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Analyse par ondelettes de processus dans des chaos de Wiener

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Analyse par ondelettes de processus dans des chaos de Wiener

Résumé: Dans cette thèse, on s'intéresse à des extensions du mouvement brownien fractionnaire qui appartiennent à des chaos de Wiener. De façon générale, un processus stochastique appartient à un chaos de Wiener (homogène) d'ordre $n \in \mathbb{N}^*$ si ce processus peut être représenté par une intégrale (stochastique) de Wiener multiple d'ordre n; il est non gaussien lorsque $n \geq 2$. La complexité nettement plus grande de l'intégrale de Wiener multiple par rapport à l'intégrale de Wiener classique rend l'étude des processus chaotiques beaucoup plus complexe que celle des processus gaussiens, et même des processus stables parfois. A titre d'exemples de difficultés, on peut mentionner que la non corrélation de deux variables aléatoires d'un chaos de Wiener non gaussien n'entraîne pas forcément leur indépendance; et que contrairement aux processus gaussiens et stables les fonctions caractéristiques des lois fini-dimensionnelles des processus chaotiques ne sont pas données par des formules explicites et exploitables. A cause de telles difficultés les méthodes d'ondelettes ne se sont guère développées dans le cadre des chaos de Wiener non gaussiens. L'un des objectifs de cette thèse est de mettre en place de nouvelles stratégies permettant de contourner ces difficultés et de développer ces méthodes. Un autre objectif est de construire des extensions chaotiques du mouvement brownien fractionnaire ayant une rugosité locale qui varie d'un point à un autre, puis d'étudier de façon précise la régularité de leurs trajectoires.
Mots clés : Autosimilarité, Mouvement brownien fractionnaire, Processus de Rosenblatt, Décompositions en ondelettes, Séries aléatoires, Chaos de Wiener, Processus multifractionnaires, Régularité Hölderienne.

Wavelet analysis of stochastic processes in Wiener chaoses

Abstract: In this PhD thesis, we are interested in extensions of the fractional Brownian motion which belong to Wiener chaoses. In general, a stochastic process belongs to a (homogeneous) Wiener chaos of order $n \in \mathbb{N}$ if this process can be represented by a multiple (stochastic) Wiener integral of order n; it is non-Gaussian when $n \geq 2$. The significantly greater complexity of the multiple Wiener integral compared to the classical Wiener integral makes the study of chaotic processes much more complex than that of Gaussian processes, and even of the stable processes sometimes. As examples of difficulties, it can be mentioned that the non-correlation of two random variables of a non-Gaussian Wiener chaos does not necessarily lead to their independence; and that, unlike Gaussian and stable processes, the characteristic functions of finite-dimensional distributions of chaotic processes are not given by explicit and exploitable formulas. Because of such difficulties wavelet methods have not been very developed in the context of non-Gaussian Wiener chaos. One of the goals of this thesis is to find new strategies allowing to circumvent these difficulties and to develop these methods. Another objective is to construct chaotic extensions of the fractional Brownian motion having a local roughness that varies from one point to another, and to study precisely the regularity of their paths.
Keywords: Self-similarity, Fractional Brownian motion, Rosenblatt process, Multifractional stochastic processes, Wavelet expansions, Random series, Wiener chaos, Hölder regularity.

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"L'avenir n'est jamais que du présent à mettre en ordre. Tu n'as pas à le prévoir, mais à le permettre."

Antoine de Saint-Exupéry

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Introduction

Du mouvement brownien au mouvement brownien fractionnaire

Au début du XIXème siècle, le botaniste Robert Brown avait observé le phénomène d'agitation désordonnée de certaines particules en suspension dans un fluide immobile, phénomène qui porte aujourd'hui le nom de mouvement brownien. Un siècle plus tard, Albert Einstein prédisait dans [28] la possibilité de calculer les dimensions moléculaires à partir de l'observation du mouvement brownien. Dans la même veine, entre 1907 et 1909, le physicien Jean Perrin réalisait une série d'expériences publiées dans [56] qui confirmaient les prédictions d'Einstein et conduisaient à la validation scientifique de la théorie atomiste, mettant ainsi fin à un débat millénaire. Un peu plus tard, le mathématicien Norbert Wiener, s'inspirant des observations de Perrin, fournissait le cadre mathématique qui permettait la modélisation théorique de phénomènes tels que le mouvement brownien. Aujourd'hui, le mouvement brownien $\{B(t)\}_{t\geq 0}$ est aussi appelé le processus de Wiener et est défini comme étant l'unique processus gaussien dont la variance des accroissements est égale au temps écoulé. Il occupe une place importante en mathématiques et est relié à plusieurs de leurs branches comme les probabilités, l'analyse complexe, l'analyse harmonique, la théorie des groupes, la géométrie fractale ou encore la théorie des graphes. L'une de ses propriétés fondamentales réside dans son apparition dans le cadre de certains théorèmes limites fonctionnels tels que le théorème de Donsker. Par exemple, pour une suite de variables aléatoires centrées $\{X_n\}_{n\in\mathbb{N}}$ indépendantes, identiquement distribuées et de variance 1, la suite normalisée des sommes partielles, interpolée en temps continu, converge au sens des lois fini-dimensionnelles vers un mouvement brownien standard. Autrement dit, lorsque N tend vers $+\infty$, on a

$$\left\{\frac{1}{N^{1/2}}\sum_{n=1}^{\lfloor Nt\rfloor} X_n\right\}_{t\geq 0} \xrightarrow{f.d.d} \{B(t)\}_{t\geq 0}. \tag{0.0.1}$$

Cette propriété remarquable fait du mouvement brownien un objet particulièrement utile dans une multitude de domaines. Il est par exemple l'outil de base des mathématiques financières, depuis son introduction par Louis Bachelier dans la modélisation de certaines fluctuations boursières.

Cependant, beaucoup de phénomènes ne peuvent être modélisés par le processus de Wiener. À titre d'exemple, en 1951, l'hydrologiste Harold E. Hurst [34] tentant de modéliser les crues annuelles du fleuve du Nil, a fait le constat d'une propriété de *longue mémoire* présente dans les données. Il en déduisait alors l'impossibilité de décrire ce phénomène par un processus dont les accroissements ne sont pas corrélés au cours du temps, tel que le mouvement brownien. Benoit

Mandelbrot faisait quelques années plus tard le même constat concernant les fluctuations du cours de certaines matières premières comme le coton. Depuis, la présence de longue mémoire a été identifiée dans plusieurs phénomènes issus des télécommunications, de la biologie, du trafic internet, de la finance, du traitement d'images, etc. Dans la littérature, il existe plusieurs définitions de la longue mémoire (cf. [58]), qui est aussi appelée forte dépendance ou long-range dependence (LRD). Ces définitions ne sont pas toutes équivalentes et sont utilisées chacune dans un contexte donné. Mais de façon générale, on dit d'un phénomène qu'il présente une longue mémoire si les corrélations du processus sous-jacent décroissent suffisamment lentement avec le temps. Une telle condition est par exemple satisfaite lorsque la fonction de corrélation d'une suite stationnaire de variables aléatoires $\{X_n\}_{n\in\mathbb{N}}$ vérifie:

$$\gamma_X(n) := \mathbb{E}(X_n X_0) - \mu_X^2 \underset{n\to+\infty}{\sim} n^{2(H-1)}, \tag{0.0.2}$$

avec H strictement supérieur à $1/2$. On dit alors que la suite $\{X_n\}_{n\in\mathbb{N}}$ présente une LRD de paramètre de Hurst H. Le paramètre H a été nommé par Mandelbrot en l'honneur de Hurst, qui avait donné une première heuristique pour l'estimation de ce paramètre à partir de l'observation. Dans ce contexte, un autre théorème limite fonctionnel semble plus approprié. Celui-ci constitue une contrepartie du théorème de Donsker dans le sens où les variables aléatoires sous-jacentes ne sont plus supposées indépendantes mais au contraire fortement dépendantes. Plus précisément, si l'on prend par exemple une suite stationnaire $\{\xi_n\}_{n\in\mathbb{N}}$ de variables aléatoires gaussiennes centrées, réduites et présentant une longue mémoire de paramètre de Hurst $H > 1/2$, alors on a la convergence au sens des lois fini-dimensionnelles suivante:

$$\left\{\frac{1}{N^H}\sum_{n=1}^{\lfloor Nt\rfloor}\xi_n\right\}_{t\geq 0} \overset{f.d.d}{\longrightarrow} \big\{B_H(t)\big\}_{t\geq 0}, \tag{0.0.3}$$

où le processus limite n'est plus un mouvement brownien mais un mouvement brownien fractionnaire d'ordre H. Le mouvement brownien fractionnaire a été initialement introduit par Andrei Kolmogorov en 1940 dans [39]. Celui-ci s'intéressait à l'étude des flots turbulents où des tourbillons concentrent l'essentiel de l'énergie du système. Kolmogorov construisit alors le mouvement brownien fractionnaire afin de générer des spirales gaussiennes dans un espace de Hilbert. Toutefois, ce processus stochastique n'a été réellement défini et étudié en tant qu'objet mathématique en soi qu'en 1968, dans l'article de Mandelbrot et Van Ness. Dans [46], les auteurs motivent l'appellation de ce processus du fait d'une de ses représentations sous la forme d'une primitive fractionnaire par rapport à une mesure brownienne :

$$\big\{B_H(t)\big\}_{t\in\mathbb{R}} \overset{d}{=} \left\{\int_{\mathbb{R}} \big((t-u)_+^{H-1/2} - (u)_+^{H-1/2}\big)B(du)\right\}_{t\in\mathbb{R}}, \tag{0.0.4}$$

où $\overset{d}{=}$ désigne l'égalité en loi des deux processus. Signalons que ce processus est gaussien à trajectoires continues et n'admet de dérivée nulle part. Il généralise le mouvement brownien dans le sens où $B_{1/2} = B$; toutefois ses accroissements ne sont pas indépendants lorsque $H \neq 1/2$ mais seulement stationnaires. En fait, il s'agit de l'unique processus gaussien *autosimilaire* d'ordre H et à accroissements stationnaires. L'autosimilarité signifie que pour tout $a > 0$,

$$\{B_H(at)\}_{t\in\mathbb{R}} \overset{d}{=} \{a^H B(t)\}_{t\in\mathbb{R}}.$$

C'est une propriété d'invariance d'échelle qui joue un rôle essentiel en statistique des processus. En effet, un théorème fondamental de Lamperti [41], obtenu en 1952, permet de *confondre* les processus autosimilaires avec les processus qui sont obtenus asymptotiquement par l'intérmédiaire de procédures de normalisation telles que les théorèmes limites. Signalons par ailleurs, que le coefficient d'autosimilarité H caractérise complètement la loi du mouvement brownien fractionnaire et que celui-ci gouverne la régularité de ses trajectoires (cf. Figure 1).

Les mathématiciens sont de plus en plus sollicités afin de fournir des modèles capables de rendre compte de la réalité de certains phénomènes irréguliers. Dans ce contexte, ils sont amenés à laisser de côté les outils basés sur la linéarisation comme le calcul différentiel ou la géometrie euclidienne. Car comme l'a fait remarquer Mandelbrot dans [47]: "*Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.*" Ces phénomènes qu'évoquent Mandelbrot impliquent des formes trop *brisées*, trop *rugueuses* pour que la géométrie euclidienne sache les décrire. Dans les années 70, Mandelbrot introduit le concept de *fractal*, c'est à dire des objets ayant une dimension de Hausdorff strictement supérieure à la dimension topologique, et renfermant une structure assez similaire à chaque échelle. Depuis, des structures fractales sont retrouvées cachées dans toute sorte de données. Il semblerait que la géométrie fractale soit impliquée, là où le chaos, la turbulence et le désordre sont présents. Les modèles qui conjuguent l'aléa avec l'autosimilarité sont de fait, massivement utilisés par les expérimentateurs. En penseur de la rugosité, Mandelbrot préconisait de *mesurer* afin de *modéliser*. Une façon de procéder pour quantifier à quel point une courbe varie brutalement est de comparer ses variations à celles d'une puissance fractionnaire de la norme euclidienne. Dans ce contexte de recherche, sont apparus les exposants de Hölder ponctuels. Par exemple, pour une fonction f continue n'admettant pas de dérivée, l'exposant de Hölder ponctuel en un point t peut être défini par la formule suivante:

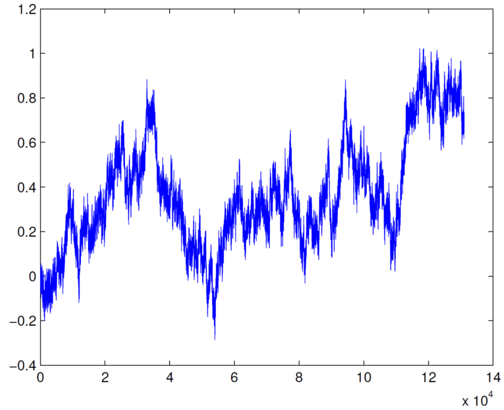
$$\alpha_f(t) = \sup\left\{\alpha \geq 0 \ : \ \limsup_{h\to 0} \frac{|f(t+h)-f(t)|}{|h|^\alpha} = 0\right\}. \tag{0.0.5}$$

Outre la popularité du mouvement brownien fractionnaire dans les modélisations, celui-ci ne parvient pas toujours à rendre compte de la réalité de certains phénomènes rugueux car celui-ci souffre des deux limitations suivantes :

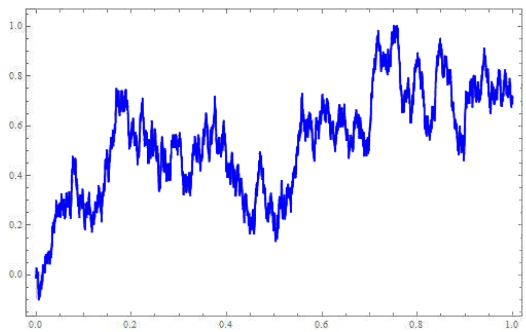
(i) C'est un processus gaussien. Or, cette hypothèse n'est pas réaliste dans plusieurs domaines comme les télécommunications, ou le traitement du signal et de l'image.

(ii) La rugosité locale, mesurée par l'exposant ponctuel de Hölder, reste la même tout le long de ses trajectoires. Autrement dit, avec probabilité 1, on a pour tout $t \in \mathbb{R}$, $\alpha_{B_H}(t) = H$. Cette propriété est essentiellement due à la stationnarité des accroissement du mouvement brownien fractionnaire.

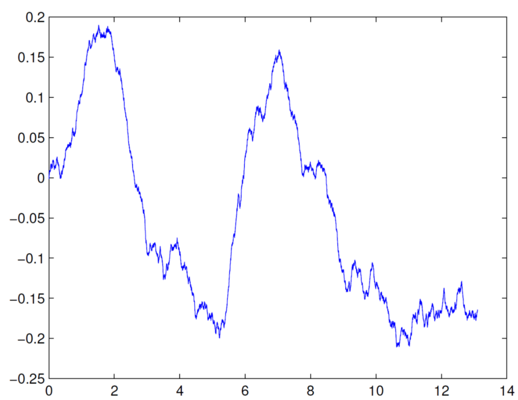
La constance de l'exposant ponctuel de Hölder est une forte limitation dans plusieurs situations. On peut citer par exemple la synthèse artificielle de montagnes, la volatilité des cours de bourse, la détection de tumeurs cancereuses à partir de l'imagerie médicale ou encore la description des traces du trafic internet. Tous ces phénomènes impliquent une rugosité variable d'un point à un autre. L'un des objectifs de cette thèse sera d'introduire des extensions non gaussiennes du



(a) H=0.3



(b) H=0.5 (Mouvement brownien)



(c) H=0.8

Figure 1: Simulations du mouvement brownien fractionnaire avec différentes valeurs du paramètre H. Plus la valeur de H augmente, plus les trajectoires deviennent lisses.

mouvement brownien fractionnaire qui outrepassent les limitations (i) et (ii). Ces extensions vivent dans ce qu'on appelle un *chaos de Wiener* d'ordre arbitraire.

Sur les chaos de Wiener

Dans le cadre des processus stochastiques non gaussiens, continues et autosimilaires, les processus chaotiques fournissent une alternative aux processus stables qui n'admettent pas de variance finie. Expliquons un peu plus ce qu'est un chaos de Wiener. Considérons un espace gaussien séparable G, défini sur un espace probabilisé $(\Omega, \mathcal{F}, \mathbb{P})$. Notons $\mathcal{F}_G$ la tribu générée par les variables aléatoires de G. Le chaos de Wiener (homogène) d'ordre d associé à G est le sous espace vectoriel fermé de $L^2(\Omega, \mathcal{F}, \mathbb{P})$ suivant

$$\mathcal{H}_n(G) := \Big\{ Q(\xi_1, \dots, \xi_d); \;\; \xi_1, \dots, \xi_d \in G, \; Q \text{ polynôme homogène d'ordre } n \Big\}.$$

Observons que $\mathcal{H}_1(G)$ n'est autre que l'espace gaussien G. Le théorème de la décomposition chaotique assure alors que

$$L^2(\Omega, \mathcal{F}_G, \mathbb{P}) = \bigoplus_{n \geq 0}^{\perp} \mathcal{H}_n(G). \tag{0.0.6}$$

Cette décomposition montre le rôle essentiel que jouent les chaos de Wiener dans la description de l'espace des variables aléatoires de carrés intégrables qui sont mesurables par rapport à la tribu $\mathcal{F}_G$. Elle illustre aussi le fait que les variables aléatoires gaussiennes ne constituent qu'un petit *fragment* des variables aléatoires, de carrés intégrables et qui sont mesurables par rapport à la tribu générée par G. Dans le cas important où l'espace gaussien G est généré par un mouvement brownien standard $\{B(t)\}_{t \in \mathbb{R}}$, autrement dit lorsque

$$G := \Big\{ \int_{\mathbb{R}} f(t) B(dt), \; f \in L^2(\mathbb{R}) \Big\},$$

le chaos de Wiener homogène d'ordre n s'identifie alors avec le sous-espace vectoriel fermé constitué des variables aléatoires qui s'écrivent comme une intégrale multiple de Wiener par rapport à une intégrande déterministe et de carré intégrable. Plus précisément,

$$\mathcal{H}_n(G) = \Big\{ I_n(f); \; f \in L^2(\mathbb{R}^n) \Big\},$$

où I_n désigne l'opérateur de l'intégrale multiple d'ordre n de Wiener. Dans ce contexte, la décomposition (0.0.6) assure que toute variable aléatoire de carré intégrable X, mesurable par rapport à $\mathcal{F}_B$, la tribu générée par le mouvement brownien, s'écrit sous la forme:

$$X = \mathbb{E}(X) + \sum_{n=1}^{+\infty} I_n(f_n), \tag{0.0.7}$$

où $f_n \in L^2(\mathbb{R}^n)$ et où la convergence de la série a lieu dans $L^2(\Omega, \mathcal{F}_B, \mathbb{P})$. De plus, la décomposition (0.0.7) est unique si l'on suppose que les fonctions f_n sont symétriques.

De façon générale, un processus stochastique appartient à un chaos de Wiener (homogène) d'ordre $n \in \mathbb{N}^*$ si ce processus peut être représenté par une intégrale (stochastique) de Wiener multiple d'ordre n ; il est non gaussien lorsque $n \geq 2$. La complexité nettement plus grande de l'intégrale de Wiener multiple par rapport à l'intégrale de Wiener classique (c'est-à-dire d'ordre 1) rend l'étude des processus chaotiques beaucoup plus complexe que celle des processus gaussiens, et même des processus stables parfois. A titre d'exemples de difficultés, on peut mentionner que la non corrélation de deux variables aléatoires d'un chaos de Wiener non gaussien n'entraîne pas forcément leur indépendance ; et que contrairement aux processus gaussiens et stables, les fonctions caractéristiques des lois fini-dimensionnelles des processus chaotiques ne sont pas données par des formules explicites et exploitables. Signalons aussi que, contrairement au chaos gaussien, les chaos d'ordre supérieur contiennent une infinité de processus qui sont autosimilaires d'un certain ordre H et à accroissements stationnaires [44].

Une autre propriété fondamentale des processus chaotiques, est leur apparition dans le cadre de généralisations non linéaires du théorème limite fonctionnel introduit précédemment. En effet, une question naturelle advient lorsqu'on observe (0.0.3). Que deviennent les objets limites lorsque l'on applique une fonction f à la suite stationnaire de variables aléatoires gaussienne $\{\xi_n\}_{n\in\mathbb{N}}$, présentant une LRD d'ordre H ? Dobrushin, Major [25] et Taqqu [69, 65] obtiennent dans les années 70, de façon indépendante et en utilisant différentes méthodes, la réponse à cette question en établissant des théorèmes limites appelés quelque fois *théorèmes limites non centraux*. Soit f une fonction de carré intégrable par rapport à la mesure gaussienne, autrement dit $f \in L^2\big(\mathbb{R}, e^{-x^2/2}dx\big)$. Rappelons que les polynômes d'Hermite $\{H_k\}_{k\in\mathbb{N}}$ forment une base orthogonale de cet espace. En conséquence, f se décompose dans $L^2\big(\mathbb{R}, e^{-x^2/2}dx\big)$ sous la forme suivante:

$$f = \sum_{k=0}^{+\infty} c_k(f) H_k. \tag{0.0.8}$$

Lorsque la fonction f n'est pas constante, on définit le rang d'Hermite comme étant le plus petit entier positif n tel que $c_n(f)$ soit non nul. Ce rang joue un rôle important dans la description des processus limites obtenus dans ces théorèmes. En effet, sous l'hypothèse que $1 - \frac{1}{2n} < H < 1$, on a la convergence au sens des lois fini-dimensionnelles suivante

$$\left\{ \frac{1}{N^{n(H-1)+1}} \sum_{k=1}^{\lfloor Nt \rfloor} \Big(f(\xi_k) - \mathbb{E}\left(f(\xi_k) \right) \Big) \right\}_{t\geq 0} \overset{f.d.d}{\rightarrow} \Big\{ c\mathcal{Z}_{H,n}(t) \Big\}_{t\geq 0},$$

pour une certaine constante $c > 0$, où le processus limite $\{\mathcal{Z}_{H,n}(t)\}_{t\geq 0}$ est appelé le *processus d'Hermite standard* d'ordre n. Ce processus vit dans un chaos de Wiener homogène d'ordre n et est défini à l'aide d'une intégrale multiple de Wiener, de la façon suivante

$$\mathcal{Z}_{H,n}(t) = c_{n,H} \int_{\mathbb{R}^n} \left\{ \int_0^t \prod_{j=1}^n (s - x_j)_+^{H-\frac{3}{2}} ds \right\} B(dx_1) \ldots B(dx_n). \tag{0.0.9}$$

Ces processus sont à trajectoires continues, autosimilaires et à accroissements stationnaires. Il a été montré en 2019 dans [10] que leurs trajectoires n'admettaient de dérivée en aucun point. Lorsque $n = 1$, le processus d'Hermite n'est autre que le mouvement brownien fractionnaire, et lorsque $n > 1$, ce processus n'est plus gaussien et appartient au chaos de Wiener homogène d'ordre n. Le plus connu parmi ces processus non gaussiens, est le processus de Rosenblatt $\{R_H(t)\}_{t\in\mathbb{R}_+}$, défini, pour tout $t \in \mathbb{R}_+$, par l'intégrale double par rapport au mouvement brownien $\{B(x)\}_{x\in\mathbb{R}}$:

$$R_H(t) := \int_{\mathbb{R}^2} K_H(t, x_1, x_2)\, B(dx_1)B(dx_2)\,, \tag{0.0.10}$$

où le paramètre $H \in]3/4, 1[$ et $K_H(t, x_1, x_2) := \int_0^t (s - x_1)_+^{H-3/2}(s - x_2)_+^{H-3/2}\, ds$, avec la convention que, pour tout $(y, \alpha) \in \mathbb{R}^2$, on a $y_+^\alpha := y^\alpha$ si $y > 0$ et $y_+^\alpha := 0$ sinon. Un des objectifs de cette thèse est d'introduire, pour une extension à deux paramètres du processus de Rosenblatt, appelée le processus de Rosenblatt généralisé, des répresentations en série aléatoire d'*ondelettes* qui sont avec probabilité 1, uniformément convergente sur tout intervalle compact.

Sur la théorie des ondelettes

Le XXème siècle a assisté à l'émergence de la théorie des ondelettes, théorie particulièrement prolifique de l'analyse mathématique. Son objectif ? Construire des bases hilbertiennes de certains espaces fonctionnels tel que $L^2(\mathbb{R}^d)$ et présentant des avantages par rapport à la classique base de Fourier. En effet, même si la base trigonométrique est bien adaptée à la résolution de nombreux problèmes issus de la physique, dont l'équation de la chaleur qui a motivé son introduction par Fourier, la convergence ponctuelle des séries correspondantes n'a pas toujours lieu. Dès 1873, le mathématicien allemand Du Bois Reymond exhibait dans [27] l'exemple d'une fonction continue dont la série de Fourier divergait à l'origine. Ce résultat avait alors beaucoup étonné ses contemporains, surtout depuis les travaux de Dirichlet. En 1909, David Hilbert adressa à son étudiant Alfred Haar le problème de savoir si ce phénomène de divergence était intrinsèque ou non à toutes les décompositions orthogonales d'un espace de Hilbert fonctionnel. Pour donner un contre exemple à cette question, Haar a construit dans sa thèse [32] une base orthonormée de $L^2([0, 1])$ connue aujourd'hui pour avoir été historiquement la première base d'ondelettes. Cette base est construite par dilatations et translations d'une unique fonction, appelée ondelette mère et qui est constante par morceaux. Haar établit alors que la suite des sommes partielles de la décomposition d'une fonction continue dans cette base était toujours uniformément convergente. Ce résultat a de quoi surprendre. En effet, une base dont les éléments sont discontinus, semble mieux adaptée à la décomposition de fonctions continues que la base trigonométrique, constituée de fonctions de classe C^∞, c'est-à-dire extêmement régulières. Il aura fallu attendre les contributions interdisciplinaires de scientifiques tels que Jean Morlet, Alex Grossmann, Stéphane Mallat, Pierre Gilles Lemarié, Yves Meyer, Jan-Olov Strömberg, Ingrid Daubechies, Stéphane Jaffard ou encore Albert Cohen, pour que le cadre général de la théorie des ondelettes soit posé, que la nature des objets qu'elle décrivait soit mieux comprise et pour que d'autres bases d'ondelettes voient le jour. Une idée importante derrière cette théorie consiste à analyser les signaux selon des niveaux de résolution et à capturer des paquets d'informations contenus localement dans une échelle donnée. Au fur et à mesure que les échelles sont passées en revue, des détails sont révélés, comme si l'on regardait au travers d'un microscope. Elle a été développée dans le domaine des mathématiques mais aussi de la physique quantique, de l'ingénierie et de la géologie sismique. Les échanges entre ces disciplines ont mené à une multitude d'applications dans des domaines très variés comme la compression d'image (JPEG2000), l'analyse de la voix humaine, la turbulence ou encore la prédiction de tremblements de terre. Signalons aussi que l'un des algorithmes qui a permis la première observation d'une onde gravitationnelle issue de la coalescence de deux trous noirs, repose sur une décomposition en ondelettes des observations [23].

Dans les années 40, Paul Levy avait déjà initié l'étude des processus stochastiques à partir d'une décomposition de type ondelette. En utilisant la base de Haar, Levy a obtenu une décomposition du mouvement brownien, uniformement convergente dans la base triangulaire de Faber-Schauder. Cette décomposition permet de simuler de façon efficiente les trajectoires du mouvement brownien, à travers la méthode qui porte le nom de *Midpoint displacement technique*. À la fin des années 90, Meyer, Sellan et Taqqu obtiennent plusieurs développements de type ondelette pour le mouvement brownien fractionnaire [49], qui généralisent la representation de Paul Levy. En 2004, Pipiras obtient à son tour plusieurs développements de type ondelette pour le processus de Rosenblatt [57], qui conduisent de façon similaire à des méthodes de simulation dans l'article [1], basées sur un algorithme pyramidal de type Mallat. En revanche, à cause des difficultés mentionnées dans la section précédente, les méthodes d'ondelettes ne se sont pas beaucoup développées dans le cadre des chaos de Wiener non gaussiens. L'un des objectifs de cette thèse est de mettre en place de nouvelles stratégies permettant de contourner ces difficultés et de développer ces méthodes.

Organisation et résultats obtenus

Ce mémoire de thèse est subdivisé en quatre chapitres. Le principal objectif du premier chapitre est de présenter de façon progressive, précise et rigoureuse le cadre scientifique dans lequel se situent les processus chaotiques qui sont étudiés dans la thèse. Etant donné que ces derniers sont des extensions du mouvement brownien fractionnaire, la première partie du chapitre concerne ce processus gaussien. La deuxième partie introduit, au moyen d'une généralisation d'un théorème limite classique qui conduit au mouvement brownien fractionnaire, l'importante classe des processus chaotiques de Hermite dont le processus de Rosenblatt fait partie. Les deux dernières parties du premier chapitre présentent des concepts et outils essentiels qui sont intimement liés à cette classe de processus, notamment l'intégrale de Wiener multiple, les chaos de Wiener et la décomposition chaotique des variables aléatoires de carré intégrable.

Les deux principaux objectifs du second chapitre sont les suivants: (a) introduire les concepts et outils de la théorie des ondelettes qui interviennent constamment et de façon cruciale dans le reste de la thèse ; (b) retracer le chemin qui va de la représentation de Meyer, Sellan et Taqqu du FBM en séries aléatoires de type ondelette, à celles du même type de Pipiras pour le processus de Rosenblatt. Signalons que dans ce chapitre, nous étendons au processus de Rosenblatt généralisé, qui sera défini par la suite, le résultat de Pipiras sur la décomposition, par une fonction d'échelle régulière et bien localisée, de la partie *basses fréquences* du processus Rosenblatt.

Nous allons maintenant présenter le chapitre 3 qui correspond en fait à l'article [6]. Comme nous l'avons déjà mentionné, Pipiras a introduit en 2004 des représentations du processus de Rosenblatt en séries aléatoires du type ondelette. Pour ce faire, il a décomposé, pour tout t fixé, la fonction noyau $(x_1, x_2) \mapsto K_H(t, x_1, x_2)$ (voir (0.0.10)) dans une base d'ondelettes classique de $L^2(\mathbb{R}^2)$, c'est-à-dire de la forme:

$$\big\{2^J\phi(2^Jx_1-k_1)\phi(2^Jx_2-k_2),\, 2^j\phi(2^jx_1-k_1)\psi(2^jx_2-k_2),\, 2^j\psi(2^jx_1-k_1)\phi(2^jx_2-k_2),\\ 2^j\psi(2^jx_1-k_1)\psi(2^jx_2-k_2) : (j,k_1,k_2)\in\mathbb{Z}^3 \text{ et } j\geq J\big\},$$

où ϕ et ψ désignent respectivement une fonction d'échelle univariée et une ondelette mère univariée correspondante qui vérifient certaines conditions, et où l'entier naturel J est arbitraire et fixé. Grâce à ces représentations du processus de Rosenblatt en séries aléatoires du type ondelette, lorsque J est suffisamment grand, $R_H(t)$ peut être approximé par $R_{H,J}(t)$ qui désigne la partie "basses fréquences" de la série ; c'est-à-dire celle associée à la suite de fonctions $\big\{2^J\phi(2^Jx_1-k_1)\phi(2^Jx_2-k_2) : (k_1,k_2)\in\mathbb{Z}^2\big\}$, autrement dit à l'espace d'approximation V_J de l'analyse multirésolution de $L^2(\mathbb{R}^2)$ sous-jacente. L'un des principaux intérêts étant que le processus $\{R_{H,J}(t)\}_{t\in\mathbb{R}_+}$ est nettement moins difficile à simuler que le processus $\{R_H(t)\}_{t\in\mathbb{R}_+}$ lui-même. Au moyen du Théorème d'Itô-Nisio, Pipiras est parvenu à montrer que, pour tout intervalle compact I, la norme uniforme par rapport à t sur cet intervalle $\|R_H-R_{H,J}\|_{I,\infty}$ converge presque sûrement vers zéro. Cependant, l'estimation de sa vitesse de convergence est restée une question ouverte ; l'une de ses difficultés majeures étant que les représentations en ondelettes du processus de Rosenblatt sont d'une forme nettement plus complexe que celles du FBM. La première motivation de l'article [6] est d'apporter une réponse à cette question dans le cadre plus large du processus de Rosenblatt généralisé $\{R_{H_1,H_2}(t)\}_{t\in\mathbb{R}_+}$ qui est défini en remplaçant dans (0.0.10) le noyau $K_H(t,x_1,x_2)$ par le noyau plus général $K_{H_1,H_2}(t,x_1,x_2)=\int_0^t(s-x_1)_+^{H_1-3/2}(s-x_2)_+^{H_2-3/2}\,ds$, où les deux paramètres H_1 et H_2 sont tels que $(H_1,H_2)\in\,]1/2,1[^2$ et $H_1+H_2>3/2$. En se plaçant sous l'hypothèse que ϕ et ψ sont une fonction d'échelle et une ondelette du type Meyer, l'article [6] prouve que l'on a presque sûrement $\|R_{H_1,H_2}-R_{H_1,H_2,J}\|_{I,\infty}=\mathcal{O}(J\,2^{-J(H_1+H_2-3/2)})$. Sa principale stratégie consiste à utiliser une base d'ondelettes non classique de $L^2(\mathbb{R}^2)$ qui est de la forme:

$$\big\{2^{(j_1+j_2)/2}\psi(2^{j_1}x_1-k_1)\psi(2^{j_2}x_2-k_2) : (j_1,j_2,k_1,k_2)\in\mathbb{Z}^4\big\},$$

et de montrer que de façon générale l'erreur d'approximation associée à un espace V_J d'une analyse multirésolution peut s'exprimer au moyen de cette base ; ce qui pourrait aussi avoir de l'intérêt en lui-même et dans d'autres contextes. Une autre contribution significative de l'article [6] est d'introduire pour le processus de Rosenblatt généralisé des représentations en séries aléatoires du type ondelette qui se distinguent de celles précédemment introduites par Pipiras dans le cadre moins général du processus de Rosenblatt. L'article [6] parvient même à établir la convergence presque sûre de ces séries, uniformément en $t\in I$ et conjointement en les 4 indices j_1, j_2, k_1 et k_2, en donnant une estimation de la vitesse de convergence. Un tel résultat de convergence conjointement en tous les indices n'avait pas été obtenu par Pipiras.

Nous allons enfin présenter le chapitre 4 qui correspond en fait à l'article [5]. Dans cet article, nous introduisons d'abord une classe de processus stochastiques appartenant à un chaos de Wiener (homogène) non gaussien, qui ne sont pas auto-similaires et dont les accroissements sont non stationnaires: le Mouvement Multifractionnaire Chaotique $\{Z(t)\}_{t\in\mathbb{R}}$ dont les trajectoires sont continues. Ce processus consiste en une extension naturelle du mouvement brownien fractionnaire. Il est défini, pour tout $t \in \mathbb{R}$, par l'intégrale de Wiener multiple:

$$Z(t) := \int_{\mathbb{R}^n} \left\{ \Big(\sum_{l=1}^{n} (t - x_l)^2 \Big)^{\frac{h(t)-n/2}{2}} - \Big(\sum_{l=1}^{n} x_l^2 \Big)^{\frac{h(t)-n/2}{2}} \right\} B(dx_1) \dots B(dx_n) \,,$$

où h est une fonction déterministe continue de $\mathbb{R}$ dans $]0,1[$. Nous obtenons ensuite une représentation en série aléatoire d'ondelettes de $\{Z(t)\}_{t\in\mathbb{R}}$, qui converge presque sûrement uniformément en t sur tout intervalle compact. Lorsque h est suffisamment régulière, nous obtenons, au moyen de cette représentation en ondelettes, de fins modules de continuités globaux et locaux du processus $\{Z(t)\}_{t\in\mathbb{R}}$, ainsi qu'un fin encadrement de son comportement à l'infini. Enfin, nous montrons que le mouvement multifractionnaire chaotique satisfait une propriété d'autosimilarité asymptotique locale, similaire à celle satisfaite par le mouvement brownien multifractionnaire.

Chapter 1

Self-similar chaotic processes with stationary increments

The aim of this chapter is to present in a progressive, precise and rigorous way the scientific framework in which the chaotic processes which are studied in the thesis, are situated. Since these are extensions of the fractional Brownian motion (FBM), the first part of the chapter concerns this Gaussian process. The second part introduces, by means of a generalization of a classical limit theorem which leads to FBM, the important class of chaotic Hermite processes of which the Rosenblatt process is a part. The last two parts of the chapter present essential concepts and tools which are intimately linked to this class of processes, especially the multiple Wiener integral, Wiener chaoses and the orthogonal chaotic decomposition of square integrable random variables.

1.1 Definitions and general properties

Definition 1.1.1. A stochastic process $\{X(t)\}_{t\in\mathbb{R}}$ is said to be *self-similar* (SS) or *H-self-similar* (H-SS) if there is $H > 0$ such that, for all $a > 0$,

$$\big\{X(at)\big\}_{t\in\mathbb{R}} \overset{d}{=} \big\{a^H X(t)\big\}_{t\in\mathbb{R}}, \tag{1.1.1}$$

where $\overset{d}{=}$ denotes equality of the finite-dimensional distributions.

Remark 1.1.2. Let $a > 0$ be arbitrary and fixed. Combining the equality $a.0 = 0$ with (1.1.1), one gets that the characteristic function $\phi_{X(0)}$ of the random variable $X(0)$ satisfies, for all $\xi \in \mathbb{R}$ and $a > 0$ $\phi_{X(0)}(\xi) = \phi_{X(0)}(a^H \xi)$; By letting a goes to 0 and by using the continuity property of the characteristic function, one derives that for all $\xi \in \mathbb{R}$, $\phi_{X(0)}(\xi) = 1$, wich entails that

$$X(0) = 0 \quad a.s. \tag{1.1.2}$$

Example 1.1.3. Let C be an arbitrary random variable. Then, it is easy to see that the stochastic process $\{t^\gamma C\}_{t\geq 0}$ is γ-SS. This trivial example shows that self-similarity does not provide any information on the finite-dimensional distributions of SS processes since the law of the random variable C can be chosen in an arbitrary way.

Example 1.1.4. Let $\{B(t)\}_{t\in\mathbb{R}_+}$ be a standard Brownian motion on the half line $\mathbb{R}_+$. Recall that it is defined as the unique (in law) centered Gaussian process whose covariance function is given by

$$\mathbb{E}\big(B(t)B(s)\big) = \min(t,s), \quad \text{for all } \ t,s \in \mathbb{R}_+. \tag{1.1.3}$$

The *two-sided* Brownian motion $\{B(t)\}_{t\in\mathbb{R}}$ is defined as $B(t) = B_1(t), t \geq 0$ and $B(t) = B_2(-t), t < 0$ where B_1 and B_2 are two independent Brownian motions on the positive half line. It is easy to check that for all $a > 0$, $\{B(at)\}_{t\in\mathbb{R}}$ and $\{a^{\frac{1}{2}}B(t)\}_{t\in\mathbb{R}}$ are two centered Gaussian processes having the same covariance function. Therefore, Brownian motion is an 1/2-SS process.

Self-similar processes appear as limits in various normalisation procedures [61, 41, 66]. For instance, one can mention the functional extension of the central limit theorem (Donsker's theorem) in which the limit process is Brownian motion. More generally, the following fundamental limit theorem was obtained by J.Lamperti in [41]. It shows that the class of SS processes can be identified with the class of processes obtained as limits in normalisation procedures. ,

Theorem 1.1.5. *Let Y be a real-valued stochastic process. Assume that there exists a stochastic process X such that the random variable $X(t)$ is non-degenerate for each $t > 0$ and that there is a Borel deterministic function f satisfying $\lim_{\xi\to+\infty} f(\xi) = +\infty$ and such that the following limit holds*

$$\left\{\frac{Y(\xi t)}{f(\xi)}\right\}_{t\in\mathbb{R}_+} \xrightarrow{f.d.d.} \{X(t)\}_{t\in\mathbb{R}_+} \ \text{as } \xi \to +\infty, \tag{1.1.4}$$

then there is $H \geq 0$ such that

$$f(\xi) = \xi^H L(\xi), \tag{1.1.5}$$

where L is a slowly varying function at infinity. Moreover, the process $\{X(t)\}_{t\geq 0}$ is $H-SS$. Conversely, if $\{X(t)\}_{t\geq 0}$ is $H-SS$ then (1.1.4) becomes an equality with $X = Y$ and $f(\xi) = \xi^H$.

Remark 1.1.6. Recall that a function L is said to be *slowly varying at infinity* if it is positive on $[c, +\infty)$ some $c \geq 0$ and one has, for any fixed $a > 0$,

$$\lim_{\xi\to+\infty} \frac{L(a\xi)}{L(\xi)} = 1. \tag{1.1.6}$$

For example, the functions $L(\xi) = \textit{constant} > 0$ and $L(\xi) = \log(\xi)$ are slowly varying at infinity.

Definition 1.1.7. A stochastic process $\{X(t)\}_{t\in\mathbb{R}}$ is said to have *stationary increments* (SI) if, for any fixed $t_0 \in \mathbb{R}$,

$$\big\{X(t+t_0) - X(t_0)\big\}_{t\in\mathbb{R}} \overset{d}{=} \big\{X(t) - X(0)\big\}_{t\in\mathbb{R}}. \tag{1.1.7}$$

Definition 1.1.8. An H-self-similar process with stationary increments will be referred to as SSSI or H-SSSI.

Proposition 1.1.9. Let $\{X(t)\}_{t\in\mathbb{R}}$ be an H-SSSI stochastic process. Then, X satisfies the following properties for all $s,t \in \mathbb{R}$:

(i) If $H \neq 1$ and $\mathbb{E}\,|X(1)| < +\infty$ then $\mathbb{E}\left(X(t)\right) = 0$.

(ii) If $\mathbb{E}\,|X(1)|^p < +\infty$ then $\mathbb{E}\left(\left|X(t) - X(s)|^p\right) = \mathbb{E}\left(|X(1)|^p\right)\left|t - s\right|^{pH}$.

(iii) If $\mathbb{E}\,|X(1)|^2 < +\infty$ then $\mathbb{E}\left(X(t)X(s)\right) = \frac{1}{2}\,\mathbb{E}\left(X(1)^2\right)\Big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big)$.

Proof. Using self-similarity, stationarity of the increments and (1.1.2), one gets for all $t \in \mathbb{R}$

$$2^H\,\mathbb{E}\,X(t) = \mathbb{E}\,X(2t) = \mathbb{E}(X(2t) - X(t)) + \mathbb{E}\,X(t) = \mathbb{E}(X(t) - X(0)) + \mathbb{E}\,X(t) = 2\,\mathbb{E}\,X(t)$$

which means that $\mathbb{E}\,X(t) = 0$ if $H \neq 1$. Next, suppose $\mathbb{E}\,|X(1)|^p < +\infty$ and consider $t \geq s$ two real numbers. Using the same arguments, one obtains

$$\mathbb{E}\left(\left|X(t) - X(s)|^p\right) = \mathbb{E}\left(\left|X(t-s)|^p\right) = \mathbb{E}\left(\left|(t-s)^H X(1)|^p\right) = |t-s|^{pH}\,\mathbb{E}\left(\left|X(1)|^p\right).$$

The property (iii) is a straightforward consequence of (ii) with $p = 2$ and the equality $ab = 2^{-1}\big(a^2 + b^2 - (a-b)^2\big)$. □

Proposition 1.1.10. Let $\{X(t)\}_{t\in\mathbb{R}}$ be an H-SSSI stochastic process. Assume that $\mathbb{E}\,|X(1)|^2 < +\infty$ and $H \in (1/2, 1)$. Then the increments of X display a *long-range dependence* in the following sense:

$$\sum_{n=0}^{+\infty} \left|\mathbb{E}\Big(\big(X(n+1) - X(n)\big)\big(X(1) - X(0)\big)\Big)\right| = +\infty. \tag{1.1.8}$$

Proof. Using self-similarity of X, one has almost surely $X(0) = 0$. Thus, for all $n \in \mathbb{N}^*$, one gets:

$$\begin{aligned}
\mathbb{E}\Big(\big(X(n+1) - X(n)\big)\big(X(1) - X(0)\big)\Big) &= \mathbb{E}\Big(X(1)\big(X(n+1) - X(n)\big)\Big) \\
&= 2^{-1}\,\mathbb{E}\left(X(1)^2\right)\left((n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\right) \\
&= 2^{-1}\,\mathbb{E}\left(X(1)^2\right)n^{2H}\left((1+n^{-1})^{2H} + (1-n^{-1})^{2H} - 2\right) \\
&= 2\,\mathbb{E}\left(X(1)^2\right)H(2H-1)n^{2H-2} + \underset{n\to+\infty}{o}(n^{2H-2}).
\end{aligned}$$

Therefore, the series in (1.1.8) diverges when $H > 1/2$. □

Remark 1.1.11. *Long-range dependance* (LRD) is also called *long memory* or *strong dependance*. There exist various definitions of LRD in the literature (see e.g. [58]), not all equivalent, and used for different contexts and purposes. In our context, we will use the following definition.

Definition 1.1.12. One says that a second-order stationary time series $X = \{X_n\}_{n\in\mathbb{Z}}$ has LRD of *Hurst index* $H \in (1/2, 1)$ if its autocovariance function satisfies for all $n \in \mathbb{Z}$

$$\gamma_X(n) := \mathbb{E}(X_n X_0) - \mu_X^2 = L(|n|)|n|^{2H-2}, \tag{1.1.9}$$

where μ_X denotes the constant mean of X and L is a *slowly varying function at infinity*.

Proposition 1.1.13. Let $\{X(t)\}_{t\in\mathbb{R}}$ be an H-SSSI stochastic process having finite moments of any order. Then there exists a modification of X whose trajectories satisfy, with probability 1, a local Hölder condition of any order $\gamma \in (0,H)$ on $\mathbb{R}$. More precisely, for all compact interval $K \subset \mathbb{R}$, there is an event Ω^*, of probability 1 such that for all $\omega \in \Omega^*$, all $\gamma \in (0,H)$ and all $t_1, t_2 \in K$

$$\big|X(t_1,\omega) - X(t_2,\omega)\big| \le C(\omega)|t_1 - t_2|^\gamma, \tag{1.1.10}$$

where C is a positive and finite random variable only depending on K and γ.

Proof. Fix $p > 1/H$. The property (ii) in Proposition 1.1.9 entails that for all $t, s \in \mathbb{R}$, one has

$$\mathbb{E}\left(\big|X(t) - X(s)|^p\right) = c_{p,H}|t-s|^{pH} \tag{1.1.11}$$

where $c_{p,H} := \mathbb{E}\left(|X(1)|^p\right)$. Then, by using the so-called Kolmogorov Centsov's theorem stated below, one obtains the existence of a modification of X whose trajectories are almost surely locally Hölder continuous on $\mathbb{R}$ of any order $\gamma \in \big[0, H - \frac{1}{p}\big)$. Since p can be arbitrary big, the result follows. □

The following theorem is an extension of the well-known Kolmogorov's continuity theorem which draws a fundamental connection between the pathwise Hölder regularity of a stochastic process and the moments of its increments. A proof of it can be found in [37].

Theorem 1.1.14. ***(Kolmogorov-Centsov's Theorem)*** *Let $\{X(t)\}_{t\in\mathbb{R}}$ be a stochastic process which satisfies, for some constants $\delta > 0$ and $\varepsilon > 0$ the following property: for any fixed $T > 0$, there exists a constant $c(T) > 0$ such that the ineqality*

$$\mathbb{E}\,|X(t) - X(s)|^\delta \le c(T)|t-s|^{1+\varepsilon}, \tag{1.1.12}$$

holds for all $t, s \in [-T, T]$. Then X has a modification whose trajectories are almost surely locally Hölder continuous on $\mathbb{R}$ of any order $\gamma \in \big[0, \varepsilon/\delta\big)$.

1.2 Fractional Brownian motion (FBM)

Fractional Brownian motion (FBM) is a Gaussian SSSI process which was first introduced in 1940 by Kolmogorov [39] as a way to generate Gaussian *spirals* in Hilbert spaces. It was implicitly considered in different frameworks by Hunt [33] in 1951, Yaglom [73] in 1958 and by Lamperti [41] in 1962. However, it was only in 1968, in Mandelbrot and Van Ness's famous paper [46], that the focus was put on FBM in its own right as a very useful mathematical model and several of its properties were derived by the authors. The name of this stochastic process comes from one of its representation as a *fractional stochastic integral* with respect to a standard Brownian motion. Since then, it has become a powerful model in many areas such as finance, hydrology, geology, telecommunications and so on.

1.2.1 Existence of FBM and paths behavior

Recall that the distribution of a centered Gaussian process is completely determined by its autocovariance function. Therefore, in order to show the existence of FBM, we only need to establish that the function in Proposition 1.1.9 (iii) is a valid autocovariance function, which is the object of the following lemma.

Lemma 1.2.1. Let $H \in (0,1)$. Then, the symmetric function

$$R_H(t,s) = |t|^{2H} + |s|^{2H} - |t-s|^{2H} \quad s,t \in \mathbb{R}$$

is positive semidefinite.

Proof. Let $t_1, \dots, t_n$ and $u_1, \dots, u_n$ be arbitrary real numbers. We want to show that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} R_H(t_i,t_j)u_iu_j \geq 0.$$

For each $i \in \{1, \cdots, n\}$, we consider u_i as a mass at the point t_i and we add a mass $u_0 = -\sum_{i=1}^{n} u_i$ at the origin $t_0 = 0$. Observe that

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} |t_i|^{2H}u_iu_j &= \sum_{i=1}^{n} |t_i|^{2H}u_i \sum_{j=1}^{n} u_j = -\sum_{i=1}^{n} |t_i|^{2H}u_iu_0 \\ &= -\sum_{i=0}^{n} |t_i - t_0|^{2H}u_iu_0.\end{aligned}$$

Similarily, one gets

$$\sum_{i=1}^{n}\sum_{j=1}^{n} |t_j|^{2H}u_iu_j = -\sum_{j=0}^{n} |t_j - t_0|^{2H}u_ju_0.$$

Therefore

$$\sum_{i=1}^{n}\sum_{j=1}^{n} R_H(t_i,t_j)u_iu_j = \sum_{i=1}^{n}\sum_{j=1}^{n} \big(|t_i|^{2H} + |t_j|^{2H} - |t_i - t_j|^{2H}\big)u_iu_j = -\sum_{i=0}^{n}\sum_{j=0}^{n} |t_i - t_j|^{2H}u_iu_j.$$

Next, we will use the fact that for all $c > 0$ the function $t \mapsto \exp\big(-c|t|^{2H}\big)$ is the characteristic function of a symmetric $2H$-stable random variable X with the scaling exponent $c^{1/2H}$. It follows that

$$\sum_{j=0}^{n}\sum_{k=0}^{n} \exp\big(-c|t_j - t_k|^{2H}\big)u_ju_k = \mathbb{E}\left(\sum_{j=0}^{n}\sum_{k=0}^{n} u_ju_k \exp\big(i(t_j - t_k)\xi\big)\right) = \mathbb{E}\left|\sum_{j=0}^{n} u_j \exp(it_j\xi)\right|^2 \geq 0.$$

The result follows by observing that as c tends to 0, one has

$$\sum_{j=0}^{n}\sum_{k=0}^{n} \exp\big(-c|t_j - t_k|^{2H}\big)u_ju_k = \sum_{j=0}^{n}\sum_{k=0}^{n} \left(\exp\big(-c|t_j - t_k|^{2H}\big) - 1\right)u_ju_k = -c\sum_{j=0}^{n}\sum_{k=0}^{n} |t_j - t_k|^{2H}u_ju_k + o(c).$$

$\square$

Definition 1.2.2. Let $H \in (0,1)$. A real valued centered Gaussian process $\{B_H(t)\}_{t\in\mathbb{R}}$ is called *Fractional Brownian Motion* (FBM) of Hurst parameter $H \in (0,1)$ if for all $t, s \in \mathbb{R}$:

$$\mathbb{E}\left(B_H(t)B_H(s)\right) = \frac{1}{2}\,\mathbb{E}\left(B_H(1)^2\right)\Big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big). \tag{1.2.1}$$

It is called *standard* when $\mathbb{E}\left(B_H(1)^2\right) = 1$. Observe that $B_{1/2}$ is Brownian motion.

Proposition 1.2.3. FBM of Hurst parameter H is, up to a multiplicative constant, the unique centered Gaussian H-SSSI process.

Proof. The uniqueness of the law is given by the mean and autocovariance functions that characterize the distribution of Gaussian processes. Let B_H be a FBM. To show stationarity of the increments and H-self-similarity, it is enough to prove that for all fixed $t_0 \in \mathbb{R}$ and $a > 0$, one has for all $s, t \in \mathbb{R}$

$$\mathbb{E}\Bigg(\big(B_H(t+t_0) - B_H(t_0)\big)\big(B_H(s+t_0) - B_H(t_0)\big)\Bigg) = \mathbb{E}\Bigg(\big(B_H(t) - B_H(0)\big)\big(B_H(s) - B_H(0)\big)\Bigg)$$

and

$$\mathbb{E}\left(B_H(at)B_H(as)\right) = \mathbb{E}\left(a^H B_H(t) a^H B_H(s)\right)$$

which is easy to verify. □

Let us focus now on some fundamental properties of sample paths of FBM. Notice that combining Proposition 1.2.3 and Proposition 1.1.13 with the fact that Gaussian processes have finite moments of any order, one obtains the existence of a modification of FBM whose sample paths on $\mathbb{R}$ satisfy, with probability 1, a local Hölder condition of any order $\gamma \in (0, H)$. In all the sequel, FBM will be identified with its Hölder continuous modification.
The following result provides a useful *time inversion formula* for FBM.

Proposition 1.2.4. Let B_H be a FBM and denote by X_H the stochastic process defined for all $t \in \mathbb{R}$ by

$$X_H(t) = \begin{cases} 0 & \text{if} \quad t = 0 \\ |t|^{2H} B_H(\frac{1}{t}) & \text{if} \quad t \neq 0, \end{cases} \tag{1.2.2}$$

then X_H is a continuous FBM of Hurst parameter H.

Proof. The processes X_H and B_H are two centered Gaussian processes having the same covariance function. Therefore, one only has to show that X_H is continuous. It is clear that X_H is continuous on $\mathbb{R}^*$. Next, using continuity of B_H at 0, one gets

$$\begin{aligned}
\mathbb{P}\big(X_H(t) \underset{t\to 0}{\longrightarrow} 0\big) &= \mathbb{P}\Bigg(\bigcap_{n\geq 1}\bigcup_{p\geq 1}\bigcap_{|t|\in(0,1/p]\cap\mathbb{Q}} |X_H(t)| \leq \frac{1}{n}\Bigg) \\
&= \mathbb{P}\Bigg(\bigcap_{n\geq 1}\bigcup_{p\geq 1}\bigcap_{|t|\in(0,1/p]\cap\mathbb{Q}} |B_H(t)| \leq \frac{1}{n}\Bigg) \\
&= \mathbb{P}\big(B_H(t) \underset{t\to 0}{\longrightarrow} 0\big) \\
&= 1,
\end{aligned}$$

and the result follows. □

The next proposition provides, almost surely, a sharp estimate of the global modulus of continuity of FBM.

Proposition 1.2.5. Let $H \in (0,1)$. One has

$$\limsup_{h \to 0^+} \sup_{0 \le t \le 1-h} \frac{|B_H(t+h) - B_H(t)|}{\sqrt{2h^{2H}\log(h^{-1})}} = 1 \quad a.s. \tag{1.2.3}$$

The asymptotic behavior of FBM is given by the following proposition.

Proposition 1.2.6. Let $H \in (0,1)$. One has

$$\limsup_{t \to +\infty} \frac{|B_H(t)|}{\sqrt{2t^{2H}\log\big(\log(t)\big)}} = 1 \quad a.s. \tag{1.2.4}$$

Using stationarity of the increments, the inversion time formula and Proposition 1.2.6, one can deduce a sharp estimate of the local modulus of continuity of FBM.

Proposition 1.2.7. Let $H \in (0,1)$. One has for all $t \in \mathbb{R}_+$

$$\limsup_{h \to 0^+} \frac{|B_H(t+h) - B_H(t)|}{\sqrt{2h^{2H}\log\big(\log(h^{-1})\big)}} = 1 \quad a.s. \tag{1.2.5}$$

Observe that (1.2.5) holds on an event of probability 1 which depends on t.

The next corollary is a straightforward consequence of Proposition 1.2.7.

Corollary 1.2.8. With probability 1, the trajectories of B_H fail to satisfy a uniform Hölder condition of any order $\gamma > H$ on any interval I with non empty interior.

Nowhere differentiability of Brownian paths was first established by Paley, Wiener and Zygmund [53]. Later, Kawana and Kono derived in [38] nowhere differentiability of sample paths for a class of Gaussian processes that includes FBM.

Theorem 1.2.9. *With probability* 1, *the paths of FBM are nowhere differentiable.*

1.2.2 Some representations of FBM

There exist many explicit representations of FBM. One gives two of them using the Wiener integral.

Proposition 1.2.10. Let $H \in (0,1)$. FBM have the following two integral representations:

(i) *Well-balanced* representation

$$\big\{B_H(t)\big\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{ \int_{\mathbb{R}} \big(|t-u|^{H-1/2} - |u|^{H-1/2}\big) B(du) \right\}_{t\in\mathbb{R}} \tag{1.2.6}$$

with the convention that $|t-u|^0 - |u|^0 = 1_{[0,t)}(u)$ if $t \ge 0$ and $|t-u|^0 - |u|^0 = -1_{(t,0]}(u)$ else.

(ii) *Non-anticipative* or *causal* representation

$$\big\{B_H(t)\big\}_{t\in\mathbb{R}} \overset{d}{=} \bigg\{\int_{\mathbb{R}} \big((t-u)_+^{H-1/2} - (u)_+^{H-1/2}\big) B(du)\bigg\}_{t\in\mathbb{R}} \tag{1.2.7}$$

where $y_+^\alpha = y^\alpha$ for $y \geq 0$ and $y_+^\alpha = 0$ else.

Proof. One only shows (1.2.6) since (1.2.7) can be obtained in a rather similar way. One denotes by X the Gaussian process considered in (1.2.6). In view of Proposition 1.2.3, for showing that X is a FBM, it is enough to prove that it is an H-SSSI process. It follows from the isometry property of the Wiener integral that for all, $t_1, t_2 \in \mathbb{R}$, one has

$$\mathbb{E}(X(t_1)X(t_2)) = \int_{\mathbb{R}} \left(|t_1 - s|^{H-1/2} - |s|^{H-1/2}\right)\left(|t_2 - s|^{H-1/2} - |s|^{H-1/2}\right) ds. \tag{1.2.8}$$

Fix $t_0 \in \mathbb{R}$ and $a > 0$ and use respectively the changes of variable $r = s/a$ and $u = s + t_0$ in (1.2.8), to get for all $t_1, t_2 \in \mathbb{R}$

$$\mathbb{E}\left(X(at_1)X(at_2)\right) = \mathbb{E}\left(a^H X(t_1) a^H X(at_2)\right). \tag{1.2.9}$$

and

$$\mathbb{E}\Big(\big(X(t_1+t_0) - X(t_0)\big)\big(X(t_2+t_0) - X(t_0)\big)\Big) = \mathbb{E}\left(X(t_1)X(t_2)\right). \tag{1.2.10}$$

Thus, combining (1.2.9) and (1.2.10) with the Gaussianity of X, one gets the equality in (1.2.6). $\square$

Remark 1.2.11. Observe that when $H \in (1/2, 1)$, the representation (1.2.7) can be expressed, up to a multiplicative constant, as

$$\big\{B_H(t)\big\}_{t\in\mathbb{R}} \overset{d}{=} \bigg\{\int_{\mathbb{R}} \bigg(\int_0^t (s-u)_+^{H-3/2} ds\bigg) B(du)\bigg\}_{t\in\mathbb{R}}. \tag{1.2.11}$$

1.2.3 On a Central Limit Theorem

Let us end this section dedicated to FBM by stating a central-limit theorem in which FBM appears as the limit process. It is a counterpart to Donsker's theorem in the sense that the underlying random variables are no longer assumed to be independent, but rather *strongly dependent* (LRD). It can also be seen as another illustration of Theorem 1.1.5.

Theorem 1.2.12. *Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a centered Gaussian LRD time series (in the sense of Definition 1.1.12) with Hurst index $H \in (1/2, 1)$. Then, as $N \to +\infty$,*

$$\left\{\frac{1}{N^H L(N)^{1/2}} \sum_{n=1}^{\lfloor Nt \rfloor} \xi_n\right\}_{t\geq 0} \xrightarrow{f.d.d} \big\{cB_H(t)\big\}_{t\geq 0}, \tag{1.2.12}$$

for some constant $c > 0$, where L is the same slowly varying function at infinity as in Definition 1.1.12.

Remark 1.2.13. The assumption of Gaussianity in the previous theorem can be weakened. Theorem 1.2.12 can be extended to the case of linear series defined by

$$\xi_n = \sum_{j=-\infty}^{+\infty} c_j \varepsilon_{n-j}$$

with i.i.d random variables $\varepsilon_n, n \in \mathbb{Z}$, with zero mean and unit variance and $\sum_{j\in\mathbb{Z}} |c_j|^2 < +\infty$. The limit process remains FBM (see Proposition 2.8.8 in [58]).

In order to prove Theorem 1.2.12, one needs the following result, known as Karamata's theorem.

Lemma 1.2.14. Let L be a slowly varying function at infinity and $p > -1$. Then, for

$$c_k = L(k)k^p, \quad k \geq 1,$$

one has that

$$\sum_{k=1}^{n} c_k \underset{+\infty}{\sim} \frac{L(n)n^{p+1}}{p+1}.$$

Proof of Theorem 1.2.12. One denotes by ξ^N the normalized partial sum process, so that for all $t \geq 0$

$$\xi^N(t) := \frac{1}{N^H L(N)^{1/2}} \sum_{n=1}^{\lfloor Nt \rfloor} \xi_n.$$

Since the process ξ^N is Gaussian and centered, it is enough to show that for all $t > s > 0$

$$\mathbb{E}\left(\xi^N(t)\xi^N(s)\right) \underset{N\to+\infty}{\longrightarrow} \mathbb{E}\left(B_H(t)B_H(s)\right). \tag{1.2.13}$$

Observe that one has

$$\begin{aligned}
\mathbb{E}\left(\xi^N(t)\xi^N(s)\right) &= \frac{1}{L(N)N^{2H}}\,\mathbb{E}\left(\sum_{n=1}^{\lfloor Nt \rfloor}\xi_n \sum_{n=1}^{\lfloor Ns \rfloor}\xi_n\right)\\
&= \frac{1}{2L(N)N^{2H}}\left\{\mathbb{E}\left(\sum_{n=1}^{\lfloor Nt \rfloor}\xi_n\right)^2 + \mathbb{E}\left(\sum_{n=1}^{\lfloor Ns \rfloor}\xi_n\right)^2 - \mathbb{E}\left(\sum_{n=\lfloor Ns \rfloor+1}^{\lfloor Nt \rfloor}\xi_n\right)^2\right\}\\
&= \frac{1}{2L(N)N^{2H}}\left\{\mathbb{E}\left(\sum_{n=1}^{\lfloor Nt \rfloor}\xi_n\right)^2 + \mathbb{E}\left(\sum_{n=1}^{\lfloor Ns \rfloor}\xi_n\right)^2 - \mathbb{E}\left(\sum_{n=1}^{\lfloor Nt \rfloor-\lfloor Ns \rfloor}\xi_n\right)^2\right\}. \qquad (1.2.14)
\end{aligned}$$

We want to estimate the quantity $\mathrm{Var}(\xi_1 + \cdots + \xi_N)$ as $N \to +\infty$. Observe that using $\gamma_\xi(h) =$

$\gamma_\xi(-h)$ for all $h \in \mathbb{Z}_+$ and Lemma 1.2.14, one gets

$$\begin{aligned}
\mathrm{Var}(\xi_1 + \cdots + \xi_N) &= \sum_{j,k=1}^{N} \gamma_\xi(j-k) = \sum_{h=-(N-1)}^{N-1} (N - |h|)\gamma_\xi(h) \\
&= N \sum_{h=-(N-1)}^{N-1} \gamma_\xi(h) - \sum_{h=-(N-1)}^{N-1} |h|\gamma_\xi(h) \\
&= N\gamma_\xi(0) + 2N \sum_{n=1}^{N-1} L(n)n^{2H-2} - 2\sum_{n=1}^{N-1} L(n)n^{2H-1} \\
&\sim \frac{L(N)N^{2H}}{H(2H-1)}. \qquad (1.2.15)
\end{aligned}$$

Thus, combining (1.2.14) and (1.2.15) one obtains (1.2.13) since for example the first term in (1.2.14) satisfies

$$\frac{1}{2L(N)N^{2H}} \mathbb{E}\Big(\sum_{n=1}^{\lfloor Nt \rfloor} \xi_n\Big)^2 = \frac{\mathrm{Var}(\xi_1 + \cdots + \xi_{\lfloor Nt \rfloor})}{2L(\lfloor Nt \rfloor)\lfloor Nt \rfloor^{2H}} \frac{L(\lfloor Nt \rfloor)}{L(N)} \Big(\frac{\lfloor Nt \rfloor}{N}\Big)^{2H} \sim \frac{1}{2}H(2H-1)t^{2H},$$

as N goes to infinity and the two other terms in (1.2.14) satisfy similar results. □

1.3 Hermite processes and multiple Wiener integral

1.3.1 On a Non-Central Limit Theorem

Definition 1.3.1. The nth *Hermite polynomial* is defined by

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n(e^{-x^2/2})}{dx^n}, \quad n \geq 1 \qquad (1.3.1)$$

and $H_0(x) = 1$. For example, the five first Hermite polynomials are given by

$$H_0(x) = 1,\ H_1(x) = x,\ H_2(x) = x^2 - 1,\ H_3(x) = x^3 - 3x,\ H_4(x) = x^4 - 6x^2 + 3.$$

Remark 1.3.2. One very handy property of Hermite polynomials is that they appear in the expansion in powers of t of the function $F(x,t) = \exp(tx - t^2/2)$. More precisely, by using the Taylor's expansion of $\exp\Big(-\frac{1}{2}(x-t)^2\Big)$ in t, one gets for all $x, t \in \mathbb{R}$:

$$\begin{aligned}
F(x,t) &= \exp(tx - t^2/2) \\
&= \exp\Big(\frac{x^2}{2} - \frac{1}{2}(x-t)^2\Big) \\
&= \exp(x^2/2) \sum_{n=0}^{+\infty} \frac{t^n}{n!} \frac{\partial(e^{-(x-t)^2/2})}{\partial t^n}\Big|_{t=0} \\
&= \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n(x), \qquad (1.3.2)
\end{aligned}$$

where we used the change of variable $y = x - t$ to obtain

$$\frac{\partial^n (e^{-(x-t)^2/2})}{\partial t^n}\bigg|_{t=0} = (-1)^n \frac{d^n (e^{-y^2/2})}{dy^n}\bigg|_{y=x} = (-1)^n \frac{d^n (e^{-x^2/2})}{dx^n} = e^{-x^2/2} H_n(x).$$

Furthermore, by observing that

$$\frac{\partial F(x,t)}{\partial t} = (x-t)F(x,t) \quad \text{and} \quad \frac{\partial F(x,t)}{\partial x} = tF(x,t) \quad \text{for all} \quad x, t \in \mathbb{R}$$

and by using (1.3.2) one can easily derive the following two useful properties

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x), \quad n \geq 1 \tag{1.3.3}$$

and

$$H_n'(x) = nH_{n-1}(x), \quad n \geq 1. \tag{1.3.4}$$

The link between Hermite polynomials and Gaussian random variables is explained by the following proposition.

Proposition 1.3.3. Let (ξ_1, ξ_2) be a standard Gaussian vector. Then, for all $p, q \in \mathbb{Z}_+$, one has:

$$\mathbb{E}\Big(H_p(\xi_1)H_q(\xi_2)\Big) = \begin{cases} q!\big(\mathbb{E}(\xi_1\xi_2)\big)^q & \text{if} \quad p = q \\ 0 & \text{else.} \end{cases} \tag{1.3.5}$$

Proof. Recall that the moment generating function of a centered Gaussian random variable ξ is given by

$$\mathbb{E}\big(\exp(u\xi)\big) = \exp\Big(\frac{1}{2}\mathbb{E}\big((u\xi)^2\big)\Big), \quad \text{for all } u \in \mathbb{R}. \tag{1.3.6}$$

Therefore, applying (1.3.6) to $\xi = s\xi_1 + t\xi_2$ with $u = 1$, one gets for all $s, t \in \mathbb{R}$

$$\begin{aligned}\mathbb{E}\big(F(\xi_1, s)F(\xi_2, t)\big) &= \mathbb{E}\Bigg(\exp\Big(s\xi_1 - \frac{s^2}{2}\Big)\exp\Big(t\xi_2 - \frac{t^2}{2}\Big)\Bigg) = \exp\Big(-\frac{t^2}{2} - \frac{s^2}{2}\Big)\mathbb{E}\big(\exp(s\xi_1 + t\xi_2)\big) \\ &= \exp\Big(-\frac{t^2}{2} - \frac{s^2}{2}\Big)\exp\Big(\frac{1}{2}\mathbb{E}\big((s\xi_1 + t\xi_2)^2\big)\Big) = \exp\big(st\,\mathbb{E}(\xi_1\xi_2)\big). \end{aligned} \tag{1.3.7}$$

By taking the $(p+q)$th partial derivative $\frac{\partial^{p+q}}{\partial s^p \partial t^q}$ at $t = s = 0$ for the left-hand side in (1.3.7), one obtains

$$\frac{\partial^{p+q}}{\partial s^p \partial t^q}\mathbb{E}\big(F(\xi_1, s)F(\xi_2, t)\big)\bigg|_{s=t=0} = \mathbb{E}\big(H_p(\xi_1)H_q(\xi_2)\big), \tag{1.3.8}$$

since

$$\mathbb{E}\big(F(\xi_1, s)F(\xi_2, t)\big) = \sum_{m,n=0}^{+\infty} \frac{s^n}{n!}\frac{t^m}{m!}\mathbb{E}\big(H_n(\xi_1)H_m(\xi_2)\big).$$

Concerning the right-hand side in (1.3.7), one has

$$\frac{\partial^{p+q}}{\partial s^p \partial t^q}\exp\big(st\,\mathbb{E}(\xi_1\xi_2)\big)\bigg|_{s=t=0} = \begin{cases} q!\big(\mathbb{E}(\xi_1\xi_2)\big)^q & \text{if} \quad p = q \\ 0 & \text{else,} \end{cases} \tag{1.3.9}$$

since

$$\exp\big(st\,\mathbb{E}(\xi_1\xi_2)\big) = \sum_{n=0}^{+\infty} \frac{(st)^n}{n!}\,\mathbb{E}(\xi_1\xi_2)^n.$$

Finally, combining (1.3.9) and (1.3.8) with (1.3.7), one derives the result. □

Proposition 1.3.4. Let ϕ be the probability density function of a standard normal distribution i.e. $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$ for all $x \in \mathbb{R}$. Then, the collection $\{H_n(x)\}_{n\geq 0}$ forms an orthogonal basis of the space $L^2(\mathbb{R},\phi)$. Recall that the space $L^2(\mathbb{R},\phi)$ consists of all Lebesgue measurable real-valued functions f such that $\int_{\mathbb{R}} |f(x)|^2\phi(x)dx < +\infty$.

Proof. First, notice that Proposition 1.3.3 entails that the collection $\{H_n(x)\}_{n\geq 0}$ is orthogonal. Therefore, to show Proposition 1.3.4, it is enough to prove that if a function $f \in L^2(\mathbb{R},\phi)$ satisfies for all $n \in \mathbb{Z}_+$

$$\langle f, H_n\rangle_{L^2(\mathbb{R},\phi)} = 0, \tag{1.3.10}$$

then necessarily

$$f = 0,\ \ \phi - a.e.$$

Let $f \in L^2(\mathbb{R},\phi)$ be a function that satisfies (1.3.10). Observe that if follows from the linearity and continuity of the scalar product $\langle .,.\rangle_{L^2(\mathbb{R},\phi)}$ and from (1.3.2) that, for all $z \in \mathbb{C}$, $\mathbb{E}\big(f(\xi)F(\xi,z)\big) = 0$, with ξ being a standard normal random variable and $F(x,z) = \exp\big(xz - \frac{z^2}{2}\big)$. Thus, for all $z \in \mathbb{C}$ one has

$$\mathbb{E}\big(f(\xi)\exp(z\xi)\big) = (2\pi)^{-1/2}\int_{\mathbb{R}} f(x)\exp\big(-2^{-1}x^2\big)\exp(zx) = 0.$$

Therefore, the Fourier transform of the function $x \to f(x)\exp\big(-2^{-1}x^2\big)$ is zero. This means that, almost everywhere, $f(x)\exp\big(-2^{-1}x^2\big) = 0$ and the result follows. □

Remark 1.3.5. By virtue of Proposition 1.3.4, every function $f \in L^2(\mathbb{R},\phi)$ can be expanded as a series in Hermite polynomials. More precisely, there exists a unique sequence of real numbers $(c_n(f))_{n\geq 0}$ such that

$$f = \sum_{n=0}^{+\infty} c_n(f)H_n, \tag{1.3.11}$$

where the convergence of the series in (1.3.11) holds in $L^2(\mathbb{R},\phi)$. Moreover, observe that for any standard Gaussian variable ξ, one has

$$\mathbb{E}\Big(f(\xi) - \sum_{n=0}^{N} c_n(f)H_n(\xi)\Big)^2 = \Big\|f - \sum_{n=0}^{N} c_n(f)H_n\Big\|^2_{L^2(\mathbb{R},\phi)} \underset{N\to+\infty}{\longrightarrow} 0.$$

Therefore,

$$f(\xi) = \sum_{n=0}^{+\infty} c_n(f)H_n(\xi), \tag{1.3.12}$$

where the convergence holds in $L^2(\Omega)$.

Definition 1.3.6. Let $f \in L^2(\mathbb{R}, \phi)$ be an arbitrary function which is not constant. The *Hermite rank* r_f of f is defined as the smallest integer $k \geq 1$ for which $c_k(f) \neq 0$, where $(c_n(f))_{n\geq 0}$ is the sequence of real numbers defined in (1.3.11).

Example 1.3.7. One will determine the Hermite rank of power functions. First, recall that the moments of a standard Gaussian random variable ξ are given by

$$\mathbb{E}(\xi^{2n}) = \frac{(2n)!}{2^n n!} \quad , \quad \mathbb{E}(\xi^{2n+1}) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{1.3.13}$$

Therefore, by setting $f_n(x) = x^n$ one gets

$$\langle f_{2n}, H_1 \rangle_{L^2(\mathbb{R},\phi)} = \mathbb{E}\left(f_{2n}(\xi)H_1(\xi)\right) = \mathbb{E}\left(\xi^{2n+1}\right) = 0,$$

and

$$\langle f_{2n}, H_2 \rangle_{L^2(\mathbb{R},\phi)} = \mathbb{E}\left(\xi^{2n}(\xi^2 - 1)\right) = \mathbb{E}\left(\xi^{2n+2}\right) - \mathbb{E}\left(\xi^{2n}\right) \neq 0.$$

Hence, the Hermite rank of the function x^{2n} equals 2. Similarly, one has

$$\langle f_{2n+1}, H_1 \rangle_{L^2(\mathbb{R},\phi)} = \mathbb{E}(\xi^{2n+2}) \neq 0,$$

so the Hermite rank of the function x^{2n+1} equals 1.

The following theorem is the main result of this subsection. It is called a *Non-Central Limit* theorem and has been obtained independently and using different methods by Taqqu in [66, 65] and by Dobrushin and Major in [25].

Theorem 1.3.8. *Let $\{\xi_n\}_{n\in\mathbb{Z}}$ be a centered Gaussian stationary time series displaying LRD of Hurst index equal to $H \in (1/2, 1)$ (see Definition 1.1.12). Let $f \in L^2(\mathbb{R}, \phi)$ be a function whose Hermite rank is an integer $d \geq 1$. One assumes that $1 - \frac{1}{2d} < H < 1$. Then*

$$\left\{ \frac{1}{L(N)^{d/2} N^{d(H-1)+1}} \sum_{n=1}^{\lfloor Nt \rfloor} \Big(f(\xi_n) - \mathbb{E}\left(f(\xi_n)\right) \Big) \right\}_{t\geq 0} \overset{f.d.d}{\to} \left\{ c\mathcal{Z}_{H,d}(t) \right\}_{t\geq 0}$$

for some constant $c > 0$, where $\{\mathcal{Z}_{H,d}(t)\}_{t\geq 0}$ is the standard Hermite process of order d.

Definition 1.3.9. Let $\{B(t), t \in \mathbb{R}\}$ be a two-sided Brownian motion. The Hermite process $\{\mathcal{Z}_{H,d}(t)\}_{t\in\mathbb{R}}$ of order d is defined as

$$\mathcal{Z}_{H,d}(t) = c_{d,H} \int_{\mathbb{R}^d} \left\{ \int_0^t \prod_{j=1}^{d} (s - x_j)_+^{H-\frac{3}{2}} ds \right\} B(dx_1) \dots B(dx_d), \tag{1.3.14}$$

for some constant $c_{d,H}$, where the Hurst parameter H belongs to $(1-\frac{1}{2d}, 1)$. It is called *standard* when $\mathbb{E}\left(\mathcal{Z}_{H,d}(1)^2\right) = 1$. When $d = 2$, it is called the *Rosenblatt process* and it will be denoted by $\{R_H(t)\}_{t\in\mathbb{R}}$. Also, notice that Hermite process of order $d = 1$ is Fractional Brownian Motion (see the representation (1.2.11) of FBM in Remark 1.2.11).

Proposition 1.3.10. The Hermite process $\mathcal{Z}_{H,d}$ is a well-defined $\{d(H-1)+1\}$-SSSI stochastic process having finite moments of any order.

In order to prove Proposition 1.3.10, one needs to recall some results on multiple Wiener integrals which is the aim of the next section.

1.3.2 Multiple Wiener integral

The notion of multiple Wiener integral was first defined by N. Wiener in [72] who named it *polynomial chaos*. Later, another construction was given by K. Itô in [35]. This latter construction has an advantage over Wiener's construction: integrals of different orders are orthogonal to each other. In this subsection, we will present Itô's construction. We mention that two well-known books on multiple Wiener integrals and related topics are [36, 52].
Let $\{W(t)\}_{t\in\mathbb{R}}$ be a real-valued centered Gaussian process with independent increments defined on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Recall that the process $\{W(t)\}_{t\in\mathbb{R}}$ generates an independently scattered Gaussian random measure W on $(\mathbb{R}, \mathcal{B}, \mu)$. The control measure μ is assumed to be without atoms and satisfies, for all $A \in \mathcal{B}_0 := \{A \in \mathcal{B};\ \mu(A) < +\infty\}$, $\mathbb{E}\,|W(A)|^2 = \mu(A)$. Fix an integer $d \geq 1$. Our aim is to give a sense to the following writing

$$\int_{\mathbb{R}^d} f(x_1, \ldots, x_d) W\big(d\mu(x_1)\big) \ldots W\big(d\mu(x_d)\big) := I_d(f), \tag{1.3.15}$$

for all real-valued deterministic functions f belonging to $L^2(\mathbb{R}^d, \mu^{\otimes d})$. To do so, one first considers a class of *elementary functions.*

Definition 1.3.11. A real-valued function $f(x_1, \ldots, x_d)$ of $L^2(\mathbb{R}^d, \mathcal{B}^d, \mu^{\otimes d})$ is called *special* or *simple* if, for some integer $n \geq 1$, it has the form

$$f(x_1, \cdots, x_d) = \sum_{i_1,\ldots,i_d=1}^{n} a_{i_1,\ldots,i_d} 1_{A_{i_1}\times\cdots\times A_{i_d}}(x_1, \ldots, x_d), \tag{1.3.16}$$

where $A_1, \ldots, A_n$ are pairwise-disjoint sets belonging to $\mathcal{B}_0$, and the coefficients $a_{i_1,\ldots,i_d}$ are zero if any two of the indices $i_1, \ldots, i_d$ are equal. The set of all *special* functions is denoted by $\mathcal{E}_d$.

For a special function f of the form (1.3.16), one defines the linear map $I_d : \mathcal{E}_d \to L^2(\Omega)$ by

$$I_d(f) := \sum_{i_1,\ldots,i_d=1}^{n} a_{i_1,\ldots,i_d} W(A_{i_1}) \times \cdots \times W(A_{i_d}), \quad \text{for all } f \in \mathcal{E}_d. \tag{1.3.17}$$

Notice that $I_d(f)$ is well-defined since it does not depend on the particular representation of f.

Proposition 1.3.12. The following properties hold for any $f \in \mathcal{E}_d$ and $g \in \mathcal{E}_q$:

(i) $I_d(f) = I_d(\tilde{f})$ with $\tilde{f}$ denoting the *symmetrization* of f defined for all $(x_1, \ldots, x_d) \in \mathbb{R}^d$ by:

$$\tilde{f}(x_1, \ldots, x_d) = \frac{1}{d!} \sum_{\sigma\in\mathcal{S}_d} f(x_{\sigma(1)}, \ldots, f(x_{\sigma(d)}),$$

where $\mathcal{S}_d$ is the set of all permutations over $\{1, \ldots, d\}$. Recall that $\mathrm{Card}(\mathcal{S}_d) = d!$.

(ii) $\mathbb{E}\big(I_d(f)\big) = 0$

(iii) For all $q \in \mathbb{N}$, one has:

$$\mathbb{E}\Big(I_d(f) I_q(g)\Big) = \begin{cases} 0 & \text{if} \quad d \neq q \\ d!\langle \tilde{f}, \tilde{g}\rangle_{L^2(\mathbb{R}^d,\mu^{\otimes d})} & \text{else.} \end{cases} \tag{1.3.18}$$

Proof. By virtue of the linearity of the map I_d, it is enough to show that the properties (i) and (ii) hold for $f = 1_{A_1\times\cdots\times A_d}$ with $A_1,\ldots,A_d$ a system of pairwise-disjoint sets. Observe that for all $\sigma \in \mathcal{S}_d$, one has

$$I_d(f) = W(A_1)\times\cdots\times W(A_d) = W(A_{\sigma(1)})\times\cdots\times W(A_{\sigma(d)}),$$

so

$$I_d(\tilde{f}) = \frac{1}{d!}\sum_{\sigma\in\mathcal{S}_d} W(A_{\sigma(1)})\times\cdots\times W(A_{\sigma(d)}) = W(A_1)\times\cdots\times W(A_d) = I_d(f).$$

Next, using the independence of the centered random variables $W(A_i)$ and $W(A_j)$ for any two indices $i\neq j$, one gets

$$\mathbb{E}\big(I_d(f)\big) = \mathbb{E}\big(W(A_1)\times\cdots\times W(A_d)\big) = 0.$$

Let us now focus on the proof of (iii). Consider two functions $f\in\mathcal{E}_d$ and $g\in\mathcal{E}_q$. We may and shall assume that the functions f and g are associated with the same system of pairwise-disjoint sets $A_1,\ldots,A_n$, so that

$$f = \sum_{i_1,\ldots,i_d=1}^{n} a_{i_1,\ldots,i_d} 1_{A_{i_1}\times\cdots\times A_{i_d}} \quad\text{and}\quad g = \sum_{j_1,\ldots,j_q=1}^{n} b_{j_1,\ldots,j_q} 1_{A_{j_1}\times\cdots\times A_{j_q}},$$

where the coefficients $a_{i_1,\ldots,i_d}$ (respectively the coefficients $b_{i_1,\ldots,i_d}$) vanish if any two of the indices $i_1,\ldots,i_d$ (respectively $j_1,\ldots,j_q$) are equal. Observe that the product $I_d(f)I_q(g)$ is a sum of terms of the form

$$a_{i_1,\ldots,i_d} b_{j_1,\ldots,j_q} W(A_{i_1})\times\cdots\times W(A_{i_d})\times W(A_{j_1})\times\cdots\times W(A_{j_q}). \tag{1.3.19}$$

Notice that if $d\neq q$, at least one indice of $\{1,\ldots,n\}$ appears exactly once in (1.3.19). Therefore, using the independence of $W(A_i)$ and $W(A_j)$ for $i\neq j$, one derives $\mathbb{E}\big(I_d(f)I_q(g)\big) = 0$. Suppose now that $d=q$. In view of (i), one can assume that f and g are symmetric, so that $a_{i_1,\ldots,i_d} = a_{i_{\sigma(1)},\ldots,i_{\sigma(d)}}$ and $b_{j_1,\ldots,j_d} = b_{j_{\sigma(1)},\ldots,j_{\sigma(d)}}$, for all $\sigma\in\mathcal{S}_d$. Consequently, one has

$$\begin{aligned}
\mathbb{E}\big(I_d(f)I_d(g)\big) &= \mathbb{E}\left(\sum_{i_1,\ldots,i_d=1}^{n}\sum_{j_1,\ldots,j_d=1}^{n} a_{i_1,\ldots,i_d} b_{j_1,\ldots,j_d} W(A_{i_1})\times\cdots\times W(A_{i_d})\times W(A_{j_1})\times\cdots\times W(A_{j_d})\right)\\
&= \mathbb{E}\left((d!)^2 \sum_{i_1<\cdots<i_d}\sum_{j_1<\cdots<j_d} a_{i_1,\ldots,i_d} b_{j_1,\ldots,j_d} W(A_{i_1})\times\cdots\times W(A_{i_d})\times W(A_{j_1})\times\cdots\times W(A_{j_d})\right)\\
&= (d!)^2\,\mathbb{E}\left(\sum_{i_1<\cdots<i_d} a_{i_1,\ldots,i_d} b_{i_1,\ldots,i_d} W(A_{i_1})^2\times\cdots\times W(A_{i_d})^2 + X\right).
\end{aligned}$$

where X denotes the random variable consisting of the sum of all the terms

$$a_{i_1,\ldots,i_d} b_{j_1\ldots j_d} W(A_{i_1})\times\cdots\times W(A_{i_d})\times W(A_{j_1})\times\cdots\times W(A_{j_d}) \tag{1.3.20}$$

over the indices $\{i_1,\ldots,i_d\}$ and $\{j_1,\ldots,j_d\}$ which satisfy

$$i_1 < \ldots < i_d \quad,\quad j_1 < \ldots < j_d \quad\text{and}\quad (i_1,\ldots,i_d)\neq(j_1,\ldots,j_d).$$

Note that $\mathbb{E}(X) = 0$ since at least one indice of $\{1, \ldots, n\}$ appears exactly once in (1.3.20). Thus,

$$\begin{aligned}\mathbb{E}\left(I_d(f)I_d(g)\right) &= (d!)^2 \sum_{i_1<\cdots<i_d} a_{i_1,\ldots,i_d} b_{i_1,\ldots,i_d} \mu(A_{i_1}) \ldots \mu(A_{i_d}) \\ &= d!\langle f, g\rangle_{L^2(\mathbb{R}^d,\mu^{\otimes d})},\end{aligned}$$

which establish (iii). □

Proposition 1.3.13. The linear space $\mathcal{E}_d$ is a dense subset of $L^2(\mathbb{R}^d, \mu^{\otimes d})$.

Proof. In order to prove Proposition 1.3.13, one only has to show that the indicator function of any set $E_1 \times \cdots \times E_d$, $E_i \in \mathcal{B}_0$ can be approximated by a *special* function. Notice that the non-existence of atoms for the measure μ entails that for any $\varepsilon > 0$, there exists, for some integer $n \geq 1$, a system of sets $\{F_1, \ldots, F_n\} \subset \mathcal{B}_0$ satisfying the following properties:

(i) $F_1, \ldots, F_n$ are pairwise-disjoint sets.

(ii) For all $i \in \{1, \ldots, n\}$, one has $\mu(F_i) < \varepsilon$.

(iii) Each E_i can be expressed as the disjoint union of some of the F_j.

Therefore, one can write the function $1_{E_1\times\cdots\times E_d}$ as

$$1_{E_1\times\cdots\times E_d} = \sum_{i_1,\ldots,i_d=1}^{n} c_{i_1,\ldots,i_d} 1_{F_{i_1}\times\cdots\times F_{i_d}}, \tag{1.3.21}$$

where $c_{i_1,\ldots,i_d}$ is 0 or 1. We divide this sum into two parts. Let I be the set of the d-uples of $\{i_1, \ldots, i_d\}$ which are all different and let J be the set of the remaining d-ples. Then, observe that the function

$$f := \sum_{(i_1,\ldots,i_d)\in I} c_{i_1,\ldots,i_d} 1_{F_{i_1}\times\cdots\times F_{i_d}}$$

belongs to the space $\mathcal{E}_d$ since $F_{i_1}, \ldots, F_{i_d}$ are pairwise-disjoint sets for all $(i_1, \ldots, i_d) \in I$. Moreover, notice that using property (iii), one derives $\bigcup_{i=1}^{d} E_i = \bigcup_{i=1}^{n} F_i$. Thus, by setting

$$\alpha := \mu\Big(\bigcup_{i=1}^{d} E_i\Big) = \mu\Big(\bigcup_{i=1}^{n} F_i\Big) = \sum_{i=1}^{n} \mu(F_j),$$

one gets

$$\begin{aligned}\|1_{E_1\times\cdots\times E_d} - f\|^2_{L^2(\mathbb{R}^d,\mu^{\otimes d})} &= \sum_{(i_1,\ldots,i_d)\in J} c_{i_1,\ldots,i_d} \mu(F_{i_1}) \ldots \mu(F_{i_d}) \\ &\leq \binom{d}{2} \sum_{i=1}^{n} \mu(F_i)^2 \sum_{j_1,\ldots,j_{d-2}=1}^{n} \mu(F_{j_1}) \ldots \mu(F_{j_{d-2}}) \\ &= \binom{d}{2} \sum_{i=1}^{n} \mu(F_i)^2 \Big(\sum_{i=1}^{n} \mu(F_i)\Big)^{d-2} \\ &\leq \binom{d}{2} \varepsilon\alpha^{d-1},\end{aligned}$$

which leads to the result.

□

Letting $f = g$ in (1.3.18), one obtains the following property which can be seen as an isometry property:

$$\mathbb{E}\left(I_d(f)^2\right) = d!\|\tilde{f}\|^2_{L^2(\mathbb{R}^d,\mu^{\otimes d})} \leq d!\|f\|^2_{L^2(\mathbb{R}^d,\mu^{\otimes d})}. \tag{1.3.22}$$

Therefore, using Proposition 1.3.13, for all $f \in L^2(\mathbb{R}^d, \mu^{\otimes d})$, there exists a sequence of special functions $(f_n)_{n\in\mathbb{N}} \in \mathcal{E}_d^{\mathbb{N}}$ such that

$$\|f - f_n\|_{L^2(\mathbb{R}^d,\mu^{\otimes d})} \underset{n\to+\infty}{\longrightarrow} 0.$$

Then, using (1.3.22) one can show that the sequence $\{I_d(f_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $L^2(\Omega)$, so it converges in $L^2(\Omega)$. One sets

$$I_d(f) \overset{L^2(\Omega)}{=} \lim_{n\to+\infty} I_d(f_n). \tag{1.3.23}$$

Observe that the isometry property ensures that the limit in (1.3.23) does not depend on the particular choice of the sequence $(f_n)_{n\in\mathbb{N}}$. Hence, the map I_d on $\mathcal{E}_d$ can be extended to a linear continuous map from $L^2(\mathbb{R}^d, \mu^{\otimes d})$ to $L^2(\Omega)$ which satisfies the properties (i), (ii) and (iii) in Proposition 1.3.12 as well as the isometry property (1.3.22). One will also denote by I_d this extension.

Notice that for all $f \in L^2(\mathbb{R}^d)$, the random variable $I_d(f)$ has finite moments of any order. More precisely, the moments of multiple Wiener integrals satisfy the following *hypercontractivity*-type inequality. For a proof of this result, one refers to Corollary 2.8.14 in the book [51].

Proposition 1.3.14. For all $p \geq 2$ and for any $f \in L^2(\mathbb{R}^d)$, one has

$$\mathbb{E}\left(I_d(f)^p\right)^{1/p} \leq (p-1)^{d/2}\,\mathbb{E}\left(I_d(f)^2\right)^{1/2}. \tag{1.3.24}$$

The next result provides a useful formula for the change of variables in multiple Wiener integrals. A proof of it can be found in [45] (Theorem 4.5).

Proposition 1.3.15. Let μ_1 and μ_2 be two non-atomic measures on $(\mathbb{R}^d, \mathcal{B}^d)$ such that μ_1 is absolutely continuous with respect to μ_2, and let $h : \mathbb{R} \to \mathbb{R}$ be a function such that

$$|h(u)|^2 = \frac{d\mu_1}{d\mu_2}(u), \quad u \in \mathbb{R}.$$

One considers W_1 and W_2 two real-valued independently scattered Gaussian measures on $(\mathbb{R}^d, \mathcal{B}^d)$ with control measures μ_1 and μ_2 respectively. Denote by $I_d^{(1)}$ and $I_d^{(2)}$ the corresponding multiple Wiener integrals. For all $k \in \{1, \dots, d\}$, let $f_k \in L^2(\mathbb{R}^k, \mu_1^{\otimes k})$ and set

$$\grave{f}_k(x_1, \dots, x_k) = f_k(x_1, \dots, x_k)h(x_1)\dots h(x_k).$$

Then, one has

$$\left(I_1^{(1)}(f_1), I_2^{(1)}(f_2), \dots, I_d^{(1)}(f_d)\right) \overset{(d)}{=} \left(I_1^{(2)}(\grave{f}_1), I_2^{(2)}(\grave{f}_2), \dots, I_d^{(2)}(\grave{f}_d)\right). \tag{1.3.25}$$

Definition 1.3.16. Let $f \in L^2(\mathbb{R}^p)$ and $g \in L^2(\mathbb{R}^q)$. For any $1 \leq r \leq \min(p,q)$, the *contraction* of r indices of f and g is the function denoted by $f \otimes_r g \in L^2(\mathbb{R}^{p+q-2r}, \mu^{\otimes p+q-2r})$ defined for all $(u_1, \dots, u_{p+q-2r}) \in \mathbb{R}^{p+q-2r}$ as

$$\begin{aligned} f \otimes_r g(u_1, \dots, u_{p+q-2r}) = \int_{\mathbb{R}^r} & f(u_1, \dots, u_{p-r}, s_1, \dots, s_r) \\ & \times g(u_{p-r+1}, \dots, u_{p+q-2r}, s_1, \dots, s_r) d\mu(s_1) \dots d\mu(s_r). \end{aligned} \tag{1.3.26}$$

For $r = 0$, $f \otimes_0 g$ is the usual *tensor product* $f \otimes g \in L^2(\mathbb{R}^{p+q})$ defined as

$$f \otimes g(u_1, \dots, u_{p+q}) = f(u_1, \dots, u_p) g(u_{p+1}, \dots, u_{p+q}). \tag{1.3.27}$$

One also denotes $f^{\otimes n} = f \otimes \cdots \otimes f$ (n times).

Remark 1.3.17. Observe that even when the functions f and g are symmetric, the tensor product and the contractions of f and g are not necessarily symmetric. One will denote by $f \tilde{\otimes} g$ and $f \tilde{\otimes}_r g$ their symmetrizations.

The following Proposition provides a formula for the multiplication of multiple Wiener integrals. A proof of this result can be found in [52] (Proposition 1.1.3).

Proposition 1.3.18. Fix two integers $p, q \geq 1$. Let $f \in L^2(\mathbb{R}^p)$ and $g \in L^2(\mathbb{R}^q)$ be two symmetric functions. Then, one has

$$I_p(f) I_q(g) = \sum_{r=0}^{\min(p,q)} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \otimes_r g). \tag{1.3.28}$$

In particular, one has

$$I_p(f) I_1(g) = I_{p+1}(f \otimes g) + p I_{p-1}(f \otimes_1 g). \tag{1.3.29}$$

One ends this section by giving the proof of Proposition 1.3.10.

Proof of Proposition 1.3.10. Let us first show that the Hermite process $\mathcal{Z}_{H,k}$ is well-defined. To do so, it is enough to check that for all $t \geq 0$, the kernel function

$$f_t(x_1, \dots, x_d) := \int_0^t \prod_{j=1}^d (s - x_j)_+^{H - \frac{3}{2}} ds, \quad (x_1, \dots, x_d) \in \mathbb{R}^d$$

is in $L^2(\mathbb{R}^d)$, which will guarantee the existence of the multiple Wiener integral $I_d(f_t)$. Notice that using successively Fubini-Tonelli's theorem, the symmetry of the function $(s_1, s_2) \to (s_1 - x)_+^{H - \frac{3}{2}} (s_2 - x)_+^{H - \frac{3}{2}}$, the change of variables $v = (s_2 - s_1)^{-1} u$ and the fact that $c_H :=$

$\int_0^{+\infty} v^{H-\frac{3}{2}}(1+v)^{H-\frac{3}{2}}dv < +\infty$ for all $H \in (1-\frac{1}{2d}, 1)$, one gets

$$\begin{aligned}
\|f_t\|^2_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}} dx_1 \cdots \int_{\mathbb{R}} dx_d \Big(\int_0^t \prod_{j=1}^d (s-x_j)_+^{H-\frac{3}{2}} ds \Big)^2 \\
&= \int_{\mathbb{R}} dx_1 \cdots \int_{\mathbb{R}} dx_d \Big(\int_0^t \prod_{j=1}^d (s_1-x_j)_+^{H-\frac{3}{2}} ds_1 \prod_{j=1}^d (s_2-x_j)_+^{H-\frac{3}{2}} ds_2 \Big) \\
&= \int_{[0,t]^2} ds_1 ds_2 \Big(\int_{\mathbb{R}} (s_1-x)_+^{H-\frac{3}{2}} (s_2-x)_+^{H-\frac{3}{2}} dx \Big)^d \\
&= 2\int_0^t ds_1 \int_{s_1}^t ds_2 \Big(\int_0^{+\infty} u^{H-\frac{3}{2}} (s_2-s_1+u)^{H-\frac{3}{2}} du \Big)^d \\
&= 2\int_0^t ds_1 \int_{s_1}^t ds_2 \Big(\int_0^{+\infty} v^{H-\frac{3}{2}} (1+v)^{H-\frac{3}{2}} dv \Big)^d (s_2-s_1)^{2d(H-1)} \\
&= 2c_H^d \int_0^t ds_1 \int_{s_1}^t (s_2-s_1)^{2d(H-1)} ds_2 \\
&= 2c_H^d \big(2d(H-1)+1)(2d(H-1)+2)\big)^{-1} t^{2d(H-1)+2} \\
&< +\infty.
\end{aligned}$$

Next, observe that the kernel function f_t satisfies for all $a>0$

$$f_{at}(ax_1,\dots,ax_d) = a^{d(H-\frac{3}{2})+1} f_t(x_1,\dots,x_d), \quad \text{for all } (x_1,\dots,x_d)\in\mathbb{R}^d.$$

Let μ_1 and μ_2 be the two control measures defined by $d\mu_1(x) = adx$ and $d\mu_2(x) = dx$. With the notations of Proposition 1.3.15 (notice that $\mathcal{Z}_{d,H}(t) = I_d^{(2)}(f_t)$) one derives

$$\begin{aligned}
\mathcal{Z}_{d,H}(at) &= \int_{\mathbb{R}^d} f_{at}(x_1,\dots,x_d) B(dx_1)\dots B(dx_d) \\
&= a^{d(H-\frac{3}{2})+1} \int_{\mathbb{R}^d} f_t(y_1,\dots,y_d) B\big(d(ay_1)\big)\dots B\big(d(ay_d)\big) \\
&= a^{d(H-\frac{3}{2})+1} I_d^{(1)}(f_t). \qquad (1.3.30)
\end{aligned}$$

Using the change of variables formula (1.3.25), one gets $I_d^{(1)}(f_t) \overset{(d)}{=} I_d^{(2)}(a^{d/2}f_t) = a^{d/2}\mathcal{Z}_{d,H}(t)$, so

$$\mathcal{Z}_{d,H}(at) \overset{(d)}{=} a^{d(H-1)+1}\mathcal{Z}_{d,H}(t). \qquad (1.3.31)$$

By the linearity of multiple Wiener integrals, one derives the self-similarity of the Hermite process.

Similarily, one obtains the stationarity of the increments by observing that for all $h>0$

$$f_{t+h}(x_1,\dots,x_d) - f_h(x_1,\dots,x_d) = f_t(x_1-h,\dots,x_d-h), \quad \text{for all } (x_1,\dots,x_d)\in\mathbb{R}^d,$$

so

$$\mathcal{Z}_{d,H}(t+h) - \mathcal{Z}_{d,H}(h) \overset{(d)}{=} \mathcal{Z}_{d,H}(t).$$

Finally, the finiteness of the moments of the Hermite process is an immediate consequence of the hypercontractivity property stated in Proposition 1.3.14. $\square$

1.4 Wiener chaos

The underlying probability space is denoted by $(\Omega, \mathcal{F}, \mathbb{P})$ and assumed to be complete.

1.4.1 Chaos decomposition

Definition 1.4.1. A *Gaussian linear space* G is a linear space consisting of real-valued centered Gaussian random variables. If the linear space G is closed in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, then G is said to be a *Gaussian Hilbert space*.

From now on, $G \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ will be a fixed Gaussian Hilbert space assumed to be separable. Moreover, one denotes by $\mathcal{F}_G$ the σ-field generated by all the Gaussian random variables belonging to G and one denotes by $\widehat{G}$ the so-called *Fock space* of G defined as

$$\widehat{G} := L^2(\Omega, \mathcal{F}_G, \mathbb{P}) \subset L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

Definition 1.4.2. Let $\mathcal{P}_0(G)$ be the subspace of $\widehat{G}$ consisting of constants. Fix an integer $n \geq 1$. The *Wiener chaos* of order n denoted by $\mathcal{P}_n(G)$ is the closure in $\widehat{G}$ of the linear subspace

$$\Big\{Q(\xi_1, \ldots, \xi_d);\ d \in \mathbb{N},\ \xi_1, \ldots, \xi_d \in G,\ Q \in \mathbb{R}[X_1, \ldots, X_d],\ \deg Q \leq n\Big\}.$$

Definition 1.4.3. Set $\mathcal{H}_0(G) := \mathcal{P}_0(G)$ and let $\mathcal{H}_n(G)$ be the orthogonal complement of $\mathcal{P}_{n-1}(G)$ in $\mathcal{P}_n(G)$ for any integer $n \geq 1$, i.e.

$$\mathcal{P}_n(G) = \mathcal{P}_{n-1}(G) \overset{\perp}{\oplus} \mathcal{H}_n(G).$$

The space $\mathcal{H}_n(G)$ is called the *homogeneous Wiener chaos* of order n.

Notice that a straightforward consequence of Definition 1.4.3 is that for all $n \in \mathbb{N}$, one has

$$\mathcal{P}_n(G) = \overset{\perp}{\bigoplus_{0 \leq k \leq n}} \mathcal{H}_k(G), \tag{1.4.1}$$

and thus

$$\overline{\bigcup_{n \geq 0} \mathcal{P}_n(G)} = \overset{\perp}{\bigoplus_{n \geq 0}} \mathcal{H}_n(G). \tag{1.4.2}$$

The following Theorem provides the *Wiener chaos decomposition* of the Fock space $\widehat{G}$.

Theorem 1.4.4. *The Fock space* $\widehat{G}$ *admits the orthogonal decomposition*

$$\widehat{G} = \overset{\perp}{\bigoplus_{n \geq 0}} \mathcal{H}_n(G). \tag{1.4.3}$$

Consequently, for each $X \in \widehat{G}$ there exists an unique orthogonal sequence $\{X_n\}_{n\in\mathbb{Z}_+}$ of random variables such that

$$X = \sum_{n=0}^{+\infty} X_n, \qquad X_n \in \mathcal{H}_n.$$

This latter decomposition is called the Wiener chaos decomposition of X.

Proof. Set

$$\mathcal{H} := \overset{\perp}{\bigoplus_{n\geq 0}} \mathcal{H}_n(G).$$

It is clear that $\mathcal{H} \subset \widehat{G}$. Conversely, one will show that $\mathcal{H}^\perp = \{0\}$. Consider $X \in \widehat{G}$ such that $X \perp \mathcal{H}$. For all $\xi \in G$ and $k \geq 0$, one has $\xi^k \in \mathcal{P}_k(G) \subset \mathcal{H}$. Thus, $\mathbb{E}(X\xi^k) = 0$ for all $\xi \in G$ and $k \geq 0$. Next, observe that combining

$$e^{i\xi} = \lim_{n\to+\infty} \sum_{k=0}^{n} \frac{(i\xi)^k}{k!} \quad \text{in } L^2(\Omega)$$

with the fact that $\mathcal{H}$ is a closed subspace of $\widehat{G}$, one gets that $e^{i\xi} \in \mathcal{H}_\mathbb{C} := \mathcal{H} + i\mathcal{H}$. Therefore

$$\mathbb{E}(Xe^{i\xi}) = 0, \quad \text{for all } \xi \in G.$$

Then, the result follows by using the next Lemma. $\square$

Lemma 1.4.5. Let X be a random variable belonging to $\widehat{G}$. If $\mathbb{E}(Xe^{i\xi}) = 0$ for all $\xi \in G$, then one almost surely has $X = 0$.

Proof. Assume that G is spanned by the random variables $\{\xi_n\}_{n\geq 0}$. Define for all $n \in \mathbb{N}$

$$V_n := \operatorname{span}\{\xi_1, \ldots, \xi_n\} \quad \text{and} \quad \mathcal{F}_n := \sigma(\xi_1, \ldots, \xi_n),$$

and set

$$V_\infty := \bigcup_{n\geq 1} V_n, \quad \text{and} \quad \mathcal{F}_\infty := \bigcup_{n\geq 1} \mathcal{F}_n.$$

Fix an integer $n \geq 1$. Let X_n be the random variable defined as $X_n := \mathbb{E}(X|\mathcal{F}_n) \in L^2(\Omega, \mathcal{F}_n, \mathbb{P})$. Using the definition of conditional expectation, one obtains that for all $\xi \in V_n$

$$\mathbb{E}(X_n e^{i\xi}) = \mathbb{E}\left(\mathbb{E}(Xe^{i\xi}|\mathcal{F}_n)\right) = \mathbb{E}(Xe^{i\xi}) = 0.$$

Next, observe that since $X_n \in L^2(\Omega, \mathcal{F}_n, \mathbb{P})$, one has for all $\omega \in \Omega$

$$X_n(\omega) = \psi_n\big(\xi_1(\omega), \ldots, \xi_n(\omega)\big)$$

for some $\psi_n \in L^2(\mathbb{R}^n, \nu)$, where ν denotes the distribution of the Gaussian vector $(\xi_1 \ldots, \xi_n)$. Therefore, for all $(t_1, \ldots, t_n) \in \mathbb{R}^n$, one obtains with $\xi = t_1\xi + \ldots t_n\xi_n$ that

$$\begin{aligned} 0 = \mathbb{E}(X_n e^{it\xi}) &= \mathbb{E}\left(\psi_n(\xi_1, \ldots, \xi_n) e^{it_1\xi_1 + \cdots + it_n\xi_n}\right) \\ &= \int_{\mathbb{R}^n} \psi_n(x_1, \ldots, x_n) e^{it_1x_1 + \cdots + it_nx_n} d\nu(x). \end{aligned}$$

In other words, the complex measure $\psi_n d\nu$ on $\mathbb{R}^n$ has a trivial Fourier transform. Hence, for all $n \in \mathbb{N}$, one has $\psi_n = 0$ ν-a.e, and thus $X_n = \mathbb{E}(X|\mathcal{F}_n) = 0$ a.s . One deduces from Doob's Martingale Convergence theorem that

$$X = \mathbb{E}(X|\mathcal{F}_\infty) \overset{a.s.}{=} \lim_{n\to+\infty} \mathbb{E}(X|\mathcal{F}_n) = 0.$$

$\square$

One ends this subsection by stating two important results on Wiener chaoses. The first provides a very useful *Hypercontractivity-type* inequality, it is analogous to the inequality stated in Proposition 1.3.14. The second theorem extends the well-known result concerning tails of Gaussian random variables. It shows that the probability distributions of random variables belonging to Wiener chaoses have exponentially bounded tails. One refers to the book [36] for the proofs of these results (see Theorem 5.10 and Theorem 6.7).

Theorem 1.4.6. *Fix an integer $n \geq 0$. For all random variable χ belonging to $\mathcal{P}_n(G)$, one has*

$$\|\chi\|_p \leq (p-1)^{n/2}\|\chi\|_2, \quad \textit{for all } p \geq 2, \tag{1.4.4}$$

where $\|\chi\|_p = \big(\mathbb{E}(|\chi|^p)\big)^{1/p}$.

Theorem 1.4.7. *For each $n \geq 1$, there exists a universal constant $c_n > 0$ such that for every $\chi \in \mathcal{P}_n(G)$ and $y \geq 2$,*

$$\mathbb{P}\big(|\chi| > y\|\chi\|_2\big) \leq \exp(-c_n y^{2/n}).$$

1.4.2 Wick polynomials

Our aim now is to give some orthogonal bases of the homogeneous Wiener chaoses $\mathcal{H}_n(G)$ and thus of the Wiener chaoes $\mathcal{P}_n(G)$ and the Fock-space $\widehat{G}$ associated to the Gaussian Hilbert space G. To this purpose, one needs to define the notion of *Wick polynomials* and *Wick products* which will play a major role in the obtention of such bases.

Definition 1.4.8. Let $Q \in \mathbb{R}[X_1, \dots, X_d]$ be a multivariate polynomial of degree $n \in \mathbb{N}$. For all (jointly) Gaussian random variables $\xi_1, \dots, \xi_d \in G$, the *Wick polynomial* denoted by $:Q(\xi_1, \dots, \xi_d):$ is defined as the orthogonal projection of the random variable $Q(\xi_1, \dots, \xi_d) \in \widehat{G}$ onto the nth homgeneous Wiener chaos $\mathcal{H}_n$. We also include the case $n = 0$ by setting $::= 1 \in \mathcal{H}_0(G)$.

Remark 1.4.9. Using the definition of the n-th Wiener chaos, one can derive an expression of the n-th homogeneous Wiener chaos in terms of Wick polynomials. More precisely, the space $\mathcal{H}_n(G)$ is spanned by the family

$$\Big\{ :Q(\xi_1, \dots, \xi_d):, \ d \in \mathbb{N}, \ \xi_1, \dots, \xi_d \in G, \ \ \deg Q \leq n \Big\}.$$

Also, observe that all terms of the form $\xi_1^{\beta_1} \dots \xi_d^{\beta_d}$ in the polynomial $Q(\xi_1, \dots, \xi_d)$ such that $\beta_1 + \dots + \beta_d < n$ has no contribution in the definition of the Wick polynomial $:Q(\xi_1, \dots, \xi_d):$. In view of this, one will only focus on homogeneous polynomials of degree n.

Denote by Λ the set of all sequences $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots)$ of non-negative integers with only finitely many elements different from 0. For each $\boldsymbol{\alpha} \in \Lambda$, set $|\boldsymbol{\alpha}| = \sum_i \alpha_i$ and let Λ_n and $\Lambda_{\leq n}$ be the subsets of all $\boldsymbol{\alpha} \in \Lambda$ such that $|\boldsymbol{\alpha}| = n$ and $|\boldsymbol{\alpha}| \leq n$ respectively. For each finite sequence of elements $\xi_1, \dots, \xi_d \in G$, set $\mathcal{P}_0(\xi_1, \dots, \xi_d) := \mathcal{P}_0(G)$ and for all $n \geq 1$ define the closed subspace $\mathcal{P}_n(\xi_1, \dots, \xi_d) \subset \mathcal{P}_n(G)$ as the set of all polynomials of the random variables $\xi_1, \dots, \xi_d$ having a degree smaller or equal to n, i.e

$$\mathcal{P}_n(\xi_1, \dots, \xi_d) := \operatorname{span}\left\{ \xi_1^{\alpha_1} \dots \xi_d^{\alpha_d}, \quad \boldsymbol{\alpha} \in \Lambda_{\leq n} \right\} \subset \mathcal{P}_n(G).$$

One also defines $\mathcal{H}_n(\xi_1, \dots, \xi_d)$ as the orthogonal complement of $\mathcal{P}_{n-1}(\xi_1, \dots, \xi_d)$ in $\mathcal{P}_n(\xi_1, \dots, \xi_d)$. The following result shows that the Wick polynomial $: Q(\xi_1 \dots, \xi_d) :$ is the same for every Gaussian Hilbert space that contains $\xi_1, \dots, \xi_d$.

Proposition 1.4.10. Let $Q \in \mathbb{R}_n[X_1, \dots, X_d]$. Then the Wick polynomial $: Q(\xi_1, \dots, \xi_d) :$ equals the orthogonal projection of $Q(\xi_1, \dots, \xi_d)$ onto $\mathcal{H}_n(\xi_1, \dots, \xi_d)$.

Proof. Fix a multivariate polynomial $Q \in \mathbb{R}[X_1, \dots, X_d]$ of degree n and let $\xi_1, \dots, \xi_d$ be d Gaussian random variables belonging to G. One denotes by $: \widetilde{Q}(\xi_1, \dots, \xi_d) :$ the orthogonal projection of $Q(\xi_1, \dots, \xi_d) \in \mathcal{P}_n(\xi_1, \dots, \xi_d)$ onto $\mathcal{H}_n(\xi_1, \dots, \xi_d)$. We want to show that $: Q(\xi_1, \dots, \xi_d) :=: \widetilde{Q}(\xi_1, \dots, \xi_d) :$. Observe that by definition

$$Q(\xi_1, \dots, \xi_d) - : \widetilde{Q}(\xi_1, \dots, \xi_d) : \in \mathcal{P}_{n-1}(\xi_1, \dots, \xi_d) \subset \mathcal{P}_{n-1}(G).$$

So, in order to derive the result, one only has to show that $: \widetilde{Q}(\xi_1, \dots, \xi_d) : \perp \mathcal{P}_{n-1}(G)$. Let $(\chi_n)_{n \in \mathbb{N}}$ be an orthonormal sequence of G satisfying the two following properties:

(i) $\mathbb{E}(\xi_i \chi_j) = 0$ for all $i \in \{1, \dots, d\}$ and $j \in \mathbb{N}$;

(ii) $G = \overline{\operatorname{span}\{\xi_1, \dots, \xi_d, \chi_1, \dots, \chi_m, \dots\}}$.

One derives from (ii) that $\mathcal{P}_{n-1}(G)$ is spanned by random variables of the form

$$\prod_i \xi_i^{\alpha_i} \prod_j \chi_j^{\beta_j} \quad \text{with} \quad \sum_i \alpha_i + \sum_j \beta_j \leq n - 1.$$

Next, observe that (i) entails that $\xi_1, \dots, \xi_d$ and $\chi_1, \chi_2, \dots$ are independent, so

$$\mathbb{E}\left(: \widetilde{Q}(\xi_1, \dots, \xi_d) : \prod_i \xi_i^{\alpha_i} \prod_j \chi_j^{\beta_j} \right) = \mathbb{E}\left(: \widetilde{Q}(\xi_1, \dots, \xi_d) : \prod_i \xi_i^{\alpha_i} \right) \mathbb{E}\left(\prod_j \chi_j^{\beta_j} \right) = 0,$$

where we used the fact that $\prod_j \xi_i^{\alpha_i} \in \mathcal{P}_{n-1}(\xi_1, \dots, \xi_d)$ so it is orthogonal to $: \widetilde{Q}(\xi_1, \dots, \xi_d) : \in \mathcal{H}_n(\xi_1, \dots, \xi_d)$. Therefore, for all $\eta \in \mathcal{P}_{n-1}(G)$, one has $\mathbb{E}\big(: \widetilde{Q}(\xi_1, \dots, \xi_d) : \eta \big) = 0$, which achieves the proof of Proposition 1.4.10. □

Remark 1.4.11. By virtue of the orthogonality of homogeneous Wiener chaoses, it is immediate to see that Wick polynomials of different degrees are orthogonal to each other. Now, let ξ be a standard Gaussian random variable belonging to G. Since $\mathcal{H}_1(G) = G$, observe that

$$: \xi := \xi. \tag{1.4.5}$$

Let us now determine $: \xi^2 :$. Observe that $\mathcal{P}_1(\xi) = \text{span}\{1, \xi\}$. and $\xi^2 - \text{Proj}_{\mathcal{H}_2(\xi)}(\xi^2) \in \mathcal{P}_1(\xi)$. So, there exist two real numbers a and b such that $\text{Proj}_{\mathcal{H}_2(\xi)}(\xi^2) = \xi^2 - a - b\xi$. Using the fact that $\text{Proj}_{\mathcal{H}_2(\xi)}(\xi^2) \perp \mathcal{P}_1(\xi)$, one gets $a = 1$ and $b = 0$. Therefore, using Proposition 1.4.10, one obtains

$$: \xi^2 := \xi^2 - 1. \tag{1.4.6}$$

In the same way, one can derive

$$: \xi^3 := \xi^3 - 3\xi. \tag{1.4.7}$$

The Wick product may be seen as a kind of Gram-Schmidt orthogonalization procedure in the context of Wiener chaoses. It allows to construct an orthogonal sequence of $\widehat{G}$ starting from usual products $\xi_1 \dots \xi_n$ of Gaussian random variables belonging to G. The following proposition provides a generalization of (1.4.5), (1.4.6) and (1.4.7) using Hermite polynomials.

Proposition 1.4.12. Let $\xi_1, \dots, \xi_d$ be an orthonormal sequence of elements in G, i.e.

$$\mathbb{E}(\xi_i \xi_j) = \delta_{i,j}, \quad \text{for all } i, j \tag{1.4.8}$$

and let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \Lambda_n$. Then

$$: \xi_1^{\alpha_1} \dots \xi_d^{\alpha_d} := H_{\alpha_1}(\xi_1) \dots H_{\alpha_d}(\xi_d). \tag{1.4.9}$$

In particular, one has for all standard Gaussian random variable $\xi \in G$

$$: \xi^n := H_n(\xi). \tag{1.4.10}$$

Proof. First, notice that the polynomial $X_1^{\alpha_1} \dots X_d^{\alpha_d} - H_{\alpha_1}(X_1) \dots H_{\alpha_d}(X_d)$ is of degree smaller or equal than $n-1$ since for all $i \in \mathbb{N}$, the dominant coefficient of the polynomial $H_i(X)$ is 1, i.e. $H_i(X) = X^i + \dots$. Therefore, one has $\xi_1^{\alpha_1} \dots \xi_d^{\alpha_d} - H_{\alpha_1}(\xi_1) \dots H_{\alpha_d}(\xi_d) \in \mathcal{P}_{n-1}(\xi_1, \dots, \xi_d)$. So, it only remains to show that $H_{\alpha_1}(\xi_1) \dots H_{\alpha_d}(\xi_d) \perp \mathcal{P}_{n-1}(\xi_1, \dots, \xi_d)$. Next, observe that each random variable χ belonging to $\mathcal{P}_{n-1}(\xi_1, \dots, \xi_d)$ can be written as a finite sum of terms of the form

$$\xi_1^{\beta_1} \dots \xi_d^{\beta_d} \quad \text{with} \quad \beta_1 + \dots + \beta_d \leq n - 1.$$

Using successively the orthogonality of the sequence $\xi_1, \dots, \xi_d$, the fact that there exists at least one index $i_0 \in \{1, \dots, d\}$ such that $\beta_{i_0} < \alpha_{i_0}$, and that the polynomial $X^{\beta_{i_0}}$ can be written in terms of Hermite polynomials with degrees less than or equal to β_{i_0} and by using Proposition 1.3.3, one derives

$$\mathbb{E}\left(\prod_{i=1}^{d} H_{\alpha_i}(\xi_i)\xi_i^{\beta_i}\right) = \prod_{i=1}^{d} \mathbb{E}\left(H_{\alpha_i}(\xi_i)\xi_i^{\beta_i}\right) = \mathbb{E}\left(H_{\alpha_{i_0}}(\xi_{i_0})\xi_{i_0}^{\beta_{i_0}}\right)\mathbb{E}\left(\prod_{i \neq i_0} H_{\alpha_i}(\xi_i)\xi_i^{\beta_i}\right) = 0,$$

which leads to the result. □

The following results is a straightforward consequence of Remark 1.4.9, Proposition 1.4.12 and Proposition 1.3.3.

Corollary 1.4.13. Let $(\xi_d)_{d\in\mathbb{N}}$ be an orthonormal basis of G. Then, the following family of Wick products of degree n

$$\Big\{ :\xi_1^{\alpha_1}\ldots\xi_d^{\alpha_d} := H_{\alpha_1}(\xi_1)\ldots H_{\alpha_d}(\xi_d), \quad d\in\mathbb{N}, \quad |\boldsymbol{\alpha}| = n\Big\}$$

forms a complete orthogonal sequence of $\mathcal{H}_n(G)$.

Using a Polarization procedure (see for instance Theorem D.1 in [36]), one can derive from Corollary 1.4.13 the following result.

Corollary 1.4.14. Let $(\xi_d)_{d\in\mathbb{N}}$ be an orthonormal basis of G. The following family of Wick powers of degree n

$$\Big\{ :\xi_d^n := H_n(\xi_d), \quad d\in\mathbb{N}\Big\}$$

forms a complete orthogonal sequence of $\mathcal{H}_n(G)$.

1.4.3 Connections between Wiener chaoses and multiple Wiener integrals

One focuses now on the particular case where the Gaussian separable Hilbert space G is spanned by a centered Gaussian process $\{W(t)\}_{t\in\mathbb{R}}$ with independent increments. Recall that the process W generates an independently scattered Gaussian random measure on $(\mathbb{R},\mathcal{B},\mu)$. Using properties of simple Wiener integral, one can define G as

$$G := \Big\{ \int_{\mathbb{R}} f(t)W(dt) = I_1(f), \ f\in L^2(\mathbb{R},\mu)\Big\}.$$

Hermite polynomials draw a fundamental connecion between multiple Wiener integrals and Wiener chaoses associated to the Gaussian Hilbert space G through the following statement.

Proposition 1.4.15. Let $h\in L^2(\mathbb{R},\mu)$ be a function of norm one, i.e. $\|h\|_{L^2(\mathbb{R},\mu)} = 1$. Then, one has for all $d\in\mathbb{N}$

$$I_d\big(h^{\otimes d}\big) = H_d\big(I_1(h)\big), \tag{1.4.11}$$

where H_d is the dth Hermite polynomial.

Proof. One proceeds by induction on d. For $d=1$, the result is obvious since $H_1(X) = X$. Fix an integer $d\geq 1$ and assume that (1.4.11) holds. By using (1.3.3), and by applying (1.3.29) to $f = h^{\otimes d}$ and $g = h$, one obtains that

$$\begin{aligned} H_{d+1}\big(I_1(h)\big) &= I_1(h)H_d\big(I_1(h)\big) - dH_{d-1}\big(I_1(h)\big) \\ &= I_1(h)I_d(h^{\otimes d}) - dI_{d-1}\big(h^{\otimes(d-1)}\big) \\ &= I_{d+1}\big(h^{\otimes(d+1)}\big), \end{aligned}$$

where we used the fact that $h^{\otimes d} \otimes_1 h = h^{\otimes d-1}$. Indeed, for all $(u_1, \ldots, u_{d-1}) \in \mathbb{R}^{d-1}$, one has

$$\begin{aligned} h^{\otimes d} \otimes_1 h(u_1, \ldots, u_{d-1}) &= \int_{\mathbb{R}} h^{\otimes d}(u_1, \ldots, u_{d-1}, s) h(s) d\mu(s) \\ &= h^{\otimes d-1}(u_1, \ldots, u_{d-1}) \int_{\mathbb{R}} h^2(s) d\mu(s) \\ &= h^{\otimes d-1}(u_1, \ldots, u_{d-1}). \end{aligned}$$

□

Corollary 1.4.16. The multiple Wiener integral I_d maps $L^2(\mathbb{R}^d, \mu^{\otimes d})$ onto the homogeneous Wiener chaos $\mathcal{H}_n(G)$.

Proof. We want to show that the image $I_d\big(L^2(\mathbb{R}^d)\big)$ equals $\mathcal{H}_d(G)$. By combining Corollary 1.4.14 with Proposition 1.4.15 one derives

$$\begin{aligned} \mathcal{H}_d(G) &= \operatorname{span}\Big\{ H_d\big(I_1(h)\big),\ h \in L^2(\mathbb{R}),\ \|h\|_{L^2(\mathbb{R})} = 1 \Big\} \\ &= \operatorname{span}\Big\{ I_d(h^{\otimes d}),\ h \in L^2(\mathbb{R}),\ \|h\|_{L^2(\mathbb{R})} = 1 \Big\} \\ &\subset I_d\big(L^2(\mathbb{R}^d)\big). \end{aligned} \tag{1.4.12}$$

Conversely, the orthogonality of multiple Wiener integrals of different orders entails that

$$I_d\big(L^2(\mathbb{R}^d)\big) \perp \bigoplus_{m \neq d}^{\perp} \mathcal{H}_m(G) = \mathcal{H}_d(G)^{\perp}.$$

Therefore,

$$\overline{I_d\big(L^2(\mathbb{R}^d)\big)} \subset \mathcal{H}_d(G). \tag{1.4.13}$$

In view of (1.4.13), to conclude it only remains to show that $I_d\big(L^2(\mathbb{R}^d)\big)$ is a closed subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$, which is easy to check since I_d satisfies on $L^2(\mathbb{R}^d)$ the isometry property $\|I_d(f)\|^2_{L^2(\Omega)} = d!\ \|\widetilde{f}\|^2_{L^2(\mathbb{R}^d)} \leq d!\ \|f\|^2_{L^2(\mathbb{R}^d)}$. □

The following proposition is a straightforward consequence of Theorem 1.4.4 and Corollary 1.4.16.

Proposition 1.4.17. For every square integrable random variable X that is measurable with respect to the $\sigma-$field generated by the process W, there exists a sequence $(f_d)_{d\in\mathbb{N}}$ such that $f_d \in L^2(\mathbb{R}^d, \mu^{\otimes d})$ and satisfying the following decomposition

$$X = \mathbb{E}(X) + \sum_{d=1}^{+\infty} I_d(f_d), \tag{1.4.14}$$

where the convergence of the series holds in $L^2(\Omega, \mathcal{F}_W, \mathbb{P})$.

1.4.4 Other chaotic processes related to Hermite processes

Fine study of path behavior of stochastic processes (regularity, laws of iterated logarithm, etc.) is a very classical research topic in theory of probability and harmonic analysis whose origins go back to well-known works of Paul Lévy on Brownian motion made in the 1940s. Since then, the more general framework of Gaussian processes has been extensively investigated in the literature. Nevertheless, the framework of non-Gaussian processes belonging to Wiener chaoses has been much less explored. The study of global and local regularity of these processes has only been partially initiated since their discovery. A main goal of our thesis is to study, using a wavelet approach, the path behavior of some chaotic multifractional extensions of the Fractional Brownian Motion.
In the paper [50] published in 1986, T. Mori and H. Oodaira introduced a large class of self-similar processes with stationary increments belonging to any arbitrary homogeneous Wiener chaos and represented by multiple Wiener integrals that widely extend Hermite processes. Namely, these processes are defined as follows

$$X_{d,H}(t) = \int_{\mathbb{R}^d} Q_t(x_1, \ldots, x_d) B(dx_1) \ldots B(dx_d), \quad \text{for all } t \geq 0$$

with $H \in (1/2, 1)$ and where the kernel function Q_t belongs, for each fixed $t \in \mathbb{R}_+$, to the space of square integrable and symmetric functions over $\mathbb{R}^d$ satisfying the three following conditions

- $Q_t(x_1, \ldots, x_d) = \int_0^t q(s - x_1, \ldots, s - x_d) ds$ for almost all $x_1, \cdots, x_d \in \mathbb{R}$,
- $q(ax_1, \ldots, ax_d) = a^{H-d/2-1} q(x_1, \ldots, x_d)$, for all $a > 0$ and for almost all $x_1, \cdots, x_d \in \mathbb{R}$,
- $\int_{\mathbb{R}^d} \left| q(x_1, \ldots, x_d) q(x_1 + 1, \ldots, x_d + 1) \right| dx_1 \ldots dx_d < +\infty$.

The third condition guarentees that the multiple Wiener integral is well-defined while the two first conditions combined with Proposition 1.3.15 and Proposition 1.3.14 ensure that the process $X_{d,H}$ is an H-SSSI process having finite moments of any order. Therefore, the process may and shall be assumed to be with continuous paths. T. Mori and H. Oodaira derived the following law of the iterated logarithm for the process $X_{d,H}$: one has almost surely

$$\limsup_{n \to +\infty} \frac{|X_{d,H}(n)|}{n^H (2 \log \log n)^{d/2}} < +\infty.$$

Notice that this result has been proved by them only in the case where n is an integer tending to $+\infty$. It may be seen as a weak generalization of the iterated logarithm law satisfied by FBM (see Proposition 1.2.6) since the integer d in the factor $(\log \log n)^{d/2}$ corresponds to the order of the homogeneous Wiener chaos that contains the process $X_{d,H}$. Unfortunately one cannot deduce from this result the local behavior at 0 since the process $X_{d,H}$ does not *a priori* satisfy an inversion time formula when $d > 1$.
In [71], the authors obtained, using chaining arguments and Malliavin calculus, an upper bound for the global modulus of continuity of self-similar processes with stationary increments represented by multiple Wiener integrals. Last year, A. Ayache derived in [10] lower bound for local

oscillations of Hermite processes which, among other things, establishes the nowhere differentiability of their paths.
In the last section of Chapter 2 and in Chapter 3, we will be interested in an extension of the Rosenblatt process depending on two parameters $H_1, H_2 \in (1/2, 1)$ satisfying $H_1 + H_2 > 3/2$ called the *generalized Rosenblatt process* and which belongs to the class of processes introduced in [50]. It is defined through the double multiple Wiener integral:

$$R_{H_1,H_2}(t) := \int_{\mathbb{R}^2} \left(\int_0^t (s-x_1)_+^{H_1-\frac{3}{2}} (s-x_2)_+^{H_2-\frac{3}{2}} ds \right) dB(x_1)dB(x_2),$$

and belongs to the second homogeneous Wiener chaos. The generalized Rosenblatt process is self-similar of order $H_1 + H_2 - 1$ with stationary increments displaying long-range dependence property. It was first introduced by Maejima and Tudor in their paper [44]. In the last few years, several articles related to it, written by Bai and Taqqu, were published (see [13, 14, 12, 11, 15]). The generalized Rosenblatt process appears in Non-Central Limit theorems as limit when considering a quadratic form involving two linear processes having long-range dependence property with different Hurst index. One also mentions that the generalized Rosenblatt process has been used in [44] to prove that in contrast with the first homogeneous Wiener chaos which contains only one H-SSSI process that is Fractional Brownian motion, there are infinitly many H-SSSI processes belonging to the second homogeneous Wiener chaos.
In 2014, B. Arras investigated in [3] another class of chaotic processes $\{Y_{d,H}(t) : t \in \mathbb{R}\}$ which belongs to the class of processes considered in [50] consisting in a very natural chaotic extension of FBM (see representation (1.2.6) of FBM). Namely, it is defined, for all $t \in \mathbb{R}$, through the multiple Wiener integral on $\mathbb{R}^d$:

$$Y_{d,H}(t) = \int_{\mathbb{R}^d} \left[\|\mathbf{t}^* - \mathbf{x}\|_2^{H-\frac{d}{2}} - \|\mathbf{x}\|_2^{H-\frac{d}{2}} \right] dB_{x_1}...dB_{x_d}\,, \tag{1.4.15}$$

where $\mathbf{t}^* = (t, ..., t) \in \mathbb{R}^d$ and $\|\cdot\|_2$ denotes the Euclidian norm over $\mathbb{R}^d$. The process $\{Y_{d,H}(t) : t \in \mathbb{R}\}$ was studied in [3] through wavelet methods inspired by the ones in [9, 7]. The author obtained estimates of the local and global behavior as well as an iterated logarithm law of the paths of this process. In Chapter 4, one will introduce *multifractional* extensions with non stationary increments of this process, and will study their local and global behavior using a wavelet approach.

Chapter 2

Wavelet series representations of chaotic SSSI stochastic processes

2.1 Some background on the wavelet theory

The primary goal of the wavelet theory is to construct Hilbertian bases of functional spaces such as $L^2(\mathbb{R}^d)$ that have some advantages over the classical Fourier basis. Indeed, despite its great utility, the trigonometric system suffers from some drawbacks. We mention for instance the fail of pointwise convergence of partial Fourier sums for some continuous functions (see [27]). In order to avoid this issue, Haar introduced in 1910 an orthonormal basis known as the first orthonormal wavelet basis.
One of the most fundamental ideas behind wavelets is to analyze signals according to scales. They provide a kind of a mathematical microscope that reveals details as scales are reviewed. The widespread development of this approach starting from the mid 80's is mainly due to pioneering works of Jan-Olov Strömberg, Jean Morlet, Alex Grossman, Yves Meyer, Stéphane Mallat, Ingrid Daubechies and Stéphane Jaffard. It offers very significant advantages over the traditional Fourier's approach in analyzing physical phenomena that involve irregularities and sharp bumps. They were developed in the fields of mathematics, quantum physics, electrical engineering, and seismic geology. Also, interchanges between these fields have led to many wavelet applications related with image compression, voice data analysis, turbulence, human vision, radar, earthquake prediction and so on. Essentially, the wavelet theory allows to decompose any square integrable function onto a set of functions, which are derived from a finite number of prototype wavelets by scaling and translating. More precisely, an orthonormal wavelet basis of the Hilbert space $L^2(\mathbb{R}^d)$ is an orthonormal basis obtained by dilations and translations of functions usually denoted by $\psi_1, \dots, \psi_{2^d-1}$ and called mother wavelets. Such a basis is of the form

$$\left\{2^{jd/2}\psi_p(2^j\mathbf{x}-\mathbf{k});\quad p\in\{1,\dots,2^d-1\},\ j\in\mathbb{Z}\text{ and }\ \mathbf{k}\in\mathbb{Z}^d\right\}.$$

The multi-index $\mathbf{k}$ refers to location and the index j to scale or resolution.

2.1.1 Multiresolution analyses and univariate wavelets

Definition 2.1.1. A *multiresolution analysis* (MRA) of the Hilbert space $L^2(\mathbb{R})$ is a sequence $(V_j)_{j\in\mathbb{Z}}$ of *closed subspaces* of $L^2(\mathbb{R})$ satisfying the following five conditions:

(i) $V_j \subset V_{j+1}$, for every $j \in \mathbb{Z}$;

(ii) The intersection $\bigcap_{j\in\mathbb{Z}} V_j$ reduces to $\{0\}$.

(iii) The union $\bigcup_{j\in\mathbb{Z}} V_j$ is a dense subspace in $L^2(\mathbb{R})$.

(iv) All the spaces V_j are scaled versions of the central space V_0. More precisely $f \in V_0 \iff f(2^j\cdot) \in V_j$.

(v) There exists a function ϕ in V_0, called *scaling function* or *father wavelet*, such that the sequence $\big\{\phi(\cdot - k) : , k \in \mathbb{Z}\big\}$ forms an orthonormal basis of V_0.

Remark 2.1.2. Observe that the properties (iv) and (v) entail that, for any $j \in \mathbb{Z}$, the sequence of functions

$$\Big\{2^{jd/2}\phi(2^j\cdot - k), \quad k \in \mathbb{Z}\Big\}$$

forms an orthonormal basis of V_j. This shows that a multiresolution analysis is completely characterized by the scaling function associated with it.

Example 2.1.3. A classical example of MRA of $L^2(\mathbb{R})$ can be constructed starting from the scaling function $\mathcal{K}(t) = 1_{[0,1)}(t)$, for all $t \in \mathbb{R}$. The corresponding MRA is defined for all $j \in \mathbb{Z}$ as

$$V_j = \left\{ f \in L^2(\mathbb{R});\;\; \forall k \in \mathbb{Z}\;\; f_{\left[2^{-j}k, 2^{-j}(k+1)\right)} = constant \right\},$$

where $f_{[a,b)}$ denotes the restriction of the function f over the interval $[a,b)$. This example is known as the *Haar multiresolution analysis*.

Notice that by virtue of the property (i), one can define for every $j \in \mathbb{Z}$, the orthogonal complement of V_j on V_{j+1} denoted by W_j, namely

$$V_{j+1} = V_j \overset{\perp}{\oplus} W_j \qquad \text{for all} \quad j \in \mathbb{Z}. \tag{2.1.1}$$

The closed subspace W_j of $L^2(\mathbb{R})$ is called the j-th *detail space* or *wavelet space*. The following proposition is one of the keystones of the wavelet theory.

Proposition 2.1.4. The space $L^2(\mathbb{R})$ admits the two following orthogonal decompositions

$$L^2(\mathbb{R}) = V_J \overset{\perp}{\oplus} \left(\overset{\perp}{\bigoplus_{j\geq J}} W_j \right) \;\; \text{for all } J \in \mathbb{Z} \tag{2.1.2}$$

and

$$L^2(\mathbb{R}) = \overset{\perp}{\bigoplus_{j\in\mathbb{Z}}} W_j. \tag{2.1.3}$$

Proof. For any $J \in \mathbb{Z}$, one sets

$$Z_J := V_J \overset{\perp}{\oplus} \Big(\overset{\perp}{\bigoplus_{j \geq J}} W_j \Big).$$

Notice that Z_J is a closed subspace of $L^2(\mathbb{R})$. By proceeding by induction on $Q \in \mathbb{N}$ and by using (2.1.1) one derives that

$$Z_{J+Q} = Z_J, \quad \text{for all } Q \in \mathbb{N}.$$

Since the integer J can be chosen arbitrarily one gets

$$Z_J = Z_0 \ \text{ for all } J \in \mathbb{Z}. \tag{2.1.4}$$

Therefore, for any $J \in \mathbb{Z}$ and each $f \in V_J$, one has that $f \in V_J \overset{\perp}{\oplus} \Big(\bigoplus_{j \geq J} W_j \Big) = Z_0$. Hence,

$$\bigcup_{j \in \mathbb{Z}} V_j \subset Z_0.$$

Using the fact that Z_0 is a closed subspace of $L^2(\mathbb{R})$ and the property (iii) of the MRA $(V_j)_{j\in\mathbb{Z}}$, one obtains (2.1.2). Next, define for all $J \in \mathbb{Z}$

$$Z := \overset{\perp}{\bigoplus_{j \in \mathbb{Z}}} W_j \qquad \text{and} \qquad Z'_J = \overset{\perp}{\bigoplus_{j \geq J}} W_j.$$

Observe that the closed subspace Z'_J is orthogonal to V_J and that $Z'_J \subset Z$. To establish (2.1.3), one shows that $Z^\perp = \{0\}$. Let $f \in Z^\perp$. One has

$$\forall g \in Z, \ \langle f, g \rangle = 0. \tag{2.1.5}$$

The decomposition (2.1.2) entails that for every fixed $J \in \mathbb{Z}$, one has $f = v_J + w_J$, where $v_J \in V_J$ and $w_J \in Z'_J$. Therefore, by taking $g = w_J$ in (2.1.5), one derives $w_J = 0$ and thus $f = v_J \in V_J$. Consequently, using the property (ii) of the MRA $(V_j)_{j\in\mathbb{Z}}$, one gets $f \in \bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ which establishes (2.1.3). $\square$

In view of the decompositions (2.1.2) and (2.1.3), to construct orthonormal wavelet bases of $L^2(\mathbb{R})$, it is crucial to investigate the detail space W_j. Observe that this latter space W_j inherits of the scaling property satisfied by the approximation space V_j. More precisely, one has

$$f \in W_0 \iff f(2^j \cdot) \in W_j, \quad \text{for all } j \in \mathbb{Z}. \tag{2.1.6}$$

The following theorem is due to Mallat and Meyer (1986). It gives information on the detail spaces and provides a way of constructing orthonormal wavelet bases of $L^2(\mathbb{R})$. A proof of this result can be found in [24] (see for instance Theorem 5.11).

Theorem 2.1.5. *Let $(V_j)_{j\in\mathbb{Z}}$ be an MRA of $L^2(\mathbb{R})$ spanned by the scaling function ϕ. Then, there exists a function ψ called mother wavelet such that the sequence*

$$\Big\{\psi(\cdot - k), \quad k \in \mathbb{Z}\Big\}$$

forms an orthonormal basis of W_0. Moreover, one possible way of constructing the mother wavelet ψ is to set

$$\psi(x) = 2^{1/2} \sum_{k\in\mathbb{Z}} (-1)^k \overline{\langle \phi, \phi_{1,-k+1}\rangle}_{L^2(\mathbb{R})} \phi(2x-k), \tag{2.1.7}$$

where the convergence of the series (2.1.7) holds in $L^2(\mathbb{R})$.

The following corollary is a straightforward consequence of Theorem 2.1.5, (2.1.2), (2.1.3) and (2.1.6).

Corollary 2.1.6. Denote respectively by ϕ and ψ a scaling function and a corresponding mother wavelet and fix an arbitrary scale $J \in \mathbb{Z}$. Then the two sequences

$$\Big\{2^{j/2}\psi(2^j\cdot - k)\ ;\ (j,k)\in\mathbb{Z}^2\Big\}$$

and

$$\Big\{2^{J/2}\phi(2^J\cdot - k)\ ;\ k\in\mathbb{Z}\Big\} \bigcup \Big\{2^{j/2}\psi(2^j\cdot - k)\ ;\ \ k\in\mathbb{Z},\ j\geq J\Big\}$$

are orthonormal bases of $L^2(\mathbb{R})$. For sake of convenience, one sets for all $(j,k)\in\mathbb{Z}^2$

$$\psi_{j,k} := 2^{j/2}\psi(2^j\cdot - k) \ \text{ and } \ \phi_{j,k} := 2^{j/2}\phi(2^j\cdot - k).$$

The term wavelet refers to the fact that the function ψ is oscillating and wave-like. It can be seen as a *brief oscillation* like one recorded by a seismograph or heart monitor. It is generaly assumed to has some *vanishing* moments. One says that ψ has $N\in\mathbb{N}$ vanishing moments if it satisfies for all $n\in\{0,\dots,N-1\}$

$$\int_{\mathbb{R}} \psi(u)u^n du = 0. \tag{2.1.8}$$

The greater number of vanishing moments a wavelet has, the more the wavelet oscillates. In contrast, the scaling function ϕ generally satisfies

$$\widehat{\phi}(0) := \int_{\mathbb{R}} \phi(u)du = 1. \tag{2.1.9}$$

Corollary 2.1.6 entails that every function $f\in L^2(\mathbb{R})$ has the two following decompositions

$$f(t) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} d_{j,k}\psi_{j,k}(t) \tag{2.1.10}$$

and for all $J\in\mathbb{Z}$,

$$f(t) = \sum_{k\in\mathbb{Z}} a_{J,k}\phi_{J,k}(t) + \sum_{j=J}^{+\infty}\sum_{k\in\mathbb{Z}} d_{j,k}\psi_{j,k}(t), \tag{2.1.11}$$

with

$$d_{j,k} := \int_{\mathbb{R}} f(u)\psi_{j,k}(u)du \quad \text{and} \quad a_{J,k} := \int_{\mathbb{R}} f(u)\phi_{J,k}(u)du,$$

and where the convergence of the series in (2.1.10) and (2.1.11) hold in $L^2(\mathbb{R})$. The term $\sum_{k\in\mathbb{Z}} a_{J,k}\phi_{J,k}(t)$ can be viewed as an *approximation* of f at the scale J and the terms $\sum_{k\in\mathbb{Z}} d_{j,k}\psi_{j,k}(t)$ as *details* at scales j. Moreover, the following proposition shows that when the function f has some global Hölder regularity, the coefficients $\{a_{J,k}\}_{k\in\mathbb{Z}}$ approximate the function f at scale J in the following sense

Proposition 2.1.7. Assume that $f \in L^2(\mathbb{R})$ satisfies, for some $\beta \in (0,1]$, a global Hölder of order β; that is one has $|f(t) - f(s)| \leq c|t-s|^\beta$, for some constant c and all $s,t \in \mathbb{R}$. Also assume that the scaling function ϕ satisfies (2.1.9) and $\int_{\mathbb{R}} |u|^\beta |\psi(u)| du < +\infty$. Then, there exists a constant $C > 0$ such that for all $J \in \mathbb{Z}_+$

$$\sup_{k\in\mathbb{Z}} \left| 2^{J/2} a_{J,k} - f\big(2^{-J}k\big) \right| \leq C 2^{-\beta J}. \tag{2.1.12}$$

Proof. Fix an arbitrary scale $J \in \mathbb{Z}_+$. One can derive from (2.1.9) and the change of variable $u = 2^J v - k$ that $\int_{\mathbb{R}} 2^J \phi(2^J v - k) dv = 1$. Therefore, using the triangle inequality and the Hölder condition satisfied by f, one obtains for each $k \in \mathbb{Z}$ that

$$\begin{aligned} \left| 2^{J/2} a_{J,k} - f\big(2^{-J}k\big) \right| &= 2^J \left| \int_{\mathbb{R}} \big(f(v) - f(2^{-J}k)\big)\phi(2^J v - k) dv \right| \\ &\leq 2^J \int_{\mathbb{R}} \big|f(v) - f(2^{-J}k)\big| \big|\phi(2^J v - k)\big| dv \\ &\leq 2^J \int_{\mathbb{R}} \big|v - 2^{-J}k\big|^\beta \big|\phi(2^J v - k)\big| dv \\ &\leq 2^{-\beta J} \int_{\mathbb{R}} |u|^\beta |\phi(u)| du, \end{aligned} \tag{2.1.13}$$

which establishes (2.1.12). □

Example 2.1.8. One considers the Haar scaling function $\mathcal{K}(t) = 1_{[0,1)}(t)$ associated with the Haar multiresolution analysis defined in the Example 2.1.3. Using Theorem 2.1.5, one can construct a mother wavelet associated to it. First, observe that for all $k \in \mathbb{Z}$

$$\langle \mathcal{K}, \mathcal{K}_{1,-k+1} \rangle_{L^2(\mathbb{R})} = 2^{1/2} \int_{\mathbb{R}} \mathcal{K}(x)\mathcal{K}(2x + k - 1) dx = \begin{cases} 2^{-1/2} & \text{if} \quad k = 0, 1 \\ 0 & \text{else.} \end{cases}$$

Therefore, by (2.1.7), one can construct the mother wavelet $\mathcal{H}$ as

$$\mathcal{H}(x) = \mathcal{K}(2x) - \mathcal{K}(2x - 1) = 1_{[0,1/2)}(x) - 1_{[1/2,1)}(x),$$

for all $x \in \mathbb{R}$. The function $\mathcal{H}$ is known as the Haar mother wavelet. Notice that it has only one vanishing moment.

Theorem 2.1.9. ***(The two variants of the Haar basis)***
For each $(j,k) \in \mathbb{Z}^2$, one sets $\mathcal{H}_{j,k} = 2^{j/2}\mathcal{H}(2^j \cdot - k)$ and $\mathcal{K}_{j,k} = 2^{j/2}\mathcal{K}(2^j \cdot - k)$.

(i) The sequence of functions

$$\Big\{\mathcal{H}_{j,k}\ ;\ (j,k)\in\mathbb{Z}^2\Big\}$$

is called the first variant of the Haar wavelet basis of $L^2(\mathbb{R})$.

(ii) For any fixed scale $J\in\mathbb{Z}$*, the sequence of functions*

$$\Big\{\mathcal{K}_{J,k}\ ;\ k\in\mathbb{Z}\Big\}\bigcup\Big\{\mathcal{H}_{j,k}\ ;\ k\in\mathbb{Z}\ ,\ j\geq J\Big\},$$

is called the second variant of the Haar wavelet basis of $L^2(\mathbb{R})$.

The main advantage of the Haar MRA is its simplicity. However, the functions $\mathcal{K}$ and $\mathcal{H}$ are discontinuous and therefore not differentiable, and this might be a drawback. In 1985, Meyer established the existence of a scaling function ϕ and a mother wavelet ψ that belong to the Schwartz class and for which the Fourier transforms $\widehat{\phi}$ and $\widehat{\psi}$ are infinitely differentiable compactly supported functions. Recall that the Schwartz class $S(\mathbb{R})$ is the space of infinitely differentiable functions f which satisfy for all $p\in\mathbb{Z}_+$ and $N\in\mathbb{Z}_+$, the following localization property

$$\sup_{x\in\mathbb{R}}\big(1+|x|\big)^N|f^{(p)}(x)|<+\infty, \tag{2.1.14}$$

where $f^{(p)}$ denotes the derivative of f of order p. One also mentions that our convention for the Fourier transform is to write

$$\widehat{f}(\xi):=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-i\xi x}f(x)\,dx\quad\text{and}\quad f(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{ix\xi}\widehat{f}(\xi)d\xi. \tag{2.1.15}$$

Remark 2.1.10. It is well known that f belongs to the Schwartz class $S(\mathbb{R})$ if and only if its Fourier transform $\widehat{f}$ belongs to $S(\mathbb{R})$.

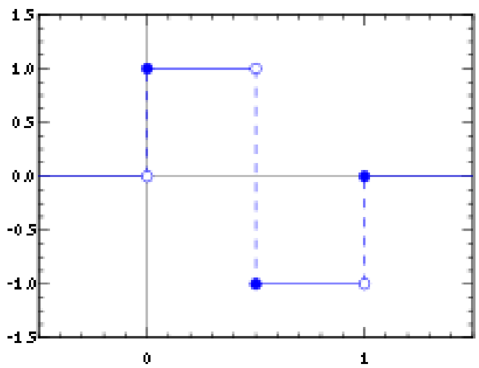
Theorem 2.1.11. ***(Meyer wavelet bases)*** *There exist a real-valued scaling function* ϕ *and a mother wavelet* ψ *which generate an orthonormal basis of* $L^2(\mathbb{R})$ *and satisfy the following properties*

(i) $\phi,\psi\in S(\mathbb{R})$.

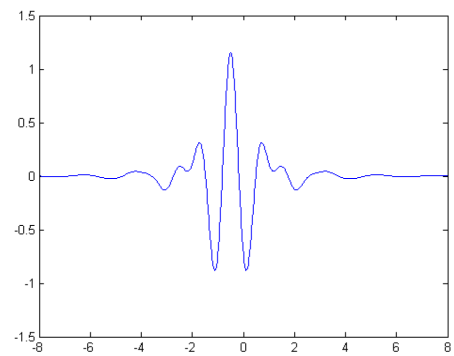
(ii) $\operatorname{supp}\widehat{\phi}\subset\big[-\frac{4\pi}{3},\frac{4\pi}{3}\big]$ *and for all* $\xi\in\big[-\frac{2\pi}{3},\frac{2\pi}{3}\big]$, $\widehat{\phi}(\xi)=1$.

(iii) $\operatorname{supp}\widehat{\psi}\subset\big\{\xi\in\mathbb{R};\ \frac{2\pi}{3}\leq|\xi|\leq\frac{8\pi}{3}\big\}$.

A wavelet basis generated by such ϕ *and* ψ *is called a Meyer wavelet basis.*



(a) Haar wavelet



(b) Meyer wavelet

2.1.2 Tensor product method and multivariate wavelets

In this subsection we present the classical tensor product method which allows to construct, for every integer $d \geq 2$ a multiresolution analysis (MRA) $(V_j^d)_{\in\mathbb{Z}}$ of $L^2(\mathbb{R}^d)$ starting from $(V_j^1)_{j\in\mathbb{Z}}$ a MRA of $L^2(\mathbb{R})$. We follow the formalism employed in the book [4].

Proposition 2.1.12. Let $(V_j^1)_{j\in\mathbb{Z}}$ be a MRA of $L^2(\mathbb{R})$ generated by a scaling function ϕ^1. For each fixed $j \in \mathbb{Z}$ and for any integer $d \geq 2$, one denotes by V_j^d the closed subspace of $L^2(\mathbb{R}^d)$ defined through the induction relation:

$$V_j^d := V_j^{d-1} \bigotimes V_j^1. \tag{2.1.16}$$

Then $(V_j^d)_{j\in\mathbb{Z}}$ is a MRA of $L^2(\mathbb{R}^d)$ associated to the scaling function $\phi^d : (x_1, \ldots, x_d) \mapsto \prod_{i=1}^d \phi^1(x_i)$.

The sequence of the corresponding detail spaces $(W_j^d)_{j\in\mathbb{Z}}$ is the sequence of closed and pairwise orthogonal subspaces of $L^2(\mathbb{R}^d)$ such that, for each $j \in \mathbb{Z}$, W_j^d is the orthogonal complement of V_j^d in V_{j+1}^d. Notice that, when $d \geq 2$, using (2.1.16), (2.1.1) and the bilinearity property of tensor product, one obtains

$$\begin{aligned}
V_{j+1}^d &= \left(V_j^{d-1} \overset{\perp}{\bigoplus} W_j^{d-1}\right) \bigotimes \left(V_j^1 \overset{\perp}{\bigoplus} W_j^1\right) \\
&= V_j^d \overset{\perp}{\bigoplus} \left((V_j^{d-1} \bigotimes W_j^1) \overset{\perp}{\bigoplus} (W_j^{d-1} \bigotimes V_j^1) \overset{\perp}{\bigoplus} (W_j^{d-1} \bigotimes W_j^1) \right),
\end{aligned}$$

which entails that

$$W_j^d = (V_j^{d-1} \bigotimes W_j^1) \overset{\perp}{\bigoplus} (W_j^{d-1} \bigotimes V_j^1) \overset{\perp}{\bigoplus} (W_j^{d-1} \bigotimes W_j^1). \tag{2.1.17}$$

Next, let E_d be the finite set of cardinality $2^d - 1$ defined as $E_d := \{0,1\}^d \backslash \{0, \ldots, 0\}$. It can be shown by induction on $d \geq 2$ and by using (2.1.17) that

$$W_j^d = \overset{\perp}{\bigoplus_{\varepsilon \in E_d}} W_j^{(\varepsilon)}, \tag{2.1.18}$$

where, for any $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in E_d$,

$$W_j^{(\varepsilon)} := \bigotimes_{i=1}^{d} W_j^{\varepsilon_i} \tag{2.1.19}$$

with the convention that $W_j^0 := V_j^1$. Furthermore, observe that by proceeding as in the proof of Proposition 2.1.4, one obtains the two following decompositions

$$L^2(\mathbb{R}^d) = V_J^d \overset{\perp}{\bigoplus} \left(\overset{\perp}{\bigoplus_{j \geq J}} W_j^d \right) \tag{2.1.20}$$

and

$$L^2(\mathbb{R}^d) = \overset{\perp}{\bigoplus_{j \in \mathbb{Z}}} W_j^d. \tag{2.1.21}$$

One ends this subsection by stating the following theorem which allows to construct an orthonormal wavelet basis of $L^2(\mathbb{R}^d)$ starting from an orthonormal wavelet basis of $L^2(\mathbb{R})$. Its proof relies on Theorem 2.1.5, Proposition 2.1.12, (2.1.18), (2.1.19), (2.1.20) and (2.1.21).

Theorem 2.1.13. *Let $(V_j^1)_{j\in\mathbb{Z}}$ be a MRA of $L^2(\mathbb{R})$ with a (univariate) scaling function ϕ^1 and corresponding (univariate) mother wavelet ψ^1. For each $\varepsilon \in E_d$, the function $\psi^{(\varepsilon)}$, called a multivariate mother wavelet, is defined as*

$$\psi^{(\varepsilon)}(x_1, \dots, x_d) := \prod_{i=1}^{d} \psi^{\varepsilon_i}(x_i), \tag{2.1.22}$$

with the convention that $\psi^0 := \phi^1$. Recall that the multivariate scaling function ϕ^d, which has been introduced in Proposition 2.1.12, is defined as

$$\phi^d(x_1, \dots, x_d) := \prod_{i=1}^{d} \phi^1(x_i) = \prod_{i=1}^{d} \psi^0(x_i). \tag{2.1.23}$$

The following three results hold

(i) For each fixed $\varepsilon \in E_d$ and $j \in \mathbb{Z}$, the sequence

$$\left\{ 2^{jd/2} \psi^{(\varepsilon)}(2^j \cdot - \mathbf{k}),\ \mathbf{k} \in \mathbb{Z}^d \right\}$$

is an orthonormal basis of the space $W_j^{(\varepsilon)}$

(ii) The sequence

$$\left\{ 2^{jd/2} \psi^{(\varepsilon)}(2^j \cdot - \mathbf{k}),\ \varepsilon \in E_d,\ j \in \mathbb{Z},\ \mathbf{k} \in \mathbb{Z}^d \right\}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$.

(iii) For every $J \in \mathbb{Z}$, the sequence

$$\left\{ 2^{Jd/2} \phi^d(2^J \cdot - \mathbf{k}),\ \mathbf{k} \in \mathbb{Z}^d \right\} \bigcup \left\{ 2^{jd/2} \psi^{(\varepsilon)}(2^j \cdot - \mathbf{k}),\ \varepsilon \in E_d,\ j \geq J,\ \mathbf{k} \in \mathbb{Z}^d \right\}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$.

2.2 Wavelet-type expansions of the FBM

In this section, we will present two of the most standard wavelet series representations of Fractional Brownian Motion which have been introduced in [49] and in another form in [18]. Following [49], we will work in the sequel with a Meyer scaling function ϕ and a corresponding mother wavelet ψ defined in Theorem 2.1.11.
Consider the non-anticipative representation of FBM introduced in (1.2.7), that is

$$B_H(t) = \frac{1}{\Gamma(H+1/2)} \int_{\mathbb{R}} \Big((t-u)_+^{H-1/2} - (-u)_+^{H-1/2} \Big) B(du) \ \text{ for all } t \in \mathbb{R},$$

where Γ is the Gamma function defined for every real number $z \in (0,+\infty)$ as $\Gamma(z) := \int_0^{+\infty} u^{z-1} e^{-u} du$. Recall that the process B_H is always identified with its modification with continuous paths. For each fixed $t \in \mathbb{R}$, one denotes by f_t the kernel function defined as

$$f_t(u) = \frac{1}{\Gamma(H+1/2)} \Big((t-u)_+^{H-1/2} - (-u)_+^{H-1/2} \Big) \quad \text{for all } u \in \mathbb{R}.$$

By expanding the square integrable function f_t into the Meyer wavelet bases $\{\psi_{j,k};\ (j,k) \in \mathbb{Z}^2\}$ and $\big\{\phi_{J,l},\ l \in \mathbb{Z}\big\} \cup \big\{\psi_{j,k},\ \ j \geq J,\ k \in \mathbb{Z}\big\}$ and by using the isometry property of the Wiener integral, one gets

$$B_H(t) = \sum_{(j,k) \in \mathbb{Z}^2} d_{j,k}(t) \varepsilon_{j,k}, \tag{2.2.1}$$

and for each fixed $J \in \mathbb{Z}$,

$$B_H(t) = \sum_{l \in \mathbb{Z}} a_{J,l}(t) \mu_{J,l} + \sum_{j=J}^{+\infty} \sum_{k \in \mathbb{Z}} d_{j,k}(t) \varepsilon_{j,k}, \tag{2.2.2}$$

with

$$d_{j,k}(t) := \langle f_t, \psi_{j,k} \rangle, \quad a_{J,l}(t) := \langle f_t, \phi_{J,l} \rangle, \tag{2.2.3}$$

and

$$\varepsilon_{j,k} := \int_{\mathbb{R}} \psi_{j,k}(u) B(du), \quad \mu_{J,l} := \int_{\mathbb{R}} \phi_{J,l}(u) B(du), \tag{2.2.4}$$

where the convergence of the series in (2.2.1) and (2.2.2) hold in $L^2(\Omega)$. One of the strengths of the wavelet approach is that it allows to decouple the time variable t from the random variables $\varepsilon_{j,k}$ and $\mu_{J,l}$. Moreover, the randomness of FBM becomes indexed by $\mathbb{Z}^2$ (which is a countable set) instead of $\mathbb{R}$. The representation (2.2.2) allows to isolate the low frequencies. Indeed, the term $\sum_{l \in \mathbb{Z}} a_{J,l}(t) \mu_{J,l}$ gives the *tendency*, while the term $\sum_{j=J}^{\infty} \sum_{k \in \mathbb{Z}} d_{j,k}(t) \varepsilon_{j,k}$ provides details that are *fluctuations* around tendency. Also, notice that using the orthonormality of the system $\big\{\phi_{J,l},\ l \in \mathbb{Z}\big\} \cup \big\{\psi_{j,k},\ \ j \geq J,\ k \in \mathbb{Z}\big\}$ and the isometry property of Wiener integral, one obtains that the random variables $\mu_{J,l},\ l \in \mathbb{Z}$ and $\varepsilon_{j,k},\ j \geq J,\ k \in \mathbb{Z}$ are independent standard Gaussian random variables. Furthermore, one can derive a *deterministic* control of these random variables by using the following useful lemma.

Lemma 2.2.1. Let $\{\epsilon_n;\ n \in \mathbb{N}\}$ be a sequence of standard random variables. Then, there exist an event Ω^* of probability 1, and an almost surely finite random variable C such that for all $n \in \mathbb{N}$ and all $\omega \in \Omega^*$, one has

$$|\epsilon_n(\omega)| \leq C(\omega)\sqrt{\log(3+n)}. \tag{2.2.5}$$

Proof. The proof of this result is classical and relies on Borel-Cantelli lemma. One knows that the tail of the standard Gaussian distribution decreases exponentially fast. More precisely, for a standard Gaussian random variable X, one has for all $a \geq 2$:

$$\mathbb{P}(|X| \geq a) \leq e^{-a^2/2}.$$

By taking $a = \gamma\sqrt{\log(3+n)}$ with $\gamma \geq 2$, one gets for all $n \in \mathbb{N}$,

$$\mathbb{P}\Big(|\varepsilon_n| \geq \gamma\sqrt{\log(3+n)}\Big) \leq e^{-2^{-1}\gamma^2\log(3+n)} = (3+n)^{-\gamma^2/2}.$$

Therefore, one has

$$\sum_{n=1}^{+\infty} \mathbb{P}\Big(|\varepsilon_n| \geq \gamma\sqrt{\log(3+n)}\Big) < +\infty.$$

Hence, by virtue of the Borel-Cantelli lemma, one has the existence of an event Ω^*, of probability 1, such that for all $\omega \in \Omega^*$:

$$\exists\ n(\omega),\quad \forall\ m \geq n(\omega):\ |\varepsilon_m(\omega)| \leq \gamma\sqrt{\log(3+m)}. \tag{2.2.6}$$

Denote by T the stopping time corresponding to the smallest n for which (2.2.6) happens, namely

$$T(\omega) := \begin{cases} \min\big\{n \in \mathbb{N},\ \forall m \geq n:\ |\varepsilon_m(\omega)| \leq \gamma\sqrt{\log(3+m)}\big\} & \text{if} \quad \omega \in \Omega^* \\ +\infty & \text{else,} \end{cases}$$

and define the random variable $\widetilde{C}$ as

$$\widetilde{C}(\omega) := \begin{cases} \sup\limits_{1\leq m\leq T(\omega)} |\varepsilon_m(\omega)| & \text{if} \quad \omega \in \Omega^* \\ +\infty & \text{else.} \end{cases}$$

Finally, one gets the result by setting for each $\omega \in \Omega^*$, $C(\omega) := \sup\big(\widetilde{C}(\omega), \gamma\big)$. □

An indexing argument allows to obtain the following corollary (see Lemma 2 in [8]).

Corollary 2.2.2. Let $\{\varepsilon_{j,k},\ (j,k) \in \mathbb{Z}^2\}$ be a sequence of standard Gaussian random variables. Then, there exist an event $\widetilde{\Omega}$, of probability 1, and an almost surely finite random variable C such that for all $(j,k) \in \mathbb{Z}$ and all $\omega \in \widetilde{\Omega}$,

$$|\varepsilon_{j,k}(\omega)| \leq C(\omega)\sqrt{\log(3+|j|+|k|)}. \tag{2.2.7}$$

Let us now provide an explicit expression for the deterministic coefficients $d_{j,k}(t)$. Observe that by using the change of variable $u = 2^j s - k$ and by splitting the integral into two parts (which is possible since $\psi \in S(\mathbb{R})$), one gets

$$\begin{aligned} d_{j,k}(t) &= \frac{1}{\Gamma(H+1/2)} 2^{j/2} \int_{\mathbb{R}} \Big((t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \Big) \psi(2^j s - k) ds \\ &= \frac{1}{\Gamma(H+1/2)} 2^{-jH} \int_{\mathbb{R}} \Big(\big(2^j t - k - u\big)_+^{H-1/2} - \big(-k-u\big)_+^{H-1/2} \Big) \psi(u) du \\ &= 2^{-jH} \big(\Psi_{H+1}(2^j t - k) - \Psi_{H+1}(-k) \big), \end{aligned} \tag{2.2.8}$$

where Ψ_{H+1} denotes the left-sided fractional primitive of ψ of order $H+1/2$ defined in the following way:

Definition 2.2.3. The *left-sided fractional primitive* of order $H+1/2$, of the Meyer mother wavelet ψ is defined for all $x \in \mathbb{R}$ through

$$\Psi_{H+1}(x) = \frac{1}{\Gamma(H+1/2)} \int_{\mathbb{R}} (x-u)_+^{H-1/2} \psi(u) du, \tag{2.2.9}$$

or equivalently through its Fourier transform by

$$\widehat{\Psi}_{H+1}(\xi) = (i\xi)^{-H-1/2} \widehat{\psi}(\xi), \quad \text{for all } \xi \in \mathbb{R}. \tag{2.2.10}$$

The following lemma is a straightforward consequence of Remark 2.1.10, Theorem 2.1.11 and (2.2.10).

Lemma 2.2.4. The function Ψ_{H+1} inherits of the nice properties satisfied by the Meyer mother wavelet. It is infinitly many times differentiable and well-localized, that is one has

$$\sup_{x \in \mathbb{R}} \big(1 + |x|\big)^N |\partial^p \Psi_{H+1}(x)| < +\infty, \ \forall p, N \in \mathbb{Z}_+.$$

Moreover, its Fourier transform is compactly supported and vanishes in a neighborhood of zero; more precisely, one has

$$\operatorname{supp} \widehat{\Psi}_{H+1} \subset \Big\{ \xi \in \mathbb{R}; \quad \frac{2\pi}{3} \le |\xi| \le \frac{4\pi}{3} \Big\}.$$

Theorem 2.2.5. ***(The first wavelet representation of FBM)*** *The FBM $\{B_H(t)\}_{t \in \mathbb{R}}$ can, almost surely, be expressed as*

$$B_H(t) = \sum_{(j,k) \in \mathbb{Z}^2} 2^{-jH} \big(\Psi_{H+1}(2^j t - k) - \Psi_{H+1}(-k) \big) \varepsilon_{j,k}, \tag{2.2.11}$$

where the wavelet series (2.2.11) converges almost surely, uniformly on each compact subset of $\mathbb{R}$.

One way for estabilishing Theorem 2.2.5 is to use the following theorem which follows from Itô-Nisio's theorem (see [40]).

Theorem 2.2.6. *Let T be a positive real number. One denotes by $\mathcal{C}([-T,T])$ the Banach space of real-valued continuous functions over $[-T,T]$ equipped with the supremum norm. Let $(U_k)_{k\in\mathbb{N}}$ be a sequence of independent stochastic processes whose paths belong to $\mathcal{C}([-T,T])$ and vanish at 0. One defines for each $n\in\mathbb{N}$ and $\omega\in\Omega$:*

$$X_n(t,\omega)=\sum_{k=1}^{n}U_k(t,\omega). \tag{2.2.12}$$

Assume that the following two properties hold :

(i) For all $t\in\mathbb{R}$, the sequence $\big(X_n(t,\cdot)\big)_{n\in\mathbb{N}}$ converges in $L^2(\Omega)$.

(ii) There exist three constants $\delta>0$, $c>0$ and $e>0$, such that for all $t_1,t_2\in[-T,T]$ and $n\in\mathbb{N}$:

$$\mathbb{E}\left(|X_n(t_1,\cdot)-X_n(t_2,\cdot)|^{\delta}\right)\le c|t_1-t_2|^{1+e}. \tag{2.2.13}$$

Then, there exists an event $\widetilde{\Omega}$, of probability 1, such that the sequence $\big(X_n(.,\omega)\big)_{n\in\mathbb{N}}$ converges in $\mathcal{C}([-T,T])$ for all $\omega\in\widetilde{\Omega}$.

Remark 2.2.7. If $(U_k)_{k\in\mathbb{N}}$ is a sequence of independant Gaussian processes, then the equivalence of Gaussian moments allows to replace the condition (2.2.13) by

$$\mathbb{E}\left(|X_n(t_1,\cdot)-X_n(t_2,\cdot)|^{2}\right)\le c|t_1-t_2|^{e}. \tag{2.2.14}$$

Proof of Theorem 2.2.5. Fix an arbitrary strictly positive real number $T>0$. Let $(D_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of $\mathbb{Z}^2$ satisfying $\bigcup_{n\in\mathbb{N}}D_n=\mathbb{Z}^2$. Set for all $n\in\mathbb{N}$ and $\omega\in\Omega$:

$$X_n(t,\omega):=\sum_{(j,k)\in D_n}2^{-jH}(\Psi_{H+1}(2^jt-k)-\Psi_{H+1}(-k))\varepsilon_{j,k}(\omega). \tag{2.2.15}$$

One will show that the sequence $(X_n)_{n\in\mathbb{N}}$ falls within the scope of Itô-Nisio theorem. Recall that $(\varepsilon_{j,k})_{j,k\in\mathbb{Z}^2}$ is a sequence of independent standard Gaussian random variables, that the deterministic coefficients $d_{j,k}(t)=2^{-jH}(\Psi_{H+1}(2^jt-k)-\Psi_{H+1}(-k))$ are real-valued continuous functions over $[-T,T]$ that satisfy $d_{j,k}(0)=0$. Also recall that for each fixed $t\in\mathbb{R}$, the sequence $\big(X_n(t,\cdot)\big)_{n\in\mathbb{N}}$ converges in $L^2(\Omega)$. Therefore, it only remains to check that the second condition in Theorem 2.2.6 is fulfilled. The independence of the random variables $\varepsilon_{j,k}$ entails that for all $t_1,t_2\in[-T,T]$ one has:

$$\mathbb{E}\,|\sum_{(j,k)\in D_n}(d_{j,k}(t_1)-d_{j,k}(t_2))\varepsilon_{j,k}|^2\le\sum_{j,k\in\mathbb{Z}}|d_{j,k}(t_1)-d_{j,k}(t_2)|^2=\sum_{j,k\in\mathbb{Z}}|\langle f_{t_1}-f_{t_2},\psi_{j,k}\rangle|^2$$

and by applying Parseval's Formula, one derives:

$$\sum_{j,k\in\mathbb{Z}}|\langle f_{t_1}-f_{t_2},\psi_{j,k}\rangle|^2=||f_{t_1}-f_{t_2}||^2_{L^2(\mathbb{R})}=\kappa_H\int_{\mathbb{R}}\left((t_1-s)_+^{H-1/2}-(t_2-s)_+^{H-1/2}\right)^2ds=c_H|t_1-t_2|^{2H}$$

where κ_H and c_H are two strictly positive constants not depending on n. Therefore, the series

$$\sum_{(j,k)\in\mathbb{Z}^2} 2^{-jH}\big(\Psi_{H+1}(2^j t-k)-\Psi_{H+1}(-k)\big)\varepsilon_{j,k}$$

is with probability 1 uniformly convergent in t on $[-T,T]$. $\square$

Remark 2.2.8. Concerning the approximation coefficients $a_{J,k}(t)$ defined in (2.2.2), one cannot proceed in a similar way as for the detail coefficients $d_{j,k}(t)$ because of the two following reasons:

(a) The function Φ_{H+1}, which is the left-sided fractional primitive of order $H+1/2$ of the scaling function ϕ, is badly localized and does not belong to the Schwartz class $S(\mathbb{R})$. For instance, when $H>1/2$, one can show that for all positive real number x big enough,

$$|\Phi_{H+1}(x)|\geq Cx^{H-1/2}, \tag{2.2.16}$$

for some constant C not depending on x, which entails that $\lim_{x\to+\infty}|\Phi_{H+1}(x)|=+\infty$.

(b) We would like to have a wavelet series where the sequence of the random coefficients $2^{J/2}\mu_{J,k}$ approximate $B_H(2^{-J}k)$ (see Proposition 2.1.7), which is not possible since the random variables $\mu_{J,k},\ k\in\mathbb{Z}$ are independent, while the random variables $B_H(2^{-J}k),\ k\in\mathbb{Z}$ are dependent.

Proof of (a) of Remark 2.2.8 Fix $x>1$. Observe that by using successively an integration by parts, the equality $\alpha\Gamma(\alpha)=\Gamma(\alpha+1)$ for all $\alpha>0$, the change of variable $v=x^{-1}s$ and by splitting the integral into two parts, one derives that

$$\begin{aligned}\Phi_{H+1}(x)&=\kappa_{H+1}\int_{-\infty}^{x}(x-s)^{H-1/2}\phi(s)ds\\&=\kappa_H\int_{-\infty}^{x}(x-s)^{H-3/2}\int_{-\infty}^{s}\phi(u)duds\\&=\kappa_H x^{H-1/2}\int_{-\infty}^{1}(1-v)^{H-3/2}\int_{-\infty}^{xv}\phi(u)dudv\\&=x^{H-1/2}f(x)+g(x),\end{aligned} \tag{2.2.17}$$

where $\kappa_H:=\Big(\Gamma(H-1/2)\Big)^{-1}$ and f and g are the two real-valued functions defined for all $x\geq 1$ by

$$f(x):=\kappa_H\int_0^1(1-v)^{H-3/2}\int_{-\infty}^{xv}\phi(u)dudv \ \text{ and } \ g(x):=\kappa_H x^{H-1/2}\int_0^{+\infty}(1+v)^{H-3/2}\int_{xv}^{+\infty}\phi(-u)dudv.$$

Using the fact that $\int_{\mathbb{R}}\phi(u)du=\widehat{\phi}(0)=1$ and the dominated convergence theorem, one easily derives that

$$\lim_{x\to+\infty}f(x)=C', \tag{2.2.18}$$

where C' is the positive and finite (since $H > 1/2$) constant defined as $C' = \kappa_H \int_0^1 (1-v)^{H-3/2} dv < +\infty$. On the other hand, observe that since $\phi \in S(\mathbb{R})$, one has

$$c_\varepsilon := \sup_{v\in\mathbb{R}} (1+|v|)^{H+1/2+\varepsilon} |\phi(v)| < +\infty,$$

for some arbitrary $\varepsilon > 0$ such that $H - 1/2 < \varepsilon < 3/2 - H$. Therefore, for every $x \geq 1$, one has

$$\begin{aligned}
|g(x)| &\leq c_\varepsilon \kappa_H x^{H-1/2} \int_0^{+\infty} (1+v)^{H-3/2} \int_{xv}^{+\infty} (1+u)^{-H-1/2-\varepsilon} du dv \\
&= c'_{\varepsilon,H} x^{H-1/2} \int_0^{+\infty} (1+v)^{H-3/2} (1+xv)^{-H+1/2-\varepsilon} dv \\
&\leq c'_{\varepsilon,H} x^{-(1+\varepsilon)} \int_0^{+\infty} (1+v)^{-(1+\varepsilon)} v^{-(H-1/2+\varepsilon)} dv \\
&\leq c''_{\varepsilon,H} x^{-(1+\varepsilon)},
\end{aligned}$$

which establishes that

$$\lim_{x\to+\infty} |g(x)| = 0. \tag{2.2.19}$$

Finally, by combining (2.2.18) and (2.2.19) with (2.2.17), one gets (2.2.16). □

In order to overcome both of the difficulties (a) and (b) outlined in Remark 2.2.8, the authors of [49] have replaced the fractional primitive of ϕ by the so-called fractional scaling function $\Phi_\Delta^{(d)}$ defined in the following way:

Definition 2.2.9. The *fractional scaling function* of order $d \in (-1/2, 1/2)$ of the Meyer scaling function, denoted by Φ_Δ^d is defined through its Fourier transform by

$$\widehat{\Phi}_\Delta^{(d)}(\xi) = \Big(\frac{1-e^{-i\xi}}{i\xi}\Big)^d \widehat{\phi}(\xi), \quad \text{for all } \xi \in \mathbb{R}^* \quad \text{and} \quad \widehat{\Phi}_\Delta^{(d)}(0) = 1, \tag{2.2.20}$$

with the convention that for any complex number z such that $z \notin (-\infty, 0]$, one has

$$z^d := \exp\big(d \log|z| + id\mathrm{Arg}(z)\big)$$

and where $\mathrm{Arg}(z)$ denotes the principal value, which is the unique value of $\arg(z)$ that belongs to $(-\pi, \pi)$.

Lemma 2.2.10. The fractional scaling function $\Phi_\Delta^{(d)}$ of the Meyer scaling function is well-defined, infinitely many times differentiable and well-localized. In other words, it belongs to the Schwartz space $S(\mathbb{R})$.

Proof. According to Remark 2.1.10, it is enough to show that the function $\widehat{\Phi}_\Delta^{(d)}$ belongs to $S(\mathbb{R})$. One denotes by g the function defined for all $\xi \in \mathbb{R}$ by

$$g(\xi) := \begin{cases} \frac{1-e^{-i\xi}}{i\xi} & \text{if} \quad \xi \neq 0 \\ 1 & \text{else.} \end{cases}$$

Notice that g is infinitely many times differentiable since it satisfies for all $\xi \in \mathbb{R}$,

$$g(\xi) = -\frac{1}{i\xi} \sum_{n=1}^{+\infty} \frac{(-i\xi)^n}{n!} = \sum_{n=1}^{+\infty} \frac{(-i\xi)^{n-1}}{n!}.$$

Observe that for all $\xi \in (-5\pi/3, 5\pi/3)$, $g(\xi) \notin (-\infty, 0]$, and thus the function $\xi \mapsto g(\xi)^d$ is infinitely many times differentiable on the interval $(-5\pi/3, 5\pi/3)$. By using the fact that the function $\widehat{\phi}$ is in $S(\mathbb{R})$ and that supp $\widehat{\phi} \subset [-4\pi/3, 4\pi/3]$, one derives that the function $\widehat{\Phi}_{\Delta}^{(d)}$ is infinitely many times differentiable on $(-5\pi/3, 5\pi/3)$ and has a compact supported included in $[-4\pi/3, 4\pi/3]$, which achieves the proof. $\square$

Remark 2.2.11. In the sequel, for any $d \neq 0$, the term $(1-e^{-i\xi})^{-d}$, will be expressed, for every $k \in \mathbb{Z}$ and $\xi \in \big(2\pi k, 2\pi(k+1)\big)$ as

$$(1-e^{-i\xi})^{-d} = \lim_{r \to 1^-} F_d(re^{-i\xi}), \tag{2.2.21}$$

where F_d is the continuous function defined for all complex number $z \notin [1, +\infty)$ as $F_d(z) = (1-z)^{-d}$. Recall that F_d is analytic on the disc $|z| < 1$ and that its Taylor expansion is given by

$$F_d(z) := \sum_{m=0}^{+\infty} \gamma_m^{(d)} z^m, \tag{2.2.22}$$

where the coefficients $\gamma_m^{(d)}$, $m \in \mathbb{Z}_+$ are defined as $\gamma_0^{(d)} := 1$ and, for $m \geq 1$, as

$$\gamma_m^{(d)} := \frac{d\, \Gamma(m+d)}{\Gamma(m+1)\Gamma(d+1)}. \tag{2.2.23}$$

Observe that using Stirling's formula

$$\Gamma(x) \underset{+\infty}{\sim} (2\pi)^{1/2} x^{x-1/2} e^{-x},$$

one can derive that

$$\gamma_m^{(d)} \sim \frac{de^{-d+1}}{\Gamma(d+1)} \frac{(m+d)^{m+d-1/2}}{(m+1)^{m+1/2}} \sim c_d m^{d-1} \quad \text{as } m \text{ goes to } +\infty, \tag{2.2.24}$$

with $c_d = d/\Gamma(d+1)$.

Notice that the use of the fractional scaling function $\Phi_{\Delta}^{(d)}$ instead of the fractional primitive Φ_{H+1} requires to modify the random coefficients $\mu_{J,l}$. From now on, our goal is to explain how they have to be modified.

Definition 2.2.12. Denote by $\mathfrak{B}$ the backshift operator defined for all time series $\{X_n\}_{n\in\mathbb{Z}}$ as $\mathfrak{B}X_n = X_{n-1}$, for each $n \in \mathbb{Z}$. Let $\{Z_n\}_{n\in\mathbb{Z}}$ be a sequence of independent identically distributed centred Gaussian random variables. Let d be a real number such that $|d| < 1/2$. The Gaussian

FARIMA$(0,d,0)$ (autoregressive fractionally integrated moving average), denoted by $\{Z_n^{(d)}\}_{n\in\mathbb{Z}}$, is defined for every $n \in \mathbb{Z}$ as

$$Z_n^{(d)} := (I-\mathfrak{B})^{-d} Z_n = \sum_{m=0}^{+\infty} \gamma_m^{(d)} \mathfrak{B}^m Z_n = \sum_{m=0}^{+\infty} \gamma_m^{(d)} Z_{n-m}, \tag{2.2.25}$$

where the convergence of the series (2.2.25) holds in $L^2(\Omega)$ and where $\gamma_m^{(d)}$, $m \in \mathbb{Z}_+$ are the coefficients defined in (2.2.23). The Gaussian FARIMA random walk time series associated to $\{Z_n^{(d)}\}_{n\in\mathbb{Z}}$, denoted by $\{S_N^{(d)}\}_{N\in\mathbb{Z}}$, is defined as follows

$$S_N^{(d)} := \begin{cases} \sum_{n=1}^N Z_n^{(d)} & \text{if} \quad N \geq 1, \\ 0 & \text{if} \quad N = 0, \\ -\sum_{n=N+1}^0 Z_n^{(d)} & \text{if} \quad N \leq -1. \end{cases} \tag{2.2.26}$$

Remark 2.2.13. The series in (2.2.25) is also almost surely convergent. Indeed, for each fixed $n \in \mathbb{Z}$, the sequence $\{\gamma_m^{(d)} Z_{n-m};\ m \in \mathbb{Z}_+\}$ is a sequence of independent random variables that satisfies for all $A > 0$, the three following conditions

(i) $\sum_{m=0}^{+\infty} \mathbb{P}\Big(\big|\gamma_m^{(d)} Z_{n-m}\big| \geq A\Big) < +\infty$,

(ii) $\sum_{m=0}^{+\infty} \mathbb{E}\left(\gamma_m^{(d)} Z_{n-m} 1_{|\gamma_m^{(d)} Z_{n-m}| \leq A}\right)$ converges,

(iii) $\sum_{m=0}^{+\infty} \mathrm{Var}\Big(\gamma_m^{(d)} Z_{n-m} 1_{|\gamma_m^{(d)} Z_{n-m}| \leq A}\Big) < +\infty$.

Therefore, the Kolmogorov's Three-Series theorem allows to conclude.

From now on, d will denote the real number

$$-1/2 < d := H - 1/2 < 1/2. \tag{2.2.27}$$

Let us return to the wavelet representation series (2.2.2). For each $J \in \mathbb{Z}$, one denotes by $\{B_{H,J}(t)\}_{t\in\mathbb{R}}$ the stochastic process defined by the first term in (2.2.2), namely,

$$B_{H,J}(t) := \sum_{l\in\mathbb{Z}} a_{J,l}(t)\mu_{J,l}, \text{ for all } t \in \mathbb{R}. \tag{2.2.28}$$

Using Plancherel formula, one gets

$$a_{J,l}(t) := \langle f_t, \phi_{J,k} \rangle = \langle \widehat{f}_t, \widehat{\phi}_{J,k} \rangle.$$

Notice that the Fourier transform of the functions f_t and $\phi_{J,l}$ are respectively given for all $\xi \in \mathbb{R}$, by

$$\widehat{f}_t(\xi) = \frac{1}{\sqrt{2\pi}} \frac{e^{-it\xi} - 1}{(-i\xi)^{H+1/2}} \quad \text{and} \quad \widehat{\phi}_{J,l}(\xi) = 2^{-J/2} e^{-i2^{-J}l\xi} \widehat{\phi}(2^{-J}\xi).$$

Next, by using successively the fact that $\frac{e^{it\xi}-1}{i\xi} = \int_0^t e^{i\xi s} ds$ (with the convention that for $t<0$, $\int_0^t = -\int_t^0$), Fubini's Theorem, the change of variable $\eta = 2^{-J}\xi$ and (2.2.20) with $d = H-1/2$, one derives

$$a_{J,l}(t) = \frac{1}{\sqrt{2\pi}} 2^{-J/2} \int_{\mathbb{R}} \int_0^t e^{i(s-2^{-J}l)\xi} \frac{\widehat{\phi}(2^{-J}\xi)}{(i\xi)^{H-1/2}} ds d\xi = \frac{1}{\sqrt{2\pi}} 2^{-J/2} \int_0^t \int_{\mathbb{R}} e^{i(s-2^{-J}l)\xi} \frac{\widehat{\phi}(2^{-J}\xi)}{(i\xi)^d} d\xi ds \tag{2.2.29}$$

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi}} 2^{-J(d-1/2)} \int_0^t \int_{\mathbb{R}} e^{i(2^J s-l)\eta} \frac{\widehat{\phi}(\eta)}{(i\eta)^d} d\eta ds \\ &= \frac{1}{\sqrt{2\pi}} 2^{-J(d-1/2)} \int_0^t \int_{\mathbb{R}} e^{i(2^J s-l)\eta} (1-e^{-i\eta})^{-d} \widehat{\Phi}_\Delta^{(d)}(\eta) d\eta ds \\ &= \frac{1}{\sqrt{2\pi}} 2^{-J(d-1/2)} \int_0^t \widetilde{a}_{J,l}^{(d)}(s) ds, \end{aligned} \tag{2.2.30}$$

where

$$\widetilde{a}_{J,l}^{(d)}(s) := \int_{\mathbb{R}} e^{i(2^J s-l)\eta} (1-e^{-i\eta})^{-d} \widehat{\Phi}_\Delta^{(d)}(\eta) d\eta. \tag{2.2.31}$$

Observe that the second equality in (2.2.29) holds since

$$\Big| \int_0^t \int_{\mathbb{R}} e^{i(s-2^{-J}l)\xi} \frac{\widehat{\phi}(2^{-J}\xi)}{(i\xi)^d} d\xi ds \Big| \le \int_0^t \int_{\mathbb{R}} \frac{|\widehat{\phi}(2^{-J}\xi)|}{|\xi|^d} d\xi ds = t 2^{-J(1-d)} \int_{\mathbb{R}} \frac{|\widehat{\phi}(\eta)|}{|\eta|^d} d\eta < +\infty.$$

Lemma 2.2.14. Let $(r_n)_{n\in\mathbb{N}}$ be an increasing sequence of real numbers belonging to $(0,1)$ such that $\lim_{n\to+\infty} r_n = 1$. Then, for all $s\in\mathbb{R}$, one has

$$\widetilde{a}_{J,l}^{(d)}(s) = \lim_{n\to+\infty} \int_{\mathbb{R}} e^{i(2^J s-l)\eta} (1-r_n e^{-i\eta})^{-d} \widehat{\Phi}_\Delta^{(d)}(\eta) d\eta. \tag{2.2.32}$$

Proof. First, recall that the function $\widehat{\Phi}_\Delta^{(d)}$ is in the Schwartz class $S(\mathbb{R})$ and that its support satisfies

$$\text{supp } \widehat{\Phi}_\Delta^{(d)} \subset \Big[-\frac{4\pi}{3}, \frac{4\pi}{3} \Big]. \tag{2.2.33}$$

Therefore,

$$\int_{\mathbb{R}} e^{i(2^J s-l)\eta} (1-r_n e^{-i\eta})^{-d} \widehat{\Phi}_\Delta^{(d)}(\eta) d\eta = \int_{-4\pi/3}^{4\pi/3} e^{i(2^J s-l)\eta} (1-r_n e^{-i\eta})^{-d} \widehat{\Phi}_\Delta^{(d)}(\eta) d\eta.$$

Observe that in the case where $-1/2 < d \le 0$ (i.e. $0 < H \le 1/2$), (2.2.32) is a straightforward consequence of the dominated convergence theorem and the following inequality

$$\Big| e^{i(2^J s-l)\eta} (1-r_n e^{-i\eta})^{-d} \widehat{\Phi}_\Delta^{(d)}(\eta) \Big| \le 2 |\widehat{\Phi}_\Delta^{(d)}(\eta)|, \text{ for all } \eta\in\mathbb{R}.$$

Now consider the case $0<d<1/2$ (i.e. $1/2<H<1$). Let A_n and A_n^c be the two sets defined for all $n\in\mathbb{N}$, by

$$A_n := \Big\{ \eta \in \Big[-\frac{4\pi}{3}, \frac{4\pi}{3}\Big];\ |1-e^{-i\eta}| \le 2(1-r_n) \Big\} \text{ and } A_n^c := \Big\{ \eta \in \Big[-\frac{4\pi}{3}, \frac{4\pi}{3}\Big];\ |1-e^{-i\eta}| > 2(1-r_n) \Big\}.$$

Observe that for all n big enough, one has that $\eta \in A_n \Rightarrow |\eta| \leq 4(1-r_n)$. Therefore,

$$\begin{aligned}\left|\int_{A_n} e^{i(2^J s-l)\eta}(1-r_n e^{-i\eta})^{-d}\widehat{\Phi}_\Delta^{(d)}(\eta)d\eta\right| &\leq \left\|\widehat{\Phi}_\Delta^{(d)}\right\|_\infty (1-r_n)^{-d}\int_R 1_{A_n}(\eta)d\eta \\ &\leq 8\left\|\widehat{\Phi}_\Delta^{(d)}\right\|_\infty (1-r_n)^{1-d},\end{aligned}$$

where we used the fact that $|1-r_n e^{-i\eta}| \geq 1-r_n$. Thus, since $1-d>0$, one obtains

$$\lim_{n\to+\infty}\int_{A_n} e^{i(2^J s-l)\eta}(1-r_n e^{-i\eta})^{-d}\widehat{\Phi}_\Delta^{(d)}(\eta)d\eta = 0. \tag{2.2.34}$$

On the other hand, notice that for all $\eta \in A_n^c$, one has

$$\begin{aligned}\left|1-r_n e^{-i\eta}\right| &= \left|1-e^{-i\eta}+(1-r_n)e^{-i\eta}\right| \\ &\geq \left|\left|1-e^{-i\eta}\right| - \left|1-r_n\right|\right| \\ &= \left|1-e^{-i\eta}\right| - (1-r_n) \\ &> \frac{1}{2}\left|1-e^{-i\eta}\right|. \end{aligned}\tag{2.2.35}$$

Therefore, for all $\eta \in \mathbb{R}$, one has

$$\left|1_{A_n^c}(\eta)e^{i(2^J s-l)\eta}(1-r_n e^{-i\eta})^{-d}\widehat{\Phi}_\Delta^{(d)}(\eta)\right| \leq 2^d\left|1-e^{-i\eta}\right|^{-d}\left|\widehat{\Phi}_\Delta^{(d)}(\eta)\right|.$$

By using the previous inequality, the fact that for all $\eta \in [-4\pi/3, 4\pi/3]\backslash\{0\}$, $\lim_{n\to+\infty} 1_{A_n^c}(\eta) = 1$ and the dominated convergence theorem, one derives that

$$\lim_{n\to+\infty}\int_{A_n^c} e^{i(2^J s-l)\eta}(1-r_n e^{-i\eta})^{-d}\widehat{\Phi}_\Delta^{(d)}(\eta)d\eta = \int_{-\frac{4\pi}{3}}^{\frac{4\pi}{3}} e^{i(2^J s-l)\eta}(1-e^{-i\eta})^{-d}\widehat{\Phi}_\Delta^{(d)}(\eta)d\eta. \tag{2.2.36}$$

Finally, by combining (2.2.34) and (2.2.36), one obtains (2.2.32). □

Lemma 2.2.15. For each fixed integer $n \in \mathbb{N}$, one has

$$\int_{\mathbb{R}} e^{i(2^J s-l)\eta}(1-r_n e^{-i\eta})^{-d}\widehat{\Phi}_\Delta^{(d)}(\eta)d\eta = \sqrt{2\pi}\sum_{m=0}^{+\infty}\gamma_m^{(d)} r_n^m \Phi_\Delta^{(d)}\Big(2^J s-(l+m)\Big). \tag{2.2.37}$$

Proof. Observe that since $r_n < 1$ for all $n \in \mathbb{N}$, one has

$$\sum_{m=0}^{+\infty}\int_{\mathbb{R}}\left|e^{i(2^J s-l)\eta}\gamma_m^{(d)} r_n^m e^{-i\eta m}\widehat{\Phi}_\Delta^{(d)}(\eta)\right|d\eta = \Big(\int_{\mathbb{R}}|\widehat{\Phi}_\Delta^{(d)}(\eta)|d\eta\Big)\sum_{m=0}^{+\infty} r_n^m\gamma_m^{(d)} < +\infty.$$

Therefore, we can switch the sum and the integral to get

$$\begin{aligned}\int_{\mathbb{R}} e^{i(2^J s-l)\eta}(1-r_n e^{-i\eta})^{-d}\widehat{\Phi}_\Delta^{(d)}(\eta)d\eta &= \sum_{m=0}^{+\infty}\gamma_m^{(d)} r_n^m\int_{\mathbb{R}} e^{i\left(2^J s-(l+m)\right)\eta}\widehat{\Phi}_\Delta^{(d)}(\eta)d\eta \\ &= \sqrt{2\pi}\sum_{m=0}^{+\infty}\gamma_m^{(d)} r_n^m \Phi_\Delta^{(d)}\Big(2^J s-(l+m)\Big). \end{aligned}\tag{2.2.38}$$

□

Lemma 2.2.16. For all $l \in \mathbb{Z}$ and all $s \in \mathbb{R}$, one has

$$\widetilde{a}_{J,l}^{(d)}(s) = \sqrt{2\pi} \sum_{m=0}^{+\infty} \gamma_m^{(d)} \Phi_\Delta^{(d)} \Big(2^J s - (l+m) \Big). \tag{2.2.39}$$

Proof. The equality (2.2.39) is a straightforwad consequence of Lemma 2.2.14, Lemma 2.2.15, and the dominated convergence theorem since for each fixed $s \in \mathbb{R}$ and $l \in \mathbb{Z}$, one has

$$\sum_{m=0}^{+\infty} \gamma_m^{(d)} \big| \Phi_\Delta^{(d)}(2^J s - (l+m)) \big| < +\infty.$$

□

One sets $\gamma_m^{(d)} = 0$ for all $m \in \mathbb{Z}_-^*$. Also, for each integer $L \geq 1$, one sets $\mu_{J,l}^{(L)} = \mu_{J,l}$ if $|l| \leq L$ and $\mu_{J,l}^{(L)} = 0$ else. Let $B_{H,J}^{(L)}(t)$ be the random variable defined, for all $t \in \mathbb{R}$, $J \in \mathbb{Z}$ and $L \in \mathbb{N}$ as

$$B_{H,J}^{(L)}(t) := \sum_{|l| \leq L} a_{J,l}(t) \mu_{J,l} = \sum_{l \in \mathbb{Z}} a_{J,l}(t) \mu_{J,l}^{(L)}. \tag{2.2.40}$$

Observe that

$$\lim_{L \to +\infty} B_{H,J}^{(L)}(t) \overset{L^2(\Omega)}{=} B_{H,J}(t).$$

As we can extract a subsequence of $\big\{ B_{H,J}^{(L)}(t); L \geq 1 \big\}$ which is almost surely convergent, we may assume that

$$\lim_{L \to +\infty} B_{H,J}^{(L)}(t) \overset{a.s.}{=} B_{H,J}(t). \tag{2.2.41}$$

Notice that using (2.2.30), (2.2.39) and the finiteness of the sum in (2.2.40), one derives

$$B_{H,J}^{(L)}(t) = \sum_{l \in \mathbb{Z}} a_{J,l}(t) \mu_{J,l}^{(L)} = \frac{1}{\sqrt{2\pi}} 2^{-J(d-1/2)} \int_0^t \sum_{l \in \mathbb{Z}} \widetilde{a}_{J,l}^{(d)}(s) \mu_{J,l}^{(L)} ds. \tag{2.2.42}$$

By using successively Lemma 2.2.16 and the change of variables $m = q - l$, by interchanging the sum over l and the sum over q, and by using the change of variables $p = q - l$, one gets

$$\begin{aligned}
\sum_{l \in \mathbb{Z}} \widetilde{a}_{J,l}^{(d)}(s) \mu_{J,l}^{(L)} &= \sqrt{2\pi} \sum_{l \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \gamma_m^{(d)} \Phi_\Delta^{(d)} \Big(2^J s - (l+m) \Big) \mu_{J,l}^{(L)} \\
&= \sqrt{2\pi} \sum_{l=-\infty}^{+\infty} \sum_{q=-\infty}^{+\infty} \gamma_{q-l}^{(d)} \Phi_\Delta^{(d)} \Big(2^J s - q \Big) \mu_{J,l}^{(L)} \\
&= \sqrt{2\pi} \sum_{q \in \mathbb{Z}} \Bigg(\sum_{p \in \mathbb{Z}} \gamma_p^{(d)} \mu_{J,q-p}^{(L)} \Bigg) \Phi_\Delta^{(d)}(2^J s - q)
\end{aligned}$$

Therefore,

$$\sum_{l \in \mathbb{Z}} \widetilde{a}_{J,l}^{(d)}(s) \mu_{J,l}^{(L)} = \sqrt{2\pi} \sum_{q \in \mathbb{Z}} \epsilon_{J,q}^{(d,L)} \Phi_\Delta^{(d)}(2^J s - q), \tag{2.2.43}$$

where $\epsilon_{J,q}^{(d,L)}, q \in \mathbb{Z}$ is the sequence of centred Gaussian random variables defined by

$$\epsilon_{J,q}^{(d,L)} := \sum_{p \in \mathbb{Z}} \gamma_p^{(d)} \mu_{J,q-p}^{(L)} = \sum_{p=\max(0,-L+q)}^{L+q} \gamma_p^{(d)} \mu_{J,q-p}. \tag{2.2.44}$$

By using the Kolmogorov three series theorem (see Remark 2.2.13), one derives that

$$\lim_{L \to +\infty} \epsilon_{J,q}^{(d,L)} \overset{a.s.}{=} \epsilon_{J,q}^{(d)}, \tag{2.2.45}$$

where $\{\epsilon_{J,q}^{(d)}\}_{q \in \mathbb{Z}}$ is the sequence of Gaussian FARIMA$(0,d,0)$ defined for all $(J,q) \in \mathbb{Z}^2$ by

$$\epsilon_{J,q}^{(d)} := \sum_{p=0}^{+\infty} \gamma_p^{(d)} \mu_{J,q-p}. \tag{2.2.46}$$

Lemma 2.2.17. There exist an event $\widetilde{\Omega}$, of probability 1, and a finite random variable $\widetilde{C}$ such that for all $(J,q,\omega) \in \mathbb{Z}^2 \times \widetilde{\Omega}$,

$$\sup_{L \geq 1} \left| \epsilon_{J,q}^{(d,L)}(\omega) \right| \leq \widetilde{C}(\omega) \sqrt{\log(3 + |J| + |q|)}. \tag{2.2.47}$$

Moreover, for all $(J,\omega,u) \in \mathbb{Z} \times \widetilde{\Omega} \times \mathbb{R}$, one has

$$\sum_{q \in \mathbb{Z}} \sup_{L \geq 1} \left| \epsilon_{J,q}^{(d,L)}(\omega) \right| \left| \Phi_{\Delta}^{(d)}(u-q) \right| \leq C(\omega) \sqrt{\log(3 + |J| + |u|)}. \tag{2.2.48}$$

The following corollary is a straighforward consequence of (2.2.42), (2.2.43), (2.2.45), (2.2.48) and the dominated convergence theorem.

Corollary 2.2.18. For each $t \in \mathbb{R}$, one has almost surely,

$$\lim_{L \to +\infty} B_{H,J}^{(L)}(t) = 2^{-J(d+1/2)} \int_0^{2^J t} \sum_{q \in \mathbb{Z}} \epsilon_{J,q}^{(d)} \Phi_{\Delta}^{(d)}(u-q) du. \tag{2.2.49}$$

Proof of Lemma 2.2.17 First, notice that (2.2.44) entails that the sequence $\{\epsilon_{J,q}^{(d,L)}\}_{q \in \mathbb{Z}}$ is a sequence of centred Gaussian random variables. Moreover, one can derive from the independence of the standard Gaussian random variables $\{\mu_{J,l}\}_{l \in \mathbb{Z}}$ that for all $(L,q) \in \mathbb{N} \times \mathbb{Z}$,

$$\operatorname{Var}\big(\epsilon_{J,q}^{(d,L)}\big) = \sum_{p=\max(0,-L+q)}^{L+q} \left(\gamma_p^{(d)}\right)^2 := \sigma_{L,q}^2 \leq \sum_{p=0}^{+\infty} \left(\gamma_p^{(d)}\right)^2 := \sigma^2 < +\infty, \tag{2.2.50}$$

since $\left(\gamma_p^{(d)}\right)^2 \underset{p \to +\infty}{\sim} cp^{2d-2}$, for some constant $c > 0$ and $2d-2 < -1$. Let $(J,q) \in \mathbb{Z}^2$ be arbitrary and fixed. For each $L \in \mathbb{N}$, one denotes by $\mathcal{F}_L$ the $\sigma-$field spanned by the random variables $\Big\{\mu_{J,q-p} \ ; \ \max(0,-L+q) \leq p \leq L+q\Big\}$. Notice that $\Big\{\epsilon_{J,q}^{(d,L)} \ ; \ L \in \mathbb{N}\Big\}$ is a $\Big(\mathcal{F}_L\Big)_{L \in \mathbb{N}}-$martingale.

Consider the function g defined for all $x \in \mathbb{R}$ as $g(x) = \exp(cx^2)$, where c is a positive constant small enough which will be defined more precisely later. Since g is a positive convex function, the sequence $\left\{ g\big(\epsilon_{J,q}^{(d,L)}\big) \ ; \ L \in \mathbb{N} \right\}$ is a positive submartingale. Hence, by using the Doob inequality, one gets that for all $t \geq 1$ and each $N \in \mathbb{N}$,

$$\mathbb{P}\Big(\sup_{1 \leq L \leq N} g\big(\epsilon_{J,q}^{(d,L)}\big) > t \Big) \leq \frac{\mathbb{E}\left(g\big(\epsilon_{J,q}^{(d,N)}\big) \right)}{t}. \tag{2.2.51}$$

By observing that g is an even function that increases over $\mathbb{R}_+$, one derives

$$\begin{aligned} \mathbb{P}\Big(\sup_{1 \leq L \leq N} g\big(\epsilon_{J,q}^{(d,L)}\big) > t \Big) &= \mathbb{P}\Big(\sup_{1 \leq L \leq N} g\Big(\big|\epsilon_{J,q}^{(d,L)}\big| \Big) > t \Big) \\ &= \mathbb{P}\Big(g\Big(\sup_{1 \leq L \leq N} \big|\epsilon_{J,q}^{(d,L)}\big| \Big) > t \Big) \\ &= \mathbb{P}\Big(\sup_{1 \leq L \leq N} \big|\epsilon_{J,q}^{(d,L)}\big| > h(t) \Big), \end{aligned} \tag{2.2.52}$$

where h is the funtion defined, for all $t \geq 1$, by $h(t) = \sqrt{\frac{\log(t)}{c}}$. Thus, by combining (2.2.51) and (2.2.52), one gets

$$\mathbb{P}\Big(\sup_{1 \leq L \leq N} \big|\epsilon_{J,q}^{(d,L)}\big| > h(t) \Big) \leq \frac{\mathbb{E}\left(g\big(\epsilon_{J,q}^{(d,N)}\big) \right)}{t}. \tag{2.2.53}$$

Our goal now is to bound uniformly in N the quantity $\mathbb{E}\left(g\big(\epsilon_{J,q}^{(d,N)}\big) \right)$. Recall that $\epsilon_{J,q}^{(d,N)} \sim \mathcal{N}(0, \sigma_{N,q}^2)$ and that $\sigma_{N,q}^2 \leq \sigma^2$ (see (2.2.50)). Therefore, by using the change of variable $u = \sigma_{N,q}^{-1} x$, and by choosing the constant c such that $0 < c < \frac{1}{2\sigma^2}$ (so that $1 - 2\sigma_{N,q}^2 c > 1 - 2\sigma^2 c > 0$), one obtains

$$\begin{aligned} \mathbb{E}\left(g\big(\epsilon_{J,q}^{(d,N)}\big) \right) &= (2\pi\sigma_{N,q}^2)^{-1/2} \int_{\mathbb{R}} \exp\Big(- \frac{1}{2\sigma_{N,q}^2} x^2 (1 - 2\sigma_{N,q}^2 c) \Big) dx \\ &= (2\pi)^{-1/2} \int_{\mathbb{R}} \exp\Big(- \frac{1}{2} u^2 (1 - 2\sigma_{N,q}^2 c) \Big) du \\ &\leq (2\pi)^{-1/2} \int_{\mathbb{R}} \exp\Big(- \frac{1}{2} u^2 (1 - 2\sigma^2 c) \Big) du = \sqrt{\frac{1}{1 - 2\sigma^2 c}} := \kappa. \end{aligned} \tag{2.2.54}$$

Hence, by combining (2.2.53) and (2.2.54), one derives that for any $(J, q, N) \in \mathbb{Z}^2 \times \mathbb{N}$ and each $t \geq 1$,

$$\mathbb{P}\Big(\sup_{1 \leq L \leq N} \big|\epsilon_{J,q}^{(d,L)}\big| > h(t) \Big) \leq \frac{\kappa}{t}.$$

Thus, letting N goes to $+\infty$, one gets

$$\mathbb{P}\Big(\sup_{L \geq 1} \big|\epsilon_{J,q}^{(d,L)}\big| > h(t) \Big) \leq \frac{\kappa}{t}. \tag{2.2.55}$$

Let $\delta > 0$ be an arbitray positive real number. By applying (2.2.55) to $t = (1 + |J| + |q|)^{\delta^2 c}$, one obtains:

$$\mathbb{P}\Big(\sup_{L\geq 1}\big|\epsilon_{J,q}^{(d,L)}\big| > \delta\sqrt{\log(3+|J|+|q|)}\Big) \leq \frac{\kappa}{(1+|J|+|q|)^{\delta^2 c}}. \tag{2.2.56}$$

Therefore, by choosing δ big enough so that $\delta^2 c > 2$, one derives that

$$\sum_{(J,q)\in\mathbb{Z}^2} \mathbb{P}\Big(\sup_{L\geq 1}\big|\epsilon_{J,q}^{(d,L)}\big| \geq \delta\sqrt{\log(3+|J|+|q|)}\Big) < +\infty.$$

By using Borel-Cantelli lemma and by proceeding as in the proof of Lemma 2.2.1, one derives the existence of an event $\widetilde{\Omega}$, of probability 1, and a finite random variable $\widetilde{C}$ such that for all $(J, q, \omega) \in \mathbb{Z} \times \mathbb{Z} \times \widetilde{\Omega}$,

$$\sup_{L\geq 1}\big|\epsilon_{J,q}^{(d,L)}(\omega)\big| \leq \widetilde{C}(\omega)\sqrt{\log(3+|J|+|q|)}. \tag{2.2.57}$$

By using successively the inequality (2.2.57), Lemma 2.2.10, the change of variable $m = q+\lfloor u\rfloor$, the inequality $\sqrt{\log(3+x+y)} \leq \sqrt{\log(3+x)}\sqrt{\log(3+y)}$, for all $x, y \in \mathbb{R}_+$, one derives that

$$\begin{aligned}
\sum_{q\in\mathbb{Z}} \sup_{L\geq 1}\big|\epsilon_{J,q}^{(d,L)}(\omega)\big|\big|\Phi_\Delta^{(d)}(u-q)\big| &\leq C_1(\omega)\sum_{q\in\mathbb{Z}}\sqrt{\log(3+|J|+|q|)}\big(2+|u-q|\big)^{-2} \\
&\leq C_1(\omega)\sum_{m\in\mathbb{Z}}\sqrt{\log(3+|J|+|m|+\big|\lfloor u\rfloor\big|)}\big(1+|m|\big)^{-2} \\
&\leq C_1(\omega)\sqrt{\log(3+|J|+|u|)}\sum_{m\in\mathbb{Z}}\sqrt{\log(4+|m|)}(1+|m|)^{-2} \\
&\leq C(\omega)\sqrt{\log(3+|J|+|u|)},
\end{aligned} \tag{2.2.58}$$

which establishes (2.2.48). □

Combining (2.2.41) and (2.2.49), one obtains that, for all $t \in \mathbb{R}$, almost surely,

$$B_{H,J}(t) = 2^{-J(d+1/2)}\int_0^{2^J t}\sum_{q\in\mathbb{Z}}\epsilon_{J,q}^{(d)}\Phi_\Delta^{(d)}(u-q)du. \tag{2.2.59}$$

We will use the following lemma borrowed from [49] to give the final expression of the second wavelet type expansion of FBM. It can be derived using an Abel transform (summation by parts).

Lemma 2.2.19. Let f be a function in the Schwartz space $S(\mathbb{R})$ and let $a_q, q \in \mathbb{Z}$, be a slowly increasing sequence, that is $|a_q| \leq C(1+|q|^m)$, for some constant C and integer m. Define A_q by $A_0 = 0$ and $A_q - A_{q-1} = a_q$, $q \in \mathbb{Z}$, and $\widetilde{f}(t) = \int_{t-1}^t f(s)ds$, for all $t \in \mathbb{R}$. Then, $\widetilde{f} \in S(\mathbb{R})$ and A_q is a slowly increasing sequence. Moreover,

$$\int_0^t \Big\{\sum_{q\in\mathbb{Z}} a_q f(s-q)\Big\}ds = \sum_{q\in\mathbb{Z}} A_q\Big(\widetilde{f}(t-q) - \widetilde{f}(-q)\Big). \tag{2.2.60}$$

Define the Gaussian FARIMA random walk time series $\{S_{J,q}^{(d)}\}_{q\in\mathbb{Z}}$ associated to $\{\epsilon_{J,q}^{(d)}\}_{q\in\mathbb{Z}}$, namely

$$S_{J,q}^{(d)} := \begin{cases} \sum_{k=1}^{q} \epsilon_{J,k}^{(d)} & \text{if} \quad q \geq 1 \\ 0 & \text{if} \quad q = 0, \\ -\sum_{k=q+1}^{0} \epsilon_{J,k}^{(d)} & \text{if} \quad q \leq -1. \end{cases} \tag{2.2.61}$$

By applying Lemma 2.2.19 with $f = \Phi_\Delta^{(d)}$, $a_q = \epsilon_{J,q}^{(d)}$ and $A_q = S_{J,q}^{(d)}$, to the sum (2.2.59), one gets

$$B_{H,J}(t) = 2^{-J(d+1/2)} \int_0^{2^J t} \sum_{q\in\mathbb{Z}} \epsilon_{J,q}^{(d)} \Phi_\Delta^{(d)}(u-q)du = 2^{-J(d+1/2)} \sum_{q\in\mathbb{Z}} S_{J,q}^{(d)} \Big(\widetilde{\Phi}_\Delta^{(d)}(2^J t - q) - \widetilde{\Phi}_\Delta^{(d)}(-q) \Big), \tag{2.2.62}$$

where the real-valued function $\widetilde{\Phi}_\Delta^{(d)}$ is defined by

$$\widetilde{\Phi}_\Delta^{(d)}(t) = \int_{t-1}^{t} \Phi_\Delta^{(d)}(s)ds.$$

Observe that using Plancherel formula and (2.1.15), one gets

$$\widetilde{\Phi}_\Delta^{(d)}(t) = \int_{\mathbb{R}} 1_{[t-1,t]}(s)\Phi_\Delta^{(d)}(s)ds = \int_{\mathbb{R}} \overline{\widehat{1}_{[t-1,t]}(\xi)} \widehat{\Phi}_\Delta^{(d)}(\xi)d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi t} \frac{1-e^{-i\xi}}{i\xi} \widehat{\Phi_\Delta^{(d)}}(\xi)d\xi. \tag{2.2.63}$$

One deduces from (2.2.63) and (2.2.20) that the Fourier transform of the function $\widetilde{\Phi}_\Delta^{(d)}$ is given for all $\xi \in \mathbb{R}$ by

$$\widehat{\widetilde{\Phi}_\Delta^{(d)}}(\xi) = \frac{1-e^{-i\xi}}{i\xi} \widehat{\Phi}_\Delta^{(d)}(\xi) = \frac{(1-e^{-i\xi})}{(i\xi)} \frac{(1-e^{-i\xi})^d}{(i\xi)^d} \widehat{\phi}(\xi) = \widehat{\Phi}_\Delta^{(d+1)}(\xi). \tag{2.2.64}$$

By combining (2.2.62) and (2.2.64), one derives that

$$B_{H,J}(t) = 2^{-J(d+1/2)} \sum_{q\in\mathbb{Z}} S_{J,q}^{(d)} \Big(\Phi_\Delta^{(d+1)}(2^J t - q) - \Phi_\Delta^{(d+1)}(-q) \Big). \tag{2.2.65}$$

We are now in a position to give the second wavelet representation of FBM.

Theorem 2.2.20. ***(The second wavelet representation of FBM)*** *The FBM $\{B_H(t)\}_{t\in\mathbb{R}}$ can be expressed as,*

$$B_H(t) = \sum_{q\in\mathbb{Z}} 2^{-JH} \big(\Phi_\Delta^{(H+1/2)}(2^J t - q) - \Phi_\Delta^{(H+1/2)}(-q)\big) S_{J,q}^{(H-1/2)} + \sum_{j=J}^{+\infty} \sum_{k\in\mathbb{Z}} 2^{-jH} \big(\Psi_H(2^j t - k) - \Psi_H(-k)\big)\varepsilon_{j,k}. \tag{2.2.66}$$

Moreover, the series in (2.2.66) are almost surely, uniformly convergent on each compact subset of $\mathbb{R}$.

Proof. First, notice that we already showed that the second series in (2.2.66) is almost surely uniformly convergent on each compact subset of $\mathbb{R}$ (see Theorem 2.2.5). Thus, it only remains to show that the first series in (2.2.65), denoted by $B_{H,J}(t)$, is almost surely, uniformly convergent on each compact subset of $\mathbb{R}$. Let T be an arbitrary positive real number. We will show that the series

$$\sum_{q\in\mathbb{Z}} \Phi_{\Delta}^{(H+1/2)}(2^J t-q)S_{J,q}^{(H-1/2)}$$

is almost surely uniformly convergent on $[-T,T]$. Observe that using Lemma 2.2.17 and (2.2.61), one can bound the random variables $S_{J,q}^{H-1/2}$, $J,q\in\mathbb{Z}$ on the event $\widetilde{\Omega}$, of probability 1, introduced in Lemma 2.2.17. More precisely, for each $(J,q,\omega)\in\mathbb{Z}^2\times\widetilde{\Omega}$, one has

$$|S_{J,q}^{(H-1/2)}(\omega)| \le C(\omega)(1+|q|)\sqrt{\log\Big(3+|J|+|q|\Big)}, \tag{2.2.67}$$

for some finite random variable C, not depending on J and q. Next, if follows from the triangle inequality, Lemma 2.2.10, the inequality (2.2.67), the change of variable $m=q+\lfloor 2^J t\rfloor$, the inequality $\sqrt{\log(4+x+y)} \le \sqrt{\log(4+x)}\sqrt{\log(4+y)}$, for all $x,y\ge 0$, and the fact that $|u-\lfloor u\rfloor|\le 1$, one derives that for any $t\in[-T,T]$ and each fixed $L>2$,

$$\begin{aligned}
&\sum_{q\in\mathbb{Z}} \big|\Phi_{\Delta}^{(H+1/2)}(2^J t-q)\big|\big|S_{J,q}^{(H-1/2)}(\omega)\big| \\
&\quad\le C_1(\omega)\sum_{q\in\mathbb{Z}}\big(1+|q|\big)\big(2+|2^J t-q|\big)^{-L}\sqrt{\log\Big(3+|J|+|q|\Big)} \\
&\quad\le C_1(\omega)\sum_{m\in\mathbb{Z}}\Big(1+|m|+\big|\lfloor 2^J t\rfloor\big|\Big)\Big(2+\big|2^J t-\lfloor 2^J t\rfloor-m\big|\Big)^{-L}\sqrt{\log\Big(3+|m|+|J|+\big|\lfloor 2^J t\rfloor\big|\Big)} \\
&\quad\le C_1(\omega)\sum_{m\in\mathbb{Z}}(2+|m|+|2^J t|)\Big(2+|m|-\big|2^J t-\lfloor 2^J t\rfloor\big|\Big)^{-L}\sqrt{\log\Big(4+|m|+|J|+|2^J t|\Big)} \\
&\quad\le C_2(\omega)\big(2+|2^J T|\big)\sqrt{\log\Big(4+|J|+|2^J T|\Big)},
\end{aligned} \tag{2.2.68}$$

where C_1 is an almost surely finite random variable, and $C_2=C_1\sum_{m\in\mathbb{Z}}(2+|m|)^{-(L-1)}\sqrt{\log(4+|m|)}$, which achieves the proof.

□

2.3 Wavelet-type expansion of the generalized Rosenblatt process

The goal of this section is to extend the results presented in the previous section to the case of the generalized Rosenblatt process, which has been briefly introduced by the end of the first chapter. Our approach is inspired from the one employed by Pipiras in [57] in the context of the Rosenblatt process.

2.3.1 Framework

The generalized Rosenblatt process, also called the generalized fractional Rosenblatt motion (gfRm), is a stochastic process that belongs to the second homogeneous Wiener chaos. It is denoted by $\left\{R_{H_1,H_2}(t)\right\}_{t\in\mathbb{R}_+}$ and depends on two parameters $H_1, H_2 \in (1/2, 1)$ which satisfy $H_1 + H_2 > 3/2$. The gfRm is defined for all $t \in \mathbb{R}_+$, through the double Wiener integral,

$$R_{H_1,H_2}(t) := \int_{\mathbb{R}^2} K_{H_1,H_2}(t, x_1, x_2)\, dB(x_1)dB(x_2), \qquad \text{for all } t \in \mathbb{R}_+\,, \tag{2.3.1}$$

where, the integrand K_{H_1,H_2} is given, for every $(t, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^2$, by:

$$K_{H_1,H_2}(t, x_1, x_2) := \frac{1}{\Gamma(H_1 - 1/2)\Gamma(H_2 - 1/2)} \int_0^t (s - x_1)_+^{H_1-3/2}(s - x_2)_+^{H_2-3/2}\, ds. \tag{2.3.2}$$

Observe that the gfRm reduces to the Rosenblatt process when $H_1 = H_2$. Also, notice that the condition $H_1 + H_2 > 3/2$ implies that, for each fixed $t \in \mathbb{R}_+$, the kernel function $(x_1, x_2) \mapsto K_{H_1,H_2}(t, x_1, x_2)$ is in $L^2(\mathbb{R}^2)$, which guarantees the existence of the double Wiener integral in (2.3.1). Moreover, by proceeding as in the proof of Proposition 1.3.10, one can establish the following proposition.

Proposition 2.3.1. The gfRm $\left\{R_{H_1,H_2}(t)\right\}_{t\in\mathbb{R}_+}$ is a $\{H_1 + H_2 - 1\}$-SSSI continuous stochastic process having finite moments of any order.

Let $(V_j)_{j\in\mathbb{Z}}$ be an MRA of $L^2(\mathbb{R})$ associated with an univariate Meyer scaling function ϕ and a corresponding mother wavelet ψ. Also, for each $j \in \mathbb{Z}$, denote by W_j the orthogonal complement of V_j in V_{j+1}. One knows from Section 2.1.2 that one can construct a MRA $(V_j^2)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^2)$ starting from $(V_j)_{j\in\mathbb{Z}}$ by setting

$$V_j^2 = V_j \otimes V_j.$$

Let $J \in \mathbb{Z}$ be arbitrary and fixed. It follows from (2.1.20) that $L^2(\mathbb{R}^2)$ admits the following orthogonal decomposition

$$L^2(\mathbb{R}^2) = \Big(V_J \otimes V_J\Big) \overset{\perp}{\bigoplus} \Big(\overset{\perp}{\bigoplus_{j\geq J}} W_j \otimes W_j\Big) \overset{\perp}{\bigoplus} \Big(\overset{\perp}{\bigoplus_{j\geq J}} V_j \otimes W_j\Big) \overset{\perp}{\bigoplus} \Big(\overset{\perp}{\bigoplus_{j\geq J}} W_j \otimes V_j\Big). \tag{2.3.3}$$

Notice that Theorem 2.1.13 provides an orthonormal basis of $L^2(\mathbb{R}^2)$ which is adapted to the decomposition (2.3.3). Namely, for each fixed $J \in \mathbb{Z}$, the sequence of functions:

$$\begin{aligned}
\widetilde{\mathcal{B}}_J := \Big\{\phi_{J,k_1} \otimes \phi_{J,k_2};\ (k_1, k_2) \in \mathbb{Z}^2\Big\} & \bigcup \Big\{\psi_{j,k_1} \otimes \psi_{j,k_2};\ j \geq J,\ (k_1, k_2) \in \mathbb{Z}^2\Big\} \\
& \bigcup \Big\{\phi_{j,k_1} \otimes \psi_{j,k_2}; j \geq J\ ,\ (k_1, k_2) \in \mathbb{Z}^2\Big\} \\
& \bigcup \Big\{\psi_{j,k_1} \otimes \phi_{j,k_2}; j \geq J\ ,\ (k_1, k_2) \in \mathbb{Z}^2\Big\}
\end{aligned}$$

forms an orthonormal basis of $L^2(\mathbb{R}^2)$, where $\phi_{j,k}(x) = 2^{j/2}\phi(2^jx - k)$ and $\psi_{j,k}(x) = 2^{j/2}\psi(2^jx - k)$, for all $j, k \in \mathbb{Z}$ and $x \in \mathbb{R}$. Recall that $\otimes$ denotes the usual tensor product. By expanding for each fixed $t \in \mathbb{R}_+$ the kernel function $(x_1, x_2) \mapsto K_{H_1,H_2}(t, x_1, x_2)$ in the orthonormal wavelet basis $\widetilde{\mathcal{B}}_J$ and by using the isometry property of the double Wiener integral, one derives that

$$\begin{aligned} R_{H_1,H_2}(t) = \sum_{(k_1,k_2)\in\mathbb{Z}^2} &\langle K_{H_1,H_2}(t,\cdot,\cdot); \phi_{J,k_1} \otimes \phi_{J,k_2}\rangle_{L^2(\mathbb{R}^2)} \mu_{J,k_1,k_2} \\ &+ \sum_{j\geq J} \sum_{(k_1,k_2)\in\mathbb{Z}^2} \langle K_{H_1,H_2}(t,\cdot,\cdot); \psi_{j,k_1} \otimes \psi_{j,k_2}\rangle_{L^2(\mathbb{R}^2)} \varepsilon^{(1)}_{j,k_1,k_2} \\ &+ \sum_{j\geq J} \sum_{(k_1,k_2)\in\mathbb{Z}^2} \langle K_{H_1,H_2}(t,\cdot,\cdot); \phi_{j,k_1} \otimes \psi_{j,k_2}\rangle_{L^2(\mathbb{R}^2)} \varepsilon^{(2)}_{j,k_1,k_2} \\ &+ \sum_{j\geq J} \sum_{(k_1,k_2)\in\mathbb{Z}^2} \langle K_{H_1,H_2}(t,\cdot,\cdot); \psi_{j,k_1} \otimes \phi_{j,k_2}\rangle_{L^2(\mathbb{R}^2)} \varepsilon^{(3)}_{j,k_1,k_2}, \end{aligned} \tag{2.3.4}$$

where the four series are unconditionally convergent in $L^2(\Omega)$ and where

$$\mu_{J,k_1,k_2} := \int_{\mathbb{R}^2} \phi_{J,k} \otimes \phi_{J,k_2}(x_1,x_2) dB_{x_1} dB_{x_2} \ , \ \varepsilon^{(1)}_{j,k_1,k_2} := \int_{\mathbb{R}^2} \psi_{j,k_1} \otimes \psi_{j,k_2}(x_1,x_2) dB_{x_1} dB_{x_2},$$
$$\varepsilon^{(2)}_{j,k_1,k_2} := \int_{\mathbb{R}^2} \phi_{j,k_1} \otimes \psi_{j,k_2}(x_1,x_2) dB_{x_1} dB_{x_2} \ \text{ and } \ \varepsilon^{(3)}_{j,k_1,k_2} := \int_{\mathbb{R}^2} \psi_{j,k_1} \otimes \phi_{j,k_2}(x_1,x_2) dB_{x_1} dB_{x_2}.$$

Remark 2.3.2. It is worth mentioning that each one of the terms in (2.3.4) can be expressed as the double Wiener integral associated with the kernel function corresponding to the orthogonal projection of the function $K(t,\cdot,\cdot)$ into the corresponding space in (2.3.3). For example, the first series in (2.3.4) can be expressed as

$$\int_{\mathbb{R}^2} \mathrm{Proj}_{V_J \otimes V_J}\Big(K_{H_1,H_2}(t,\cdot,\cdot)\Big)(x_1,x_2) dB(x_1) dB(x_2).$$

Remark 2.3.3. Notice that Proposition 1.3.18 entails that for all $f, g \in L^2(\mathbb{R})$, one has

$$\int_{\mathbb{R}^2} f \otimes g(x_1,x_2) dB_{x_1} dB_{x_2} = \int_{\mathbb{R}} f(x) dB_x \int_{\mathbb{R}} g(x) dB_x - \int_{\mathbb{R}} f(x) g(x) dx. \tag{2.3.5}$$

By using (2.3.5) one derives that for all $(j, J) \in \mathbb{Z}$ and $(k_1, k_2) \in \mathbb{Z}^2$,

$$\begin{aligned} \mu_{J,k_1,k_2} &= \mu_{J,k_1} \mu_{J,k_2} - \delta_{k_1,k_2}, \\ \varepsilon^{(1)}_{j,k_1,k_2} &= \varepsilon_{j,k_1} \varepsilon_{j,k_2} - \delta_{k_1,k_2}, \\ \varepsilon^{(2)}_{j,k_1,k_2} &= \mu_{j,k_1} \varepsilon_{j,k_2}, \\ \varepsilon^{(3)}_{j,k_1,k_2} &= \varepsilon_{j,k_1} \mu_{j,k_2}, \end{aligned}$$

where $\mu_{J,k}, k \in \mathbb{Z}$ and $\varepsilon_{j,k}, j \geq J, k \in \mathbb{Z}$ are the independent standard Gaussian random variables defined in (2.2.4), and where δ_{k_1,k_2} denotes the Kronecker symbol, i.e. $\delta_{k_1,k_2} = 1$ if $k_1 = k_2$ and $\delta_{k_1,k_2} = 0$ else. Let us now give explicit expressions for the coefficients $\langle K_{H_1,H_2}(t,\cdot,\cdot); \phi_{J,k_1} \otimes \phi_{J,k_2}\rangle_{L^2(\mathbb{R}^2)}$, $\langle K_{H_1,H_2}(t,\cdot,\cdot); \psi_{j,k_1} \otimes \psi_{j,k_2}\rangle_{L^2(\mathbb{R}^2)}$, $\langle K_{H_1,H_2}(t,\cdot,\cdot); \phi_{j,k_1} \otimes \psi_{j,k_2}\rangle_{L^2(\mathbb{R}^2)}$ and $\langle K_{H_1,H_2}(t,\cdot,\cdot); \psi_{j,k_1} \otimes \phi_{j,k_2}\rangle_{L^2(\mathbb{R}^2)}$. Observe that by using successively Fubini's theorem (which is possible since the function $(x_1, x_2) \to K(t, x_1, x_2)\phi_{J,k_1} \otimes \phi_{J,k_2}(x_1, x_2)$ is an integrable function over $\mathbb{R}^2$ by using Cauchy-Schwarz's inequality) and by using the change of variables $u_1 = 2^J x_1 - k_1$ and $u_2 = 2^J x_2 - k_2$, we derive

$$\begin{aligned}
\langle K_{H_1,H_2}(t,\cdot,\cdot); \phi_{J,k_1} \otimes \phi_{J,k_2}\rangle_{L^2(\mathbb{R}^2)} &= 2^J \kappa_{H_1,H_2} \int_{\mathbb{R}^2}\int_0^t (s-x_1)_+^{H_1-3/2}(s-x_2)_+^{H_2-3/2} \\
&\qquad \times \phi(2^J x_1 - k_1)\phi(2^J x_2 - k_2)ds dx_1 dx_2 \\
&= 2^{-J}\kappa_{H_1,H_2} \int_0^t \int_{\mathbb{R}^2} \left(s - 2^{-J}(u_1+k_1)\right)_+^{H_1-3/2}\left(s - 2^{-J}(u_2+k_2)\right)_+^{H_2-3/2} \\
&\qquad \times \phi(u_1)\phi(u_2)du_1 du_2 ds \\
&= 2^{-J(H_1+H_2-2)} \int_0^t \Phi_{H_1}(2^J s - k_1)\Phi_{H_2}(2^J s - k_2)ds,
\end{aligned}$$

where $\kappa_{H_1,H_2} := \Big(\Gamma(H_1-1/2)\Gamma(H_2-1/2)\Big)^{-1}$, and Φ_H denotes the left-sided fractional primitive of order $H-1/2$ of the scaling function ϕ (see Definition 2.2.3) defined for all $x \in \mathbb{R}$, by

$$\Phi_H(x) := \frac{1}{\Gamma(H-1/2)} \int_{\mathbb{R}} (x-u)_+^{H-3/2}\phi(u)du. \tag{2.3.6}$$

By proceeding in a similar way, we obtain for all $(j, k_1, k_2) \in \mathbb{Z}^3$,

$$\begin{aligned}
\langle K_{H_1,H_2}(t,\cdot,\cdot); \psi_{j,k_1} \otimes \psi_{j,k_2}\rangle_{L^2(\mathbb{R}^2)} &= 2^{-j(H_1+H_2-2)} \int_0^t \Psi_{H_1}(2^j s - k_1)\Psi_{H_2}(2^j s - k_2)ds, \\
\langle K_{H_1,H_2}(t,\cdot,\cdot); \phi_{j,k_1} \otimes \psi_{j,k_2}\rangle_{L^2(\mathbb{R}^2)} &= 2^{-j(H_1+H_2-2)} \int_0^t \Phi_{H_1}(2^j s - k_1)\Psi_{H_2}(2^j s - k_2)ds, \\
\langle K_{H_1,H_2}(t,\cdot,\cdot); \psi_{j,k_1} \otimes \phi_{j,k_2}\rangle_{L^2(\mathbb{R}^2)} &= 2^{-j(H_1+H_2-2)} \int_0^t \Psi_{H_1}(2^j s - k_1)\Phi_{H_2}(2^j s - k_2)ds.
\end{aligned}$$

To summarize, for each fixed $t \in \mathbb{R}$ and $J \in \mathbb{Z}$, we have

$$R_{H_1,H_2}(t) = R_{H_1,H_2,J}(t) + \sum_{j=J}^{+\infty} \mathcal{R}_{1,j}(t) + \sum_{j=J}^{+\infty} \mathcal{R}_{2,j}(t) + \sum_{j=J}^{+\infty} \mathcal{R}_{3,j}(t), \tag{2.3.7}$$

where

$$R_{H_1,H_2,J}(t) := 2^{-J(H_1+H_2-2)} \sum_{(k_1,k_2)\in\mathbb{Z}^2} \int_0^t \Phi_{H_1}(2^J s - k_1)\Phi_{H_2}(2^J s - k_2)ds\Big(\mu_{J,k_1}\mu_{J,k_2} - \delta_{k_1,k_2}\Big), \tag{2.3.8}$$

$$\mathcal{R}_{1,j}(t) := 2^{-j(H_1+H_2-2)} \sum_{(k_1,k_2)\in\mathbb{Z}^2} \int_0^t \Psi_{H_1}(2^j s - k_1)\Psi_{H_2}(2^j s - k_2)ds\Big(\varepsilon_{j,k_1}\varepsilon_{j,k_2} - \delta_{k_1,k_2}\Big), \tag{2.3.9}$$

$$\mathcal{R}_{2,j}(t) := 2^{-j(H_1+H_2-2)} \sum_{(k_1,k_2)\in\mathbb{Z}^2} \int_0^t \Phi_{H_1}(2^j s - k_1)\Psi_{H_2}(2^j s - k_2)ds\mu_{j,k_1}\varepsilon_{j,k_2}, \tag{2.3.10}$$

$$\mathcal{R}_{3,j}(t) := 2^{-j(H_1+H_2-2)} \sum_{(k_1,k_2)\in\mathbb{Z}^2} \int_0^t \Psi_{H_1}(2^j s - k_1)\Phi_{H_2}(2^j s - k_2)ds\varepsilon_{j,k_1}\mu_{j,k_2}. \tag{2.3.11}$$

Similarly to the representation (2.2.2) of FBM, the representation (2.3.7) allows to isolate the low from the high frequencies of the gfRm. The series $R_{H_1,H_2,J}(t)$ is the approximation process and gives the tendency, while the other three processes provide details that are fluctuations around tendency.

2.3.2 Approximation process

Let us first focus on the approximation series $R_{H_1,H_2,J}(t)$. Our first goal is to provide a more convenient expression for it, as we did for the Fractional Brownian Motion (see Remark 2.2.8). To this end, one needs to introduce some additional notations. One sets

$$d_1 := H_1 - 1/2 \ \text{ and } \ d_2 := H_2 - 1/2. \tag{2.3.12}$$

Observe that $d_1 + d_2 > 1/2$. Let $\widetilde{\Phi}^{(d_1,d_2)}_{\Delta,n}$ be the real-valued function defined for all $z \in \mathbb{R}$, by

$$\widetilde{\Phi}^{(d_1,d_2)}_{\Delta,n}(z) := \int_{z-1}^{z} \Phi^{(d_1)}_{\Delta}(u)\Phi^{(d_2)}_{\Delta}(u-n)du. \tag{2.3.13}$$

Also, let $S^{(d_1,d_2)}_{J,q,n}$ be the centred random variable defined, for all $(J,q,n) \in \mathbb{Z}^3$, by

$$S^{(d_1,d_2)}_{J,q,n} := \begin{cases} \sum_{k=1}^{q} \epsilon^{(d_1)}_{J,k}\epsilon^{(d_2)}_{J,k+n} - \mathbb{E}\left(\epsilon^{(d_1)}_{J,k}\epsilon^{(d_2)}_{J,k+n}\right) & \text{if} \quad q \geq 1, \\ 0 & \text{if} \quad q = 0, \\ -\sum_{k=q+1}^{0} \epsilon^{(d_1)}_{J,k}\epsilon^{(d_2)}_{J,k+n} - \mathbb{E}\left(\epsilon^{(d_1)}_{J,k}\epsilon^{(d_2)}_{J,k+n}\right) & \text{if} \quad q \leq -1, \end{cases} \tag{2.3.14}$$

with $\{\epsilon^{(d)}_{J,i},\ i \in \mathbb{Z}\}$ being the FARIMA$(0,d,0)$ sequence of random variables defined in (2.2.46). The following proposition is the main result of this subsection.

Proposition 2.3.4. For each fixed $t \in \mathbb{R}_+$, one has almost surely,

$$R_{H_1,H_2,J}(t) = 2^{-J(d_1+d_2)} \sum_{n\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} \Big(\widetilde{\Phi}_{\Delta,n}^{(d_1,d_2)}(2^J t - q) - \widetilde{\Phi}_{\Delta,n}^{(d_1,d_2)}(-q) \Big) S_{J,q,n}^{(d_1,d_2)}. \tag{2.3.15}$$

Moreover, the series (2.3.15) is almost surely uniformly convergent on compact intervals.

The proof of Proposition 2.3.4 follows the ideas which have previously allowed to obtain the second wavelet representation series of FBM (see Section 2.2). First, notice that the function Φ_H is a square integrable function since it satisfies for each $x \in \mathbb{R}$, $|\Phi_H(x)| \le c(1+|x|)^{H-3/2}$, for some positive constant c (see (3.5) in [57]), and since $H \in (1/2, 1)$. Also, notice that its Fourier transform $\widehat{\Phi}_H(\xi) = (i\xi)^{-(H-1/2)}\widehat{\phi}(\xi)$ is in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ since $\widehat{\phi}$ is compactly supported. Therefore, one derives that for almost every $x \in \mathbb{R}$,

$$\Phi_H(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \widehat{\Phi}_H(\xi) d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{ix\xi}}{(i\xi)^{H-1/2}} \widehat{\phi}(\xi) d\xi. \tag{2.3.16}$$

Thus, for almost every $s \in \mathbb{R}$, one has

$$\Phi_H(2^J s - k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i(2^J s-k)\xi} (1-e^{-i\xi})^{-d} \widehat{\Phi}_{\Delta}^{(d)}(\xi) d\xi = \frac{1}{\sqrt{2\pi}} \widetilde{a}_{J,k}^{(d)}(s), \tag{2.3.17}$$

where $d = H - 1/2$ and $\widetilde{a}_{J,k}^{(d)}(s)$ is defined similarly to (2.2.31). Next, by combining (2.3.17) and (2.3.12) with (2.3.8), one can write

$$R_{H_1,H_2,J}(t) = \frac{2^{-J(d_1+d_2-1)}}{2\pi} \sum_{(k_1,k_2)\in\mathbb{Z}^2} \int_0^t \widetilde{a}_{J,k_1}^{(d_1)}(s) \widetilde{a}_{J,k_2}^{(d_2)}(s) \Big(\mu_{J,k_1}\mu_{J,k_2} - \delta_{k_1,k_2} \Big) ds. \tag{2.3.18}$$

Now, we truncate the sum in (2.3.18) by defining for any arbitrary integer $L \in \mathbb{N}$, the random variable $R_{H_1,H_2,J}(t)$ as the following finite sum,

$$R_{H_1,H_2,J}(t) := \frac{2^{-J(d_1+d_2-1)}}{2\pi} \Bigg\{ \sum_{(k_1,k_2)\in\mathbb{Z}^2} \int_0^t \widetilde{a}_{J,k_1}^{(d_1)}(s) \widetilde{a}_{J,k_2}^{(d_2)}(s) \mu_{J,k_1}^{(L)} \mu_{J,k_2}^{(L)} ds - \sum_{|k|\le L} \int_0^t \widetilde{a}_{J,k}^{(d_1)}(s) \widetilde{a}_{J,k}^{(d_2)}(s) ds \Bigg\}, \tag{2.3.19}$$

where $\mu_{J,k}^{(L)}$, $k \in \mathbb{Z}$ are the independent Gaussian random variables defined by

$$\mu_{J,k}^{(L)} := \mu_{J,k} \ \text{ if } \ |k| \le L \quad \text{and} \quad \mu_{J,k}^{(L)} = 0 \ \text{ else.}$$

Observe that similarly to (2.2.41), one can assume that for each fixed $t \in \mathbb{R}_+$,

$$\lim_{L\to+\infty} R_{H_1,H_2,J}^{(L)}(t) \overset{a.s.}{=} R_{H_1,H_2,J}(t). \tag{2.3.20}$$

On the other hand, notice that the equality (2.2.43) entails that

$$\sum_{(k_1,k_2)\in\mathbb{Z}^2} \int_0^t \widetilde{a}_{J,k_1}^{(d_1)}(s) \widetilde{a}_{J,k_2}^{(d_2)}(s) \mu_{J,k_1}^{(L)} \mu_{J,k_2}^{(L)} ds = 2\pi \int_0^t \sum_{q_1\in\mathbb{Z}} \sum_{q_2\in\mathbb{Z}} \Phi_{\Delta}^{(d_1)}(2^J u - q_1) \Phi_{\Delta}^{(d_2)}(2^J u - q_2) \epsilon_{J,q_1}^{(d_1,L)} \epsilon_{J,q_2}^{(d_2,L)} du, \tag{2.3.21}$$

where $\epsilon_{J,q}^{(d,L)}, q \in \mathbb{Z}$ is the sequence of the centred Gaussian random variables defined in (2.2.44). Thus, by taking expectations of both sides of the equality (2.3.21), one gets

$$\sum_{|k|\leq L} \int_0^t \widetilde{a}_{J,k}^{(d_1)}(s)\widetilde{a}_{J,k}^{(d_2)}(s)ds = 2\pi \int_0^t \sum_{q_1\in\mathbb{Z}} \sum_{q_2\in\mathbb{Z}} \Phi_\Delta^{(d_1)}(2^J u - q_1)\Phi_\Delta^{(d_2)}(2^J u - q_2)\, \mathbb{E}\left(\epsilon_{J,q_1}^{(d_1,L)}\epsilon_{J,q_2}^{(d_2,L)}\right)du \tag{2.3.22}$$

Therefore, combining (2.3.21) and (2.3.22) with (2.3.19), one gets

$$R_{H_1,H_2,J}^{(L)}(t) = 2^{-J(d_1+d_2-1)} \int_0^t \sum_{q_1\in\mathbb{Z}} \sum_{q_2\in\mathbb{Z}} \Phi_\Delta^{(d_1)}(2^J u-q_1)\Phi_\Delta^{(d_2)}(2^J u-q_2)\left(\epsilon_{J,q_1}^{(d_1,L)}\epsilon_{J,q_2}^{(d_2,L)}-\mathbb{E}\left(\epsilon_{J,q_1}^{(d_1,L)}\epsilon_{J,q_2}^{(d_2,L)}\right)\right)du. \tag{2.3.23}$$

The following lemma can be derived from Lemma 2.2.17 and allows to bound uniformly in L the series

$$\sum_{(q_1,q_2)\in\mathbb{Z}^2} \left|\Phi_\Delta^{(d_1)}(u-q_1)\right|\left|\Phi_\Delta^{(d_1)}(u-q_2)\right|\left|\epsilon_{J,q_1}^{(d_1,L)}\epsilon_{J,q_2}^{(d_2,L)} - \mathbb{E}\left(\epsilon_{J,q_1}^{(d_1,L)}\epsilon_{J,q_2}^{(d_2,L)}\right)\right|.$$

Lemma 2.3.5. Let $\widetilde{\Omega}$ be the event of probability 1, which has been introduced in Lemma 2.2.17. There exists an almost surely finite random variable C such that, for all $(\omega, J, q_1, q_2) \in \widetilde{\Omega} \times \mathbb{Z}^3$, one has

$$\sup_{L\geq 1}\left|\epsilon_{J,q_1}^{(d_1,L)}(\omega)\epsilon_{J,q_2}^{(d_2,L)}(\omega) - \mathbb{E}\left(\epsilon_{J,q_1}^{(d_1,L)}\epsilon_{J,q_2}^{(d_2,L)}\right)\right| \leq C(\omega)\sqrt{\log(3+|J|+|q_1|)\log(3+|J|+|q_2|)}. \tag{2.3.24}$$

Consequently, one has almost surely, for every $u \in \mathbb{R}$,

$$\sum_{(q_1,q_2)\in\mathbb{Z}^2} \left|\Phi_\Delta^{(d_1)}(u-q_1)\right|\left|\Phi_\Delta^{(d_1)}(u-q_2)\right|\sup_{L\geq 1}\left|\epsilon_{J,q_1}^{(d_1,L)}\epsilon_{J,q_2}^{(d_2,L)} - \mathbb{E}\left(\epsilon_{J,q_1}^{(d_1,L)}\epsilon_{J,q_2}^{(d_2,L)}\right)\right| \leq C' \log\left(3+|J|+|u|\right), \tag{2.3.25}$$

where C' is an almost surely finite positive random variable not depending on J and u.

Next, observe that when L goes to $+\infty$, (2.3.23), Lemma 2.3.5, (2.2.45) and the dominated convergence theorem allow to obtain

$$\lim_{L\to+\infty} R_{H_1,H_2,J}^{(L)}(t) = 2^{-J(d_1+d_2-1)} \int_0^t \sum_{q_1\in\mathbb{Z}} \sum_{q_2\in\mathbb{Z}} \Phi_\Delta^{(d_1)}(2^J u-q_1)\Phi_\Delta^{(d_2)}(2^J u-q_2)\left(\epsilon_{J,q_1}^{(d_1)}\epsilon_{J,q_2}^{(d_2)}-\mathbb{E}\left(\epsilon_{J,q_1}^{(d_1)}\epsilon_{J,q_2}^{(d_2)}\right)\right)du. \tag{2.3.26}$$

By combining (2.3.20) with (2.3.26) and by reindexing the sum in (2.3.26), one derives that for every $t \in \mathbb{R}_+$, one has almost surely,

$$R_{H_1,H_2,J}(t) = 2^{-J(d_1+d_2)} \int_0^{2^J t} \sum_{n\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} \Phi_\Delta^{(d_1)}(u-q)\Phi_\Delta^{(d_2)}(u-q-n)\left(\epsilon_{J,q}^{(d_1)}\epsilon_{J,n+q}^{(d_2)}-\mathbb{E}\left(\epsilon_{J,q}^{(d_1)}\epsilon_{J,n+q}^{(d_2)}\right)\right)du. \tag{2.3.27}$$

Notice that the upper bound (2.3.25) also allows to write

$$R_{H_1,H_2,J}(t) = 2^{-J(d_1+d_2)} \sum_{n\in\mathbb{Z}} \int_0^{2^J t} \sum_{q\in\mathbb{Z}} \Phi_\Delta^{(d_1)}(u-q)\Phi_\Delta^{(d_2)}(u-q-n)\left(\epsilon_{J,q}^{(d_1)}\epsilon_{J,n+q}^{(d_2)} - \mathbb{E}\left(\epsilon_{J,q}^{(d_1)}\epsilon_{J,n+q}^{(d_2)}\right)\right) du. \tag{2.3.28}$$

For each fixed $n \in \mathbb{Z}$, one applies the Abel transform, i.e. Lemma 2.2.19 with

$$f = \Phi_\Delta^{(d_1)}(\cdot)\Phi_\Delta^{(d_2)}(\cdot - n) \quad \text{and} \quad a_q = \epsilon_{J,q}^{(d_1)}\epsilon_{J,q+n}^{(d_2)} - \mathbb{E}\left(\epsilon_{J,q}^{(d_1)}\epsilon_{J,q+n}^{(d_2)}\right),$$

to get almost surely

$$R_{H_1,H_2,J}(t) = 2^{-J(d_1+d_2)} \sum_{n\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} \left(\widetilde{\Phi}_{\Delta,n}^{(d_1,d_2)}(2^J t - q) - \widetilde{\Phi}_{\Delta,n}^{(d_1,d_2)}(-q)\right) S_{J,q,n}^{(d_1,d_2)}, \tag{2.3.29}$$

where the function $\widetilde{\Phi}_{\Delta,n}^{(d_1,d_2)}$ and the random variables $S_{J,q,n}^{(d_1,d_2)}$ are respectively defined through (2.3.13) and (2.3.14). Moreover, the function $\widetilde{\Phi}_{\Delta,n}^{(d_1,d_2)}$ is in the Schwartz class $S(\mathbb{R})$.

End of the Proof of Proposition 2.3.4 It only remains to prove that the series (2.3.15) is almost surely uniformly convergent on compact intervals. Let $T > 0$ be arbitrary and fixed and assume without loss of generality that a compact interval is $[0, T]$. We will show that the series

$$\sum_{n\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} \widetilde{\Phi}_{\Delta,n}^{(d_1,d_2)}(2^J t - q) S_{J,q,n}^{(d_1,d_2)}$$

is almost surely uniformly convergent on the compact interval $[0, T]$. Observe that by combining (2.3.24) and (2.3.14) with (2.2.44), one derives that almost surely, for all $(J, q, n) \in \mathbb{Z}^3$, one has

$$\left|S_{J,q,n}^{(d_1,d_2)}\right| \le C(1+|q|)\log(3+|J|+|q|)\sqrt{\log(3+|n|)}, \tag{2.3.30}$$

where C is an almost surely finite positive random variable. Next, using the fact that the function $\Phi_\Delta^{(d)}$ is in the Schwartz class, one derives that for any arbitraty fixed $L > 3$, one has

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} \left|\widetilde{\Phi}_{\Delta,n}^{(d_1,d_2)}(2^J t - q)\right|\left|S_{J,q,n}^{(d_1,d_2)}\right| &\le 2^J \sum_{n\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} \int_{t-2^{-J}}^t \left|\Phi_\Delta^{(d_1)}(2^J s - q)\right|\left|\Phi_\Delta^{(d_2)}(2^J s - q - n)\right|\left|S_{J,q,n}^{(d_1,d_2)}\right| ds \\
&\le C 2^J \int_{t-2^{-J}}^t \sum_{q\in\mathbb{Z}} \frac{(1+|q|)\log(3+|J|+|q|)}{(2+|2^J s - q|)^L} \sum_{n\in\mathbb{Z}} \frac{\sqrt{\log(3+|n|)}}{(2+|2^J s - q - n|)^L} ds \\
&\le C' 2^J \int_{t-2^{-J}}^t \sum_{q\in\mathbb{Z}} \frac{(1+|q|)\log(3+|J|+|q|)}{(2+|2^J s - q|)^{L-1}} ds \\
&\le C''(1+|2^J T|)\log\left(3+|J|+2^J T\right),
\end{aligned} \tag{2.3.31}$$

where we used, for all $u \in \mathbb{R}$, the following two bounds,

$$\sum_{n\in\mathbb{Z}} \frac{\sqrt{\log(3+|n|)}}{(2+|u-n|)^L} \le c_1\sqrt{\log(3+|u|)} \le c_1(2+|u|),$$

and

$$\sum_{q\in\mathbb{Z}} \frac{(1+|q|)\log(3+|J|+|q|)}{(2+|u-q|)^{L-1}} \leq c_2(1+|u|)\log(3+|J|+|u|),$$

where c_1 and c_2 are two strictly positive deterministic constants.

2.3.3 Detail processes

For each $n \in \mathbb{Z}$, one denotes by $\Psi^{(n)}_{H_1,H_2}$, $\big(\Psi\Phi\big)^{(n)}_{H_1,H_2}$ and $\big(\Phi\Psi\big)^{(n)}_{H_1,H_2}$ the three real-valued functions defined for all $z \in \mathbb{R}$ by

$$\Psi^{(n)}_{H_1,H_2}(z) := \int_{-\infty}^{z} \Psi_{H_1}(v)\Psi_{H_2}(v-n)dv, \tag{2.3.32}$$

$$\big(\Phi\Psi\big)^{(n)}_{H_1,H_2}(z) := \int_{-\infty}^{z} \Phi^{(H_1-1/2)}_{\Delta}(v)\Psi_{H_2}(v-n)dv, \tag{2.3.33}$$

and

$$\big(\Psi\Phi\big)^{(n)}_{H_1,H_2}(z) := \int_{-\infty}^{z} \Psi_{H_1}(v)\Phi^{(H_2-1/2)}_{\Delta}(v-n)dv. \tag{2.3.34}$$

The following proposition provides the detail process associated with the bi-variate Meyer mother wavelet $\psi \otimes \psi$.

Proposition 2.3.6. For every $j \geq J$, the process $\mathcal{R}_{1,j}$ can be expressed in terms of $\Psi^{(n)}_{H_1,H_2}$ as

$$\mathcal{R}_{1,j}(t) = 2^{-j(H_1+H_2-1)} \sum_{n\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} \Big(\Psi^{(n)}_{H_1,H_2}(2^j t - q) - \Psi^{(n)}_{H_1,H_2}(-q)\Big)\Big(\varepsilon_{j,q}\varepsilon_{j,q+n} - \delta_{q,q+n}\Big), \tag{2.3.35}$$

for all $t \in \mathbb{R}_+$, and the series (2.3.35) is almost surely uniformly convergent on compact intervals (jointly in the indices n and q). Moreover, the series $\sum_{j=J}^{\infty} \mathcal{R}_{1,j}(t)$ is also, almost surely uniformly convergent in t on compact intervals (with respect to the index j).

Proof. Using the fact that the function Ψ_H belongs to $S(\mathbb{R})$ and using the bound (2.2.7) in Lemma 2.2.2, one derives that,

$$\sum_{k_1,k_2} \int_0^t |\Psi_{H_1}(2^j s - k_1)||\Psi_{H_2}(2^j s - k_2)|\Big|\varepsilon_{j,k_1}\varepsilon_{j,k_2} - \delta_{k_1,k_2}\Big| ds \leq C(1+t)\log\Big(3+|j|+2^j t\Big),$$

which establishes that $\mathcal{R}_{1,j}(t)$ is uniformly convergent on compact intervals. Notice that this implies that the process $\mathcal{R}_{1,j}$ has almost surely continuous paths. Also, notice that $\mathcal{R}_{1,j}(0) = 0$. From the independence of the random variables $\varepsilon_{j,k}, j \in \mathbb{Z}$, we derive that the processes $\mathcal{R}_{1,j},\ j \geq J$ are independent. We will use the Itô-Nisio's theorem stated in Theorem 2.2.6 to prove that the series $\sum_{j=J}^{+\infty} \mathcal{R}_{1,j}(t)$ is almost surely uniformly convergent on compact intervals. We assume that a compact interval is $[0,T]$ for some arbitrary strictly positive real number T. Let $\mathcal{R}^*_{1,N}$ be the stochastic process over $[0,T]$ defined, for each integer $N \geq J$, by

$$\mathcal{R}^*_{1,N}(t) := \sum_{j=J}^{N} \mathcal{R}_{1,j}(t).$$

We want to show that the sequence $(\mathcal{R}^*_{1,N})_{N\geq J}$ is weakly relatively compact in the Banach space $\mathcal{C}([0,T])$ equipped with the supremum norm. To this end, we only have to show that for each $t_1,t_2\in[0,T]$, one has

$$\mathbb{E}\left|\mathcal{R}^*_{1,N}(t_1)-\mathcal{R}^*_{1,N}(t_2)\right|^2\leq c|t_1-t_2|^{\alpha}, \tag{2.3.36}$$

for some $\alpha>1$ and $c>0$ not depending on N,t_1 and t_2. Notice that following Remark 2.3.2, the process $\mathcal{R}^*_{1,N}(t)$ may be expressed as

$$\mathcal{R}^*_{1,N}(t)=\int_{\mathbb{R}^2}\mathrm{Proj}_{\bigoplus_{J\leq j\leq N}W_j\otimes W_j}\Big(K_{H_1,H_2}(t,\cdot,\cdot)\Big)(x_1,x_2)dB(x_1)dB(x_2). \tag{2.3.37}$$

Using successively (2.3.37), the isometry property of the double Wiener integral, Parseval's formula and Proposition 2.3.1, one derives that

$$\begin{aligned}\mathbb{E}\left|\mathcal{R}^*_{1,N}(t_1)-\mathcal{R}^*_{1,N}(t_2)\right|^2&\leq 2\int_{\mathbb{R}^2}\bigg(\mathrm{Proj}_{\bigoplus_{J\leq j\leq N}W_j\otimes W_j}\Big(K_{H_1,H_2}(t_1,\cdot,\cdot)-K_{H_1,H_2}(t_2,\cdot,\cdot)\Big)(x_1,x_2)\bigg)^2dx_1dx_2\\&\leq 2\int_{\mathbb{R}^2}\Big(K_{H_1,H_2}(t_1,x_1,x_2)-K_{H_1,H_2}(t_2,x_1,x_2)\Big)^2dx_1dx_2\\&=2\,\mathbb{E}\left|R_{H_1,H_2}(t_1)-R_{H_1,H_2}(t_2)\right|^2\leq c_{H_1,H_2}|t_1-t_2|^{2(H_1+H_2-1)},\end{aligned} \tag{2.3.38}$$

which establishes (2.3.36) since $H_1+H_2>3/2$ and thus $2(H_1+H_2-1)>1$. □

The following proposition provides the other two detail processes associated with the two bivariate Meyer mother wavelets $\phi\otimes\psi$ and $\psi\otimes\phi$.

Proposition 2.3.7. For each fixed $t\in\mathbb{R}_+$, one has almost surely,

$$\mathcal{R}_{2,j}(t)=2^{-j(H_1+H_2-1)}\sum_{n\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\Big(\big(\Phi\Psi\big)^{(n)}_{H_1,H_2}(2^jt-q)-\big(\Phi\Psi\big)^{(n)}_{H_1,H_2}(-q)\Big)\epsilon^{(H_1-1/2)}_{j,q}\varepsilon_{j,q+n} \tag{2.3.39}$$

and

$$\mathcal{R}_{3,j}(t)=2^{-j(H_1+H_2-1)}\sum_{n\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\Big(\big(\Psi\Phi\big)^{(n)}_{H_1,H_2}(2^jt-q)-\big(\Psi\Phi\big)^{(n)}_{H_1,H_2}(-q)\Big)\varepsilon_{j,q}\epsilon^{(H_2-1/2)}_{j,q+n}, \tag{2.3.40}$$

and the two series (2.3.39) and (2.3.40) are almost surely uniformly convergent in t on compact intervals (jointly in the indices n and q). Moreover, the two series $\sum_{j=J}^{+\infty}\mathcal{R}_{2,j}$ and $\sum_{j=J}^{+\infty}\mathcal{R}_{3,j}$ are almost surely uniformly convergent in t on compact intervals.

Proof. We will only prove that the result holds for the process $\mathcal{R}_{2,j}$. The result concerning the process $\mathcal{R}_{3,j}$ can be established in a rather similar way. First, observe that by arguing as in the proof of Proposition 2.3.4, we can show that for each fixed $t\in\mathbb{R}_+$, one has almost surely

$$\sum_{(k_1,k_2)\in\mathbb{Z}^2}\int_0^t\Phi_{H_1}(2^js-k_1)\Psi_{H_2}(2^js-k_2)ds\mu_{j,k_1}\varepsilon_{j,k_2}=\sum_{(k_1,k_2)\in\mathbb{Z}^2}\int_0^t\Phi^{(d_1)}_{\Delta}(2^js-k_1)\Psi_{H_2}(2^js-k_2)ds\epsilon^{(d_1)}_{j,k_1}\varepsilon_{j,k_2}, \tag{2.3.41}$$

where $d_1 := H_1 - 1/2$ and $\{\epsilon_{j,k}^{(d)}, k \in \mathbb{Z}\}$ is the FARIMA$(0, d, 0)$ sequence of random variables defined in (2.2.46). By observing that the two functions $\Phi_\Delta^{(d_1)}$ and Ψ_{H_2} are in the Schwartz class and by using the bounds (2.2.7) and $|\epsilon_{j,k}^{(d)}| \leq C\sqrt{\log(3 + |j| + |k|)}$, for some almost surely finite random variable C, and by proceeding as in the end of the proof of Proposition 2.3.4, we may show that the series

$$\sum_{(k_1,k_2)\in\mathbb{Z}^2} \int_0^t \Phi_\Delta^{(d_1)}(2^j s - k_1)\Psi_{H_2}(2^j s - k_2) ds \epsilon_{j,k_1}^{(d_1)} \varepsilon_{j,k_2}$$

is almost surely uniformly convergent on compact intervals. Moreover, by combining (2.3.33), (2.3.10) and (2.3.41), one derives that for each fixed $t \in \mathbb{R}_+$, the equality (2.3.39) holds almost surely (on an event of probability 1 depending on t). Therefore, to prove Proposition 2.3.7, it only remains to show that the series $\sum_{j \geq J} \mathcal{R}_{2,j}$ is almost surely uniformly convergent (in t) on each compact interval. Let $K = [0, T]$ be an arbitrary compact interval and define for each integer $N \geq J$ the stochastic process $\mathcal{R}_{2,N}^*$ over K as

$$\mathcal{R}_{2,N}^*(t) := \sum_{j=J}^{N} \mathcal{R}_{2,j}(t).$$

Unfortunately, one cannot use the Itô-Nisio's theorem because the processes $\mathcal{R}_{2,j}, j \geq J$ are correlated. In order to overcome such a difficulty, the author of [57] has established his Lemma 7 whose proof is inspired by that of Theorem 6.6.1 in [40]. By adapting this latter lemma, one can show that the series $\sum_{j=J}^{N} \mathcal{R}_{2,j}$ is almost surely uniformly convergent on the compact interval K if it is weakly relatively compact in the space of continuous functions over K equipped with supremum norm. It is then enough to show that

$$\mathbb{E}\left|\mathcal{R}_{2,N}^*(t_1) - \mathcal{R}_{2,N}^*(t_2)\right|^2 \leq c|t_1 - t_2|^\alpha, \tag{2.3.42}$$

for some $\alpha > 1$ and $c > 0$ not depending on N, t_1 and t_2. By expressing the random variable $\mathcal{R}_{2,N}^*(t)$ as

$$\mathcal{R}_{2,N}^*(t) = \int_{\mathbb{R}^2} \mathrm{Proj}_{\bigoplus_{J\leq j\leq N} V_j \otimes W_j}\Big(K_{H_1,H_2}(t, \cdot, \cdot)\Big)(x_1, x_2) dB(x_1) dB(x_2),$$

and by arguing as in the proof of the inequality (2.3.38), one derives that

$$\begin{aligned}\mathbb{E}\left|\mathcal{R}_{2,N}^*(t_1) - \mathcal{R}_{2,N}^*(t_2)\right|^2 &\leq 2\,\mathbb{E}\left|R_{H_1,H_2}(t_1) - R_{H_1,H_2}(t_2)\right|^2 \\ &\leq c_{H_1,H_2}|t_1 - t_2|^{2(H_1+H_2-1)},\end{aligned}$$

which achieves the proof of Proposition 2.3.7.

□

2.3.4 Conclusion

The results obtained in the previous subsections are synthesized by the following theorem, which is the main result of this chapter. It extends the wavelet-type expansion of the fractional Brownian motion given in Theorem 2.2.20, as well as the wavelet-type expansion of the Rosenblatt process obtained in [57], to the case of the generalized Rosenblatt process, under the assumption that the underlying multiresolution analysis is of Meyer type. It is derived by combining Proposition 2.3.4, Proposition 2.3.6 and Proposition 2.3.7 with (2.3.7).

Theorem 2.3.8. ***(Wavelet-type expansion of the generalized Rosenblatt process)*** *The generalized Rosenblatt process $\{R_{H_1,H_2}(t),\ t \in \mathbb{R}_+\}$ defined by (2.3.1) has the following wavelet-type expansion: for each fixed $J \in \mathbb{Z}$,*

$$R_{H_1,H_2}(t) = R_{H_1,H_2,J}(t) + \sum_{i=1}^{3} \sum_{j=J}^{+\infty} \mathcal{R}_{i,j}(t). \tag{2.3.43}$$

The approximation process is given by

$$R_{H_1,H_2,J}(t) = 2^{-J(d_1+d_2)} \sum_{n\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} \Big(\widetilde{\Phi}_{\Delta,n}^{(d_1,d_2)}(2^J t - q) - \widetilde{\Phi}_{\Delta,n}^{(d_1,d_2)}(-q) \Big) S_{J,q,n}^{(d_1,d_2)}, \tag{2.3.44}$$

where $d_1 = H_1 - 1/2$ and $d_2 = H_2 - 1/2$. The detail processes $\mathcal{R}_{i,j}, i = 1,2,3$ are given by

$$\mathcal{R}_{1,j}(t) = 2^{-j(H_1+H_2-1)} \sum_{n\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} \Big(\Psi_{H_1,H_2}^{(n)}(2^j t - q) - \Psi_{H_1,H_2}^{(n)}(-q) \Big) \Big(\varepsilon_{j,q}\varepsilon_{j,q+n} - \delta_{q,q+n} \Big), \tag{2.3.45}$$

$$\mathcal{R}_{2,j}(t) = 2^{-j(H_1+H_2-1)} \sum_{n\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} \Big(\big(\Phi\Psi\big)_{H_1,H_2}^{(n)}(2^j t - q) - \big(\Phi\Psi\big)_{H_1,H_2}^{(n)}(-q) \Big) \epsilon_{j,q}^{(H_1-1/2)} \varepsilon_{j,q+n} \tag{2.3.46}$$

and

$$\mathcal{R}_{3,j}(t) = 2^{-j(H_1+H_2-1)} \sum_{n\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} \Big(\big(\Psi\Phi\big)_{H_1,H_2}^{(n)}(2^j t - q) - \big(\Psi\Phi\big)_{H_1,H_2}^{(n)}(-q) \Big) \varepsilon_{j,q} \epsilon_{j,q+n}^{(H_2-1/2)}. \tag{2.3.47}$$

Notice that the series in (2.3.44) is almost surely uniformly convergent on compact intervals. Also, notice that, for each fixed j and J, the series in (2.3.45), (2.3.46) and (2.3.47) are almost surely uniformly convergent on compact intervals.

Remark 2.3.9.
- Thanks to Theorem 2.3.8, when J is a positive integer large enough, the generalized Rosenblatt process R_{H_1,H_2} can be almost surely approximated uniformly on any compact interval by the process $R_{H_1,H_2,J}$ with continuous paths issued from a multiresolution analysis of $L^2(\mathbb{R}^2)$ of Meyer type. In other words, for any arbitrary fixed compact interval $K \subset \mathbb{R}$, one has almost surely

  $$\|R_{H_1,H_2} - R_{H_1,H_2,J}\|_{K,\infty} := \sup_{t\in K} \big| R_{H_1,H_2}(t) - R_{H_1,H_2,J}(t) \big| \underset{J\to+\infty}{\longrightarrow} 0.$$

  Notice that the strategy which has been employed for obtaining Theorem 2.3.8 as well as Theorem 1 in [57] does not allow to get any explicit estimate of the uniform approximation error $\|R_{H_1,H_2} - R_{H_1,H_2,J}\|_{K,\infty}$. In the next chapter, our main goal will be to precisely estimate this approximation error by making use of a new strategy.

- Notice that a stronger type of convergence than the one obtained in Theorem 2.3.8 is that the series involved in (2.3.43) converge almost surely and uniformly on compact intervals jointly in the three indices j, q and n. Such a stronger type of convergence was not obtained by Meyer, Sellan and Taqqu in [49] for the fractional Brownian motion, nor by Pipias in [57] for the Rosenblatt process. The convergence proved by them as well as the one obtained in Theorem 2.3.8 only holds in the following sense: the series in the indices of location q, n converge almost surely and uniformly on compact intervals and then the same convergence holds for the series in the index of resolution j . The joint convergence in the case of the generalized Ronsenblatt process will be studied in the next chapter.

Chapter 3

Rate of convergence of wavelet-type expansions of the generalized Rosenblatt process

Pipiras introduced in the early 2000s an almost surely and uniformly convergent (on compact intervals) wavelet-type expansion of the classical Rosenblatt process. Yet, the issue of estimating, almost surely, its uniform rate of convergence remained an open question. The main goal of this chapter is to provide an answer to it in the more general framework of the generalized Rosenblatt process, under the assumption that the underlying wavelet basis belongs to the class due to Meyer. The main ingredient of our strategy consists in expressing in a non-classical (new) way the approximation errors related with the approximation spaces of a multiresolution analysis of $L^2(\mathbb{R}^2)$. Such a non-classical expression may also be of interest in its own right.

3.1 Introduction and statement of the two main results

The Rosenblatt process is a non-Gaussian extension of the classical fractional Brownian motion (see e.g. [61, 29]) which was first introduced in the pioneering article [59]. Later, the well-known papers [69, 25, 67] drew important connections between it and Non-Central Limit theorem. The history behind the Rosenblatt process is outlined in the review article [68], and a detailed presentation of various properties of this process and many other related topics can be found in the recent book [58]. For about two decades there has been increasing interest in its study, we refer to the works [57, 1, 16, 43, 70, 2, 22] to cite only a few. More precisely, it is a real-valued non-Gaussian self-similar stochastic process with stationary increments which belongs to the second order Wiener chaos and has continuous paths. We mention in passing that two classical books on Wiener chaoses, multiple Wiener integrals and related topics are [36, 52]; a more recent book on them is [54]. Throughout this chapter, using the same terminology as in [57], the Rosenblatt process is called fractional Rosenblatt motion (fRm in short) and denoted by $\{R_H(t)\}_{t\in\mathbb{R}_+}$ since it depends on a parameter $H \in (3/4, 1)$. Notice that this parameter H is replaced by the parameter $\kappa := H - 1/2$ by some authors, because of the fact that 2κ corresponds to the self-similarity exponent of fRm. $\{R_H(t)\}_{t\in\mathbb{R}_+}$ is defined through the following double Wiener integral with respect to a given Brownian motion $\{B(x)\}_{x\in\mathbb{R}}$:

$$R_H(t) := \int_{\mathbb{R}^2}' K_H(t, x_1, x_2)\, dB(x_1)dB(x_2), \quad \text{for all } t \in \mathbb{R}_+ . \tag{3.1.1}$$

The symbol $\int_{\mathbb{R}^2}'$ in (3.1.1) denotes integration over $\mathbb{R}^2$ excluding the diagonal, and the integrand K_H is given, for every $(t, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^2$, by:

$$K_H(t, x_1, x_2) := \big(\Gamma(H-1/2)\big)^{-2} \int_0^t (s-x_1)_+^{H-3/2}(s-x_2)_+^{H-3/2}\, ds\,, \tag{3.1.2}$$

with the convention that, for each $(y, \alpha) \in \mathbb{R}^2$, one has $y_+^\alpha := y^\alpha$ if y is positive, and $y_+^\alpha := 0$ else. Observe that Γ in (3.1.2) is the usual "Gamma function" defined, for every real number $z \in (0, +\infty)$, as $\Gamma(z) := \int_0^{+\infty} u^{z-1} e^{-u}\, du$. Also, observe that the inequalities $3/4 < H < 1$ imply, for each fixed $t \in \mathbb{R}_+$, that the kernel function $(x_1, x_2) \mapsto K_H(t, x_1, x_2)$ belongs to the Hilbert space $L^2(\mathbb{R}^2)$; this is why the double Wiener integral in (3.1.1) is well-defined.

In the early 2000s, a wavelet-type series expansion of $\{R_H(t)\}_{t\in\mathbb{R}_+}$, which is almost surely and uniformly convergent in t on any compact interval $I \subset \mathbb{R}_+$, was introduced by Pipiras in [57]. Thanks to it, this process can be almost surely approximated, uniformly in $t \in I$, by a sequence of processes $\big(\{R_{H,J}(t)\}_{t\in\mathbb{R}_+}\big)_{J\in\mathbb{N}}$ with continuous paths stemming from a multiresolution analysis (MRA) of $L^2(\mathbb{R}^2)$ and for which efficient simulation methods are available (see [1, 57]). Yet, the issue of estimating, almost surely, the uniform norm over I of the approximation error, that is $\|R_H - R_{H,J}\|_{I,\infty} := \sup_{t\in I} \big|R_H(t) - R_{H,J}(t)\big|$, remained an open question. The primary motivation behind this chapter is to provide an answer to it. In order to explain the main focus of our strategy employed to this end, we need to briefly present the fundamental notion of MRA which is the keystone of the wavelet theory. Let us mention that two very classical references on this theory are the books [24, 48].

A MRA of the Hilbert space $L^2(\mathbb{R}^d)$, the integer $d \geq 1$ being arbitrary, is a sequence $(V_j^d)_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)$ satisfying the following four conditions:

(i) $V_j^d \subset V_{j+1}^d$, for every $j \in \mathbb{Z}$;

(ii) $\bigcap_{j\in\mathbb{Z}} V_j^d = \{0\}$ and $\bigcup_{j\in\mathbb{Z}} V_j^d$ is dense in $L^2(\mathbb{R}^d)$;

(iii) for any function $f(x) \in L^2(\mathbb{R}^d)$, $f(x)$ belongs to V_0^d if and only if $f(2^j x)$ belongs to V_j^d, for all $j \in \mathbb{Z}$, in other words one has $V_j^d := \big\{ f(2^j x) : \ f(x) \in V_0^d \big\}$;

(iv) There exists a function $\phi^d(x)$ in V_0^d, called scaling function, such that the sequence $\big\{\phi^d(x - l) : \ l \in \mathbb{Z}^d\big\}$ forms an orthonormal basis of V_0^d.

Notice that it can easily be derived from (iii) and (iv) that the sequence $\big\{2^{jd/2}\, \phi^d(2^j x - l) : \ l \in \mathbb{Z}^d\big\}$ is an orthonormal basis of V_j^d, for every fixed $j \in \mathbb{Z}$. The closed subspace W_j^d of $L^2(\mathbb{R}^d)$ denotes the orthogonal complement of V_j^d in V_{j+1}^d, that is one has

$$V_{j+1}^d = V_j^d \overset{\perp}{\oplus} W_j^d, \quad \text{for all } j \in \mathbb{Z}\,. \tag{3.1.3}$$

Observe that it follows from (3.1.3) and (ii) that, for any fixed $J \in \mathbb{Z}$, the following three fundamental equalities hold:

$$V_J^d = \overset{\perp}{\bigoplus_{-\infty<j<J}} W_j^d \tag{3.1.4}$$

and

$$L^2(\mathbb{R}^d) = V_J^d \overset{\perp}{\oplus} \Big(\overset{\perp}{\bigoplus_{J\le j<+\infty}} W_j^d \Big) = \overset{\perp}{\bigoplus_{-\infty<j<+\infty}} W_j^d \,. \tag{3.1.5}$$

It is known that there are 2^d-1 functions $\psi^{d,p}$, $p \in \{1,\dots,2^d-1\}$, belonging to W_0^d, called mother wavelets, such that the sequence $\big\{\psi^{d,p}(x-k) : \ (p,k) \in \{1,\dots,2^d-1\} \times \mathbb{Z}^d\big\}$ is an orthonormal basis of W_0^d. A straightforward consequence is that the sequence $\big\{2^{jd/2}\,\psi^{d,p}(2^jx-k) : \ (p,k) \in \{1,\dots,2^d-1\} \times \mathbb{Z}^d\big\}$ is an orthonormal basis of W_j^d, for every fixed $j \in \mathbb{Z}$. Thus, using (3.1.4) and (3.1.5), it can be shown that:

Theorem 3.1.1 (see e.g. [48, 24])**.** *Assume that $J \in \mathbb{Z}$ is arbitrary and fixed.*

1. *Similarly to the sequence of functions $\big\{2^{Jd/2}\,\phi^d(2^Jx-l) \ : \ l \in \mathbb{Z}^d\big\}$, the sequence of functions*

$$\big\{2^{jd/2}\,\psi^{d,p}(2^jx-k) : \ (p,j,k) \in \{1,\dots,2^d-1\} \times \mathbb{Z} \times \mathbb{Z}^d \text{ and } j<J\big\}$$

is an orthonormal basis of V_J^d.

2. *The sequences of functions*

$$\big\{2^{Jd/2}\,\phi^d(2^Jx-l) : \ l \in \mathbb{Z}^d\big\} \\ \cup\big\{2^{jd/2}\,\psi^{d,p}(2^jx-k) : \ (p,j,k) \in \{1,\dots,2^d-1\} \times \mathbb{Z} \times \mathbb{Z}^d \text{ and } j \ge J\big\}$$

and

$$\big\{2^{jd/2}\,\psi^{d,p}(2^jx-k) : \ (p,j,k) \in \{1,\dots,2^d-1\} \times \mathbb{Z} \times \mathbb{Z}^d\big\}$$

are two orthonormal bases of $L^2(\mathbb{R}^d)$. Such bases of $L^2(\mathbb{R}^d)$ are called wavelet bases.

Usually, for any $d \ge 2$, a MRA $(V_j^d)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^d)$ is obtained starting from a MRA $(V_j^1)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ by making use of the tensor product method. It is important for our purposes to briefly describe this method in the case where $d=2$. In all the sequel, for the sake of simplicity, $(V_j^1)_{j\in\mathbb{Z}}$ is denoted by $(V_j)_{j\in\mathbb{Z}}$ and corresponding scaling function and mother wavelet are denoted by ϕ and ψ. Throughout this chapter, these two functions are assumed to be real-valued. The tensor product method simply consists in defining, for every $j \in \mathbb{Z}$, the space V_j^2 as the tensor product:

$$V_j^2 := V_j \otimes V_j. \tag{3.1.6}$$

In other words, V_j^2 is the closed subspace of $L^2(\mathbb{R}^2)$ spanned by the set of functions $\big\{g(x_1)h(x_2) : g \in V_j \text{ and } h \in V_j\big\}$. Then a scaling function associated with $(V_j^2)_{j\in\mathbb{Z}}$ is $(x_1,x_2) \mapsto \phi(x_1)\phi(x_2)$, and three corresponding mother wavelets are $(x_1,x_2) \mapsto \phi(x_1)\psi(x_2)$, $(x_1,x_2) \mapsto \psi(x_1)\phi(x_2)$ and $(x_1,x_2) \mapsto \psi(x_1)\psi(x_2)$. Such a MRA $(V_j^2)_{j\in\mathbb{Z}}$ was implicitly used in Section 4 of [57] to construct the stochastic processes $\{R_{H,J}(t)\}_{t\in\mathbb{R}_+}$, $J \in \mathbb{N}$, which approximate the fRm $\{R_H(t)\}_{t\in\mathbb{R}_+}$. More precisely, for any fixed $t \in \mathbb{R}_+$, the functions $(x_1, x_2) \mapsto K_{H,J}(t, x_1, x_2)$ and $(x_1, x_2) \mapsto K^{\perp}_{H,J}(t, x_1, x_2)$ respectively denote the orthogonal projections in $L^2(\mathbb{R}^2)$ of the kernel function $(x_1, x_2) \mapsto K_H(t, x_1, x_2)$ (see (3.1.2)) respectively on the space V_J^2 and on its orthogonal complement $(V_J^2)^{\perp}$. The random variable $R_{H,J}(t)$ is defined as:

$$R_{H,J}(t) := \int'_{\mathbb{R}^2} K_{H,J}(t, x_1, x_2)\, dB(x_1)dB(x_2). \tag{3.1.7}$$

Observe that one can derive from (3.1.1) and (3.1.7) that

$$R_H(t) - R_{H,J}(t) := \int'_{\mathbb{R}^2} K^{\perp}_{H,J}(t, x_1, x_2)\, dB(x_1)dB(x_2). \tag{3.1.8}$$

In order to estimate almost surely the approximation error $\|R_H - R_{H,J}\|_{I,\infty}$, it is crucial to express $R_H(t) - R_{H,J}(t)$ in a convenient way. Under some general conditions on ϕ and ψ, an expression for it as a random series is given in the statement of Theorem 1 in [57]. Basically, this expression for $R_H(t) - R_{H,J}(t)$ is issued from the expansion of the function $(x_1, x_2) \mapsto K^{\perp}_{H,J}(t, x_1, x_2)$ in the classical orthonormal basis of $(V_J^2)^{\perp}$, namely:

$$\begin{aligned}\big\{2^j\phi(2^jx_1 - k_1)\psi(2^jx_2 - k_2),\ &2^j\psi(2^jx_1 - k_1)\phi(2^jx_2 - k_2),\\ &2^j\psi(2^jx_1 - k_1)\psi(2^jx_2 - k_2) : (j, k_1, k_2) \in \mathbb{Z}^3 \text{ and } j \geq J\big\}.\end{aligned}$$

Unfortunately, this expression has a drawback: the terms of the random series providing it, that is the random variables $Z^{(j)}_{\kappa,d}(t)$, $j \geq J$ defined in relation (4.9) of [57], are correlated. In order to avoid such a drawback, our strategy consists in expanding the function $(x_1, x_2) \mapsto K^{\perp}_{H,J}(t, x_1, x_2)$ in another much less classical orthonormal basis of $(V_J^2)^{\perp}$, namely:

$$\mathcal{B}''_J := \Big\{2^{\frac{j_1+j_2}{2}}\psi(2^{j_1}x_1 - k_1)\psi(2^{j_2}x_2 - k_2) :\ (j_1, j_2, k_1, k_2) \in \mathbb{Z}^4 \text{ and } j_1 \vee j_2 := \max\{j_1, j_2\} \geq J\Big\}. \tag{3.1.9}$$

We mention in passing that the fact that $\mathcal{B}''_J$ is an orthonormal basis of $(V_J^2)^{\perp}$ can be shown in the following way. The first part of Theorem 3.1.1 (with $d = 1$) and (3.1.6) imply that

$$\mathcal{B}'_J := \Big\{2^{\frac{j_1+j_2}{2}}\psi(2^{j_1}x_1 - k_1)\psi(2^{j_2}x_2 - k_2) :\ (j_1, j_2, k_1, k_2) \in \mathbb{Z}^4 \text{ and } j_1 \vee j_2 < J\Big\}$$

is an orthonormal basis of V_J^2. Moreover, the second part of Theorem 3.1.1 (with $d = 1$) and the equality $L^2(\mathbb{R}^2) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ entail that

$$\mathcal{B} := \Big\{2^{\frac{j_1+j_2}{2}}\psi(2^{j_1}x_1 - k_1)\psi(2^{j_2}x_2 - k_2) :\ (j_1, j_2, k_1, k_2) \in \mathbb{Z}^4\Big\} \tag{3.1.10}$$

is an orthonormal basis of $L^2(\mathbb{R}^2)$. Combining these two results with the equality $L^2(\mathbb{R}^2) = V_J^2 \overset{\perp}{\oplus} (V_J^2)^{\perp}$, it follows that $\mathcal{B}''_J$ is an orthonormal basis of $(V_J^2)^{\perp}$.

Our strategy also works in the more general framework of the generalized Rosenblatt process, that we call generalized fractional Rosenblatt motion (gfRm). It was first introduced by

Maejima and Tudor in their paper [44]. In the last few years, several articles related to it, written by Bai and Taqqu, were published (see [13, 14, 12, 11, 15]). The gfRm depends on two parameters H_1 and H_2 satisfying

$$H_1, H_2 \in (1/2, 1) \quad \text{and} \quad H_1 + H_2 > 3/2\,. \tag{3.1.11}$$

It is denoted by $\{R_{H_1,H_2}(t)\}_{t\in\mathbb{R}_+}$ and defined as:

$$R_{H_1,H_2}(t) := \int'_{\mathbb{R}^2} K_{H_1,H_2}(t,x_1,x_2)\, dB(x_1)dB(x_2), \qquad \text{for all } t \in \mathbb{R}_+\,, \tag{3.1.12}$$

where, the integrand K_{H_1,H_2} is given, for every $(t,x_1,x_2) \in \mathbb{R}_+ \times \mathbb{R}^2$, by:

$$K_{H_1,H_2}(t,x_1,x_2) := \frac{1}{\Gamma(H_1-1/2)\Gamma(H_2-1/2)} \int_0^t (s-x_1)_+^{H_1-3/2}(s-x_2)_+^{H_2-3/2}\, ds. \tag{3.1.13}$$

Notice that the condition (3.1.11) implies that, for each fixed $t \in \mathbb{R}_+$, the kernel function $(x_1,x_2) \mapsto K_{H_1,H_2}(t,x_1,x_2)$ is in $L^2(\mathbb{R}^2)$, which guarantees the existence of the double Wiener integral in (3.1.12). Also notice that the gfRm reduces to the fRm when $H_1 = H_2$. Similarly to the fRm, the gfRm belongs to the second order Wiener chaos, it has continuous paths, and it is self-similar with stationary increments; the self-similarity exponent being the quantity $H_1 + H_2 - 1$. For any fixed $J \in \mathbb{N}$ and $t \in \mathbb{R}_+$, the functions $(x_1,x_2) \mapsto K_{H_1,H_2,J}(t,x_1,x_2)$ and $(x_1,x_2) \mapsto K^{\perp}_{H_1,H_2,J}(t,x_1,x_2)$ are defined in the same way as $(x_1,x_2) \mapsto K_{H,J}(t,x_1,x_2)$ and $(x_1,x_2) \mapsto K^{\perp}_{H,J}(t,x_1,x_2)$. Namely, they respectively are the orthogonal projections in $L^2(\mathbb{R}^2)$ of the kernel function $(x_1,x_2) \mapsto K_{H_1,H_2}(t,x_1,x_2)$ (see (3.1.13)) respectively on the space V_J^2 and on its orthogonal complement $(V_J^2)^{\perp}$. Similarly to (3.1.7) and (3.1.8), the random variable $R_{H_1,H_2,J}(t)$ is defined as:

$$R_{H_1,H_2,J}(t) := \int'_{\mathbb{R}^2} K_{H_1,H_2,J}(t,x_1,x_2)\, dB(x_1)dB(x_2), \tag{3.1.14}$$

and one can derive from (3.1.12) and (3.1.14) that

$$R_{H_1,H_2}(t) - R_{H_1,H_2,J}(t) := \int'_{\mathbb{R}^2} K^{\perp}_{H_1,H_2,J}(t,x_1,x_2)\, dB(x_1)dB(x_2). \tag{3.1.15}$$

We mention in passing that it results from the isometry property of double Wiener integrals and from the Kolmogorov's continuity theorem that, for any fixed $J \in \mathbb{N}$, the stochastic processes $\{R_{H_1,H_2,J}(t)\}_{t\in\mathbb{R}_+}$ and $\{R_{H_1,H_2}(t) - R_{H_1,H_2,J}(t)\}_{t\in\mathbb{R}_+}$ have continuous paths.

By expanding, for each fixed $J \in \mathbb{N}$ and $t \in \mathbb{R}_+$, the function $(x_1,x_2) \mapsto K^{\perp}_{H_1,H_2,J}(t,x_1,x_2)$ in the basis $\mathcal{B}''_J$ (see (3.1.9)), and by using (3.1.15) as well as the isometry property of double Wiener integrals, one obtains that

$$R_{H_1,H_2}(t) - R_{H_1,H_2,J}(t) = \sum_{(j_1,j_2)\in\mathbb{Z}^2,\, j_1\vee j_2 \geq J} \; \sum_{(k_1,k_2)\in\mathbb{Z}^2} \mathcal{K}^{k_1,k_2}_{j_1,j_2}(t)\varepsilon^{k_1,k_2}_{j_1,j_2}\,. \tag{3.1.16}$$

Notice that, for all $(j_1, j_2, k_1, k_2) \in \mathbb{Z}^4$, the real-valued deterministic coefficient $\mathcal{K}_{j_1,j_2}^{k_1,k_2}(t)$ is given by:

$$\mathcal{K}_{j_1,j_2}^{k_1,k_2}(t) := 2^{\frac{j_1+j_2}{2}} \int_{\mathbb{R}^2} K_{H_1,H_2}(t, x_1, x_2)\psi(2^{j_1}x_1 - k_1)\psi(2^{j_2}x_2 - k_2)\, dx_1 dx_2\,; \tag{3.1.17}$$

we mention that (3.1.13), (3.1.17) and the dominated convergence theorem imply that $\mathcal{K}_{j_1,j_2}^{k_1,k_2}$ is a continuous function on $\mathbb{R}_+$. Also notice that $\varepsilon_{j_1,j_2}^{k_1,k_2}$ is the random variable of the second order Wiener chaos defined, for all $(j_1, j_2, k_1, k_2) \in \mathbb{Z}^4$, as:

$$\varepsilon_{j_1,j_2}^{k_1,k_2} := 2^{\frac{j_1+j_2}{2}} \int_{\mathbb{R}^2}' \psi(2^{j_1}x_1 - k_1)\psi(2^{j_2}x_2 - k_2)\, dB(x_1)dB(x_2). \tag{3.1.18}$$

Observe that our previous arguments for deriving the equality (3.1.16) only allow to assert that the random series in it is, for each fixed $t \in \mathbb{R}_+$ and $J \in \mathbb{N}$, unconditionally convergent in $L^2(\Omega)$, where Ω denotes the underlying probability space. Yet, in the proof of our Theorem 3.1.2, stated below, we show that this random series is also almost surely convergent uniformly in $t \in I$. Let us point out that one may assume without any restriction that the latter compact interval is of the form $I = [0, T]$, where the real number $T > 2$ is arbitrary and fixed. This assumption is systematically made from now on.

The goal of Section 3.2 is to derive the following theorem which provides an almost sure estimate of the approximation error $\|R_{H_1,H_2} - R_{H_1,H_2,J}\|_{I,\infty}$.

Theorem 3.1.2. *Assume that ψ in (3.1.9) is a Meyer's mother wavelet; that is ψ belongs to the Schwartz class $S(\mathbb{R})$ and its Fourier transform $\widehat{\psi}$ is compactly supported with support satisfying*

$$\text{supp } \widehat{\psi} \subseteq \Big\{\xi \in \mathbb{R} : \ \frac{2\pi}{3} \le |\xi| \le \frac{8\pi}{3}\Big\}. \tag{3.1.19}$$

Then, for all compact interval $I \subset \mathbb{R}_+$, there exists an almost surely finite random variable C (depending on I) for which one has, almost surely, for each $J \in \mathbb{N}$,

$$\|R_{H_1,H_2} - R_{H_1,H_2,J}\|_{I,\infty} := \sup_{t\in I} \big|R_{H_1,H_2}(t) - R_{H_1,H_2,J}(t)\big| \le C J\, 2^{-J(H_1+H_2-3/2)}. \tag{3.1.20}$$

Remark 3.1.3. Recall that throughout our thesis, the Fourier transform $\widehat{f}$ of an arbitrary function f in the class Schwartz $S(\mathbb{R})$ is defined, for all $\xi \in \mathbb{R}$, as $\widehat{f}(\xi) := (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-i\xi x} f(x)\, dx$.

Let us now present the goal of Section 3.3 of this chapter. Using (3.1.12) and the wavelet basis $\mathcal{B}$ (see (3.1.10)), similarly to (3.1.16) one can derive, for each fixed $t \in \mathbb{R}_+$, that

$$R_{H_1,H_2}(t) = \sum_{(j_1,j_2,k_1,k_2)\in\mathbb{Z}^4} \mathcal{K}_{j_1,j_2}^{k_1,k_2}(t)\varepsilon_{j_1,j_2}^{k_1,k_2}, \tag{3.1.21}$$

where the random series is unconditionally convergent in $L^2(\Omega)$. The main goal of Section 3.3 is to show that it is also convergent in a much stronger sense: almost surely, uniformly in $t \in I := [0, T]$, and jointly in the four indices j_1, j_2, k_1 and k_2; and to obtain an estimate of the almost sure rate of convergence. More precisely, the following theorem is derived in Section 3.3:

Theorem 3.1.4. *Let $T > 2$, $b > 0$, $d > 0$ and $g > 0$ be arbitrary and fixed real numbers. For all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$, let $\widetilde{R}_{H_1,H_2,n}(t)$ be the random variable of the second order Wiener chaos defined as the finite random sum:*

$$\widetilde{R}_{H_1,H_2,n}(t) := \sum_{(j_1,j_2,k_1,k_2)\in\mathcal{S}_n^+\cup\mathcal{S}_n^-} \mathcal{K}_{j_1,j_2}^{k_1,k_2}(t)\varepsilon_{j_1,j_2}^{k_1,k_2}\,, \tag{3.1.22}$$

where $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ are the two finite disjoint sets such that

$$\mathcal{S}_n^+ := \big\{(j_1,j_2,k_1,k_2) \in \mathbb{Z}^4 : \, -2^{nb} \le j_1 \wedge j_2\,,\ 0 \le j_1 \vee j_2 < n\,,\ |k_1| \le 2^{n+1}T\,,\ |k_2| \le 2^{n+1}T\big\} \tag{3.1.23}$$

and

$$\mathcal{S}_n^- := \big\{(j_1,j_2,k_1,k_2) \in \mathbb{Z}^4 : \, -2^{nd} \le j_1 \wedge j_2 \le j_1 \vee j_2 < 0\,,\ |k_1| \le 2^{ng}\,,\ |k_2| \le 2^{ng}\big\}; \tag{3.1.24}$$

recall that $j_1 \wedge j_2 := \min\{j_1,j_2\}$ and $j_1 \vee j_2 := \max\{j_1,j_2\}$. Then, under the assumption that ψ in (3.1.10) is a Meyer's mother wavelet, there exists an almost surely finite random variable C (depending on T,b,d,g) for which one has, almost surely, for each $n \in \mathbb{N}$,

$$\|R_{H_1,H_2} - \widetilde{R}_{H_1,H_2,n}\|_{I,\infty} := \sup_{t\in I}\big|R_{H_1,H_2}(t) - \widetilde{R}_{H_1,H_2,n}(t)\big| \le Cn\,2^{-n(H_1+H_2-3/2)}\,, \tag{3.1.25}$$

where $I := [0,T]$.

3.2 Proof of Theorem 3.1.2

In order to prove Theorem 3.1.2 one needs to obtain several intermediary results.

First one shows that, for all $t \in \mathbb{R}_+$ and $(j_1,j_2,k_1,k_2) \in \mathbb{Z}^4$, the deterministic coefficient $\mathcal{K}_{j_1,j_2}^{k_1,k_2}(t)$ (see (3.1.17)) can be nicely expressed in terms of Ψ_{H_1} and Ψ_{H_2}, the two left-sided fractional primitives of orders $H_1 - 1/2$ and $H_2 - 1/2$ of the Meyer's mother wavelet ψ. One refers to the book [60] for a classical reference on the fundamental notions of fractional primitives and derivatives. Recall that, for any $H \in (1/2,1)$, the left-sided fractional primitive of ψ of order $H - 1/2$ is the real-valued function, we denote by Ψ_H, defined as:

$$\Psi_H(s) := \frac{1}{\Gamma(H-1/2)}\int_{\mathbb{R}} (s-y)_+^{H-3/2}\psi(y)\,dy\,, \quad \text{for all } s \in \mathbb{R}. \tag{3.2.1}$$

It inherits some important properties of the Meyer's mother wavelet ψ, namely:

Proposition 3.2.1. The function Ψ_H belongs to the Schwartz class $S(\mathbb{R})$. This means that it is infinitely differentiable on the real line, and that itself and its derivative of any order have rapid decrease at infinity; in other words, for each fixed (positive) real number L, one has

$$\sup_{x\in\mathbb{R}}\Big\{\big(3+|x|\big)^L\,|\Psi_H(x)|\Big\} < +\infty, \tag{3.2.2}$$

and (3.2.2) remains valid when Ψ_H is replaced by its derivative of an arbitrary order. Moreover, its Fourier transform $\widehat{\Psi}_H$ is the infinitely differentiable function on the real line given by:

$$\widehat{\Psi}_H(0) = 0 \quad \text{and} \quad \widehat{\Psi}_H(\xi) = (i\xi)^{-(H-1/2)}\,\widehat{\psi}(\xi), \quad \text{for every } \xi \in \mathbb{R} \setminus \{0\}. \tag{3.2.3}$$

Thus, in view of (3.1.19), $\widehat{\Psi}_H$ has a compact support satisfying:

$$\operatorname{supp} \widehat{\Psi}_H \subseteq \Big\{\xi \in \mathbb{R} : \quad \frac{2\pi}{3} \leq |\xi| \leq \frac{8\pi}{3}\Big\}. \tag{3.2.4}$$

The proof of Proposition 3.2.1 has been skipped since it is very classical. The following proposition shows that the deterministic coefficient $\mathcal{K}_{j_1,j_2}^{k_1,k_2}(t)$ can be nicely expressed in terms of the functions Ψ_{H_1} and Ψ_{H_2}.

Proposition 3.2.2. For all $t \in \mathbb{R}_+$ and $(j_1, j_2, k_1, k_2) \in \mathbb{Z}^4$, one has

$$\mathcal{K}_{j_1,j_2}^{k_1,k_2}(t) = 2^{j_1(1-H_1)+j_2(1-H_2)} A_{j_1,j_2}^{k_1,k_2}(t), \tag{3.2.5}$$

where

$$A_{j_1,j_2}^{k_1,k_2}(t) = \int_0^t \Psi_{H_1}(2^{j_1}s - k_1)\Psi_{H_2}(2^{j_2}s - k_2)\, ds. \tag{3.2.6}$$

Proof of Proposition 3.2.2 The proposition can easily be obtained by using (3.1.17), (3.2.1), Fubini's theorem and the change of variables $y_1 = 2^{j_1}x_1 - k_1$ and $y_2 = 2^{j_2}x_2 - k_2$. $\square$

Let us now provide, for every $(j_1, j_2, k_1, k_2) \in \mathbb{Z}^4$, a nice expression for the random variable $\varepsilon_{j_1,j_2}^{k_1,k_2}$ (see (3.1.18)). The following proposition shows that it is the product of two independent standard Gaussian random variables except in the particular case where $j_1 = j_2$ and $k_1 = k_2$.

Proposition 3.2.3. One has that

$$\varepsilon_{j,j}^{k,k} = \left(2^{j/2} \int_{\mathbb{R}} \psi(2^j x - k)\, dB(x)\right)^2 - 1\,, \quad \text{for all } (j,k) \in \mathbb{Z}^2, \tag{3.2.7}$$

and, for every $(j_1, j_2, k_1, k_2) \in \mathbb{Z}^4$,

$$\varepsilon_{j_1,j_2}^{k_1,k_2} = \left(2^{j_1/2} \int_{\mathbb{R}} \psi(2^{j_1} x - k_1)\, dB(x)\right)\left(2^{j_2/2} \int_{\mathbb{R}} \psi(2^{j_2} x - k_2)\, dB(x)\right), \quad \text{if } j_1 \neq j_2 \text{ or } k_1 \neq k_2, \tag{3.2.8}$$

where $\int_{\mathbb{R}} (\cdot)\, dB$ denotes the usual Wiener integral on $\mathbb{R}$. Moreover, the orthonormality property of the wavelets $2^{j/2}\psi(2^j x - k)$, $(j,k) \in \mathbb{Z}^2$, and elementary properties of the usual Wiener integral entail that the real-valued random variables $2^{j/2} \int_{\mathbb{R}} \psi(2^j x - k)\, dB(x)$, $(j,k) \in \mathbb{Z}^2$, are independent with the same $\mathcal{N}(0,1)$ distribution.

Proof of Proposition 3.2.3 It is well-known (see e.g. Lemma A.1 in [57] or [36, 52]) that, for every real-valued functions f and g belonging to $L^2(\mathbb{R})$, one has

$$\int_{\mathbb{R}^2}' f(x_1)g(x_2)\, dB(x_1)dB(x_2) = \int_{\mathbb{R}} f(x)\, dB(x) \int_{\mathbb{R}} g(x)\, dB(x) - \int_{\mathbb{R}} f(x)g(x)\, dx\,. \tag{3.2.9}$$

Thus, using (3.1.18), (3.2.9) and the orthonormality property of the wavelets $2^{j/2}\psi(2^j x-k)$, $(j,k)\in\mathbb{Z}^2$, one obtains (3.2.7) and (3.2.8). $\square$

The following crucial lemma easily results from Proposition 3.2.3 and from Lemma 2 in [8].

Lemma 3.2.4. There exist Ω^* an event of probability 1 and C a positive random variable of finite moment of any order, such that, for all $\omega\in\Omega^*$ and for each $(j_1,j_2,k_1,k_2)\in\mathbb{Z}^4$, one has

$$\left|\varepsilon_{j_1,j_2}^{k_1,k_2}(\omega)\right|\le C(\omega)\sqrt{\log\big(3+|j_1|+|k_1|\big)\log\big(3+|j_2|+|k_2|\big)}. \tag{3.2.10}$$

Let us now introduce a partition of the set of integers $\mathbb{Z}$ which will play a fundamental role in the proof of Theorem 3.1.2.

Definition 3.2.5. The parameter $a\in(1/2,1)$ is arbitrary and fixed. For every $(j,k)\in\mathbb{Z}_+\times\mathbb{Z}$, one denotes by $B_{j,k}$ the compact interval of the real line centered at the dyadic number $2^{-j}k$ and of radius 2^{-ja}, that is

$$B_{j,k}:=\big[2^{-j}k-2^{-ja},2^{-j}k+2^{-ja}\big]. \tag{3.2.11}$$

For each fixed $j\in\mathbb{Z}_+$ and $t\in\mathbb{R}_+$, $D_j^1(t)$, $D_j^2(t)$ and $D_j^3(t)$ are the three disjoint subsets of $\mathbb{Z}$ defined as:

$$D_j^1(t):=\big\{k\in\mathbb{Z}:B_{j,k}\subseteq[0,t]\big\}, \tag{3.2.12}$$

$$D_j^2(t):=\big\{k\in\mathbb{Z}\backslash D_j^1(t):B_{j,k}\cap[0,t]\neq\emptyset\big\}, \tag{3.2.13}$$

and

$$D_j^3(t):=\big\{k\in\mathbb{Z}:B_{j,k}\cap[0,t]=\emptyset\big\}. \tag{3.2.14}$$

It can easily be seen that these three disjoints sets, which depend on t and also on the parameter a, form a partition of $\mathbb{Z}$, that is:

$$\mathbb{Z}=D_j^1(t)\cup D_j^2(t)\cup D_j^3(t). \tag{3.2.15}$$

Remark 3.2.6. Notice that, $D_j^3(t)$ is always an infinite set; while $D_j^2(t)$ and $D_j^1(t)$ are always finite sets which can even be empty. Their cardinalities satisfy, for each fixed positive real number T and for all $j\in\mathbb{Z}_+$,

$$\sup_{t\in[0,T]}\big\{\operatorname{card}(D_j^1(t))\big\}\le c'\,2^j \tag{3.2.16}$$

and

$$\sup_{t\in[0,T]}\big\{\operatorname{card}(D_j^2(t))\big\}\le c''2^{j(1-a)}, \tag{3.2.17}$$

where $c'\ge 1$ and $c''\ge 1$ are two positive finite constants. Notice that c' depends on T while c'' does not depend on it.

Definition 3.2.7. For all $J\in\mathbb{N}$, one denotes by $\aleph_{1,J}$ and $\aleph_{2,J}$ the two infinite subsets of $\mathbb{Z}^2$ defined as:

$$\aleph_{1,J}:=\big\{(j_1,j_2)\in\mathbb{Z}^2:\ j_1\ge j_2 \text{ and } j_1\ge J\big\} \tag{3.2.18}$$

and

$$\aleph_{2,J}:=\big\{(j_1,j_2)\in\mathbb{Z}^2:\ j_2\ge j_1 \text{ and } j_2\ge J\big\}. \tag{3.2.19}$$

Lemma 3.2.8. Let $T > 2$ and $L \geq 3/2$ be arbitrary and fixed real numbers. For every $J \in \mathbb{N}$, let $\mathcal{H}^1_{1,J}$ and $\mathcal{H}^1_{2,J}$ be the positive random variables defined as:

$$\mathcal{H}^1_{1,J} := \sum_{(j_1,j_2)\in\aleph_{1,J}} 2^{j_1(1-H_1)+j_2(1-H_2)} \sum_{k_1\in\mathbb{Z}} \sum_{|k_2|>2^{j_1+1}T} \sup_{t\in[0,T]} \Big\{ \big| A^{k_1,k_2}_{j_1,j_2}(t) \big| \Big\} \big| \varepsilon^{k_1,k_2}_{j_1,j_2} \big| \tag{3.2.20}$$

and

$$\mathcal{H}^1_{2,J} := \sum_{(j_1,j_2)\in\aleph_{2,J}} 2^{j_1(1-H_1)+j_2(1-H_2)} \sum_{|k_1|>2^{j_2+1}T} \sum_{k_2\in\mathbb{Z}} \sup_{t\in[0,T]} \Big\{ \big| A^{k_1,k_2}_{j_1,j_2}(t) \big| \Big\} \big| \varepsilon^{k_1,k_2}_{j_1,j_2} \big|. \tag{3.2.21}$$

Then, there exists a positive almost surely finite random variable C such that, for all $J \in \mathbb{N}$ and $l \in \{1,2\}$, the following inequality holds on the event Ω^*:

$$\mathcal{H}^1_{l,J} \leq C\, 2^{-J(H_1+H_2+L-3)}\, J \sqrt{\log(3+J)}\,. \tag{3.2.22}$$

Proof of Lemma 3.2.8 One only shows that (3.2.22) is satisfied when $l = 1$, the case where $l = 2$ can be treated in the same way. Let $(j_1, j_2) \in \aleph_{1,J}$ be arbitrary and fixed. Using (3.2.6), (3.2.2), (3.2.10), (3.4.1), the triangle inequality, the inequality $j_1 \geq j_2$, Lemma 3.4.2, the fact that $y \mapsto (2+y)^{-L}\sqrt{\log(2+y)}$ is a decreasing function on $\mathbb{R}_+$, and (3.4.2) one gets that

$$\begin{aligned}
&\sum_{k_1\in\mathbb{Z}} \sum_{|k_2|>2^{j_1+1}T} \sup_{t\in[0,T]} \Big\{ \big| A^{k_1,k_2}_{j_1,j_2}(t) \big| \Big\} \big| \varepsilon^{k_1,k_2}_{j_1,j_2} \big| \\
&\leq C_0 \sum_{k_1\in\mathbb{Z}} \sum_{|k_2|>2^{j_1+1}T} \int_0^T \frac{\sqrt{\log(3+j_1+|k_1|)}}{\big(3+|2^{j_1}s-k_1|\big)^L} \times \frac{\sqrt{\log(3+|j_2|+|k_2|)}}{\big(3+|2^{j_2}s-k_2|\big)^L}\, ds \\
&\leq C_0 \int_0^T \left(\sum_{k_1\in\mathbb{Z}} \sum_{|k_2|>2^{j_1+1}T} \frac{\sqrt{\log(3+j_1+|k_1|)}}{\big(3+|2^{j_1}s-k_1|\big)^L} \times \frac{\sqrt{\log(3+|j_2|+|k_2|)}}{\big(3+|k_2|-2^{j_2}s\big)^L} \right) ds \\
&\leq C_1 T\, 2^L \sqrt{\log(3+j_1+2^{j_1}T)\log(3+|j_2|)} \sum_{|k_2|>2^{j_1+1}T} \frac{\sqrt{\log(3+|k_2|)}}{\big(3+|k_2|\big)^L} \\
&\leq C_1 T\, 2^{L+1} \sqrt{\log(3+j_1+2^{j_1}T)\log(3+|j_2|)} \int_{2^{j_1+1}T}^{+\infty} \frac{\sqrt{\log(2+y)}}{(2+y)^L}\, dy \\
&\leq C_2 \sqrt{\log(3+|j_2|)}\, j_1\, 2^{-j_1(L-1)},
\end{aligned} \tag{3.2.23}$$

where C_0, C_1 and C_2 are 3 positive almost surely finite random variables not depending on

(j_1, j_2) and J. Next, it follows from (3.2.18), (3.2.23), the triangle inequality, and (3.4.1) that

$$\begin{aligned}
&\sum_{(j_1,j_2)\in\aleph_{1,J}} 2^{j_1(1-H_1)+j_2(1-H_2)} \sum_{k_1\in\mathbb{Z}} \sum_{|k_2|>2^{j_1+1}T} \sup_{t\in[0,T]} \Big\{ \big|A_{j_1,j_2}^{k_1,k_2}(t)\big| \Big\} \big|\varepsilon_{j_1,j_2}^{k_1,k_2}\big| \\
&\le C_2 \sum_{j_1=J}^{+\infty} \sum_{j_2=-\infty}^{j_1} \sqrt{\log(3+|j_2|)}\, 2^{j_2(1-H_2)}\, j_1\, 2^{-j_1(H_1+L-2)} \\
&\le C_2 \sum_{j_1=J}^{+\infty} \sum_{p=0}^{+\infty} \sqrt{\log(3+|j_1-p|)}\, 2^{(j_1-p)(1-H_2)}\, j_1\, 2^{-j_1(H_1+L-2)} \\
&\le C_2 \sum_{j_1=J}^{+\infty} j_1\, 2^{-j_1(H_1+H_2+L-3)} \sum_{p=0}^{+\infty} \sqrt{\log(3+j_1+p)}\, 2^{-p(1-H_2)} \\
&\le C_3 \sum_{j_1=J}^{+\infty} j_1 \sqrt{\log(3+j_1)}\, 2^{-j_1(H_1+H_2+L-3)} \\
&\le C_4 J \sqrt{\log(3+J)}\, 2^{-J(H_1+H_2+L-3)},
\end{aligned}$$

where the positive almost surely finite random variables C_3 and C_4 are defined as:

$$C_3 := C_2 \sum_{p=0}^{+\infty} \sqrt{\log(3+p)}\, 2^{-p(1-H_2)} \quad \text{and} \quad C_4 := C_3 \sum_{q=0}^{+\infty} (1+q) \sqrt{\log(3+q)}\, 2^{-q(H_1+H_2+L-3)}.$$

Thus, one obtains (3.2.22) when $l = 1$. $\square$

Lemma 3.2.9. Let $T > 2$ and $L \ge 2^{-1}(1-a)^{-1} + 1$ be arbitrary and fixed real numbers. For every $J \in \mathbb{N}$, let $\mathcal{H}_{1,J}^2$ and $\mathcal{H}_{2,J}^2$ be the positive random variables defined as:

$$\mathcal{H}_{1,J}^2 := \sum_{(j_1,j_2)\in\aleph_{1,J}} 2^{j_1(1-H_1)+j_2(1-H_2)} \sup_{t\in[0,T]} \Big\{ \sum_{k_1\in D_{j_1}^3(t)} \sum_{k_2\in\mathbb{Z}} \big|A_{j_1,j_2}^{k_1,k_2}(t)\big| \big|\varepsilon_{j_1,j_2}^{k_1,k_2}\big| \Big\} \tag{3.2.24}$$

and

$$\mathcal{H}_{2,J}^2 := \sum_{(j_1,j_2)\in\aleph_{2,J}} 2^{j_1(1-H_1)+j_2(1-H_2)} \sup_{t\in[0,T]} \Big\{ \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in D_{j_2}^3(t)} \big|A_{j_1,j_2}^{k_1,k_2}(t)\big| \big|\varepsilon_{j_1,j_2}^{k_1,k_2}\big| \Big\}. \tag{3.2.25}$$

Then, there exists a positive almost surely finite random variable C such that, for all $J \in \mathbb{N}$ and $l \in \{1, 2\}$, the following inequality holds on the event Ω^*:

$$\mathcal{H}_{l,J}^2 \le C\, 2^{-J((L-1)(1-a)+H_1+H_2-2)}\, J^2 \sqrt{\log(3+J)}\,. \tag{3.2.26}$$

Proof of Lemma 3.2.9 One only shows that (3.2.26) is satisfied when $l = 1$, the case where $l = 2$ can be treated in the same way. Let $t \in [0, T]$ and $(j_1, j_2) \in \aleph_{1,J}$ be arbitrary and fixed.

Using (3.2.6), (3.2.10), (3.2.2), Lemma 3.4.2 and Lemma 3.4.3, one gets that

$$
\begin{aligned}
&\sum_{k_1 \in D^3_{j_1}(t)} \sum_{k_2 \in \mathbb{Z}} \big|A^{k_1,k_2}_{j_1,j_2}(t)\big|\big|\varepsilon^{k_1,k_2}_{j_1,j_2}\big| \\
&\le C_0 \int_0^T \left(\sum_{k_1 \in D^3_{j_1}(t)} \sum_{k_2 \in \mathbb{Z}} \frac{\sqrt{\log(3 + j_1 + |k_1|)}}{\big(3 + |2^{j_1}s - k_1|\big)^L} \times \frac{\sqrt{\log(3 + |j_2| + |k_2|)}}{\big(3 + |2^{j_2}s - k_2|\big)^L} \right) ds \\
&\le C_1 \sqrt{\log(3 + |j_2| + 2^{j_2}T)} \int_0^T \left(\sum_{k_1 \in D^3_{j_1}(t)} \frac{\sqrt{\log(3 + j_1 + |k_1|)}}{\big(3 + |2^{j_1}s - k_1|\big)^L} \right) ds \\
&\le C_2 (j_1 + 1) 2^{-j_1(L-1)(1-a)} \sqrt{\log\big(3 + |j_2| + 2^{j_2}T\big)},
\end{aligned} \tag{3.2.27}
$$

where C_0, C_1 and C_2 are 3 positive almost surely finite random variables not depending on t, (j_1, j_2) and J. Next, it follows from (3.2.18), (3.2.27), the triangle inequality and (3.4.1) that

$$
\begin{aligned}
&\sum_{(j_1,j_2) \in \aleph_{1,J}} 2^{j_1(1-H_1)+j_2(1-H_2)} \sup_{t \in [0,T]} \left\{ \sum_{k_1 \in D^3_{j_1}(t)} \sum_{k_2 \in \mathbb{Z}} \big|A^{k_1,k_2}_{j_1,j_2}(t)\big|\big|\varepsilon^{k_1,k_2}_{j_1,j_2}\big| \right\} \\
&\le C_2 \sum_{j_1=J}^{+\infty} \sum_{j_2=-\infty}^{j_1} (j_1+1) 2^{-j_1((L-1)(1-a)+H_1-1)}\, 2^{j_2(1-H_2)} \sqrt{\log\big(3 + |j_2| + 2^{j_2}T\big)} \\
&\le C_2 \sum_{j_1=J}^{+\infty} \sum_{p=0}^{+\infty} (j_1+1) 2^{-j_1((L-1)(1-a)+H_1-1)}\, 2^{(j_1-p)(1-H_2)} \sqrt{\log\big(3 + |j_1 - p| + 2^{j_1-p}T\big)} \\
&\le C_2 \sum_{j_1=J}^{+\infty} (j_1+1) \sqrt{\log\big(3 + 2^{j_1}T\big)}\, 2^{-j_1((L-1)(1-a)+H_1+H_2-2)} \sum_{p=0}^{+\infty} \sqrt{\log(3 + j_1 + p)}\, 2^{-p(1-H_2)} \\
&\le C_3 \sum_{j_1=J}^{+\infty} j_1^2 \sqrt{\log(3 + j_1)}\, 2^{-j_1((L-1)(1-a)+H_1+H_2-2)} \\
&\le C_4 J^2 \sqrt{\log(3 + J)}\, 2^{-J((L-1)(1-a)+H_1+H_2-2)},
\end{aligned}
$$

where the two positive almost surely finite random variables C_3 and C_4 are defined as:

$$
C_3 := 16 C_2 (2 + T) \sum_{p=0}^{+\infty} \sqrt{\log(3 + p)}\, 2^{-p(1-H_2)}
$$

and

$$
C_4 := C_3 \sum_{q=0}^{+\infty} (1 + q)^2 \sqrt{\log(3 + q)}\, 2^{-q((L-1)(1-a)+H_1+H_2-2)}.
$$

Thus one obtains (3.2.26) when $l = 1$. $\square$

Lemma 3.2.10. Let $T > 2$ be an arbitrary and fixed real number. For every $J \in \mathbb{N}$, let $\mathcal{H}^3_{1,J}$ and $\mathcal{H}^3_{2,J}$ be the positive random variables defined as:

$$\mathcal{H}^3_{1,J} := \sum_{(j_1,j_2)\in\aleph_{1,J}} 2^{j_1(1-H_1)+j_2(1-H_2)} \sup_{t\in[0,T]} \Big\{ \sum_{k_1\in D^2_{j_1}(t)} \sum_{k_2\in\mathbb{Z}} \big|A^{k_1,k_2}_{j_1,j_2}(t)\big|\big|\varepsilon^{k_1,k_2}_{j_1,j_2}\big| \Big\} \tag{3.2.28}$$

and

$$\mathcal{H}^3_{2,J} := \sum_{(j_1,j_2)\in\aleph_{2,J}} 2^{j_1(1-H_1)+j_2(1-H_2)} \sup_{t\in[0,T]} \Big\{ \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in D^2_{j_2}(t)} \big|A^{k_1,k_2}_{j_1,j_2}(t)\big|\big|\varepsilon^{k_1,k_2}_{j_1,j_2}\big| \Big\}. \tag{3.2.29}$$

Then, there exists a positive almost surely finite random variable C such that, for all $J \in \mathbb{N}$ and $l \in \{1,2\}$, the following inequality holds on the event Ω^*:

$$\mathcal{H}^3_{l,J} \le C\, 2^{-J(H_1+H_2+a-2)}\, J\sqrt{\log(3+J)}. \tag{3.2.30}$$

Proof of Lemma 3.2.10 One only shows that (3.2.30) is satisfied when $l = 1$, the case where $l = 2$ can be treated in the same way. Let $L \in (1,+\infty)$, $t \in [0,T]$ and $(j_1,j_2) \in \aleph_{1,J}$ be arbitrary and fixed. Using (3.2.6), (3.2.10), (3.2.2), Lemma 3.4.2, the inequality $|k_1| \le 2^{j_1(1-a)} + 2^{j_1}T$, for all $k_1 \in D^2_{j_1}(t)$ (see (3.2.13), (3.2.12) and (3.2.11)), the change of variable $z = 2^{j_1}s - k_1$, (3.2.17) and (3.4.2), one gets that

$$\begin{aligned}
&\sum_{k_1\in D^2_{j_1}(t)} \sum_{k_2\in\mathbb{Z}} \big|A^{k_1,k_2}_{j_1,j_2}(t)\big|\big|\varepsilon^{k_1,k_2}_{j_1,j_2}\big| \\
&\le C_0 \int_0^T \Bigg(\sum_{k_1\in D^2_{j_1}(t)} \sum_{k_2\in\mathbb{Z}} \frac{\sqrt{\log(3+j_1+|k_1|)}}{\big(3+|2^{j_1}s-k_1|\big)^L} \times \frac{\sqrt{\log(3+|j_2|+|k_2|)}}{\big(3+|2^{j_2}s-k_2|\big)^L} \Bigg)\, ds \\
&\le C_1 \sqrt{\log\big(3+|j_2|+2^{j_2}T\big)} \int_0^T \Bigg(\sum_{k_1\in D^2_{j_1}(t)} \frac{\sqrt{\log(3+j_1+|k_1|)}}{\big(3+|2^{j_1}s-k_1|\big)^L} \Bigg)\, ds \\
&\le C_1 \sqrt{\log\big(3+|j_2|+2^{j_2}T\big)\log\big(3+j_1+2^{j_1(1-a)}+2^{j_1}T\big)} \sum_{k_1\in D^2_{j_1}(t)} \int_{\mathbb{R}} \frac{ds}{\big(3+|2^{j_1}s-k_1|\big)^L} \\
&= C_1 \Bigg(\int_{\mathbb{R}} \frac{ds}{\big(3+|z|\big)^L} \Bigg) \sqrt{\log\big(3+|j_2|+2^{j_2}T\big)\log\big(3+j_1+2^{j_1(1-a)}+2^{j_1}T\big)}\, \operatorname{card}(D^2_{j_1}(t))\, 2^{-j_1} \\
&\le C_2\, 2^{-j_1 a} \sqrt{(j_1+1)\log\big(3+|j_2|+2^{j_2}T\big)},
\end{aligned} \tag{3.2.31}$$

where C_0, C_1 and C_2 are 3 positive almost surely finite random variables not depending on t, (j_1,j_2) and J. Next, it follows from (3.2.18), (3.2.31), (3.4.1) and the inequalities $a > 1/2$ and

$H_1 + H_2 > 3/2$, that

$$\begin{aligned}
&\sum_{(j_1,j_2)\in\aleph_{1,J}} 2^{j_1(1-H_1)+j_2(1-H_2)} \sup_{t\in[0,T]} \Big\{ \sum_{k_1\in D^2_{j_1}(t)} \sum_{k_2\in\mathbb{Z}} \big|A^{k_1,k_2}_{j_1,j_2}(t)\big|\big|\varepsilon^{k_1,k_2}_{j_1,j_2}\big| \Big\} \\
&\le C_2 \sum_{j_1=J}^{+\infty} 2^{-j_1(H_1+a-1)}\sqrt{j_1+1} \sum_{j_2=-\infty}^{j_1} 2^{j_2(1-H_2)}\sqrt{\log\big(3+|j_2|+2^{j_2}T\big)} \\
&\le C_2 \sum_{j_1=J}^{+\infty} 2^{-j_1(H_1+a-1)}\sqrt{j_1+1} \sum_{p=0}^{+\infty} 2^{(j_1-p)(1-H_2)}\sqrt{\log\big(3+|j_1-p|+2^{j_1-p}T\big)} \\
&\le C_2 \sum_{j_1=J}^{+\infty} 2^{-j_1(H_1+H_2+a-2)}\sqrt{(j_1+1)\log\big(3+2^{j_1}T\big)} \sum_{p=0}^{+\infty} 2^{-p(1-H_2)}\sqrt{\log(3+j_1+p)} \\
&\le C_3 \sum_{j_1=J}^{+\infty} 2^{-j_1(H_1+H_2+a-2)}\, j_1\,\sqrt{\log(3+j_1)} \\
&\le C_4 2^{-J(H_1+H_2+a-2)}\, J\,\sqrt{\log(3+J)}\,,
\end{aligned}$$

where the 2 positive almost surely finite random variables C_3 and C_4 are defined as:

$$C_3 := 2C_2\sqrt{\log(3+T)}\sum_{p=0}^{+\infty} 2^{-p(1-H_2)}\sqrt{\log(3+p)}$$

and

$$C_4 := C_3 \sum_{q=0}^{+\infty} 2^{-q(H_1+H_2+a-2)}\,(1+q)\,\sqrt{\log(3+q)}\,.$$

Thus, one obtains (3.2.30) when $l = 1$. $\square$

Definition 3.2.11. For all $(j_1, j_2, k_1, k_2) \in \mathbb{Z}^4$, one sets

$$F^{k_1,k_2}_{j_1,j_2} := \int_{\mathbb{R}} \Psi_{H_1}(2^{j_1}s - k_1)\Psi_{H_2}(2^{j_2}s - k_2)\, ds. \tag{3.2.32}$$

Notice that the latter integral is well-defined and finite since one knows from Proposition 3.2.1 that the real-valued functions Ψ_{H_1} and Ψ_{H_2} belong to the Schwartz class $S(\mathbb{R})$.

Remark 3.2.12. It results from the Plancherel formula and from elementary property of Fourier transform that

$$F^{k_1,k_2}_{j_1,j_2} = 2^{-j_1-j_2}\int_{\mathbb{R}} \exp\big(-i(2^{-j_1}k_1 - 2^{-j_2}k_2)\xi\big)\widehat{\Psi}_{H_1}(2^{-j_1}\xi)\overline{\widehat{\Psi}_{H_2}(2^{-j_2}\xi)}\, d\xi. \tag{3.2.33}$$

Observe that one can easily derive from (3.2.4) and (3.2.33) that

$$F^{k_1,k_2}_{j_1,j_2} = 0, \quad \text{for all } (j_1, j_2, k_1, k_2) \in \mathbb{Z}^4 \text{ such that } |j_1 - j_2| \ge 2. \tag{3.2.34}$$

In other words, a necessary condition for having $F_{j_1,j_2}^{k_1,k_2} \neq 0$ is that $j_2 \in \{j_1-1, j_1, j_1+1\}$. This leads us to introduce the following notations: for every $(j,k_1,k_2) \in \mathbb{Z}^3$, one sets

$$\widetilde{F}_j^{k_1,k_2} := F_{j,j}^{k_1,k_2}, \qquad F_{j,-}^{k_1,k_2} := F_{j,j-1}^{k_1,k_2} \quad \text{and} \quad F_{j,+}^{k_1,k_2} := F_{j,j+1}^{k_1,k_2}.$$

Thus, using (3.2.33) and the change of variables $\eta = 2^{-j}\xi$ and $\eta' = 2^{-j-1}\xi$, one gets that

$$\widetilde{F}_j^{k_1,k_2} := F_{j,j}^{k_1,k_2} = 2^{-j} \int_{\mathbb{R}} \exp\big(-i(k_1-k_2)\eta\big) \widehat{\Psi}_{H_1}(\eta) \overline{\widehat{\Psi}_{H_2}(\eta)}\, d\eta\,, \tag{3.2.35}$$

$$F_{j,-}^{k_1,k_2} := F_{j,j-1}^{k_1,k_2} = 2^{1-j} \int_{\mathbb{R}} \exp\big(-i(k_1-2k_2)\eta\big) \widehat{\Psi}_{H_1}(\eta) \overline{\widehat{\Psi}_{H_2}(2\eta)} d\eta \tag{3.2.36}$$

and

$$F_{j,+}^{k_1,k_2} := F_{j,j+1}^{k_1,k_2} = 2^{-j} \int_{\mathbb{R}} \exp\big(-i(2k_1-k_2)\eta'\big) \widehat{\Psi}_{H_1}(2\eta') \overline{\widehat{\Psi}_{H_2}(\eta')}\, d\eta'\,. \tag{3.2.37}$$

Remark 3.2.13. Let Λ_1, Λ_2 and Λ_3 be the three functions defined, for all $\eta \in \mathbb{R}$, as:

$$\Lambda_1(\eta) := \nu_0 \widehat{\Psi}_{H_1}(\eta) \overline{\widehat{\Psi}_{H_2}(\eta)}\,, \quad \Lambda_2(\eta) := \nu_0 \widehat{\Psi}_{H_1}(\eta) \overline{\widehat{\Psi}_{H_2}(2\eta)} \quad \text{and} \quad \Lambda_3(\eta) := \nu_0 \widehat{\Psi}_{H_1}(2\eta) \overline{\widehat{\Psi}_{H_2}(\eta)}, \tag{3.2.38}$$

where the constant $\nu_0 := (2\pi)^{1/2}$. These 3 functions as well as their Fourier transforms $\widehat{\Lambda}_1$, $\widehat{\Lambda}_2$ and $\widehat{\Lambda}_3$ belong to the Schwartz class $S(\mathbb{R})$ since Ψ_{H_1} and Ψ_{H_2} are in this class. Thus, for any arbitrary fixed positive real number L, there exists a positive finite constant c such that

$$\big|\widehat{\Lambda}_m(v)\big| \le c\big(3+|v|\big)^{-L}, \quad \text{for all } (m,v) \in \{1,2,3\} \times \mathbb{R}. \tag{3.2.39}$$

On the other hand, one knows from (3.2.35), (3.2.36), (3.2.37), (3.2.38) and Remark 3.1.3, that, for all $(j,k_1,k_2) \in \mathbb{Z}^3$, one has

$$\widetilde{F}_j^{k_1,k_2} = 2^{-j}\, \widehat{\Lambda}_1(k_1-k_2)\,, \qquad F_{j,-}^{k_1,k_2} = 2^{1-j}\, \widehat{\Lambda}_2(k_1-2k_2) \quad \text{and} \quad F_{j,+}^{k_1,k_2} = 2^{-j}\, \widehat{\Lambda}_3(2k_1-k_2). \tag{3.2.40}$$

Thus, one can derive from (3.2.39) and (3.2.40) that, for all $(j,k_1,k_2) \in \mathbb{Z}^3$, the following three inequalities, in which c denotes the same constant as in (3.2.39), hold:

$$\big|\widetilde{F}_j^{k_1,k_2}\big| \le c\, 2^{-j}\big(3+|k_1-k_2|\big)^{-L}, \tag{3.2.41}$$

$$\big|F_{j,-}^{k_1,k_2}\big| \le c\, 2^{1-j}\big(3+|k_1-2k_2|\big)^{-L} \quad \text{and} \quad \big|F_{j,+}^{k_1,k_2}\big| \le c\, 2^{-j}\big(3+|2k_1-k_2|\big)^{-L}. \tag{3.2.42}$$

Lemma 3.2.14. Let $T>2$ and $L>2$ be arbitrary and fixed real numbers. For every $J \in \mathbb{N}$, let $\mathcal{H}_{1,J}^4$ and $\mathcal{H}_{2,J}^4$ be the positive random variables defined as:

$$\mathcal{H}_{1,J}^4 := \sum_{(j_1,j_2) \in \aleph_{1,J}} 2^{j_1(1-H_1)+j_2(1-H_2)} \sup_{t\in[0,T]} \Bigg\{ \sum_{k_1 \in D_{j_1}^1(t)} \sum_{k_2 \in \mathbb{Z}} \big|A_{j_1,j_2}^{k_1,k_2}(t) - F_{j_1,j_2}^{k_1,k_2}\big| \big|\varepsilon_{j_1,j_2}^{k_1,k_2}\big| \Bigg\} \tag{3.2.43}$$

and

$$\mathcal{H}_{2,J}^4 := \sum_{(j_1,j_2) \in \aleph_{2,J}} 2^{j_1(1-H_1)+j_2(1-H_2)} \sup_{t\in[0,T]} \Bigg\{ \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in D_{j_2}^1(t)} \big|A_{j_1,j_2}^{k_1,k_2}(t) - F_{j_1,j_2}^{k_1,k_2}\big| \big|\varepsilon_{j_1,j_2}^{k_1,k_2}\big| \Bigg\}. \tag{3.2.44}$$

Then, there exists a positive almost surely finite random variable C such that, for all $J \in \mathbb{N}$ and $l \in \{1,2\}$, the following inequality holds on the event Ω^*:

$$\mathcal{H}^4_{l,J} \leq C J^{3/2}\, 2^{-J((L-2)(1-a)+H_1+H_2-1)}. \tag{3.2.45}$$

Proof of Lemma 3.2.14 One only shows that (3.2.45) is satisfied when $l=1$, the case where $l=2$ can be treated in the same way. Let $t \in [0,T]$ and $(j_1,j_2) \in \aleph_{1,J}$ be arbitrary and fixed. Using (3.2.6), (3.2.32), (3.2.11), (3.2.12), (3.2.2), (3.2.10), the inequality $|k_1| \leq 2^{j_1}T$ when $k_1 \in D^1_{j_1}(t)$, the inequality $2^{j_1}T \geq j_1$, Lemma 3.4.2, the inequality $j_2 \leq j_1$ and (3.4.1), one gets that

$$\begin{aligned}
&\sum_{k_1 \in D^1_{j_1}(t)} \sum_{k_2 \in \mathbb{Z}} \big|A^{k_1,k_2}_{j_1,j_2}(t) - F^{k_1,k_2}_{j_1,j_2}\big|\big|\varepsilon^{k_1,k_2}_{j_1,j_2}\big| \\
&\leq C_0 \sqrt{\log(3+2^{j_1+1}T)} \int_{\mathbb{R}\setminus[0,t]} \Bigg(\sum_{k_1 \in D^1_{j_1}(t)} \frac{1}{\big(3+|2^{j_1}s-k_1|\big)^L}\Bigg)\Bigg(\sum_{k_2\in\mathbb{Z}} \frac{\sqrt{\log(3+|j_2|+|k_2|)}}{\big(3+|2^{j_2}s-k_2|\big)^L}\Bigg)\, ds \\
&\leq C_1 \sqrt{\log(3+2^{j_1+1}T)} \int_{\mathbb{R}\setminus[0,t]} \Bigg(\sum_{k_1 \in D^1_{j_1}(t)} \frac{\sqrt{\log(3+|j_2|+|2^{j_2}s|)}}{\big(3+|2^{j_1}s-k_1|\big)^L}\Bigg)\, ds \\
&\leq C_1 \sqrt{\log(3+2^{j_1+1}T)} \int_{\mathbb{R}\setminus[0,t]} \Bigg(\sum_{k_1 \in D^1_{j_1}(t)} \frac{\sqrt{\log(3+|j_2|+|2^{j_1}s|)}}{\big(3+|2^{j_1}s-k_1|\big)^L}\Bigg)\, ds \\
&\leq C_1 \log(3+2^{j_1+1}T) \int_{\mathbb{R}\setminus[0,t]} \Bigg(\sum_{k_1 \in D^1_{j_1}(t)} \frac{\sqrt{\log(3+|j_2|+|2^{j_1}s-k_1|)}}{\big(3+|2^{j_1}s-k_1|\big)^L}\Bigg)\, ds \\
&\leq C_1 \log(3+2^{j_1+1}T)\big(U_{j_1,j_2}(t)+V_{j_1,j_2}\big),
\end{aligned} \tag{3.2.46}$$

where C_0 and C_1 are 2 positive almost surely finite random variables not depending on t, T, (j_1,j_2) and J, and where

$$U_{j_1,j_2}(t) := \int_t^{+\infty} \Bigg(\sum_{k_1 \leq 2^{j_1}t-2^{j_1(1-a)}} \frac{\sqrt{\log(3+|j_2|+2^{j_1}s-k_1)}}{\big(3+2^{j_1}s-k_1\big)^L}\Bigg)\, ds \tag{3.2.47}$$

and

$$V_{j_1,j_2} := \int_{-\infty}^{0} \Bigg(\sum_{k_1 \geq 2^{j_1(1-a)}} \frac{\sqrt{\log(3+|j_2|-2^{j_1}s+k_1)}}{\big(3-2^{j_1}s+k_1\big)^L}\Bigg)\, ds. \tag{3.2.48}$$

Next, the change of variable $y = 2^{j_1}(s-t)$ in (3.2.47), the fact that, for any fixed $j_2 \in \mathbb{Z}$, the function $y \mapsto (2+y)^{-L}\sqrt{\log(2+|j_2|+y)}$ is decreasing on $\mathbb{R}_+$, and integrations by parts entail that

$$
\begin{aligned}
U_{j_1,j_2}(t) &= 2^{-j_1}\int_0^{+\infty}\Bigg(\sum_{k_1\le 2^{j_1}t-2^{j_1(1-a)}}\frac{\sqrt{\log(3+|j_2|+y+2^{j_1}t-k_1)}}{\big(3+y+2^{j_1}t-k_1\big)^L}\Bigg)\,dy\\
&\le 2^{-j_1}\int_0^{+\infty}\Bigg(\sum_{m=0}^{+\infty}\frac{\sqrt{\log(3+|j_2|+y+2^{j_1(1-a)}+m)}}{\big(3+y+2^{j_1(1-a)}+m\big)^L}\Bigg)\,dy\\
&\le 2^{-j_1}\int_0^{+\infty}\Bigg(\int_0^{+\infty}\frac{\sqrt{\log(2+|j_2|+y+2^{j_1(1-a)}+z)}}{\big(2+y+2^{j_1(1-a)}+z\big)^L}\,dz\Bigg)\,dy\\
&\le 2^{1-j_1}\int_0^{+\infty}\frac{\sqrt{\log(2+|j_2|+y+2^{j_1(1-a)})}}{\big(2+y+2^{j_1(1-a)}\big)^{L-1}}\,dy\\
&\le 2^{2-j_1}(L-2)^{-1}\,\frac{\sqrt{\log(2+|j_2|+2^{j_1(1-a)})}}{\big(2+2^{j_1(1-a)}\big)^{L-2}}\,.
\end{aligned}
\tag{3.2.49}
$$

Similar arguments allow to derive from (3.2.48) that

$$
V_{j_1,j_2}\le 2^{2-j_1}(L-2)^{-1}\,\frac{\sqrt{\log(2+|j_2|+2^{j_1(1-a)})}}{\big(2+2^{j_1(1-a)}\big)^{L-2}}\,.
\tag{3.2.50}
$$

Next, it follows from (3.2.18), (3.2.46), (3.2.49), (3.2.50), the triangle inequality, (3.4.1) and (3.1.11) that

$$
\begin{aligned}
&\sum_{(j_1,j_2)\in\aleph_{1,J}} 2^{j_1(1-H_1)+j_2(1-H_2)}\sup_{t\in[0,T]}\Bigg\{\sum_{k_1\in D^1_{j_1}(t)}\sum_{k_2\in\mathbb{Z}}\big|A^{k_1,k_2}_{j_1,j_2}(t)-F^{k_1,k_2}_{j_1,j_2}\big|\big|\varepsilon^{k_1,k_2}_{j_1,j_2}\big|\Bigg\}\\
&\le C_2\sum_{j_1=J}^{+\infty}\sum_{j_2=-\infty}^{j_1}(j_1+1)\,2^{-j_1((L-2)(1-a)+H_1)}\,2^{j_2(1-H_2)}\sqrt{\log(2+|j_2|+2^{j_1(1-a)})}\\
&\le C_2\sum_{j_1=J}^{+\infty}\sum_{p=0}^{+\infty}(j_1+1)\,2^{-j_1((L-2)(1-a)+H_1)}\,2^{(j_1-p)(1-H_2)}\sqrt{\log(2+|j_1-p|+2^{j_1(1-a)})}\\
&\le C_2\sum_{j_1=J}^{+\infty}(j_1+1)\,2^{-j_1((L-2)(1-a)+H_1+H_2-1)}\sum_{p=0}^{+\infty}2^{-p(1-H_2)}\,\sqrt{\log(2+j_1+2^{j_1(1-a)}+p)}\\
&\le C_3\sum_{j_1=J}^{+\infty}j_1^{3/2}\,2^{-j_1((L-2)(1-a)+H_1+H_2-1)}\\
&\le C_4J^{3/2}\,2^{-J((L-2)(1-a)+H_1+H_2-1)},
\end{aligned}
$$

where C_2, C_3 and C_4 are 3 positive almost surely finite random variables not depending on t and J. Thus one obtains (3.2.45) when $l=1$. □

Definition 3.2.15. For all $J \in \mathbb{N}$, one denotes by $\mathcal{M}_{1,J}$ and $\mathcal{M}_{2,J}$ the positive random variables defined as:

$$\mathcal{M}_{1,J} := \sum_{(j_1,j_2)\in\aleph_{1,J}} 2^{j_1(1-H_1)+j_2(1-H_2)} \sup_{t\in[0,T]} \Big\{ \Big| \sum_{k_1\in D^1_{j_1}(t)} \sum_{|k_2|\le 2^{j_1+1}T} F^{k_1,k_2}_{j_1,j_2}\, \varepsilon^{k_1,k_2}_{j_1,j_2} \Big| \Big\} \tag{3.2.51}$$

and

$$\mathcal{M}_{2,J} := \sum_{(j_1,j_2)\in\aleph_{2,J}} 2^{j_1(1-H_1)+j_2(1-H_2)} \sup_{t\in[0,T]} \Big\{ \Big| \sum_{|k_1|\le 2^{j_2+1}T} \sum_{k_2\in D^1_{j_2}(t)} F^{k_1,k_2}_{j_1,j_2}\, \varepsilon^{k_1,k_2}_{j_1,j_2} \Big| \Big\}, \tag{3.2.52}$$

where the real number $T > 2$ is arbitrary and fixed. Recall that $F^{k_1,k_2}_{j_1,j_2}$ has been introduced in (3.2.32).

Lemma 3.2.16. There exists Ω^{**} an event of probability 1, and there is C^{**} a positive finite random variable, such that, for all $(\omega, J) \in \Omega^{**} \times \mathbb{N}$ and $l \in \{1, 2\}$, one has

$$\mathcal{M}_{l,J}(\omega) \le C^{**}(\omega) J\, 2^{-J(H_1+H_2-3/2)}. \tag{3.2.53}$$

Our next goal is to show that Lemma 3.2.16 holds; one focuses on the case $l = 1$, the other case $l = 2$ can be treated similarly.

Remark 3.2.17. Observe that setting $j = j_1$ in (3.2.51), and using Remark 3.2.12 and (3.2.18) one has that

$$\mathcal{M}_{1,J} = \sum_{j=J}^{+\infty} 2^{j(2-H_1-H_2)} \Big(\sup_{t\in[0,T]} \big|\widetilde{M}_j(t)\big| + 2^{H_2-1} \sup_{t\in[0,T]} \big|M_{j,-}(t)\big| \Big), \tag{3.2.54}$$

where, for all $j \in \mathbb{N}$ and $t \in [0, T]$,

$$\widetilde{M}_j(t) := \sum_{k_1\in D^1_j(t)} \sum_{|k_2|\le 2^{j+1}T} \widetilde{F}^{k_1,k_2}_j\, \varepsilon^{k_1,k_2}_{j,j} \tag{3.2.55}$$

and

$$M_{j,-}(t) := \sum_{k_1\in D^1_j(t)} \sum_{|k_2|\le 2^{j+1}T} F^{k_1,k_2}_{j,-}\, \varepsilon^{k_1,k_2}_{j,j-1}\,. \tag{3.2.56}$$

In fact, in the case where $l = 1$, Lemma 3.2.16 is a straightforward consequence of (3.2.54) and the following lemma.

Lemma 3.2.18. There exist $\widetilde{\Omega}^{**}$ and Ω^{**}_- two events of probability 1, and there are $\widetilde{C}^{**}$ and C^{**}_- two positive finite random variables, such that one has:

$$\sup_{t\in[0,T]} \big\{|\widetilde{M}_j(t,\omega)|\big\} \le \widetilde{C}^{**}(\omega)\, j\, 2^{-j/2}\,, \quad \text{for all } (\omega, j) \in \widetilde{\Omega}^{**} \times \mathbb{N} \tag{3.2.57}$$

and

$$\sup_{t\in[0,T]} \big\{|M_{j,-}(t,\omega)|\big\} \le C^{**}_-(\omega)\, j\, 2^{-j/2}\,, \quad \text{for all } (\omega, j) \in \Omega^{**}_- \times \mathbb{N}. \tag{3.2.58}$$

In order to show that Lemma 3.2.18 is satisfied, one needs some preliminary results.

Remark 3.2.19. Let $j \in \mathbb{Z}_+$ and $t \in \mathbb{R}_+$ be arbitrary and fixed. Using (3.2.11) and (3.2.12) it can easily be shown that

$$D_j^1(t) = \begin{cases} \emptyset\,, & \text{if } t \in [0, 2^{1-ja}), \\ D_j^1\big(m_{j,t}\, 2^{-j} + 2^{-ja}\big)\,, & \text{if } t \in [2^{1-ja}, +\infty), \end{cases} \tag{3.2.59}$$

where $m_{j,t}$ is the positive integer defined as $m_{j,t} := \lfloor 2^j t - 2^{j(1-a)} \rfloor$. Thus, denoting by $\mathcal{I}_j$ the finite set of positive integers defined as

$$\mathcal{I}_j := \mathbb{N} \cap \big(2^{j(1-a)} - 1, 2^j T - 2^{j(1-a)}\big], \tag{3.2.60}$$

in view of (3.2.59), one has, for all $j \in \mathbb{N}$, that

$$\sup_{t\in[0,T]} \big\{ |\widetilde{M}_j(t)| \big\} = \sup_{m\in\mathcal{I}_j} \big\{ \big|\widetilde{M}_j\big(m\, 2^{-j} + 2^{-ja}\big)\big| \big\} \tag{3.2.61}$$

and

$$\sup_{t\in[0,T]} \big\{ |M_{j,-}(t)| \big\} = \sup_{m\in\mathcal{I}_j} \big\{ \big|M_{j,-}\big(m\, 2^{-j} + 2^{-ja}\big)\big| \big\}. \tag{3.2.62}$$

Observe that the cardinality of the set $\mathcal{I}_j$, satisfies, for some finite constant $c \geq 1$ only depending on T,

$$\operatorname{card}(\mathcal{I}_j) \leq c\, 2^j\,, \quad \text{for all } j \in \mathbb{Z}_+. \tag{3.2.63}$$

The following lemma is a straightforward consequence of Theorem 6.7 in [36].

Lemma 3.2.20. There exists a positive finite universal constant $\breve{c}$ such that, for every random variable χ belonging to the second order Wiener chaos and for each real number $y \geq 2$, one has

$$\mathbb{P}\Big(|\chi| > y \|\chi\|_{L^2(\Omega)} \Big) \leq \exp\big(-\breve{c}\, y \big)\,, \tag{3.2.64}$$

where $\|\chi\|_{L^2(\Omega)} := \Big(\mathbb{E}\big[|\chi|^2\big] \Big)^{1/2}$.

Lemma 3.2.21. There exists a finite constant $c > 0$, depending on T, such that one has

$$\sup_{t\in[0,T]} \Big\{ \big\| \widetilde{M}_j(t) \big\|_{L^2(\Omega)} \Big\} \leq c\, 2^{-j/2}\,, \quad \text{for all } j \in \mathbb{N}, \tag{3.2.65}$$

and

$$\sup_{t\in[0,T]} \Big\{ \big\| M_{j,-}(t) \big\|_{L^2(\Omega)} \Big\} \leq c\, 2^{-j/2}\,, \quad \text{for all } j \in \mathbb{N}. \tag{3.2.66}$$

Proof of Lemma 3.2.21 We only give the proof of (3.2.65) since (3.2.66) can be shown in the same way except that the first inequality in (3.2.42) has to be used instead of (3.2.41). Let $t \in [0, T]$ and $j \in \mathbb{N}$ be arbitrary and fixed. Observe that, in view of the inclusion

$D_j^1(t) \subset \{k_2 \in \mathbb{Z} : |k_2| \le 2^{j+1}T\}$ the set $D_j^1(t) \times \{k_2 \in \mathbb{Z} : |k_2| \le 2^{j+1}T\}$ can be expressed as the disjoint union:

$$D_j^1(t) \times \{k_2 \in \mathbb{Z} : |k_2| \le 2^{j+1}T\} = \Delta_j^<(t) \cup \Delta_j^=(t) \cup \Delta_j^>(t) \cup \Delta_j^c(t)\,,$$

where, for any $\mathcal{R} \in \{<\,,\,=\,,\,>\}$,

$$\Delta_j^{\mathcal{R}}(t) := \{(k_1,k_2) \in D_j^1(t) \times D_j^1(t) : \ k_1 \mathcal{R} k_2\} \tag{3.2.67}$$

and

$$\Delta_j^c(t) := D_j^1(t) \times \{k_2 \in \mathbb{Z} : \ |k_2| \le 2^{j+1}T \text{ and } k_2 \notin D_j^1(t)\}. \tag{3.2.68}$$

Thus, using (3.2.55), one gets that

$$\widetilde{M}_j(t) = \widetilde{M}_j^<(t) + \widetilde{M}_j^=(t) + \widetilde{M}_j^>(t) + \widetilde{M}_j^c(t)\,, \tag{3.2.69}$$

where, for any $\mathcal{R} \in \{<\,,\,=\,,\,>\,,\,c\}$,

$$\widetilde{M}_j^{\mathcal{R}}(t) := \sum_{(k_1,k_2) \in \Delta_j^{\mathcal{R}}(t)} \widetilde{F}_j^{k_1,k_2}\, \varepsilon_{j,j}^{k_1,k_2}\,. \tag{3.2.70}$$

Next, it follows from (3.2.69) and the triangle inequality that

$$\|\widetilde{M}_j(t)\|_{L^2(\Omega)} \le \|\widetilde{M}_j^<(t)\|_{L^2(\Omega)} + \|\widetilde{M}_j^=(t)\|_{L^2(\Omega)} + \|\widetilde{M}_j^>(t)\|_{L^2(\Omega)} + \|\widetilde{M}_j^c(t)\|_{L^2(\Omega)}\,. \tag{3.2.71}$$

The advantage in using the sets $\Delta_j^{\mathcal{R}}(t)$, $\mathcal{R} \in \{<\,,\,=\,,\,>\,,\,c\}$, is that one can derive from (3.2.67), (3.2.68) and Proposition 3.2.3, that, for any $(k_1',k_2') \in \Delta_j^{\mathcal{R}}(t)$ and $(k_1'',k_2'') \in \Delta_j^{\mathcal{R}}(t)$, the two centered random variables $\varepsilon_{j,j}^{k_1',k_2'}$ and $\varepsilon_{j,j}^{k_1'',k_2''}$ are uncorrelated except when $(k_1',k_2') = (k_1'',k_2'')$. Then, it follows from (3.2.70), (3.2.7), (3.2.8), classical properties of Wiener integral, the fact that the fourth moment of a standard Gaussian random variable equals 3, and (3.2.41) that

$$\begin{aligned} \|\widetilde{M}_j^{\mathcal{R}}(t)\|_{L^2(\Omega)}^2 \;&:=\; \mathbb{E}\Big(\big(\widetilde{M}_j^{\mathcal{R}}(t)\big)^2\Big) \qquad\qquad (3.2.72)\\ &=\; \sum_{(k_1,k_2) \in \Delta_j^{\mathcal{R}}(t)} \big(\widetilde{F}_j^{k_1,k_2}\big)^2\, \mathbb{E}\Big(\big(\varepsilon_{j,j}^{k_1,k_2}\big)^2\Big) \le c_1 2^{-2j} \sum_{(k_1,k_2) \in \Delta_j^{\mathcal{R}}(t)} \big(3 + |k_1 - k_2|\big)^{-2L}\,, \end{aligned}$$

where the arbitrary real number $L > 1/2$ is fixed, and $c_1 > 0$ is a finite constant only depending on L. Then, denoting by c_2 the finite constant $c_2 := 2c_1 \sum_{n=0}^{+\infty} (3+n)^{-2L}$ and using (3.2.72) and the inclusion

$$\Delta_j^{\mathcal{R}}(t) \subset \mathbb{Z} \times \{k_2 \in \mathbb{Z} : |k_2| \le 2^{j+1}T\}\,,$$

one obtains that

$$\|\widetilde{M}_j^{\mathcal{R}}(t)\|_{L^2(\Omega)}^2 \le c_1 2^{-2j} \sum_{|k_2| \le 2^{j+1}T} \sum_{k_1 \in \mathbb{Z}} \big(3 + |k_1 - k_2|\big)^{-2L} \le 8Tc_2 2^{-j}. \tag{3.2.73}$$

Finally, one can derive from (3.2.73) and (3.2.71) that (3.2.65) holds. □
We are now in a position to prove Lemma 3.2.18.

Proof of Lemma 3.2.18 We only give the proof of (3.2.57) since (3.2.58) can be shown in the same way. Let $\kappa \geq 2$ be a constant which will be defined more precisely later. Using (3.2.61) and (3.2.60), for all $j \in \mathbb{N}$, one has

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in[0,T]} \big\{ |\widetilde{M}_j(t)| \big\} > \kappa\, j \sup_{t\in[0,T]} \Big\{ \big\| \widetilde{M}_j(t) \big\|_{L^2(\Omega)} \Big\} \Big) \\
&= \mathbb{P}\Big(\sup_{m\in\mathcal{I}_j} \Big\{ \big| \widetilde{M}_j\big(m\, 2^{-j} + 2^{-ja}\big) \big| \Big\} > \kappa\, j \sup_{t\in[0,T]} \Big\{ \big\| \widetilde{M}_j(t) \big\|_{L^2(\Omega)} \Big\} \Big) \\
&\leq \sum_{m\in\mathcal{I}_j} \mathbb{P}\Big(\big| \widetilde{M}_j\big(m\, 2^{-j} + 2^{-ja}\big) \big| > \kappa\, j \sup_{t\in[0,T]} \Big\{ \big\| \widetilde{M}_j(t) \big\|_{L^2(\Omega)} \Big\} \Big) \\
&\leq \sum_{m\in\mathcal{I}_j} \mathbb{P}\Big(\big| \widetilde{M}_j\big(m\, 2^{-j} + 2^{-ja}\big) \big| > \kappa\, j \, \big\| \widetilde{M}_j\big(m\, 2^{-j} + 2^{-ja}\big) \big\|_{L^2(\Omega)} \Big). \qquad (3.2.74)
\end{aligned}$$

Next, observe that, for any $t \in [0, T]$ (and in particular for $t = m\, 2^{-j} + 2^{-ja}$) the random variable $\widetilde{M}_j(t)$ belongs to the second order Wiener chaos, since it is (see (3.2.55)) a linear combination of random variables $\varepsilon_{j,j}^{k_1,k_2}$ which are in this chaos. Thus, one can make use of Lemma 3.2.20 to bound from above the probabilities in (3.2.74). In this way, for all $m \in \mathcal{I}_j$, one gets

$$\mathbb{P}\Big(\big| \widetilde{M}_j\big(m\, 2^{-j} + 2^{-ja}\big) \big| > \kappa\, j \big\| \widetilde{M}_j\big(m\, 2^{-j} + 2^{-ja}\big) \big\|_{L^2(\Omega)} \Big) \leq \exp\big(- \breve{c}\, \kappa\, j \big), \qquad (3.2.75)$$

where $\breve{c}$ is the same positive finite constant as in (3.2.64). Next combining (3.2.74) and (3.2.75) with (3.2.63), one obtains that

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in[0,T]} \big\{ |\widetilde{M}_j(t)| \big\} > \kappa\, j \sup_{t\in[0,T]} \Big\{ \big\| \widetilde{M}_j(t) \big\|_{L^2(\Omega)} \Big\} \Big) \\
&\leq c_1 2^j \exp\big(- \breve{c}\, \kappa\, j \big) = c_1 \exp\big(- (\breve{c}\, \kappa - \log 2) j \big), \qquad (3.2.76)
\end{aligned}$$

where c_1 denotes the positive finite constant c in (3.2.63). One can assume that the finite constant κ is chosen such that $\kappa > (\log 2)/\breve{c}$. Then, one can derive from (3.2.76) that

$$\sum_{j=1}^{+\infty} \mathbb{P}\Big(\sup_{t\in[0,T]} \big\{ |\widetilde{M}_j(t)| \big\} > \kappa\, j \sup_{t\in[0,T]} \Big\{ \big\| \widetilde{M}_j(t) \big\|_{L^2(\Omega)} \Big\} \Big) < +\infty.$$

Thus, the Borel-Cantelli lemma implies that there exist $\widetilde{\Omega}^{**}$ an event of probability 1 and $\widetilde{C}_2^{**}$ a positive finite random variable, such that, for all $\omega \in \widetilde{\Omega}^{**}$ and for every $j \in \mathbb{N}$, one has

$$\sup_{t\in[0,T]} \big\{ |\widetilde{M}_j(t,\omega)| \big\} \leq \widetilde{C}_2^{**}(\omega)\, j \sup_{t\in[0,T]} \Big\{ \big\| \widetilde{M}_j(t) \big\|_{L^2(\Omega)} \Big\}. \qquad (3.2.77)$$

Finally, (3.2.57) results from (3.2.65) and (3.2.77). $\square$

We are now in a position to complete the proof of Theorem 3.1.2.

End of the proof of Theorem 3.1.2 Recall that the compact interval I is assumed to be of the form $I = [0, T]$, where the real number $T > 2$ is arbitrary and fixed. Also recall that, for each $(j_1, j_2, k_1, k_2) \in \mathbb{Z}^4$, one denotes by $A_{j_1,j_2}^{k_1,k_2}$ the deterministic real-valued continuous function over $\mathbb{R}_+$ defined through (3.2.6), and that one denotes by Ω^* the event of probability 1 which was introduced in Lemma 3.2.4. Let us first show that, for all fixed $(j_1, j_2) \in \mathbb{Z}^2$ and for every $\omega \in \Omega^*$, the series of continuous functions $\sum_{(k_1,k_2)\in\mathbb{Z}^2} A_{j_1,j_2}^{k_1,k_2}(\bullet)\, \varepsilon_{j_1,j_2}^{k_1,k_2}(\omega)$ is normally convergent with respect to the uniform norm $\|\cdot\|_{I,\infty}$, that is one has

$$\sum_{(k_1,k_2)\in\mathbb{Z}^2} \big\|A_{j_1,j_2}^{k_1,k_2}\big\|_{I,\infty} \big|\varepsilon_{j_1,j_2}^{k_1,k_2}(\omega)\big| < +\infty\,. \tag{3.2.78}$$

Using (3.2.2), (3.2.6), (3.2.10), the definition of $\|\cdot\|_{I,\infty}$, and the triangle inequality, one gets, for some positive finite constant C_1 depending on T, $(j_1, j_2) \in \mathbb{Z}^2$ and ω, that

$$\begin{aligned}
&\sum_{(k_1,k_2)\in\mathbb{Z}^2} \big\|A_{j_1,j_2}^{k_1,k_2}\big\|_{I,\infty} \big|\varepsilon_{j_1,j_2}^{k_1,k_2}(\omega)\big| \\
&\le C_1 \sum_{(k_1,k_2)\in\mathbb{Z}^2} \int_0^T \frac{\sqrt{\log(3+|j_1|+|k_1|)\log(3+|j_2|+|k_2|)}}{\big(1+2^{j_1}T+|2^{j_1}s-k_1|\big)^2\big(1+2^{j_2}T+|2^{j_2}s-k_2|\big)^2}\, ds \\
&\le C_1 \sum_{(k_1,k_2)\in\mathbb{Z}^2} \int_0^T \frac{\sqrt{\log(3+|j_1|+|k_1|)\log(3+|j_2|+|k_2|)}}{\big(1+2^{j_1}T+|k_1|-|2^{j_1}s|\big)^2\big(1+2^{j_2}T+|k_2|-|2^{j_2}s|\big)^2}\, ds \\
&\le C_1 T \sum_{(k_1,k_2)\in\mathbb{Z}^2} \frac{\sqrt{\log(3+|j_1|+|k_1|)\log(3+|j_2|+|k_2|)}}{\big(1+|k_1|\big)^2\big(1+|k_2|\big)^2} < +\infty\,,
\end{aligned}$$

which shows that (3.2.78) holds. Next, for each fixed $(j_1, j_2) \in \mathbb{Z}^2$, one denotes by $\{X_{j_1,j_2}(t)\}_{t\in I}$ the stochastic process with continuous paths which vanishes outside of the event Ω^*, and which is defined on Ω^*, for all $(t, \omega) \in I \times \Omega^*$, as

$$X_{j_1,j_2}(t,\omega) = \sum_{(k_1,k_2)\in\mathbb{Z}^2} A_{j_1,j_2}^{k_1,k_2}(t)\, \varepsilon_{j_1,j_2}^{k_1,k_2}(\omega)\,. \tag{3.2.79}$$

Let us now show that in order to derive (3.1.20) it is enough to prove that there are $\check{\Omega}$ an event of probability 1 included in Ω^*, and $\check{C}$ a positive finite random variable such that one has on $\check{\Omega}$, for every $J \in \mathbb{N}$,

$$\sum_{(j_1,j_2)\in\mathbb{Z}^2,\, j_1\vee j_2\ge J} 2^{j_1(1-H_1)+j_2(1-H_2)}\, \|X_{j_1,j_2}\|_{I,\infty} \le \check{C}\, J\, 2^{-J(H_1+H_2-3/2)}. \tag{3.2.80}$$

Assuming that (3.2.80) is true, then it entails, for all fixed $J \in \mathbb{N}$ and for every $\omega \in \check{\Omega}$, that

$$\sum_{(j_1,j_2)\in\mathbb{Z}^2,\, j_1\vee j_2\ge J} 2^{j_1(1-H_1)+j_2(1-H_2)}\, \big\|X_{j_1,j_2}(\bullet,\omega)\big\|_{I,\infty} < +\infty,$$

which means that the series of continuous functions

$$X_J(\bullet,\omega) := \sum_{(j_1,j_2)\in\mathbb{Z}^2,\, j_1\vee j_2\ge J} 2^{j_1(1-H_1)+j_2(1-H_2)}\, X_{j_1,j_2}(\bullet,\omega) \tag{3.2.81}$$

is normally convergent with respect to the norm $\|\cdot\|_{I,\infty}$; thus, $X_J(\bullet,\omega)$ is a continuous function on I. In the sequel, one denotes by $\{X_J(t)\}_{t\in I}$ the stochastic process with continuous paths which vanishes outside of $\check{\Omega}$ and which is defined through (3.2.81) on $\check{\Omega}$. It can easily be seen that (3.2.80) and the triangle inequality imply, almost surely for all $J\in\mathbb{N}$, that

$$\|X_J\|_{I,\infty} \le \check{C}\, J\, 2^{-J(H_1+H_2-3/2)}. \tag{3.2.82}$$

On the other hand, one knows from (3.2.79), (3.2.5) and (3.1.16), that, for all fixed $J\in\mathbb{N}$ and $t\in I$, the random series

$$\sum_{(j_1,j_2)\in\mathbb{Z}^2,\, j_1\vee j_2\ge J} 2^{j_1(1-H_1)+j_2(1-H_2)}\, X_{j_1,j_2}(t)$$

converges to $R_{H_1,H_2}(t)-R_{H_1,H_2,J}(t)$ in $L^2(\Omega)$. Combining the later fact with (3.2.81), one obtains, for all $t\in I$ almost surely,

$$X_J(t) = R_{H_1,H_2}(t) - R_{H_1,H_2,J}(t)\,.$$

The latter equality and the fact that the stochastic processes $\{R_{H_1,H_2}(t)-R_{H_1,H_2,J}(t)\}_{t\in I}$ and $\{X_J(t)\}_{t\in I}$ have continuous paths imply that they are indistinguishable. Thus, (3.1.20) is nothing else than (3.2.82).

Finally, let us show that (3.2.80) holds. Using (3.2.79), the triangle inequality, (3.2.20), (3.2.21), (3.2.24), (3.2.25), (3.2.28), (3.2.29), (3.2.43), (3.2.44), (3.2.51) and (3.2.52), one gets, for all $J\in\mathbb{N}$, that

$$\sum_{(j_1,j_2)\in\mathbb{Z}^2,\, j_1\vee j_2\ge J} 2^{j_1(1-H_1)+j_2(1-H_2)}\, \|X_{j_1,j_2}\|_{I,\infty} \le \sum_{l=1}^{2}\Big(\mathcal{M}_{l,J} + \sum_{m=1}^{4}\mathcal{H}_{l,J}^{m}\Big).$$

Therefore, combining Lemmas 3.2.8, 3.2.9 and 3.2.14 (in which L is assumed to be large enough) with Lemmas 3.2.10 and 3.2.16, one obtains (3.2.80). □

3.3 Proof of Theorem 3.1.4

In order to prove Theorem 3.1.4 one needs to obtain some intermediary results.

Lemma 3.3.1. Let $T>2$ and $L\ge 3/2$ be arbitrary and fixed real numbers. For every $n\in\mathbb{N}$, let $\mathcal{L}_{1,n}^1$ and $\mathcal{L}_{2,n}^1$ be the positive random variables defined as:

$$\mathcal{L}_{1,n}^1 := \sum_{j_1\vee j_2<n} 2^{j_1(1-H_1)+j_2(1-H_2)} \sum_{k_1\in\mathbb{Z}}\ \sum_{|k_2|>2^{n+1}T}\ \sup_{t\in[0,T]}\Big\{\big|A_{j_1,j_2}^{k_1,k_2}(t)\big|\Big\}\big|\varepsilon_{j_1,j_2}^{k_1,k_2}\big| \tag{3.3.1}$$

and

$$\mathcal{L}_{2,n}^1 := \sum_{j_1\vee j_2<n} 2^{j_1(1-H_1)+j_2(1-H_2)} \sum_{|k_1|>2^{n+1}T}\ \sum_{k_2\in\mathbb{Z}}\ \sup_{t\in[0,T]}\Big\{\big|A_{j_1,j_2}^{k_1,k_2}(t)\big|\Big\}\big|\varepsilon_{j_1,j_2}^{k_1,k_2}\big|. \tag{3.3.2}$$

Then, there exists a positive almost surely finite random variable C such that, for all $n\in\mathbb{N}$ and $l\in\{1,2\}$, the following inequality holds on the event Ω^* of probability 1 which was introduced in Lemma 3.2.4:

$$\mathcal{L}_{l,n}^1 \le C\, 2^{-n(H_1+H_2+L-3)}\, n\, \log(3+n)\,. \tag{3.3.3}$$

Proof of Lemma 3.3.1 One only shows that (3.3.3) is satisfied when $l=1$, the case where $l=2$ can be treated in the same way. Let $n\in\mathbb{N}$ and $(j_1,j_2)\in\mathbb{Z}^2$ be arbitrary, fixed and such that

$$j_1\vee j_2<n. \tag{3.3.4}$$

Using (3.2.6), (3.2.2), (3.2.10), (3.4.1), the triangle inequality, (3.3.4), Lemma 3.4.2, the fact that $y\mapsto(2+y)^{-L}\sqrt{\log(2+y)}$ is a decreasing function on $\mathbb{R}_+$, and (3.4.2), one gets that

$$\begin{aligned}
&\sum_{k_1\in\mathbb{Z}}\sum_{|k_2|>2^{n+1}T}\sup_{t\in[0,T]}\Big\{\big|A^{k_1,k_2}_{j_1,j_2}(t)\big|\Big\}\big|\varepsilon^{k_1,k_2}_{j_1,j_2}\big|\\
&\le C_0\sum_{k_1\in\mathbb{Z}}\sum_{|k_2|>2^{n+1}T}\int_0^T\frac{\sqrt{\log(3+|j_1|+|k_1|)}}{\big(3+|2^{j_1}s-k_1|\big)^L}\times\frac{\sqrt{\log(3+|j_2|+|k_2|)}}{\big(3+|2^{j_2}s-k_2|\big)^L}\,ds\\
&\le C_0\int_0^T\left(\sum_{k_1\in\mathbb{Z}}\sum_{|k_2|>2^{n+1}T}\frac{\sqrt{\log(3+|j_1|+|k_1|)}}{\big(3+|2^{j_1}s-k_1|\big)^L}\times\frac{\sqrt{\log(3+|j_2|+|k_2|)}}{\big(3+|k_2|-2^{j_2}s\big)^L}\right)ds\\
&\le C_1T\,2^L\sqrt{\log(3+|j_1|+2^{j_1}T)\log(3+|j_2|)}\sum_{|k_2|>2^{n+1}T}\frac{\sqrt{\log(3+|k_2|)}}{\big(3+|k_2|\big)^L}\\
&\le C_1T\,2^{L+1}\sqrt{\log(3+|j_1|+2^{j_1}T)\log(3+|j_2|)}\int_{2^{n+1}T}^{+\infty}\frac{\sqrt{\log(2+y)}}{(2+y)^L}\,dy\\
&\le C_2\sqrt{\log(3+|j_1|)\log(3+|j_2|)}\,n\,2^{-n(L-1)},
\end{aligned}\tag{3.3.5}$$

where C_0, C_1 and C_2 are 3 positive almost surely finite random variables not depending on (j_1,j_2) and n. Next, it follows from (3.3.1), (3.3.5) and (3.4.1) that

$$\begin{aligned}
\mathcal{L}^1_{1,n} &\le C_2\,n\,2^{-n(L-1)}\sum_{j_1=-\infty}^{n-1}\sum_{j_2=-\infty}^{n-1}2^{j_1(1-H_1)+j_2(1-H_2)}\sqrt{\log(3+|j_1|)\log(3+|j_2|)}\\
&\le C_3\,2^{-n(H_1+H_2+L-3)}\,n\,\log(3+n)\,,
\end{aligned}$$

where C_3 is a positive almost surely finite random variable not depending on n. Thus, one obtains (3.3.3) when $l=1$. □

Lemma 3.3.2. Let $T>2$, $L>1$ and $g>0$ be arbitrary and fixed real numbers. For every $n\in\mathbb{N}$, let $\mathcal{L}^2_{1,n}$ and $\mathcal{L}^2_{2,n}$ be the positive random variables defined as:

$$\mathcal{L}^2_{1,n}:=\sum_{(j_1,j_2)\in\mathbb{N}^2}2^{-j_1(1-H_1)-j_2(1-H_2)}\sum_{k_1\in\mathbb{Z}}\sum_{|k_2|>2^{ng}}\sup_{t\in[0,T]}\Big\{\big|A^{k_1,k_2}_{-j_1,-j_2}(t)\big|\Big\}\big|\varepsilon^{k_1,k_2}_{-j_1,-j_2}\big| \tag{3.3.6}$$

and

$$\mathcal{L}^2_{2,n}:=\sum_{(j_1,j_2)\in\mathbb{N}^2}2^{-j_1(1-H_1)-j_2(1-H_2)}\sum_{|k_1|>2^{ng}}\sum_{k_2\in\mathbb{Z}}\sup_{t\in[0,T]}\Big\{\big|A^{k_1,k_2}_{-j_1,-j_2}(t)\big|\Big\}\big|\varepsilon^{k_1,k_2}_{-j_1,-j_2}\big|. \tag{3.3.7}$$

Then, there exists a positive almost surely finite random variable C such that, for all $n\in\mathbb{N}$ and $l\in\{1,2\}$, the following inequality holds on the event Ω^*:

$$\mathcal{L}^2_{l,n}\le C\,2^{-n(L-1)g}\sqrt{n}\,. \tag{3.3.8}$$

Proof of Lemma 3.3.2 One only shows that (3.3.8) is satisfied when $l = 1$, the case where $l = 2$ can be treated in the same way. Let $n \in \mathbb{N}$ and $(j_1, j_2) \in \mathbb{N}^2$ be arbitrary and fixed. Using (3.2.6), (3.2.2), (3.2.10), (3.4.1), the triangle inequality, and the fact that $y \mapsto (2 + y)^{-L}\sqrt{\log(2+y)}$ is a decreasing function on $\mathbb{R}_+$, one gets that

$$
\begin{aligned}
&\sum_{k_1\in\mathbb{Z}}\sum_{|k_2|>2^{ng}} \sup_{t\in[0,T]}\Big\{\big|A^{k_1,k_2}_{-j_1,-j_2}(t)\big|\Big\}\big|\varepsilon^{k_1,k_2}_{-j_1,-j_2}\big| \\
&\le C_0 \sum_{k_1\in\mathbb{Z}}\sum_{|k_2|>2^{ng}} \int_0^T \frac{\sqrt{\log(3+j_1+|k_1|)}}{\big(3+T+|2^{-j_1}s-k_1|\big)^L} \times \frac{\sqrt{\log(3+j_2+|k_2|)}}{\big(3+T+|2^{-j_2}s-k_2|\big)^L}\, ds \\
&\le C_0 \sqrt{\log(3+j_1)\log(3+j_2)} \int_0^T \Bigg(\sum_{k_1\in\mathbb{Z}}\sum_{|k_2|>2^{ng}} \frac{\sqrt{\log(3+|k_1|)}}{\big(3+|k_1|\big)^L} \times \frac{\sqrt{\log(3+|k_2|)}}{\big(3+|k_2|\big)^L}\Bigg) ds \\
&\le C_1 \sqrt{\log(3+j_1)\log(3+j_2)} \sum_{|k_2|>2^{ng}} \frac{\sqrt{\log(3+|k_2|)}}{\big(3+|k_2|\big)^L} \\
&\le 2C_1 \sqrt{\log(3+j_1)\log(3+j_2)} \int_{2^{ng}}^{+\infty} \frac{\sqrt{\log(2+y)}}{(2+y)^L}\, dy \\
&\le C_2 \sqrt{\log(3+j_1)\log(3+j_2)}\; 2^{-n(L-1)g}\sqrt{n}, \qquad (3.3.9)
\end{aligned}
$$

where C_0, C_1 and C_2 are 3 positive almost surely finite random variables not depending on (j_1, j_2) and n. Next, it follows from (3.3.6) and (3.3.9) that

$$
\begin{aligned}
\mathcal{L}^2_{1,n} &\le C_2 2^{-n(L-1)g}\sqrt{n} \sum_{(j_1,j_2)\in\mathbb{N}^2} 2^{-j_1(1-H_1)-j_2(1-H_2)}\sqrt{\log(3+j_1)\log(3+j_2)} \\
&= C_3 2^{-n(L-1)g}\sqrt{n}\,,
\end{aligned}
$$

where C_3 is a positive almost surely finite random variable not depending on n. Thus, one obtains (3.3.8) when $l = 1$. □

Lemma 3.3.3. Let $T > 2$ and $b > 0$ be arbitrary and fixed real numbers. For every $n \in \mathbb{N}$, let $\mathcal{L}^3_{1,n}$ and $\mathcal{L}^3_{2,n}$ be the positive random variables defined as:

$$
\mathcal{L}^3_{1,n} := \sum_{j_1=0}^{n-1}\sum_{j_2>2^{nb}} 2^{j_1(1-H_1)-j_2(1-H_2)} \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} \sup_{t\in[0,T]}\Big\{\big|A^{k_1,k_2}_{j_1,-j_2}(t)\big|\Big\}\big|\varepsilon^{k_1,k_2}_{j_1,-j_2}\big| \qquad (3.3.10)
$$

and

$$
\mathcal{L}^3_{2,n} := \sum_{j_1>2^{nb}}\sum_{j_2=0}^{n-1} 2^{-j_1(1-H_1)+j_2(1-H_2)} \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} \sup_{t\in[0,T]}\Big\{\big|A^{k_1,k_2}_{-j_1,j_2}(t)\big|\Big\}\big|\varepsilon^{k_1,k_2}_{-j_1,j_2}\big|. \qquad (3.3.11)
$$

Then, there exists a positive almost surely finite random variable C such that, for all $n \in \mathbb{N}$ and $l \in \{1, 2\}$, the following inequality holds on the event Ω^*:

$$
\mathcal{L}^3_{l,n} \le C\, n\, 2^{(1-H_1\wedge H_2)n-(1-H_1\vee H_2)2^{nb}}. \qquad (3.3.12)
$$

Proof of Lemma 3.3.3 One only shows that (3.3.12) is satisfied when $l=1$, the case where $l=2$ can be treated in the same way. Let $n \in \mathbb{N}$ and $(j_1,j_2) \in \mathbb{Z}_+^2$ be arbitrary, fixed and such that

$$0 \le j_1 < n \quad \text{and} \quad j_2 > 2^{nb}. \tag{3.3.13}$$

Using (3.2.6), (3.2.2), (3.2.10), (3.4.1), the triangle inequality, (3.3.13), Lemma 3.4.2 and (3.4.2), one gets that

$$\begin{aligned}
&\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} \sup_{t\in[0,T]}\Big\{\big|A^{k_1,k_2}_{j_1,-j_2}(t)\big|\Big\}\big|\varepsilon^{k_1,k_2}_{j_1,-j_2}\big| \\
&\le C_0 \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} \int_0^T \frac{\sqrt{\log(3+j_1+|k_1|)}}{\big(3+|2^{j_1}s-k_1|\big)^2} \times \frac{\sqrt{\log(3+j_2+|k_2|)}}{\big(3+T+|2^{-j_2}s-k_2|\big)^2}\, ds \\
&\le C_0\sqrt{\log(3+j_2)} \int_0^T \left(\sqrt{\log(3+j_1+2^{j_1}T)} \sum_{k_2\in\mathbb{Z}} \frac{\sqrt{\log(3+|k_2|)}}{\big(3+|k_2|\big)^2}\right) ds \\
&\le C_1\sqrt{n\log(3+j_2)}\,,
\end{aligned} \tag{3.3.14}$$

where C_0 and C_1 are 2 positive almost surely finite random variables not depending on (j_1,j_2) and n. Next, it follows from (3.3.10) and (3.3.14) that

$$\begin{aligned}
\mathcal{L}^3_{1,n} &\le C_1\sqrt{n}\sum_{j_1=0}^{n-1}\sum_{j_2>2^{nb}} 2^{j_1(1-H_1)-j_2(1-H_2)}\sqrt{\log(3+j_2)} \\
&\le C_2\, n\, 2^{(1-H_1)n-(1-H_2)2^{nb}} \le C_2\, n\, 2^{(1-H_1\wedge H_2)n-(1-H_1\vee H_2)2^{nb}}\,,
\end{aligned}$$

where C_2 is a positive almost surely finite random variable not depending on n. Thus, one obtains (3.3.12) when $l=1$. $\square$

Lemma 3.3.4. Let $T>2$ and $d>0$ be arbitrary and fixed real numbers. For every $n\in\mathbb{N}$, let $\mathcal{Q}_n$ be the positive random variable defined as:

$$\mathcal{Q}_n := \sum_{(j_1,j_2)\in\mathbb{N}^2,\, j_1\vee j_2>2^{nd}} 2^{-j_1(1-H_1)-j_2(1-H_2)} \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} \sup_{t\in[0,T]}\Big\{\big|A^{k_1,k_2}_{-j_1,-j_2}(t)\big|\Big\}\big|\varepsilon^{k_1,k_2}_{-j_1,-j_2}\big|. \tag{3.3.15}$$

Then, there exists a positive almost surely finite random variable C such that, for all $n\in\mathbb{N}$, the following inequality holds on the event Ω^*:

$$\mathcal{Q}_n \le C\, 2^{-(1-H_1\vee H_2)2^{nd}}\sqrt{n}\,. \tag{3.3.16}$$

Proof of Lemma 3.3.4 Let $(j_1,j_2)\in\mathbb{N}^2$ be arbitrary and fixed. Using (3.2.6), (3.2.2), (3.2.10), (3.4.1) and the triangle inequality, one gets that

$$
\begin{aligned}
&\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}\sup_{t\in[0,T]}\Big\{\big|A^{k_1,k_2}_{-j_1,-j_2}(t)\big|\Big\}\big|\varepsilon^{k_1,k_2}_{-j_1,-j_2}\big| \\
&\le C_0\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}\int_0^T \frac{\sqrt{\log(3+j_1+|k_1|)}}{\big(3+T+|2^{-j_1}s-k_1|\big)^2}\times\frac{\sqrt{\log(3+j_2+|k_2|)}}{\big(3+T+|2^{-j_2}s-k_2|\big)^2}\,ds \\
&\le C_0\sqrt{\log(3+j_1)\log(3+j_2)}\int_0^T\Bigg(\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}\frac{\sqrt{\log(3+|k_1|)}}{\big(3+|k_1|\big)^2}\times\frac{\sqrt{\log(3+|k_2|)}}{\big(3+|k_2|\big)^2}\Bigg)\,ds \\
&= C_1\sqrt{\log(3+j_1)\log(3+j_2)}\,, \qquad (3.3.17)
\end{aligned}
$$

where C_0 and C_1 are 2 positive almost surely finite random variables not depending on (j_1,j_2). Next, it follows from (3.3.15) and (3.3.17) that, for all $n\in\mathbb{N}$, one has

$$
\begin{aligned}
\mathcal{Q}_n &\le C_1\sum_{(j_1,j_2)\in\mathbb{N}^2,\, j_1\vee j_2>2^{nd}} 2^{-j_1(1-H_1)-j_2(1-H_2)}\sqrt{\log(3+j_1)\log(3+j_2)} \\
&\le C_2 2^{-(1-H_1\vee H_2)2^{nd}}\sqrt{n}\,,
\end{aligned}
$$

where C_2 is a positive almost surely finite random variable not depending on n. Thus, one obtains (3.3.16). □

We are now in a position to complete the proof of Theorem 3.1.4.

End of the proof of Theorem 3.1.4 Let Ω^* and Ω^{**} be the same events of probability 1 as in Lemmas 3.2.4 and 3.2.16. Let us first show that in order to derive (3.1.25) it is enough to prove that there exists a positive almost surely finite random variable C such that the inequality

$$
\big\|\widetilde{R}_{H_1,H_2,n+p}-\widetilde{R}_{H_1,H_2,n}\big\|_{I,\infty}\le Cn\,2^{-n(H_1+H_2-3/2)} \qquad (3.3.18)
$$

holds, for all $(n,p)\in\mathbb{N}^2$, on the event $\Omega^*\cap\Omega^{**}$ of probability 1. Assuming that (3.3.18) is true, then it turns out that, for every fixed $\omega\in\Omega^*\cap\Omega^{**}$, the sequence of continuous functions $\big(\widetilde{R}_{H_1,H_2,n}(\bullet,\omega)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space of the continuous functions over I equipped with the norm $\|\cdot\|_{I,\infty}$. Thus, for any fixed $\omega\in\Omega^*\cap\Omega^{**}$, it converges to a continuous function over I denoted by $\widetilde{R}_{H_1,H_2}(\bullet,\omega)$. On the other hand, when $\omega\notin\Omega^*\cap\Omega^{**}$, one assumes that $\widetilde{R}_{H_1,H_2}(\bullet,\omega)$ is identically zero. Next observe that one has

$$
\widetilde{R}_{H_1,H_2}(t)=R_{H_1,H_2}(t),\quad \text{for all } t\in I,\text{ almost surely,} \qquad (3.3.19)
$$

since one knows from (3.1.21) and (3.1.22) that, for any fixed $t\in I$, the sequence of random variables $\big(\widetilde{R}_{H_1,H_2,n}(t)\big)_{n\in\mathbb{N}}$ converges to $R_{H_1,H_2}(t)$ in $L^2(\Omega)$. The equality (3.3.19) implies that the stochastic processes $\{\widetilde{R}_{H_1,H_2}(t)\}_{t\in I}$ and $\{R_{H_1,H_2}(t)\}_{t\in I}$ are indistinguishable since they have continuous paths. Thus, letting p in (3.3.18) tend to $+\infty$, one obtains (3.1.25).

Finally, let us show that (3.3.18) holds. Using (3.1.22), (3.1.23), (3.1.24), the equality $I:=[0,T]$, (3.2.12), the inclusion

$$
D^1_j(t)\subset\Big\{k\in\mathbb{Z}:|k|\le 2^{n+p+1}T\Big\},\quad \text{for all } t\in I \text{ and } (j,n,p)\in\mathbb{N}^3 \text{ s.t. } n\le j<n+p,
$$

the triangle inequality, (3.2.20), (3.2.21), (3.2.24), (3.2.25), (3.2.28), (3.2.29), (3.2.43), (3.2.44), (3.2.51), (3.2.52), (3.3.1), (3.3.2), (3.3.6), (3.3.7), (3.3.10), (3.3.11), and (3.3.15), one gets, for all $(n,p) \in \mathbb{N}^2$, that

$$\left\| \widetilde{R}_{H_1,H_2,n+p} - \widetilde{R}_{H_1,H_2,n} \right\|_{I,\infty} \leq \mathcal{Q}_n + \sum_{l=1}^{2} \Big(\mathcal{M}_{l,n} + \sum_{m=1}^{4} \mathcal{H}_{l,n}^{m} + \sum_{m'=1}^{3} \mathcal{L}_{l,n}^{m'} \Big).$$

Therefore, combining Lemmas 3.2.8, 3.2.9, 3.2.14, 3.3.1, and 3.3.2 (in which L is assumed to be large enough) with Lemmas 3.2.10, 3.2.16, 3.3.3, and 3.3.4, one obtains (3.3.18). □

3.4 Appendix

Lemma 3.4.1. For every $(x,y) \in \mathbb{R}_+^2$, one has

$$\log(3+x+y) \leq \log(3+x)\log(3+y). \tag{3.4.1}$$

Moreover, for any fixed positive real number T there exists a constant $c > 0$ such that, for all $x \in \mathbb{R}_+$, the following inequality holds:

$$\log(3+x+2^x T) \leq c(1+x). \tag{3.4.2}$$

The proof of Lemma 3.4.1 is standard and easy this is why it has been omitted.

Lemma 3.4.2. For any fixed real number $L > 1$, there exists a constant $c > 0$ such that, for all $j \in \mathbb{Z}$ and for each $s \in \mathbb{R}$, one has:

$$\sum_{k\in\mathbb{Z}} \frac{\sqrt{\log(3+|j|+|k|)}}{(3+|2^j s-k|)^L} \leq c\sqrt{\log\left(3+|j|+2^j|s|\right)}. \tag{3.4.3}$$

Proof of Lemma 3.4.2 Setting $m = k - \lfloor 2^j s \rfloor$, where $\lfloor 2^j s \rfloor$ denotes the integer part of $2^j s$, and using the triangle inequality and (3.4.1), one obtains that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \frac{\sqrt{\log(3+|j|+|k|)}}{(3+|2^j s-k|)^L} &= \sum_{m\in\mathbb{Z}} \frac{\sqrt{\log\left(3+|j|+|m+\lfloor 2^j s\rfloor|\right)}}{(3+|2^j s-\lfloor 2^j s\rfloor-m|)^L} \\ &\leq \sqrt{\log\left(3+|j|+2^j|s|\right)} \sum_{m\in\mathbb{Z}} \frac{\sqrt{\log(4+|m|)}}{(3+|2^j s-\lfloor 2^j s\rfloor-m|)^L}.\end{aligned}$$

Then, noticing that

$$3+|2^j s-\lfloor 2^j s\rfloor-m| \geq 2+|m|,$$

one gets that

$$\sum_{k\in\mathbb{Z}} \frac{\sqrt{\log(3+|j|+|k|)}}{(3+|2^j s-k|)^L} \leq c\sqrt{\log(3+|j|+2^j|s|)},$$

where the constant

$$c := \sum_{m\in\mathbb{Z}} \frac{\sqrt{\log(4+|m|)}}{(2+|m|)^L} < +\infty.$$

□

Lemma 3.4.3. For each fixed real number $L > 1$, there exists a constant $c > 0$ such that, for every $t \in \mathbb{R}_+$, for all $s \in [0,t]$ and for any $j \in \mathbb{Z}_+$, one has

$$\sum_{k\in D_j^3(t)} \frac{\sqrt{\log(3+j+|k|)}}{(3+|2^j s-k|)^L} \le c(j+1)2^{-j(L-1)(1-a)}\sqrt{\log(3+t)}\,, \tag{3.4.4}$$

where $D_j^3(t)$ is defined through (3.2.14) and (3.2.11).

Proof of Lemma 3.4.3 In view of (3.2.14) and (3.2.11), one has $D_j^3(t) = D_j^{3,+}(t) \cup D_j^{3,-}(t)$, where $D_j^{3,+}(t)$ and $D_j^{3,-}(t)$ are the two disjoint sets defined as:

$$D_j^{3,+}(t) = \left\{k\in\mathbb{Z},\ k > 2^j t + 2^{j(1-a)}\right\}$$

and

$$D_j^{3,-}(t) = \left\{k\in\mathbb{Z},\ k < -2^{j(1-a)}\right\}.$$

Thus, one gets that

$$\sum_{k\in D_j^3(t)} \frac{\sqrt{\log(3+j+|k|)}}{(3+|2^j s-k|)^L} = \sum_{k>2^jt+2^{j(1-a)}} \frac{\sqrt{\log(3+j+k)}}{(3+|2^j s-k|)^L} + \sum_{k<-2^{j(1-a)}} \frac{\sqrt{\log(3+j+|k|)}}{(3+|2^j s-k|)^L}. \tag{3.4.5}$$

Let us now provide an appropriate upper bound for the first term in the right-hand side of (3.4.5). One denotes by $\lfloor\cdot\rfloor$ the integer part function. Using the change of variable $m = k-\lfloor 2^j t\rfloor$, the triangle inequality, (3.4.1), the inequality $\lfloor 2^j t\rfloor - 2^j s > -1$, the inequality $\log(3+m) \le 2+m$, and the fact that $x \mapsto (1+x)^{-L}\sqrt{\log(2+x)}$ is a decreasing function on $\mathbb{R}_+$, one obtains that

$$\begin{aligned}
\sum_{k>2^jt+2^{j(1-a)}} \frac{\sqrt{\log(3+j+k)}}{(3+|2^j s-k|)^L} &= \sum_{k>2^jt+2^{j(1-a)}} \frac{\sqrt{\log(3+j+k)}}{(3+k-2^j s)^L} \\
&= \sum_{m>2^jt-\lfloor 2^jt\rfloor+2^{j(1-a)}} \frac{\sqrt{\log(3+j+\lfloor 2^jt\rfloor+|m|)}}{(3+\lfloor 2^jt\rfloor-2^js+m)^L} \\
&\le \sqrt{\log(3+j+2^jt)} \sum_{m>2^{j(1-a)}} \frac{\sqrt{\log(3+m)}}{(2+m)^L} \\
&\le \sqrt{\log(3+j+2^jt)} \int_{2^{j(1-a)}}^{+\infty} \frac{\sqrt{\log(2+x)}}{(1+x)^L}\,dx \\
&\le c_1\sqrt{(j+1)\log(3+t)}\; 2^{-j(L-1)(1-a)}\sqrt{\log\left(2+2^{j(1-a)}\right)} \\
&\le c_2(j+1)2^{-j(L-1)(1-a)}\sqrt{\log(3+t)}\,,
\end{aligned} \tag{3.4.6}$$

where c_1 and c_2 are two positive finite constants not depending on j, t, s and a. Similarly to (3.4.6), it can be shown that

$$\sum_{k<-2^{j(1-a)}} \frac{\sqrt{\log(3+j+|k|)}}{(3+|2^j s-k|)^L} \leq c_3 2^{-j(L-1)(1-a)} \sqrt{(j+1)\log(3+j)}\,, \tag{3.4.7}$$

where c_3 is a positive finite constant not depending on j, t, s and a. Finally, putting together (3.4.5), (3.4.6), (3.4.7) and (3.4.2), it follows that (3.4.4) holds. □

Chapter 4

Wavelet analysis of a multifractional process in an arbitrary Wiener chaos

The well-known multifractional Brownian motion (mBm) is the paradigmatic example of a continuous Gaussian process with non-stationary increments whose local regularity changes from point to point. In this chapter, using a wavelet approach, we construct a natural extension of mBm which belongs to a homogeneous Wiener chaos of an arbitrary order. Then, we study its global and local behavior.

4.1 Introduction

Fractional Brownian motion (fBm) of an arbitrary Hurst parameter $H \in (0,1)$, denoted by $\{B_H(t) : t \in \mathbb{R}\}$, is defined, up to a multiplicative constant, as the unique (in distribution) Gaussian process with stationary increments which is globally self-similar of order H. Recall that a stochastic process $\{F(t) : t \in \mathbb{R}\}$ is said to be with stationary increments if, for any fixed point $t_0 \in \mathbb{R}$, one has:

$$\big\{F(t_0+u) - F(t_0) : u \in \mathbb{R}\big\} \overset{law}{=} \big\{F(u) - F(0) : u \in \mathbb{R}\big\}\,; \tag{4.1.1}$$

and it is said to be globally self-similar of order H if, for each fixed positive real number ν, one has:

$$\big\{\nu^{-H} F(\nu t) : t \in \mathbb{R}\big\} \overset{law}{=} \big\{F(t) : t \in \mathbb{R}\big\}\,. \tag{4.1.2}$$

The representation of fBm as a well-balanced moving average is given, for every $t \in \mathbb{R}$, by the Wiener integral over $\mathbb{R}$:

$$B_H(t) = \int_{\mathbb{R}} \Big[|t-s|^{H-1/2} - |s|^{H-1/2} \Big] dB(s), \tag{4.1.3}$$

with the convention that $|t-s|^0 - |s|^0 = \log|t-s| - \log|s|$. FBm was first introduced by Kolmogorov in 1940 as a way for generating Gaussian spirals in Hilbert spaces [39]. Later, in 1968, the well-known article [46] by Mandelbrot and Van Ness emphasised its importance as a model in several areas of application: hydrology, geology, finance, and so on. Since then many applied and theoretical aspects of this stochastic process have been extensively explored in the literature and, among many other things, its sample behavior has been well understood. Despite its importance in modeling, fBm does not always succeed in giving a sufficiently reliable description of real-life signals. Indeed, fBm suffers from two main limitations:

(a) its Gaussian character,

(b) local roughness of its sample paths remains everywhere the same; more precisely, their local and pointwise Hölder exponents are everywhere equal to the Hurst parameter H.

In order to overcome the limitation (b) of fBm, the so-called *multifractional Brownian motion* (mBm) was introduced, about twenty years ago, independently by Benassi, Jaffard and Roux [18] and by Lévy Vehel and Peltier [55]. The latter continuous Gaussian process with non-stationary increments can be obtained by substituting to the constant Hurst parameter H in (4.1.3) a deterministic continuous function $H(\cdot)$ depending on t and with values in the open interval $(0,1)$. Nowadays mBm has become a quite useful model in the fields of financial modeling and signal processing (see for instance [19, 21, 20]).

In order to overcome the limitations (a) and (b) together, extensions of mBm whose Hurst parameter is a stochastic process or more generally a sequence of stochastic processes were introduced in [9, 7]. Other extensions of mBm to frames of heavy-tailed stable distributions were proposed in [64, 63, 26]. More recently, [62] constructed a multifractional generalized Rosenblatt process belonging to the second order homogeneous Wiener chaos. In this chapter, we construct a multifractional process, denoted by $\{Z(t) : t \in \mathbb{R}\}$, which belongs to a homogeneous Wiener chaos of an arbitrary integer order $d \geq 2$. The latter multifractional process is not a generalization of the Rosenblatt process but of a process $\{Y_H(t) : t \in \mathbb{R}\}$ consisting in a very natural chaotic extension of the fBm in (4.1.3). Namely, it is defined, for all $t \in \mathbb{R}$, through the multiple Wiener integral on $\mathbb{R}^d$:

$$Y_H(t) = \int_{\mathbb{R}^d} \left[\|\mathbf{t}^* - \mathbf{x}\|_2^{H-\frac{d}{2}} - \|\mathbf{x}\|_2^{H-\frac{d}{2}} \right] dB_{x_1} ... dB_{x_d} \,, \tag{4.1.4}$$

where $\mathbf{t}^* = (t, ..., t) \in \mathbb{R}^d$ and $\|\cdot\|_2$ denotes the Euclidian norm over $\mathbb{R}^d$. A class of chaotic self-similar processes with stationary increments, which implicitly includes $\{Y_H(t) : t \in \mathbb{R}\}$, had been first introduced and investigated in [50]. Long time later, $\{Y_H(t) : t \in \mathbb{R}\}$ was explicitly introduced and studied in its own right in [3] through wavelet methods inspired by the ones in [9, 7].

Recall that a centred non-Gaussian square integrable real-valued random variable, on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, belongs to the homogeneous Wiener chaos of an arbitrary integer order $d \geq 2$ when it can be represented by a multiple Wiener integral over $\mathbb{R}^d$. We always denote by $I_d(\cdot)$ this stochastic integral, and use the classical convention that, for every $f \in L^2(\mathbb{R}^d)$, one has $I_d(f) = I_d(\tilde{f})$; the function $\tilde{f}$ being the symmetrization of f, defined, for all $(t_1, ..., t_d) \in \mathbb{R}^d$, as $\tilde{f}(t_1, ..., t_d) = \frac{1}{d!} \sum_{\sigma \in \mathcal{S}_d} f(t_{\sigma(1)}, ..., f(t_{\sigma(d)}))$, where $\mathcal{S}_d$ refers to the set of all permutations of $\{1, ..., d\}$ (observe that $d!$ is the cardinality of $\mathcal{S}_d$). A very important property of multiple Wiener integrals, which somehow can be viewed as an *isometry property*, is that, for all function $f \in L^2(\mathbb{R}^d)$, one has

$$\mathbb{E}\left(|I_d(f)|^2\right) = d!\, \|\tilde{f}\|^2_{L^2(\mathbb{R}^d)} \leq d!\, \|f\|^2_{L^2(\mathbb{R}^d)} \,. \tag{4.1.5}$$

Before ending these very short recalls on multiple Wiener integrals, it is worth mentioning that two well-known books on them and related topics are [36, 52].

Roughly speaking, we define $Z = \{Z(t) : t \in \mathbb{R}\}$, the multifractional generalization of $Y_H = \{Y_H(t) : t \in \mathbb{R}\}$, as $Z(t) = Y_{H(t)}(t)$, for all $t \in \mathbb{R}$, where $H(\cdot)$ is an arbitrary deterministic continuous function over $\mathbb{R}$ with values in the open interval $(0,1)$. More precisely, let us consider the chaotic stochastic field $X = \{X(u,v) : (u,v) \in \mathbb{R} \times (0,1)\}$, such that, for every $(u,v) \in \mathbb{R} \times (0,1)$, one has

$$X(u,v) = \int_{\mathbb{R}^d} \left[\|\mathbf{u}^* - \mathbf{x}\|_2^{v-\frac{d}{2}} - \|\mathbf{x}\|_2^{v-\frac{d}{2}} \right] dB_{x_1} \dots dB_{x_d}. \tag{4.1.6}$$

We mention in passing that, for each fixed $v \in (0,1)$, the stochastic processes $X(\cdot, v) = \{X(u,v) : u \in \mathbb{R}\}$ and $Y_v = \{Y_v(u) : u \in \mathbb{R}\}$ have the same law. The multifractional process $Z = \{Z(t) : t \in \mathbb{R}\}$ is defined, for all $t \in \mathbb{R}$, as

$$Z(t) = X(t, H(t)). \tag{4.1.7}$$

By expanding, for each fixed $(t,H) \in \mathbb{R} \times (0,1)$, the kernel function $\mathbf{x} \mapsto \|\mathbf{t}^* - \mathbf{x}\|_2^{H-\frac{d}{2}} - \|\mathbf{x}\|_2^{H-\frac{d}{2}}$ in (4.1.4) into a Meyer wavelet basis of $L^2(\mathbb{R}^d)$ (see e.g. [42, 48, 24]), a random series representation for the chaotic fractional process $\{Y_H(t) : t \in \mathbb{R}\}$ has been constructed in [3], which also has shown that this series is almost surely absolutely convergent, for each fixed $(t,H) \in \mathbb{R} \times (0,1)$. The main goal of Section 4.2 is to transpose these two results into the setting of the chaotic stochastic field $\{X(u,v) : (u,v) \in \mathbb{R} \times (0,1)\}$, and more importantly to show that the random series representation of this field is almost surely uniformly convergent in (u,v) on each compact subset of $\mathbb{R} \times (0,1)$. Then, thanks to this nice representation, global and local behavior of $\{X(u,v) : (u,v) \in \mathbb{R} \times (0,1)\}$ and $\{Z(t) : t \in \mathbb{R}\}$ are studied in Sections 4.3 and 4.4, by using a wavelet methodology which is, to a certain extent, inspired by the one introduced in [9, 7].

4.2 Uniformly convergent random series representation

First, we need to introduce some additional notations. We denote by $S(\mathbb{R}^d)$ the *Schwartz class*, that is the space of the infinitely differentiable complex-valued functions over $\mathbb{R}^d$ which, as well as all their partial derivatives of any order, vanish at infinity faster than any power function.

Let $E = \{0,1\}^d \backslash \{(0,\dots,0)\}$. A Meyer wavelet basis of $L^2(\mathbb{R}^d)$ is an orthonormal (or Hilbertian) basis of $L^2(\mathbb{R}^d)$ of the form:

$$\left\{ 2^{\frac{jd}{2}} \psi^{(\epsilon)}(2^j \mathbf{x} - \mathbf{k}) : j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, \epsilon \in E \right\}; \tag{4.2.1}$$

for the sake of convenience, one sets:

$$\psi_{j,\mathbf{k}}^{(\epsilon)}(\mathbf{x}) = 2^{\frac{jd}{2}} \psi^{(\epsilon)}(2^j \mathbf{x} - \mathbf{k}). \tag{4.2.2}$$

The $2^d - 1$ real-valued functions $\psi^{(\epsilon)}$, $\epsilon \in E$, which generate the basis are called *the d-variate Meyer mother wavelets*. They can be expressed as tensor products of ψ^0 and ψ^1 which respectively denote a 1-variate Meyer father and mother wavelet. More precisely, for each

$\epsilon = (\epsilon_1, \ldots, \epsilon_d) \in E$ and $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, one has that:

$$\psi^{(\epsilon)}(\mathbf{x}) = \prod_{l=1}^{d} \psi^{\epsilon_l}(x_l)\,. \tag{4.2.3}$$

Let us emphasize that the 1-variate Meyer father and mother wavelet belong to $S(\mathbb{R})$. Morever, their Fourier transforms $\mathcal{F}(\psi^0)$ and $\mathcal{F}(\psi^1)$ are infinitely differentiable compactly supported functions satisfying: supp $\mathcal{F}(\psi^0) \subset [-\frac{4\pi}{3}, \frac{4\pi}{3}]$, supp $\mathcal{F}(\psi^1) \subset \big\{\xi \in \mathbb{R} : \frac{2\pi}{3} \leq |\xi| \leq \frac{8\pi}{3}\big\}$, and $\mathcal{F}(\psi^0)(\xi) = 1$, for all $\xi \in [-\frac{2\pi}{3}, \frac{2\pi}{3}]$. Thus, in view of (4.2.3), the d-variate Meyer mother wavelets $\psi^{(\epsilon)}$, $\epsilon \in E$, belong to $S(\mathbb{R}^d)$ and have infinitely differentiable compactly supported Fourier transforms which vanish in a neighborhood of 0. Using these nice properties of the d-variate Meyer mother wavelets, for each $\epsilon \in E$, it can be shown in $\mathbb{R}^d \times [0, 1]$, as

$$\Psi^\epsilon(\mathbf{u}, v) = \int_{\mathbb{R}^d} \|\mathbf{u} - \mathbf{s}\|_2^{v-d/2}\, \psi^{(\epsilon)}(\mathbf{s}) d\mathbf{s}\,,$$

is infinitely differentiable on $\mathbb{R}^d \times (0, 1)$ and satisfies, as well as all its partial derivatives of any order, the following very useful localization property:

$$\forall\,(n, \mathbf{p}, q) \in \mathbb{Z}_+ \times \mathbb{Z}_+^d \times \mathbb{Z}_+\,, \quad \sup\left\{ \big(\alpha + \|\mathbf{u}\|_2\big)^n \left|(\partial_{\mathbf{u}}^{\mathbf{p}} \partial_v^q \Psi^\epsilon)(\mathbf{u}, v)\right| : (\mathbf{u}, v) \in \mathbb{R}^d \times (0, 1)\right\} < +\infty\,, \tag{4.2.4}$$

where α is an arbitrary positive fixed real number.

Before giving the random series representation of the field X derived from the Meyer wavelet basis (4.2.1), let us state the following important lemma borrowed from [3].

Lemma 4.2.1. For each $(j, \mathbf{k}, \epsilon) \in \mathbb{Z} \times \mathbb{Z}^d \times E$, let $I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})$ be the multiple Wiener integral over $\mathbb{R}^d$ of the wavelet function defined in (4.2.2). That is one has

$$I_d(\psi_{j,\mathbf{k}}^{(\epsilon)}) = \int_{\mathbb{R}^d} \psi_{j,\mathbf{k}}^{(\epsilon)}(\mathbf{x}) dB_{x_1} \ldots dB_{x_d}. \tag{4.2.5}$$

Then, there exists an event Ω^* of probability 1 and a finite positive random variable C_d such that, for all $\omega \in \Omega^*$ and for each $(j, \mathbf{k}, \epsilon) \in \mathbb{Z} \times \mathbb{Z}^d \times E$, one has

$$|I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})(\omega)| \leq C_d(\omega) \big(\log(e + |j| + \|\mathbf{k}\|_1)\big)^{\frac{d}{2}}, \tag{4.2.6}$$

where $\|\cdot\|_1$ denotes the 1-norm over $\mathbb{R}^d$; that is $\|\mathbf{k}\|_1 = \sum_{l=1}^d |k_l|$, the k_l's being the coordinates of $\mathbf{k}$.

The following proposition provides the random series representation of the field X derived from the Meyer wavelet basis (4.2.1). The proposition has been obtained in [3] with Y_H in place of X (see (4.1.4) and (4.1.6)).

Proposition 4.2.2. For each fixed $(u, v, \omega) \in \mathbb{R} \times (0, 1) \times \Omega^*$, one has

$$\sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\epsilon \in E} 2^{-jv} \left|(\Psi^\epsilon(2^j \mathbf{u}^* - \mathbf{k}, v) - \Psi^\epsilon(-\mathbf{k}, v)) I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})(\omega)\right| < +\infty\,.$$

This means that the series of real numbers

$$\sum_{j\in\mathbb{Z}}\sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{\epsilon\in E}2^{-jv}(\Psi^\epsilon(2^j\mathbf{u}^*-\mathbf{k},v)-\Psi^\epsilon(-\mathbf{k},v))I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})(\omega)$$

is absolutely convergent, and consequently that it converges to a finite limit not depending on the way the terms of the series are ordered. Moreover, for all fixed $(u,v)\in\mathbb{R}\times(0,1)$, one has, almost surely,

$$X(u,v)=\sum_{j\in\mathbb{Z}}\sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{\epsilon\in E}2^{-jv}(\Psi^\epsilon(2^j\mathbf{u}^*-\mathbf{k},v)-\Psi^\epsilon(-\mathbf{k},v))I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})\,. \tag{4.2.7}$$

Remark 4.2.3. From now on and till the end of this chapter, the chaotic stochastic field $X=\{X(u,v):(u,v)\in\mathbb{R}\times(0,1)\}$ will be systematically identified with its modification

$$\Big\{\sum_{j\in\mathbb{Z}}\sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{\epsilon\in E}2^{-jv}(\Psi^\epsilon(2^j\mathbf{u}^*-\mathbf{k},v)-\Psi^\epsilon(-\mathbf{k},v))I_d(\psi_{j,\mathbf{k}}^{(\epsilon)}):(u,v)\in\mathbb{R}\times(0,1)\Big\}\,,$$

which has just been introduced in Proposition 4.2.2. Also, we will always assume that X vanishes outside of Ω^*, the event of probability 1 introduced in Lemma 4.2.1. The low and high frequency parts of X are the two chaotic stochastic fields, denoted respectively by $X^{lf}=\{X^{lf}(u,v):(u,v)\in\mathbb{R}\times(0,1)\}$ and $X^{hf}=\{X^{hf}(u,v):(u,v)\in\mathbb{R}\times(0,1)\}$, which vanish outside of Ω^*, and are defined, for every $(u,v,\omega)\in\mathbb{R}\times(0,1)\times\Omega^*$, as:

$$X^{lf}(u,v,\omega)=\sum_{j=-\infty}^{-1}\sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{\epsilon\in E}2^{-jv}(\Psi^\epsilon(2^j\mathbf{u}^*-\mathbf{k},v)-\Psi^\epsilon(-\mathbf{k},v))I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})(\omega) \tag{4.2.8}$$

and

$$X^{hf}(u,v,\omega)=\sum_{j=0}^{+\infty}\sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{\epsilon\in E}2^{-jv}(\Psi^\epsilon(2^j\mathbf{u}^*-\mathbf{k},v)-\Psi^\epsilon(-\mathbf{k},v))I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})(\omega). \tag{4.2.9}$$

One clearly has, for all $(u,v,\omega)\in\mathbb{R}\times(0,1)\times\Omega$, that

$$X(u,v,\omega)=X^{lf}(u,v,\omega)+X^{hf}(u,v,\omega). \tag{4.2.10}$$

Recall that Ω is the underlying probability space.

Let us now state the main result of the present section.

Theorem 4.2.4. *The random series in the right-hand side of (4.2.7) is, on the event Ω^* of probability 1 , uniformly convergent in (u,v), on each compact subset of $\mathbb{R}\times(0,1)$.*

The following lemma will play a major role in the proof of Theorem 4.2.4 and in other important proofs in this chapter.

Lemma 4.2.5. For all fixed $(\mathbf{p},q)\in\mathbb{Z}_+^d\times\mathbb{Z}_+$, there exists a positive finite random variable $C_{\mathbf{p},q}$ such that, for all $(j,\mathbf{u},\omega)\in\mathbb{Z}\times\mathbb{R}^d\times\Omega^*$, one has

$$\sum_{\epsilon\in E}\sum_{\mathbf{k}\in\mathbb{Z}^d}|I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})(\omega)|\sup_{v\in(0,1)}\left|(\partial_{\mathbf{u}}^{\mathbf{p}}\partial_v^q\Psi^\epsilon)(\mathbf{u}-\mathbf{k},v)\right|\le C_{\mathbf{p},q}(\omega)(\log(e+|j|+\|\mathbf{u}\|_1))^{\frac{d}{2}}\,. \tag{4.2.11}$$

As a straightforward consequence, for all $(j,\mathbf{u},v,\omega)\in\mathbb{Z}\times\mathbb{R}^d\times(0,1)\times\Omega^*$, the series

$$\Phi_j(\mathbf{u},v,\omega)=\sum_{\epsilon\in E}\sum_{\mathbf{k}\in\mathbb{Z}^d}I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})(\omega)\Psi^\epsilon(\mathbf{u}-\mathbf{k},v) \tag{4.2.12}$$

is absolutely convergent, and the real-valued function $\Phi_j(\cdot,\cdot,\omega):(\mathbf{u},v)\mapsto\Phi_j(\mathbf{u},v,\omega)$ is well-defined and infinitely differentiable on $\mathbb{R}^d\times(0,1)$. Moreover, for each $(j,\mathbf{u},v,\omega)\in\mathbb{Z}\times\mathbb{R}^d\times(0,1)\times\Omega^*$, one has

$$(\partial_{\mathbf{u}}^{\mathbf{p}}\partial_v^q\Phi_j)(\mathbf{u},v,\omega)=\sum_{\epsilon\in E}\sum_{\mathbf{k}\in\mathbb{Z}^d}I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})(\omega)(\partial_{\mathbf{u}}^{\mathbf{p}}\partial_v^q\Psi^\epsilon)(\mathbf{u}-\mathbf{k},v),$$

and the following inequality holds:

$$\sup_{v\in(0,1)}\left|(\partial_{\mathbf{u}}^{\mathbf{p}}\partial_v^q\Phi_j)(\mathbf{u},v,\omega)\right|\le C_{\mathbf{p},q}(\omega)(\log(e+|j|+\|\mathbf{u}\|_1))^{\frac{d}{2}}. \tag{4.2.13}$$

Proof of Lemma 4.2.5. It easily follows from (4.2.4) and from the finiteness of the set E that, for all $(\mathbf{p},q)$ fixed in $\mathbb{Z}_+^d\times\mathbb{Z}_+$, there is a positive finite deterministic constant $c_{\mathbf{p},q}$ for which the following inequality holds for every $(\mathbf{u},\mathbf{k},\epsilon)\in\mathbb{R}^d\times\mathbb{Z}^d\times E$:

$$\sup_{v\in(0,1)}\left|(\partial_{\mathbf{u}}^{p}\partial_v^q\Psi^\epsilon)(\mathbf{u}-\mathbf{k},v)\right|\le\frac{c_{\mathbf{p},q}}{(\sqrt{d}+1+\|\mathbf{u}-\mathbf{k}\|_2)^{2d}}\,.$$

In the sequel, one sets $\lfloor\mathbf{u}\rfloor=(\lfloor u_1\rfloor,\dots,\lfloor u_d\rfloor)$, where $\lfloor u_l\rfloor$ denotes the integer part of the l-th coordinate u_l of the vector $\mathbf{u}$. Then, using Lemma 4.2.1, the change of variable $\mathbf{m}=\mathbf{k}+\lfloor\mathbf{u}\rfloor$, the triangle inequality and the inequality

$$\forall\,(x,y)\in\mathbb{R}_+^2,\quad(\log(e+x+y))^{\frac{d}{2}}\le(\log(e+x))^{\frac{d}{2}}(\log(e+y))^{\frac{d}{2}}\,, \tag{4.2.14}$$

one gets, for some positive finite constant $C'_{\mathbf{p},q}(\omega)$ and for all $(j,\mathbf{u},\omega)\in\mathbb{Z}\times\mathbb{R}^d\times\Omega^*$, that:

$$\begin{aligned}
&\sum_{\epsilon\in E}\sum_{\mathbf{k}\in\mathbb{Z}^d}|I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})(\omega)|\sup_{v\in(0,1)}\left|(\partial_{\mathbf{u}}^{\mathbf{p}}\partial_v^q\Psi^\epsilon)(\mathbf{u}-\mathbf{k},v)\right|\\
&\qquad\le C'_{\mathbf{p},q}(\omega)\sum_{\mathbf{k}\in\mathbb{Z}^d}\frac{(\log(e+|j|+\|\mathbf{k}\|_1))^{\frac{d}{2}}}{(\sqrt{d}+1+\|\mathbf{u}-\mathbf{k}\|_2)^{2d}}\\
&\qquad\le C'_{\mathbf{p},q}(\omega)\sum_{\mathbf{m}\in\mathbb{Z}^d}\frac{(\log(e+|j|+\|\mathbf{m}+\lfloor\mathbf{u}\rfloor\|_1))^{\frac{d}{2}}}{(\sqrt{d}+1+\|\mathbf{u}-\lfloor\mathbf{u}\rfloor-\mathbf{m}\|_2)^{2d}}\\
&\qquad\le C'_{\mathbf{p},q}(\omega)\sum_{\mathbf{m}\in\mathbb{Z}^d}\frac{(\log(e+|j|+\|\mathbf{m}\|_1+\|\lfloor\mathbf{u}\rfloor\|_1))^{\frac{d}{2}}}{(\sqrt{d}+1-\|\mathbf{u}-\lfloor\mathbf{u}\rfloor\|_2+\|\mathbf{m}\|_2)^{2d}}\\
&\qquad\le C'_{\mathbf{p},q}(\omega)(\log(e+|j|+\|\lfloor\mathbf{u}\rfloor\|_1))^{\frac{d}{2}}\sum_{\mathbf{m}\in\mathbb{Z}^d}\frac{(\log(e+\|\mathbf{m}\|_1))^{\frac{d}{2}}}{(1+\|\mathbf{m}\|_2)^{2d}}.
\end{aligned}$$

Thus, in view of the fact that

$$\sum_{\mathbf{m}\in\mathbb{Z}^d} \frac{(\log(e+\|\mathbf{m}\|_1))^{\frac{d}{2}}}{(1+\|\mathbf{m}\|_2)^{2d}} < +\infty\,,$$

it turns out that (4.2.11) is satisfied. $\square$

Proof of Theorem 4.2.4. First, observe that, in view of (4.2.7) and (4.2.12), $X(u,v,\omega)$ can be expressed, for all $(u,v,\omega)\in\mathbb{R}\times(0,1)\times\Omega^*$, as:

$$X(u,v,\omega) = \sum_{j\in\mathbb{Z}} A_j(u,v,\omega)\,, \tag{4.2.15}$$

where, for each $j\in\mathbb{Z}$, $A_j(\cdot,\cdot,\omega)$ is the infinitely differentiable function on $\mathbb{R}\times(0,1)$ defined as:

$$\forall\,(u,v)\in\mathbb{R}\times(0,1),\quad A_j(u,v,\omega) = 2^{-jv}\big(\Phi_j(2^j\mathbf{u}^*,v,\omega)-\Phi_j(\mathbf{0},v,\omega)\big)\,. \tag{4.2.16}$$

Thus, for proving the theorem, one has to show that the convergence of the series in (4.2.15) holds uniformly in (u,v) on each compact subset of $\mathbb{R}\times(0,1)$. To this end, it is enough to prove that, for all fixed positive real numbers ν and $a<b<1$, one has

$$\sum_{j\in\mathbb{Z}} \sup_{(u,v)\in[-\nu,\nu]\times[a,b]} |A_j(u,v,\omega)| < +\infty. \tag{4.2.17}$$

Let us sets

$$T_1(\omega) = \sum_{j=0}^{+\infty} \sup_{(u,v)\in[-\nu,\nu]\times[a,b]} |A_j(u,v,\omega)|$$

and

$$T_2(\omega) = \sum_{j=-\infty}^{-1} \sup_{(u,v)\in[-\nu,\nu]\times[a,b]} |A_j(u,v,\omega)|.$$

One can derive from (4.2.16) and (4.2.13) that

$$T_1(\omega) \le 2C_{\mathbf{0},0}(\omega)\sum_{j=0}^{+\infty} 2^{-ja}(\log(e+j+2^jd\nu))^{\frac{d}{2}} < +\infty\,. \tag{4.2.18}$$

On another hand, using the mean value theorem and (4.2.13), one obtains, for every $(j,u,v,\omega)\in\mathbb{Z}_-\times[-\nu,\nu]\times[a,b]\times\Omega^*$, that

$$\begin{aligned}
|A_j(u,v,\omega)| &\le 2^{-ja}|\Phi_j(2^j\mathbf{u}^*,v,\omega)-\Phi_j(\mathbf{0},v,\omega)|\\
&\le 2^{j(1-a)}|u|\sum_{i=1}^{d}\sup_{x\in[0\wedge 2^ju,0\vee 2^ju]}|\partial_{u_i}\Phi_j(\mathbf{x}^*,v,\omega)|\\
&\le d\nu C_{\mathbf{1},0}(\omega)2^{j(1-a)}\left(\log(e+|j|+2^jd\nu)\right)^{\frac{d}{2}}.
\end{aligned}$$

Thus, one gets that

$$T_2(\omega) \le d\nu C_{\mathbf{1},0}(\omega)\sum_{j=1}^{\infty} 2^{-j(1-a)}(\log(e+j+2^{-j}d\nu))^{\frac{d}{2}} < +\infty \tag{4.2.19}$$

Finally, combining (4.2.18) and (4.2.19), it follows that (4.2.17) is satisfied. $\square$

4.3 Global behavior

First, we state the main results of the section and then we give their proofs.

Theorem 4.3.1. *Let X^{lf} and X^{hf} be the low and high frequency parts of the field X which were introduced in Remark 4.2.3. The following two results hold, for all $\omega \in \Omega^*$.*

(i) The function $X^{lf}(\cdot,\cdot,\omega) : (u,v) \mapsto X^{lf}(u,v,\omega)$ is infinitly many times differentiable on $\mathbb{R} \times (0,1)$.

(ii) For all fixed $u \in \mathbb{R}$, the function $X^{hf}(u,\cdot,\omega) : v \mapsto X^{hf}(u,v,\omega)$ is infinitly many times differentiable on $(0,1)$. Moreover, for each fixed $q \in \mathbb{Z}_+$, the function $(\partial_v^q X)(\cdot,\cdot,\omega)$: $(u,v) \mapsto (\partial_v^q X)(u,v,\omega)$ is continuous on $\mathbb{R} \times (0,1)$.

Corollary 4.3.2. For each $(\omega, q) \in \Omega^* \times \mathbb{Z}_+$, and for all non degenerate compact intervals $\mathcal{J} \subset \mathbb{R}$ and $\mathcal{H} \subset (0,1)$, one has:

$$\sup_{(u,v_1,v_2)\in\mathcal{J}\times\mathcal{H}^2} \left\{ \frac{|(\partial_v^q X)(u,v_1,\omega) - (\partial_v^q X)(u,v_2,\omega)|}{|v_1 - v_2|} \right\} < +\infty. \tag{4.3.1}$$

Theorem 4.3.3. *For each $(\omega, q) \in \Omega^* \times \mathbb{Z}_+$, and for all non degenerate compact intervals $\mathcal{J} \subset \mathbb{R}$ and $\mathcal{H} \subset (0,1)$, one has:*

$$\sup_{(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2} \left\{ \frac{|(\partial_v^q X)(u_1,v_1,\omega) - (\partial_v^q X)(u_2,v_2,\omega)|}{\big|u_1 - u_2\big|^{v_1\vee v_2}\Big(1 + \big|\log|u_1-u_2|\big|\Big)^{q+\frac{d}{2}} + \big|v_1 - v_2\big|} \right\} < +\infty. \tag{4.3.2}$$

Corollary 4.3.4. Let $H(\cdot)$ be the continuous functional parameter of the chaotic multifractional process $\{Z(t) : t \in \mathbb{R}\}$ (see (4.1.7)). Let $\mathcal{L} \subset \mathbb{R}$ be an arbitrary non degenerate compact interval. One sets

$$\underline{H}(\mathcal{L}) := \min\{H(t) : t \in \mathcal{L}\} \quad \text{and} \quad \overline{H}(\mathcal{L}) := \max\{H(t) : t \in \mathcal{L}\}. \tag{4.3.3}$$

Assuming that

$$H(\cdot) \in C^{\gamma_{\mathcal{L}}}(\mathcal{L}) \quad \text{for some} \quad \gamma_{\mathcal{L}} \in [\underline{H}(\mathcal{L}), 1), \tag{4.3.4}$$

where $C^{\gamma_{\mathcal{L}}}(\mathcal{L})$ denotes the global space of Hölder on $\mathcal{L}$ of order $\gamma_{\mathcal{L}}$. Then, for all $\omega \in \Omega^*$, one has:

$$\sup_{(t_1,t_2)\in\mathcal{L}^2} \left\{ \frac{|Z(t_1,\omega) - Z(t_2,\omega)|}{\big|t_1 - t_2\big|^{\underline{H}(\mathcal{L})}\Big(1 + \big|\log|t_1-t_2|\big|\Big)^{\frac{d}{2}}} \right\} < +\infty\,. \tag{4.3.5}$$

Proof of Theorem 4.3.1. First, we point out that one knows from the proof of Theorem 4.2.4 that, for all $\omega \in \Omega^*$, one has

$$X^{lf}(u,v,\omega) = \sum_{j=-\infty}^{-1} A_j(u,v,\omega) \tag{4.3.6}$$

and

$$X^{hf}(u,v,\omega)=\sum_{j=0}^{+\infty}A_j(u,v,\omega)\,, \tag{4.3.7}$$

where the series in (4.3.6) and (4.3.7) are uniformly convergent in (u,v) on each compact subset of $\mathbb{R}\times(0,1)$. Moreover, one knows that, for each $j\in\mathbb{Z}$, the function $A_j(\cdot,\cdot,\omega)$ is infinitely differentiable on $\mathbb{R}\times(0,1)$. Thus, in order to prove the theorem, it is enough to show that, for all $(m,q,\omega)\in\mathbb{Z}_+\times\mathbb{Z}_+\times\Omega^*$, and for each positive real numbers ν and $a<b$, one has

$$\sum_{j=-\infty}^{-1}\sup_{(u,v)\in[-\nu,\nu]\times[a,b]}\big|(\partial_u^m\partial_v^q A_j)(u,v,\omega)\big|<+\infty$$

and

$$\sum_{j=0}^{+\infty}\sup_{(u,v)\in[-\nu,\nu]\times[a,b]}\big|(\partial_v^q A_j)(u,v,\omega)\big|<+\infty.$$

This can be done by following the main lines of the proof of (4.2.18) and (4.2.19). □

Proof of Corollary 4.3.2. It follows from Theorem 4.3.1, that for all fixed $(u,\omega)\in\mathbb{R}\times\Omega^*$, the function $X(u,\cdot,\omega)$ is infinitely differentiable on $(0,1)$; and for all $q\in\mathbb{Z}_+$, the function $(\partial_v^q X)(\cdot,\cdot,\omega)$ is continuous. That is enough to prove that (4.3.1) holds. □

Corollary 4.3.2 and the following lemma are the two main ingredients of the proof of Theorem 4.3.3.

Lemma 4.3.5. For each $(\omega,q)\in\Omega^*\times\mathbb{Z}_+$, and for all non degenerate compact intervals $\mathcal{J}\subset\mathbb{R}$ and $\mathcal{H}\subset(0,1)$, one has

$$\sup_{(u_1,u_2,v)\in\mathcal{J}^2\times\mathcal{H}}\left\{\frac{|(\partial_v^q X)(u_1,v,\omega)-(\partial_v^q X)(u_2,v,\omega)|}{\big|u_1-u_2\big|^v\Big(1+\big|\log|u_1-u_2|\big|\Big)^{q+\frac{d}{2}}}\right\}<+\infty. \tag{4.3.8}$$

Proof of Lemma 4.3.5. First, notice that Theorem 4.3.1 entails that the lemma holds when X in (4.3.8) is replaced by X^{lf}. Thus, one only has to prove that the lemma is true when X in (4.3.8) is replaced by X^{hf}. Using the continuity property of the function $(\partial_v^q X)(\cdot,\cdot,\omega)$ (see Theorem 4.3.1), one has that

$$\sup_{(u_1,u_2,v)\in\mathcal{K}'}\left\{\frac{|(\partial_v^q X^{hf})(u_1,v,\omega)-(\partial_v^q X^{hf})(u_2,v,\omega)|}{\big|u_1-u_2\big|^v\Big(1+\big|\log|u_1-u_2|\big|\Big)^{q+\frac{d}{2}}}\right\}<+\infty, \tag{4.3.9}$$

where $\mathcal{K}'$ is the compact subset of $\mathbb{R}^2\times(0,1)$ defined as

$$\mathcal{K}'=\big\{(u_1,u_2,v)\in\mathcal{J}^2\times\mathcal{H}:\ \ |u_1-u_2|\geq 2^{-1}\big\}.$$

Thus, in order to derive the lemma, it is enough to prove that:

$$\sup_{(u_1,u_2,v)\in\mathcal{K}}\left\{\frac{\big|(\partial_v^q X^{hf})(u_1,v,\omega)-(\partial_v^q X^{hf})(u_2,v,\omega)\big|}{\big|u_1-u_2\big|^v\Big(1+\big|\log|u_1-u_2|\big|\Big)^{q+\frac{d}{2}}}\right\}<+\infty, \tag{4.3.10}$$

where $\mathcal{K}$ is the compact subset of $\mathbb{R}^2\times(0,1)$ defined as

$$\mathcal{K}=\big\{(u_1,u_2,v)\in\mathcal{J}^2\times\mathcal{H}:\quad |u_1-u_2|\le 2^{-1}\big\}.$$

We will show (4.3.10) for $q=0$; the proof can be done in a rather similar way in the general case where q is an arbitrary nonegative integer. There is no restriction to assume that $\mathcal{J}=[-\nu,\nu]$ and $\mathcal{H}=[a,b]\subset(0,1)$, where ν and $a<b$ are fixed positive real numbers. Let $(u_1,u_2,v)\in\mathcal{K}$ be arbitrary; there is no restriction to assume that $u_1\neq u_2$ since (4.3.10) is clearly satisfied when $u_1=u_2$. Then, denote by j_0 be the biggest nonegative integer satisfying $|u_1-u_2|\le 2^{-j_0}$. Observe that $j_0\ge 1$ and that one has:

$$2^{-(j_0+1)}<|u_1-u_2|\le 2^{-j_0}, \tag{4.3.11}$$

which means that

$$j_0=\left\lfloor\frac{\log\big(|u_1-u_2|^{-1}\big)}{\log 2}\right\rfloor. \tag{4.3.12}$$

Notice that one knows from Lemma 4.2.5 and (4.2.16) that the function $A_0(\cdot,\cdot,\omega)$ is infinitely differentiable on $\mathbb{R}\times(0,1)$, which implies that it satisfies (4.3.10). This allows to assume that the sum over j in (4.3.7) starts from $j=1$ instead of $j=0$. Thus, one has that

$$|X^{hf}(u_1,v,\omega)-X^{hf}(u_2,v,\omega)|\le S_1(u_1,u_2,v,\omega)+S_2(u_1,u_2,v,\omega), \tag{4.3.13}$$

where

$$S_1(u_1,u_2,v,\omega)=\sum_{j=1}^{j_0}|A_j(u_1,v,\omega)-A_j(u_2,v,\omega)| \tag{4.3.14}$$

and

$$S_2(u_1,u_2,v,\omega)=\sum_{j=j_0+1}^{+\infty}|A_j(u_1,v,\omega)-A_j(u_2,v,\omega)|. \tag{4.3.15}$$

In order to derive appropriate upper bounds for $S_1(u_1,u_2,v,\omega)$ and $S_2(u_1,u_2,v,\omega)$, notice that there exists a deterministic positive finite constant c such that:

$$\forall\, x\ge 1,\quad \log(e+x+2^x d\nu)\le cx. \tag{4.3.16}$$

Using (4.3.15), (4.2.16), the triangle inequality, Lemma 4.2.5, (4.3.16), the inequality

$$\forall\,(x,y)\in\mathbb{R}_+^2,\quad (1+x+y)^{\frac{d}{2}}\le(1+x)^{\frac{d}{2}}(1+y)^{\frac{d}{2}},$$

(4.3.11) and (4.3.12), one gets:

$$\begin{aligned}
S_2(u_1,u_2,v,\omega) &\le 2\sum_{j=j_0+1}^{+\infty} 2^{-jv} \sup_{(u,v)\in[-\nu,\nu]\times[a,b]} |\Phi_j(2^j\mathbf{u}^*,v,\omega)| \\
&\le C_2(\omega)\sum_{j=j_0+1}^{+\infty} 2^{-jv}\big(\log(e+j+2^j d\nu)\big)^{\frac{d}{2}} \\
&\le C_2'(\omega)\sum_{j=j_0+1}^{+\infty} 2^{-jv} j^{\frac{d}{2}} \\
&\le C_2'(\omega)2^{-(j_0+1)v}(1+j_0)^{\frac{d}{2}}\sum_{j=0}^{+\infty} 2^{-ja}(1+j)^{\frac{d}{2}} \\
&\le C_2''(\omega)2^{-(j_0+1)v}(1+j_0)^{\frac{d}{2}} \\
&\le C_2'''(\omega)|u_1-u_2|^v\Big(1+\big|\log|u_1-u_2|\big|\Big)^{\frac{d}{2}}, \qquad (4.3.17)
\end{aligned}$$

where C_2 is a positive finite random variable not depending on (u_1,u_2,v) derived from Lemma 4.2.5, $C_2'=C_2c^{d/2}$, $C_2''=C_2'\sum_{j=0}^{+\infty}2^{-ja}(1+j)^{\frac{d}{2}}<+\infty$ and $C_2'''=(\log 2)^{-\frac{d}{2}}C_2''$.

On another hand, using (4.3.14), (4.2.16), the mean value theorem, the triangle inequality, Lemma 4.2.5, (4.3.16), (4.3.11) and (4.3.12), one gets:

$$\begin{aligned}
S_1(u_1,u_2,v,\omega) &\le \sum_{j=1}^{j_0} 2^{j(1-v)}|u_1-u_2|\sum_{i=1}^{d}\sup_{(u,v)\in[-\nu,\nu]\times[a,b]}|\partial_{u_i}\Phi_j(2^j\mathbf{u}^*,v,\omega)| \\
&\le C_1(\omega)|u_1-u_2|\sum_{j=1}^{j_0} 2^{j(1-v)}\big(\log(e+j+2^j d\nu)\big)^{\frac{d}{2}} \\
&\le C_1'(\omega)|u_1-u_2|\sum_{j=1}^{j_0} 2^{j(1-v)}(j+1)^{\frac{d}{2}} \\
&\le C_1''(\omega)|u_1-u_2|(1+j_0)^{\frac{d}{2}}2^{j_0(1-v)} \\
&\le C_1'''(\omega)|u_1-u_2|^v\Big(1+\big|\log|u_1-u_2|\big|\Big)^{\frac{d}{2}}, \qquad (4.3.18)
\end{aligned}$$

where C_1 is a positive finite random variable not depending on (u_1,u_2,v) derived from Lemma 4.2.5, $C_1'=C_1c^{d/2}$, $C_1''=2^{1-a}(2^{1-b}-1)^{-1}C_1'$ and $C_1'''=(\log 2)^{-\frac{d}{2}}C_1''$.

Finally, puting together (4.3.13), (4.3.17) and (4.3.18), one obtains (4.3.10). $\square$

Proof of Theorem 4.3.3. For all $(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2$, one sets:

$$f(u_1,u_2,v_1,v_2)=\frac{|(\partial_v^q X)(u_1,v_1,\omega)-(\partial_v^q X)(u_2,v_2,\omega)|}{\big|u_1-u_2\big|^{v_1\vee v_2}\Big(1+\big|\log|u_1-u_2|\big|\Big)^{q+\frac{d}{2}}+|v_1-v_2|},$$

with the convention that $\frac{0}{0} = 0$. Observe that one has:

$$f(u_1, u_2, v_1, v_2) = f(u_2, u_1, v_2, v_1).$$

Thus, one gets that:

$$\sup_{(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2} f(u_1, u_2, v_1, v_2) = \sup_{(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2} f(u_1, u_2, v_1 \vee v_2, v_1 \wedge v_2). \tag{4.3.19}$$

Moreover, using the triangle inequality, one obtains that:

$$\begin{aligned}
&\sup_{(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2} f(u_1, u_2, v_1 \vee v_2, v_1 \wedge v_2) \\
&\quad\le \sup_{(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2} \left\{ \frac{|(\partial_v^q X)(u_1, v_1 \vee v_2, \omega) - (\partial_v^q X)(u_2, v_1 \vee v_2, \omega)|}{|u_1 - u_2|^{v_1\vee v_2}\Big(1 + \big|\log|u_1 - u_2|\big|\Big)^{q+\frac{d}{2}} + |v_1 - v_2|} \right\} \\
&\qquad + \sup_{(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2} \left\{ \frac{|(\partial_v^q X)(u_2, v_1 \vee v_2, \omega) - (\partial_v^q X)(u_2, v_1 \wedge v_2, \omega)|}{|u_1 - u_2|^{v_1\vee v_2}\Big(1 + \big|\log|u_1 - u_2|\big|\Big)^{q+\frac{d}{2}} + |v_1 - v_2|} \right\} \\
&\quad\le \sup_{(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2} \left\{ \frac{|(\partial_v^q X)(u_1, v_1 \vee v_2, \omega) - (\partial_v^q X)(u_2, v_1 \vee v_2, \omega)|}{|u_1 - u_2|^{v_1\vee v_2}\Big(1 + \big|\log|u_1 - u_2|\big|\Big)^{q+\frac{d}{2}}} \right\} \\
&\qquad + \sup_{(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2} \left\{ \frac{|(\partial_v^q X)(u_2, v_1 \vee v_2, \omega) - (\partial_v^q X)(u_2, v_1 \wedge v_2, \omega)|}{|v_1 - v_2|} \right\} \\
&\quad\le \sup_{(u_1,u_2,v)\in\mathcal{J}^2\times\mathcal{H}} \left\{ \frac{|(\partial_v^q X)(u_1, v, \omega) - (\partial_v^q X)(u_2, v, \omega)|}{|u_1 - u_2|^{v}\Big(1 + \big|\log|u_1 - u_2|\big|\Big)^{q+\frac{d}{2}}} \right\} \\
&\qquad + \sup_{(u,v_1,v_2)\in\mathcal{J}\times\mathcal{H}^2} \left\{ \frac{|(\partial_v^q X)(u, v_1, \omega) - (\partial_v^q X)(u, v_2, \omega)|}{|v_1 - v_2|} \right\}.
\end{aligned} \tag{4.3.20}$$

Finally, putting together (4.3.19), (4.3.20), Corollary 4.3.2 and Lemma 4.3.5, one gets that

$$\sup_{(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2} f(u_1, u_2, v_1, v_2) < +\infty,$$

which shows that (4.3.2) holds. □

Proof of Corollary 4.3.4. Using (4.1.7) and Theorem 4.3.3, in the case where $q = 0$, $\mathcal{J} = \mathcal{L}$ and $\mathcal{H} = [\underline{H}(\mathcal{L}), \overline{H}(\mathcal{L})]$ (see (4.3.3)), one obtains, for all $\omega \in \Omega^*$, that:

$$\sup_{(t_1,t_2)\in\mathcal{L}^2} \left\{ \frac{|Z(t_1, \omega) - Z(t_2, \omega)|}{|t_1 - t_2|^{H(t_1)\vee H(t_2)}\Big(1 + \big|\log|t_1 - t_2|\big|\Big)^{\frac{d}{2}} + |H(t_1) - H(t_2)|} \right\} < +\infty. \tag{4.3.21}$$

Then, (4.3.4) and (4.3.21) imply that (4.3.5) holds. □

4.4 Local and asymptotic behavior

First, we state the main results of the section and then we give their proofs.

Theorem 4.4.1. *Let $u_0 \in \mathbb{R}$ be an arbitrary fixed point. Then, one has, almost surely, for every $q \in \mathbb{Z}_+$ and non degenerate compact interval $\mathcal{H} \subset (0,1)$, that:*

$$\sup_{(u,v)\in[u_0-1,u_0+1]\times\mathcal{H}} \left\{ \frac{|(\partial_v^q X)(u,v) - (\partial_v^q X)(u_0,v)|}{|u-u_0|^v \Big(1 + \big|\log|u-u_0|\big|\Big)^q \Big(\log\Big(e + \big|\log|u-u_0|\big|\Big)\Big)^{\frac{d}{2}}} \right\} < +\infty. \tag{4.4.1}$$

Corollary 4.4.2. Let $t_0 \in \mathbb{R}$ be an arbitrary fixed point. Assume that there exists a constant $\gamma_{t_0} \in [H(t_0), 1)$ such that the continuous function $H(\cdot)$ satisfies

$$\sup_{t\in\mathbb{R}} \left\{ \frac{|H(t) - H(t_0)|}{|t-t_0|^{\gamma_{t_0}}} \right\} < +\infty. \tag{4.4.2}$$

Then, one has, almost surely:

$$\sup_{t\in[t_0-1,t_0+1]} \left\{ \frac{|Z(t) - Z(t_0)|}{|t-t_0|^{H(t_0)} \Big(\log\Big(e + \big|\log|t-t_0|\big|\Big)\Big)^{\frac{d}{2}}} \right\} < +\infty. \tag{4.4.3}$$

The following theorem shows that the chaotic multifractional process $\{Z(t) : t \in \mathbb{R}\}$ has *a local asymptotic self-similarity* property rather similar to the one satisfied by the classical Gaussian multifractional Brownian motion (see [17, 30, 31]).

Theorem 4.4.3. *Let $t_0 \in \mathbb{R}$ be an arbitrary fixed point such that the condition (4.4.2) holds. Then, the stochastic process $\{Z(t) : t \in \mathbb{R}\}$ is at t_0, strongly locally asymptotically self-similar of order $H(t_0)$ and the tangent process is $\big\{X(s, H(t_0)) : s \in \mathbb{R}\big\}$. More precisely, let $(\nu_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive real numbers which converges to 0. For each $n \in \mathbb{N}$, let $T_{t_0,\nu_n} Z = \big\{(T_{t_0,\nu_n} Z)(s) : s \in \mathbb{R}\big\}$ be the stochastic process with continuous paths, defined, for all $s \in \mathbb{R}$, as*

$$(T_{t_0,\nu_n} Z)(s) = \frac{Z(t_0 + \nu_n s) - Z(t_0)}{\nu_n^{H(t_0)}}\,. \tag{4.4.4}$$

Then, when n goes to $+\infty$, the probability measure induced on $\mathcal{C}(\mathcal{J})$ by $\big\{(T_{t_0,\nu_n} Z)(s) : s \in \mathbb{R}\big\}$ converges to the one induced on $\mathcal{C}(\mathcal{J})$ by $\big\{X(s, H(t_0)) : s \in \mathbb{R}\big\}$, where $\mathcal{C}(\mathcal{J})$ denotes the usual Banach space of the real-valued continuous functions over an arbitrary non degenerate compact interval $\mathcal{J}$ of the real line equipped with the uniform norm.

Remark 4.4.4. One can derive from Theorem 4.4.3 and zero-one law that, for any fixed arbitrarily small positive real number η, one has, almost surely,

$$\sup_{t\in[t_0-1,t_0+1]} \left\{ \frac{|Z(t) - Z(t_0)|}{|t-t_0|^{H(t_0)+\eta}} \right\} = +\infty\,,$$

which means that the exponent $H(t_0)$ in (4.4.3) is optimal. Moreover, when $\gamma_{\mathcal{L}}$ in (4.3.3) belongs to $\big[\overline{H}(\mathcal{L}),1\big)$, then, using similar arguments, it can be shown that the exponent $\underline{H}(\mathcal{L})$ in (4.3.5) is optimal: one has, almost surely,

$$\sup_{(t_1,t_2)\in\mathcal{L}^2}\left\{\frac{|Z(t_1)-Z(t_2)|}{|t_1-t_2|^{\underline{H}(\mathcal{L})+\eta}}\right\}=+\infty\,.$$

Theorem 4.4.5. *Let $\delta\in(0,+\infty)$ be arbitrary and fixed. One sets $\mathbb{R}_\delta^+=\{u\in\mathbb{R};\quad |u|\ge\delta\}$. Then, for all $\omega\in\Omega^*$ and non degenerate compact interval $\mathcal{H}\subset(0,1)$, one has:*

$$\sup_{(u,v)\in\mathbb{R}_\delta^+\times\mathcal{H}}\left\{\frac{|X^{lf}(u,v,\omega)|}{|u|^v\Big(\log\Big(e+\big|\log|u|\big|\Big)\Big)^{\frac{d}{2}}}\right\}<+\infty \tag{4.4.5}$$

and

$$\sup_{(u,v)\in\mathbb{R}_\delta^+\times\mathcal{H}}\left\{\frac{|X^{hf}(u,v,\omega)|}{\Big(\big(\log\big(e+|u|\big)\Big)^{\frac{d}{2}}}\right\}<+\infty. \tag{4.4.6}$$

Notice that a straightforward consequence of (4.4.5), (4.4.6) and (4.2.10) is that:

$$\sup_{(u,v)\in\mathbb{R}_\delta^+\times\mathcal{H}}\left\{\frac{|X(u,v,\omega)|}{|u|^v\Big(\log\Big(e+\big|\log|u|\big|\Big)\Big)^{\frac{d}{2}}}\right\}<+\infty.$$

Corollary 4.4.6. Assume that the continuous function $H(\cdot)$ is with values in a compact interval included in $(0,1)$ (this means that $\inf_{t\in\mathbb{R}}H(t)>0$ and $\sup_{t\in\mathbb{R}}H(t)<1$). Then, for each fixed $\omega\in\Omega^*$ and $\delta>0$, one has:

$$\sup_{|t|\ge\delta}\left\{\frac{|Z(t,\omega)|}{|t|^{H(t)}\Big(\log\Big(e+\big|\log|t|\big|\Big)\Big)^{\frac{d}{2}}}\right\}<+\infty. \tag{4.4.7}$$

The following lemma will play a crucial role in the proof of Theorem 4.4.1.

Lemma 4.4.7. For all fixed integer $j\ge1$ and $(\mathbf{u},\theta)\in\mathbb{R}^d\times[1,+\infty)$, let $D_j(\mathbf{u},\theta)$ be the finite non-empty set defined as:

$$D_j(\mathbf{u},\theta)=\Big\{(\epsilon,\mathbf{k})\in E\times\mathbb{Z}^d:\ \|\mathbf{u}-2^{-j}\mathbf{k}\|_1\le d\,j^\theta 2^{-j}\Big\}. \tag{4.4.8}$$

Then, for each fixed $(\mathbf{u},\theta)\in\mathbb{R}^d\times[1,+\infty)$ there is a deterministic positive finite constant c_*, only depending on (u,θ,d), such that one has, almost surely:

$$\limsup_{j\to+\infty}\left\{\frac{\max_{\mathbf{k}\in D_j(\mathbf{u},\theta)}\big|I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})\big|}{\big(\log(2+j)\big)^{\frac{d}{2}}}\right\}\le c_*\,. \tag{4.4.9}$$

Proof of Lemma 4.4.7. The lemma can be derived from the Borel-Cantelli Lemma by showing that for some fixed well-chosen deterministic positive finite constant $a \geq 2$, one has

$$\sum_{j=1}^{+\infty} \mathbb{P}\Big(\max_{\mathbf{k} \in D_j(\mathbf{u},\theta)} |I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})| > a\sqrt{d!}\big(\log(2+j)\big)^{\frac{d}{2}} \Big) < +\infty. \tag{4.4.10}$$

It can easily be seen that, for all $j \geq 1$, the following inequality holds:

$$\mathbb{P}\Big(\max_{\mathbf{k} \in D_j(\mathbf{u},\theta)} |I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})| > a\sqrt{d!}\big(\log(2+j)\big)^{\frac{d}{2}} \Big) \leq \sum_{\mathbf{k} \in D_j(\mathbf{u},\theta)} \mathbb{P}\Big(|I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})| > a\sqrt{d!}\big(\log(2+j)\big)^{\frac{d}{2}} \Big). \tag{4.4.11}$$

Let us conveniently bound from above the probabilities in the right-hand side of (4.4.11). Observe that (4.1.5) and the equality $\|\psi_{j,\mathbf{k}}^{(\epsilon)}\|_{L^2(\mathbb{R}^d)} = 1$ imply, for all $(j,\mathbf{k},\epsilon) \in \mathbb{Z} \times \mathbb{Z}^d \times E$, that

$$\mathbb{E}\big(|I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})|^2\big) = d!\|\tilde{\psi}_{j,\mathbf{k}}^{(\epsilon)}\|_{L^2(\mathbb{R}^d)}^2 \leq d!\|\psi_{j,\mathbf{k}}^{(\epsilon)}\|_{L^2(\mathbb{R}^d)}^2 = d!\,. \tag{4.4.12}$$

Then (4.4.12) and Theorem 6.7 in [36] entail that, for all real number $\alpha \geq 2$, one has

$$\mathbb{P}\big(|I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})| > \alpha\sqrt{d!}\big) \leq \mathbb{P}\Big(|I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})| > \alpha\big\|I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})\big\|_{L^2(\Omega)}\Big) \leq \exp\big(-\kappa_d \alpha^{\frac{2}{d}}\big)\,, \tag{4.4.13}$$

where κ_d is a deterministic positive finite constant only depending on d. Thus, setting in (4.4.13) $\alpha = a\big(\log(2+j)\big)^{\frac{d}{2}}$, one gets that

$$\mathbb{P}\Big(|I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})| > a\sqrt{d!}\big(\log(2+j)\big)^{\frac{d}{2}}\Big) \leq \exp\Big(-\kappa_d\, a^{\frac{2}{d}} \log(2+j)\Big) = (2+j)^{-\kappa_d a^{\frac{2}{d}}}\,. \tag{4.4.14}$$

On another hand, one can easily derives from (4.4.8) that there exists a deterministic positive finite constant c such that, for all $(\mathbf{u},\theta) \in \mathbb{R}^d \times [1,+\infty)$, one has

$$\operatorname{card}\big(D_j(\mathbf{u},\theta)\big) \leq c\, j^{d\theta}. \tag{4.4.15}$$

Putting together (4.4.11), (4.4.14) and (4.4.15), one obtains, for all $j \geq 1$, that

$$\mathbb{P}\Big(\max_{\mathbf{k} \in D_j(\mathbf{u},\theta)} |I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})| > a\sqrt{d!}\big(\log(2+j)\big)^{\frac{d}{2}} \Big) \leq c\, j^{d\theta - \kappa_d a^{\frac{2}{d}}}\,. \tag{4.4.16}$$

Thus, assuming that the constant a has been chosen big enough so that $d\theta - \kappa_d a^{\frac{2}{d}} < -1$, then it follows from (4.4.16) that (4.4.10) holds. □

Proof of Theorem 4.4.1. Using the same arguments as in the proof of Lemma 4.3.5, it turns out that in order to derive the theorem it is enough to show that (4.4.1) holds when X in it is repalced by X^{hf}, and one can assume that the sums over j in (4.3.7) and in (4.2.9) start from $j = 1$ instead of $j = 0$. Also, for the sake of simplicity one focuses on the case where $q = 0$. The proof can be done in a rather similar way in the general case where q is an arbitrary nonegative integer.

Let us express the compact interval $\mathcal{H}$ as $\mathcal{H} = [a, b]$, where the real numbers a and b are such that $0 < a < b < 1$. Let then $(u, v) \in [u_0 - 1, u_0 + 1] \times [a, b] \subset \mathbb{R} \times (0, 1)$ be arbitrary and fixed. There is no restriction to assume that $0 < |u - u_0| \le 2^{-15}$, since sample paths of X^{hf} are almost surely continuous functions. One denotes by j_1 be the biggest nonegative integer which satisfies $|u - u_0| \le 2^{-(j_1-1)}$. Then, one has:

$$2^{-j_1} < |u - u_0| \le 2^{-(j_1-1)}\,, \tag{4.4.17}$$

which means that

$$j_1 = 1 + \left\lfloor \frac{\log\left(|u-u_0|^{-1}\right)}{\log 2} \right\rfloor. \tag{4.4.18}$$

One sets

$$j_2 = j_1 + \left\lfloor \frac{d \log j_1}{2a \log 2} \right\rfloor. \tag{4.4.19}$$

Observe that one has $j_1 \ge 16$ and $j_2 \ge j_1 + 2$. Moreover, for any $j \in \{j_1 + 1, ..., j_2\}$, the following inequality holds:

$$j^{\frac{d}{a}} \ge 2^{j-j_1+2}. \tag{4.4.20}$$

Next, for all integer $j \ge 1$, let $\mathcal{D}_j(u_0)$ be the finite non-empty set defined as $\mathcal{D}_j(u_0) = D_j(\mathbf{u}_0^*, \frac{d}{a})$, where, as usual, $\mathbf{u}_0^*$ denotes the vector of $\mathbb{R}^d$ whose coordinates are all equal to the real number u_0, and $D_j(\mathbf{u}_0^*, \frac{d}{a})$ is defined through (4.4.8) with $\mathbf{u} = \mathbf{u}_0^*$ and $\theta = \frac{d}{a}$. That is:

$$\mathcal{D}_j(u_0) = \left\{ (\epsilon, \mathbf{k}) \in E \times \mathbb{Z}^d : \|\mathbf{u}_0^* - 2^{-j}\mathbf{k}\|_1 \le d\, j^{\frac{d}{a}} 2^{-j} \right\}. \tag{4.4.21}$$

Then, Lemma 4.4.7 entails that one has almost surely, for all integer $j \ge 1$,

$$\max_{(\epsilon,\mathbf{k})\in\mathcal{D}_j(u_0)} |I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})| \le C(\log(2+j))^{\frac{d}{2}}\,, \tag{4.4.22}$$

where C is a positive almost surely finite random variable not depending on j. Next, one denotes by $\mathcal{D}_j^{co}(u_0)$ the complement of $\mathcal{D}_j(u_0)$ in $E \times \mathbb{Z}^d$, that is:

$$\mathcal{D}_j^{co}(u_0) = \left\{ (\epsilon, \mathbf{k}) \in E \times \mathbb{Z}^d : \|\mathbf{u}_0^* - 2^{-j}\mathbf{k}\|_1 > d\, j^{\frac{d}{a}} 2^{-j} \right\}. \tag{4.4.23}$$

Let us mention in passing that

$$\mathcal{D}_j^{co}(u_0) \subset \bigcup_{l=1}^{d} \left\{ (\epsilon, \mathbf{k}) \in E \times \mathbb{Z}^d : |u_0 - 2^{-j}k_l| > j^{\frac{d}{a}} 2^{-j} \right\}\,, \tag{4.4.24}$$

where k_l is the l-th coordinate of $\mathbf{k}$. One can derive from (4.2.9) (where the sum over j is assumed to start from $j = 1$ instead of $j = 0$), (4.4.21), (4.4.23) and the triangle inequality that

$$|X^{hf}(u, v) - X^{hf}(u_0, v)| \le R_1(u, u_0, v) + R_2(u, u_0, v) + R_3(u, u_0, v) + R_4(u, u_0, v), \tag{4.4.25}$$

where

$$R_1(u,u_0,v) = \sum_{j=1}^{j_1} \sum_{(\epsilon,\mathbf{k})\in\mathcal{D}_j(u_0)} 2^{-jv} |I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})| |\Psi^{\epsilon}(2^j\mathbf{u}^* - \mathbf{k}, v) - \Psi^{\epsilon}(2^j\mathbf{u}_0^* - \mathbf{k}, v)|, \tag{4.4.26}$$

$$R_2(u,u_0,v) = \sum_{j=j_1+1}^{+\infty} \sum_{(\epsilon,\mathbf{k})\in\mathcal{D}_j(u_0)} 2^{-jv} |I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})| |\Psi^{\epsilon}(2^j\mathbf{u}^* - \mathbf{k}, v) - \Psi^{\epsilon}(2^j\mathbf{u}_0^* - \mathbf{k}, v)|, \tag{4.4.27}$$

$$R_3(u,u_0,v) = \sum_{j=1}^{j_2} \sum_{(\epsilon,\mathbf{k})\in\mathcal{D}_j^{co}(u_0)} 2^{-jv} |I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})| |\Psi^{\epsilon}(2^j\mathbf{u}^* - \mathbf{k}, v) - \Psi^{\epsilon}(2^j\mathbf{u}_0^* - \mathbf{k}, v)|, \tag{4.4.28}$$

and

$$R_4(u,u_0,v) = \sum_{j=j_2+1}^{+\infty} \sum_{(\epsilon,\mathbf{k})\in\mathcal{D}_j^{co}(u_0)} 2^{-jv} |I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})| |\Psi^{\epsilon}(2^j\mathbf{u}^* - \mathbf{k}, v) - \Psi^{\epsilon}(2^j\mathbf{u}_0^* - \mathbf{k}, v)|. \tag{4.4.29}$$

From now on, our goal is to derive an appropriate upper bound for each term in the right-hand side of (4.4.25). In all the sequel, one assumes that L is an arbitrary large fixed positive integer. Therefore, one has

$$c := \sup_{y\in\mathbb{R}} \left\{ \sum_{\mathbf{k}\in\mathbb{Z}^d} \frac{\Big(\log\big(e + \|\mathbf{y}^* - \mathbf{k}\|_2\big)\Big)^{\frac{d}{2}}}{\big(1 + \|\mathbf{y}^* - \mathbf{k}\|_2\big)^L} \right\} < +\infty. \tag{4.4.30}$$

Putting together (4.4.27), (4.4.22), (4.2.4), (4.4.30), (4.2.14), (4.4.17) and (4.4.18), one obtains that almost surely:

$$\begin{aligned} R_2(u,u_0,v) &\le K_2' \sum_{j=j_1+1}^{+\infty} 2^{-jv} \Big(\log(2+j)\Big)^{\frac{d}{2}} \\ &\le K_2' \sum_{j=0}^{+\infty} 2^{-(j+j_1+1)v} \Big(\log(2+j+j_1+1)\Big)^{\frac{d}{2}} \\ &\le K_2' 2^{-(j_1+1)v} \Big(\log(e+j_1+1)\Big)^{\frac{d}{2}} \sum_{j=0}^{+\infty} 2^{-ja} \Big(\log(e+j)\Big)^{\frac{d}{2}} \\ &\le K_2 |u-u_0|^v \Big(\log(e + |\log|u-u_0||\Big)^{\frac{d}{2}}, \end{aligned} \tag{4.4.31}$$

where K_2' and K_2 are two positive almost surely finite random variables not depending on (u,v).

Next, using (4.4.29), (4.2.11), the fact that $|u| \le |u_0| + 1$, the inequality $2^j > j$ for all

$j \in \mathbb{Z}_+$, (4.2.14), (4.4.17), (4.4.18) and (4.4.19), one gets, on the event of probability 1 Ω^*, that:

$$
\begin{aligned}
R_4(u,u_0,v) &\le K_4' \sum_{j=j_2+1}^{+\infty} 2^{-jv} \Big(\log\big(e + (|u_0|+d+1)2^j\big)\Big)^{\frac{d}{2}} \\
&\le K_4'' 2^{-(j_2+1)v} (j_2+1)^{\frac{d}{2}} \sum_{j=0}^{+\infty} 2^{-ja} \Big(\log\big(e + (|u_0|+d+1)2^j\big)\Big)^{\frac{d}{2}} \\
&\le K_4''' 2^{-(j_2+1)v} (j_2+1)^{\frac{d}{2}} \\
&= K_4''' 2^{-j_1 v} \exp\left(-(v\log 2)\left(1 + \left\lfloor \frac{d \log j_1}{2a\log 2}\right\rfloor\right) + \frac{d\log(j_2+1)}{2}\right) \\
&\le K_4''' |u-u_0|^v \exp\left(\frac{d}{2}\log\left(\frac{j_2+1}{j_1}\right)\right) \\
&\le K_4 |u-u_0|^v\,,
\end{aligned}
\tag{4.4.32}
$$

where K_4', K_4'' and K_4''' are three positive finite random variables not depending on (u,v), and where $K_4 = K_4'''\big(2 + \frac{d}{2a\log 2}\big)^{\frac{d}{2}}$.

Next, observe that using the mean value theorem, it can be shown that, for all fixed $(\epsilon, j, \mathbf{k}) \in E \times \mathbb{N} \times \mathbb{Z}^d$, there exists a real number $\lambda^{\epsilon}_{j,\mathbf{k}}(u,u_0) \in (0,1)$ such that:

$$
\Psi^{\epsilon}(2^j\mathbf{u}^* - \mathbf{k}, v) - \Psi^{\epsilon}(2^j\mathbf{u}_0^* - \mathbf{k}, v) = 2^j(u-u_0)\sum_{n=1}^{d} (\partial_{y_n}\Psi^{\epsilon})\big(2^j\mathbf{u}_0^* + \lambda^{\epsilon}_{j,\mathbf{k}}(u,u_0)2^j(\mathbf{u}^*-\mathbf{u}_0^*)\big). \tag{4.4.33}
$$

Then, combining (4.4.26), (4.4.33), (4.4.22), (4.2.4), (4.4.30), (4.4.17) and (4.4.18), one gets almost surely that

$$
\begin{aligned}
R_1(u,u_0,v) &\le K_1'|u-u_0| \sum_{j=1}^{j_1} 2^{j(1-v)}\Big(\log(2+j)\Big)^{\frac{d}{2}} \\
&\le K_1'|u-u_0|\big(\log(2+j_1)\big)^{\frac{d}{2}} \sum_{j=1}^{j_1} 2^{(j_1-j+1)(1-v)} \\
&\le K_1'|u-u_0| 2^{(j_1+1)(1-v)}\Big(\log(2+j_1)\Big)^{\frac{d}{2}} \sum_{j=1}^{+\infty} 2^{-j(1-b)} \\
&\le K_1|u-u_0|^v\Big(\log(2+|\log|u-u_0||)\Big)^{\frac{d}{2}},
\end{aligned}
\tag{4.4.34}
$$

where K_1' and K_1 are two positive almost surely finite random variables not depending on (u,v).

It only remains to obtain a convenient upper bound for $R_3(u,u_0,v)$. Notice that, using the equivalence of all norms on $\mathbb{R}^d$, one deduces from (4.2.4) that:

$$
\forall\,(n,\mathbf{p},q) \in \mathbb{N}\times\mathbb{Z}^d\times\mathbb{Z}, \quad \sup\left\{ \big(\alpha + \|\mathbf{u}\|_1\big)^n \,|\partial^{\mathbf{p}}_{\mathbf{u}}\partial^q_v \Psi^{\epsilon}(\mathbf{u},v)| : \mathbf{u}\in\mathbb{R}^d, v\in[a,b]\right\} < +\infty\,, \tag{4.4.35}
$$

where α is an abritrary positive real number. Next, combining (4.4.28), (4.4.33) and (4.4.35), one obtains that

$$R_3(u,u_0,v) \le \kappa_3|u-u_0| \sum_{j=1}^{j_2} \sum_{(\epsilon,\mathbf{k})\in\mathcal{D}_j^{co}(u_0)} 2^{j(1-v)} \frac{|I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})|}{\big(2d+1+\|2^j\mathbf{u}_0^*-\mathbf{k}+\lambda_{j,\mathbf{k}}^{\epsilon}(u,u_0)2^j(\mathbf{u}^*-\mathbf{u}_0^*)\|_1\big)^L} \tag{4.4.36}$$

where κ_3 denotes a positive finite and deterministic constant. Observe that using the triangle inequality, (4.4.17) and the fact that $\lambda_{j,\mathbf{k}}^{\epsilon}(u,u_0)\in(0,1)$ one has, for all $j\in\{1,...,j_1\}$ and $(\epsilon,\mathbf{k})\in\mathcal{D}_j^{co}(u_0)$, that

$$\begin{aligned}&2d+1+\|2^j\mathbf{u}_0^*-\mathbf{k}+\lambda_{j,\mathbf{k}}^{\epsilon}(u,u_0)2^j(\mathbf{u}^*-\mathbf{u}_0^*)\|_1\\ &\ge \|2^j\mathbf{u}_0^*-\mathbf{k}\|_1-\lambda_{j,\mathbf{k}}^{\epsilon}(u,u_0)\,d\,2^j|u-u_0|+2d+1\\ &\ge 1+\|2^j\mathbf{u}_0^*-\mathbf{k}\|_1.\end{aligned} \tag{4.4.37}$$

Also observe that using the triangle inequality, the fact that $\lambda_{j,\mathbf{k}}^{\epsilon}(u,u_0)\in(0,1)$, (4.4.17), (4.4.23) and (4.4.20), one obtains, for all $j\in\{j_1+1,...,j_2\}$ and all $(\epsilon,\mathbf{k})\in\mathcal{D}_j^{co}(u_0)$, that

$$\begin{aligned}&2d+1+\|2^j\mathbf{u}_0^*-\mathbf{k}+\lambda_{j,\mathbf{k}}^{\epsilon}(u,u_0)2^j(\mathbf{u}^*-\mathbf{u}_0^*)\|_1\\ &\ge \|2^j\mathbf{u}_0^*-\mathbf{k}\|_1-\lambda_{j,\mathbf{k}}^{\epsilon}(u,u_0)\,d\,2^j|u-u_0|+2d+1\\ &\ge \frac{1}{2}\|2^j\mathbf{u}_0^*-\mathbf{k}\|_1+\frac{1}{2}\|2^j\mathbf{u}_0^*-\mathbf{k}\|_1-d\,2^{j-j_1+1}+2d+1\\ &\ge \frac{1}{2}\|2^j\mathbf{u}_0^*-\mathbf{k}\|_1+\frac{d}{2}j^{\frac{d}{a}}-d\,2^{j-j_1+1}+2d+1\\ &\ge \frac{1}{2}(1+\|2^j\mathbf{u}_0^*-\mathbf{k}\|_1).\end{aligned} \tag{4.4.38}$$

Thus, combining (4.4.37), (4.4.38) and (4.4.36), one gets that

$$R_3(u,u_0,v) \le 2\kappa_3|u-u_0| \sum_{j=1}^{j_2} \sum_{(\epsilon,\mathbf{k})\in\mathcal{D}_j^{co}(u_0)} 2^{j(1-v)} \frac{|I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})|}{\big(1+\|2^j\mathbf{u}_0^*-\mathbf{k}\|_1\big)^L}\,. \tag{4.4.39}$$

Next, using (4.4.39), Lemma 4.2.1, the inequalities

$$\begin{aligned}\log\big(e+j+\|\mathbf{k}\|_1\big) &\le \log\big(e+j+\|2^j\mathbf{u}_0^*\|_1+\|2^j\mathbf{u}_0^*-\mathbf{k}\|_1\big)\\ &\le \log\big(e+j+\|2^j\mathbf{u}_0^*\|_1\big)\prod_{l=1}^{d}\log\big(e+|2^ju_0-k_l|\big)\,,\end{aligned}$$

$$\big(1+\|2^j\mathbf{u}_0^*-\mathbf{k}\|_1\big)^L \ge \prod_{l=1}^{d}\big(1+|2^ju_0-k_l|\big)^{\frac{L}{d}}\,,$$

(4.4.24), and (4.4.30) with $\mathbb{Z}$ in place of $\mathbb{Z}^d$, it turns out that, on the event of probability 1 Ω^*, one has

$$R_3(u,u_0,v) \le K_3'|u-u_0| \sum_{j=1}^{j_2} j^{\frac{d}{2}}\,2^{j(1-v)} \sum_{k\in\mathcal{D}_{1,j}^{co}(u_0)} \frac{\big(\log(e+|2^ju_0-k|)\big)^{\frac{d}{2}}}{\big(1+|2^ju_0-k|\big)^{\frac{L}{d}}}\,, \tag{4.4.40}$$

where K'_3 is a positive finite random variable not depending on (u,v), and where

$$\mathcal{D}^{co}_{1,j}(u_0) = \left\{k \in \mathbb{Z} : |2^j u_0 - k| > j^{\frac{d}{a}}\right\}. \tag{4.4.41}$$

Next, let us assume that η is an arbitrarily small fixed positive real number. using (4.4.41), and the fact that $x \mapsto \log(e+x)$ and $x \mapsto x$ are increasing functions over $\mathbb{R}_+$, one gets that

$$\sum_{k\in\mathcal{D}^{co}_{1,j}(u_0)} \frac{\big(\log(e+|2^j u_0 - k|)\big)^{\frac{d}{2}}}{\big(1+|2^j u_0 - k|\big)^{\frac{L}{d}}} \le 2\int_{j^{\frac{d}{a}}}^{+\infty} \frac{\big(\log(e+1+x)\big)^{\frac{d}{2}}}{x^{\frac{L}{d}}}\,dx \le \kappa'_3\, j^{-(\frac{L-d}{a}-\eta)}, \tag{4.4.42}$$

where κ'_3 is a positive finite deterministic constant not depending on j. Moreover the assumption that L is an arbitrarily large integer allows to assume that

$$\frac{L-d}{a} - \eta - \frac{d}{2} > \frac{d}{2a} > 0. \tag{4.4.43}$$

Thus, using the fact that $v \in [a,b] \subset (0,1)$, one obtains that

$$\begin{aligned}
\sum_{j=1}^{j_2} j^{-(\frac{L-d}{a}-\eta-\frac{d}{2})}\, 2^{j(1-v)} &\le \sum_{j=1}^{\lfloor j_2/2\rfloor} 2^{j(1-v)} + \big(j_2/2\big)^{-(\frac{L-d}{a}-\eta-\frac{d}{2})} \sum_{\lfloor j_2/2\rfloor+1}^{j_2} 2^{j(1-v)} \\
&\le 4\big(2^{1-b}-1\big)^{-1}\Big(2^{j_2(1-v)/2} + \big(j_2/2\big)^{-(\frac{L-d}{a}-\eta-\frac{d}{2})}\, 2^{j_2(1-v)}\Big) \\
&\le \kappa''_3\, j_2^{-(\frac{L-d}{a}-\eta-\frac{d}{2})}\, 2^{j_2(1-v)},
\end{aligned} \tag{4.4.44}$$

where the finite deterministic constant

$$\kappa''_3 = 4\big(2^{1-b}-1\big)^{-1}\Big(2^{(\frac{L-d}{a}-\eta-\frac{d}{2})} + \sup_{n\in\mathbb{N}}\big\{2^{-n(1-b)/2}\, n^{(\frac{L-d}{a}-\eta-\frac{d}{2})}\big\}\Big).$$

Moreover, one can derive from (4.4.18), (4.4.19) and (4.4.43) that

$$\begin{aligned}
j_2^{-(\frac{L-d}{a}-\eta-\frac{d}{2})}\, 2^{j_2(1-v)} &\le 4|u-u_0|^{v-1}\, j_2^{-(\frac{L-d}{a}-\eta-\frac{d}{2})}\, 2^{d\log(j_1)/(2a\log 2)} \\
&\le 4|u-u_0|^{v-1}\, j_2^{-(\frac{L-d}{a}-\eta-\frac{d}{2}-\frac{d}{2a})} \le 4|u-u_0|^{v-1}.
\end{aligned} \tag{4.4.45}$$

Next, putting together (4.4.40), (4.4.42), (4.4.44) and (4.4.45), it turns out that, on the event of probability 1 Ω^*, one has

$$R_3(u,u_0,v) \le K_3|u-u_0|^{v}, \tag{4.4.46}$$

where K_3 is a positive finite random variable not depending on (u,v).

Finally, combining (4.4.25), (4.4.31), (4.4.32), (4.4.34) and (4.4.46), one obtains the theorem. □

Proof of Corollary 4.4.2. Using (4.1.7) and the triangle inequality, one gets that

$$\sup_{t\in[t_0-1,t_0+1]}\left\{\frac{|Z(t)-Z(t_0)|}{|t-t_0|^{H(t_0)}\big(\log(e+\big|\log|t-t_0|\big|)\big)^{\frac{d}{2}}}\right\} \le U_1(t_0)+U_2(t_0), \tag{4.4.47}$$

where

$$U_1(t_0) = \sup_{t\in[t_0-1,t_0+1]} \left\{ \frac{|X(t,H(t)) - X(t,H(t_0))|}{|t-t_0|^{H(t_0)}\big(\log(e+\big|\log|t-t_0|\big|\big)^{\frac{d}{2}}} \right\} \tag{4.4.48}$$

and

$$U_2(t_0) = \sup_{t\in[t_0-1,t_0+1]} \left\{ \frac{|X(t,H(t_0)) - X(t_0,H(t_0))|}{|t-t_0|^{H(t_0)}\big(\log(e+\big|\log|t-t_0|\big|\big)^{\frac{d}{2}}} \right\}. \tag{4.4.49}$$

Next, observe that it follows from the assumption (4.4.2) that

$$\sup_{t\in[t_0-1,t_0+1]} \left\{ \frac{|H(t) - H(t_0)|}{|t-t_0|^{H(t_0)}\big(\log(e+\big|\log|t-t_0|\big|\big)^{\frac{d}{2}}} \right\} < +\infty\,. \tag{4.4.50}$$

On the other hand, denoting by $\mathcal{H}$ the compact interval included in $(0,1)$ defined as

$$\mathcal{H} = H([t_0-1,t_0+1]) = \big\{H(t) : t\in[t_0-1,t_0+1]\big\}\,,$$

one clearly has that

$$\sup_{t\in[t_0-1,t_0+1]} \left\{ \frac{|X(t,H(t)) - X(t,H(t_0))|}{|H(t)-H(t_0)|} \right\} \le \sup_{(u,v_1,v_2)\in[t_0-1,t_0+1]\times\mathcal{H}^2} \left\{ \frac{|X(u,v_1) - X(u,v_2)|}{|v_1-v_2|} \right\}. \tag{4.4.51}$$

Thus, combining (4.4.48), (4.4.50) and (4.4.51) with Corollary 4.3.2, one almost surely gets that:

$$U_1(t_0) < +\infty. \tag{4.4.52}$$

On the other hand, it easily follows from (4.4.49) and Theorem 4.4.1 that:

$$U_2(t_0) \le \sup_{(u,v)\in[t_0-1,t_0+1]\times\mathcal{H}} \left\{ \frac{|X(u,v) - X(u_0,v)|}{|u-u_0|^{v}\big(\log(e+\big|\log|u-u_0|\big|\big)^{\frac{d}{2}}} \right\} < +\infty. \tag{4.4.53}$$

Finally, putting together (4.4.52), (4.4.53) and (4.4.47), one obtains (4.4.3). □

Proof of Theorem 4.4.3. It easily follows from (4.4.4) and (4.1.7) that, for every $n\in\mathbb{N}$, the stochastic process $T_{t_0,\nu_n}Z = \big\{(T_{t_0,\nu_n}Z)(s) : s\in\mathbb{R}\big\}$ can be expressed as the sum of the two stochastic processes $T^1_{t_0,\nu_n}X = \big\{(T^1_{t_0,\nu_n}X)(s) : s\in\mathbb{R}\big\}$ and $T^2_{t_0,\nu_n}X = \big\{(T^2_{t_0,\nu_n}X)(s) : s\in\mathbb{R}\big\}$, defined, for all $s\in\mathbb{R}$, as:

$$(T^1_{t_0,\nu_n}X)(s) = \frac{X\big(t_0+\nu_n s, H(t_0)\big) - X\big(t_0, H(t_0)\big)}{\nu_n^{H(t_0)}} \tag{4.4.54}$$

and

$$(T^2_{t_0,\nu_n}X)(s) = \frac{X\big(t_0+\nu_n s, H(t_0+\nu_n s)\big) - X\big(t_0+\nu_n s, H(t_0)\big)}{\nu_n^{H(t_0)}}\,. \tag{4.4.55}$$

Next, using (4.4.54), the stationary increments property of the stochastic process $X(\cdot, H(t_0)) = \big\{X(u, H(t_0)) : u \in \mathbb{R}\big\}$ (see (4.1.1)), its global self-similar property of order $H(t_0)$ (see (4.1.2)), and the equality $X(0, H(t_0)) \overset{a.s.}{=} 0$, one gets that

$$\big\{(T^1_{t_0,\nu_n} X)(s) : s \in \mathbb{R}\big\} \overset{law}{=} \big\{X(s, H(t_0)) : s \in \mathbb{R}\big\}\,.$$

This equatility in the sense of finite-dimensional distributions and the fact that $\big\{(T^1_{t_0,\nu_n} X)(s) : s \in \mathbb{R}\big\}$ and $\big\{X(s, H(t_0)) : s \in \mathbb{R}\big\}$ have continuous paths imply that these two processes induce the same probability distribution on the space of continuous functions $\mathcal{C}(\mathcal{J})$. Thus, in order to derive the theorem, it is enough to show that $T^2_{t_0,\nu_n} X$, viewed as a random variable with values in the space $\mathcal{C}(\mathcal{J})$, converges to 0 in this space, when n goes to $+\infty$. That is

$$\lim_{n\to+\infty} \sup_{s\in\mathcal{J}} \big|(T^2_{t_0,\nu_n} X)(s)\big| \overset{a.s.}{=} 0\,. \tag{4.4.56}$$

There is non rectriction to assume that $\mathcal{J} = [-M, M]$ for some fixed positive real number M, and that $\nu_n \in (0, 1]$, for every $n \in \mathbb{N}$. Let then $\mathcal{I}$ and $\mathcal{H}$ be the compact intervals defined as $\mathcal{I} = [t_0 - M, t_0 + M]$ and $\mathcal{H} = H(\mathcal{I}) = \big\{H(t) : t \in \mathcal{I}\big\}$. It follows from Corollary 4.3.2 that the positive random variable A defined as

$$A = \sup_{(u,v_1,v_2)\in\mathcal{I}\times\mathcal{H}^2} \left\{\frac{|X(u, v_1) - X(u, v_2)|}{|v_1 - v_2|}\right\} \tag{4.4.57}$$

is finite on the event of probability 1 Ω^*. Moreover, one can derive from (4.4.55) and (4.4.57) that on Ω^*, for all $n \in \mathbb{N}$, one has

$$\sup_{s\in\mathcal{J}} \big|(T^2_{t_0,\nu_n} X)(s)\big| \le \nu_n^{-H(t_0)} A \sup_{s\in\mathcal{J}} \Big\{\big|H(t_0 + \nu_n s) - H(t_0)\big|\Big\}\,. \tag{4.4.58}$$

Finally, combining (4.4.2) and (4.4.58) one obtains (4.4.56). □

Proof of Theorem 4.4.5 In view of the fact that on the event of probability 1 Ω^* the fields X^{lf} and X^{hf} are with continuous paths, one can assume without any restriction that $\delta = 2$. Let $\mathcal{H} = [a, b] \subset (0, 1)$ be an arbitrary compact interval and let u be an arbitrary real number such that $|u| \ge 2$. One denotes by j_3 the biggest positive integer satisfying: $|u| \ge 2^{j_3}$. Then, one gets that:

$$2^{j_3} \le |u| < 2^{j_3+1}\,, \tag{4.4.59}$$

which means that

$$j_3 = \left\lfloor \frac{\log|u|}{\log 2} \right\rfloor. \tag{4.4.60}$$

First, one shows that (4.4.5) holds. Recall that, for all $(v, \omega) \in (0, 1) \times \Omega^*$, one has:

$$X^{lf}(u, v, \omega) = \sum_{j=1}^{+\infty} 2^{jv}\big(\Phi_{-j}(2^{-j}\mathbf{u}^*, v, \omega) - \Phi_{-j}(\mathbf{0}, v, \omega)\big)\,, \tag{4.4.61}$$

where $\Phi_{-j}(\cdot,\cdot,\omega)$ is the infinitely differentiable function on $\mathbb{R}^d \times (0,1)$, introduced in (4.2.12). Also, recall that the series in (4.4.61) is uniformly convergent in (u,v) on each compact subset of $\mathbb{R}\times(0,1)$. Using the mean value theorem, one gets, for all $(j,v)\in\mathbb{N}\times\mathcal{H}$, that:

$$|\Phi_{-j}(2^{-j}\mathbf{u}^*,v,\omega)-\Phi_{-j}(\mathbf{0},v,\omega)| \le 2^{-j}|u|\sum_{n=1}^{d}\sup_{(y,v)\in[0\wedge 2^{-j}u,0\vee 2^{-j}u]\times\mathcal{H}}|\partial_{y_n}\Phi_{-j}(\mathbf{y}^*,v,\omega)|.$$

Then, Lemma 4.2.5 entails that:

$$|\Phi_{-j}(2^{-j}\mathbf{u}^*,v,\omega)-\Phi_{-j}(\mathbf{0},v,\omega)| \le C_d(\omega)2^{-j}|u|(\log(e+j+2^{-j}|u|))^{\frac{d}{2}}\,, \tag{4.4.62}$$

where C_d is a positive finite random variable not depending on u. Thus, one can derive from (4.4.62), (4.2.14), (4.4.59) and (4.4.60) that:

$$\begin{aligned}
&\sum_{j=j_3+1}^{+\infty}2^{jv}|\Phi_{-j}(2^{-j}\mathbf{u}^*,v,\omega)-\Phi_{-j}(\mathbf{0},v,\omega)|\\
&\qquad\le C_d(\omega)|u|\sum_{j=j_3+1}^{+\infty}2^{-j(1-v)}(\log(e+1+j))^{\frac{d}{2}}\\
&\qquad\le C_d(\omega)|u|2^{-(j_3+1)(1-v)}\sum_{l=0}^{+\infty}2^{-l(1-v)}(\log(e+2+l+j_3))^{\frac{d}{2}}\\
&\qquad\le C_d'(\omega)2^{j_3v}(\log(e+j_3))^{\frac{d}{2}}\\
&\qquad\le C_d''(\omega)|u|^v(\log(e+\big|\log|u|\big|)^{\frac{d}{2}}\,,
\end{aligned} \tag{4.4.63}$$

where

$$C_d' = 2^b\,C_d\sum_{l=0}^{+\infty}2^{-l(1-b)}(\log(e+2+l))^{\frac{d}{2}} < +\infty\,,$$

and $C_d''=C_d'(\log(e+\frac{1}{\log 2}))^{\frac{d}{2}}$. On the other hand, Lemma 4.2.5, (4.4.59), (4.2.14), and (4.4.60) imply that:

$$\begin{aligned}
&\sum_{j=1}^{j_3}2^{jv}|\Phi_{-j}(2^{-j}\mathbf{u}^*,v,\omega)-\Phi_{-j}(\mathbf{0},v,\omega)|\\
&\qquad\le \tilde{C}_d(\omega)\sum_{j=1}^{j_3}2^{jv}(\log(e+j+2^{-j}|u|))^{\frac{d}{2}}\\
&\qquad= \tilde{C}_d(\omega)2^{(j_3+1)v}\sum_{j=1}^{j_3}2^{-jv}(\log(e+(j_3+1-j)+2^{-(j_3+1-j)}|u|))^{\frac{d}{2}}\\
&\qquad\le \tilde{C}_d'(\omega)|u|^v\sum_{j=1}^{j_3}2^{-jv}(\log(e+j_3+2^j))^{\frac{d}{2}}\\
&\qquad\le \tilde{C}_d''(\omega)|u|^v(\log(e+j_3))^{\frac{d}{2}}\\
&\qquad\le \tilde{C}_d'''(\omega)|u|^v(\log(e+\big|\log|u|\big|)^{\frac{d}{2}}\,,
\end{aligned} \tag{4.4.64}$$

where $\tilde{C}_d$ is a positive finite random variable not depending on u, $\tilde{C}'_d = 2^b \tilde{C}_d$,

$$\tilde{C}''_d = \tilde{C}'_d \sum_{l=1}^{+\infty} 2^{-la} (\log(e + 2^l))^{\frac{d}{2}} \,,$$

and $\tilde{C}'''_d = (\log(e + \frac{1}{\log 2}))^{\frac{d}{2}} \tilde{C}''_d$. Finally, (4.4.63) and (4.4.64) entail that (4.4.5) holds.

Now, let us show that (4.4.6) is satisfied. Recall that, for all $(v, \omega) \in (0,1) \times \Omega^*$, one has:

$$X^{hf}(u, v, \omega) = \sum_{j=0}^{+\infty} 2^{-jv} \big(\Phi_j(2^j \mathbf{u}^*, v, \omega) - \Phi_j(\mathbf{0}, v, \omega)\big) \,, \tag{4.4.65}$$

where $\Phi_j(\cdot, \cdot, \omega)$ is the infinitely differentiable function on $\mathbb{R}^d \times (0,1)$, introduced in (4.2.12). Also, recall that the series in (4.4.65) is uniformly convergent in (u, v) on each compact subset of $\mathbb{R} \times (0,1)$. Next, let us mention that thanks to the convexity property of the function $z \mapsto z^{\frac{d}{2}}$, one has the following inequaty:

$$\forall\, (x, y) \in \mathbb{R}_+^2, \quad (x+y)^{\frac{d}{2}} \le 2^{\frac{d}{2}-1} (x^{\frac{d}{2}} + y^{\frac{d}{2}}). \tag{4.4.66}$$

Also, one mentions that the inequality

$$\forall\, (x, y, z) \in \mathbb{R}_+^3, \quad \log(e + x + yz) \le \log(e + x + y) + \log(e + z) \tag{4.4.67}$$

holds, since $(e + x + yz) \le (e + x + y)(e + z)$ and the logarithm is an increasing function. Using (4.4.65), the triangle inequality, Lemma 4.2.5, (4.4.67) and (4.4.66), one obtains:

$$\begin{aligned}
|X^{hf}(u, v, \omega)| &\le \hat{C}_d(\omega) \sum_{j=0}^{+\infty} 2^{-ja} \big[\log(e + j + 2^j |u|)\big]^{\frac{d}{2}} \\
&\le \hat{C}_d(\omega) \sum_{j=0}^{+\infty} 2^{-ja} \Big[\log(e + |u|) + \log(e + j + 2^j)\Big]^{\frac{d}{2}} \\
&\le \hat{C}'_d(\omega) \sum_{j=0}^{+\infty} 2^{-ja} \Big[\big(\log(e + |u|)\big)^{\frac{d}{2}} + \big(\log(e + j + 2^j)\big)^{\frac{d}{2}}\Big] \\
&\le \hat{C}''_d(\omega) \big(\log(e + |u|)\big)^{\frac{d}{2}} + \hat{C}'''_d(\omega),
\end{aligned} \tag{4.4.68}$$

where $\hat{C}_d$ is a positive finite random variable not depending on u, $\hat{C}'_d = 2^{\frac{d}{2}-1} \hat{C}_d$, $\hat{C}''_d = \frac{2^a}{2^a - 1} \hat{C}'_d$, and

$$\hat{C}'''_d = \hat{C}'_d \sum_{j=0}^{+\infty} 2^{-ja} \big(\log(e + j + 2^j)\big)^{\frac{d}{2}} < +\infty \,.$$

It easily results from (4.4.68) that (4.4.6) holds. □

Proof of Corollary 4.4.6. The corollary is a straightforward consequence of Theorem 4.4.5 and (4.1.7). □

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