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Generalized stable distributions and free stable distributions

Lois stables généralisées et lois stables libres

par

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In memory of my father, Tian-Shun.

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Résumé

Cette thèse porte sur les lois stables réelles au sens large et comprend deux parties indépendantes.

La première partie concerne les lois stables généralisées introduites par Schneider [81] dans un contexte physique et étudiées ensuite par Pakes [74]. Elles sont définies par une équation différentielle fractionnaire dont on caractérise ici l'existence et l'unicité des solutions densité à l'aide de deux paramètres positifs, l'un de stabilité et l'autre de biais. On montre ensuite diverses identités en loi pour les variables aléatoires sous-jacentes. On étudie le comportement asymptotique précis de la densité aux deux extrémités du support. Dans certains cas, on donne des représentations exactes de ces densités comme fonctions de Fox. Enfin, on résoud entièrement les questions ouvertes autour de l'infinie divisibilité des lois stables généralisées qui avaient été posées dans [74].

La seconde partie, plus longue, porte sur l'analyse classique des lois α-stables libres réelles. Introduites par Bercovici et Pata [13], ces lois ont ensuite étudiées par Biane [14], Demni [29] et Hasebe-Kuznetsov [44] sous divers points de vue. Nous montrons qu'elles sont classiquement infiniment divisibles pour $\alpha \le 1$ et qu'elles appartiennent à la classe de Thorin étendue pour $\alpha \le 3/4$. La mesure de Lévy est calculée explicitement pour $\alpha = 1$ et ce calcul entraîne que les lois 1-stables libres n'appartiennent pas à la classe de Thorin, sauf dans le cas de la loi de Cauchy avec dérive. Dans le cas symétrique, nous montrons que les densités $\alpha-$stables libres ne sont pas infiniment divisibles quand $\alpha > 1$. Dans le cas de signe constant nous montrons que les densités stables libres ont une courbe en baleine, autrement dit que leurs dérivées successives ne s'annulent qu'une seule fois sur leurs supports, ce qui constitue un raffinement de l'unimodalité et fait écho à la courbe en cloche des densités stables classiques récemment montrée rigoureusement dans [83] et [58]. Nous établissons enfin plusieurs propriétés précises des densités stables libres spectralement de signe constant, parmi lesquelles une analyse détaillée de la variable aléatoire de Kanter, des expansions asymptotiques complètes en zéro, ainsi que plusieurs propriétés intrinsèques des courbes en baleine. Nous montrons enfin une nouvelle identité en loi pour l'algèbre Beta-Gamma, diverses propriétés d'ordre stochastique et nous étudions le problème classique de Van Dantzig pour la loi semi-circulaire généralisée.

Mots-clés

Algèbre bêta-gamma; Convolution gamma généralisée; Courbe en baleine; Courbe en cloche; Divisibilité infinie; Equation différentielle fractionnaire; Expansion asymptotique; Fonction double Gamma; Fonction de Fox; Fonction de Wright; Monotonie complète hyperbolique; Loi de Kanter; Loi stable généralisée; Loi stable libre; Ordre stochastique; Problème de Van Dantzig.

Abstract

This thesis deals with real stable laws in the broad sense and consists of two independent parts.

The first part concerns the generalized stable laws introduced by Schneider [81] in a physical context and then studied by Pakes [74]. They are defined by a fractional differential equation, whose existence and uniqueness of the density solutions is here characterized via two positive parameters, a stability parameter and, a bias parameter. We then show various identities in law for the underlying random variables. The precise asymptotic behavior of the density at both ends of the support is investigated. In some cases, exact representations as Fox functions of these densities are given. Finally, we solve entirely the open questions on the infinite divisibility of the generalized stable laws which had been raised in [74].

The second and longer part deals with the classical analysis of the free α−stable laws. Introduced by Bercovici and Pata [13], these laws were then studied by Biane [14], Demni [29] and Hasebe-Kuznetsov [44], from various points of view. We show that they are classically infinitely divisible for $\alpha \le 1$ and that they belong to the extended Thorin class extended for $\alpha \le 3/4$. The Lévy measure is explicitly computed for $\alpha = 1$, showing that free 1-stable distributions are not in the Thorin class except in the drifted Cauchy case. In the symmetric case, we show that free α-stable densities are not infinitely divisible when $\alpha > 1$. In the one-sided case we prove, refining unimodality, that the densities are whale-shaped, that is their successive derivatives vanish exactly once on their support. This echoes the bell shape property of the classical stable densities recently rigorously shown in [83] and [58]. We also derive several fine properties of spectrally one-sided free stable densities, including a detailed analysis of the Kanter random variable, complete asymptotic expansions at zero, and several intrinsic features of whale-shaped functions. Finally, we display a new identity in law for the Beta-Gamma algebra, various stochastic order properties, and we study the classical Van Danzig problem for the generalized semi-circular law.

Keywords

Asymptotic expansion; Bell shape; Beta-gamma algebra; Double Gamma function; Fox function; Fractional differential equation; Free stable law; Generalized Gamma convolution; Generalized stable law; Infinite divisibility; Hyperbolic complete monotonicity; Kanter law; Stochastic order; Van Dantzig problem; Whale shape; Wright function.

Contents

Symbols and abbreviations

Various

$\mathbb{N}$	set of nonnegtive integers
$\mathbb{N}^*$	set of strictly positive integers
r.v.	random variable
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of nonnegtive real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{Z}$	set of integers
$\stackrel{d}{=}$	equality in distribution
iff	if and only if

Random variables

$\mathbf{U}$	uniformly distributed r.v. on $(0,1)$
$\mathbf{L}$	unit exponential r.v.
$\mathbf{B}_{a,b}$	standard Beta(a,b) r.v.
$\mathbf{\Gamma}_t$	standard Gamma$(t,1)$ r.v. with rate parameter
$\mathbf{K}_\alpha$	Kanter random variable
$\mathbf{Z}_{\alpha,\rho}$	classical strictly stable r.v.
$\mathbf{Z}_\alpha$	$\mathbf{Z}_{\alpha,1}$
$\mathbf{W}_{\alpha,\rho}$	$\mathbf{Z}_{\alpha,\rho} \vert \mathbf{Z}_{\alpha,\rho} > 0$
$\mathbf{S}_\beta$	classical non-strictly stable r.v.
$\mathbf{S}$	exceptional classical 1-stable r.v. satisfying $\mathbb{E}[e^{s\mathbf{S}}] = s^s, s > 0$.
$X_{m,\alpha}$	generalized stable r.v.
$Y_{m,\alpha}$	$X_{m,\alpha}^{-1}$
$\mathbf{X}_{\alpha,\rho}$	free strictly stable r.v.
$\mathbf{X}_\alpha$	$\mathbf{X}_{\alpha,1}$
$\mathbf{C}_{a,b}$	free non-strictly 1-stable r.v.
$\mathbf{T}$	exceptional free 1-stable r.v. having Voiculescu transform $-\log z$
$\mathbf{W}$	$\frac{\sin(\pi\mathbf{U})}{\pi\mathbf{U}} e^{\pi\mathbf{U}\cot(\pi\mathbf{U})}$
$\mathbf{Y}_\alpha$	$\begin{cases} \mathbf{X}_\alpha - b_\alpha & \text{if } \alpha \in (0,1) \\ 1 - \mathbf{T} & \text{if } \alpha = 1 \\ b_{1/\alpha}^{-1/\alpha} - \mathbf{X}_{\alpha,1/\alpha} & \text{if } \alpha \in (1,2] \end{cases}$

Abbreviations

BF	Bernstein Function
BS	Bell-Shaped
BS_n	Bell-Shaped of order n
CM	Completely Monotone
EGGC	Extended Generalized Gamma Convolution
FID	Freely Infinitely Divisible
GGC	Generalized Gamma Convolution
Gst	Generalized STable law with parameters m and α
HCM	Hyperbolically Completely Monotone
ID	Infinitely Divisible or Infinitely Divisible distributions
ME	Mixture of Exponential distributions
rGst	Reciprocal Generalized STable law with parameters m and α
SD	Self-Decomposable
TBF	Thorin-Bernstein Function
WS	Whale-Shaped

Chapter 1

Infinitely divisible distributions

1.1 A brief historical view

The concept of infinite divisibility was introduced by Bruno de Finetti [35] in 1929 in the context of processes with stationary independent increments. Infinitely divisible distributions were systematically studied in 1937 by Paul Lévy [61], and soon later in 1938 by Alexander Yakovlevich Khintchine [53]. For this reason, the canonical form of the characteristic function of infinitely divisible distributions is called the *Lévy-Khintchine representation*. Around the same time, questions about the central limit theorem led Lévy to the introduction of the self-decomposable distributions which are also called nowaday distributions of class L. Starting from Olof Thorin's 1977 paper [85] on the infinite divisibility of the Lognormal distribution, Lennart Bondesson developed a theory of generalized gamma convolutions (GGC), a subclass of infinitely divisible distributions, which is displayed in his 1992 monograph [18]. The GGC property is fulfilled by the stable distributions, an older class of probability laws dating back to Gauss, Cauchy and, Pólya, which was also systematically by Lévy, and which will be discussed in the next chapter. Our references for a full treatment of infinitely divisible distributions are the standard treatises by Ken-iti Sato [79] and by Fred W. Steutel and Klaas Van Harn [84].

1.2 Completely monotone functions and Bernstein functions

Definition 1.1. *A function $f : (0,\infty) \to [0,\infty)$ is a* completely monotone (CM) *function if it is of class C^∞ and if*

$$(-1)^n f^{(n)}(\lambda) \geq 0 \quad \textit{for all } n \in \mathbb{N} \textit{ and } \lambda > 0.$$

The following can be found e.g. in Theorem 1.4 of [80].

Theorem 1.1 (Bernstein's Theorem)**.** *A function f is CM if and only if it is the Laplace transform of a unique positive measure μ on $[0,\infty)$, i.e. for all $\lambda > 0$ one has*

$$f(\lambda) = \mathcal{L}(\mu;\lambda) = \int_{[0,\infty)} e^{-\lambda t}\mu(dt).$$

Definition 1.2. *A function* $f : (0, \infty) \to [0, \infty)$ *is a* Bernstein function (BF) *if it is of class* C^∞ *and if*

$$(-1)^{n-1} f^{(n)}(\lambda) \geq 0 \quad \text{for all } n \in \mathbb{N}^* \text{ and } \lambda > 0.$$

From the definition, it is clear that the derivative of a Bernstein function is CM. On the other hand, the primitive of a completely monotone function is a Bernstein function whenever it is positive. This leads to the following integral representation of Bernstein functions:

$$f(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t}) \nu(dt) \tag{1.1}$$

where $a, b \geq 0$ and ν is a positive measure on $[0, \infty)$ integrating $1 \wedge x$. The class of CM functions is closed under addition, multiplication and pointwise convergence, but not closed under composition. The class of BF functions is closed under addition, composition and pointwise convergence, but not closed under multiplication. The following can be found e.g. in Theorem 3.7 of [80] and is an easy consequence of the Faà di Bruno's formula.

Theorem 1.2. *Let* f *be a positive function on* $(0, \infty)$*. Then the following assertions are equivalent:*
(i) $f \in \mathrm{BF}$.
(ii) $g \circ f \in \mathrm{CM}$ *for every* $g \in \mathrm{CM}$.
(iii) $e^{-tf} \in \mathrm{CM}$ *for every* $t > 0$.

Example 1.1. (a) The function $x \mapsto x^\alpha$ is BF if and only if $\alpha \in [0, 1]$ is a BF and for every $\alpha \in (0, 1)$ we have

$$x^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_{(0,\infty)} (1 - e^{-xt}) \frac{dt}{t^{\alpha+1}}. \tag{1.2}$$

(b) The function $x \mapsto \log(1 + x)$ is BF and

$$\log(1 + x) = \int_0^\infty (1 - e^{-xt}) \frac{e^{-t}}{t} dt. \tag{1.3}$$

We will call (1.3) the Frullani identity.

For $t > 0$, denote by $\mathbf{\Gamma}_t$ the Gamma random variable of parameter $t > 0$, with density

$$f_{\mathbf{\Gamma}_t}(x) = \frac{x^{t-1} e^{-x}}{\Gamma(t)}, \qquad x > 0.$$

The density of $\mathbf{\Gamma}_t$ is CM if and only if $t \in (0, 1]$, and is never BF. The case $t = 1$ is particularly interesting and leads to the following definition.

Definition 1.3. *A positive random variable* $\mathbf{X}$ *is called a* mixture of exponentials *(ME) if its law has the form*

$$\mathbb{P}_{\mathbf{X}}(dx) = c\delta_0(dx) + f(x)dx$$

with $c \in [0, 1]$ *and* $f \in$ *a CM function. When* $c = 0$*, we will use the notation* $\mathbf{X} \in ME^*$.

The following can be found e.g. in Theorem 9.5 in [80] and Theorem 51.12 in [79].

Theorem 1.3 (Steutel's Theorem). *Let μ be a probability measure on $[0,\infty)$. The following conditions are equivalent:*
(i) $\mu \in \mathrm{ME}^*$.
(ii) *There exists a measurable function $\eta : (0,\infty) \to [0,1]$ satisfying $\int_0^1 \eta(t)t^{-1}dt < \infty$ such that $\forall\, \lambda > 0$,*

$$\mathcal{L}(\mu;\lambda) = \exp\left[-\int_0^\infty \left(\frac{1}{t} - \frac{1}{\lambda+t}\right)\eta(t)dt\right] = \exp\left[-\int_0^\infty \left(1-e^{-\lambda x}\right) l(x)dx\right]$$

with $l(x) = \int_0^\infty e^{-xt}\eta(t)dt$.

1.3 Infinite divisibility on the real line

Definition 1.4. *A real random variable $\mathbf{X}$ is said to be* infinitely divisible (ID) *if for every $n \in \mathbb{N}^*$ there exists a real random variable $\mathbf{X}_n$ such that*

$$\mathbf{X} \stackrel{d}{=} \mathbf{X}_{n,1} + \cdots + \mathbf{X}_{n,n},$$

where $\mathbf{X}_{n,1},\dots,\mathbf{X}_{n,n}$ are mutually independent with the same law as $\mathbf{X}_n$.

In other words, a distribution function F is infinitely divisible iff for every $n \in \mathbb{N}^*$ it is the $n-$th fold convolution of a distribution function F_n with itself:

$$F = F_n^{*n} \text{ for all } n \in \mathbb{N}^*,$$

and a characteristic function ϕ is infinitely divisible iff for every $n \in \mathbb{N}^*$ it is the $n-$th power of a characteristic function ϕ_n:

$$\phi(u) = \{\phi_n(u)\}^n \text{ for all } n \in \mathbb{N}^*.$$

Here F_n and ϕ_n are respectively called the $n-$th order factor of F and of ϕ. The following representation result is the most important result in the theory of ID distributions. We refer e.g. to Theorem 8.1 in [79] for a proof.

Theorem 1.4 (Lévy-Khintchine representation). *A probability measure μ on $\mathbb{R}$ is* ID *iff*

$$\int_{\mathbb{R}} e^{itx}\mu(dx) = \exp\left[iat - \frac{1}{2}\sigma^2 t^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1 - itx\mathbf{1}_{[-1,1]}(x))\nu(dx)\right] \qquad (1.4)$$

where $a \in \mathbb{R}, \sigma^2 \geq 0$ and ν is a measure on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}\setminus\{0\}} (|x|^2 \wedge 1)\nu(dx) < \infty.$$

We call (a,σ^2,ν) in Theorem 1.4 the *generating triplet* of μ. The measure ν is called the *Lévy measure* of μ. The function $\mathbf{1}_{[-1,1]}(x)$ can be replaced by any bounded function $c(x)$ satisfying

$$c(x) = 1 + O(x) \quad \text{as } |x| \to 0, \qquad c(x) = O(1/|x|) \quad \text{as } |x| \to \infty.$$

Other examples of truncating function are $c(x) = 1/(1+x^2)$ and $c(x) = \sin(x)/x$. The following standard results are useful to prove that a given distribution is not ID. They are given e.g. in Proposition IV.2.4 resp. in Corollary IV.8.5 of [84]

Proposition 1.1. *The characteristic function of an ID distribution has no real zeros.*

Proposition 1.2. *A continuous ID distribution function F is supported either by a half-line or by $\mathbb{R}$.*

We now give some results which are more specific to the one-sided case. Observe first that for a positive random variable, the Laplace transform of its distribution is more convenient than the characteristic function. The following theorem is the Lévy-Khintchine representation theorem for Laplace transforms of ID distributions, and is de Finetti's original result. It says that the logarithm of this Laplace transform is the opposite of a Bernstein function. The proof of this result can be found e.g. in Theorem 24.11 of [79].

Theorem 1.5. *A probability measure μ on $\mathbb{R}_+$ is* ID *if and only if there exist $b \geq 0$ and a measure ν on $(0,\infty)$ satisfying $\int_{(0,\infty)}(1\wedge x)\nu(dx) < \infty$ such that*

$$-\log \mathcal{L}(\mu;\lambda) = b\lambda + \int_{(0,\infty)} (1-e^{-\lambda x})\nu(dx).$$

The measure ν is called the Lévy measure of μ and the non-negative coefficient b is called its drift coefficient.

The following characterization of positive ID measures is due to Steutel and is given e.g. in Theorem 51.1 of [79]. It is given in terms of the probability measure itself instead of its Laplace transform, by means of a certain integro-differential equation.

Theorem 1.6 (Steutel's integro-differential equation)**.** *A probability measure μ on $\mathbb{R}_+$ is* ID *if and only if there exist $b \geq 0$ and a measure ν on $(0,\infty)$ satisfying $\int_{(0,\infty)}(1\wedge x)\nu(dx) < \infty$ such that*

$$\int_{[0,x]} y\mu(dy) = \int_{(0,x]} \mu([0,x-y])y\nu(dy) + b\mu([0,x]), \quad \text{for } x > 0. \tag{1.5}$$

Taking the Laplace transform on both sides, it is easy to recover the Lévy-Khintchine formula from the Steutel integro-differential equation. When μ has a density function f, Steutel's equation reads

$$xf(x) = \int_0^x f(x-y)\, y\nu(dy) \; + \; bf(x)$$

and is a true integro-differential equation. Let us check the validity of this equation for the Gamma random variable $\mathbf{\Gamma}_t$. Taking $b=0$ and simplifying the e^{-x} and the $\Gamma(t)$, we need to check that there exists a positive measure ν such that

$$x^t = \int_0^x (x-y)^{t-1}\, ye^y \nu(dy), \qquad x > 0.$$

It is clear by direct integration that the solution is the measure with density te^{-y}/y, as obtained from the log-Laplace exponent $t\log(1+\lambda)$.

In spite of the two above theorems, it can be difficult to check that a random variable having either an explicit Laplace transform or an explicit density is ID. For this reason, various criteria have been given over the years and we now give two of them which are important in this thesis. The first one can be found e.g. in Theorem 51.6 of [79], as a consequence of a more general log-convexity criterion.

Proposition 1.3 (Goldie-Steutel). *One has* ME $\subset$ ID. *In particular, if a given random variable* X *has a* CM *density, then it is* ID.

This theorem and the following standard independent factorization

$$\mathbf{\Gamma}_a \stackrel{d}{=} \mathbf{B}_{a,b} \times \mathbf{\Gamma}_{a+b}, \quad \text{for } a > 0,\, b > 0$$

where, here and throughout, $\mathbf{B}_{a,b}$ stands for a standard $\beta(a,b)$ random variable with density

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\, x^{a-1}(1-x)^{b-1}$$

on $(0,1)$, show that any mixture $X \times \mathbf{\Gamma}_\alpha$ is ID for $0 < \alpha \leq 1$. The following theorem shows that the property can be extended up to any $0 < \alpha \leq 2$ and is much more difficult to prove. See Theorem VI.4.5 in [84].

Theorem 1.7 (Kristiansen). *The independent product* $X \times \mathbf{\Gamma}_2$ *is* ID *for any positive random variable* X.

It should be noted that this result is optimal in the sense that there exist $\mathbf{\Gamma}_t$ mixtures with $t > 2$ which are not ID. For example, the Laplace transform

$$\frac{1}{2}(1 + (1+\lambda)^{-t})$$

does not correspond to an ID distribution for every $t > 2$. See Example VI.12.1 in [84] for details.

The following result implies that the tail distribution of a positive ID random variable cannot be thinner than e^{-x^r} for any $r > 1$. It is often used to disprove that a given random variable is ID. We refer e.g. to Theorem III.9.1 in [84] for a proof.

Theorem 1.8. *Let* F *be a non-degenerate infinitely divisible distribution function on* $\mathbb{R}_+$ *with Lévy measure* ν. *Then the tail function* $\bar{F}(x) = 1 - F(x)$ *satisfies*

$$\lim_{x\to+\infty} \frac{-\log \bar{F}(x)}{x \log x} = \frac{1}{r_\nu} \tag{1.6}$$

where r_ν *is the right extremity of* $\operatorname{supp}(\nu)$ *and the right-hand side is meant to be zero if* $r_\nu = \infty$.

We notice that contrary to the tail of the distribution functions, the behavior of infinitely divisible densities f at infinity can be much more chaotic. In fact, there exist ID laws on $\mathbb{R}^+$ having an infinite and unbounded sequence of modes, as shown in the following example.

Example 1.2. Consider the independent sum $Z = X + Y$ where X has a geometric distribution with parameter $1/2$ and Y has a $\mathbf{\Gamma}_{1/2}$ distribution. The random variable Z is ID, and its density function is easily computed as

$$f(x) = \frac{1}{2\sqrt{\pi}} e^{-x} \sum_{k=0}^{[x]} \frac{1}{\sqrt{x-k}} \left(\frac{e}{2}\right)^k, \qquad x \in \mathbb{R}/\mathbb{N}.$$

One has $\lim_{x\downarrow n} f(x) = \infty$ for any $n \in \mathbb{N}$.

1.4 Self-decomposability

In this section, we briefly discuss a classic refinement of infinite divisibility which is due to Lévy.

Definition 1.5. *A real random variable* $\mathbf{X}$ *is said to be* $\boxed{\text{self-decomposable}}$ (SD) *if for every* $c \in (0,1)$*, there exist a random variable* $\mathbf{X}_c$ *independent of* $\mathbf{X}$ *such that*

$$\mathbf{X} \stackrel{d}{=} c\mathbf{X} + \mathbf{X}_c. \tag{1.7}$$

It is easy to see by the Lévy-Khintchine formula that SD random variables are ID. More precisely, one has the following characterization of SD within the ID class. See Corollary 15.11 in [79].

Theorem 1.9. *A probability measure* μ *on* $\mathbb{R}$ *is self-decomposable if and only if it is ID and its Lévy measure has density*

$$\frac{k(x)}{|x|} \tag{1.8}$$

on $\mathbb{R}^*$*, where* $k(x)$ *is non-decreasing on* $(-\infty,0)$ *and non-increasing on* $(0,\infty)$*.*

In the one-sided case, the SD property of a given distribution can also be characterized within Steutel's integro-differential equation, as shows the following. See Theorem 2.16 in Steutel [84] for a proof.

Theorem 1.10. *A distribution function on* $[0,\infty)$ *is SD if and only if it has a density function and this density satisfies the integro-differential equation*

$$xf(x) = \int_0^x f(x-y)k(y)dy, \quad \text{for } x > 0. \tag{1.9}$$

for some non-increasing function k *on* $(0,\infty)$*.*

It can be difficult to show that a given distribution is SD and several criteria are available in the literature. Let us mention the following one which can be viewed as randomization of the original definition of self-decomposability and is due to Vervaat - see Remark 4.9 in [86].

Theorem 1.11. *Suppose that a positive random variable* X *satisfies the random contraction equation*

$$X \stackrel{d}{=} \mathbf{U}(X+A)$$

where A *is any positive random variable,* $\mathbf{U}$ *has uniform distribution on* $(0,1)$*, and the three random variables on the right-hand side are independent. Then* X *is self-decomposable.*

1.5 Generalized Gamma Convolution

We now consider a certain subclass of positive SD random variables, which is in one-to-one correspondence with a certain subclass of Bernstein functions.

Definition 1.6. *A Bernstein function f is called a* Thorin-Bernstein function (TBF) *if its spectral measure ν in (1.1) has a density $t^{-1}k(t)$ where $k(t)$ is a* CM *functions. In other words,*

$$f(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t}) \frac{k(t)}{t} dt \tag{1.10}$$

where $a, b > 0$, $\int_0^\infty (1 \wedge t^{-1}) k(t) dt < \infty$ and $k(t)$ is CM.

Definition 1.7. *A probability μ on $\mathbb{R}_+$ is called a* Generalized Gamma Convolution (GGC) *if*

$$\mathcal{L}(\mu; \lambda) = e^{-f(\lambda)}$$

for some $f \in$ TBF *and $f(0) = 0$.*

By Bernstein's theorem and the Frullani identity, we see that a positive random variable $\mathbf{X}$ with law μ has a GGC distribution if and only if its Laplace exponent reads

$$-\log \mathbb{E}[e^{-\lambda \mathbf{X}}] = b\lambda + \int_0^\infty (1 - e^{-\lambda x}) \, k(x) \frac{dx}{x} = b\lambda + \int_0^\infty \log(1 + \lambda u^{-1}) \rho(du) \tag{1.11}$$

for some $b \geq 0$ and where

$$k(x) = \int_0^\infty e^{-xu} \rho(du)$$

is a CM function with Bernstein measure ρ. Henceforth, this measure ρ will be called the Thorin measure of $\mathbf{X}$.

The following characterization of GGC random variables is an easy consequence of the second equality in (1.11) and an integration by parts. See Proposition 9.10 [80] for details.

Theorem 1.12. *A random variable $\mathbf{X} \sim \mu$ is a* GGC *iff*

$$-\log \mathbb{E}[e^{-\lambda \mathbf{X}}] = b\lambda + \int_0^\infty \frac{\lambda}{\lambda + u} \frac{\omega(u)}{u} dt$$

where $b \geq 0$ and $\omega : (0, \infty) \to [0, \infty)$ is a non-decreasing function such that $\int_0^\infty (1 + u)^{-1} u^{-1} \omega(u) dt < \infty$. Moreover, one has $\omega(u) = \rho(0, u]$ for every $u > 0$, where ρ is the Thorin measure of $\mathbf{X}$.

The denomination comes from the fact that GGC is the smallest class of probability measures on $[0, \infty)$ which contains all gamma distributions and which is closed under convolutions and vague limits. This class can also be identified as that of the Wiener-Gamma perpetuities

$$\int_0^\infty a(t) d\mathbf{\Gamma}_t$$

where $\{\mathbf{\Gamma}_t, \, t \geq 0\}$ is the Gamma subordinator and $a(t)$ a suitably integrable positive and deterministic function. The point of view of Wiener-Gamma perpetuities is further developed in the survey paper [48], but it will be barely touched upon in this thesis.

The following result establishes a link between the GGC family and that of Gamma mixtures. It is due to Bondesson - see Theorem 4.1.1 in [18].

Theorem 1.13. *Let* $\mathbf{X}$ *be a non-degenerate random variable having a GGC distribution with a finite Thorin measure* ρ*. Let* $\beta = \int_0^\infty \rho(du) \in (0,\infty)$ *be the total mass of* ρ*. Then, there exists a factorization*

$$\mathbf{X} \overset{d}{=} \mathbf{\Gamma}_\beta \times \mathbf{Y}$$

for some positive random variable $\mathbf{Y}$ *independent of* $\mathbf{\Gamma}_\beta$*.*

Bondesson also showed that if the density f of a GGC distribution is such that $f(x) \sim cx^{\beta-1}$ as $x \to 0$ for some $\beta, c > 0$, then the Thorin measure of the distribution must have total mass β. This fact can be used to show that certain positive ID random variables do not belong to the GGC class. For example, the half-Cauchy distribution with density

$$\frac{2}{\pi(1+x^2)}$$

is easily seen to be ID by Kristiansen's theorem. If it were a GGC, then one would have $\beta = 1$ and the above theorem would imply that the density would be CM, which is clearly false. Hence, the half-Cauchy distribution is not a GGC.

More recently, Bondesson also proved that the GGC class is stable by independent multiplication. See the main theorem in [19].

Theorem 1.14. *Let* $\mathbf{X} \in \mathrm{GGC}$ *and* $\mathbf{Y} \in \mathrm{GGC}$ *be independent random variables. Then* $\mathbf{X} \times \mathbf{Y} \in \mathrm{GGC}$*.*

This remarkable property raises the question of whether other ID subclasses are stable by independent multiplication. It is known that this stability is not true for the ID class itself. For example, the independent product of two Poisson distributions with parameter 1 does not satisfy Steutel's integro-differential equation and hence cannot be ID - see Example VI.12.15 in [84]. On the other hand, there is no such counterexample when one factor has an absolutely continuous distribution. In particular, it is natural to raise the following conjecture, for which we, unfortunately, found no answer.

Conjecture 1.1. *Let* $\mathbf{X} \in \mathrm{SD}$ *and* $\mathbf{Y} \in \mathrm{SD}$ *be independent random variables. Then* $\mathbf{X} \times \mathbf{Y} \in \mathrm{SD}$*.*

Another long-standing conjecture on GGC random variables made by Bondesson, which can be viewed as a companion to Theorem 1.14, is the following.

Conjecture 1.2. *Let* $\mathbf{X} \in \mathrm{GGC}$*. Then* $\mathbf{X}^q$ *has a* GGC *distribution for every* $q \geq 1$*.*

The notion of GGC can be extended to distributions on the real line.

Definition 1.8. *An ID probability distribution* μ *on* $\mathbb{R}$ *such that its Lévy measure has a density* $m(x)$ *such that* $xm(x)$ *and* $xm(-x)$ *are CM as a function of* x *on* $(0,+\infty)$ *is called an* Extended Generalized Gamma Convolution (EGGC).

Observe that the strict GGC property corresponds to the case where $m(x)$ vanishes on $(-\infty, 0)$. The following is the most basic example. It corresponds to a real stable random variable.

Example 1.3. *Let* $\nu(dx)$ *from (1.4) be*

$$\nu(dx) = \begin{cases} c_1|x|^{-\alpha-1}dx, & \text{if } x < 0; \\ c_2 x^{-\alpha-1}dx, & \text{if } x > 0. \end{cases} \tag{1.12}$$

with $c_1 \geq 0, c_2 \geq 0, c_1 + c_2 > 0,$ *and* $0 < \alpha < 2$*.*

1.6 Hyperbolic complete monotonicity

We last introduce an important subclass of densities and functions.

Definition 1.9. *A smooth function $f : (0,\infty) \to [0,\infty)$ such that $f(uv)f(u/v)$ is* CM *as a function of $v + \frac{1}{v}$ for every $u > 0$ is said to be* Hyperbolically Completely Monotone (HCM)*.*

Definition 1.10. *A positive random variable $\mathbf{X}$ is called* HCM *if it has a density which is HCM.*

There is a tight connection between HCM and generalized Gamma convolutions, as shown in the next two remarkable theorems, both due to Bondesson. See Theorems 5.1.2 and 6.1.1 in [18], respectively.

Theorem 1.15. *Any* HCM *random variable has a* GGC *distribution.*

Theorem 1.16. *A function $\varphi : [0,\infty) \to (0,\infty)$ is the Laplace transform of a* GGC *if and only if $\varphi(0) = 1$ and φ is* HCM.

We also have the following closure result, given as Theorem 5.1.3 in [18].

Theorem 1.17. *The class* HCM *is closed with respect to weak non-degenerate limits.*

We now list certain properties of HCM functions and densities, all to be found in Bondesson's booklet [18]. Assuming that the functions f_1, f_2, are HCM, we have

(i) The functions $f_1(cx), c > 0$ is HCM.

(ii) The pointwise product $f_1 \cdot f_2$ is HCM.

(iii) The functions $x^\beta f_1(x^\alpha)$ are HCM for $|\alpha| \le 1$ and $\beta \in \mathbb{R}$.

(iv) $f_1(0+) > 0$ if and only if f_1 is CM.

(v) If f_1 is decreasing, then the functions $f(x+\delta)$ and $x \mapsto \int_x^\infty (y-x)^{\gamma-1} f(y)dy$ are HCM for all $\gamma, \delta > 0$.

Observe that by (iii), Conjecture 1.2 is true for HCM random variables. Observe also, still by (iii), that

$$\mathbf{X} \in \mathrm{HCM} \Leftrightarrow \mathbf{X}^{-1} \in \mathrm{HCM}. \tag{1.13}$$

This implies that HCM is a true subclass of GGC since the inverse of an element of GGC may not even be ID. For example, if $\mathbf{U}$ is uniformly distributed on $(0,1)$, then $\mathbf{U}^{-1} = (\mathbf{U}^{-1} - 1) + 1$ is a GGC as the sum of an HCM random variable and a positive constant, but $\mathbf{U}$ is not ID since it has compact support. In view of (1.13), the following conjecture is natural

Conjecture 1.3. *For any positive random variable, one has $\mathbf{X} \in$ HCM if and only if $\mathbf{X} \in$ GGC and $\mathbf{X}^{-1} \in$ GGC.*

1.7 Stable distributions

The central limit theorem (CLT) states that the sum of a number of i.i.d. random variables with a finite variance will tend to a normal distribution as the number of variables grows. When the variance is infinite, the limit was called exceptional distributions by P. Lévy. Nowadays, we say that

Definition 1.11. *A random variable* $\mathbf{X}$ *is* $\boxed{\text{stable}}$ *iff it can be obtained as*

$$\frac{1}{b_n}(\mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_n - a_n) \xrightarrow{d} \mathbf{X}, \quad n \to \infty, \tag{1.14}$$

with $\mathbf{Y}_1, \mathbf{Y}_2, \cdots$ *i.i.d.,* $a_n \in \mathbb{R}$ *and* $b_n > 0$.

If $\mathrm{Var}[\mathbf{Y}_1] = \sigma^2 < \infty$, let $a_n = n\mathbb{E}[\mathbf{Y}_1], b_n = \sqrt{n\sigma^2}$, the CLT shows that normal distributions are all stable. Two equivalent definitions are formulated as follows.

Definition 1.12. *A random variable* $\mathbf{X}$ *is* $\boxed{\text{stable}}$ *iff for any* $n \geq 2$*, there exist a positive number* c_n *and a real number* d_n *such that*

$$\mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n \overset{d}{=} c_n\mathbf{X} + d_n \tag{1.15}$$

where $\mathbf{X}_1 \overset{d}{=} \mathbf{X}_2 \overset{d}{=} \cdots \overset{d}{=} \mathbf{X}$ *and independent. If* $d_n \equiv 0$*, we say that* $\mathbf{X}$ *is strictly stable.*

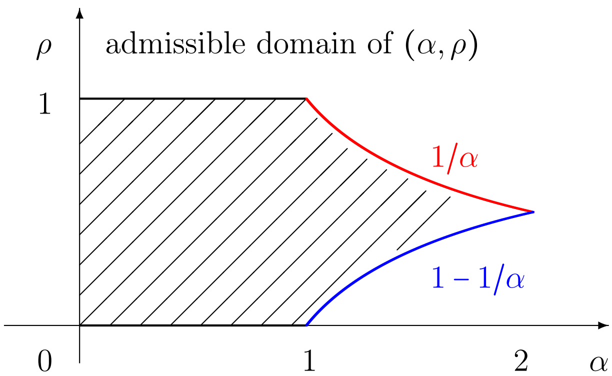
Definition 1.13. *A random variable* $\mathbf{X}$ *is* $\boxed{\text{stable}}$ *iff for any positive numbers* a *and* b*, there exist a positive number* c *and a real number* d *such that*

$$a\mathbf{X}_1 + b\mathbf{X}_2 \overset{d}{=} c\mathbf{X} + d \tag{1.16}$$

where $\mathbf{X}_1 \overset{d}{=} \mathbf{X}_2 \overset{d}{=} \mathbf{X}$ *and independent.*

Starting from the definition, we can get an expression of the characteristic functions, then using the formula of Fourier transform, we can get an expression of the density by the parameter of similarity α and the parameter of positivity ρ. Since a density is always non-negative, we can obtain an admissible domain of the parameters:

$$(\alpha, \rho) \in \mathcal{D} := \{\alpha \in (0,1], \rho \in [0,1]\} \cup \{\alpha \in (1,2], \rho \in [1 - 1/\alpha, 1/\alpha]\}$$



For $(\alpha, \rho) \in \mathcal{D}$, we will denote by $\mathbf{Z}_{\alpha,\rho}$ an $\mathbb{R}$-random variable having strictly α-stable distribution $\mu_{\alpha,\rho}$, and set $f_{\alpha,\rho}$ for its density, we will use the shorter notations $\mathbf{Z}_\alpha = \mathbf{Z}_{\alpha,1}$ and $f_\alpha = f_{\alpha,1}$. $\mathbf{Z}_{\alpha,\rho}$ is characterized by its charcteristic function

$$\mathbb{E}\left[e^{it\mathbf{Z}_{\alpha,\rho}}\right] = \exp\left[-|t|^\alpha e^{i\pi \mathrm{sgn}(t)(\frac{\alpha}{2} - \alpha\rho)}\right], \ \forall\, (\alpha, \rho) \in \mathcal{D}, \tag{1.17}$$

from which, we have

$$\mathbf{Z}_{\alpha,\rho} \stackrel{d}{=} -\mathbf{Z}_{\alpha,1-\rho}, \quad \forall\,(\alpha,\rho) \in \mathcal{D},$$

and we have necessarily

$$c_n = n^{1/\alpha} \text{ in (1.15)} \quad \text{and} \quad c^\alpha = a^\alpha + b^\alpha \text{ in (1.16)}$$

for some $\alpha \in (0,2]$.
Another useful characterization is the Mellin transform, which is of the form

$$\mathbb{E}[\mathbf{Z}_{\alpha,\rho}^{-s}\mathbf{1}_{\{\mathbf{Z}_{\alpha,\rho}>0\}}] = \frac{\Gamma(1-s)}{\alpha\pi}\sin(\pi\rho s)\Gamma(s/\alpha), \quad s \in (-\alpha,1). \tag{1.18}$$

Letting $s \to 0$, we obtain that

$$\mathbb{P}[\mathbf{Z}_{\alpha,\rho} \geq 0] \;=\; \rho, \quad \forall\;(\alpha,\rho) \in \mathcal{D},$$

that is why we call ρ the parameter of positivity. Observe that when $\rho = 1/2$, we have $\mathbf{Z}_{\alpha,1/2} \stackrel{d}{=} -\mathbf{Z}_{\alpha,1/2}$, which means $\mathbf{Z}_{\alpha,1/2}$ is symmetric for every $\alpha \in (0,2]$.

The density $f_{\alpha,\rho}$ has an explicit function in three specific situations only, which are:

- $f_{2,1/2}(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ for $x \in \mathbb{R}$, (standard gaussian density),
- $f_{1/2}(x) = \frac{1}{2\sqrt{\pi x^3}}e^{-\frac{1}{4x}}$ for $x \geq 0$, (inverse Gamma density),
- $f_{1,\rho}(x) = \dfrac{\sin(\pi\rho)}{\pi(x^2 + 2\cos(\pi\rho)x + 1)}$ for $x \in \mathbb{R}$, (standard Cauchy density with drift).

For general case, we have a convergent series representation, $\forall\alpha \in (0,1], \rho \in [0,1], \forall x > 0$,

$$f_{\alpha,\rho}(x) = \frac{1}{\pi}\sum_{n\geq 1}(-1)^{n-1}\frac{\Gamma(1+\alpha n)}{n!}\sin(n\pi\alpha\rho)x^{-\alpha n-1}; \tag{1.19}$$

and $\forall\alpha \in (1,2], \rho \in [1-1/\alpha, 1/\alpha], \forall x > 0$,

$$f_{\alpha,\rho}(x) = x^{-1-\alpha}f_{\frac{1}{\alpha},\alpha\rho}(x^{-\alpha}).$$

The relation in the last formula is the so-called duality law, it is equivalent to

$$\mathbf{Z}^+_{\alpha,\rho} \stackrel{d}{=} \left(\mathbf{Z}^+_{\frac{1}{\alpha},\alpha\rho}\right)^{-\frac{1}{\alpha}}, \quad \forall\,\alpha \in (1,2], \rho \in [1-1/\alpha, 1/\alpha], \tag{1.20}$$

where $\mathbf{X}^+$ is the *cutoff* of $\mathbf{X}$, i.e. the positive random variable with distribution function

$$F_{\mathbf{X}^+}(x) = P(\mathbf{X} \leq x | \mathbf{X} \geq 0), \quad x \geq 0.$$

For $\beta \in [-1,1]$, we will denote by $\mathbf{S}_\beta$ ($\beta \neq 0$) the 1-stable random variable having the density g_β defined in (A.23), and denote $\mathbf{S}_0$ the cauchy distribuiton with $c_0 = \frac{\pi}{2}$, and $c_1 = 0$. The characteristic function of $\mathbf{S}_\beta$ is of the form

$$\mathbb{E}\left[e^{it\mathbf{S}_\beta}\right] = \exp\left(-\frac{\pi}{2}|t| - i\beta t\log|t|\right).$$

Definition 1.14. *We say that two random variables* $\mathbf{X}, \mathbf{Y}$ *are equivalent if there exist* $a > 0, b \in \mathbb{R}$ *such that:* $\mathbf{X} \overset{d}{=} a\mathbf{Y} + b$*, and we denote by* $\mathbf{X} \sim \mathbf{Y}$*.*

Every α-stable random variable with $\alpha \neq 1$ is equivalent to some $\mathbf{Z}_{\alpha,\rho}$. But this is not the case for non-strictly 1-stable random variables, i.e. a non-strictly 1-stable random variable is never equivalent to a strictly stable one.

The proofs of all results in this section are given in Appendix A.

1.7.A Lévy-Khintchine representations of stable distributions

We need the Lemma 14.11 in Sato [79] which is formulated as follows,

Lemma 1.1.

$$\int_0^\infty (e^{ix} - 1)\frac{\alpha}{\Gamma(1-\alpha)x^{1+\alpha}}dx = -e^{-i\pi\alpha/2}, \quad \textit{for} \quad 0 < \alpha < 1, \tag{1.21}$$

$$\int_0^\infty (e^{ix} - 1 - ix)\frac{\alpha}{\Gamma(1-\alpha)x^{1+\alpha}}dx = -e^{-i\pi\alpha/2}, \quad \textit{for} \quad 1 < \alpha < 2, \tag{1.22}$$

$$\int_0^\infty (e^{ixz} - 1 - ixz\mathbf{1}_{(0,1]}(x))\frac{dx}{x^2} = -\frac{\pi z}{2} - iz\log z + icz, \quad \textit{for} \quad z > 0 \quad \textit{with} \tag{1.23}$$

$$c = \int_1^\infty x^{-2}\sin x dx + \int_0^1 x^{-2}(\sin x - x)dx.$$

Theorem 1.18. (i)*For* $\alpha \in (0,1), \rho \in [0,1]$,

$$\mathbb{E}\left[e^{it\mathbf{Z}_{\alpha,\rho}}\right] = \exp\left(\int_{-\infty}^\infty (e^{itx} - 1)\nu(dx)\right)$$

where

$$\nu(dx) = \begin{cases} c_1\frac{\alpha}{\Gamma(1-\alpha)x^{1+\alpha}}dx, & \textit{if} \quad x > 0; \\ c_2\frac{\alpha}{\Gamma(1-\alpha)|x|^{1+\alpha}}dx, & \textit{if} \quad x < 0. \end{cases} \tag{1.24}$$

with $c_1 = \sin(\pi\alpha\rho)/\sin(\pi\alpha)$ *and* $c_2 = \sin(\pi\alpha(1-\rho))/\sin(\pi\alpha)$*.*
(ii) *For* $\alpha \in (1,2), \rho \in [1 - 1/\alpha, 1/\alpha]$,

$$\mathbb{E}\left[e^{it\mathbf{Z}_{\alpha,\rho}}\right] = \exp\left(\int_{-\infty}^\infty (e^{itx} - 1 - itx)\nu(dx)\right)$$

where

$$\nu(dx) = \begin{cases} c_1\frac{\alpha}{\Gamma(1-\alpha)x^{1+\alpha}}dx, & \textit{if} \quad x > 0; \\ c_2\frac{\alpha}{\Gamma(1-\alpha)|x|^{1+\alpha}}dx, & \textit{if} \quad x < 0. \end{cases} \tag{1.25}$$

with $c_1 = \sin(\pi\alpha\rho)/\sin(\pi(2-\alpha))$ *and* $c_2 = \sin(\pi\alpha(1-\rho))/\sin(\pi(2-\alpha))$*.*
(iii)*For* $\beta \in [-1,1]$,

$$\mathbb{E}\left[e^{it\mathbf{S}_\beta}\right] = \exp\left(iat + \int_{-\infty}^\infty (e^{itx} - 1 - ixt\mathbf{1}_{[-1,1]}(x))\nu(dx)\right),$$

where $a \in \mathbb{R}$ *and*

$$\nu(dx) = \begin{cases} c_1 x^{-2}dx, & \textit{if} \quad x > 0, \\ c_2|x|^{-2}dx, & \textit{if} \quad x < 0, \end{cases} \tag{1.26}$$

with $c_1 = \frac{1+\beta}{2}, c_2 = \frac{1-\beta}{2}$.
(iv) *For* $\rho \in [0,1]$,

$$\mathbb{E}\left[e^{it\mathbf{Z}_{1,\rho}}\right] = \exp\left(i\tilde{a}t + \int_{-\infty}^{\infty}(e^{itx} - 1 - ixt\mathbf{1}_{[-1,1]}(x))\nu(dx)\right), \tag{1.27}$$

where $\tilde{a} \in \mathbb{R}$ *and* $\nu(dx) = \frac{\sin(\pi\rho)}{\pi x^2}\mathbf{1}_{x\neq 0}dx$.

Proof. The proof of (i), (ii) and (iii) are similar, we first derive the representations from the above lemma for the extreme cases, i.e. $\{\alpha \in (0,1), \rho = \pm 1\}$, $\{\alpha \in (1,2), \rho = \frac{1}{\alpha} \text{ or } 1 - \frac{1}{\alpha}\}$ and $\{\alpha = 1, \beta = \pm 1\}$. Then we can obtain the other cases by a simple linear combination. We do only (i) here.
From (1.21), we have

$$\mathbb{E}\left[e^{it\mathbf{Z}_\alpha}\right] = \exp(-e^{-i\pi \operatorname{sgn}(t)\alpha/2}|t|^\alpha) = \exp(\int_0^\infty (e^{itx} - 1)\frac{\alpha}{\Gamma(1-\alpha)x^{1+\alpha}}dx),$$

$$\mathbb{E}\left[e^{it\mathbf{Z}_{\alpha,0}}\right] = \exp(-e^{i\pi \operatorname{sgn}(t)\alpha/2}|t|^\alpha) = \exp(\int_{-\infty}^0 (e^{itx} - 1)\frac{\alpha}{\Gamma(1-\alpha)|x|^{1+\alpha}}dx).$$

For $\theta \in (-\alpha/2, \alpha/2)$, there exist $c_1 > 0, c_2 > 0$, such that $e^{i\pi\theta} = c_1 e^{-i\pi\alpha/2} + c_2 e^{i\pi\alpha/2}$, then $e^{i\pi(\theta+\alpha/2)} = c_1 + c_2 e^{i\pi\alpha}$, $c_2 = \sin(\pi(\alpha/2+\theta))/\sin(\pi\alpha)$, similarly, $c_1 = \sin(\pi(\alpha/2-\theta))/\sin(\pi\alpha)$. Substitute θ by $\frac{\alpha}{2} - \alpha\rho$, we obtain (i).
(iv) is a consequence of (iii), since $\mathbf{Z}_{1,\rho}$ with $\rho \in (0,1)$ is equivalent to $\mathbf{S}_0$. Note that $\mathbf{Z}_{1,1} \equiv 1$ and $\mathbf{Z}_{1,0} \equiv -1$.

□

Remark 1.1. (a) From the Lévy-Khintchine representation we see that $\forall\, \alpha \in (0,1), \rho \in [0,1]$,

$$\mathbf{Z}_{\alpha,\rho} \overset{d}{=} c_1^{1/\alpha}\mathbf{Z}_\alpha - c_2^{1/\alpha}\tilde{\mathbf{Z}}_\alpha$$

and $\forall\, \alpha \in (1,2), \rho \in [1-1/\alpha, 1/\alpha]$,

$$\mathbf{Z}_{\alpha,\rho} \overset{d}{=} c_1^{1/\alpha}\mathbf{Z}_{\alpha,\frac{1}{\alpha}} - c_2^{1/\alpha}\tilde{\mathbf{Z}}_{\alpha,\frac{1}{\alpha}},$$

where $\tilde{\mathbf{Z}}_{\alpha,\rho}$ is an independent copie of $\mathbf{Z}_{\alpha,\rho}$, c_1 and c_2 are the same as those in the above theorem.
(b) For $\beta \in [-1,1]$,

$$\mathbf{S}_\beta \overset{d}{=} -\mathbf{S}_{-\beta} \quad \text{and} \quad \mathbf{S}_\beta \overset{d}{=} \frac{1+\beta}{2}\mathbf{S}_1 - \frac{1-\beta}{2}\tilde{\mathbf{S}}_1,$$

where $\tilde{\mathbf{S}}_1$ is an independent copie of $\mathbf{S}_1$.

1.7.B Asymptotic expansions for the positive stable densities

Recall that the density f_α of $\mathbf{Z}_\alpha$ is $g_{\alpha,-\alpha/2}$ which has been studied in section A.1.B,

$$f_\alpha(x) = \frac{\alpha}{x^{1+\alpha}}\phi(-\alpha, 1-\alpha; -x^{-\alpha}) = \frac{\alpha}{x^{1+\alpha}}M_\alpha(x^{-\alpha}). \tag{1.28}$$

The function

$$M_\alpha(z) := \phi(-\alpha, 1-\alpha; -z), \quad 0 < \alpha < 1,$$

is the so-called M-Wright function which has been considered in details in [63, 64]. Mainardi and Tomirotti [63] have shown that for $\alpha = 1/q$, where $q \geq 2$ is a positive integer, the M-Wright function can be expressed as a sum of simpler $(q-1)$ entire functions. In the particular case $q = 2$ and $q = 3$, they obtained

$$M_{\frac{1}{2}}(z) = \frac{1}{\sqrt{\pi}} \exp(-z^2/4),$$

and

$$M_{\frac{1}{3}}(z) = 3^{2/3} \mathrm{Ai}(z/3^{1/3}),$$

where Ai denotes the Airy function:

$$\mathrm{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt \equiv \frac{1}{\pi} \lim_{b\to\infty} \int_0^b \cos\left(\frac{t^3}{3} + xt\right) dt.$$

Thus $f_{\frac{1}{2}}$ and $f_{\frac{1}{3}}$ can be expressed as

$$f_{\frac{1}{2}}(x) = \frac{1}{2\sqrt{\pi x^3}} \exp(-\frac{1}{4x}),\ x > 0, \quad \text{and} \quad f_{\frac{1}{3}}(x) = (3x^4)^{-\frac{1}{3}} \mathrm{Ai}((3x)^{-\frac{1}{3}}),\ x > 0.$$

They also gave their asymptotic representation for large $|z|$ using the saddle point method,

$$M_\alpha(z) \sim \frac{1}{\sqrt{2\pi(1-\alpha)}} (\alpha z)^{\frac{\alpha-1/2}{1-\alpha}} \exp\left(-\frac{1-\alpha}{\alpha} (\alpha z)^{\frac{1}{1-\alpha}}\right), \quad \text{as } |z| \to +\infty. \tag{1.29}$$

Therefore we have the following propositions.

Proposition 1.4. $\forall\, \alpha \in (0,1)$, *one has*

$$f_\alpha(x) \sim \frac{\alpha}{\Gamma(1-\alpha) x^{1+\alpha}} \quad \text{as } x \to +\infty \qquad \text{and} \qquad f_\alpha(x) \sim c_\alpha\, x^{-\frac{2-\alpha}{2-2\alpha}} e^{-(1-\alpha)(x/\alpha)^{-\frac{\alpha}{1-\alpha}}} \quad \text{as } x \to 0+,$$

with

$$c_\alpha = \frac{\alpha^{\frac{1}{2(1-\alpha)}}}{\sqrt{2\pi(1-\alpha)}}.$$

Remark 1.2. This proposition is a special case of proposition 2.1. Their relation is that $f_{1-\alpha}(x) = c f_{1,\alpha}(cx)$ with $c = (1-\alpha)^{\frac{1}{\alpha-1}}$.

1.7.C Kanter's factorization

The formula (4.5) in [64] stated that

$$\int_0^\infty x^s M_\alpha(x) dx = \frac{\Gamma(s+1)}{\Gamma(\alpha s+1)}, \quad s > -1, \quad \alpha \in (0,1),$$

which implies that

$$\mathbb{E}[\mathbf{Z}_\alpha^s] = \frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)}, \quad \text{for } s < \alpha.$$

Kanter [51, Corollary 4.1] found an independent factorization of the positive stable distributions

$$\mathbf{Z}_\alpha \overset{d}{=} \mathbf{L}^{1-1/\alpha} \times \mathbf{K}_\alpha, \tag{1.30}$$

where $\mathbf{L}$ has unit exponential distribution and $\mathbf{K}_\alpha$ is the so-called Kanter random variable having fractional moments

$$\mathbb{E}[\mathbf{K}_\alpha^s] = \frac{\Gamma(1-s/\alpha)}{\Gamma(1-(1/\alpha-1)s)\Gamma(1-s)}, \quad \text{for } s<\alpha, \tag{1.31}$$

and in particular has a support $[b_\alpha, +\infty)$ which is bounded away from zero, with

$$b_\alpha^{-1} = \alpha^{-1}(1-\alpha)^{1-\frac{1}{\alpha}} = \lim_{n\to+\infty} \mathbb{E}[\mathbf{K}_\alpha^{-n}]^{1/n},$$

by Stirling's formula (B.5).

Several analytical properties of the density of $\mathbf{K}_\alpha$ have been obtained in [50, 82]. In particular, Corollary 3.2 in [50] shows that

Proposition 1.5. *For every $s>0$, there exists $c_{\alpha,s}>0$ such that $\mathbf{K}_\alpha^s - c_{\alpha,s}$ is a mixture of exponential distribution. Particularly, the density of $\mathbf{K}_\alpha - b_\alpha$ is CM.*

We will use this fact repeatedly in the chapter 3.

1.7.D HCM property of positive stable distributions

Pierre Bosch and Thomas Simon [23] proved that

The density f_α is HCM if and only if $\alpha \le 1/2$.

This result was conjectured in 1977 by Bondesson. The only if part is easy to establish because the random variable $\mathbf{Z}_\alpha^{-1}$ is not infinitely divisible, and hence cannot have a HCM density by (1.13), when $\alpha > 1/2$. The if part is based on the following three lemmas.

Lemma 1.2. The density of the product

$$\mathbf{\Gamma}_c \times \mathbf{B}_{a_1,b_1} \times \cdots \times \mathbf{B}_{a_n,b_n}$$

is HCM for every $n \ge 1$, $a_i, b_i > 0$ and $c < \min\{a_i\}$.

Lemma 1.3. For every $\alpha \in (0,1)$, one has the a.s. convergent factorization

$$\mathbf{Z}_\alpha^{-1} \stackrel{d}{=} e^{\gamma(1-\alpha^{-1})} \times \prod_{n=0}^{\infty} a_n \mathbf{B}_{\alpha+n\alpha,1-\alpha}$$

where γ is the Euler's constant, and $a_n = e^{\psi(1+n\alpha)-\psi(\alpha+n\alpha)}$, ψ is the digamma function.

Lemma 1.4. For every $a, b > 0$, one has the a.s. convergent factorization

$$\mathbf{\Gamma}_a \stackrel{d}{=} e^{\psi(a)} \times \prod_{n=0}^{\infty} b_n \mathbf{B}_{a+nb,b}$$

with $b_n = e^{\psi(a+b+nb)-\psi(a+nb)}$.

Combining lemmas 1.3 and 1.4 with the elementary factorization

$$\mathbf{B}_{a,b+c} \stackrel{d}{=} \mathbf{B}_{a,b} \times \mathbf{B}_{a+b,c}$$

one has

$$\mathbf{Z}_\alpha^{-1} \stackrel{d}{=} e^{\gamma(1-\alpha^{-1})-\psi(\alpha)} \times \mathbf{\Gamma}_\alpha \times \prod_{n=0}^{\infty} e^{\psi(1+n\alpha)-\psi(2\alpha+n\alpha)} \mathbf{B}_{2\alpha+n\alpha,1-2\alpha}.$$

Applying lemma 1.2 and theorem 1.17 concludes the proof.
We will mimic this proof to prove a similar result for generalized stable distributions in section 2.2.B. Pierre Bosch and Thomas Simon conjectured at the end of [23] that

Conjecture 1.4. *The density of* $\mathbf{Z}_\alpha^q$ *is* HCM *if and only if* $\alpha \le \frac{1}{2}$ *and* $|q| \ge \frac{\alpha}{1-\alpha}$.

We have proved in Chapter 3 that $\mathbf{Z}^{-\frac{\alpha}{1-\alpha}}$ is not HCM for $\alpha < 1/5$, see Remark 3.9 (b). Thus this conjecture is not true in general.

1.7.E Shape of densities of stable distributions

It is known that two-sided stable densities are real analytic on $\mathbb{R}$, never vanishes, and that all their derivatives tend to zero at infinity. Hence, their n-th derivative vanishes at least n times on $\mathbb{R}$ by Rolle's theorem.

Definition 1.15. *A smooth non-negative function on some open interval* $I \subset \mathbb{R}$ *is said to be* *bell-shaped* (BS) *if it vanishes at both ends of* I*, and if*

$$\#\{x \in \operatorname{Supp} f,\ f^{(n)}(x) = 0\} \;=\; n$$

for every $n \ge 1$.

W. Gawronski proved in [40] that two-sided α-stable densities are bell-shaped when $\alpha = 2$ or $1/n$ for some $n = 1, 2, 3, \cdots$. T. Simon proved in [83] that positive stable distributions have bell-shaped density functions. Recently, M. Kwasnicki proved in [58] that

Theorem 1.19 (Corollary 1.3 in [58])**.** *All stable distributions on* $\mathbb{R}$ *have bell-shaped density functions.*

Bell-shaped (BS) functions

Moreover, Kwasnicki has discovered a large class of functions f that are bell-shaped, including all smooth densities of EGGCs.

Theorem 1.20 (Theorem 1.1 in [58])**.** *Suppose that* f *is a locally integrable function which converges to zero at* $\pm\infty$*, and which is decreasing near* ∞ *and increasing near* $-\infty$*. Suppose furthermore that for* $\xi \in \mathbb{R}\backslash\{0\}$ *the fourier transform of* f *satisfies*

$$\begin{aligned}\mathcal{L}(f; i\xi) &:= \int_{-\infty}^{\infty} e^{-i\xi x} f(x)dx \\ &= \exp\left[-a\xi^2 - ib\xi + c + \int_{-\infty}^{\infty}\left(\frac{1}{i\xi + s} - \left(\frac{1}{s} - \frac{i\xi}{s^2}\right)\mathbf{1}_{\mathbb{R}\backslash(-1,1)}(s)\right)\varphi(s)ds\right]\end{aligned}$$

with $a \ge 0, b, c \in \mathbb{R}$*, and* $\varphi : \mathbb{R} \to \mathbb{R}$ *is a function with the following properties:*

1. *for every* $k \in \mathbb{Z}$ *the function* $\varphi(s) - k$ *changes its sign at most once, and for* $k = 0$ *this change takes place at* $s = 0 : s\varphi(s) \ge 0$*;*
2. *we have*
$$\left(\int_{-\infty}^{-1} + \int_1^{\infty}\right)\frac{|\varphi(s)|}{|s^3|}ds < \infty;$$
3. *we have*
$$\int_{-1}^{1} \Re\mathcal{L}(f; i\xi)d\xi < \infty, \qquad \textit{and} \qquad \lim_{\xi\to 0} \Im\mathcal{L}(f; i\xi) = 0.$$

If in addition f *is smooth, then* f *is bell-shaped.*

We will use this theorem to prove Theorem 3.4 (d).

Bell-shaped of order n (BS$_n$) densities

For a given smooth density f on $(0,\infty)$ and $n \geq 0$, let us introduce the following property: one has $f \in \mathrm{BS}_n$ if

$$\begin{cases} \sharp\{x > 0, \;\; f^{(i)}(x) = 0\} \; = \; i & \text{for } i \leq n, \\ \sharp\{x > 0, \;\; f^{(i)}(x) = 0\} \; = \; n & \text{for } i > n. \end{cases}$$

For $n \geq 0$, this property was introduced in [83] under the less natural denomination WBS$_{n-1}$ - see the definition therein. Clearly, one has BS$_0$ = CM and BS$_1$ corresponds to whale-shaped (see Definition 3.1) functions supported on $(0,+\infty)$, which plays a role in chapter 3. Since the density of $\mathbf{\Gamma}_t$ has $m-$th derivative

$$(-1)^m \left(\sum_{p=0}^{m} \binom{m}{p} (1-t)_p \, x^{-p} \right) \frac{x^{t-1} e^{-x}}{\Gamma(t)}$$

on $(0,\infty)$, it is an easy exercise using Rolle's theorem and Descartes' rule of signs to show that $\mathbf{\Gamma}_t \in \mathrm{BS}_n$ for $t \in (n, n+1]$. In this respect, the class BS$_n$ can be thought of as an extension of the densities of $\mathbf{\Gamma}_t$ for $t \in (n, n+1]$. Moreover, we have just seen that the set of densities of $\Gamma_{n+1}-$mixtures contains the class BS$_n$ for $n = 0, 1$. We actually believe that this is true for all $n \geq 0$. The following proposition gives us more BS$_n$ densities.

Proposition 1.6 (Proposition in [83]). *Let $X \in \mathrm{ME}^*$ and $\lambda_i > 0$ for all $i \in \mathbb{N}^*$. For every $n \geq 0$, the independent sum $X + Exp(\lambda_1) + \cdots + Exp(\lambda_n)$ has a BS$_n$ density.*

We will use this proposition to prove Theorem 3.4 (c).

Chapter 2

Density solutions to a class of integro-differential equations

2.1 Introduction and statement of the results

In this paper, we are concerned with the following integro-differential equation

$$x^m f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-v)^{\alpha-1} f(v)\, dv \tag{2.1}$$

on $(0, \infty)$, with $\alpha > 0$ and $m \in \mathbb{R}$. This equation can be written in a more compact way as

$$\mathrm{I}^{\alpha}_{0+} f = x^m f,$$

where I^{α}_{0+} is the left-sided Riemann-Liouville fractional integral on the half-axis. We refer to the comprehensive monograph [55] for more details on fractional operators and the corresponding differential equations. We are interested in density solutions to (2.1), that is we are searching for such f satisfying (2.1) which are also probability densities on $(0, \infty)$. In this framework, the identities (2.1.31) and (2.1.38) in [55] imply that the auxiliary function $h = \mathrm{I}^{\alpha}_{0+} f$ is a solution to the fractional differential equation

$$\mathrm{D}^{\alpha}_{0+} h = x^{-m} h, \tag{2.2}$$

where D^{α}_{0+} is the left-sided Riemann-Liouville fractional derivative. This latter equation can be solved in the case $m = 1$ in terms of the classical Wright function - see Theorem 5.10 in [55], and we will briefly come back to this example in Section 3.

Observe that density solutions to (2.1) may not exist. If $\alpha = m$ for example, then (2.1) becomes

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-v)^{\alpha-1} f(xv)\, dv,$$

and the integral of the right-hand side over $(0, +\infty)$ is infinite while the integral of the left-hand side is 1 if f is a density function. In this respect, let us also notice that the arbitrary constant $\Gamma(\alpha)$ in (2.1) was chosen without loss of generality: if $f_{m,\alpha}$ is a density solution to (2.1), then $f_{c,m,\alpha}(x) = c f_{m,\alpha}(cx)$ is for every $c > 0$ a density solution to

$$x^m f(x) = \frac{c^{\alpha-m}}{\Gamma(\alpha)} \int_0^x (x-v)^{\alpha-1} f(v) dv.$$

Let us start with a few examples. When $\alpha = n$ is a positive integer, then (2.1) becomes an ODE of order n satisfied by the $n-$th cumulative distribution function

$$F_n(x) \;=\; \int_{0<x_1<\ldots<x_n<x} f(x_1)\,dx_1\ldots dx_n,$$

which is

$$F_n \;=\; x^m F_n^{(n)}.$$

- For $\alpha = 1$, we solve $F_1 = x^m F_1'$ with F_1 bounded and vanishing at zero. This implies that F_1' is a density iff $m > 1$ with $F_1(x) = e^{-\frac{x^{1-m}}{(m-1)}}$, that is $f_{m,1} = F_1'$ is the density of the Fréchet random variable $((m-1)\boldsymbol{\Gamma}_1)^{\frac{1}{1-m}}$ where, here and throughout, $\boldsymbol{\Gamma}_t$ denotes a Gamma random variable of parameter $t > 0$, with density

$$f_{\boldsymbol{\Gamma}_t}(x) \;=\; \frac{x^{t-1}e^{-x}}{\Gamma(t)}, \qquad x > 0.$$

- For $\alpha = 2$, we solve $F_2 = x^m F_2''$ with F_2 having linear growth at infinity and vanishing at zero. Supposing $m > 2$ and making the substitution $K(x) = x^\nu F_2((x/2\nu)^{-2\nu})$ with $\nu = 1/(m-2)$, we obtain Bessel's modified differential equation

$$x^2K'' + xK' - (x^2+\nu^2)K \;=\; 0,$$

whose solutions satisfying the required properties for F_2 are constant multiples of the Macdonald function K_ν. A density solution to (2.1) is then

$$f_{m,2}(x) \;=\; F_2''(x) \;=\; x^{-m}F_2(x) \;=\; c_\nu\, x^{-3/2-1/\nu} K_\nu(2\nu x^{-1/2\nu}),$$

where c_ν is the normalizing constant. On the other hand, a computation using e.g. the formula 7.12(23) p.82 in [34] shows that the independent product $\boldsymbol{\Gamma}_1 \times \boldsymbol{\Gamma}_{\nu+1}$ has density

$$\frac{2\,x^{\frac{\nu}{2}}}{\Gamma(\nu+1)}\,K_\nu(2\sqrt{x})\,\mathbf{1}_{(0,\infty)}(x).$$

By a change of variable, this implies that $f_{m,2}$ is the density of $((m-2)\sqrt{\boldsymbol{\Gamma}_1 \times \boldsymbol{\Gamma}_{\nu+1}})^{\frac{2}{2-m}}$.

For $\alpha \geq 3$, the resulting ODE's have higher order and do not seem to exhibit any classical special function. In section 3, however, we will see that the density solutions to (2.1) can be characterized in terms of the Gamma distribution for all integer values of α.

When α is not a positive integer (2.1) is a true integro-differential equation, which can be handled via the Laplace transform

$$\mathcal{L}(\lambda) \;=\; \int_0^\infty e^{-\lambda x} f(x)\,dx.$$

In particular, when $m = n$ is a positive integer, the latter satisfies an ODE of order n analogous to the above, which is

$$\mathcal{L} \;=\; (-1)^n \lambda^\alpha \mathcal{L}^{(n)}.$$

- For $m = 1$, we solve $\mathcal{L} = -\lambda^\alpha \mathcal{L}'$ with $\mathcal{L}$ a completely monotone (CM) function satisfying $\mathcal{L}(0) = 1$. This implies that there is a density solution to (2.1) iff $\alpha \in (0,1)$ with $\mathcal{L}(\lambda) = e^{-\frac{\lambda^{1-\alpha}}{(1-\alpha)}}$, that is $f_{1,\alpha}$ is the density of $(1-\alpha)^{\frac{1}{\alpha-1}} \mathbf{Z}_{1-\alpha}$ where, here and throughout, $\mathbf{Z}_\beta$ is the standard positive β-stable random variable with Laplace transform
$$\mathbb{E}[e^{-\lambda \mathbf{Z}_\beta}] = e^{-\lambda^\beta}, \qquad \lambda \geq 0. \tag{2.3}$$
- For $m = 2$, we solve $\mathcal{L} = \lambda^\alpha \mathcal{L}''$ with the same restrictions on $\mathcal{L}$. Supposing $\alpha < 2$ and setting $\nu = 1/(2-\alpha)$, the same reasoning as above leads to
$$\mathcal{L}(\lambda) = \frac{2\nu^\nu}{\Gamma(\nu)} \sqrt{\lambda}\, K_\nu(2\nu\lambda^{\frac{1}{2\nu}}) = \mathbb{E}[e^{-\nu^2 \lambda^{\frac{1}{\nu}} \mathbf{\Gamma}_\nu^{-1}}],$$
where the second equality follows again from the formula 7.12(23) p.82 in [34]. For $\alpha = 1 = \nu$, we recover the above Fréchet density $f_{2,1}(x) = x^{-2} e^{-\frac{1}{x}}$. For $\alpha \in (1,2)$, it follows from (2.3) that $f_{2,\alpha}$ is the density of the independent product $\nu^{2\nu} \mathbf{\Gamma}_\nu^{-\nu} \times \mathbf{Z}_{\frac{1}{\nu}}$. For $\alpha \in (0,1)$, it is not clear from classical integral formulæ on the Macdonald function that the above $\mathcal{L}$ is indeed a CM function viz. $f_{2,\alpha}$ is a density. In Section 3, we will see that $f_{2,\alpha}$ is for all $\alpha \in (0,2)$ the density of a certain random variable involving two independent copies of $\mathbf{Z}_{1-\frac{\alpha}{2}}$.

The study of density solutions to (2.1) for m a positive integer was initiated in [81] and then pursued in [74], where the corresponding random variables are called "generalized stable". Apart from the classical stable case $m = 1$, these random variables are of interest in the case $\{m = 2, \alpha \in (0,1)\}$ which is especially investigated in Section 3 of [81] and Section 7 of [74], because of its connections to particle transport along the one-dimensional lattice - see [15]. The paper [81] takes the point of view of Fox functions and shows that for all $m \in \mathbb{N}^*, \alpha \in (0,1)$ there exists a density solution to (2.1) having a convergent power series representation at infinity and a Fréchet-like behavior at zero - see (2.12) and (2.15) therein. The paper [74] takes the point of view of size-biasing and shows that for all $m \in \mathbb{N}^*, \alpha \in (0,m)$ there exists a unique density solution to (2.1), whose corresponding random variable can be represented in the case $\alpha \in (m-1,m)$ as a finite independent product involving the random variables $\mathbf{Z}_\beta$ and $\mathbf{\Gamma}_t$ - see Theorems 4.3 and 4.2 therein.

In this paper, we characterize the existence and unicity of density solutions to (2.1) for all $m \in \mathbb{R}$ and $\alpha > 0$, and we obtain a representation of the corresponding random variables as two infinite products involving the Beta random variable $\mathbf{B}_{a,b}$, whose density is recalled to be
$$f_{\mathbf{B}_{a,b}}(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \qquad x \in (0,1).$$
Here and throughout, all infinite products are assumed to be independent and a.s. convergent. Our main result reads as follows.

Theorem 2.1. *The equation (2.1) has a density solution if and only if $m > \alpha$. This solution is unique, and it is the density of*
$$X_{m,\alpha} \stackrel{d}{=} \left(a^{\frac{m-a}{a}} \Gamma(\frac{m}{a}) \prod_{n=0}^\infty \left(\frac{m+an}{a+an}\right) \mathbf{B}_{a+an,m-a} \right)^{-1} \stackrel{d}{=} \left(\frac{\Gamma(m)}{\Gamma(a)} \prod_{n=0}^\infty \left(\frac{m+n}{a+n}\right) \mathbf{B}_{1+\frac{n}{a},\frac{m}{a}-1} \right)^{-\frac{1}{a}},$$
with the notation $a = m - \alpha$.

The fact that the two above infinite products are actually a.s. convergent is an easy consequence of the martingale convergence theorem - see the beginning of Section 2.1 in [59] and the references therein. The proof of the above theorem relies on the Mellin transform of f, which is by (2.1) the solution to a functional equation of first order given as (2.9) below. This kind of equation has often been encountered in the recent literature, with various point of views - see e.g. [87, 73, 57, 75]. If f is assumed to be a density, then (2.1) or (2.9) amount to a random contraction equation

$$Y \stackrel{d}{=} \mathbf{B}_{m-\alpha,\alpha} \times \hat{Y}_{m-\alpha} \tag{2.4}$$

connecting a random variable Y and its size-bias $\hat{Y}_{m-\alpha}$, with the notations of the beginning of Section 2 in [74]. Our two product representations are then essentially a consequence of Theorem 3.5 in [73] and Lemma 3.2 in [74]. However, these simple representations do not seem to have been observed as yet, see in this respect the bottom of p.208 in [74].

Throughout, motivated by precise asymptotics analogous to (2.15) in [81], we will also connect the Mellin transform of the solution to (2.4) to the double Gamma function $G(z;\tau), z,\tau > 0$. This function, also known as the Barnes function for $\tau = 1$, was introduced in [8] as a generalization of the Gamma function. It fulfils the functional equations

$$G(z+1;\tau) = \Gamma(z\tau^{-1})G(z;\tau) \quad \text{and} \quad G(z+\tau;\tau) = (2\pi)^{\frac{\tau-1}{2}}\tau^{\frac{1}{2}-z}\Gamma(z)G(z;\tau) \tag{2.5}$$

with normalization $G(1;\tau) = 1$. The link between these two functional equations and that of (2.9) was thoroughly investigated in [57] in the framework of Lévy perpetuities - see Section 3 therein. The normalization also implies

$$G(\tau;\tau) = \frac{(2\pi)^{\frac{\tau-1}{2}}}{\sqrt{\tau}} \tag{2.6}$$

for every $\tau > 0$, which will be used henceforth.

A consequence of our main result is the solution to open problems related to the infinite divisibility of the density solutions to (2.1), recently formulated in [74]. Recall that the law of a positive random variable X is called a generalized Gamma convolution ($X \in$ GGC for short) iff its log-Laplace exponent reads

$$-\log \mathbb{E}[e^{-\lambda X}] = a\lambda + \int_0^\infty (1-e^{-\lambda x})k(x)\frac{dx}{x} = a\lambda + \int_0^\infty \log(1+\lambda u^{-1})\mu(du) \tag{2.7}$$

with $a \geq 0$ and

$$k(x) = \int_0^\infty e^{-xu}\,\mu(du)$$

is a CM function, whose Bernstein measure μ is called the Thorin measure of X.

We now suppose $m > \alpha$ and denote by $X_{m,\alpha}$ the positive random variable whose density $f_{m,\alpha}$ is the unique density solution to (2.1). The law of $X_{m,\alpha}$ will be denoted by $Gst(m,\alpha)$ and called generalized stable with parameters m and α, whereas the law of $Y_{m,\alpha} = X_{m,\alpha}^{-1}$ will be denoted by r$Gst(m,\alpha)$, in accordance with the terminology of [74]. Observe from (2.1) that the density $g_{m,\alpha}(x) = x^{-2}f_{m,\alpha}(x^{-1})$ of $Y_{m,\alpha}$ is such that $h_{m,\alpha}(x) = x^{1-\alpha}g_{m,\alpha}(x)$ is a positive solution to

$$\mathrm{I}_-^\alpha h = x^{2\alpha-m}h,$$

where I_-^α is the right-sided Riemann-Liouville fractional integral on the half-axis. This dual equation to (2.1) is the one appearing in a physical context for $m = 2$ - see (27) in [15].

Corollary 2.1. *With the above notations, one has:*

(a) $X_{m,\alpha} \in GGC$ *for all* $m > \alpha$.

(b) $X_{m,\alpha} \in HCM \Leftrightarrow Y_{m,\alpha} \in ID \Leftrightarrow m \le 2\alpha$.

Part (a) of the corollary is a generalization of Theorem 5.3 in [74], solving the open questions formulated thereafter. Besides, by e.g. Theorem III.4.10 in [84], it shows that the density solution $f_{m,\alpha}$ to (2.1) is also the unique density solution to

$$xf(x) = \int_0^x k_{m,\alpha}(x-y)\, f(y)\, dy \tag{2.8}$$

where $k_{m,\alpha}$ is the CM function associated to $X_{m,\alpha}$ through (2.7). This latter equation is known as Steutel's integro-differential equation for infinitely divisible densities. Except in the obvious case $m = 1$, the link between the two convolution kernels $(x-y)^{\alpha-1}$ and $k_{m,\alpha}(x-y)$ is mysterious, and the function $k_{m,\alpha}$ is not explicit in general. See however Section 3 for an analytical treatment of $k_{m,\alpha}$ in the cases $m = 2\alpha$ and $m = 2$. Part (b) gives a characterization of the infinite divisibility of the law r$Gst(m,\alpha)$, and it is an extension of Theorem 5.1 in [74]. It can also be viewed as a generalization of the main result of [23], which handles the case $m = 1$. As will be observed in Remark 2.1 (a) below, its proof also allows us to solve entirely the open question stated after Theorem 3.2 in [74].

We now turn to the asymptotic behavior of the densities $f_{m,\alpha}$ at zero and infinity. This is a basic question for the classical special functions, which is investigated e.g. all along [34]. When m is an integer, the densities $f_{m,\alpha}$ are Fox functions and in [81], the general results of [24] are used in order to derive convergent power series representations at infinity, with an exact first order polynomial term, as well as a non-trivial exponentially small behavior at zero - see (2.20) and (2.21) therein. In general, $f_{m,\alpha}$ is not a Fox function. Indeed, following from (2.10) (B.6) and (B.7), $f_{m,\alpha}$ is a Fox function iff there exist $a_1,\cdots,a_M,c_1,\cdots,c_N \ge 0$ and $b_1,\cdots,b_M,d_1,\cdots,d_N > 0$ such that

$$\frac{1-z^\alpha}{(1-z)(1-z^{m-\alpha})} = \sum_{i=1}^{M}\frac{z^{a_i}}{1-z^{b_i}} - \sum_{i=1}^{N}\frac{z^{c_i}}{1-z^{d_i}}, \quad \forall\, z \in (0,1).$$

The answer is positive if $\alpha \in \mathbb{N}$, if $m \in \mathbb{N}$, or if $\frac{m}{m-\alpha} \in \mathbb{N}$, which will be studied in section 2.3.A. But it is not always right, for example, $\alpha = 1/2, m = 3/2$. However, we can show the following estimates, which generalize (2.20) and (2.21) in [81].

Proposition 2.1. *With the above notation, one has*

$$f_{m,\alpha}(x) \sim \frac{x^{\alpha-m-1}}{\Gamma(\alpha)} \quad \text{as } x\to\infty \qquad \text{and} \qquad f_{m,\alpha}(x) \sim c_{m,\alpha}\, x^{-\frac{m(1+\alpha)}{2\alpha}} e^{-(\frac{\alpha}{m-\alpha})x^{\frac{\alpha-m}{\alpha}}} \quad \text{as } x\to 0,$$

with

$$c_{m,\alpha} = \frac{(2\pi)^{\frac{m-2}{2}}(m-\alpha)^{\frac{\alpha(1-m)}{2(m-\alpha)}}}{\sqrt{\alpha}\, G(m,m-\alpha)}.$$

The estimate at infinity is an elementary consequence of (2.1). The derivation of the estimate at zero, much more delicate, is centered around the exact case $m = 2\alpha$ which corresponds to the Fréchet random variable $\mathbf{\Gamma}_\alpha^{-1}$. When $m > 2\alpha$, the underlying random variable is the exponential functional of a Lévy process without negative jumps, and we can apply the recent Tauberian results of [75]. To handle the case $m < 2\alpha$ which has fat exponential tails, we perform an induction based on a multiplicative identity in law involving the above $\mathbf{\Gamma}_\alpha^{-1}$.

The next section is devoted to the proof of the three above results. In the last section, we display several remarkable factorizations generalizing those of [74], and we investigate the corresponding Fox function representations and convergent power series expansions. We also discuss some explicit Thorin measures coming from (2.7) and the behavior of the laws $Gst(m,\alpha)$ when $\alpha \to 0$ and $\alpha \to m$.

2.2 Proofs

2.2.A Proof of the theorem

We begin with the only if part. Introduce the Mellin transform

$$\mathcal{M}(s) = \int_0^\infty x^{-s} f(x) dx$$

which is well-defined for all $s \in \mathbb{R}$ with possibly infinite values, since f is non-negative. By Fubini's theorem - see also Lemma 2.15 in [55], we readily deduce from (2.1) the functional equation

$$\frac{\mathcal{M}(s+a)}{\mathcal{M}(s)} = \frac{\Gamma(m+s)}{\Gamma(a+s)}, \qquad s > -a. \tag{2.9}$$

By strict log-convexity of the Gamma function and since $m > m - \alpha = a$, we observe that the right-hand side of (2.9) is increasing in s. Since $s \mapsto \mathcal{M}(s)$ is also log-convex by Hölder's inequality, the left-hand side of (2.9) is non-increasing in s when $a \le 0$. All of this shows that there is a density solution to (2.1) only if $a > 0 \Leftrightarrow m > \alpha$.

We now proceed to the if part. On the one hand, the functional identity (2.5) shows that for all $a \in (0,m)$, the function

$$\mathcal{M}(s) \;=\; a^{\frac{(m-a)s}{a}} \times \frac{G(m+s,a)G(a,a)}{G(a+s,a)G(m,a)}, \qquad s > -a, \tag{2.10}$$

is a solution to (2.9). On the other hand, Proposition 2 in [59] implies that this function equals

$$\mathbb{E}\left[\left(a^{\frac{m-a}{a}} \Gamma(\frac{m}{a}) \prod_{n=0}^{\infty} \left(\frac{m+an}{a+an}\right) \mathbf{B}_{a+an,m-a}\right)^s\right], \qquad s > -a.$$

This shows that there is a density solution to (2.1) for $m > \alpha$, which is that of $X_{m,\alpha}$ defined by the first product representation. To obtain uniqueness, we use the same argument as in [74]. If f is a density solution to (2.1) and if Y is the random variable with density $g(x) = x^{-2} f(x^{-1})$, we deduce from (2.1) the identity

$$g(x) \;=\; \frac{\Gamma(m)}{\Gamma(a)\Gamma(m-a)} \int_0^1 t^{a-1}(1-t)^{m-a-1} \left(\frac{(xt^{-1})^a}{\mathcal{M}_g(a)} g(xt^{-1})\right) \frac{dt}{t}$$

with the notation

$$\mathcal{M}_g(s) = \int_0^\infty x^{-s} f(x) dx \;=\; \mathbb{E}[Y^s].$$

This translates into the random contraction equation

$$Y \stackrel{d}{=} \mathbf{B}_{a,m-a} \times \hat{Y}_a$$

where $\hat{Y}_a$ is the size-bias of order a of Y, having density $\frac{x^a g(x)}{\mathcal{M}_g(a)}$. By Theorem 3.5 in [73], the solutions to this random equation are unique up to scale transformation. Since (2.9) implies the normalization $\mathbb{E}[Y^a] = \frac{\Gamma(m)}{\Gamma(a)}$, we finally obtain the uniqueness of g, and that of f as well.

To conclude the proof, it remains to show the identity in law between the two product representations of $X_{m,\alpha}$. This is actually given as Lemma 3.2 in [74], but we provide here a simple and separate argument. Setting $\mathcal{M}_a(s) = \mathcal{M}(as)$ transforms (2.9) into

$$\frac{\mathcal{M}_a(s+1)}{\mathcal{M}_a(s)} = \frac{\Gamma(m+as)}{\Gamma(a+as)},$$

whose solution is unique thanks to the main result of [87] and the log-convexity of the Gamma function. The functional identity (2.5) shows that this solution is given by

$$\mathcal{M}_a(s) = \frac{G(\frac{m}{a}+s,\frac{1}{a})G(1,\frac{1}{a})}{G(1+s,\frac{1}{a})G(\frac{m}{a},1)} = \mathbb{E}\left[\left(\frac{\Gamma(m)}{\Gamma(a)}\prod_{n=0}^{\infty}\left(\frac{m+n}{a+n}\right)\mathbf{B}_{1+\frac{n}{a},\frac{m}{a}-1}\right)^s\right], \qquad s > -1,$$

where the second equality follows again from Proposition 2 in [59]. this completes the proof.

□

2.2.B Proof of the Corollary

It is well-known and easy to see from the expression of its density that $\mathbf{B}_{b,c}^{-1} - 1 \in \mathrm{HCM} \subset \mathrm{GGC}$ for every $b, c > 0$, so that $\mathbf{B}_{b,c}^{-1} \in \mathrm{GGC}$ as well. The first infinite product representation in the Theorem and the main result of [19] imply that $X_{m,\alpha} \in \mathrm{GGC}$, which concludes the proof of Part (a).

The first inclusion $X_{m,\alpha} \in \mathrm{HCM} \Rightarrow Y_{m,\alpha} \in \mathrm{ID}$ of Part (b) is an obvious consequence of (1.13). As in Theorem 5.1 (b) of [74], the second inclusion $Y_{m,\alpha} \in \mathrm{ID} \Rightarrow m \le 2\alpha$ follows from well-known bounds on the upper tails of positive ID distributions - see e.g. Theorem III.9.1 in [84], and the small-ball estimate

$$x^{\frac{\alpha-m}{\alpha}} \log \mathbb{P}[X_{m,\alpha} < x^{-1}] = x^{\frac{\alpha-m}{\alpha}} \log \mathbb{P}[Y_{m,\alpha} > x] \to -\left(\frac{\alpha}{m-\alpha}\right), \qquad x \to \infty. \tag{2.11}$$

When m is a positive integer, the latter estimate is a consequence of (2.15) in [81], taking into account the normalization (2.1) therein. To prove (2.11) in the general case, we consider the random variable $Z_{m,\alpha} = (Y_{m,\alpha})^{\frac{m-\alpha}{\alpha}}$ and we study the behavior of its positive entire moments through the quantities

$$a_n = \frac{(\mathbb{E}[(Z_{m,\alpha})^n])^{\frac{1}{n}}}{n} = \frac{a}{n}\left(\frac{G(m+bn,a)G(a,a)}{G(a+bn,a)G(m,a)}\right)^{\frac{1}{n}},$$

where the second equality follows from (2.10), recalling the notation $a = m-\alpha$ and having set $b = \frac{a}{m-a}$. By Stirling's formula and the estimate (4.5) in [17], we obtain

$$\lim_{n\to\infty} a_n = \frac{b}{\mathrm{e}} \tag{2.12}$$

Lemma 3.2 in [28] states that for an almost surely non-negative random variable, if

$$\lim_{n\to\infty} \frac{(\mathbb{E}[(X)^n])^{\frac{1}{n}}}{n} = a$$

for some constant $a \in (0, \infty)$, then

$$\log \mathbb{P}(X > x) \sim -\frac{x}{ae}, \quad x \to \infty,$$

which implies (2.11).
In order to show the last inclusion $m \le 2\alpha \Rightarrow X_{m,\alpha} \in \text{HCM}$, we will use the argument of the main result in [23]. If $m = 2\alpha$, then the first product representation and Lemma 3 in [23] imply

$$Y_{2\alpha,\alpha} \stackrel{d}{=} \alpha \prod_{n=0}^{\infty} \left(\frac{n+2}{n+1}\right) \mathbf{B}_{\alpha(n+1),\alpha} \stackrel{d}{=} c_\alpha \mathbf{\Gamma}_\alpha$$

for some normalizing constant c_α which is here one, since the infinite product has unit expectation. Hence

$$X_{2\alpha,\alpha} \stackrel{d}{=} \mathbf{\Gamma}_\alpha^{-1} \in \text{HCM}. \tag{2.13}$$

If $m < 2\alpha$ viz. $m > 2a$, the same argument shows that

$$\begin{aligned} Y_{m,\alpha} &\stackrel{d}{=} a^{\frac{m-a}{a}} \Gamma(\frac{m}{a}) \prod_{n=0}^{\infty} \left(\frac{m+an}{a+an}\right) \mathbf{B}_{a+an,m-a} \\ &\stackrel{d}{=} a^{\frac{m-a}{a}} \Gamma(\frac{m}{a}) \left(\prod_{n=0}^{\infty} \left(\frac{n+2}{n+1}\right) \mathbf{B}_{a(n+1),a}\right) \times \left(\prod_{n=0}^{\infty} \left(\frac{m+an}{2a+an}\right) \mathbf{B}_{2a+an,m-2a}\right) \\ &\stackrel{d}{=} a^{\frac{m-2a}{a}} \Gamma(\frac{m}{a})\, \mathbf{\Gamma}_a \times \left(\prod_{n=0}^{\infty} \left(\frac{m+an}{2a+an}\right) \mathbf{B}_{2a+an,m-2a}\right), \end{aligned}$$

which belongs to HCM by Lemma 1 in [23].

□

Remark 2.1. (a) The above proof makes it also possible to characterize the infinite divisibility of the class $\mathcal{L}(a, b, r)$ defined in Section 3 of [74] as the solutions in law to the random contraction equation

$$X \stackrel{d}{=} \mathbf{B}_{a,b} \times \hat{X}_r,$$

with the notation of Section 2 in [74]. By (3.7) in [74], these solutions are constant multiples of the infinite product

$$\prod_{n=0}^{\infty} \left(\frac{a+b+rn}{a+rn}\right) \mathbf{B}_{a+rn,b},$$

and our argument shows similarly that this product is in HCM as soon as $b \ge r$. By the second statement of Theorem 3.2 in [74], this entails the characterization

$$\mathcal{L}(a, b, r) \in \text{ID} \Leftrightarrow \mathcal{L}(a, b, r) \in \text{HCM} \Leftrightarrow b \ge r$$

for every $a > 0$, providing an answer to the open question stated after Theorem 3.2 in [74].

(b) The second identity in our main result and Example VI.12.21 in [84] readily imply that $\log Gst(m,\alpha) \in \text{SD}$ for every $m > \alpha$. Theorem 3.1 in [74] also shows that $\log \mathcal{L}(a,b,r) \in$ ID for every $a,b,r > 0$, and that $\log \mathcal{L}(a,b,r) \in \text{SD}$ if and only if the function

$$x \mapsto \frac{x^a(1-x^b)}{(1-x)(1-x^r)}$$

is non-decreasing on $(0,1)$, a property which neither holds for all $a,b,r > 0$ nor seems to be characterized cosily in terms of a,b,r.

2.2.C Proof of the Proposition

The asymptotics at infinity is read off immediately in the integro-differential equation (2.1) itself, since

$$\begin{aligned} f_{m,\alpha}(x) &= \left(\int_0^\infty (1-vx^{-1})\mathbf{1}_{\{v\le x\}} f_{m,\alpha}(v)\,dv\right)\frac{x^{\alpha-m-1}}{\Gamma(\alpha)} \\ &\sim \frac{x^{\alpha-m-1}}{\Gamma(\alpha)} \quad \text{as } x\to\infty, \end{aligned}$$

where the estimate follows from dominated convergence and the fact that $f_{m,\alpha}$ is a density on $(0,\infty)$.

□

The derivation of the asymptotics at zero is more involved, and we have to consider three cases separately. Observe that at the logarithmic level, the asymptotic was already obtained in (2.11).

The case $m = 2\alpha$

Here, the identity (2.13) implies

$$f_{2\alpha,\alpha}(x) = \frac{x^{-\alpha-1}}{\Gamma(\alpha)}\, e^{-\frac{1}{x}},$$

which shows the desired asymptotic behavior in an exact formula, since

$$c_{2\alpha,\alpha} = \frac{(2\pi)^{\alpha-1}\alpha^{-\alpha}}{G(2\alpha,\alpha)} = \frac{(2\pi)^{\frac{\alpha-1}{2}}}{\sqrt{\alpha}\Gamma(\alpha)G(\alpha,\alpha)} = \frac{1}{\Gamma(\alpha)}.$$

□

The case $m > 2\alpha$

We first show the estimate

$$f_{m,\alpha}(x) \sim c\,x^{-\frac{m(1+\alpha)}{2\alpha}} e^{-(\frac{\alpha}{m-\alpha})x^{\frac{\alpha-m}{\alpha}}} \tag{2.14}$$

for some positive constant c which will be identified afterwards. Recall the notation $a = m-\alpha$ and introduce the parameter $\beta = \frac{m-a}{a} \in (0,1)$. From (2.10) and the first equation in (2.5), we get

$$\phi_{m,\alpha}(s) = \frac{\mathcal{M}(s+1)}{\mathcal{M}(s)} = a^\beta \frac{\Gamma(1+\beta+\frac{s}{a})}{\Gamma(1+\frac{s}{a})} = a^\beta \frac{\Gamma(1+\beta(1+u))}{\Gamma(1+\beta u)}$$

with the notation $s = a\beta u$. Using e.g. Lemma 1 in [21], this implies

$$\frac{s\,\mathcal{M}(s+1)}{\mathcal{M}(s)} = \psi_{m,\alpha}(s)$$

where

$$\psi_{m,\alpha}(s) = a^\beta \left(\Gamma(\beta+1)s + \int_{-\infty}^0 (e^{sx} - 1 - sx) \left(\frac{am\beta e^{mx}}{\Gamma(1-\beta)(1-e^{ax})^{\beta+2}} \right) dx \right)$$

is the Laplace exponent of a Lévy process without positive jumps $\{L_t^{(m,\alpha)}, t \geq 0\}$, that is $\psi_{m,\alpha}(s) = \log \mathbb{E}[e^{sL_1^{(m,\alpha)}}]$. By the Bertoin-Yor criterion - see Proposition 2 in [16] and its proof, we deduce

$$X_{m,\alpha} \stackrel{d}{=} \int_0^\infty e^{-L_t^{(m,\alpha)}}\, dt.$$

The required estimate will now follow from a recent general result of Patie and Savov on exponential functionals of Lévy processes without positive jumps. We first write

$$\phi_{m,\alpha}(s) = a^\beta\, \Phi_\beta(sa^{-1})$$

where

$$\Phi_\beta(u) = \left(\frac{u+\beta}{u}\right) \frac{\Gamma(\beta+u)}{\Gamma(u)} = u^\beta \left(1 + \frac{\beta(\beta+1)}{2u} + \frac{\beta(\beta^2-1)(3\beta+2)}{24u^2} + \mathrm{O}(u^{-3}) \right),$$

the expansion being e.g. a consequence of Formulæ (4) and (5') in [33]. This expansion also shows, after some algebra, that

$$\Phi_\beta^{-1}(u) = u^{\frac{1}{\beta}} - \frac{(\beta+1)}{2} + \mathrm{O}(u^{-\frac{1}{\beta}})$$

and, from the concavity of Φ_β and the monotone density theorem, that $\Phi_\beta'(u) \sim \beta u^{\beta-1}$. This implies

$$\phi_{m,\alpha}^{-1}(s) = a\,\Phi_\beta^{-1}(sa^{-\beta}) = s^{\frac{1}{\beta}} - \frac{m}{2} + \mathrm{O}(s^{-\frac{1}{\beta}})$$

and

$$(\phi_{m,\alpha}^{-1})'(s) = \frac{1}{\phi_{m,\alpha}'(\phi_{m,\alpha}^{-1}(s))} \sim \frac{s^{\frac{1-\beta}{\beta}}}{\beta}.$$

Putting everything together with Formula (5.47) in [75], we finally obtain (2.14), and it remains to identify the constant c. To do so, we introduce the random variable $U_{m,\alpha} = \beta X_{m,\alpha}^{-\frac{1}{\beta}} = \beta Z_{m,\alpha}$, with density

$$h_{m,\alpha}(x) = (\beta x^{-1})^{\beta+1} f_{m,\alpha}((\beta x^{-1})^\beta) \sim c\,(\beta x^{-1})^{\frac{m(1-\alpha)}{2(m-\alpha)}}\, e^{-x} \quad \text{as } x \to \infty.$$

A standard approximation using Laplace's method and Stirling's formula implies

$$\frac{\mathbb{E}[U_{m,\alpha}^n]}{n!} \sim c\,(\beta n^{-1})^{\frac{m(1-\alpha)}{2(m-\alpha)}} \quad \text{as } n \to \infty.$$

On the other hand, we have

$$\frac{\mathbb{E}[U_{m,\alpha}^n]}{n!} = \frac{\alpha^n}{n!} \left(\frac{G(m+bn,a)G(a,a)}{G(a+bn,a)G(m,a)} \right) \sim c_{m,\alpha}\,(\beta n^{-1})^{\frac{m(1-\alpha)}{2(m-\alpha)}} \quad \text{as } n \to \infty,$$

where the estimate follows from (4.5) in [17], Stirling's formula, and some algebra. This completes the proof.

□

The case $m < 2\alpha$

In this case, the small ball estimate (2.11) shows that $Y_{m,\alpha}$ does not have exponential moments, so that $X_{m,\alpha}$ is not distributed as the exponential functional of a Lévy process without positive jumps, by Proposition 2 in [16]. Hence, we cannot use the estimate (5.47) in [75]. We will first prove (2.14) via an induction on n, where

$$(n+1)a \;<\; m \;\le\; (n+2)a. \tag{2.15}$$

The case $n = 0$ follows from the previous cases $m \ge 2\alpha \Leftrightarrow m \le 2a$. To prove the induction step, we first observe the identity in law

$$Y_{m,m-a} \overset{d}{=} Y_{m-a,m-2a} \times \mathbf{\Gamma}_\alpha, \tag{2.16}$$

which is a consequence of (2.10), the second equation in (2.5), and fractional moment identification. The multiplicative convolution formula leads then to

$$f_{m,\alpha}(x) \;=\; \frac{1}{\Gamma(\alpha)}\int_0^\infty f_{\alpha,2\alpha-m}(xy)\, y^\alpha\, e^{-y}\, dy.$$

Setting again $b = \frac{a}{m-a} < 1$, we choose $\delta \in (b,1)$ and we decompose

$$f_{m,\alpha}(x) \;=\; \frac{1}{\Gamma(\alpha)}\int_0^{x^{-\delta}} f_{\alpha,2\alpha-m}(xy)\, y^\alpha\, e^{-y}\, dy \;+\; \mathrm{o}(e^{-x^{-\eta}}), \qquad x\to 0, \tag{2.17}$$

for every $\eta \in (b,\delta)$, where the Landau estimate follows readily from the bounded character of $f_{\alpha,2\alpha-m}$. To estimate the integral, we use the induction hypothesis on $f_{\alpha,2\alpha-m}$, and the fact that $\delta < 1$, in order to obtain

$$\begin{aligned}\int_0^{x^{-\delta}} f_{\alpha,2\alpha-m}(xy)\, y^\alpha\, e^{-y}\, dy \;&\sim\; c\,x^{-\frac{\alpha(1+2\alpha-m)}{2(2\alpha-m)}}\int_0^{x^{-\delta}} y^{\frac{\alpha(2\alpha-m-1)}{2(2\alpha-m)}}\, e^{-y-(\frac{2\alpha-m}{m-\alpha})(xy)^{\frac{\alpha-m}{2\alpha-m}}}\, dy\\ &\sim\; c\,x^{-(b+\frac{m+1}{2})}\int_0^{x^{b-\delta}} z^{\frac{\alpha(2\alpha-m-1)}{2(2\alpha-m)}}\, e^{-x^{-b}(z+(\frac{2\alpha-m}{m-\alpha})z^{\frac{\alpha-m}{2\alpha-m}})}\, dz.\end{aligned}$$

Using Laplace's approximation, we deduce that there exists a positive constant $\tilde c$ such that

$$\int_0^{x^{-\delta}} f_{\alpha,2\alpha-m}(xy)\, y^\alpha\, e^{-y}\, dy \;\sim\; \tilde c\, x^{-\frac{b+m+1}{2}}\, e^{-b^{-1}x^{-b}} \;=\; \tilde c\, x^{-\frac{m(1+\alpha)}{2\alpha}}\, e^{-(\frac{\alpha}{m-\alpha})x^{\frac{\alpha-m}{\alpha}}}.$$

By (2.17), this completes the proof of (2.14) by induction. The identification of the constant $\tilde c$ is done exactly in the same way as in the case $m > 2\alpha$.

□

Remark 2.2. (a) The derivation of the asymptotics at infinity follows also, in a more complicated way similar to the argument of Theorem 4.4 in [74], from the behavior of $\mathcal{M}(s)$ at its first pole $s = -a$. More precisely, by (2.9), we have

$$\mathcal{M}(s) \;\sim\; \left(\frac{a^{a-m}G(m-a,a)G(a,a)}{G(m,a)}\right)\times\frac{1}{G(a+s,a)} \;=\; \frac{1}{a\Gamma(m-a)G(a+s,a)} \qquad \text{as } s\downarrow -a,$$

where the equality comes from (2.6) and the second equation in (2.5). The latter also imply

$$\frac{1}{G(a+s,a)} \;=\; \frac{(2\pi)^{\frac{a-1}{2}}a^{\frac{1}{2}-a-s}\Gamma(s+a)}{G(2a+s,a)} \;\sim\; \frac{a}{a+s} \qquad \text{as } s\downarrow -a,$$

showing that this first pole is simple and isolated. Putting everything together and using e.g. Theorem 4 in [36], we obtain the required asymptotic

$$f_{m,\alpha}(x) \sim \frac{x^{-a-1}}{\Gamma(m-a)} = \frac{x^{\alpha-m-1}}{\Gamma(\alpha)} \qquad \text{as } x \to \infty.$$

In principle, the exact expression of $\mathcal{M}(s)$ and Theorem 4 in [36] should make it possible to derive a more complete expansion of $f_{m,\alpha}$ at infinity. As mentioned in the introduction, an absolutely convergent power series expansion exists when m is an integer, as a consequence of a Fox representation of order m for $g_{m,\alpha}(x) = x^{-2} f_{m,\alpha}(x^{-1})$ - see (2.11) and (2.12) in [81]. We will consider some examples in Section 3.1, revisiting in particular the case when m is an integer. See also Theorem 3 in [57] for some results in this vein, which apply to some cases when $m > 2\alpha$ is not an integer.

(b) In the strict stable case $m = 1$, our asymptotic at zero reads simply

$$f_{1,\alpha}(x) \sim \frac{x^{-\frac{1+\alpha}{2\alpha}}}{\sqrt{2\pi\alpha}} e^{-(\frac{\alpha}{1-\alpha})x^{\frac{\alpha-1}{\alpha}}} = \frac{x^{-\frac{2-a}{2(1-a)}}}{\sqrt{2\pi(1-a)}} e^{-(\frac{1-a}{a})x^{\frac{-a}{1-a}}},$$

in accordance with $X_{1,\alpha} \stackrel{d}{=} a^{-\frac{1}{a}} \mathbf{Z}_a$ and the first order term of (2.4.30) in [46]. The latter formula displays actually a complete expansion of the density $f_{1,\alpha}$ at zero, with non-explicit coefficients. The detailed argument for this expansion, which relies on (2.3), Fourier inversion, and the method of steepest descent, is in the proof of Theorem 2.4.6 in [46]. In the absence of explicit Laplace transform, a complete expansion at zero for $f_{m,\alpha}$ seems difficult to derive in general when $m \neq 1$.

(c) The multiplicative identity (2.16) has a more general formulation, which is

$$Y_{m,\alpha} \stackrel{d}{=} Y_{q,q-a} \times Y^{(q-a)}_{m+a-q,m-q} \tag{2.18}$$

for every $q \in (m-a, m)$, with the alternative notation $X^{(t)} = \hat{X}_t$. Notice that (2.18) boils down to (2.16) for $q = m - a$ and that, contrary to the self-similar identity

$$X_{1,1-a} \stackrel{d}{=} b^{\frac{a-b}{ab}} X_{1,1-b} \times X^{\frac{1}{b}}_{1,1-\frac{a}{b}}$$

which is valid for every $b \in (a, 1)$, it is not a subordination formula. As in Corollary 4 (a) of [59], it can also be shown that $X_{m,\beta}$ is a multiplicative factor of $X_{m,\alpha}$ for every $0 < \beta < \alpha < m$.

2.3 Further remarks

2.3.A Some particular factorizations

In this paragraph we consider three situations where the law $Gst(m,\alpha)$ has simpler expressions as a finite product involving the Gamma or the positive stable distribution. This expression is derived from rewriting (2.10) as a moment of Gamma type, thanks to the concatenation formulæ of (2.5). We refer to [49] for a survey on moments of Gamma type. In our three cases, the density $f_{m,\alpha}$ is also a Fox $H-$function and we display the convergent power series representations, when it is possible. Throughout, we use again the notations $a = m - \alpha$ and $X^{(t)} = \hat{X}_t$ in order to have simpler formulæ. Our reference for Fox functions is Section 1.12 in [55], especially (1.12.1) and (1.12.19) therein.

The case $\alpha = n \in \mathbb{N}$

We have

$$\mathcal{M}(s) = a^{\frac{ns}{a}} \prod_{i=1}^{n} \left(\frac{\Gamma(1 + \frac{i-1}{a} + \frac{s}{a})}{\Gamma(1 + \frac{i-1}{a})} \right), \qquad s > -a.$$

This shows that $X_{m,n}$ is a finite independent product of generalized Fréchet random variables, as was already observed in the introduction for $\alpha = 1, 2$: one has

$$X_{m,n} \stackrel{d}{=} \left(a^n \, \mathbf{\Gamma}_1 \times \cdots \times \mathbf{\Gamma}_{1+\frac{n-1}{a}} \right)^{-\frac{1}{a}}.$$

The Fox function representation of $f_{m,n}$ is then

$$f_{m,n}(x) = \left(\frac{a^{\frac{n}{a}}}{\prod_{i=1}^{n} \Gamma(1 + \frac{i-1}{a})} \right) H^{0,n}_{n,0} \left[a^{\frac{n}{a}} x \,\middle|\, \begin{matrix} (\frac{-i}{a}, \frac{1}{a})_{i=1,\dots,n} \\ \hline \end{matrix} \right].$$

When $a \notin \mathbb{Q}$ or $a \in \mathbb{Q}$ with $a = \frac{p}{q}$ irreducible and $p \geq n$, the following convergent power series representation holds:

$$f_{m,n}(x) = \left(\frac{a^{\frac{n}{a}+1}}{\prod_{i=1}^{n} \Gamma(1 + \frac{i-1}{a})} \right) \sum_{r=1}^{n} \sum_{k=0}^{\infty} \frac{(-1)^k a^{-(\frac{rn}{a} + n(k+1))}}{k!} \left(\hat{\prod}_{j=1}^{n} \Gamma(\tfrac{j-r}{a} - k) \right) x^{-(r+a(k+1))},$$

where the hat product indicates omission of $j = r$. For $n = 1$, this simplifies into

$$f_{m,1}(x) = x^{-a-1} \sum_{k \geq 0} \frac{(-1)^k}{k!} \frac{1}{(ax^a)^k} = x^{-a-1} e^{-\frac{1}{ax^a}}$$

as expected, since $X_{m,1} \stackrel{d}{=} (a\mathbf{\Gamma}_1)^{-\frac{1}{a}}$. For $n = 2$ and $\nu = \frac{1}{a} \notin \mathbb{N}$, this simplifies into

$$\begin{aligned} f_{m,2}(x) &= \frac{x^{-a-1}}{\Gamma(\nu)} \sum_{k \geq 0} \frac{(-1)^k}{k!} \left(\Gamma(\nu - k) + \Gamma(-\nu - k)(ax^{a/2})^{-2\nu} \right) (ax^{a/2})^{-2k} \\ &= \frac{2x^{-a-3/2}}{a^{\nu}\Gamma(\nu)} K_{\nu}\left(\frac{2}{ax^{a/2}}\right) \end{aligned}$$

as expected, since $X_{m,2} \stackrel{d}{=} (a\sqrt{\mathbf{\Gamma}_1 \times \mathbf{\Gamma}_{1+\nu}})^{-\frac{2}{a}}$ - see the second example in the introduction. Notice that the representation of $f_{m,2}$ in terms of the Macdonald function K_{ν} also holds for $\nu \in \mathbb{N}$, but then the convergent series representation has a logarithmic term - see Formula 7.2.5(37) in [34].

The case $m = an, n \in \{2,3,\dots\}$

We have

$$\mathcal{M}(s) = \prod_{i=1}^{n-1}\left(\frac{\Gamma(ia+s)}{\Gamma(ia)}\right), \quad s > -a.$$

This shows that $X_{an,a(n-1)}$ is a finite independent product of inverse Gamma random variables, as was already observed in (2.13) for $n = 2$: one has

$$X_{m,n} \stackrel{d}{=} \left(\mathbf{\Gamma}_a \times \cdots \times \mathbf{\Gamma}_{a(n-1)}\right)^{-1}.$$

The Fox function representation of $f_{an,a(n-1)}$ is

$$f_{an,a(n-1)}(x) = \left(\frac{1}{\prod_{i=1}^{n-1}\Gamma(ia)}\right) H^{0,n-1}_{n-1,0}\left[x \,\middle|\, \begin{matrix}(-ia,1)_{i=1,\dots,n} \\ \hline \end{matrix}\right].$$

When $n = 2$ or $a,\dots,(n-2)a \notin \mathbb{N}$, the following convergent power series representation holds:

$$f_{an,a(n-1)}(x) = \left(\frac{1}{\prod_{i=1}^{n-1}\Gamma(ia)}\right)\sum_{r=1}^{n-1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\hat{\prod}_{j=1}^{n-1}\Gamma((j-r)a-k)\right)x^{-ra-k-1}.$$

For $n = 2$, this simplifies into

$$f_{2a,a}(x) = \frac{x^{-a-1}}{\Gamma(a)}\sum_{k\geq 0}\frac{(-1)^k x^{-k}}{k!} = \frac{x^{-a-1}e^{-\frac{1}{x}}}{\Gamma(a)},$$

as expected from (2.13). For $n = 3$ and $a \notin \mathbb{N}$, similarly as above we get

$$f_{3a,2a}(x) = \frac{x^{-a-1}}{\Gamma(a)\Gamma(2a)}\sum_{k\geq 0}\frac{(-1)^k}{k!}\left(\Gamma(a-k)+\Gamma(-a-k)x^{-a}\right)x^{-k} = \frac{x^{-3a/2-1}}{\Gamma(a)\Gamma(2a)}K_a(2x^{-1/2}),$$

the representation on the right-hand side in terms of the Macdonald function holding for $a \in \mathbb{N}$ as well.

The case $m = n$

We have

$$\mathcal{M}(s) = a^{\frac{ns}{a}}\frac{\Gamma(1+\frac{s}{a})}{\Gamma(1+s)} \times \prod_{i=1}^{n-1}\left(\frac{\Gamma(\frac{i+s}{a})}{\Gamma(\frac{i}{a})}\right) = \left(\frac{a^{\frac{n}{a}}}{n}\right)^s \times \prod_{i=0}^{n-1}\left(\frac{\Gamma(1+\frac{i+s}{a})\Gamma(1+\frac{i}{n})}{\Gamma(1+\frac{i+s}{n})\Gamma(1+\frac{i}{a})}\right), \quad s > -a,$$

where the second equality comes from the Legendre-Gauss multiplication formula for the Gamma function. Observe also that the first equality is Theorem 4.1 in [74], with a different normalization. This shows that $X_{n,\alpha}$ is a finite independent product of power transforms of size-biased stable random variables, as was already mentioned in the introduction for $m = 1,2$: one has

$$X_{n,\alpha} \stackrel{d}{=} na^{-\frac{n}{a}}\left(\mathbf{Z}_{\frac{a}{n}} \times \mathbf{Z}_{\frac{a}{n}}^{(\frac{-1}{n})} \times \cdots \times \mathbf{Z}_{\frac{a}{n}}^{(\frac{1-n}{n})}\right)^{\frac{1}{n}}.$$

This factorization may look more satisfactory than that of Theorem 4.2 in [74], and it is also valid in the full range $a \in (0, m)$. The Fox function representation of $f_{m,n}$ is derived similarly as (2.11) in [81] - beware again our different normalization: one has

$$f_{n,\alpha}(x) = \left(\frac{a^{\frac{n}{a}-1}}{\prod_{i=1}^{n-1}\Gamma(\frac{i}{a})}\right) H^{0,n}_{n,1}\left[a^{\frac{n}{a}}x \,\middle|\, \begin{matrix}(1-\frac{i}{a},\frac{1}{a})_{i=1,\dots,n}\\ \\ (0,1)\end{matrix}\right].$$

When $a \notin \mathbb{Q}$ or $a \in \mathbb{Q}$ with $a = \frac{p}{q}$ irreducible and $p \geq n$, the following convergent power series representation holds:

$$f_{n,\alpha}(x) = \left(\frac{a^{\frac{n}{a}}}{\prod_{i=1}^{n-1}\Gamma(\frac{i}{a})}\right)\sum_{r=1}^{n}\sum_{k=0}^{\infty}\frac{(-1)^k a^{-(\frac{rn}{a}+nk)}}{k!\Gamma(1-r-ak)}\left(\hat{\prod}_{j=1}^{n}\Gamma(\tfrac{j-r}{a}-k)\right)x^{-r-ak}.$$

For $n = 1$, this simplifies into

$$f_{1,\alpha}(x) = \sum_{k\geq 1}\frac{(-a)^{-k}}{k!\Gamma(-ak)}\,x^{-1-ak}$$

as expected from e.g. Theorem 2.4.1 in [46], since $X_{1,\alpha} \stackrel{d}{=} a^{-\frac{1}{a}}\mathbf{Z}_a$. By (2.2.35) in [55], the auxiliary function $h_{1,\alpha} = \mathrm{I}^{\alpha}_{0+}f_{1,\alpha}$ has Laplace transform

$$(\mathcal{L}h_{1,\alpha})(\lambda) = \lambda^{-\alpha}\exp\left(-\frac{\lambda^{1-\alpha}}{1-\alpha}\right),$$

in accordance with (5.2.143) in [55], which leads to (5.2.139) therein, and our above equation (2.2) which is for $m = 1$ the fractional differential equation (5.2.137) in [55] with $\lambda = 1$ therein.

In the physically relevant case $n = 2$ and for $\nu = \frac{1}{a} \notin \mathbb{N}$, the series representation simplifies into

$$f_{2,\alpha}(x) = \frac{1}{x\Gamma(\nu)}\sum_{k\geq 0}\frac{(-1)^k}{k!\Gamma(-ak)}\left(\Gamma(\nu-k)-(1+ak)\Gamma(-\nu-k)(ax^{a/2})^{-2\nu}\right)(ax^{a/2})^{-2k}.$$

Observe the striking formal resemblance with $f_{m,2}$, although no expression in terms of a classical special function seems here available. Notice also that for $\nu \in \mathbb{N}$, there is no convergent power series representation for $f_{2,\alpha}$ in general, save for $\nu = a = \alpha = 1$ where the reduction formula (1.12.43) in [55] yields

$$f_{2,1}(x) = H^{0,2}_{2,1}\left[x \,\middle|\, \begin{matrix}(1-i,1)_{i=1,2}\\ \\ (0,1)\end{matrix}\right] = H^{0,1}_{1,0}\left[x \,\middle|\, \begin{matrix}(-1,1)\\ \hline \end{matrix}\right] = x^{-2}e^{-\frac{1}{x}},$$

as again expected from (2.13).

2.3.B Some explicit Thorin measures

As mentioned in the introduction, it follows from Part (a) of the Corollary that the density solutions to (2.1) are also solution to the Steutel's integro-differential equation (2.8), whose convolution kernel $k_{m,\alpha}(x-y)$ is such that

$$k_{m,\alpha}(x) = \int_0^{\infty} e^{-xu}\mu_{m,\alpha}(du)$$

is a CM function. In the literature, the measure $\mu_{m,\alpha}$ is called the Thorin measure associated to the random variable $X_{m,\alpha} \in \mathrm{GGC}$, and we refer to [48] - see also Chapter 3 in [18] - for more on this topic. From (2.7), the measure $\mu_{m,\alpha}$ is related to the Laplace transform $\mathcal{L}_{m,\alpha}$ of $X_{m,\alpha}$ via its Stieltjes transform:

$$\int_0^\infty \frac{\mu_{m,\alpha}(du)}{u+\lambda} = -(\log \mathcal{L}_{m,\alpha})'(\lambda).$$

Recall that when $m=1$, we have

$$k_{1,\alpha}(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} = \int_0^\infty e^{-xu}\left(\frac{\sin(\pi\alpha)}{\pi u^\alpha}\right) du,$$

so that $\mu_{1,\alpha}$ has a simple explicit density. Let us mention two other cases where $\mu_{1,\alpha}$ has a more or less explicit density.

The case $m = 2\alpha$

This case was already discussed at the end of Section 3.2 in [21], but we do it again here for completeness. From (2.13) we have $X_{2\alpha,\alpha} \stackrel{d}{=} \mathbf{\Gamma}_\alpha^{-1}$, whose Laplace transform is computed similarly as in the introduction:

$$\mathcal{L}_{2\alpha,\alpha}(\lambda) = \frac{2\,\lambda^{\frac{\alpha}{2}}}{\Gamma(\alpha)} K_\alpha(2\sqrt{\lambda}).$$

Using Formulæ 7.11.(25-26) in [34], we deduce

$$-(\log \mathcal{L}_{2\alpha,\alpha})'(\lambda) = \frac{K_{\alpha-1}(2\sqrt{\lambda})}{\sqrt{\lambda}K_\alpha(2\sqrt{\lambda})} = \int_0^\infty \left(\frac{1}{4\pi^2 u\left((J_\alpha(2\sqrt{u}))^2 + (Y_\alpha(2\sqrt{u}))^2\right)}\right)\frac{du}{u+\lambda} \tag{2.19}$$

where the second, non-trivial, equality follows from the main result of [41] - see also [47] for a simpler argument using the Perron-Stieltjes inversion formula and the Wronskian of Hankel functions. This shows that $\mu_{2\alpha,\alpha}$ has an explicit density $\varphi_{2\alpha,\alpha}$ which is expressed in terms of the classical Bessel functions J_α and Y_α :

$$\varphi_{2\alpha,\alpha}(u) = \frac{1}{4\pi^2 u\left((J_\alpha(2\sqrt{u}))^2 + (Y_\alpha(2\sqrt{u}))^2\right)}.$$

Remark 2.3. For every $\alpha > 0, t \neq 0$, the Laplace transform of $\mathbf{\Gamma}_\alpha^{-\frac{1}{t}}$ is computed formally as

$$\mathbb{E}[e^{-\lambda \mathbf{\Gamma}_\alpha^{-\frac{1}{t}}}] = \frac{1}{t\Gamma(\alpha)}\int_0^\infty x^{\alpha t-1}\exp\left(-x^t - \frac{\lambda}{x}\right)dx = \frac{Z_t^{\alpha t}(\lambda)}{t\Gamma(\alpha)},$$

where Z_ρ^ν is the so-called Krätzel function - see (1.7.42) in [55]. On the other hand, we know by Theorem 4 in [22] and the discussion therebefore that

$$\mathbf{\Gamma}_\alpha^{-\frac{1}{t}} \in \mathrm{GGC} \Leftrightarrow \mathbf{\Gamma}_\alpha^{-\frac{1}{t}} \in \mathrm{ID} \Leftrightarrow t \geq -1.$$

This shows that $-(\log Z_\rho^\nu)'$ is the Stieltjes transform of a positive measure $\mu_{\rho,\nu}$ for all $\rho \geq -1$ and $\nu\rho \geq 0$, and that it is not CM for $\rho < -1$. The measure $\mu_{\rho,\nu}$ is not explicit in general, except for $\rho = 1$ by the preceding discussion and Formula (1.7.43) in [55]. The case $\rho = \nu > 0$ corresponds to the Fréchet random variable $\mathbf{\Gamma}_1^{-\frac{1}{\rho}}$ and to our above special case $\alpha = 1$. It is also discussed in Section 3.4 of [21] for $\rho = \nu \in (0,1)$, from the point of view of Bochner's subordination.

The case $m = 2$

As seen in the introduction, we have

$$\mathcal{L}_{2,\alpha}(\lambda) = \frac{2\nu^\nu}{\Gamma(\nu)} \sqrt{\lambda}\, K_\nu(2\nu\lambda^{\frac{1}{2\nu}})$$

with the notation $\nu = \frac{1}{2-\alpha} \in (1/2, \infty)$. The same computation as above and the Perron-Stieltjes inversion formula lead to

$$-(\log \mathcal{L}_{2,\alpha})'(\lambda) = \frac{\lambda^{\frac{1}{2\nu}-1} K_{\nu-1}(2\nu\lambda^{\frac{1}{2\nu}})}{K_\nu(2\nu\lambda^{\frac{1}{2\nu}})} = \int_0^\infty \left(\frac{1}{2\pi\nu u} \Im\left(\frac{zK_{\nu-1}(z)}{K_\nu(z)} \right) \right) \frac{du}{u+\lambda}$$

with $z = 2\nu(e^{\mathrm{i}\pi}u)^{\frac{1}{2\nu}}$, which shows a semi-explicit expression for the density $\varphi_{2,\alpha}$ of $\mu_{2,\alpha}$.

When $\nu > 1 \Leftrightarrow \alpha \in (1,2)$, we have $\arg(z^2) \in (0,\pi)$ and we can apply the second equality in (2.19) which holds on the complex plane cut along the negative real axis. This yields, after some algebra, an explicit integral representation connecting $\varphi_{2,\alpha}$ to $\varphi_{2\alpha,\alpha}$:

$$\varphi_{2,\alpha}(u) = \frac{1}{\nu^2} \int_0^\infty f_{\mathbf{X}_\nu}\left(\frac{u}{x}\right) \left(x^{\frac{1}{\nu}-1} \varphi_{2\alpha,\alpha}(x^{\frac{1}{\nu}}) \right) \frac{dx}{x}$$

where $f_{\mathbf{X}_\nu}$ is the density of $\mathbf{X}_\nu \stackrel{d}{=} \nu^{-2\nu}(\mathbf{C}_{\frac{1}{\nu}})^\nu$ and $\mathbf{C}_\mu$ is for every $\mu \in (0,1)$ the half-Cauchy random variable with density

$$\frac{\sin(\pi\mu)}{\pi\mu(x^2 + 2\cos(\pi\mu)x + 1)}.$$

Observe that since $\mathbf{X}_\nu \xrightarrow{d} 1$ as $\nu \to 1$, the above representation boils down to the tautological identity $\varphi_{2,1} = \varphi_{2,1}$ when $\nu = \alpha = 1$, a special case of Paragraph 3.2.1 above.

When $\nu \in (1/2, 1) \Leftrightarrow \alpha \in (0,1)$, we can write $z = \mathrm{i}Z$ with $Z = 2\nu e^{\mathrm{i}\pi(\frac{1}{2\nu}-\frac{1}{2})} u^{\frac{1}{2\nu}}$ such that $\arg(Z^2) \in (0,\pi)$. By Formula 7.2.2(16) in [34], we obtain

$$\varphi_{2,\alpha}(u) = \frac{-1}{2\pi\nu u} \Im\left(\frac{ZH^{(2)}_{\nu-1}(Z)}{H^{(2)}_\nu(Z)} \right),$$

a complex expression which does not seem to lead to any particular real simplification.

2.3.C Some limit behaviors

In this last paragraph we study the limit behavior of $Gst(m,\alpha)$ when the parameters (m,α) reach their admissibility boundary.

- When $\alpha \to 0 \Leftrightarrow a \to m$, our main result shows immediately that

  $$X_{m,\alpha} \xrightarrow{d} 1,$$

  an extension of the case $m = 1$ where it is obvious from (2.3) that $\mathbf{Z}_a \xrightarrow{d} 1$ as $a \to 1$.

- When $\alpha \to m \Leftrightarrow a \to 0$, the situation is a bit more involved. The second identity of our main result shows that

$$aX^a_{m,\alpha} \stackrel{d}{=} \frac{\Gamma(a+1)}{\Gamma(m)} \times \left(\frac{m}{a} \times \mathbf{B}_{1,\frac{m}{a}-1}\right)^{-1} \times \left(\prod_{n=0}^{\infty} \left(\frac{m+n+1}{a+n+1}\right) \mathbf{B}_{1+\frac{1+n}{a},\frac{m}{a}-1}\right)^{-1}$$
$$\xrightarrow{d} \frac{1}{\Gamma(m)\mathbf{\Gamma}_1},$$

  where the convergence follows from (2.5) in [59]. This is again an extension of the case $m = 1$ where $\mathbf{Z}_a^{-a} \xrightarrow{d} \mathbf{\Gamma}_1$ as $a \to 0$. See also [27] for the behavior of real stable laws with small self-similarity parameter.

Remark 2.4. When $m \to \infty$ and $a = m - \alpha$ is fixed, putting together (2.10) and (4.5) in [17] shows after some comparison with Theorem 1.4 and Remark 1.5 in [68] that $Gst(m, \alpha)$ exhibits a mod-Gaussian convergence. We have not investigated the full details, leaving them to further research.

Chapter 3

Some properties of the free stable distributions

3.1 Introduction

In this paper, we investigate certain properties of real free stable random variables. We say that a real random variable $\mathbf{X}$ is free stable, if for any $a, b > 0$ there exists $c > 0, d \in \mathbb{R}$ such that

$$a\mathbf{X}_1 + b\mathbf{X}_2 \stackrel{d}{=} c\mathbf{X} + d, \tag{3.1}$$

where $\mathbf{X}_1, \mathbf{X}_2$ are free copies of $\mathbf{X}$. As in the classical framework, when $\mathbf{X}$ is not constant it turns out that there exist solutions to (3.1) only if $c = (a^\alpha + b^\alpha)^{1/\alpha}$ for some fixed $\alpha \in (0, 2]$ which is called the stability parameter.

We will be mostly concerned with free strictly stable densities, which correspond to the case $d = 0$. Every free strictly stable distribution turns out to be equivalent to a distribution whose Voiculescu transform is of the form

$$\phi_{\alpha,\rho}(z) = -e^{\mathrm{i}\pi\alpha\rho} z^{-\alpha+1}, \quad \Im(z) > 0, \tag{3.2}$$

where (α, ρ) belongs to the following set $\mathcal{D}$ of admissible parameters:

$$\mathcal{D} = \{\alpha \in (0, 1], \rho \in [0, 1]\} \cup \{\alpha \in (1, 2], \rho \in [1 - 1/\alpha, 1/\alpha]\}.$$

Above, we have used the standard terminology that two measures μ, ν on the line are equivalent if there exist real numbers $a > 0, b \in \mathbb{R}$ such that $\mu(S) = \nu(aS + b)$ for every Borel set S. We refer e.g. to [71] for some background on the free additive convolution, to [12] for the original solution to the equation (3.1), and to the introduction of [44] for the above parametrization (α, ρ), which mimics that of the strict classical framework. Let us also recall that free stable laws appear as limit distributions of spectra of large random matrices with possibly unbounded variance - see [11, 25], and that their domains of attraction have been fully characterized in [13, 14]. In the following, we will denote by $\mathbf{X}_{\alpha,\rho}$ the random variable whose Voiculescu transform is given by (3.2), and set $f_{\alpha,\rho}$ for its density. The analogy with the classical case extends to the fact, observed in Corollary 1.3 of [44], that with our parametrization one has

$$\mathbb{P}[\mathbf{X}_{\alpha,\rho} \geq 0] = \rho.$$

For this reason, we will call ρ the positivity parameter of the free strictly random variable $\mathbf{X}_{\alpha,\rho}$. Clearly one has $\mathbb{P}[\mathbf{X}_{\alpha,\rho} \le 0] = 1-\rho$ and the Voiculescu transform also shows that $\mathbf{X}_{\alpha,\rho} \stackrel{d}{=} -\mathbf{X}_{\alpha,1-\rho}$. In this paper, some focus will be put on the one-sided case and we will use the shorter notations $\mathbf{X}_{\alpha,1} = \mathbf{X}_\alpha$ and $f_{\alpha,1} = f_\alpha$. Throughout, the random variable $\mathbf{X}_{\alpha,\rho}$ will be mostly handled as a classical random variable via its usual Fourier, Laplace and Mellin transforms, except for a few situations where the free independence is discussed.

Several analytical properties of free stable densities have been derived in the Appendix to [14], where it was shown in particular that they can be expressed in closed form via the inverse of certain trigonometric functions. It is also indicated in [14] that every free stable distribution of stability index $\alpha \neq 1$ is equivalent to a free strictly stable distribution of the same index. The density $f_{\alpha,\rho}$ turns out to be a truly explicit function in three specific situations only, which is again reminiscent of the classical case:

- $f_{2,1/2}(x) = \dfrac{\sqrt{4-x^2}}{2\pi}$ for $x \in [-2,2]$, (semi-circular density),
- $f_{1/2}(x) = \dfrac{\sqrt{4x-1}}{2\pi x^2}$ for $x \ge 1/4$, (inverse Beta density),
- $f_{1,\rho}(x) = \dfrac{\sin(\pi\rho)}{\pi(x^2+2\cos(\pi\rho)x+1)}$ for $x \in \mathbb{R}$, (standard Cauchy density with drift).

The study of $f_{\alpha,\rho}$ was carried on further in [42, 44] where, among other results, several factorizations and series representations were obtained. Our purpose in this paper is to deduce from these results several new and non-trivial properties. Our first findings deal with the infinite divisibility of $\mathbf{X}_{\alpha,\rho}$. Since this random variable is freely infinitely divisible (FID), it is a natural question whether it is also classically infinitely divisible (ID).

Theorem 3.1. *One has*

(a) *For every $\alpha \in (0,1]$ and $\rho \in [0,1]$, the random variable $\mathbf{X}_{\alpha,\rho}$ is* ID.
(b) *For every $\alpha \in (1,2]$, the random variable $\mathbf{X}_{\alpha,1/2}$ is not* ID.

Above, the non ID character of $\mathbf{X}_{2,1/2}$ is plain from the compactness of its support. Observe also that by continuity of the law of $\mathbf{X}_{\alpha,\rho}$ in (α,ρ) and closedness in law of the ID property - see e.g. Lemma 7.8 in [79], for every $\alpha \in (1,2)$ there exists some $\epsilon(\alpha) > 0$ such that $\mathbf{X}_{\alpha,\rho}$ is not ID for all $\rho \in [1/2-\epsilon(\alpha), 1/2+\varepsilon(\alpha)]$. We believe that one can take $\varepsilon(\alpha) = 1/\alpha - 1/2$, that is our above result is optimal with respect to the ID property. Unfortunately, we found no evidence for this fact as yet - see Remark 3.3 for possible approaches.

As it will turn out in the proof, for $\alpha \le 1$ the ID random variables $\mathbf{X}_{\alpha,\rho}$ have no Gaussian component. A natural question is then the structure of their Lévy measure. We will say that the law of a positive ID random variable is a generalized Gamma convolution (GGC) if its Lévy measure has a density φ such that $x\varphi(x)$ is a completely monotonic (CM) function on $(0,+\infty)$. There exists an extensive literature on such positive distributions, starting from the seventies with the works of O. Thorin. The denomination comes from the fact that up to translation, these laws are those of the random integrals

$$\int_0^\infty a(t)\, d\mathbf{\Gamma}_t$$

where $a(t)$ is a suitable deterministic function and $\{\boldsymbol{\Gamma}_t,\, t \geq 0\}$ is the standard Gamma subordinator. We refer to [18] for a comprehensive monograph with an accent on the Pick functions representation and to the more recent survey [48] for the above Wiener-Gamma integral representation, among other topics. See also Chapters 8 and 9 in [80] for their relationship with Stieltjes functions. In Chapter 7 of [18], this notion is extended to distributions on the real line. Following (7.1.5) therein, we will say that the law of a real ID random variable is an extended GGC if its Lévy measure has a density φ such that $x\varphi(x)$ and $x\varphi(-x)$ are CM as a function of x on $(0,+\infty)$. In order to simplify our presentation, we will also use the notation GGC for extended GGC.

Theorem 3.2. *For every $\alpha \in (0,3/4]$ and $\rho \in [0,1]$, the law of $\mathbf{X}_{\alpha,\rho}$ is a* GGC.

Contrary to the above, we think that this result is not optimal and that the random variable $\mathbf{X}_{\alpha,\rho}$ has a GGC law at least for every $\alpha \in (0,4/5]$ and $\rho \in [0,1]$ - see Conjecture 3.1. During our proof, we will see that for every $\alpha, \rho < 1$ the GGC character of $\mathbf{X}_{\alpha,\rho}$ is a consequence of that of $\mathbf{X}_\alpha$. Unfortunately this simpler question, which is connected to the hyperbolically completely monotonic (HCM) character of negative powers of the classical positive stable distribution, is rather involved. Moreover, we will see in Corollary 3.1 that the law of $\mathbf{X}_\alpha$ is not a GGC for α close enough to 1.

Our next result deals with the case $\alpha = 1$. According to the Appendix of [14], every free 1-stable distribution is equivalent to a unique distribution whose Voiculescu transform writes

$$\phi_\rho(z) \;=\; -2\rho\mathrm{i} \;-\; \frac{2(1-2\rho)}{\pi}\log z, \quad \Im(z) > 0,$$

for some $\rho \in [0,1]$. By (3.2), this means that a free 1-stable distribution is equivalent to the law of the free sum

$$\mathbf{C}_{a,b} \stackrel{d}{=} a\mathbf{X}_{1,1/2} \;+\; b\mathbf{T},$$

for $a \geq 0, b \in \mathbb{R}$, where $\mathbf{T}$ has Voiculescu transform $-\log z$ and will be called henceforth the exceptional free 1-stable random variable. More precisely, for any $a > 0$ and $b \neq 0$, the random variable $C_{a,b}$ is equivalent to the 1-free non strictly stable random variable whose Voiculescu transform is ϕ_ρ, where $\rho \neq 1/2$ is determined by

$$\frac{a}{\pi b} = \begin{cases} \frac{\rho}{1-2\rho}, & b > 0, \\ \frac{1-\rho}{1-2\rho}, & b < 0. \end{cases}$$

The case $b = 0$ corresponds to $\rho = 1/2$ and to the 1-free symmetric strictly stable random variable, which is the standard Cauchy random variable: one sees from (3.2) that $\phi_{1/2}$ is the Voiculescu transform of $\mathbf{X}_{1,1/2}$. Observe also that ϕ_0 is the Voiculescu transform of $\frac{2}{\pi}(\mathbf{T} + \log(\pi/2))$ whereas ϕ_1 is that of $-\frac{2}{\pi}(\mathbf{T} + \log(\pi/2))$. Notice finally that in the above parametrization of free 1-stable distributions, the parameter ρ is not a positivity parameter. Actually, as in the classical framework there does not seem to exist a closed formula for $\mathbb{P}[\mathbf{C}_{a,b} \geq 0]$ when $b \neq 0$.

The density of $\mathbf{C}_{a,b}$ can be retrieved from Proposition A.1.3 of [14], in an implicit way. In this paper, taking advantage of a factorization due to Zolotarev for the exceptional classical 1-stable random variable, we obtain the following explicit result.

Theorem 3.3. *The random variable $\mathbf{C}_{a,b}$ is* ID *without Gaussian component and with Lévy measure*

$$\frac{1}{x^2}\left(\frac{a}{\pi}\,\mathbf{1}_{\{x\neq 0\}} \;+\; |b|\left(1-\frac{|b^{-1}x|\,e^{-2|b^{-1}x|}}{1-e^{-|b^{-1}x|}}\right)\mathbf{1}_{\{bx<0\}}\right)dx,$$

where the second term is assumed to be zero if $b=0$.

This computation implies that the random variable $\mathbf{C}_{a,b}$ is self-decomposable (SD) and that the associated Lévy process has CM jumps, but that its law is not a GGC except for $b=0$ - see Remark 3.7. A key-tool for the proof is an identity connecting $\mathbf{T}$ and the free Gumbel random variable - see Proposition 3.2, providing an analogue of Zolotarev's factorization in the free setting, and which is interesting in its own right.

Our last main result concerns the shape of the densities $f_{\alpha,\rho}$. It was shown in the Appendix to [14] that the latter are analytic on the interior of their support, and strictly unimodal i.e. they have a unique local maximum. These basic properties mimic those of the classical stable densities displayed in the monograph [93]. A refinement of strict unimodality was recently investigated in [58], where it is shown that all classical stable densities are bell-shaped (BS), that is their $n-$th derivative vanishes exactly n times on the interior of their support, as is the case for the standard Gaussian density. The freely strictly 1-stable density $f_{1,\rho}$ is BS, but it is visually clear that this property is not fulfilled neither by $f_{2,1/2}$ nor by $f_{1/2}$. Let us introduce the following alternative refinement of strict unimodality.

Definition 3.1. *A non-negative function f on $\mathbb{R}$ is said to be* whale-shaped *if its support is a closed half-line, if it is smooth in the interior of its support and vanishes at both ends of its support, and if*

$$\sharp\{x\in \operatorname{Supp} f,\; f^{(n)}(x)=0\} \;=\; 1$$

for every $n\geq 1$.

The denomination comes from the visual aspect of such functions - see Figure 3.1 and compare with the visual aspect of a bell-shaped density given in Figure 3.2. We will denote by WS the whale-shaped property and set WS_+ (resp. WS_-) for those whale-shaped functions whose support is a positive half-line $[x_0,+\infty)$ for some $x_0\in\mathbb{R}$, resp. a negative half-line $(-\infty,x_0]$. Observe that if $f\in\mathrm{WS}_+$, then $x\mapsto f(-x)$ belongs to WS_-. It is easy to see that if $f\in\mathrm{WS}_+$ has support $[x_0,+\infty)$, then f is positive on $(x_0,+\infty)$, $f^{(n)}(+\infty)=0$ and $(-1)^{n-1}f^{(n)}(x_0+)>0$ for every $n\geq 1$. In particular, the class WBS_0 introduced in the main definition of [83] corresponds to those WS_+ functions whose support is $(0,+\infty)$. Observe finally that the sequence of vanishing places of the successive derivatives of a function in WS_+ increases, by Rolle's theorem. Other less immediate properties of WS functions will be established in Section 3.3.H.

Theorem 3.4. *One has*

(a) *For every $\alpha\in(0,1)$, the density f_α is* WS_+.
(b) *For every $\alpha\in(0,3/4]$ and $\rho\in(0,1)$, the density $f_{\alpha,\rho}$ is* BS.
(c) *The density of $\mathbf{T}$ is* WS_-.
(d) *For $a\neq 0$ and for $b=0$ or $ab^{-1}\in\pi\mathbb{Z}$, the density of $\mathbf{C}_{a,b}$ is* BS.

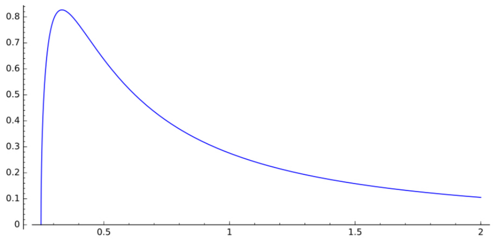


Figure 3.1: The free positive 1/2-stable density (WS).

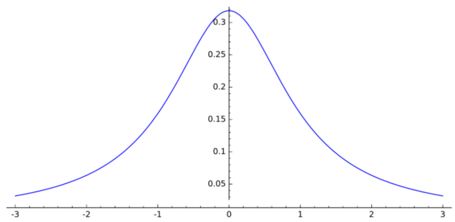


Figure 3.2: The free symmetric 1-stable density (BS).

This result leaves open the question of the exact shape of the density for all $\alpha > 1$. Observe that the limiting case $\alpha = 2$ is rather peculiar since it can be elementally shown that its even derivatives never vanish, whereas its odd derivatives vanish only once and at zero. But since the BS property is not closed under pointwise limits, it might be true that $f_{\alpha,\rho}$ is BS whenever its support is $\mathbb{R}$. On the other hand, in spite of Theorem 3.4 (c) we think that for $\alpha \in (1,2)$ the visually whale-shaped density $f_{\alpha,1/\alpha}$, whose support is a negative half-line, is not WS_-. Indeed, we will see in Proposition 3.15 that otherwise it would be ID, and we know that this is not true at least for α close enough to 2.

Our four theorems are proved in Section 3.2. In the last section, we derive further results related to the analysis of the one-sided free stable densities. First, we analyze in more detail the Kanter random variable $\mathbf{K}_\alpha$, which plays an important role in the proof of all four theorems. The range $\alpha < 1/5$ is particularly investigated, and two conjectures made in [50] and [23] are answered in the negative. A curious Airy-type function is displayed in the case $\alpha = 1/5$. We also derive the full asymptotic expansion of the densities of $\mathbf{X}_\alpha$, $\mathbf{X}_{\alpha,1-1/\alpha}$ and $1-\mathbf{T}$ at the left end of their support, completing the series representation at infinity (1.16) in [44]. We then provide some explicit finite factorizations of $\mathbf{X}_\alpha$ and $\mathbf{K}_\alpha$ with α rational in terms of the Beta random variable, and an identity in law for random discriminants on the unit circle is briefly discussed. These factorizations motivate a new identity for the Beta-Gamma algebra, which is derived thanks to a formula of Thomae on the generalized hypergeometric function. Stochastic and convex orderings are obtained for certain negative powers of $\mathbf{X}_\alpha$, where the free Gumbel law and the exceptional free 1-stable law appear naturally at the limit. We show that some generalizations of the semi-circular random variable $\mathbf{X}_{2,1/2}$ provide a family of examples solving the so-called van Dantzig's problem. Finally, we display some striking properties of whale-shaped functions and densities.

Notation. Throughout, unless otherwise explicitly stated, in any factorization of the type $X \stackrel{d}{=} Y + Z$ or $X \stackrel{d}{=} Y \times Z$, the random variables Y, Z on the right-hand side will be assumed to be classically independent.

3.2 Proofs of the main results

3.2.A Preliminaries

The proofs of all four theorems rely on the following result by Haagerup and Möller [42] who, using a general property of the $S-$transform, have computed the fractional moments of $\mathbf{X}_\alpha$. They obtained

$$\begin{aligned}\mathbb{E}[\mathbf{X}_\alpha^s] &= \frac{\Gamma(1-s/\alpha)}{\Gamma(2-(1/\alpha-1)s)\Gamma(1-s)}\\ &= \left(\frac{1}{1+(1-1/\alpha)s}\right)\times\left(\frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)\Gamma(1-(1/\alpha-1)s)}\right)\end{aligned}$$

for $s<\alpha$. Identifying the two factors, we get the following multiplicative identity in law

$$\mathbf{X}_\alpha \overset{d}{=} \mathbf{U}^{1-1/\alpha}\times\mathbf{K}_\alpha, \tag{3.3}$$

where $\mathbf{U}$ is uniform on $(0,1)$ and $\mathbf{K}_\alpha$ is the so-called Kanter random variable. The latter appears in the following factorization due to Kanter - see Corollary 4.1 in [51]:

$$\mathbf{Z}_\alpha \overset{d}{=} \mathbf{L}^{1-1/\alpha}\times\mathbf{K}_\alpha, \tag{3.4}$$

where $\mathbf{L}$ has unit exponential distribution and $\mathbf{Z}_\alpha$ is a classical positive $\alpha-$stable random variable with Laplace transform $\mathbb{E}[e^{-\lambda\mathbf{Z}_\alpha}]=e^{-\lambda^\alpha}$ and fractional moments

$$\mathbb{E}[\mathbf{Z}_\alpha^s] = \frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)}$$

for $s<\alpha$. Observe that the random variable $\mathbf{K}_\alpha$ has fractional moments

$$\mathbb{E}[\mathbf{K}_\alpha^s] = \frac{\Gamma(1-s/\alpha)}{\Gamma(1-(1/\alpha-1)s)\Gamma(1-s)} \tag{3.5}$$

for $s<\alpha$, and in particular a support $[b_\alpha,+\infty)$ which is bounded away from zero, with

$$b_\alpha^{-1} = \alpha^{-1}(1-\alpha)^{1-\frac{1}{\alpha}} = \lim_{n\to+\infty}\mathbb{E}[\mathbf{K}_\alpha^{-n}]^{1/n},$$

by Stirling's formula. The density of $\mathbf{K}_\alpha$ is explicit for $\alpha=1/2$, with

$$\mathbf{K}_{\frac{1}{2}} \overset{d}{=} \frac{1}{4\cos^2(\pi\mathbf{U}/2)} \overset{d}{=} \frac{1}{4\mathbf{B}_{\frac{1}{2},\frac{1}{2}}}$$

and where, here and throughout, $\mathbf{B}_{a,b}$ stands for a standard $\beta(a,b)$ random variable with density

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}$$

on $(0,1)$. Plugging this in (3.3) yields easily

$$\mathbf{X}_{\frac{1}{2}} \overset{d}{=} \frac{1}{4\mathbf{B}_{\frac{1}{2},\frac{3}{2}}}$$

and we retrieve the aforementioned closed expression of $f_{1/2}$. Several analytical properties of the density of $\mathbf{K}_\alpha - b_\alpha$ have been obtained in [50, 82]. In particular, Corollary 3.2 in [50] shows that it is CM, a fact which we will use repeatedly in the sequel.

Remark 3.1. (a) Specifying Haagerup and Möller's result to the negative integers yields

$$\mathbb{E}[\mathbf{X}_\alpha^{-n}] = \frac{1}{n\alpha^{-1}+1}\binom{n\alpha^{-1}+1}{n}, \quad n \geq 0.$$

The latter is a so-called Fuss-Catalan sequence, and it falls within the scope of more general positive-definite sequences studied in [65, 66]. With the notations of these papers, one has $\mathbf{X}_\alpha \stackrel{d}{=} W_{1/\alpha,1}^{-1}$. This implies that f_α can be written explicitly, albeit in complicated form, for $\alpha = 1/3$ and $\alpha = 2/3$ - see (40) and (41) in [66]. It is also interesting to mention that $\mathbf{X}_{\frac{1}{2}}^{-1}$ has Marchenko-Pastur (or free Poisson) distribution, with density

$$\frac{1}{2\pi}\sqrt{\frac{4-x}{x}}$$

on $(0,4]$. More generally, Proposition A.4.3 in [14] - see also (8) in [66] - shows that $\mathbf{X}_{\frac{1}{n}}^{-1}$ is distributed for each $n \geq 2$ as the $(n-1)$-th free multiplicative convolution power of the Marchenko-Pastur distribution.

(b) The negative integer moments of $\mathbf{K}_\alpha$ are given by the simple binomial formula

$$\mathbb{E}[\mathbf{K}_\alpha^{-n}] = \binom{n\alpha^{-1}}{n}, \quad n \geq 0.$$

This shows that the law of $\mathbf{K}_\alpha^{-1}$ is of the type studied in [67], more precisely it is $\nu(1/\alpha, 0)$ with the notations therein. By Gauss's multiplication formula - see e.g. Theorem 1.5.2 in [1] - and Mellin inversion, this also implies the identity

$$\mathbf{K}_{\frac{1}{3}}^{-1} \stackrel{d}{=} \mathbf{K}_{\frac{2}{3}}^{-2} \stackrel{d}{=} 27\,\mathbf{B}_{\frac{1}{3},\frac{2}{3}}(1-\mathbf{B}_{\frac{1}{3},\frac{2}{3}})$$

in terms of a single random variable $\mathbf{B}_{\frac{1}{3},\frac{2}{3}}$. In particular, the density of $\mathbf{K}_\alpha$ can be written in closed form for $\alpha = 1/3$ and $\alpha = 2/3$ as a two-to-one transform of the density of $\mathbf{B}_{\frac{1}{3},\frac{2}{3}}$ - see also Theorems 5.1 and 5.2 in [67]. As seen above, $\mathbf{K}_{\frac{1}{2}}^{-1} \stackrel{d}{=} 4\mathbf{B}_{\frac{1}{2},\frac{1}{2}}$ is arc-sine distributed, with density

$$\frac{1}{\pi\sqrt{x(4-x)}}$$

on $(0,4]$. It is well-known that this is the distribution of the rescaled free sum of two Bernoulli random variables with parameter $1/2$. It turns out that in general, $\mathbf{K}_{\frac{1}{n}}^{-1}$ is distributed for each $n \geq 2$ as the $(n-1)$-th free multiplicative convolution power of a free Bernoulli process at time $n/(n-1)$ - see (6.9) in [67].

(c) The random variable $\mathbf{K}_\alpha$ can be expressed as the following explicit deterministic transformation of a single uniform variable $\mathbf{U}$ on $(0,1)$:

$$\mathbf{K}_\alpha \stackrel{d}{=} \frac{\sin(\pi\alpha\mathbf{U})\sin^{\frac{1-\alpha}{\alpha}}(\pi(1-\alpha)\mathbf{U})}{\sin^{\frac{1}{\alpha}}(\pi\mathbf{U})}. \tag{3.6}$$

This is Kanter's original observation - see Section 4 in [51], and it will play an important role in the proof of Theorem 3. Notice that the deterministic transformation involved in

(3.6) appears in the implicit expression of the densities f_α, which is given in the second part of Proposition A.I.4 in [14] - see also (11) in [66] for the case when α is the reciprocal of an integer. There does not seem to exist any computational explanation of this fact. We refer to equation (1) in [29], and also to Propositions 1 and 2 therein for further results on this transformation.

3.2.B Proof of Theorem 1

The case $\alpha \le 1$

We begin with the one-sided situation $\rho = 1$. We deduce from (3.3) and the multiplicative convolution formula that, for any $x > 0$,

$$\begin{aligned} f_\alpha(x+b_\alpha) &= \frac{\alpha}{1-\alpha}\int_1^{1+\frac{x}{b_\alpha}} y^{-\frac{2-\alpha}{1-\alpha}} f_{\mathbf{K}_\alpha}(y^{-1}(x+b_\alpha))\,dy \\ &= \frac{\alpha b_\alpha^{\frac{1}{1-\alpha}}}{1-\alpha}\int_0^1 \frac{x}{(b_\alpha+tx)^{\frac{2-\alpha}{1-\alpha}}}\, f_{\mathbf{K}_\alpha}\Big(b_\alpha + \frac{b_\alpha(1-t)x}{b_\alpha+tx}\Big)\,dt. \end{aligned}$$

On the one hand, for every $t \in (0,1)$, the function

$$x \mapsto \frac{(1-t)x}{b_\alpha+tx}$$

is a Bernstein function - see [80]. On the other hand, by the aforementioned Corollary 3.2 in [50], the function $z \mapsto f_{\mathbf{K}_\alpha}(b_\alpha+z)$ is CM. Hence, by e.g. Theorem 3.7 in [80], the function

$$x \mapsto f_{\mathbf{K}_\alpha}\Big(b_\alpha + \frac{b_\alpha(1-t)x}{b_\alpha+tx}\Big)$$

is CM, and so is

$$x \mapsto (b_\alpha+tx)^{-\frac{2-\alpha}{1-\alpha}} f_{\mathbf{K}_\alpha}\Big(b_\alpha + \frac{b_\alpha(1-t)x}{b_\alpha+tx}\Big)$$

as the product of two CM functions. Integrating in t shows that $x \mapsto x^{-1} f_\alpha(x+b_\alpha)$ is CM on $(0,\infty)$ and it is easy to see from Bernstein's theorem that this implies the independent factorization

$$\mathbf{X}_\alpha \overset{d}{=} b_\alpha \;+\; \mathbf{\Gamma}_2 \times \mathbf{Y}_\alpha$$

for some positive random variable $\mathbf{Y}_\alpha$ where, here and throughout, $\mathbf{\Gamma}_t$ stands for a standard $\Gamma(t)$ random variable with density

$$\frac{x^{t-1}e^{-x}}{\Gamma(t)}$$

on $(0,+\infty)$. By Kristiansen's theorem [56], this shows that $\mathbf{X}_\alpha$ is ID.

To handle the two-sided situation $\rho \in (0,1)$, we appeal to the following identity in law which was observed in [44] - see (2.8) therein:

$$\mathbf{X}_{\alpha,\rho} \overset{d}{=} \mathbf{X}_{1,\rho} \times \mathbf{X}_\alpha. \tag{3.7}$$

Since $\mathbf{X}_{1,\rho}$ has a drifted Cauchy law and since the underlying Cauchy process $\{\mathbf{X}_t^{(1,\rho)},\, t \ge 0\}$ is self-similar with index one, the latter identity transforms into

$$\mathbf{X}_{\alpha,\rho} \overset{d}{=} \mathbf{X}^{(1,\rho)}_{\mathbf{X}_\alpha} \tag{3.8}$$

which is a Bochner's subordination identity. By e.g. Theorem 30.1 in [79], this finally shows that $\mathbf{X}_{\alpha,\rho}$ is ID for every $\alpha \in (0,1]$ and $\rho \in [0,1]$.

□

Remark 3.2. (a) The above proof shows that

$$s \mapsto \frac{\mathbb{E}[(\mathbf{X}_\alpha - b_\alpha)^s]}{\Gamma(2+s)}$$

is the Mellin transform of some positive random variable. On the other hand, it seems difficult to find a closed formula for the Mellin transform $\mathbb{E}[(\mathbf{X}_\alpha - b_\alpha)^s]$, except in the case $\alpha = 1/2$ where

$$\mathbb{E}[(\mathbf{X}_{\frac{1}{2}} - b_{\frac{1}{2}})^s] = \frac{2^{1-2s}}{\pi}\Gamma(3/2+s)\,\Gamma(1/2-s), \qquad s \in (-3/2, 1/2).$$

When α is the reciprocal of an integer, there is an expression in terms of the terminating value of a generalized hypergeometric function - see Remark 3.15 (c), but we are not sure whether this always transforms into a ratio of products of Gamma functions, as is the case for $\mathbf{X}_\alpha$.

(b) We believe that $\mathbf{X}_\alpha - b_\alpha$ is a $\Gamma_{3/2}$–mixture for every $\alpha \in (0,1)$, that is

$$s \mapsto \frac{\mathbb{E}[(\mathbf{X}_\alpha - b_\alpha)^s]}{\Gamma(3/2+s)}$$

is the Mellin transform of some positive random variable. This more stringent property is actually true for $\alpha \le 3/4$, as a consequence of the above proof and Theorem 3.2 - see Remark 3.10 (b).

The case $\alpha > 1$ and $\rho = 1/2$

We first derive a closed expression for the Fourier transform of $\mathbf{X}_{\alpha,\rho}$, which has independent interest. It was already obtained as Theorem 1.8 in [44] in a slightly different manner. Our proof is much simpler and so we include it here. Introduce the so-called Wright function

$$\phi(a,b,z) = \sum_{n\ge 0} \frac{z^n}{\Gamma(b+an)\, n!}$$

with $a > -1, b \in \mathbb{R}$ and $z \in \mathbb{C}$. This function was thoroughly studied in the original articles [89, 90, 91] for various purposes, and is referenced in Formula 18.1(27) in the encyclopedia [34]. It will play a role in other parts of the present paper.

Lemma 3.1. *$\forall\,(\alpha,\rho) \in \mathcal{D}$, one has*

$$\mathbb{E}[e^{\mathrm{i}t\mathbf{X}_{\alpha,\rho}}] = \phi(\alpha-1, 2, -(\mathrm{i}t)^\alpha e^{-\mathrm{i}\pi\alpha\rho\,\mathrm{sgn}(t)}), \qquad t \in \mathbb{R}.$$

Proof. The case $\alpha = 1$ is an easy and classic computation, since $\mathbf{X}_{1,\rho}$ has a drifted Cauchy distribution and $\phi(0,2,z) = e^z$. When $\alpha \ne 1$, we first observe that since $\mathbf{X}_{\alpha,\rho} \stackrel{d}{=} -\mathbf{X}_{\alpha,1-\rho}$, it is enough to consider the case $t > 0$. Combining e.g. Theorem 14.19 in [79] and Corollary 1.5 in [44] yields

$$e^{-(\mathrm{i}x)^\alpha e^{-\mathrm{i}\pi\alpha\rho}} = \int_0^\infty t\,e^{-t}\,\mathbb{E}[e^{\mathrm{i}xt^{1-1/\alpha}\mathbf{X}_{\alpha,\rho}}]\,dt = x^{\frac{2\alpha}{1-\alpha}}\int_0^\infty t\,e^{-tx^{\frac{\alpha}{1-\alpha}}}\,\mathbb{E}[e^{\mathrm{i}t^{1-1/\alpha}\mathbf{X}_{\alpha,\rho}}]\,dt$$

for all $x > 0$. On the other hand, a straightforward computation implies

$$x^{\frac{2\alpha}{1-\alpha}} \int_0^\infty t\, e^{-tx^{\frac{\alpha}{1-\alpha}}}\, \phi(\alpha-1, 2, -(\mathrm{i}t^{1-1/\alpha})^\alpha e^{-\mathrm{i}\pi\alpha\rho})\, dt \;=\; e^{-(\mathrm{i}x)^\alpha e^{-\mathrm{i}\pi\alpha\rho}}, \qquad x > 0.$$

The result follows then by uniqueness of the Laplace transform.

□

We can now finish the proof of the case $\alpha > 1, \rho = 1/2$, where the above lemma reads

$$\mathbb{E}[e^{\mathrm{i}t\mathbf{X}_{\alpha,1/2}}] \;=\; \phi(\alpha-1, 2, -|t|^\alpha), \qquad t \in \mathbb{R}.$$

Applying Theorem 1 in [89] and some trigonometry, we obtain the asymptotic behavior

$$\phi(\alpha-1, 2, -t^\alpha) \;\sim\; \kappa_\alpha\, t^{-3/2}\, e^{\cos(\pi/\alpha)\,\alpha(\alpha-1)^{1/\alpha-1}\,t} \cos(3\pi/2\alpha \,+\, \sin(\pi/\alpha)\,\alpha(\alpha-1)^{1/\alpha-1}\,t)$$

as $t \to +\infty$, for some $\kappa_\alpha > 0$. This implies that $t \mapsto \mathbb{E}[e^{\mathrm{i}t\mathbf{X}_{\alpha,1/2}}]$ vanishes (an infinite number of times) on $\mathbb{R}$, and hence cannot be the characteristic function of an ID distribution - see e.g. Lemma 7.5 in [79].

□

Remark 3.3. (a) It was recently shown in Theorem 1 of [7] that for any $a, \beta > 0$, the function $\phi(a, \beta, -z)$ has only positive zeroes on $\mathbb{C}$. Combined with Lemma 3.1, this entails that the function $t \mapsto \mathbb{E}[e^{\mathrm{i}t\mathbf{X}_{\alpha,\rho}}]$ never vanishes on $\mathbb{R}$ for $\alpha > 1$ and $\rho \neq 1/2$, so that the above simple argument cannot be applied. Nevertheless, we conjecture that $\mathbf{X}_{\alpha,\rho}$ is not ID for all $\alpha > 1$ and $\rho \in [1-1/\alpha, 1/\alpha]$.

(b) When $\rho = 1/\alpha$, Lemma 3.1 also gives the moment generating function

$$\mathbb{E}[e^{\lambda\mathbf{X}_{\alpha,1/\alpha}}] \;=\; \phi(\alpha-1, 2, \lambda^\alpha) \;=\; \prod_{n\geq 1}\left(1 + \frac{\lambda^\alpha}{\lambda_{\alpha,n}}\right), \qquad \lambda \geq 0,$$

where $0 < \lambda_{\alpha,1} < \lambda_{\alpha,2} \ldots$ are the positive zeroes of $\phi(\alpha-1, 2, -z)$. Above, the product representation is a consequence of the Hadamard factorization for the entire function $\phi(\alpha-1, 2, z)$ which is of order < 1 - see again Theorem 1 in [89], whereas the simplicity of the zeroes follows from the Laguerre theorem on the separation of zeroes for $\phi(\alpha-1, 2, z)$, which has genus 0.

Consider now the random variable

$$\mathbf{Y}_\alpha \;=\; b_{1/\alpha}^{-1/\alpha} \;-\; \mathbf{X}_{\alpha,1/\alpha} \;=\; \alpha(\alpha-1)^{1/\alpha-1} \;-\; \mathbf{X}_{\alpha,1/\alpha},$$

whose support is $(0,\infty)$ by Proposition A.1.2 in [14], and whose infinite divisibility amounts to that of $\mathbf{X}_{\alpha,1/\alpha}$. Its log-Laplace transform reads

$$\begin{aligned} -\log \mathbb{E}[e^{-\lambda\mathbf{Y}_\alpha}] \;&=\; \alpha(\alpha-1)^{1/\alpha-1}\lambda \;-\; \sum_{n\geq 1}\log\left(1 + \frac{\lambda^\alpha}{\lambda_{\alpha,n}}\right) \\ &=\; \int_0^\infty (1 - e^{-\lambda^\alpha x})\left(\frac{(\alpha-1)^{1/\alpha-1}x^{-1/\alpha}}{\Gamma(1-1/\alpha)} \;-\; \sum_{n\geq 1} e^{-\lambda_{\alpha,n}x}\right)\frac{dx}{x} \end{aligned} \tag{3.9}$$

where in the second equality we have used Frullani's identity repeatedly and the well-known formula (1) p.viii in [80]. Putting everything together shows that $\mathbf{X}_{\alpha,1/\alpha}$ is ID if and only if the function on the right-hand side is Bernstein. Unfortunately, this property seems difficult to check at first sight. Observe by Corollary 3.7 (iii) in [80] that this function is not Bernstein if the function

$$x \mapsto \frac{(\alpha-1)^{1/\alpha-1}x^{-1/\alpha}}{\Gamma(1-1/\alpha)} - \sum_{n\geq 1} e^{-\lambda_{\alpha,n}x}$$

takes negative values on $(0,\infty)$, but this property seems also difficult to study. A lengthy asymptotic analysis which will not be included here, shows that it converges at zero to some positive constant.

(c) Rewriting equation (3.9) as

$$\lambda^{1/\alpha} = b_{1/\alpha}\left(\sum_{n\geq 1}\log\left(1+\frac{\lambda}{\lambda_{\alpha,n}}\right) - \log\mathbb{E}[e^{-\lambda^{1/\alpha}\mathbf{Y}_\alpha}]\right),$$

we obtain the factorization

$$\mathbf{Z}_{1/\alpha} \stackrel{d}{=} b_{1/\alpha}\left(\sum_{n\geq 1}\mathrm{Exp}(\lambda_{\alpha,n}) + \mathbf{Z}^{(1/\alpha)}_{\mathbf{Y}_\alpha}\right)$$

where $\{\mathbf{Z}^{(1/\alpha)}_t, t\geq 0\}$ is the $(1/\alpha)$−stable subordinator and all quantities on the right-hand side are independent. This identity is similar to that of the Lemma in [83], except that the parameters $\lambda_{\alpha,n}$ of the exponential random variables are not explicit.

3.2.C Proof of Theorem 2

The case $\rho = 1$

Here, we need to show that the law of $\mathbf{X}_\alpha$ is a true GGC. To do so, we first observe that by (3.3) and some rearrangements, one has

$$\mathbb{E}[e^{-\lambda\mathbf{X}_\alpha}] = \frac{\alpha\,\lambda^{\frac{\alpha}{1-\alpha}}}{1-\alpha}\int_\lambda^\infty \mathbb{E}[e^{-x\mathbf{K}_\alpha}]\,x^{\frac{1}{\alpha-1}}\,dx, \qquad \lambda\geq 0. \tag{3.10}$$

A combination of Theorem 6.1.1 and Properties (iv) and (xi) p.68 in [18] imply then that it is enough to show that the law of $\mathbf{K}_\alpha$ itself is a GGC. Alternatively, one can use the main result of [19], since it is easily seen that $\mathbf{U}^{1-1/\alpha}$ has a GGC distribution. To analyze the law of $\mathbf{K}_\alpha$, we use the identity in law

$$\mathbf{K}_\alpha \stackrel{d}{=} \mathbf{K}_{1-\alpha}^{\frac{1}{\alpha}-1}, \tag{3.11}$$

a consequence of (3.5) which shows that both random variables have the same fractional moments. Plugging (3.11) again into (3.4) implies that the Laplace transform of $\mathbf{K}_{1-\alpha}$ is the survival function of the power transformation $\mathbf{Z}_\alpha^{\frac{-\alpha}{1-\alpha}}$. In other words, one has

$$\mathbb{E}[e^{-x\mathbf{K}_{1-\alpha}}] = \mathbb{P}[\mathbf{L}\geq x\mathbf{K}_{1-\alpha}] = \mathbb{P}[\mathbf{Z}_\alpha^{-\frac{\alpha}{1-\alpha}}\geq x], \qquad x\geq 0. \tag{3.12}$$

Setting $F_\alpha(x)$ for the function defined in (3.12), we next observe that since $\mathbf{K}_\alpha$ has a CM density and support $[b_\alpha,+\infty)$, this function F_α has by Theorem 9.5 in [80] an analytic extension on $\mathbb{C}\setminus(-\infty,0]$ which is given by

$$F_\alpha(z) = \exp -\left[b_{1-\alpha}z + \int_0^\infty \frac{z}{t+z}\,\frac{\theta_\alpha(t)}{t}\,dt\right] \tag{3.13}$$

for some measurable function $\theta_\alpha : (0,\infty) \to [0,1]$ such that $\int_0^1 \theta_\alpha(t)t^{-1}\,dt < \infty$. See also Theorem 51.12 in [79]. Applying now Theorem 8.2 (v) in [80], we see that the GGC property of $\mathbf{K}_{1-\alpha}$ is equivalent to the non-decreasing character of θ_α on $(0,\infty)$, and the following proposition allows us to conclude the proof of the case $\rho = 1$.

Proposition 3.1. *The function θ_α has a continuous version on $(0,\infty)$, which is non-decreasing for every $\alpha \in [1/4, 1)$.*

Proof. The analysis of θ_α depends, classically, on the behavior of F_α near the cut $(-\infty, 0]$. Assume for a moment that θ_α is continuous. For every $r > 0$ and $\delta \in (0,1)$, we have, after some simple rearrangements,

$$\begin{aligned}\frac{F_\alpha(re^{\mathrm{i}\pi(1-\delta)})}{F_\alpha(re^{\mathrm{i}\pi(\delta-1)})} &= \exp -2\mathrm{i}\left[\sin(\pi\delta)b_{1-\alpha}r + \int_0^\infty \frac{\sin(\pi(1-\delta))\,\theta_\alpha(rt)}{1+2\cos(\pi(1-\delta))t+t^2}\,dt\right]\\ &= \exp -\left[2\mathrm{i}\sin(\pi\delta)b_{1-\alpha}r + 2\mathrm{i}\pi\mathbb{E}\left[\theta_\alpha(r(\mathbf{X}_{1,1-\delta})^+)\right]\right]\\ &\to \exp -2\mathrm{i}\pi\theta_\alpha(r)\end{aligned}$$

as $\delta \to 0$ since $\mathbf{X}_{1,1-\delta} \to \mathbf{1}$ in law as $\delta \to 0$ and θ_α is bounded continuous. On the other hand, it follows from the third expression of F_α in (3.12) and the first formula of Corollary 1 p.71 in [93], after a change of variable, that

$$F_\alpha(x) = 1 + \frac{1}{2\mathrm{i}\pi}\int_0^\infty e^{-t}\left(e^{-e^{\mathrm{i}\pi\alpha}t^\alpha x^{1-\alpha}} - e^{-e^{-\mathrm{i}\pi\alpha}t^\alpha x^{1-\alpha}}\right)\frac{dt}{t}, \qquad x > 0.$$

The analytic continuations of F_α near the cut are then expressed, changing the variable backwards, as

$$F_\alpha(re^{\mathrm{i}\pi}) = 1 + \frac{1}{2\mathrm{i}\pi}\int_0^\infty e^{-ru}\left(e^{ru^\alpha} - e^{ru^\alpha e^{-2\mathrm{i}\pi\alpha}}\right)\frac{du}{u}$$

and

$$F_\alpha(re^{-\mathrm{i}\pi}) = 1 + \frac{1}{2\mathrm{i}\pi}\int_0^\infty e^{-ru}\left(e^{ru^\alpha e^{2\mathrm{i}\pi\alpha}} - e^{ru^\alpha}\right)\frac{du}{u} = \overline{F_\alpha(re^{\mathrm{i}\pi})}.$$

Therefore, we obtain

$$\frac{F_\alpha(re^{\mathrm{i}\pi})}{F_\alpha(re^{-\mathrm{i}\pi})} = e^{-2\mathrm{i}\pi\eta_\alpha(r)}$$

for every $r > 0$, with the notation

$$\eta_\alpha(r) = \frac{1}{\pi}\arg[F_\alpha(re^{-\mathrm{i}\pi})].$$

Since

$$\Im(F_\alpha(re^{-\mathrm{i}\pi})) = \frac{1}{2\pi}\int_0^\infty e^{-ru}\,\Re\left(e^{ru^\alpha} - e^{ru^\alpha e^{2\mathrm{i}\pi\alpha}}\right)\frac{du}{u} > 0$$

for every $r > 0$, the function η_α takes its values in $[0,1]$ and is clearly continuous. By construction, the functions $t^{-1}\eta_\alpha(t)$ and $t^{-1}\theta_\alpha(t)$ have the same Stieltjes transform, and it follows by uniqueness that θ_α has a continuous version, which is η_α.

It remains to study the monotonous character of η_α on $(0,\infty)$. A first observation is that, expanding the exponentials inside the brackets and using the complement formula for the Gamma function, the following absolutely convergent series representation holds:

$$F_\alpha(re^{-\mathrm{i}\pi}) = \sum_{n\geq 0}\frac{z^n e^{\mathrm{i}\pi n\alpha}}{n!\Gamma(1-n\alpha)} = \phi(-\alpha, 1, ze^{\mathrm{i}\pi\alpha}) \tag{3.14}$$

with $z = r^{1-\alpha}$. In particular, the function

$$r \;\mapsto\; r^{\alpha-1}\,\mathfrak{I}\big(F_\alpha(re^{-\mathrm{i}\pi})\big) \;=\; \frac{1}{\pi}\sum_{n\geq 1}\frac{\Gamma(n\alpha)}{n!}\,r^{(n-1)(1-\alpha)}$$

is absolutely monotonous on $(0,\infty)$, and the non-decreasing character of θ_α will hence be established as soon as $r\mapsto r^{\alpha-1}\mathfrak{R}(F_\alpha(re^{-\mathrm{i}\pi}))$ is non-increasing on $(0,\infty)$. We use the representation

$$\mathfrak{R}(F_\alpha(re^{-\mathrm{i}\pi})) \;=\; 1 \;+\; \frac{1}{2\pi}\int_0^\infty e^{-r^{1-1/\alpha}u}\,\mathfrak{I}(e^{u^\alpha e^{2\mathrm{i}\pi\alpha}})\,\frac{du}{u}$$

and divide this last part of the proof into three parts.

- The case $\alpha\in[1/2,1)$. If $\alpha = 1/2$, we simply have $\mathfrak{R}(F_{1/2}(re^{-\mathrm{i}\pi}))\equiv 1$. If $\alpha > 1/2$ we rewrite, using again the first part of Corollary 1 p.71 in [93],

$$\begin{aligned}\mathfrak{R}(F_\alpha(re^{-\mathrm{i}\pi})) \;&=\; \frac{1}{2} \;+\; \frac{1}{2}\left(1-\frac{1}{\pi}\int_0^\infty e^{-r^{1-1/\alpha}u}\,\mathfrak{I}(e^{-u^\alpha e^{-i\pi\rho\alpha}})\,\frac{du}{u}\right)\\ &=\; \frac{1}{2}\big(1+\mathbb{P}[\mathbf{Z}_{\alpha,\rho}\leq r^{1-1/\alpha}]\big)\end{aligned}$$

where $\rho = 2-1/\alpha\in(0,1)$ and $\mathbf{Z}_{\alpha,\rho}$ is as in Lemma 3.1 a real α−stable random variable with positivity parameter ρ. Thus, $\mathfrak{R}(F_\alpha(re^{-\mathrm{i}\pi}))$ decreases (from 1 to $1/2\alpha$) on $(0,\infty)$ and $r^{\alpha-1}\mathfrak{R}(F_\alpha(re^{-\mathrm{i}\pi}))$ also decreases on $(0,\infty)$, as required.

- The case $\alpha\in[1/3,1/2)$. Setting $\rho = 1/\alpha-2\in(0,1]$ and using the same notation as in the previous case, we rewrite

$$\begin{aligned}\mathfrak{R}(F_\alpha(re^{-\mathrm{i}\pi})) \;&=\; 1 \;+\; \frac{1}{2\pi}\int_0^\infty e^{-r^{1-1/\alpha}u}\,\mathfrak{I}(e^{-u^\alpha e^{-i\pi\rho\alpha}})\,\frac{du}{u}\\ &=\; 1 \;+\; \frac{1}{2}\,\mathbb{P}[\mathbf{Z}_{\alpha,\rho}\geq r^{1-1/\alpha}]\\ &=\; 1 \;+\; \frac{\rho}{2}\,\mathbb{P}[\mathbf{W}_{\alpha,\rho}^{-\alpha}\leq r^{1-\alpha}]\end{aligned}$$

where $\mathbf{W}_{\alpha,\rho}\overset{d}{=}\mathbf{Z}_{\alpha,\rho}\,|\,\mathbf{Z}_{\alpha,\rho}>0$ is the cut-off random variable defined in Chapter 3 of [93]. Observe that here, the function $r\mapsto\mathfrak{R}(F_\alpha(re^{-\mathrm{i}\pi}))$ increases. Setting $h_{\alpha,\rho}$ for the density function of $\mathbf{W}_{\alpha,\rho}^{-\alpha}$ on $(0,\infty)$, we get after a change of variable

$$r^{\alpha-1}\,\mathfrak{R}(F_\alpha(re^{-\mathrm{i}\pi})) \;=\; r^{\alpha-1} \;+\; \frac{\rho}{2}\int_0^1 h_{\alpha,\rho}(r^{1-\alpha}x)\,dx,$$

and it is hence sufficient to prove that the function $h_{\alpha,\rho}$ is non-increasing on $(0,\infty)$. Using the expression for the Mellin transform of $\mathbf{W}_{\alpha,\rho}$ given at the bottom of p.186 in [93] together with the complement and multiplication formulæ for the Gamma function, we obtain

$$\begin{aligned}\mathbb{E}[\mathbf{W}_{\alpha,\rho}^{-\alpha s}] \;&=\; \frac{\Gamma(1+s)}{\Gamma(1+\alpha\rho s)}\times\frac{\Gamma(1-\alpha s)}{\Gamma(1-\alpha\rho s)}\\ &=\; \frac{2^s}{1+s}\times\frac{\Gamma(3/2+s/2)}{\Gamma(3/2)}\times\frac{\Gamma(1+s/2)}{\Gamma(1+\alpha\rho s)}\times\frac{\Gamma(1-\alpha s)}{\Gamma(1-\alpha\rho s)}\end{aligned}$$

for every $s \in (-1, 1/\alpha)$. Identifying the factors and using $\alpha\rho < 1/2$, this implies the identity in law

$$\mathbf{W}_{\alpha,\rho}^{-\alpha} \stackrel{d}{=} 2\mathbf{U} \times \sqrt{\mathbf{\Gamma}_{3/2}} \times \left(\frac{\mathbf{Z}_\rho}{\mathbf{Z}_{2\alpha\rho}}\right)^{\alpha\rho}$$

where all factors on the right hand side are assumed independent. Hence, $\mathbf{W}_{\alpha,\rho}^{-\alpha}$ admits $\mathbf{U}$ as a multiplicative factor and by Khintchine's theorem, its density is non-increasing on $(0,\infty)$.

- The case $\alpha \in [1/4, 1/3)$. Contrary to the above, the argument is here entirely analytic. We consider

$$\begin{aligned} G_\alpha(r) = \Re(F_\alpha(r^{\frac{\alpha}{\alpha-1}} e^{-\mathrm{i}\pi})) &= 1 + \frac{1}{2\pi}\int_0^\infty e^{-ru}\,\Im(e^{u^\alpha e^{2\mathrm{i}\pi\alpha}})\,\frac{du}{u} \\ &= 1 + \frac{1}{2\pi}\int_0^\infty e^{\cos(2\pi\alpha)u^\alpha - ru}\sin(\sin(2\pi\alpha)u^\alpha)\,\frac{du}{u} \\ &= 1 + \frac{1}{2\pi\alpha}\int_0^\infty g_{\alpha,r}(t)\,\sin(t)\,dt \end{aligned}$$

where

$$t \mapsto g_{\alpha,r}(t) = t^{-1} e^{\cot(2\pi\alpha)t - r(\sin(2\pi\alpha))^{-1/\alpha} t^{1/\alpha}}$$

decreases on $(0, +\infty)$. For every $k \ge 0$ we have

$$\int_{2k\pi}^{2(k+1)\pi} g_{\alpha,r}(t)\,\sin(t)\,dt = \int_0^\pi (g(t+2k\pi) - g(t+(2k+1)\pi))\,\sin(t)\,dt > 0,$$

so that $G_\alpha(r) > 1$ for every $r > 0$. We next compute

$$\begin{aligned} (r^\alpha G_\alpha(r))' &= \alpha r^{\alpha-1}(G_\alpha(r) + \frac{r}{\alpha}G'_\alpha(r)) \\ &> \alpha r^{\alpha-1}\left(1 - \frac{r}{2\pi\alpha}\int_0^\infty e^{\cos(2\pi\alpha)u^\alpha - ru}\sin(\sin(2\pi\alpha)u^\alpha)du\right) \\ &= \frac{r^\alpha}{2\pi}\int_0^\infty e^{-ru}\left(2\pi\alpha - e^{\cos(2\pi\alpha)u^\alpha}\sin(\sin(2\pi\alpha)u^\alpha)\right)du > 0, \end{aligned}$$

since $2\pi\alpha > 1 \ge e^{\cos(2\pi\alpha)u^\alpha}\sin(\sin(2\pi\alpha)u^\alpha)$ for every $u > 0$. Changing the variable backwards, this finally shows that $r \mapsto r^{\alpha-1}\Re(F_\alpha(re^{-\mathrm{i}\pi}))$ decreases on $(0,\infty)$.

□

Remark 3.4. (a) The above argument shows that the survival function $x \mapsto \mathbb{P}[\mathbf{Z}_\alpha^{-\frac{\alpha}{1-\alpha}} \ge x]$ is HCM for every $\alpha \ge 1/4$, with the terminology of [18]. A consequence of Corollary 3.2 is that this is not true anymore for $\alpha < 1/5$, and we believe - see Conjecture 3.1 - that the right domain of validity of this property is $\alpha \in [1/5, 1)$. The more stringent property that $\mathbf{Z}_\alpha^{-\frac{\alpha}{1-\alpha}}$ is a HCM random variable for $\alpha \le 1/2$ was conjectured in [20] and some partial results were obtained in [20, 23]. In [37], it is claimed that this latter property holds true if and only if $\alpha \in [1/3, 1/2]$.

(b) The analytical proof for the case $\alpha \in [1/4, 1/3)$ conveys to the case $\alpha \in [1/3, 1/2)$. Nevertheless, it is informative to mention the probabilistic interpretation of $\Re(F_\alpha(re^{-\mathrm{i}\pi}))$ for $\alpha \in [1/3, 1/2)$. Simulations show that this function oscillates for $\alpha < 1/3$. See also Section 4.2 for a striking similarity between the cases $\alpha = 1/3$ and $\alpha = 1/5$.

(c) We do not know if the representation (3.13) holds for the Laplace transform of $\mathbf{X}_\alpha$. Since the latter is a Γ_2-mixture we obtain, similarly as above,

$$\mathbb{E}[e^{-x\mathbf{X}_\alpha}] \;=\; e^{-b_{1-\alpha}x}\int_0^\infty \frac{\nu_\alpha(dt)}{(x+t)^2}$$

for some positive measure ν_α on $[0,+\infty)$. This representation would suffice if we could show that the generalized Stieltjes functions on the right-hand side is the product of two standard Stieltjes functions, applying Theorem 6.17 in [80] as in the proof of Theorem 9.5 therein. However, this is not true in general, for example when ν_α is the sum of two Dirac masses. Observe that in the other direction, the product of two Stieltjes functions is a generalized Stieltjes function of order 2 - see Theorem 7 in [52]. With the notation of [52], we believe that the exact Stieltjes order of $\mathbb{E}[e^{-x\mathbf{X}_\alpha}]$ is actually 3/2, which however does not seem of any particular help for (3.13). Alternatively, because of (3.10) one would like to prove that if f has representation (3.13), then so has $x\mapsto \int_x^\infty f(y)dy$. This is true in the GGC case by Property xi) p.68 in [18], but we were not able to prove this in general.

The case $\rho < 1$

The case $\rho = 0$ follows from $\mathbf{X}_{\alpha,0} \stackrel{d}{=} -\mathbf{X}_\alpha$. For $\rho \in (0,1)$ we appeal to (3.8), the previous case, and the Huff-Zolotarev subordination formula which is given e.g. in Theorem 30.1 of [79]. Since the law of $\mathbf{X}_\alpha$ is a GGC for $\alpha \le 3/4$, its Laplace transform reads

$$\mathbb{E}[e^{-\lambda\mathbf{X}_\alpha}] \;=\; \exp-\left[b_\alpha\lambda \;+\; \int_0^\infty (1-e^{-\lambda x})\, k_\alpha(x)\,\frac{dx}{x}\right]$$

for some CM function k_α. Formula (30.8) in [79] and the closed expression of the density of $\mathbf{X}_{1,\rho}$ imply that the Lévy measure $\nu_{\alpha,\rho}$ of $\mathbf{X}_{\alpha,\rho}$ has density

$$\begin{aligned}\psi_{\alpha,\rho}(x) \;&=\; b_\alpha\psi_{1,\rho}(x) \;+\; \frac{\sin(\pi\rho)}{\pi}\int_0^\infty \frac{k_\alpha(u)}{x^2+2\cos(\pi\rho)xu+u^2}\,du\\ &=\; \frac{\sin(\pi\rho)}{\pi|x|}\left(\frac{b_\alpha}{|x|} \;+\; \int_0^\infty \frac{k_\alpha(|x|u)}{1+2\cos(\pi\rho)\,\mathrm{sgn}(x)\,u+u^2}\,du\right)\end{aligned}$$

over $\mathbb{R}^*$, where the closed expression for $\psi_{1,\rho}$ can be deduced e.g. from Theorem 14.10 and Lemma 14.11 in [79]. Both functions $x\psi_{\alpha,\rho}(x)$ and $x\psi_{\alpha,\rho}(-x)$ are hence CM on $(0,\infty)$.

□

Remark 3.5. Since $b_\alpha > 0$ and the ID random variable $\mathbf{X}_{1,\rho}$ has no Gaussian component, the Huff-Zolotarev subordination formula shows that $\mathbf{X}_{\alpha,\rho}$ does not have a Gaussian component either, and that for $\rho\in(0,1)$ its Lévy measure is such that

$$\int_{|x|\le 1} |x|\,\nu_{\alpha,\rho}(dx) \;=\; +\infty.$$

With the terminology of [79] - see Definition 11.9 therein, this means that the Lévy process associated with $\mathbf{X}_{\alpha,\rho}$ is of type C. This contrasts with the classical $\alpha-$stable Lévy process which is of type B for $\alpha < 1$. When $\rho = 1$ and $\alpha \le 3/4$, the GGC property shows that the Lévy process corresponding to $\mathbf{X}_\alpha$ is of type B. We believe that this is true for all $\alpha\in(0,1)$, but this cannot be deduced from the sole Γ_2-mixture property established in Theorem 1.

3.2.D Proof of Theorem 3

It is well-known and easy to see from the Voiculescu transform

$$\phi_{1,1/2}(z) = -\mathrm{i}$$

that the free sum of $\mathbf{X}_{1,1/2}$ with any random variable is also a classically independent sum. Hence, the ID character of $\mathbf{C}_{a,b}$ follows from that of $\mathbf{T}$, which is a consequence of Theorem 3.1 and the convergence in law

$$\frac{(1-\alpha)^{1-\alpha} - \mathbf{X}_\alpha}{1-\alpha} \xrightarrow{d} \mathbf{T} \qquad \text{as } \alpha \uparrow 1, \tag{3.15}$$

the latter being easily obtained in comparing the two Voiculescu transforms. This concludes the first part of the theorem. Moreover, it is clear that neither $\mathbf{X}_{1,1/2}$ nor $\mathbf{T}$, whose support is a half-line by Proposition A.1.3 in [14], have a Gaussian component, and this property conveys hence to $\mathbf{C}_{a,b}$. Finally, since the Lévy measure of $\mathbf{X}_{1,1/2}$ is

$$\frac{1}{\pi x^2} \mathbf{1}_{\{x \neq 0\}}$$

as seen in the above proof, we are reduced to show by independence and scaling that the Lévy measure of $\mathbf{T}$ has density

$$\frac{1}{x^2}\left(1 - \frac{|x|\, e^{-2|x|}}{1 - e^{-|x|}}\right) \mathbf{1}_{\{x<0\}}.$$

This last computation will be done in two steps. Consider the random variable

$$\mathbf{W} = \frac{\sin(\pi \mathbf{U})}{\pi \mathbf{U}}\, e^{\pi \mathbf{U} \cot(\pi \mathbf{U})}$$

and the exceptional 1-stable random variable $\mathbf{S}$ characterized by

$$\mathbb{E}[e^{s\mathbf{S}}] = s^s, \qquad s > 0.$$

Proposition 3.2. *One has the identities*

$$\mathbf{S} \stackrel{d}{=} \log \mathbf{L} + \log \mathbf{W} \qquad \textit{and} \qquad \mathbf{T} \stackrel{d}{=} \log \mathbf{U} + \log \mathbf{W}.$$

Proof. We begin with the first identity. Using (3.4), we decompose

$$\frac{(1-\alpha)^{1-\alpha} - \mathbf{Z}_\alpha}{1-\alpha} \stackrel{d}{=} \mathbf{K}_\alpha \times \left(\frac{1 - \mathbf{L}^{1-\frac{1}{\alpha}}}{1-\alpha}\right) + \left(\frac{(1-\alpha)^{1-\alpha} - \mathbf{K}_\alpha}{1-\alpha}\right). \tag{3.16}$$

On the one hand, a comparison of the two moment generating functions yields

$$\frac{(1-\alpha)^{1-\alpha} - \mathbf{Z}_\alpha}{1-\alpha} \xrightarrow{d} \mathbf{S} \qquad \text{as } \alpha \uparrow 1.$$

On the other hand, the right-hand side of (3.16) is a deterministic transformation, depending on α, of $(\mathbf{L}, \mathbf{U})$ independent. It is easy to see from (3.6) that

$$\mathbf{K}_\alpha \times \left(\frac{1 - \mathbf{L}^{1-\frac{1}{\alpha}}}{1-\alpha}\right) \xrightarrow{a.s.} \log \mathbf{L} \qquad \text{as } \alpha \uparrow 1.$$

To study the second term, we use the elementary expansions

$$\begin{aligned}
\sin(\pi\alpha\mathbf{U}) &= \sin(\pi\mathbf{U}) + (\alpha-1)\pi\mathbf{U}\cos(\pi\mathbf{U}) + O((1-\alpha)^3)\\
\sin^{\frac{1-\alpha}{\alpha}}(\pi(1-\alpha)\mathbf{U}) &= 1 + (1-\alpha)\log\sin(\pi\mathbf{U}(1-\alpha)) + O((1-\alpha)^2\log^2(1-\alpha))\\
\sin^{\frac{1}{\alpha}}(\pi\mathbf{U}) &= \sin(\pi\mathbf{U})(1+(1-\alpha)\log\sin(\pi\mathbf{U})) + O((1-\alpha)^2)\\
(1-\alpha)^{1-\alpha} &= 1 + (1-\alpha)\log(1-\alpha) + O((1-\alpha)^2\log^2(1-\alpha))
\end{aligned}$$

which, combined with (3.6), yield the almost sure asymptotics

$$\frac{(1-\alpha)^{1-\alpha} - \mathbf{K}_\alpha}{1-\alpha} = \log\left(\frac{\sin(\pi\mathbf{U})}{\pi\mathbf{U}} e^{\pi\mathbf{U}\cot(\pi\mathbf{U})}\right) + O((1-\alpha)\log^2(1-\alpha)).$$

Putting everything together completes the proof of the first identity. The second one is derived exactly in the same way, using (3.3) and (3.15).

□

Remark 3.6. (a) The first identity in Proposition 3.2 is actually the consequence of an integral transformation due to Zolotarev - see (2.2.19) with $\beta = 1$ in [93]. We have offered a separate proof which is perhaps clearer, and which enhances the similarities between the free and the classical case echoing those between (3.3) and (3.4). Observe in particular the identity

$$\mathbf{S} \overset{d}{=} \mathbf{T} + \log \mathbf{\Gamma}_2 \tag{3.17}$$

reminiscent of Corollary 1.5 in [44], and which is a consequence of Proposition 3.2 and the standard identities

$$\mathbf{L}^\beta \overset{d}{=} \mathbf{U}^\beta \times \mathbf{\Gamma}_2^\beta \tag{3.18}$$

valid for every $\beta \in \mathbb{R}^*$ and their limit as $\beta \to 0$, which is

$$\log \mathbf{L} \overset{d}{=} \log \mathbf{U} + \log \mathbf{\Gamma}_2. \tag{3.19}$$

(b) It is interesting to look at these standard identities (3.18) and (3.19) in the context of extreme value distributions. Indeed, the three classical extreme distributions are Fréchet $\mathbf{L}^\beta$ for $\beta < 0$, Weibull $-\mathbf{L}^\beta$ for $\beta > 0$ and Gumbel $-\log \mathbf{L}$ for $\beta \to 0$, whereas the free counterparts are $\mathbf{U}^\beta$ for $\beta < 0$, $-\mathbf{U}^\beta$ for $\beta > 0$ and $-\log \mathbf{U}$ for $\beta \to 0$ according to the classification of [10].

(c) Recently Vargas and Voiculescu have introduced Boolean extreme value distributions [39]. The result is the Dagum distribution, which is indexed by $\beta > 0$ and has density function

$$\frac{x^{1/\beta - 1}}{\beta(1+x^{1/\beta})^2}$$

on $(0,\infty)$. Hence, the Dagum distribution is the law of

$$(\mathbf{U}^{-1} - 1)^\beta \overset{d}{=} \left(\frac{\mathbf{L}}{\mathbf{L}}\right)^\beta$$

which is the independent quotient of two Fréchet distributions, and an example of the generalized Beta distribution of the second kind (GB2). On the other hand, by Proposition

4.12 (b) in [2], the Boolean α−stable distribution has for $\alpha \le 1$ the law of the independent quotient

$$\frac{\mathbf{Z}_{\alpha,\rho}}{\mathbf{Z}_\alpha}$$

and it is interesting to notice that by Zolotarev's duality - see (3.3.16) in [93] - and scaling, the positive part of this random variable is distributed as

$$\left(\frac{\mathbf{Z}_{\alpha\rho}}{\mathbf{Z}_{\alpha\rho}}\right)^\rho \xrightarrow{d} \left(\frac{\mathbf{L}}{\mathbf{L}}\right)^{\frac{1}{\alpha}} \qquad \text{as } \rho \to 0.$$

Finding an interpretation about why such quotients appear in those two Boolean cases is left to future work.

(d) The second identity in Proposition 3.2 can be rewritten as

$$e^{\mathbf{T}} \stackrel{d}{=} \mathbf{U} \times \mathbf{W}.$$

In [3], it is pointed out that the law of $e^{\mathbf{T}}$ is the Dykema-Haagerup distribution, which appears as the eigenvalue distribution of $A_N^* A_N$ as $N \to \infty$, where A_N is an $N \times N$ upper-triangular random matrix with independent complex Gaussian entries - see [32].

(e) It follows from Euler's product and summation formulæ for the sine and the cotangent that $\log \mathbf{W}$ is a decreasing concave deterministic transformation of $\mathbf{U}$. This implies easily that $\log \mathbf{W}$ has an increasing density on its support which is $(-\infty, 1]$. In particular, $\log \mathbf{W}$ is unimodal. Besides, since the densities of $\log \mathbf{U}$ and $\log \mathbf{L}$ are clearly log-concave on the interior of their support, applying Theorem 52.3 in [79] we retrieve the known facts that $\mathbf{S}$ and $\mathbf{T}$ are unimodal random variables.

Our second step is to compute the Mellin transform of $\mathbf{W}$.

Proposition 3.3. *One has*

$$\mathbb{E}[\mathbf{W}^s] = \frac{s^s}{\Gamma(1+s)} = \exp\left[s - \int_0^\infty (1 - e^{-sx})\left(1 - \frac{x}{e^x - 1}\right)\frac{dx}{x^2}\right]$$

for all $s > 0$.

Proof. The first equality follows from

$$s^s = \mathbb{E}[e^{s\mathbf{S}}] = \mathbb{E}[\mathbf{L}^s]\,\mathbb{E}[\mathbf{W}^s] = \Gamma(1+s)\,\mathbb{E}[\mathbf{W}^s], \qquad s > 0,$$

a consequence of the first identity in Proposition 3.2. To get the second one, we proceed as in the proof of Lemma 14.11 of [79] and start from Frullani's identity

$$\log s = \int_0^\infty \frac{e^{-x} - e^{-sx}}{x}\, dx$$

which transforms, dividing the integral at 1 and making an integration by parts, into

$$s \log s = \int_0^\infty (e^{-sx} - 1 + sx\mathbf{1}_{\{x\le 1\}})\frac{dx}{x^2} - s\left(\int_0^\infty (e^{-x} - 1 + x\mathbf{1}_{\{x\le 1\}})\frac{dx}{x^2}\right).$$

On the other hand, it is well-known - see e.g. Proposition 4 (a) in [92] - that

$$\log \Gamma(1+s) = -\gamma s + \int_0^\infty (e^{-sx} - 1 + sx)\frac{dx}{x(e^x - 1)}$$

where $\gamma = -\Gamma'(1)$ is Euler's constant. Combining the two formulæ yields

$$\begin{aligned}\log \mathbb{E}[\mathbf{W}^s] &= cs + \int_0^\infty (e^{-sx} - 1 + sx\mathbf{1}_{\{x\le 1\}})\left(1 - \frac{x}{e^x-1}\right)\frac{dx}{x^2}\\ &= \tilde{c}s - \int_0^\infty (1-e^{-sx})\left(1 - \frac{x}{e^x-1}\right)\frac{dx}{x^2}\end{aligned}$$

where $c, \tilde{c}$ are two constants to be determined. But it is clear that $\tilde{c}$ is the right end of the support of $\log \mathbf{W}$ which we know, by Remark 3.6 (c), to be one. Alternatively, one can use Binet's formula

$$\gamma = \int_0^\infty \left(\frac{e^{-x}}{1-e^{-x}} - \frac{e^{-x}}{x}\right) dx,$$

which is 1.7.2(22) in [34] for $z = 1$, and rearrange the different integrals, to retrieve $\tilde{c} = 1$. This completes the proof.

□

We can now finish the proof of Theorem 3.3. Putting together Propositions 3.2 and 3.3, we get

$$\begin{aligned}\log \mathbb{E}[e^{s\mathbf{T}}] = \log \mathbb{E}[\mathbf{U}^s] + \log \mathbb{E}[\mathbf{W}^s] &= -\log(1+s) + \log \mathbb{E}[\mathbf{W}^s]\\ &= s - \int_0^\infty (1-e^{-sx})\left(1 - \frac{x\,e^{-2x}}{1-e^{-x}}\right)\frac{dx}{x^2}\end{aligned}$$

where the third equality follows from rearranging Frullani's identity and the second equality in Proposition 3.3. All of this shows that the ID random variable $\mathbf{T}$ has support $(-\infty, 1]$ - in accordance with Proposition A.1.3 in [14], and that its Lévy measure has density

$$\frac{1}{x^2}\left(1 - \frac{|x|\,e^{-2|x|}}{1-e^{-|x|}}\right)\mathbf{1}_{\{x<0\}}$$

as required.

□

Remark 3.7. (a) The first equality in Proposition 3.3 shows that $\mathbf{W}$ has the distribution ν_0 studied in Theorem 6.1 of [65]. This distribution also appears in Sakuma and Yoshida's limit theorem - see [78]. Finally, combining this equality and the second identity in Proposition 3.2 implies

$$\mathbb{E}[e^{s\mathbf{T}}] = \frac{s^s}{\Gamma(2+s)}$$

for all $s > 0$, which was previously obtained in [3] by other methods, and will be used henceforth.

(b) It is easy to see that the function

$$x \mapsto \frac{1}{x} - \frac{1}{e^x - 1}$$

decreases from 1/2 to zero on $(0, \infty)$. By Corollary 15.11 in [79], this shows that $\log \mathbf{W}$ is SD. A further computation yields

$$\frac{1}{x^2}\left(1 - \frac{x}{e^x-1}\right) = \int_0^\infty e^{-ux}(u - [u])\,du, \qquad x > 0. \tag{3.20}$$

This implies that the lévy process associated to $\log \mathbf{W}$ has CM jumps. By Theorem 3, so does $\mathbf{T}$ whose Lévy measure has density

$$\frac{e^{-|x|}}{|x|} + \frac{1}{x^2}\left(1 - \frac{|x|}{e^{|x|}-1}\right) = \int_0^\infty e^{-u|x|}(u - [u-1]_+)\, du, \qquad x < 0.$$

By Theorem 51.12 in [79], the latter computation also implies that the law of the positive random variable $1 - \log \mathbf{W}$ is a mixture of exponentials (ME) viz. it has a CM density, which improves on Remark 3.6 (e) and will be used henceforth. Reasoning as in Corollary 3.2 in [50] finally implies that the law of

$$\frac{1}{\mathbf{W}} - \frac{1}{\mathrm{e}}$$

is an ME as well.

(c) Making an integration by parts in (3.20) yields

$$\frac{1}{x}\left(1 - \frac{x}{e^x-1}\right) = \int_0^\infty e^{-ux}\left(du - \sum_{n\geq 1}\delta_n(du)\right)$$

where δ stands for the Dirac mass. By (7.1.5) in [18], this implies that the law of $\log \mathbf{W}$ is not a GGC, and the same is true for $\mathbf{T}$ because

$$\frac{1}{x}\left(1 - \frac{x\, e^{-2x}}{1-e^{-x}}\right) = \int_0^\infty e^{-ux}\left(du - \sum_{n\geq 2}\delta_n(du)\right).$$

By (3.15) and Theorem 7.1.1 in [18], this yields the following negative counterpart to Theorem 3.2.

Corollary 3.1. *There exists $\alpha_0 < 1$ such that for every $\alpha \in (\alpha_0, 1)$, the law of $\mathbf{X}_\alpha$ is not a* GGC*.*

This also implies that there is a function $\delta\colon (\alpha_0, 1) \to [0,1)$ such that $\mathbf{X}_{\alpha,\rho}$ is not a GGC for $\alpha \in (\alpha_0, 1)$ and $\rho \in [\delta(\alpha), 1]$. Observe on the other hand that it does not seem possible to apply our methods to $\mathbf{X}_{\alpha,\rho}$ with a fixed $\rho \in (0,1)$. Indeed, as in the classical case, the possible limit laws of affine transformations of $\mathbf{X}_{\alpha,\rho}$ with $\rho \in (0,1)$ fixed and $\alpha \to 1$ are given only in terms of $\mathbf{X}_{1,\rho}$, whose law is a GGC.

3.2.E Proof of Theorem 4

The one-sided case

By (3.3) and Corollary 3.2 in [50], we have the independent factorisation

$$\mathbf{X}_\alpha \stackrel{d}{=} b_\alpha\, \mathbf{U}^{-1/\beta}(1 + \mathbf{X})$$

where $1/\beta = 1/\alpha - 1$ and $\mathbf{X}$ has a CM density on $(0,\infty)$. We will now show the WS property for all positive random variables of the type

$$\mathbf{Y} = \mathbf{U}^{-1/\beta}(1 + \mathbf{X}) - 1$$

with $\beta > 0$ and $\mathbf{X}$ having a CM density on $(0,\infty)$. Setting f, g for the respective densities of $\mathbf{Y}, \mathbf{X}$, the multiplicative convolution formula shows that

$$f(x) = \frac{\beta}{(x+1)^{\beta+1}} \int_0^x (y+1)^\beta g(y)\, dy = \frac{\beta}{(1+1/x)^{\beta+1}} \int_0^1 (y+1/x)^\beta g(xy)\, dy$$

for every $x > 0$. In particular, one has $f(0+) = f(+\infty) = 0$. Moreover, the first equality and an induction on n imply that f is smooth with

$$(x+1) f^{(n+1)}(x) = \beta g^{(n)}(x) - (\beta + n + 1) f^{(n)}(x) \tag{3.21}$$

for every $n \geq 0$. Hence, we also have $f^{(n)}(+\infty) = 0$ for all $n \geq 0$ and a successive application of Rolle's theorem yields

$$\#\{x \in (0,\infty) \mid f^{(n)}(x) = 0\} \geq 1$$

for every $n \geq 1$. Fix now $n \geq 1$ and suppose that there exist $0 < x_n^{(1)} < x_n^{(2)} < \infty$ such that

$$f^{(n)}(x_n^{(1)}) = f^{(n)}(x_n^{(2)}) = 0.$$

By (3.21) and the complete monotonicity of g, we have

$$(-1)^n f^{(n+1)}(x_n^{(i)}) > 0$$

for $i = 1, 2$. An immediate analysis based on the intermediate value theorem shows then that there must exist $x_n^{(3)} \in (x_n^{(1)}, x_n^{(2)})$ with

$$f^{(n)}(x_n^{(3)}) = 0 \quad \text{and} \quad (-1)^n f^{(n+1)}(x_n^{(3)}) \leq 0,$$

which is impossible again by (3.21) and the complete monotonicity of g. All in all, we have proved that

$$\#\{x \in (0,\infty) \mid f^{(n)}(x) = 0\} = 1$$

for all $n \geq 1$, which is the WS property.

□

The two-sided case

We know by Proposition A.1.4 in [14] that $f_{\alpha,\rho}$ is an analytic integrable function on $\mathbb{R}$, and by Theorem 1.7 in [44] that it converges to zero at $\pm\infty$, decreases near $+\infty$ and increases near $-\infty$. Moreover, we have shown in Theorem 2 that if $\alpha \leq 3/4$, it is the density of an ID distribution on $\mathbb{R}$ with Lévy measure $\varphi_{\alpha,\rho}(x)\, dx$ such that $x\varphi_{\alpha,\rho}(x)$ and $x\varphi_{\alpha,\rho}(-x)$ are CM on $(0,\infty)$. We are hence in position to apply Corollary 1.2 in [58], which shows that $f_{\alpha,\rho}$ is BS.

□

The exceptional 1-stable case

We use the second identity in Proposition 3.2, which rewrites

$$1 - \mathbf{T} \stackrel{d}{=} (1 - \log \mathbf{W}) + \mathbf{L}.$$

We have seen in Remark 3.7 (b) that the random variable $1 - \log \mathbf{W}$ has a CM density on $(0,+\infty)$, in other words that it belongs to the class ME^* with the notations of [83]. Applying the Proposition in [83] with $n = 1$ shows that $1 - \mathbf{T}$ has a WBS_0 density, with the notation of the main definition in [83]. As mentioned in the introduction, this means that the density of $\mathbf{T}$ is WS_-.

□

The two-sided 1-stable case with $b = 0$ or $ab^{-1} \in \pi\mathbb{Z}$

We may suppose $a > 0$ by symmetry. If $b = 0$ the statement is clear since it is elementally shown that the Cauchy density

$$\frac{1}{\pi(1+x^2)}$$

is BS - see also Corollary 1.3 in [58]. If $b \neq 0$, we may suppose $b < 0$ by symmetry. By independence, we have

$$\log \mathbb{E}[e^{-\mathrm{i}\xi \mathbf{C}_{a,b}}] \;=\; -a|\xi| \;+\; \log \mathbb{E}[e^{\mathrm{i}|b|\xi \mathbf{T}}], \qquad \xi \in \mathbb{R}.$$

A further computation using Lemma 14.11 in [79] and Remark 3.7 (b) yields

$$\log \mathbb{E}[e^{-\mathrm{i}\xi \mathbf{C}_{a,b}}] \;=\; c_1 \;+\; c_2 \mathrm{i}\xi \;+\; \int_{\mathbb{R}} \left(\frac{1}{\mathrm{i}\xi + s} - \left(\frac{1}{s} - \frac{\mathrm{i}\xi}{s^2} \right) \mathbf{1}_{\mathbb{R}\setminus(-1,1)}(s) \right) \varphi_{a,b}(s)\, ds$$

for some $c_1, c_2 \in \mathbb{R}$ and

$$\varphi_{a,b}(s) \;=\; \frac{a}{\pi}\, s \;+\; (|b|s - [|b|s - 1]_+)\, \mathbf{1}_{\{s \geq 0\}}.$$

This function satisfies (1.1) and (1.2) in [58] and is such that $s\varphi_{a,b}(s) \geq 0$. Moreover, for $ab^{-1} \in \pi\mathbb{Z}$ the function $\varphi_{a,b}(s) - k$ changes its sign only once for every $k \in \mathbb{Z}$. Finally, we know from Propositions A.1.3 and A.2.1 in [14] that the density of $\mathbf{C}_{a,b}$ is smooth, converges to zero at $\pm\infty$, decreases near $+\infty$ and increases near $-\infty$. We can hence apply Theorem 1.1 in [58] and conclude the proof.

□

Remark 3.8. (a) If the random variable $1 - \log \mathbf{W}$ had a PF_∞ density as $\mathbf{L}$ does, then the BS character of $f_{1,1/2}$ and the additive total positivity arguments used in [58, 83] would show that $\mathbf{C}_{a,b}$ has a BS density on $\mathbb{R}$ for $a \neq 0$. But $1 - \log \mathbf{W}$ cannot have a PF_∞ density, since its law is not a GGC - see e.g. Example 3.2.2 in [18].

(b) If $ab^{-1} \notin \pi\mathbb{Z}$, the function $\varphi_{a,b}(s) - k$ changes its sign at least three times for every negative integer k, so that we cannot use Theorem 1.1 in [58]. It is not clear to the authors whether the density of $\mathbf{C}_{a,b}$ is always BS for $a \neq 0$, and the case $ab^{-1} \in \pi\mathbb{Z}$ might be more the exception than the rule.

3.3 Further results

3.3.A Some properties of the function θ_α

In this paragraph we consider further aspects of the function

$$\theta_\alpha(r) \;=\; \frac{1}{\pi} \arg[F_\alpha(re^{-\mathrm{i}\pi})], \tag{3.22}$$

whose non-decreasing character amounts to the GGC property for the law of $\mathbf{K}_{1-\alpha}$. We first prove the following asymptotic result.

Proposition 3.4. *For every* $\alpha \in [1/5, 1)$*, one has*

$$\lim_{r\to+\infty} \Re(F_\alpha(re^{-\mathrm{i}\pi})) = \frac{1}{2\alpha}\cdot$$

For every $\alpha \in (0, 1/5)$*, one has*

$$\liminf_{r\to+\infty} \Re(F_\alpha(re^{-\mathrm{i}\pi})) = -\infty \qquad \text{and} \qquad \limsup_{r\to+\infty} \Re(F_\alpha(re^{-\mathrm{i}\pi})) = +\infty.$$

The second part of this proposition has an immediate corollary, which answers in the negative an open problem stated in [50] - see Conjecture 3.1 therein.

Corollary 3.2. *The function* θ_α *is not monotonous on* $(0,\infty)$ *for* $\alpha < 1/5$*. In particular, the law of* $\mathbf{K}_\alpha$ *is not a* GGC *for* $\alpha > 4/5$*.*

Proof of Proposition 3.4. We have seen during the proof of Proposition 3.1 that $\Re(F_{1/2}(re^{-\mathrm{i}\pi})) \equiv 1$ and that

$$\Re(F_\alpha(re^{-\mathrm{i}\pi})) \to \frac{1}{2\alpha}$$

as $r \to +\infty$ for all $\alpha \in (1/2, 1)$. We next consider the case $\alpha \in [1/5, 1/2)$ introducing, as above, the function

$$G_\alpha(r) = \Re(F_\alpha(r^{\frac{\alpha}{\alpha-1}}e^{-\mathrm{i}\pi})) = 1 + \frac{1}{2\pi}\Im\left(\int_0^\infty e^{-ru+u^\alpha e^{2\mathrm{i}\pi\alpha}}\,\frac{du}{u}\right).$$

Setting $\theta = \frac{5}{6}(1-2\alpha) \in (0, 1/2]$, we have $2\alpha + \alpha\theta \in [1/2, 1)$ and by Cauchy's theorem, we can rewrite

$$G_\alpha(r) = 1 + \frac{\theta}{2} + \frac{1}{2\pi}\Im\left(\int_0^\infty e^{-rue^{i\pi\theta}+u^\alpha e^{i\pi(2\alpha+\alpha\theta)}}\,\frac{du}{u}\right).$$

The latter converges to

$$1 + \frac{\theta}{2} + \frac{1}{2\pi}\Im\left(\int_0^\infty e^{u^\alpha e^{i\pi(2\alpha+\alpha\theta)}}\,\frac{du}{u}\right) = 1 + \frac{\theta}{2} + \frac{1}{2\pi\alpha}\Im\left(\int_0^\infty e^{-ue^{-i\pi(1-2\alpha-\alpha\theta)}}\,\frac{du}{u}\right)$$

as $r \to 0$. The evaluation of the oscillating integral on the right-hand side is given e.g. in Formula 1.6(36) p.13 in [34], and we finally obtain

$$\lim_{r\to 0} G_\alpha(r) = 1 + \frac{\theta}{2} + \frac{1}{2\alpha}(1-2\alpha-\alpha\theta) = \frac{1}{2\alpha}\cdot$$

We finally consider the case $\alpha \in (0, 1/5)$, which is much more technical and requires several steps. Setting $\theta = 2\alpha/(1-\alpha) \in (0, 1/2)$, we have $2\alpha + \alpha\theta = \theta$ and the same argument as above implies

$$G_\alpha(r) = 1 + \frac{\theta}{2} + \frac{1}{2\pi}\Im\left(\int_0^\infty e^{(-rt+t^\alpha)e^{\mathrm{i}\pi\theta}}\,\frac{dt}{t}\right).$$

Hence, we are reduced to show that

$$\liminf_{r\to 0} H_\alpha(r) = -\infty \qquad \text{and} \qquad \limsup_{x\to 0} H_\alpha(r) = +\infty$$

with the notations $f_r(t) = \sin(\pi\theta)(-rt + t^\alpha)$ and

$$H_\alpha(r) = \Im\left(\int_0^\infty e^{(-rt+t^\alpha)e^{\mathrm{i}\pi\theta}}\,\frac{dt}{t}\right) = \int_0^\infty e^{\cot(\pi\theta)f_r(t)}\sin(f_r(t))\,\frac{dt}{t}\cdot$$

Let us begin with the liminf. Setting

$$r_k \;=\; \alpha\left(\frac{(1-\alpha)\sin(\pi\theta)}{2k\pi}\right)^{1/\alpha-1} \qquad \text{and} \qquad m_k \;=\; \left(\frac{2k\pi}{(1-\alpha)\sin(\pi\theta)}\right)^{1/\alpha},$$

it is clear that the function $f_{r_k}(t)$ increases on $(0, m_k)$ and decreases on $(m_k, +\infty)$, and that its global maximum equals $f_{r_k}(m_k) = 2k\pi$. This yields

$$\int_{m_k}^{\infty} e^{\cot(\pi\theta) f_{r_k}(t)} \sin(f_{r_k}(t))\,\frac{dt}{t} \;<\; 0 \qquad \text{for every } k \ge 1.$$

Considering now the unique $a_k \in (0, m_k)$ such that $f_{r_k}(a_k) = \pi$, we have $\lim_{k\to\infty} a_k = (\pi/\sin(\pi\theta))^{1/\alpha}$, so that

$$\int_0^{a_k} e^{\cot(\pi\theta) f_{r_k}(t)} \sin(f_{r_k}(t))\,\frac{dt}{t} \;\to\; \int_0^{(\pi/\sin(\pi\theta))^{1/\alpha}} e^{\cos(\pi\theta)t^\alpha} \sin(t^\alpha \sin(\pi\theta))\,\frac{dt}{t} \;<\; \infty$$

as $k \to +\infty$. Hence it suffices to show that $A_k \to -\infty$ as $k \to +\infty$, with

$$A_k \;=\; \int_{a_k}^{m_k} e^{\cot(\pi\theta) f_{r_k}(t)} \sin(f_{r_k}(t))\,\frac{dt}{t} \;=\; \frac{1}{\sin(\pi\theta)} \int_\pi^{2k\pi} \frac{e^{\cot(\pi\theta)u}}{-r_k\varphi_k(u) + \alpha(\varphi_k(u))^\alpha} \sin(u)\,du,$$

where the second equality comes from a change of variable, having set $\varphi_k(u)$ for the inverse function of f_{r_k} on $[\pi, 2k\pi]$ and written

$$\varphi_k'(u) \;=\; \frac{1}{f_{r_k}'(\varphi_k(u))} \;=\; \frac{1}{\sin(\pi\theta)(-r_k + \alpha(\varphi_k(u))^{\alpha-1})} \;>\; 0.$$

We next define $p_k(u) := e^{-\cot(\pi\theta)u}(-r_k\varphi_k(u) + \alpha(\varphi_k(u))^\alpha)$ and prove its strict unimodality on $[\pi, 2k\pi]$, computing

$$p_k'(u) = e^{-\cot(\pi\theta)u}\frac{\varphi_k'(u)}{\varphi_k(u)}(-r_k t + \alpha^2 t^\alpha - \cos(\pi\theta)(-r_k t + \alpha t^\alpha)^2)$$

with $t = \varphi_k(u)$. The strict unimodality of $p_k(u)$ on $(\pi, 2k\pi)$ amounts to the fact that

$$q_k(t) \;=\; -r_k t + \alpha^2 t^\alpha - \cos(\pi\theta)(-r_k t + \alpha t^\alpha)^2$$

has at most one zero point on $[a_k, m_k]$. It is clear by construction that there exists $c_k \in (0, m_k)$ such that $g_k(t) = -r_k t + \alpha t^\alpha$ increases on $(0, c_k)$ and decreases on (c_k, m_k), and for all $t \in (c_k, m_k)$ we have $q_k(t) = t g_k'(t) - \cos(\pi\theta)(-r_k t + \alpha t^\alpha)^2 < 0$. On the other hand, the function $g_k(t)$ is increasing and concave on $[0, c_k)$, so that its inverse function $\psi_k(v)$ is increasing and convex on $[0, g_k(c_k))$. Now since

$$q_k(t) \;=\; 0 \;\Leftrightarrow\; \cos(\pi\theta)v^2 \;-\; \alpha v \;+\; (1-\alpha)r_k\psi_k(v) \;=\; 0,$$

we see that there are at most two solutions of $q_k(t) = 0$ on $[0, c_k)$, one of them being zero, and hence at most one solution on $[a_k, m_k)$, as required. We now denote by z_k the unique mode of $p_k(u)$ on $[a_k, m_k]$ and, setting $l_k = \inf\{l \ge 1,\ z_k \le 2l\pi\}$, decompose

$$A_k \;=\; \frac{1}{\sin(\pi\theta)}\left(\int_\pi^{2l_k\pi} p_k^{-1}(u)\sin(u)\,du \;+\; \int_{2l_k\pi}^{2k\pi} p_k^{-1}(u)\sin(u)\,du\right).$$

Since $z_k \to \tan(\pi\theta)$ viz. $l_k \to l_\infty < +\infty$ as $k \to \infty$, it is easy to see that the first term in the decomposition is bounded, and we are finally reduced to show that

$$B_k \;=\; \int_{2l_k\pi}^{2k\pi} p_k^{-1}(u)\,\sin(u)\,du \;\to\; -\infty \qquad \text{as } k \to +\infty.$$

Since $p_k^{-1}(u)$ increases on $[2l_k\pi, 2k\pi]$, we have

$$B_k \;=\; \sum_{j=l_k}^{k-1}\left(\int_{2j\pi}^{(2j+1)\pi} (p_k^{-1}(u) - p_k^{-1}(u+\pi))\,\sin(u)\,du\right)$$

for every $k \geq 1$ and since $p_k(u) \to \frac{\alpha}{\sin(\pi\theta)} u e^{-\cot(\pi\theta)u}$ pointwise as $k \to +\infty$, Fatou's lemma implies

$$\limsup_{k\to+\infty} B_k \;\leq\; \frac{\sin(\pi\theta)}{\alpha}\sum_{j=l_\infty}^{\infty}\int_{2j\pi}^{(2j+1)\pi}\left(\frac{e^{\cot(\pi\theta)u}}{u} - \frac{e^{\cot(\pi\theta)(u+\pi)}}{u+\pi}\right)\sin(u)\,du.$$

Using the inequality

$$\frac{1+e^{\pi\cot\pi\theta}}{2u} \leq \frac{e^{\pi\cot\pi\theta}}{u+\pi}$$

which holds for $u \geq \frac{\pi(e^{\pi\cot\pi\theta}+1)}{e^{\pi\cot\pi\theta}-1}$, we deduce that for j_∞ large enough, one has

$$\begin{aligned}\limsup_{k\to+\infty} B_k \;&\leq\; \frac{\pi\sin(\pi\theta)}{\alpha}\sum_{j=j_\infty}^{\infty}\frac{1-e^{\pi\cot\pi\theta}}{2}\int_{2j\pi}^{(2j+1)\pi}\frac{e^{\cot(\pi\theta)u}}{u}\sin(u)\,du\\ &\leq\; -\frac{(e^{\pi\cot\pi\theta}-1)\pi\sin(\pi\theta)}{4\alpha}\sum_{j=j_\infty}^{\infty}\int_{2j\pi+\pi/6}^{2j\pi+5\pi/6}\frac{1}{u}\,du \;=\; -\infty.\end{aligned}$$

All of this shows that

$$\liminf_{r\to 0} H_\alpha(r) \;=\; -\infty.$$

The argument for the limsup follows exactly along the same lines, considering the subsequence

$$\tilde{r}_k \;=\; \alpha\left(\frac{(1-\alpha)\sin(\pi\theta)}{(2k+1)\pi}\right)^{1/\alpha-1}.$$

□

Remark 3.9. (a) In the case $\alpha \in [1/3, 1/2)$ we have seen in the proof of Proposition 3.1 that

$$\Re(F_\alpha(re^{-\mathrm{i}\pi})) \;=\; 1 \;+\; \frac{\rho}{2}\,\mathbb{P}[\mathbf{Z}_{\alpha,\rho}^{-\alpha} \leq r^{1-\alpha}]$$

with $\rho = 1/\alpha - 2$, which does converge to $1/(2\alpha)$ as $r \to +\infty$. In the case $\alpha \in [1/4, 1/3)$, the proof of Proposition 3.1 shows that

$$\lim_{r\to 0} G_\alpha(r) \;=\; 1 \;+\; \frac{1}{2\pi\alpha}\int_0^\infty e^{\cos(2\pi\alpha)u}\sin(\sin(2\pi\alpha)u)\,\frac{du}{u} \;=\; \frac{1}{2\alpha}$$

again by Formula 1.6(36) in [34]. The above contour argument is hence only necessary for $\alpha \in [1/5, 1/4)$.

(b) As mentioned in Remark 3.4 (a), the above proof shows that $x \mapsto \mathbb{P}[\mathbf{Z}_\alpha^{-\frac{\alpha}{1-\alpha}} \geq x]$ is not HCM for every $\alpha < 1/5$. By Theorem 6.3.5 in [18], this implies that $\mathbf{Z}_\alpha^{-\frac{\alpha}{1-\alpha}}$ is not HCM for $\alpha < 1/5$ either. This shows that Conjecture 1.2 in [20] is not true in general.

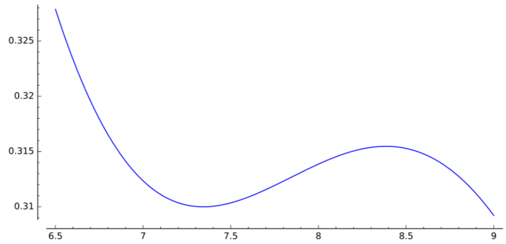


Figure 3.3: $x^{-1}G_{1/5}(x^{-5})$

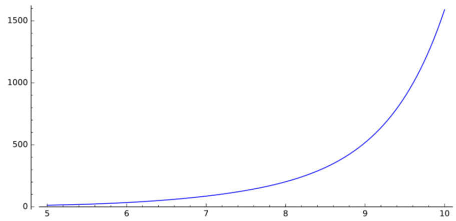


Figure 3.4: $\tilde{G}_{1/5}(x)$

We believe that θ_a is non-decreasing for $\alpha \in [1/5, 1)$, which is equivalent to the following

Conjecture 3.1. *The law of* $\mathbf{K}_\alpha$ *is a* GGC *if and only if* $\alpha \le 4/5$.

The above Corollary 3.2 shows the only if part, and in the proof of Theorem 2 we have shown the if part for $\alpha \le 3/4$. However, it seems that our methods fail to handle the remaining case $\alpha \in (3/4, 4/5]$, because some simulations show that $r^{\alpha-1}G_\alpha(r^{\frac{\alpha-1}{\alpha}}) = r^{\alpha-1}\Re(F_\alpha(re^{-i\pi}))$ is not monotonous anymore, at least for α close enough to $1/5$ - see Figure 3.3. Observe from (3.14) that the problem can be reformulated in terms of the monotonicity of the ratio of two power series, the non-decreasing character of θ_α being equivalent to that of

$$\tilde{G}_\alpha : x \longmapsto \frac{\Im(F_\alpha(xe^{-i\pi}))}{\Re(F_\alpha(xe^{-i\pi}))} = \frac{\displaystyle\sum_{n\ge 0} \frac{\sin(n\pi\alpha)}{n!\Gamma(1-n\alpha)} x^n}{\displaystyle\sum_{n\ge 0} \frac{\cos(n\pi\alpha)}{n!\Gamma(1-n\alpha)} x^n}$$

on $(0, \infty)$. A necessary condition for $\tilde{G}_\alpha$ to be non-decreasing is that its denominator does not vanish on $(0, \infty)$, which is false for $\alpha < 1/5$ by Proposition 3.4 and true for $\alpha \ge 1/4$ by the proof of Theorem 2. But the case $\alpha \in [1/5, 1/4)$ still eludes us. Let us mention that monotonicity properties of ratios of power series are studied in the literature on special functions - see e.g. Chapter 3.1 in [6]. For example, one could be tempted to apply Theorem 4.3 in [45] since $x \mapsto \tan(x\pi\alpha)$ is locally increasing. However, we could not find any clue in this literature for our problem, and it is not easy to understand why the value $\alpha = 1/5$ should be critical for the monotonicity of the above ratio. See Figure 4 for a convincing simulation. Let us finally mention [70] for an operator-theoretic approach to the above power series.

We finally turn to the behavior of $F_\alpha(re^{-i\pi})$ at infinity, which implies that of $\theta_\alpha(r)$.

Proposition 3.5. *One has*

$$F_\alpha(re^{-i\pi}) \sim \frac{i\, c_\alpha e^{b_{1-\alpha} r}}{\sqrt{r}} \qquad \text{as } r \to +\infty,$$

with $c_\alpha = \frac{\alpha^{\frac{1}{2(\alpha-1)}}}{\sqrt{2\pi(1-\alpha)}}$. *In particular, one has* $\theta_\alpha(r) \to 1/2$ *as* $r \to +\infty$.

Proof. From (3.14), we can write

$$F_\alpha(re^{-i\pi}) = \phi(-\alpha, 1, r^{1-\alpha}e^{i\pi\alpha}), \qquad r > 0.$$

We now use the asymptotic expansion for large $z \in \mathbb{C}$ and $a \in (-1,0)$ of the Wright function $\phi(a,b,z)$, which has been obtained in [91]. Applying therein Theorem 1 for $\alpha \le 1/3$ resp. Theorem 5 for $\alpha > 1/3$ and taking the first term in (1.3) implies the required asymptotic for $F_\alpha(re^{-i\pi})$, since we have here

$$A_0 = \frac{1}{\sqrt{2\pi\alpha}} \qquad \text{and} \qquad Y = b_{1-\alpha}re^{-i\pi}$$

in the notation of [91], the first equality being a consequence of Stirling's formula. From (3.22), we then readily deduce that $\theta_\alpha(r) \to 1/2$ as $r \to +\infty$.

□

Remark 3.10. (a) Taking the first two terms in the series representation (3.14) yields at once the asymptotic behavior of $\theta_\alpha(r)$ at zero, which is

$$\theta_\alpha(r) \sim \frac{r^{1-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)^2}.$$

On the other hand, the complete asymptotic expansion (1.3) in [91] has only purely imaginary terms in our framework, so that we cannot deduce from it the asymptotics of $\theta_\alpha(r) - 1/2$ at infinity. It follows from Proposition 3.1 that $\theta_\alpha(r) \in [0,1/2)$ for $\alpha \ge 1/4$, and from Proposition 3.4 that $\theta_\alpha(r) - 1/2$ crosses zero an infinite number of times for $\alpha < 1/5$, as $r \mapsto +\infty$. For $\alpha \in [1/5, 1/4)$, we are currently unable to prove that $\theta_\alpha(r) \in [0,1/2)$ for every $r > 0$, which would be a first step to show that it increases from 0 to 1/2. Recall that the latter is equivalent to the fact that the denominator of the above $\tilde{G}_\alpha$ does not vanish on $(0,\infty)$.

(b) If $\alpha \le 3/4$, it follows from (3.13), Theorem 8.2 and Remark 8.3 in [80], and the above proposition, that the Thorin mass of the GGC random variable $\mathbf{K}_\alpha$ equals 1/2. Hence, $\mathbf{K}_\alpha - b_\alpha$ is a $\Gamma_{1/2}$–mixture by Theorem 4.1.1. in [18], which is a refinement of Corollary 3.2 in [50]. Since this property amounts to the CM character of $x \mapsto \sqrt{x}\, f_{\mathbf{K}_\alpha}(b_\alpha + x)$, a perusal of the proof of Theorem 3.1 shows that $\mathbf{X}_\alpha - b_\alpha$ is a $\Gamma_{3/2}$–mixture as soon as $\alpha \le 3/4$. We believe that this is true for every $\alpha \in (0,1)$.

3.3.B An Airy-type function

In this paragraph, we discuss a curious connection between the two cases $\alpha = 1/3$ and $\alpha = 1/5$ in the analysis of the function

$$G_\alpha(r) = 1 + \frac{1}{2\pi}\Im\left(\int_0^\infty e^{-ru+u^\alpha e^{2i\pi\alpha}}\frac{du}{u}\right).$$

The latter was important during the proofs of Theorem 2 and Proposition 3.4. For $\alpha = 1/3$, a contour integration as in Proposition 3.4 with $\theta = -1/2$ implies, making the change of variable $s = (3r)^{-1/3}$,

$$\begin{aligned} G_{\frac{1}{3}}(r) &= \frac{3}{4} + \frac{3}{2\pi}\int_0^\infty \sin(u^3/3 + us)\,\frac{du}{u} \\ &= \frac{3}{4} + \frac{3}{2\pi}\left(\int_0^\infty \sin(u^3/3)\,\frac{du}{u} + \int_0^\infty\left(\int_0^s \cos(u^3/3+uz)\,dz\right)du\right) \\ &= 1 + \frac{3}{2\pi}\int_0^s\left(\int_0^\infty \cos(u^3/3+uz)\,du\right)dz \\ &= 1 + \frac{3}{2}\int_0^s \mathrm{Ai}(z)\,dz \end{aligned}$$

where Ai stands for the classic Airy function - see e.g. Paragraph 7.3.7 in [34]. In particular, we retrieve the fact that

$$r \mapsto (3r)^{\frac{1}{3}} G_{\frac{1}{3}}(r) = \frac{1}{s} + \frac{3}{2}\int_0^1 \mathrm{Ai}(sz)\,dz$$

increases, by the well-known decreasing character of Ai on $(0,\infty)$. For $\alpha = 1/5$, the contour integration of Proposition 3.4 with $\theta = 1/2$ yields, with the change of variable $s = (5r)^{-1/5}$,

$$\begin{aligned} G_{\frac{1}{5}}(r) &= 1 + \frac{5}{2}\int_0^s\left(\int_0^\infty \cos(u^5/5 - uz)\,du\right)dz \\ &= 1 + \frac{5}{2}\int_0^s \mathrm{Ai}^{(5)}(-z)\,dz \end{aligned}$$

where we have defined, for every integer $k \ge 3$, the semi-converging integral

$$\mathrm{Ai}^{(k)}(x) = \frac{1}{\pi}\int_0^\infty \cos(u^k/k + ux)\,du, \qquad x \in \mathbb{R}.$$

We did not find any reference on the above Airy-type functions in the literature, which are solution to some linear ODE of higher order. Observe that similarly as above, one has

$$(5r)^{\frac{1}{5}} G_{\frac{1}{5}}(r) = \frac{1}{s} + \frac{5}{2}\int_0^1 \mathrm{Ai}^{(5)}(-sz)\,dz$$

but here we cannot deduce any conclusion on the monotonicity of $r^{\frac{1}{5}} G_{\frac{1}{5}}(r)$ because of the negative sign in the Airy-type function. The simulation displayed in Figure 5 shows indeed that $\mathrm{Ai}^{(5)}(-x)$ exhibits on $(0,\infty)$ exactly the same damped oscillating behavior as $\mathrm{Ai}(-x)$. It could be interesting for our purposes to perform a rigorous study of the functions $\mathrm{Ai}^{(k)}$, as in the case $k = 3$ with the Bessel functions. We leave this analysis for future research.

3.3.C Asymptotic expansions for the free extreme stable densities

In this paragraph we derive the full asymptotic expansion at zero of the density $f_{\mathbf{Y}_\alpha}$ of the random variable

$$\mathbf{Y}_\alpha = \begin{cases} \mathbf{X}_\alpha - b_\alpha & \text{if } \alpha \in (0,1), \\ \mathbf{X}_{\alpha,1-1/\alpha} + b_{1/\alpha}^{-1/\alpha} \stackrel{d}{=} b_{1/\alpha}^{-1/\alpha} - \mathbf{X}_{\alpha,1/\alpha} & \text{if } \alpha \in (1,2], \end{cases}$$

and $\mathbf{Y}_1 = 1 - \mathbf{T}$. We will use the standard notation of Definition C.1.1 in [1] for asymptotic expansions. Our expansions complete the estimates of Proposition A.1.2 in [14] and the

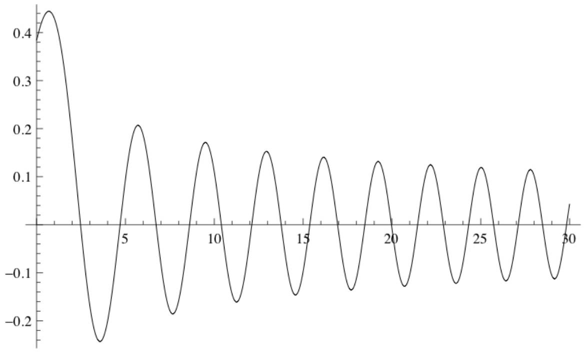


Figure 3.5: $\mathrm{Ai}^{(5)}(-x)$

series representations of Theorem 1.7 in [44], from which one can only infer that the random variable $\mathbf{Y}_\alpha$ is positive. They can also be viewed as free analogues of Linnik's expansions (14.35) in [79] - see also Theorem 2.5.3 in [93] - for the classical extreme stable distributions. Observe that in the classical case, the expansion for $\alpha > 1$ is deduced from that of the case $\alpha \in [1/2, 1)$ by the Zolotarev's duality which is discussed in Section 2.3 of [93]. Even though the very same duality relationship holds in the free case - see Proposition A.3.1 in [14] and Corollary 1.4 in [44], for $\mathbf{Y}_\alpha$ this duality only yields

$$f_{\mathbf{Y}_{1/\alpha}}(x) = \frac{1}{\alpha}(b_\alpha^{-\alpha} - x)^{-1/\alpha-1} f_{\mathbf{Y}_\alpha}((b_\alpha^{-\alpha} - x)^{1/\alpha} - b_\alpha)$$

for every $\alpha \in [1/2, 1)$, and does not seem particularly helpful to connect explicitly the two expansions at zero. When $\alpha \neq 1$, our method hinges on Wright's original papers [89] for the case $\alpha > 1$ and [91] for the case $\alpha < 1$. It is remarkable that the two expansions turn out to have the same parametrization.

Proposition 3.6. *For every $\alpha \in (0,1) \cup (1,2]$, one has*

$$f_{\mathbf{Y}_\alpha}(x) \sim \sum_{n=0}^{\infty} a_n(\alpha)\, x^{n+1/2} \qquad \text{as } x \to 0,$$

with

$$a_n(\alpha) = \left(\frac{2}{\alpha}\right)^{n+1/2} \frac{(-1)^n}{\pi\,|\alpha-1|^{(n+3/2)/\alpha}\,(2n+1)!} \times \frac{\mathrm{d}^{2n}}{\mathrm{d}v^{2n}}\left((1-v)^{-2}{}_2F_1\left[\begin{matrix}\alpha+1 & 1\\ & 3\end{matrix}; v\right]^{-n-1/2}\right)_{v=0}.$$

Proof. We begin with the case $\alpha > 1$, writing down first $f_{\mathbf{Y}_\alpha}$ with the help of Bromwich's integral formula

$$f_{\mathbf{Y}_\alpha}(x) = \frac{1}{2\pi\mathrm{i}} \int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} e^{zx} \mathcal{L}_\alpha(z)\, dz,$$

where

$$\mathcal{L}_\alpha(z) = \mathbb{E}[e^{-z\mathbf{Y}_\alpha}] = e^{-\alpha(\alpha-1)^{\frac{1}{\alpha}-1} z} \times \mathbb{E}[e^{z\mathbf{X}_{\alpha,1/\alpha}}]$$

is well-defined and analytic on the open right half-plane. Combining next Theorem 1.8 in [44] and Theorem 2 in [89], we obtain

$$\mathcal{L}_\alpha(z) = e^{-\alpha(\alpha-1)^{\frac{1}{\alpha}-1} z} \times \phi(\alpha-1, 2, z^\alpha) = \mathrm{O}(|z|^{-3/2})$$

uniformly on the right half-plane. Making a change of variable and applying Cauchy's theorem, we deduce

$$f_{\mathbf{Y}_\alpha}(x) = \frac{1}{2\pi \mathrm{i}\, x}\int_{x-\mathrm{i}\infty}^{x+\mathrm{i}\infty} e^z \mathcal{L}_\alpha(zx^{-1})\, dz = \frac{1}{2\pi \mathrm{i}\, x}\int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} e^z \mathcal{L}_\alpha(zx^{-1})\, dz.$$

Using now the full asymptotic expansion of Theorem 2 in [89], we get

$$f_{\mathbf{Y}_\alpha}(x) \sim \sum_{n=0}^{\infty} a_n(\alpha)\, x^{n+1/2} \qquad \text{as } x \to 0,$$

where

$$a_n(\alpha) = \frac{(-1)^n\, a_n}{(\alpha-1)^{(n+3/2)/\alpha}}\left(\frac{1}{2\pi \mathrm{i}}\int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} e^z z^{-3/2-n}\, dz\right) = \frac{(-1)^n\, a_n}{(\alpha-1)^{(n+3/2)/\alpha}\Gamma(n+3/2)}$$

and a_n is defined at the beginning of p.258 in [89] for $\rho = \alpha - 1$ and $\beta = 2$. Above, the interchanging of the contour integral and the expansion is easily justified - alternatively one can use the generalized Watson's lemma which is mentioned at the top of p.615 in [1], whereas the second equality follows from Hankel's formula - see e.g. Exercise 1.22 in [1]. To conclude the proof of the case $\alpha > 1$, it remains to evaluate the coefficients $a_n(\alpha)$, which is done in observing that the function in (1.21) of [89] is here

$$\sqrt{{}_2F_1\left[\begin{matrix}\alpha+1 & 1\\ 3 & \end{matrix}; v\right]},$$

and making some simplifications.

We now consider the case $\alpha < 1$. The argument is analogous but it depends on the expansions of [91] which, the author says, cannot be simply deduced from those of [89]. We again write

$$f_{\mathbf{Y}_\alpha}(x) = \frac{1}{2\pi \mathrm{i}}\int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} e^{zx} \mathcal{L}_\alpha(z)\, dz,$$

where

$$\mathcal{L}_\alpha(z) = e^{\alpha(1-\alpha)^{\frac{1}{\alpha}-1} z} \times \mathbb{E}[e^{-z\mathbf{X}_\alpha}] = e^{\alpha(1-\alpha)^{\frac{1}{\alpha}-1} z} \times \phi(\alpha-1, 2, -z^\alpha) = \mathrm{O}(|z|^{-3/2})$$

uniformly in the open right half-plane, the second equality following from Theorem 1.8 in [44] and the estimate from the Lemma p.39 in [91]. Reasoning as above, we get

$$f_{\mathbf{Y}_\alpha}(x) = \frac{1}{2\pi \mathrm{i}\, x}\int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} e^z \mathcal{L}_\alpha(zx^{-1})\, dz \sim \sum_{n=0}^{\infty} a_n(\alpha)\, x^{n+1/2} \qquad \text{as } x \to 0,$$

where

$$a_n(\alpha) = \frac{a_n}{(\alpha(1-\alpha)^{\frac{1}{\alpha}-1})^{n+3/2}\,\Gamma(n+3/2)}$$

and a_n is defined at the bottom of p.38 in [91] for $\sigma = 1-\alpha$ and $\beta = 2$. After some simplifications, we also obtain the required expression for $a_n(\alpha)$.

□

Remark 3.11. (a) It does not seem that a simple closed formula can be obtained for the coefficients $a_n(\alpha)$ in general. We can compute

$$a_0(\alpha) = \sqrt{\frac{2}{\alpha}} \times \frac{1}{\pi|\alpha-1|^{3/(2\alpha)}} \quad \text{and} \quad a_1(\alpha) = -\sqrt{\frac{2}{\alpha}} \times \left(\frac{2\alpha^2-23\alpha+47}{36\pi\,\alpha\,|\alpha-1|^{5/(2\alpha)}}\right).$$

Observe that $a_1(\alpha)$ is always negative. We believe that in general, one has

$$a_n(\alpha) = \sqrt{\frac{2}{\alpha}} \times \frac{Q_{2n}(\alpha)}{\pi\,\alpha^n\,|\alpha-1|^{(2n+3)/(2\alpha)}}$$

for some $Q_{2n} \in \mathbb{Q}_{2n}[X]$. This would again mimic the classical situation, save for the fact that here the polynomial Q_{2n} does not seem to have symmetric coefficients - see Remark 2 p.101 in [93].

(b) For $\alpha = 2$, the involved hypergeometric function becomes the standard geometric series and we simply get

$$a_n(2) = \frac{(-1)^n}{\pi\,(2n+1)!} \times \frac{\mathrm{d}^{2n}}{\mathrm{d}v^{2n}}\left((1-v)^{n-3/2}\right)_{v=0} = \frac{-1}{\pi\,(2n-1)\,16^n}\binom{2n}{n},$$

which is always negative except for $n = 0$. Of course, this can be retrieved via the binomial theorem for the explicit density

$$f_{\mathbf{Y}_2}(x) = \frac{\sqrt{x}}{\pi}\sqrt{1-\frac{x}{4}}.$$

(c) For $\alpha = 1/2$, the involved hypergeometric function simplifies with the help of Exercise 3.39 in [1], and we get

$$a_n(1/2) = \frac{(-1)^n 4^{n+2}}{2\pi\,(2n+1)!} \times \frac{\mathrm{d}^{2n}}{\mathrm{d}v^{2n}}\left((1+\sqrt{1-v})^{2n+1}(1-v)^{-2}\right)_{v=0} = \frac{(-1)^n(n+1)\,4^{n+2}}{\pi},$$

whose signs alternate. This again can be retrieved via the binomial theorem for the explicit density

$$f_{\mathbf{Y}_{1/2}}(x) = \frac{16\sqrt{x}}{\pi(1+4x)^2}.$$

(d) As already observed in Remark 3.1 (a), the densities of $\mathbf{Y}_{1/3}$ and $\mathbf{Y}_{2/3}$ can be written in closed form with the help of formulæ (40) and (41) in [66]. In principle, a full asymptotic expansion can also be derived from these expressions, but the task seems too painful. Notice that here, the involved hypergeometric functions do not seem to simplify.

(e) The above proof shows that the following functions

$$\lambda \mapsto e^{-\alpha(\alpha-1)^{\frac{1}{\alpha}-1}\lambda}\,\phi(\alpha-1,2,\lambda^\alpha) \qquad \text{resp.} \qquad \lambda \mapsto e^{\alpha(1-\alpha)^{\frac{1}{\alpha}-1}\lambda}\,\phi(\alpha-1,2,-\lambda^\alpha)$$

on $(0,\infty)$, which are obtained in removing Wright's exponential term at infinity, are CM functions for $\alpha \in (1,2]$ resp. for $\alpha \in (0,1)$.

(f) For $\alpha > 1$, we can also compute the Mellin transform of $\mathbf{Y}_\alpha$, starting from the formula

$$\mathbb{E}[\mathbf{Y}_\alpha^{-s}] \;=\; \frac{1}{\Gamma(s)}\int_0^\infty \mathbb{E}[e^{-\lambda \mathbf{Y}_\alpha}]\,\lambda^{s-1}\,d\lambda$$

which is valid for every $s > 0$ with possible infinite terms on both sides. This becomes here

$$\mathbb{E}[\mathbf{Y}_\alpha^{-s}] \;=\; \frac{b_\alpha^{-s}}{\Gamma(s)}\sum_{n\ge 0}\frac{\Gamma(s+\alpha n)}{n!\,\Gamma(2+(\alpha-1)n)}\,b_\alpha^{-\alpha n}$$

with the notation $b_\alpha = \alpha(\alpha-1)^{\frac{1}{\alpha}-1}$ and has, by Stirling's formula, an analytic extension for $-\alpha < s < 3/2$. Formally, this rewrites

$$\mathbb{E}[\mathbf{Y}_\alpha^{s}] \;=\; \frac{b_\alpha^{-s}}{\Gamma(s)}\,{}_1\Psi_1\left[\begin{array}{c}(-s,\alpha)\\(2,\alpha-1)\end{array}\middle|\, b_\alpha^{-\alpha}\right], \qquad -3/2 < s < \alpha,$$

where ${}_r\Psi_s$ is the generalized hypergeometric function originally studied in [38, 90], which is sometimes coined as a generalized Wright function, and which should not be confused with the ${}_r\psi_s$ hypergeometric series defined in (10.9.4) of [1]. For $\alpha = 2$, Gauss's multiplication and summation formulæ for the Gamma and the hypergeometric function - see Theorems 1.5.1 and 2.2.2 in [1], respectively - transform this expression into

$$\mathbb{E}[\mathbf{Y}_2^{-s}] \;=\; 2^s\,{}_2F_1\left[\begin{array}{c}-s/2\;\;(1-s)/2\\2\end{array};1\right] \;=\; \frac{4^{s+1}}{\sqrt{\pi}}\times\frac{\Gamma(3/2+s)}{\Gamma(3+s)},$$

in accordance with $\mathbf{Y}_2 \stackrel{d}{=} 4\mathbf{B}_{3/2,3/2}$.

We now complete the picture and derive the asymptotic expansion of $\mathbf{Y}_1 = 1-\mathbf{T}$. To state our result, we need to introduce the Stirling series $\{c_n,\, n\ge 0\}$ appearing in the expansion

$$\left(\frac{\mathrm{e}}{x}\right)^x\sqrt{\frac{x}{2\pi}}\;\Gamma(x)\;\sim\;\sum_{n\ge 0}c_n x^{-n} \qquad \text{as } x\to+\infty,$$

which is given e.g. in Exercise 23 p.267 of [26] - see also Lemma 1 in [38] . One has $c_0 = 1, c_1 = 1/12, c_2 = 1/288$ and $c_3 = -139/51840$. In general, c_n is a rational number and the corresponding sequences of numerators and denominators are referenced under A00163 and A00164 in the online version of [76].

Proposition 3.7. *One has*

$$f_{\mathbf{Y}_1}(x)\;\sim\;\sum_{n=0}^{\infty}a_n(1)\,x^{n+1/2} \qquad \textit{as } x\to 0,$$

with

$$a_n(1) \;=\; \frac{(-1)^n 2^{2n+1/2}\,n!\,(c_0+\cdots+c_n)}{\pi\,(2n+1)!}.$$

Proof. Applying Remark 3.7 (a), we first compute the Laplace transform

$$\mathbb{E}[e^{-z\mathbf{Y}_1}] \;=\; \frac{1}{z(1+z)}\left(\frac{z}{\mathrm{e}}\right)^z\frac{1}{\Gamma(z)}$$

for every z in the open right half-plane. Comparing next (2.15) and (2.21) in [69], we get the expansion

$$\begin{aligned}\mathbb{E}[e^{-z\mathbf{Y}_1}] &\sim \frac{1}{\sqrt{2\pi z}(1+z)} \sum_{n\geq 0}(-1)^n c_n\, z^{-n} \\ &\sim \frac{1}{\sqrt{2\pi z^3}}\left(\sum_{n\geq 0}(-1)^n z^{-n}\right)\left(\sum_{n\geq 0}(-1)^n c_n\, z^{-n}\right) \\ &\sim \frac{1}{\sqrt{2\pi z^3}} \sum_{n\geq 0}(-1)^n (c_0+\cdots+c_n)\, z^{-n}.\end{aligned}$$

as $|z| \to \infty$, uniformly in the open right half-plane. Reasoning as in Proposition 3.6, we finally obtain

$$f_{\mathbf{Y}_1}(x) \sim \sum_{n=0}^{\infty} a_n(1)\, x^{n+1/2} \qquad \text{as } x \to 0,$$

with

$$a_n(1) = \frac{(-1)^n(c_0+\cdots+c_n)}{\sqrt{2\pi}\,\Gamma(n+3/2)} = \frac{(-1)^n 2^{2n+1/2}\, n!\,(c_0+\cdots+c_n)}{\pi\,(2n+1)!}.$$

□

Remark 3.12. It is easy to see from (3.15) that

$$(1-\alpha)^{-1}\mathbf{Y}_\alpha \xrightarrow{d} \mathbf{Y}_1 \qquad \text{as } \alpha \uparrow 1,$$

and it is natural to infer from this and Proposition 3.6 that

$$\begin{aligned} a_n(1) &= \frac{(-1)^n\, 2^{n+1/2}}{\pi\,(2n+1)!} \times \frac{\mathrm{d}^{2n}}{\mathrm{d}v^{2n}}\left((1-v)^{-2}{}_2F_1\left[\begin{matrix}2 \;\; 1\\ 3\end{matrix}; v\right]^{-n-1/2}\right)_{v=0} \\ &= \frac{(-1)^n}{\pi\,(2n+1)!} \times \frac{\mathrm{d}^{2n}}{\mathrm{d}v^{2n}}\left(\frac{v^{2n+1}}{(1-v)^2(-v-\log(1-v))^{n+1/2}}\right)_{v=0},\end{aligned}$$

except that we cannot interchange a priori the asymptotic expansion at zero and the convergence in law. We have checked the correspondence for $n = 0$ and $n = 1$, with

$$a_0(1) = \frac{\sqrt{2}}{\pi} \qquad \text{and} \qquad a_1(1) = -\frac{13\sqrt{2}}{18\pi}$$

to be compared with Remark 3.11 (a). We believe that this formula is true for every $n \geq 1$. Observe that this is equivalent to the following expression of the Stirling series:

$$c_n = b_n - b_{n-1}, \qquad n \geq 1,$$

with

$$b_n = \frac{1}{2^{2n+1/2}\, n!} \times \frac{\mathrm{d}^{2n}}{\mathrm{d}v^{2n}}\left(\frac{v^{2n+1}}{(1-v)^2(-v-\log(1-v))^{n+1/2}}\right)_{v=0},$$

which is different from the combinatorial expression given in Exercise 23 p.267 of [26], and which we could not locate in the literature.

3.3.D Product representations for $\mathbf{K}_\alpha$ and $\mathbf{X}_\alpha$ with α rational

In the classical framework, the following independent factorization of the positive stable random variable was observed in [88]:

$$\mathbf{Z}_{\frac{1}{n}}^{-1} \stackrel{d}{=} n^n \times \mathbf{\Gamma}_{\frac{1}{n}} \times \mathbf{\Gamma}_{\frac{2}{n}} \times \cdots \times \mathbf{\Gamma}_{\frac{n-1}{n}}, \qquad n \geq 2. \tag{3.23}$$

A further finite factorization of $\mathbf{Z}_\alpha$ for α rational has been obtained in Formula (2.4) of [82], and reads as follows.

$$\mathbf{Z}_{\frac{p}{n}}^{-p} \stackrel{d}{=} \frac{n^n}{p^p(n-p)^{n-p}} \mathbf{L}^{n-p} \times \prod_{j=0}^{p-1} \left(\prod_{i=q_j+1}^{q_{j+1}-1} \mathbf{B}_{\frac{i}{n}, \frac{i-j}{n-p} - \frac{i}{n}} \right) \times \prod_{j=1}^{p-1} \mathbf{B}_{\frac{q_j}{n}, \frac{j}{p} - \frac{q_j}{n}} \tag{3.24}$$

for every $n > p \geq 1$, where we have set $q_0 = 0, q_p = n$ and $q_j = \sup\{i \geq 1, ip < jn\}$ for all $j = 1, \ldots p-1$. We refer to the paragraph before Theorem 1 in [82] for more detail on this notation.

For $\mathbf{K}_{\frac{p}{n}}$ and $\mathbf{X}_{\frac{p}{n}}$ we can obtain a finite factorization in terms of Beta random variables only, as a simple consequence of (3.24). These factorizations are actually consequences of the more general Theorem 2.3 in [66] and Theorem 3.1 in [67]. We omit the proof.

Proposition 3.8. *With the above notation, for every $n > p \geq 1$ one has*

$$\mathbf{K}_{\frac{p}{n}}^{-p} \stackrel{d}{=} \frac{n^n}{p^p(n-p)^{n-p}} \prod_{j=0}^{p-1} \left(\prod_{i=q_j+1}^{q_{j+1}-1} \mathbf{B}_{\frac{i}{n}, \frac{i-j}{n-p} - \frac{i}{n}} \right) \times \prod_{j=1}^{p-1} \mathbf{B}_{\frac{q_j}{n}, \frac{j}{p} - \frac{q_j}{n}}$$

and

$$\mathbf{X}_{\frac{p}{n}}^{-p} \stackrel{d}{=} \frac{n^n}{p^p(n-p)^{n-p}} \prod_{j=0}^{p-1} \left(\prod_{i=q_j+1}^{q_{j+1}-1} \mathbf{B}_{\frac{i}{n}, \frac{i-j+1}{n-p} - \frac{i}{n}} \right) \times \prod_{j=1}^{p-1} \mathbf{B}_{\frac{q_j}{n}, \frac{j}{p} - \frac{q_j}{n}}.$$

Remark 3.13. (a) For $p = 1$, the above factorizations simplify into

$$\mathbf{K}_{\frac{1}{n}}^{-1} \stackrel{d}{=} \frac{n^n}{(n-1)^{n-1}} \times \mathbf{B}_{\frac{1}{n}, \frac{1}{n(n-1)}} \times \mathbf{B}_{\frac{2}{n}, \frac{2}{n(n-1)}} \times \cdots \times \mathbf{B}_{\frac{n-1}{n}, \frac{1}{n}}$$

and

$$\mathbf{X}_{\frac{1}{n}}^{-1} \stackrel{d}{=} \frac{n^n}{(n-1)^{n-1}} \times \mathbf{B}_{\frac{1}{n}, \frac{n+1}{n(n-1)}} \times \mathbf{B}_{\frac{2}{n}, \frac{n+2}{n(n-1)}} \times \cdots \times \mathbf{B}_{\frac{n-1}{n}, \frac{2n-1}{n(n-1)}}.$$

By the main result of [19], they hence directly show that the law of $\mathbf{K}_{\frac{1}{n}}$ resp. $\mathbf{X}_{\frac{1}{n}}$ is a GGC. These Beta factorizations should also be compared to the free factorizations for $\mathbf{K}_{\frac{1}{n}}^{-1}$ and $\mathbf{X}_{\frac{1}{n}}^{-1}$ mentioned in Remark 3.1 (a) and (b).

(b) In Lemma 2 of [23], an infinite factorization of $\mathbf{Z}_\alpha^{-1}$ has also been derived in terms of Beta random variables with the help of Malmsten's formula for the Gamma function. Using this result, Corollary 1.5 in [44] and the factorization

$$\Gamma_2^\beta \stackrel{d}{=} \Gamma(\beta+2) \times \prod_{n \geq 0} \left(\frac{n+2+\beta}{n+2} \right) \mathbf{B}_{\frac{n+2}{\beta}, 1}$$

for every $\beta > 0$, which is obtained similarly as Lemma 3 in [23], one could be tempted to derive an infinite factorization of $\mathbf{X}_\alpha^{-1}$ in terms of Beta random variables for the values

$\alpha \in (0,1)$ corresponding to the GGC property. If we try to do as in Proposition 3.8, this amounts to find factorizations of the type

$$\mathbf{B}_{\alpha+n\alpha,1-\alpha} \stackrel{d}{=} \mathbf{B}_{\frac{\alpha(n+2)}{1-\alpha},1} \times \mathbf{B}_{a_n,b_n}$$

for some $a_n, b_n > 0$. However, it can be shown that such a factorization is never possible. The existence of a suitable multiplicative factorization of $\mathbf{X}_\alpha$ which would characterize its GGC property is an open question.

In the following proposition we briefly mention a connection between $\mathbf{K}_{\frac{1}{n}}, \mathbf{X}_{\frac{1}{n}}$ and two random Vandermonde determinants, which is similar to the observations made in Section 2 of [92]. We use the notation

$$\mathcal{V}(z_1,\ldots,z_n) = \prod_{1\le i<j\le n} (z_j - z_i)$$

for the Vandermonde determinant of n complex numbers $z_1,\ldots,z_n$. Let us also consider the random variable

$$\mathbf{R}_n = \left(\mathbf{U}_1 \times \mathbf{U}_2^2 \times \cdots \times \mathbf{U}_n^n\right)^{\frac{1}{n(n+1)}},$$

where $(\mathbf{U}_1,\ldots,\mathbf{U}_n)$ is a sample of size n of the uniform random variable on $(0,1)$.

Proposition 3.9. *For every $n \ge 2$, let $(\boldsymbol{\Theta}_1,\ldots,\boldsymbol{\Theta}_n)$ resp. $(\tilde{\boldsymbol{\Theta}}_1,\ldots,\tilde{\boldsymbol{\Theta}}_n)$ be a sample of size n of the uniform random variable on the unit circle resp. the uniform random variable on the circle of independent random radius $\mathbf{R}_{n-1}$. One has the identities*

$$\begin{cases} |\mathcal{V}(\boldsymbol{\Theta}_1,\ldots,\boldsymbol{\Theta}_n)|^2 \stackrel{d}{=} \mathbf{K}_{\frac{1}{2}}^{-1} \times \mathbf{K}_{\frac{1}{3}}^{-1} \times \cdots \times \mathbf{K}_{\frac{1}{n}}^{-1} \\ |\mathcal{V}(\tilde{\boldsymbol{\Theta}}_1,\ldots,\tilde{\boldsymbol{\Theta}}_n)|^2 \stackrel{d}{=} \mathbf{X}_{\frac{1}{2}}^{-1} \times \mathbf{X}_{\frac{1}{3}}^{-1} \times \cdots \times \mathbf{X}_{\frac{1}{n}}^{-1}. \end{cases}$$

Proof. To obtain the first identity, we appeal to the trigonometric version of Selberg's integral formula - see e.g. Remark 8.7.1 in [1], which yields

$$\mathbb{E}[|\mathcal{V}(\boldsymbol{\Theta}_1,\ldots,\boldsymbol{\Theta}_n)|^{2s}] = \frac{\Gamma(1+ns)}{\Gamma(1+s)^n} = \prod_{k=2}^n \left(\frac{\Gamma(1+ks)}{\Gamma(1+(k-1)s)\Gamma(1+s)}\right) = \prod_{k=2}^n \mathbb{E}[\mathbf{K}_{\frac{1}{k}}^{-s}]$$

for every $s \ge 0$, where the third equality follows at once from (3.4) and (3.5). The result follows then by Mellin inversion. The second identity is a consequence of the first one, the fact that $|\mathcal{V}(rz_1,\ldots,rz_n)|^2 = r^{n(n-1)}|\mathcal{V}(z_1,\ldots,z_n)|^2$ for every $r > 0$ and $z_1,\ldots,z_n \in \mathbb{C}$, and (3.3).

□

Remark 3.14. (a) If $(\mathbf{N}_1,\ldots,\mathbf{N}_n)$ is a sample of size n of the standard Gaussian random variable, the Dyson-Mehta's integral formula - see e.g. Corollary 8.2.3 in [1] - implies at once the identity

$$\mathcal{V}(\mathbf{N}_1,\ldots,\mathbf{N}_n)^2 \stackrel{d}{=} \mathbf{Z}_{\frac{1}{2}}^{-1} \times \mathbf{Z}_{\frac{1}{3}}^{-1} \times \cdots \times \mathbf{Z}_{\frac{1}{n}}^{-1},$$

which is given in Proposition 3 of [92]. Observe in passing that the case $n = 2$ amounts to the standard χ_2–identity $\mathbf{N}_1^2 \stackrel{d}{=} 2\boldsymbol{\Gamma}_{\frac{1}{2}}$. By (3.23), $\mathcal{V}(\mathbf{N}_1,\ldots,\mathbf{N}_n)^2$ is distributed as a finite independent product of Gamma random variables and is hence ID - see Example 5.6.3 in [18]. Moreover, Theorem 1.3 in [20] and Theorem 5.1.1 in [18] imply that $|\mathcal{V}(\mathbf{N}_1,\ldots,\mathbf{N}_n)|$

is also ID for every $n = 4p$ or $n = 4p + 1$. Since $|\mathcal{V}(\mathbf{N}_1, \ldots, \mathbf{N}_n)|$ is clearly not ID for $n = 2$ - see e.g. 4.5.IV in [18], one may wonder if this negative property does not hold true for every $n = 4p + 2$ or $n = 4p + 3$. The infinite divisibility of $\mathcal{V}(\mathbf{N}_1, \ldots, \mathbf{N}_n)$ on the line seems also an open question. The logarithmic infinite divisibility of $|\mathcal{V}(\mathbf{N}_1, \ldots, \mathbf{N}_n)|$, which is easily established with explicit Lévy-Khintchine exponent, is discussed in Section 3 of [92].

(b) Setting $\mathcal{V}_n(a,b)$ for the Vandermonde determinant of n independent copies of $\mathbf{B}_{a,b}$, a combination of the true Selberg's integral formula - see e.g. Theorems 8.1.1 in [1] - and Gauss's multiplication formula implies easily that $\mathcal{V}_n(a,b)^2 \stackrel{d}{=} \mathcal{V}_n(b,a)^2$ has a law of the type $G^{(N,N)}$ studied in Section 6 of [31], with $N = 3n(n-1)/2$. More precisely, one has

$$\mathbb{E}[\mathcal{V}_n(a,b)^{2s}] = \frac{(a_1)_s \cdots (a_N)_s}{(b_1)_s \cdots (b_N)_s} \tag{3.25}$$

for every $s \geq 0$, with explicit parameters a_i, b_i depending on a and b. For $n = 2$ and $b \geq a$ this yields the curious factorization

$$\mathcal{V}_2(a,b)^2 \stackrel{d}{=} \mathbf{B}_{b,a} \times \mathbf{B}_{\frac{1}{2},\frac{a+b}{2}} \times \mathbf{B}_{a,\frac{b-a}{2}}.$$

However, it does not seem that such simple Beta factorizations always exist for $n \geq 3$. See (6) in [92] for a related identity, and also [72] for another point of view on (3.25), where s is interpreted as a parameter of a so-called Barnes Beta distribution.

(c) Another consequence of Proposition 3.9 and Remark 3.13 (a) is the cyclic identity

$$|\mathcal{V}(\mathbf{\Theta}_1, \ldots, \mathbf{\Theta}_n)|^2 \stackrel{d}{=} n^n \, \mathbf{B}_{\frac{1}{n},\frac{n-1}{n}} \times \mathbf{B}_{\frac{2}{n},\frac{n-2}{n}} \times \cdots \times \mathbf{B}_{\frac{n-1}{n},\frac{1}{n}}.$$

Let us finally mention the convergence in law

$$|\mathcal{V}(\mathbf{\Theta}_1, \ldots, \mathbf{\Theta}_n)|^{\frac{2}{n}} \stackrel{d}{\longrightarrow} e^{\gamma} \, \mathbf{L}$$

where $\gamma = -\Gamma'(1)$ is Euler's constant, a simple consequence of Remark 8.7.1 in [1].

3.3.E An identity for the Beta-Gamma algebra

In this paragraph we prove a general identity in law which applies to the case $n = 3$ in the factorizations of Proposition 3.8, and which can be viewed as a further instance of the so-called Beta-Gamma algebra - see [31] and the references therein. We use the standard notation for the size-bias $X^{(t)}$ of real order t of a positive random variable X, that is

$$\mathbb{E}[f(X^{(t)})] = \frac{\mathbb{E}[X^t f(X)]}{\mathbb{E}[X^t]}$$

for every f bounded measurable, as soon as $\mathbb{E}[X^t] < \infty$.

Proposition 3.10. *For every $a, b, c, d > 0$ with $a < c + d$, one has*

$$\frac{1}{\mathbf{B}_{a,b}\mathbf{B}_{c,d}} - 1 \stackrel{d}{=} \frac{\mathbf{\Gamma}_{b+d}}{\mathbf{\Gamma}_c} \times \left(\frac{1}{1 - \mathbf{B}_{b,d}\mathbf{B}_{c+d-a,a+b}}\right)^{(b+d)}.$$

Proof. A direct computation using Euler's integral formula for the generalized hypergeometric functions - see e.g. (2.2.2) in [1] - yields

$$\mathbb{E}\left[\left(\frac{1}{\mathbf{B}_{a,b}\mathbf{B}_{c,d}}-1\right)^s\right] = \mathbb{E}[\mathbf{B}_{a,b}^{-s}]\,\mathbb{E}[\mathbf{B}_{c,d}^{-s}]\,{}_3F_2\left[\begin{matrix}-s & a-s & c-s\\ a+b-s & c+d-s\end{matrix};1\right]$$

where we have supposed $-b-d<s<\min(a,c)$, so that the right-hand side is finite. We next appeal to Thomae's formula:

$${}_3F_2\left[\begin{matrix}a_1 & a_2 & a_3\\ b_1 & b_2\end{matrix};1\right] = \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(c_1)}{\Gamma(a_1)\Gamma(c_1+a_2)\Gamma(c_1+a_3)}\,{}_3F_2\left[\begin{matrix}b_1-a_1 & b_2-a_1 & c_1\\ c_1+a_2 & c_1+a_3\end{matrix};1\right]$$

with $c_1 = b_1+b_2-a_1-a_2-a_3$, which is (1) in Chapter 3.2 of [5], and which holds true whenever all involved parameters are positive. Setting $a_1 = a-s$, we deduce that for every $s\in(-b-d,0)$ one has

$$\mathbb{E}\left[\left(\frac{1}{\mathbf{B}_{a,b}\mathbf{B}_{c,d}}-1\right)^s\right] = \frac{\Gamma(b+d+s)\Gamma(c-s)\Gamma(a+b)\Gamma(c+d)}{\Gamma(b+d)\Gamma(c)\Gamma(a)\Gamma(b+c+d)}\,{}_3F_2\left[\begin{matrix}b & c+d-a & b+d+s\\ b+d & b+c+d\end{matrix};1\right],$$

and the formula extends by analyticity to $s\in(-b-d,a)$. Using again Euler's formula, the right-hand side transforms into

$$\mathbb{E}[\mathbf{\Gamma}_{b+d}^s]\,\mathbb{E}[\mathbf{\Gamma}_c^{-s}]\,\frac{\Gamma(b+d)\Gamma(c+d)}{\Gamma(a)\Gamma(b)\Gamma(d)\Gamma(c+d-a)}\int_0^1\!\!\int_0^1 t^{c+d-a-1}(1-t)^{a+b-1}u^{b-1}(1-u)^{d-1}(1-ut)^{-b-d-s}dtdu$$

and we finally recognize

$$\mathbb{E}\left[\left(\frac{1}{\mathbf{B}_{a,b}\mathbf{B}_{c,d}}-1\right)^s\right] = \mathbb{E}[\mathbf{\Gamma}_{b+d}^s]\,\mathbb{E}[\mathbf{\Gamma}_c^{-s}]\,\frac{\Gamma(a+b)\Gamma(c+d)}{\Gamma(a)\Gamma(b+c+d)}\,\mathbb{E}\left[(1-\mathbf{B}_{b,d}\mathbf{B}_{c+d-a,a+b})^{-b-d-s}\right]$$

for every $s\in(-b-d,a)$, which implies the required identity in law.

□

Remark 3.15. (a) Under the symmetric assumption $c<a+b$, we obtain the identity

$$\frac{1}{\mathbf{B}_{a,b}\mathbf{B}_{c,d}}-1 \overset{d}{=} \frac{\mathbf{\Gamma}_{b+d}}{\mathbf{\Gamma}_a}\times\left(\frac{1}{1-\mathbf{B}_{d,b}\mathbf{B}_{a+b-c,c+d}}\right)^{(b+d)}.$$

If both assumptions $a<c+d$ and $c<a+b$ hold, we deduce, identifying the factors and remembering $\mathbf{\Gamma}_s^{(t)} \overset{d}{=} \mathbf{\Gamma}_{t+s}$, the identity

$$\mathbf{\Gamma}_{b+c+d}(1-\mathbf{B}_{b,d}\mathbf{B}_{c+d-a,a+b}) \overset{d}{=} \mathbf{\Gamma}_{a+b+d}(1-\mathbf{B}_{d,b}\mathbf{B}_{a+b-c,c+d})$$

which we could not locate in the literature on the Beta-Gamma algebra, and which boils down to the elementary $\mathbf{\Gamma}_{c+d} \overset{d}{=} \mathbf{\Gamma}_{a+d}\times\mathbf{B}_{c+d,a-c}$ when $b=0$. Observe on the other hand that by Proposition 4.2 (b) in [31], this identity is equivalent to $\mathbf{B}_{b,d}\mathbf{\Gamma}_{a+b-c}+\mathbf{\Gamma}_{c+d} \overset{d}{=} \mathbf{B}_{d,b}\mathbf{\Gamma}_{c+d-a}+\mathbf{\Gamma}_{a+b}$, which is easily obtained in comparing the two Laplace transforms with the help of Euler's formula (2.2.7) in [1].

(b) Combining Propositions 3.8 and 3.10 yields the two identities

$$\mathbf{X}_{\frac{1}{3}} - b_{\frac{1}{3}} \stackrel{d}{=} \frac{4}{27} \times \frac{\mathbf{\Gamma}_{\frac{3}{2}}}{\mathbf{\Gamma}_{\frac{1}{3}}} \times \left(\frac{1}{1 - \mathbf{B}_{\frac{5}{6},\frac{2}{3}} \mathbf{B}_{\frac{1}{3},\frac{3}{2}}} \right)^{(\frac{3}{2})}$$

and

$$\mathbf{K}_{\frac{1}{3}} - b_{\frac{1}{3}} \stackrel{d}{=} \frac{4}{27} \times \frac{\mathbf{\Gamma}_{\frac{1}{2}}}{\mathbf{\Gamma}_{\frac{2}{3}}} \times \left(\frac{1}{1 - \mathbf{B}_{\frac{1}{6},\frac{1}{3}} \mathbf{B}_{\frac{2}{3},\frac{1}{2}}} \right)^{(\frac{1}{2})}.$$

Observe that these represent $\mathbf{X}_{\frac{1}{3}} - b_{\frac{1}{3}}$ resp. $\mathbf{K}_{\frac{1}{3}} - b_{\frac{1}{3}}$ as an explicit $\Gamma_{\frac{3}{2}}$−mixture resp. $\Gamma_{\frac{1}{2}}$−mixture, in accordance with Remark 3.10 (b).

(c) Iterating Euler's integral formula (2.2.2) in [1] yields the general representation

$$\mathbb{E}\left[\left(\frac{1}{\mathbf{B}_{a_1,b_1} \ldots \mathbf{B}_{a_n,b_n}} - 1\right)^s\right] = \mathbb{E}[\mathbf{B}_{a_1,b_1}^{-s}] \cdots \mathbb{E}[\mathbf{B}_{a_n,b_n}^{-s}]\; {}_{n+1}F_n\left[\begin{matrix} -s & a_1 - s & \ldots & a_n - s \\ a_1 + b_1 - s & \ldots & a_n + b_n - s \end{matrix}; 1\right]$$

for $-(b_1 + \ldots + b_n) < s < \min\{a_1, \ldots, a_n\}$. It would be interesting to know if there exists some hypergeometric transformation changing the right-hand side into

$$K\, \Gamma(b_1 + \cdots + b_n + s)\Gamma(\max\{a_1, \ldots, a_n\} - s)\; {}_{n+1}F_n\left[\begin{matrix} c_1 & \ldots & c_n & b_1 + \cdots + b_n + s \\ c_1 + d_1 & \ldots & c_n + d_n \end{matrix}; 1\right]$$

for some parameters $c_i, d_i > 0$ and an integration constant K. This would imply the identity

$$\frac{1}{\mathbf{B}_{a_1,b_1} \ldots \mathbf{B}_{a_n,b_n}} - 1 \stackrel{d}{=} \frac{\mathbf{\Gamma}_{b_1+\cdots+b_n}}{\mathbf{\Gamma}_{\max\{a_1,\ldots,a_n\}}} \times \left(\frac{1}{1 - \mathbf{B}_{c_1,d_1} \ldots \mathbf{B}_{c_n,d_n}} \right)^{(b_1+\cdots+b_n)}, \tag{3.26}$$

which would generalize that of Proposition 3.10. Observe that in the framework of Proposition 3.8 we always have $b_1 + \cdots + b_n = 3/2$ resp. $1/2$ for the left-hand side of (3.26) corresponding to $\mathbf{X}^p_{\frac{p}{n+1}} - 1$ resp. $\mathbf{K}^p_{\frac{p}{n+1}} - 1$. Unfortunately, for $n \geq 3$ we are not aware of any such hypergeometric transformation.

3.3.F Stochastic orderings

In this paragraph we come back to certain random variables appearing in the proof of Theorem 3. We establish some comparison results for the rescaled random variables $\mathbf{V}_\alpha = a_\alpha \mathbf{X}_\alpha$ with support in $[1, +\infty)$, in the spirit of those in [82]. For two positive random variables X, Y we write $X \leq_{st} Y$ if $\mathbb{P}[X \geq x] \leq \mathbb{P}[Y \geq x]$ for every $x \geq 0$, and

$$X \prec_{st} Y$$

if $X \leq_{st} Y$ and there is no $c > 1$ such that $cX \leq_{st} Y$. The relationship $\prec_{st}$ can be viewed as an optimal stochastic order.

Proposition 3.11. *For every $0 < \beta < \alpha < 1$ one has*

$$\frac{1}{\mathrm{e}} \times \mathbf{U} \times \mathbf{W} \prec_{st} \mathbf{V}_\alpha^{\frac{-\alpha}{1-\alpha}} \prec_{st} \mathbf{V}_\beta^{\frac{-\beta}{1-\beta}} \prec_{st} \mathbf{U}.$$

Proof. The argument is analogous to that of (1.3) in [82] and relies on (3.6) therein which, in our notation, yields

$$(a_\alpha \mathbf{K}_\alpha)^{\frac{-\alpha}{1-\alpha}} \leq_{st} (a_\beta \mathbf{K}_\beta)^{\frac{-\beta}{1-\beta}}$$

whence, by (3.3) and direct integration,

$$\mathbf{V}_\alpha^{\frac{-\alpha}{1-\alpha}} \leq_{st} \mathbf{V}_\beta^{\frac{-\beta}{1-\beta}}$$

for every $0 < \beta < \alpha < 1$. Moreover, it is easy to see by Haagerup-Möller's evaluation of $\mathbb{E}[\mathbf{X}_\alpha^s]$ and Stirling's formula that

$$\mathbb{E}[\mathbf{V}_\beta^{\frac{-s\beta}{1-\beta}}] \to \frac{1}{1+s} \quad \text{as } \beta \to 0 \qquad \text{and} \qquad \mathbb{E}[\mathbf{V}_\alpha^{\frac{-s\alpha}{1-\alpha}}] \to \frac{s^s}{e^s \Gamma(2+s)} \quad \text{as } \alpha \to 1$$

for every $s > 0$. By Proposition 3.3, we obtain

$$\frac{1}{\mathrm{e}} \times \mathbf{U} \times \mathbf{W} \leq_{st} \mathbf{V}_\alpha^{\frac{-\alpha}{1-\alpha}} \leq_{st} \mathbf{V}_\beta^{\frac{-\beta}{1-\beta}} \leq_{st} \mathbf{U}.$$

To conclude the proof, by the definition of $\prec_{st}$ it is enough to observe that $\mathbb{P}[\mathbf{W} \leq \mathrm{e}] = 1$, a consequence of Remark 3.6 (d).

□

Remark 3.16. (a) Multiplying all factors by an independent $\mathbf{\Gamma}_2$ random variable and using the second identity in Proposition 3.2 and (3.17), we immediately retrieve Theorem A in [82].

(b) Proposition 3.2 implies the limits in law

$$\left(\frac{\alpha}{\alpha-1}\right) \log \mathbf{V}_\alpha \xrightarrow{d} \log \mathbf{U} \quad \text{as } \alpha \to 0 \qquad \text{and} \qquad \left(\frac{\alpha}{\alpha-1}\right) \log \mathbf{V}_\alpha \xrightarrow{d} \mathbf{T} - 1 \quad \text{as } \alpha \to 1,$$

to be compared with that of (3.15). This shows that the distribution of the free Gumbel random variable $-\log \mathbf{U}$ and that of the drifted exceptional 1-free stable random variable $\mathbf{T} - 1$ can be viewed as "log free stable" distributions.

(c) Specifying Proposition 3.11 to $\alpha = 1/2$ yields

$$\frac{1}{\mathrm{e}} \times \mathbf{U} \times \mathbf{W} \prec_{st} \mathbf{B}_{\frac{1}{2},\frac{3}{2}} \prec_{st} \mathbf{U},$$

whose second ordering can be observed via a single intersection property of the densities - see e.g. Lemma 1.9 (a) in [30]. We believe the above stochastic orderings between non-explicit densities are a consequence of such a single intersection property.

Our next result deals with the classical convex ordering. For two real random variables X, Y, we say that Y dominates X for the convex order and write

$$X \prec_{cx} Y$$

if $\mathbb{E}[\varphi(X)] \leq \mathbb{E}[\varphi(Y)]$ for every convex function such that the expectations exist.

Proposition 3.12. *For every* $0 < \beta < \alpha < 1$, *one has*

$$\mathbf{U} \prec_{cx} (1-\beta)\mathbf{X}_\beta^{\frac{-\beta}{1-\beta}} \prec_{cx} (1-\alpha)\mathbf{X}_\alpha^{\frac{-\alpha}{1-\alpha}} \prec_{cx} \mathbf{U} \times \mathbf{W}.$$

We omit the proof, which is analogous to that of (1.4) in [82] and a consequence of (3.7) therein. By Kellerer's theorem, this result implies that for every $t \in (0,1)$, the law of $(1-t)\mathbf{X}_t^{\frac{-t}{1-t}}$ is the marginal distribution at time t of a martingale $\{M_t,\, t \in [0,1]\}$ starting at $\mathbf{U}$ and ending at $\mathbf{U} \times \mathbf{W}$. It would be interesting to have a constructive explanation of this curious martingale connecting free extreme and free stable distributions.

3.3.G The power semicircle distribution and van Dantzig's problem

In this paragraph we consider the power semicircle distribution with density

$$h_\alpha(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}(1-x^2)^{\alpha-1/2}\mathbf{1}_{(-1,1)}(x),$$

where $\alpha > -1/2$ is the index parameter. Up to affine transformation, this law can be viewed as an extension of the arcsine, uniform and semicircle distributions which correspond to $\alpha = 0, \alpha = 1/2$ and $\alpha = 1$ respectively. It was recently studied in [4] as a non ID factor of the standard Gaussian distribution, see also the references therein for other aspects of this distribution.

The characteristic function is computed in Formula (4.7.5) of [1] in terms of the Bessel function of the first kind J_α: one has

$$\hat{h}_\alpha(t) = \frac{\Gamma(\alpha+1)}{(t/2)^\alpha} J_\alpha(t), \qquad t > 0.$$

By the Hadamard factorization - see (4.14.4) in [1], we obtain

$$\hat{h}_\alpha(z) = \prod_{n\ge1}\left(1 - \frac{z^2}{j_{\alpha,n}^2}\right), \qquad z \in \mathbb{C},$$

where $0 < j_{\alpha,1} < j_{\alpha,2} < \ldots$ are the positive zeroes of J_α and the product is absolutely convergent on every compact set of $\mathbb{C}$.

Let now $\{X_n,\, n \ge 1\}$ be an infinite sample of the Laplace distribution with density $e^{-|x|}/2$ on $\mathbb{R}$ and characteristic function

$$\mathbb{E}[e^{\mathrm{i}tX_1}] = \frac{1}{1+t^2}.$$

By (4.14.3) in [1] and Kolmogorov's one-series theorem, the random series

$$\Sigma_\alpha = \sum_{n\ge1} \frac{X_n}{j_{\alpha,n}}$$

is a.s. convergent. Its characteristic function is

$$\mathbb{E}[e^{\mathrm{i}t\Sigma_\alpha}] = \prod_{n\ge1}\left(1 + \frac{t^2}{j_{\alpha,n}^2}\right)^{-1} = \frac{1}{\hat{h}_\alpha(\mathrm{i}t)}.$$

With the terminology of [62], this means that the pair

$$\left(\hat{h}_\alpha(t), \frac{1}{\hat{h}_\alpha(\mathrm{i}t)}\right)$$

of characteristic functions is a van Dantzig pair. The case $\alpha = 1/2$ corresponds to the well-known pair

$$\left(\frac{\sin t}{t}, \frac{t}{\sinh t}\right)$$

which is one of the starting examples of [62] and, from the point of view of the Hadamard factorization, amounts to Euler's product formula for the sine - recall from (4.6.3) in [1] that $j_{1/2,n} = n\pi$. The case $\alpha = 0$ is also explicitly mentioned in [62] as an example pertaining to Theorem 5 therein - observe that this theorem covers actually the whole range $\alpha \in (-1/2, 1/2)$. In general, one has $\hat{h}_\alpha \in \mathcal{D}_1$ for all $\alpha > -1/2$ with the notation of [62], and our pairs can hence be viewed as further explicit examples of van Dantzig pairs corresponding to $\mathcal{D}_1$. The case $\alpha = 1$ is particularly worth mentioning because it shows that the semicircle characteristic function belongs to a van Dantzig pair, as does the Gaussian characteristic function.

Remark 3.17. (a) The random variable Σ_α is ID as a convolution of Laplace distributions, and is not Gaussian. Hence, by the corollary p.117 in [62], we retrieve the fact that $X_{2,1/2}$ is not ID. Unfortunately, this method does not seem to give any insight on the non ID character of $X_{\alpha,\rho}$ for $\alpha \in (1,2)$ and $\rho \neq 1/2$.

(b) Following the notation of [62], the characteristic function

$$\hat{g}_\alpha(t) = \frac{\hat{h}_\alpha(t)}{\hat{h}_\alpha(\mathrm{i}t)} = \frac{J_\alpha(t)}{I_\alpha(t)},$$

where I_α is the modified Bessel function of the first kind, is self-reciprocal. In other words, one has

$$\hat{g}_\alpha(t)\hat{g}_\alpha(\mathrm{i}t) = 1.$$

Observe that again, the distribution corresponding to $\hat{g}_\alpha(t)$ is not ID.

3.3.H Further properties of whale-shaped functions

In this paragraph we prove five analytical properties of WS functions and densities. Those five easy pieces apply all to the densities f_α, and have an independent interest. We restrict the study to the class WS_+, the corresponding properties for WS_- being deduced at once.

Proposition 3.13. *Let f be a WS_+ density with unique mode M. Then f is perfectly skew to the right, that is*

$$f(M+x) > f(M-x) \qquad \text{for every } x > 0.$$

Proof. Let x_0 be the left-extremity of $\operatorname{Supp} f$ and $M = x_1 < x_2 < x_3$ be the vanishing places of the three first derivatives of f. Suppose first $M - x_0 > x_2 - M$. Taylor's formula with integral remainder implies

$$f(M+x) - f(M-x) = \int_0^x (x-t)\,(f''(M+t) - f''(M-t))\,dt.$$

On the one-hand, we have $f''(M+t) - f''(M-t) > 0$ for all $t > x_2 - M$ since $f''(M-t) \leq 0$ for all $t \geq 0$ and $f''(M+t) > 0$ for all $t > x_2 - M$. On the other hand, writing

$$f''(M+t) - f''(M-t) = \int_0^t (f^{(3)}(M+s) + f^{(3)}(M-s))\,ds,$$

which is valid for all $t < M - x_0$, we also have $f''(M+t) - f''(M-t) > 0$ for all $t \le x_2 - M$ since $f^{(3)}(u) > 0$ for all $x_0 < u < x_3$, by the WS_+ property. Putting everything together shows $f(M+x) > f(M-x)$ for all $x > 0$. Supposing next $M - x_0 \le x_2 - M$, the proof is analogous and easier; we just need to delete the corresponding arguments for $t > x_2 - M$. □

Remark 3.18. If we denote by $M_{\alpha,\rho}$ the unique mode of $f_{\alpha,\rho}$, the function

$$x \mapsto f_{\alpha,\rho}(M_{\alpha,\rho} + x) - f_{\alpha,\rho}(M_{\alpha,\rho} - x)$$

has constant and possibly zero sign on $(0,\infty)$ for $\rho = 0, 1/2, 1$ and for $\alpha = 1$, as seen from the above proposition, the explicit drifted Cauchy case and the symmetric case. One might wonder if this property of perfect skewness remains true in general. The perfect skewness of classical stable densities is a challenging open problem, which had been stated in the introduction to [43].

Proposition 3.14. *Let f be a* WS_+ *density and M, m, μ be its respective mode, median and mean. Then f satisfies the strict mean-median-mode inequality*

$$M < m < \mu.$$

Proof. We use the same notation of the proof of the previous proposition. First, the latter clearly implies $M < m$. To obtain the two strict inequalities together, let us now consider the function

$$g(x) = f(m+x) - f(m-x)$$

on $[0, m-x_0]$. If $m < x_2$, then the WS_+ property implies $f^{(2)}(m+x) > f^{(2)}(m) > f^{(2)}(m-x)$ for every $x \in (0, m-x_0]$, so that g is strictly convex on $(0, m-x_0]$. Since $g(0) = 0, g'(0) < 0$ and $g(m-x_0) > 0$, this shows that g vanishes only once on $(0, m-x_0]$ and from below, and hence also on the whole $(0,\infty)$. If $m \ge x_2$, then g is negative on $(0, m-M]$ and strictly convex on $[m-M, m-x_0]$ and we arrive at the same conclusion. We are hence in position to apply Lemma 1.9 (a) and (a strict, easily proved version of) Theorem 1.14 in [30], which implies the strict mean-median-mode inequality for f. □

Remark 3.19. (a) It is well-known and can be seen e.g. from Theorem 1.7 in [44] that $\mathbf{X}_\alpha$ has infinite mean. Hence, in this framework the above result only reads $M < m < \infty$, and it is readily obtained from the previous proposition. This mode-median inequality is also conjectured to hold true for classical positive stable densities. See Proposition 5 and Remark 11 (b) in [82] for partial results.

(b) In the relevant case $\alpha \in (1,2)$ it is natural to conjecture that the strict mean-median-mode inequality holds, in one or the other direction, for both free and classical stable densities. Observe that the three parameters clearly coincide for $\rho = 1/2$, whereas for $\rho = 1/\alpha$, easy computations show that the mean is zero and the mode and median are positive, so that it is enough to prove $m < M$. In general, this problem is believed to be challenging and beyond the scope of the present paper. We refer to [9] for a series of results on this interesting question, which however do not apply to non-explicit densities.

Proposition 3.15. *Let f be a* WS *density on $(0,+\infty)$ and X be the corresponding random variable. Then X is a Γ_2−mixture. In particular, it is* ID.

Proof. As in Theorem 1, we need to show that $g(x) = x^{-1}f(x)$ is a CM function, in other words that $(-1)^n g^{(n)}(x) > 0$ on $(0,\infty)$. By Leibniz's formula, we first compute

$$g^{(n)}(x) \;=\; n! \sum_{p=0}^{n} \frac{(-1)^p f^{(n-p)}(x)}{(n-p)!\, x^{p+1}}.$$

This implies, after some simple rearrangements,

$$h_n'(x) \;=\; (-1)^n x^n f^{(n+1)}(x), \tag{3.27}$$

where $h_n(x) = (-1)^n x^{n+1} g^{(n)}(x)$ has the same sign as $(-1)^n g^{(n)}(x)$. By the WS property, we see that

$$h_n(x) \;=\; n! \sum_{p=0}^{n} \frac{(-1)^{n-p} f^{(n-p)}(x)}{(n-p)!}\, x^{n-p}$$

is positive on $[x_n,\infty)$ since $(-1)^i f^{(i)}(x) > 0$ when $x \in (x_i,\infty)$ for all $i \geq 0$. Moreover, it follows from (3.27) and the whale-shape that $h_n'(x) > 0$ for $x \in (0, x_{n+1}]$. It is hence enough to show that $h_n(0+) = 0$ in order to conclude the proof, because $(0,\infty) = (0,x_{n+1}] \cup [x_n,\infty)$. But the whale-shape shows again that

$$0 \;\leq\; (-1)^{i-1} x^i f^{(i)}(x) \;\leq\; 2\,(-1)^{i-1} x^{i-1} \big(f^{(i-1)}(x) - f^{(i-1)}(x/2)\big)$$

for all $x \in (0,x_1]$ and an induction on i, starting from $f(0+) = 0$, implies $(x^i f^{(i)})(0+) = 0$ for all $i \geq 0$, so that $h_n(0+) = 0$ as well.

□

Remark 3.20. (a) The WS property is not satisfied by all densities of Γ_2−mixtures vanishing at zero. A simulation shows for example that the derivative of the density

$$f(x) \;=\; x(ta^2 e^{-ax} + (1-t)e^{-x})$$

vanishes three times for $a = 20$ and $t = 4/5$. This contrasts with the densities of Γ_1−mixtures, which are characterized by their complete monotonicity - see e.g. Proposition 51.8 in [79].

(b) The above proposition entails that the infinite divisibility of $\mathbf{X}_\alpha$ is a consequence of Theorem 3.4. On the other hand, as we saw above, the proof of Theorem 3.1 also shows that $\mathbf{X}_\alpha$ is a $\Gamma_{3/2}$−mixture for $\alpha \leq 3/4$, which is not a consequence of the whale-shape.

We next study the stability of the WS property under exponential tilting. Within ID densities on $\mathbb{R}$, this transformation amounts to the multiplication of the Lévy measure by $e^{-c|x|}$, allowing one for models with finite positive moments and analogous small jumps. This is a particular instance of the general tempering transformation, where the exponential perturbation is replaced by a CM function, and we refer to [77] for a thorough study on tempered stable densities. If we restrict to ID densities on a positive half-line, it is seen from the Lévy-Khintchine formula that exponential tilting amounts to multiplying the density by the same e^{-cx} and renormalizing. In particular, the set of densities of Γ_t−mixtures with $t \in (0,2]$ is also stable under exponential tilting.

Proposition 3.16. *If* $f \in \mathrm{WS}_+$, *then* $e^{-x} f \in \mathrm{WS}_+$.

Proof. It is enough to consider the case Supp $f = (0, \infty)$. Set $g(x) = e^{-x} f(x)$. Considering $h_n(x) = (-1)^n e^x g^{(n)}(x)$ for each $n \geq 0$, we have $h_{n+1} = h_n - h_n'$ and an easy induction starting from $h_0 = f$ implies

$$(-1)^{p-1} h_{n+1}^{(p)}(0+) > 0 \qquad \text{and} \qquad h_{n+1}^{(p)}(+\infty) = 0$$

for all $n, p \geq 0$. We will now show that $h_{n+1}^{(p)}$ vanishes once on $(0, \infty)$ for all $n, p \geq 0$, and that the sequence $\{x_{p,n+1},\ p \geq 0\}$ defined by $h_{n+1}^{(p)}(x_{p,n+1}) = 0$ is increasing. This is sufficient for our purpose, in taking $p = 0$.

Consider first the case $n = 0$, with $h_1^{(p)} = f^{(p)} - f^{(p+1)}$. It is clear that $(-1)^p h_1^{(p)}(x) > 0$ for $x \in [x_{p+1}, \infty)$ and that $(-1)^p h_1^{(p+1)}(x) > 0$ for $x \in (0, x_{p+1}]$. Since $(-1)^p h_1^{(p)}(0+) < 0$, this implies that $h_1^{(p)}$ vanishes once on $(0, \infty)$ for all $p \geq 0$, and Rolle's theorem entails that the sequence $\{x_{p,1},\ p \geq 0\}$ defined by $h_1^{(p)}(x_{p,1}) = 0$ is increasing.

The induction step is obtained analogously from $h_{n+2}^{(p)} = h_{n+1}^{(p)} - h_{n+1}^{(p+1)}$, since $(-1)^p h_{n+2}^{(p)}(0+) < 0$ and, by the induction hypothesis, $(-1)^p h_{n+2}^{(p)}(x) > 0$ for $x \in [x_{p+1,n+1}, \infty)$ and $(-1)^p h_{n+2}^{(p+1)}(x) > 0$ for $x \in (0, x_{p+1,n+1}]$. □

Remark 3.21. (a) The above proposition implies that $e^{-x} f_\alpha$, the "tilted free positive stable density", is WS_+ and ID. It would be interesting to know if it is also FID.

(b) The class WS_+ is not stable under the general tempering transformation introduced in [77]. For example, the random variable obtained from $\mathbf{\Gamma}_2$ in multiplying its Lévy measure by te^{-x} is easily seen to be $(1/2)\mathbf{\Gamma}_{2t}$, whose density belongs to WS_+ only for $t \in (1/2, 1]$.

Proposition 3.17. *Let $f \in \mathrm{WS}_+$ and $\{x_n,\ n \geq 0\}$ be the vanishing places of $\{f^{(n)},\ n \geq 0\}$. Then f is analytic on (x_0, ∞) and $x_n \to \infty$.*

Proof. Again we may suppose $x_0 = 0$. If f is a density, then Proposition 3.15 implies that $f = xg$ where g is CM and hence analytic on $(0, \infty)$, so that f is analytic on $(0, \infty)$ as well. If f is not a density, then Proposition 3.16 shows that $g = e^{-cx} f$ is a WS_+ density on $(0, \infty)$ for some normalizing $c > 0$, and f inherits the analyticity of g on $(0, \infty)$.

The second property is an easy consequence of the first one. Let x_∞ be the increasing limit of $\{x_n,\ n \geq 0\}$ and suppose $x_\infty < \infty$. By the whale-shape, we would then have $(-1)^n f^{(n)}(x) > 0$ for $x > x_\infty$, so that f would be CM on (x_∞, ∞), and hence also on $(0, \infty)$ by Bernstein's theorem and analytic continuation, a contradiction since $f(0+) = 0$.

□

Appendix A

Admissible domain of classical stable distributions

We look for the stable distributions except normal distributions and degenerate distributions starting from the definition.

A.1 Strictly stable distributions

From definition 1.12, we see that a random varible is strictly stable iff for any $n \geq 2$, there exists a positive number c_n such that

$$\mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n \overset{d}{=} c_n \mathbf{X} \tag{A.1}$$

where $\mathbf{X}_1 \overset{d}{=} \mathbf{X}_2 \overset{d}{=} \cdots \overset{d}{=} \mathbf{X}$ and independent.

Define by $\phi(t)$ the logarithm of the characteristic function of $\mathbf{X}$. Then (A.1) implies

$$n\phi(t) = \phi(c_n t), \quad \forall\, n \geq 2,\ \forall\, t \in \mathbb{R}, \tag{A.2}$$

c_n is positive for every $n \geq 2$, $c_1 = 1$ and $c_{mn} = c_m c_n$, thus the function g defined on positive rationals by $g(p/q) = c_p/c_q$ is well-defined and positive. Moreover, $s\phi(t) = \phi(g(s)t)$ for all positive rational numbers s. If we extend g by left continuity (or right continuity) to $(0, +\infty)$, the identity $s\phi(t) = \phi(g(s)t)$ still holds. The identity determines g uniquely. In fact, if there exist $a > 0$ and $a \neq 1$, such that $\phi(t) = \phi(at)$ for every t positive, then $\phi(t) = \phi(a^n t) = \phi(a^{-n} t)$ for all $t > 0$ and $n \in \mathbb{N}^*$. Let n tend to $+\infty$, we have $\phi(t) \equiv \phi(0) = 0$, which is the excluded degenerate case. We can conclude that g extends to a continuous function on $(0, +\infty)$ such that $g(xy) = g(x)g(y)$. This is equivalent to the well-known Cauchy functional equation, and the continuous solutions are of the form $g(s) = s^\beta$ for some $\beta \in \mathbb{R}$. Hence $\phi(s^\beta t) = s\phi(t)$ for all $s > 0$. Letting $\alpha = 1/\beta$, we have

$$\phi(s) = s^\alpha \phi(1) \quad \text{and} \quad \phi(-s) = s^\alpha \phi(-1), \quad \text{for} \quad \mathrm{s} > 0.$$

Since $|e^{\phi(t)}| \leq 1, \phi(0) = 0$, and $\phi(t) = \overline{\phi(-t)}$, there exist $c_0 \geq 0, c_1 \in \mathbb{R}$, and $\alpha > 0$ s.t.

$$\phi(t) = -(c_0 + c_1 j)|t|^\alpha, \tag{A.3}$$

where $j = \begin{cases} +i, \text{ if } t > 0, \\ -i, \text{ if } t < 0. \end{cases}$ We assume without loss of generality that $|c_0 + c_1 i| = 1$. There exists $\theta \in [-1/2, 1/2]$ s.t. $c_0 + c_1 i = e^{i\pi\theta}$.

We consider the stable distributions of infinite variance, thus by (A.3) the positive exponent α can only be smaller than 2 (the case $\alpha = 2$ corresponds to normal distributions). Combine (A.2) and (A.3), we have $c_n = n^{\frac{1}{\alpha}}$.

According to the formule of Fourier transform (cf. p.168 (27) in [60]), if we note g the density function of stable distribuion, then

$$g_{\alpha,\theta}(x) = \frac{1}{\pi}\Re\int_0^\infty e^{-e^{i\pi\theta}t^\alpha - ixt}dt = \frac{1}{\pi}(I_{1,\epsilon}(x) + I_{2,\epsilon}(x)) \tag{A.4}$$

where

$$I_{1,\epsilon}(x) = \Re\int_0^\epsilon e^{-e^{i\pi\theta}t^\alpha - ixt}dt, \quad I_{2,\epsilon}(x) = \Re\int_\epsilon^\infty e^{-e^{i\pi\theta}t^\alpha - ixt}dt.$$

Clearly, $g_{\alpha,\theta}(x) = g_{\alpha,-\theta}(-x)$ and $I_{1,\epsilon} \to 0$ as $\epsilon \to 0$. It suffices to consider $\theta \in [-1/2, 0]$. We have three cases.

A.1.A $\alpha = 1$.

The Cauchy distribution has the probability density function

$$f(x) = \frac{1}{\pi}\left[\frac{c_0}{(x+c_1)^2 + {c_0}^2}\right], \; c_0 > 0, c_1 \in \mathbb{R}, x \in \mathbb{R}.$$

Its characteristic function is

$$\int_{\mathbb{R}} e^{itx}f(x)dx = e^{-(c_0 + c_1 j)|t|}.$$

When $c_0 = 0$, (A.3) corresponds to δ_{c_1}.
Thus, the strictly 1-stable distributions are equivalent to the Cauchy distribution ($\theta \in (-1/2, 1/2)$) or Dirac measures ($\theta = \pm 1/2$).

A.1.B $0 < \alpha < 1$.

Firstly, we suppose $\theta = -\alpha/2$. By Cauchy's integral theorem, for all $x < 0$ we change the integration path to the semi-axis $\{ue^{i\pi/2}, u > 0\}$, it is easy to verify that $I_{2,\epsilon}(x) = 0$. For $x > 0$ we change the integration path to the semi-axis $\{ue^{-i\pi/2}, u > 0\}$

$$I_{2,\epsilon}(x) \;\; = \;\; \Im\int_\epsilon^\infty e^{-e^{-i\pi\alpha}u^\alpha - xu}du \tag{A.5}$$

Let $\epsilon \to 0$, we obtain

$$\begin{aligned} g_{\alpha,-\alpha/2}(x) &= \frac{1}{\pi}\Im\int_0^\infty e^{-e^{-i\pi\alpha}u^\alpha - xu}du && \text{(A.6)} \\ &= \frac{1}{\pi}\sum_{n\geq 1}(-1)^{n-1}\frac{\Gamma(1+\alpha n)}{n!}\sin(n\alpha\pi)x^{-\alpha n - 1} && \text{(A.7)} \\ &= \sum_{n\geq 1}(-1)^{n-1}\frac{\alpha}{(n-1)!\Gamma(1-\alpha n)}x^{-\alpha n - 1} && \text{(A.8)} \\ &= \frac{\alpha}{x^{1+\alpha}}\phi(-\alpha, 1-\alpha; -x^{-\alpha}) && \text{(A.9)} \end{aligned}$$

where

$$\phi(\rho, \beta; z) = \sum_{n\geq 0} \frac{z^n}{n!\Gamma(\rho n + \beta)} \tag{A.10}$$

with $\rho > -1, \beta \in \mathbb{C}$, is an entire function. Wright [91] said that, if $-1 < \rho < -1/3$, the zeros of $\phi(\rho, \beta; z)$ lie near the two lines $\arg(z) = \pm\frac{\pi}{2}(3\rho + 1)$, while, if $-1/3 \leq \rho < 0$, the zeros lie near the positive half of the real axis. Therefore, $\phi(-\alpha, 1-\alpha; -x^{-\alpha})$ does not change its sign for $x > 0$. Besides, $\lim_{x\to+\infty} \phi(-\alpha, 1-\alpha; -x^{-\alpha}) = \phi(-\alpha, 1-\alpha; 0) = \frac{1}{\Gamma(1-\alpha)} > 0$. We have that

$$g_{\alpha,-\alpha/2}(x) > 0, \quad \forall x > 0, \tag{A.11}$$

and that

$$g_{\alpha,-\alpha/2}(x) \sim \frac{\alpha}{\Gamma(1-\alpha)x^{1+\alpha}}, \quad x \to +\infty. \tag{A.12}$$

Similarly, for any $\theta \in [-1/2, 0]$, for $x > 0$,

$$\begin{aligned} g_{\alpha,\theta}(x) &= \frac{1}{\pi}\Im \int_0^\infty e^{-e^{-i\pi(\alpha/2-\theta)}u^\alpha - xu} du \\ &= \frac{1}{\pi}\sum_{n\geq 1}(-1)^{n-1}\frac{\Gamma(1+\alpha n)}{n!}\sin(n\pi(\frac{\alpha}{2}-\theta))x^{-\alpha n-1} \\ &= \frac{\alpha/2-\theta}{x^{\alpha+1}}\,{}_1\Psi_2\left[\begin{matrix} & (1+\alpha,\alpha) & \\ (1-\alpha/2+\theta, -\alpha/2+\theta) & & (1+\alpha/2-\theta, \alpha/2-\theta)\end{matrix}; -x^{-\alpha}\right] \end{aligned} \tag{A.13}$$

while, for any $\theta \in [-1/2, 0]$, for $x < 0$,

$$\begin{aligned} g_{\alpha,\theta}(x) &= \frac{1}{\pi}\sum_{n\geq 1}(-1)^{n-1}\frac{\Gamma(1+\alpha n)}{n!}\sin(n\pi(\frac{\alpha}{2}+\theta))|x|^{-\alpha n-1} \\ &= \frac{\alpha/2+\theta}{|x|^{\alpha+1}}\,{}_1\Psi_2\left[\begin{matrix} & (1+\alpha,\alpha) & \\ (1-\alpha/2-\theta, -\alpha/2-\theta) & & (1+\alpha/2+\theta, \alpha/2+\theta)\end{matrix}; -|x|^{-\alpha}\right] \end{aligned}$$

where

$${}_p\Psi_q\left[\begin{matrix}(a_1, A_1) & (a_2, A_2) & \ldots & (a_p, A_p) \\ (b_1, B_1) & (b_2, B_2) & \ldots & (b_q, B_q)\end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{\Gamma(a_1 + A_1 n)\cdots\Gamma(a_p + A_p n)}{\Gamma(b_1 + B_1 n)\cdots\Gamma(b_q + B_q n)} \frac{z^n}{n!},$$

defined for $z, a_i, b_j \in \mathbb{C}$ and $A_i, B_j \in \mathbb{R} \setminus \{0\}$, is the Fox-Wright function or generalized Wright function. ${}_p\Psi_q(z)$ is an entire function of z if $\sum_{j=1}^q B_j - \sum_{i=1}^p A_i > -1$ (firstly proved by Wright, for the proof see e.g. [54, Theorem 1]). Thus for both x positive and negative, these two Fox-Wright functions are well-defined.

Let $|x| \to \infty$,

$$g_{\alpha,\theta}(x) \sim \frac{\alpha/2-\theta}{x^{1+\alpha}} \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha/2+\theta)\Gamma(1+\alpha/2-\theta)}, \quad x \to +\infty. \tag{A.14}$$

$$g_{\alpha,\theta}(x) \sim \frac{\alpha/2+\theta}{|x|^{1+\alpha}} \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha/2-\theta)\Gamma(1+\alpha/2+\theta)}, \quad x \to -\infty. \tag{A.15}$$

Being a density function, $g_{\alpha,\theta}(x)$ should be non-negative, from (A.15), θ should be in $[-\alpha/2, 0]$. Symmetrically, for $0 < \alpha < 1$, $\theta \in [-\alpha/2, \alpha/2]$.

Together with $g_{\alpha,\theta}(x) = g_{\alpha,-\theta}(-x)$, we conclude that $\forall \theta \in [-\alpha/2, \alpha/2], \forall x \in \mathbb{R}$,

$$g_{\alpha,\theta}(x) = \frac{1}{\pi}\sum_{n\geq 1}(-1)^{n-1}\frac{\Gamma(1+\alpha n)}{n!}\sin(n\pi(\frac{\alpha}{2} - \operatorname{sgn}(x)\,\theta))|x|^{-\alpha n-1}. \tag{A.16}$$

For $s \in (-\alpha, 1)$, the Mellin transform of $g_{\alpha,\theta}(x)\mathbf{1}_{x>0}$ is

$$\begin{aligned} M(s) &:= \int_0^\infty x^{-s} g_{\alpha,\theta}(x)dx \\ &= \frac{1}{\pi}\Im \int_0^\infty e^{-e^{-i\pi(\alpha/2-\theta)}u^\alpha} \int_0^\infty x^{-s}e^{-xu}dxdu \\ &= \frac{\Gamma(1-s)}{\alpha\pi} \sin(\pi(1/2-\theta/\alpha)s)\Gamma(s/\alpha) \\ &\to \frac{1}{2} - \frac{\theta}{\alpha}, \quad \text{as } s \to 0. \end{aligned} \tag{A.17}$$

We denote $\rho := \frac{1}{2} - \frac{\theta}{\alpha}$, and call it the parameter of positivity since $\int_0^\infty g_{\alpha,\theta}(x)dx = \rho$.

Remark A.1. On the other hand, for $0 < \alpha < 1$, $\theta \in [-\alpha/2, \alpha/2]$, $x > 0$,

$$g_{\alpha,\theta}(x) = \left(\frac{1}{2} - \frac{\theta}{\alpha}\right) \frac{d}{dx} {}_1\Psi_2 \left[\begin{matrix} (1,\alpha) \\ (1, \theta - \alpha/2) \quad (1, -\theta + \alpha/2) \end{matrix} ; -x^{-\alpha} \right], \tag{A.18}$$

and, for $0 < \alpha < 1$, $\theta \in [-\alpha/2, \alpha/2]$, $x < 0$,

$$g_{\alpha,\theta}(x) = -\left(\frac{1}{2} + \frac{\theta}{\alpha}\right) \frac{d}{dx} {}_1\Psi_2 \left[\begin{matrix} (1,\alpha) \\ (1, -\theta - \alpha/2) \quad (1, \theta + \alpha/2) \end{matrix} ; -|x|^{-\alpha} \right]. \tag{A.19}$$

$$\int_0^\infty g_{\alpha,\theta}(x)dx = \tfrac{1}{2} - \tfrac{\theta}{\alpha} \quad \text{and} \quad \int_{-\infty}^0 g_{\alpha,\theta}(x)dx = \tfrac{1}{2} + \tfrac{\theta}{\alpha}$$

imply that

$$\lim_{x\to-\infty} {}_1\Psi_2 \left[\begin{matrix} (1,\alpha) \\ (1, \theta - \alpha/2) \quad (1, -\theta + \alpha/2) \end{matrix} ; x \right] = 0,$$

and that

$$\lim_{x\to-\infty} {}_1\Psi_2 \left[\begin{matrix} (1,\alpha) \\ (1, -\theta - \alpha/2) \quad (1, \theta + \alpha/2) \end{matrix} ; x \right] = 0,$$

i.e.

$$\lim_{x\to-\infty} {}_1\Psi_2 \left[\begin{matrix} (1,\alpha) \\ (1, \gamma) \quad (1, -\gamma) \end{matrix} ; x \right] = 0 \quad \text{for every } \gamma \in (-\alpha, \alpha) \text{ and } \alpha \in (0,1).$$

A.1.C $\quad 1 < \alpha < 2$.

We use again the Cauchy's integral theorem for $I_{2,\epsilon}(x)$, $x \in \mathbb{R}$. Let's change the integration path to the half-line $\{ue^{-i\pi\theta/\alpha}, u > 0\}$, we then obtain

$$I_{2,\epsilon}(x) = \Re \int_\epsilon^\infty e^{-u^\alpha - xue^{i\pi(1/2-\theta/\alpha)}} e^{-i\pi\theta/\alpha} du.$$

It follows that

$$\begin{aligned} g_{\alpha,\theta}(x) &= \frac{1}{\pi}\Re \int_0^\infty e^{-u^\alpha - xue^{i\pi(1/2-\theta/\alpha)}} e^{-i\pi\theta/\alpha} du \\ &= \frac{1}{\pi}\Re \int_0^\infty \sum_{k\ge0} \frac{(-xue^{i\pi(1/2-\theta/\alpha)})^k}{k!} e^{-i\pi\theta/\alpha} e^{-u^\alpha} du \\ &= \frac{1}{\pi} \sum_{n\ge1} (-1)^{n-1} \frac{\Gamma(1+\frac{n}{\alpha})}{n!} \sin(n\pi(\frac{1}{2} - \frac{\theta}{\alpha})) x^{n-1} \end{aligned}$$

The infinite sum expression gives us the following relations:

$$g_{\alpha,\theta}(x) = x^{-1-\alpha} g_{\alpha',\theta'}(x^{-\alpha}), \quad \forall\, x > 0,\, \forall\, \alpha \in (1,2), \tag{A.20}$$

with $\alpha' = \frac{1}{\alpha}, 1 - \frac{2}{\alpha'}\theta' = \alpha(1 - \frac{2}{\alpha}\theta)$, and

$$g_{\alpha,\theta}(-x) = g_{\alpha,-\theta}(x) = x^{-1-\alpha} g_{\alpha'',\theta''}(x^{-\alpha}), \quad \forall\, x > 0,\, \forall\, \alpha \in (1,2), \tag{A.21}$$

with $\alpha'' = \frac{1}{\alpha}, 1 - \frac{2}{\alpha''}\theta'' = \alpha(1 + \frac{2}{\alpha}\theta)$. A simple calculation shows

$$\theta' \in [-\frac{1}{2\alpha}, \frac{1}{2\alpha}] \Rightarrow \theta = \frac{\alpha + 2\alpha\theta' - 1}{2} \in [\frac{\alpha - 2}{2}, \frac{\alpha}{2}]$$

$$\theta'' \in [-\frac{1}{2\alpha}, \frac{1}{2\alpha}] \Rightarrow \theta = \frac{1 - 2\alpha\theta'' - \alpha}{2} \in [-\frac{\alpha}{2}, \frac{2 - \alpha}{2}]$$

Therefore, we know that for $1 < \alpha < 2,\, \theta \in [\alpha/2 - 1, 1 - \alpha/2]$.

The formula (A.20) implies that for $\alpha \in (1,2)$, the parameter of positivity

$$\rho = \int_0^\infty g_{\alpha,\theta}(x)dx = \int_0^\infty x^{-1-\alpha} g_{\alpha',\theta'}(x^{-\alpha})dx = \frac{1}{\alpha}\int_0^\infty g_{\alpha',\theta'}(x)dx = \frac{1}{\alpha}(\frac{1}{2} - \frac{\theta'}{\alpha'}) = \frac{1}{2} - \frac{\theta}{\alpha} \in [1 - \frac{1}{\alpha}, \frac{1}{\alpha}].$$

A.1.D Conclusion

We observe that we can fix a strictly stable random variable with two parameters (α, ρ), and

$$(\alpha, \rho) \in \{\alpha \in (0,1], \rho \in [0,1]\} \cup \{\alpha \in (1,2], \rho \in [1 - 1/\alpha, 1/\alpha]\}.$$

The density function $f_{\alpha,\rho}$ in Chapter 1 is $g_{\alpha,\theta}$ with $\theta = \frac{\alpha}{2} - \alpha\rho$.

A.2 Non-strictly 1-stable distributions: $c_n = n$ and $d_n \neq 0$.

(1.15) implies

$$n\phi(t) = \phi(nt) + i d_n t, \quad n \geq 2, \forall\, t \in \mathbb{R}, \tag{A.22}$$

$\Rightarrow$

$$\phi'(t) = \phi'(nt) + i\frac{d_n}{n}, \quad n \geq 2, \forall\, t \in \mathbb{R},$$

$\Rightarrow$

$$\phi''(t) = n\phi''(nt), \quad n \geq 2, \forall\, t \in \mathbb{R},$$

using the argument after (A.2), we obtain soon that for $t > 0$, $\phi''(t) = ict^{-1}, \phi'(t) = ic\log t + \gamma$, and then $\phi(t) = ict\log t + (\gamma - ic)t$ (without constant term since $\phi(0) = 0$), with $c = -\frac{d_n}{n\log n} \in \mathbb{R}$. We know also that $f(t) = a|t| + ibt$ is the solution of $nf(t) = f(nt), f(0) = 0$. In addition $|e^{\phi(t)}| \leq 1, \phi(t) = \overline{\phi(-t)}$, hence $\phi(t)$ must be of the form

$$\phi(t) = -c_0|t| - i\beta t\log|t| + ic_1 t$$

for some $c_0 > 0, c_1 \in \mathbb{R}$ and $\beta = \frac{d_n}{n \log n} \in \mathbb{R}$. Note that c_0 can not be 0. If $c_0 = 0$, we suppose that there exist a random variable $\mathbf{X}$ such that

$$\mathbb{E}[e^{it\mathbf{X}}] = e^{-i\beta t \log |t|},$$

for its independent copie $\tilde{\mathbf{X}}$,

$$\mathbb{E}[e^{-it\tilde{\mathbf{X}}}] = e^{i\beta t \log |t|},$$

then

$$\mathbb{E}[e^{it(\mathbf{X}-\tilde{\mathbf{X}})}] = 1,$$

which implies $\mathbf{X} - \tilde{\mathbf{X}} = \delta_0$, so do $\mathbf{X}$. We then have $\beta = 0$, but here we consider non-strictly 1-stable distributions $d_n \neq 0, \beta \neq 0$. Thus $c_0 > 0$. According to the equivalence among random variables, we can set, without loss of generality, $c_0 = \frac{\pi}{2}, c_1 = 0$. Then we want to determine the domain of β. The density function is

$$g_\beta(x) := \frac{1}{\pi}\Re \int_0^\infty e^{-\frac{\pi}{2}t - i\beta t \log t - ixt} dt, \quad x \in \mathbb{R}. \tag{A.23}$$

Proposition A.1.

$$g_\beta(x) \sim \frac{1+\beta}{2x^2}, \qquad as\ x \to +\infty; \tag{A.24}$$

$$g_\beta(x) \sim \frac{1-\beta}{2x^2}, \qquad as\ x \to -\infty. \tag{A.25}$$

Therefore, $\beta \in [-1, 1]$.

Proof. $g_\beta(x) = g_{-\beta}(-x)$, we consider only $\beta > 0$.
For $x > 0$: We can change the integral path from $(0, +\infty)$ to $\{ue^{-\frac{\pi}{2}i}, u > 0\}$ and then obtain

$$\begin{aligned} g_\beta(x) &= \frac{1}{\pi}\Im \int_0^\infty e^{-\beta u \log u + i(\beta+1)u\pi/2 - xu} du \\ &= \frac{1}{\pi}\int_0^\infty e^{-xu} e^{-\beta u \log u} \sin((\beta+1)u\pi/2) du. \end{aligned}$$

Then

$$\lim_{x\to+\infty} x^2 g_\beta(x) = \lim_{x\to+\infty} \frac{\beta+1}{2}\int_0^\infty te^{-t}e^{-\beta\frac{t}{x}\log\frac{t}{x}} \frac{\sin(\pi(\beta+1)t/(2x))}{\pi(\beta+1)t/(2x)} dt, \tag{A.26}$$

by Lebesgue's dominated convergence theorem, we have

$$g_\beta(x) \sim \frac{1+\beta}{2x^2}, \qquad \text{as } x \to +\infty.$$

For $x < 0$: We change the intgral path from $(0, +\infty)$ to the contour $L = L_1 \cup L_2$ where

$$L_1 = \{z : \Re(z) = 0, \Im(z) \text{ from 0 to 1 }\}, \quad L_2 = \{z : \Im(z) = 1, \Re(z) \text{ from 0 to } +\infty\}.$$

Then

$$\begin{aligned} g_\beta(x) &= \frac{1}{\pi}\Re\int_{L_1} e^{-\frac{\pi}{2}t - i\beta t\log t - ixt} dt + \frac{1}{\pi}\Re\int_{L_2} e^{-\frac{\pi}{2}t - i\beta t \log t - ixt} dt \\ &= \frac{-1}{\pi}\Im\int_0^1 e^{-\frac{\pi}{2}iu + \beta u(\log u + \pi i/2) - |x|u} du + \frac{1}{\pi}\Re\int_0^\infty e^{-\frac{\pi}{2}t - i\beta(i+t)\log(i+t) - ix(i+t)} dt \\ &= I_1(x) + I_2(x). \end{aligned}$$

We observe that

$$\begin{aligned} x^2 I_1(x) &= -\frac{|x|}{\pi}\Im \int_0^{|x|} e^{-t-\frac{\pi}{2}i\frac{t}{|x|}+\beta\frac{t}{|x|}(\log\frac{t}{|x|}+\pi i/2)}dt \\ &= \frac{1-\beta}{2}\int_0^{|x|} te^{-t+\beta\frac{t}{|x|}\log\frac{t}{|x|}}\frac{\sin(\pi(1-\beta)t/|2x|)}{\pi(1-\beta)t/|2x|}dt \\ &\to \frac{1-\beta}{2}, \qquad x\to-\infty, \end{aligned}$$

and that

$$e^{|x|}I_2(x) = \frac{1}{\pi}\Re\int_0^{\infty} e^{-\frac{\pi}{2}t-i\beta(i+t)\log(i+t)-ixt}dt$$

is bounded for $x<0$. Consequently,

$$g_\beta(x) \sim \frac{1-\beta}{2x^2}, \qquad \text{as } x\to-\infty.$$

□

Appendix B

Special functions

B.1 Gamma function and double gamma function

The *Euler gamma function* $\Gamma(z)$ is defined by the Euler integral of the second kind:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0, \tag{B.1}$$

where $t^{z-1} = e^{(z-1)\log t}$. This integral is convergent for all complex $z \in \mathbb{C}$ with $\Re(z) > 0$. For this function the reduction formula

$$\Gamma(z+1) = z\Gamma(z), \quad \Re(z) > 0,$$

holds; it is obtained from (B.1) by integration by parts. Using this relation, the Euler gamma function is extended to the half-plane $\Re(z) \le 0$ except $\{0, -1, -2, \cdots\}$. It follows that the gamma function is analytic everywhere in the complex plane $\mathbb{C}$ except at $z = 0, -1, -2, \cdots$, where $\Gamma(z)$ has simple poles.
We also indicate some other properties of the gamma function such as the *functional equation*:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad 0 < \Re(z) < 1; \qquad \Gamma(1/2) = \sqrt{\pi}; \tag{B.2}$$

the *Legendre duplication formula*:

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma(z + \frac{1}{2}), \quad z \notin \{-\frac{n}{2} : n \in \mathbb{N}\}; \tag{B.3}$$

and the more general *Gauss multiplication formula*:

$$\Gamma(mz) = \frac{m^{mz-1/2}}{(2\pi)^{(m-1)/2}} \prod_{k=0}^{m-1} \Gamma(z + \frac{k}{m}), \quad z \notin \{-\frac{n}{m} : n \in \mathbb{N}\}; \; m \in \{2, 3, 4, \cdots\}; \tag{B.4}$$

the Stirling asymptotic formula:

$$\Gamma(z) = \sqrt{2\pi} z^{z-1/2} e^{-z} \left[1 + O(1/z)\right], \quad |\arg(z)| < \pi; |z| \to \infty. \tag{B.5}$$

In particular,

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left[1 + O(1/n)\right], \quad n \in \mathbb{N}; n \to \infty.$$

The Malmsten formula for the Gamma function is

$$\frac{\Gamma(z+a)}{\Gamma(a)} = \exp\left(\psi(a)z + \int_0^\infty (e^{-zx} - 1 + zx)\frac{e^{-ax}}{x(1-e^{-x})}dx\right), \quad z > -a,\ a > 0. \tag{B.6}$$

where ψ is the digamma function.

The definition for the gamma function due to Karl Weierstrass is also valid for all complex numbers z except the non-positive integers:

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$$

where $\gamma \approx 0.577216$ is the Euler–Mascheroni constant.

The double Gamma function

$$G(z;\tau) = \frac{z}{\tau}e^{a\frac{z}{\tau}+b\frac{z^2}{2\tau^2}}\prod_{\substack{(m,n)\in\mathbb{N}^2\\(m,n)\neq(0,0)}}\left[(1+\frac{z}{m\tau+n})e^{-\frac{z}{m\tau+n}+\frac{z^2}{2(m\tau+n)^2}}\right],$$

where $z \in \mathbb{C}, \tau \in \mathbb{C}\backslash(-\infty, 0]$ and a, b are functions of τ only. This function, also known as the Barnes function for $\tau = 1$, was introduced in [8] as a generalization of the Gamma function. It is holomorphic on $\mathbb{C}$ and admits the following Malmsten type representation:

$$G(z;\tau) = \exp\int_0^\infty\left[\frac{1-e^{-zx}}{(1-e^{-x})(1-e^{-\tau x})} - \frac{ze^{-\tau x}}{1-e^{-\tau x}} + (z-1)(\frac{z}{2\tau}-1)e^{-\tau x} - 1\right]\frac{dx}{x}, \mathcal{R}(z) > 0. \tag{B.7}$$

It fulfils the functional equations

$$G(z+1;\tau) = \Gamma(z\tau^{-1})G(z;\tau) \qquad \text{and} \qquad G(z+\tau;\tau) = (2\pi)^{\frac{\tau-1}{2}}\tau^{\frac{1}{2}-z}\Gamma(z)G(z;\tau) \tag{B.8}$$

with normalization $G(1;\tau) = 1$. The normalization also implies

$$G(\tau;\tau) = \frac{(2\pi)^{\frac{\tau-1}{2}}}{\sqrt{\tau}} \tag{B.9}$$

for every $\tau > 0$. Billingham and King [17] have developed uniform asymptotic approximations for double gamma function for various parameter values. We state here the formula (4.5) in [17] for $|z| \to \infty$,

$$\ln G(z;\tau) = \frac{1}{2\tau}z^2\ln z - \frac{1}{\tau}(\frac{3}{4}+\frac{1}{2}\ln\tau)z^2 - \frac{1}{2}(\frac{1}{\tau}+1)z\ln z + \frac{1}{2}(\frac{\ln\tau}{\tau}+\frac{1}{\tau}+\ln\tau+1+\ln(2\pi))z$$
$$+ (\frac{\tau}{12}+\frac{1}{4}+\frac{1}{12\tau})\ln z + C(\tau) + O(\frac{1}{z}), \tag{B.10}$$

where $C(\tau)$ is an $O(1)$ function of τ alone. We don't have a concrete expression of $C(\tau)$, but we know that it is bounded.

B.2 Wright function

The Wright function (named after the British mathematician E.M. Wright)

$$\phi(\rho, \beta; z) = \sum_{n\geq 0} \frac{z^n}{n!\Gamma(\rho n + \beta)} \tag{B.11}$$

with $\rho > -1, \beta \in \mathbb{C}$, is an entire function. This function was thoroughly studied in the original articles [89, 90, 91] for various purposes, and is referenced in Formula 18.1(27) in the encyclopedia [34].

Theorem B.1 (Theorem 1 in [91]). *For* $\rho \in (-1, 0)$*, choose* $\arg z$ *to satisfy*

$$-\pi < \arg z < \pi,$$

let $y = -z$*, and write* $Y = (1+\rho)((-\rho)^{-\rho} y)^{1/(1+\rho)}$*, if* $|\arg y| \leq \min\{\frac{3}{2}\pi(1+\rho), \pi\} - \epsilon$*, then*

$$\phi(z) = Y^{1/2-\beta} e^{-Y} \{\sum_{m=0}^{M} A_m Y^{-m} + O(Y^{-M-1})\}, \quad Y \to \infty. \tag{B.12}$$

Define $\tilde{a}_m$ as the coefficient of v^{2m} in the expansion of

$$\frac{(-1)^m 2^{m-1/2} \Gamma(m+1/2)}{\pi(-\rho)^{m-1/2+\beta}(1+\rho)^{1-\beta}} (1-v)^{-\beta} \{g(v)\}^{-2m-1}$$

where $g(0) = 1,\ g(v) = \{1 + \frac{2+\rho}{3} v + \frac{(2+\rho)}{3}\frac{(3+\rho)}{4} v^2 + \cdots\}^{1/2}$.
Wright [91] proved that $A_m = \tilde{a}_m$.

Theorem B.2 (Theorem 1 in [89]). *If* $\rho > 0,\ \arg z = \theta,\ |\theta| \leq \pi$*, and*

$$Z_1 = (\rho|z|)^{\frac{1}{1+\rho}} e^{\frac{i(-\theta+\pi)}{1+\rho}}, \quad Z_2 = (\rho|z|)^{\frac{1}{1+\rho}} e^{\frac{i(-\theta-\pi)}{1+\rho}}$$

then we have

$$\phi(\rho, \beta; z) = H(Z_1) + H(Z_2)$$

where

$$H(Z) = Z^{1/2-\beta} e^{\frac{1+\rho}{\rho} Z} \{\sum_{m=0}^{M} (-1)^m l_m Z^{-m} + O(|Z|^{-M-1})\}, \quad Z \to \infty,$$

and we define l_m as the coefficient of v^{2m} in the expansion of

$$\frac{\Gamma(m+1/2)}{2\pi} \left(\frac{2}{1+\rho}\right)^{1/2+m} (1-v)^{-\beta} \{g(v)\}^{-2m-1}$$

where $g(0) = 1,\ g(v) = \{1 + \frac{2+\rho}{3} v + \frac{(2+\rho)}{3}\frac{(3+\rho)}{4} v^2 + \cdots\}^{1/2}$. In particular $l_0 = \frac{1}{\sqrt{2\pi(\rho+1)}}$.

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