\lambda(M)$ is contained in $\mathcal{H}(\lambda, -1/(4d))$ in the first case, and is contained in

$$\mathcal{H}(\lambda, -1/d) - \sum_{2\nu=\lambda} \mathcal{H}(\nu, -1/(4d))$$

in the second case. In particular, when the lattice M is of prime level p, a primitive vector $l \in M$ is reflective if and only if $(l,l) = -2$ and $\mathrm{div}(l) = 1$, or $(l,l) = -2p$ and $\mathrm{div}(l) = p$.

Following [GN97, GN98b], we define reflective modular forms and 2-reflective modular forms in the following way.

Definition 2.1.1. Let F be a non-constant holomorphic modular form on $\mathcal{D}(M)$ with respect to a finite index subgroup $\Gamma < \mathrm{O}^+(M)$ and a character (of finite order) $\chi : \Gamma \to \mathbb{C}$. The function F is called **reflective** if its support of zero divisor is set-theoretically contained in the union of reflective divisors associated to M. A reflective modular form F is called **strongly reflective** if the multiplicity of every irreducible component of $\mathrm{div}(F)$ is equal to one. A lattice M is called **reflective** if it admits a reflective modular form.

We define

$$\mathcal{H} = \mathcal{H}(0,-1) = \bigcup_{\substack{v \in M \\ (v,v)=-2}} \mathcal{D}_v(M) \tag{2.1.3}$$

as the Heegner divisor of $\mathcal{D}(M)$ generated by the (-2)-vectors in M.

Definition 2.1.2. A non-constant holomorphic modular form on $\mathcal{D}(M)$ is called **2-reflective** if its support of divisor is contained in $\mathcal{H}$. A 2-reflective modular form F is called a modular form with **complete 2-divisor** if $\operatorname{div}(F) = \mathcal{H}$. A lattice M is called **2-reflective** if it admits a 2-reflective modular form.

Note that all (-2)-vectors are reflective. Therefore, 2-reflective modular forms are particular case of reflective modular forms.

In order to narrow the class of reflective modular forms that we actually deal with, we introduce many basic reductions following Ma [Ma17, §2]. Firstly, we can kill the characters. In fact, if F is a reflective (resp. 2-reflective) modular form with respect to $\Gamma < \mathrm{O}^+(M)$ and a character $\chi : \Gamma \to \mathbb{C}$, then F^d is a reflective (resp. 2-reflective) modular form with respect to Γ and the trivial character where d is the order of χ. Secondly, we are free to change the arithmetic group Γ inside $\mathrm{O}^+(M)$.

Lemma 2.1.3. *Assume that M admits a reflective (resp. 2-reflective) modular form with respect to some $\Gamma < \mathrm{O}^+(M)$. Then M also has a reflective (resp. 2-reflective) modular form with respect to any other finite index subgroup $\Gamma' < \mathrm{O}^+(M)$.*

Proof. Let F be a given reflective (resp. 2-reflective) modular form with respect to Γ. We set $\Gamma'' = \Gamma \cap \Gamma'$. We choose representatives $\gamma_1, ..., \gamma_s \in \Gamma'$ of the coset $\Gamma'' \backslash \Gamma'$ and take the product

$$F' = \prod_{i=1}^{s} (F|_{\gamma_i}).$$

This is a modular form with respect to Γ'. For any $\gamma \in \mathrm{O}^+(M)$, we have that $\gamma(\mathcal{H}) = \mathcal{H}$ because γ preserves M. In addition, for a reflective vector v of M,

$$\begin{aligned}\sigma_{\gamma(v)}(x) &= x - \frac{2(x,\gamma(v))}{(\gamma(v),\gamma(v))}\gamma(v) \\ &= \gamma\left(\gamma^{-1}(x) - \frac{2(\gamma^{-1}(x),v)}{(v,v)}v\right) \in M\end{aligned}$$

for any $x \in M$, which implies that $\gamma(v)$ is also reflective. Therefore, F' is a reflective (resp. 2-reflective) modular form with respect to Γ'. □

By Lemma 2.1.3, a lattice M is reflective (resp. 2-reflective) if and only if it admits a reflective (resp. 2-reflective) modular form with respect to $\widetilde{\mathrm{O}}^+(M)$. In view of the Eichler criterion (see Proposition 1.4.21), throughout this chapter, we only consider reflective (resp. 2-reflective) modular forms with respect to $\widetilde{\mathrm{O}}^+(M)$.

Lemma 2.1.4. *Assume that M admits a 2-reflective modular form. Then any even overlattice M' of M has a 2-reflective modular form too.*

Proof. Let F be the given 2-reflective modular form for M. Since $M \otimes \mathbb{Q} = M' \otimes \mathbb{Q}$, we can identify $\mathcal{D}(M)$ with $\mathcal{D}(M')$ canonically. Since (-2)-vectors in M are also (-2)-vectors in M', the (-2)-Heegner divisor on $\mathcal{D}(M)$ is contained in the (-2)-Heegner divisor on $\mathcal{D}(M')$ under this identification. Hence we may view as F as a 2-reflective modular form on $\mathcal{D}(M')$ with respect to $\widetilde{\mathrm{O}}^+(M)$. Since $M < M' < M'^\vee < M^\vee$, the group $\widetilde{\mathrm{O}}^+(M)$ can be viewed as a finite index subgroup of $\widetilde{\mathrm{O}}^+(M')$. Thus, the claim follows from Lemma 2.1.3 applied to M'. □

Remark that Lemma 2.1.4 does not hold for reflective modular forms because $\mathrm{O}^+(M)$ is not contained in $\mathrm{O}^+(M')$ in general and a reflective divisor $\mathcal{D}_v$ in $\mathcal{D}(M)$ is usually not a reflective divisor in $\mathcal{D}(M')$.

Remark 2.1.5. Let Γ be a finite index subgroup of $\mathrm{O}^+(M)$. It was proved in [GHS07, Corollary 2.13] that the branch divisor of the modular projection

$$\pi_\Gamma : \mathcal{D}(M) \longrightarrow \Gamma \backslash \mathcal{D}(M)$$

is the union of the reflective divisors with respect to Γ:

$$\mathrm{Bdiv}(\pi_\Gamma) = \bigcup_{\substack{r \in M^\vee \\ \sigma_r \in \Gamma \cup -\Gamma}} \mathcal{D}_r(M). \tag{2.1.4}$$

In view of this fact, Gritsenko and Nikulin called a holomorphic modular form F for Γ reflective if its support of zero divisor is contained in $\mathrm{Bdiv}(\pi_\Gamma)$. Our definition (Definition 2.1.1) is more convenient for classification. However, the definition of Gritsenko and Nikulin is more convenient for geometric application. The existence of cusp forms of small weight ($< n$) with big divisor ($\mathrm{div}(F) \geq \mathrm{Bdiv}(\pi_\Gamma)$) may imply that the modular variety $\mathcal{F}_M(\Gamma)$ is of general type (see [GHS07, Theorem 1.1]). The existence of modular forms of big weight ($\geq n$) with small divisor ($\mathrm{div}(F) \leq \mathrm{Bdiv}(\pi_\Gamma)$) may yield that the modular variety $\mathcal{F}_M(\Gamma)$ is uniruled or has Kodaira dimension 0 (see [Gri10, Theorem 1.5] and [GH14, Theorem 2.1]).

2.2 Known classification of reflective modular forms

It is a very interesting and difficult problem to find all reflective lattices and all their reflective automorphic forms. In this section we introduce many known results in this direction.

Gritsenko–Nikulin's results

In 1998, in view of the Koecher principle for orthogonal modular forms, Gritsenko and Nikulin [GN98a, Conjecture 2.2.1] first made the following conjecture.

Conjecture 2.2.1. *Up to scaling there are only finitely many reflective lattices of signature* $(2, n)$ *with* $n \geq 3$.

The above conjecture was first formulated in [Nik96] for 2-reflective modular forms. We remark that Gritsenko and Nikulin [GN98a, GN00] also formulated a conjecture called "*Arithmetic Mirror Symmetry Conjecture*" which relates the hyperbolic reflective lattices to reflective lattices. When $n = 3$, Gritsenko and Nikulin gave some classification in [GN98b, §5.2] and [GN02, §2]. Looijenga [Loo03] proved one part of the arithmetic mirror symmetry conjecture, which might give a new approach to classify reflective modular forms.

Scheithauer's results

In his series of works, Scheithauer gave a complete classification of strongly reflective modular forms of singular weight on lattices of prime level [Sch04, Sch06, Sch15, Sch17]. We next introduce the main classification results of Scheithauer. We first describe his way to construct reflective modular forms following [Sch06].

Let Co_0 be the Conway's group i.e. the automorphism group of the Leech lattice Λ. Let $g \in Co_0$ be an element of order n. The characteristic polynomial of g on $\Lambda \otimes \mathbb{Q}$ can be written as $\prod (x^k - 1)^{b_k}$ where k ranges over the positive divisors of n. The symbol $\prod_{k|n} k^{b_k}$ is called the cycle shape of g. We define the eta product associated to g as

$$\eta_g(\tau) = \prod_{k|n} \eta(k\tau)^{b_k},$$

which is a modular function of trivial character for a group of level N. The smallest N with this property is called the level of g. We further assume that the level N of g is squarefree and the fixpoint lattice Λ^g of g is nontrivial. In this case, g has order N, the lattice Λ^g is of level N and η_g is a modular form for $\Gamma_0(N)$ of weight $\operatorname{rank}(\Lambda^g)/2$ and character χ_D, where D is the discriminant form of the lattice

$$M = U \oplus U(N) \oplus \Lambda^g.$$

The function $f_g = 1/\eta_g$ can be lifted to a modular form $F_g = F_{\Gamma_0(N),f_g,0}$ of weight $-\operatorname{rank}(\Lambda^g)/2$ for the Weil representation ρ_D. Then, the function F_g can be lifted to a Borcherds product, which turns out to be strongly reflective and to have singular weight. Further possible reflective modular forms can be obtained by applying Aktin-Lehner involutions to f_g and M.

We now state Scheithauer's classification result. We first recall the following definition.

If a modular form can be constructed as a Borcherds product of a vector-valued modular form invariant under $\mathrm{O}(D(M))$, it is called **symmetric**, otherwise it is called **non-symmetric**.

Theorem 2.2.2 (Theorem 6.28 in [Sch17]). *Let M be a lattice of prime level p and signature $(2,n)$ with $n \geq 3$ and let Ψ be a strongly reflective Borcherds product of singular weight on M. Then, as a function on the corresponding Hermitian symmetric domain $\mathcal{D}(M)$, the modular form Ψ can be identical to the Borcherds product of one of the following vector-valued modular forms in Table 2.1.*

With three exceptions, all these functions come from symmetric modular forms. Moreover, at a suitable cusp Ψ is the twisted denominator identity of the fake monster algebra by the corresponding element in the Conway's group.

The functions F_1, F_2 and F_3 are not invariant under $\mathrm{O}(D(M))$ and are constructed using other type of liftings (see [Sch17]). The Gram matrices of $\Lambda^{1^2 11^2}$ and $\Lambda^{1^1 23^1}$ are respectively

$$\Lambda^{1^2 11^2} = \begin{pmatrix} 4 & 1 & 0 & -2 \\ 1 & 4 & 2 & 0 \\ 0 & 2 & 4 & 1 \\ -2 & 0 & 1 & 4 \end{pmatrix}, \qquad \Lambda^{1^1 23^1} = \begin{pmatrix} 4 & 1 \\ 1 & 6 \end{pmatrix}.$$

The number t in the column associated with "zeros" indicates that the corresponding modular form has zeros of order one coming from the reflective vectors of norm $2t$. When the modular form is symmetric, its zeros contain all reflective divisors associated to vectors of norm $2t$. When the modular form is non-symmetric, its zeros only contain partial reflective divisors associated to vectors of norm $2t$.

We remark that Barnes-Wall lattice, $E_8(2)$, Coxeter-Todd lattice, $E_6^\vee(3)$, Maass lattice, $A_4^\vee(5)$, Barnes-Craig lattice, $\Lambda^{1^2 11^2}$ and $\Lambda^{1^1 23^1}$ are the fixpoint lattices of the elements of cycle shapes $1^8 2^8$, $1^{-8} 2^{16}$, $1^6 3^6$, $1^{-3} 3^9$, $1^4 5^4$, $1^{-1} 5^5$, $1^3 7^3$, $1^2 11^2$ and $1^1 23^1$, respectively. For these lattices, the corresponding symmetric modular forms are constructed as the Borcherds products of the vector-valued modular forms lifted from the eta products associated to the cycle shapes. The remaining three symmetric reflective modular forms in Table 2.1 are obtaining by taking Atkin-Lehner involutions. We next construct them in a different way. It is easy to check that

$$\begin{aligned}(U \oplus U(2) \oplus E_8(2))^\vee (2) &= \left(U \oplus U\left(\frac{1}{2}\right) \oplus E_8\left(\frac{1}{2}\right)\right)(2) \\ &= U(2) \oplus U \oplus E_8,\end{aligned}$$

which gives

$$\mathrm{O}^+(U \oplus U(2) \oplus E_8(2)) = \mathrm{O}^+(U \oplus U(2) \oplus E_8).$$

Therefore, the reflective modular form for $U \oplus U(2) \oplus E_8(2)$ can be viewed as reflective modular form for $U \oplus U(2) \oplus E_8$. The other two cases are similar.

Table 2.1: Strongly reflective Borcherds products of singular weight on lattices of prime level

p	genus	M	F	zeros	symmetric
2	$\mathrm{II}_{18,2}(2_{\mathrm{II}}^{+10})$	$U \oplus U(2) \oplus$ Barnes-Wall lattice	$F_{\Gamma_0(2),\eta_{1^{-8}2^{-8}},0}$	$1, 2$	Yes
	$\mathrm{II}_{10,2}(2_{\mathrm{II}}^{+2})$	$U \oplus U(2) \oplus E_8$	$F_{\Gamma_0(2),16\eta_{1^{-16}2^{8}},0}$	2	Yes
	$\mathrm{II}_{10,2}(2_{\mathrm{II}}^{+10})$	$U \oplus U(2) \oplus E_8(2)$	$F_{\Gamma_0(2),16\eta_{1^{8}2^{-16}},0}$	1	Yes
	$\mathrm{II}_{6,2}(2_{\mathrm{II}}^{-6})$	$U(2) \oplus U(2) \oplus D_4$	F_1	2	Non
3	$\mathrm{II}_{14,2}(3^{-8})$	$U \oplus U(3) \oplus$ Coxeter-Todd lattice	$F_{\Gamma_0(3),\eta_{1^{-6}3^{-6}},0}$	$1, 3$	Yes
	$\mathrm{II}_{8,2}(3^{-3})$	$U \oplus U(3) \oplus E_6$	$F_{\Gamma_0(3),9\eta_{1^{-9}3^{3}},0}$	3	Yes
	$\mathrm{II}_{8,2}(3^{-7})$	$U \oplus U(3) \oplus E_6^\vee(3)$	$F_{\Gamma_0(3),\eta_{1^{3}3^{-9}},0}$	1	Yes
	$\mathrm{II}_{6,2}(3^{+6})$	$U(3) \oplus U(3) \oplus 2A_2$	F_2	3	Non
	$\mathrm{II}_{4,2}(3^{-5})$	$U(3) \oplus U(3) \oplus A_2$	F_3	3	Non
5	$\mathrm{II}_{10,2}(5^{+6})$	$U \oplus U(5) \oplus$ Maass lattice	$F_{\Gamma_0(5),\eta_{1^{-4}5^{-4}},0}$	$1, 5$	Yes
	$\mathrm{II}_{6,2}(5^{+3})$	$U \oplus U(5) \oplus A_4$	$F_{\Gamma_0(5),5\eta_{1^{-5}5^{1}},0}$	5	Yes
	$\mathrm{II}_{6,2}(5^{+5})$	$U \oplus U(5) \oplus A_4^\vee(5)$	$F_{\Gamma_0(5),\eta_{1^{1}5^{-5}},0}$	1	Yes
7	$\mathrm{II}_{8,2}(7^{-5})$	$U \oplus U(7) \oplus$ Barnes-Craig lattice	$F_{\Gamma_0(7),\eta_{1^{-3}7^{-3}},0}$	$1, 7$	Yes
11	$\mathrm{II}_{6,2}(11^{-4})$	$U \oplus U(11) \oplus \Lambda^{1^2 11^2}$	$F_{\Gamma_0(11),\eta_{1^{-2}11^{-2}},0}$	$1, 11$	Yes
23	$\mathrm{II}_{4,2}(23^{-3})$	$U \oplus U(23) \oplus \Lambda^{1^1 23^1}$	$F_{\Gamma_0(23),\eta_{1^{-1}23^{-1}},0}$	$1, 23$	Yes

Remark 2.2.3. Scheithauer's reflective modular forms of singular weight have some applications. The reflective modular forms of singular weight for

$$2U \oplus D_8 \cong U \oplus U(2) \oplus E_8$$
$$2U \oplus 3A_2 \cong U \oplus U(3) \oplus E_6$$
$$2U \oplus A_4^\vee(5) \cong U \oplus U(5) \oplus A_4$$

can be also constructed as the additive liftings of some holomorphic Jacobi forms of singular weight at the one-dimensional cusps related to $2U$ (see [Gri10, Gri18, GSZ18, GW17, GW18b]). Their pull-backs will give infinite families of holomorphic Siegel paramodular forms which are simultaneously Borcherds products and additive liftings, which support the "Theta-Block Conjecture for Paramodular Forms" formulated in [GPY15]. Besides, the pull-back of the reflective modular forms of singular weight for

$$2U \oplus A_6^\vee(7) \cong U \oplus U(7) \oplus \text{Barnes-Craig lattice}$$
$$2U \oplus 2A_4^\vee(5) \cong U \oplus U(5) \oplus \text{Maass lattice}$$

at the one-dimensional cusps related to $2U$ will give infinite families of antisymmetric holomorphic Siegel paramodular forms of weights 3 and 4 (see [GW19a]).

Using the Riemann–Roch theorem, Scheithauer also gave bounds on the signature for non-symmetric reflective modular forms and for reflective modular forms without (-2)-divisor.

Theorem 2.2.4 (Theorem 6.5 in [Sch17]). *Let M be an even lattice of prime level p and signature $(2,n)$ with $n > 2$ carrying a non-symmetric reflective Borcherds product. Then $p \leq 11$ and $n \leq 2 + 24/p$.*

Lemma 2.2.5 (Proposition 6.1 in [Sch17]). *Let M be a lattice of signature $(2,n)$ and prime level p. If M admits a reflective Borcherds product which has only reflective divisors of norm $2p$, then $n \leq 2 + 24/(p+1)$.*

Ditmann's results

Scheithauer also constructed many strongly reflective modular forms of singular weight for lattices of squarefree level (see the table after Theorem 10.3 in [Sch06]). Recently, Dittmann [Dit18, Theorem 1.2] proved that these modular forms are the only strongly reflective modular forms of singular weight for lattices of squarefree level N which can be written as $U \oplus U(N) \oplus L(-1)$. Dittmann [Dit18, Theorem 1.1] also proved that there are only finitely many reflective lattices of signature $(2,n)$ with $n \geq 4$ and squarefree level N that split $U \oplus U(N) \oplus L(-1)$. For our purpose, we introduce a particular case of Dittmann's result.

Lemma 2.2.6 (Lemma 4.5 in [Dit18]). *Let M be a reflective lattice of signature $(2,n)$ and prime level p. If M can be expressed as $U \oplus U(p) \oplus L(-1)$, then $n \leq 2 + 48/(p+1)$.*

Proof. We suppose that F is a symmetric reflective modular form for M. Since $M = U \oplus U(p) \oplus L(-1)$, we derive from Theorem 1.3.3 that F is a Borcherds product. By Proposition 1.3.7, the corresponding vector-valued modular form can be constructed as a lifting in Theorem 1.3.4. Thus there exists a nearly holomorphic modular form f of weight $1 - n/2$ for $\Gamma_0(p)$ with a character. We note that $f = c_0 q^{-1} + a + O(q)$. We see from the expression of M that there are nontrivial vectors of norm 0 in the discriminant group of M. By Theorem 1.3.4, we conclude that $f|_S = c_p q^{-1/p} + b + O(q^{1/p})$, otherwise there will be another type of principal part Foureir coefficients which give non-reflective divisors. Here c_0, a, c_p, b are constants. The Riemann–Roch theorem applied to f gives

$$-2 \leq p\nu_0(f) + \nu_\infty(f) \leq \frac{p+1}{12}\left(1 - \frac{n}{2}\right),$$

which proves the lemma. □

Remark 2.2.7. From the form of f in the above proof, we see that if $M = U \oplus U(p) \oplus L(-1)$ has discriminant p^a and is reflective then any lattice of the same signature and same level with discriminant p^b is also reflective, where $a - b$ is even. Moreover, the corresponding reflective modular form can be constructed by the lifting of f.

Ma's results

We next introduce Ma's classification results following [Ma17, Ma18].

Theorem 2.2.8 (Theorem 1.1 in [Ma17]). *There are only finitely many 2-reflective lattices of signature $(2,n)$ with $n \geq 7$. In particular, there is no 2-reflective lattice when $n \geq 26$ except the even unimodular lattice of signature $(2,26)$.*

Theorem 2.2.9 (Proposition 3.2 in [Ma17]). *There is no reflective lattices of signature $(2,n)$ with $n \geq 26$ containing $2U$ except the even unimodular lattice of signature $(2,26)$.*

If a modular form F has weight α and every component of $\mathrm{div}(F)$ has multiplicity $\leq \beta$, we say that F has slope $\leq \beta/\alpha$.

Theorem 2.2.10 (Corollary 1.9 in [Ma18]). *Let $r > 0$ be a fixed rational number. Then up to scaling there are only finitely many lattices of signature $(2, n)$ with $n \geq 4$ which carries a reflective modular form of slope $\leq r$. In particular, for a fixed natural number β, there are up to scaling only finitely many lattices with $n \geq 4$ which carries a reflective modular form of vanishing order $\leq \beta$.*

As a corollary of the above theorem, Ma proved the following result.

Theorem 2.2.11 (Corollary 1.10 in [Ma18]). *Up to scaling there are only finitely many lattices of signature $(2, n)$ with $n \geq 4$ which carries a strongly reflective modular form.*

2.3 Construction of reflective modular forms

In this section we construct many reflective modular forms.

2.3.1 Quasi pull-backs

Borcherds [Bor95] constructed a modular form Φ_{12} of singular weight 12 and character det with respect to $\mathrm{O}^+(\mathrm{II}_{2,26})$

$$\Phi_{12} \in M_{12}(\mathrm{O}^+(\mathrm{II}_{2,26}), \det)$$

where $\mathrm{II}_{2,26}$ is the unique even unimodular lattice of signature $(2, 26)$. The function Φ_{12} is constructed as the Borcherds product of the following nearly holomorphic modular form of weight -12

$$1/\Delta(\tau) = q^{-1} + 24 + 324q + 3200q^2 + \cdots$$

and it is a modular form with complete 2-divisor i.e.

$$\mathrm{div}(\Phi_{12}) = \sum_{\substack{v \in \mathrm{II}_{2,26}/\{\pm 1\} \\ (v,v)=-2}} \mathcal{D}_v(\mathrm{II}_{2,26}).$$

By Eichler criterion (Proposition 1.4.21), all (-2)-vectors in $\mathrm{II}_{2,26}$ form only one orbit with respect to $\mathrm{O}^+(\mathrm{II}_{2,26})$. We next introduce the quasi pull-back of the Borcherds modular form Φ_{12}.

First we give a general property of rational quadratic divisors. Let M be an even lattice of signature $(2, n)$ and let T be a primitive sublattice of signature $(2, m)$ with $m < n$. Then $T_M^\perp$ is negative definite and we have the usual inclusions

$$T \oplus T_M^\perp < M < M^\vee < T^\vee \oplus (T_M^\perp)^\vee.$$

For $v \in M$ with $v^2 < 0$ we write

$$v = \alpha + \beta, \quad \alpha \in T^\vee,\ \beta \in (T_M^\perp)^\vee.$$

Then we have

$$\mathcal{D}(T) \cap \mathcal{D}_v(M) = \begin{cases} \mathcal{D}_\alpha(T), & \text{if } \alpha^2 < 0, \\ \varnothing, & \text{if } \alpha^2 \geq 0,\ \alpha \neq 0, \\ \mathcal{D}(T), & \text{if } \alpha = 0, \text{ i.e. } v \in T_M^\perp. \end{cases}$$

The statements of the next theorem were proved in [BKP+98, Theorem 1.2] and [GHS13, Theorems 8.3 and 8.18].

Theorem 2.3.1. *Let $T \hookrightarrow \mathrm{II}_{2,26}$ be a primitive nondegenerate sublattice of signature $(2,n)$ with $n \geq 3$, and let $\mathcal{D}(T) \hookrightarrow \mathcal{D}(\mathrm{II}_{2,26})$ be the corresponding embedding of the Hermitian symmetric domains. The set of (-2)-roots*

$$R_{-2}(T^\perp) = \{r \in \mathrm{II}_{2,26} : r^2 = -2,\ (r,T) = 0\}$$

in the orthogonal complement is finite. We put $N(T^\perp) = \# R_{-2}(T^\perp)/2$. Then the function

$$\Phi_{12}|_T = \frac{\Phi_{12}(Z)}{\prod_{r \in R_{-2}(T^\perp)/\pm 1}(Z,r)}\Bigg|_{\mathcal{D}(T)^\bullet} \in M_{12+N(T^\perp)}(\widetilde{\mathrm{O}}^+(T), \det), \tag{2.3.1}$$

where in the product over r we fix a finite system of representatives in $R_{-2}(T^\perp)/\pm 1$. The modular form $\Phi_{12}|_T$ vanishes only on rational quadratic divisors of type $\mathcal{D}_v(T)$ where $v \in T^\vee$ is the orthogonal projection of a (-2)-root $r \in \mathrm{II}_{2,26}$ on $T^\vee$ satisfying $-2 \leq v^2 < 0$. If the set $R_{-2}(T^\perp)$ is non-empty then the quasi pull-back $\Phi_{12}|_T$ is a cusp form.

In general, the quasi pull-back $\Phi_{12}|_T$ is not a reflective modular form. To determine its divisor, we must do explicit calculations. We refer to [Gra09] for this type of calculations. We next introduce several arguments which can be used to seek reflective modular forms without complicated calculations.

In [Gri12] Gritsenko proposed 24 Jacobi type constructions of the Borcherds modular form Φ_{12} based on the 24 one dimensional boundary components of the Baily-Borel compactification of the modular variety $\mathrm{O}^+(\mathrm{II}_{2,26})\backslash\mathcal{D}(\mathrm{II}_{2,26})$. These components correspond exactly to the classes of positive definite even unimodular lattices of rank 24. They are the 23 Niemeier lattices $N(R)$ uniquely determined by their root sublattices R of rank 24

$3E_8$	$E_8 \oplus D_{16}$	D_{24}	$2D_{12}$	$3D_8$	$4D_6$
$6D_4$	A_{24}	$2A_{12}$	$3A_8$	$4A_6$	$6A_4$
$8A_3$	$12A_2$	$24A_1$	$E_7 \oplus A_{17}$	$2E_7 \oplus D_{10}$	$4E_6$
$E_6 \oplus D_7 \oplus A_{11}$	$A_{15} \oplus D_9$	$2A_9 \oplus D_6$	$2A_7 \oplus 2D_5$	$4A_5 \oplus D_4$	

and the Leech lattice Λ_{24} without 2-roots (see [SC98, Chapter 18]). We next construct a lot of reflective modular forms by quasi pull-backs of Φ_{12} in different one dimensional boundary components, some already known, some new.

For convenience, we fix the discriminant groups of irreducible root lattices. Let $e_1, ..., e_n$ be the standard basis of $\mathbb{R}^n$

1. For $A_1 = \mathbb{Z}$ with the bilinear form $2x^2$, we fix $A_1^\vee/A_1 = \{0, 1/2\}$.

2. For $D_n = \{x \in \mathbb{Z}^n : \sum_{i=1}^n x_i \in 2\mathbb{Z}\}$ with $n \geq 4$, we fix $D_n^\vee/D_n = \{[0],[1],[2],[3]\}$, where $[1] = \frac{1}{2}\sum_{i=1}^n e_i$, $[2] = e_1$, $[3] = \frac{1}{2}\sum_{i=1}^n e_i - e_n$. Then $[1]^2 = [3]^2 = \frac{n}{4}$ and $[2]^2 = 1$.

3. For $A_n = \{x \in \mathbb{Z}^{n+1} : \sum_{i=1}^{n+1} x_i = 0\}$ with $n \geq 2$, we fix $A_n^\vee/A_n = \{[i] : 0 \leq i \leq n\}$, where $[i] = (\frac{i}{n+1}, ..., \frac{i}{n+1}, \frac{-j}{n+1}, ..., \frac{-j}{n+1})$ with j components equal to $\frac{i}{n+1}$ and $i + j = n+1$. The norm of $[i]$ is $\frac{ij}{n+1}$.

4. The lattice E_6 is of level 3. Its discriminant group is of order 3 and generated by one element $[1]$ of norm $4/3$.

5. The lattice E_7 is of level 4. Its discriminant group is of order 2 and generated by one element $[1]$ of norm $3/2$.

The first argument:

This argument was due to Gritsenko and Nikulin. In [GN18], they constructed modular forms with complete 2-divisor by quasi pull-backs of Φ_{12}. We recall their main ideas such that readers can understand the other arguments better.

For an even positive definite lattice L, we define Norm_2 condition as

$$\mathrm{Norm}_2: \quad \forall\, \bar{c} \in L^\vee/L, \quad \exists\, h_c \in \bar{c} \quad \text{s.t.} \quad 0 \le h_c^2 \le 2. \tag{2.3.2}$$

The reason why we formulate Norm_2 condition is the following. If the lattice L satisfies Norm_2 condition and ϕ is a weakly holomorphic Jacobi form of index L, then its singular Fourier coefficients are totally determined by the q^n-terms with non-positive n.

Proof of the claim. It is known that $f(n,\ell)$ depends only on $2n-(\ell,\ell)$ and $\ell \mod L$. Suppose that $f(n,\ell)$ is singular, i.e. $2n-(\ell,\ell)<0$. There exists a vector $\ell_1 \in L^\vee$ such that $(\ell_1,\ell_1)\le 2$ and $\ell-\ell_1 \in L$ because L satisfies Norm_2 condition. It is clear that $(\ell,\ell)-(\ell_1,\ell_1)$ is an even integer. If $-2\le 2n-(\ell,\ell)<0$, it follows that $2n-(\ell,\ell)=-(\ell_1,\ell_1)$ and $f(n,\ell)=f(0,\ell_1)$. If $2n-(\ell,\ell)<-2$, then there exists a negative integer n_1 satisfying $2n-(\ell,\ell)=2n_1-(\ell_1,\ell_1)$. Thus there exists a Fourier coefficient $f(n_1,\ell_1)$ with negative n_1 such that $f(n_1,\ell_1)=f(n,\ell)$. □

Remark 2.3.2. The following lattices satisfy Norm_2 condition

A_1	A_2	A_3	A_4	A_5	A_6	A_7
D_4	D_5	D_6	D_7	D_8		
E_6	E_7	E_8	$2E_8$	N_8		
$2A_1$	$3A_1$	$4A_1$	$2A_2$	$3A_2$	$2A_3$	$2D_4$
$A_1(2)$	$A_1(3)$	$A_1(4)$	$2A_1(2)$	$A_2(2)$		
$A_2(3)$	$A_3(2)$	$D_4(2)$	$E_8(2)$.			

Note that $A_1(2)\cong D_1$, $2A_1\cong D_2$ and $A_3\cong D_3$. Remark that the fact that the lattice $A_4^\vee(5)$ satisfies Norm_2 condition was proved in [GW17]. Here N_8 is the Nikulin's lattice defined as (see [GN18, Example 4.3])

$$N_8=\langle 8A_1, h=(a_1+\cdots+a_8)/2\rangle \cong D_8^\vee(2),$$

where $(a_i,a_j)=2\delta_{ij}$, $(h,h)=4$. The root sublattice generated by roots of N_8 is $8A_1$.

Let K be a primitive sublattice of $N(R)$ containing a direct summand of the same rank of the root lattice R or a primitive sublattice of the Leech lattice $\Lambda_{24}=N(\varnothing)$. We assume that K satisfies the Norm_2 condition. Let $T=2U\oplus K(-1)$.

The theta function $\vartheta_{N(R)}$ of the Niemeier lattice $N(R)$ is a holomorphic Jacobi form of weight 12 and index 1 for $N(R)$. Then we have

$$\varphi_{0,N(R)}(\tau,\mathfrak{z})=\frac{\vartheta_{N(R)}(\tau,\mathfrak{z})}{\Delta(\tau)}=q^{-1}+24+\sum_{r\in R, r^2=2}e^{2\pi i(r,\mathfrak{z})}+O(q)\in J^{w.h.}_{0,N(R),1}.$$

We write $\mathfrak{z}=\mathfrak{z}_1+\mathfrak{z}_2$ with $\mathfrak{z}_1\in K\otimes\mathbb{C}$ and $\mathfrak{z}_2\in K^\perp_{N(R)}\otimes\mathbb{C}$ and define the pull-back of $\varphi_{0,N(R)}$ on the lattice $K\hookrightarrow N(R)$ as

$$\begin{aligned}\varphi_{0,K}(\tau,\mathfrak{z}_1)&=\varphi_{0,N(R)}(\tau,\mathfrak{z})|_{\mathfrak{z}_2=0}\\&=q^{-1}+24+n_K+\sum_{r\in K, r^2=2}e^{2\pi i(r,\mathfrak{z})}+O(q)\in J^{w.h.}_{0,K,1},\end{aligned}$$

where n_K is the number of 2-roots in R orthogonal to K. Since K satisfies the Norm_2 condition, the singular Fourier coefficients of $\varphi_{0,K}$ are completely represented by its q^{-1} and q^0-terms. Thus,

Table 2.2: Reflective cusp forms with complete 2-divisor

lattice	A_1	$2A_1$	$3A_1$	$4A_1$	N_8	A_2	$2A_2$	$3A_2$	A_3	$2A_3$	A_4	A_5
weight	35	34	33	32	28	45	42	39	54	48	62	69
lattice	A_6	A_7	D_4	$2D_4$	D_5	D_6	D_7	D_8	E_6	E_7	E_8	$2E_8$
weight	75	80	72	60	88	102	114	124	120	165	252	132

the quasi pull-back $\Phi_{12}|_T$ is equal to $\operatorname{Borch}(\varphi_{0,K})$ up to a constant and it is a modular form with complete 2-divisor. In this way, Gritsenko and Nikulin [GN18, Theorems 4.3, 4.4] constructed the following modular forms with complete 2-divisor.

When K is one of the following 10 sublattices of the Leech lattice Λ_{24}

$$A_1(2) \quad A_1(3) \quad A_1(4) \quad 2A_1(2) \quad A_2(2) \quad A_2(3) \quad A_3(2) \quad D_4(2) \quad E_8(2) \quad A_4^\vee(5)$$

there exists a (non cusp) modular form of weight 12 with complete 2-divisor for the lattice $2U \oplus K(-1)$.

The second argument:

This argument was formulated in [Gri18]. Here we describe it in a more understandable way and use it to construct much more reflective modular forms. This argument is based on the following observation.

Observation: The vector of minimum norm in any non-trivial class of discriminant group of the lattice A_1 corresponds to 2-reflective divisor. The vector of minimum norm in any non-trivial class of discriminant group of the lattice $2A_1$, A_2, D_4 or $A_1(2)$ is reflective.

Let $K = K_0 \oplus K_1 \oplus K_2$ be a primitive sublattice of $N(R)$. The lattice K_0 contains a direct summand of the same rank of R. The lattices K_1, K_2 take A_1, $2A_1$, A_2, D_4 or $A_1(2)$, and they are contained in different direct summands of R. The second lattice K_2 is allowed to be empty. We further assume that K satisfies Norm_2 condition. Let $T = 2U \oplus K(-1)$.

Again, we consider the pull-back of $\varphi_{0,N(R)}$ on the lattice $K \hookrightarrow N(R)$. The above assumptions guarantee that the singular Fourier coefficients of $\varphi_{0,K}$ are totally determined by its q^{-1}, q^0-terms and correspond to reflective divisors. Therefore, the quasi pull-back $\Phi_{12}|_T = \operatorname{Borch}(\varphi_{0,K})$ is a reflective modular form.

We first use this argument to construct 2-reflective modular forms. To do this, we can only take $K_1, K_2 = \varnothing, A_1$. Let $R = 3E_8$, $K_0 = 2E_8$ and $K_1 = A_1$ contained in the third copy of E_8. Then the quasi pull-back $\Phi_{12}|_T$ will give a 2-reflective modular form for $2U \oplus 2E_8(-1) \oplus A_1(-1)$. Similarly, when K takes one of the following 16 lattices, the quasi pull-back $\Phi_{12}|_T$ will give a 2-reflective modular form on T

$$\begin{array}{llllllll} 2E_8 \oplus A_1 & E_8 \oplus A_1 & E_8 \oplus 2A_1 & D_6 \oplus A_1 & D_4 \oplus A_1 & D_4 \oplus 2A_1 & A_4 \oplus A_1 & A_3 \oplus A_1 \\ A_3 \oplus 2A_1 & A_2 \oplus A_1 & A_2 \oplus 2A_1 & 2A_2 \oplus A_1 & D_5 \oplus A_1 & A_5 \oplus A_1 & E_7 \oplus A_1 & E_6 \oplus A_1. \end{array}$$

We then use this argument to construct reflective modular forms. For example, let $R = 3E_8$, $K_0 = E_8$ and $K_1 = K_2 = D_4$ contained in the second and the third copy of E_8 respectively. Then the quasi pull-back $\Phi_{12}|_T$ gives a reflective modular form for $2U \oplus E_8(-1) \oplus 2D_4(-1)$. When K takes one of the following 33 lattices, the quasi pull-back $\Phi_{12}|_T$ is a reflective modular form on T.

1. When $R = 3E_8$, the lattice K can take

$$\begin{array}{ll} \{E_8, 2E_8\} \oplus \{2A_1, A_2, D_4, A_1(2)\} & E_8 \oplus A_1 \oplus \{2A_1, A_2, D_4, A_1(2)\} \\ E_8 \oplus 2A_1 \oplus \{2A_1, A_2, D_4, A_1(2)\} & E_8 \oplus A_2 \oplus \{A_2, D_4, A_1(2)\} \\ E_8 \oplus D_4 \oplus \{D_4, A_1(2)\} & E_8 \oplus 2A_1(2). \end{array}$$

2. When $R = 6D_4$, the lattice K can take $D_4 \oplus \{A_2, D_4, A_1(2)\}$.

3. When $R = 6A_4$, the lattice K can take $A_4 \oplus A_2$.

4. When $R = 8A_3$, the lattice K can take $A_3 \oplus \{A_2, A_1(2)\}$.

5. When $R = 12A_2$, the lattice K can take $A_2 \oplus A_1(2)$.

6. When $R = 24A_1$, the lattice K can take $\{A_1, 2A_1\} \oplus A_1(2)$.

7. When $R = 4E_6$, the lattice K can take $E_6 \oplus A_2$.

8. When $R = 2A_7 \oplus 2D_5$, the lattice K can take $D_5 \oplus A_2$.

For (5) and (6), the constructions are a bit different. We take $A_1(2)$ in ()5 as a sublattice of $2A_2$ and take $A_1(2)$ in (6) as a sublattice of $2A_1$. They can also be constructed in another way. For example, to construct a reflective modular form for $2U \oplus A_2 \oplus A_1(2)$, we can use the pull-back $A_2 \oplus A_1(2) < N(6D_4)$. Remark that for one lattice we may construct different reflective modular forms. In the above, we focus on reflective lattices and only construct one reflective modular form for a certain lattice.

The third argument:

We now consider the general case, i.e. the lattice K does not satisfy Norm_2 condition. Assume that the lattice K satisfies the condition in the second argument. We further assume that the minimum norm of vector in non-trivial class of discriminant group of K is less than 4 and all the vectors (noted by v) of minimum norm larger than 2 satisfy the condition: the vector $(0, 1, v, 1, 0)$ is reflective i.e. the reflection associated to this vector belongs to $\mathrm{O}^+(T)$. In this case, the singular Fourier coefficients of $\varphi_{0,K}$ are determined by its q^{-1}, q^0, q^1-terms and correspond to reflective divisors. Therefore, the quasi pull-back $\Phi_{12}|_T = \mathrm{Borch}(\varphi_{0,K})$ is a reflective modular form.

1. When the lattice K take one of the following 8 lattices, we get 2-reflective lattices

$$\begin{array}{lllll} 5A_1 & D_{10} & N_8 \oplus A_1 & D_8 \oplus A_1 & D_6 \oplus 2A_1 \\ 2D_4 \oplus A_1 & E_7 \oplus 2A_1 & D_4 \oplus 3A_1 & & \end{array}$$

   For the last lattice, we use $R = 6D_4$ and take $3A_1$ from three different copies of D_4.

2. When the lattice K take one of the following 12 lattices, we get reflective lattices

$$4A_2 \quad 3D_4 \quad 2E_6 \quad 2E_7 \quad A_8 \quad A_9 \quad D_9 \quad A_5 \oplus D_4 \quad D_{12} \quad 2A_4 \quad 2D_5 \quad 2D_6.$$

   There are a lot of this type of reflective lattices. In the above, we only consider the simplest case $K = K_0$. By [Sch04, §9], we know that the lattice $2A_2(2)$ is a primitive sublattice of the Leech lattice and it satisfies our condition. Thus the quasi pull-back gives a reflective modular form of weight 12 for $2U \oplus 2A_2(-2)$.

The fourth argument:

We can also consider the quasi pull-backs of some other reflective modular forms. We have known that the lattice $2U \oplus 2E_8 \oplus D_4$ is a reflective lattice. It is easy to check that

$$2U \oplus 2E_8 \oplus D_4 \cong 2U \oplus E_8 \oplus D_{12}$$

because they have the same discriminant form. For $4 \le n \le 10$, we have $D_n \oplus D_{12-n} < D_{12}$. In a similar way, we show that the quasi pull-back of $2U \oplus E_8 \oplus D_{12}$ into $2U \oplus E_8 \oplus D_n$ will give a reflective modular form for the lattice $2U \oplus E_8 \oplus D_n$ with $4 \le n \le 10$.

The fifth argument:

This argument relies on the construction of the Niemeier lattice $N(R)$ from the root lattice R. We will explain the main idea by considering several interesting examples.

(1) Let $R = 6D_4$. We consider its sublattice $K = D_4 \oplus 5A_1$, where every A_1 is contained in a different copy of D_4. The singular Fourier coefficients of the weakly holomorphic Jacobi form $\varphi_{0,K}$ are determined by its q^{-1}, q^0, q^1-terms. It is clear that the q^{-1}, q^0-terms correspond to 2-reflective divisors. We next consider the q^1-term. The q^1-term is the pull-back of the vectors of norm 4 in $N(6D_4)$. Since the pull-backs of vectors of norm 4 in $6D_4$ gives either non-singular Fourier coefficients or singular Fourier coefficients equivalent to that of q^0-term, we only need to consider the pull-backs of vectors of norm 4 in $N(6D_4)$ and not in $6D_4$. This type of vectors is of the form $[i_1] \oplus [i_2] \oplus [i_3] \oplus [i_4] \oplus [i_5] \oplus [i_6]$, here four of the six indices are non-zero. Its pull-back to $D_4 \oplus 5A_1$ only gives the singular Fourier coefficients of type $[i] \oplus (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0)$, which correspond to 2-reflective divisors. Therefore, the function $\mathrm{Borch}(\varphi_{0,K})$ is a 2-reflective modular form for $2U \oplus D_4 \oplus 5A_1$.

Similarly, the quasi pull-back on $6A_2 < N(6D_4)$ gives a reflective modular form for $2U \oplus 6A_2$.

(2) Let $R = 8A_3$. We consider its sublattice $K = 8A_1$, where every A_1 is contained in a different copy of A_3. The singular Fourier coefficients of the weakly holomorphic Jacobi form $\varphi_{0,K}$ are determined by its q^{-1}, q^0, q^1-terms. It is obvious that the q^{-1}, q^0-terms correspond to 2-reflective divisors. We next consider the q^1-term. The q^1-term is the pull-back of the vectors of norm 4 in $N(8A_3)$. We only need to consider the pull-back of vectors of norm 4 in $N(8A_3)$ and not in $8A_3$. This type of vectors of norm 4 is of the form $[2] \oplus [2] \oplus [2] \oplus [2] \oplus 0^4$ or $[2] \oplus [1] \oplus [1] \oplus [1] \oplus [1] \oplus 0^3$. Its pull-back to $8A_1$ only gives the singular Fourier coefficients of type $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0)$, which correspond to 2-reflective divisors. Therefore, the function $\mathrm{Borch}(\varphi_{0,K})$ is a 2-reflective modular form for $2U \oplus 8A_1$.

(3) Let $R = 12A_2$. We consider its sublattice $K = 12A_1$, where every A_1 is contained in a different copy of A_2. The singular Fourier coefficients of the weakly holomorphic Jacobi form $\varphi_{0,K}$ are determined by its q^{-1}, q^0, q^1, q^2-terms. Firstly, the q^{-1}, q^0-terms correspond to 2-reflective divisors. We next consider the q^1 and q^2-terms. The q^1-term is the pull-back of the vectors of norm 4 in $N(12A_2)$. We only need to consider the pull-back of vectors of norm 4 in $N(12A_2)$ and not in $12A_2$. This type of vectors of norm 4 is of the form $[1]^6 \oplus 0^6$. Its pull-back to $12A_1$ only gives the singular Fourier coefficients of type $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0^7)$ or $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0^6)$, which all correspond to reflective divisors. The q^2-term is the pull-back of the vectors of norm 6 in $N(12A_2)$. This type of vectors of norm 6 is of the form $[1]^9 \oplus 0^3$. Its pull-back to $12A_1$ only gives the singular Fourier coefficients of type $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0^3)$, which all correspond to 2-reflective divisors. Therefore, the function $\mathrm{Borch}(\varphi_{0,K})$ is a reflective modular form for $2U \oplus 12A_1$.

Using the idea of pull-backs, it is easy to prove the following result.

Lemma 2.3.3. *Let M be an even lattice of signature of $(2,n)$ and L be an even positive definite lattice. If $M\oplus L(-1)$ is reflective (resp. 2-reflective), then M is reflective (resp. 2-reflective) too.*

As an application of the above lemma, we further construct some reflective modular forms. We can check

$$U\oplus U(2)\oplus D_4 < U\oplus U(2)\oplus D_4\oplus D_4 \cong 2U\oplus N_8,$$

which yields that $U\oplus U(2)\oplus D_4$ is a 2-reflective lattice. Similarly, we claim that $U\oplus U(3)\oplus A_2$ is a reflective lattice because

$$U\oplus U(3)\oplus A_2 < U\oplus U(3)\oplus A_2\oplus E_6 \cong 2U\oplus 4A_2.$$

2.3.2 Constructing reflective modular forms from Jacobi forms

There are 4 known reflective modular forms of singular weight for lattices of non-squarefree level

$$2U\oplus L(-1): \quad L = A_1(4),\ 2A_1(2),\ A_2(3),\ 4A_1. \tag{2.3.3}$$

They were all known to Gritsenko. We next present their constructions.

1. From [Gri99, GN98b], we know

$$\phi_{0,4}(\tau,z) = \frac{\vartheta(\tau,3z)}{\vartheta(\tau,z)} = \zeta^{\pm 1} + 1 + O(q) \in J^w_{0,A_1,4}. \tag{2.3.4}$$

   Then the function $\mathrm{Borch}(\phi_{0,4})$ is a strongly reflective modular form of weight $1/2$ for $\mathrm{O}^+(2U\oplus A_1(-4))$ with divisor coming from reflective vectors of norm -8 and divisor 8. This modular form can also be constructed as an additive lifting of $\vartheta(\tau,z)$ (see [GN98b]).

2. First, we have

$$\vartheta(\tau,z_1)\vartheta(\tau,z_2)\in J_{1,2A_1,\frac{1}{2}}(\upsilon_\eta^6\times\nu).$$

   By Proposition 1.4.17, we have

$$\begin{aligned}\phi_{0,2A_1,2}(\tau,\mathfrak{z}) &= -\frac{[\vartheta(\tau,z_1)\vartheta(\tau,z_2)]|T_-^{(4)}(5)}{\vartheta(\tau,z_1)\vartheta(\tau,z_2)}\\ &= \zeta_1^{\pm1}+\zeta_2^{\pm1}+2+O(q)\in J^w_{0,2A_1,2}.\end{aligned}$$

   Then the function $\mathrm{Borch}(\phi_{0,2A_1,2})$ is a strongly reflective modular form of weight 1 for $\mathrm{O}^+(2U\oplus 2A_1(-2))$ with divisor coming from reflective vectors of norm -4 and divisor 4. Note that the function $\phi_{0,2A_1,2}$ can also be constructed in another way

$$\phi_{0,2A_1,2}(\tau,\mathfrak{z}) = \frac{\vartheta(\tau,2z_1+z_2)\vartheta(\tau,2z_2-z_1)+\vartheta(\tau,2z_2+z_1)\vartheta(\tau,2z_1-z_2)}{\vartheta(\tau,z_1)\vartheta(\tau,z_2)}$$

   and we have

$$\mathrm{Borch}(\phi_{0,2A_1,2}) = \mathrm{Grit}(\vartheta(\tau,z_1)\vartheta(\tau,z_2)).$$

3. Recall that

$$A_2^\vee/A_2 = \{\overline{\mu_i} : 0\le i\le 2\}$$

   where $\mu_0=0$, $\mu_1 = (\frac{1}{3},\frac{1}{3},-\frac{2}{3})$ and $\mu_2 = (\frac{2}{3},-\frac{1}{3},-\frac{1}{3})$. The norms of μ_1 and μ_2 are all $\frac{2}{3}$ and correspond to reflective divisors (In fact, $3\mu_1$ and $3\mu_2$ are reflective vectors in $2U\oplus A_2(-1)$). We can check

$$\Theta(\tau,z_1,z_2) = \frac{\vartheta(\tau,z_1)\vartheta(\tau,z_1-z_2)\vartheta(\tau,z_2)}{\eta}\in J_{1,A_2,1}(\upsilon_\eta^8).$$

Then we have

$$\begin{aligned}\phi_{0,A_2,3}(\tau,\mathfrak{z}) &= -\frac{\Theta(\tau,z_1,z_2)|T_-^{(3)}(4)}{\Theta(\tau,z_1,z_2)}\\ &= \zeta_1^{\pm 1}+\zeta_2^{\pm 1}+(\zeta_1\zeta_2^{-1})^{\pm 1}+2+O(q)\in J^w_{0,A_2,3}.\end{aligned}$$

Then the function $\mathrm{Borch}(\phi_{0,A_2,3})$ is a strongly reflective modular form of weight 1 for $\widetilde{\mathrm{O}}^+(2U\oplus A_2(-3))$ with divisor coming from reflective vectors of norm -18 and divisor 9. Note that

$$\mathrm{Borch}(\phi_{0,A_2,3})=\mathrm{Grit}(\Theta).$$

4. We can check

$$\begin{aligned}\phi_{0,4A_1,1}(\tau,\mathfrak{z}) &= -\frac{[\vartheta(\tau,z_1)\vartheta(\tau,z_2)\vartheta(\tau,z_3)\vartheta(\tau,z_4)]|T_-^{(2)}(3)}{\vartheta(\tau,z_1)\vartheta(\tau,z_2)\vartheta(\tau,z_3)\vartheta(\tau,z_4)}\\ &= \zeta_1^{\pm 1}+\zeta_2^{\pm 1}+\zeta_3^{\pm 1}+\zeta_4^{\pm 1}+4+O(q)\in J^w_{0,4A_1,1}.\end{aligned}$$

Then the function $\mathrm{Borch}(\phi_{0,4A_1,1})$ is a strongly reflective modular form of weight 2 for $\mathrm{O}^+(2U\oplus 4A_1(-1))$ with divisor coming from reflective vectors of norm -2 and divisor 2. Note that

$$\mathrm{Borch}(\phi_{0,4A_1,1})=\mathrm{Grit}(\vartheta(\tau,z_1)\vartheta(\tau,z_2)\vartheta(\tau,z_3)\vartheta(\tau,z_4)).$$

We also construct a reflective modular form for the lattice $2U\oplus 2E_8\oplus A_1(2)$. We recall the theta function for root lattice E_8

$$\vartheta_{E_8}(\tau,\mathfrak{z})=1+q\sum_{\substack{r\in E_8\\ r^2=2}}e^{2\pi i(r,\mathfrak{z})}+O(q^2)\in J_{4,E_8,1} \tag{2.3.5}$$

and the Jacobi-Eisenstein series of weight 4 and index 2 introduced in [EZ85, §2]

$$E_{4,2}(\tau,z)=1+q(14\zeta^{\pm 2}+64\zeta^{\pm 1}+84)+O(q^2)\in J_{4,A_1,2}.$$

Then the function $\vartheta_{E_8}\otimes\vartheta_{E_8}\otimes E_{4,2}$ gives a holomorphic Jacobi form of weight 12 and index 1 for $2E_8\oplus A_1(2)$ and we have

$$\frac{\vartheta_{E_8}\otimes\vartheta_{E_8}\otimes E_{4,2}}{\Delta}=q^{-1}+108+14\zeta^{\pm 2}+64\zeta^{\pm 1}+\sum_{\substack{v\in 2E_8\\ v^2=2}}e^{2\pi i(r,\mathfrak{z})}+O(q)\in J^w_{0,2E_8\oplus A_1(2),1}.$$

Since the lattice $2E_8\oplus A_1(2)$ satisfies Norm_2 condition, the singular Fourier coefficients of the above Jacobi form are determined only by its q^{-1}-term and q^0-term. Therefore, its Borcherds product gives a reflective modular form of weight 54 for $\mathrm{O}^+(2U\oplus 2E_8(-1)\oplus A_1(-2))$. Its divisors coming from reflective vectors of norm -2 and divisor 1 have order one. Its divisors coming from reflective vectors of norm -4 and divisor 4 have order 78. Its divisors coming from reflective vectors of norm -4 and divisor 2 have order 14.

2.4 Nonexistence of 2-reflective modular forms

Let M be an even integral lattice of signature $(2,n)$ with $n\geq 3$. Following [Ma17], we consider the decomposition (2.4.1) of the (-2)-Heegner divisor $\mathcal{H}$ defined in (2.1.3). Let $D(M)=M^\vee/M$ be the discriminant group of M. Let $\pi_M\subset D(M)$ denote the subset of elements of order 2 and norm $-1/2$. For each $\mu\in\pi_M$ we abbreviate $\mathcal{H}(\mu,-1/4)$ by $\mathcal{H}_\mu$. We also set

$$\mathcal{H}_0=\bigcup_{\substack{l\in M,(l,l)=-2\\ \mathrm{div}(l)=1}} l^\perp\cap\mathcal{D}(M)$$

to be the principal (-2)-Heegner divisor. Then we have the following decomposition

$$\mathcal{H} = \mathcal{H}_0 + \sum_{\mu\in\pi_M} \mathcal{H}_\mu. \tag{2.4.1}$$

Next, we assume that the lattice M contains $2U$ i.e. $M = 2U \oplus L(-1)$. In this case, each $\mathcal{H}_*$ is an $\widetilde{\mathrm{O}}^+(M)$-orbit of a single quadratic divisor $l^\perp \cap \mathcal{D}(M)$ and it is irreducible. We can write each element of π_M in the form $\mu = (0, n_\mu, \mu_0/2, 1, 0)$, where $n_\mu \in \mathbb{Z}$, $\mu_0 \in L$ and $2n_\mu - \frac{1}{4}(\mu_0,\mu_0) = -\frac{1}{2}$. If M admits a 2-reflective modular form F of weight k, then its divisor can be written as

$$\begin{aligned}\mathrm{div}(F) &= \beta_0\mathcal{H}_0 + \sum_{\mu\in\pi_M} \beta_\mu\mathcal{H}_\mu \\ &= \beta_0\mathcal{H} + \sum_{\mu\in\pi_M} (\beta_\mu - \beta_0)\mathcal{H}_\mu,\end{aligned} \tag{2.4.2}$$

where β_* are non-negative integers. By Theorem 1.3.3, there exists a nearly holomorphic vector-valued modular form f of weight $-\mathrm{rank}(L)/2$ with respect to the Weil representation ρ_M of $\mathrm{Mp}_2(\mathbb{Z})$ on the group ring $\mathbb{C}[D(M)]$ with principal part

$$\beta_0 q^{-1}\mathbf{e}_0 + \sum_{\mu\in\pi_M} (\beta_\mu - \beta_0) q^{-1/4}\mathbf{e}_\mu,$$

such that F is the Borcherds product of f. In view of the isomorphism between vector-valued modular forms and Jacobi forms (see Theorem 1.4.8), there exists a weakly holomorphic Jacobi form ϕ_L of weight 0 and index 1 for L with singular Fourier coefficients of the form

$$\mathrm{Sing}(\phi_L) = \beta_0 \sum_{r\in L} q^{(r,r)/2-1}\zeta^r + \sum_{\mu\in\pi_M} (\beta_\mu - \beta_0) \sum_{s\in L+\mu_0/2} q^{(s,s)/2-1/4}\zeta^s \tag{2.4.3}$$

where $\zeta^l = e^{2\pi i (l,\mathfrak{z})}$. Thus, we have

$$\phi_L(\tau,\mathfrak{z}) = \beta_0 q^{-1} + \beta_0 \sum_{r\in R_L} \zeta^r + 2k + \sum_{u\in\pi_M} (\beta_\mu - \beta_0) \sum_{s\in R_\mu(L)} \zeta^s + O(q) \tag{2.4.4}$$

here and subsequently, R_L denotes the set of 2-roots in L and

$$R_\mu(L) = \{s \in L^\vee : 2s \in R_L, s - \mu_0/2 \in L\}. \tag{2.4.5}$$

With the help of equation (2.4.4) and Lemma 1.4.12, we get the following theorem.

Theorem 2.4.1. *Let L be a positive-definite even lattice and $M = 2U \oplus L(-1)$. Suppose that F is a 2-reflective modular form of weight k with divisor of the form (2.4.2). Then the weight k of F is given by the following formula*

$$\begin{aligned}k =& \beta_0\left[12 + |R_L|\left(\frac{12}{\mathrm{rank}(L)} - \frac{1}{2}\right)\right] \\ &+ \left(\frac{3}{\mathrm{rank}(L)} - \frac{1}{2}\right)\sum_{\mu\in\pi_M} (\beta_\mu - \beta_0)|R_\mu(L)|.\end{aligned} \tag{2.4.6}$$

Remark 2.4.2. From the above theorem, we have

1. If $R(L)$ is empty, then the weight of 2-reflective modular form is $12\beta_0$.
2. When $\mathrm{rank}(L) \geq 6$, there is no 2-reflective modular form with $\beta_0 = 0$. This fact can also be proved by Riemann–Roch theorem as the proof of [Sch17, Proposition 6.1].

3. When $\operatorname{rank}(L) = 6$, the weight of 2-reflective modular form is $\beta_0(12 + \frac{3}{2}|R(L)|)$ and the modular form is not of singular weight.

We next study modular forms with complete 2-divisor (i.e. $\operatorname{div}(F) = \mathcal{H}$), which are the simplest 2-reflective modular forms.

Theorem 2.4.3. *If there exists a modular form with complete* 2*-divisor for* $M = 2U \oplus L(-1)$*, then either* $\operatorname{rank}(L) \leq 8$*, or* L *is a unimodular lattice of rank* 16 *or* 24*. Moreover, the weight of the corresponding modular form is*

$$k = 12 + |R_L| \left(\frac{12}{\operatorname{rank}(L)} - \frac{1}{2} \right). \tag{2.4.7}$$

Proof. Firstly, the above formula is a direct result of Theorem 2.4.1. Let F be a modular form with complete 2-divisor. Then there exists a weakly holomorphic Jacobi form of weight 0 and index L such that

$$\phi(\tau, \mathfrak{z}) = q^{-1} + \sum_{r \in R(L)} \zeta^r + 2k + O(q)$$

whose singular Fourier coefficients are (see formula (2.4.3))

$$\operatorname{Sing}(\phi) = \sum_{n \geq -1} \sum_{\substack{l \in L \\ (l,l) = 2n+2}} q^n \zeta^l,$$

with $\operatorname{Borch}(\phi) = F$. By Theorem 2.2.8, it is known that $\operatorname{rank}(L) < 24$ or L is a unimodular lattice of rank 24. Let us assume that $\operatorname{rank}(L) \leq 23$ and let us construct two special Jacobi forms by using the differential operators introduced in Lemma 1.4.11. For simplicity, we set $R = |R_L|$ and $n_0 = \operatorname{rank}(L)$.

$$\begin{aligned} f_2(\tau, \mathfrak{z}) &= \frac{24}{n_0 - 24} H_0(\phi)(\tau, \mathfrak{z}) \\ &= q^{-1} + \sum_{r \in R(L)} \zeta^r - R + O(q) \in J^{w.h.}_{2,L,1} \\ f_4(\tau, \mathfrak{z}) &= \frac{24}{n_0 - 28} H_2(f_2)(\tau, \mathfrak{z}) \\ &= q^{-1} + \sum_{r \in R(L)} \zeta^r - \frac{(R + 24)(n_0 - 4)}{n_0 - 28} + O(q) \in J^{w.h.}_{4,L,1} \end{aligned}$$

Let E_4 and E_6 denote the Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})$ of weight 4 and 6, respectively. Then we can check that

$$\begin{aligned} g(\tau, \mathfrak{z}) &= \frac{n_0 - 28}{48} \left[E_4(\tau)\phi(\tau, \mathfrak{z}) - f_4(\tau, \mathfrak{z}) \right] \\ &= R\left(1 - \frac{14}{n_0}\right) + 6(n_0 - 26) + O(q) \in J_{4,L,1} \end{aligned}$$

and

$$\begin{aligned} h(\tau, \mathfrak{z}) &= E_6(\tau)\phi(\tau, \mathfrak{z}) - E_4(\tau) f_2(\tau, \mathfrak{z}) \\ &= \frac{24R}{n_0} - 720 + O(q) \in J_{6,L,1} \end{aligned}$$

are holomorphic Jacobi forms of weight 4 and 6, respectively. In fact, the singular Fourier coefficients are stable under the actions of the differential operators, so the singular Fourier coefficients of f_2 and f_4 come from $\operatorname{Sing}(\phi)$. In order to check g and h are holomorphic Jacobi forms, we only need to check that g and h have no singular Fourier coefficient i.e. the singular part $\mathrm{Sing}(\phi)$ has been cancelled by the above combinations of ϕ, f_2 and f_4.

Since the singular weight of holomorphic Jacobi form of index L is $\frac{n_0}{2}$, we deduce that $g = 0$ if $n_0 > 8$ and $h = 0$ if $n_0 > 12$. By direct calculations, we have

- when $R = 0$, $g \neq 0$ if $n_0 < 24$.
- when $R > 0$, $g \neq 0$ if $n_0 \leq 14$.
- when $n_0 \geq 15$, $g = 0$ and $h = 0$ if and only if $n_0 = 16$ and $R = 480$.

When $n_0 = 16$, from $h = 0$, it follows that the Fourier coefficients of ϕ satisfy: $c(n,l) = 0$ if $2n - (l,l) = 0$ and $l \notin L$. Otherwise, there exists a Fourier coefficient $c(n,l) \neq 0$ with $2n - (l,l) = 0$ and $l \notin L$. Assume that $c(n,l)$ is such Fourier coefficient with the smallest n. Then the coefficient of $q^n\zeta^l$ in $E_6\phi$ is $c(n,l)$ and the coefficient of $q^n\zeta^l$ in $E_4 f_2$ is $-2c(n,l)$. Thus, the coefficient of $q^n\zeta^l$ in h is not zero and then $h \neq 0$, which leads to a contradiction. Therefore, the following holomorphic Jacobi form of singular weight 8

$$E_8\phi - E_6 f_2 = 1728 + \sum_{\substack{n>0, l\in L^\vee \\ 2n=(l,l)}} a(n,l) q^n \zeta^l \in J_{8,L,1}$$

satisfies the same condition: $a(n,l) = 0$ if $2n - (l,l) = 0$ and $l \notin L$. We then obtain

$$E_8\phi - E_6 f_2 = 1728 \sum_{l\in L} q^{\frac{(l,l)}{2}} \zeta^l$$

and L has to be unimodular. The proof is completed. □

Remark 2.4.4. It is worth pointing out that there exist lattices L which admit a modular form with complete 2-divisor when $1 \leq \mathrm{rank}(L) \leq 8$ (see Table 2.2). For example, the Igusa cusp form χ_{35} is a modular form with complete 2-divisor.

We are going to generalize our method to prove the nonexistence of 2-reflective modular forms in higher dimensions.

Theorem 2.4.5. *Suppose that $M = 2U \oplus L(-1)$ is a 2-reflective lattice satisfying $\mathrm{rank}(L) \geq 12$. Then either $\mathrm{rank}(L) = 17$, or L is a unimodular lattice of rank 16 or 24. Furthermore, when $\mathrm{rank}(L) = 17$, the weight of the corresponding 2-reflective modular form is $75\beta_0$, where β_0 is the multiplicity of the divisor $\mathcal{H}_0$.*

Proof. If M has a 2-reflective modular form F of weight k with divisor of the form (2.4.2), then there exists a weakly holomorphic Jacobi form ϕ of weight 0 and index L with singular Fourier coefficients of the form (2.4.3). Let us assume that $\mathrm{rank}(L) \leq 23$. We next construct a holomorphic Jacobi form of weight 6 from ϕ. We write $\phi = S_1 + d + S_2 + \cdots$, where S_1 and S_2 are the first and second terms in (2.4.3), respectively, and $d = 2k$. It is clear that $\phi - S_1 - S_2$ does not have the term with negative hyperbolic norm. We can construct ($n_0 = \mathrm{rank}(L)$)

$$f_2 = \frac{24}{n_0 - 24} H_0(\phi) = S_1 + d_1 + c_1 S_2 + \cdots \in J^{w.h.}_{2,L,1},$$
$$f_4 = \frac{24}{n_0 - 28} H_2(f_2) = S_1 + d_2 + c_2 S_2 + \cdots \in J^{w.h.}_{4,L,1},$$
$$f_6 = \frac{24}{n_0 - 32} H_4(f_4) = S_1 + d_3 + c_3 S_2 + \cdots \in J^{w.h.}_{6,L,1},$$

where

$$d_1 = \frac{n_0(d-24\beta_0)}{n_0-24}, \qquad c_1 = \frac{n_0-6}{n_0-24},$$
$$d_2 = \frac{(n_0-4)(d_1-24\beta_0)}{n_0-28}, \qquad c_2 = c_1\frac{n_0-10}{n_0-28},$$
$$d_3 = \frac{(n_0-8)(d_2-24\beta_0)}{n_0-32}, \qquad c_3 = c_2\frac{n_0-14}{n_0-32}.$$

The function

$$\varphi_6 = (c_1-c_3)E_6\phi + (c_3-1)E_4f_2 + (1-c_1)f_6 = u + O(q) \in J_{6,L,1}$$

where

$$u = (d-504\beta_0)(c_1-c_3) + (d_1+240\beta_0)(c_3-1) + d_3(1-c_1)$$

is a holomorphic Jacobi form of weight 6 because the potential singular Fourier coefficients S_1 and S_2 have been cancelled.

In view of the singular weight, $\varphi_6 = 0$ if $\mathrm{rank}(L) \geq 13$. From Remark 2.4.2, we know that if F exists then $d = 2k \geq \mathrm{rank}(L)$ and $\beta_0 > 0$ when $\mathrm{rank}\, L \geq 6$. By direct calculations, when $n_0 = 13$ or 14, $u \neq 0$, which is impossible.

We next assume that $15 \leq \mathrm{rank}(L) \leq 23$. We construct

$$g = E_4\phi - f_4 = (d+240\beta_0) - d_2 + (1-c_2)S_2 + \cdots \in J^{w.h.}_{4,L,1}.$$

Since g only has singular Fourier coefficients of the form S_2, the minimum possible hyperbolic norm of its Fourier coefficients is $-\frac{1}{2}$. Therefore, $\eta^6 g$ is a holomorphic Jacobi form of weight 7 and index L with character, where η is the Dedekind eta function. In view of the singular weight, we have that $\eta^6 g = 0$ and then $g = 0$. If $S_2 = 0$, then F is a 2-reflective modular form with complete 2-divisor, which gives that L is a unimodular lattice of rank 16 by Theorem 2.4.3. If $S_2 \neq 0$, then $1-c_2 = 0$, which gives $n_0 = 17$. By $(d+240\beta_0)-d_2 = 0$ and $n_0 = 17$, we get $d = 150\beta_0$.

We now consider the case $\mathrm{rank}(L) = 12$. Without loss of generality, we can assume that L is maximal, namely, L has no any proper even overlattice. By direct calculation, the constant term of φ_6 is not zero and we assume it to be 1. The function φ_6 has singular weight 6. Thus, it is a $\mathbb{C}$-linear combination of theta-functions for L defined as (1.4.10). Since L is maximal, there is no $\gamma \in L^\vee$ such that $\gamma \notin L$ and $(\gamma,\gamma) = 2$. Hence, the q^1-term of Fourier expansion of φ_6 comes only from the theta-function Θ_0^L. But $\varphi_6(\tau,0) = E_6(\tau) = 1 - 504q + \ldots$, this leads to a contradiction.

The proof is completed. □

Remark 2.4.6. Firstly, there exist 2-reflective lattices when $1 \leq \mathrm{rank}(L) \leq 8$. When $\mathrm{rank}(L) = 11$, we do not know if there exists any 2-reflective lattice. When $\mathrm{rank}(L) = 9, 10, 17$, there exist 2-reflective lattices, such as $L = E_8 \oplus A_1$, $E_8 \oplus 2A_1$, $2E_8 \oplus A_1$ (see §2.3.1).

We now consider the general case, this means that M does not contain two hyperbolic planes. For our purpose, we need the follwing lemma proved in [Ma18, Lemma 1.7].

Lemma 2.4.7. *Let M be an even lattice of signature $(2,n)$ with $n \geq 8$. There exists an even overlattice M' of M containing $2U$ such that $\delta(M) = \delta(M')$, where $\delta(M)$ is the exponent of $M^\vee/M$ i.e. the maximal order of elements in $M^\vee/M$.*

Theorem 2.4.8. *Let M be a 2-reflective lattice of signature $(2,n)$ with $n \geq 14$. Then $n = 19$ or M is isomorphic to the unique even unimodular lattices of signature $(2,18)$ or $(2,26)$.*

Proof. The proof is similar to the proof of [Ma17, Proposition 3.1]. Let M be a 2-reflective lattice of signature $(2,n)$ with $n \geq 14$. By the above lemma, there exists an even overlattice M' of M containing $2U$. By Lemma 2.1.4, the lattice M' is also 2-reflective. We thus complete the proof by Theorem 2.4.5. □

2.5 Nonexistence of reflective modular forms

In this section we use similar arguments to show the nonexistence of reflective modular forms of large rank. As an application of our arguments in the previous section, we attempt to classify reflective modular forms on lattices of prime level and large rank.

Let $M = 2U \oplus L(-1)$ be an even lattice of prime level p and F be a reflective modular form of weight k with respect to $\widetilde{\mathrm{O}}^+(M)$. The divisor of F can be represented as

$$\mathrm{div}(F) = \beta_0 \mathcal{H}_0 + \sum_{\gamma \in \pi_{M,p}} \beta_\gamma \mathcal{H}(\gamma, -1/p), \tag{2.5.1}$$

where $\pi_{M,p} \subset D(M)$ is the subset of elements of norm $-2/p$. By Theorem 1.3.3, there exists a nearly holomorphic modular form with principal part

$$\beta_0 q^{-1}\mathbf{e}_0 + \sum_{\gamma \in \pi_{M,p}} \beta_\gamma q^{-1/p}\mathbf{e}_\gamma.$$

Then there exists a weakly holomorphic Jacobi form of weight 0 with singular Fourier coefficients

$$\mathrm{Sing}(\psi_L) = \beta_0 \sum_{r\in L} q^{(r,r)/2-1}\zeta^r + \sum_{\gamma\in\pi_{M,p}} \beta_\gamma \sum_{s \in L+\gamma} q^{(s,s)/2-1/p}\zeta^s. \tag{2.5.2}$$

Then the q^0-term of ψ_L can be written as

$$\psi_L(\tau,\mathfrak{z}) = \beta_0 q^{-1} + \beta_0 \sum_{r\in R_L}\zeta^r + 2k + \sum_{\gamma\in\pi_{M,p}}\beta_\gamma \sum_{s\in C_\gamma(L)} \zeta^s + O(q),$$

where

$$C_\gamma(L) = \{s \in L^\vee : (s,s) = 2/p, s-\gamma \in L\}. \tag{2.5.3}$$

Thus, we get a formula related to the weight of the above reflective modular form

$$\begin{aligned} k =& \beta_0\left[12 + |R_L|\left(\frac{12}{\mathrm{rank}(L)} - \frac{1}{2}\right)\right] \\ &+ \left(\frac{12}{p\cdot \mathrm{rank}(L)} - \frac{1}{2}\right)\sum_{\gamma\in\pi_{M,p}} \beta_\gamma |C_\gamma(L)|. \end{aligned} \tag{2.5.4}$$

It is possible to find a similar formula for the weight of reflective modular forms for general lattices.

Remark 2.5.1. Let $M = 2U \oplus L(-1)$ be an even lattice of prime level p and F be a reflective modular form of weight k for M. From (2.5.4), when $\mathrm{rank}(L) = 12$ and $p = 2$, then $k = \beta_0(12 + \frac{1}{2}|R(L)|)$ so the function F is not of singular weight. When $\mathrm{rank}(L) = 8$ and $p = 3$, $k = \beta_0(12 + |R(L)|)$ and F is not of singular weight.

By Theorem 2.2.9, when a reflective modular form F exists, we have that either $\mathrm{rank}(L) \leq 23$ or L is a unimodular lattice of rank 24. We next give a finer classification of reflective modular forms on lattices of prime level.

Theorem 2.5.2. *Let $M = 2U \oplus L(-1)$ be an even lattice of prime level p. If M admits a reflective modular form of weight k for $\widetilde{\mathrm{O}}^+(M)$, then we have*

1. *when $p = 2$, either $\mathrm{rank}(L) \leq 16$ or $\mathrm{rank}(L) = 20$ and $k = 24\beta_0$.*
2. *when $p = 3$, either $\mathrm{rank}(L) \leq 12$ or $\mathrm{rank}(L) = 18$ and $k = 48\beta_0$.*
3. *when $p \geq 5$, $\mathrm{rank}(L) \leq 8 + 24/(p+1)$.*

Proof. Similar to the proof of Theorem 2.4.5, there exists a weakly holomorphic Jacobi form ϕ of weight 0 and index L with singular Fourier coefficients of the form (2.5.2). Assume that $\mathrm{rank}(L) \leq 23$. We write $\phi = S_1 + d + S_2 + \cdots$, where S_1 and S_2 are the first and second terms in (2.5.2), respectively, and $d = 2k$. We can construct ($n_0 = \mathrm{rank}(L)$ and $a = 24/p$)

$$f_2 = \frac{24}{n_0 - 24} H_0(\phi) = S_1 + d_1 + c_1 S_2 + \cdots \in J^{w.h.}_{2,L,1},$$
$$f_4 = \frac{24}{n_0 - 28} H_2(f_2) = S_1 + d_2 + c_2 S_2 + \cdots \in J^{w.h.}_{4,L,1},$$
$$f_6 = \frac{24}{n_0 - 32} H_4(f_4) = S_1 + d_3 + c_3 S_2 + \cdots \in J^{w.h.}_{6,L,1},$$

where

$$d_1 = \frac{n_0(d - 24\beta_0)}{n_0 - 24}, \qquad c_1 = \frac{n_0 - a}{n_0 - 24},$$
$$d_2 = \frac{(n_0 - 4)(d_1 - 24\beta_0)}{n_0 - 28}, \qquad c_2 = c_1 \frac{n_0 - a - 4}{n_0 - 28},$$
$$d_3 = \frac{(n_0 - 8)(d_2 - 24\beta_0)}{n_0 - 32}, \qquad c_3 = c_2 \frac{n_0 - a - 8}{n_0 - 32}.$$

We can check that the function

$$\varphi_6 = (c_1 - c_3) E_6 \phi + (c_3 - 1) E_4 f_2 + (1 - c_1) f_6 = u + O(q) \in J_{6,L,1}$$

has no term of the form S_1 or S_2 and then it has no singular Fourier coefficient, so it is a holomorphic Jacobi form of weight 6, where

$$u = (d - 504\beta_0)(c_1 - c_3) + (d_1 + 240\beta_0)(c_3 - 1) + d_3(1 - c_1).$$

We also construct a weakly holomorphic Jacobi form of weight 4

$$g = E_4 \phi - f_4 = (d + 240\beta_0) - d_2 + (1 - c_2) S_2 + \cdots \in J^{w.h.}_{4,L,1}.$$

By Theorem 2.4.3, we have $S_2 \neq 0$ when $n_0 > 8$. By direct calculations, we get

$$c_2 = 1 \iff \mathrm{rank}(L) = 14 + \frac{12}{p}. \tag{2.5.5}$$

Therefore, when $p = 2$, $c_2 = 1$ if and only if $n_0 = 20$, when $p = 3$, $c_2 = 1$ if and only if $n_0 = 18$, when $p > 3$, $c_2 - 1 \neq 0$. We thus obtain

- when $p = 2$, if $g = 0$ then $n_0 = 20$ and $d = 48\beta_0$;
- when $p = 3$, if $g = 0$ then $n_0 = 18$ and $d = 96\beta_0$;
- when $p \geq 5$, $g \neq 0$.

Suppose $g \neq 0$ and $n_0 > 8$. Then $c_2 \neq 1$, otherwise g will be a holomorphic Jacobi form of weight 4, which contradicts the singular weight. The weakly holomorphic Jacobi form g corresponds to a nearly holomorphic vector-valued modular form

$$F = \sum_{\gamma \in A_M} F_\gamma \mathbf{e}_\gamma$$

of weight $4 - n_0/2$. Note that $F_0(\tau)$ has no term q^n with negative n and for any nonzero $\gamma \in D(M)$, the possible term q^n with negative n of F_γ is $q^{-1/p}$. We know from Proposition 1.3.6 that $F_0 \neq 0$

because the function $\sum_{\sigma\in \mathrm{O}(D(M))}\sigma\cdot F$ is invariant under the orthogonal group $\mathrm{O}(D(M))$ of the discriminant group $D(M)$ and it is not zero. In addition, F_0 is a nearly holomorphic modular form of weight $4-n_0/2$ with respect to $\Gamma_0(p)$ and its expansion at the cusp 0 is a linear combination of F_γ. As in the proof of [Sch17, Proposition 6.1], the Riemann-Roch theorem applied to F_0 gives

$$-1 \le p\nu_0(F_0) + \nu_\infty(F_0) \le \left(4 - \frac{n_0}{2}\right)\frac{p+1}{12}.$$

This implies

$$n_0 \le 8 + \frac{24}{p+1}.$$

It remains to prove that M is not reflective if $n_0 = 14$ and $p = 3$. But in this case, $u \neq 0$ and then $\varphi_6 \neq 0$, which gives a contradiction. The proof is completed. □

Note that when $\mathrm{rank}(L) = 16$ and $p = 2$, we have $u \equiv 0$. Therefore, our argument cannot determine the weight of the corresponding reflective modular form.

Remark 2.5.3. From the oddity formula, the rank of any lattice of prime level is even.

1. The rank of an even positive-definite lattice of level 2 is known to be divisible by 4. Thus, by Theorem 2.5.2, if $2U \oplus L(-1)$ is a reflective lattice of level 2, then $\mathrm{rank}(L)$ must be 4, 8, 12, 16 or 20. We note that there exist reflective lattices of such ranks. When $L = D_4$, $2D_4$, $E_8 \oplus D_4$, $E_8 \oplus 2D_4$ or $2E_8 \oplus D_4$, $2U \oplus L(-1)$ admits a reflective modular form.

2. If L is an even positive-definite lattice of level 3 and of rank n with determinant $\det(L) = |L^\vee/L| = 3^r$, then either $r \in \{0, n\}$ and $8|n$, or $0 < r < n$ and $2r \equiv n \mod 4$. Theorem 2.5.2 says that $2U \oplus L(-1)$ is a reflective lattice of level 3 only if $\mathrm{rank}(L) = 2, 4, 6, 8, 10, 12$ or 18. Reflective lattices of such ranks exist. When $L = A_2$, $2A_2$, $3A_2$, $4A_2$, $E_8 \oplus A_2$, $E_8 \oplus 2A_2$ or $2E_8 \oplus A_2$, the lattice $2U \oplus L(-1)$ admits a reflective modular form.

We next extend the above classification results to the general case. The following lemma introduced in [Ma18, Corollary 3.2] is very useful for our purpose.

Lemma 2.5.4. *Let M be a lattice of signature $(2, n)$ with $n \ge 11$. There exists a lattice M_1 on $M \otimes \mathbb{Q}$ such that $\mathrm{O}^+(M) \subset \mathrm{O}^+(M_1)$ and that M_1 is a scaling of an even lattice containing $2U$.*

We remark that the above lattice M_1 is usually not an overlattice of M and it is constructed as a sublattice of an overlattice of M. We next use the above lemma to extend Ma's result [Ma17, Proposition 3.2] to the general case.

Theorem 2.5.5.

1. *There is no reflective lattice of signature $(2, n)$ with $n > 26$.*

2. *Let M be an even lattice of signature $(2, 26)$. If it admits a reflective modular form which can be constructed as a Borcherds product, then it is isomorphic to the unique even unimodular lattice of signature $(2, 26)$.*

Proof. We first prove the statement (1). By contradiction, assume that there is a reflective lattice M of signature $(2, n)$ with $n > 26$ and we denote the corresponding reflective modular form by F. By Lemma 2.5.4, there exists a lattice M_1 on $M \otimes \mathbb{Q}$ such that $\mathrm{O}^+(M) \subset \mathrm{O}^+(M_1)$ and that M_1 is a scaling of an even lattice M_2 containing $2U$. Here, we have a natural isomorphism

$$\mathcal{D}(M) \cong \mathcal{D}(M_1) \cong \mathcal{D}(M_2),$$

where the first comes from the equality $M \otimes \mathbb{Q} = M_1 \otimes \mathbb{Q}$ and the second from the identification $M_1 = M_2$ as $\mathbb{Z}$-modules. Moreover, the inclusion $\mathrm{O}^+(M) \subset \mathrm{O}^+(M_1) \cong \mathrm{O}^+(M_2)$ is compatible with this isomorphism and the isomorphism preserves the reflective divisors. Thus, F is a reflective modular form for M_1 and then also a reflective modular form for M_2, which contradicts Proposition 2.2.9.

We next prove the statement (2). The proof is similar to Proposition 2.2.9. Assume that the corresponding reflective modular form F is a Borcherds product of a nearly holomorphic modular form f. Then Δf is a holomorphic modular form of weight 0 and hence must be an $\mathrm{Mp}_2(\mathbb{Z})$-invariant vector in $\mathbb{C}[D(M)]$. Then we get $M = II_{2,26}$ because Δf does not transform correctly under the matrix S when $|D(M)| \neq 1$. This completes the proof. □

Remark 2.5.6. We do not know if there is any other reflective lattice of signature $(2,26)$ except the scalings of $\mathrm{II}_{2,26}$. By the second statement in Theorem 2.5.5 and Theorem 1.3.3, such reflective lattice is not of the form $U \oplus U(m) \oplus L(-1)$. This question is related to the general question if all reflective modular forms come from Borcherds products.

We now extend Theorem 2.5.2 to the general case.

Theorem 2.5.7. *Let M be a reflective lattice of signature $(2,n)$ and of prime level p.*

1. *If $p = 2$ and $n > 18$ and $n \neq 22$, then M is isomorphic to $\mathrm{II}_{2,26}(2)$.*
2. *If $p = 3$ and $n > 14$ and $n \neq 20$, then M is isomorphic to $\mathrm{II}_{2,18}(3)$ or $\mathrm{II}_{2,26}(3)$.*
3. *If $p > 3$ and $n > 10 + 24/(p+1)$, then M is isomorphic to $\mathrm{II}_{2,18}(p)$ or $\mathrm{II}_{2,26}(p)$.*

Proof. Let F be a reflective modular form for M. In this case, we have $n \geq 11$. Assume that the determinant of M is p^a, where $1 \leq a \leq n+2$ is an integer. Since M is of prime level p, the discriminant group of M is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^a$ and the minimum number of generators of this group is $l(M) = a$.

If $n > a+2$, by Lemma 1.1.3, the lattice M can be represented as $2U \oplus L$. Thus, the function F is also a reflective modular form for $2U \oplus L$. We then prove this theorem by Theorem 2.5.2.

If $n \leq a+2$, then $a \geq 9$ because $n \geq 11$. Since M is of prime level p, the lattice $M^\vee(p)$, which is a scaling of the dual lattice of M, is even and of determinant p^{2+n-a}. Moreover, if $M^\vee(p)$ is not unimodular, then it is of level p. In view of the natural isomorphism

$$\mathrm{O}^+(M) \cong \mathrm{O}^+(M^\vee) \cong \mathrm{O}^+(M^\vee(p)),$$

the function F is a reflective modular form for $M^\vee(p)$. Since $n > (2+n-a)+2$, we can prove the case as the previous case. If $M^\vee(p)$ is unimodular, then $M = (M^\vee(p))^\vee(p)$ is a scaling of an even unimodular lattice. Thus, the proof is completed. □

Remark 2.5.8. Theorem 2.2.4 gives bounds on the signature for non-symmetric reflective modular forms. But the bounds do not hold in the symmetric case. However, the above result gives bounds in the symmetric case. It is also a supplement to Ditmann's result (see Lemma 2.2.6).

Theorem 2.5.9. *There is no any reflective lattice of signature $(2,n)$ with $23 \leq n \leq 25$.*

Proof. Let M be an even lattice of signature $(2,n)$ with $23 \leq n \leq 25$. Firstly, we assume that M contains $2U$. By contradiction, assume that M is reflective. Then, there exists a weakly holomorphic Jacobi form ϕ of weight 0 and we can construct a weakly holomorphic Jacobi form f_4 of weight 4 from ϕ as in the proof of Theorem 2.5.2. We define $g = E_4\phi - f_4$. Then g has no singular Fourier coefficient of hyperbolic norm -2. From (2.5.5), we get $g \neq 0$. Moreover, the minimum possible hyperbolic norm of the Fourier coefficients of g is -1. Then $\eta^{12} g$ is a holomorphic Jacobi form of weight 10 with character, which leads to a contradiction due to the singular weight. The general case can be proved as the proof of Theorem 2.5.5. □

Proposition 2.5.10. *If $M = 2U \oplus L(-1)$ is a reflective lattice of signature $(2,22)$ and F is a reflective modular form for $\widetilde{\mathrm{O}}^+(M)$, then the weight of F is $24\beta_0$ and the divisor of F is given by*

$$\mathrm{div}(F) = \beta_0\mathcal{H} + \sum_v \beta_v \mathcal{H}(v,-1/2), \tag{2.5.6}$$

where the sum takes over all vectors of norm -1 and order 2 in the discriminant group of M, β_0, β_v are natural numbers.

Proof. Under the notations of the proof of Theorem 2.5.9, if F has other type of divisors, then the Jacobi form g of weight 4 will have singular Fourier coefficients of hyperbolic norm > -1. Thus $\eta^{12}g$ is a holomorphic Jacobi form of singular weight 10, which contradicts the fact that the hyperbolic norm of non-zero Fourier coefficients of any holomorphic Jacobi form of singular weight is always zero. This also forces that the constant term of g is zero, which yields $k = 24\beta_0$. □

Corollary 2.5.11. *Assume that $2U \oplus L(-1)$ is a reflective lattice of signature $(2,22)$. Let R_L denote the set of 2-roots in L. We set*

$$R_1(L) = \{v \in L^\vee : (v,v) = 1,\ 2v \in L\}.$$

Then $|R_L| \geq 120$ and the set $R_L \cup R_1(L)$ generates the vector space $L \otimes \mathbb{R}$.

Proof. Suppose that F is a reflective modular form for the lattice. Then F has weight $24\beta_0$ and its divisor is of the form (2.5.6). We denote the associated weakly holomorphic Jacobi form of weight 0 and index L by ϕ. We define $R = |R_L|$ and $R_1 = \sum_{v\in R_1(L)} \beta_v$. By (1.4.17), we have

$$\frac{1}{24}\left(\beta_0(R+48) + R_1\right) - \beta_0 = \frac{1}{40}(2\beta_0 R + R_1),$$

which yields

$$R = 120 + \frac{2}{\beta_0}R_1.$$

Then we have $R \geq 120$. The second assertion follows from (1.4.18). □

Remark 2.5.12. When $1 \leq \mathrm{rank}(L) \leq 20$ and $\mathrm{rank}(L) \neq 19$, there exist reflective lattices $2U \oplus L(-1)$, such as A_n for $1 \leq n \leq 7$, D_8, $E_8 \oplus A_1$, $E_8 \oplus A_2$, $E_8 \oplus A_2 \oplus A_1$, $E_8 \oplus D_4$, $E_8 \oplus D_4 \oplus A_1$, $E_8 \oplus D_4 \oplus A_2$, $E_8 \oplus D_7$, $E_8 \oplus 2D_4$, $2E_8 \oplus A_1$, $2E_8 \oplus A_2$, $2E_8 \oplus D_4$. But we do not know if there exists reflective lattice $2U \oplus L(-1)$ with $\mathrm{rank}(L) = 19$.

2.6 Classification of 2-reflective modular forms

2.6.1 More refined results

In this subsection, we prove the following main result. We use the notations in §2.4.

Theorem 2.6.1. *Let $M = 2U \oplus L(-1)$ and F be a 2-reflective modular form of weight k with divisor of the form (2.4.2) for M. Let $R(L)$ be the root sublattice generated by 2-roots of L. If $R(L)$ is empty, then $k = 12\beta_0$. If $R(L)$ is not empty, then $R(L)$ and L have the same rank, which is denoted by n. Furthermore, the lattice $R(L)$ satisfies one of the following conditions*

(a) *$R(L) = nA_1$. In this case, all β_μ satisfying $R_\mu(L) \neq \varnothing$ are the same.*

(b) *The lattice A_1 is not an irreducible component of $R(L)$. In this case, all the irreducible components of $R(L)$ have the same Coxeter numbers, which is denoted by h. In addition, the sets $R_\mu(L)$ are all empty and the weight k is given by*

$$k = \beta_0\left(12 + 12h - \frac{1}{2}nh\right).$$

(c) $R(L) = mA_1 \oplus R$, *where* $1 \le m \le n-2$ *and the lattice* A_1 *is not an irreducible component of* R*. In this case, all the irreducible components of* R *have the same Coxeter numbers, which is denoted by* h*. All* β_μ *satisfying* $R_\mu(L) \neq \varnothing$ *are the same, which is denoted by* β_1*. We have*

$$\begin{aligned} \beta_1 &= (2h-3)\beta_0, \\ k &= \beta_0\left[\left(12 - \frac{n+3m}{2}\right)h + 12 + 3m\right]. \end{aligned}$$

Moreover, the lattice L *can be represented as* $mA_1 \oplus L_0$*, where* L_0 *is an overlattice of* R*.*

Proof. First, if $R(L) = \varnothing$ then we derive from (2.4.6) that $k = 12\beta_0$.

We next assume that $R(L) \neq \varnothing$. Let $R(L) = \oplus R_i$ be the decomposition of irreducible components i.e. R_i are irreducible root lattices. We write $\mathfrak{z} = \sum_i \mathfrak{z}_i \in L \otimes \mathbb{C}$, where $\mathfrak{z}_i \in R_i \otimes \mathbb{C}$. For irreducible root lattices, only the lattice A_1 satisfies the property that there is a root v such that $v/2$ is in the dual lattice. By (1.4.18) and (2.4.4), we conclude that $R(L)$ and L have the same rank. Otherwise, there exists a vector in $L \otimes \mathbb{C}$ orthogonal to $R(L) \otimes \mathbb{C}$, which contradicts the identity (1.4.18) because the number C is not equal to 0. In a similar way, we can prove the statement (a).

We next prove the statement (b). Since there is no R_i equal to A_1, the sets $R_\mu(L)$ are all empty. By Lemma 1.4.12 and (2.4.4), we have

$$\sum_i \sum_{\substack{r \in R_i \\ r^2=2}} \beta_0 (r, \mathfrak{z}_i)^2 = \frac{2}{n} \sum_i \beta_0 h_i \operatorname{rank}(R_i) \sum_i (\mathfrak{z}_i, \mathfrak{z}_i),$$

where h_i are the Coxeter numbers of R_i. On the other hand, by Proposition 1.1.4, we have

$$\sum_{\substack{r \in R_i \\ r^2=2}} (r, \mathfrak{z}_i)^2 = 2h_i(\mathfrak{z}_i, \mathfrak{z}_i).$$

Thus, all the Coxeter numbers h_i are the same. The weight formula is obtained by (2.4.6).

We now prove the statement (c). Firstly, all non-empty $R_\mu(L)$ are contained in the components mA_1. We write $R = \oplus R_j$, where R_j are irreducible root lattices. For $1 \le t \le m$, if the dual lattice of the t-th copy of A_1 is contained in $L^\vee$, then the corresponding $R_\mu(L)$ have two elements and we note the corresponding β_μ by β_t. If not, we set $\beta_t = \beta_0$. We also denote the ellptic parameter associated to the t-th copy of A_1 by $\mathfrak{z}_t$. By Lemma 1.4.12 and (2.4.4), we have

$$\sum_j \sum_{\substack{r \in R_j \\ r^2=2}} \beta_0 (r, \mathfrak{z}_j)^2 + \sum_t 2((\beta_t - \beta_0) + 4\beta_0)\mathfrak{z}_t^2 = 2C\left[\sum_j (\mathfrak{z}_j, \mathfrak{z}_j) + \sum_t 2\mathfrak{z}_t^2\right].$$

In the above identity, we use the standard model of A_1: $A_1 = \mathbb{Z}\alpha$ with $\alpha^2 = 2$. Let h_j denote the Coxeter number of R_j. Then we have

$$C = \beta_0 h_j = \frac{1}{2}\beta_t + \frac{3}{2}\beta_0.$$

Therefore, all h_j are the same and the dual lattice of every copy of A_1 is contained in $L^\vee$. It follows that $\beta_1 = (2h-3)\beta_0$. Combining the formula $\beta_1 = (2h-3)\beta_0$ and (2.4.6) together, we deduce the weight formula. We set $L_0 = \{v \in L : (v, x) = 0, \forall x \in mA_1\}$. Then we have

$$mA_1 \oplus L_0 < L < L^\vee < mA_1^\vee \oplus L_0^\vee.$$

For any $l \in L$, we can write $l = l_1 + l_2$ with $l_1 \in mA_1^\vee$ and $l_2 \in L_0^\vee$. Since $mA_1^\vee < L^\vee$, we have $(l, mA_1^\vee) \in \mathbb{Z}$. Thus $(l_1, mA_1^\vee) \in \mathbb{Z}$, which yields $l_1 \in mA_1$. Therefore, $l_2 = l - l_1 \in L$ and then $l_2 \in L_0$ due to $(l_2, mA_1) = 0$. We thus prove $L = mA_1 \oplus L_0$. □

As the classification of even unimodular lattices, we define the following classes of 2-reflective lattices.

Definition 2.6.2. A 2-reflective lattice $M = 2U \oplus L(-1)$ is called Leech type if $R(L) = \varnothing$. The lattice M is called Niemeier type if it satisfies the condition in statement (b) and called quasi-Niemeier type if it satisfies the condition in statement (c).

In a similar way, we can prove the following necessary condition for a lattice to be reflective. This condition would be useful to classify reflective lattices containing two hyperbolic planes.

Proposition 2.6.3. *If the lattice $2U \oplus L(-1)$ is reflective, then either $\mathfrak{R}_L$ is empty, or $\mathfrak{R}_L$ generates the space $L \otimes \mathbb{R}$, where $\mathfrak{R}$ is the root system of L*

$$\mathfrak{R}_L = \{r \in L : r \text{ is primitive}, \sigma_r \in \mathrm{O}(L)\}.$$

In Theorem 2.4.5, we have shown that if $M = 2U \oplus L(-1)$ is a 2-reflective lattice of signature $(2, 19)$ then the weight of the corresponding 2-reflective modular form is $75\beta_0$. We next use the above theorem to prove the following refined classification.

Theorem 2.6.4. *If M is a 2-reflective lattice of signature $(2, 19)$, then it is isomorphic to the lattice $2U \oplus 2E_8(-1) \oplus A_1(-1)$.*

Proof. We first prove the assertion under the assumption that M contains $2U$ i.e. $M = 2U \oplus L(-1)$. It is clear that $R(L)$ is non-empty because the weight is $75\beta_0$. In addition, we have $R(L) \neq 17A_1$, otherwise the weight of the associated 2-reflective modular form is

$$k \leq \beta_0\left(12 + 2\cdot 17\left(\frac{12}{17} - \frac{1}{2}\right)\right) - 2\cdot 17\beta_0\left(\frac{3}{17} - \frac{1}{2}\right) = 30\beta_0.$$

If M is of quasi-Niemeier type, then we have

$$k = \beta_0\left[\left(12 - \frac{17+3m}{2}\right)h + 12 + 3m\right] = 75\beta_0,$$

which follows that $h(7-3m)/2 + 3m = 63$. Since $1 \leq m \leq 15$, the only solution is $m = 1$ and $h = 30$. By Table 1.1, $R(L) = 2E_8 \oplus A_1$ or $D_{16} \oplus A_1$. But the lattices D_{16} and $E_8 \oplus D_8$ are in the same genus. Thus, $2U \oplus D_{16} \oplus A_1 \cong 2U \oplus E_8 \oplus D_8 \oplus A_1$. If $L = D_{16} \oplus A_1$, then the lattice $2U \oplus E_8 \oplus D_8 \oplus A_1$ is 2-reflective, which gives a contradiction by Theorem 2.6.1 (c). The only non-trivial even overlattice of D_{16} is the unimodular lattice D_{16}^+. Since $2U \oplus D_{16}^+ \oplus A_1 \cong 2U \oplus 2E_8 \oplus A_1$, we prove the theorem in this case.

If M is of Niemeier type, then we have

$$k = \beta_0\,(12 + 12h - 17h/2) = 75\beta_0,$$

which implies $h = 18$. By Table 1.1, we have $R(L) = A_{17}$ or $R(L) = D_{10} \oplus E_7$. If $L = D_{10} \oplus E_7$, then the lattice $2U \oplus E_8 \oplus E_7 \oplus 2A_1$ is 2-reflective, as the lattices D_{10} and $E_8 \oplus 2A_1$ are in the same genus. This leads to a contradiction by the previous case. If $L = A_{17}$, the 2-reflective vector v with $\mathrm{div}(v) = 2$ is represented as $(0, 2, [9], 1, 0)$ which appears in the q^2-term of the corresponding Jacobi form $\phi_{0,A_{17}}$ of weight 0. In addition, the q^1-term of $\phi_{0,A_{17}}$ has no singular Fourier coefficient of hyperbolic norm $-1/2$. On the other hand, the Niemeier lattice $N(A_{17} \oplus E_7)$ is generated by the isotropic subgroup

$$G = \{[0] \oplus [0], [3] \oplus [1], [6] \oplus [0], [9] \oplus [1], [12] \oplus [0], [15] \oplus [1]\}$$

over $A_{17} \oplus E_7$. Thus, the pull-back of $\varphi_{0,N(A_{17}\oplus E_7)}$ on A_{17} will give a weakly holomorphic Jacobi form $\psi_{0,A_{17}}$ of weight 0 which has the same q^{-1} and q^0-terms as $\phi_{0,A_{17}}$ and its singular Fourier coefficients in q^1-term are represented by [3] and [6]. The reason why these two Jacobi forms have the same q^{-1} and q^0-terms is that they have the same type of 2-reflective divisors and the corresponding coefficients are determined by the formulas in Lemma 1.4.12. We will often use this argument later. Thus, $\phi := (\phi_{0,A_{17}} - \psi_{0,A_{17}})/\Delta$ is a weak Jacobi form of weight -12 and index 1 for A_{17}. We can assume that it is invariant under the orthogonal group $\mathrm{O}(A_{17})$ by considering its symmetriction. The q^0-term of ϕ contains five $\mathrm{O}(A_{17})$-orbits i.e. [0], [1], [2], [3], [6]. By [Wir92], the space of weak Jacobi forms of index 1 for A_{17} invariant under $\mathrm{O}(A_{17})$ is a free module generated by ten Jacobi forms of weights 0, -2, -4, ..., -18 over the ring of $\mathrm{SL}_2(\mathbb{Z})$ modular forms. The ten generators were constructed in [Ber99]. Note that there are ten independent $\mathrm{O}(A_{17})$-orbits appearing in q^0-terms of these generators, i.e. $[i]$ for $0 \le i \le 9$. There are only three independent weak Jacobi forms of weight -12. But the q^0-term of ϕ has only five orbits. By direct calculations, we can show that the q^0-term of a weak Jacobi form of weight -12 contains at least eight orbits, which leads to a contradiction.

We complete the proof of the particular case by the fact that if L_1 is a non-trivial even overlattice of $R(L) = A_{17}$ or $D_{10} \oplus E_7$ then $2U \oplus L_1$ is of determinant 2 and isomorphic to $2U \oplus 2E_8 \oplus A_1$.

We now consider the remaining case that M does not contain $2U$. By Lemma 2.4.7, there exists an even overlattice M' of M containing $2U$. By Lemma 2.1.4, M' is also 2-reflective and then it is isomorphic to $2U \oplus 2E_8(-1) \oplus A_1(-1)$. We claim that the order of the group M'/M is not a prime, otherwise the discriminant group of M will be $2p^2$ and M will contain $2U$ by Lemma 1.1.3. Thus, there exists an even lattice M_1 such that $M \subset M_1 \subset M'$ and M'/M_1 is a nontrivial cyclic group. Then M_1 contains $2U$ by Lemma 1.1.3. It follows that M_1 is 2-reflective but not isomorphic to $2U \oplus 2E_8(-1) \oplus A_1(-1)$, which contradicts the previous case. This completes the proof. □

2.6.2 Classification of 2-reflective lattices of Niemeier type

In this subsection we classify 2-reflective lattices of Niemeier type. We first consider the case of $L = R(L)$ and then consider their overlattices. We discuss case by case. Let $M = 2U \oplus L(-1)$. By Theorem 2.4.5 and Theorem 2.6.4, if M is 2-reflective, then either M is isomorphic to the even unimodular lattices of signature $(2, 26)$ and $(2, 18)$ or $2U \oplus 2E_8(-1) \oplus A_1(-1)$, or we have $\mathrm{rank}(L) \le 11$. Therefore, we only need to consider the case of $\mathrm{rank}(L) \le 11$.

(1) h=3: The unique irreducible root lattice of Coxeter number 3 is A_2. By §2.3.1, the lattice M is 2-reflective if $L = A_2$, $2A_2$, $3A_2$. The lattice M is not 2-reflective for $L = mA_2$ with $m \ge 4$. Otherwise, since $4A_2 < E_6 \oplus A_2$, the lattice $2U \oplus E_6 \oplus A_2$ is also 2-reflective, which is impossible because E_6 and A_2 have different Coxeter numbers. Then we prove this claim by Lemma 2.3.3.

(2) h=4: The unique irreducible root lattice of Coxeter number 4 is A_3. By §2.3.1, the lattice M is 2-reflective if $L = A_3$, $2A_3$. The lattice M is not 2-reflective for $L = mA_3$ with $m \ge 3$ because we observe from their extended Coxeter-Dynkin diagrams that $3A_3 < D_6 \oplus A_3$.

(3) h=5: The unique irreducible root lattice of Coxeter number 5 is A_4. By §2.3.1, the lattice M is 2-reflective if $L = A_4$. We claim that the lattice M is not 2-reflective for $L = mA_4$ with $m \ge 2$. Otherwise, the lattice $2U(5) \oplus 2A_4^\vee(5) \cong U \oplus U(5) \oplus E_8(5)$ will has a reflective modular form with 10-reflective divisors because $(2U \oplus 2A_4)^\vee(5) = 2U(5) \oplus 2A_4^\vee(5)$ and the 2-reflective modular form for $2U \oplus 2A_4$ can be viewed as a 10-reflective modular form for $2U(5) \oplus 2A_4^\vee(5)$. By Theorem 1.3.3, the corresponding 10-reflective modular form should be a Borcherds product. This contradicts Lemma 2.2.5 because $10 > 2 + 24/(5+1)$.

(4) h=6: The irreducible root lattices of Coxeter number 6 are A_5 and D_4. By §2.3.1, the lattice M is 2-reflective if $L = A_5$, D_4, $2D_4$. We claim that the lattice M is not 2-reflective for $L = 2A_5$ and $L = A_5 \oplus D_4$.

Firstly, there is no 2-reflective vector of norm $1/2$ (this type of vectors should have order 2) in the discriminant group of $2A_5$. If there is a 2-reflective modular form for $2U \oplus 2A_5$ then it is a modular form with complete 2-divisor. But we have proved in Theorem 2.4.3 that if M has a modular form with complete 2-divisor then $\operatorname{rank}(L) \le 8$. Hence we get a contradiction.

Finally, we have known in §2.3.1 that the quasi pull-back of $A_5 \oplus D_4 < N(4A_5 \oplus D_4)$ gives a reflective modular form. As in the last part of the proof of Theorem 2.6.4, we can check that this modular form has additional reflective divisor $\mathcal{H}([2] \oplus [2], -1/6)$ given by the pull-backs of the vectors of norm 4 of type $[5] \oplus [2] \oplus [1] \oplus [0] \oplus [2]$. We denote the corresponding Jacobi form of weight 0 by ϕ_1. Suppose that the lattice $2U \oplus A_5 \oplus D_4$ is 2-reflective and we denote the corresponding Jacobi form of weight 0 by ϕ_2. Then ϕ_1 and ϕ_2 have the same q^{-1} and q^0 terms. Moreover, $\phi := \phi_1 - \phi_2$ is a weakly holomorphic Jacobi form of weight 0 whose singular Fourier coefficients appear in its q^1-term are represented by $[3] \oplus [2]$ and $[2] \oplus [2]$. Thus, the minimum hyperbolic norm of singular Fourier coefficients of ϕ is $-1/2$. Thus $\eta^6\phi$ is a holomorphic Jacobi form of weight 3 with a character. In view of the singular weight, this leads to a contradiction.

(5) h=7: The unique irreducible root lattice of Coxeter number 7 is A_6. The lattice M is 2-reflective if $L = A_6$.

(6) h=8: The irreducible root lattices of Coxeter number 8 are A_7 and D_5. By §2.3.1, the lattice M is 2-reflective if $L = A_7$, D_5. We claim that the lattice M is not 2-reflective for $L = 2D_5$. This claim can be proved as in the case of $2A_5$.

(7) h=9: The unique irreducible root lattice of Coxeter number 9 is A_8. The lattice M is not 2-reflective if $L = A_8$. We can prove this claim in a similar way as the case of $A_5 \oplus D_4$.

(8) h=10: The irreducible root lattices of Coxeter number 10 are A_9 and D_6. By §2.3.1, the lattice M is 2-reflective if $L = D_6$. We can prove that the lattice M is not 2-reflective for $L = A_9$ in a similar way as the case of A_8 and $A_5 \oplus D_4$.

(9) h=11: The unique irreducible root lattice of Coxeter number 11 is A_{10} (level 11). The lattice M is not 2-reflective if $L = A_{10}$. Since A_{10} is of prime level 11, if $2U \oplus A_{10}(-1)$ is 2-reflective then the corresponding 2-reflective modular form is a modular form with complete 2-divisor. This leads to a contradiction.

(10) h=12: The irreducible root lattices of Coxeter number 12 are A_{11}, E_6 and D_7. The lattice M is 2-reflective if $L = D_7$, E_6. The lattice M is not 2-reflective for $L = A_{11}$, which can be proved as the case of $2A_5$.

(11) h is larger than 12: In view of $\operatorname{rank}(L) \le 11$, the rest cases are $L = D_m$ with $8 \le m \le 11$, E_7, E_8. The lattice M is 2-reflective if $L = D_8$, D_{10}, E_7, E_8. The lattice M is not 2-reflective for $L = D_9$, D_{11}, which can be proved as the case of $2A_5$.

(12) The case of overlattices: Let L_1 be a non-trivial even overlattice of $R(L)$ whose root sublattice generated by 2-roots is $R(L)$. In this case, the minimum norm of vectors in nontrivial class of $L_1/R(L)$ is an even integer larger than 2. It is easy to show that there is no such $R(L)$.

By the discussions above, we have thus proved the following theorem.

Theorem 2.6.5. *Let $M = 2U \oplus L(-1)$ be a 2-reflective lattice of Niemeier type. Then L can only take one of the following 21 lattices up to genus*

$3E_8$	$2E_8$	E_8	E_7	E_6	A_2	$2A_2$	$3A_2$	A_3	$2A_3$	A_4
A_5	A_6	A_7	D_4	$2D_4$	D_5	D_6	D_7	D_8	D_{10}.	

2.6.3 Classification of 2-reflective lattices of quasi-Niemeier type

In this subsection we classify 2-reflective lattices of quasi-Niemeier type. We use the notations in Theorem 2.6.1. Let $R(L) = mA_1 \oplus R$ and $L = mA_1 \oplus L_0$. We assume $\operatorname{rank}(L) \le 11$. Let

$M = 2U \oplus L(-1)$. By Lemma 2.3.3, if M is 2-reflective then $M = 2U \oplus L_0(-1) \oplus (m-1)A_1(-1)$ is also 2-reflective. Therefore, we only need to consider the root lattices formulated in Theorem 2.6.5 for L_0. We prove the following theorem.

Theorem 2.6.6. *Let $M = 2U \oplus L(-1)$ be a 2-reflective lattice of quasi-Niemeier type. Then L is in one of the genus of the following 21 lattices*

$$\begin{aligned} A_1 &\oplus \{E_8, E_7, E_6, A_2, 2A_2, A_3, A_4, A_5, D_4, 2D_4, D_5, D_6, D_8\} \\ 2A_1 &\oplus \{E_8, A_2, A_3, D_4, D_6\} \\ D_4 &\oplus \{3A_1, 4A_1, 5A_1\}. \end{aligned}$$

Note that $3A_1 \oplus D_6$ and $A_1 \oplus 2D_4$, $2A_1 \oplus E_7$ and $A_1 \oplus D_8$ have the isomorphic discriminant forms respectively.

Proof. By §2.3.1, when L takes one of the above lattices, the lattice M is 2-reflective. We next prove that M is not 2-reflective for other lattices.

(1) Since $4A_1 < D_4$ and $6A_1 < D_6$, we have $m \le 5$. In addition, when $m = 4$ or 5, $L_0 = D_4$ or A_5. But when $L_0 = A_5$, $4A_1 \oplus A_5 < A_5 \oplus D_4$, which is impossible because $2U \oplus A_5 \oplus D_4$ is not 2-reflective. Thus when $m \ge 4$, the lattice L_0 can only take D_4.

(2) The lattice L is not equal to $2D_4 \oplus mA_1$ for $m \ge 2$ because $2D_4 \oplus 2A_1 < D_4 \oplus D_6$.

(3) The lattice L is not equal to $E_8 \oplus 3A_1$. If $2U \oplus E_8 \oplus 3A_1$ is 2-reflective, then we have by Theorem 2.6.1 that $\beta_1 = 57$ and $k = 81\beta_0$. By Theorem 1.4.19, the q^0-term of the corresponding Jacobi form of weight 0 will define a holomorphic Jacobi form for $E_8 \oplus 3A_1$ as a theta block (see (1.4.19)). Thus the function corresponding to any copy of A_1

$$\eta^{(162-8)/3}\left(\frac{\vartheta(\tau,2z)}{\eta(\tau)}\right)\left(\frac{\vartheta(\tau,z)}{\eta(\tau)}\right)^{56}$$

is a holomorphic Jacobi form of index 30 for A_1. We calculate its hyperbolic norm of the first Fourier coefficient

$$4 \times \frac{1}{24}\left(\frac{154}{3} - 57 + 57 \times 3\right) \times 30 - 29^2 = -14.333... < 0,$$

which contradicts the definition of holomorphic Jacobi forms.

Since D_{10} and $E_8 \oplus 2A_1$ are in the same genus, the lattice M is not 2-reflective if $L = D_{10} \oplus mA_1$ with $m \ge 1$.

(4) $L \ne D_8 \oplus mA_1$ for $m \ge 2$. It is because that $D_8 \oplus 2A_1$ and $D_4 \oplus D_6$ are in the same genus. Furthermore, $L \ne E_7 \oplus 3A_1$ because $U \oplus E_7 \oplus 3A_1 \cong U \oplus D_8 \oplus 2A_1$.

(5) $L \ne A_1 \oplus 3A_2$. Otherwise, there exists a 2-reflective modular form for $2U \oplus A_1 \oplus 3A_2$ and we note the corresponding Jacobi form of weight 0 by ϕ. On the other hand, the pull-back of $\varphi_{0,N(R)}$ on $A_1 \oplus 3A_2 < N(12A_2)$ will also give a Jacobi form of weight 0 which is noted by ϕ_1. Using the idea in this section, we conclude that ϕ and ϕ_1 have the same q^0-term and the difference $\psi := \phi - \phi_1$ will give a Jacobi form of weight 0 without q^{-1} and q^0-terms for $A_1 \oplus 3A_2$. This function is not zero and its singular Fourier coefficients are represented by $[1]^3 \oplus (\frac{1}{2})$ which has hyperbolic norm $-1/2$ and does not correspond to 2-reflective divisor. Thus $\eta^6\psi$ is a holomorphic Jacobi form of weight 3 with a character for $A_1 \oplus 3A_2$, which contradicts the singular weight.

$L \ne 3A_1 \oplus A_3$. Suppose that the lattice $2U \oplus 3A_1 \oplus A_3$ is 2-reflective and we denote the corresponding Jacobi form of weight 0 by ϕ. The pull-back of $\varphi_{0,N(R)}$ on $3A_1 \oplus A_3 < N(8A_3)$ gives a Jacobi form of weight 0 (noted by ϕ_1). The functions ϕ and ϕ_1 have the same q^0-term and their difference $f := \phi - \phi_1 = O(q)$ is a Jacobi form of weight 0 for $3A_1 \oplus A_3$. This function is not zero and its singular Fourier coefficients are represented by $v_1 := (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \oplus [1]$ (with hyperbolic norm $-1/4$) and $v_2 := (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \oplus [2]$ (with hyperbolic norm $-1/2$). Hence $\eta^6 f$ is a holomorphic Jacobi form of singular weight 3 with a character for $3A_1 \oplus A_3$. This contradicts the singular weight because there is a Fourier coefficient with non-zero hyperbolic norm i.e. $q^{1/4}\zeta^{(v_1,\mathfrak{z})}$ with hyperbolic norm $1/4$.

All other cases can be proved in a similar way. Since the pull-back of $\varphi_{0,N(R)}$ has additional singular Fourier coefficients in its q^1-term which does not correspond to 2-reflective divisor, we can construct a holomorphic Jacobi form of low weight with a character, which will contradicts the singular weight. The proof is completed. $\square$

2.6.4 Classification of 2-reflective lattices of other type

In this subsection, we discuss the final case i.e. $R(L) = nA_1$. Firstly, if $\beta_0 = 0$, the only possible case is $L = nA_1$. In this case the weight k is equal to $(6-n)\beta_1$. In view of the singular weight, we have $k \geq n\beta_1/2$ since $\eta^{2k/n}(\vartheta(\tau,z)/\eta)^{\beta_1}$ is a holomorphic Jacobi form. Therefore we get $1 \leq n \leq 4$. The corresponding 2-reflective modular forms can be constructed as the quasi pull-backs of the 2-reflective modular form of singular weight 2 for $2U \oplus 4A_1$ (see the case 4 in §2.3.2). In view of Theorem 2.6.1, we thus prove the following.

Theorem 2.6.7. *If $M = 2U \oplus L(-1)$ has a 2-reflective modular form with $\beta_0 = 0$ in its zero divisor, then $L = nA_1$ with $1 \leq n \leq 4$.*

By §2.3.1, when $L = nA_1$ with $1 \leq n \leq 8$, the lattice M is 2-reflective. For the overlattices, the lattices $2U \oplus N_8$ and $2U \oplus N_8 \oplus A_1$ are 2-reflective. The lattice $2U \oplus N_8 \oplus 2A_1$ is not 2-reflective because

$$2A_1 \oplus N_8 < 2A_1 \oplus 2D_4 < D_4 \oplus D_6.$$

To complete the classification, we show that the lattice $2U \oplus 9A_1$ is not 2-reflective. Conversely, suppose that there exists a 2-reflective modular form for $2U \oplus 9A_1$. From §2.3.1, the lattice $2U \oplus 8A_1$ is 2-reflective and the 2-reflective modular form is constructed as a quasi pull-back on $8A_1 < N(8A_3)$. For this 2-reflective modular form, we have $\beta_1 = 5$. We claim that this function is the unique 2-reflective modular form for $2U \oplus 8A_1$ up to constant. Otherwise, by considering the difference of the two 2-reflective modular forms, we will get a weak Jacobi form of weight 0 for $8A_1$ whose minimal hyperbolic norm of singular Fourier coefficients is $-1/2$. Thus, its product with η^6 will give a holomorphic Jacobi form of weight 3 for $8A_1$, which is impossible due to the singular weight.

The quasi pull-back of the 2-reflective modular form for $2U \oplus 9A_1$ will be the 2-reflective modular form for $2U \oplus 8A_1$. Therefore, in the case of $9A_1$ we have $\beta_1 = 5\beta_0$. Thus the weight is given by

$$k = \beta_0\left(12 + 18\left(\frac{12}{9} - \frac{1}{2}\right)\right) + \left(\frac{3}{9} - \frac{1}{2}\right) \times 18 \times 4\beta_0 = 15\beta_0.$$

The q^0-term of the corresponding Jacobi form of weight 0 defines a holomorphic Jacobi form for $9A_1$. Then the part related to each copy of A_1

$$\eta^{30/9}(\tau)\left(\frac{\vartheta(\tau,2z)}{\eta(\tau)}\right)\left(\frac{\vartheta(\tau,z)}{\eta(\tau)}\right)^4$$

is a holomorphic Jacobi form of index 4 for A_1. But its hyperbolic norm of the first Fourier coefficient is

$$4 \times \frac{1}{24}\left(\frac{30}{9} + 2 \times 5\right) \times 4 - 3^2 = -\frac{1}{9} < 0,$$

which gives a contradiction.

2.6.5 Final classification

Combining Theorem 2.4.8, Theorem 2.6.1, Theorem 2.6.4, Theorem 2.6.5, Theorem 2.6.6 and §2.6.4 together, we prove the following two classification results.

Theorem 2.6.8. *Let M be a 2-reflective lattice of signature $(2,n)$ with $n \geq 14$. Then it is isomorphic to $\mathrm{II}_{2,18}$, or $2U \oplus 2E_8(-1) \oplus A_1(-1)$, or $\mathrm{II}_{2,26}$.*

Theorem 2.6.9. *There are only three types of 2-reflective lattices containing two hyperbolic planes:*

(a) $\mathrm{II}_{2,26}$;

(b) $2U \oplus L(-1)$: *every lattice in the genus of L has no 2-root. In this case, the corresponding 2-reflective modular form has a Weyl vector of norm zero and has weight $12\beta_0$, where β_0 is the multiplicity of the principal Heegner divisor $\mathcal{H}_0$;*

(c) $2U \oplus L(-1)$: *every lattice in the genus of L has 2-roots and the 2-roots generate a sublattice of the same rank. In this case, L is in the genus of one of the following 50 lattices*

n	L								
3	A_1								
4	$2A_1$,	A_2							
5	$3A_1$,	$A_1 \oplus A_2$,	A_3						
6	$4A_1$,	$2A_1 \oplus A_2$,	$A_1 \oplus A_3$,	A_4,	D_4,	$2A_2$			
7	$5A_1$,	$2A_1 \oplus A_3$,	$A_1 \oplus 2A_2$,	$A_1 \oplus A_4$,	$A_1 \oplus D_4$,	A_5,	D_5		
8	$6A_1$,	$2A_1 \oplus D_4$,	$A_1 \oplus A_5$,	$A_1 \oplus D_5$,	E_6,	$3A_2$,	$2A_3$,	A_6,	D_6
9	$7A_1$,	$3A_1 \oplus D_4$,	$A_1 \oplus D_6$,	$A_1 \oplus E_6$,	E_7,	A_7,	D_7		
10	$8A_1$,	$4A_1 \oplus D_4$,	$2A_1 \oplus D_6$,	$A_1 \oplus E_7$,	E_8,	$2D_4$,	D_8,	N_8	
11	$5A_1 \oplus D_4$,	$A_1 \oplus 2D_4$,	$A_1 \oplus D_8$,	$A_1 \oplus E_8$					
12	$2A_1 \oplus E_8$								
18	$2E_8$								
19	$2E_8 \oplus A_1$								

Note that $5A_1 \oplus D_4$ and $A_1 \oplus N_8$, $A_1 \oplus 2D_4$ and $3A_1 \oplus D_6$, $A_1 \oplus D_8$ and $2A_1 \oplus E_7$, $2A_1 \oplus E_8$ and D_{10} are in the same genus, respectively. Here, $N_8 \cong D_8^\vee(2)$ is the Nikulin lattice. Moreover, every lattice has a 2-reflective modular form with a positive norm Weyl vector. Thus, every associated Lorentzian lattice $U \oplus L(-1)$ is hyperbolic 2-reflective.

The following theorem gives a classification of modular forms with complete 2-divisor.

Theorem 2.6.10. *Assume that $M = 2U \oplus L(-1)$ has a modular form with complete 2-divisor and the set of 2-roots of L is non-empty. Then L is in the genus of $3E_8$ or one of the lattices formulated in Table 2.2.*

Proof. Firstly, from the formula $\beta_1 = (2h-3)\beta_0$ in Theorem 2.6.1, we see that there is no modular form with complete 2-divisor for 2-reflective lattice of quasi-Niemeier type. For the lattice of type mA_1, there exists a 2-reflective modular form for $2U \oplus 5A_1$ whose 2-reflective divisor of type $(0,1,(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}),1,0)$ has multiplicity 9. This means that the lattices $2U \oplus mA_1$ with $m \geq 5$ do not have a modular form with complete 2-divisor. We next construct the 2-reflective modular form for $2U \oplus 5A_1$. Let $8A_1 = \oplus_{i=1}^8 \mathbb{Z}\alpha_i$ with $\alpha_i^2 = 2$. The Nikulin's lattice is $N_8 = \langle 8A_1, h\rangle$, where $h = \frac{1}{2}\sum_{i=1}^8 \alpha_i$. It is known that there is a modular form with complete 2-divisor for $2U \oplus N_8$ (see §2.3.1). Thus there is a weakly holomorphic Jacobi form ϕ_{0,N_8} of weight 0 for N_8 with the singular Fourier coefficients

$$\mathrm{Sing}(\phi_{0,N_8}) = q^{-1} + 56 + \sum_{n\in\mathbb{N}} \sum_{\substack{r\in N_8\\ r^2=2n+2}} q^n e^{2\pi i(\mathfrak{z},r)}$$

We consider the pull-back of $5A_1 < N_8$

$$\phi_{0,5A_1}(\tau,\mathfrak{z}_5) = q^{-1} + 62 + \sum_{i=1}^{5} \zeta_i^{\pm 2} + O(q),$$

where $\mathfrak{z}_5 = \sum_{i=1}^5 z_i\alpha_i$ and $\zeta_i = e^{2\pi i z_i}$. We need to determine the singular Fourier coefficients in q^1-term. This type of Fourier coefficients is of the form $\frac{1}{2}\sum_{i=1}^5 \alpha_i$ and it comes from the pull-back of vectors of norm 4 in N_8 of type $\frac{1}{2}\sum_{i=1}^5 \alpha_i \pm \frac{1}{2}\alpha_6 \pm \frac{1}{2}\alpha_7 \pm \frac{1}{2}\alpha_8$. Thus the coefficient of $q\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5$ is 8. The Borcherds product of $\phi_{0,5A_1}$ gives the claimed 2-reflective modular form. □

Note that there are in fact two different 2-reflective modular forms for $2U \oplus 5A_1$. The second one can be constructed as the quasi pull-back of Φ_{12} on $5A_1 < N(8A_3)$ (see §2.3.1)

Remark that there are lattices not of type $2U \oplus L$ which have a modular form with complete 2-divisor, such as the lattices $U(2) \oplus \langle -2\rangle \oplus (k+1)\langle 2\rangle$ with $1 \le k \le 7$ (see [GN18, Theorem 6.1]).

Proposition 2.6.11. *If M is a maximal lattice of signature $(2,10)$ having a modular form with complete 2-divisor, then it is isomorphic to $\mathrm{II}_{2,10}$.*

Proof. It is a refinement of the proof of Theorem 2.4.3. By [Ma17, Proposition 3.1], the lattice M can be written as $M = 2U \oplus L(-1)$. Thus, there exists a weakly holomorphic Jacobi form of weight 0 and index L. As the proof of Theorem 2.4.3, we can construct a holomorphic Jacobi form of weight 4 and index L, denoted by g. It is easy to check that the constant term of g is not zero and we assume it to be 1. The function g has singular weight 4. Thus, it is a $\mathbb{C}$-linear combination of theta-functions for L defined as (1.4.10). Since L is maximal, there is no $\gamma \in L^\vee$ such that $\gamma \notin L$ and $(\gamma,\gamma) = 2$. Hence, the q^1-term of Fourier expansion of g comes only from the theta-function Θ_0^L. In view of $g(\tau,0) = E_4(\tau) = 1 + 240q + ...$, the number of 2-roots in L is 240. By Theorem 2.6.1, the Coxeter number of L is 30, which forces that L is isomorphic to E_8. The proof is completed. □

The following theorem gives a classification of 2-reflective modular forms of singular weight.

Theorem 2.6.12. *Assume that $M = 2U \oplus L(-1)$ has a 2-reflective modular form of singular weight. Then L is in the genus of $3E_8$ or $4A_1$.*

Proof. This result is a direct consequence of Theorem 2.6.9 and our weight formula. □

We close this section with two remarks.

Remark 2.6.13. Theorem 2.6.8 holds for meromorphic 2-reflective modular forms. Firstly, from its proof, we see that Theorem 2.6.1 is still true for meromorphic 2-reflective modular forms. Secondly, in the proof of Theorem 2.4.5, we only need to make minor correction for the case of $\mathrm{rank}(L) = 12, 13, 14$. In these cases, we need to show that the constant u of holomorphic Jacobi form ϕ_6 is not zero. This can be done using Theorem 2.6.1.

Remark 2.6.14. For any 2-reflective lattice of type $2U \oplus L(-1)$ with $\mathrm{rank}(L) > 6$, the corresponding 2-reflective modular form is unique up to a constant. Indeed, if there are two independent 2-reflective modular forms, then there will exist a weakly holomorphic Jacobi form ψ of weight 0 and index L without q^{-1}-term. Then $\eta^6\psi$ will be a holomorphic Jacobi form of weight 3 with a character, which contradicts the singular weight.

Similarly, for any reflective lattice of type $2U \oplus L(-1)$ with $\mathrm{rank}(L) > 12$, the corresponding reflective modular form is unique up to a constant.

Remark 2.6.15. Similar to Theorem 2.6.9, replacing 2-roots with root system $\mathcal{R}_L$ (see Proposition 2.6.3), we find that there are also exactly three types of reflective lattices containing $2U$. But $\mathrm{II}_{2,26}$ is not the unique reflective lattice of type (a). In fact, the lattice

$$U \oplus \text{Coxeter-Todd lattice} \cong U \oplus 6A_2 \cong U \oplus E_6 \oplus E_6^\vee(3)$$

is also a reflective lattice of type (a). Besides, the reflective lattice of type (c) may have no a reflective modular form with positive norm Weyl vector, for example, the lattice $2U \oplus A_6^\vee(7)$ has a unique reflective modular form and this modular form has singular weight 3 (see [GW19a]). Thus, the case of reflective is different from the case of 2-reflective. We remark that every lattice having a reflective modular form of singular weight is of type (c).

2.6.6 Automorphic correction of 2-reflective hyperbolic lattices

An even lattice S of signature $(1,n)$ is called **hyperbolic 2-reflective** if the subgroup generated by 2-reflections is of finite index in the orthogonal group of S, i.e. $W^{(2)} = \langle \sigma_r : r \in S, r^2 = -2 \rangle < \mathrm{O}(S)$ is of finite index. The lattice S is called **hyperbolic reflective** if the subgroup generated by all reflections is of finite index in $\mathrm{O}(S)$. Hyperbolic reflective lattices are closely related to reflective modular forms. In [Bor98, Theorem 12.1], Borcherds proved that if the lattice $U \oplus S$ has a reflective (resp. 2-reflective) modular form with a Weyl vector of positive norm then S is hyperbolic reflective (resp. 2-reflective).

Hyperbolic 2-reflective lattices are of a special interest because of its close connection with the theory of K3-surfaces. The classification of such lattices is now available thanks to the work of Nikulin and Vinberg (see [Nik81] for $n \geq 4$, [Nik84] for $n = 2$, [Vin07] for $n = 3$, and a survey [Bel16]). Table 2.3 gives the number of hyperbolic 2-reflective lattices. The models of all these lattices can be found in [GN18, §3.2]. For $\mathrm{rank}(S) = 10$, we need to add the lattice $U \oplus D_4 \oplus 4A_1$ to the table in [GN18, §3.2].

Table 2.3: Hyperbolic 2-reflective lattices

$n+1$	3	4	5	6	7	8	9	10	11	12	13	14	$15, \ldots, 19$	≥ 20
Number of lattices	26	14	9	10	9	12	10	9	4	4	3	3	1	0

In [Bor00], Borcherds suggested that interesting hyperbolic reflective lattices should be associated to reflective modular forms. In view of this suggestion, Gritsenko and Nikulin considered the following automorphic correction of hyperbolic 2-reflective lattices in [GN18].

Definition 2.6.16. Let S be a hyperbolic 2-reflective lattice. If there exists a positive integer m such that $U(m) \oplus S$ has a 2-reflective modular form, then we say that S has an automorphic correction.

By means of our classification results in this section, we prove the following.

Theorem 2.6.17. *Let S be a hyperbolic 2-reflective lattice of signature $(1,n)$ with $n \geq 5$. If S is one of the following 18 lattices*

$U \oplus E_8 \oplus E_7$	$U \oplus E_8 \oplus D_6$	$U \oplus E_8 \oplus D_4 \oplus A_1$	$U \oplus E_8 \oplus D_4$	$U \oplus D_8 \oplus D_4$
$U \oplus E_8 \oplus 4A_1$	$U \oplus E_8 \oplus 3A_1$	$U \oplus D_8 \oplus 3A_1$	$U \oplus E_8 \oplus A_3$	$U \oplus D_8 \oplus 2A_1$
$U \oplus 2D_4 \oplus 2A_1$	$U \oplus E_8 \oplus A_2$	$U \oplus E_6 \oplus A_2$	$U \oplus D_4 \oplus A_3$	$U \oplus D_5 \oplus A_2$
$U \oplus D_4 \oplus A_2$	$U \oplus A_4 \oplus A_2$	$U \oplus A_3 \oplus A_2$		

then it has no automorphic correction. If S is one of the other 51 *lattices, it has at least one automorphic correction.*

Proof. If $U(m) \oplus S$ is 2-reflective, then $U \oplus S$ is also 2-reflective. We then prove the result by Theorem 2.6.1 (b) and (c). The automorphic corrections of S can be found in §2.3.1 and [GN18]. □

For $2 \leq n \leq 4$, there are a lot of hyperbolic 2-reflective lattices not of type $U \oplus L(-1)$. Our argument does not work well in this case.

Remark 2.6.18. It is possible to use the classification of hyperbolic 2-reflective lattices to prove Theorem 2.6.9. The Weyl vector of a Borcherds product is given by $(A, \vec{B}, C)$ in Theorem 1.4.19. For a 2-reflective modular form for the lattice of type $2U \oplus mA_1 \oplus L_0$ with $m \geq 0$ and $L_0 \neq \varnothing$ (see notations in Theorem 2.6.1), we have

$$(A, \vec{B}, C) = \left(h + 1, \sum \rho_i + \frac{h-1}{2} \sum \alpha_j, h\right),$$

where ρ_i is the Weyl vector of the irreducible components of the root sublattice of L_0 and α_j are the positive roots of mA_1. We thus calculate the norm of the Weyl vector as

$$2AC - (\vec{B}, \vec{B}) = \frac{h(h+1)}{12}\left(24 - n - 5m + \frac{6m(3h-1)}{h(h+1)}\right).$$

If the Weyl vector has positive norm, then the lattice $U \oplus mA_1 \oplus L_0$ is hyperbolic 2-reflective. Remark that all lattices in Theorem 2.6.9 except $\mathrm{II}_{2,26}$ have a 2-reflective modular form with a Weyl vector of positive norm. Thus all the corresponding Lorentzian lattices are hyperbolic 2-reflective.

Furthermore, we can show that for almost all lattices determined by Theorem 2.6.1 the norm of Weyl vectors are positive. For example, when $m = 0$ and $n < 24$, we have $2AC - (\vec{B}, \vec{B}) = \frac{h(h+1)}{12}(24 - n) > 0$. Thus they are all hyperbolic 2-reflective and we may use the classification of hyperbolic 2-reflective lattices to classify 2-reflective lattices.

Remark 2.6.19. Let L be a primitive sublattice of a Niemeier lattice $N(R)$. If the orthogonal complement of L on $N(R)$ has 2-roots, then every reflective modular form for $2U \oplus L(-1)$ constructed as the quasi pull-back of Φ_{12} is a cusp form (see Theorem 2.3.1) and then has a Weyl vector of positive norm, which yields that the corresponding Lorentzian lattice is hyperbolic reflective.

Note that the sublattices $6A_2 < N(6D_4)$ and $12A_1 < N(12A_2)$ do not satisfy the above assumption. By direct calculations, the Weyl vectors of the corresponding reflective modular forms have zero norm.

It is now easy to see that the lattice $U \oplus L(-1)$ are hyperbolic reflective for some L in §2.3.1, such as $L = 2E_8 \oplus D_4$, $2E_8 \oplus 2A_1$, $2E_8 \oplus A_1(2)$, $E_8 \oplus D_9$, $E_8 \oplus 2D_4$, $E_8 \oplus D_7$, $2E_7$, $E_8 \oplus D_4 \oplus A_1(2)$, $11A_1$, $5A_2$, $A_5 \oplus D_4$, $D_5 \oplus A_2$, and so on.

2.7 Application: moduli space of K3 surfaces

As a first application, we consider the family of lattices

$$T_n = U \oplus U \oplus E_8(-1) \oplus E_8(-1) \oplus \langle -2n \rangle \tag{2.7.1}$$

where $n \in \mathbb{N}$. The modular variety $\widetilde{\mathrm{O}}^+(T_n)\backslash \mathcal{D}(T_n)$ is the moduli space of polarized K3 surfaces of degree $2n$. The subset

$$Discr = \bigcup_{\substack{l \in T_n \\ (l,l)=-2}} l^\perp \cap \mathcal{D}(T_n) \tag{2.7.2}$$

is the discriminant of this moduli space. Nikulin [Nik96] asked the question whether the discriminant is equal to the set of zeros of certain automorphic form. This question is equivalent to whether T_n is 2-reflective. Nikulin showed that for any N there exists $n > N$ such that T_n is not 2-reflective. Gritsenko and Nikulin [GN02] proved that the lattice T_n is not 2-reflective if $n > \left(\frac{32}{3} + \sqrt{128+8N}\right)^2$, where N is the integer such that any even integer larger than N can be represented as the sum of the squares of eight different positive integers. Finally, Looijenga [Loo03] proved that T_n is not 2-reflective if $n \geq 2$. Now this result is immediately by Theorem 2.6.1 because the set of 2-roots in $2E_8 \oplus \langle 2n \rangle$ does not span the whole space $\mathbb{R}^{17}$ when $n \geq 2$. Moreover, Theorem 2.6.4 gives a generalization of this result.

We next use the arguments developed in this chapter to prove the following theorem.

Theorem 2.7.1. *The lattice T_n is reflective if and only if $n = 1, 2$.*

Proof. We have known from §2.3.1 that T_1 and T_2 are reflective. We next suppose that $n \geq 3$ and T_n is reflective. Then there exists a weakly holomorphic Jacobi form of weight 0 and index 1 for $2E_8 \oplus \langle 2n \rangle$ with first Fourier coefficients of the form

$$\phi(\tau, \mathfrak{z}) = c_0 q^{-1} + c_0 \sum_{\substack{r \in 2E_8 \\ r^2=2}} e^{2\pi i (\mathfrak{z}, r)} + c_1 \zeta^{\pm \frac{1}{n}} + c_2 \zeta^{\pm \frac{1}{2n}} + 2k + O(q),$$

where c_0, $c_1 \in \mathbb{N}$, $c_2 \in \mathbb{Z}$ satisfying $c_1 + c_2 \geq 0$, $\zeta^{\pm\frac{1}{n}} = \exp(2\pi i(\mathfrak{z}, \frac{1}{n}\alpha))$, $\zeta^{\pm\frac{1}{2n}} = \exp(2\pi i(\mathfrak{z}, \frac{1}{2n}\alpha))$, α is the basis of the lattice $\langle 2n \rangle$ with $\alpha^2 = 2n$. The reason we have $c_1 + c_2 \geq 0$ is that it is the multiplicity of the Heegner divisor $\mathcal{H}(\frac{1}{2n}\alpha, -\frac{1}{4n})$. By Lemma 1.4.12, we get

$$\begin{aligned} 60nc_0 &= 4c_1 + c_2, \\ k + c_1 + c_2 &= 132c_0. \end{aligned}$$

We can assume $c_0 = 1$. The q^0-term of ϕ defines a holomorphic Jacobi form for $2E_8 \oplus \langle 2n \rangle$ as a theta block. In particular, the part related to $\langle 2n \rangle$

$$\eta^{2k-16}(\tau) \left(\frac{\vartheta(\tau, 2z)}{\eta(\tau)} \right)^{c_1} \left(\frac{\vartheta(\tau, z)}{\eta(\tau)} \right)^{c_2}$$

is a holomorphic Jacobi form of index $30n$ for A_1. Thus, the hyperbolic norm of its first Fourier coefficient should be non-negative. We calculate it as

$$\begin{aligned} &4 \times \frac{2k - 16 + 2c_1 + 2c_2}{24} \times 30n - \left(\frac{2c_1 + c_2}{2}\right)^2 \\ =&1240n - \left(\frac{4c_1 + c_2}{6} + \frac{c_1 + c_2}{3}\right)^2 \\ \leq& 1240n - 100n^2, \end{aligned}$$

which implies that $1240n \geq 100n^2$ i.e. $n \leq 12$. The last inequality follows from $c_1 + c_2 \geq 0$.

If ϕ has no singular Fourier coefficient of hyperbolic norm -1, then as in §2.4 and §2.5, by using the differential operators to kill the term q^{-1} (consider a linear combination of $E_4\phi$ and $H_2(H_0(\phi))$), we can construct a non-zero weak Jacobi form ϕ_4 of weight 4 whose hyperbolic norms of singular Fourier coefficients are > -1, more precisely $\geq -2/3$ (see the description of reflective vectors in §2.1). Then $\eta^8\phi_4$ is a holomorphic Jacobi form of weight 8 with a character for $2E_8 \oplus \langle 2n \rangle$, which contradicts the singular weight.

Thus ϕ must have singular Fourier coefficients of hyperbolic norm -1. When $n \leq 12$, the singular Fourier coefficients of ϕ are determined by q^{-1}, q^0, q^1 and q^2-terms. Since the singular Fourier coefficients of ϕ correspond to reflective divisors, the singular Fourier coefficients of hyperbolic norm -1 are represented by $\frac{1}{2}\alpha$ with $\frac{\alpha^2}{4} = 1 (\operatorname{mod} 2)$ because the order must be 2. The only possible case is $n = 6$ or 10.

In the case $n = 6$, the possible singular Fourier coefficients of ϕ are: q^{-1}, $\zeta^{\pm 1/6}$, $\zeta^{\pm 1/12}$ and $q\zeta^{\pm 1/2}$ with hyperbolic norms -2, $-1/3$, $-1/12$ and -1, respectively. Similarly, by using the differential operators to kill the terms q^{-1} and $q\zeta^{\pm 1/2}$ (consider a linear combination of $E_6\phi$, $E_4H_0(\phi)$ and $H_4(H_2(H_0(\phi)))$), we can construct a non-zero weak Jacobi form ϕ_6 of weight 6 with only singular Fourier coefficients of types $\zeta^{\pm 1/6}$ and $\zeta^{\pm 1/12}$. Then $\eta^4\phi_6$ gives a holomorphic Jacobi form of weight 8 for $2E_8 \oplus \langle 12 \rangle$, which contradicts the singular weight. Therefore T_6 is not reflective.

We can prove the case $n = 10$ in a similar way. The proof is completed. □

2.8 Application: classification of dd-modular forms

Our arguments in the previous sections are also applicable to some other questions. In this section we use similar arguments to classify the modular forms with the simplest reflective divisors, i.e. the dd-modular forms defined in [CG11].

Let nA_1 denote the lattice of n copies of $A_1 = \langle 2 \rangle$, $n \in \mathbb{N}$. Let $\{e_1, ..., e_n\}$ denote the standard basis of $\mathbb{R}^n$ with standard scalar product $(\cdot,\cdot)$. We choose the following model for the lattice $nA_1(m)$:

$$(\langle e_1, ..., e_n \rangle_{\mathbb{Z}},\ 2m(\cdot,\cdot)) \tag{2.8.1}$$

and set $\mathfrak{z}_n = \sum_{i=1}^n z_i e_i \in nA_1 \otimes \mathbb{C}$, $\zeta_i = e^{2\pi i z_i}$, for $1 \leq i \leq n$. We define

$$\Gamma_{n,m} = \mathrm{O}^+(2U \oplus nA_1(-m)) \tag{2.8.2}$$

and the definition of dd-modular forms is as follows.

Definition 2.8.1. A holomorphic modular form with respect to $\Gamma_{n,m}$ is called a dd-modular form if it vanishes exactly along the $\Gamma_{n,m}$-orbit of the diagonal $\{\mathfrak{z}_n = 0\}$. The $\Gamma_{n,m}$-orbit of the diagonal $\{\mathfrak{z}_n = 0\}$, denoted by $\Gamma_{n,m}\{\mathfrak{z}_n = 0\}$, is called the diagonal divisor.

It is well-known that the Igusa form Δ_5 which is the product of the ten even theta constants vanishes precisely along the diagonal divisor $\{z = 0\}$. Therefore, the dd-modular form is a natural generalization of Δ_5. Gritsenko and Hulek [GH99] proved that the dd-modular form exists for the lattice $A_1(m)$ if and only if $1 \leq m \leq 4$. Cléry and Gritsenko [CG11] developed the arguments in [GH99] and gave the full classification of the dd-modular forms with respect to the Hecke subgroups of the Siegel paramodular groups. But their approach is hard to generalize to higher dimensions. Since dd-modular forms are crucial in determining the structure of the fixed space of modular forms and have applications in physics, as an important application of our arguments, we prove the following classification results for all dd-modular forms for lattices of the shape nA_1.

Theorem 2.8.2. *The dd-modular form exists if and only if the pair (n,m) takes one of the eight values*

$$(1,1),\qquad (1,2),\qquad (1,3),\qquad (1,4),\qquad (2,1),\qquad (2,2),\qquad (3,1),\qquad (4,1).$$

Proof. Suppose that F_c is a modular form of weight k with respect to $\Gamma_{n,m}$ with the divisor $c\cdot\Gamma_{n,m}\{z_n=0\}$, where c is the multiplicity of the diagonal divisor and it is a positive integer. The diagonal divisor $\Gamma_{n,m}\{z_n=0\}$ is the union of the primitive Heegner divisors $\mathcal{P}(\pm e_i/(2m),-1/(4m))$, $1\le i\le n$, where the primitive Heegner divisor of discriminant (μ,d) is defined as

$$\mathcal{P}(\mu,d)=\bigcup_{\substack{M+\mu\ni l \text{ primitive}\\ (l,l)=2d}} l^{\perp}\cap\mathcal{D}(M).$$

It is clear that we have

$$\mathcal{P}(\mu,y)=\mathcal{H}(\mu,y)-\sum_{d>y}x_d\mathcal{H}(\lambda_d,d),$$

where x_d are integers and $\lambda_d\in M^\vee$ (we refer to [BM17, Lemma 4.2] for an explicit formula). For arbitrary Heegner divisor $\mathcal{H}(\lambda,d)$ with $\lambda=(0,n_1,\lambda_0,n_2,0)\in A_M$, the principal part of the corresponding nearly holomorphic modular form of weight $-\operatorname{rank}(L)/2$ with respect to the Weil representation ρ_M of $\mathrm{Mp}_2(\mathbb{Z})$ is $q^d\mathbf{e}_\lambda$. Hence the singular Fourier coefficients of the corresponding weakly holomorphic Jacobi form of weight 0 are represented as

$$\sum_{r\in L+\lambda_0}q^{(r,r)/2+d}e^{2\pi i(r,\mathfrak{z})}.$$

Since $\{\pm e_i/(2m):1\le i\le n\}$ is the set of vectors in $nA_1(m)^\vee$ with the minimum norm $1/(2m)$ in $nA_1(m)^\vee/nA_1(m)$, through the previous explanations, there exists a weak Jacobi form $f_{nA_1,m}$ of weight 0 and index $nA_1(m)$ satisfying

$$f_{nA_1,m}=c\cdot\sum_{1\le i\le n}\zeta_i^{\pm1}+2k+O(q)$$

such that F_c is the Borcherds product of $f_{nA_1,m}$. We next apply Lemma 1.4.12 to our case. In this case, $c(n,\ell)=0$ for $n<0$, $\operatorname{rank}(L)=n$. For each term $c\zeta_i$ or $c\zeta_i^{-1}$, the corresponding $\ell=\pm\frac{1}{2m}e_i$, $c(0,\ell)(\ell,\ell)=c\cdot 2m\cdot\frac{1}{4m^2}=\frac{c}{2m}$. Therefore, we get

$$m(2nc+2k)=12c,$$

then $nm\le5$. It is not hard to show that a weak Jacobi form for $nA_1(m)$ has integral Fourier coefficients if its q^0-term is integral when $nm\le5$.

When $m\le4$, $2k$ is integral if $c=1$. Hence the existence of F_c is equivalent to the existence of F_1. In view of $k\ge n/2$, then the triplet (m,n,k) can only take one of the eight values

$$(1,1,5),(1,2,4),(1,3,3),(1,4,2),(2,1,2),(2,2,1),(3,1,1),(4,1,\frac{1}{2}).$$

When $m=5$, we only need to consider the case of $c=5$, and we obtain the unique solution $(5,1,1)$. But the unique weak Jacobi form of weight 0 and index 5 for A_1 is $\psi_{0,5}^{(1)}=5\zeta^{\pm1}+2+q(-\zeta^5+\cdots)$ (see [Gri99, formula (1.12)]). The corresponding Borcherds product is not holomorphic, that is, F_c does not exist in the case. We have thus proved the theorem. □

Remark 2.8.3. Similarly, we can define dd-modular forms with respect to the lattices $A_n(m)$ or $D_n(m)$. Using the same methodology, we can easily classify these dd-modular forms. In fact, dd-modular forms with respect to the lattices $L(m)$, where $L = A_n, n \geq 2$ or $L = D_n, n \geq 4$, exist if and only if the pair (L, m) takes one of the following fifteen values

$$\begin{array}{cccccccc} (A_2,1) & (A_3,1) & (A_4,1) & (A_5,1) & (A_6,1) & (A_7,1) & (A_2,2) & (A_2,3) \\ (A_3,2) & (D_4,1) & (D_5,1) & (D_6,1) & (D_7,1) & (D_8,1) & (D_4,2). & \end{array}$$

Note that all dd-modular forms in Theorem 2.8.2 and in the above list do exist and can be found in [CG11, Gri10, GN98b, GN18].

2.9 Open questions

In this section, we would like to formulate some interesting questions related to our work.

1. Are there 2-reflective lattices of signature $(2, 13)$? (see Remark 2.4.6)

   By Theorem 2.6.9, there is no 2-reflective lattice of signature $(2, 13)$ which can be represented as $2U \oplus L(-1)$ such that L has 2-roots. If $M = 2U \oplus L(-1)$ is a 2-reflective lattices of signature $(2, 13)$, then every lattice in the genus of L has no 2-root. It is very possible that such L does not exist. This suggests us that there might be no 2-reflective lattice of signature $(2, 13)$.

2. Are there reflective lattices of signature $(2, 21)$? (see Remark 2.5.12)

   By [Ess96], there is no hyperbolic reflective lattice of signature $(1, 20)$. In view of the relation between reflective modular forms and hyperbolic reflective lattices, we conjecture that the above question has a negative answer.

3. Let $M = 2U \oplus L(-1)$ be a reflective lattice of signature $(2, 22)$. Is L in the genus of $2E_8 \oplus D_4$?

   By [Ess96], $U \oplus 2E_8 \oplus D_4$ is the unique maximal hyperbolic reflective lattice of signature $(1, 21)$. Proposition 2.5.10 and [Ess96] indicates that the answer may be positive.

4. Classify all 2-reflective lattices of type $2U \oplus L(-1)$ satisfying that every lattice in the genus of L has no 2-root.

   This type of lattices is hard to classify using the Jacobi forms approach. The classification is related to the question: *"Classify even positive-definite lattices satisfying that every lattice in the genus has no 2-root"*. It is possible that such lattices of large rank (≥ 9) do not exist.

   Corresponding to Theorem 2.6.10, we conjecture that if the lattice $2U \oplus L(-1)$ has a modular form with complete 2-divisor and L has no 2-root then L is a primitive sublattice of the Leech lattice satisfying the Norm_2 condition.

5. Classify all reflective modular forms on lattices of prime level.

   This question is a continuation of Scheithauer's work [Sch06, Sch17], Dittmann's result (see Lemma 2.2.6) and our Theorem 2.5.7. By Lemma 2.2.6 and Theorem 2.5.7, it is easy to prove that $2U \oplus 2E_8 \oplus D_4$ and $2U(2) \oplus 2E_8(2) \oplus D_4$ are the only reflective lattices of level 2 and signature $(22, 2)$ because other lattices can be written as $U \oplus U(2) \oplus L$. In addition, $2U \oplus 2E_8 \oplus A_2$, $2U(3) \oplus 2E_8(3) \oplus A_2$, $\mathrm{II}_{18,2}(3)$, $\mathrm{II}_{26,2}(3)$ are the only reflective lattices of level 3 and signature $(n, 2)$ with $n \geq 15$. Using Remark 2.2.7 and known reflective modular forms, we can prove that all lattices of level 2 and signature $(2, n)$ with $n \leq 18$ are reflective, and all lattices of level 3 and signature $(2, n)$ with $n \leq 14$ are reflective. By means of the similar Jacobi forms approach, it is possible to prove the conjecture that there is no reflective lattice of prime level p when $p > 23$ except the scalings of unimodular lattices. We have proved this when $p \equiv 3 \mod 4$. We hope to complete the final classification in the near future.

6. Are there any other reflective modular forms of signature $(2, 26)$ except the scalings of $\mathrm{II}_{2,26}$?

   This question is related to the general question if all modular forms whose divisors are linear combinations of rational quadratic divisors can be constructed as Borcherds products of modular forms for the Weil representation. In general, not all such modular forms are Borcherds products in the rigorous sense. For example, the lattice $\mathrm{II}_{2,26}(2)$ is reflective but the corresponding reflective modular form (i.e. Φ_{12}) is not a Borcherds product of any modular form for the Weil representation associated to the discriminant form of $\mathrm{II}_{2,26}(2)$. Therefore, it seems that we should ask if any such modular forms come from Borcherds products. More precisely, if F is a modular form for a lattice M of signature $(2, n)$ with $n > 2$ whose divisor is a linear combination of rational quadratic divisors, then is there a lattice M' such that $M' \otimes \mathbb{Q} = M \otimes \mathbb{Q}$, F can be viewed as a modular form for M' and F is a Borcherds product of a modular form for the Weil representation associated to the discriminant form of M' ?

7. Are Borcherds products of singular weight reflective?

   This question was mentioned in [Sch17]. At present, all known Borcherds products of singular weight are reflective except some pull-backs. For example, the modular form Φ_{12} for $\mathrm{II}_{2,26} = 2U \oplus 3E_8$ is a Borcherds product of

$$\frac{\vartheta_{3E_8}(\tau, \mathfrak{z})}{\Delta(\tau)} = q^{-1} + 24 + \sum_{\substack{v \in 3E_8 \\ v^2=2}} e^{2\pi i (v, \mathfrak{z})} + O(q) \in J^{w.h.}_{0,3E_8,1}.$$

   It is of singular weight and reflective. We consider the pull-back of $\frac{\vartheta_{3E_8}}{\Delta}$ on the lattice $3D_8 < 3E_8$, which gives a Borcherds product of singular weight. By Theorem 2.5.5, it is not reflective. Therefore, we suggest formulating the following conjecture

   Conjecture 2.9.1. *Let F be a Borcherds product of singular weight for a lattice M of signature $(2, n)$ with $n > 2$. Then there exists an even lattice M' such that $M' \otimes \mathbb{Q} = M \otimes \mathbb{Q}$ and F can be viewed as a reflective modular form for M'.*

   The first step towards this conjecture is due to Scheithauer. It is known that Φ_{12} is the unique reflective modular form of singular weight for unimodular lattice. In [Sch17, Theorem 4.5], Scheithauer proved that Φ_{12} is the unique Borcherds product of singular weight for unimodular lattice. It means that the above conjecture is true for unimodular lattices. Besides, in [DHS15, OS18] the authors gave a classification of Borcherds products of singular weight on simple lattices. All Borcherds products of singular weight in their papers are reflective, which aslo supports the conjecture.

Chapter 3

Weyl invariant E_8 Jacoi forms

3.1 Background: Weyl invariant Jacobi forms

In this section we define the Weyl invariant Jacobi forms and recall Wirthmüller's structure theorem. Assume that R is an irreducible root system of rank r. Then R is one of the following types (see [Bou60])

$$A_n(n \geq 1), \quad B_n(n \geq 2), \quad C_n(n \geq 2), \quad D_n(n \geq 3), \quad E_6, \quad E_7, \quad E_8, \quad G_2, \quad F_4.$$

Let $L(R)$ be the root lattice generated by R. When $L(R)$ is an odd lattice, we equip $L(R)$ with the new bilinear form $2(\cdot,\cdot)$ rescaled by 2. We denote the normalized bilinear form of $L(R)$ by $\langle\cdot,\cdot\rangle$. In what follows, let $L(R)^\vee$ denote the dual lattice of $L(R)$ and $W(R)$ denote the Weyl group of R. As $L(R)$ are now even positive-definite lattices, we can define $W(R)$-invariant Jacobi forms with respect to $L(R)$ in the following way.

Definition 3.1.1. Let $\varphi : \mathbb{H} \times (L(R) \otimes \mathbb{C}) \to \mathbb{C}$ be a holomorphic function and $k \in \mathbb{Z}$, $t \in \mathbb{N}$. If φ satisfies the following properties

- Weyl invariance:
$$\varphi(\tau, \sigma(\mathfrak{z})) = \varphi(\tau, \mathfrak{z}), \quad \sigma \in W(R), \tag{3.1.1}$$
- Quasi-periodicity:
$$\varphi(\tau, \mathfrak{z} + x\tau + y) = \exp\left(-t\pi i[\langle x, x\rangle\tau + 2\langle x, \mathfrak{z}\rangle]\right)\varphi(\tau, \mathfrak{z}), \quad x, y \in L(R), \tag{3.1.2}$$
- Modularity:
$$\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{\mathfrak{z}}{c\tau+d}\right) = (c\tau+d)^k \exp\left(t\pi i\frac{c\langle\mathfrak{z},\mathfrak{z}\rangle}{c\tau+d}\right)\varphi(\tau,\mathfrak{z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \tag{3.1.3}$$
- $\varphi(\tau, \mathfrak{z})$ admits a Fourier expansion of the form
$$\varphi(\tau, \mathfrak{z}) = \sum_{n\in\mathbb{N}} \sum_{\ell \in L(R)^\vee} f(n,\ell) e^{2\pi i(n\tau + \langle \ell, \mathfrak{z}\rangle)}, \tag{3.1.4}$$

then φ is called a $W(R)$-invariant weak Jacobi form of weight k and index t. If φ further satisfies the condition
$$f(n,\ell) \neq 0 \Longrightarrow 2nt - \langle \ell, \ell\rangle \geq 0$$
then φ is called a $W(R)$-invariant holomorphic Jacobi form. If φ further satisfies the stronger condition
$$f(n,\ell) \neq 0 \Longrightarrow 2nt - \langle \ell, \ell\rangle > 0$$

then φ is called a $W(R)$-invariant Jacobi cusp form. We denote by

$$J^{\mathrm{w},W(R)}_{k,L(R),t} \supsetneq J^{W(R)}_{k,L(R),t} \supsetneq J^{\mathrm{cusp},W(R)}_{k,L(R),t}$$

the vector spaces of $W(R)$-invariant weak Jacobi forms, holomorphic Jacobi forms and Jacobi cusp forms of weight k and index t.

We next introduce many notations about root systems following [Bou60]. The dual root system of R is defined as

$$R^\vee = \{r^\vee : r \in R\}, \tag{3.1.5}$$

where $r^\vee = \frac{2}{(r,r)}r$ is the coroot of r. The weight lattice of R is defined as

$$\Lambda(R) = \{v \in R \otimes \mathbb{Q} : (r^\vee, v) \in \mathbb{Z}, \ \forall\, r \in R\}. \tag{3.1.6}$$

Let $\widetilde{\alpha}$ denote the highest root of $R^\vee$. In 1992, Wirthmüller proved the following structure theorem.

Theorem 3.1.2 (see Theorem 3.6 in [Wir92])**.** *If R is not of type E_8, then the bigraded ring of $W(R)$-invariant weak Jacobi forms over the ring of $\mathrm{SL}_2(\mathbb{Z})$ modular forms is the polynomial algebra in $r+1$ basic $W(R)$-invariant weak Jacobi forms of weight $-k(j)$ and index $m(j)$*

$$\varphi_{-k(j),m(j)}(\tau,\mathfrak{z}), \quad j = 0,1,...,r.$$

Apart from $k(0) = 0$ and $m(0) = 1$, the indices $m(j)$ are the coefficients of $\widetilde{\alpha}^\vee$ written as a linear combination of the simple roots of R. The integers $k(j)$ are the degrees of the generators of the ring of $W(R)$-invariant polynomials and also the exponents of the Weyl group $W(R)$ increased by 1.

We formulate the weights and indices of these generators in Table 3.1. For a lattice L, we write $\mathrm{O}(L)$ for the integral orthogonal group of L. We have $A_3 \cong D_3$. Since C_2 is isomorphic to B_2 via scaling by $\sqrt{2}$ and a 45 degree rotation, the spaces of Jacobi forms for them are isomorphic. Hence we only consider B_n for $n \geq 2$, C_n for $n \geq 3$, D_n for $n \geq 4$ in Table 3.1. Note that $W(C_n)/W(D_n) = \mathbb{Z}/2\mathbb{Z}$ and $W(C_n) = \mathrm{O}(D_n)$ if $n \neq 4$. The generators for root systems of types A_n, B_n and D_4 were constructed in [Ber99]. The generators for root systems E_6 and E_7 can be found in [Sak17b, Sat98].

Table 3.1: Weights and indices of the generators of Weyl invariant weak Jacobi forms

R	$L(R)$	$W(R)$	$(k(j),m(j))$
A_n	A_n	$W(A_n)$	$(0,1),(j,1) : 2 \leq j \leq n+1$
B_n	nA_1	$\mathrm{O}(nA_1)$	$(2j,1) : 0 \leq j \leq n$
C_n	D_n	$W(C_n)$	$(0,1),(2,1),(4,1),(2j,2) : 3 \leq j \leq n$
D_n	D_n	$W(D_n)$	$(0,1),(2,1),(4,1),(n,1),(2j,2) : 3 \leq j \leq n-1$
E_6	E_6	$W(E_6)$	$(0,1),(2,1),(5,1),(6,2),(8,2),(9,2),(12,3)$
E_7	E_7	$W(E_7)$	$(0,1),(2,1),(6,2),(8,2),(10,2),(12,3),(14,3),(18,4)$
G_2	A_2	$\mathrm{O}(A_2)$	$(0,1),(2,1),(6,2)$
F_4	D_4	$\mathrm{O}(D_4)$	$(0,1),(2,1),(6,2),(8,2),(12,3)$

Example 3.1.3. The $W(A_1)$-invariant Jacobi forms are the classical Jacobi forms in the sense of Eichler and Zagier [EZ85]. Let $R = A_1 = \mathbb{Z}\alpha$ with $\alpha^2 = 2$. In this case, the root lattice $L(A_1)$ is even and the dual root system of A_1 is itself. The Weyl group $W(A_1)$ is generated by the sign change $z \mapsto -z$. Thus, the Weyl invariant A_1 Jacobi forms are in fact the classical Jacobi forms of even weight in the sense of Eichler-Zagier [EZ85]. The highest root is $\alpha = 1 \cdot \alpha$. This means that the two generators are all of index 1. The $W(A_1)$-invariant polynomials is generated by the basic polynomial X^2 which is of degree 2. Thus the generators have weights 0 and -2 respectively. This coincides with the structure theorem in [EZ85].

Wirthmüller's theorem does not cover the case $R = E_8$. In the next sections, we will focus on $W(E_8)$-invariant Jacobi forms and present a proper extension of Wirthmüller's theorem to this case.

3.2 $W(E_8)$-invariant Jacobi forms

In this section we give a brief overview of root lattice E_8 and introduce many useful facts about $W(E_8)$-invariant Jacobi forms.

3.2.1 Definitions and basic properties

We first introduce the Jacobi theta functions. Let $q = e^{2\pi i\tau}$ and $\zeta = e^{2\pi i z}$, where $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$. The Jacobi theta functions of level two (see [Mum83, Chapter 1]) are defined as

$$\vartheta_{00}(\tau,z) = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}}\zeta^n, \qquad \vartheta_{01}(\tau,z) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n^2}{2}}\zeta^n,$$
$$\vartheta_{10}(\tau,z) = q^{\frac{1}{8}}\zeta^{\frac{1}{2}} \sum_{n\in\mathbb{Z}} q^{\frac{n(n+1)}{2}}\zeta^n, \qquad \vartheta_{11}(\tau,z) = iq^{\frac{1}{8}}\zeta^{\frac{1}{2}} \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n(n+1)}{2}}\zeta^n.$$

Note that $\vartheta(\tau,z) = -i\vartheta_{11}(\tau,z)$.

We next recall some standard facts about root lattice E_8. For a more careful treatment of this important lattice we refer to [Bou60, SC98]. The lattice E_8 is the unique even positive-definite unimodular lattice of rank 8 and one of its construction is as follows

$$\left\{(x_1,\ldots,x_8) \in \frac{1}{2}\mathbb{Z}^8 : x_1 \equiv \cdots \equiv x_8 \bmod 1,\ x_1 + \cdots + x_8 \equiv 0 \bmod 2\right\}.$$

The following eight vectors

$$\alpha_1 = \frac{1}{2}(1,-1,-1,-1,-1,-1,-1,1) \qquad \alpha_2 = (1,1,0,0,0,0,0,0)$$
$$\alpha_3 = (-1,1,0,0,0,0,0,0) \qquad \alpha_4 = (0,-1,1,0,0,0,0,0)$$
$$\alpha_5 = (0,0,-1,1,0,0,0,0) \qquad \alpha_6 = (0,0,0,-1,1,0,0,0)$$
$$\alpha_7 = (0,0,0,0,-1,1,0,0) \qquad \alpha_8 = (0,0,0,0,0,-1,1,0)$$

are the simple roots of E_8 and

$$w_1 = (0,0,0,0,0,0,0,2) \qquad w_2 = \frac{1}{2}(1,1,1,1,1,1,1,5)$$
$$w_3 = \frac{1}{2}(-1,1,1,1,1,1,1,7) \qquad w_4 = (0,0,1,1,1,1,1,5)$$
$$w_5 = (0,0,0,1,1,1,1,4) \qquad w_6 = (0,0,0,0,1,1,1,3)$$
$$w_7 = (0,0,0,0,0,1,1,2) \qquad w_8 = (0,0,0,0,0,0,1,1)$$

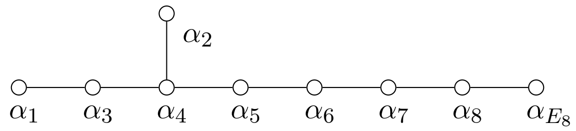

Figure 3.1: Extended Coxeter-Dynkin diagram of E_8

are the fundamental weights of E_8. The fundamental weights w_j form the dual basis, so $(\alpha_i, w_j) = \delta_{ij}$. We remark that the highest root α_{E_8} of E_8 is w_8, which can be written as a linear combination of the simple roots

$$\alpha_{E_8} = w_8 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8. \tag{3.2.1}$$

The exponents of the Weyl group $W(E_8)$ are 1, 7, 11, 13, 17, 19, 23, 29. In Figure 3.1 we give the extended Coxeter-Dynkin diagram of E_8.

By [SC98], Weyl group $W(E_8)$ is of order $2^{14}3^5 5^2 7 = 696729600$ and it is generated by all permutations of 8 letters, all even sign changes, and the matrix $\mathrm{diag}\{H_4, H_4\}$, where H_4 is the Hadamard matrix

$$H_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

In the sequel, we introduce many basic properties of $W(E_8)$-invariant Jacobi forms. We first remark that $W(E_8)$-invariant weak Jacobi forms are all of even weight because the operator $\mathfrak{z} \mapsto -\mathfrak{z}$ belongs to $W(E_8)$. In view of the singular weight, we have

$$J_{k,E_8,t}^{W(E_8)} = \{0\} \quad \text{if} \quad k < 4 \tag{3.2.2}$$

and this fact is also true for holomorphic Jacobi forms with character.

The following fact is very standard. For a proof, we refer to [Gri94, Lemma 2.1].

Lemma 3.2.1. *Let $\varphi \in J_{k,E_8,t}^{w,W(E_8)}$. Then the coefficients $f(n,\ell)$ in (3.1.4) depend only on the class of ℓ in E_8/tE_8 and on the value of $2nt - (\ell,\ell)$. Besides,*

$$f(n,\ell) \neq 0 \Longrightarrow 2nt - (\ell,\ell) \geq -\min\{(v,v) : v \in \ell + tE_8\}.$$

Using the same technique as in the proof of [EZ85, Theorem 8.4], we can prove the following result.

Lemma 3.2.2. *Let $t \in \mathbb{N}$. The graded algebra $J_{*,E_8,t}^{\mathrm{w},W(E_8)}$ (resp. $J_{*,E_8,t}^{W(E_8)}$, $J_{*,E_8,t}^{\mathrm{cusp},W(E_8)}$) of $W(E_8)$-invariant weak Jacobi forms (resp. holomorphic Jacobi forms, Jacobi cusp forms) of index t is a free module over M_*. Moreover, these three modules have the same rank over M_*.*

We next study carefully the Fourier expansion of $W(E_8)$-invariant Jacobi forms. For any $n \in \mathbb{N}$, we define the q^n-term of φ as

$$[\varphi]_{q^n} = \sum_{\ell \in E_8} f(n,\ell) e^{2\pi i(\ell,\mathfrak{z})}$$

and for any $m \in E_8$, we denote the Weyl orbit of m by

$$\mathrm{orb}(m) = \sum_{\sigma \in W(E_8)/W(E_8)_m} e^{2\pi i(\sigma(m),\mathfrak{z})}, \tag{3.2.3}$$

where $W(E_8)_m$ is the stabilizer subgroup of $W(E_8)$ with respect to m.

It is clear by Lemma 3.2.1 that $[\varphi]_{q^n}$ is an exponential $W(E_8)$-invariant polynomial. By [Bou60, Théorème VI.3.1] or [Lor05, Theorem 3.6.1], we obtain the next lemma.

Lemma 3.2.3. *Let* $\varphi \in J^{\mathrm{w},W(E_8)}_{k,E_8,t}$ *and* $n \in \mathbb{N}$. *We have*

$$[\varphi]_{q^n} \in \mathbb{C}[\operatorname{orb}(w_i) : 1 \leq i \leq 8].$$

Moreover, $\operatorname{orb}(w_i)$, $1 \leq i \leq 8$, *are algebraically independent over* $\mathbb{C}$.

3.2.2 Constructions of Jacobi forms

In this subsection we introduce several techniques to construct Jacobi forms. We recall the following weight raising differential operator (see Lemma 1.4.11).

Lemma 3.2.4. *Let* $\varphi(\tau,\mathfrak{z}) = \sum f(n,\ell)e^{2\pi i(n\tau+(\ell,\mathfrak{z}))}$ *be a* $W(E_8)$*-invariant weak Jacobi form of weight* k *and index* t. *Then* $H_k(\varphi)$ *is a* $W(E_8)$*-invariant weak Jacobi form of weight* $k+2$ *and index* t, *where*

$$H_k(\varphi)(\tau,\mathfrak{z}) = H(\varphi)(\tau,\mathfrak{z}) + \frac{4-k}{12}E_2(\tau)\varphi(\tau,\mathfrak{z}),$$
$$H(\varphi)(\tau,\mathfrak{z}) = \sum_{n\in\mathbb{N}}\sum_{\ell\in E_8}\left[n - \frac{1}{2t}(\ell,\ell)\right]f(n,\ell)e^{2\pi i(n\tau+(\ell,\mathfrak{z}))},$$

and $E_2(\tau) = 1 - 24\sum_{n\geq 1}\sigma(n)q^n$ *is the Eisenstein series of weight* 2 *on* $\mathrm{SL}_2(\mathbb{Z})$.

The next lemma gives a quite useful identity related to q^0-term of any Jacobi form of weight zero. It is a particular case of Lemma 1.4.12.

Lemma 3.2.5. *Let* $\varphi(\tau,\mathfrak{z}) = \sum f(n,\ell)e^{2\pi i(n\tau+(\ell,\mathfrak{z}))}$ *be a* $W(E_8)$*-invariant weak Jacobi form of weight* 0 *and index* t. *Then we have the following identity*

$$2t\sum_{\ell\in E_8} f(0,\ell) = 3\sum_{\ell\in E_8} f(0,\ell)(\ell,\ell).$$

Lemma 3.2.6. *Let* s *be a positive integer and* $\varphi \in J^{\mathrm{w},W(E_8)}_{k,E_8,t}$. *Then we have*

$$\varphi|_k T_-(s) = s^{-1}\sum_{\substack{ad=s\\ b \bmod d}} a^k\varphi\left(\frac{a\tau+b}{d}, a\mathfrak{z}\right) \in J^{\mathrm{w},W(E_8)}_{k,E_8,st}.$$

Moreover, the function $\varphi|_k T_-(s)$ *has a Fourier expansion of the form*

$$(\varphi|_k T_-(s))(\tau,\mathfrak{z}) = \sum_{\substack{n\in\mathbb{N}\\ \ell\in E_8}}\sum_{\substack{d\in\mathbb{N}\\ d|(n,\ell,s)}} d^{k-1}f\left(\frac{ns}{d^2},\frac{\ell}{d}\right)e^{2\pi i(n\tau+(\ell,\mathfrak{z}))},$$

where $f(n,\ell)$ *are the Fourier coefficients of* φ *and the notation* $d|(n,\ell,s)$ *means that* $d|(n,s)$ *and* $d^{-1}\ell \in E_8$.

Since weight 4 is the singular weight, it is impossible to construct $W(E_8)$-invariant holomorphic Jacobi form of weight 6 from Jacobi forms of weight 4 by the differential operators introduced in Lemma 3.2.4. Next, we give a resultful method to construct $W(E_8)$-invariant holomorphic Jacobi forms of weight 6. It is well-known that the theta function of the root lattice E_8 defined as

$$\begin{aligned}\vartheta_{E_8}(\tau,\mathfrak{z}) &= \sum_{\ell\in E_8}\exp\left(\pi i(\ell,\ell)\tau + 2\pi i(\ell,\mathfrak{z})\right)\\ &= \frac{1}{2}\left[\prod_{j=1}^{8}\vartheta(\tau,z_j) + \prod_{j=1}^{8}\vartheta_{00}(\tau,z_j) + \prod_{j=1}^{8}\vartheta_{01}(\tau,z_j) + \prod_{j=1}^{8}\vartheta_{10}(\tau,z_j)\right]\end{aligned} \tag{3.2.4}$$

is a $W(E_8)$-invariant holomorphic Jacobi form of weight 4 and index 1. One can check that $\vartheta_{E_8}(t\tau, t\mathfrak{z})$ is a $W(E_8)$-invariant holomorphic Jacobi form of weight 4 and index t for the congruence subgroup $\Gamma_0(t)$. We take a modular form of weight 2 on $\Gamma_0(t)$ and note it by $g(\tau)$. Then $g(\tau)\vartheta_{E_8}(t\tau, t\mathfrak{z})$ is a $W(E_8)$-invariant holomorphic Jacobi form of weight 6 and index t for $\Gamma_0(t)$. Therefore, the trace operator of $g(\tau)\vartheta_{E_8}(t\tau, t\mathfrak{z})$ defined by

$$Tr_{\mathrm{SL}_2(\mathbb{Z})}(g(\tau)\vartheta_{E_8}(t\tau, t\mathfrak{z})) = \sum_{\gamma\in\Gamma_0(t)\backslash\mathrm{SL}_2(\mathbb{Z})} (g(\tau)\vartheta_{E_8}(t\tau, t\mathfrak{z}))|_{6,t}\gamma \tag{3.2.5}$$

is a $W(E_8)$-invariant holomorphic Jacobi form of weight 6 and index t, where $|_{k,t}\gamma$ is the slash action of $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ defined by

$$(\phi|_{k,t}\gamma)(\tau, \mathfrak{z}) = (c\tau + d)^{-k} \exp\left(-t\pi i \frac{c(\mathfrak{z}, \mathfrak{z})}{c\tau + d}\right) \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{\mathfrak{z}}{c\tau + d}\right).$$

By the index raising operators introduced in Lemma 3.2.6, one can construct a $W(E_8)$-invariant holomorphic Jacobi form of weight 4 and index $t \geq 1$:

$$X_t(\tau, \mathfrak{z}) = *(\vartheta_{E_8}|_4 T_-(t))(\tau, \mathfrak{z}) = 1 + O(q) \in J_{4,E_8,t}^{W(E_8)}, \tag{3.2.6}$$

where $*$ is a constant such that $X_t(\tau, 0) = E_4(\tau)$. Sakai [Sak17a] constructed five $W(E_8)$-invariant holomorphic Jacobi forms of weight 4

$$A_i(\tau, \mathfrak{z}) = X_i(\tau, \mathfrak{z}),\ i = 1, 2, 3, 5, \tag{3.2.7}$$

$$A_4(\tau, \mathfrak{z}) = A_1(\tau, 2\mathfrak{z}). \tag{3.2.8}$$

Since $E_2(\tau) - pE_2(p\tau) \in M_2(\Gamma_0(p))$ if p is a prime number, one can construct $W(E_8)$-invariant holomorphic Jacobi forms of weight 6 and prime index. Furthermore, one can construct $W(E_8)$-invariant holomorphic Jacobi forms of weight 6 and index $t \geq 2$ using the index raising operators. These Jacobi forms may reduce to Eisenstein series $E_6(\tau)$ by taking $\mathfrak{z} = 0$. Sakai [Sak17a, Appendix A.1] also constructed $W(E_8)$-invariant holomorphic Jacobi forms of weight 6 and index 2, 3, 4, 6 by choosing particular modular forms on congruence subgroups of weight 2. They are constructed in the following way

$$B_2(\tau, \mathfrak{z}) = *Tr_{\mathrm{SL}_2(\mathbb{Z})}\left[(2E_2(2\tau) - E_2(\tau))\vartheta_{E_8}(2\tau, 2\mathfrak{z})\right], \tag{3.2.9}$$

$$B_3(\tau, \mathfrak{z}) = *Tr_{\mathrm{SL}_2(\mathbb{Z})}\left[(3E_2(3\tau) - E_2(\tau))\vartheta_{E_8}(3\tau, 3\mathfrak{z})\right], \tag{3.2.10}$$

$$B_4(\tau, \mathfrak{z}) = *Tr_{\mathrm{SL}_2(\mathbb{Z})}\left[\vartheta_{01}^4(2\tau)\vartheta_{E_8}(4\tau, 4\mathfrak{z})\right], \tag{3.2.11}$$

$$B_6(\tau, \mathfrak{z}) = *Tr_{\mathrm{SL}_2(\mathbb{Z})}\left[(3E_2(3\tau) - E_2(\tau))\vartheta_{E_8}(6\tau, 6\mathfrak{z})\right], \tag{3.2.12}$$

here, these constants $*$ are chosen such that $B_j(\tau, 0) = E_6(\tau)$.

3.2.3 Lifting elliptic modular forms to Jacobi forms

In this subsection we give another way to construct $W(E_8)$-invariant Jacobi forms. For our purpose, we focus on the lattices $E_8(p)$, which is the group E_8 equipped with the following rescaled bilinear form

$$\langle \cdot, \cdot \rangle_p = p(\cdot, \cdot),$$

where p is a prime number. Let $D(p) = E_8(p)^\vee / E_8(p)$ be the discriminant group of $E_8(p)$. Then $D(p)$ is of level p. By Theorem 1.3.4, we have

Proposition 3.2.7. *Let $f \in M_k(\Gamma_0(p))$ be a scalar valued holomorphic modular form for $\Gamma_0(p)$ of weight k. Then*

$$F_{\Gamma_0(p),f,0}(\tau) := \sum_{M \in \Gamma_0(p)\backslash \mathrm{SL}_2(\mathbb{Z})} f|_M(\tau)\rho_{D(p)}(M^{-1})e_0 \in M_k^{inv}(\rho_{D(p)}). \tag{3.2.13}$$

If we write

$$f|_S(\tau) = \sum_{t=0}^{p-1} g_t(\tau), \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where

$$g_t(\tau+1) = \exp\left(\frac{2t\pi i}{p}\right) g_t(\tau), \quad 0 \le t \le p-1,$$

then we have

$$F_{\Gamma_0(p),f,0}(\tau) = f(\tau)e_0 + \frac{1}{p^3}\sum_{\gamma \in D(p)} g_{j_\gamma}(\tau)e_\gamma, \tag{3.2.14}$$

here $j_\gamma/p = -\langle \gamma,\gamma\rangle_p/2 \mod 1$ for $\gamma \in D(p)$.

Moreover, the map

$$f \in M_k(\Gamma_0(p)) \mapsto F_{\Gamma_0(p),f,0} \in M_k^{inv}(\rho_{D(p)}) \tag{3.2.15}$$

is an isomorphism.

Recall that the theta function for the lattice $E_8(p)$ are defined by

$$\Theta_\gamma^{E_8(p)}(\tau,\mathfrak{z}) = \sum_{\ell \in \gamma + E_8} \exp\left(\pi i \langle \gamma,\gamma\rangle_p \tau + 2\pi i \langle \gamma, \mathfrak{z}\rangle_p\right), \quad \gamma \in D(p). \tag{3.2.16}$$

Proposition 3.2.8. *Under the assumptions of Proposition 3.2.7, if we write*

$$F_{\Gamma_0(p),f,0}(\tau) = \sum_{\gamma \in D(p)} F_{\Gamma_0(p),f,0;\gamma}(\tau)e_\gamma,$$

then the function

$$\Phi_{\Gamma_0(p),f,0}(\tau,\mathfrak{z}) = \sum_{\gamma \in D(p)} F_{\Gamma_0(p),f,0;\gamma}(\tau)\Theta_\gamma^{E_8(p)}(\tau,\mathfrak{z}) \tag{3.2.17}$$

is a $W(E_8)$-invariant holomorphic Jacobi form of weight $k+4$ and index p. Moreover, the application maps cusp forms to Jacobi cusp forms.

As applications of the above map, we can construct $W(E_8)$-invariant Jacobi forms of small weights

$$\Phi_{\Gamma_0(p),1,0}(\tau,\mathfrak{z}) \in J_{4,E_8,p}^{W(E_8)}, \tag{3.2.18}$$

$$\Phi_{\Gamma_0(p),pE_2(p\tau)-E_2(\tau),0}(\tau,\mathfrak{z}) \in J_{6,E_8,p}^{W(E_8)}. \tag{3.2.19}$$

We remark that the above map is injective. But it is not surjective in general unless the homomorphism $\mathrm{O}(E_8(p)) = W(E_8) \to \mathrm{O}(D(p))$ is surjective. The map $\mathrm{O}(E_8(p)) = W(E_8) \to \mathrm{O}(D(p))$ is surjective if and only if $p = 2$. Therefore, as an analogue of the natural isomorphism

$$\begin{aligned} M_k(\mathrm{SL}_2(\mathbb{Z})) &\longrightarrow J_{k+4,E_8,1}^{W(E_8)} \\ f(\tau) &\longmapsto f(\tau)\vartheta_{E_8}(\tau,\mathfrak{z}), \end{aligned}$$

we can build the following isomorphism.

Corollary 3.2.9. *We have the following isomorphism*

$$\begin{aligned} M_k(\Gamma_0(2)) &\longrightarrow J_{k+4,E_8,2}^{W(E_8)} \\ f(\tau) &\longmapsto \Phi_{\Gamma_0(2),f,0}(\tau,\mathfrak{z}) \end{aligned}$$

and it induces an isomorphism between the subspaces of cusp forms.

3.3 The ring of $W(E_8)$-invariant Jacobi forms

In this section we study the ring of $W(E_8)$-invariant weak Jacobi forms. Lemma 3.2.2 says that the space of $W(E_8)$-invariant weak Jacobi forms of fixed index is a free module over M_*. In what follows we give an explicit formula to compute the number of generators. The following is our main theorem, which is an explicit version of the Sakai conjecture in [Sak17b, Page4].

Theorem 3.3.1. *Let t be a positive integer. The space $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ of $W(E_8)$-invariant weak Jacobi forms of index t is a free module of rank $r(t)$ over M_*, where $r(t)$ is given by the generating series*

$$\frac{1}{(1-x)(1-x^2)^2(1-x^3)^2(1-x^4)^2(1-x^5)(1-x^6)} = \sum_{t\geq 0} r(t)x^t. \tag{3.3.1}$$

Equivalently, the bigraded algebra $J^{\mathrm{w},W(E_8)}_{*,E_8,*}$ of $W(E_8)$-invariant weak Jacobi forms is contained in the polynomial algebra of nine variables over the fractional field of $\mathbb{C}[E_4,E_6]$. More precisely,

$$J^{\mathrm{w},W(E_8)}_{*,E_8,*} \subsetneq \mathbb{C}(E_4,E_6)[A_1,A_2,A_3,A_4,A_5,B_2,B_3,B_4,B_6]. \tag{3.3.2}$$

and the functions A_1, A_2, A_3, A_4, A_5, B_2, B_3, B_4, B_6 are algebraically independent over M_*.

Proof. We denote the rank of $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ over M_* by $R(t)$. It is sufficient to show $R(t)=r(t)$ in order to prove the theorem. On the one hand, the nine Jacobi forms A_i and B_j are algebraically independent over M_*. In fact, using the nine Jacobi forms $\alpha_{k,t}$ in [Sak17a, Appendix A.2] and [Sak17b, pages 11, 19], we can construct nine $W(E_8)$-invariant weak Jacobi forms as polynomials in A_i and B_j over M_*: one Jacobi form of index 1 with q^0-term 1, two Jacobi forms of index 2 with q^0-terms $\mathrm{orb}(w_1)$ and $\mathrm{orb}(w_8)$ respectively, two Jacobi forms of index 3 with q^0-terms $\mathrm{orb}(w_2)$ and $\mathrm{orb}(w_7)$ respectively, two Jacobi forms of index 4 with q^0-terms $\mathrm{orb}(w_3)$ and $\mathrm{orb}(w_6)$ respectively, one Jacobi form of index 5 with q^0-term $\mathrm{orb}(w_5)$, one Jacobi form of index 6 with q^0-term $\mathrm{orb}(w_4)$. Since $\mathrm{orb}(w_i)$, $1\leq i\leq 8$, are algebraically independent over $\mathbb{C}$, we conclude that these nine weak Jacobi forms are algebraically independent over M_*. Therefore the nine Jacobi forms A_i and B_j are also algebraically independent over M_*. This fact yields $R(t)\geq r(t)$. On the other hand, Lemma 3.3.3 below gives $R(t)\leq r(t)$. Then the proof is completed. □

The next lemma is crucial to the proof of the above theorem. It also has its own interest because the values of $W(E_8)$-invariant Jacobi forms at $q=0$ are very interesting in quantum field theory.

Lemma 3.3.2. *Assume that*

$$\phi_t(\tau,\mathfrak{z}) = \sum_{n\geq 0}\sum_{\ell\in E_8} f(n,\ell)e^{2\pi i(n\tau+(\ell,\mathfrak{z}))}$$

is a $W(E_8)$-invariant weak Jacobi form of index t. Then we have

$$\sum_{\ell\in E_8} f(0,\ell)e^{2\pi i(\ell,\mathfrak{z})} = \sum_{\substack{X\in\mathbb{N}^8\\ T(X)\leq t}} c(X)\prod_{i=1}^{8}\mathrm{orb}(w_i)^{x_i}, \tag{3.3.3}$$

where $c(X)\in\mathbb{C}$ are constants, $X=(x_1,x_2,\dots,x_8)\in\mathbb{N}^8$ and

$$T(X) = 2x_1+3x_2+4x_3+6x_4+5x_5+4x_6+3x_7+2x_8.$$

Moreover, $\mathrm{orb}(w_i)$, $1\leq i\leq 8$, are algebraically independent over $\mathbb{C}$.

Proof. By Lemma 3.2.3, we know that

$$[\phi_t]_{q^0} = \sum_{\ell\in E_8} f(0,\ell)e^{2\pi i(\ell,\mathfrak{z})} \in \mathbb{C}[\mathrm{orb}(w_i), 1\le i\le 8]$$

and $\mathrm{orb}(w_i)$, $1\le i\le 8$, are algebraically independent over $\mathbb{C}$. We put

$$\Lambda_+ = \{m\in E_8 : (\alpha_i, m)\ge 0, 1\le i\le 8\} = \bigoplus_{i=1}^{8}\mathbb{N}w_i,$$

which is the closure of a Weyl chamber. We recall the following standard facts:

(a) Every $W(E_8)$-orbit in E_8 meets the set Λ_+ in exactly one point ([Bou60, Théorème VI.1.2(ii)]).

(b) Define a partial order on E_8 by $m\ge m'$ if $m-m'\in\bigoplus_{i=1}^{8}\mathbb{R}_+\alpha_i$. Then $m\ge\sigma(m)$ holds for all $m\in\Lambda_+$ and $\sigma\in W(E_8)$ (see [Bou60, Prop.VI.1.18]).

(c) For each $m\in\Lambda_+$, there are only finitely many $m'\in\Lambda_+$ satisfying $m\ge m'$ (see [Bou60, p.187]).

By the fact (a), we have

$$[\phi_t]_{q^0} = \sum_{m\in\Lambda_+} c(m)\,\mathrm{orb}(m). \tag{3.3.4}$$

For $m\in\Lambda_+$ with $m=\sum_{i=1}^{8}x_iw_i$, $x_i\in\mathbb{N}$, we define

$$\begin{aligned} T(m) =&(m,w_8)\\ =&2x_1+3x_2+4x_3+6x_4+5x_5+4x_6+3x_7+2x_8. \end{aligned}$$

When $T(m)>t$, we have

$$(m-tw_8, m-tw_8) = (m,m)-2t(m,w_8)+2t^2 < (m,m).$$

In this case, by Lemma 3.2.1, there will be a Fourier coefficient of type $q^{n_0}e^{2\pi i(m-tw_8,\mathfrak{z})}$ with $n_0<0$, which contradicts the definition of weak Jacobi forms. Thus, we obtain

$$[\phi_t]_{q^0} = \sum_{\substack{m\in\Lambda_+\\ T(m)\le t}} c(m)\,\mathrm{orb}(m). \tag{3.3.5}$$

Define

$$f_m = \prod_{i=1}^{8}\mathrm{orb}(w_i)^{x_i}, m=\sum_{i=1}^{8}x_iw_i.$$

By the facts (b) and (c), the product f_m can be written as a finite sum

$$f_m = \mathrm{orb}(m) + \sum_{\substack{m_1\in\Lambda_+\\ m_1<m}} c_{m,m_1}\,\mathrm{orb}(m_1). \tag{3.3.6}$$

We note that $m_1<m$ implies $T(m_1)\le T(m)$ because $m-m_1>0$ and w_8 is the highest root of E_8, which yield

$$T(m)-T(m_1) = (m,w_8)-(m_1,w_8) = (m-m_1,w_8)\ge 0.$$

We therefore establish the desired formula by equations (3.3.5), (3.3.6) and (c). □

Lemma 3.3.3. *The space $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ of $W(E_8)$-invariant weak Jacobi forms of index t is a free module of rank $\leq r(t)$ over M_*, where $r(t)$ is defined by (3.3.1).*

Proof. Conversely, suppose that the rank $R(t)$ of $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ over M_* is larger than $r(t)$. We assume that $\{\psi_i : 1 \leq i \leq R(t)\}$ is a basis of $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ over M_* and the weight of ψ_i is a_i. We put $a = \max\{a_i : 1 \leq i \leq R(t)\}$. According to Lemma 3.3.2, q^0-terms of $W(E_8)$-invariant weak Jacobi forms of index t can be written as $\mathbb{C}$-linear combinations of $r(t)$ fundamental elements $f_m = \prod_{i=1}^8 \mathrm{orb}(w_i)^{x_i}, m = \sum_{i=1}^8 x_i w_i$ with $T(m) \leq t$. Since $R(t) > r(t)$, there exists a homogenous polynomial $P \neq 0$ of degree one over M_* such that

$$P(\psi_i, 1 \leq i \leq R(t)) = \sum_{i=1}^{R(t)} c_i E_{a+4-a_i} \psi_i = O(q) \in J^{\mathrm{w},W(E_8)}_{a+4,E_8,t},$$

where c_i are constants and E_{a+4-a_i} are Eisenstein series of weight $a+4-a_i$ on $\mathrm{SL}_2(\mathbb{Z})$. Hence $P(\psi_i, 1 \leq i \leq R(t))/\Delta \neq 0 \in J^{\mathrm{w},W(E_8)}_{a-8,E_8,t}$ and thus it is a linear combination of ψ_i over M_*, which is impossible. □

Some values of rank $r(t)$ are shown in Table 3.2.

Table 3.2: Rank of $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ over M_*

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$r(t)$	1	3	5	10	15	27	39	63	90	135	187	270	364	505

Remark 3.3.4. If Wirthmüller's theorem holds for $R = E_8$, then the weights and indices of the nine generators are as follows (see Theorem 3.1.2)

$$\begin{array}{lllll} (0,1) & (-2,2) & (-8,2) & (-12,3) & (-14,3) \\ (-18,4) & (-20,4) & (-24,5) & (-30,6). & \end{array}$$

Therefore, Theorem 3.3.1 implies that the statement about the indices of generators in Wirthmüller's theorem holds for $R = E_8$.

We note that Lemma 3.3.2 can be extended to any irreducible root system by means of the following facts:

- The weight lattice $\Lambda(R^\vee)$ of $R^\vee$ is isomorphic to the dual lattice of $L(R)$.
- The multiplicative invariant algebra $\mathbb{Z}[\Lambda(R^\vee)]^{W(R)}$ is a polynomial algebra over $\mathbb{Z}$: the Weyl orbits of the fundamental weights of $R^\vee$ are algebraically independent generators (see [Bou60, Théorème VI.3.1 and Exemple 1]).
- $T(l)$ can be defined as $(l, \widetilde{\alpha}^\vee)$ (see §3.1). If $T(l)$ is greater than the index t, then the norm of $l - t\widetilde{\alpha}^\vee$ is smaller than the norm of l.
- $l_1 < l_2 \Longrightarrow T(l_1) \leq T(l_2)$, for any l_1, l_2 in the dual lattice of $L(R)$.

Thus, by virtue of the analogues of Lemmas 3.3.2, 3.3.3, we can give a new proof of the fact about the indices of generators in Wirthmüller's theorem.

The fact about the weights of generators is related to the Taylor expansion of basic weak Jacobi forms at the point $\mathfrak{z} = 0$. We next explain it more precisely. Given an irreducible root system R of rank r, if we could find $r+1$ basic weak Jacobi forms ϕ_j of expected indices with

q^0-terms containing the corresponding Weyl orbits of the fundamental weights, then we may show by the above arguments that these Jacobi forms are algebraically independent over M_* and for each Jacobi form φ, there exist a modular form f with non-zero constant term and a non-zero polynomial $P \in M_*[X_j, 0 \le j \le r]$ such that $f\varphi = P(\phi_j, 0 \le j \le r)$. If f is not constant, then f vanishes at a point $\tau_0 \in \mathbb{H}$. We observe that the leading terms of the Taylor expansion of ϕ_j are a homogeneous $W(R)$-invariant polynomial of degree equal to the absolute value of weight of ϕ_j. Therefore, if these ϕ_j further have the expected weights and the generators of $W(R)$-invariant polynomials appear in their leading terms of the Taylor expansions, then it is possible to prove that $\phi_j(\tau_0, \mathfrak{z})$ are algebraically independent over $\mathbb{C}$ using the fact that the eight generators of $W(R)$-invariant polynomials are algebraically independent over $\mathbb{C}$, which gives a contradiction. Then f is a constant and we deduce that the ring of Jacobi forms is the polynomial algebra generated by ϕ_j over M_*. The above discussions give an explanation of why the ring of Weyl invariant weak Jacobi forms is possible to be a polynomial algebra.

Remark 3.3.5. Our main theorem shows that every $W(E_8)$-invariant weak Jacobi form can be expressed uniquely as a polynomial in A_i and B_j with coefficients which are meromorphic $\mathrm{SL}_2(\mathbb{Z})$ modular forms (quotients of holomorphic modular forms). By the structure results in the next section, these meromorphic modular forms are in fact holomorphic except at infinity when the index is less than or equal to 3. In other words, we have

$$J^{\mathrm{w},W(E_8)}_{*,E_8,t} \subsetneq M_*\left[\frac{1}{\Delta}\right][A_1, A_2, A_3, A_4, A_5, B_2, B_3, B_4, B_6], \quad t = 1, 2, 3.$$

But when index $t \ge 4$, it is very likely that the above meromorphic modular forms have a pole at one point $\tau_0 \in \mathbb{H}$, which is different from the case in [EZ85]. In [ZGH+18], the authors checked numerically that the $W(E_8)$-invariant holomorphic Jacobi form of weight 16 and index 5 defined as

$$P = 864A_1^3A_2 + 21E_6^2A_5 - 770E_6A_3B_2 + 3825A_1B_2^2 - 840E_6A_2B_3 + 60E_6A_1B_4$$

vanishes at the zero points $\tau = \pm\frac{1}{2} + \frac{\sqrt{3}}{2}i$ of E_4 for general E_8 elliptic parameters. If the zeros of P and E_4 do indeed coincide, then P/E_4 will be a $W(E_8)$-invariant holomorphic Jacobi form of weight 12 and index 5.

Remark 3.3.6. In some sense, the choice of generators A_i and B_j in our main theorem is optimal. By the structure theorems in the next section, it is very natural to choose A_1, A_2, A_3, A_5, B_2, B_3 as generators because the corresponding spaces are all one-dimensional. There are 2 independent $W(E_8)$-invariant holomorphic Jacobi forms of weight 4 and index 4. One is A_4 and the other is X_4 (see (3.2.6)). But ΔX_4 can be expressed as a polynomial in our generators A_1, A_2, A_3, A_4, B_2, B_3 and Eisenstein series E_4, E_6. Therefore, we cannot choose X_4 instead of B_4. Besides, B_6 cannot be replaced by X_6 because $\Delta^m X_6$ with sufficiently large integer m can be expressed as a polynomial in our generators A_1, A_2, A_3, A_4, A_5, B_2, B_3, B_4 and E_4, E_6.[1]

3.4 $W(E_8)$-invariant Jacobi forms of small index

It is well-known that the space $J^{\mathrm{w},W(E_8)}_{*,E_8,1} = J^{W(E_8)}_{*,E_8,1}$ of $W(E_8)$-invariant weak (or holomorphic) Jacobi forms of index 1 is a free module over M_* generated by the theta function ϑ_{E_8}. In this big section, we give explicit descriptions of the structure of $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ and construct the generators when $t = 2, 3, 4$. The cases of index 5 and 6 are also discussed. We show two approachs to do this. The first one is based on the differential operators and the second relies on the pull-backs from $W(E_8)$-invariant Jacobi forms to the classical Jacobi forms for A_1.

[1]The author is grateful to Kazuhiro Sakai for explaining this point.

3.4.1 Notations and basic lemmas

In this subsection we present a new way to characterize q^0-terms of $W(E_8)$-invariant Jacobi forms. This new way is convenient to calculate q^0-terms of Jacobi forms under the action of the differential operators.

Let us denote by R_{2n} the set of all vectors $\ell \in E_8$ with $(\ell,\ell) = 2n$. The Weyl group $W(E_8)$ acts on R_{2n} in the usual way. The next lemma shows the orbits of R_{2n} under $W(E_8)$.

Lemma 3.4.1. *The orbits of R_{2n} under the action of $W(E_8)$ are given by*

$$\begin{aligned}
&W(E_8)\backslash R_2 = \{w_8\} && W(E_8)\backslash R_4 = \{w_1\}\\
&W(E_8)\backslash R_6 = \{w_7\} && W(E_8)\backslash R_8 = \{2w_8, w_2\}\\
&W(E_8)\backslash R_{10} = \{w_1 + w_8\} && W(E_8)\backslash R_{12} = \{w_6\}\\
&W(E_8)\backslash R_{14} = \{w_3, w_7 + w_8\} && W(E_8)\backslash R_{16} = \{2w_1, w_2 + w_8\}\\
&W(E_8)\backslash R_{18} = \{w_1 + w_7, 3w_8\} && W(E_8)\backslash R_{20} = \{w_5, w_1 + 2w_8\}\\
&W(E_8)\backslash R_{22} = \{w_6 + w_8, w_1 + w_2\} && W(E_8)\backslash R_{24} = \{2w_7, w_3 + w_8\}.
\end{aligned}$$

Proof. Applying the fact (a) in the proof of Lemma 3.3.2, we can prove the lemma by direct calculations. □

Corresponding to the above orbits, we define the following Weyl orbits.

$$\begin{aligned}
&\sum_{2} = \mathrm{orb}(w_8) && \sum_{4} = *\,\mathrm{orb}(w_1) && \sum_{6} = *\,\mathrm{orb}(w_7)\\
&\sum_{8'} = *\,\mathrm{orb}(w_2) && \sum_{8''} = *\,\mathrm{orb}(2w_8) && \sum_{10} = *\,\mathrm{orb}(w_1 + w_8)\\
&\sum_{12} = *\,\mathrm{orb}(w_6) && \sum_{14'} = *\,\mathrm{orb}(w_3) && \sum_{14''} = *\,\mathrm{orb}(w_7 + w_8)\\
&\sum_{16'} = *\,\mathrm{orb}(2w_1) && \sum_{16''} = *\,\mathrm{orb}(w_2 + w_8) && \sum_{18'} = *\,\mathrm{orb}(w_1 + w_7)\\
&\sum_{18''} = *\,\mathrm{orb}(3w_8) && \sum_{20'} = *\,\mathrm{orb}(w_5) && \sum_{20''} = *\,\mathrm{orb}(w_1 + 2w_8)\\
&\sum_{22'} = *\,\mathrm{orb}(w_1 + w_2) && \sum_{22''} = *\,\mathrm{orb}(w_6 + w_8) && \sum_{24'} = *\,\mathrm{orb}(2w_7)\\
&\sum_{24''} = *\,\mathrm{orb}(w_3 + w_8) && \sum_{26'} = *\,\mathrm{orb}(2w_1 + w_8) && \sum_{26''} = *\,\mathrm{orb}(w_2 + w_7)\\
&\sum_{28'} = *\,\mathrm{orb}(w_1 + w_6) && \sum_{30'} = *\,\mathrm{orb}(w_4) && \sum_{32'} = *\,\mathrm{orb}(w_1 + w_3)\\
&\sum_{32''} = *\,\mathrm{orb}(2w_2) && \sum_{36'} = *\,\mathrm{orb}(3w_1)
\end{aligned}$$

The normalizations of these Weyl orbits are chosen such that they reduce to 240 if one takes $\mathfrak{z} = 0$. By Lemma 3.2.1 and equation (3.3.5), it is easy to prove the next three lemmas.

Lemma 3.4.2.

$$\begin{aligned}
\max\{\min\{(v,v) : v \in l + 2E_8\} : l \in E_8\} &= 4,\\
\max\{\min\{(v,v) : v \in l + 3E_8\} : l \in E_8\} &= 8,\\
\max\{\min\{(v,v) : v \in l + 4E_8\} : l \in E_8\} &= 16,\\
\max\{\min\{(v,v) : v \in l + 5E_8\} : l \in E_8\} &= 22,\\
\max\{\min\{(v,v) : v \in l + 6E_8\} : l \in E_8\} &= 36.
\end{aligned}$$

Lemma 3.4.3. *Let φ_t be a $W(E_8)$-invariant weak Jacobi form of index t. Then its q^0-term can be written as*

$$\begin{aligned}
[\varphi_2]_{q^0} =&240c_0 + c_1 \sum_2 + c_2 \sum_4,\\
[\varphi_3]_{q^0} =&240c_0 + c_1 \sum_2 + c_2 \sum_4 + c_3 \sum_6 + c_4 \sum_{8'},\\
[\varphi_4]_{q^0} =&240c_0 + c_1 \sum_2 + c_2 \sum_4 + c_3 \sum_6 + c_4' \sum_{8'} + c_4'' \sum_{8''} + c_5 \sum_{10} + c_6 \sum_{12} + c_7 \sum_{14'} + c_8 \sum_{16'},\\
[\varphi_5]_{q^0} =&240c_0 + c_1 \sum_2 + c_2 \sum_4 + c_3 \sum_6 + c_4' \sum_{8'} + c_4'' \sum_{8''} + c_5 \sum_{10} + c_6 \sum_{12} + c_7' \sum_{14'} + c_7'' \sum_{14''}\\
&+ c_8' \sum_{16'} + c_8'' \sum_{16''} + c_9 \sum_{18'} + c_{10} \sum_{20'} + c_{11} \sum_{22'},\\
[\varphi_6]_{q^0} =&240c_0 + c_1 \sum_2 + c_2 \sum_4 + c_3 \sum_6 + c_4' \sum_{8'} + c_4'' \sum_{8''} + c_5 \sum_{10} + c_6 \sum_{12} + c_7' \sum_{14'} + c_7'' \sum_{14''}\\
&+ c_8' \sum_{16'} + c_8'' \sum_{16''} + c_9' \sum_{18'} + c_9'' \sum_{18''} + c_{10}' \sum_{20'} + c_{10}'' \sum_{20''} + c_{11}' \sum_{22'} + c_{11}'' \sum_{22''} + c_{12}' \sum_{24'}\\
&+ c_{12}'' \sum_{24''} + c_{13}' \sum_{26'} + c_{13}'' \sum_{26''} + c_{14} \sum_{28'} + c_{15} \sum_{30'} + c_{16}' \sum_{32'} + c_{16}'' \sum_{32''} + c_{18} \sum_{36'},
\end{aligned}$$

where $c_i \in \mathbb{C}$ are constants.

Lemma 3.4.4. *Assume that φ is a $W(E_8)$-invariant weak Jacobi form of index t.*

1. *Let $t = 2$. Then φ is a holomorphic Jacobi form if and only if its q^0-term is a constant. Moreover, φ is a Jacobi cusp form if and only if its q^0-term is 0 and its q^1-term is of the form $c_0 + c_1 \sum_2$.*

2. *Let $t = 3$. Then φ is a holomorphic Jacobi form if and only if its q^0-term is a constant and its q^1-term is of the form*
$$240c_0 + c_1 \sum_2 + c_2 \sum_4 + c_3 \sum_6 .$$
Moreover, a holomorphic Jacobi form φ is a Jacobi cusp form if and only if $c_3 = 0$ and its q^0-term is 0.

3. *Let $t = 4$. Then φ is a holomorphic Jacobi form if and only if its q^0-term is a constant and its q^1-term is of the form*
$$240c_0 + c_1 \sum_2 + c_2 \sum_4 + c_3 \sum_6 + c_4' \sum_{8'} + c_4'' \sum_{8''} .$$
Moreover, a holomorphic Jacobi form φ is a Jacobi cusp form if and only if $c_4' = c_4'' = 0$ and its q^0-term is 0 and its q^2-term does not contain the term $\sum_{16'}$.

4. *Let $t = 5$. Then φ is a holomorphic Jacobi form if and only if its q^0-term is a constant and its q^1-term is of the form*
$$240c_0 + c_1 \sum_2 + c_2 \sum_4 + c_3 \sum_6 + c_4' \sum_{8'} + c_4'' \sum_{8''} + c_5 \sum_{10}$$
and its q^2-term does not contain the term $\sum_{22'}$. Moreover, a holomorphic Jacobi form φ is a Jacobi cusp form if and only if $c_5 = 0$ and its q^0-term is 0 and its q^2-term does not contain the term $\sum_{20'}$.

5. *Let $t = 6$. Then φ is a holomorphic Jacobi form if and only if its q^0-term is a constant and its q^1-term is of the form*

$$240c_0 + c_1 \sum_{2} + c_2 \sum_{4} + c_3 \sum_{6} + c_4' \sum_{8'} + c_4'' \sum_{8''} + c_5 \sum_{10} + c_6 \sum_{12}$$

and its q^2-term does not contain the terms $\sum_{26'}$, $\sum_{26''}$, $\sum_{28'}$, $\sum_{30'}$, $\sum_{32'}$, $\sum_{32''}$, $\sum_{36'}$. Moreover, a holomorphic Jacobi form φ is a Jacobi cusp form if and only if $c_6 = 0$ and its q^0-term is 0 and its q^2-term does not contain the terms $\sum_{24'}$, $\sum_{24''}$ and its q^3-term does not contain the term $\sum_{36'}$.

We next explain how to determine holomorphic Jacobi forms of singular weight. Let φ_t be a $W(E_8)$-invariant holomorphic Jacobi form of weight 4 and index t. In view of the singular weight, we have

$$\varphi_t(\tau, \mathfrak{z}) = \sum_{n\in\mathbb{N}} \sum_{\substack{\ell\in E_8 \\ (\ell,\ell)=2nt}} f(n,\ell) e^{2\pi i (n\tau + (\ell,\mathfrak{z}))}.$$

Therefore the coefficients $f(n,\ell)$ depend only on the class of ℓ in E_8/tE_8. Let $n \geq 1$ and assume that

$$\phi_t(\tau, \mathfrak{z}) = q^n \sum_{\substack{\ell\in E_8 \\ (\ell,\ell)=2nt}} f(n,\ell) e^{2\pi i (\ell,\mathfrak{z})} + O(q^{n+1}) \in J_{4,E_8,t}^{W(E_8)}.$$

Since $\phi_t(\tau, 0) = 0$, if there exists $\ell \in E_8$ such that $f(n,\ell) \neq 0$, then there exist $\ell_1, \ell_2 \in E_8$ satisfying $(\ell_1,\ell_1) = (\ell_2,\ell_2) = 2nt$, $\mathrm{orb}(\ell_1) \neq \mathrm{orb}(\ell_2)$ and $(\ell_i,\ell_i) = \min\{(v,v) : v \in \ell_i + tE_8\}$, for $i = 1,2$. From this, we deduce

$$\begin{aligned} J_{4,E_8,t}^{W(E_8)} &= \mathbb{C}A_t, t = 1,2,3,5, \\ J_{4,E_8,4}^{W(E_8)} &= \mathbb{C}A_4 \oplus \mathbb{C}X_4, \\ 1 \leq \dim J_{4,E_8,6}^{W(E_8)} &\leq 2. \end{aligned}$$

If $\dim J_{4,E_8,6}^{W(E_8)} = 2$, then there exists a $W(E_8)$-invariant holomorphic Jacobi form of weight 4 and index 6 with Fourier expansion of the form

$$F_{4,6}(\tau, \mathfrak{z}) = q^2 \Big(\sum_{24'} - \sum_{24''}\Big) + O(q^3).$$

Let v_2 be a vector of norm 2 in E_8 and $z \in \mathbb{C}$. By direct calculations, we see that

$$\frac{F_{4,6}(\tau, zv_2)}{\Delta^2(\tau)} = \zeta^{\pm 8} + \sum_{1\leq i\leq 7} c(i)\zeta^{\pm i} + c(0) + O(q)$$

is a non-zero weak Jacobi form of weight -20 and index 6 in the sense of Eichler and Zagier [EZ85], where $\zeta = e^{2\pi i z}$ and $c(i) \in \mathbb{C}$ are constants. It is obvious that $J_{-20,6}^{\mathrm{w}} = \{0\}$ by [EZ85], which leads to a contradiction. Thus, we have proved the following.

Lemma 3.4.5.

$$\begin{aligned} J_{4,E_8,t}^{W(E_8)} &= \mathbb{C}A_t, t = 1,2,3,5, \\ J_{4,E_8,4}^{W(E_8)} &= \mathbb{C}A_4 \oplus \mathbb{C}X_4, \\ J_{4,E_8,6}^{W(E_8)} &= \mathbb{C}X_6. \end{aligned}$$

3.4.2 The case of index 2

In this subsection we discuss the structure of the space of $W(E_8)$-invariant Jacobi forms of index 2. Firstly, Theorem 3.3.1 shows that $J^{W(E_8)}_{*,E_8,2}$ is a free M_*-module of rank 3. It is obvious that A_2 and B_2 must be generators of weight 4 and weight 6 respectively. As A_1^2 and E_4A_2 are linearly independent, A_1^2 is a generator of weight 8. Hence $J^{W(E_8)}_{*,E_8,2}$ is a free M_*-module generated by A_2, B_2 and A_1^2. This fact can also be proved by Corollary 3.2.9.

If $\phi \in J^{\mathrm{w},W(E_8)}_{k,E_8,2}$, then $\Delta\phi \in J^{W(E_8)}_{k+12,E_8,2}$ by Lemma 3.4.4. From $J^{W(E_8)}_{4,E_8,2} = \mathbb{C}A_2$ and $J^{W(E_8)}_{6,E_8,2} = \mathbb{C}B_2$, we obtain $k+12 \geq 8$. Thus, $\dim J^{\mathrm{w},W(E_8)}_{k,E_8,2} = 0$ for $k \leq -6$. We next construct many basic Jacobi forms of index 2.

$$\varphi_{-4,2} = \frac{\vartheta_{E_8}^2 - \frac{1}{9}E_4\left(\vartheta_{E_8}|T_-(2)\right)}{\Delta} = 2\sum_2 - \sum_4 -240 + O(q) \in J^{\mathrm{w},W(E_8)}_{-4,E_8,2} \tag{3.4.1}$$

$$\varphi_{-2,2} = 3H_{-4}(\varphi_{-4,2}) = \sum_2 + \sum_4 -480 + O(q) \in J^{\mathrm{w},W(E_8)}_{-2,E_8,2} \tag{3.4.2}$$

$$\varphi_{0,2} = \frac{1}{2}E_4\varphi_{-4,2} - H_{-2}(\varphi_{-2,2}) = \sum_2 +120 + O(q) \in J^{\mathrm{w},W(E_8)}_{0,E_8,2} \tag{3.4.3}$$

Remark 3.4.6. There is another construction of $\varphi_{0,2}$

$$\varphi_{0,2} = * \sum_{\sigma\in W(E_8)} f(\tau,\sigma(\mathfrak{z})),$$

where $*$ is a constant and

$$f(\tau,\mathfrak{z}) = -\frac{[\vartheta(\tau,z_1+z_2)\vartheta(\tau,z_1-z_2)\cdots\vartheta(\tau,z_7+z_8)\vartheta(\tau,z_7-z_8)]|T_-(2)}{\vartheta(\tau,z_1+z_2)\vartheta(\tau,z_1-z_2)\cdots\vartheta(\tau,z_7+z_8)\vartheta(\tau,z_7-z_8)}.$$

It is easy to check the following constructions.

$$\begin{aligned} A_2 =& \frac{1}{9}\vartheta_{E_8}|T_-(2) = \frac{8}{9}\Phi_{\Gamma_0(2),1,0} = \frac{1}{1080}\left(3E_4\varphi_{0,2} - E_4^2\varphi_{-4,2} - E_6\varphi_{-2,2}\right) \\ =& 1 + q\cdot\sum_4 + O(q^2). \end{aligned} \tag{3.4.4}$$

$$\begin{aligned} B_2 =& \frac{16}{15}\Phi_{\Gamma_0(2),2E_2(2\tau)-E_2(\tau),0} = \frac{1}{1080}\left(3E_6\varphi_{0,2} - E_4E_6\varphi_{-4,2} - E_4^2\varphi_{-2,2}\right) \\ =& 1 + q\left[-\frac{8}{5}\sum_2 - \frac{3}{5}\sum_4 +24\right] \\ &+ q^2\left[\sum_{8''} - \frac{24}{5}\sum_{8'} - \frac{224}{5}\sum_6 - \frac{72}{5}\sum_4 - \frac{32}{5}\sum_2 +24\right] + O(q^3) \end{aligned} \tag{3.4.5}$$

$$U_{12,2} = \Delta\varphi_{0,2} = q\left(\sum_2 +120\right) + O(q^2) \in J^{\mathrm{cusp},W(E_8)}_{12,E_8,2} \tag{3.4.6}$$

$$V_{14,2} = \frac{1}{3}\Delta\left(E_6\varphi_{-4,2} + E_4\varphi_{-2,2}\right) = q\left[\sum_2 -240\right] + O(q^2) \in J^{\mathrm{cusp},W(E_8)}_{14,E_8,2} \tag{3.4.7}$$

$$W_{16,2} = \frac{1}{3}\Delta\left(E_4^2\varphi_{-4,2} + E_6\varphi_{-2,2}\right) = q\left[\sum_2 -240\right] + O(q^2) \in J^{\mathrm{cusp},W(E_8)}_{16,E_8,2} \tag{3.4.8}$$

It is easily seen that $\varphi_{-4,2}$ and $\varphi_{-2,2}$ must be generators of $J^{\mathrm{w},W(E_8)}_{*,E_8,2}$ over M_*. Since $[\varphi]_{q^0}(\tau,0) = 0$ if φ is a weak Jacobi form of negative weight, we claim that $\varphi_{0,2}$ is also a generator of $J^{\mathrm{w},W(E_8)}_{*,E_8,2}$ over M_* due to $[\varphi_{0,2}]_{q^0}(\tau,0) = 360$. We have thus proved the following structure theorem.

Theorem 3.4.7. *The spaces* $J^{\mathrm{w},W(E_8)}_{*,E_8,2}$, $J^{W(E_8)}_{*,E_8,2}$ *and* $J^{\mathrm{cusp},W(E_8)}_{*,E_8,2}$ *are all free* M_**-module generated by three Jacobi forms. More exactly, we have*

$$\begin{aligned} J^{\mathrm{w},W(E_8)}_{*,E_8,2} &= M_*\langle \varphi_{-4,2}, \varphi_{-2,2}, \varphi_{0,2}\rangle, \\ J^{W(E_8)}_{*,E_8,2} &= M_*\langle A_2, B_2, \vartheta_{E_8}^2\rangle, \\ J^{\mathrm{cusp},W(E_8)}_{*,E_8,2} &= M_*\langle U_{12,2}, V_{14,2}, W_{16,2}\rangle. \end{aligned}$$

Proof. It remains to prove the third claim. The third claim can be covered by Corollary 3.2.9. But, we here use another way to prove it. For arbitrary $f \in J^{W(E_8)}_{2k,E_8,2}$ with $k \geq 4$, there exist two complex numbers c_1, c_2 such that

$$f - c_1 E_{2k-4}A_2 - c_2 E_{2k-8}\Delta\varphi_{-4,2} \in J^{\mathrm{cusp},W(E_8)}_{2k,E_8,2},$$

we replace $E_{2k-8}\Delta\varphi_{-4,2}$ with $\Delta\varphi_{-2,2}$ when $k = 5$. From this, we have

$$\dim J^{\mathrm{cusp},W(E_8)}_{2k,E_8,2} = \dim J^{W(E_8)}_{2k,E_8,2} - 2, \ k \geq 4.$$

We then assert that $\dim J^{\mathrm{cusp},W(E_8)}_{2k,E_8,2} = 0$ for $k \leq 5$, and $\dim J^{\mathrm{cusp},W(E_8)}_{2k,E_8,2} = 1$ for $2k = 12, 14$, and $\dim J^{\mathrm{cusp},W(E_8)}_{16,E_8,2} = 2$. Since $W_{16,2}$ is independent of $E_4U_{12,2}$, we complete the proof. □

As an application of our results, we prove that Wirthmüller's theorem does not hold for E_8.

Theorem 3.4.8. *The bigraded ring* $J^{\mathrm{w},W(E_8)}_{*,E_8,*}$ *over* M_* *is not a polynomial algebra.*

Proof. Suppose, contrary to our claim, that $J^{\mathrm{w},W(E_8)}_{*,E_8,*}$ is a polynomial algebra over M_*. Then there exists a finite set S such that $J^{\mathrm{w},W(E_8)}_{*,E_8,*} = M_*[S]$ and the elements of S are algebraically independent over M_*. This contradicts the fact that ϑ_{E_8}, $\varphi_{-4,2}$, $\varphi_{-2,2}$, $\varphi_{0,2} \in S$ and the following algebraic relation

$$\vartheta_{E_8}^2 = \frac{1}{1080}E_4\left(3E_4\varphi_{0,2} - E_4^2\varphi_{-4,2} - E_6\varphi_{-2,2}\right) + \Delta\varphi_{-4,2}.$$

□

3.4.3 The case of index 3

In this subsection we continue to discuss the structure of the module of $W(E_8)$-invariant Jacobi forms of index 3. We first claim that the possible minimum weight of $W(E_8)$-invariant weak Jacobi forms of index 3 is -8. If there exists a $W(E_8)$-invariant weak Jacobi form ϕ of weight $2k < -8$ and index 3 whose q^0-term is not zero, then we can construct a weak Jacobi form of weight -10 and index 3 whose q^0-term is not zero. In fact, this function can be constructed as $E_{-10-2k}\phi$ if $2k \leq -14$, or $H_{-12}(\phi)$ if $2k = -12$. We now assume that there exists a $W(E_8)$-invariant weak Jacobi form ϕ of weight -10 and index 3 whose q^0-term is represented as

$$[\phi]_{q^0} = 240c_0 + c_1\sum_2 + c_2\sum_4 + c_3\sum_6 + c_4\sum_{8'}.$$

Then $c_4 \neq 0$, otherwise $\Delta\phi$ will be a $W(E_8)$-invariant holomorphic Jacobi form of weight 2, which is impossible. By means of the differential operators, we construct $\phi_{-8} = H_{-10}(\phi)$, $\phi_{-6} = H_{-8}(\phi_{-8})$, $\phi_{-4} = H_{-6}(\phi_{-6})$ and $H_{-4}(\phi_{-4})$. They are respectively weak Jacobi forms of weight -8, -6, -4, -2 with q^0-term of the form (order: $240c_0, c_1\sum_2, c_2\sum_4, c_3\sum_6, c_4\sum_{8'}$)

$$\begin{aligned} &\text{weight} \quad -10: && (a_{1,j})_{j=1}^9 = (1,1,1,1,1) \\ &\text{weight} \quad -10+2(i-1): && a_{i,j} = \left(\frac{18-2i}{12} - \frac{j-1}{3}\right)a_{i-1,j} \end{aligned}$$

where $2 \le i \le 5$, $1 \le j \le 5$. For these Jacobi forms, if we take $\mathfrak{z} = 0$ then their q^0-terms will be zero. We thus get a system of 5 linear equations with 5 unknowns

$$Ax = 0, \quad A = (a_{i,j})_{5\times 5}, \quad x = (c_0, c_1, c_2, c_3, c_4)^t.$$

By direct calculations, this system has only trivial solution, which contradicts our assumption. Hence the possible minimum weight is -8. Indeed, there exists the unique $W(E_8)$-invariant weak Jacobi form of weight -8 and index 3 up to a constant. Suppose that ϕ is a non-zero weak Jacobi form of weight -8 with q^0-term of the form

$$240c_0 + c_1 \sum_2 + c_2 \sum_4 + c_3 \sum_6 + c_4 \sum_{8'}.$$

Similarly, we can construct weak Jacobi forms of weight -6, -4, -2 with q^0-term of the form

$$\begin{aligned} &\text{weight} \quad -8: && (b_{1,j})_{j=1}^9 = (1,1,1,1,1) \\ &\text{weight} \quad -8+2(i-1): && b_{i,j} = \left(\frac{16-2i}{12} - \frac{j-1}{3}\right) b_{i-1,j} \end{aligned}$$

where $2 \le i \le 4$, $1 \le j \le 5$. Then we can build a system of 4 linear equations with 5 unknowns

$$Bx = 0, \quad B = (b_{i,j})_{4\times 5}, \quad x = (c_0, c_1, c_2, c_3, c_4)^t. \tag{3.4.9}$$

We found that $(c_0, c_1, c_2, c_3, c_4) = (1, -4, 6, -4, 1)$ is the unique nontrivial solution of the above system. Therefore, the weak Jacobi form of weight -8 and index 3 is unique if it exists. Next, we construct many weak Jacobi forms of index 3.

$$\begin{aligned} B_{-2,3} =& -5\frac{\vartheta_{E_8} B_2 - \frac{1}{28} E_6 (\vartheta_{E_8} | T_-(3))}{\Delta} = 3\sum_2 + 3\sum_4 + 5\sum_6 - 11 \times 240 + O(q) \in J^{\mathrm{w}, W(E_8)}_{-2, E_8, 3} \\ \varphi_{-4,3} =& \frac{\vartheta_{E_8} A_2 - \frac{1}{28} E_4 [\vartheta_{E_8} | T_-(3)]}{\Delta} = \sum_2 + \sum_4 - \sum_6 - 240 + O(q) \in J^{\mathrm{w}, W(E_8)}_{-4, E_8, 3} \\ A_{0,3} =& \vartheta_{E_8} \varphi_{-4,2} = 2\sum_2 - \sum_4 - 240 + O(q) \in J^{\mathrm{w}, W(E_8)}_{0, E_8, 3} \\ \varphi_{-2,3} =& 3H_{-4}(\varphi_{-4,3}) = \sum_2 + \sum_6 - 480 + O(q) \in J^{\mathrm{w}, W(E_8)}_{-2, E_8, 3} \\ \varphi_{0,3} =& \frac{3}{8}\left(A_{0,3} + E_4 \varphi_{-4,3} - 2H_{-2}(\varphi_{-2,3})\right) = \sum_2 + O(q) \in J^{\mathrm{w}, W(E_8)}_{0, E_8, 3} \end{aligned}$$

Remark 3.4.9. There is another construction of $\varphi_{0,3}$

$$\varphi_{0,3} = * \sum_{\sigma \in W(E_8)} g(\tau, \sigma(\mathfrak{z}))$$

where $*$ is a constant and the function g is defined as

$$g(\tau, \mathfrak{z}) = \prod_{i=1}^8 \frac{\vartheta(\tau, 2z_i)}{\vartheta(\tau, z_i)} + \prod_{i=1}^8 \frac{\vartheta(\tau, 2z_i)}{\vartheta_{00}(\tau, z_i)} + \prod_{i=1}^8 \frac{\vartheta(\tau, 2z_i)}{\vartheta_{01}(\tau, z_i)} + \prod_{i=1}^8 \frac{\vartheta(\tau, 2z_i)}{\vartheta_{10}(\tau, z_i)}.$$

We next construct the $W(E_8)$-invariant weak Jacobi form of weight -8 and index 3. Firstly, we can check

$$E_4^2 \varphi_{-4,3} + 6E_6 \varphi_{-2,3} - 2E_4 A_{0,3} - E_6 B_{-2,3} = O(q) \in J^{\mathrm{w}, W(E_8)}_{4, E_8, 3}.$$

If $E_4^2\varphi_{-4,3}+6E_6\varphi_{-2,3}-2E_4A_{0,3}-E_6B_{-2,3}=0$, then we have

$$\begin{aligned} f_{-6,3} &= \frac{E_4\varphi_{-4,3}-2A_{0,3}}{E_6} = -3\sum_2 +3\sum_4 -\sum_6 +240+O(q)\in J^{\mathrm{w},W(E_8)}_{-6,E_8,3}, \\ H_{-6}(f_{-6,3}) &= -\frac{3}{2}\sum_2+\frac{1}{2}\sum_4+\frac{1}{6}\sum_6+200+O(q)\in J^{\mathrm{w},W(E_8)}_{-4,E_8,3}. \end{aligned}$$

It is easy to see that $f_{-6,3}$, $H_{-6}(f_{-6,3})$, $\varphi_{-4,3}$ are free over M_* because $E_4\phi_{-8,3}$, $H_{-6}(f_{-6,3})$ and $\varphi_{-4,3}$ are independent. However

$$E_6f_{-6,3}-3E_4H_{-6}(f_{-6,3})-\frac{3}{2}E_4\varphi_{-4,3}=O(q).$$

Hence we can get a non-zero weak Jacobi form of index 3 and weight -12, which is impossible. It follows that $E_4^2\varphi_{-4,3}+6E_6\varphi_{-2,3}-2E_4A_{0,3}-E_6B_{-2,3}\neq 0$, and we can construct

$$\begin{aligned} \varphi_{-8,3} &= * \,\frac{E_4^2\varphi_{-4,3}+6E_6\varphi_{-2,3}-2E_4A_{0,3}-E_6B_{-2,3}}{\Delta} \\ &= \sum_{8'}-4\sum_6+6\sum_4-4\sum_2+240+O(q)\in J^{\mathrm{w},W(E_8)}_{-8,E_8,3}. \end{aligned} \tag{3.4.10}$$

$$\varphi_{-6,3}=-3H_{-8}(\varphi_{-8,3})=\sum_{8'}-6\sum_4+8\sum_2-720+O(q)\in J^{\mathrm{w},W(E_8)}_{-6,E_8,3}. \tag{3.4.11}$$

In fact, we get the coefficients of the q^0-term of $\varphi_{-8,3}$ from the solution of the system of linear equations (3.4.9). We now arrive at our main theorem in this subsection.

Theorem 3.4.10. *The space $J^{\mathrm{w},W(E_8)}_{*,E_8,3}$ is a free M_*-module generated by five weak Jacobi forms. More precisely, we have*

$$J^{\mathrm{w},W(E_8)}_{*,E_8,3}=M_*\langle\varphi_{-8,3},\varphi_{-6,3},\varphi_{-4,3},\varphi_{-2,3},\varphi_{0,3}\rangle.$$

Proof. We first claim that there is no weak Jacobi form of weight -6 and index 3 independent of $\varphi_{-6,3}$. Conversely, suppose that there exists a weak Jacobi form of weight -6 which is linearly independent of $\varphi_{-6,3}$, noted by f. Without loss of generality we can assume

$$[f]_{q^0}=240c_0+c_1\sum_2+c_2\sum_4+c_3\sum_6\neq 0.$$

Once again, we can construct weak Jacobi forms of weight -4, -2 and 0 by the differential operators, respectively. They have q^0-terms of the form (order: $240c_0, c_1\sum_2, c_2\sum_4, c_3\sum_6$)

$$\begin{aligned} &\text{weight} \quad -6: && (c_{1,j})_{j=1}^4=(1,1,1,1) \\ &\text{weight} \quad -6+2(i-1): && c_{i,j}=\frac{9-i-2j}{6}c_{i-1,j} \end{aligned}$$

where $2\leq i\leq 4$, $1\leq j\leq 4$. For each Jacobi form of negative weight, if we take $\mathfrak{z}=0$ then its q^0-term will be zero. Hence we have

$$\sum_{j=1}^4 c_{i,j}c_{j-1}=0,\quad 1\leq i\leq 3.$$

By Lemma 3.2.5, we have

$$\sum_{j=1}^4(12-6j)c_{4,j}c_{j-1}=0.$$

We thus get a system of linear equations of 4×4. By direct calculations, we obtain $c_j = 0$ for $0 \le j \le 3$, which contradicts our assumption.

Theorem 3.3.1 shows that $J^{\mathrm{w},W(E_8)}_{*,E_8,3}$ is a free M_*-module generated by five weak Jacobi forms. It is obvious that $\varphi_{-8,3}$, $\varphi_{-6,3}$ and $\varphi_{-4,3}$ are generators. Since $\varphi_{-2,3}$ is independent of $E_6\varphi_{-8,3}$ and $E_4\varphi_{-6,3}$, the function $\varphi_{-2,3}$ must be a generator. Moreover, $\varphi_{0,3}$ is also a generator on account of $[\varphi_{0,3}]_{q^0}(\tau,0) \neq 0$. We then conclude the eager result. $\square$

In the rest of this subsection, we investigate the spaces of holomorphic Jacobi forms and Jacobi cusp forms of index 3, respectively. Let $k \ge 2$. It is easy to see that the five dimensional space $\mathfrak{A}$ generated by $E_{2k+8}\varphi_{-8,3}$, $E_{2k+6}\varphi_{-6,3}$, $E_{2k+4}\varphi_{-4,3}$, $E_{2k+2}\varphi_{-2,3}$, $E_{2k-4}\Delta\varphi_{-8,3}$ (if $k=3$, we replace $E_{2k-4}\Delta\varphi_{-8,3}$ with $\Delta\varphi_{-6,3}$) does not contain non-zero holomorphic Jacobi form of weight $2k$. Moreover, for any $\phi \in J^{\mathrm{w},W(E_8)}_{2k,E_8,3}$, there exists a Jacobi form $f \in \mathfrak{A}$ such that $\phi - f$ is a holomorphic Jacobi form. We then assert

$$\dim J^{W(E_8)}_{2k,E_8,3} = \dim J^{\mathrm{w},W(E_8)}_{2k,E_8,3} - 5, \quad k \ge 2.$$

It is clear that $\dim J^{W(E_8)}_{2k,E_8,3} = 1$, for $k = 2,3$. Thus, we deduce

$$\begin{aligned} A_3 =& \frac{1}{28}E_4(\vartheta_{E_8}|T_-(3)) = \frac{27}{28}\Phi_{\Gamma_0(3),1,0} = 1 + q\sum_6 + O(q^2),\\ B_3 =& \frac{81}{160}\Phi_{\Gamma_0(3),3E_2(3\tau)-E_2(\tau),0} = 1 + q\left[-\frac{7}{20}\sum_6 - \frac{27}{20}\sum_4 - \frac{9}{20}\sum_2 + 12\right] + O(q^2). \end{aligned}$$

We further construct

$$A_2\vartheta_{E_8} = 1 + q\left[\sum_2 + \sum_4\right] + O(q^2) \in J^{W(E_8)}_{8,E_8,3} \tag{3.4.12}$$

$$B_2\vartheta_{E_8} = 1 + q\left[-\frac{3}{5}\sum_2 - \frac{3}{5}\sum_4 + 24\right] + O(q^2) \in J^{W(E_8)}_{10,E_8,3}, \tag{3.4.13}$$

$$\vartheta_{E_8}^3 = 1 + 3q\sum_2 + O(q^2) \in J^{W(E_8)}_{12,E_8,3}. \tag{3.4.14}$$

It is easy to check that the following vector spaces have the corresponding basis.

$$\begin{aligned} J^{W(E_8)}_{8,E_8,3} &= \mathbb{C}\{E_4A_3,\ A_2\vartheta_{E_8}\}\\ J^{W(E_8)}_{10,E_8,3} &= \mathbb{C}\{E_6A_3,\ E_4B_3,\ B_2\vartheta_{E_8}\}\\ J^{W(E_8)}_{12,E_8,3} &= \mathbb{C}\{E_4^2A_3,\ E_6B_3,\ E_4A_2\vartheta_{E_8},\ \vartheta_{E_8}^3\} \end{aligned}$$

From the above discussions, we claim that $A_3, B_3, A_2\vartheta_{E_8}, B_2\vartheta_{E_8}, \vartheta_{E_8}^3$ are free over M_*. This proves the following theorem.

Theorem 3.4.11. *The space $J^{W(E_8)}_{*,E_8,3}$ is a free M_*-module generated by five holomorphic Jacobi forms. More precisely, we have*

$$J^{W(E_8)}_{*,E_8,3} = M_*\langle A_3,\ B_3,\ A_2\vartheta_{E_8},\ B_2\vartheta_{E_8},\ \vartheta_{E_8}^3\rangle.$$

In the end, we determine the structure of Jacobi cusp forms of index 3. We first construct many basic Jacobi cusp forms.

$$U_{10,3} = -\frac{35}{54}E_6A_3 - \frac{50}{27}E_4B_3 + \frac{5}{2}B_2\vartheta_{E_8} = q\left[\sum_4 - \frac{2}{3}\sum_2 - 80\right] + O(q^2) \in J^{\mathrm{cusp},W(E_8)}_{10,E_8,3}$$

$$\begin{aligned}
U_{12,3} =& E_4A_2\vartheta_{E_8} - \vartheta_{E_8}^3 = q\left[\sum_4 -2\sum_2 +240\right] + O(q^2) \in J_{12,E_8,3}^{\text{cusp},W(E_8)} \\
U_{14,3} =& \Delta(E_4\varphi_{-2,3} + E_6\varphi_{-4,3}) = q\left[\sum_4 +2\sum_2 -720\right] + O(q^2) \in J_{14,E_8,3}^{\text{cusp},W(E_8)} \\
V_{12,3} =& \Delta\varphi_{0,3} = q\cdot\sum_2 + O(q^2) \in J_{12,E_8,3}^{\text{cusp},W(E_8)} \\
U_{16,3} =& \Delta^2\varphi_{-8,3} = O(q^2) \in J_{16,E_8,3}^{\text{cusp},W(E_8)}
\end{aligned}$$

For arbitrary $k \geq 4$, we can show that the two dimensional space $\mathfrak{B}$ generated by $E_{2k-4}A_3$ and $E_{2k-8}\Delta\varphi_{-4,3}$ (if $k = 5$, we replace $E_{2k-8}\Delta\varphi_{-4,3}$ with E_4B_3) does not contain non-zero Jacobi cusp form of weight $2k$. Moreover, for any $\phi \in J_{2k,E_8,3}^{W(E_8)}$, there exists a Jacobi form $g \in \mathfrak{B}$ such that $\phi - g$ is a Jacobi cusp form. We thus deduce

$$\dim J_{2k,E_8,3}^{\text{cusp},W(E_8)} = \dim J_{2k,E_8,3}^{W(E_8)} - 2, \quad k \geq 4. \tag{3.4.15}$$

In a similar argument, we prove the next theorem.

Theorem 3.4.12. *The space $J_{*,E_8,3}^{\text{cusp},W(E_8)}$ is a free M_*-module generated by five Jacobi cusp forms. More exactly, we have*

$$J_{*,E_8,3}^{\text{cusp},W(E_8)} = M_*\langle U_{10,3},\ U_{12,3},\ V_{12,3},\ U_{14,3},\ U_{16,3}\ \rangle.$$

3.4.4 The case of index 4

In this subsection we study the structure of the space of $W(E_8)$-invariant Jacobi forms of index 4. We first assert that the possible minimum weight of $W(E_8)$-invariant weak Jacobi forms of index 4 is -16. If there exists a non-zero weak Jacobi form ϕ of index 4 and weight $k < -16$, then its q^0-term is not zero and is not of the form

$$c_4'\sum_{8'} + c_4''\sum_{8''} + 240c_0,$$

otherwise we can construct a non-zero holomorphic Jacobi form of wight less than 4 (i.e. $\Delta\phi$). As in the case of index 3, by the Eisenstein series and the differential operators, we can construct a weak Jacobi form of weight -18 with non-zero q^0-term and note it by f. For convenience, we write

$$c_4'\sum_{8'} + c_4''\sum_{8''} = (c_4' + c_4'')\sum_8 = c_4\sum_8.$$

We can assume that f has q^0-term of the form

$$240c_0 + c_1\sum_2 + c_2\sum_4 + c_3\sum_6 + c_4\sum_8 + c_5\sum_{10} + c_6\sum_{12} + c_7\sum_{14'} + c_8\sum_{16'},$$

where c_i are not all zero. We then construct weak Jacobi forms of weight -16, -14, -12, -10, -8, -6, -4, -2, respectively. They have q^0-terms of the following form (order: $240c_0, c_1\sum_2, \cdots, c_8\sum_{16'}$)

$$\begin{aligned}
&\text{weight} \quad -18: && (a_{1,j})_{j=1}^9 = (1,1,1,1,1,1,1,1,1) \\
&\text{weight} \quad -18 + 2(i-1): && a_{i,j} = \frac{29 - 2i - 3j}{12}a_{i-1,j}
\end{aligned}$$

where $2 \leq i \leq 9$, $1 \leq j \leq 9$. For these Jacobi forms, if we put $\mathfrak{z} = 0$, then their q^0-terms will become zero. Hence we can get a system of linear equations:

$$Ax = 0, \quad A = (a_{i,j})_{9\times 9}, \quad x = (c_0, c_1, \cdots, c_8)^t.$$

By direct calculations, we know that the determinant of the matrix A is not zero, it follows that the q^0-term of f is zero, which leads to a contradiction. Hence the possible minimum weight is -16. One weak Jacobi form of index 4 and weight -16 can be constructed as

$$\begin{aligned}\varphi_{-16,4} =& c\sum_{\sigma\in W(E_8)} h(\tau,\sigma(\mathfrak{z}))\\ =&\sum_{16'}-8\sum_{14'}+28\sum_{12}-56\sum_{10}+14\sum_{8''}+56\sum_{8'}-56\sum_{6}+28\sum_{4}-8\sum_{2}+240+O(q),\end{aligned}\tag{3.4.16}$$

where c is a constant and the function h is defined as

$$h(\tau,\mathfrak{z})=\frac{1}{\Delta^2}\prod_{i=1}^{4}\vartheta(\tau,z_{2i-1}+z_{2i})^2\vartheta(\tau,z_{2i-1}-z_{2i})^2.$$

Next, we show that $\varphi_{-16,4}$ is the unique weak Jacobi form of index 4 and weight -16 up to a constant. Similarly, suppose that there exists a weak Jacobi form $\phi_{-16,4}$ of weight -16 with q^0-term of the form

$$240c_0+c_1\sum_{2}+c_2\sum_{4}+c_3\sum_{6}+c_4\sum_{8}+c_5\sum_{10}+c_6\sum_{12}+c_7\sum_{14'}+c_8\sum_{16'}.$$

Then we can construct weak Jacobi forms of weight -14, -12, -10, -8, -6, -4, -2, respectively. They have q^0-terms of the form (order: $240c_0, c_1\sum_2,\cdots,c_8\sum_{16'}$)

$$\begin{aligned}&\text{weight}\quad -16: && (b_{1,j})_{j=1}^{9}=(1,1,1,1,1,1,1,1,1)\\ &\text{weight}\quad -16+2(i-1): && b_{i,j}=\frac{27-2i-3j}{12}b_{i-1,j}\end{aligned}$$

where $2\le i\le 8$, $1\le j\le 9$. For these Jacobi forms, if we take $\mathfrak{z}=0$, their q^0-terms will be zero. We then get a system of linear equations:

$$Bx=0,\quad B=(b_{i,j})_{8\times 9},\quad x=(c_0,c_1,\cdots,c_8)^t.$$

By direct calculations, it has the unique nontrivial solution

$$x=(1,-8,28,-56,70,-56,28,-8,1).$$

We see at once that $[\varphi_{-16,4}-\phi_{-16,4}]_{q^0}=*(\sum_{8'}-\sum_{8''})$. Thus $\Delta(\varphi_{-16,4}-\phi_{-16,4})$ is a holomorphic Jacobi form of weight -4, which yields $\varphi_{-16,4}=\phi_{-16,4}$.

Applying differential operators to $\varphi_{-16,4}$, we construct the following basic weak Jacobi forms.

$$\begin{aligned}\varphi_{-14,4}=&-3H_{-16}(\varphi_{-16,4})\\ =&\sum_{16'}-2\sum_{14'}-14\sum_{12}+70\sum_{10}-28\sum_{8''}-112\sum_{8'}+154\sum_{6}-98\sum_{4}+34\sum_{2}-1200+O(q)\\ \varphi_{-12,4}=&-\frac{2}{7}H_{-14}(\varphi_{-14,4})-\frac{1}{7}E_4\varphi_{-16,4}\\ =&\sum_{14'}-4\sum_{12}+3\sum_{10}+2\sum_{8''}+8\sum_{8'}-25\sum_{6}+24\sum_{4}-11\sum_{2}+480+O(q)\in J^{\mathrm{w},W(E_8)}_{-12,E_8,4}\\ \varphi_{-10,4}=&-\frac{4}{9}H_{-12}(\varphi_{-12,4})-\frac{5}{162}(E_4\varphi_{-14,4}-E_6\varphi_{-16,4})\\ =&\sum_{12}-4\sum_{10}+\sum_{8''}+4\sum_{8'}-5\sum_{4}+4\sum_{2}-240+O(q)\in J^{\mathrm{w},W(E_8)}_{-10,E_8,4}\\ \varphi_{-8,4}=&-\frac{3}{5}H_{-10}(\varphi_{-10,4})-\frac{1}{15}E_4\varphi_{-12,4}+\frac{1}{90}E_6\varphi_{-14,4}-\frac{1}{90}E_4^2\varphi_{-16,4}\\ =&\sum_{10}-\frac{7}{10}\sum_{8''}-\frac{28}{10}\sum_{8'}+4\sum_{6}-\sum_{4}-\sum_{2}+120+O(q)\in J^{\mathrm{w},W(E_8)}_{-8,E_8,4}\end{aligned}$$

$$\begin{aligned}
\varphi_{-6,4} =& -\frac{1}{2}E_4\varphi_{-10,4} + \frac{1}{6}E_6\varphi_{-12,4} - \frac{1}{36}(E_4^2\varphi_{-14,4} - E_4E_6\varphi_{-16,4}) - 4H_{-8}(\varphi_{-8,4})\\
=& \sum_{8''} + 4\sum_{8'} - 14\sum_{6} + 12\sum_{4} - 2\sum_{2} - 240 + O(q) \in J^{\mathrm{w},W(E_8)}_{-6,E_8,4}\\
\varphi_{-4,4} =& -\frac{10}{81}E_4\varphi_{-8,4} + \frac{5}{81}E_6\varphi_{-10,4} + \frac{5}{1458}(E_4E_6\varphi_{-14,4} - E_4^3\varphi_{-16,4})\\
& - \frac{5}{243}E_4^2\varphi_{-12,4} - \frac{2}{9}H_{-6}(\varphi_{-6,4})\\
=& \sum_{6} - 2\sum_{4} + \sum_{2} + O(q) \in J^{\mathrm{w},W(E_8)}_{-4,E_8,4}\\
\varphi_{-2,4} =& -\frac{5}{9}E_6\varphi_{-8,4} + \frac{5}{18}E_4^2\varphi_{-10,4} + \frac{5}{324}(E_4^3\varphi_{-14,4} - E_4^2E_6\varphi_{-16,4})\\
& - \frac{5}{54}E_4E_6\varphi_{-12,4} + \frac{1}{6}E_4\varphi_{-6,4} + 12H_{-4}(\varphi_{-4,4})\\
=& -7\sum_{4} + 8\sum_{2} - 240 + O(q) \in J^{\mathrm{w},W(E_8)}_{-2,E_8,4}\\
\varphi_{0,4} =& H_{-2}(\varphi_{-2,4}) = 2\sum_{2} - 120 + O(q) \in J^{\mathrm{w},W(E_8)}_{0,E_8,4}
\end{aligned}$$

$$\begin{aligned}
\psi_{-8,4} =& \frac{\vartheta_{E_8}|T_-(4) - 73\vartheta_{E_8}(\tau,2\mathfrak{z})}{72\Delta} = * \sum_{\sigma\in W(E_8)}\left[-\frac{1}{\Delta}\prod_{i=1}^{8}\vartheta(\tau,2z_i)\right](\tau,\sigma(\mathfrak{z}))\\
=& \sum_{8'} - \sum_{8''} + O(q) \in J^{\mathrm{w},W(E_8)}_{-8,E_8,4}.
\end{aligned} \tag{3.4.17}$$

We now arrive at our main theorem in this subsection.

Theorem 3.4.13. *The space $J^{\mathrm{w},W(E_8)}_{*,E_8,4}$ is a free M_*-module generated by ten weak Jacobi forms. More precisely, we have*

$$J^{\mathrm{w},W(E_8)}_{*,E_8,4} = M_*\langle \varphi_{-2k,4},\ 0 \le k \le 8;\ \psi_{-8,4}\rangle.$$

Proof. It is sufficient to show that there is no any other weak Jacobi forms of weight less than -4 which are independent of $\varphi_{-2k,4}$, $0 \le k \le 8$, and $\psi_{-8,4}$. We only prove that there is no weak Jacobi forms of weight -14 independent of $\varphi_{-14,4}$ because other cases are similar. Suppose that there exists a weak Jacobi form of weight -14 which is linearly independent of $\varphi_{-14,4}$, noted by f. We can assume

$$[f]_{q^0} = 240c_0 + c_1\sum_{2} + c_2\sum_{4} + c_3\sum_{6} + c_4\sum_{8} + c_5\sum_{10} + c_6\sum_{12} + c_7\sum_{14'} \neq 0.$$

Once again, we can construct weak Jacobi forms of weight -12, -10, -8, -6, -4, -2 and 0, respectively. They have q^0-terms of the following form (order: $240c_0, c_1\sum_2, \cdots, c_7\sum_{14'}$)

$$\begin{aligned}
&\text{weight} \quad -14: && (c_{1,j})_{j=1}^{8} = (1,1,1,1,1,1,1,1,1)\\
&\text{weight} \quad -14+2(i-1): && c_{i,j} = \frac{25-2i-3j}{12}c_{i-1,j}
\end{aligned}$$

where $2 \le i \le 8$, $1 \le j \le 8$. For each Jacobi form of negative weight, if we put $\mathfrak{z} = 0$ then its q^0-term will become zero. For the Jacobi form of weight zero, we can modify $c_{8,j}$ to $(14-6j)c_{8,j}$ by Lemma 3.2.5. We then get a system of linear equations:

$$Cx = 0, \quad C = (c_{i,j})_{8\times 8}, \quad x = (c_0, c_1, \cdots, c_7)^t.$$

By direct calculations, it has only trivial solution. Therefore q^0-term of f is of the form

$$[f]_{q^0} = * \left(\sum_{8'} - \sum_{8''} \right).$$

Therefore, Δf is a holomorphic Jacobi form of weight -2 and then we have $* = 0$, which contradicts our assumption.

In the end of the proof, we explain why there is no weak Jacobi form $\psi_{-6,4}$ of weight -6 with $[\psi_{-6,4}]_{q^0} = \sum_{8'} - \sum_{8''}$. If $\psi_{-6,4}$ exists, then $\psi_{-8,4}$, $\psi_{-6,4}$ and $\varphi_{-14,4}$ are free over M_*. This contradicts the fact that $(E_6\psi_{-8,4} - E_4\psi_{-6,4})/\Delta \in J^{\mathrm{w},W(E_8)}_{-14,E_8,4}$. □

In the rest of this subsection, we study the spaces of holomorphic Jacobi forms and Jacobi cusp forms of index 4. We first construct many basic Jacobi forms.

$$A_4 = \vartheta_{E_8}(\tau, 2\mathfrak{z}) = 1 + q \sum_{8''} + O(q^2) \in J^{W(E_8)}_{4,E_8,4} \tag{3.4.18}$$

$$B_4 = \frac{1}{33} B_2 | T_-(2) + \frac{2}{55} \Delta\varphi_{-6,4} = 1 + q \left[\frac{1}{15} \sum_{8''} - \frac{28}{15} \sum_{6} - \frac{4}{15} \sum_{2} - 8 \right] + O(q^2) \tag{3.4.19}$$

$$\begin{aligned} C_{8,4} =& \frac{1}{54} \Delta (E_4^3 \varphi_{-16,4} - E_4 E_6 \varphi_{-14,4} + 6E_4^2 \varphi_{-12,4} - 18E_6 \varphi_{-10,4} + 36E_4 \varphi_{-8,4}) \\ =& q \left[\frac{1}{5} \sum_{8''} + \frac{4}{5} \sum_{8'} - 4 \sum_{6} + 6 \sum_{4} - 4 \sum_{2} + 240 \right] + O(q^2) \in J^{W(E_8)}_{8,E_8,4} \end{aligned} \tag{3.4.20}$$

$$\begin{aligned} U_{10,4} =& -\frac{5}{324} \Delta (E_4^2 E_6 \varphi_{-16,4} - E_4^3 \varphi_{-14,4} + 6E_4 E_6 \varphi_{-12,4} - 18E_4^2 \varphi_{-10,4} \\ &+ 36E_6 \varphi_{-8,4} - \frac{54}{5} E_4 \varphi_{-6,4}) - *\Delta^2 \varphi_{-14,4} \\ =& q \left[\sum_{6} - 3 \sum_{4} + 3 \sum_{2} - 240 \right] + O(q^2) \in J^{\mathrm{cusp},W(E_8)}_{10,E_8,4} \end{aligned} \tag{3.4.21}$$

$$\begin{aligned} U_{12,4} =& -\frac{5}{324} \Delta (E_4 E_6^2 \varphi_{-16,4} - E_4^2 E_6 \varphi_{-14,4} + 6E_6^2 \varphi_{-12,4} - 18E_4 E_6 \varphi_{-10,4} \\ &+ 36E_4^2 \varphi_{-8,4} - \frac{54}{5} E_6 \varphi_{-6,4}) - *\Delta^2 E_4 \varphi_{-16,4} \\ =& q \left[\sum_{6} - 3 \sum_{4} + 3 \sum_{2} - 240 \right] + O(q^2) \in J^{\mathrm{cusp},W(E_8)}_{12,E_8,4} \end{aligned} \tag{3.4.22}$$

Similar to the case of index 3, we can show the following identities

$$\dim J^{W(E_8)}_{2k,E_8,4} = \dim J^{\mathrm{w},W(E_8)}_{2k,E_8,4} - 13, \quad k \geq 2 \tag{3.4.23}$$

$$\dim J^{\mathrm{cusp},W(E_8)}_{2k,E_8,4} = \dim J^{W(E_8)}_{2k,E_8,4} - 4, \quad k \geq 4, \tag{3.4.24}$$

and use them to prove the next theorem.

Theorem 3.4.14.

1. *The space* $J^{W(E_8)}_{*,E_8,4}$ *is a free* M_**-module generated by the following ten holomorphic Jacobi forms*

weight 4 :	A_4	$\Delta\psi_{-8,4}$	
weight 6 :	B_4	$\Delta\varphi_{-6,4}$	
weight 8 :	$C_{8,4}$	$\Delta\varphi_{-4,4}$	$\Delta^2\varphi_{-16,4}$
weight 10 :	$\Delta\varphi_{-2,4}$	$\Delta^2\varphi_{-14,4}$	
weight 12 :	$\Delta^2\varphi_{-12,4}$		

2. *The space* $J^{\mathrm{cusp},W(E_8)}_{*,E_8,4}$ *is a free* M_**-module generated by the following ten Jacobi cusp forms*

weight 8 :	$\Delta\varphi_{-4,4} - *\Delta^2\varphi_{-16,4}$		
weight 10 :	$\Delta\varphi_{-2,4} - *\Delta^2\varphi_{-14,4}$	$U_{10,4}$	
weight 12 :	$\Delta\varphi_{0,4} - *\Delta^2 E_4\varphi_{-16,4}$	$U_{12,4}$	$\Delta^2\varphi_{-12,4}$
weight 14 :	$\Delta^2(E_4\varphi_{-14,4} - E_6\varphi_{-16,4})$	$\Delta^2\varphi_{-10,4}$	
weight 16 :	$\Delta^2\varphi_{-8,4}$	$\Delta^2\psi_{-8,4}$	

Remark that these constants $*$ are chosen to cancel the term $q^2\sum_{16'}$ in the above constructions of Jacobi cusp forms.

3.4.5 Isomorphisms between spaces of Jacobi forms

In this subsection we use the embeddings of lattices to build two isomorphisms between the spaces of certain Jacobi forms. We denote by $J^{\mathrm{w},\mathrm{O}(L)}_{k,L,t}$ the space of weak Jacobi forms of weight k and index t for the lattice L which are invariant under the action of the integral orthogonal group $\mathrm{O}(L)$.

We first consider the case of index 2. Recall that the Nikulin's lattice is defined as (see [GN18, Example 4.3])

$$N_8 = \langle 8A_1, h = (a_1 + \cdots + a_8)/2\rangle \cong D_8^\vee(2),$$

where $(a_i, a_j) = 2\delta_{ij}$, $(h,h) = 4$. It is easy to check that N_8 is a sublattice of E_8. We then have

$$N_8 < E_8 \Rightarrow E_8 < N_8^\vee \Rightarrow E_8(2) < N_8^\vee(2) \cong D_8.$$

Thus, we arrive at the following isomorphism.

Proposition 3.4.15. *The natural map*

$$\begin{aligned} J^{\mathrm{w},\mathrm{O}(D_8)}_{k,D_8,1} &\longrightarrow J^{\mathrm{w},W(E_8)}_{k,E_8,2} \\ \phi(\tau,Z) &\longmapsto \frac{1}{|W(E_8)|}\sum_{\sigma\in W(E_8)} \widehat{\phi}(\tau,\sigma(\mathfrak{z})) \end{aligned}$$

is a M_**-modules isomorphism. Here,* $Z = \sum_{i=1}^8 z_i e_i$*, the set* $\{e_i : 1 \le i \le 8\}$ *is the standard basis of* $\mathbb{R}^8$*, and* $\widehat{\phi}(\tau,\mathfrak{z})$ *is defined as*

$$\widehat{\phi}(\tau,\mathfrak{z}) = \phi(\tau, z_1+z_2, z_1-z_2, z_3+z_4, z_3-z_4, z_5+z_6, z_5-z_6, z_7+z_8, z_7-z_8).$$

Proof. Under the discriminant groups, the $\mathrm{O}(D_8)$-orbit of $(\frac{1}{2},...,\frac{1}{2})$ corresponds to $\sum_4$. The $\mathrm{O}(D_8)$-orbit of $(1,0,...,0)$ corresponds to $\sum_2$. From this, we assert that the above map is injective by comparing q^0-terms of Jacobi forms. The space $J^{\mathrm{w},\mathrm{O}(D_8)}_{*,D_8,*}$ is in fact the space the Weyl invariant weak Jacobi forms for the root system C_8. By Table 3.1, there exist weak Jacobi forms $\phi_{-4,D_8,1}$, $\phi_{-2,D_8,1}$, $\phi_{0,D_8,1}$. Thus, by Theorem 3.4.7, we prove the surjectivity of the above map. □

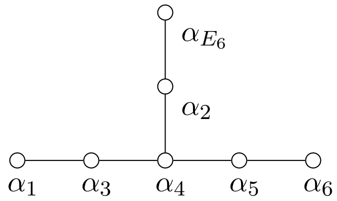

Figure 3.2: Extended Coxeter-Dynkin diagram of E_6

Remark that
$$N_8 < 2D_4 < E_8 \Rightarrow E_8(2) < 2D_4^\vee(2) = 2D_4 < D_8$$
and the induced map
$$J^{\mathrm{w},\mathrm{O}(2D_4)}_{k,2D_4,1} \longrightarrow J^{\mathrm{w},W(E_8)}_{k,E_8,2}$$
is also a M_*-modules isomorphism.

We next consider the case of index 3. We see from the extended Coxeter-Dynkin diagram of E_8 (see Figure 3.1) that $A_2 \oplus E_6$ is a sublattice of E_8. Observing the extended Coxeter-Dynkin diagram of E_6 (see Figure 3.2), we find that $3A_2$ is a sublattice of E_6. Then we have
$$4A_2 < E_8 \Rightarrow E_8 < 4A_2^\vee \Rightarrow E_8(3) < 4A_2^\vee(3) \cong 4A_2.$$

In a similar way, we can prove the following result.

Proposition 3.4.16. *The natural map*
$$\begin{aligned} J^{\mathrm{w},\mathrm{O}(4A_2)}_{k,4A_2,1} &\longrightarrow J^{\mathrm{w},W(E_8)}_{k,E_8,3} \\ \varphi(\tau,Z) &\longmapsto \frac{1}{|W(E_8)|}\sum_{\sigma\in W(E_8)} \widetilde{\varphi}(\tau,\sigma(\mathfrak{z})) \end{aligned}$$
is a M_*-modules isomorphism. Here, we fix the standard model of A_2
$$A_2 = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2,\ (\beta_i,\beta_i) = 2,\ i = 1,2,\ (\beta_1,\beta_2) = 1,$$
and $\widetilde{\varphi}(\tau,\mathfrak{z})$ is defined as
$$\begin{aligned} \widetilde{\varphi}(\tau,\mathfrak{z}) =& \phi(\tau, z_6 - z_7, z_7 + z_8, (z_1 - z_2 - z_3 - z_4 - z_5 - z_6 - z_7 + z_8)/2, -z_1 + z_2, \\ & - z_1 - z_2, (z_1 + z_2 + z_3 + z_4 + z_5 - z_6 - z_7 + z_8)/2, -z_3 + z_4, -z_4 + z_5). \end{aligned}$$

It is known that $J^{\mathrm{w},\mathrm{O}(A_2)}_{*,A_2,1}$ is generated by $a_{0,1}$ and $a_{-2,1}$ over M_*, here $a_{0,1} \in J^{\mathrm{w},\mathrm{O}(A_2)}_{0,A_2,1}$ and $a_{-2,1} \in J^{\mathrm{w},\mathrm{O}(A_2)}_{-2,A_2,1}$ (see Table 3.1). By the two functions, we can construct
$$\begin{aligned} \varphi_{0,4A_2,1} &= a_{0,1}\otimes a_{0,1}\otimes a_{0,1}\otimes a_{0,1} \in J^{\mathrm{w},\mathrm{O}(4A_2)}_{0,4A_2,1}, \\ \varphi_{-2,4A_2,1} &= \sum a_{-2,1}\otimes a_{0,1}\otimes a_{0,1}\otimes a_{0,1} \in J^{\mathrm{w},\mathrm{O}(4A_2)}_{-2,4A_2,1}, \\ \varphi_{-4,4A_2,1} &= \sum a_{-2,1}\otimes a_{-2,1}\otimes a_{0,1}\otimes a_{0,1} \in J^{\mathrm{w},\mathrm{O}(4A_2)}_{-4,4A_2,1}, \\ \varphi_{-6,4A_2,1} &= \sum a_{-2,1}\otimes a_{-2,1}\otimes a_{-2,1}\otimes a_{0,1} \in J^{\mathrm{w},\mathrm{O}(4A_2)}_{-6,4A_2,1}, \\ \varphi_{-8,4A_2,1} &= a_{-2,1}\otimes a_{-2,1}\otimes a_{-2,1}\otimes a_{-2,1} \in J^{\mathrm{w},\mathrm{O}(4A_2)}_{-8,4A_2,1}, \end{aligned}$$
here the sums take over all permutations of 4 copies of A_2. Then we conclude from the above isomorphism that $J^{\mathrm{w},\mathrm{O}(4A_2)}_{*,4A_2,1}$ is generated by $\varphi_{2j,4A_2,1}$, $0 \le j \le 4$, over M_*. Moreover, the image of $\varphi_{-8,4A_2,1}$ gives a new construction of our generator of index 3 i.e. $\varphi_{-8,3}$.

We can also consider the case of index 5. It is easy to see $2A_4 < E_8$, which yields $E_8(5) < 2A_4^\vee(5)$. Unfortunately, the induced map

$$J^{\mathrm{w},\mathrm{O}(2A_4)}_{k,2A_4^\vee(5),1} \longrightarrow J^{\mathrm{w},W(E_8)}_{k,E_8,5} \tag{3.4.25}$$

is not an isomorphism. In fact, for $\phi \in J^{\mathrm{w},\mathrm{O}(2A_4)}_{k,2A_4^\vee(5),1}$, the function $\Delta^2\phi$ is always a holomorphic Jacobi form. But, for $\psi \in J^{\mathrm{w},W(E_8)}_{k,E_8,5}$, the function $\Delta^2\psi$ is not a holomorphic Jacobi form in general. It is becauce that the lattice $A_4^\vee(5)$ satisfies the Norm_2 condition in (2.3.2) (see [GW17, Lemma 2]).

3.4.6 Pull-backs of Jacobi forms

We have seen from §3.4.3 and §3.4.4 that our approach based on differential operators works well when the absolute value of minimal weight equals the maximal norm of Weyl orbits appearing in q^0-terms of Jacobi forms. But when the index is larger than 4, the absolute value of minimal weight will be less than the maximal norm of Weyl orbits, which causes our previous approach to not work well because there are not enough linear equations in this case. In order to study $W(E_8)$-invariant Jacobi forms of index larger than 4, we introduce a new approach relying on pull-backs of Jacobi forms.

For convenience, we first recall some results on classical Jacobi forms introduced by Eichler and Zagier in [EZ85]. Let $J^{\mathrm{w}}_{2k,t}$ be the space of weak Jacobi forms of weight $2k$ and index t. It is well-known that the bigraded ring of weak Jacobi forms of even weight and integral index is a polynomial algebra over M_* generated by two basic weak Jacobi forms ($\zeta = e^{2\pi i z}$)

$$\phi_{-2,1}(\tau,z) = \zeta + \zeta^{-1} - 2 + O(q) \in J^{\mathrm{w}}_{-2,1},$$
$$\phi_{0,1}(\tau,z) = \zeta + \zeta^{-1} + 10 + O(q) \in J^{\mathrm{w}}_{0,1}.$$

Let $\phi \in J^{\mathrm{w},W(E_8)}_{k,E_8,t}$ and v_4 be a vector of norm 4 in E_8. Then the function $\phi(\tau, zv_4)$ is a weak Jacobi form of weight k and index $2t$. In order to compute the Fourier coefficients of $\phi(\tau,zv_4)$, we consider the pull-backs of Weyl orbits. Let $\sum_v$ be a Weyl orbit associated to v defined in §3.4.1. Recall that

$$\sum_v = \frac{240}{|W(E_8)|}\sum_{\sigma\in W(E_8)} \exp(2\pi i(\sigma(v),\mathfrak{z})).$$

Since the Weyl group $W(E_8)$ acts transitively on the set R_4 of vectors of norm 4 in E_8 (see Lemma 3.4.1), we have

$$\begin{aligned}\sum_v(zv_4) &= \frac{240}{|W(E_8)|}\sum_{\sigma\in W(E_8)}\exp(2\pi i(\sigma(v),v_4)z)\\ &= \frac{240}{|W(E_8)|}\sum_{\sigma\in W(E_8)}\exp(2\pi i(v,\sigma(v_4))z)\\ &= \frac{240}{|R_4|}\sum_{l\in R_4}\exp(2\pi i(v,l)z).\end{aligned}$$

In view of this fact, we define

$$\max(\sum_v, v_4) := \max(v,R_4) = \max\{(v,l) : l\in R_4\}. \tag{3.4.26}$$

It is easy to check that

$$\begin{array}{llll} \max(w_1,R_4)=4 & \max(w_2,R_4)=5 & \max(w_3,R_4)=7 & \max(w_4,R_4)=10\\ \max(w_5,R_4)=8 & \max(w_6,R_4)=6 & \max(w_7,R_4)=4 & \max(w_8,R_4)=2 \end{array}$$

and the maximal value can be obtained at $l = (0,0,0,0,0,0,0,2)$. Thus, we get

$$\begin{array}{llll}
\max(\sum_{2}, v_4) = 2 & \max(\sum_{4}, v_4) = 4 & \max(\sum_{6}, v_4) = 4 & \max(\sum_{8'}, v_4) = 5 \\
\max(\sum_{8''}, v_4) = 4 & \max(\sum_{10}, v_4) = 6 & \max(\sum_{12}, v_4) = 6 & \max(\sum_{14'}, v_4) = 7 \\
\max(\sum_{14''}, v_4) = 6 & \max(\sum_{16'}, v_4) = 8 & \max(\sum_{16''}, v_4) = 7 & \max(\sum_{18'}, v_4) = 8 \\
\max(\sum_{18''}, v_4) = 6 & \max(\sum_{20'}, v_4) = 8 & \max(\sum_{20''}, v_4) = 8 & \max(\sum_{22'}, v_4) = 9 \\
\max(\sum_{22''}, v_4) = 8 & \max(\sum_{24'}, v_4) = 8 & \max(\sum_{24''}, v_4) = 9 & \max(\sum_{26'}, v_4) = 10 \\
\max(\sum_{26''}, v_4) = 9 & \max(\sum_{28'}, v_4) = 10 & \max(\sum_{30'}, v_4) = 10 & \max(\sum_{32'}, v_4) = 11 \\
\max(\sum_{32''}, v_4) = 10 & \max(\sum_{36'}, v_4) = 12. & &
\end{array}$$

This new approach can be used to recover some cases of index 3 and 4.

Index 3: Assume that $\phi = \sum_{8'} + \cdots + O(q) \in J^{\mathrm{w},W(E_8)}_{-2k,E_8,3}$, where $\cdots$ stands for the Weyl orbits of norm less than 8. Then we have

$$\phi(\tau, zv_4) = \zeta^{\pm 5} + \cdots + O(q) \in J^{\mathrm{w}}_{-2k,6},$$

here $\cdots$ stands for the terms of type $\zeta^{\pm i}$ with $0 \leq i \leq 4$. Note that in the above equation the term $\zeta^{\pm 5}$ may have positive coefficient different from 1. But this does not affect our discussion. Thus, we always omit this type of coefficient hereafter. We claim that $-2k \geq -8$. If $-2k < -8$ i.e. $k > 4$, then $J^{\mathrm{w}}_{-2k,6} = \phi^k_{-2,1} \cdot J^{\mathrm{w}}_{0,6-k}$. Since $J^{\mathrm{w}}_{-2k,6} \neq \{0\}$, we have $k \leq 6$. But $J^{\mathrm{w}}_{-10,6}$ is generated by $\phi^5_{-2,1}\phi_{0,1} = \zeta^{\pm 6} + \cdots$ and $J^{\mathrm{w}}_{-12,6}$ is generated by $\phi^6_{-2,1} = \zeta^{\pm 6} + \cdots$. This contradicts the Fourier expansion of $\phi(\tau, zv_4)$.

Assume that $\phi \in J^{\mathrm{w},W(E_8)}_{-2k,E_8,3}$ has no Fourier coefficient $\sum_{8'}$ in its q^0-term. By Lemma 3.4.4, we have $\Delta\phi \in J^{W(E_8)}_{12-2k,E_8,3}$. Thus we get $12 - 2k > 4$ i.e. $-2k > -8$.

Conclusion: The possible minimum weight in this case is ≥ -8 and the dimension of $J^{\mathrm{w},W(E_8)}_{-8,E_8,3}$ is at most one.

Index 4: Assume that $\phi = \sum_{16'} + \cdots + O(q) \in J^{\mathrm{w},W(E_8)}_{-2k,E_8,4}$ with $k > 0$. Then we have

$$\phi(\tau, zv_4) = \zeta^{\pm 8} + \cdots + O(q) \in J^{\mathrm{w}}_{-2k,8}.$$

Since $J^{\mathrm{w}}_{-2k,8} = \phi^k_{-2,1} \cdot J^{\mathrm{w}}_{0,8-k}$, we have $8 - k \geq 0$ i.e. $-2k \geq -16$.

Assume that $\phi \in J^{\mathrm{w},W(E_8)}_{-2k,E_8,4}$ has no Fourier coefficient $\sum_{16'}$ in its q^0-term. By Lemma 3.2.1 and Lemma 3.4.2, the function $\eta^{42}\phi$ is a $W(E_8)$-invariant holomorphic Jacobi form of weight $21 - 2k$ and index 4 with a character. In view of the singular weight, we get $21 - 2k \geq 4$ i.e. $-2k \geq -16$.

Conclusion: The possible minimum weight in this case is ≥ -16.

Assume that $\phi = \sum_{14'} + \cdots + O(q) \in J^{\mathrm{w},W(E_8)}_{-2k,E_8,4}$ with $k > 0$. Then we have

$$\phi(\tau, zv_4) = \zeta^{\pm 7} + \cdots + O(q) \in J^{\mathrm{w}}_{-2k,8}.$$

Since $J^{\mathrm{w}}_{-2k,8} = \phi^k_{-2,1} \cdot J^{\mathrm{w}}_{0,8-k}$, the spaces $J^{\mathrm{w}}_{-16,8}$ and $J^{\mathrm{w}}_{-14,8}$ are all generated by one function with leading Fourier coefficient $\zeta^{\pm 8}$. Thus $k \leq 6$ i.e. $-2k \geq -12$.

Assume that $\phi \in J^{\mathrm{w},W(E_8)}_{-2k,E_8,4}$ has no Fourier coefficients $\sum_{16'}$ and $\sum_{14'}$ in its q^0-term. By Lemma 3.2.1 and Lemma 3.4.2, the function $\eta^{36}\phi$ is a $W(E_8)$-invariant holomorphic Jacobi form of weight $18-2k$ and index 4 with a character. In view of the singular weight, we get $18-2k \geq 4$ i.e. $-2k \geq -14$.

Conclusion: The dimension of $J^{\mathrm{w},W(E_8)}_{-16,E_8,4}$ is at most one.

3.4.7 The case of index 5

In this subsection we use the approach in §3.4.6 to dertermine the possible minimum weight of the generators of $J^{\mathrm{w},W(E_8)}_{*,E_8,5}$.

(I) Assume that $\phi = \sum_{22'} + \cdots + O(q) \in J^{\mathrm{w},W(E_8)}_{-2k,E_8,5}$ with $k>0$. Then we have

$$\phi(\tau, zv_4) = \zeta^{\pm 9} + \cdots + O(q) \in J^{\mathrm{w}}_{-2k,10}.$$

Since $J^{\mathrm{w}}_{-2k,10} = \phi^k_{-2,1}\cdot J^{\mathrm{w}}_{0,10-k}$, we have $10-k \geq 0$. But when $k=9$ or 10, the spaces $J^{\mathrm{w}}_{-20,10}$ and $J^{\mathrm{w}}_{-18,10}$ are all generated by one function with leading Fourier coefficient $\zeta^{\pm 10}$, which contradicts the Fourier expansion of $\phi(\tau, zv_4)$. Therefore, we get $k \leq 8$ i.e. $-2k \geq -16$.

(II) Assume that $\phi \in J^{\mathrm{w},W(E_8)}_{-2k,E_8,5}$ has no Fourier coefficient $\sum_{22'}$ in its q^0-term. By Lemma 3.4.4, the function $\Delta^2\phi \in J^{W(E_8)}_{24-2k,E_8,5}$. By Lemma 3.4.5, we have $24-2k \geq 6$ i.e. $-2k \geq -18$.

(III) Assume that $\phi \in J^{\mathrm{w},W(E_8)}_{-2k,E_8,5}$ has no Fourier coefficients $\sum_{22'}$ and $\sum_{20'}$ in its q^0-term. By Lemma 3.2.1 and Lemma 3.4.2, the function $\eta^{44}\phi$ is a $W(E_8)$-invariant Jacobi cusp form of weight $22-2k$ and index 5 with a character. From the singular weight, it follows that $22-2k > 4$ i.e. $-2k \geq -16$.

(IV) Assume that $\phi \in J^{\mathrm{w},W(E_8)}_{-2k,E_8,5}$ has no Fourier coefficients $\sum_{22'}$, $\sum_{20'}$ and $\sum_{18'}$ in its q^0-term. By Lemma 3.2.1 and Lemma 3.4.2, the function $\eta^{40}\phi$ is a $W(E_8)$-invariant Jacobi cusp form of weight $20-2k$ and index 5 with a character. It follows that $20-2k>4$ i.e. $-2k \geq -14$.

By the discussions above, we get the following result.

Proposition 3.4.17.

$$\begin{aligned}
&\dim J^{\mathrm{w},W(E_8)}_{-2k,E_8,5} = 0, \quad \textit{if} \quad -2k \leq -20,\\
&\dim J^{\mathrm{w},W(E_8)}_{-18,E_8,5} \leq 1,\\
&\dim J^{\mathrm{w},W(E_8)}_{-16,E_8,5} \leq 3.
\end{aligned}$$

Moreover, if the $W(E_8)$-invariant weak Jacobi form of weight -18 and index 5 exists, then its q^0-term has no Fourier coefficient $\sum_{22'}$ but must contain Fourier coefficient $\sum_{20'}$.

We do not know if the $W(E_8)$-invariant weak Jacobi form of weight -18 and index 5 exists. But the $W(E_8)$-invariant weak Jacobi forms of weight -16 and index 5 do indeed exist. We next show how to construct one such Jacobi form.

We first construct a weak Jacobi form of weight -8 and index 1 for $A_4^\vee(5)$ invariant under the integral orthogonal group. It is known that

$$M_2\left(\Gamma_0(5), \left(\frac{\cdot}{5}\right)\right) = \mathbb{C}\eta^5(\tau)/\eta(5\tau) + \mathbb{C}\eta^5(5\tau)/\eta(\tau).$$

Using Proposition 1.4.9, we can construct two independent holomorphic Jacobi forms f_1 and f_2 of weight 4 from the space $M_2(\Gamma_0(5), (\frac{\cdot}{5}))$. Then the function $(*f_1 - *f_2)/\Delta$ will be a weak Jacobi form of weight -8 for the lattice $A_4^\vee(5)$.

The tensor product of the above weak Jacobi form of weight -8 defines a weak Jacobi form of weight -16 and index 1 for $2A_4^\vee(5)$, whose image under map (3.4.25) gives a $W(E_8)$-invariant weak Jacobi form of weight -16 and index 5 with the following q^0-term

$$\begin{aligned}\varphi_{-16,5} =&240-6\sum_2+13\sum_4-8\sum_6-14\sum_8+28\sum_{10}-14\sum_{12}\\ &-8\sum_{14}+13\sum_{16}-6\sum_{18'}+\sum_{20'},\end{aligned}\tag{3.4.27}$$

here and subquently, we use the following notations

$$c_j{}'\sum_{2j'}+c_j{}''\sum_{2j''}=(c_j{}'+c_j{}'')\sum_{2j}=c_j\sum_{2j},\quad j=4,7,8,9,10,11,12,13.$$

In general, we only know the coefficients c_j and it is hard to calculate the exact values of c_j' and $c_j{}''$. Thus, the above notations are convenient for us.

3.4.8 The case of index 6

In this subsection we discuss the possible minimum weight of the generators of $J^{\mathrm{w},W(E_8)}_{*,E_8,6}$.

(I) Assume that $\phi=\sum_{36'}+\cdots+O(q)\in J^{\mathrm{w},W(E_8)}_{-2k,E_8,6}$ with $k>0$. Then we have

$$\phi(\tau,zv_4)=\zeta^{\pm 12}+\cdots+O(q)\in J^{\mathrm{w}}_{-2k,12}.$$

From $J^{\mathrm{w}}_{-2k,12}=\phi_{-2,1}^k\cdot J^{\mathrm{w}}_{0,12-k}$, we obtain $12-k\geq 0$ i.e. $-2k\geq -24$.

(II) Assume that $\phi=\sum_{32'}+*\sum_{32''}+\cdots+O(q)\in J^{\mathrm{w},W(E_8)}_{-2k,E_8,6}$ with $k>0$. Similarly, we have

$$\phi(\tau,zv_4)=\zeta^{\pm 11}+\cdots+O(q)\in J^{\mathrm{w}}_{-2k,12},$$

and then $k\leq 12$. But when $k=11$ or 12, the space $J^{\mathrm{w}}_{-2k,12}$ is generated by one function with leading q^0-term $\zeta^{\pm 12}$, which gives a contradiction. Thus, $k\leq 10$ i.e. $-2k\geq -20$.

(III) Assume that $\phi\in J^{\mathrm{w},W(E_8)}_{-2k,E_8,6}$ has no Fourier coefficient $\sum_{36'}$ in its q^0-term. Similarly, the function $\eta^{64}\phi$ is a $W(E_8)$-invariant holomorphic Jacobi form of weight $32-2k$ and index 6 with a character. Hence, $32-2k\geq 4$. But when $2k=28$, the Jacobi form $\eta^{64}\phi$ has singular weight, which yields that $\eta^{64}\phi$ has a Fourier expansion of the form

$$q^{8/3}\left(\sum_{32'}-\sum_{32''}\right)+O(q^{11/3}).$$

This contradicts the above argument **(II)**. Thus, we have $-2k\geq -26$.

(IV) Assume that $\phi\in J^{\mathrm{w},W(E_8)}_{-2k,E_8,6}$ has no Fourier coefficients $\sum_{36'}$, $\sum_{32'}$ and $\sum_{32''}$ in its q^0-term. Then the function $\eta^{60}\phi$ is a $W(E_8)$-invariant holomorphic Jacobi form of weight $30-2k$ and index 6 with a character. Hence, $30-2k\geq 4$ i.e. $-2k\geq -26$. Similarly, when $2k=26$, the Jacobi form $\eta^{60}\phi$ has singular weight, which implies that $\eta^{60}\phi$ has a Fourier expansion of the form

$$q^{5/2}\sum_{30'}+O(q^{7/2}).$$

This is impossible because $\phi(\tau,0)=0$. Thus, we have $-2k\geq -24$.

(V) Assume that $\phi\in J^{\mathrm{w},W(E_8)}_{-2k,E_8,6}$ has no Fourier coefficients $\sum_{36'}$, $\sum_{32'}$, $\sum_{32''}$ and $\sum_{30'}$ in its q^0-term. Then the function $\eta^{56}\phi$ is a $W(E_8)$-invariant holomorphic Jacobi form of weight $28-2k$ and index 6 with a character. Hence, $-2k\geq -24$. Similarly, when $2k=24$, the Jacobi form $\eta^{56}\phi$ has singular weight, which forces that $\eta^{56}\phi$ has a Fourier expansion of the form

$$q^{7/3}\sum_{28'}+O(q^{10/3}).$$

This is impossible because $\phi(\tau, 0) = 0$. Thus, we have $-2k \geq -22$.

Combining the above arguments together, we have

Proposition 3.4.18.

$$\begin{aligned}
&\dim J^{\mathrm{w},W(E_8)}_{-2k,E_8,6} = 0, \quad \textit{if} \quad -2k \leq -28,\\
&\dim J^{\mathrm{w},W(E_8)}_{-26,E_8,6} \leq 1,\\
&\dim J^{\mathrm{w},W(E_8)}_{-24,E_8,6} \leq 3.
\end{aligned}$$

Moreover, if the $W(E_8)$-invariant weak Jacobi form of weight -26 and index 6 exists, then its q^0-term has no Fourier coefficients $\sum_{36'}$ and $\sum_{32'}$ but must contain Fourier coefficient $\sum_{32''}$.

It is easy to continue the above discussions and prove that

$$\dim J^{\mathrm{w},W(E_8)}_{-22,E_8,6} \leq 4, \qquad \dim J^{\mathrm{w},W(E_8)}_{-20,E_8,6} \leq 6.$$

We next construct two independent $W(E_8)$-invariant weak Jacobi forms of weight -24 and index 6.

From the embeddings of lattices $4A_2 < E_8$ and $2D_4 < E_8$, we see

$$E_8(6) < 4A_2(2), \qquad E_8(6) < 2D_4(3).$$

By Table 3.1, there exist an $\mathrm{O}(A_2)$-invariant weak Jacobi form $\phi_{-6,A_2,2}$ of weight -6 and index 2 for the lattice A_2 and an $\mathrm{O}(D_4)$-invariant weak Jacobi form $\phi_{-12,D_4,3}$ of weight -12 and index 3 for the lattice D_4. The function $\phi_{-6,A_2,2}$ can be constructed as

$$\frac{\vartheta^2(\tau,z_1)\vartheta^2(\tau,z_1-z_2)\vartheta^2(\tau,z_1)}{\eta^{18}(\tau)}$$

and its q^0-term is rather simple. As a tensor product of $\phi_{-6,A_2,2}$, we can construct a $W(E_8)$-invariant weak Jacobi form of weight -24 and index 6:

$$\begin{aligned}
\varphi_{-24,6} =&240 - 8\sum_2 + 24\sum_4 - 24\sum_6 - 36\sum_8 + 120\sum_{10} - 88\sum_{12} - 88\sum_{14} + 198\sum_{16}\\
&- 88\sum_{18} - 88\sum_{20} + 120\sum_{22} - 36\sum_{24} - 24\sum_{26} + 24\sum_{28'} - 8\sum_{30'} + \sum_{32''}.
\end{aligned} \tag{3.4.28}$$

The function $\phi_{-12,D_4,3}$ is a linear combination of $\phi^3_{-4,D_4,1}$ and $\phi_{-4,D_4,1}\psi^2_{-4,D_4,1}$, where $\phi_{-4,D_4,1}$ is the generator of $J^{\mathrm{w},W(D_4)}_{-4,D_4,1}$ invariant under the odd sign change (i.e. $z_1 \mapsto -z_1$) and $\psi_{-4,D_4,1}$ is the generator of $J^{\mathrm{w},W(D_4)}_{-4,D_4,1}$ anti-invariant under the odd sign change. We refer to [Ber99] for their constructions. The q^0-term of $\phi_{-12,D_4,3}$ is quite complicated but it is easy to see that it contains only one $\mathrm{O}(D_4)$-orbit of vectors of norm 18. Thus, the tensor product of $\phi_{-12,D_4,3}$ gives a $W(E_8)$-invariant weak Jacobi form of weight -24 and index 6 with leading Fourier coefficient $\sum_{36'}$ in its q^0-term. We note this function by $\psi_{-24,6}$.

If the unqiue $W(E_8)$-invariant weak Jacobi form $\varphi_{-26,6}$ of weight -26 and index 6 exists, then it is possible to prove that the above inequalities of dimensions will be equalities and the generators can be constructed by applying the differential operators to $\varphi_{-26,6}$, $\varphi_{-24,6}$ and $\psi_{-24,6}$. In addition, the function $\varphi_{-4,2}\varphi_{-16,4}$ should be also a generator of weight -20.

3.4.9 Further remarks

We close this section with the remarks below.

Remark 3.4.19. A holomorphic function is called a E_8 Jacobi form if it only satisfies the last three conditions in Definition 3.1.1. The space of $W(E_8)$-invariant Jacobi forms is much smaller than the space of E_8 Jacobi forms. By [FM07], the dimension of the space of E_8 holomorphic Jacobi forms of weight 4 and index 2 is 51. It follows that the dimension of the space of E_8 weak Jacobi forms of weight -8 and index 2 is 50. One such function is

$$\frac{1}{\Delta(\tau)}\prod_{i=1}^{4}\vartheta(\tau,z_{2i-1}+z_{2i})\vartheta(\tau,z_{2i-1}-z_{2i}).$$

But we have proved that there is no non-zero $W(E_8)$-invariant weak Jacobi form of weight -8 and index 2. We refer to [EK94] for the dimensional formulas of the spaces of E_8 Jacobi forms of large weights.

Remark 3.4.20. The methods described in this paper would be useful to study the ring of Jacobi forms for other lattices. For example, using similar methods, we can show that the space $J^{\mathrm{w},\mathrm{O}(A_4)}_{*,A_4^\vee(5),1}$ is a free module over M_* generated by six weak Jacobi forms of weights -8, -6, -4, -4, -2 and 0, here $J^{\mathrm{w},\mathrm{O}(A_4)}_{*,A_4^\vee(5),1}$ is the space of weak Jacobi forms of index 1 for the lattice $A_4^\vee(5)$ which are invariant under the integral orthogonal group of $A_4^\vee(5)$.

Remark 3.4.21. One question still unanswered in this paper is whether the bigraded ring of $W(E_8)$-invariant weak Jacobi forms is finitely generated over M_*. This question is at present far from being solved. If this ring is finitely generated, then it will contain a lot of generators (more than 20) and there are plenty of algebraic relations among generators. Moreover, there might be generators of index larger than 6 because the generator of degree 30 of the ring of $W(E_8)$-invariant polynomials does not appear in the leading term of Taylor expansion of any $W(E_8)$-invariant weak Jacobi form of index ≤ 6

3.5 Modular forms in ten variables

In this section we investigate modular forms with respect to the orthogonal group $\mathrm{O}^+(2U\oplus E_8(-1))$. In [HU14], Hashimoto and Ueda proved that the graded ring of modular forms with respect to $\mathrm{O}^+(2U\oplus E_8(-1))$ is a polynomial ring in modular forms of weights 4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42. The dimension of the space of modular forms of fixed weight can be computed by their results. In [DKW18], the authors showed that one can choose the additive liftings of Jacobi-Eisenstein series as generators. We next give an upper bound of $\dim M_k(\mathrm{O}^+(2U\oplus E_8))$ based on our theory of $W(E_8)$-invariant Jacobi forms.

Let F be a modular form of weight k with respect to $\mathrm{O}^+(2U\oplus E_8(-1))$ with trivial character and $\mathfrak{z}=(z_1,\cdots,z_8)\in E_8\otimes\mathbb{C}$. We write

$$\begin{aligned}F(\tau,\mathfrak{z},\omega)&=\sum_{\substack{n,m\in\mathbb{N},\ell\in E_8\\ 2nm-(\ell,\ell)\geq 0}}a(n,\ell,m)\exp(2\pi i(n\tau+(\ell,\mathfrak{z})+m\omega))\\&=\sum_{m=0}^{\infty}f_m(\tau,\mathfrak{z})p^m,\end{aligned}$$

where $p=\exp(2\pi i\omega)$. Then $f_m(\tau,\mathfrak{z})$ is a $W(E_8)$-invariant holomorphic Jacobi form of weight k and index m and we have

$$a(n,\ell,m)=a(m,\ell,n),\quad\forall(n,\ell,m)\in\mathbb{N}\oplus E_8\oplus\mathbb{N}.$$

For $r \in \mathbb{N}$, we set

$$M_k(\mathrm{O}^+(2,10))(p^r) = \{F \in M_k(\mathrm{O}^+(2U \oplus E_8(-1))) : f_m = 0,\ m < r\}$$

and

$$J_{k,E_8,m}^{W(E_8)}(q^r) = \{f \in J_{k,E_8,m}^{W(E_8)} : f(\tau,\mathfrak{z}) = O(q^r)\}.$$

We then have the following exact sequence

$$0 \longrightarrow M_k(\mathrm{O}^+(2,10))(p^{r+1}) \longrightarrow M_k(\mathrm{O}^+(2,10))(p^r) \xrightarrow{P_r} J_{k,E_8,r}^{W(E_8)}(q^r),$$

where $r \geq 0$ and P_r maps F to f_r. From this, we obtain the following estimation

$$\dim M_k(\mathrm{O}^+(2,10))(p^r) - \dim M_k(\mathrm{O}^+(2,10))(p^{r+1}) \leq \dim J_{k,E_8,r}^{W(E_8)}(q^r).$$

It is clear that $J_{k,E_8,r}^{W(E_8)}(q^r) = \{0\}$ for sufficiently large r, and then $M_k(\mathrm{O}^+(2,10))(p^r) = M_k(\mathrm{O}^+(2,10))(p^{r+1}) = \cdots = \{0\}$. We thus deduce

$$\dim M_k(\mathrm{O}^+(2U \oplus E_8)) \leq \sum_{r=0}^{\infty} \dim J_{k,E_8,r}^{W(E_8)}(q^r) \leq \sum_{r=0}^{\infty} \dim J_{k-12r,E_8,r}^{\mathrm{w},W(E_8)}. \tag{3.5.1}$$

We use the pull-back from $W(E_8)$-invariant Jacobi forms to $W(E_7)$-invariant Jacobi forms to improve the inequality (3.5.1).

Proposition 3.5.1. *Let $E_7 = \{v \in E_8 : (v, w_8) = 0\}$. We have the following homomorphism*

$$\begin{aligned} \Phi :\ J_{2k,E_8,m}^{\mathrm{w},W(E_8)} &\longrightarrow J_{2k,E_7,m}^{\mathrm{w},W(E_7)}, \\ f(\tau,\mathfrak{z}) &\longmapsto f(\tau, z_1, \cdots, z_6, z_7, -z_7). \end{aligned}$$

If $\Phi(f) = 0$, then we have $f/G^2 \in J_{2k+240,E_8,m-60}^{\mathrm{w},W(E_8)}$, where

$$G(\tau,\mathfrak{z}) = \prod_{u \in R_2^+(E_8)} \frac{\vartheta(\tau,(u,\mathfrak{z}))}{\eta^3(\tau)}$$

is a weak E_8 Jacobi form of weight -120 and index 30 which is anti-invariant under $W(E_8)$. The symbol $R_2^+(E_8)$ denotes the set of all positive roots of E_8.

By the following structure theorem in [Wir92] (see Table 3.1)

$$J_{2*,E_7,*}^{\mathrm{w},W(E_7)} = M_*[\phi_{0,1}, \phi_{-2,1}, \phi_{-6,2}, \phi_{-8,2}, \phi_{-10,2}, \phi_{-12,3}, \phi_{-14,3}, \phi_{-18,4}],$$

we get

$$J_{2k,E_8,m}^{\mathrm{w},W(E_8)} = \{0\} \quad \text{if} \quad 2k < -5m. \tag{3.5.2}$$

As a consequence of the above proposition and (3.5.1), we deduce

$$\dim M_k(\mathrm{O}^+(2U \oplus E_8(-1))) \leq \sum_{0 \leq r \leq \frac{k}{7}} \dim J_{k-12r,E_8,r}^{\mathrm{w},W(E_8)}. \tag{3.5.3}$$

We note that $\dim M_6(\mathrm{O}^+(2U \oplus E_8(-1))) = 0$ due to $J_{6,E_8,1}^{W(E_8)} = \{0\}$.

By inequality (3.5.3), we get upper bounds of $\dim M_{2k}(\mathrm{O}^+(2U \oplus E_8(-1)))$ for small k. By [HU14, Corollary 1.3], we can calculate the exact values of dimension. We list them in Table 3.3.

$$x = 23 + \dim J_{-18,E_8,5}^{\mathrm{w},W(E_8)}.$$

Table 3.3: Dimension of modular forms for $\mathrm{O}^+(\mathrm{II}_{2,10})$

weight	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42
bound	1	0	1	1	2	1	3	2	4	4	6	5	9	8	12	**13**	17	**17**	24	x
dim.	1	0	1	1	2	1	3	2	4	4	6	5	9	8	12	12	17	16	24	23

We see at once that the upper bound is equal to the exact dimension when weight is less than 42 and not equal to 34 or 38.

By means of the same technique, we can assert

$$\dim M_4(\mathrm{O}^+(2U\oplus E_8(-2))) = 1, \qquad \dim M_6(\mathrm{O}^+(2U\oplus E_8(-2))) = 1.$$

These two modular forms can be constructed as additive liftings of holomorphic Jacobi forms A_2 and B_2, respectively.

In the end of this section, we construct two reflective modular forms. It is easy to check that

$$\phi_{0,2} = \frac{E_4^2 A_2}{\Delta} + E_4\varphi_{-4,2} - 2\varphi_{0,2} = q^{-1} + 24 + O(q)$$

is a $W(E_8)$-invariant weakly holomorphic Jacobi form of weight 0 and index 2. Its Borcherds product $\mathrm{Borch}(\phi_{0,2}) \in M_{12}(\mathrm{O}^+(2U\oplus E_8(-2)), \det)$ is a reflective modular form of weight 12 with complete 2-reflective divisor for $\mathrm{O}^+(2U\oplus E_8(-2))$. Another construction of this modular form can be found in [GN18].

The Borcherds product of $\varphi_{0,2}$ is a strongly reflective modular form of weight 60 with respect to $\mathrm{O}^+(2U\oplus E_8(-2))$. Its divisor is determined by reflective vectors of norm -4 and divisor 2. It is anti-invariant under the Weyl group $W(E_8)$ and it is not an additive lifting. Moreover, its first Fourier–Jacobi coefficient is given by the theta block $\prod_{r\in R_2^+(E_8)} \vartheta(\tau,(r,\mathfrak{z}))$.

Bibliography

[Bai66] W. L. Baily, *Fourier-Jacobi series,* in: Algebraic Groups and Discontinuous Subgroups, 296–300. Proc. Sympos. Pure Math., **9**, Amer. Math. Soc., Providence, 1966.

[Bel16] M. Belolipetsky, *Arithmetic hyperbolic reflection groups.* Bull. Amer. Math. Soc. (N. S.) **53** (2016) 437–475.

[Ber99] M. Bertola, *Jacobi groups, Jacobi forms and their applications.* PhD thesis, SISSA, Trieste, 1999.

[Ber00a] M. Bertola, *Frobenius manifold structure on orbit space of Jacobi group; Part I.* Differential Geom. Appl., **13** (2000) 19–41.

[Ber00b] M. Bertola, *Frobenius manifold structure on orbit space of Jacobi group; Part II.* Differential Geom. Appl., **13** (2000) 213–233.

[BL00] L. A. Borisov, A. Libgober, *Elliptic genera of toric varieties and applications to mirror symmetry.* Invent. Math. **140** (2000), no. 2, 453–485.

[BM17] J. H. Bruinier, M. Möller, *Cones of Heegner divisor.* arXiv: 1705.05534.

[Bor95] R. E. Borcherds, *Automorphic forms on* $\mathrm{O}_{s+2,2}$ *and infinite products.* Invent. Math., **120** (1995), no. 1, 161–213.

[Bor98] R. E. Borcherds, *Automorphic forms with singularities on Grassmannians.* Invent. Math., **123** (1998), no. 3, 491–562.

[Bor00] R. E. Borcherds, *Reflection groups of Lorentzian lattices.* Duck Math. J. **104** (2000), no.2, 319–366.

[Bou60] N. Bourbaki, *Groupes et algèbres de Lie.* Chapter 4,5 et 6.

[BKP+98] R. E. Borcherds, L. Katzarkov, T. Pantev, N. I. Shepherd-Barron, *Families of* $K3$ *surfaces.* J. Algebra Geom. **7** (1998), 183–193.

[Bru02] J. H. Bruinier, *Borcherds products on* $\mathrm{O}(2,n)$ *and Chern classes of Heegner divisors.* Lecture Notes in Math. **1780**. Springer-Verlag, 2002.

[Bru14] J. H. Bruinier, *On the converse theorem for Borcherds products.* J. Algebra **397** (2014), 315–342.

[Bru17] J. H. Bruinier, *Borcherds products with prescribed divisor.* Bull. Lond. Math. Soc. **49** (2017) no. 6, 979–987.

[BS06] J. Bernstein, O. Schwarzman, *Chevalley's theorem for the complex crystallographic groups.* J. Nonlinear Math. Phys. **13** (2006), no. 3, 323–351.

[CG11] F. Cléry, V. Gritsenko, *Siegel modular forms of genus 2 with the simplest divisor.* Proc. London Math. Soc. **102** (2011), 1024–1052.

[CG13] F. Cléry, V. Gritsenko, *Modular forms of orthogonal type and Jacobi theta-series.* Abh. Math. Semin. Univ. Hambg **83** (2013), 187–217.

[CK00] Y. Choie, H. Kim, *Differential operators and Jacobi forms of several variables.* J. Number Theory, **82** (2000) 140–163.

[Dit18] M. Dittmann, *Reflective automorphic forms on lattices of squarefree level.* to appear in Trans. Amer. Math. Soc. DOI: https://doi.org/10.1090/tran/7620. arXiv: 1805.08996.

[DHS15] M. Dittmann, H. Hagemeier, M. Schwagenscheidt, *Automorphic products of singular weight for simple lattices.* Math. Z. **279** (2015), no. 1-2, 585–603.

[DKW18] C. Dieckmann, A. Krieg, M. Woitalla, *The graded ring of modular forms on the Cayley half-space of degree two.* Ramanujan J (2018).

[Dub96] B. Dubrovin, *Geometry of 2D topological field theories.* Integrable Systems and Quantum Groups, (Montecatini Terme, 1993), Lecture Notes in Math., **1620**, Springer, Berlin, 1996, 120–384.

[Dub98] B. Dubrovin, *Differential geometry of the space of orbits of a Coxeter group.* Surv. Diff. Geom. **4** (1999) 181–211.

[Ebe02] W. Ebeling, *Lattices and Codes,* volume 21. Vieweg, Braunschweig Wiesbaden, 2002.

[Eic52] M. Eichler, *Quadratische Formen und orthogonale Gruppen.* Grundlehren der mathematischen Wissenschaften **63**. Springer-Verlag, Berlin-New York, 1952.

[EK94] M. Eie, A. Krieg, *The theory of Jacobi forms over the Cayley numbers.* Trans. Amer. Math. Soc. **342** (2): 793–805, 1994.

[Ess96] F. Esselmann, *Über die maximale Dimension von Lorentz-Gittern mit coendlicher Spiegelungsgruppe.* J. Number Theory **61** (1996), no. 1, 103–144.

[EZ85] M. Eichler, D. Zagier, *The theory of Jacobi forms.* Progress in Mathematics **55**. Birkhäuser, Boston, Mass., 1985.

[FM07] E. Freitag, R. Salvati Manni, *Modular forms for the even unimodular lattice of signature* $(2, 10)$. J. Algebraic Geom. **16**, 753–791, 2007.

[Gra09] B. Grandpierre, *Produits automorphes, classification des reseaux et theorie du codage.* PhD. dissertation, Lille1, 2009.

[Gri91] V. Gritsenko, *Jacobi functions of n variables.* J. Soviet Math. **53** (1991) 243–252.

[Gri94] V. Gritsenko, *Irrationality of the moduli spaces of polarized abelian surfaces.* Int. Math. Res. Not. IMRN **6** (1994), 235–243.

[Gri95] V. Gritsenko, *Modular forms and moduli spaces of Abelian and K3 surfaces.* Algerba i Analyz **6** (1994), 65–102; English translation: St. Petersburg Math. Journal **6** (1995), 1179–1208.

[Gri99] V. Gritsenko, *Elliptic genus of Calabi–Yau manifolds and Jacobi and Siegel modular forms.* St. Petersburg Math. J. **11** (1999), 781–804.

[Gri99b] V. Gritsenko, *Complex vector bundles and Jacobi forms.* Proc. of Symposium “Automorphic forms and L-functions”, vol. **1103**, RIMS, Kyoto, 1999, 71–85.

[Gri10] V. Gritsenko, *Reflective modular forms in algebraic geometry.* arXiv:1005.3753.

[Gri12] V. Gritsenko, 24 *faces of the Borcherds modular form* Φ_{12}. arXiv: 1203.6503.

[Gri18] V. Gritsenko, *Reflective modular forms and their applications.* Russian Math. Surveys **73** (2018), no. 5.

[GH98] V. Gritsenko, K. Hulek, *Minimal Siegel modular threefolds.* Math. Proc. Cambridge Philos. Soc. **123** (1998) 461–485.

[GH99] V. Gritsenko, K. Hulek, *The modular form of the Barth-Nieto quintic.* Intern. Math. Res. Notices **17** (1999), 915–938.

[GH14] V. Gritsenko, K. Hulek, *Uniruledness of orthogonal modular varieties.* J. Algebraic Geom. **23** (2014), 711–725.

[GHS07] V. Gritsenko, K. Hulek, G. K. Sankaran, *The Kodaira dimension of the moduli of K3 surfaces.* Invent. Math. **169** (2007), 519–567.

[GHS09] V. Gritsenko, K. Hulek, G. K. Sankaran, *Abelianisation of orthogonal groups and the fundamental group of modular varieties.* J. Algebra **322** (2009), no.2, 463–478.

[GHS13] V. Gritsenko, K. Hulek, G. K. Sankaran, *Moduli of K3 surfaces and irreducible symplectic manifolds,* from: "Handbook of moduli, I" (editors G Farkas, I Morrison), Adv. Lect. Math. 24, International Press (2013) 459–526.

[GN97] V. Gritsenko, V. V. Nikulin, *Siegel automorphic form corrections of some Lorentzian Kac-Moody Lie algebras.* Amer. J. Math. **119** (1997), no. 1, 181–224.

[GN98a] V. Gritsenko, V. V. Nikulin, *Automorphic forms and Lorentzian Kac-Moody algebras. I.* Internat. J. Math. **9** (1998) no.2, 153–200.

[GN98b] V. Gritsenko, V. Nikulin, *Automorphic forms and Lorentzian Kac–Moody algebras. Part II.* Internat. J. Math. **9** (1998), 201–275.

[GN00] V. Gritsenko, V. V. Nikulin, *The arithmetic mirror symmetry and Calabi-Yau manifolds.* Comm. Math. Phys. **210** (2000), no.1, 1–11.

[GN02] V. Gritsenko, V. V. Nikulin, *On the classification of Lorentzian Kac-Moody algebras.* Russian Math. Surveys **57** (2002), no. 5, 921–979.

[GN18] V. Gritsenko, V. V. Nikulin, *Lorentzian Kac-Moody algebras with Weyl groups of* 2-*reflections.* Proc. London Math. Soc. **116** (2018), no.3, 485–533.

[GPY15] V. Gritsenko, C. Poor, D. Yuen, *Borcherds products everywhere.* J. Number Theory**148** (2015), 164–195.

[GPY16] V. Gritsenko, C. Poor, D. S. Yuen, *Antisymmetric paramodular forms of weights* 2 *and* 3. arXiv: 1609.04146, to appear in IMRN.

[GSZ18] V. Gritsenko, N. P. Skoruppa, D. Zagier, *Theta-blocks*, Preprint 2018.

[GW17] V. Gritsenko, H. Wang, *Conjecture on theta-blocks of order* 1. Uspekhi Matematicheskikh Nauk 72:5(437) (2017), 191–192. English version: Russian Mathematical Surveys, Volume 72, no. 5 (2017), 968–970.

[GW18] V. Gritsenko, H. Wang, *Powers of Jacobi triple product, Cohen's numbers and the Ramanujan Δ-function.* European Journal of Mathematics, Volume 4, Issue 2 (2018), 561–584.

[GW18a] V. Gritsenko, H. Wang, *Graded ring of integral Jacobi forms.* arXiv:1810.09392.

[GW18b] V. Gritsenko, H. Wang, *Theta block conjecture for paramodular forms of weight* 2. arXiv:1812.08698.

[GW18c] V. Gritsenko, H. Wang, *Orbits of Jacobi forms and theta relations.* preprint 2018.

[GW18d] V. Gritsenko, H. Wang, *Integral Jacobi forms for* $2A_1$. preprint 2018.

[GW19a] V. Gritsenko, H. Wang, *Antisymmetric paramodular forms of weight* 3. to appear in Sbornik: Mathematics.

[HU14] H. Hashimoto, K. Ueda, *The ring of modular forms for the even unimodular lattice of signature* $(2, 10)$. arXiv:1406.0332.

[Kac90] V. Kac, *Infinite dimensional Lie algebras.* 3rd ed., Cambridge Univ. Press, 1990.

[Loo76] E. Looijenga, *Root systems and elliptic curves.* Invent. Math. **38** (1976), 17–32.

[Loo80] E. Looijenga, *Invariant theory for generalized root systems.* Invent. Math. **61** (1980), 1–32.

[Loo03] E. Looijenga, *Compactifications defined by arrangements. II. Locally symmetric varieties of type IV.* Duke Math. J. **119** (2003), no. 3, 527–588.

[Lor05] M. Lorenz, *Multiplicative invariant theory.* Invariant Theory and Algebraic Transformation Groups, VI, Encyclopaedia Math. Sci., vol. **135**, Springer-Verlag, Berlin (2005).

[Ma17] S. Ma, *Finiteness of* 2*-reflective lattices of signature* $(2, n)$. Amer. J. Math. **139** (2017), 513–524.

[Ma18] S. Ma, *On the Kodaira dimension of orthogonal modular varieties.* Invent. math. (2018).

[MNV+98] J. A. Minahan, D. Nemeschansky, C. Vafa, and N. P. Warner, *E-Strings and* $N = 4$ *Topological Yang-Mills Theories.* Nucl. Phys. **B527** (1998), 581–623.

[Moh02] K. Mohri, *Exceptional String: Instanton Expansions and Seiberg-Witten Curve.* Rev. Math. Phys. **14** (2002) 913–975.

[Mum83] D. Mumford, *Tata lectures on theta I.* Progress in Mathem. **28**, Birkhäuser, Boston, Basel, Mass, 1983.

[Nik80] V. V. Nikulin, *Integer symmetric bilinear forms and some of their geometric applications.* Math. USSR Izv. **14**, 103–167 (1980).

[Nik81] V. V. Nikulin, *Quotient-groups of groups of automorphisms of hyperbolic forms by subgroups generated by 2-reflections.* Algebro-geometric applications (Russian), Current problems in mathematics, Vol. 18, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1981, pp. 3–114.

[Nik84] V. V. Nikulin, *K3 surfaces with a finite group of automorphisms and a Picard group of rank three.* (Russian) Trudy Mat. Inst. Steklov. **165** (1984), 119–142. Algebraic geometry and its applications.

[Nik96] V. V. Nikulin, *A remark on discriminants for moduli of K3 surfaces as sets of zeros of automorphic forms.* J. Math. Sci. **81** (1996), no. 3, 2738–2743.

[Oda77] T. Oda, *On modular forms associated with indefinite quadratic forms of signature* $(2, n-2)$. Math. Ann. **231** (1977), 97–144.

[OP17] G. Oberdieck, A. Pixton, *Gromov-Witten theory of elliptic fibrations: Jacobi forms and holomorphic anomaly equations.* arXiv: 1709.01481.

[OS18] S. Opitz, M. Schwagenscheidt, *Holomorphic Borcherds products of singular weight for simple lattices of arbitrary level.* arXiv:1810.06290.

[PSY17] C. Poor, J. Shurman, D. S. Yuen, *Siegel paramodular forms of weight* 2 *and squarefree level.* Int. J. Number Theory **13** (2017), 2627–2652.

[Sai90] K. Saito, *Extended affine root systems. II. Flat invariants.* Publ. Res. Inst. Math. Sci., **26** (1990), 15–78.

[Sak17a] K. Sakai, *Topological string amplitudes for the local* $\frac{1}{2}K3$ *surface.* PTEP. Prog. Theor. Exp. Phys. 2017, no. 3.

[Sak17b] K. Sakai, E_n *Jacobi forms and Seiberg–Witten curves.* arXiv:1706.04619v2, to appear in Communications in Number Theory and Physics.

[Sat98] I. Satake, *Flat structure for the simple elliptic singularity of type* $\widetilde{E_6}$ *and Jacobi form.* Proc. Japan Acad. Ser. A Math. Sci., **69** (1993), 247–251.

[SC98] N. J. A. Sloane, J. H. Conway, *Sphere Packings, Lattices and Groups,* volume 21. Spring, New York, Berlin, Heidelberg, London, Paris, Tokyo, 1998.

[Sch04] N. R. Scheithauer, *Generalized Kac-Moody algebras, automorphic forms and Conway's group I.* Adv. Math. **183** (2004), 240–270.

[Sch06] N. R. Scheithauer, *On the classification of automorphic products and generalized Kac-Moody algebras.* Invent. Math. **164** (2006), 641–678.

[Sch09] N. R. Scheithauer, *The Weil representation of* $\mathrm{SL}_2(\mathbb{Z})$ *and some applications.* Int. Math. Res. Not. IMRN (2009) no.8, 1488–1545.

[Sch15] N. R. Scheithauer, *Some constructions of modular forms for the Weil representation of* $\mathrm{SL}(2, \mathbb{Z})$. Nagoya Math. J. **220**(2015), 1–43.

[Sch17] N. R. Scheithauer, *Automorphic products of singular weight.* Compos. Math. **153** (9), 1855–1892 (2017).

[Tot00] B. Totaro, *Chern numbers for singular varieties and elliptic homology.* Ann. of Math. (2) **151** (2000), no. 2, 757–791.

[Vin72] E. B. Vinberg, *On unimodular integral quadratic forms.* Funct. Anal. Appl. **6** (1972), 105–111.

[Vin07] E. B. Vinberg, *Classification of* 2*-reflective hyperbolic lattices of rank* 4. Tr. Mosk. Mat. Obs. **68** (2007), 44–76; English transl., Trans. Moscow Math. Soc. (2007), 39–66.

[Wan18a] H. Wang, *Reflective Modular Forms: A Jacobi Forms Approach.* arXiv:1801.09590, to appear in Int. Math. Res. Not. IMRN (Doi: https://doi.org/10.1093/imrn/rnz070)

[Wan18b] H. Wang, *Weyl invariant* E_8 *Jacobi forms.* arXiv:1801.08462.

[Wan19] H. Wang, *The classification of* 2*-reflective modular forms.* Preprint 2019, 33 pages.

[Wir92] K. Wirthmüller, *Root systems and Jacobi forms.* Compos. Math. **82** (1992) 293–354.

[ZGH+18] M. Zotto, J.Gu, M. Huang, A. Poor, A. Klemm, G. Lockhart, *Topological Strings on Singular Elliptic Calabi-Yau* 3*-folds and Minimal* 6*d SCFTs.* J. High Energ. Phys. (2018) 2018: 156.