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Universalité pour les permutations aléatoires

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[1] À l'exception de la période de la COVID-19
[2] C'est le jour de la fête nationale en France mais c'est du pur hasard

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[3] J'espère qu'ils ne me détestent pas à cause de ça.
[4] Région oblige
[5] But please without Friday's rule

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Résumé

On présente dans cette thèse des techniques de preuve d'universalité pour les permutations aléatoires. La principale méthode utilise une marche aléatoire sur le groupe symétrique. Cette technique nous permet de généraliser plusieurs résultats de convergence connus pour le cas uniforme, entre autres, le résultat de Baik, Deift et Johansson sur les fluctuations de la longueur de la plus longue sous-suite croissante. Cette technique n'est pas spécifique aux permutations aléatoires. On présente ainsi une généralisation à d'autres groupes.

Une deuxième partie de la thèse est consacrée à l'utilisation de la méthode des moments ; on étudie la structure en cycle de produits de permutations indépendantes ayant une loi stable sous conjugaison. On montre qu'un simple contrôle des points fixes et des cycles de longueur 2 garantit une universalité pour les lois jointes des petits cycles du produit.

Mots-clés : Permutations alétoires, marches aléatoires, universalité, distribution de Tracy-Widom.

Abstract

We present, in this work, universality techniques for random permutations. The main method uses a random walk on the symmetric group. This technique allows us to generalize several results of known convergences for the uniform case. We generalize for example the result of Baik, Deift and Johansson on the fluctuations of the length of the longest increasing subsequence. This technique is not specific to random permutations. We present then a generalization to other groups.

Using the method of moments, we study the cycle structure of the product of two independent conjugation invariant random permutations. We show that a simple control of fixed points and cycles of length 2 guarantees universality for the joint distribution of the small cycles of the product of the two permutations.

Keywords: Random permutations, random walks, universality, Tracy-Widom distribution.

Notations and abbreviations

- Sets:
  - $\mathbb{N}^*$: The set of positive natural integers (p.12).
  - $\mathbb{Z}$: The set of integers (p.24).
  - $\mathbb{R}$: The set of real numbers (p.24).
  - $\mathbb{R}^d$: The set of $(x_1, x_2, \dots, x_d)$ such that $x_1, \dots, x_d$ are real numbers (p.24).
  - $\mathscr{C}^0(\mathbb{R})$ the set of continuous real functions (p.24).
  - $\mathfrak{S}_n$: The symmetric group of size n (p.12).
  - $\mathfrak{S}_n^0$: The set of permutation of $\mathfrak{S}_n$ of order n (p.24).
  - $\mathfrak{S}_\infty := \cup_{n\geq 1}\mathfrak{S}_n$ (p.24).
  - $\mathbb{Y}_n$: The set of integer partitions of size n (p.42).
  - $\mathbb{Y} := \cup_{n\geq 0}\mathbb{Y}_n$ (p.42).
- Functions:
  - $\mathrm{card}(A)$: The cardinal of the finite set A (p.33).
  - $\#(\sigma)$: The total number of cycles of σ (p.13).
  - $\#_k(\sigma)$: The number of cycles of σ of length k (p.15).
  - $\mathrm{tr}(\sigma)$: The number of fixed points of σ (p.13).
  - $\hat{\lambda}(\sigma)$: The cycle structure of σ (p.42).
  - $\lambda(\sigma)$: The shape of the RSK image of λ (p.46).
  - λ': The transpose of the partition λ (p.42).
  - $|\lambda|$: The size of the partition λ (p.42).
  - $\ell(\lambda)$: The length of the partition λ (p.43).

- $\mathrm{LIS}(\sigma)$: The length of the longest increasing subsequence of σ (p.14).
- $\mathrm{LDS}(\sigma)$: The length of the longest decreasing subsequence of σ (p.14).
- $\mathrm{LCS}(\sigma, \rho)$: The length of the longest common subsequence between σ and ρ (p.15).
- $\mathrm{LICS}(\sigma)$: The length of the longest increasing circular subsequence of σ (p.31).
- $\mathrm{LDCS}(\sigma)$: The length of the longest decreasing circular subsequence of σ (p.31).
- $\mathrm{LAS}(\sigma)$: The length of the longest alternating subsequence of σ (p.32).
- $\mathcal{N}_D(\sigma)$: The number of peaks of σ (p.34).
- $\mathcal{N}_{exc_j}(\sigma)$: The number of j-exceedances of σ (p.34).
- $\mathcal{N}_{D_j}(\sigma)$: The number of j-descents of σ (p.35).
- $\mathcal{N}_D(\sigma)$: The number of descents of σ (p.35).
- $\mathcal{K}_j$ the number of j-clicks of graph of the permutation σ (p.35).
- $D(\sigma)$: The set of descents of σ (p.35).
- F_2: The CDF of the Tracy-Widom distribution (p.14).
- $F_{2,k}$: The CDF of the top k right particles of the Airy ensemble (p.69).
- K_{sin}: The kernel of the sine process (p.68).
- K_{Airy}: The kernel of the Airy ensemble (p.68).
- $K_{sin,\alpha}$: The kernel of the discrete sine process (p.69).

- Other symbols and random variables:
  - $\mathcal{H}_{inv}$, $\mathcal{H}^{\mathbb{P}}_{inv,\alpha}$, $\mathcal{H}^{\mathbb{L}^p}_{inv,\alpha}$, $\mathcal{H}^{\mathrm{tr},p}_{inv,\alpha}$: Classes of sequences of conjugation invariant random permutations (p.13).
  - $\stackrel{d}{=}$: Equality in distribution (p.13).
  - $\xrightarrow[n\to\infty]{\mathbb{P}}$: Convergence in probability (p.13).
  - $\xrightarrow[n\to\infty]{d}$: Convergence in distribution (p.25).
  - $\xrightarrow[n\to\infty]{\mathbb{L}^p}$: Convergence in $\mathbb{L}^p$ (p.13).
  - $\sigma_{unif,n}$: Uniform random permutation on $\mathfrak{S}_n$ (p.14).
  - $\sigma_{Ew,\theta,n}$: Ewens random permutation on $\mathfrak{S}_n$ with parameter θ (p.119).
  - $\sigma_{Kig,\mu} = (\sigma_{Kig,\mu,n})_{n\geq 1}$: Kingman virtual permutation with parameter μ (p.124).
- Abbreviations:
  - TW: Tracy-Widom (p.14).
  - KPZ: Kardar–Parisi–Zhang (p.69).

- GUE: Gaussian Unitary Ensemble (p.14).
- GOE: Gaussian Orthogonal Ensemble (p.14).
- GSE: Gaussian Symplectic Ensemble (p.14).
- DPP: Determinantal Point Process (p.68).
- RSK: Robinson–Schensted–Knuth (p.41).
- CDF: Cumulative Distribution Function (p.14).
- VKLS: Vershik-Kerov-Logan-Shepp (p.99).

Résumé substantiel

Durant le XXe siècle, les théorèmes limites en théorie des probabilités étaient principalement liés à l'indépendance ou à la faible dépendance. Par conséquent, la distribution gaussienne apparaît comme limite de nombreux modèles. À la fin du XXe siècle et au début du XXIe siècle, de grands progrès ont été réalisés dans la compréhension des modèles à forte dépendance. Plusieurs exemples intéressants comme les modèles de croissance, les matrices aléatoires, les systèmes de particules en interaction et les permutations aléatoires ont été étudiés. Pour ces modèles, on remarque l'apparition, avec le bon scaling, des mêmes loi limites (Tracy-Widom, processus sinus, loi semi-circulaire, processus d'Airy). Néanmoins, nous n'avons pas une image globale de cette classe d'universalité. C'est pourquoi, de nombreux mathématiciens s'intéressent aujourd'hui à la compréhension de cette classe.

Dans cette thèse, on s'intéresse, en particulier, aux phénomènes d'universalité pour des permutations aléatoires. Dans la littérature, l'étude de la plupart des fonctions de permutations n'est effectuée que pour certaines lois, en particulier, pour la loi uniforme, certaines lois de Mallows, les lois d'Ewens, etc. Dans cette thèse, on essaye de proposer des résultats d'universalité pour les permutations ayant une loi stable sous conjugaison. On montre entre-autre, sous un contrôle du nombre total de cycles, que des fluctuations de type Tracy-Widom apparaissent pour la longueur de la plus longue sous-suite croissante et de la plus longue sous-suite commune de deux copies indépendantes et identiquement distribuées (i.i.d.). Sous un contrôle plus faible, on obtient la forme limite de Vershik-Kerov-Logan-Shepp. Les mêmes techniques de preuves donnent une réponse pour d'autres fonctions et aussi sont utiles prouver des phénomènes d'universalité à d'autres groupes. On présente dans la suite nos principaux résultats.

Plus longue sous-suite croissante

On note dans la suite par $\mathfrak{S}_n$ le groupe des permutations de $\{1, \ldots, n\}$. Pour $\sigma \in \mathfrak{S}_n$, une sous-suite $(\sigma(i_1), \ldots, \sigma(i_k))$ de σ de longueur k est dite une sous-suite croissante (resp. décroissante) si $i_1 < \cdots < i_k$ et $\sigma(i_1) < \cdots < \sigma(i_k)$ (resp. $\sigma(i_1) > \cdots > \sigma(i_k)$). Soit $\mathrm{LIS}(\sigma)$ (resp. $\mathrm{LDS}(\sigma)$) la longueur de la plus longue sous-suite croissante (resp. décroissante) de σ. Par exemple,

$$\text{pour} \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}, \ \mathrm{LIS}(\sigma) = 2 \ \text{et} \ \mathrm{LDS}(\sigma) = 4.$$

L'étude du comportement asymptotique de la plus longue sous-suite croissante d'une permutation uniforme est connue sous le nom de problème d'Ulam. Ulam (1961) a conjecturé que la limite de

$$\frac{\mathbb{E}(\mathrm{LIS}(\sigma_{unif,n}))}{\sqrt{n}}$$

existe. Ici, $\sigma_{unif,n}$ est une permutation choisie uniformément parmi les permutations de $\mathfrak{S}_n$. En utilisant un argument de sous-additivité, Hammersley (1972) a résolu cette conjecture. En utilisant la théorie des représentations, Vershik et Kerov (1977) et Logan et Shepp (1977) ont montré que cette limite est égale à 2. On trouve dans la littérature une preuve alternative probabiliste dans (Aldous and Diaconis, 1995). Cette preuve reprend l'argument de Hammersley (1972). Les fluctuations ont été étudiées par Baik, Deift et Johansson qui ont montré le résultat suivant.

Théorème. (Baik, Deift, and Johansson, 1999) Pour tout $s \in \mathbb{R}$,

$$\mathbb{P}\left(\frac{\mathrm{LIS}(\sigma_{unif,n}) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s \right) \xrightarrow[n\to\infty]{} F_2(s),$$

où F_2 est la fonction de répartition de la distribution de Tracy-Widom (pour $\beta = 2$).

Pour plus de détails historiques et pour les preuves de ces résultats, on recommande (Romik, 2015). À part le cas uniforme, Mueller et Starr (2013) ont étudié le cas de la loi de Mallows. Le cas des involutions aléatoires est étudié par Baik et Rains (2001) qui ont montré que la loi limite dépend du nombre de points fixes et dans certains régimes, les lois de Tracy-Widom, pour $\beta = 1$ et $\beta = 4$, apparaissent. Ils ont également montré l'apparition d'une famille de lois qui interpolent entre ces deux lois. Mueller et Starr ont montré que pour la loi de Mallows avec la distance de Kendall tau, il y a une transition de phase entre un régime gaussien et un régime de Tracy-Widom.

Dans cette thèse, on présente des résultats d'universalité pour les permutations aléatoires stables sous conjugaison. Dans la suite, on considère une suite de permutations aléatoires $(\sigma_n)_{n\geq 1}$ tels que σ_n est supportée dans $\mathfrak{S}_n$ pour tout $n \geq 1$. On dit qu'elle est stable sous conjugaison si pour tout entier strictement positif $n \geq 1$ et pour tout $\sigma \in \mathfrak{S}_n$, on a $\sigma^{-1}\sigma_n\sigma \stackrel{d}{=} \sigma_n$.

Définition. En notant $\#(\sigma)$ le nombre de cycles de σ et $\mathrm{tr}(\sigma)$ le nombre de ses points fixes, pour $\alpha \geq 1$ et $p \in [1, \infty]$, on dit que la suite $(\sigma_n)_{n\geq 1}$ satisfait $\mathcal{H}^{\mathbb{P}}_{inv,\alpha}$ si

$$(\mathcal{H}^{\mathbb{P}}_{inv,\alpha}) \qquad (\sigma_n)_{n\geq 1} \text{ est stable sous conjugaison et } \quad \frac{\#(\sigma_n)}{n^{\frac{1}{\alpha}}} \xrightarrow[n\to\infty]{\mathbb{P}} 0,$$

qu'elle satisfait $\mathcal{H}^{\mathbb{L}^p}_{inv,\alpha}$ si

$$(\mathcal{H}^{\mathbb{L}^p}_{inv,\alpha}) \qquad (\sigma_n)_{n\geq 1} \text{ est stable sous conjugaison et } \quad \frac{\#(\sigma_n)}{n^{\frac{1}{\alpha}}} \xrightarrow[n\to\infty]{\mathbb{L}^p} 0,$$

et qu'elle satisfait $\mathcal{H}^{\mathrm{tr},p}_{inv,\alpha}$ si

$$(\mathcal{H}^{\mathrm{tr},p}_{inv,\alpha}) \qquad (\sigma_n)_{n\geq 1} \text{ est stable sous conjugaison et } \quad \frac{\mathrm{tr}(\sigma_n)}{n^{\frac{1}{\alpha}}} \xrightarrow[n\to\infty]{\mathbb{L}^p} 0.$$

En particulier, ($\mathcal{H}^{\mathrm{tr},1}_{inv,\alpha}$) est équivalent à

$$(\sigma_n)_{n\geq 1} \text{ est stable sous conjugaison et } \quad \mathbb{P}(\sigma_n(1)=1)n^{\frac{\alpha-1}{\alpha}} \xrightarrow[n\to\infty]{} 0.$$

On montre alors le résultat suivant.

Théorème. Sous ($\mathcal{H}^{\mathbb{P}}_{inv,2}$),

$$\frac{\mathrm{LIS}(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} 2 \quad \text{and} \quad \frac{\mathrm{LDS}(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} 2.$$

De plus, pour tout $p \in [1,\infty)$, sous ($\mathcal{H}^{\mathbb{L}^p}_{inv,2}$),

$$\frac{\mathrm{LIS}(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{L}^p} 2 \quad \text{and} \quad \frac{\mathrm{LDS}(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{L}^p} 2.$$

La convergence en probabilité est évoquée sans preuve dans (Kammoun, 2018) vu que celle-ci est similaire à celle de (Kammoun, 2018, Theorem 1.2). On détaillera la preuve dans la Sous-section 2.2.1. Pour les fluctuations, on a le résultat suivant.

Théorème. Si $(\sigma_n)_{n\geq 1}$ est stable sous conjugaison et

$$\frac{1}{n^{\frac{1}{6}}} \min_{1\leq i\leq n} \left(\left(\sum_{j=1}^{i} \#_j(\sigma_n) \right) + \frac{\sqrt{n}}{i} \sum_{j=i+1}^{n} \#_j(\sigma_n) \right) \xrightarrow[n\to\infty]{\mathbb{P}} 0,$$

alors pour tout $s \in \mathbb{R}$,

$$(\mathrm{TW}) \quad \mathbb{P}\left(\frac{\mathrm{LIS}(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s \right) \xrightarrow[n\to\infty]{} F_2(s) \text{ et } \mathbb{P}\left(\frac{\mathrm{LDS}(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s \right) \xrightarrow[n\to\infty]{} F_2(s).$$

Ce résultat généralise un résultat prouvé dans (Kammoun, 2018, Theorem 1.2) qui est le suivant.

Corollaire. ($\mathcal{H}^{\mathbb{P}}_{inv,6}$) implique (TW).

L'idée de la preuve, dans la Sous-section 5.2.1, est de construire un couplage entre toute distribution satisfaisant ces hypothèses et une permutation uniforme afin d'utiliser le résultat de Baik, Deift et Johansson.

Plus longue sous-suite commune

Étant donné $\sigma \in \mathfrak{S}_n$, $(\sigma(i_1), \ldots, \sigma(i_k))$ est une sous-suite de σ de longueur k si $i_1 < i_2 < \cdots < i_k$. On note $\mathrm{LCS}(\sigma_1, \sigma_2)$ la longueur de la plus longue sous-suite commune (LCS) de deux permutations. La plus longue sous-suite commune de mots a d'abord été étudiée dans le contexte de l'informatique théorique (Maier, 1978). L'une des utilisations naturelles est de comprendre les modifications dans les différentes versions du même ficher (Masek and Paterson, 1980). Dans la suite, on suppose que σ_n et ρ_n sont indépendantes et supportées dans $\mathfrak{S}_n$. L'étude du cas de permutations aléatoires indépendantes a été initiée par Houdré and Işlak (2014). Récemment, Houdré et Xu (2018) ont montré que pour les permutations aléatoires i.i.d.

$$\mathbb{E}(\mathrm{LCS}(\sigma_n, \rho_n)) \geq n^{\frac{1}{3}}.$$

Il est conjecturé par Bukh and Zhou (2016) que pour les permutations aléatoires i.i.d.,

$$\mathbb{E}(\mathrm{LCS}(\sigma_n, \rho_n)) \geq \sqrt{n}.$$

Avant de donner une solution partielle de cette conjecture, on donne une borne inférieure asymptotique pour la plus longue sous-suite commune de deux permutations aléatoires indépendantes (pas forcément de même loi).

Théorème. On suppose que $n \geq 1$, σ_n et ρ_n sont indépendantes et leurs lois sont stables sous conjugaison. On a alors

$$\liminf_{n\to\infty} \frac{\mathbb{E}(\mathrm{LCS}(\sigma_n, \rho_n))}{\sqrt{n}} \geq 2\sqrt{13} - 6 \simeq 1.21\ldots.$$

Cette borne améliore la borne dans (Kammoun, 2020) qui est la suivante.

(1) $$\liminf_{n\to\infty} \frac{\mathbb{E}(\mathrm{LCS}(\sigma_n, \rho_n))}{\sqrt{n}} \geq 2\sqrt{\theta} \simeq 0.564\ldots,$$

où θ est l'unique solution de $G(2\sqrt{x}) = \frac{2+x}{12}$,

(2) $$\begin{aligned} G := [0,2] &\to \left[0, \frac{1}{2}\right] \\ x &\mapsto \int_{-1}^{1} \left(\Omega(s) - \left|s + \frac{x}{2}\right| - \frac{x}{2}\right)_+ \mathrm{d}s, \end{aligned}$$

et

$$\Omega(s) := \begin{cases} \frac{2}{\pi}(s\arcsin(s) + \sqrt{1-s^2}) & \text{si } |s| < 1 \\ |s| & \text{si } |s| \geq 1 \end{cases}.$$

Sous un contrôle de nombre de points fixes, on obtient le résultat suivant.

Proposition. On suppose que pour tout $n \geq 1$, σ_n et ρ_n sont indépendantes et stables sous conjugaison

- Si $(\sigma_n)_{n\geq 1}$ et $(\rho_n)_{n\geq 1}$ vérifient ($\mathcal{H}^{\mathrm{tr},1}_{inv,1}$), alors

$$\liminf_{n\to\infty} \frac{\mathbb{E}(\mathrm{LCS}(\sigma_n,\rho_n))}{\sqrt{n}} \geq 2. \tag{3}$$

- Si $\liminf_{n\to\infty} \sqrt{n}\mathbb{P}(\sigma_n(1)=1)\mathbb{P}(\rho_n(1)=1) \geq \alpha$ pour un certain $0<\alpha\leq 2$, alors

$$\liminf_{n\to\infty} \frac{\mathbb{E}(\mathrm{LCS}(\sigma_n,\rho_n))}{\sqrt{n}} \geq \alpha. \tag{4}$$

Ce qui donne une réponse partielle à la conjecture de départ.

Corollaire. On suppose que pour tout $n \geq 1$, σ_n et ρ_n sont i.i.d. et stables sous conjugaison. On a alors

$$\liminf_{n\to\infty} \frac{\mathbb{E}(\mathrm{LCS}(\sigma_n,\rho_n))}{\sqrt{n}} \geq 2.$$

En particulier, il existe $n_0 \in \mathbb{N}$ tels que pour tout $n > n_0$ et pour tout couple de permutations aléatoires i.i.d. et stables sous conjugaison supportées dans $\mathfrak{S}_n$, on a

$$\mathbb{E}(\mathrm{LCS}(\sigma_n,\rho_n)) \geq \sqrt{n}.$$

On présente la preuve de ces deux derniers résultats dans la Sous-section 7.4.2.

Quand ρ_n n'est pas stable sous conjugaison, on peut quand même donner une borne inférieure de $\frac{\mathbb{E}(\mathrm{LCS}(\sigma_n,\rho_n))}{\sqrt{n}}$.

Théorème. Supposons que pour tout $n \geq 1$, σ_n et ρ_n sont indépendantes et σ_n est stable sous conjugaison. Alors

$$\liminf_{n\to\infty} \frac{\mathbb{E}(\mathrm{LCS}(\sigma_n,\rho_n))}{\sqrt{n}} \geq G^{-1}\left(\liminf_{n\to\infty} \frac{\mathbb{E}(\#(\sigma_n))}{2n}\right).$$

En particulier, si $(\sigma_n)_{n\geq 1}$ vérifie ($\mathcal{H}^{\mathbb{P}}_{inv,1}$), alors

$$\liminf_{n\to\infty} \frac{\mathbb{E}\left(\mathrm{LCS}(\sigma_n,\rho_n)\right)}{\sqrt{n}} \geq 2.$$

Sous un contrôle additionnel, on obtient un comportement similaire au cas uniforme.

Proposition. Supposons que pour tout $n \geq 1$, σ_n et ρ_n sont indépendantes.

- Si $(\sigma_n)_{n\geq 1}$ vérifie $(\mathcal{H}^{\mathbb{P}}_{inv,2})$ alors

$$\frac{\mathrm{LCS}(\sigma_n,\rho_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} 2.$$

- Si $(\sigma_n)_{n\geq 1}$ vérifie $(\mathcal{H}^{\mathbb{L}^1}_{inv,2})$ alors

$$\frac{\mathrm{LCS}(\sigma_n,\rho_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{L}^1} 2.$$

- Si $(\sigma_n)_{n\geq 1}$ vérifie $(\mathcal{H}^{\mathbb{P}}_{inv,6})$ alors $\forall s\in\mathbb{R}$,

$$\mathbb{P}\left(\frac{\mathrm{LCS}(\sigma_n,\rho_n)-2\sqrt{n}}{n^{\frac{1}{6}}}\leq s\right) \xrightarrow[n\to\infty]{} F_2(s).$$

On prouve ce résultat dans la Sous-section 7.4.3.

Petits cycles

La structure en cycles d'une permutation du groupe symétrique $\mathfrak{S}_n$ choisie uniformément est bien comprise (voir (Arratia, Barbour, and Tavaré, 2003) pour des résultats détaillés). En particulier, le résultat classique suivant est valable :

Théorème. (Arratia, Barbour, and Tavaré, 2000, Théorème 3.1) Pour tout $k\geq 1$,

$$(\#_1(\sigma_{unif,n}),\dots,\#_k(\sigma_{unif,n})) \xrightarrow[n\to\infty]{d} (\xi_1,\xi_2,\dots,\xi_k), \tag{5}$$

où $\#_k$ est le nombre de cycles de longueur k, $\xrightarrow[n\to\infty]{d}$ désigne la convergence en loi, $\xi_1,\xi_2,\dots,\xi_k$ sont indépendants et la loi de ξ_d est Poisson de paramètre $\frac{1}{d}$.

En effet, dans (Arratia, Barbour, and Tavaré, 2000), il y a un contrôle de la convergence lorsque n,k vont tous les deux vers l'infini tel que $\frac{k}{n}\to 0$. Mais nous limiterons notre étude au cas où k est fixe. Dans ce travail conjoint avec Mylène Maïda, nous nous sommes interrogés sur la classe d'universalité de cette convergence. Nous montrons qu'un produit de permutations stables sous conjugaison qui n'ont pas trop de points fixes ni de cycles de taille 2 appartient à cette classe. Plus précisément, nous avons le résultat suivant.

Théorème. (Kammoun and Maïda, 2020) Soit $m\geq 2$. Pour $1\leq \ell\leq m$, soit $(\sigma_{\ell,n})_{n\geq 1}$ une suite de permutations aléatoires telle que pour tout $n\geq 1$, $\sigma_{\ell,n}\in\mathfrak{S}_n$. Pour tout $k\geq 1$, soit $t_k^n := \#_k(\prod_{\ell=1}^m \sigma_{\ell,n})$. Supposons que

(H_1) Pour tout $n\geq 1$, $(\sigma_{1,n},\dots,\sigma_{m,n})$ sont indépendantes.

(H_2) Pour tout $n\geq 1$ et $1\leq\ell\leq m$, $\sigma_{\ell,n}$ est stable sous conjugaison sauf peut-être pour un $\ell\in\{1,\dots,m\}$.

- Il existe $1 \leq i < j \leq m$ tel que pour tout $k \geq 1$,

(H_3) $\sigma_{i,n}, \sigma_{j,n}$ vérifient $(\mathcal{H}^{\mathrm{tr},k}_{inv,2})$,

(H_4) $\sigma^2_{i,n}, \sigma^2_{j,n}$ vérifient $(\mathcal{H}^{\mathrm{tr},1}_{inv,1})$.

Alors pour tout $k \geq 1$,

$$(t^n_1, t^n_2, \ldots, t^n_k) \xrightarrow[n\to\infty]{d} (\xi_1, \xi_2, \ldots, \xi_k). \tag{6}$$

Mukherjee (2016) a obtenu des résultats similaires pour des permutations équicontinues et convergentes en permuton. Dans le cas de deux permutations, une convergence plus forte (en variation totale) a été établie par Chmutov et Pittel (2016) lorsque l'une des permutations a tous ses cycles de longueur au moins 3 (voir aussi (Gamburd, 2006)). Aucun de ces résultats ne couvre par exemple le produit de deux distributions d'Ewens. Il est facile de vérifier qu'ils satisfont les convergences requises dans (H_3) et (H_4) et notre résultat indique par conséquent que les petits cycles d'un produit de (au moins deux) permutations d'Ewens se comportent comme ceux d'une permutation uniforme. Dans notre cadre, dans le cas de deux permutations, un résultat plus faible peut être obtenu sans aucune hypothèse sur les cycles de taille 2. On présente une preuve de ce résultat dans le Chapitre 6.

Proposition. (Kammoun and Maïda, 2020) Pour $m = 2$. Supposons que

(H_1) Pour tout $n \geq 1$, σ_n et ρ_n sont indépendantes.

(H_2) Pour tout $n \geq 1$, σ_n ou ρ_n est stable sous conjugaison.

(H_3) Pour tout $k \geq 1$, σ_n et ρ_n vérifient $(\mathcal{H}^{\mathrm{tr},k}_{inv,2})$.

Alors pour tout $v \geq 1$,

$$\lim_{n\to\infty} \mathbb{E}(t^n_v) = \frac{1}{v}.$$

Notre motivation pour comprendre la structure en cycles des permutations aléatoires est la relation, dans le cas des permutations stables sous conjugaison, avec la plus longue sous-suite commune de deux permutations. Par exemple, pour $m = 2$, si $\sigma_n^{-1}\rho_n$ est stable sous conjugaison et

$$\frac{\#(\sigma_n^{-1}\rho_n)}{n^{\frac{1}{6}}} \xrightarrow[n\to\infty]{d} 0$$

alors pour tout $s \in \mathbb{R}$,

$$\mathbb{P}\left(\frac{\mathrm{LCS}(\sigma_n, \rho_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) \xrightarrow[n\to\infty]{} F_2(s),$$

Une autre motivation vient de la théorie des "trafic distributions", une théorie des probabilités non-commutatives introduite par Male (2011) pour comprendre les moments de matrices aléatoires invariantes par conjugaison par une matrice de permutation. Comme le montre Male (2011), la limite au sens de la convergence en "traffic distributions" pour les matrices de permutations est triviale. Il est donc naturel de s'interroger sur les fluctuations limites pour un produit de plusieurs matrices de permutations, ce qui est un cas vraiment non commutatif. Ainsi en écrivant notre résultat comme suit, on peut voir notre résultat comme l'analyse de second ordre dans ce cadre :

$$\left(\operatorname{tr}\left(\prod_{\ell=1}^{m}\sigma_{\ell,n}\right),\dots,\operatorname{tr}\left(\left(\prod_{\ell=1}^{m}\sigma_{\ell,n}\right)^{k}\right)\right)\xrightarrow[n\to\infty]{d}\left(\xi_1,\xi_1+2\xi_2,\dots,\sum_{d|k}d\xi_d\right),$$

où, nous rappelons que $\xi_1,\xi_2,\dots$ sont indépendants et la distribution de ξ_d est la loi de Poisson de paramètre $\frac{1}{d}$.

Cela signifie que dans ce cadre, les fluctuations ne sont régies que par une hypothèse sur des points fixes et des cycles de taille 2.

Technique du ping-pong

On présente ici une technique de preuve d'universalité pour les permutations aléatoires. Étant donné $n\geq 1$ et $E\subset\mathfrak{S}_n$, on définit

$$\operatorname{next}(E):=\{\rho\circ(i,j);\rho\in E\text{ et }\#(\rho\circ(i,j))=\#(\rho)-1\}\cup\{\rho\in E;\#(\rho)=1\}$$

et

$$\operatorname{final}(\sigma):=\begin{cases}\operatorname{next}^{\#(\sigma)-1}(\{\sigma\}) & \text{si }\#(\sigma)>1\\ \{\sigma\} & \text{sinon}\end{cases}.$$

En d'autres termes, $\operatorname{next}(E)$ est l'ensemble des permutations obtenues en concaténant, si possible, deux cycles d'une certaine permutation $\sigma\in E$ et $\operatorname{final}(\sigma)$ est l'ensemble des permutations obtenues en concaténant tous les cycles de σ. En particulier, $\operatorname{final}(\sigma)\subset\mathfrak{S}_n^0$ l'ensemble des permutations avec un seul cycle. Soit $\mathcal{G}_{\mathfrak{S}_n}$ le digraphe dont les arêtes sont $\{(\sigma,\rho)\in\mathfrak{S}_n\times\mathfrak{S}_n\,;\rho\in\operatorname{next}(\{\sigma\})\}$. On représente $\mathcal{G}_{\mathfrak{S}_3}$ dans la figure 1. $\mathcal{G}_{\mathfrak{S}_n}$ peut être vu comme une version dirigée du graphe de Cayley de $\mathfrak{S}_n$ généré par des transpositions où les arêtes sont orientées vers la permutation avec moins de cycles (la plus éloignée de l'identité pour la distance de graphe) pour lequel nous avons ajouté des boucles pour les permutations de $\mathfrak{S}_n^0$. On s'intéresse à la marche aléatoire uniforme sur ce graphe.

Soit f une fonction définie sur $\mathfrak{S}_\infty=\cup_{i=1}^{\infty}\mathfrak{S}_n$ et ayant des valeurs sur un espace métrique (F,d_F), par exemple $\mathbb{Z}^d$, $\mathbb{R}^d$ ou $\mathscr{C}^0(\mathbb{R})$. Pour $1\leq k\leq n$, on note

$$\begin{aligned}\varepsilon'_{n,k}(f)&:=\max_{\sigma\in\mathfrak{S}_n,\#(\sigma)=k}\ \max_{\rho\in\operatorname{final}(\sigma)}d_F(f(\sigma),f(\rho))\\ \varepsilon_{n,k}(f)&:=\max_{\sigma\in\mathfrak{S}_n,\#(\sigma)=k}\ \max_{\rho\in\operatorname{next}(\{\sigma\})}d_F(f(\sigma),f(\rho))\\ \varepsilon_n(f)&:=\max_{1\leq k<n}\varepsilon_{k,n}(f).\end{aligned}$$

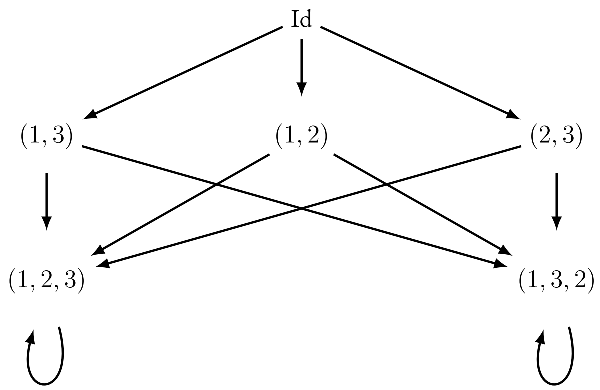

Figure 1: Le graphe $\mathcal{G}_{\mathfrak{S}_n}$ pour $n = 3$

Notre principal résultat est le suivant.

Théorème. Si $(\sigma_n)_{n\geq 1}$ et $(\sigma_{ref,n})_{n\geq 1}$ sont stables sous conjugaison et s'il existe un certain $x \in F$ tel que

$$f(\sigma_{ref,n}) \xrightarrow[n\to\infty]{\mathbb{P}} x,$$

$$\varepsilon'_{n,\#(\sigma_{ref,n})}(f) \xrightarrow[n\to\infty]{\mathbb{P}} 0 \quad \text{et} \quad \varepsilon'_{n,\#(\sigma_n)}(f) \xrightarrow[n\to\infty]{\mathbb{P}} 0, \tag{7}$$

alors $f(\sigma_n) \xrightarrow[n\to\infty]{\mathbb{P}} x$.
La convergence en probabilité peut être remplacée par une convergence dans $\mathbb{L}^p$ si les contrôles sont vrais dans $\mathbb{L}^p$.

Lorsque $F = \mathbb{R}^d$, on obtient également la convergence en loi.

Théorème. Supposons que $F = \mathbb{R}^d$, $(\sigma_n)_{n\geq 1}$ et $(\sigma_{ref,n})_{n\geq 1}$ sont stables sous conjugaison. On suppose qu'on a les convergences dans (7) et qu'il existe une variable aléatoire X supportée dans F telle que

$$f(\sigma_{ref,n}) \xrightarrow[n\to\infty]{d} X.$$

Alors

$$f(\sigma_n) \xrightarrow[n\to\infty]{d} X. \tag{8}$$

La preuve utilise des techniques markoviennes généralisables facilement sur d'autres groupes. On détaillera dans le Chapitre 2 quelques généralisations et quelques applications de ce résultat pour des fonctions de permutations connues.

1 Introduction and main results

"…que le but unique de la science, c'est l'honneur de l'esprit humain et que sous ce titre, une question de nombres vaut autant qu'une question de système du monde"

Jacobi

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1.1 Introduction

In the twentieth century, limit theorems in probability were mainly related to independence or weak dependence. As a consequence, the Gaussian distribution appears as a limiting distribution for many models. The case where the variables are strongly correlated is much more difficult since classical techniques are not helpful. At the end of the twentieth century and the beginning of the twenty-first century, there was great progress in the understanding of models with strong dependence. Several interesting examples as random growth models, random matrices, interacting particle systems and random permutations have been studied. Nevertheless, we do not have a global picture of this class of universality. That is why many mathematicians are excited nowadays to understand this class. For random permutations, in general, the study of most statistics is done only for some specific laws especially for the uniform law, Mallows with Kendall tau distance, (generalized) Ewens. In this thesis, we are interested in universality for conjugation invariant permutations. We show in particular that, under a

Statistic	Chapters	Uniform case
Longest increasing subsequence	1,2,4,5,7	Vershik and Kerov (1977) Logan and Shepp (1977) Baik, Deift, and Johansson (1999)
Edge of RSK	5	Borodin, Okounkov, and Olshanski (2000) Johansson (2001)
Limiting shape of RSK	7	Vershik and Kerov (1977) Logan and Shepp (1977)
Longest common subsequence	1,7	Baik, Deift, and Johansson (1999) Houdré and Işlak (2014)
Small cycles	6	Arratia, Barbour, and Tavaré (2000)
Descents	2,8,9	Borodin, Diaconis, and Fulman (2010)
Longuest alterning subsequence	2	Stanley (2010) and Romik (2011)
Peaks, exceedences	2	Fulman, Kim, and Lee (2019)
Graph of the permutation	2	Gürerk, Islak, and Yildiz (2019)

Table 1.1: Some examples

control on the total number of cycles, we have Tracy-Widom fluctuations for the length of the longest increasing subsequence and of the longest common subsequence of two independent and identically distributed (i.i.d.) copies. Under a weaker control, we have the Vershik-Kerov-Logan-Shepp limiting shape. For general statistics, we give different techniques to prove universality. We summarize statistics that we are interested in proving universality results in Table 1.1. We do not assume that the reader is familiar with those statistics. They will be introduced progressively when needed.

1.2 Some functions on random permutations and our main results

We prove our universality results for conjugation invariant random permutations. To do so, we introduce some notations. Given $n, m \in \mathbb{N}^* = \{1, 2, \dots\}$ such that $n \geq m$, we denote by

- $\mathfrak{S}_n$: the group of permutations of $\{1, \dots, n\}$, also known as the symmetric group. We will use the two classic notations for permutation: the two vectors notation and the representation as a disjoint cycles products. For example,

$$\mathfrak{S}_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$$
$$= \{Id_3, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}.$$

We recall here that $(i_1, i_2, \ldots, i_k)$ is the cyclic permutation of order k defined by

$$\sigma(i) = \begin{cases} i_{j+1} & \text{if } i = i_j, \quad j \leq k-1 \\ i_1 & \text{if } i = i_k \\ i & \text{otherwise} \end{cases}.$$

Here, $1 \leq i_1, i_2, \ldots, i_k \leq n$ are pairwise distinct and $k \geq 2$. In particular, up to a permutation of the cycles, any $\sigma \in \mathfrak{S}_n$ admit a unique representation as a product of disjoint cycles. For example, for

$$\sigma_{ex} := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 2 & 1 & 4 & 6 \end{pmatrix}, \tag{1.1}$$

we have

$$\sigma_{ex} = (1,5,4)(2,3) = (2,3)(1,5,4)$$

are the two possible representations of σ_{ex} as a product of disjoint cycles.

- $\#(\sigma)$: the number of cycles of σ. For example, $\#(\sigma_{ex}) = 3$.

- $\mathrm{tr}(\sigma)$: the number of fixed points of σ.

Definition 1.1. A sequence of random permutations $(\sigma_n)_{n\geq 1}$ is said to be conjugation invariant if

$$\forall n \geq 1,\ \forall \sigma \in \mathfrak{S}_n,\ \sigma_n \overset{d}{=} \sigma^{-1}\sigma_n\sigma. \tag{$\mathcal{H}_{inv}$}$$

Definition 1.2. For $\alpha \geq 1$ and $p \in [1, \infty]$, we say that the sequences of random permutations $(\sigma_n)_{n\geq 1}$ satisfies $\mathcal{H}^{\mathbb{P}}_{inv,\alpha}$ if

$$(\sigma_n)_{n\geq 1} \text{ is conjugation invariant and } \quad \frac{\#(\sigma_n)}{n^{\frac{1}{\alpha}}} \xrightarrow[n\to\infty]{\mathbb{P}} 0, \tag{$\mathcal{H}^{\mathbb{P}}_{inv,\alpha}$}$$

we say that it satisfies $\mathcal{H}^{\mathbb{L}^p}_{inv,\alpha}$ if

$$(\sigma_n)_{n\geq 1} \text{ is conjugation invariant and } \quad \frac{\#(\sigma_n)}{n^{\frac{1}{\alpha}}} \xrightarrow[n\to\infty]{\mathbb{L}^p} 0, \tag{$\mathcal{H}^{\mathbb{L}^p}_{inv,\alpha}$}$$

and we say that it satisfies $\mathcal{H}^{\mathrm{tr},p}_{inv,\alpha}$ if

$$(\sigma_n)_{n\geq 1} \text{ is conjugation invariant and } \quad \frac{\mathrm{tr}(\sigma_n)}{n^{\frac{1}{\alpha}}} \xrightarrow[n\to\infty]{\mathbb{L}^p} 0. \tag{$\mathcal{H}^{\mathrm{tr},p}_{inv,\alpha}$}$$

In particular, ($\mathcal{H}^{\mathrm{tr},1}_{inv,\alpha}$) is equivalent to

$$(\sigma_n)_{n\geq 1} \text{ is conjugation invariant and } \quad \mathbb{P}(\sigma_n(1)=1)n^{\frac{\alpha-1}{\alpha}} \xrightarrow[n\to\infty]{} 0.$$

We will present in this chapter some of our key results.

1.2.1 Monotonous subsequences

Given $\sigma \in \mathfrak{S}_n$, a subsequence $(\sigma(i_1),\dots,\sigma(i_k))$ is an increasing (resp. decreasing) subsequence of σ of length k if $i_1 < \dots < i_k$ and $\sigma(i_1) < \dots < \sigma(i_k)$ (resp. $\sigma(i_1) > \dots > \sigma(i_k)$). We denote by $\mathrm{LIS}(\sigma)$ (resp. $\mathrm{LDS}(\sigma)$) the length of the longest increasing (resp. decreasing) subsequence of σ. There is a language abuse here: a longest increasing subsequence may not be unique but its length is always defined. For example,

$$\text{if} \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}, \ \mathrm{LIS}(\sigma) = 2 \text{ and } \mathrm{LDS}(\sigma) = 4.$$

The study of the limiting behavior of $\mathrm{LIS}(\sigma_{unif,n})$, where $\sigma_{unif,n}$ is a uniform random permutation on $\mathfrak{S}_n$, is known as the Ulam's problem (or the Ulam-Hammersley problem): Ulam (1961) conjectured that the limit as n goes to infinity of

$$\frac{\mathbb{E}(\mathrm{LIS}(\sigma_{unif,n}))}{\sqrt{n}}$$

exists. Using a subadditivity argument, Hammersley (1972) proved this conjecture. He also proved that this convergence holds in probability. Vershik and Kerov (1977) and Logan and Shepp (1977) proved that this limit is equal to 2. An alternative proof is given by Aldous and Diaconis (1995). The asymptotic fluctuations were studied by Baik, Deift and Johansson. They proved the following result:

Theorem 1.3. (Baik, Deift, and Johansson, 1999) For all $s \in \mathbb{R}$,

$$\mathbb{P}\left(\frac{\mathrm{LIS}(\sigma_{unif,n}) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) \xrightarrow[n\to\infty]{} F_2(s),$$

where F_2 is the cumulative distribution function (CDF) of the GUE Tracy-Widom distribution.

We will define F_2 in (5.2). For historical details, full proofs and applications, we strongly recommend (Romik, 2015). Apart from the uniform case, Mueller and Starr (2013) studied the longest increasing subsequence for Mallows' distribution. The case of random involutions is studied by Baik and Rains (2001) who showed that the limiting distribution depends on the number of fixed points and in some regimes, the GOE/GSE Tracy-Widom distributions appear. They also showed the appearance of a family of probability distributions that interpolate between the GOE and the GSE Tracy-Widom distribution. Mueller and Starr showed that for Mallows' distribution, there is a phase transition between the Gaussian and the Tracy-Widom regimes. In this thesis, we prove universality results for conjugation invariant random permutations.

Theorem 1.4. Under ($\mathcal{H}^{\mathbb{P}}_{inv,2}$),

$$\frac{\mathrm{LIS}(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} 2 \quad \text{and} \quad \frac{\mathrm{LDS}(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} 2.$$

Moreover, for any $p \in [1, \infty)$, under ($\mathcal{H}^{\mathbb{L}^p}_{inv,2}$),

$$\frac{\mathrm{LIS}(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{L}^p} 2 \quad \text{and} \quad \frac{\mathrm{LDS}(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{L}^p} 2.$$

The convergence in probability is stated without proof in (Kammoun, 2018) as it is similar to the proof of (Kammoun, 2018, Theorem 1.2). We will give a full proof in Subsection 2.2.1. For the fluctuations, we have the following result.

Theorem 1.5. Assume that $(\sigma_n)_{n\geq 1}$ is conjugation invariant and

$$\frac{1}{n^{\frac{1}{6}}} \min_{1\leq i\leq n} \left(\left(\sum_{j=1}^{i} \#_j(\sigma_n) \right) + \frac{\sqrt{n}}{i} \sum_{j=i+1}^{n} \#_j(\sigma_n) \right) \xrightarrow[n\to\infty]{\mathbb{P}} 0. \tag{1.2}$$

Then for all $s \in \mathbb{R}$,

$$\mathbb{P}\left(\frac{\mathrm{LIS}(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s \right) \xrightarrow[n\to\infty]{} F_2(s) \text{ and } \mathbb{P}\left(\frac{\mathrm{LDS}(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s \right) \xrightarrow[n\to\infty]{} F_2(s). \tag{TW}$$

Here, $\#_j(\sigma)$ is the number of cycles of σ of length j.

The idea of the proof we give in Subsection 5.2.1 is to construct a coupling between any distribution satisfying these hypothesises and the uniform distribution in order to use Theorem 1.3 to obtain first the lower bound then the upper bound. This theorem generalizes the following result.

Corollary 1.6. (Kammoun, 2018, Theorem 1.2) If ($\mathcal{H}^{\mathbb{P}}_{inv,6}$) is satisfied then (TW) holds.

We will prove Corollary 1.6 in Subsection 2.2.1.

1.2.2 The longest common subsequence

Given $\sigma \in \mathfrak{S}_n$, $(\sigma(i_1), \ldots, \sigma(i_k))$ is a subsequence of σ of length k if $i_1 < i_2 < \cdots < i_k$. We denote by $\mathrm{LCS}(\sigma_1, \sigma_2)$ the length of the longest common subsequence (LCS) of two permutations. The LCS is deeply related to the LIS since

$$\mathrm{LCS}(\sigma, \rho) = \mathrm{LIS}(\sigma \circ \rho^{-1}). \tag{1.3}$$

The understanding of the LCS of words was studied first in the context of theoretical computer science (Maier, 1978). The LCS is related to the understanding of the modifications in the different versions of the same file (Masek and Paterson, 1980). In the sequel, we suppose that σ_n and ρ_n are independent random permutations on $\mathfrak{S}_n$. The study of the LCS of independent random permutations was initiated by Houdré and Işlak (2014). It is conjectured by Bukh and Zhou (2016) that for i.i.d. random permutations,

$$\mathbb{E}(\mathrm{LCS}(\sigma_n, \rho_n)) \geq \sqrt{n}.$$

Recently, Houdré and Xu (2018) showed that for i.i.d. random permutations,

$$\mathbb{E}(\mathrm{LCS}(\sigma_n, \rho_n)) \geq n^{\frac{1}{3}}.$$

We will show an asymptotic bound in the scale $\sqrt{n}$ in the case where the law of at least one of the two permutations is conjugation invariant. In Theorem 1.7, we give an asymptotic lower bound for the LCS of two independent random permutations. Under a good control of the number of fixed points, we give a better bound in Proposition 1.8. Finally, as an application of Proposition 1.8, we give an asymptotically optimal lower bound for conjugation invariant i.i.d. random permutations in Corollary 1.9.

Theorem 1.7. Assume that for any $n \geq 1$, σ_n and ρ_n are independent and their distributions are conjugation invariant. Then

$$\liminf_{n\to\infty} \frac{\mathbb{E}(\mathrm{LCS}(\sigma_n, \rho_n))}{\sqrt{n}} \geq 2\sqrt{13} - 6 \simeq 1.21\ldots.$$

Note that this improves (Kammoun, 2020) where we proved that under the same hypothesis, we have

$$\liminf_{n\to\infty} \frac{\mathbb{E}(\mathrm{LCS}(\sigma_n, \rho_n))}{\sqrt{n}} \geq 2\sqrt{\theta} \simeq 0.564\ldots, \tag{1.4}$$

where θ is the unique solution of $G(2\sqrt{x}) = \frac{2+x}{12}$,

$$\begin{aligned} G := [0,2] &\to \left[0, \frac{1}{2}\right] \\ x &\mapsto \int_{-1}^{1} \left(\Omega(s) - \left|s + \frac{x}{2}\right| - \frac{x}{2}\right)_+ \mathrm{d}s, \end{aligned} \tag{1.5}$$

and

$$\Omega(s) := \begin{cases} \frac{2}{\pi}(s \arcsin(s) + \sqrt{1-s^2}) & \text{if } |s| < 1 \\ |s| & \text{if } |s| \geq 1 \end{cases}.$$

We will recall the proof of (1.4) in Subsection 7.4.3 and we will prove Theorem 1.7 in Section 7.5. The function Ω appears as the Vershik-Kerov-Logan-Shepp limiting shape. For more details, one can see Figure 7.1 and Theorem 7.4.

Assuming an extra assumption on fixed points, we obtain a better bound.

Proposition 1.8. Assume that for any $n \geq 1$, σ_n and ρ_n are independent and their distributions are conjugation invariant.

$$\text{If} \qquad (\sigma_n)_{n\geq 1} \text{ et } (\rho_n)_{n\geq 1} \text{ satisfy } (\mathcal{H}^{\mathrm{tr},1}_{inv,1}), \tag{1.6}$$

$$\text{then} \qquad \liminf_{n\to\infty} \frac{\mathbb{E}(\mathrm{LCS}(\sigma_n, \rho_n))}{\sqrt{n}} \geq 2. \tag{1.7}$$

Moreover,

$$\text{if} \qquad \liminf_{n\to\infty} \sqrt{n}\mathbb{P}(\sigma_n(1) = 1)\mathbb{P}(\rho_n(1) = 1) \geq \alpha \quad \text{for some } 0 < \alpha \leq 2, \tag{1.8}$$

$$\text{then} \qquad \liminf_{n\to\infty} \frac{\mathbb{E}(\mathrm{LCS}(\sigma_n, \rho_n))}{\sqrt{n}} \geq \alpha. \tag{1.9}$$

Consequently, we obtain the following result for i.i.d. random permutations.

Corollary 1.9. Assume that for any $n \geq 1$, σ_n and ρ_n are two i.i.d. conjugation invariant random permutations. Then

$$\liminf_{n\to\infty} \frac{\mathbb{E}(\mathrm{LCS}(\sigma_n, \rho_n))}{\sqrt{n}} \geq 2.$$

We conjecture that we can get rid of (1.6) and (1.8); the conjugation invariance is sufficient to obtain (1.7) which is equivalent to replace $2\sqrt{\theta}$ by 2 in Theorem 1.7. We will prove Proposition 1.8 and Corollary 1.9 in Subsection 7.4.2. The idea of the proof is to understand the cycle structure of $\sigma_n^{-1} \circ \rho_n$.

When ρ_n is not conjugation invariant, we give an asymptotic lower bound of $\frac{\mathbb{E}(\mathrm{LCS}(\sigma_n,\rho_n))}{\sqrt{n}}$ in Theorem 1.10.

Theorem 1.10. Assume that for any $n \geq 1$, σ_n and ρ_n are independent and σ_n is conjugation invariant. Then

$$\liminf_{n\to\infty} \frac{\mathbb{E}(\mathrm{LCS}(\sigma_n, \rho_n))}{\sqrt{n}} \geq G^{-1}\left(\liminf_{n\to\infty} \frac{\mathbb{E}(\#(\sigma_n))}{2n}\right).$$

We recall that G is defined in (1.5).

In particular, if $(\sigma_n)_{n\geq 1}$ satisfies $(\mathcal{H}^{\mathbb{P}}_{inv,1})$, we have

$$\liminf_{n\to\infty} \frac{\mathbb{E}\left(\mathrm{LCS}(\sigma_n,\rho_n)\right)}{\sqrt{n}} \geq 2.$$

We prove in Proposition 1.11 that under a good control of the number of cycles of σ_n,

$$\lim_{n\to\infty} \frac{\mathbb{E}(\mathrm{LCS}(\sigma_n,\rho_n))}{\sqrt{n}} = 2$$

and under a stronger control, we have Tracy-Widom fluctuations for $\mathrm{LCS}(\sigma_n,\rho_n)$.

Proposition 1.11. Assume that for any $n \geq 1$, σ_n and ρ_n are independent.

- If $(\sigma_n)_{n\geq 1}$ satisfies $(\mathcal{H}^{\mathbb{P}}_{inv,2})$ then

$$\frac{\mathrm{LCS}(\sigma_n,\rho_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} 2.$$

- If $(\sigma_n)_{n\geq 1}$ satisfies $(\mathcal{H}^{\mathbb{L}^1}_{inv,2})$ then

$$\frac{\mathrm{LCS}(\sigma_n,\rho_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{L}^1} 2.$$

- If $(\sigma_n)_{n\geq 1}$ satisfies $(\mathcal{H}^{\mathbb{P}}_{inv,6})$ then $\forall s \in \mathbb{R}$,

$$\mathbb{P}\left(\frac{\mathrm{LCS}(\sigma_n,\rho_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) \xrightarrow[n\to\infty]{} F_2(s).$$

Note that in Theorem 1.10 and in Proposition 1.11, we do not make any assumption on the distribution of ρ_n. The proof in Subsection 7.4.3 is based on a coupling argument between σ_n and a uniform permutation in order to use the result of Baik, Deift and Johansson.

1.2.3 Small cycles

The cycle structure of a permutation chosen uniformly on the symmetric group $\mathfrak{S}_n$ is well understood (see (Arratia, Barbour, and Tavaré, 2003) for detailed results). In particular, the following classical result holds:

Theorem 1.12. (Arratia, Barbour, and Tavaré, 2000, Theorem 3.1) For any $k \geq 1$,

$$(\#_1(\sigma_{unif,n}),\dots,\#_k(\sigma_{unif,n})) \xrightarrow[n\to\infty]{d} (\xi_1,\xi_2,\dots,\xi_k), \tag{1.10}$$

where $\xrightarrow[n\to\infty]{d}$ denotes the convergence in distribution, $\xi_1,\xi_2,\dots,\xi_k$ are independent and the distribution of ξ_d is Poisson of parameter $\frac{1}{d}$.

In fact, Arratia, Barbour, and Tavaré (2000) study the setting where both n and k go to infinity such that $\frac{k}{n} \to 0$. But, we will limit our study to the case where k is fixed. In a joint work with Mylène Maïda, we questioned the universality class of this convergence. We show that a product of conjugation invariant permutations that do not have too many fixed points and cycles of size 2 lies within this class. More precisely, we have the following.

Theorem 1.13. (Kammoun and Maïda, 2020) Let $m \geq 2$. For $1 \leq \ell \leq m$, let $(\sigma_{\ell,n})_{n\geq 1}$ be a sequence of random permutations such that for any $n \geq 1$, $\sigma_{\ell,n}$ is a random permutation on $\mathfrak{S}_n$. For any $k \geq 1$, let $t_k^n := \#_k(\prod_{\ell=1}^m \sigma_{\ell,n})$. Assume that

(H_1) For any $n \geq 1$, $(\sigma_n, \ldots, \sigma_{m,n})$ are independent.

(H_2) For any $n \geq 1$ and $1 \leq \ell \leq m$, $\sigma_{\ell,n}$ is conjugation invariant except maybe for one $\ell \in \{1, \ldots, m\}$.

- There exists $1 \leq i < j \leq m$ such that for any $k \geq 1$,

(H_3) $\sigma_{i,n}, \sigma_{j,n}$ satisfy $(\mathcal{H}_{inv,2}^{\mathrm{tr},k})$,

(H_4) $\sigma_{i,n}^2, \sigma_{j,n}^2$ satisfy $(\mathcal{H}_{inv,1}^{\mathrm{tr},1})$.

Then for any $k \geq 1$,

$$(t_1^n, t_2^n, \ldots, t_k^n) \xrightarrow[n\to\infty]{d} (\xi_1, \xi_2, \ldots, \xi_k).$$

Similar results have been obtained by Mukherjee (2016) for permutations that are equicontinuous and converging as a permuton (see definitions there) and a stronger convergence (in total variation distance) was established by Chmutov and Pittel (2016) when one of the permutations has all its cycles of length at least 3 (see also (Gamburd, 2006)). None of thesis results covers for example the product of two Ewens distributions. They are known to satisfy the convergences required in (H_3) and (H_4) so that our result tells that the product of (at least two) Ewens distributions behaves like a uniform permutation, as far as small cycles are concerned. In our framework, in the case of two permutations, a weaker result can be obtained without any hypothesis on the cycles of size 2.

Proposition 1.14. (Kammoun and Maïda, 2020) For $m = 2$. Assume that

(H_1) For any $n \geq 1$, σ_n and ρ_n are independent.

(H_2) For any $n \geq 1$, σ_n or ρ_n is conjugation invariant.

- For any $k \geq 1$,

(H_3) σ_n, ρ_n satisfy $(\mathcal{H}_{inv,2}^{\mathrm{tr},k})$.

Then $\forall v \geq 1$,

$$\lim_{n\to\infty} \mathbb{E}(t_v^n) = \frac{1}{v}.$$

Note that under the independence hypothesis and when one of the permutations $\sigma_{\ell,n}$ follows the uniform distribution, the product also follows the uniform distribution and Theorem 1.13 is a direct consequence of Theorem 1.12.

Our motivation to understand the cycle structure of random permutations is the relation, in the case of conjugation invariant permutations, to the longest common subsequence (LCS) of two permutations. For example, for $m = 2$, if $\rho_n\sigma_n^{-1}$ is conjugation invariant and

$$\frac{\#(\rho_n\sigma_n^{-1})}{n^{\frac{1}{6}}} \xrightarrow[n\to\infty]{d} 0,$$

then by (1.3) and Corollary 1.6, for any $s \in \mathbb{R}$.

$$\mathbb{P}\left(\frac{\mathrm{LCS}(\sigma_n, \rho_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) \xrightarrow[n\to\infty]{} F_2(s),$$

Another motivation comes from traffic distributions, a non-commutative probability theory introduced by Male (2011) to understand the moments of permutation invariant random matrices. As shown in (Male, 2011), the limit in traffic distribution of uniform permutation matrices is trivial but Theorem 1.12 can be seen as a second-order result in this framework. It is therefore natural to ask about limiting joint fluctuations for the product of several permutation matrices, which is a really non-commutative case. To emphasize this relation, we rewrite Theorem 1.13 as follows.

Corollary 1.15. Under the same hypothesis as in Theorem 1.13, for any $k \geq 1$,

$$\left(\mathrm{tr}\left(\prod_{\ell=1}^{m}\sigma_{\ell,n}\right), \ldots, \mathrm{tr}\left(\left(\prod_{\ell=1}^{m}\sigma_{\ell,n}\right)^k\right)\right) \xrightarrow[n\to\infty]{d} \left(\xi_1, \xi_1 + 2\xi_2, \ldots, \sum_{d|k} d\xi_d\right),$$

where, we recall that $\xi_1, \xi_2, \ldots$ are independent and the distribution of ξ_d is Poisson of parameter $\frac{1}{d}$.

It means that in this framework, the fluctuations are only governed by a hypothesis on fixed points and cycles of order two.

1.3 Organization

We presented in this chapter some of our key results. In Chapter 2, we give a Markov technique to prove universality for random permutations. It uses a random walk on the Cayley graph of the symmetric group generated by transpositions. For a good choice of initial distribution, this walk converges rapidly to the Ewens distribution with parameter 0. We will apply our results to various functions of permutations. In Chapter 4, we will generalize this technique to other Cayley graphs of permutations and to other groups. The results in Chapters 2 and 4 are new even if we can recover some results already proved in our papers (Kammoun, 2018; Kammoun, 2020) as applications. In Chapter 3, we recall some known results on the RSK correspondence and Schur measures. This chapter does not contain any new result but gather many technical tools used in the sequel. We prove in Chapter 5 universality results related to the length of the longest increasing, decreasing and common subsequences. We give also a more general convergence dealing with the first rows of the tableau obtained via the RSK correspondence. This chapter contains already published results as well some improvements for some bounds. Using different techniques, we show in Chapter 6 that the limiting joint moments of small cycles of products of conjugation invariant random permutations depend only on the law of fixed points and cycles of length 2. The proof uses a modified version of the Wigner's moment method. This chapter contains already published results in a joint work with Mylène Maïda. In Chapter 7, we conjecture that the shape of a typical Young diagram obtained by the RSK correspondence depends only on the proportion of its fixed points. We prove this result in some particular cases. As an application, we obtain lower bounds of the longest common subsequence of two independent random permutations when at least one of the permutations is conjugation invariant. This chapter contains already published results as well some improvements for some bounds. In Chapter 8, we give some examples of known conjugation invariant random permutations. We will present in particular the Ewens distribution, some generalizations and the Kingman virtual permutations. We will apply the results obtained in the previous chapters to those permutations. This chapter does not contain any new result. Finally, in Chapter 9, we suggest some possible improvements of our results as well as some possible ways to use the same techniques to tackle other problems. Here is the dependency graph for reading this thesis.

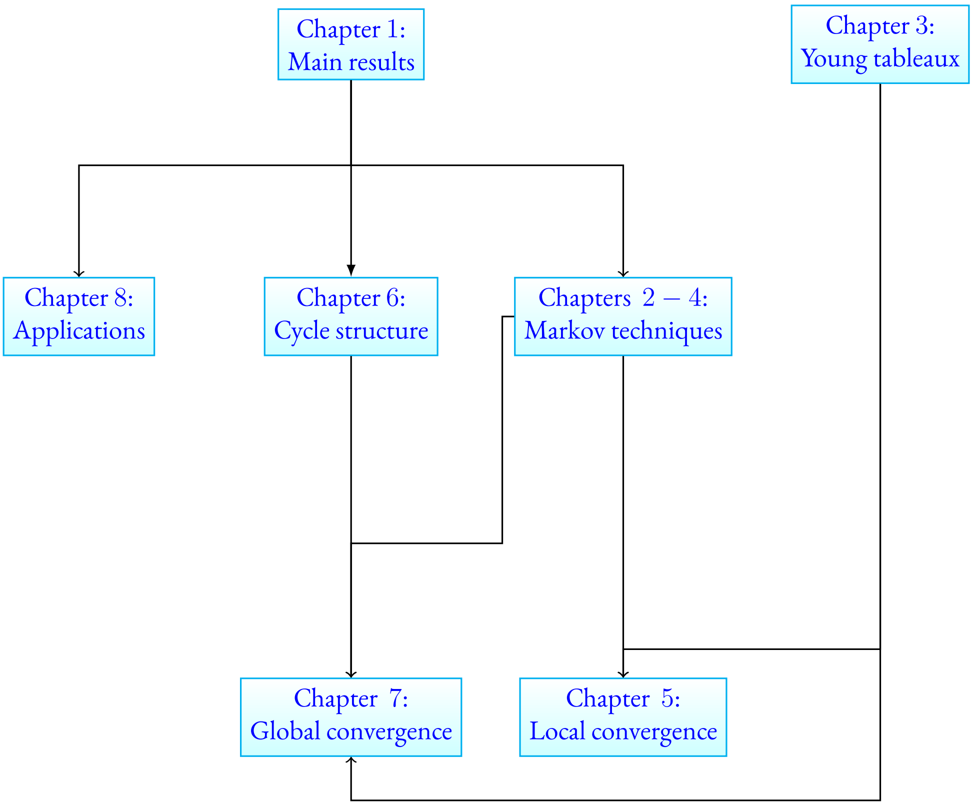

Figure 1.1: The dependency graph

2

Random walks and universality for random permutations

"En mathématiques, les noms sont arbitraires. Libre à chacun d'appeler un opérateur auto-adjoint un éléphant et une décomposition spectrale une trompe. On peut alors démontrer un théorème suivant lequel tout éléphant a une trompe. Mais on n'a pas le droit de laisser croire que ce résultat a quelque chose à voir avec de gros animaux gris."

Gerald Sussman

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In this chapter, we present some Markovian approaches to prove universality results for some functions on the symmetric group $\mathfrak{S}_n$. The main approach uses a coupling between conjugation invariant random permutations and the uniform distribution, for which the functions were already studied. Some of these functions are already studied in (Kammoun, 2018; Kammoun, 2020) but not the general case.

2.1 First method: the ping-pong

2.1.1 Rebound on the Ewens zero distribution [1]

Given $n \geq 1$ and $E \subset \mathfrak{S}_n$, we define

$$\mathrm{next}(E) := \{\rho \circ (i,j); \rho \in E,\ \#(\rho \circ (i,j)) = \#(\rho) - 1\} \cup \{\rho \in E; \#(\rho) = 1\}$$

and

$$\mathrm{final}(\sigma) := \begin{cases} \mathrm{next}^{\#(\sigma)-1}(\{\sigma\}) & \text{if } \#(\sigma) > 1 \\ \{\sigma\} & \text{otherwise} \end{cases},$$

where we recall that $\#(\sigma)$ is the number of cycles of σ. In other words, $\mathrm{next}(E)$ is the set of permutations obtained by concatenating, if possible, two cycles of some $\sigma \in E$, and $\mathrm{final}(\sigma)$ is the set of permutations obtained by concatenating all the cycles of σ. In particular,

$$\mathrm{final}(\sigma) \subset \mathfrak{S}_n^0 := \{\sigma \in \mathfrak{S}_n;\ \#(\sigma) = 1\}.$$

Let $\mathcal{G}_{\mathfrak{S}_n}$ be the directed graph with vertices $\mathfrak{S}_n$ and edges $\{(\sigma,\rho); \sigma \in \mathfrak{S}_n, \rho \in \mathrm{next}(\{\sigma\})\}$. We represent $\mathcal{G}_{\mathfrak{S}_3}$ in Figure 2.1. $\mathcal{G}_{\mathfrak{S}_n}$ can be seen as a directed version of the Cayley graph of $\mathfrak{S}_n$ generated by transpositions where the edges are oriented toward the permutations with fewer cycles (the further from the identity according to the graph distance), for which we added loops at the permutations of $\mathfrak{S}_n^0$. In this first part of this section, we will examine the uniform random walk on $\mathcal{G}_{\mathfrak{S}_n}$.

Let f be a function defined on $\mathfrak{S}_\infty := \cup_{i=1}^\infty \mathfrak{S}_n$ and taking its values in some metric space (F, d_F), for example $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^d$ or $\mathscr{C}^0(\mathbb{R})$. It turns out that the uniform distribution on $\mathfrak{S}_n^0$, also known as the Ewens distribution with parameter 0, [2] is useful to obtain universality results for conjugation invariant permutations if f does not change too much by merging two cycles. More precisely, we define for $1 \leq k \leq n$,

$$\varepsilon'_{n,k}(f) := \max_{\sigma \in \mathfrak{S}_n, \#(\sigma)=k} \ \max_{\rho \in \mathrm{final}(\sigma)} d_F(f(\sigma), f(\rho)).$$

[1] The ping-pong table here is $\mathfrak{S}_n^0$ and the ball is moving over $\mathfrak{S}_n$.

[2] See Definition 8.1 for more details.

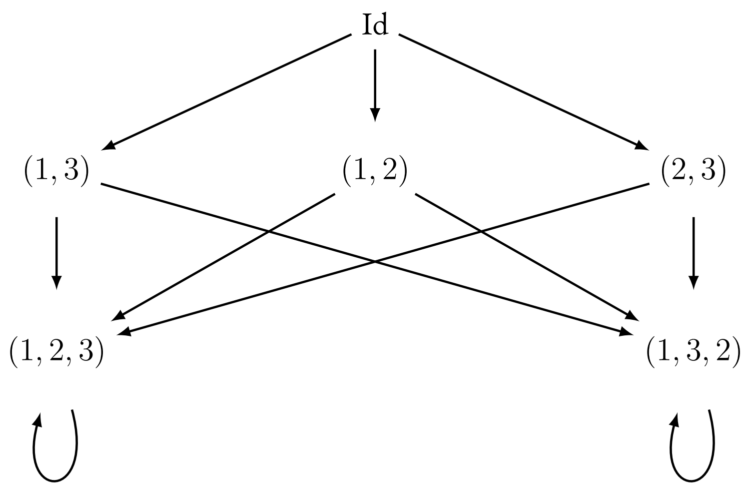

Figure 2.1: The directed graph $\mathcal{G}_{\mathfrak{S}_3}$

We present now the main result of this thesis.

Theorem 2.1. Assume that $(\sigma_n)_{n\geq 1}$ and $(\sigma_{ref,n})_{n\geq 1}$ satisfy ($\mathcal{H}_{inv}$). Suppose that there exists $x \in F$ such that

(2.1) $$f(\sigma_{ref,n}) \xrightarrow[n\to\infty]{\mathbb{P}} x,$$

(2.2) $$\varepsilon'_{n,\#(\sigma_{ref,n})}(f) \xrightarrow[n\to\infty]{\mathbb{P}} 0$$

(2.3) $$\text{and that} \quad \varepsilon'_{n,\#(\sigma_n)}(f) \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

Then

(2.4) $$f(\sigma_n) \xrightarrow[n\to\infty]{\mathbb{P}} x.$$

Moreover, if the assumptions (2.1)–(2.3) hold true for the $\mathbb{L}^p$ convergence for some $p \geq 1$ instead of the convergence in probability, then so does (2.4).

When $F = \mathbb{R}^d$, we obtain also the convergence in distribution.

Theorem 2.2. Assume that $F = \mathbb{R}^d$ and that $(\sigma_n)_{n\geq 1}$ and $(\sigma_{ref,n})_{n\geq 1}$ satisfy ($\mathcal{H}_{inv}$). Suppose that (2.2) and (2.3) hold true and that there exists a random variable X supported on F such that

$$f(\sigma_{ref,n}) \xrightarrow[n\to\infty]{d} X.$$

Then

(2.5) $$f(\sigma_n) \xrightarrow[n\to\infty]{d} X.$$

The idea of the proof is to compare both $f(\sigma_n)$ and $f(\sigma_{ref,n})$ with $f(\sigma_{Ew,0,n})$. In general, the choice $\sigma_{ref,n} \stackrel{d}{=} \sigma_{unif,n}$ is interesting since, the convergence in (2.1) is known for many statistics. Moreover, using Proposition 8.2, we have immediately the following result.

Corollary 2.3. If $\sigma_{ref,n} \stackrel{d}{=} \sigma_{unif,n}$, in both theorems 2.1 and 2.2, the hypothesis (2.2) can be replaced by the existence of $\kappa > 0$ such that

$$\max_{\left|\frac{k}{\log(n)}-1\right|<\kappa} \varepsilon'_{n,k}(f) \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

We chose to give a very simple version that can be checked easily for many statistics. We will improve some results in the remainder of this thesis. For almost sure convergence, one can obtain similar results after defining properly the spaces. We will not discuss here this type of convergence. We will give many applications using the following observation.

Remark 2.4. By the triangle inequality, we have

$$\varepsilon'_{n,k}(f) \leq \sum_{i=2}^{k} \varepsilon_{n,i}(f) \leq (k-1)\varepsilon_n(f),$$

where

$$\varepsilon_{n,k}(f) := \max_{\sigma\in\mathfrak{S}_n, \#(\sigma)=k} \max_{\rho\in \mathrm{next}(\{\sigma\})} d_F(f(\sigma), f(\rho)) \quad \text{and} \quad \varepsilon_n(f) := \max_{1\leq k<n} \varepsilon_{k,n}(f).$$

Consequently, if there exists some $\alpha \leq 1$ such that

$$\varepsilon_n(f) = O\left(\frac{1}{n^{\frac{1}{\alpha}}}\right)$$

then $(\mathcal{H}^{\mathbb{P}}_{inv,\alpha})$ implies (2.3) and $(\mathcal{H}^{\mathbb{L}^p}_{inv,\alpha})$ implies the equivalent hypothesis in $\mathbb{L}^p$. Moreover, if $\sigma_{ref,n} \stackrel{d}{=} \sigma_{unif,n}$, then Proposition 8.5 implies (2.2). We will give some direct applications of this observation in the next subsection.

2.1.2 Some applications

In the next corollary, we will give some applications. The first column of Table 2.1 contains the function to study. We apologize to the reader because some of these statistics are not defined yet. One can check the corresponding result in the fifth column for more details.

Corollary 2.5. For the functions f the distribution X and the real α in Table 2.1, if $(\mathcal{H}^{\mathbb{P}}_{inv,\alpha})$ is satisfied, then

$$f(\sigma_n) \xrightarrow[n\to\infty]{d} X$$

except for the sixth example where the convergence holds in probability.[3] For the first and the forth examples the convergence holds also in $\mathbb{L}^p$ under $(\mathcal{H}_{inv,\alpha}^{\mathbb{L}^p})$. For the fifth example please check the corresponding theorem for more details about the type of convergence.

Note that:

- We give in the third column the inequality we used to obtain our results. Except for the cases where we study the RSK image of the permutation (see Section 3.1), the longest alternating subsequence and the descent process, the inequality is trivial, but we will prove all the inequalities in the sequel of this thesis.
- We want to emphasize that these results are just a direct application of theorems 2.1 and 2.2. Using more sophisticated controls of the error, one could obtain larger classes of universality. One can check Table 9.1 where we summarize our "best" results.
- For all our examples, the special case of Ewens distribution satisfies the hypothesis.

2.1.3 Proof of theorems 2.1 and 2.2

Let ρ_n be a conjugation invariant random permutation. To prove theorems 2.1 and 2.2, the idea is to modify ρ_n to obtain a conjugation invariant random permutation supported on $\mathfrak{S}_n^0$. We define the following Markov operator T associated to the uniform random walk over $\mathcal{G}_{\mathfrak{S}_n}$. Another way to see it is the following :[4]

- If the realization σ of ρ_n has one cycle, σ remains unchanged ($T(\sigma) = \sigma$).
- Otherwise, we choose a couple (i, j) uniformly from the nonempty set

$$\{(i, j); j \notin \mathcal{C}_i(\sigma)\}$$

  and we take $T(\sigma) = \sigma \circ (i, j)$. Here $\mathcal{C}_i(\sigma)$ is the cycle of σ containing i.

For example, for $n = 3$, transition probabilities of T are given in Figure 2.2.

We denote by $T^k(\rho_n)$ the random permutation obtained after applying k times the operator T. It is the random permutation obtained after k steps of the uniform random walk on $\mathcal{G}_{\mathfrak{S}_n}$ with initial state ρ_n. Table 2.2 sums up the evolution of the random walk if we start from the uniform distribution on $\mathfrak{S}_3$. Remark that the condition $j \notin \mathcal{C}_i(\sigma)$ guarantees that $\#(\sigma \circ (i, j)) = \#(\sigma) - 1$ since the cycles containing i and j are merged and the remaining of cycles are the same for σ and $\sigma \circ (i, j)$.

[3] In the space of continual diagrams i.e. the set of 1-Lipschitz real functions f such that outside one compact, $f(x) = |x - a|$. One can see (Kerov, 1993; Sodin, 2017) for more details for continual diagrams. We will use as distance, $d_F(f, g) = \sup_{x\in\mathbb{R}} |f(x) - g(x)|$ which is finite since both functions are continuous and outside one compact of $\mathbb{R}$, $f - g$ is constant.

[4] Slightly different Markov operators have already been studied in (Kammoun, 2018; Kammoun, 2020), we modify a little the two operators presented in the cited papers to obtain a uniform random walk easy to generalize to other sets. The three operators coincide when $n \leq 3$.

$f(\sigma)$	X	Error	Hypotheses	Theorem
$\frac{\mathrm{LIS}(\sigma)}{\sqrt{n}}, \frac{\mathrm{LDS}(\sigma)}{\sqrt{n}}$	2	$\varepsilon_n \leq \frac{2}{\sqrt{n}}$	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$ $(\mathcal{H}^{L^p}_{inv,2})$	Theorem 1.4
$\frac{\mathrm{LISC}(\sigma)}{\sqrt{n}}, \frac{\mathrm{LDSC}(\sigma)}{\sqrt{n}}$	2	$\varepsilon_n \leq \frac{2}{\sqrt{n}}$	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$	Corollary 2.9
$\frac{\mathrm{LIS}(\sigma)-2\sqrt{n}}{n^{\frac{1}{6}}}$, $\frac{\mathrm{LDS}(\sigma)-2\sqrt{n}}{n^{\frac{1}{6}}}$	Tracy-Widom	$\varepsilon_n \leq \frac{2}{n^{\frac{1}{6}}}$	$(\mathcal{H}^{\mathbb{P}}_{inv,6})$	Corollary 1.6
$\frac{\lambda_i(\sigma)}{\sqrt{n}}$	2	$\varepsilon_n \leq \frac{4}{\sqrt{n}}$	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$ $(\mathcal{H}^{L^p}_{inv,2})$	Proposition 5.9
$\left(\frac{\lambda_i(\sigma)-2\sqrt{n}}{n^{\frac{1}{6}}}\right)_{1\leq i\leq d}$	Airy ensemble	$\varepsilon_n \leq \frac{4}{n^{\frac{1}{6}}}$	$(\mathcal{H}^{\mathbb{P}}_{inv,6})$	Theorem 5.7
$s \to \frac{L_{\lambda(\sigma)}(s\sqrt{2n})}{\sqrt{2n}}$	Ω	$\varepsilon'_{n,k} \leq \frac{2\sqrt{k-1}}{\sqrt{n}}$	$(\mathcal{H}^{\mathbb{P}}_{inv,1})$	Theorem 7.4
$\frac{\mathrm{LCS}(\sigma,\rho)}{\sqrt{n}}$	2	$\varepsilon_n \leq \frac{2}{\sqrt{n}}$	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$	Proposition 1.11
$\frac{\mathrm{LCS}(\sigma,\rho)-\sqrt{n}}{n^{\frac{1}{6}}}$	Tracy-Widom	$\varepsilon_n \leq \frac{2}{n^{\frac{1}{6}}}$	$(\mathcal{H}^{\mathbb{P}}_{inv,6})$	Proposition 1.11
$\frac{\mathcal{K}_j(\sigma)}{n^j}$	$\frac{1}{j!^2}$	$\varepsilon_n \leq \frac{2j}{n}$	$(\mathcal{H}^{\mathbb{P}}_{inv,1})$	Corollary 2.19
$\frac{\mathcal{K}_j(\sigma)-\frac{n^j}{(j!)^2}}{\sqrt{n}}$	$\mathcal{N}\left(0, \frac{\binom{4j-2}{2j-1}-2\binom{2j-1}{j}^2}{2((2m-1)!)^2}\right)$	$\varepsilon_n \leq \frac{2j}{\sqrt{n}}$	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$	Corollary 2.19
$\frac{\mathcal{N}_{exc}(\sigma)}{n}$	$\frac{1}{2}$	$\varepsilon_n \leq \frac{4}{n}$	$(\mathcal{H}^{\mathbb{P}}_{inv,1})$	Corollary 2.19
$\frac{\mathcal{N}_{exc}(\sigma)-\frac{n}{2}}{\sqrt{n}}$	$\mathcal{N}(0, \frac{1}{12})$	$\varepsilon_n \leq \frac{4}{\sqrt{n}}$	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$	Corollary 2.19
$\mathbb{1}_{D(\sigma)\subset A}$	$Ber(\det([k_0(j-i)]_A))$	Proposition 2.20	$(\mathcal{H}^{\mathbb{P}}_{inv,1})$	Corollary 2.22

The results below are fully understood in the conjugation invariant case.

$\frac{\mathcal{N}_D(\sigma)}{n}$	$\frac{1}{2}$	$\varepsilon_n \leq \frac{4}{n}$	$(\mathcal{H}^{\mathbb{P}}_{inv,1})$	Corollary 2.19
$\frac{\mathcal{N}_D(\sigma)-\frac{n}{2}}{\sqrt{n}}$	$\mathcal{N}(0, \frac{1}{12})$	$\varepsilon_n \leq \frac{4}{\sqrt{n}}$	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$	Corollary 2.19
$\frac{\mathcal{N}_{peak}(\sigma)}{n}$	$\frac{1}{3}$	$\varepsilon_n \leq \frac{6}{n}$	$(\mathcal{H}^{\mathbb{P}}_{inv,1})$	Corollary 2.19
$\frac{\mathcal{N}_{peak}(\sigma)-\frac{n}{2}}{\sqrt{n}}$	$\mathcal{N}(0, \frac{2}{45})$	$\varepsilon_n \leq \frac{6}{\sqrt{n}}$	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$	Corollary 2.19
$\frac{\mathrm{LAS}(\sigma)}{n}$	$\frac{2}{3}$	$\varepsilon_n \leq \frac{6}{n}$	$(\mathcal{H}^{\mathbb{P}}_{inv,1})$	Corollary 2.15
$\frac{\mathrm{LAS}(\sigma)-\frac{2n}{3}}{\sqrt{n}}$	$\mathcal{N}(0, \frac{8}{45})$	$\varepsilon_n \leq \frac{6}{\sqrt{n}}$	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$	Corollary 2.15

Table 2.1: Some examples

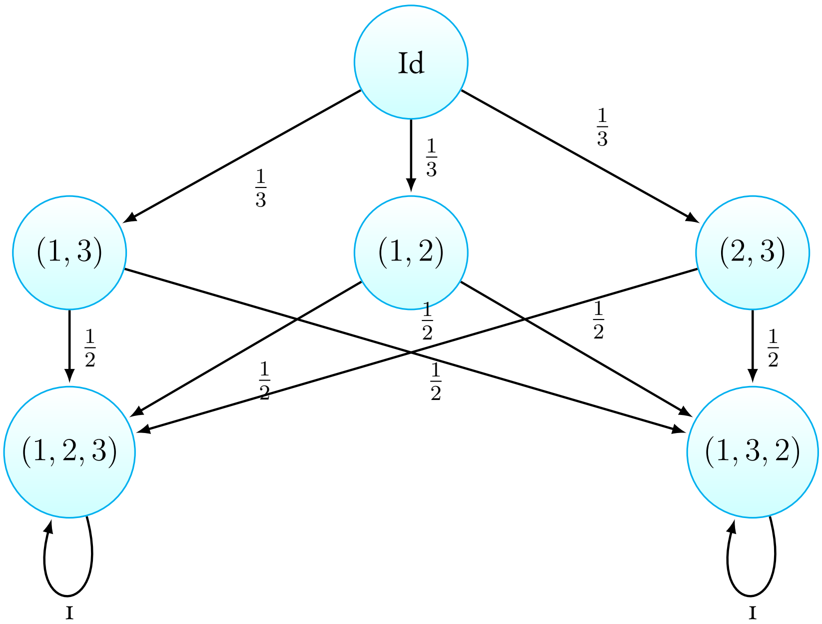

Figure 2.2: The transition probabilities of T for $n = 3$

	$\sigma_{unif,3}$	$T(\sigma_{unif,3})$	$T^2(\sigma_{unif,3})$
Id	$1/6$	0	0
$(1,2)$	$1/6$	$1/18$	0
$(1,3)$	$1/6$	$1/18$	0
$(2,3)$	$1/6$	$1/18$	0
$(1,2,3)$	$1/6$	$5/12$	$1/2$
$(1,3,2)$	$1/6$	$5/12$	$1/2$

Table 2.2: Transitions for the $\sigma_{unif,3}$

In particular,

$$\#(T^i(\rho_n)) \overset{a.s}{=} \max(\#(\rho_n) - i, 1). \tag{2.6}$$

The invariant measure of this walk (for conjugation invariant permutations) is trivial.

Lemma 2.6. If ρ_n is a conjugation invariant random permutation of $\mathfrak{S}_n$ then the law of $T^{n-1}(\rho_n)$ [5] is the uniform distribution on $\mathfrak{S}_n^0$ i.e.

$$T^{n-1}(\rho_n) \overset{d}{=} \sigma_{Ew,0,n}.$$

Proof. First, by construction, if ρ_n is conjugation invariant then $T(\rho_n)$ is also conjugation invariant. Indeed, one can see that $T(\rho_n)$ is conjugation invariant since the construction depends only on the cycle structure of ρ_n and all the integers between 1 and n play a symmetric role. By iteration, $T^{n-1}(\rho_n)$

[5] After all, a drunk and lost man who is driving on a two-way road (the Cayley graph of $\mathfrak{S}_n$) needs $n\log(n)$ steps to be close to his destination and will never attend it but if he drives in a one-way road, he needs at most n step to be sure to arrive to destination. In both cases, it is dangerous for a drunk man to drive.

is conjugation invariant. Moreover, using (2.6),

$$\#(T^{n-1}(\sigma_n)) \overset{a.s}{=} 1. \tag{2.7}$$

Knowing that all the elements of $\mathfrak{S}_n^0$ belong to the same conjugacy class, they are equally distributed and Lemme 2.6 follows from (2.7). □

We now prove theorems 2.1 and 2.2.

Proof of theorems 2.1 and 2.2. Equality (2.6) implies that

$$T^{n-1}(\rho_n) \overset{a.s}{=} T^{\#(\rho_n)-1}(\rho_n).$$

Therefore, almost surely,

$$d_F(f(T^{n-1}(\rho_n)), f(\rho_n)) = d_F(f(T^{\#(\rho_n)-1}(\rho_n)), f(\rho_n)) \le \varepsilon'_{n,\#(\rho_n)}.$$

Thus, if $\varepsilon'_{n,\#(\rho_n)} \xrightarrow[n\to\infty]{\mathbb{P}} 0$, then for any $\varepsilon > 0$,

$$\mathbb{P}\left(d_F\left(f(T^{n-1}(\rho_n)), f(\rho_n)\right) > \varepsilon\right) \xrightarrow[n\to\infty]{} 0. \tag{2.8}$$

According to Lemma 2.6, $T^{n-1}(\rho_n)$ does not depend on the law of ρ_n. By choosing at first $\rho_n = \sigma_{ref,n}$, (2.2) then yields

$$f(\sigma_{Ew,0,n}) \xrightarrow[n\to\infty]{\mathbb{P}} x.$$

By choosing at a second step $\rho_n = \sigma_n$, we obtain (2.4) for any σ_n satisfying the hypothesis of Theorem 2.1. One can prove Theorem 2.2 using the same argument. □

2.2 Proof of Corollary 2.5

2.2.1 First application: Longest Increasing Subsequence

Direct applications of the previous universality result are Theorem 1.4 and Corollary 1.6. The key argument of our proofs is the following lemma:

Lemma 2.7. For any permutation σ and for any transposition τ,

$$|\mathrm{LIS}(\sigma\circ\tau) - \mathrm{LIS}(\sigma)| \le 2, \quad |\mathrm{LDS}(\sigma\circ\tau) - \mathrm{LDS}(\sigma)| \le 2.$$

Proof. Let σ be a permutation. By definition of $\mathrm{LIS}(\sigma)$, there exists $i_1 < i_2 < \cdots < i_{\mathrm{LIS}(\sigma)}$ such that $\sigma(i_1) < \cdots < \sigma(i_{\mathrm{LIS}(\sigma)})$. Let $\tau = (j,k)$ be a transposition and $i'_1, i'_2, \ldots, i'_m$ be the same sequence as $i_1, i_2, \ldots, i_{\mathrm{LIS}(\sigma)}$ after removing j and k if needed. We have $\sigma(i'_1) < \cdots < \sigma(i'_m)$. In particular,

$\text{LIS}(\sigma) - 2 \leq m \leq \text{LIS}(\sigma)$. Knowing that $\forall i \notin \{j, k\}, \sigma \circ \tau(i) = \sigma(i)$, then

$$\sigma \circ \tau(i'_1) < \cdots < \sigma \circ \tau(i'_m).$$

Therefore,

$$\text{LIS}(\sigma) - \text{LIS}(\sigma \circ \tau) \leq 2. \tag{2.9}$$

We obtain the second inequality by replacing σ by $\sigma \circ \tau$ in (2.9). For $\text{LDS}(\sigma)$ the proof is similar. □

Proof of Theorem 1.4 and Corollary 1.6 . The main functions we want to study are

$$f_{\text{LIS1}}(\sigma) := \frac{\text{LIS}(\sigma)}{\sqrt{n}} \text{ and } f_{\text{LIS2}}(\sigma) := \frac{\text{LIS}(\sigma) - 2\sqrt{n}}{n^{\frac{1}{6}}}.$$

Using Lemma 2.7, we have for all $n \geq 3$,

$$\varepsilon_n(f_{\text{LIS1}}) = \frac{2}{\sqrt{n}} \text{ and } \varepsilon_n(f_{\text{LIS2}}) = \frac{2}{n^{\frac{1}{6}}},$$

and one can conclude using theorems 2.1 and 2.2 with $\sigma_{ref,n} = \sigma_{unif,n}$ since the uniform case is already studied. Indeed, one can see (Vershik and Kerov, 1977; Logan and Shepp, 1977) for the convergence of f_{LIS1} in probability, (Baik, Deift, and Suidan, 2016) for the convergence in $\mathbb{L}^p$ of f_{LIS1} and (Baik, Deift, and Johansson, 1999) for the convergence of f_{LIS2} in probability. For the $\text{LDS}(\sigma)$, the proof is similar. □

Definition 2.8. Given $\sigma \in \mathfrak{S}_\infty$, a subsequence is said to be increasing (resp. decreasing) circular if it is increasing (resp. decreasing) up to a circular permutation. One can see (Albert et al., 2007) for rigorous definition and more details. We denote by $\text{LICS}(\sigma)$ (resp. $\text{LDCS}(\sigma)$) the length of the longest increasing (resp. decreasing) circular subsequence.

Corollary 2.9. If $(\mathcal{H}^{\mathbb{P}}_{inv,2})$ is satisfied then

$$\frac{\text{LICS}(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} 2 \quad \text{and} \quad \frac{\text{LDCS}(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} 2.$$

Proof. The uniform case is proved in (Albert et al., 2007, Theorem 1) for the LICS and the case of the LDCS can be obtained by composition by the permutation $i \mapsto n - i + 1$. Moreover, using the same argument as for the LIS in Lemma 2.7, we have

$$|\text{LICS}(\sigma \circ \tau) - \text{LICS}(\sigma)| \leq 2 \quad \text{and} \quad |\text{LDCS}(\sigma \circ \tau) - \text{LDCS}(\sigma)| \leq 2,$$

which concludes the proof using Theorem 2.1. □

2.2.2 Second application: Longest Alternating Subsequence

A more tricky application is the length of the Longest Alternating Subsequence. This is a special case of a large class of statistics we will present in the next subsection.

Definition 2.10. Given $\sigma \in \mathfrak{S}_n$, $(\sigma(i_1), \sigma(i_2), \ldots, \sigma_n(i_k))$ is said to be an alternating subsequence of σ of length k if $i_1 < i_2 < \cdots < i_k$ and $\sigma(i_1) > \sigma(i_2) < \sigma(i_3) > \ldots \sigma(i_k)$. We denote by $\mathrm{LAS}(\sigma)$ the length of the longest alternating subsequence of σ.

The uniform case is already studied in (Stanley, 2010; Romik, 2011). We have the two following results.

Proposition 2.11. (Stanley, 2010, Page 17) For $n \geq 2$,

$$\mathbb{E}(\mathrm{LAS}(\sigma_{unif,n})) = \frac{2n}{3} + \frac{1}{6}$$

and for $n \geq 4$,

$$\mathbb{V}\mathrm{ar}(\mathrm{LAS}(\sigma_{unif,n})) = \frac{8n}{45} - \frac{13}{180}.$$

Proposition 2.12. (Romik, 2011, Proposition 4)

$$\frac{\mathrm{LAS}(\sigma_{unif,n}) - \frac{2}{3}n}{\sqrt{n}} \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0, \frac{8}{45}\right).$$

Here, $\mathcal{N}(m, \sigma^2)$ is the normal distribution. We also make use of the following result.

Proposition 2.13. (Romik, 2011, Corollary 2)

$$\mathrm{LAS}(\sigma) = 1 + \sum_{i=1}^{n-1} M_k(\sigma),$$

where

$$M_1(\sigma) = \mathbb{1}_{\sigma(1)>\sigma(2)}$$

and for $1 < k < n$,

$$M_k(\sigma) = \mathbb{1}_{\sigma(k-1)>\sigma(k)<\sigma(k+1)} + \mathbb{1}_{\sigma(k-1)<\sigma(k)>\sigma(k+1)}.$$

This yields the following.

Lemma 2.14. For any $\sigma \in \mathfrak{S}_n$ and $1 \leq i, j \leq n$,

$$|\mathrm{LAS}(\sigma) - \mathrm{LAS}(\sigma \circ (i, j))| \leq 6.$$

Proof. Let $1 \leq k < n$. If $\min(|k-i|, |k-j|) \geq 2$, then $M_k(\sigma) = M_k(\sigma \circ (i,j))$ and consequently,

$$\begin{aligned}
|\mathrm{LAS}(\sigma) - \mathrm{LAS}(\sigma \circ (i,j))| &= \left| \sum_{k \in (\{i-1,i,i+1\} \cup \{j-1,j,j+1\}) \cap \{1,\dots,n-1\}} M_k(\sigma) - M_k(\sigma \circ (i,j)) \right| \\
&\leq \sum_{k \in (\{i-1,i,i+1\} \cup \{j-1,j,j+1\}) \cap \{1,\dots,n-1\}} |M_k(\sigma) - M_k(\sigma \circ (i,j))| \\
&\leq \sum_{k \in (\{i-1,i,i+1\} \cup \{j-1,j,j+1\}) \cap \{1,\dots,n-1\}} 1 \\
&= \mathrm{card}((\{i-1,i,i+1\} \cup \{j-1,j,j+1\}) \cap \{1,\dots,n-1\}) \\
&\leq 6.
\end{aligned}$$

□

Consequently, we have the next corollary.

Corollary 2.15. • Under ($\mathcal{H}^{\mathbb{P}}_{inv,1}$), we have

(2.10) $$\frac{\mathrm{LAS}(\sigma_n)}{n} \xrightarrow[n\to\infty]{\mathbb{P}} \frac{2}{3}$$

and

(2.11) $$\mathbb{E}(\mathrm{LAS}(\sigma_n)) = \frac{2}{3}n + o(n).$$

• Under ($\mathcal{H}^{\mathbb{P}}_{inv,2}$), we have

(2.12) $$\frac{\mathrm{LAS}(\sigma_n) - \frac{2}{3}n}{\sqrt{n}} \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0, \frac{8}{45}\right).$$

Proof of Corollary 2.15. Let f_{LAS1} and f_{LAS2} be the two functions defined on $\mathfrak{S}_\infty$ by:
For $\sigma \in \mathfrak{S}_n$,

$$f_{LAS1}(\sigma) := \frac{\mathrm{LAS}(\sigma)}{n} \quad \text{and} \quad f_{LAS2}(\sigma) := \frac{\mathrm{LAS}(\sigma) - \frac{2}{3}n}{\sqrt{n}}.$$

By Lemma 2.14, we obtain $\varepsilon_n(f_{LAS1}) \leq \frac{6}{n}$ and $\varepsilon_n(f_{LAS2}) \leq \frac{6}{\sqrt{n}}$. Thus (2.10) and (2.12) follow from theorems 2.1 and 2.2. Moreover, since $\frac{\mathrm{LAS}(\sigma_n)}{n} \in (0,1]$, (2.11) is a direct consequence of (2.10).

□

2.3 Local statistics

2.3.1 Definition and examples

Definition 2.16. Given $k \geq 1$, we call a function f defined on $\mathfrak{S}_\infty$ a local function of type k, and we write $f \in \mathcal{L}oc_k$, if there exist a positive integer $m \geq 1$, a Boolean function g defined on $\mathbb{N}^{(m+1)k}$ such that, for any $n \geq k+m-1$ and any $\sigma \in \mathfrak{S}_n$,

$$f(\sigma) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} g(i_1, \ldots, i_k, \sigma(i_1), \sigma(i_1 - 1), \ldots, \sigma(i_1 - m + 1), \sigma(i_2), \ldots, \sigma(i_k - m + 1)).$$

We used the convention $\sigma(i) = 0$ when $i \leq 0$.

Here are some examples of local statistics.

- The number of fixed points:
  By choosing $k = m = 1$ and $g(x, y) = \mathbb{1}_{x=y}$, we obtain that $\mathrm{tr} \in \mathcal{L}oc_1$.
- $\#_k \in \mathcal{L}oc_k$ and $\sigma \mapsto \mathrm{tr}(\sigma^k) \in \mathcal{L}oc_k$.
- The number of j-exceedances[6]:
  For $j \in \mathbb{N}$ fixed, we define for $\sigma \in \mathfrak{S}_n$ and, we define

$$\mathcal{N}_{exc_j}(\sigma) := \mathrm{card}(\{i, \sigma_i \geq i + j\}).$$

  We choose again $k = m = 1$ and $g(x, y) = \mathbb{1}_{x+j \leq y}$ and we obtain again $\mathcal{N}_{exc_j} \in \mathcal{L}oc_1$.
- Longest alternating subsequence (LAS):
  $\mathrm{LAS} \in \mathcal{L}oc_1$. This is a direct application of Proposition 2.13. Here, $k = 1, m = 3$ and

$$g(i, y_1, y_2, y_3) = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i = 1 \\ \mathbb{1}_{y_2 > y_1} & \text{if } i = 2 \\ \mathbb{1}_{l<k>j} + \mathbb{1}_{y_3 > y_2 < y_1} & \text{if } i > 2 \end{cases}.$$

- Number of peaks:
  For $\sigma \in \mathfrak{S}_n$, we define

$$\mathcal{N}_{peak}(\sigma) := \mathrm{card}(\{1 < i < n, \sigma(i-1) < \sigma(i) > \sigma(i+1)\}).$$

  We choose again $k = 1, m = 3$ and $g(x, y_1, y_2, y_3) = \mathbb{1}_{x \geq 3} \mathbb{1}_{y_1 < y_2 > y_3}$ and we obtain again $\mathcal{N}_{peak} \in \mathcal{L}oc_1$.

[6] In the literature, j-exceedances is sometimes defined by the condition $\sigma_i \geq i + j$ and othertimes by $\sigma_i = i + j$. In both cases, the number j-exceedances is a local statistic but only the first case is in interest for our purpose.

- Number of j-descents:
  For $j \geq 1, \sigma \in \mathfrak{S}_n$, we define

$$\mathcal{N}_{D_j}(\sigma) := \text{card}\{1 \leq i \leq n-1, \sigma(i+1) + j \leq \sigma(i)\}.$$

  We choose $k = 1, m = 2$ and $g(x, y_1, y_2) = \mathbb{1}_{x \geq 2} \mathbb{1}_{y_2 \geq y_1 + j}$ and we obtain again $\mathcal{N}_{D_j} \in \mathcal{L}oc_1$.

  When $j = 1$, the $1-$descents are known as the descents. We also set

$$\mathcal{N}_D(\sigma) := \text{card}\{1 \leq i \leq n-1, \sigma(i+1) < \sigma(i)\} = \mathcal{N}_{D_1}(\sigma).$$

- Number of inversions and $m-$clicks of the permutation graph:

  Definition 2.17. Let $\sigma \in \mathfrak{S}_n$. Let $\mathfrak{G}(\sigma) = (V_{\mathfrak{G}(\sigma)}, E_{\mathfrak{G}(\sigma)})$[7] be the permutation graph of σ defined by

$$V_{\mathfrak{G}(\sigma)} = \{1, \ldots, n\} \text{ and } E_{\mathfrak{G}(\sigma)} = \{(i, j) \in \{1, 2, \ldots, n\}; (\sigma(i) - \sigma(j))(i - j) < 0\}.$$

  For example, $E_{\mathfrak{G}(\sigma)} = \varnothing$ if and only if $\sigma = Id_n$ and for the permutation $\sigma : i \mapsto n - i + 1$, $\mathfrak{G}(\sigma)$ is the complete graph with n vertices.

  Given $j \geq 2$, we denote by

$$\mathcal{K}_j(\sigma) := \text{card}(\{(i_1, i_2, \ldots, i_j); 1 \leq i_1 < \cdots < i_j \leq n, \sigma(i_1) > \cdots > \sigma(i_j)\})$$

  the number of j-clicks of $\mathfrak{G}(\sigma)$[8]. In particular, $\mathcal{K}_2(\sigma)$ is the number of inversions of σ. One can easily check that with $\mathcal{K}_j \in \mathcal{L}oc_j$. Here,

$$g(x_1, \ldots, x_j, y_1, \ldots, y_j) = \mathbb{1}_{y_1 > y_2 > \cdots > y_j}.$$

- Let $d_k(\sigma) := \text{card}(\{i; (i, k) \in E_{\mathfrak{G}(\sigma)}\})$ be the degree of the vertex k in $\mathfrak{G}(\sigma)$. We have $d_k(\sigma) \in \mathcal{L}oc_2$.

2.3.2 First universality results for local statistics

Proposition 2.18. Given $k \geq 1$, $f \in \mathcal{L}oc_k$, a random real variable X, $k - 1 < \gamma \leq k$ and $(a_n)_{n \geq 0} \in \mathbb{R}^{\mathbb{N}}$ such that

$$\frac{f(\sigma_{unif,n}) - a_n}{n^\gamma} \xrightarrow[n \to \infty]{d} X,$$

[7] Fun fact 1: the application $\sigma \mapsto \mathfrak{G}(\sigma)$ is injective.

[8] This a special case of the number of occurrences of a pattern in a permutation. In general, the number of occurrences of any pattern is a local statistic.

if ($\mathcal{H}^{\mathbb{P}}_{inv,\frac{1}{\gamma-k+1}}$) holds then

$$\frac{f(\sigma_n)-a_n}{n^{\gamma}} \xrightarrow[n\to\infty]{d} X.$$

Proof. By counting the number of possible choices of $1 \le i_1 < i_2, \cdots < i_k \le n$ such that $\{i,j\} \cap \{i_1,\dots,i_1-m+1, i_2,\dots,i_k-m+1\} \neq \emptyset$, it is easy to see that for any permutation $\sigma \in \mathfrak{S}_n$ and any transposition (i,j) we have

$$|f(\sigma(i,j)) - f(\sigma)| \le \frac{2km(n-1)!}{(k-1)!(n-k)!} \le 2kmn^{k-1}.$$

Consequently for $h = \frac{f-a_n}{n^{\gamma}}$, $\varepsilon_n(h) \le 2kn^{k-\gamma-1}m$ and one can conclude using Remark 2.4. □

One can then easily apply this result combined with the discussion in the previous subsection to our local statistics.

Corollary 2.19. Under ($\mathcal{H}^{\mathbb{P}}_{inv,1}$), we have for any $j \ge 2$,

$$\frac{\mathcal{N}_{D_j}(\sigma_n)}{n} \xrightarrow[n\to\infty]{\mathbb{L}^1} \frac{1}{2},$$
$$\frac{\mathcal{N}_{D}(\sigma_n)}{n} \xrightarrow[n\to\infty]{\mathbb{L}^1} \frac{1}{2},$$
$$\frac{\mathcal{K}_{j}(\sigma_n)}{n^m} \xrightarrow[n\to\infty]{\mathbb{L}^1} \frac{1}{(m!)^2},$$
$$\frac{\mathcal{N}_{exc_j}(\sigma_n)}{n} \xrightarrow[n\to\infty]{\mathbb{L}^1} \frac{1}{2},$$
$$\frac{\mathcal{N}_{peak(\sigma_n)}}{n} \xrightarrow[n\to\infty]{\mathbb{L}^1} \frac{1}{3}.$$

Moreover, under ($\mathcal{H}^{\mathbb{P}}_{inv,2}$), we have for any $j \ge 2$,

$$\frac{\mathcal{N}_{D_j}(\sigma_n)-\frac{n}{2}}{\sqrt{n}} \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0,\frac{1}{12}\right),$$
$$\frac{\mathcal{N}_{D}(\sigma_n)-\frac{n}{2}}{\sqrt{n}} \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0,\frac{1}{12}\right),$$
$$\frac{\mathcal{K}_{j}(\sigma_n)-\frac{n^j}{(j!)^2}}{n^{j-\frac{1}{2}}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0,v_j),$$
$$\frac{\mathcal{N}_{exc_j}(\sigma_n)-\frac{n}{2}}{\sqrt{n}} \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0,\frac{1}{12}\right),$$
$$\frac{\mathcal{N}_{peak}(\sigma_n)-\frac{n}{2}}{\sqrt{n}} \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0,\frac{2}{45}\right),$$

where

$$v_j = \frac{\binom{4j-2}{2j-1} - 2\binom{2j-1}{j}^2}{2((2m-1)!)^2}.$$

The uniform case for $\mathcal{N}_D, \mathcal{N}_{peak}, \mathcal{K}_j$ and $\mathcal{N}_{exc_1}$ has already been studied. One can find a proof respectively in (Kim and Lee, 2020), (Fulman, Kim, and Lee, 2019), (Gürerk, Islak, and Yildiz, 2019) and (Féray, 2013). For the conjugation invariant case, as we explained before, $\mathcal{N}_D$ and $\mathcal{N}_{peak}$ are fully understood but, to the best knowledge of the author, it is not the case for $\mathcal{K}_j$ and $\mathcal{N}_{exc_1}$. For $\mathcal{N}_{exc_1}$, the special case of the Ewens distribution is studied in (Féray, 2013). Moreover, the results for $\mathcal{N}_{D_j}$ and $\mathcal{N}_{exc_j}$ are direct consequences of respectively $\mathcal{N}_D$ and $\mathcal{N}_{exc_1}$ since for any conjugation invariant random permutation σ_n,

$$0 \leq \mathbb{E}(\mathcal{N}_D(\sigma_n) - \mathcal{N}_{D_j}(\sigma_n)) = \frac{(j-1)(n-j-1)(1-\mathbb{P}(\sigma_n(1)=1))}{n-1} \leq j-1$$

and

$$0 \leq \mathbb{E}(\mathcal{N}_{exc_1}(\sigma_n) - \mathcal{N}_{exc_j}(\sigma_n)) \leq j-1.$$

2.4 Further discussion and improved bounds

2.4.1 Universality for $\widetilde{\mathcal{L}oc}$

Also, we denote by $\widetilde{\mathcal{L}oc}$ the set of local functions f of any type associated with a Boolean function g such that

$$\text{card}(\{i \in \mathbb{N}^*; \max_{I \in \mathbb{N}^{k-1}} \max_{J \in \mathbb{N}^{mk}} g(I,i,J) = 1\}) < \infty. \tag{2.13}$$

For this class, it is simple to obtain the convergence of the expectation. It can be seen as a macroscopic universality result.

Let $A \subset \mathbb{N}^*$ be finite, $n > \max(A)$ and $(\sigma_n)_{n\geq 1}$ satisfying $(\mathcal{H}_{inv})$. Using again the random walk associated to T and seeing that

$$\mathbb{P}(\exists i \in \{i_1 - i_2; i_1 \in A, 0 \leq i_2 < m-1\}, (T^{n-1}(\sigma_n))(i) \neq \sigma_n(i)) \leq \frac{2\#(\sigma_n)\text{card}(A)m}{n},$$

we obtain the following.

Proposition 2.20. Given $f \in \widetilde{\mathcal{L}oc}$ and assuming that $(\sigma_n)_{n\geq 1}$ and $(\sigma_{ref,n})_{n\geq 1}$ satisfy $(\mathcal{H}^{\mathbb{P}}_{inv,1})$ we have

$$\mathbb{E}(f(\sigma_n)) - \mathbb{E}(f(\sigma_{ref,n})) \xrightarrow[n\to\infty]{} 0.$$

Moreover, if $f(\sigma_{ref,n})$ converges in distribution then $f(\sigma_n)$ does also converge to the same limit.

We give now an application: Given n be a positive integer and $\sigma \in \mathfrak{S}_n$, we define

(2.14) $$D(\sigma) := \{i \in \{1, \ldots, n-1\};\ \sigma(i+1) < \sigma(i)\}.$$

When σ is random, $D(\sigma)$ is known as a descent process.

Given $A \subset \mathbb{N}^*$ finite, if we introduce

(2.15) $$D^A(\sigma) := \mathbb{1}_{A \subset D(\sigma)},$$

then $D^A \in \mathcal{L}oc_{|A|} \cap \widetilde{\mathcal{L}oc}$. Here,

$$g(x_1, x_2, \ldots, x_{|A|}, y_1, y'_1, y_2, \ldots, y_{|A|}, y'_{|A|}) = \mathbb{1}_{A=\{x_i-1, 1\le i\le |A|\}} \prod_{i=1}^{|A|} \mathbb{1}_{y_i<y'_i}.$$

We further investigate the descent process. First, the descent process is well understood in the uniform case.

Theorem 2.21. (Borodin, Diaconis, and Fulman, 2010, Theorem 5.1) For any positive integer n and any $A \subset \{1, 2, \ldots, n-1\}$,

$$\mathbb{P}(A \subset D(\sigma_{unif,n})) = \det([k_0(j-i)]_{i,j\in A}),$$

where,

$$\sum_{i\in\mathbb{Z}} k_0(i) z^i = \frac{1}{1-e^z}.$$

We say that the descent process is determinantal with kernel $K_0(i,j) := k_0(j-i)$.

In the non-uniform setting, the descent process is already studied for the Mallows law with Kendall tau metric: it is also determinantal with different kernels, see (Borodin, Diaconis, and Fulman, 2010, Proposition 5.2). We showed in (Kammoun, 2018) that for a large class of random permutations, the limiting descent process is determinantal with the same kernel as the uniform setting. We will detail first a weaker result than (Kammoun, 2018).

Corollary 2.22. Under ($\mathcal{H}^{\mathbb{P}}_{inv,1}$), for any finite set $A \subset \mathbb{N}^*$,

(DPP) $$\lim_{n\to\infty} \mathbb{P}(A \subset D(\sigma_n)) = \det([k_0(j-i)]_{i,j\in A}).$$

Proof. Just apply Proposition 2.20 for the statistic D^A defined in (2.15) □

The same argument can be applied for other local statistics but not necessarily in $\widetilde{\mathcal{L}oc}$. For example, we have similar results for the degree of vertices of the permutation graph.

Proposition 2.23. Under $(\mathcal{H}^{\mathbb{P}}_{inv,1})$,

$$\frac{d_k(\sigma_n)}{n} \xrightarrow[n\to\infty]{\mathbb{P}} \frac{1}{2}, \quad \frac{d_{\frac{n}{2}}(\sigma_n)}{n} \xrightarrow[n\to\infty]{\mathbb{P}} \frac{1}{2}, \quad \frac{d_n(\sigma_n)}{n} \xrightarrow[n\to\infty]{\mathbb{P}} \frac{1}{2}.$$

Moreover, under $(\mathcal{H}^{\mathbb{P}}_{inv,2})$,

$$\frac{d_{\frac{n}{2}}(\sigma_n) - \frac{n}{2}}{2\sqrt{n}} \xrightarrow[n\to\infty]{d} \mathcal{N}(U, 1-U), \quad \frac{d_n(\sigma_n) - \frac{n}{2}}{\sqrt{n}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0,6), \quad \frac{d_k(\sigma_n) - \frac{n}{2}}{\sqrt{n}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0,6),$$

where U is a uniform random variable on $[0,1]$.

Note that d_k is a local statistic for fixed k but it is not the case for d_n. The uniform case is already studied by Gürerk, Islak, and Yildiz (2019). The problem for d_n is that for any $2 < k < n$, $\varepsilon_n(d_n) = n-1$ since $d_n(Id_n) = 0$ and $d_n((n,1)) = n-1$ and thus we cannot apply directly our previous approach. The idea of the proof is the following. If we condition on the event

$$E_n = \{T^1, T^2, \dots, T^n \text{ do not change } \sigma_n(n)\},$$

then d_j changes at most by 2 every time we apply T and one concludes easily since

$$\mathbb{P}(E_n) \geq 1 - 2\frac{\mathbb{E}(\#(\sigma_n))}{n}.$$

2.4.2 Improved bounds

For some statistics, one can obtain a better lower bound by using a different way to go from $\sigma_{Ew,0,n}$ to σ_n. Unlike the previous examples, the control of the error may depend on the statistic. Our first example is the longest increasing subsequence. We give a lower bound for the expectation for a conjugation invariant random permutation. Using this inverse walk one can obtain the following results.

Proposition 2.24. If $(\sigma_n)_{n\geq 1}$ is conjugation invariant then for any $\varepsilon > 0$,

$$\mathbb{P}\left(\mathrm{LIS}(\sigma_n) > (2\sqrt{13} - 6 - \varepsilon)\sqrt{n}\right) \xrightarrow[n\to\infty]{} 1.$$

This yields the following lower bound

$$\liminf_{n\to\infty} \frac{\mathbb{E}(\mathrm{LIS}(\sigma_n))}{\sqrt{n}} \geq 2\sqrt{13} - 6 \simeq 1.21\dots$$

We will prove this result in Chapter 7. Moreover, under an addition assumption of cycle structure, the uniform permutation minimizes asymptotically the LIS of invariant random permutations.

Proposition 2.25. If ($\mathcal{H}^{\mathbb{P}}_{inv,\frac{3}{2}}$) is satisfied, then for any $k \geq 1$, for any $s \in \mathbb{R}$,

$$\limsup_{n\to\infty} \mathbb{P}\left(\frac{\mathrm{LIS}(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) \leq F_2(s).$$

We will define properly F_2 in (5.1).

We will prove these results properly in Subsection 5.3.3 as well as a generalization of Proposition 2.25

Finally, remark that:

- For some statistics, one can obtain better bound using the same technique with better control. One of the possibilities is to control separately small and big cycles. We will apply this technique to prove Theorem 1.5.
- When the distribution has a large proportion of fixed points, it is sometimes more efficient to apply the strategy on reference laws other than the uniform distribution. One can see for example the proof of Proposition 5.10.

3

Schur measures and monotone subsequences

"May not music be described as the mathematics of the sense, mathematics as music of the reason? The soul of each is the same! Thus, the musician feels mathematics, the mathematician thinks music ..."
James Joseph Sylvester

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3.2.6 Some examples of Schur measures . . . 53

Historically, the study of the length of the longest increasing subsequence of permutations has been done through the Robinson–Schensted–Knuth map, aka the RSK (or RS) map, a one-to-one map between the symmetric group and the set of couples of Young tableaux of the same shape. This map has nice properties, in particular, the Greene's Theorem. The shape of the image of $\sigma_{unif,n}$ by RSK is a random partition distributed according to the Plancherel measure which is, up to Poissonization, a particular case of Schur measures which are a set of well-understood probability measures on Young diagrams thanks to their so-called integrable properties. In this chapter, we will introduce the RSK map and the Schur measures. We will not give any new results in this chapter but the concepts are necessary to understand many proofs. Readers who are familiar with the representations of the symmetric group may skip this chapter.

3.1 The Robinson–Schensted–Knuth map

Before presenting the RSK map, we recall the definitions of Young diagrams and Young Tableaux.

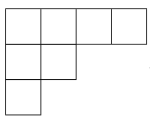
3.1.1 Young diagrams

We start with a few definitions and notations.

- $\lambda = \{\lambda_i\}_{i\geq 1}$: a Young diagram of size n. A Young diagram is an integer partition of n i.e.

  * $\forall i \geq 1, \lambda_i \in \mathbb{N}$,
  * $\forall i \geq 1, \lambda_{i+1} \leq \lambda_i$,
  * $\sum_{i=1}^{\infty} \lambda_i = n$.

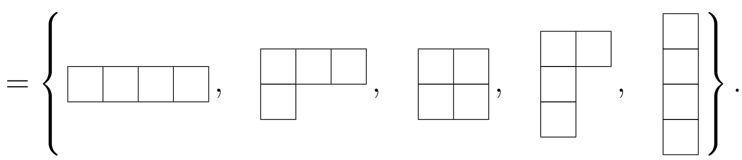
  We can represent a Young diagram by boxes of size 1×1 such that the row i contains exactly λ_i boxes. For example, if $\lambda = (4, 2, 1, \underline{0})$, we have the diagram

  ,

  where $\underline{0} = (0)_{i\geq 1}$.

- $\mathbb{Y}_n$: the set of Young diagrams of size n. For example,

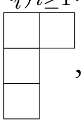
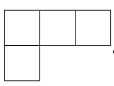
$$\mathbb{Y}_4 = \{(4, \underline{0}),\ (3, 1, \underline{0}),\ (2, 2, \underline{0}),\ (2, 1, 1, \underline{0}),\ (1, 1, 1, 1, \underline{0})\}$$

- $|\lambda| := \sum_{i\geq} \lambda_i$: the size of the partition. For example, $|(2, 1, 1, \underline{0})| = 4$.

- $\mathbb{Y} := \cup_{n\geq 0} \mathbb{Y}_n$: the set of Young diagrams[1].

- $\lambda' = (\lambda'_i)_{i\geq 1}$: the transpose of λ i.e. $\lambda'_i = \mathrm{card}(j; \lambda_j \geq i)$. For example,

  If $\lambda =$ , then $\lambda' =$ .

[1]also called Ferrers diagrams

- $\ell(\lambda) := \mathrm{card}\{i; \lambda_i > 0\} = \lambda'_1$: the length of the partition. For example, $\ell((2,1,1,\underline{0})) = 3$.
- $\hat{\lambda}(\sigma) \in \mathbb{Y}_n$: the cycle structure of σ i.e. the integer partition obtained by ordering the lengths of the cycles of σ and by adding as many 1 as fixed points. For example, $\hat{\lambda}(\sigma_{ex}) = (3,2,1,\underline{0})$. We recall that
$$\sigma_{ex} := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 2 & 1 & 4 & 6 \end{pmatrix} = (1,5,4)(2,3).$$
In particular, we have $\#(\sigma) := \ell(\hat{\lambda}(\sigma))$.

3.1.2 Young tableaux

Definition 3.1. Given an integer partition (a Young diagram) $\lambda \in \mathbb{Y}_n$, a (standard) Young tableau of shape λ is a filling of the boxes of λ using the entries $\{1,2,\dots,n\}$ increasing in each row and each column.

For example, standard Young tableaux of shape $\begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1}\end{array}$ are $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$.

Let $\dim(\lambda) = \mathrm{card}(\{$ Young tableaux of shape $\lambda\})$. For example, $\dim\left(\begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1}\end{array}\right) = 3$. We have the following classic result.

Proposition 3.2. $\dim(\mu)$ is equal to the dimension of the irreducible representation of $\mathfrak{S}_n$ indexed by μ i.e. the dimension of the irreducible representation associated to the conjugacy class
$$\{\sigma \in \mathfrak{S}_n; \hat{\lambda}(\sigma) = \mu\}.$$

One can find a proof of this result in (Sagan, 2001).

Corollary 3.3. (Burnside identity)
$$\sum_{\lambda \in \mathbb{Y}_n} \dim(\lambda)^2 = \mathrm{card}(\mathfrak{S}_n) = n!.$$

Definition 3.4. Given a Young diagram $\lambda \in \mathbb{Y}_n$ and a positive integer N, a semi-standard Young tableau of shape λ is a filling of the boxes of λ using the entries $\{1,2,\dots,N\}$, weakly increasing in each row and increasing in each column.

For exemple,. for $N = 2$, semi-standard Young tableaux of shape $\begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1}\end{array}$ are
$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

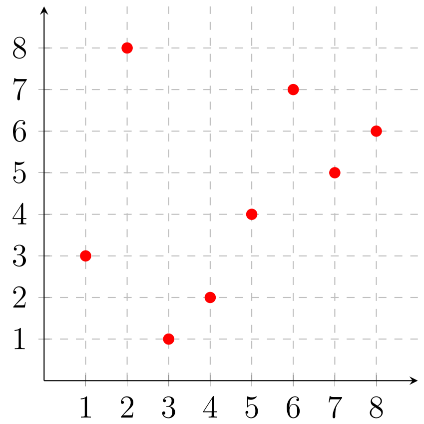
3.1.3 Viennot's geometric construction

Corollary 3.3 guarantees the existence of a one-to-one map between the symmetric group $\mathfrak{S}_n$ and the set of couples of Young tableaux with same shape in $\mathbb{Y}_n$. One of those maps is known as the

Robinson–Schensted map (Robinson, 1938; Schensted, 1961) or the Robinson–Schensted–Knuth map (Knuth, 1970). We will present the RSK map using the Viennot geometric construction (Viennot, 1977). For other equivalent constructions, we recommend (Sagan, 2001). For sake of simplicity, we will apply every step directly on

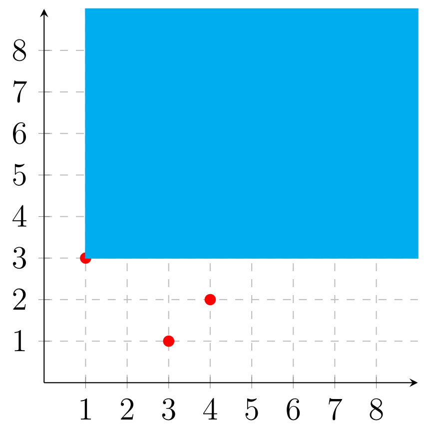
$$\sigma_{ex,2} := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix}.$$

- Step 1 : We draw the points $\{(i, \sigma(i)), 1 \leq i \leq n\}$.



- Step 2: Starting from the first point $(1, \sigma(1))$, we shadow the top-right region i.e.

$$\{(x, y); x \geq 1, y \geq \sigma(1)\}.$$

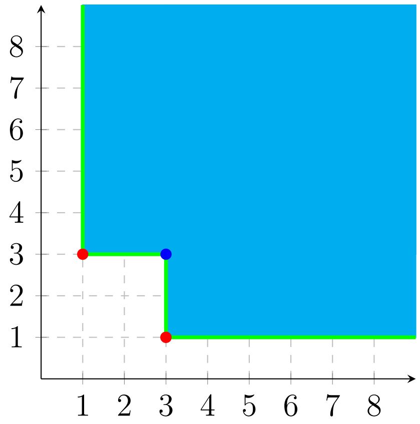


- Step 3: We continue shadowing the top right region of every point i.e. we shadow the region

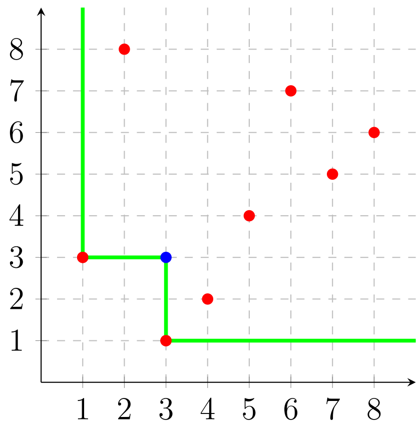
$$\cup_{i=1}^{n}\{(x, y); x \geq i, y \geq \sigma(i)\}.$$

We draw the boundary of the shadowed region. We add a point with a different color (blue here) in every corner of type ⌉.

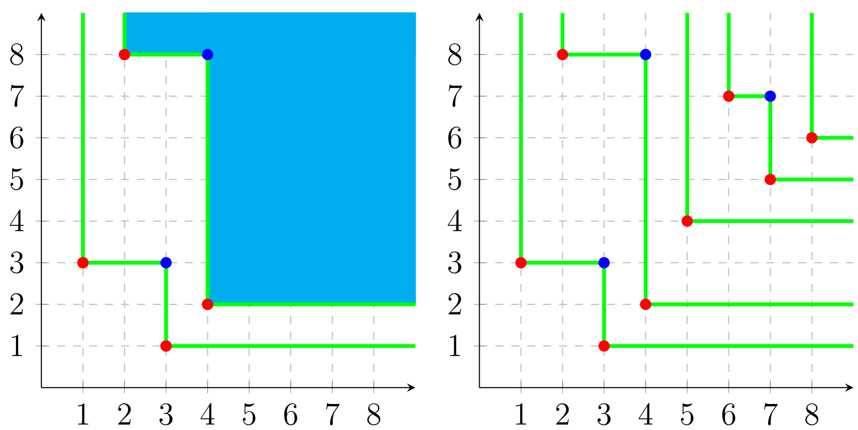
In the following example, we have one ⌉ corner in (3,3) and two L corners in $(1, 3)$ and $(3, 1)$.



- Step 4: We remove the shadowed region and we draw again the non-used points.

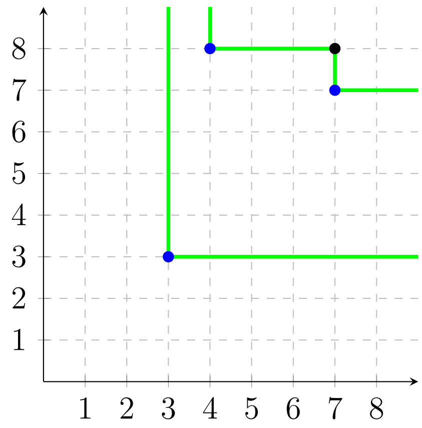


- Step 5: We repeat steps 2-3-4 for unused points until using all original (red) points. The length of the first row of both Young tableaux is then the number of times we repeated steps 2-3-4. The entries of the first row of the first Young tableau are just the y-coordinate of the horizontal lines going to infinity and those of the second Young tableau is the x-coordinate of the vertical lines going to infinity.



| 1 | 2 | 4 | 5 | 6 |, | 1 | 2 | 5 | 6 | 8 |

- Step 6: We repeat steps 2-3-4-5 for the previous points we added (blue points) to obtain the second array of the two tableaux. We choose a different color for new ⅂ corners (black here).

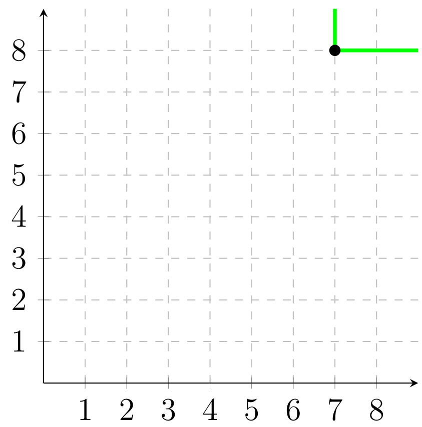


1	2	4	5	6
3	7			

,

1	2	5	6	8
3	4			

- Step 7: We continue till using all points i.e. we do not have any ⅂ corner.



1	2	4	5	6
3	7			
8				

,

1	2	5	6	8
3	4			
7				

The two tableaux are obviously of the same shape and we denote by

$$\lambda(\sigma) = \{\lambda_i(\sigma)\}_{i\geq 1}.$$

the shape of the image of σ by this map.

3.1.4 Greene's Theorem

Greene's theorem is a way to understand the $\lambda(\sigma)$ in term of increasing subsequences of σ. Let $\sigma \in \mathfrak{S}_n$. We denote by

$$\begin{aligned}
\mathfrak{I}_1(\sigma) &:= \{s \subset \{1,2,\dots,n\};\ \forall i,j \in s,\ (i-j)(\sigma(i)-\sigma(j)) \geq 0\},\\
\mathfrak{D}_1(\sigma) &:= \{s \subset \{1,2,\dots,n\};\ \forall i,j \in s,\ (i-j)(\sigma(i)-\sigma(j)) \leq 0\},\\
\mathfrak{I}_{k+1}(\sigma) &:= \{s \cup s',\ s \in \mathfrak{I}_k,\ s' \in \mathfrak{I}_1\},\\
\mathfrak{D}_{k+1}(\sigma) &:= \{s \cup s',\ s \in \mathfrak{D}_k,\ s' \in \mathfrak{D}_1\}.
\end{aligned}$$

For example, for

$$\sigma_{ex,3} := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \mathfrak{I}_1(\sigma_{ex,3}) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}\}$$

and

$$\mathfrak{I}_2(\sigma_{ex,3}) = \mathfrak{D}_2(\sigma_{ex,3}) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

We have the following.

Proposition 3.5. (Greene, 1974) For any permutation $\sigma \in \mathfrak{S}_n$,

$$\max_{s \in \mathfrak{I}_i(\sigma)} \operatorname{card}(s) = \sum_{k=1}^{i} \lambda_k(\sigma), \qquad \max_{s \in \mathfrak{D}_i(\sigma)} \operatorname{card}(s) = \sum_{k=1}^{i} \lambda'_k(\sigma).$$

In particular,

$$\max_{s \in \mathfrak{I}_1(\sigma)} \operatorname{card}(s) = \lambda_1(\sigma) = \mathrm{LIS}(\sigma), \qquad \max_{s \in \mathfrak{D}_1(\sigma)} \operatorname{card}(s) = \lambda'_1(\sigma) = \mathrm{LDS}(\sigma).$$

This result is obtained first by Greene (1974) (see also (Sagan, 2001, Theorem 3.7.3)). The case where $i = 1$, is attributed to Schensted (1961). An alternative and recent construction to understand increasing subsequences is given by Houdré and Litherland (2011).

3.1.5 RSK for words

Let $\mathcal{A} = a_1 < a_2 < \cdots < a_N$ be a finite alphabet and $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ be a finite word of $\mathcal{A}$ i.e. $\forall 1 \leq i \leq n, \omega_i \in \mathcal{A}$. In this case, the RSK map is a one-to-one map between words of length n and couples of standard Young tableaux and semi-standard Young tableaux of same shape and using the entries $1, 2, \dots, N$. The RSK algorithm is similar to that we described before. For example, for the word $(1,1,2,1)$, we obtain

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & & \\ \cline{1-1} \end{array}\ , \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & & \\ \cline{1-1} \end{array}$$

as its image. We will not detail here an example, we recommend (Sagan, 2001) for more details.

Remark 3.6. Note that the Greene Theorem also holds for the RSK image of words. The only difference is that subsequences are weakly increasing (or decreasing).

3.2 Schur measures

This section is highly inspired by (Borodin and Gorin, 2016). Let $\Lambda_n := \mathbb{C}_{sym}[x_1, \ldots, x_n]$ be the ring of symmetric polynomials with n variables with coefficients in $\mathbb{C}$.

Definition 3.7. Given $P \in \Lambda_n$ such that

$$P(x_1, \ldots, x_n) = \sum_{j=1}^{N} c_j \prod_{i=1}^{n} x_i^{\alpha_{i,j}},$$

with $c_j \neq 0$ and $\left\{(\alpha_{i,j})_{i\in\{1,\ldots,n\}}\right\}_{j\in\{1,\ldots,N\}}$ are pairwise different. We define,

$$\deg(P) := \max_{1\leq j\leq N} \sum_{i=1}^{n} \alpha_{i,j}.$$

Here are some usual examples of symmetric polynomials of degree k.

- Complete homogeneous functions:

$$h_{k,n}(x_1, x_2, \ldots, x_n) = \sum_{1\leq i_1\leq i_2\cdots\leq i_k\leq n} \prod_{j=1}^{k} x_{i_j}.$$

- Elementary homogeneous functions (of degree $-\infty$ when $k > n$):

$$e_{k,n}(x_1, x_2, \ldots, x_n) = \sum_{1\leq i_1< i_2\cdots< i_k\leq n} \prod_{j=1}^{k} x_{i_j}.$$

- Power sums (Newton polynomials) :

$$p_{k,n}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i^k.$$

For example,

$$\begin{aligned} p_{2,3}(x_1, x_2, x_3) &= x_1^2 + x_2^2 + x_3^2 \\ e_{2,3}(x_1, x_2, x_3) &= x_1x_2 + x_2x_3 + x_3x_1 \\ h_{2,3}(x_1, x_2, x_3) &= e_2(x_1, x_2, x_3) + p_2(x_1, x_2, x_3). \end{aligned}$$

We will use the conventions: $e_{0,n} = h_{0,n} = p_{0,n} = 1$ and $\forall z < 0$, $h_{z,n} = e_{z,n} = p_{z,n} = 0$.

3.2.1 The ring of symmetric polynomials

The functions given above are compatible with projection in the following sense. Given $n \in \mathbb{N}^*$, we define

$$\begin{aligned} \pi_n : \Lambda_n &\to \Lambda_{n-1} \\ f(x_1, x_2, \ldots, x_n) &\mapsto f(x_1, x_2, \ldots, x_{n-1}, 0). \end{aligned}$$

Definition 3.8. (The ring of symmetric polynomials)
The ring of symmetric polynomials Λ is defined as the projective limit of

$$\Lambda := \Lambda_0 \overset{\pi_1}{\longleftarrow} \Lambda_1 \overset{\pi_2}{\longleftarrow} \Lambda_2, \ldots$$

i.e. Λ is the set of sequences $(f_i)_{i\in\mathbb{N}}$ such that

- $\forall i \in \mathbb{N}, f_i \in \Lambda_i$.
- $\forall i \geq 1, \pi_i(f_i) = f_{i-1}$.
- $\max_{i\in\mathbb{N}} deg(f_i) < \infty$.

For example, for any $k \geq 0$, $p_k, e_k, h_k \in \Lambda$ where

$$p_k(x_1, \ldots, x_n) = p_{k,n}(x_1, \ldots x_n),$$

$$e_k(x_1, \ldots, x_n) = e_{k,n}(x_1, \ldots x_n),$$
$$h_k(x_1, \ldots, x_n) = h_{k,n}(x_1, \ldots x_n).$$

Theorem 3.9.
$$\Lambda = \mathbb{C}[\{h_i\}_{i\geq 1}] = \mathbb{C}[\{e_i\}_{i\geq 1}] = \mathbb{C}[\{p_i\}_{i\geq 1}].$$

In other words, every symmetric polynomial can be written in a unique way as a sum of products of complete (resp. elementary) homogeneous functions. One can find a rigorous proof of this classic result in (Macdonald, 1995, Chapitre 1,Section 2).

3.2.2 Schur polynomials

Definition 3.10. Given $N \geq 1$ and a Young diagram λ such that $\ell(\lambda) \leq N$, the Schur polynomial s_λ is defined by:

$$s_\lambda(x_1, x_2, \ldots, x_N) = \frac{\det\left[x_i^{\lambda_j+N-j}\right]_{1\leq, i,j\leq N}}{\det[x_i^{N-j}]_{1\leq i,j\leq N}} = \frac{\det\left[x_i^{\lambda_j+N-j}\right]_{1\leq, i,j\leq N}}{\prod_{i<j}(x_i - x_j)}.$$

When $\ell(\lambda) > N$, we use the convention $s_\lambda(x_1, x_2, \ldots, x_N) = 0$.

Proposition 3.11.

$$\forall \lambda \in \mathbb{Y}, s_\lambda \in \Lambda.$$

This is a direct application of the Jacubi-Trudi formula and the anti-symmetry of the determinant.

Proposition 3.12. (Jacobi–Trudi formula)

$$s_\lambda = \det[h_{\lambda_i - i + j}]_{1 \le i,j \le \ell(\lambda)} = \det[e_{\lambda'_i - i + j}]_{1 \le i,j \le \ell(\lambda')}.$$

Proposition 3.13.
For any, $x_1, x_2, \ldots, x_k \in \mathbb{C}$, $s_\lambda(x_1, \ldots, x_k)$ is the sum over all semi-standard Young tableaux of shape λ using the entries $1, 2, \ldots, k$ of $\prod_{i=1}^k x_i^{\text{the number of entries equal to } i}$.

For example, the semi-standard Young tableau $\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$ gives the polynomial $x_1^1 x_2^3$ and

$$s_{(3,1,\underline{0})}(x_1, x_2) = x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3.$$

3.2.3 Schur's positive specializations

Definition 3.14. A specialization ρ is a map from Λ to $\mathbb{C}$ such that $\forall \lambda \in \mathbb{C}, \forall f, g \in \Lambda$,

$$\begin{aligned} \rho(f + g) &= \rho(f) + \rho(g), \\ \rho(f.g) &= \rho(f)\rho(g) \\ \text{and } \rho(\lambda f) &= \lambda \rho(f). \end{aligned}$$

In the sequel of this thesis, we will use the same notations as in (Borodin and Gorin, 2016); we will denote by $f(\rho)$ the quantity $\rho(f)$. One can see the legitimacy of this notation because for all $(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ the map

$$f \mapsto \rho(f) = f(x_1, \ldots, x_n)$$

is a specialization.

Definition 3.15. A specialization ρ is said to be Schur positive if

$$\forall \lambda \in \mathbb{Y}; s_\lambda(\rho) \ge 0.$$

for exemple, for any non-negative real numbers $x_1, \ldots, x_n$, the specialization $f \mapsto f(x_1, \ldots, x_n)$ is Schur positive. This is direct application of Proposition 3.13.

Theorem 3.16. (Thoma's theorem, Schoenberg-Edrei Theorem)
A specialization ρ is Schur positive if and only if there exists a tuple (α, β, γ) where γ is a non-negative real number and $\alpha = (\alpha_i)_{\geq 1}$ and $\beta = (\beta_i)_{\geq 1}$ are two sequences of non-negative real numbers satisfying

$$\sum_{i=1}^{\infty} \alpha_i + \beta_i < \infty$$

such that

$$p_1(\rho) = \gamma + \sum_{i=1}^{\infty} (\alpha_i + \beta_i),$$
$$\forall k \geq 2,\ p_k(\rho) = \sum_{i=1}^{\infty} (\alpha_i^k - (-1)^k \beta_i^k).$$

This result is known in the literature as the Schoenberg-Edrei theorem (Edrei, 1952). But many equivalent versions have been proved for this result. The best known is the Thoma's theorem (Thoma, 1964). Simpler proofs are obtained in (Vershik and Kerov, 1981; Kerov, Okounkov, and Olshanski, 1997). Those specializations are related to the representations of the infinite symmetric group. For more details, one can see for example (Kerov, 2003, Introduction, Page 27) and (Kerov, 2003, Chapter 2,Section 2,Theorem 2).
Remark that using the fundamental theorem of symmetric functions, a specialization is totally characterized by its evaluations on Newton polynomials (power sums). In particular, for Schur polynomials, we have the following formula.

Theorem 3.17. (Frobenius' Formula)

$$s_\lambda = \sum_{\sigma \in \mathfrak{S}_{|\lambda|}} \chi_\sigma^\lambda \prod_k \frac{p_k^{\#_k \sigma}}{(\#_k \sigma)! k^{\#_k \sigma}}, \tag{3.1}$$

where χ_σ^ρ is the character of the irreducible representation of the symmetric group indexed by λ and evaluated on σ. We recall that $\#_k \sigma$ is the number of cycles of σ of length k.

In the sequel of this chapter, we will denote a Schur positive specialization by $\rho = [\alpha, \beta, \gamma]$. We have immediately the following result.

Proposition 3.18. (Frobenius' Formula) (Ram, 1991)
Let $\rho = [\alpha, \beta, \gamma]$ be a positive Schur specialization. For any $\lambda \in \mathbb{Y}$,

$$s_\lambda(\rho) = \sum_{\sigma \in \mathfrak{S}_{|\lambda|}} \chi_\sigma^\lambda \prod_k \frac{(\gamma \delta_{1k} + p_k(\alpha) + (-1)^{k-1} p_k(\beta))^{\#_k \sigma}}{(\#_k \sigma)! k^{\#_k \sigma}}.$$

3.2.4 Some examples of Schur positive specializations

- $\rho = [\underline{0}, \underline{0}, \gamma]$

Proposition 3.19.
If $\rho = [\underline{0}, \underline{0}, \gamma]$ with $\gamma > 0$ then

$$\forall \lambda \in \mathbb{Y}, s_\lambda(\rho) = \frac{dim(\lambda)\gamma^{|\lambda|}}{|\lambda|!}. \tag{3.2}$$

Proof. Assume that $\rho = [\underline{0}, \underline{0}, \gamma]$. One can see easily that $\forall k > 1, p_k(\rho) = 0$. Using Proposition 3.18, the only term of the sum that does not vanish is when $\sigma = Id_{\mathfrak{S}_{|\lambda|}}$. Consequently,

$$s_\lambda(\rho) = \chi^\lambda_{Id_{\mathfrak{S}_{|\lambda|}}} \frac{p_1(\rho)^{|\lambda|}}{|\lambda|!},$$

Since

$$\chi^\lambda_{Id_{\mathfrak{S}_{|\lambda|}}} = dim(\lambda),$$

one can conclude that

$$s_\lambda(\rho) = \frac{dim(\lambda)\gamma^{|\lambda|}}{|\lambda|!}.$$

□

- $\rho = [\alpha, \underline{0}, 0]$

$$s_\lambda(\rho) = s_\lambda(\alpha) = \lim_{n\to\infty} s_\lambda(\alpha_1, \dots, \alpha_n).$$

Indeed, $(p_k(\alpha_1, \dots, \alpha_n))_{n\geq 1}$ is monotonous and

$$p_k(\alpha_1, \dots, \alpha_n) \leq (p_1(\alpha_1, \dots, \alpha_n))^k \leq (p_1(\alpha))^k < \infty$$

and then

$$p_k(\alpha) = \lim_{n\to\infty} p_k(\alpha_1, \dots, \alpha_n)$$

and one conclude by Frobenius Formula.

- $\rho = [\underline{0}, \beta, 0]$

$$s_\lambda(\rho) = s_{\lambda'}([\beta, \underline{0}, 0]) = s_{\lambda'}(\beta) = \lim_{n\to\infty} s_{\lambda'}(\beta_1, \dots, \beta_n).$$

3.2.5 Schur measures

We denote by $\mathscr{P}(\mathbb{Y})$ the set of subsets of $\mathbb{Y}$.

Definition 3.20. Let (ρ_1,ρ_2) be a couple of Schur positive specializations such that

$$h(\rho_1,\rho_2) := \sum_{\lambda} s_\lambda(\rho_1)s_\lambda(\rho_2) < \infty. \tag{3.3}$$

One can define a probability measure on $(\mathbb{Y}, \mathscr{P}(\mathbb{Y}))$ such that

$$\forall \lambda \in \mathbb{Y}; \mathbb{S}_{\rho_1,\rho_2}(\lambda) := \frac{s_\lambda(\rho_1)s_\lambda(\rho_2)}{h(\rho_1,\rho_2)}. \tag{3.4}$$

Theorem 3.21. (Cauchy equality)

$$h(\rho_1,\rho_2) = \exp\left(\sum_{k>0} \frac{p_k(\rho_1)p_k(\rho_2)}{k}\right).$$

In particular, if

$$\exists (r,C) \in]0,1[\times\mathbb{R}; \forall k \geq 1,\ \forall i \in \{1,2\},\ p_k(\rho_i) \leq Cr^k, \tag{3.5}$$

then $h(\rho_1,\rho_2) < \infty$.

3.2.6 Some examples of Schur measures

- Poissonized Plancherel measure:
  If $\rho_1 = \rho_2 = [\underline{0}, \underline{0}, \sqrt{t}]$. The distribution of a random Young diagram according to the Schur measure $\mathbb{S}_{\rho_1,\rho_2}$ is just the shape of the image by RSK correspondence of a random permutation σ obtained as follows. Let n be a random Poisson variable with size t. Conditionally on n, σ is a uniform permutation over $\mathfrak{S}_n$.

  Proof. In this case, for any $k > 1$,

  $$p_k(\rho_1) = p_k(\rho_2) = 0.$$

  Consequently,

  $$h(\rho_1,\rho_2) = \exp(p_1(\rho_1)p_1(\rho_2)) = \exp(t)$$

  and

  $$\begin{aligned} \mathbb{S}_{[\underline{0},\underline{0},\sqrt{t}],[\underline{0},\underline{0},\sqrt{t}]}(\mathbb{Y}_n) = \sum_{\lambda\in\mathbb{Y}_n} \frac{s_\lambda(\rho_1)s_\lambda(\rho_2)}{h(\rho_1,\rho_2)} &= \frac{t^n \sum_{\lambda\in\mathbb{Y}_n} dim(\lambda)^2}{n!n!\exp(t)} \\ &= \frac{t^n n!}{n!n!\exp(t)} \\ &= \frac{t^n \exp(-t)}{n!}. \end{aligned}$$

□

- Random uniform word with Poisson length: Let $\mathcal{A}$ be an ordered and finite alphabet of length n and $(w(t))_{t\geq 0}$ be the random Markov word obtained as follows. At $t = 0$, $w = \varnothing$. At each step, after a random exponential time, we add to the existent word one letter chosen uniformly from $\mathcal{A}$. For fixed $t > 0$, the shape of the RSK image of $w(t)$ is distributed according to the Schur measure

$$\mathbb{S}_{\left[\left(\underbrace{\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n}}_{\text{n times}},\underline{0}\right),\underline{0},0\right],[\underline{0},\underline{0},t]}.$$

4 Universality techniques for other sets

"$\forall x \in$ Mathematician $\exists y \in$ MathsProblems s.t. y causes x to cry"

Anonymous

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4.1 General idea and main results

The same technique of proof we presented in Chapter 2 can be applied to other sets having a similar structure to the symmetric group. We will give applications in the next subsection. In general, one can apply the same techniques when there exists a "nice" sequence of undirected graphs[1] $G := (G_n = (V_n, E_n))_{n\geq 1}$[2] such that [3].

$$\forall n \geq 1,\ G_n \text{ is locally finite.} \tag{4.1}$$

$$\forall n \geq 1,\ \text{there exists a countable set } I_n \text{ and finite sets } (V_n^i)_{i\in I_n} \text{ such that } V_n = \sqcup_{i\in I_n} V_n^i. \tag{4.2}$$

For any $n \geq 1$, for any $i, j \in I_n$, for any $\sigma_1, \sigma_2 \in V_n^i$,

$$\operatorname{card}(\{\sigma' \in V_n^j; (\sigma', \sigma_1) \in E_n\}) = \operatorname{card}(\{\sigma' \in V_n^j; (\sigma', \sigma_2) \in E_n\}) =: \mathrm{e}_{j,i}. \tag{4.3}$$

[1] Unlike Chapter 2, we will start from an undirected graph.
[2] We use the usual notations i.e. V_n is the set of vertices and E_n is the set of edges.
[3] We use $\sqcup$ to denote disjoint union.

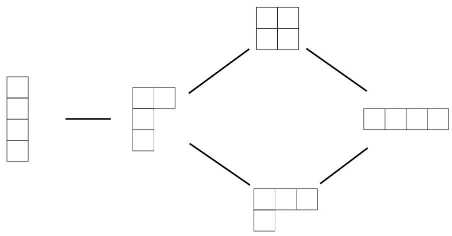

Figure 4.1: The classes graph for the Cayley graph of $\mathfrak{S}_n$ generated by transpositions for $n = 4$

i.e. the number of neighbors in V_n^j of any element of V_n^i only depends on (i, j); we denote it by $\mathrm{e}_{i,j}$. We denote by $\widetilde{E_n} := \{(i, j) \in I_n^2; \mathrm{e}_{i,j} > 0\}$ and by $\widetilde{G_n} := (I_n, \widetilde{E_n})$ the classes graph. We need moreover in the sequel of this chapter that

$$\forall n \geq 1, \text{ the classes graph } \widetilde{G_n} \text{ is connected.} \tag{4.4}$$

In the sequel of this chapter, we assume (4.1)– (4.4).

For example, if G_n is the Cayley graph of the symmetric group generated by transpositions we have

- $V_n = \mathfrak{S}_n$
- $E_n = \{(\sigma, \sigma \circ (i, j)); \sigma \in \mathfrak{S}_n,\ i \neq j\}$
- $I_n = \mathbb{Y}_n$,
- $V_n^i = \{\sigma \in \mathfrak{S}_n; \hat{\lambda}(\sigma) = i\}$,
- $\widetilde{E_n}$ the set of couples of Young tableaux such that one can obtain one from the other by concatenating two arrows. For example, for $n = 4$, we obtain the classes graph in Figure 4.1.

With analogy with Chapter 2, we will now construct a new directed graph for which we will consider the uniform random walk. Let d_{G_n} be the usual graph distance and for $\sigma \in V_n$, we denote by $Class(\sigma)$ the unique $i \in I_n$ such that $j \in V_n^i$.

Let $(i_n^*)_{n\geq 1} \in \prod_{n\geq 1} I_n$ be a "nice" sequence of classes[4]. We denote by $\underline{d}(\sigma) := \min_{\rho \in V_n^{i_n^*}} d_{G_n}(\sigma, \rho)$. The random walk we use to prove universality will be the uniform random walk on the directed graph $G'_n := (V_n, E'_n)$ where

$$E'_n = \{(\sigma_1, \sigma_2) \in E_n\,; \underline{d}(\sigma_2) = \underline{d}(\sigma_1) - 1\} \cup \{(\sigma, \sigma), \sigma \in V_n^{i_n^*}\}.$$

Back to the example of the Cayley graph of the symmetric group generated by transpositions we have

- $Class(\sigma) = \hat{\lambda}(\sigma)$,

[4] We will give some possible choices in the remainder of this chapter

- $i_n^* = (n, \underline{0})$ is the Young diagram with a unique row of length n,
- $(V_n, E_n') = \mathcal{G}_{\mathfrak{S}_n}$[5],
- $\underline{d}(\sigma) = \#(\sigma) - 1$.

With analogy with the symmetric group, let $T_{G_n'}$ be the Markov operator associated to the uniform random walk on G_n', $V_\infty := \cup_{n\geq 1} V_n$ and f be a function defined on V_∞ and having values on some metric space (F, d_F). With analogy with Chapter 2, for $S \subset V_n$ and $\sigma \in V_n$, let

$$\text{next}(S) := \{\sigma_2; \sigma_1 \in S \text{ and } (\sigma_1, \sigma_2) \in E_n'\},$$

$$\text{final}(\sigma) := \begin{cases} \text{next}^{\underline{d}(\sigma)}(\{\sigma\}) & \text{if } \underline{d}(\sigma) > 1 \\ \{\sigma\} & \text{otherwise} \end{cases}$$

and for $i \in I_n$ and $p \geq 1$, we define

$$\begin{aligned}
\varepsilon_{n,i,p}(f) &:= \left(\sum_{\sigma \in V_n^i} \sum_{\rho \in \text{next}(\{\sigma\})} \frac{(d_F(f(\sigma), f(\rho)))^p}{\text{card}(V_n^i)\text{card}(\text{next}(\{\sigma\}))} \right)^{\frac{1}{p}} \\
\varepsilon_{n,p}(f) &:= \sup_{i \in I_n} \varepsilon_{n,i,p}(f) \\
\varepsilon_{n,i,\infty}(f) &:= \max_{\sigma \in V_n^i} \max_{\rho \in \text{next}(\{\sigma\})} d_F(f(\sigma), f(\rho)) \\
\varepsilon_{n,\infty}(f) &:= \sup_{i \in I_n} \varepsilon_{n,i,\infty}(f) \\
\varepsilon'_{n,i,p}(f) &:= \left(\sum_{\sigma \in V_n^i} \sum_{\rho \in \text{final}(\sigma)} \frac{(d_F(f(\sigma), f(\rho)))^p}{\text{card}(V_n^i)\text{card}(\text{final}(\sigma))} \right)^{\frac{1}{p}} \\
\varepsilon'_{n,i,\infty}(f) &:= \max_{\sigma \in V_n^i} \max_{\rho \in \text{final}(\sigma)} d_F(f(\sigma), f(\rho)).
\end{aligned}$$

Finally, let $(\sigma_n)_{n\geq 1}$ be a sequence of random variables such that σ_n is supported on V_n. We say that σ_n is G_n invariant (with respect to the partition $\{V_n^i\}_{i \in I_n}$)[6] if for any $i \in I_n$ and any $\sigma, \rho \in V_n^i$

$$\mathbb{P}(\sigma_n = \sigma) = \mathbb{P}(\sigma_n = \rho),$$

and we say that $(\sigma_n)_{n\geq 1}$ is G-invariant if σ_n is G_n-invariant $\forall n \geq 1$.

Definition 4.1. For $\alpha > 0$ and $p \in [1, \infty]$, we say that $(\sigma_n)_{n\geq 1}$ satisfies $\mathcal{H}^{\mathbb{P}}_{G-inv,\alpha}$ if

$(\mathcal{H}^{\mathbb{P}}_{G-inv,\alpha})$ $\quad$ $(\sigma_n)_{n\geq 1}$ is G-invariant and $\quad \dfrac{\underline{d}(\sigma_n)}{n^{\frac{1}{\alpha}}} \xrightarrow[n\to\infty]{\mathbb{P}} 0,$

[5] $\mathcal{G}_{\mathfrak{S}_n}$ is defined in Chapter 2

[6] we omit this precision when it is clear from the context.

we say that it satisfies $\mathcal{H}^{\mathbb{L}^p}_{G-inv,\alpha}$ if

$$(\mathcal{H}^{\mathbb{L}^p}_{G-inv,\alpha}) \qquad (\sigma_n)_{n\geq 1} \text{ is G-invariant and } \quad \frac{\underline{d}(\sigma_n)}{n^{\frac{1}{\alpha}}} \xrightarrow[n\to\infty]{\mathbb{L}^p} 0.$$

Interesting results can be obtained if the graph satisfies an additional symmetry property:

For any $\sigma_1 \in V_n$, for any $\sigma_2, \sigma_3 \in \text{final}(\sigma_1)$, the number of paths in G'_n of length $\underline{d}(\sigma)$ from σ_1 to σ_2 is equal to that from σ_1 to σ_3 i.e. $A_{G'_n}$ the adjacency matrix of G'_n satisfies the following:

$$(4.5) \qquad \forall \sigma_1 \in \mathfrak{S}_n, \exists\, c_{\sigma_1} \in \mathbb{N} \text{ such that } \forall \rho \in \mathfrak{S}_n,\ A^{\underline{d}(\sigma)}_{G'_n}(\sigma_1,\rho) = c_{\sigma_1}\mathbb{1}_{\rho\in\text{final}(\sigma_1)}.$$

In particular, we have the following:

Lemma 4.2. Under (4.1)–(4.5), for any $(\sigma_n)_{n\geq 1}$ G-invariant, for any $\mathbb{P} \in [1,\infty[$,

$$\mathbb{E}\left(\left(d_F\left(f(\sigma_n), f\left(T^{\underline{d}(\sigma_n)}_{G'_n}(\sigma_n)\right)\right)\right)^p\right) = \mathbb{E}((\underline{\varepsilon}'_{n,Class(\sigma_n),p})^p).$$

Proof. For any random variable σ_n, we have

$$\begin{aligned}\mathbb{E}\left(\left(d_F\left(f(\sigma_n), f\left(T^{\underline{d}(\sigma_n)}_{G'_n}(\sigma_n)\right)\right)\right)^p\right) &= \mathbb{E}\left(\mathbb{E}\left(\left(d_F\left(f(\sigma_n), f\left(T^{\underline{d}(\sigma_n)}_{G'_n}(\sigma_n)\right)\right)\right)^p \Big| \sigma_n\right)\right)\\ &= \sum_{i\in I_n}\sum_{\sigma\in V_n^i} \mathbb{P}(\sigma_n=\sigma)\mathbb{E}\left(\left(d_F\left(f(\sigma_n), f\left(T^{\underline{d}(\sigma_n)}_{G'_n}(\sigma_n)\right)\right)\right)^p \Big| \sigma_n=\sigma\right).\end{aligned}$$

If $(\sigma_n)_{n\geq 1}$ is G-invariant, then $\mathbb{P}(\sigma_n = \sigma) = \frac{1}{\text{card}(Class(\sigma))}\mathbb{P}(Class(\sigma_n)Class(\sigma))$. Moreover, under (4.5),

$$\mathbb{E}\left(\left(d_F(f(\sigma_n), f\left(T^{\underline{d}(\sigma_n)}_{G'_n}(\sigma_n)\right)\right)^p \Big| \sigma_n=\sigma\right) = \frac{\sum_{\rho\in\text{final}(\sigma)}(d_F(f(\sigma), f(\rho)))^p}{\text{card}(\text{final}(\sigma))}.$$

Consequently, one can conclude since

$$\begin{aligned}\mathbb{E}((\underline{\varepsilon}'_{n,Class(\sigma_n),p})^p) =& \mathbb{E}\left(\mathbb{E}((\underline{\varepsilon}'_{n,Class(\sigma_n),p})^p)\Big| Class(\sigma_n)\right)\\ &= \sum_{i\in I_n}\mathbb{P}(Class(\sigma_n)=i)(\underline{\varepsilon}'_{n,i,p})^p\\ &= \sum_{i\in I_n}\mathbb{P}(Class(\sigma_n)=i)\sum_{\sigma\in V_n^i}\sum_{\rho\in\text{final}(\sigma)}\frac{(d_F(f(\sigma), f(\rho)))^p}{\text{card}(V_n^i)\text{card}(\text{final}(\sigma))}.\end{aligned}$$

□

Similarly, one can prove the following.

Lemma 4.3. Under (4.1)–(4.4), $(\sigma_n)_{n\geq 1}$ is G-invariant, for $n \geq 1$, for any $\mathbb{P} \in [1,\infty[$,

$$\mathbb{E}\left(\left(d_F(f(\sigma_n), f(T_{G'_n})(\sigma_n))\right)^p\right) = \mathbb{E}((\underline{\varepsilon}_{n,Class(\sigma_n),p})^p).$$

This gives as a universality result.

Theorem 4.4. Assume that (4.1)–(4.4) and that $(\sigma_n)_{n\geq 1}$ and $(\sigma_{ref,n})_{n\geq 1}$ are G-invariant. Suppose that there exists some deterministic $x \in F$ and $p \in [1, \infty[$ such that

$$f(\sigma_{ref,n}) \xrightarrow[n\to\infty]{\mathbb{P}} x \quad (\text{ resp. } f(\sigma_{ref,n}) \xrightarrow[n\to\infty]{\mathbb{L}^p} x),$$

$$\varepsilon_{n,Class(\sigma_{ref,n}),\infty}(f) \xrightarrow[n\to\infty]{\mathbb{P}} 0 \quad (\text{ resp. } \varepsilon_{n,Class(\sigma_{ref,n}),\infty}(f) \xrightarrow[n\to\infty]{\mathbb{L}^p} 0) \tag{4.6}$$

and

$$\varepsilon_{n,Class(\sigma_n),\infty}(f) \xrightarrow[n\to\infty]{\mathbb{P}} 0 \quad (\text{ resp. } \varepsilon_{n,Class(\sigma_n),\infty}(f) \xrightarrow[n\to\infty]{\mathbb{L}^p} 0). \tag{4.7}$$

Then

$$f(\sigma_n) \xrightarrow[n\to\infty]{\mathbb{P}} x \quad (\text{resp. } f(\sigma_n) \xrightarrow[n\to\infty]{\mathbb{L}^p} x).$$

Moreover, under (4.5), (4.6) and (4.7) can be replaced by

$$\varepsilon'_{n,Class(\sigma_{ref,n}),1}(f) \xrightarrow[n\to\infty]{\mathbb{P}} 0 \quad (\text{ resp. } \varepsilon'_{n,Class(\sigma_{ref,n}),p}(f) \xrightarrow[n\to\infty]{\mathbb{L}^p} 0)$$

and

$$\varepsilon'_{n,Class(\sigma_n),1}(f) \xrightarrow[n\to\infty]{\mathbb{P}} 0 \quad (\text{ resp. } \varepsilon'_{n,Class(\sigma_n),p}(f) \xrightarrow[n\to\infty]{\mathbb{L}^p} 0).$$

Idea of the proof. The proof is identical to that of theorems 2.1 and 2.2. Indeed, (4.3) guarantees that under the G-invariance, for any $n \geq 1$, $T_{G'_n}(\sigma_n)$ is G_n invariant and by construction almost surely

$$\underline{d}(T_{G'_n}(\sigma_n)) = \max(0, \underline{d}(\sigma_n) - 1).$$

Consequently, by induction, $T_{G'_n}^{\underline{d}(\sigma_n)}(\sigma_n)$ is distributed according to the uniform distribution on $V_n^{i_n^*}$ and almost surely

$$d_F(f(T_{G'_n}^{\underline{d}(\sigma_n)}(\sigma_n)), f(\sigma_n)) \leq \varepsilon_{n,class(\sigma_n),\infty}(f).$$

□

Similarly to Remark 2.4, by the triangle inequality and using that the arithmetic mean is smaller than the p-mean, we have [7]

$$(\varepsilon_{n,k,p}(f))^p \leq \sum_{i=1}^{\underline{d}(k)} \max_{j;\underline{d}(j)=i} (\varepsilon^p_{n,j,p}(f)) \leq \underline{d}(k)\varepsilon^p_{n,p}(f).$$

[7] There is here a notation abuse. Since $\underline{d}$ is constant in any class, we denote by $\underline{d}(k)$, $\underline{d}(\sigma)$ for some $\sigma \in k$.

Consequently, if there exists $\alpha > 0$ such that

$$\varepsilon_{n,p}^p(f) = O\left(\frac{1}{n^{\frac{1}{\alpha}}}\right),$$

then one can obtain (4.6) and (4.7) for the equivalent classes of $(\mathcal{H}_{G-inv,\alpha}^{\mathbb{P}})$ (resp.$(\mathcal{H}_{G-inv,\alpha}^{\mathbb{L}^p})$).

4.2 Some examples of finite graphs

In general, Cayley graphs are good candidates. An interesting case is when there exists $(i_n^*)_{n\geq 1} \in \prod_{n\geq 1} I_n$ such that

$$\frac{1}{\text{card}(V_n)} \sum_{\sigma\in V_n} \min_{\sigma'\in V_n^{i_n^*}} d_{G_n}(\sigma,\sigma') = o\left(\max_{\sigma_1,\sigma_2\in V_n} d_{G_n}(\sigma_1,\sigma_2)\right),$$

in this case, the comparison with the uniform distribution can be done for reasonable statistics. The first four examples we give are different ways to apply our results to the symmetric group. The other four examples are different graphs. Our eight examples satisfy (4.1)– (4.5). In the first two examples we will give in details the different objects, for the other we will give only G_n, I_n V_n^i and i_n^*. The others can be obtained easily by applying the definitions.

- The Cayley graph of symmetric group generated by transpositions: We recall that

  * $(V_n, E_n') = \mathcal{G}_{\mathfrak{S}n}$,
  * I_n=$\mathbb{Y}_n$,
  * $V_n^i = \{\sigma \in \mathfrak{S}_n; \hat{\lambda}(\sigma) = i\}$,
  * $Class(\sigma) = \hat{\lambda}(\sigma)$,
  * $i_n^* = (n, \underline{0})$ the Young diagram with a unique row of length n,
  * $\underline{d}(\sigma) = \#(\sigma) - 1$,

  We have then the following.

$$\frac{1}{\text{card}(V_n)} \sum_{\sigma\in V_n} \min_{\sigma'\in V_n^{i_n^*}} d_{G_n}(\sigma,\sigma*) = \mathbb{E}(\#(\sigma_{unif,n}) - 1)$$
$$= \sum_{k=2}^{n} \frac{1}{k} = o(n-1) = o\left(\max_{\sigma_1,\sigma_2\in V_n} d_{G_n}(\sigma_1,\sigma_2)\right).$$

- Even permutations: A permutation $\sigma \in \mathfrak{S}_n$ is said to be even if $n - \#(\sigma)$ is even. Cycles of length 3 are a generator of $\mathfrak{S}_n$. When n is odd, $\mathfrak{S}_n^0$ is a subset of the set of even permutations. One can choose for example.

  * G_n the Cayley graph of $\mathfrak{S}_{2n+1}$ generated by cycles of length 3

- $*$ I_n=$\{\lambda \in \mathbb{Y}_{2n+1}; \ell(\lambda) \equiv 1 \pmod 2\}$,
- $*$ $V_n^i = \{\sigma \in \mathfrak{S}_{2n+1}, \hat{\lambda}(\sigma) = i\}$,
- $*$ $Class(\sigma) = \hat{\lambda}(\sigma)$,
- $*$ $i_n^* = (2n+1, \underline{0})$,
- $*$ $\underline{d}(\sigma) = \frac{\#(\sigma)+1}{2}$.

- $\mathfrak{S}_n$ seen as a Coxeter group: Here we take the right (or the left) Cayley graph generated by transpositions of type $(i, i+1)$. [8] In this case we have:
  - $*$ G_n the right (or the left) Cayley graph of $\mathfrak{S}_n$ generated by $\{(i, i+1); 1 \leq i \leq n-1\}$.
  - $*$ $I_n = \{0, 1, \dots, \frac{n(n-1)}{2}\}$,
  - $*$ $V_n^i = \{\sigma; \mathcal{K}_2(\sigma) = i\}$, where we recall that $\mathcal{K}_2(\sigma)$ is the number of inversions of σ.
  - $*$ $Class(\sigma) = \mathcal{K}_2(\sigma)$,
  - $*$ $i_n^* = \lceil \frac{n^2}{4} \rceil$,
  - $*$ $\underline{d}(\sigma) = |\lceil \frac{n^2}{4} \rceil - \mathcal{K}_2(\sigma)|$.

  For example, G_3' is represented in Figure 4.2. Corollary 2.19 guarantees that $i_n^* = \lceil \frac{n^2}{4} \rceil$ is a good candidate if we want to compare with the uniform distribution. But also it is possible to choose $i_n^* = 0$ when looking for universality results for random permutations close to the identity. For this graph, the Mallows law with Kendall tau distance is G_n-invariant and one can obtain a first order universality for all local statistics we already studied in the previous chapter and for the limiting shape[9]. The second order fails.

- Using the same previous graph (same G_n) but with only two classes even and odd permutations[10] i.e. $I_n = \{\text{even}, \text{odd}\}$ we obtain that, if $f(\sigma_n)$ converges in probability (or $\mathbb{L}^1$) when σ_n follows one of these three distributions
  - $*$ Uniform law of $\mathfrak{S}_n$
  - $*$ Uniform law of even permutations
  - $*$ Uniform law of odd permutations

  it converges also for the two others as soon as

$$\frac{\min\left(\sum_{\sigma \in \mathfrak{S}_n, 1 \leq i < n} d_F(f(\sigma \circ (i, i+1)), f(\sigma)); \sum_{\sigma \in \mathfrak{S}_n, 1 \leq i < n} d_F(f((i, i+1) \circ \sigma), f(\sigma))\right)}{n!(n-1)} = o(1).$$

[8] Fun fact 2: depending on the choose of the right or the left composition, one can obtain a different universality theorem. The classes are the same but the graph (and consequently error controls) are different.

[9] We apologize again to the reader because it is not defined yet.

[10] Here, the choice of i_n^* is not important but the reader can take $i_n^* = \text{even}$.

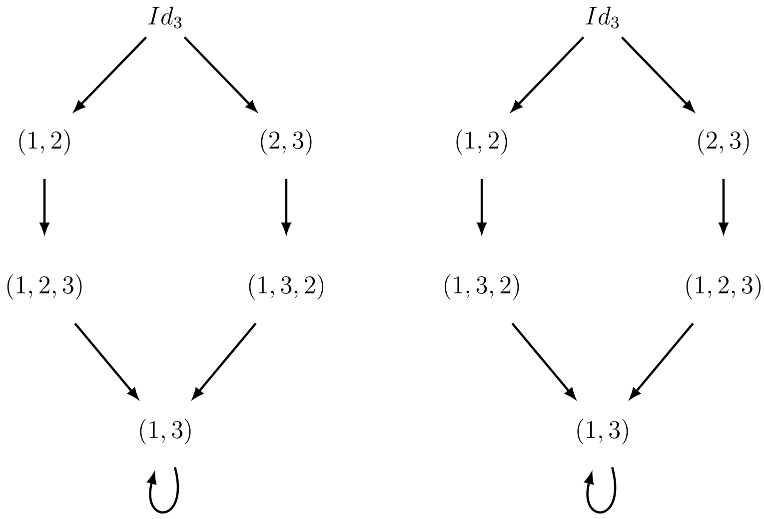

Figure 4.2: G'_3 obtained by the transpositions $(1,2)$ and $(2,3)$

- Another possible application is the hypercube $(\mathbb{Z}/2\mathbb{Z})^{2n}$. In this case, we set

  * $G_n = (\mathbb{Z}/2\mathbb{Z})^{2n}$
  * $I_n = \{0, 1, \dots, 2n\}$
  * V_n^i is the set of edges of the graph such that the graph distance from $(0, \dots, 0)$ is i.
  * $i_n^* = n$.

  In this case,

$$\frac{1}{\mathrm{card}(V_n)} \sum_{\sigma \in V_n} \min_{\sigma' \in V_n^{i_n^*}} d_{G_n}(\sigma, \sigma*) = \frac{\sum_{k=0}^{2n} |\binom{2n}{k}(k-n)|}{4^n}$$
$$\leq \sqrt{\frac{\sum_{k=0}^{2n} \binom{2n}{k}(k-n)^2}{4^n}}$$
$$= \sqrt{\frac{n}{2}}$$
$$= o(n) = o\left(\max_{\sigma_1, \sigma_2 \in V_n} d_{G_n}(\sigma_1, \sigma_2)\right).$$

- $(\mathbb{Z}/d\mathbb{Z})^{nd}$: Let $\mathcal{R}_n$ be the equivalent relation defined as follows: For any

$$x = (x_i)_{1 \leq i \leq nd}, y = (y_i)_{1 \leq i \leq nd} \in (\mathbb{Z}/d\mathbb{Z})^{nd},\ x\mathcal{R}_n y \Leftrightarrow \exists \sigma \in \mathfrak{S}_{nd}, y = (x_{\sigma(i)})_{1 \leq i \leq nd}.$$

  $\mathcal{R}_n$ define naturally the classes of the vertices. The central limit theorem guarantees that the class i_n^* where we have exactly n coordinates equal to k for any k in $\mathbb{Z}/d\mathbb{Z}$. is a good candidate[11].

[11] Fun fact 3: by choosing fixed and different proportions of every element of $\mathbb{Z}/d\mathbb{Z}$ for i_n^*, one can obtain different universality result.

- Let $(H_n)_{n\geq 1}$ be a sequence of non-commutative and finite groups and $(A_n)_{n\geq 1}$ such that A_n is a conjugation invariant subset of H_n[12].

  * G_n be the Cayley graph generated by H_n.
  * I_n is the set of conjugacy classes
  * $V_n^i = i$

  In this case, G-invariant random variables are conjugation invariant variables. The choice of i_n^* is specific to the choice of G_n.

- Dihedral group $\mathbb{D}_{2n}$ [13] with $n \geq 3$: The Dihedral group $\mathbb{D}_{2n}$ is defined via its representation $< \sigma, \mu | \sigma^2, \mu^2, (\mu\sigma)^n >$[14]. This representation shows that $\mathbb{D}_{2n}$ is a Coxeter group. For our study, one can admit that

$$\mathbb{D}_{2n} = \{\mathrm{s}_0, \ldots, \mathrm{s}_{n-1}, \mathrm{r}_0, \ldots, \mathrm{r}_{n-1}\}$$

  and

$$\mathrm{r}_i\,\mathrm{r}_j = \mathrm{r}_{i+j}, \quad \mathrm{r}_i\,\mathrm{s}_j = \mathrm{s}_{i+j}, \quad \mathrm{s}_i\,\mathrm{r}_j = \mathrm{s}_{i-j}, \quad \mathrm{s}_i\,\mathrm{s}_j = \mathrm{r}_{i-j}.$$

  Here, (i, j) are in $\mathbb{Z}/n\mathbb{Z}$. One can choose either

  * G_n: the Cayley graph generated by $\{s_i, 0 \leq i \leq n\}$,
  * $I_n = \{$r,s$\}$,
  * $V_n^s = \{s_i, 0 \leq i \leq n\}$ is the set of transpositions and $V_n^r = \{r_i, 0 \leq i \leq n\}$ is the set of rotations,
  * $i_n^* = r$

  or keep the same graph and choose conjugacy classes as classes (as in the previous examples)[15]. In the second case, we require that $f(\sigma_n)$ convergences for any sequence of rotations and a transposition does not change a lot the statistic.

- Colored permutations: A less trivial example is the set of signed permutations and more generally the set of colored permutations. Given two positive integers n and m, a colored permutation is a map $\pi = (\sigma, \varphi)$ such that $\sigma \in \mathfrak{S}_n$ and $\varphi \in \{1, \ldots, n\}^{\{1,\ldots,m\}}$. A subsequence $\pi(x_1), \ldots, \pi(x_k)$ of π is called increasing of length $m(k-1)+p$ if $\sigma(x_1) < \sigma(x_2) < \cdots < \sigma(x_k)$ and $\varphi(x_1) = \varphi(x_2) = \cdots = \varphi(x_k) = p$. We denote by $\mathrm{LIS}(\pi)$ the length of a longest increasing subsequence.

[12] i.e. if $\sigma \in A_n$ then $\bar{\sigma} \subset H_n$.

[13] This is a typical "bad" Cayley graph since its diameter is bounded (equal to 2) and consequently the universality result is trivial.

[14] This notation is classic to define groups. It means in our case that $\mathbb{D}_{2n}$ is isomorphic to the group generated by σ and μ such that $\sigma^2 = \mu^2 = (\mu\sigma)^n = 1$

[15] There are $n+1$ or $n+2$ depending on the parity of n.

Theorem 4.5. Let $(\pi_n = (\sigma_n, \varphi_n))_{n\geq 1}$ be a sequence of random colored permutations and assume that:

- σ_n is independent of φ_n,
- φ_n is distributed according to the uniform distribution,
- σ_n is conjugation invariant,
- $\frac{\#\sigma_n}{n^{\frac{1}{6}}} \xrightarrow{\mathbb{P}} 0$.

then,

$$\mathbb{P}\left(\frac{\mathrm{LIS}(\pi_n) - 2\sqrt{nm}}{m^{\frac{2}{3}}\sqrt[6]{nm}} < s\right) \to F_2^m(s). \tag{4.8}$$

Proof. The uniform case is proved by Borodin (1999). To apply our theorem, choose the graph where two colored permutations are related by an edge if only the first components differ by a transposition i.e.

- $E_n := \{((\sigma, \varphi), (\sigma \circ (i,j), \varphi)); i \neq j\}$,
- $I_n = \mathbb{Y}_n$,
- $V_n^i = \{(\sigma, \varphi); \bar{\sigma} = i\}$,
- $i_n^* = (n, \underline{0})$.

□

For our examples, a trivial example of $G-$invariant elements is the uniform measure, or the uniform measure on a given class. Since $\underline{d}$ is constant in classes, a natural way to generalize Ewens measures is the following. Given $q \in \mathbb{R}_+$, the probability measure satisfying

$$\mathbb{P}(\rho_{G,q,n} = \sigma) = \frac{q^{\underline{d}(\sigma)}}{\sum_{\sigma' \in V_n} q^{\underline{d}(\sigma')}}$$

is $G-$invariant and for any statistic f such that $f(\rho_{G,0,n})$ converges, one can obtain a non-empty universality result around $\rho_{G,0,n}$ since

$$err_n := q \mapsto \mathbb{E}(d_F(f(T^{\underline{d}(\rho_{G,q,n})}(\rho_{G,q,n}), f(\rho_{G,q,n}))))$$

is continuous and $err_n(0) = 0$. In fact, in the case of permutations, Ewens and Mallows measures with Kendall tau distance are particular case of $\rho_{G,q,n}$.

4.3 Infinite case

We take now $G_n = G$ an infinite graph. Example of "nice graphs":

- The infinite d-regular tree $\mathfrak{T}_d$.
- The set of words of a finite alphabet of length d.
- The free group $\mathcal{F}_d$ with its natural Cayley graph.
- The Cayley graph of $\mathcal{B}_d$, the Artin Braid group.
- The Cayley graph of an infinite and finitely generated group $H =< x_1, x_2, \ldots, x_n >$.

The classes here are indexed by $\mathbb{N}$ according to the distance to the root (or the identity). Example of universality: Let G be such that

$$0 < \liminf_{n\to\infty} \frac{\log(card(\{x; \underline{d}(x) = n\}))}{n} = \limsup_{n\to\infty} \frac{\log(card(\{x; \underline{d}(x) = n\}))}{n} = \log(\lambda) < \infty.$$

It is the case for the first three examples. Let f be a statistic such that $f(\sigma_n)$ convergences for the uniform law on $V^n = V_n^n$ and $\sum_{i=1}^{\infty} \underline{\varepsilon}'_{n,i,\infty}(f) < \infty$. We obtain then that $f(\sigma_n)$ converges for the Mallows law when its parameter goes to λ. More generally, it converges for any distribution such that $Class(i)$ converges in probability to infinity.

5

Local Universality for RSK of invariant random permutations

"La vie n'est bonne qu'à étudier et à enseigner les mathématiques."

Blaise Pascal

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The goal of this chapter is to prove the results we have presented in the first two chapters that are related to the point process obtained the RSK correspondence. we first present some known asymptotic results for the uniform permutation and recall how they are related to universality results. We apply then our results to conjugation invariant random permutations.

5.1 Known results for the uniform permutation

Before stating the results, we need to introduce the notion of determinantal point processes.

Determinantal point processes (DPP) were first introduced by Macchi (1975) to describe fermions in quantum mechanics. They appear naturally in problems related to Random Matrix Theory. For further reading, we refer for example to (Johansson, 2006). Let $\mathfrak{X} \in \{\{1,2,\dots,n\},\mathbb{R}^d,\mathbb{Z}\}$. For simplicity, we will restrict our study to those spaces but there is a general theory for Polish spaces.

Definition 5.1. A configuration is a locally compact subset of $\mathfrak{X}$. We denote by $conf(\mathfrak{X})$ the set of configurations.

Definition 5.2. A point process is a random configuration. Here, we will use the smallest σ-algebra containing

$$\cup_{K\subset\mathfrak{X}\text{ compact}}\cup_{i\in\mathbb{N}}\{X\in conf(\mathfrak{X});\operatorname{card}(X\cap K)=i\}.$$

Definition 5.3. A point process γ defined on $\mathbb{X}$ is said to be determinantal for the measure μ if there exists K defined on $\mathbb{X}^2$ with values on $\mathbb{C}$ such that, $\forall k>0$, $f:\mathbb{X}^k\to C$ continuous and compactly supported,

$$\mathbb{E}\left(\sum_{\substack{(x_1,\dots x_k)\in\gamma\\ \forall i\neq j, x_i\neq x_j}} f(x_1,\dots,x_n)\right)=\int_{\mathbb{X}^k} f(x_1,\dots,x_k)\det([K(x_i,x_j)]_{1\le i,j\le k})d\mu(x_1)...d\mu(x_k).$$

In this case K, is called the kernel[1] and

$$\rho_k(x_1,x_2,\dots,x_k):=\det([K(x_i,x_j)]_{1\le i,j\le k})$$

are the k-correlation functions.

In our context, we will be interested to DPP on $\mathbb{Z}+\frac{1}{2}$ on some Schur measures introduced in Chapter 3. Let ρ_1,ρ_2 be two Schur positive specializations satisfying (3.5). Let $\lambda=(\lambda_i)_{i\ge1}$ be a random Young diagram distributed according to the Schur measure $\mathbb{S}_{\rho_1,\rho_2}$.

Proposition 5.4. (Borodin and Gorin, 2016, Theorem 5.3)(Okounkov, 2001) The point process $(\lambda_i-i+\frac{1}{2})_{i\ge1}$ is a determinantal point processes with kernel

$$K(i,j)=\frac{1}{(2i\pi)^2}\oint_{\Gamma_1}\oint_{\Gamma_1}\frac{\mathscr{H}(\rho_1;v)\mathscr{H}(\rho_2;w^{-1})}{\mathscr{H}(\rho_1;v^{-1})\mathscr{H}(\rho_2;w)}\frac{\sqrt{vw}}{v-w}\frac{dvdw}{v^{i+1}w^{-j+1}}$$

where

$$\mathscr{H}(\rho;z)=\exp\left(\sum_{i=1}^{\infty}p_k(\rho)\frac{z^k}{k}\right)$$

and Γ_1 and Γ_2 are two circular contours with respective radius R_1,R_2 satisfying $r<R_2<R_1<\frac{1}{r}$.[2]

In particular, the Poissonized Plancherel measure is determinantal. From there we can study the asymptotic behavior in the bulk. Before stating the result we introduce the sine and the discrete sine process.

[1] It is not unique!

[2] r in the same as in (3.5)

The (continuous) sine process appears as a limit in the bulk for β-ensembles. The case $\beta = 2$ is determinantal and the associated kernel is

$$K_{sin}(x, y) = \frac{sin(x-y)}{x-y}.$$

For more details about the universality of the sine process one can see for example (Tao and Vu, 2014; Erdhos and Yau, 2012). For the discrete case, a stationary determinantal point process can be defined via the kernel

$$K_{sin,\alpha}(x, y) = \frac{\sin(\alpha(x-y))}{x-y}\mathbb{1}_{x\neq y} + \alpha\mathbb{1}_{x=y}.$$

We have the following asymptotic in the bulk

Theorem 5.5. (Borodin, Okounkov, and Olshanski, 2000, Theorem 3)(Borodin and Gorin, 2016, Theorem 5.5)
For the Poissonized Plancherel measure, for any $|u| < 2$ and $(x, y) \in \mathbb{Z}^2$,

$$\lim_{\theta\to\infty} K\left(\lfloor u\theta\rfloor + x + \frac{1}{2}, \lfloor u\theta\rfloor + y + \frac{1}{2}\right) = \begin{cases} \frac{\varphi}{\pi} & \text{if } x = y \\ \frac{\sin(\varphi(x-y))}{\pi(x-y)} & \text{otherwise} \end{cases} =: K_{sin,u}(x, y),$$

where $2cos(\varphi) = u$.

Note that since the space here is $\mathbb{Z}$, this convergence implies the weak convergence. The de-Poissonization also goes well in this case. One can see for example (Baik, Deift, and Suidan, 2016).
One can also study the asymptotic at the edge i.e. the first rows of the process. It involves the Airy ensemble, a the determinantal point process associated to the kernel,

$$K_{Airy}(x, y) = \int_0^\infty Ai(x+\lambda)Ai(y+\lambda)d\lambda,$$

where Ai is the Airy function, one can see for example (Quastel and Remenik, 2014) for rigorous definition of the Airy function . The Airy process appears as a limiting distribution at the edge for many models of the Kardar-Parisi-Zhang (KPZ) dimension 1+1 universality class. For examples, the limiting joint distribution of the top eigenvalues of the Gaussian Unitary Ensemble (GUE) is the Airy Ensemble. For universality results one can see for example (Tracy and Widom, 1994; Bowick and Brézin, 1991; Forrester, 1993). In the remainder of this paper, we denote by

$$F_{2,k}(s_1, s_2, \ldots, s_k) := \mathbb{P}(\forall i \leq k,\ \xi_i \leq s_i) \tag{5.1}$$

the CDF of the top right k particles of the Airy ensemble $(\xi_i)_{i\geq 1}$. In particular,

$$\begin{aligned} F_2(s) := F_{2,1}(s) &= \det(1 - K_{Airy})_{L_2(s,\infty)} \\ &= 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \int_s^\infty \cdots \int_s^\infty \det[K_{Airy}(x_j, x_k)]_{1\leq,j,k\leq i} dx_1 \ldots dx_i \end{aligned} \tag{5.2}$$

is the CDF of the celebrated Tracy-Widom distribution. The Tracy-Widom distribution appears in many problems of random growth, integrable probability and as the distribution of the rescaled largest eigenvalue of many models of random matrices (Corwin, 2012; Borodin and Gorin, 2016). F_2 can be expressed in terms of the Hastings-McLeod solution of the Painlevé II equation (Tracy and Widom, 1994). A more general version of the result of Theorem 1.3 is the following.

Theorem 5.6. (Borodin, Okounkov, and Olshanski, 2000, Theorem 5)(Johansson, 2001, Theorem 1.4) For all real numbers $s_1, s_2, \dots, s_k$,

$$\lim_{n\to\infty} \mathbb{P}\left(\forall i \leq k,\ \frac{\lambda_i(\sigma_{unif,n}) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s_i\right) = F_{2,k}(s_1, s_2, \dots, s_k).$$

Indeed the Poissonized Plancherel processes converges at the edge[3]. It turns out that the de-Poissonization works well in this case. For further details about de-Poissonization techniques, we strongly recommend (Baik, Deift, and Suidan, 2016, Chapter 2) and (Johansson, 2001).

5.2 Statement of results for conjugation invariant random permutations

It is natural to ask if what we see in previous section is specific to the uniform case. It turns out that in the case of conjugation invariant random permutation with few cycles, the convergence will be the same at the edge. We conjecture that it is also true for the bulk but we do not have any idea how to prove it.

5.2.1 Main results

For the permutations satisfying the same assumptions as in Theorem 1.5, we have the same asymptotic as in the uniform setting at the edge.

Theorem 5.7. Assume that $(\sigma_n)_{n\geq 1}$ is conjugation invariant and

$$\frac{1}{n^{\frac{1}{6}}} \min_{1\leq i\leq n} \left(\left(\sum_{j=1}^{i} \#_j(\sigma_n) \right) + \frac{\sqrt{n}}{i} \sum_{j=i+1}^{n} \#_j(\sigma_n) \right) \xrightarrow[n\to\infty]{\mathbb{P}} 0. \tag{5.3}$$

Then for all positive integer k, for all real numbers $s_1, s_2, \dots, s_k$,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}\left(\forall i \leq k,\ \frac{\lambda_i(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s_i\right) &= \lim_{n\to\infty} \mathbb{P}\left(\forall i \leq k,\ \frac{\lambda'_i(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s_i\right)\\ &= F_{2,k}(s_1, s_2, \dots, s_k).\end{aligned} \tag{Ai}$$

[3] We will avoid to introduce the type of convergence, but it guarantees the convergence of top k-right particles to the k-right particles of the Airy ensemble.

Before proving this result, we will first prove a simple weaker version.

Proposition 5.8. If ($\mathcal{H}^{\mathbb{P}}_{inv,6}$) is satisfied then (Ai) holds true.

Under weaker assumptions, we can still prove the first order convergence.

Proposition 5.9. If ($\mathcal{H}^{\mathbb{P}}_{inv,2}$) is satisfied then for any $i \geq 1$

$$\frac{\lambda_i(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} 2 \quad \text{and} \quad \frac{\lambda_i'(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} 2.$$

Moreover, for any $p \in [1, \infty)$, under ($\mathcal{H}^{\mathbb{L}^p}_{inv,2}$),

$$\frac{\lambda_i(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{L}^p} 2 \quad \text{and} \quad \frac{\lambda_i'(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{L}^p} 2.$$

Corollary 1.6 (resp. Theorem 1.4) is a direct application of Proposition 5.8 (resp. Proposition 5.9) for $k = 1$ (resp. $i = 1$). The proof we provide is a generalization of the proof we gave in Chapter 2 of Corollary 1.6 (resp. Theorem 1.4). We give separate proofs because the proof of Corollary 1.6 and (resp. Theorem 1.4) is simpler and does not require any knowledge of the representations of the symmetric group.

5.2.2 Extension to virtual permutations

In Chapter 8, we will study virtual permutations. To describe the behavior at their soft edge we will need the following result.

Proposition 5.10. Let $(\mathbb{P}_n)_{n\geq 1}$ be a sequence of conjugation invariant probability measures. Assume that there exists a positive integer k such that for all real numbers $s_1, s_2, \dots, s_k$,

$$\lim_{n\to\infty} \mathbb{P}_n\left(\left\{\sigma \in \mathfrak{S}_n,\ \forall 1 \leq i \leq k,\ \frac{\lambda_i'(\sigma) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s_i\right\}\right) = F_{2,k}(s_1, \dots, s_k). \tag{5.4}$$

Let $0 \leq x_0 < 1$, $\mathbb{P}$ a probability measure on $\mathfrak{S}_\infty$ and $(\sigma_n)_{n\geq 1}$ be a sequence of random permutations such that for all positive integer n, for all $\sigma \in \mathfrak{S}_n$,

$$\mathbb{P}(\sigma_n = \sigma) := \sum_{j=0}^{l} \binom{l}{j} x_0^j (1 - x_0)^{n-j} \mathbb{P}_{n-j}(\sigma^j), \tag{5.5}$$

where l is the number of fixed points of σ and σ^j is the permutation obtained by removing j fixed points of σ. Then for all real numbers $s_1, s_2, \dots, s_k$,

$$\lim_{n\to\infty} \mathbb{P}\left(\forall 1 \leq i \leq k,\ \frac{\lambda_i'(\sigma_n) - 2\sqrt{(1-x_0)n}}{(1-x_0)n^{\frac{1}{6}}} \leq s_i\right) = F_{2,k}(s_1, \dots, s_k).$$

Let we make some remarks:

- We prove this result in Subsection 5.3.2.
- Note that in Theorem 1.5 we require that the number of small cycles is smaller than $n^{\frac{1}{6}}$. It may not be the optimal scale. The best counterexample we found is when the number of small cycles is of order $\sqrt{n}$ for the general case and of order n for virtual random permutations. Nevertheless, the bound condition on the number of cycles for $\frac{\mathrm{LIS}(\sigma_n)}{\sqrt{n}}$ is sharp.
- For the bulk, we conjecture that the class of universality is larger. We think that we need only a condition that $n - \mathrm{tr}(\sigma)$ goes in probability to infinity to obtain the sine kernel. The scale depends of course on $n - \mathrm{tr}(\sigma)$ i.e. we conjecture that for any f compactly supported on $\mathbb{Z}^d$.

$$\lim_{n\to\infty} \sum_{i_1<i_2<\dots<i_k} \mathbb{E}(f(\lambda_{i_1}(\sigma_n) - i_1 - \lfloor u\sqrt{n-\mathrm{tr}(\sigma)}\rfloor, \dots, \lambda_{i_k}(\sigma_n) - i_k - \lfloor u\sqrt{n-\mathrm{tr}(\sigma)}\rfloor))$$
$$= \sum_{j_1>j_2>\dots>j_k} f(j_1, j_2, \dots, j_k)\det([K_{sin,u}(j_i, j_\ell)]_{1\le i,\ell\le k}).$$

5.3 Proof of results

We will prove first propositions 5.8 and 5.9 as they are direct applications of Theorem 2.1. We prove then a generalization to virtual permutations. Finally, we will prove Propositions 2.25, Theorems 1.5 and of Proposition 5.7 as they need the introduction of new Markov operators.

5.3.1 Proof of propositions 5.8 and 5.9

Lemma 5.11. For any permutation σ and any transposition τ,

$$\left|\sum_{k=1}^{i} \lambda_k(\sigma) - \lambda_k(\sigma\circ\tau)\right| \le 2, \quad \left|\sum_{k=1}^{i} \lambda'_k(\sigma) - \lambda'_k(\sigma\circ\tau)\right| \le 2. \tag{5.6}$$

Moreover,

$$|\lambda_i(\sigma) - \lambda_i(\sigma\circ\tau)| \le 4, \quad |\lambda'_i(\sigma) - \lambda'_i(\sigma\circ\tau)| \le 4. \tag{5.7}$$

Proof. Let σ be a permutation and $\tau = (l, m)$ be a transposition. We have then for all integer i,

$$\{s \setminus \{l,m\}, s \in \mathfrak{I}_i(\sigma)\} \subset \mathfrak{I}_i(\sigma\circ\tau)$$

and similarly

$$\{s \setminus \{l,m\}, s \in \mathfrak{D}_i(\sigma)\} \subset \mathfrak{D}_i(\sigma\circ\tau).$$

Consequently, by Lemma 3.5,

$$\sum_{k=1}^{i} \lambda_k(\sigma) - \lambda_k(\sigma\circ\tau) \ge -2, \quad \sum_{k=1}^{i} \lambda'_k(\sigma) - \lambda'_k(\sigma\circ\tau) \ge -2.$$

Using the same argument with $\sigma \circ \tau$ instead of σ, (5.6) follows. Moreover, since

$$\lambda_{i+1} = \sum_{k=1}^{i+1} \lambda_k - \sum_{k=1}^{i} \lambda_k, \quad \lambda'_{i+1} = \sum_{k=1}^{i+1} \lambda'_k - \sum_{k=1}^{i} \lambda'_k,$$

the triangle inequality yields (5.7). □

Using (5.7), Propositions 5.8 and 5.9 are direct applications of Theorem 2.1.

5.3.2 Proof of Proposition 5.10

Proof of Proposition 5.10. An interpretation of the random permutation defined by equation (5.5) is the following. Let n be a positive integer. We construct a subset A_n of $\{1, 2, \ldots, n\}$ as follows: for every $1 \leq i \leq n$, with probability x_0, $i \in A_n$ independently of other points. The points of A_n are then fixed points of σ_n. After that, we permute the elements of $\{1, 2, \ldots, n\} \setminus A_n$ according to the probability distribution $\mathbb{P}_{n-|A_n|}$. In particular, A_n is a subset of all fixed points of σ_n. The main idea is that a decreasing subsequence cannot have more than one element belonging to A_n. Moreover, a decreasing subsequence of the restriction of σ_n on $\{1, 2, \ldots, n\} \setminus A_n$ is a decreasing subsequence of σ_n. In other words, for all real number s, for all $1 \leq j \leq n$,

$$\begin{aligned}\mathbb{P}_j(\{\sigma \in \mathfrak{S}_j, \mathrm{LDS}(\sigma) \leq s-1\}) &\leq \mathbb{P}(\mathrm{LDS}(\sigma_n) \leq s \,|\, |A_n| = n-j) \\ &\leq \mathbb{P}_j(\{\sigma \in \mathfrak{S}_j, \mathrm{LDS}(\sigma) \leq s\}).\end{aligned}$$

More generally, using Lemma 3.5, we have for all real numbers $s_1, \ldots, s_k$,

$$\begin{aligned}\mathbb{P}_j(\{\sigma \in \mathfrak{S}_j, \forall i < k, \lambda'_i(\sigma) \leq s_i - 2i + 1\}) &\leq \mathbb{P}(\forall i < k, \lambda'_i(\sigma_n) \leq s_i \,|\, |A_n| = n-j) \\ &\leq \mathbb{P}_j(\{\sigma \in \mathfrak{S}_j, \forall i < k, \lambda'_i(\sigma) \leq s_i\}).\end{aligned}$$

Consequently,

$$\begin{aligned}\mathbb{P}_j(\{\sigma \in \mathfrak{S}_j, \forall i < k, \lambda'_i(\sigma) \leq s_i - 2k + 1\}) &\leq \mathbb{P}(\forall i < k, \lambda'_i(\sigma_n) \leq s_i \,|\, |A_n| = n-j) \\ &\leq \mathbb{P}_j(\{\sigma \in \mathfrak{S}_j, \forall i < k, \lambda'_i(\sigma) \leq s_i\}).\end{aligned}$$

In the sequel of the proof, let $s_1, \ldots, s_k$ be k real numbers and $\varepsilon > 0$. As $|A_n|$ is a random binomial variable with parameters n and x_0, and using the central limit theorem, there exist n_0, $\alpha > 0$ such that, $n_0 > \frac{\alpha^2}{(1-x_0)^2}$ and $\forall n > n_0$,

$$\mathbb{P}(||A_n| - nx_0| < \alpha\sqrt{n}) > 1 - \varepsilon. \tag{5.8}$$

We denote by $p_j^n := \mathbb{P}(|A_n| = n-j)$, $\tilde{x}_0 := 1 - x_0$ and $\tilde{k} := 2k-1$. As

$$\mathbb{P}\left(\forall i \leq k, \frac{\lambda_i'(\sigma_n) - 2\sqrt{n\tilde{x}_0}}{(n\tilde{x}_0)^{\frac{1}{6}}} \leq s_i\right) = \sum_{j=0}^{n} \mathbb{P}\left(\forall i \leq k, \frac{\lambda_i'(\sigma_n) - 2\sqrt{n\tilde{x}_0}}{(n\tilde{x}_0)^{\frac{1}{6}}} \leq s_i \middle| |A_n| = n-j\right) p_j^n,$$

we have

$$\mathbb{P}\left(\forall i \leq k, \frac{\lambda_i'(\sigma_n) - 2\sqrt{n\tilde{x}_0}}{(n\tilde{x}_0)^{\frac{1}{6}}} \leq s_i\right) \leq \varepsilon + \sum_{j=\lceil n\tilde{x}_0 - \alpha\sqrt{n}\rceil}^{\lfloor n\tilde{x}_0 + \alpha\sqrt{n}\rfloor} \mathbb{P}_j\left(\left\{\sigma \in \mathfrak{S}_j, \forall i \leq k, \frac{\lambda_i'(\sigma) - 2\sqrt{n\tilde{x}_0}}{(n\tilde{x}_0)^{\frac{1}{6}}} \leq s_i\right\}\right) p_j^n \tag{5.9}$$

and

$$\mathbb{P}\left(\forall i \leq k, \frac{\lambda_i'(\sigma_n) - 2\sqrt{n\tilde{x}_0}}{(n\tilde{x}_0)^{\frac{1}{6}}} \leq s_i\right) \geq \sum_{j=\lceil n\tilde{x}_0 - \alpha\sqrt{n}\rceil}^{\lfloor n\tilde{x}_0 + \alpha\sqrt{n}\rfloor} \mathbb{P}_j\left(\left\{\sigma \in \mathfrak{S}_j, \forall i \leq k, \frac{\lambda_i'(\sigma) - 2\sqrt{n\tilde{x}_0} + \tilde{k}}{(n\tilde{x}_0)^{\frac{1}{6}}} \leq s_i\right\}\right) p_j^n. \tag{5.10}$$

Here, $\lfloor x \rfloor$ and $\lceil x \rceil$ are respectively the floor and the ceiling functions. If $|j - n\tilde{x}_0| < \alpha\sqrt{n}$, then

$$\left|\sqrt{j} - \sqrt{n\tilde{x}_0}\right| \leq \frac{\alpha\sqrt{n}}{\sqrt{j} + \sqrt{n\tilde{x}_0}} \leq \frac{\alpha}{\sqrt{\tilde{x}_0}}.$$

Thus,

$$\mathbb{P}_j\left(\left\{\sigma \in \mathfrak{S}_j, \forall i \leq k, \frac{\lambda_i'(\sigma) - 2\sqrt{n\tilde{x}_0} + \tilde{k}}{(n\tilde{x}_0)^{\frac{1}{6}}} \leq s_i\right\}\right) \geq \mathbb{P}_j\left(\left\{\sigma \in \mathfrak{S}_j, \forall i \leq k, \frac{\lambda_i'(\sigma) - 2\sqrt{j}}{j^{\frac{1}{6}}} \leq h(s_i, n) - \frac{2\alpha + \tilde{k}}{j^{\frac{1}{6}}}\right\}\right)$$

and

$$\mathbb{P}_j\left(\left\{\sigma \in \mathfrak{S}_j, \forall i \leq k, \frac{\lambda_i'(\sigma) - 2\sqrt{n\tilde{x}_0}}{(n\tilde{x}_0)^{\frac{1}{6}}} \leq s_i\right\}\right) \leq \mathbb{P}_j\left(\left\{\sigma \in \mathfrak{S}_j, \forall i \leq k, \frac{\lambda_i'(\sigma) - 2\sqrt{j}}{j^{\frac{1}{6}}} \leq -h(-s_i, n) + \frac{2\alpha}{j^{\frac{1}{6}}}\right\}\right).$$

where, $h(s,n) = s(1 - \frac{\alpha}{\sqrt{n}})^{\frac{1}{6}}$ if $s > 0$ and $h(s,n) = s(1 + \frac{\alpha}{\sqrt{n}})^{\frac{1}{6}}$ otherwise.

By the continuity and the monotony on each variable of $F_{2,k}$, there exists $\delta > 0$ such that:

$$F_{2,k}(s_1,\dots,s_k) - \varepsilon < F_{2,k}(s_1-\delta,\dots,s_k-\delta) < F_{2,k}(s_1+\delta,\dots,s_k+\delta) < F_{2,k}(s_1,\dots,s_k) + \varepsilon.$$

Moreover, there exists $n_1 > n_0$ such that for all $n > n_1$, for all $j > n\tilde{x}_0 - \alpha\sqrt{n}$, for all $i < k$,

$$s_i - \delta \le h(s_i, n) - \frac{2\alpha + \tilde{k}}{j^{\frac{1}{6}}}$$

and

$$s_i + \delta > -h(-s_i, n) + \frac{2\alpha}{j^{\frac{1}{6}}}.$$

Consequently, if $n > n_1$, inequalities (5.9) and (5.10) become respectively:

$$\mathbb{P}\left(\forall i \le k, \frac{\lambda_i'(\sigma_n) - 2\sqrt{n\tilde{x}_0}}{(n\tilde{x}_0)^{\frac{1}{6}}} \le s_i\right) \le \varepsilon + \sum_{j=\lceil n\tilde{x}_0 - \alpha\sqrt{n}\rceil}^{\lfloor n\tilde{x}_0 + \alpha\sqrt{n}\rfloor} \mathbb{P}_j\left(\left\{\sigma \in \mathfrak{S}_j, \forall i \le k, \frac{\lambda_i'(\sigma) - 2\sqrt{j}}{j^{\frac{1}{6}}} \le s_i + \delta\right\}\right) p_j^n \tag{5.11}$$

and

$$\mathbb{P}\left(\forall i \le k, \frac{\lambda_i'(\sigma_n) - 2\sqrt{n\tilde{x}_0}}{(n\tilde{x}_0)^{\frac{1}{6}}} \le s_i\right) \ge \sum_{j=\lceil n\tilde{x}_0 - \alpha\sqrt{n}\rceil}^{\lfloor n\tilde{x}_0 + \alpha\sqrt{n}\rfloor} \mathbb{P}_j\left(\left\{\sigma \in \mathfrak{S}_j, \forall i \le k, \frac{\lambda_i'(\sigma) - 2\sqrt{j}}{j^{\frac{1}{6}}} \le s_i - \delta\right\}\right) p_j^n. \tag{5.12}$$

Under (5.4),

$$\mathbb{P}_j\left(\left\{\sigma \in \mathfrak{S}_j, \forall i \le k, \frac{\lambda_i'(\sigma) - 2\sqrt{j}}{j^{\frac{1}{6}}} \le s_i + \delta\right\}\right) \xrightarrow[j\to\infty]{} F_{2,k}(s_1+\delta,\dots,s_k+\delta),$$

and

$$\mathbb{P}_j\left(\left\{\sigma \in \mathfrak{S}_j, \forall i \le k, \frac{\lambda_i'(\sigma) - 2\sqrt{j}}{j^{\frac{1}{6}}} \le s_i - \delta\right\}\right) \xrightarrow[j\to\infty]{} F_{2,k}(s_1-\delta,\dots,s_k-\delta).$$

Therefore, since $\lceil n\tilde{x}_0 - \alpha\sqrt{n}\rceil \to \infty$, there exists $n_2 > n_1$ such that $\forall n > n_2, \forall j \geq \lceil n\tilde{x}_0 - \alpha\sqrt{n}\rceil$,

$$\begin{aligned} F_{2,k}(s_1-\delta,\dots,s_k-\delta)-\varepsilon &< \mathbb{P}_j\left(\left\{\sigma\in\mathfrak{S}_j, \forall i\leq k, \frac{\lambda_i'(\sigma)-2\sqrt{j}}{j^{\frac{1}{6}}}\leq s_i-\delta\right\}\right)\\ &< \mathbb{P}_j\left(\left\{\sigma\in\mathfrak{S}_j, \forall i\leq k, \frac{\lambda_i'(\sigma)-2\sqrt{j}}{j^{\frac{1}{6}}}\leq s_i+\delta\right\}\right)\\ &< F_{2,k}(s_1-\delta,\dots,s_k-\delta)+\varepsilon. \end{aligned}$$

Finally, if $n > n_2$, using (5.8), inequalities (5.11) and (5.12) become

$$(F_{2,k}(s_1,\dots,s_k)-2\varepsilon)(1-\varepsilon) < \mathbb{P}\left(\forall i\leq k, \frac{\lambda_i'(\sigma_n)-2\sqrt{n\tilde{x}_0}}{(n\tilde{x}_0)^{\frac{1}{6}}}\leq s_i\right) < F_{2,k}(s_1,\dots,s_k)+3\varepsilon,$$

and the proof of the proposition is therefore complete. □

5.3.3 Proof of Propositions 2.25

We will prove a more general version of proposition 2.25.

Proposition 5.12. If $(\mathcal{H}^{\mathbb{P}}_{inv,\frac{3}{2}})$ is satisfied, then for any $k \geq 1$, for any $s_1,\dots,s_k \in \mathbb{R}$,

$$\limsup_{n\to\infty} \mathbb{P}\left(\forall i\leq k', \frac{\lambda_i(\sigma_n)-2\sqrt{n}}{n^{\frac{1}{6}}}\leq s_i\right) \leq F_2(s_1,s_2,\dots,s_k).$$

To do so, we define a new Markov operator. Let $\sigma \in \mathfrak{S}_n^0$, $\lambda \in \mathbb{Y}_n$ and $i \in \{1,\dots,n\}$[4]. We define $\mathfrak{T}_{i,\lambda}(\sigma) := \left(\sigma^{\lambda_1+1}(i),\dots,\sigma^{\lambda_1+\lambda_2}(i)\right)\dots\left(\sigma^{\sum_{j=1}^{\ell(\lambda)-1}\lambda_j}(i),\dots,\sigma^n(i)\right)$. Now let σ_n be a conjugation invariant random permutation and let T_{σ_n} be the Markov operator defined on $\mathfrak{S}_n^0$ as follows. Starting from $\sigma \in \mathfrak{S}_n^0$, choose i uniformly in $\{1,\dots,n\}$ and λ randomly according to the distribution of $\hat{\lambda}(\sigma_n)$[5] and then $T_{\sigma_n}(\sigma)$ returns $\mathfrak{T}_{i,\lambda}(\sigma)$.[6] For example, the transition probabilities of $T_{\sigma_{unif,3}}$. are shown in Figure 5.1. By construction, $\hat{\lambda}(T_{\sigma_n}(\sigma)) = \lambda$ and thus, for any cyclic permutation $\sigma \in \mathfrak{S}_n^0$,

$$\hat{\lambda}(T_{\sigma_n}(\sigma)) \stackrel{d}{=} \hat{\lambda}(\sigma_n).$$

This yields,

$$\hat{\lambda}(T_{\sigma_n}(\sigma_{Ew,0,n})) \stackrel{d}{=} \hat{\lambda}(\sigma_n).$$

Finally, since the construction depends only on the cycle structure, $T_{\sigma_n}(\sigma_{Ew,0,n})$ is conjugation invariant and

$$T_{\sigma_n}(\sigma_{Ew,0,n}) \stackrel{d}{=} \sigma_n. \tag{5.13}$$

[4] We recall that $\mathfrak{S}_n^0$ is the set of cyclic permutations.
[5] $\hat{\lambda}(\sigma)$ is the cycle structure of σ.
[6] Here we define a different Markov operator for every distribution.

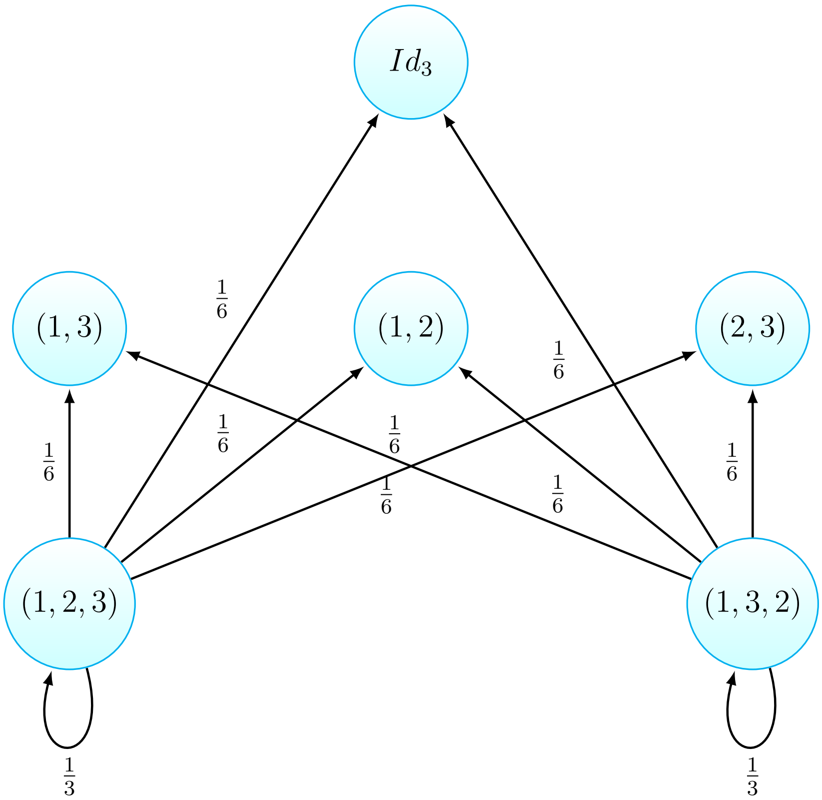

Figure 5.1: The transition probabilities of $T_{\sigma_{unif,3}}$

Our main argument is the following lemma.

Lemma 5.13. For any permutation $\rho \in \mathfrak{S}_n^0$, for any conjugation invariant random permutation σ_n, for any positive integer k, almost surely

$$\mathbb{E}\left(\left(\sum_{i=1}^{k} \lambda_j(T_{\sigma_n}(\rho)) - \lambda_j(\rho)\right)_{-} \middle| \#(T_{\sigma_n}(\rho))\right) \leq \frac{\#(T_{\sigma_n}(\rho))}{n} \sum_{j=1}^{k} \lambda_i(\rho)$$
$$\stackrel{d}{=} \frac{\#(\sigma_n)}{n} \sum_{i=1}^{k} \lambda_i(\rho).$$

Proof. Let $i_1 < i_2 < \cdots < i_{\sum_{i=1}^k \lambda_i(\rho)}$ such that $\left\{i_1, i_2, \cdots < i_{\sum_{i=1}^k \lambda_i(\rho)}\right\} \subset \mathfrak{I}_k(\rho)$. We have then for any permutation ρ',

$$\left\{i_1, i_2, \cdots < i_{\sum_{i=1}^k \lambda_i(\rho)}\right\} \cap \{i, \rho'(i) = \rho(i)\} \subset \mathfrak{I}_k(\rho')$$

and then

$$\left(\sum_{j=1}^{k} \lambda_j(\rho') - \lambda_j(\rho)\right)_{-} \leq \operatorname{card}\left\{j \leq \sum_{i=1}^{k} \lambda_i(\rho); \rho(i_j) \neq \rho'(i_j)\right\}. \tag{5.14}$$

Consequently, almost surely

$$\mathbb{E}\left(\left(\sum_{j=1}^{k}\lambda_j(T_{\sigma_n}(\rho))-\lambda_j(\rho)\right)_-\middle|\#(T_{\sigma_n}(\rho))\right)\leq\sum_{j=1}^{\sum_{i=1}^{k}\lambda_i(\rho)}\mathbb{E}(\mathbb{1}_{\rho(j)\neq\rho'(j)}|\#(T_{\sigma_n}(\rho)))$$
$$=\sum_{i=1}^{k}\lambda_i(\rho)\frac{\#(T_{\sigma_n}(\rho))}{n}.$$

□

Proof of Proposition 5.12. For any $\varepsilon>0$ there exists n_0 such that

$$\mathbb{P}\left(\sum_{i=1}^{k}\lambda_i(\sigma_{Ew,0,n})<9k\sqrt{n}\right)\geq\sqrt{1-\varepsilon}$$

and by hypothesis for any $\varepsilon'>0$ there exist $n_1>$ such that for any $n>n_1$

$$\mathbb{P}\left(\#(\sigma_n)<\varepsilon'\frac{n^{\frac{2}{3}}}{9k}\right)>\sqrt{1-\varepsilon}.$$

Consequently,

$$\mathbb{P}\left(\frac{\mathbb{E}\left(\left(\sum_{j=1}^{k}\lambda_j(T_{\sigma_n}(\sigma_{Ew,0,n}))-\lambda_j(\sigma_{Ew,0,n})\right)_-\middle|\#(T_{\sigma_n}(\sigma_{Ew,0,n}))\right)}{n^{\frac{1}{6}}}<\varepsilon'\right)>1-\varepsilon.$$

This yields

$$\frac{\left(\sum_{j=1}^{k}\lambda_j(T_{\sigma_n}(\sigma_{Ew,0,n}))-\lambda_j(\sigma_{Ew,0,n})\right)_-}{n^{\frac{1}{6}}}\xrightarrow[n\to\infty]{\mathbb{P}}0,$$

which concludes the proof since $T_{\sigma_n}(\sigma_{Ew,0,n})\stackrel{d}{=}\sigma_n$. □

5.3.4 Proof of Theorems 1.5 and of Proposition 5.7

Since Theorems 1.5 is the particular case $k=1$ of Proposition 5.7, we will prove only Proposition 5.7. Moreover ($\mathcal{H}^{\mathbb{P}}_{inv,\frac{3}{2}}$) implies clearly (5.3) and consequently, the first bound of Proposition 5.7 is a direct application of Proposition 5.12. So it is sufficient to prove that under ($\mathcal{H}_{inv}$) and (5.3), we have

$$\liminf_{n\to\infty}\mathbb{P}\left(\forall i\leq k',\frac{\lambda_i(\sigma_n)-2\sqrt{n}}{n^{\frac{1}{6}}}\leq s_i\right)\geq F_2(s_1,s_2,\dots,s_k).\tag{5.15}$$

Sketch of proof. We will not go trough all the details since we have already presented similar techniques many times. The idea is to modify the random walk associated to T as following. Given $1\leq j\leq n-1$, we define $\hat{T}_j$ the Markov operator as following. $\hat{T}_j(\sigma)$ is a permutation chosen uniformly at random among the permutations obtained by merging all cycles of length less than j to (one of) the biggest cycles of σ to obtain a permutation with cycles of length more than j. Since this construction depends only on the cycle structure, under ($\mathcal{H}_{inv}$), $\hat{T}_j(\sigma_n)$ is conjugation invariant. Therefore $T^n(\hat{T}_j(\sigma_n))$ is distributed according to $Ew(0)$[7]. Similarly to the proof in the previous subsection, we have

$$\mathbb{E}\left(\left(\sum_{i=1}^{k}\lambda_j(T^n(\hat{T}_j(\sigma_n))) - \lambda_j(\hat{T}_j(\sigma_n)))\right)_- \middle| \#(\hat{T}_j(\sigma_n))\right) \leq \frac{\#(\hat{T}_j(\sigma_n))}{j}\sum_{i=1}^{k}\lambda_i(\hat{T}_j(\sigma_n)).$$

Let $(j_n)_{n>1}$ be such that

$$\frac{1}{n^{\frac{1}{6}}}\left(\left(\sum_{k=1}^{j_n}\#_k(\sigma_n)\right) + \frac{\sqrt{n}}{j_n}\sum_{k=j_n+1}^{n}\#_k(\sigma_n)\right) \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

We have then $(T_{j_n}(\sigma_n))_{n\geq 1}$ satisfies ($\mathcal{H}^{\mathbb{P}}_{inv,6}$),

$$\frac{\sum_{i=1}^{k}\lambda_i(\hat{T}_{j_n}(\sigma_n))}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} 2k$$

and

$$\frac{\mathbb{E}\left(\left(\sum_{i=1}^{k}\lambda_j(T^n(\hat{T}_j(\sigma_n))) - \lambda_j(\hat{T}_j(\sigma_n)))\right)_- \middle| \#(\hat{T}_j(\sigma_n))\right)}{n^{\frac{1}{6}}} \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

This yields (5.15). □

[7] T is defined in Chapter 2

6 Cycles' structure

"Do not worry about your difficulties in Mathematics. I can assure you mine are still greater."

Albert Einstein

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In this chapter, we prove, using the method of moments that under a good control of fixed points, the number of small cycles is well controlled. In particular, we prove that:

Lemma 6.1. For any $k \geq 2$, there exists $C, C' > 0$ such that for any $n \geq 1$, for any independent random permutations σ_n and ρ_n with distributions invariant under conjugation,

$$\mathbb{P}\left(c_1\left((\sigma_n) \circ \rho_n\right) = k\right) \leq \frac{C}{n} + C'(\mathbb{P}(\sigma_n(1) = 1) + \mathbb{P}(\rho_n(1) = 1)),$$

where $c_m(\sigma)$ is the length of the cycle of σ containing m.

This lemma has an interest to have a lower bound for the longest common subsequence. Moreover, in a joint work with Mylène Maïda, we proved also that the joint distribution of the number of small cycles of product of conjugation invariant permutations is asymptotically the same as the uniform case.

We begin with a few preliminary remarks and simplifications.

First of all, the equivalence between Theorem 1.13 and Corollary 1.15 is due to the following classical argument. For any $\sigma \in \mathfrak{S}_n$, if $c_i(\sigma)$ denotes the length of the cycle of σ containing i,

$$\operatorname{tr}(\sigma^k) = \sum_{i=1}^{n} \mathbb{1}_{\sigma^k(i)=i} = \sum_{i=1}^{n} \mathbb{1}_{c_i(\sigma)|k} = \sum_{j|k} \sum_{i=1}^{n} \mathbb{1}_{c_i(\sigma)=j} = \sum_{j|k} j \, \#_j \sigma. \tag{6.1}$$

In the hypothesis H_2, we assume that one of the permutations, say σ_n, may not have a conjugation invariant distribution. In fact, it is enough to prove of Theorem 1.13 in the case where all permutations are conjugation invariant. Indeed, if we choose τ_n uniform and independent of the σ-algebra generated by $(\sigma_{\ell,n})_{1\le \ell \le m}$, the cycle structure of $\prod_{\ell=1}^{m} \sigma_{\ell,n}$ is the same as

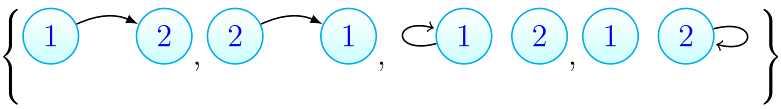
$$\tau_n^{-1} \left(\prod_{\ell=1}^{m} \sigma_{\ell,n} \right) \tau_n = (\tau_n^{-1}\sigma_{1,n}\tau_n) \prod_{\ell=2}^{m} (\tau_n^{-1}\sigma_{\ell,n}\tau_n) \stackrel{d}{=} (\tau_n^{-1}\sigma_{1,n}\tau_n) \prod_{\ell=2}^{m} \sigma_{\ell,n}$$

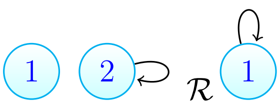
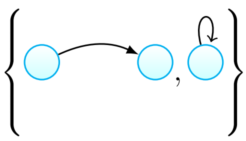
and $(\tau_n^{-1}\sigma_{1,n}\tau_n)$ is also conjugation invariant.

6.1 Preliminary results

To prove this result, we will introduce some new objects. To a couple of permutations, we will associate a couple of graphs.

We denote by $\mathbb{G}^n_k$ the set of oriented graphs with vertices $[n]$ and having exactly k edges. We allow here loops but not multiple edges. For example,

$\mathbb{G}^2_1 =$ 

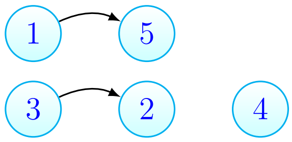
Given $g \in \mathbb{G}^n_k$, we denote by E_g the set of its edges and by $A_g := [\mathbb{1}_{(i,j)\in E_g}]_{1\le i,j\le n}$ its adjacency matrix. A connected component of g is called *trivial* if it does not have any edge and a vertex i of g is called *isolated* if E_g does not contain any edge of the form (i,j) or (j,i). We say that two oriented simple graphs g_1 and g_2 are *isomorphic* if one can obtain g_2 by changing the labels of the vertices of g_1. In particular, if $g_1, g_2 \in \mathbb{G}^n_k$ then g_1, g_2 are isomorphic if and only if there exists a permutation matrix σ such that $A_{g_1}\sigma = \sigma A_{g_2}$. Let $g \in \mathbb{G}^n_k$, we denote by $\tilde{g}$ the graph obtained from g after removing isolated vertices. Let $\mathcal{R}$ be the equivalence relation such that $g_1 \mathcal{R} g_2$ if $\tilde{g}_1$ and $\tilde{g}_2$ are isomorphic. We denote by $\hat{\mathbb{G}}_k := \cup_{n\ge 1}\mathbb{G}^n_k / \mathcal{R}$ the set of equivalence classes of $\cup_{n\ge 1}\mathbb{G}^n_k$ for the relation $\mathcal{R}$.For example,  and $\hat{\mathbb{G}}_1 =$ .

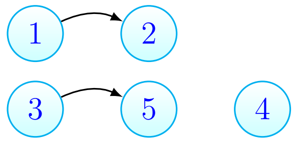
Let n be a positive integer and $\sigma, \rho \in \mathfrak{S}_n$. Let $k_m := c_m(\sigma^{-1}\circ\rho)$, $(i_1^m = m, i_2^m, \dots, i_{k_m}^m)$ be the cycle of $\sigma^{-1}\circ\rho$ containing m and $j_l^m := \rho(i_l^m)$. In particular, $i_1^m, i_2^m, \dots, i_{k_m}^m$ are pairwise distinct and $j_1^m, j_2^m, \dots, j_{k_m}^m$ are pairwise distinct. We denote by $\mathcal{G}_1^m(\sigma,\rho) \in \mathbb{G}^n_{k_m}$ the graph such that

$E_{\mathcal{G}_1^m(\sigma,\rho)} = \{(i_1^m, j_{k_m}^m)\} \cup \left(\cup_{l=1}^{k_m-1} \{(i_{l+1}^m, j_l^m)\}\right)$. We denote also by $\mathcal{G}_2^m(\sigma,\rho) \in \mathbb{G}_{k_m}^n$ the graph such that $E_{\mathcal{G}_2^m(\sigma,\rho)} = \cup_{l=1}^{k_m} \{(i_l^m, j_l^m)\}$. *In particular,* $\mathcal{G}_1^m(\sigma,\rho)$ *and* $\mathcal{G}_2^m(\sigma,\rho)$ *have the same set of non-isolated vertices.* For $i \in \{1,2\}$, let $\hat{\mathcal{G}}_i^m(\sigma,\rho)$ be the equivalence class of $\mathcal{G}_i^m(\sigma,\rho)$.
For example, if

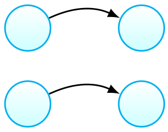
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix} \quad \text{and} \quad \rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \end{pmatrix},$$

we obtain $E_{\mathcal{G}_1^1(\sigma,\rho)} = \{(1,5),(3,2)\}$, $E_{\mathcal{G}_2^1(\sigma,\rho)} = \{(1,2),(3,5)\}$,

$\mathcal{G}_1^1(\sigma,\rho) =$ 

, $\mathcal{G}_2^1(\sigma,\rho) =$ 

and

$\hat{\mathcal{G}}_1^1(\sigma,\rho) = \hat{\mathcal{G}}_2^1(\sigma,\rho) =$  .

Finally, given $g \in \mathbb{G}_k^n$, we denote by $\mathfrak{S}_{n,g} := \{\sigma \in \mathfrak{S}_n : \forall (i,j) \in E_g, \sigma(i) = j\}$. It is not difficult to prove the two following results.

Lemma 6.2. If $m_1 \in \{i_l^{m_2} : 1 \leq l \leq k_{m_2}\}$, then $\mathcal{G}_1^{m_1}(\sigma,\rho) = \mathcal{G}_1^{m_2}(\sigma,\rho)$ and $\mathcal{G}_2^{m_1}(\sigma,\rho) = \mathcal{G}_2^{m_2}(\sigma,\rho)$.

Proof. If $m_1 \in \{i_l^{m_2} : 1 \leq l \leq k_{m_2}\}$, then there exists $1 \leq l \leq k_{m_1}$ such that $(\sigma_1^{-1} \circ \rho)^l(m_1) = m_2$. Consequently, $k_{m_1} = k_{m_2}$, $(i_1^{m_2}, i_2^{m_2}, \ldots, i_{k_{m_2}}^{m_2}) = (i_l^{m_1}, i_{l+1}^{m_1}, \ldots, i_{k_{m_1}}^{m_1}, i_1^{m_1}, \ldots, i_{l-1}^{m_1})$ and $(j_1^{m_2}, j_2^{m_2}, \ldots, j_{k_{m_2}}^{m_2}) = (j_l^{m_1}, j_{l+1}^{m_1}, \ldots, j_{k_{m_1}}^{m_1}, j_1^{m_1}, \ldots, j_{l-1}^{m_1})$ and we can check easily that $\mathcal{G}_1^{m_1}(\sigma,\rho) = \mathcal{G}_1^{m_2}(\sigma,\rho)$ and $\mathcal{G}_2^{m_1}(\sigma,\rho) = \mathcal{G}_2^{m_2}(\sigma,\rho)$. □

To obtain a combinatorial control, we prove first the following result.

Proposition 6.3. Let $g_1, g_2 \in \mathbb{G}_k^n$. Assume that there exists $\rho \in \mathfrak{S}_n$ such that $A_{g_2}\rho = \rho A_{g_1}$. If ρ has a fixed point on any non-trivial connected component of g_1, then $\mathfrak{S}_{n,g_1} \cap \mathfrak{S}_{n,g_2} = \varnothing$ or $A_{g_1} = A_{g_2}$.

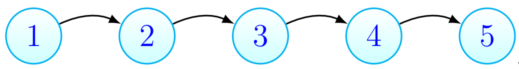
Proof. Let $\rho \in \mathfrak{S}_n$ be a permutation having a fixed point on any non-trivial connected component of g_1 such that $A_{g_2}\rho = \rho A_{g_1}$. Assume that $A_{g_1} \neq A_{g_2}$. There exists necessarily $(i,j) \in E_{g_1}$ such that $\rho(i) = i$ and $\rho(j) \neq j$ or $\rho(j) = j$ and $\rho(i) \neq i$. This is true because if we choose any connected component of g_1 having a non fixed point of ρ, this component contains by hypothesis at least one fixed point of ρ. Since this component contains both fixed and non-fixed points of ρ, one can choose two adjacent points one a fixed and the other a non-fixed point ρ. In the first case ($\rho(i) = i$ and $\rho(j) \neq j$), $\mathfrak{S}_{n,g_1} \cap \mathfrak{S}_{n,g_2} \subset \{\sigma \in \mathfrak{S}_n : \sigma(i) = j, \sigma(i) = \rho(j)\} = \varnothing$. In the second case, $\mathfrak{S}_{n,g_1} \cap \mathfrak{S}_{n,g_2} \subset \{\sigma \in \mathfrak{S}_n : \sigma(i) = j, \sigma(\rho(i)) = j\} = \varnothing$. □

This yields the following.

Corollary 6.4. For any graph $g \in \mathbb{G}_k^n$ having p non-trivial connected components and v non-isolated vertices, for any conjugation invariant random permutation σ_n on $\mathfrak{S}_n$,

$$\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g}) \leq \frac{(n-v)!}{(n-p)!}.$$

Proof. If there exist i, j, l, with $j \neq l$ such that $\{(i,j) \cup (i,l)\} \subset E_g$ or $\{(j,i) \cup (l,i)\} \subset E_g$ then $\mathfrak{S}_{n,g} = \varnothing$. Therefore, if $\mathfrak{S}_{n,g} \neq \varnothing$, then non-trivial connected components of g having w vertices are either cycles of length w or isomorphic to $\overline{g}_w$, where $A_{\overline{g}_w} = [\mathbb{1}_{j=i+1}]_{1 \leq i,j \leq w}$.
For example,

$\overline{g}_5 =$  .

Let $g \in \mathbb{G}_k^n$ such that $\mathfrak{S}_{n,g} \neq \varnothing$. Fix p vertices $x_1, x_2, \dots, x_p$ each belonging to a different non-trivial connected components of g. Let $\{x_1, x_2, \dots, x_p, \dots, x_v\}$ be the set of non-isolated vertices of g. Let

$$F = \{(y_i)_{p+1 \leq i \leq v}; y_i \in [n] \setminus \{x_1, \dots, x_p\} \text{ pairwise distinct}\}.$$

Given $y = (y_i)_{p+1 \leq i \leq v} \in F$, we denote by $g_y \in \mathbb{G}_k^n$ the graph isomorphic to g obtained by fixing the labels of $x_1, x_2, \dots, x_p$ and by changing the labels of x_i by y_i for $p+1 \leq i \leq v$. Since non-trivial connected components of g of length w are either cycles or isomorphic to $\bar{g}_w$, if $y \neq y' \in F$, then $g_y \neq g_{y'}$ and by Proposition 6.3, $\mathfrak{S}_{n,g_y} \cap \mathfrak{S}_{n,g_{y'}} = \varnothing$. Since σ_n is conjugation invariant, we have $\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g_y}) = \mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g_{y'}}) = \mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g})$. Therefore,

$$\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g}) = \frac{\sum_{y \in F} \mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g_y})}{\mathrm{card}(F)} = \frac{\mathbb{P}(\sigma_n \in \cup_{y \in F} \mathfrak{S}_{n,g_y})}{\mathrm{card}(F)} \leq \frac{1}{\mathrm{card}(F)} = \frac{(n-v)!}{(n-p)!}.$$

□

We define now some new objects.

- Let $I = (s_1, s_2, \dots, s_l)$ a set of distinct indices of $\{1, \dots, n\}$. We denote by

$$\mathcal{G}^I(\sigma,\rho) = (\mathcal{G}_1^{s_1}(\sigma,\rho), \mathcal{G}_2^{s_1}(\sigma,\rho), \mathcal{G}_1^{s_2}(\sigma,\rho), \dots, \mathcal{G}_1^{s_l}(\sigma,\rho), \mathcal{G}_2^{s_l}(\sigma,\rho))$$

  and

$$\hat{\mathcal{G}}^I(\sigma,\rho) = (\hat{\mathcal{G}}_1^{s_1}(\sigma,\rho), \hat{\mathcal{G}}_2^{s_1}(\sigma,\rho), \hat{\mathcal{G}}_1^{s_2}(\sigma,\rho), \dots, \hat{\mathcal{G}}_1^{s_l}(\sigma,\rho), \hat{\mathcal{G}}_2^{s_l}(\sigma,\rho)).$$

- For $i \in \{1,2\}$, let $\mathcal{G}_i^{\{1,2,\dots,k\}}(\sigma,\rho)$ be the graph such that $E_{\mathcal{G}_i^{\{1,2,\dots,k\}}(\sigma,\rho)} = \cup_{l=1}^k E_{\mathcal{G}_i^\ell(\sigma,\rho)}$ and $\hat{\mathcal{G}}_i^{\{1,2,\dots,k\}}(\sigma,\rho)$ be the equivalence class of $\mathcal{G}_i^{\{1,2,\dots,k\}}(\sigma,\rho)$.

Using the conjugation invariance and the relation (6.1), Theorem 1.13 is equivalent to the following:

under the same hypothesis, for any $v_1, v_2, v_3, \ldots, v_k \geq 1$,

$$(*) \qquad \lim_{n\to\infty} \sum_{\hat{g}_i, \hat{g}'_i \in \hat{\mathbb{G}}_{v_i},\, 1\leq i\leq k} n^k \mathbb{P}\left(\hat{\mathcal{G}}^{\{1,2,\ldots,k\}}(\sigma_n,\rho_n) = (\hat{g}_1, \hat{g}'_1, \hat{g}_2, \ldots \hat{g}'_k)\right) = C_{v_1,v_2,\ldots,v_k},$$

where $C_{v_1,v_2,\ldots,v_k}$ is a constant independent of the laws of the permutations. Note that, for any $v_i \geq 1$, $\hat{\mathbb{G}}_{v_i}$ and therefore the number of terms of the sum is finite. Let us explain briefly why (*) implies Theorem 1.13. First, for any $e_1, \ldots, e_\ell$, we have

$$(6.2) \quad \mathbb{P}\left(c_1(\sigma_n^{-1}\rho_n) = e_1, \ldots, c_\ell(\sigma_n^{-1}\rho_n) = e_\ell\right) = \sum_{\hat{g}_i, \hat{g}'_i \in \widehat{\mathbb{G}}_{e_i},\, 1\leq i\leq \ell} \mathbb{P}\left(\widehat{\mathcal{G}}^{\{1,2,\ldots,\ell\}}(\sigma_n,\rho_n) = (\hat{g}_1, \hat{g}'_1, \hat{g}_2, \ldots \hat{g}'_\ell)\right).$$

Moreover, using the relation (6.1), one can check that the joint moments of $(\tilde{t}_1^n, \ldots, \tilde{t}_k^n)$ can be expressed as follows : for any $\ell_1, \ldots, \ell_m$ in $\{1, ..., k\}$,

$$\begin{aligned}\mathbb{E}(\tilde{t}_{\ell_1}^n, \ldots \tilde{t}_{\ell_m}^n) &= \frac{1}{\ell_1 \ldots \ell_m} \mathbb{E}\left(\prod_{p=1}^m \sum_{i_p=1}^n \mathbb{1}_{c_{i_p}(\sigma_n^{-1}\rho_n)=\ell_p}\right) \\ &= \frac{1}{\ell_1 \ldots \ell_m} \sum_{\mathbf{i}\in\{1,\ldots,n\}^m} \mathbb{P}\left(c_{i_1}(\sigma_n^{-1}\rho_n) = \ell_1, \ldots, c_{i_m}(\sigma_n^{-1}\rho_n) = \ell_m\right).\end{aligned}$$

For $\mathbf{i} := (i_1, \ldots, i_m) \in \{1, \ldots, n\}^m$, denote $\ker(\mathbf{i})$ the partition of $\{1, \ldots, m\}$ such that p and q are in the same block whenever $i_p = i_q$. By conjugation invariance, for any $\mathbf{i} \in \{1, \ldots, n\}^m$, the quantity $\mathbb{P}\left(c_{i_1}(\sigma_n^{-1}\rho_n) = \ell_1, \ldots, c_{i_m}(\sigma_n^{-1}\rho_n) = \ell_m\right)$ depends only on $\ker(\mathbf{i})$ and is denoted $\mathbb{P}(\ker(\mathbf{i}))$. For any partition λ of $\{1, \ldots, m\}$, if $|\lambda|$ denotes the number of blocks of a partition λ, there exist $e_1, \ldots, e_{|\lambda|}$ such that

$$\mathbb{P}(\lambda) = \mathbb{P}\left(c_1(\sigma_n^{-1}\rho_n) = e_1, \ldots, c_{|\lambda|}(\sigma_n^{-1}\rho_n) = e_{|\lambda|}\right),$$

so that

$$\mathbb{E}(\tilde{t}_{\ell_1}^n, \ldots \tilde{t}_{\ell_m}^n) = \frac{1}{\ell_1 \ldots \ell_m} \sum_\lambda n(n-1)\ldots(n-|\lambda|+1)\mathbb{P}(\lambda),$$

which together with (6.2) makes the link between (*) and Theorem 2. Indeed, let us look at the (joint) moments. For example, if we take $P(x) = x^2$, we have

$$\begin{aligned}\mathbb{E}\left(P\left(\tilde{t}_1^n\right)\right) &= \mathbb{E}\left(\left(\sum_{i=1}^n \mathbb{1}_{c_i(\sigma^{-1}\circ\rho)=1}\right)^2\right) \\ &= \sum_{i=1}^n \mathbb{E}\left(\mathbb{1}_{c_i(\sigma^{-1}\circ\rho)=1}\right) + \sum_{i\neq j}^n \mathbb{E}\left(\mathbb{1}_{c_i(\sigma^{-1}\circ\rho)=1}\mathbb{1}_{c_j(\sigma^{-1}\circ\rho)=1}\right) \\ &= n\mathbb{E}\left(\mathbb{1}_{c_1(\sigma^{-1}\circ\rho)=1}\right) + (n^2-n)\mathbb{E}\left(\mathbb{1}_{c_1(\sigma^{-1}\circ\rho)=1}\mathbb{1}_{c_2(\sigma^{-1}\circ\rho)=1}\right) \\ &\xrightarrow[n\to\infty]{} C_1 + C_{1,1} = 1 + 1 = 2\end{aligned}$$

Similarly, if we take $P(x,y) = xy$, we obtain

$$\mathbb{E}(P(\tilde{t}_1^n, \tilde{t}_2^n)) \xrightarrow[n\to\infty]{d} C_{1,2} = C_{2,1} = 1.$$

Before getting into the proof of (*), let us gather some useful combinatorial and then probabilistic results.

Lemma 6.5. For any $m \leq n$, for any permutation $\sigma, \rho \in \mathfrak{S}_n$,

$$\begin{aligned}
&k_m(\rho,\sigma) = k_m(\sigma,\rho),\\
&j^m_\ell(\rho,\sigma) = j^m_{k_m(\sigma,\rho)-\ell+1}(\sigma,\rho),\ \forall 1 \leq \ell \leq k_m(\sigma,\rho),\\
&i^m_\ell(\rho,\sigma) = i^m_{k_m(\sigma,\rho)-\ell+2}(\sigma,\rho),\ \forall 2 \leq \ell \leq k_m(\sigma,\rho),\\
&i^m_1(\rho,\sigma) = i^m_1(\sigma,\rho) = m,\\
&A_{\mathcal{G}_1^m(\sigma,\rho)} = A^T_{\mathcal{G}_2^{\rho(m)}(\rho^{-1},\sigma^{-1})}.
\end{aligned}$$

Lemma 6.6. If all non-trivial connected components of $\mathcal{G}_1^{m_1}(\sigma,\rho)$ and $\mathcal{G}_2^{m_1}(\sigma,\rho)$ have 2 vertices then both $\mathcal{G}_1^{m_1}(\sigma,\rho)$ and $\mathcal{G}_2^{m_1}(\sigma,\rho)$ have no 2-cycles.

Proof. Using the symmetries of the problem (Lemmas 6.2 and 6.5), it suffices to prove that if all non-trivial connected components of $\mathcal{G}_1^1(\sigma,\rho)$ and $\mathcal{G}_2^1(\sigma,\rho)$ have 2 vertices then it is impossible to have at the same time $(1,2) \in \mathcal{G}_2^1(\sigma,\rho)$ and $(2,1) \in \mathcal{G}_2^1(\sigma,\rho)$. To simplify notations, let $k_1 := k_1(\sigma,\rho) = c_1(\sigma^{-1}\circ\rho)$, $i_o^1 := i_o^1(\sigma,\rho)$ and $j_o^1 := j_o^1(\sigma,\rho)$.

Let $A = \{\eta > 1; j^1_\eta \in \{i^1_1, i^1_2, \dots, i^1_{\eta-1}\}$ or $i^1_\eta \in \{j^1_1, j^1_2, \dots, j^1_{\eta-1}\}\}$. Suppose that $(1,2) \in \mathcal{G}_2^1(\sigma,\rho)$ and $(2,1) \in \mathcal{G}_2^1(\sigma,\rho)$ then $k_1 \geq 2$ and there exists a unique $1 < l \leq k_1$ such that $i^1_l = 2$ and $j^1_l = 1$ so that A is non-empty. Let $\ell' := inf(A) \geq 2$. Assume that $\ell' > 2$. If $j^1_{\ell'} \in \{i^1_1, i^1_2, \dots, i^1_{\ell'-1}\}$, then there exists $\ell'' < \ell'$ such that $j^1_{\ell'} = i^1_{\ell''}$ and since the component of $\mathcal{G}_2^1(\sigma,\rho)$ containing $i^1_{\ell'}$ has two vertices and by definition $(i^1_{\ell'}, j^1_{\ell'})$ and $(i^1_{\ell''}, j^1_{\ell''})$ are two edges of $\mathcal{G}_2^1(\sigma,\rho)$, then $j^1_{\ell''} = i^1_{\ell'}$. Since $(i^1_{\ell'}, j^1_{\ell'-1}) = (j^1_{\ell''}, j^1_{\ell'-1})$ and $(i^1_{\ell''+1}, j^1_{\ell''})$ are edges of $\mathcal{G}_1^1(\sigma,\rho)$ and since $\mathcal{G}_1^1(\sigma,\rho)$ has only connected components of size 2, we have necessarily $i^1_{\ell''+1} = j^1_{\ell'-1}$. One can check easily that $\ell'' < \ell' - 2$ otherwise either $\mathcal{G}_1^1(\sigma,\rho)$ or $\mathcal{G}_2^1(\sigma,\rho)$ has a loop. Indeed, if $\ell'' = \ell' - 2$, then $(i^1_{\ell''+1}, j^1_{\ell''+1}) = (j^1_{\ell'-1}, j^1_{\ell''+1}) = (j^1_{\ell'-1}, j^1_{\ell'-1})$ is an edge of $\mathcal{G}_2^1(\sigma,\rho)$ and if $\ell'' = \ell' - 1$, then $(i^1_{\ell''+1}, j^1_{\ell''}) = (j^1_{\ell'-1}, j^1_{\ell''}) = (j^1_{\ell'-1}, j^1_{\ell'-1})$ is an edge of $\mathcal{G}_1^1(\sigma,\rho)$. This implies that $\ell' - 1 \in A$, which is absurd. $i^1_{\ell'} \in \{j^1_1, j^1_2, \dots, j^1_{\ell'-1}\}$ can be treated using the same techniques and one can extend easily to $\ell' = 2$. □

We now introduce the following notation : given $g \in \mathbb{G}_k^n$, we denote by

$$\mathfrak{S}_{n,g} := \{\sigma \in \mathfrak{S}_n; \forall (i,j) \in E_g, \sigma(i) = j\}.$$

In other words, $\mathfrak{S}_{n,g}$ is the set of permutations σ such that g is a sub-graph of g_σ. It is not difficult to prove the two following lemmas.

Lemma 6.7. Let $g_1, g_1', g_2, \ldots, g_k' \in \cup_\ell \mathbb{G}_\ell^n$ and let g, g' be such that $E_g = \cup_{\ell=1}^k E_{g_i}$ and $E_{g'} = \cup_{\ell=1}^k E_{g_i'}$. Assume that there exists ρ, σ such that

$$\mathcal{G}^{\{1,2,\ldots,k\}}(\sigma, \rho) = (g_1, g_1', g_2, \ldots, g_k').$$

Then for any random permutation ρ_n, σ_n,

$$\begin{aligned}\mathbb{P}\left(\bigcap_{i=1}^k \{\sigma_n \in \mathfrak{S}_{n,g_i}, \rho_n \in \mathfrak{S}_{n,g_i'}\}\right) &= \mathbb{P}\left(\mathcal{G}^{\{1,2,\ldots,k\}}(\sigma_n, \rho_n) = (g_1, g_1', g_2, \ldots, g_k')\right) \\ &= \mathbb{P}\left(\mathcal{G}_1^{\{1,2,\ldots,k\}}(\sigma_n, \rho_n) = g, \mathcal{G}_2^{\{1,2,\ldots,k\}}(\sigma_n, \rho_n) = g'\right).\end{aligned}$$

Proof. We will only prove the first equality. The second one can be obtained using the same argument. Let σ', ρ' be two permutations. We have seen that $\mathcal{G}_2^m(\sigma', \rho')$ is a subset of $g_{\rho'}$, so that

$$\mathcal{G}_2^m(\sigma', \rho') = g_m' \Rightarrow \rho' \in \mathfrak{S}_{n,g_m'},$$

and that $\mathcal{G}_1^m(\sigma', \rho')$ is a subset of $g_{\sigma'}$, so that

$$\mathcal{G}_1^m(\sigma', \rho') = g_m \Rightarrow \sigma' \in \mathfrak{S}_{n,g_m}.$$

Consequently,

$$\mathbb{P}\left(\mathcal{G}^{\{1,2,\ldots,k\}}(\sigma_n, \rho_n) = (g_1, g_1', g_2, \ldots, g_k')\right) \leq \mathbb{P}\left(\bigcap_{i=1}^k \{\sigma_n \in \mathfrak{S}_{n,g_i}, \rho_n \in \mathfrak{S}_{n,g_i'}\}\right).$$

Now suppose that there exists ρ', σ' such that

$$\mathcal{G}^{\{1,2,\ldots,k\}}(\sigma', \rho') = (g_1, g_1', g_2, \ldots, g_k').$$

Let σ, ρ such that $\sigma \in \cap_{i=1}^k \mathfrak{S}_{n,g_i}$ and $\rho \in \cap_{i=1}^k \mathfrak{S}_{n,g_i'}$. By definition and by iteration on ℓ, one can check that for any $\ell' \leq k$, $i_{\ell'}^\ell(\sigma', \rho') = i_{\ell'}^\ell(\sigma, \rho)$ and $j_{\ell'}^\ell(\sigma', \rho') = j_{\ell'}^\ell(\sigma, \rho)$. Consequently,

$$\mathcal{G}^{\{1,2,\ldots,k\}}(\sigma, \rho) = (g_1, g_1', g_2, \ldots, g_k').$$

Finally we obtain

$$\mathbb{P}\left(\mathcal{G}^{\{1,2,\ldots,k\}}(\sigma_n, \rho_n) = (g_1, g_1', g_2, \ldots, g_k')\right) \geq \mathbb{P}\left(\bigcap_{i=1}^k \{\sigma_n \in \mathfrak{S}_{n,g_i}, \rho_n \in \mathfrak{S}_{n,g_i'}\}\right).$$

□

Lemma 6.8. For any graph $g \in \mathbb{G}_k^n$ having f loops, p non-trivial connected components and v non-isolated vertices, for any random permutation σ_n with conjugation invariant distribution on $\mathfrak{S}_n$,

$$\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g}) \leq \frac{\mathbb{P}(\sigma_n(1) = 1, \dots, \sigma_n(f) = f)}{\binom{n-p}{v-p}(v-p)!} \leq \frac{1}{\binom{n-p}{v-p}(v-p)!}.$$

Proof. By conjugation invariance, one can suppose without loss of generality that the loops of g are $(1,1), (2,2), \dots (f,f)$ and the set of non isolated vertices of g are $\{1, 2, \dots, v\}$.
If there exist i, j, l, with $j \neq l$ such that $\{(i,j) \cup (i,l)\} \subset E_g$ or $\{(j,i) \cup (l,i)\} \subset E_g$ then $\mathfrak{S}_{n,g} = \varnothing$. Therefore, if $\mathfrak{S}_{n,g} \neq \varnothing$, then non-trivial connected components of g having w vertices are either cycles of length w or isomorphic to $\overline{g}_w$, where $A_{\overline{g}_w} = [\mathbb{1}_{j=i+1}]_{1 \leq i,j \leq w}$.
Let $g \in \mathbb{G}_k^n$ such that $\mathfrak{S}_{n,g} \neq \varnothing$. Fix p vertices $x_1 = 1, x_2 = 2, \dots, x_f = f, x_{f+1}, \dots, x_p$ each belonging to a different non-trivial connected components of g. Let $x_{p+1} < x_{p+2} < \cdots < x_v$ be such that $\{x_{p+1}, \dots, x_v\} = \{1, 2, \dots, v\} \setminus \{x_1, \dots x_p\}$ be the other non-isolated vertices. Let

$$F = \{(y_i)_{p+1 \leq i \leq v}; y_i \in \{1, 2, \dots, n\} \setminus \{x_1, \dots x_p\} \text{ pairwise distinct}\}.$$

Given $y = (y_i)_{p+1 \leq i \leq v} \in F$, we denote by $g_y \in \mathbb{G}_k^n$ the graph isomorphic to g obtained by fixing the labels of $x_1, x_2, \dots, x_p$ and by changing the labels of x_i by y_i for $p+1 \leq i \leq v$. Since non-trivial connected components of g of length w are either cycles or isomorphic to $\bar{g}_w$, if $y \neq y' \in F$, then $g_y \neq g_{y'}$ and by Proposition 6.3, $\mathfrak{S}_{n,g_y} \cap \mathfrak{S}_{n,g_{y'}} = \varnothing$. Since σ_n is conjugation invariant, we have $\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g_y}) = \mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g_{y'}}) = \mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g})$. Remark also that for any $y \in F$ and any $i \leq f$, (i,i) is a loop of g_y. Thus, $\mathfrak{S}_{n,g_y} \subset \{\sigma \in \mathfrak{S}_n; \forall i \leq f, \sigma_n(i) = i\}$ and thus

$$\begin{aligned}\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g}) = \frac{\sum_{y \in F} \mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g_y})}{\text{card}(F)} = \frac{\mathbb{P}(\sigma_n \in \cup_{y \in F} \mathfrak{S}_{n,g_y})}{\text{card}(F)} &\leq \frac{\mathbb{P}(\sigma_n(1) = 1, \dots, \sigma_n(f) = f)}{\binom{n-p}{v-p}(v-p)!} \\ &\leq \frac{1}{\binom{n-p}{v-p}(v-p)!}.\end{aligned}$$

□

Lemma 6.9. Let σ_n be a random permutation with conjugation invariant distribution on $\mathfrak{S}_n$ such that, for any $k \geq 1$, $\lim_{n \to \infty} \mathbb{E}\left(\left(\frac{\#_1 \sigma_n}{\sqrt{n}}\right)^k\right) = 0$. Then, for any $f \geq 1$,

$$\mathbb{P}(\sigma_n^1(1) = 1, \dots, \sigma_n^1(f) = f) = o(n^{-\frac{f}{2}}).$$

Lemma 6.10. For any $p \geq 1$, let g be a graph with p non-trivial components each having 2 vertices. Assume that at least one of these components is a cycle. Then for any random permutation σ_n with conjugation invariant distribution on $\mathfrak{S}_n$,

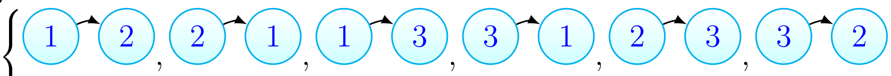
$$\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g}) \leq \frac{\mathbb{P}(c_1(\sigma_n) = 2)}{\binom{n-p}{p} p!}.$$

Proof. Remark that by conjugation invariance, one can suppose without loss of generality that the set of non isolated vertices of g are $\{1, 2, \dots, 2p\}$ and that $(1, 2), (2, 1) \in E_g$. Using the same definitions as the previous proof with $f = 0$ and $v = 2p$ and by choosing $x_1 = 1$, we have $\mathfrak{S}_{n,g_y} \subset \{\sigma \in \mathfrak{S}_n; c_1(\sigma) = 2\}$. Thus,

$$\begin{aligned}\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g}) = \frac{\sum_{y\in F} \mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g_y})}{\mathrm{card}(F)} &= \frac{\mathbb{P}(\sigma_n \in \cup_{y\in F}\mathfrak{S}_{n,g_y})}{\mathrm{card}(F)} \\ &\leq \frac{\mathbb{P}(c_1(\sigma_n) = 2)}{\mathrm{card}(F)} \\ &= \frac{\mathbb{P}(c_1(\sigma_n) = 2)}{\binom{n-p}{p} p!}.\end{aligned}$$

□

By the previous combinatorial lemmas, we get that the main contribution will come from the following subset of graphs. Let $\mathcal{T}_k^n \subset \mathbb{G}_k^n$ be the set of graphs g having exactly k non-trivial component each having one edge and two vertices.

For example, $\mathcal{T}_1^3 = \Big\{$  $\Big\}$. Let $\widehat{\mathcal{T}}_k$ be the equivalence class of the graphs of $\cup_n \mathcal{T}_k^n$.

Their contribution is as follows.

Lemma 6.11. For any $p \geq 1$, $n \geq 2p$ and any graph $g \in \mathcal{T}_p^n$, for any random permutation σ_n with conjugation invariant distribution on $\mathfrak{S}_n$,

$$\frac{1}{\binom{n-p}{p}p!}\left(1 - \frac{p^2 - p}{n-1} - p\mathbb{P}(\sigma_n(1) = 1)\right) \leq \mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g}) \leq \frac{1}{\binom{n-p}{p}p!}.$$

Proof. The upper bound is due to Lemma 6.8 with $v = 2p$. Using the conjugation invariance, one can suppose without loss of generality that $E_g = \{(1, i_1), (2, i_2), \dots, (p, i_p)\}$ where $i_j > p$ are all distinct. Let

$$\mathfrak{S}_n^p = \{\sigma \in \mathfrak{S}_n, \forall i \leq p, \sigma(i) > p\}.$$

Remark that $\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g} | \sigma_n \in \mathfrak{S}_n \setminus \mathfrak{S}_n^p) = 0$. If $\mathbb{P}(\sigma_n \in \mathfrak{S}_n^p) = 0$, then necessarily by conjugation invariance, $1 - \frac{p^2-p}{n-1} - p\mathbb{P}(\sigma_n(1) = 1) \leq 0$.

Suppose now that $\mathbb{P}(\sigma_n \in \mathfrak{S}_n^p) \neq 0$. We obtain

$$\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g}) = \mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g} | \sigma_n \in \mathfrak{S}_n^p)\mathbb{P}(\sigma_n \in \mathfrak{S}_n^p).$$

Using again the conjugation invariance, we obtain

$$\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g} | \sigma_n \in \mathfrak{S}_n^p) = \frac{1}{\binom{n-p}{p} p!}$$

and

$$\begin{aligned}\mathbb{P}(\sigma_n \in \mathfrak{S}_n^p) &= 1 - \mathbb{P}(\sigma_n \in \mathfrak{S}_n \setminus \mathfrak{S}_n^p) \\ &\geq 1 - \sum_{i=1}^{p} \mathbb{P}(\sigma_n(i) \leq p) \\ &= 1 - p\left(\mathbb{P}(\sigma_n(1) = 1) + \frac{(1 - \mathbb{P}(\sigma_n(1) = 1))(p-1)}{n}\right) \\ &\geq 1 - \frac{p^2 - p}{n-1} - p\mathbb{P}(\sigma_n(1) = 1).\end{aligned}$$

□

6.2 Proof of Lemma 6.1

Proof of Lemma 6.1. Note that $\hat{\mathbb{G}}_k$ is finite. Therefore, it is sufficient to prove that for any $\hat{g}_1, \hat{g}_2 \in \hat{\mathbb{G}}_k$ having the same number of vertices, there exist two constants $C_{\hat{g}_1,\hat{g}_2}$ and $C'_{\hat{g}_1,\hat{g}_2}$ such that for any integer n,

$$\mathbb{P}((\hat{\mathcal{G}}_1^1(\sigma_n, \rho_n), \hat{\mathcal{G}}_2^1(\sigma_n, \rho_n)) = (\hat{g}_1, \hat{g}_2))) \leq \frac{C_{\hat{g}_1,\hat{g}_2}}{n} + C'_{\hat{g}_1,\hat{g}_2}(\mathbb{P}(\sigma_n(1) = 1) + \mathbb{P}(\rho_n(1) = 1)).$$

Let $\hat{g}_1, \hat{g}_2 \in \hat{\mathbb{G}}_k$ be two unlabeled graphs having respectively p_1 and p_2 connected component and $v \leq 2k$ vertices. Let $B^n_{\hat{g}_1,\hat{g}_2}$ be the set of couples $(g_1, g_2) \in (\mathbb{G}_k^n)^2$ having the same non-isolated vertices such that 1 is a non-isolated vertex of both graphs and, for $i \in \{1, 2\}$, the equivalence class of g_i is $\hat{g}_i$.

- Suppose that $\hat{g}_1$ and $\hat{g}_2$ do not contain any loop i.e. no edges of type (i, i). Then $p_1 \leq \frac{v}{2}$ and

$p_2 \leq \frac{v}{2}$. Consequently,

$$\begin{aligned}
&\mathbb{P}((\hat{\mathcal{G}}_1^1(\sigma_n,\rho_n),\hat{\mathcal{G}}_2^1(\sigma_n,\rho_n)) = (\hat{g}_1,\hat{g}_2)))\\
=&\sum_{(g_1,g_2)\in B^n_{\hat{g}_1,\hat{g}_2}} \mathbb{P}((\mathcal{G}_1^1(\sigma_n,\rho_n),\mathcal{G}_2^1(\sigma_n,\rho_n)) = (g_1,g_2))\\
\leq&\sum_{(g_1,g_2)\in B^n_{\hat{g}_1,\hat{g}_2}} \mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g_1}, \rho_n \in \mathfrak{S}_{n,g_2})\\
=&\sum_{(g_1,g_2)\in B^n_{\hat{g}_1,\hat{g}_2}} \mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g_1})\mathbb{P}(\rho_n \in \mathfrak{S}_{n,g_2})\\
\leq&\sum_{(g_1,g_2)\in B^n_{\hat{g}_1,\hat{g}_2}} \frac{1}{\binom{n-p_1}{v-p_1}(v-p_1)!}\frac{1}{\binom{n-p_2}{v-p_2}(v-p_2)!}\\
=&\frac{card(B^n_{\hat{g}_1,\hat{g}_2})}{\binom{n-p_1}{v-p_1}(v-p_1)!\binom{n-p_2}{v-p_2}(v-p_2)!}\\
\leq&\frac{\binom{n-1}{v-1}v!^2}{\binom{n-p_1}{v-p_1}(v-p_1)!\binom{n-p_2}{v-p_2}(v-p_2)!}\\
\leq& C_{g_1,g_2}n^{v-1-(v-p_1+v-p_2)} = C_{g_1,g_2}n^{p_1+p_2-v-1} \leq \frac{C_{g_1,g_2}}{n}.
\end{aligned}$$

- Suppose that $\hat{g}_1$ contains a loop. By Lemma 6.2, if $\hat{\mathcal{G}}_1^m(\sigma_1,\sigma_2) = \hat{g}_1$, then there exists j, a fixed point of σ_1, such that $k_j = k$ and $j \in \{i_l^m, 1 \leq l \leq k\}$. Thus, almost surely,

$$\sum_{i=1}^n \mathbf{1}_{\hat{\mathcal{G}}_1^i(\sigma_n,\rho_n)=\hat{g}_1} \leq k\, card(\{i \in fix(\sigma_n); k_i = k\}) \leq k\,\#_1(\sigma_n),$$

where $fix(\sigma) := \{i, \sigma(i) = i\}$ is the set of fixed points of σ. Consequently, since σ_n is conjugation invariant,

$$\begin{aligned}
\mathbb{P}\left(\left(\hat{\mathcal{G}}_1^1(\sigma_n,\rho_n),\hat{\mathcal{G}}_2^1(\sigma_n,\rho_n)\right) = (\hat{g}_1,\hat{g}_2)\right) &\leq \mathbb{P}\left(\hat{\mathcal{G}}_1^1(\sigma_n,\rho_n) = \hat{g}_1\right)\\
&= \frac{\sum_{i=1}^n \mathbb{P}\left(\hat{\mathcal{G}}_1^i(\sigma_n,\rho_n) = \hat{g}_1\right)}{n}\\
&\leq k\frac{\mathbb{E}(\#_1(\sigma_n))}{n}\\
&= k\mathbb{P}(\sigma_n(1) = 1).
\end{aligned}$$

Similarly, if $\hat{g}_2$ contains a loop, then

$$\mathbb{P}\left(\left(\hat{\mathcal{G}}_1^1(\sigma_n,\rho_n),\hat{\mathcal{G}}_2^1(\sigma_n,\rho_n)\right) = (\hat{g}_1,\hat{g}_2)\right) \leq k\mathbb{P}(\rho_n(1) = 1).$$

□

6.3 Proof of Proposition 1.14

We will prove in full details first the case $m = 2$ and extend the proof to a larger number of permutations in a second step. In the sequel, σ_n and ρ_n will be denoted respectively by σ_n and ρ_n.

Proof. We will adapt the proof in the previous subsection. Let $v_1 \geq 1$ be fixed. In the case $k = 1$, since $C_1 = 1$, (*) holds if we have:

$$\forall \hat{g}_1, \hat{g}_2 \in \hat{\mathbb{G}}_{v_1}, \mathbb{P}\left(\left(\hat{\mathcal{G}}_1^1(\sigma_n, \rho_n), \hat{\mathcal{G}}_2^1(\sigma_n, \rho_n)\right) = (\hat{g}_1, \hat{g}_2)\right) = \frac{C_{\hat{g}_1, \hat{g}_2}}{n} + o\left(\frac{1}{n}\right)$$

and

$$\sum_{\hat{g}_1, \hat{g}_2 \in \hat{\mathbb{G}}_{v_1}} C_{\hat{g}_1, \hat{g}_2} = C_1 = 1.$$

Let $\hat{g}_1, \hat{g}_2 \in \hat{\mathbb{G}}_{v_1}$ be two unlabeled graphs having respectively p_1 and p_2 connected components and $v \leq 2v_1$ vertices. We denote by

$$p_n(\hat{g}_1, \hat{g}_2) := \mathbb{P}((\hat{\mathcal{G}}_1^1(\sigma_n, \rho_n), \hat{\mathcal{G}}_2^1(\sigma_n, \rho_n)) = (\hat{g}_1, \hat{g}_2)).$$

Let $B^n_{\hat{g}_1, \hat{g}_2}$ be the set of couples $(g_1, g_2) \in (\mathbb{G}^n_{v_1})^2$ having the same non-isolated vertices such that

- 1 is a non-isolated vertex of both graphs.
- For $i \in \{1, 2\}$, the equivalence class of g_i is $\hat{g}_i$.
- There exists σ, ρ such that $\mathcal{G}_1^1(\sigma, \rho) = g_1$ and $\mathcal{G}_2^1(\sigma, \rho) = g_2$.

By Lemma 6.7 and H_1, we have

$$\begin{aligned} p_n(\hat{g}_1, \hat{g}_2) &= \sum_{(g_1, g_2) \in B^n_{\hat{g}_1, \hat{g}_2}} \mathbb{P}((\mathcal{G}_1^1(\sigma_n, \rho_n), \mathcal{G}_2^1(\sigma_n, \rho_n)) = (g_1, g_2)) \\ &= \sum_{(g_1, g_2) \in B^n_{\hat{g}_1, \hat{g}_2}} \mathbb{P}(\sigma_n \in \mathfrak{S}_{n, g_1}, \rho_n \in \mathfrak{S}_{n, g_2}) \\ &= \sum_{(g_1, g_2) \in B^n_{\hat{g}_1, \hat{g}_2}} \mathbb{P}(\sigma_n \in \mathfrak{S}_{n, g_1}) \mathbb{P}(\rho_n \in \mathfrak{S}_{n, g_2}) \end{aligned} \tag{6.3}$$

Starting from (6.3), we now distinguish different cases, depending on the structure of $\hat{g}_1$ and $\hat{g}_2$.

- Case 1: $\hat{g}_1$ and $\hat{g}_2$ have respectively f_1 and f_2 loops i.e. edges of type (i, i) with $f_1 + f_2 > 0$.

Then $2p_1 - f_1 \leq v$ and $2p_2 - f_2 \leq v$. Consequently, by Lemmas 6.8 and 6.9,

$$\begin{aligned}
p_n(\hat{g}_1, \hat{g}_2) &= o\left(n^{\frac{-f_1-f_2}{2}}\right) \sum_{(g_1,g_2)\in B^n_{\hat{g}_1,\hat{g}_2}} \frac{1}{\binom{n-p_1}{v-p_1}(v-p_1)!} \frac{1}{\binom{n-p_2}{v-p_2}(v-p_2)!} \\
&= \frac{\mathrm{card}(B^n_{\hat{g}_1,\hat{g}_2})}{\binom{n-p_1}{v-p_1}(v-p_1)!\binom{n-p_2}{v-p_2}(v-p_2)!} o\left(n^{\frac{-f_1-f_2}{2}}\right) \\
&\leq \frac{\binom{n-1}{v-1} v!^2 o\left(n^{\frac{-f_1-f_2}{2}}\right)}{\binom{n-p_1}{v-p_1}(v-p_1)!\binom{n-p_2}{v-p_2}(v-p_2)!} \\
&= n^{v-1-(v-p_1+v-p_2)} o\left(n^{\frac{-f_1-f_2}{2}}\right) \\
&= o(n^{-1}).
\end{aligned}$$

- Case 2: $\hat{g}_1$ and $\hat{g}_2$ do not contain any loop, so that $p_1 \leq \frac{v}{2}$ and $p_2 \leq \frac{v}{2}$. Then, again by Lemma 6.8,

$$\begin{aligned}
p_n(\hat{g}_1, \hat{g}_2) &\leq \sum_{(g_1,g_2)\in B^n_{\hat{g}_1,\hat{g}_2}} \frac{1}{\binom{n-p_1}{v-p_1}(v-p_1)!} \frac{1}{\binom{n-p_2}{v-p_2}(v-p_2)!} \\
&= \frac{\mathrm{card}(B^n_{\hat{g}_1,\hat{g}_2})}{\binom{n-p_1}{v-p_1}(v-p_1)!\binom{n-p_2}{v-p_2}(v-p_2)!} \\
&\leq \frac{\binom{n-1}{v-1} v!^2}{\binom{n-p_1}{v-p_1}(v-p_1)!\binom{n-p_2}{v-p_2}(v-p_2)!} = O\left(n^{v-1-(v-p_1+v-p_2)}\right).
\end{aligned}$$

Therefore, if $p_1 < \frac{v}{2}$, as $p_1 \leq \frac{v-1}{2}$ we have

$$p_n(\hat{g}_1, \hat{g}_2) = O(n^{-\frac{3}{2}}).$$

The same holds if $p_2 < \frac{v}{2}$ and the only remaining terms are the cases when $p_1 = \frac{v}{2} = v_1$ and $p_2 = \frac{v}{2} = v_1$. In this case, both graphs have necessarily connected components having two vertices. By Lemma 6.6, we obtain that the only non-trivial contribution comes from $\hat{g}_1 = \hat{g}_2 = \widehat{\mathcal{T}}_{v_1}$. By Lemma 6.11, we obtain

$$\begin{aligned}
\frac{\mathrm{card}\left(B^n_{\widehat{\mathcal{T}}_{v_1},\widehat{\mathcal{T}}_{v_1}}\right)}{\binom{n-p_1}{v-p_1}(v-p_1)!\binom{n-p_2}{v-p_2}(v-p_2)!}\left(1 - O\left(\frac{1}{n}\right)\right) &\leq p_n(\widehat{\mathcal{T}}_{v_1}, \widehat{\mathcal{T}}_{v_1}) \\
&\leq \frac{\mathrm{card}\left(B^n_{\widehat{\mathcal{T}}_{v_1},\widehat{\mathcal{T}}_{v_1}}\right)}{\binom{n-p_1}{v-p_1}(v-p_1)!\binom{n-p_2}{v-p_2}(v-p_2)!}.
\end{aligned}$$

Moreover, each element of $B^n_{\widehat{\mathcal{T}}_{v_1},\widehat{\mathcal{T}}_{v_1}}$ can be characterized by a choice of $i^1_2, i^1_3, \dots i^1_{v_1}, j^1_1, \dots j^1_{v_1}$

pairwise distinct in $\{2, 3, \ldots, n\}$, so that

$$\operatorname{card}\left(B^n_{\widehat{\mathcal{T}}_{v_1},\widehat{\mathcal{T}}_{v_1}}\right) = \binom{n-1}{2v_1-1}(2v_1-1)!.$$

Since $v = 2p_1 = 2p_2 = 2v_1$, we get that

$$p_n(\widehat{\mathcal{T}}_{v_1}, \widehat{\mathcal{T}}_{v_1}) = \frac{1+o(1)}{n}.$$

Summarizing all cases, we get that $C_{\hat{g}_1,\hat{g}_2} = 0$ unless $\hat{g}_1 = \hat{g}_2 = \widehat{\mathcal{T}}_{v_1}$, in which case

$$C_{\widehat{\mathcal{T}}_{v_1},\widehat{\mathcal{T}}_{v_1}} = 1.$$

□

6.4 Proof of Theorem 1.13 for $m = 2$

The proof of Theorem 1.13 is similar to that of Proposition 1.14. Instead of studying $\mathcal{G}_i^1$, we study $\mathcal{G}_i^{\{1,2,\ldots,k\}}$. We will prove using the same argument that only the event

$$\left\{\sigma, \rho; \forall i \in \{1,2\}, \mathcal{G}_i^{\{1,2,\ldots,k\}}(\sigma,\rho) \in \cup_{p\geq 1}\mathcal{T}_p^n\right\}$$

will contribute to the limit.

Proof of Theorem 1.13 in the case $m = 2$. Let $\mathbf{v}$=$(v_1, v_2, \ldots v_k)$ be fixed. If $\forall i \leq k, c_i(\sigma^{-1}\rho) = v_i$, then

$$\mathcal{G}_1^{\{1,2,\ldots,k\}}(\sigma,\rho), \mathcal{G}_2^{\{1,2,\ldots,k\}}(\sigma,\rho) \in \bigcup_{p\leq\sum_{i=1}^k v_k} \hat{\mathbb{G}}_p.$$

Since $\bigcup_{p\leq\sum_{i=1}^k v_k} \hat{\mathbb{G}}_p$ is finite, it is sufficient to prove that for any pair $\hat{g}_1, \hat{g}_2 \in \bigcup_{p\leq\sum_{i=1}^k v_k} \hat{\mathbb{G}}_p$ having the same number of non-isolated vertices, there exists a constant $C_{\hat{g}_1,\hat{g}_2,\mathbf{v}}$ such that under the assumptions of Theorem 1.13,

$$\mathbb{P}\left((\hat{\mathcal{G}}_1^{\{1,2,\ldots,k\}}(\sigma_n,\rho_n), \hat{\mathcal{G}}_2^{\{1,2,\ldots,k\}}(\sigma_n,\rho_n)) = (\hat{g}_1,\hat{g}_2) \cap A_{\mathbf{v}}\right) = \frac{C_{\hat{g}_1,\hat{g}_2,\mathbf{v}}}{n^k} + o\left(\frac{1}{n^k}\right),$$

where $A_{\mathbf{v}} := \{\forall i \leq k, c_i(\sigma_n^{-1}\rho_n) = v_i\}$.

Let $\hat{g}_1, \hat{g}_2 \in \bigcup_{p\leq\sum_{i=1}^k v_k} \hat{\mathbb{G}}_p$ be two unlabeled graphs having respectively p_1 and p_2 connected components and v vertices. Let $B^{n,\mathbf{v}}_{\hat{g}_1,\hat{g}_2}$ be the set of couples (g_1, g_2) with n vertices, having the same non-isolated vertices such that

- $1, 2, \ldots, k$ are non-isolated vertices of both graphs,

- for $i \in \{1,2\}$, the equivalence class of g_i is $\hat{g}_i$,

- there exists σ, ρ such that for $i \in \{1,2\}$, $\mathcal{G}_i^{\{1,2,\dots k\}}(\sigma,\rho) = g_i$ and $c_i(\sigma^{-1}\rho) = v_i$.

As before, we denote by

$$p_{n,\mathbf{v}}(\hat{g}_1,\hat{g}_2) := \mathbb{P}\left((\hat{\mathcal{G}}_1^{\{1,2,\dots,k\}}(\sigma_n,\rho_n), \hat{\mathcal{G}}_2^{\{1,2,\dots,k\}}(\sigma_n,\rho_n)) = (\hat{g}_1,\hat{g}_2) \cap A_{\mathbf{v}} \right)$$

and we have

$$\begin{aligned} p_{n,\mathbf{v}}(\hat{g}_1,\hat{g}_2) &= \sum_{(g_1,g_2)\in B^{n,\mathbf{v}}_{\hat{g}_1,\hat{g}_2}} \mathbb{P}((\mathcal{G}_1^{\{1,2,\dots,k\}}(\sigma_n,\rho_n), \mathcal{G}_2^{\{1,2,\dots,k\}}(\sigma_n,\rho_n)) = (g_1,g_2)) \\ &= \sum_{(g_1,g_2)\in B^{n,\mathbf{v}}_{\hat{g}_1,\hat{g}_2}} \mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g_1}, \rho_n \in \mathfrak{S}_{n,g_2}) \\ &= \sum_{(g_1,g_2)\in B^{n,\mathbf{v}}_{\hat{g}_1,\hat{g}_2}} \mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g_1})\mathbb{P}(\rho_n \in \mathfrak{S}_{n,g_2}). \end{aligned}$$

Starting from there, we distinguish different cases:

- Case 1: $\hat{g}_1$ and $\hat{g}_2$ have respectively f_1 and f_2 loops i.e. edges of type (i,i) with $f_1+f_2>0$. Then $2p_1 - f_1 \le v$ and $2p_2 - f_2 \le v$. Consequently, by Lemmas 6.8 and 6.9,

$$\begin{aligned} p_{n,\mathbf{v}}(\hat{g}_1,\hat{g}_2) &= \frac{\mathrm{card}(B^{n,\mathbf{v}}_{\hat{g}_1,\hat{g}_2})}{\binom{n-p_1}{v-p_1}(v-p_1)!\binom{n-p_2}{v-p_2}(v-p_2)!} o\left(n^{\frac{-f_1-f_2}{2}}\right) \\ &\le \frac{\binom{n-k}{v-k} v!^2 o\left(n^{\frac{-f_1-f_2}{2}}\right)}{\binom{n-p_1}{v-p_1}(v-p_1)!\binom{n-p_2}{v-p_2}(v-p_2)!} = n^{v-k-(v-p_1+v-p_2)} o\left(n^{\frac{-f_1-f_2}{2}}\right) = o(n^{-k}). \end{aligned}$$

- Case 2: $\hat{g}_1$ and $\hat{g}_2$ do not contain any loop. Then $p_1 \le \frac{v}{2}$ and $p_2 \le \frac{v}{2}$. Consequently,

$$\begin{aligned} p_{n,\mathbf{v}}(\hat{g}_1,\hat{g}_2) &\le \frac{\mathrm{card}(B^{n,\mathbf{v}}_{\hat{g}_1,\hat{g}_2})}{\binom{n-p_1}{v-p_1}(v-p_1)!\binom{n-p_2}{v-p_2}(v-p_2)!} \\ &\le \frac{\binom{n-k}{v-k} v!^2}{\binom{n-p_1}{v-p_1}(v-p_1)!\binom{n-p_2}{v-p_2}(v-p_2)!} \\ &\le C n^{v-k-(v-p_1+v-p_2)}. \end{aligned}$$

  Therefore, if $p_1 < \frac{v}{2}$ or $p_2 < \frac{v}{2}$ then $p_{n,\mathbf{v}}(\hat{g}_1,\hat{g}_2) = o(n^{-k})$. The only remaining terms are the cases when $p_1 = \frac{v}{2}$ and $p_2 = \frac{v}{2}$. In this case, both graphs have necessarily only connected components having two vertices. Assume that one of the two graphs has a cycle. Then, by Lemma 6.10, we have

$$p_{n,\mathbf{v}}(\hat{g}_1,\hat{g}_2) \le \sum_{(g_1,g_2)\in B^{n,\mathbf{v}}_{\hat{g}_1,\hat{g}_2}} \frac{(\mathbb{P}(c_1(\sigma_n)=2)+\mathbb{P}(c_1(\rho_n)=2))}{\binom{n-p_1}{v-p_1}(v-p_1)!\binom{n-p_2}{v-p_2}(v-p_2)!}$$
$$\le C(\mathbb{P}(c_1(\sigma_n)=2)+\mathbb{P}(c_1(\rho_n)=2))n^{-k}.$$

Under H_4, we have $\mathbb{P}(c_1(\sigma_n)=2)+\mathbb{P}(c_1(\rho_n)=2)) = o(1)$ so that $p_{n,\mathbf{v}}(\hat{g}_1,\hat{g}_2) = o(n^{-k})$ as soon as one of the graph has a cycle.

As before, the only non-trivial contributions come from the cases when $\hat{g}_1 = \hat{g}_2 = \widehat{\mathcal{T}}_p$ for some $p \le \sum_{i=1}^k v_i$ and by Lemma 6.11, we obtain

$$\frac{\mathrm{card}\left(B^{n,\mathbf{v}}_{\widehat{\mathcal{T}}_p,\widehat{\mathcal{T}}_p}\right)}{\binom{n-p_1}{v-p_1}(v-p_1)!\binom{n-p_2}{v-p_2}(v-p_2)!}\left(1-O\left(\frac{1}{n}\right)\right) \le p_{n,\mathbf{v}}\left(\widehat{\mathcal{T}}_p,\widehat{\mathcal{T}}_p\right)$$
$$\le \frac{\mathrm{card}\left(B^{n,\mathbf{v}}_{\widehat{\mathcal{T}}_p,\widehat{\mathcal{T}}_p}\right)}{\binom{n-p_1}{v-p_1}(v-p_1)!\binom{n-p_2}{v-p_2}(v-p_2)!}.$$

One can conclude since, for any $n \ge 2p$,

$$\mathrm{card}\left(B^{n,\mathbf{v}}_{\widehat{\mathcal{T}}_p,\widehat{\mathcal{T}}_p}\right) = \mathrm{card}\left(B^{2p,\mathbf{v}}_{\widehat{\mathcal{T}}_p,\widehat{\mathcal{T}}_p}\right)\binom{n-k}{2p-k}$$

and consequently, for any $p \le \sum_{i=1}^k v_i$,

$$C_{\widehat{\mathcal{T}}_p,\widehat{\mathcal{T}}_p,\mathbf{v}} = \frac{\mathrm{card}\left(B^{2p,\mathbf{v}}_{\widehat{\mathcal{T}}_p,\widehat{\mathcal{T}}_p}\right)}{(2p-k)!},$$

and $C_{\hat{g}_1,\hat{g}_2,\mathbf{v}} = 0$, as soon as $(\hat{g}_1,\hat{g}_2) \notin \left\{(\widehat{\mathcal{T}}_p,\widehat{\mathcal{T}}_p), p \le \sum_{i=1}^k v_i\right\}$. As the constants $C_{\hat{g}_1,\hat{g}_2,\mathbf{v}}$ do not depend on the distributions of σ_n and ρ_n, this concludes the proof of Theorem 1.13 in the case of two permutations. □

6.5 Proof of Theorem 1.13 for $m > 2$

To extend to $m > 2$, we will proceed by induction on the number m of permutations. Our main argument is the following lemma.

Lemma 6.12. Let $(\sigma_n^1)_{n\ge1}, (\sigma_n^2)_{n\ge1}$ be two sequences of random permutations such that for any $n \ge 1$, $\sigma_n^1, \sigma_n^2 \in \mathfrak{S}_n$. Assume that

- For any $n \ge 1$, σ_n^1 and σ_n^2 are independent.
- For any $n \ge 1$ and $\ell \in \{1,2\}$, for any $\sigma \in \mathfrak{S}_n$, $\sigma^{-1}\sigma_n^\ell\sigma \stackrel{d}{=} \sigma_n^\ell$.

- For any $k \geq 1$,

$$\frac{\#_1\,\sigma_n^1}{\sqrt{n}} \xrightarrow[n\to\infty]{L^k} 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\mathbb{E}(\#_2\,\sigma_n^1)}{n} = 0.$$

Then,

$$\frac{\#_1(\sigma_n^1\sigma_n^2)}{\sqrt{n}} \xrightarrow[n\to\infty]{L^k} 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\mathbb{E}(\#_2(\sigma_n^1\sigma_n^2))}{n} = 0. \qquad . \tag{6.4}$$

Sketch of proof. We will only give a sketch of the proof. The idea is to repeat the same study as in the case $m = 2$ in the two particular quantities.

- Take $k \geq 1$ and $v_1 = v_2 = \cdots = v_k = 1$. One can show that, under the hypothesis of Lemma 6.12,

$$\lim_{n\to\infty} \sum_{\hat{g}_i,\hat{g}'_i \in \hat{\mathbb{G}}_1,\, 1\leq i\leq k} n^{\frac{k}{2}} \mathbb{P}(\hat{\mathcal{G}}^{\{1,2,\ldots,k\}}(\sigma_n^1, \sigma_n^2) = (\hat{g}_1, \hat{g}'_1, \hat{g}_2, \ldots \hat{g}'_k)) = 0.$$

  This leads to the first limit in (6.4).

- Take $k = 1$ and $v_1 = 2$. One can show that, under the hypothesis of Lemma 6.12,

$$\forall \hat{g}_1, \hat{g}_2 \in \hat{\mathbb{G}}_2, \lim_{n\to\infty} \mathbb{P}((\hat{\mathcal{G}}_1^1(\sigma_n^1, \sigma_n^2), \hat{\mathcal{G}}_2^1(\sigma_n^1, \sigma_n^2)) = (\hat{g}_1, \hat{g}_2))) = 0.$$

  This leads to the second limit in (6.4).

□

7

Global convergence for RSK and longest common subsequence

"Tout mathématicien digne de ce nom a connu, parfois seulement à de rares intervalles, ces états d'exaltation lucide où les pensées s'enchaînent comme par miracle, et où l'inconscient (quel que soit le sens qu'on attache à ce mot) paraît aussi avoir sa part. [...] A la différence du plaisir sexuel, celui-là peut durer plusieurs heures, voire plusieurs jours ; qui l'a connu en désire le renouvellement mais est impuissant à le provoquer, sinon tout au plus par un travail opiniâtre dont il apparaît alors comme la récompense ; il est vrai que le plaisir qu'on en ressent est sans rapport avec la valeur des découvertes auxquelles il s'associe."

André Weil

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In this chapter, we recall the Vershik-Kerov-Logan-Shepp convergence. We generalize this result to conjugation invariant random permutations with few number of cycles. For a general conjugation invariant random permutation, we conjecture that the limiting shape depends only on the distribution of fixed points. The four first sections of this chapter contain already known results as well as proofs in (Kammoun, 2018; Kammoun, 2020). The proof in Section 7.5 is new.

7.1 Uniform case

The typical shape under the Plancherel measure was studied separately by Logan and Shepp (1977) and Vershik and Kerov (1977). Stronger results are proved by Vershik and Kerov (1985). In 1993, Kerov studied the limiting fluctuations but did not publish his results. See Ivanov and Olshanski (2002) for further details. Let $L_{\lambda(\sigma)}$ be the height function of $\lambda(\sigma)$ rotated by $\frac{3\pi}{4}$ and extended by the function $x \mapsto |x|$ to obtain a function defined on $\mathbb{R}$. For example, if $\lambda(\sigma) = (7, 5, 2, 1, 1, \underline{0})$ the associated function $L_{\lambda(\sigma)}$ is represented by Figure 7.1. For the Plancherel measure we have the following result.

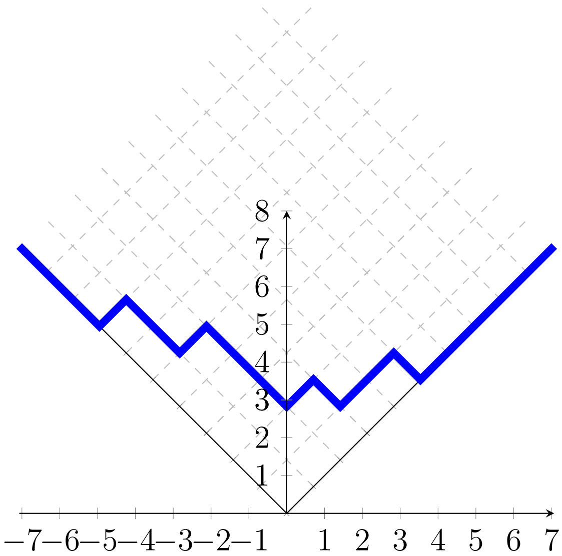

Figure 7.1: $L_{(7,5,2,1,1,\underline{0})}$

Theorem 7.1. (Vershik and Kerov, 1985, Theorem 4)

$$\sup_{s\in\mathbb{R}} \left| \frac{1}{\sqrt{2n}} L_{\lambda(\sigma_{unif,n})}\left(s\sqrt{2n}\right) - \Omega(s) \right| \xrightarrow[n\to\infty]{\mathbb{P}} 0,$$

where

$$\Omega(s) := \begin{cases} \frac{2}{\pi}(s\arcsin(s) + \sqrt{1-s^2}) & \text{if } |s| < 1 \\ |s| & \text{if } |s| \geq 1 \end{cases}.$$

Under weaker conditions than those of Theorem 5.7, we show a similar result. For the remainder of this paper, we will refer to this limiting shape as the Vershik-Kerov-Logan-Shepp shape. This convergence is closely related to the Wigner's semi-circular law. For further details, we recommend (Sodin, 2017) and (Kerov, 2003, Chapter 4 Section 2).

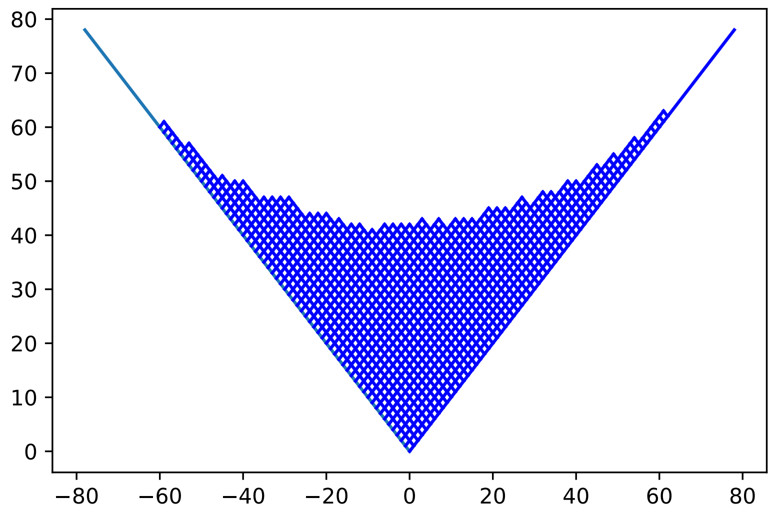

Figure 7.2: A simulation for shape of a random Young tableau uniformly chosen according to the Plancherel measure.

7.2 Invariant random permutations: a conjecture and some proofs

We conjecture the following.

Conjecture 7.2. Let $(\sigma_n)_{n\geq 1}$ be a sequence of conjugation invariant random permutations. We have

$$\sup_{s\in\mathbb{R}}\left|\frac{1}{2\sqrt{n}}L_{\lambda(\sigma_n)}\left(2s\sqrt{n-\#_1(\sigma_n)}\right)-\sqrt{1-\frac{\#_1(\sigma_n)}{n}}\Omega\left(s\right)\right|\xrightarrow[n\to\infty]{\mathbb{P}}0 \tag{7.1}$$

where

$$\Omega(s):=\begin{cases}\frac{2}{\pi}(s\arcsin(s)+\sqrt{1-s^2}) & \text{if } |s|<1\\ |s| & \text{if } |s|\geq 1\end{cases}.$$

Conjecture 7.2 is equivalent to the following conjecture.

Conjecture 7.3. Let $(\sigma_n)_{n\geq 1}$ be a sequence of conjugation invariant random permutations. Assume that

$$\frac{\#_1(\sigma_n)}{n}\xrightarrow[n\to\infty]{d}0. \tag{7.2}$$

Then

$$\sup_{s\in\mathbb{R}}\left|\frac{1}{\sqrt{2n}}L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right)-\Omega(s)\right|\xrightarrow[n\to\infty]{\mathbb{P}}0.$$

This result is proved in the case of the uniform distribution on involutions by Méliot (2011). We prove the particular case of $(\mathcal{H}^{\mathbb{P}}_{inv,1})$.

Theorem 7.4. Under $(\mathcal{H}^{\mathbb{P}}_{inv,1})$,

$$\text{(VKLS)} \qquad \sup_{s\in\mathbb{R}} \left| \frac{1}{\sqrt{2n}} L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right) - \Omega(s) \right| \xrightarrow[n\to\infty]{\mathbb{P}} 0,$$

Consequently we obtain the following

Proposition 7.5. Assume that the sequence $(\sigma_n)_{n\geq 1}$ of random permutations is conjugation invariant and

$$\frac{\#(\sigma_n) - \#_1(\sigma_n)}{n} \xrightarrow[n\to\infty]{\mathbb{P}} 0. \tag{7.3}$$

Then

$$\sup_{s\in\mathbb{R}} \left| \frac{1}{2\sqrt{n}} L_{\lambda(\sigma_n)}\left(2s\sqrt{n - \#_1(\sigma_n)}\right) - \sqrt{1 - \frac{\#_1(\sigma_n)}{n}}\,\Omega\left(s\right) \right| \xrightarrow[n\to\infty]{\mathbb{P}} 0, . \tag{7.4}$$

Using the same technique, the result of Méliot (2011) implies this corollary.

Proposition 7.6. Assume that the sequence $(\sigma_n)_{n\geq 1}$ of conjugation invariant random permutations and

$$1 - \frac{\#_1(\sigma_n^2)}{n} \xrightarrow[n\to\infty]{\mathbb{P}} 0. \tag{7.5}$$

Then

$$\sup_{s\in\mathbb{R}} \left| \frac{1}{2\sqrt{n}} L_{\lambda(\sigma_n)}\left(2s\sqrt{n - \#_1(\sigma_n)}\right) - \sqrt{1 - \frac{\#_1(\sigma_n)}{n}}\,\Omega\left(s\right) \right| \xrightarrow[n\to\infty]{\mathbb{P}} 0. \tag{7.6}$$

We will not prove the equivalence between Theorem 7.4 and propositions 7.5 and between the result of Méliot (2011) and Proposition 7.6 since they can be obtained by the same technique as the proof in Section 8.5.

7.3 Proof of Theorem 7.4

The statement is the same as in (Kammoun, 2018) and the proof is almost the same. Let $(O, \vec{x}, \vec{y})$ be the canonical frame of the Euclidean plane and $\vec{u} := \frac{\sqrt{2}}{2}(\vec{x}+\vec{y}), \vec{v} := \frac{\sqrt{2}}{2}(\vec{y}-\vec{x})$. Let $\lambda \in \mathbb{Y}_n$. Using the convention $\lambda_0 = \infty$, let $\mathscr{C}_\lambda$ be the curve obtained by connecting the points with coordinates $(0, \lambda_0), (0, \lambda_1), (1, \lambda_1), (1, \lambda_2), \dots, (i, \lambda_i), (i, \lambda_{i+1}), \dots$ in the axes system $(O, \vec{u}, \vec{v})$ as in Figure 7.3. By construction $\mathscr{C}_\lambda$ is the curve of L_λ. This yields the following.

Lemma 7.7. Let $\alpha, \beta \in \mathbb{N}$ and A the point such that $\overrightarrow{OA} = \alpha\vec{u} + \beta\vec{v}$. If $A \in \mathscr{C}_\lambda$, then

$$\lambda_{\alpha+1} \leq \beta \leq \lambda_\alpha. \tag{7.7}$$

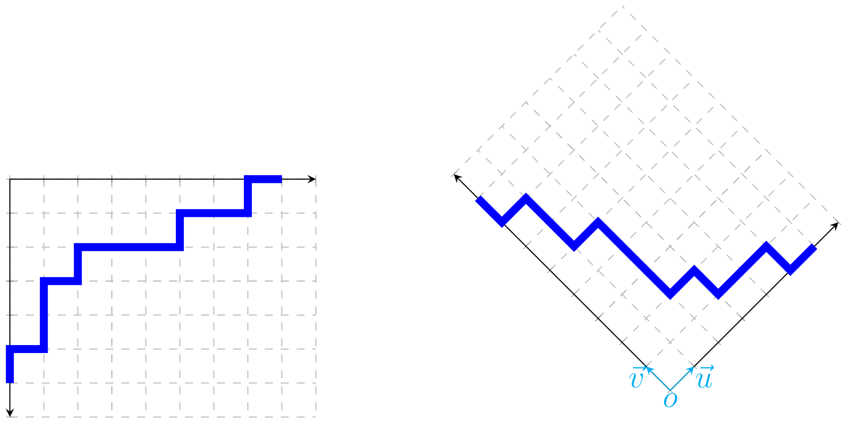

Figure 7.3: $\mathscr{C}_\lambda$ for $\lambda = (7, 5, 2, 1, 1, \underline{0})$

We have also the following result.

Lemma 7.8. For all $i \in \mathbb{Z}$,

$$\frac{\sqrt{2}}{2} L_\lambda\left(\frac{\sqrt{2}}{2}i\right) \pm \frac{i}{2} \in \mathbb{N}, \tag{7.8}$$

Proof. Let M be such that $\overrightarrow{OM} = s_1\vec{u} + s_2\vec{v}$. By construction, if $M \in \mathscr{C}_\lambda$ then $s_1, s_2 \geq 0$ and either $s_1 \in \mathbb{N}$ or $s_2 \in \mathbb{N}$. If we apply this observation to M defined by

$$\overrightarrow{OM} := \frac{\sqrt{2}}{2} i\vec{x} + L_\lambda\left(\frac{\sqrt{2}}{2}i\right)\vec{y} = \left(\frac{\sqrt{2}}{2}L_\lambda\left(\frac{\sqrt{2}}{2}i\right) + \frac{i}{2}\right)\vec{u} + \left(\frac{\sqrt{2}}{2}L_\lambda\left(\frac{\sqrt{2}}{2}i\right) - \frac{i}{2}\right)\vec{v},$$

we obtain (7.8). □

To prove Theorem 7.4, our main lemma is the following.

Lemma 7.9. Let $n, m \in \mathbb{N}^*$, $\lambda = (\lambda_i)_{i\geq 1} \in \mathbb{Y}_n$, $\mu = (\mu_i)_{i\geq 1} \in \mathbb{Y}_m$. Then,

$$\sup_{s\in\mathbb{R}} \left(L_\lambda(s) - L_\mu(s)\right)^2 \leq 4 \max_{i\geq 1} \left|\sum_{k=1}^{i} (\lambda_k - \mu_k)\right|. \tag{7.9}$$

Proof. Note that for any $i \in \mathbb{Z}$, $s \mapsto L_\lambda(s)$ and $s \mapsto L_\mu(s)$ are affine functions on $\left[\frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2}(i+1)\right]$ and thus (7.9) is equivalent to

$$\sup_{i\in\mathbb{Z}} \left(L_\lambda\left(\frac{\sqrt{2}}{2}i\right) - L_\mu\left(\frac{\sqrt{2}}{2}i\right)\right)^2 \leq 4 \max_{i\geq 1} \left|\sum_{k=1}^{i} (\lambda_k - \mu_k)\right|.$$

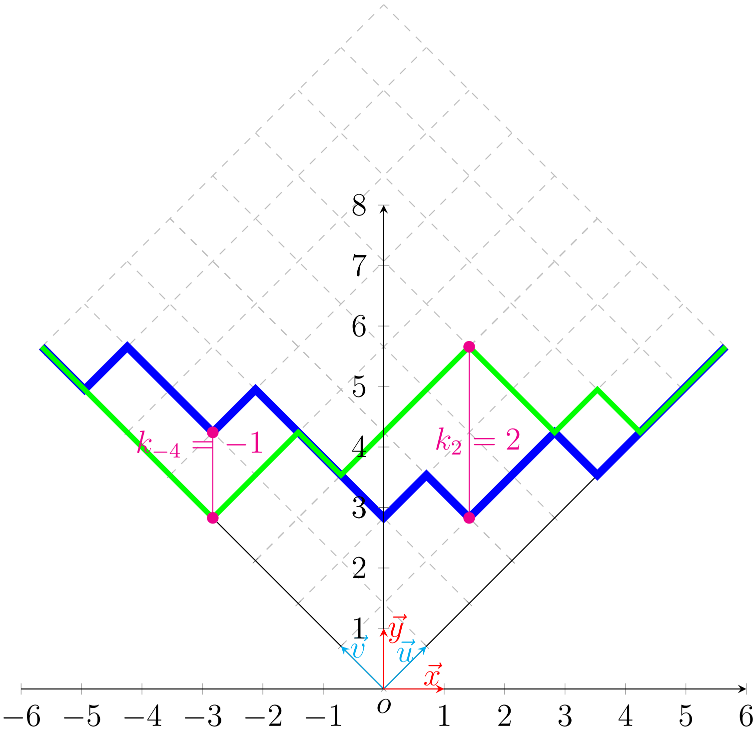

Figure 7.4: An example where $\lambda = (7, 5, 2, 1, 1, \underline{0})$ and $\mu = (4, 4, 3, 3, 3, 1, \underline{0})$

Let $i \in \mathbb{Z}$. It follows from Lemma 7.8 that there exists $k_i \in \mathbb{Z}$ such that,

$$L_\mu\left(\frac{\sqrt{2}}{2}i\right) - L_\lambda\left(\frac{\sqrt{2}}{2}i\right) = k_i\sqrt{2}.$$

To simplify notations, we denote

$$j := \sqrt{2}L_\lambda\left(\frac{\sqrt{2}}{2}i\right) \in \mathbb{N}.$$

Let A and B be the points such that

$$\overrightarrow{OA} := \frac{\sqrt{2}}{2}(i\vec{x} + j\vec{y}) = \frac{i+j}{2}\vec{u} + \frac{j-i}{2}\vec{v},$$
$$\overrightarrow{OB} = \frac{\sqrt{2}}{2}(i\vec{x} + (j+2k_i)\vec{y}) = \frac{i+j+2k_i}{2}\vec{u} + \frac{j-i+2k_i}{2}\vec{v}.$$

Clearly, $A \in \mathscr{C}_\lambda$ and $B \in \mathscr{C}_\mu$. By Lemma 7.8, $\frac{i+j}{2}, \frac{j-i}{2} \in \mathbb{N}$. We can then apply Lemma 7.7. In the case where $k_i > 0$, we have

$$\lambda_{\frac{i+j}{2}+1} \leq \frac{j-i}{2}, \quad \mu_{\frac{i+j}{2}+k_i} \geq \frac{j-i}{2} + k_i.$$

Using the fact that $(\lambda_l)_{l\geq 1}$ and $(\mu_l)_{l\geq 1}$ are decreasing, we have,

$$2\max_{l\geq 1}\left|\sum_{k=1}^{l}(\lambda_k-\mu_k)\right| \geq \sum_{l=\frac{i+j}{2}+1}^{\frac{i+j}{2}+k_i} \mu_l-\lambda_l \geq \sum_{l=\frac{i+j}{2}+1}^{\frac{i+j}{2}+k_i} \mu_{\frac{i+j}{2}+k_i}-\lambda_{\frac{i+j}{2}+1} \geq k_i^2.$$

Similarly, in the case where $k_i<0$,

$$-2\max_{l\geq 1}\left|\sum_{k=1}^{l}(\lambda_k-\mu_k)\right| \leq \sum_{l=\frac{i+j}{2}+1+k_i}^{\frac{i+j}{2}} \mu_l-\lambda_l \leq \sum_{l=\frac{i+j}{2}+1+k_i}^{\frac{i+j}{2}} \mu_{\frac{i+j}{2}+k_i+1}-\lambda_{\frac{i+j}{2}} \leq -k_i^2.$$

This yields

$$4\max_{i\geq 1}\left|\sum_{k=1}^{i}(\lambda_k-\mu_k)\right| \geq \max_{i\in\mathbb{Z}}\left(\sqrt{2}k_i\right)^2 = \sup_{s\in\mathbb{R}}\left(L_\lambda(s)-L_\mu(s)\right)^2.$$

□

Proof of Theorem 7.4. Using Lemma 5.11, for any $\sigma\in\mathfrak{S}_n$, for any $\rho\in \text{final}(\sigma)$ [1],

$$\max_{i\geq 1}\left|\sum_{k=1}^{i}(\lambda_k(\sigma)-\lambda_k(\rho))\right| \leq 2(\#(\sigma)-1). \tag{7.10}$$

We wan to apply Theorem 2.1. Let now F be the set of continual diagrams i.e the set of 1-Lipschitz real functions g from $\mathbb{R}$ to $\mathbb{R}_+$ such that $\exists a,b>0$ s.t. $\forall x\notin[-b,b], g(x)=|x-a|$. For $g,h\in F$, we denote by $d_F(g,h)=\sup_{\mathbb{R}}|h-g|$. For $\sigma\in\mathfrak{S}_n$, $f(\sigma)$ is the function $s\to\frac{L_{\lambda(\sigma)}\left(s\sqrt{2n}\right)}{\sqrt{2n}}$. So that f is a function from $\mathfrak{S}_\infty$ taking values in the metric space (F,d_F). If we choose $\sigma_{ref,n}=\sigma_{unif,n}$ and x to be the function γ, we have by Theorem 7.1 that

$$f(\sigma_{ref,n}) \xrightarrow[n\to\infty]{\mathbb{P}} x.$$

By Lemma 7.9, for any $1\leq k\leq n$

$$\varepsilon'_{n,k}(f) \leq 2\sqrt{\frac{k-1}{n}}. \tag{7.11}$$

So that Theorem 2.1 gives the conclusion. □

7.4 Longest common subsequence

We will recall here the proofs in (Kammoun, 2020). In particular, we prove (1.4), propositions 1.8 and 1.11, Corollary 1.9 and Theorem 1.10.

[1] $final(\sigma)$ is defined in Chapter 2

7.4.1 General tools

We will present in this subsection some control results related to the longest increasing subsequences. Those controls will be the main tools to prove the results presented in Chapter 1. The study of the longest common subsequence is strongly related to the notion of the longest increasing subsequence. More precisely, we have the following classical result. For any $\sigma, \rho \in \mathfrak{S}_n$,

(7.12)
$$\mathrm{LCS}(\sigma,\rho) = \mathrm{LCS}(\sigma^{-1}\circ\sigma, \sigma^{-1}\circ\rho) = \mathrm{LCS}(Id_n, \sigma^{-1}\circ\rho) = \mathrm{LIS}(\sigma^{-1}\circ\rho) = \mathrm{LIS}(\rho^{-1}\circ\sigma).$$

To prove our results, we will use the Markov operator T' defined on $\mathfrak{S}_n$ and associated to the stochastic matrix $\left[\frac{1_{A_\sigma}(\rho)}{\mathrm{card}(A_\sigma)}\right]_{\sigma,\rho\in\mathfrak{S}_n}$ where

$$A_\sigma = \begin{cases} \{\sigma\} & \text{if } \#(\sigma) = 1 \\ \{\rho \in \mathfrak{S}_n, \sigma^{-1}\circ\rho = (i_1,i_2)\circ(i_1,i_3)\cdots\circ(i_1,i_{\#(\sigma)}) \text{ and } \#(\rho)=1\} & \text{if } \#(\sigma) > 1 \end{cases}.$$

T' is then the Markov operator mapping a permutation σ to a permutation uniformly chosen at random among the permutations obtained by merging the cycles of σ using transpositions having all a common point. Note that A_σ is not empty since any choice of one point in each cycle gives a possible $(i_1, i_2, \dots, i_{\#(\sigma)})$ and a resulting permutation ρ. We obtain then the following control.

Lemma 7.10. For any permutation $\sigma \in \mathfrak{S}_n$, almost surely,

(7.13)
$$\max_{k\geq 1}\left|\sum_{i=1}^{k}(\lambda_i(\sigma) - \lambda_i(T'(\sigma))\right| \leq \#(\sigma).$$

In particular, almost surely,

(7.14)
$$|\mathrm{LIS}(T'(\sigma)) - \mathrm{LIS}(\sigma)| \leq \#(\sigma).$$

Moreover, for any conjugation invariant random permutation σ_n on $\mathfrak{S}_n$, the law of $T'(\sigma_n)$ is $Ew(0)$.

Note that using Theorem 7.4,

(7.15)
$$\sup_{s\in\mathbb{R}}\left|\frac{1}{\sqrt{2n}}L_{\lambda(\sigma_{Ew,0,n})}\left(s\sqrt{2n}\right) - \Omega(s)\right| \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

Proof of Lemma 7.10. Let The Markov operator is not exactly the same but the proof is similar to that of Theorem 2.1. Let $\sigma \in \mathfrak{S}_n$ be a permutation. Let $\rho \in \mathrm{final}(\sigma)$. By definition of $\mathrm{LIS}(\sigma)$, there exists $i_1 < i_2 < \cdots < i_{\mathrm{LIS}(\sigma)}$ such that $\sigma(i_1) < \cdots < \sigma(i_{\mathrm{LIS}(\sigma)})$. Let $\rho = \sigma\circ(j_1,j_2)\circ(j_1,j_3)\circ\cdots\circ(j_1,j_{\#(\sigma)})$ be a permutation with a unique cycle and $i'_1, i'_2, \dots, i'_m$ be the same sequence as $i_1, i_2, \dots, i_{\mathrm{LIS}(\sigma)}$ after removing $j_1, j_2, \dots, j_{\#(\sigma)}$ if needed. We have $\mathrm{LIS}(\sigma) - \#(\sigma) \leq m$ and $\sigma(i'_1) < \cdots < \sigma(i'_m)$. As $\forall i \notin \{j_1, j_2, \dots, j_{\#(\sigma)}\}$, $\rho(i) = \sigma(i)$, so that $\rho(i'_1) < \cdots < \rho(i'_m)$. Therefore,

$m \leq \text{LIS}(\rho)$ and $\text{LIS}(\sigma) - \text{LIS}(\rho) \leq \#(\sigma)$. We can obtain the reverse inequality in (7.14) using the same techniques. Now let $k \geq 1$ and $\left\{i_1, i_2, \ldots, i_{\sum_{i=1}^{l} \lambda_i(\sigma)}\right\} \in \mathfrak{I}_k(\sigma)$. Greene's theorem guarantees the existence of such integers. Let $i_1'', i_2'', \ldots, i_m''$ be the same sequence as $i_1, i_2, \ldots, i_{\sum_{k=1}^{l} \lambda_k(\sigma)}$ after removing $j_1, j_2, \ldots, j_{\#(\sigma)}$ if needed. We have $\{i_1', i_2', \ldots, i_m'\} \in \mathfrak{I}_k(\rho)$ and we conclude as in the proof of (7.14). To prove the last part of this result, one can check that the law of $T'(\sigma_n)$ is clearly conjugation invariant. Indeed, let $\sigma, \rho \in \mathfrak{S}_n$.

$$\begin{aligned}
\mathbb{P}(T'(\sigma_n) = \sigma) &= \mathbf{1}_{\#(\sigma)=1} \sum_{\hat{\sigma} \in \mathfrak{S}_n} \mathbf{1}_{\sigma \in A_{\hat{\sigma}}} \frac{\mathbb{P}(\sigma_n = \hat{\sigma})}{\text{card}(A_{\hat{\sigma}})} \\
&= \mathbf{1}_{\#(\sigma)=1} \sum_{\hat{\sigma} \in \mathfrak{S}_n} \mathbf{1}_{\rho\circ\sigma\circ\rho^{-1} \in A_{\rho\circ\hat{\sigma}\circ\rho^{-1}}} \frac{\mathbb{P}(\rho \circ \sigma_n \circ \rho^{-1} = \rho \circ \hat{\sigma} \circ \rho^{-1})}{\text{card}(A_{\rho\circ\hat{\sigma}\circ\rho^{-1}})} \\
&= \mathbf{1}_{\#(\sigma)=1} \sum_{\hat{\sigma} \in \mathfrak{S}_n} \mathbf{1}_{\rho\circ\sigma\circ\rho^{-1} \in A_{\hat{\sigma}}} \frac{\mathbb{P}(\rho \circ \sigma_n \circ \rho^{-1} = \hat{\sigma})}{\text{card}(A_{\hat{\sigma}})} \\
&= \mathbf{1}_{\#(\rho\circ\sigma\circ\rho^{-1})=1} \sum_{\hat{\sigma} \in \mathfrak{S}_n} \mathbf{1}_{\rho\circ\sigma\circ\rho^{-1} \in A_{\hat{\sigma}}} \frac{\mathbb{P}(\sigma_n = \hat{\sigma})}{\text{card}(A_{\hat{\sigma}})} \\
&= \mathbb{P}(T'(\sigma_n) = \rho \circ \sigma \circ \rho^{-1}).
\end{aligned}$$

Moreover, by construction, almost surely, $\#(T'(\sigma_n)) = 1$. Consequently, the law of $T'(\sigma_n)$ is $Ew(0)$. □

The key control lemma will be the following.

Lemma 7.11. For any permutation $\sigma \in \mathfrak{S}_n$, for any $\alpha \geq 0$, almost surely,

$$\left|\sum_{i=1}^{\infty}(\lambda_i(\sigma) - \alpha\sqrt{n})_+ - \sum_{i=1}^{\infty}(\lambda_i(T'(\sigma)) - \alpha\sqrt{n})_+\right| \leq \#(\sigma), \tag{7.16}$$

$$\sup\left\{k \in \mathbb{N}, \sum_{i=1}^{\infty}(\lambda_i(T'(\sigma)) - k)_+ \geq \#(\sigma)\right\} \leq \text{LIS}(\sigma) \tag{7.17}$$

$$\text{and} \quad \sup\left\{k \in \mathbb{N}, \sum_{i=1}^{\infty}(\lambda_i(\sigma) - k)_+ \geq \#(\sigma)\right\} \leq \text{LIS}(T'(\sigma)). \tag{7.18}$$

Proof. We prove first that

$$\sum_{i=1}^{\infty}(\lambda_i(\sigma) - \alpha\sqrt{n})_+ - \sum_{i=1}^{\infty}(\lambda_i(T'(\sigma)) - \alpha\sqrt{n})_+ \leq \#(\sigma).$$

If $\lambda_1(\sigma) \leq \alpha\sqrt{n}$, the inequality is trivial as the right-hand side is non-negative and the left-hand side is non-positive. Otherwise, let $k := \max\{j \geq 1, \lambda_j(\sigma) > \alpha\sqrt{n}\}$. We have

$$\sum_{i=1}^{\infty}(\lambda_i(\sigma) - \alpha\sqrt{n})_+ = \sum_{i=1}^{k}(\lambda_i(\sigma) - \alpha\sqrt{n})_+ + \sum_{i=k+1}^{\infty}(\lambda_i(\sigma_n) - \alpha\sqrt{n})_+ = \sum_{i=1}^{k}(\lambda_i(\sigma) - \alpha\sqrt{n}),$$

and

$$\sum_{i=1}^{\infty}(\lambda_i(T'(\sigma)) - \alpha\sqrt{n})_+ \geq \sum_{i=1}^{k}(\lambda_i(T'(\sigma)) - \alpha\sqrt{n})_+ \geq \sum_{i=1}^{k}(\lambda_i(T'(\sigma)) - \alpha\sqrt{n}).$$

We obtain then

$$\sum_{i=1}^{\infty}(\lambda_i(\sigma) - \alpha\sqrt{n})_+ - \sum_{i=1}^{\infty}(\lambda_i(T'(\sigma)) - \alpha\sqrt{n})_+ \leq \sum_{i=1}^{k}\lambda_i(\sigma) - \lambda_i(T'(\sigma)) \leq \#(\sigma).$$

The reverse inequality in (7.16) is obtained by exchanging the role of σ and $T'(\sigma)$. Finally, Using the equivalence between $\{\mathrm{LIS}(\sigma) > k\}$ and $\{\sum_{i=1}^{\infty}(\lambda_i(\sigma) - k)_+ > 0\}$, (7.17) and (7.18) are a direct application of (7.16). □

Lemma 7.11 implies the following asymptotic control.

Lemma 7.12. For any $0 \leq \gamma \leq 2$, for any $\varepsilon > 0$,

$$\mathbb{P}\left(\frac{\sum_{i=1}^{n}(\lambda_i(\sigma_{Ew,0,n}) - \gamma\sqrt{n})_+}{n} > 2G(\gamma) - \varepsilon\right) \to 1, \tag{7.19}$$

where we recall that G is defined in (1.5). Consequently, for any $\alpha < 2$, there exist $\beta > 0$ and $n_\alpha > 0$ such that for any $n > n_\alpha$, for any conjugation invariant random permutation σ_n satisfying $\mathbb{E}(\#\sigma_n) < n\beta$, we have

$$\mathbb{E}(\mathrm{LIS}(\sigma_n)) \geq \alpha\sqrt{n}.$$

Proof. This is a direct application of (7.15). One can see that $\frac{\sum_{i=1}^{n}(\lambda_i(\sigma) - \gamma\sqrt{n})_+}{2n}$ is the area of the region delimited by the curves of the functions $x \mapsto |x|$, $x \mapsto \gamma + x$ and $x \mapsto \frac{L_{\lambda(\sigma)}(x\sqrt{2n})}{\sqrt{2n}}$, see Figure 7.5. By construction, this area is equal to

$$\int_{-\infty}^{\infty}\left(\frac{L_{\lambda(\sigma)}(s\sqrt{2n})}{\sqrt{2n}} - \left|s + \frac{\gamma}{2}\right| - \frac{\gamma}{2}\right)_+ \mathrm{d}s.$$

By (7.15),

$$\int_{-1}^{1}\left(\frac{L_{\lambda(\sigma_{Ew,0,n})}(s\sqrt{2n})}{\sqrt{2n}} - \left|s + \frac{\gamma}{2}\right| - \frac{\gamma}{2}\right)_+ \mathrm{d}s \xrightarrow[n\to\infty]{\mathbb{P}} G(\gamma).$$

We can conclude then that

$$\begin{aligned}\frac{\sum_{i=1}^{n}(\lambda_i(\sigma_{Ew,0,n}) - \gamma\sqrt{n})_+}{n} &= 2\int_{-\infty}^{\infty}\left(\frac{L_{\lambda(\sigma_{Ew,0,n})}(s\sqrt{2n})}{\sqrt{2n}} - \left|s + \frac{\gamma}{2}\right| - \frac{\gamma}{2}\right)_+ \mathrm{d}s \\ &\geq 2\int_{-1}^{1}\left(\frac{L_{\lambda(\sigma_{Ew,0,n})}(s\sqrt{2n})}{\sqrt{2n}} - \left|s + \frac{\gamma}{2}\right| - \frac{\gamma}{2}\right)_+ \mathrm{d}s \xrightarrow[n\to\infty]{\mathbb{P}} 2G(\gamma).\end{aligned}$$

This yields (7.19).

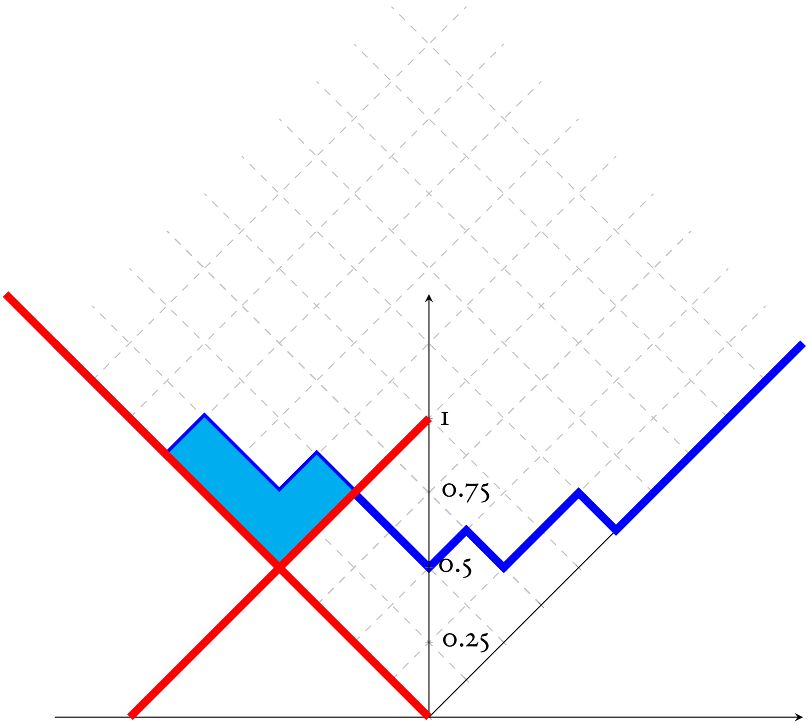

Figure 7.5: $\lambda = (7, 2, 2, 1, 1, \underline{0})$ and $\gamma = 1$

Note that it is not difficult to prove that

$$\frac{\sum_{i=1}^{n} (\lambda_i(\sigma_n) - \gamma\sqrt{n})_+}{n} \xrightarrow[n\to\infty]{\mathbb{P}} 2G(\gamma).$$

We skip the proof here as we only need (7.19) in the sequel.

Now let $\alpha < \gamma < 2$, $\varepsilon > 0$ and $\beta > 0$ such that $1 - \frac{\beta}{G(\gamma)} - \varepsilon > \frac{\alpha}{\gamma}$. Using (7.19), we obtain the existence of n_α such that for any $n > n_\alpha$,

$$\mathbb{P}\left(\frac{\sum_{i=1}^{n} (\lambda_i(T'(\sigma_n)) - \gamma\sqrt{n})_+}{n} > G(\gamma)\right) > 1 - \varepsilon.$$

Since $\{\mathrm{LIS}(\sigma) > k\}$ is equivalent to $\{\sum_{i=1}^{\infty} (\lambda_i(\sigma) - k)_+ > 0\}$ and by Markov inequality, we obtain

$$\begin{aligned}
\mathbb{E}(\mathrm{LIS}(\sigma_n)) &\geq \gamma\sqrt{n}\mathbb{P}(\mathrm{LIS}(\sigma_n) \geq \gamma\sqrt{n}) \\
&\geq \gamma\sqrt{n}\mathbb{P}\left(\frac{\sum_{i=1}^{n} (\lambda_i(T'(\sigma_n)) - \gamma\sqrt{n})_+}{n} > G(\gamma), \frac{\#(\sigma_n)}{n} < G(\gamma)\right) \\
&\geq \gamma\sqrt{n}\left(1 - \frac{\beta}{G(\gamma)} - \varepsilon\right) \geq \alpha\sqrt{n}.
\end{aligned}$$

□

7.4.2 Proof of Proposition 1.8 and Corollary 1.9

The key Lemma to prove Proposition 1.8 and Corollary 1.9 is the following.

Lemma 7.13. For any $k \geq 2$, there exists $C, C' > 0$ such that for any $n \geq 1$, for any independent random permutations σ_n and ρ_n with conjugation invariant distributions,

$$\mathbb{P}\left(c_1\left((\sigma_n)^{-1} \circ \rho_n\right) = k\right) \leq \frac{C}{n} + C'(\mathbb{P}(\sigma_n(1) = 1) + \mathbb{P}(\rho_n(1) = 1)),$$

where $c_m(\sigma)$ is the length of the cycle of σ containing m.

Proof. We will use in this proof the graphs we have defined in Chapter 6. Note that $\hat{\mathbb{G}}_k$ is finite. Therefore, it is sufficient to prove that for any $\hat{g}_1, \hat{g}_2 \in \hat{\mathbb{G}}_k$ having the same number of vertices, there exist two constants $C_{\hat{g}_1,\hat{g}_2}$ and $C'_{\hat{g}_1,\hat{g}_2}$ such that for any integer n,

$$\mathbb{P}((\hat{\mathcal{G}}_1^1(\sigma_n, \rho_n), \hat{\mathcal{G}}_2^1(\sigma_n, \rho_n)) = (\hat{g}_1, \hat{g}_2))) \leq \frac{C_{\hat{g}_1,\hat{g}_2}}{n} + C'_{\hat{g}_1,\hat{g}_2}(\mathbb{P}(\sigma_n(1) = 1) + \mathbb{P}(\rho_n(1) = 1)).$$

Let $\hat{g}_1, \hat{g}_2 \in \hat{\mathbb{G}}_k$ be two unlabeled graphs having respectively p_1 and p_2 connected components and $v \leq 2k$ vertices. Let $B^n_{\hat{g}_1,\hat{g}_2}$ be the set of couples $(g_1, g_2) \in (\mathbb{G}^n_k)^2$ having the same non-isolated vertices such that 1 is a non-isolated vertex of both graphs and, for $i \in \{1, 2\}$, the equivalence class of g_i is $\hat{g}_i$.

- Suppose that $\hat{g}_1$ and $\hat{g}_2$ do not contain any loop i.e. no edges of type (i, i). Then $p_1 \leq \frac{v}{2}$ and $p_2 \leq \frac{v}{2}$. Consequently,

$$\begin{aligned}
\mathbb{P}((\hat{\mathcal{G}}_1^1(\sigma_n, \rho_n), \hat{\mathcal{G}}_2^1(\sigma_n, \rho_n)) = (\hat{g}_1, \hat{g}_2))) &= \sum_{(g_1,g_2)\in B^n_{\hat{g}_1,\hat{g}_2}} \mathbb{P}((\mathcal{G}_1^1(\sigma_n, \rho_n), \mathcal{G}_2^1(\sigma_n, \rho_n)) = (g_1, g_2)) \\
&\leq \sum_{(g_1,g_2)\in B^n_{\hat{g}_1,\hat{g}_2}} \mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g_1}, \rho_n \in \mathfrak{S}_{n,g_2}) \\
&= \sum_{(g_1,g_2)\in B^n_{\hat{g}_1,\hat{g}_2}} \mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g_1})\mathbb{P}(\rho_n \in \mathfrak{S}_{n,g_2}) \\
&\leq \sum_{(g_1,g_2)\in B^n_{\hat{g}_1,\hat{g}_2}} \frac{(n-v)!}{(n-p_1)!}\frac{(n-v)!}{(n-p_2)!} \\
&= \text{card}(B^n_{\hat{g}_1,\hat{g}_2})\frac{(n-v)!}{(n-p_1)!}\frac{(n-v)!}{(n-p_2)!} \\
&\leq \binom{n-1}{v-1} v!^2 \frac{(n-v)!}{(n-p_1)!}\frac{(n-v)!}{(n-p_2)!} \\
&\leq C_{g_1,g_2} n^{v-1-(v-p_1+v-p_2)} = C_{g_1,g_2} n^{p_1+p_2-v-1} \leq \frac{C_{g_1,g_2}}{n}.
\end{aligned}$$

- Suppose that $\hat{g}_1$ contains a loop. By Lemma 6.2, if $\hat{\mathcal{G}}_1^m(\sigma, \rho) = \hat{g}_1$, then there exists j a fixed point of σ such that $k_j = k$ and $j \in \{i^m_l, 1 \leq l \leq k\}$. Thus, almost surely,

$$\sum_{i=1}^{n} \mathbf{1}_{\hat{\mathcal{G}}_1^i(\sigma_n,\rho_n)=\hat{g}_1} \leq k \,\text{card}(\{i \in fix(\sigma_n) : k_j = k\}) \leq k \,\#_1(\sigma_n),$$

where $fix(\sigma)$ is the set of fixed points of σ. Consequently, since σ_n is conjugation invariant,

$$\begin{aligned}\mathbb{P}\left(\left(\hat{\mathcal{G}}_1^1(\sigma_n,\rho_n),\hat{\mathcal{G}}_2^1(\sigma_n,\rho_n)\right)=(\hat{g}_1,\hat{g}_2)\right) &\leq \mathbb{P}\left(\hat{\mathcal{G}}_1^1(\sigma_n,\rho_n)=\hat{g}_1\right)\\ &= \frac{\sum_{i=1}^n \mathbb{P}\left(\hat{\mathcal{G}}_1^i(\sigma_n,\rho_n)=\hat{g}_1\right)}{n}\\ &\leq k\frac{\mathbb{E}(\#_1(\sigma_n))}{n} = k\mathbb{P}(\sigma_n(1)=1).\end{aligned}$$

Similarly, if $\hat{g}_2$ contains a loop, then

$$\mathbb{P}\left(\left(\hat{\mathcal{G}}_1^1(\sigma_n,\rho_n),\hat{\mathcal{G}}_2^1(\sigma_n,\rho_n)\right)=(\hat{g}_1,\hat{g}_2)\right) \leq k\mathbb{P}(\rho_n(1)=1).$$

This concludes the proof.

□

We will now prove Proposition 1.8.

Proof of Proposition 1.8. Suppose that for any $n \geq 1$, σ_n and ρ_n are independent and that they are both conjugation invariant.

- Assume that

$$\liminf_{n\to\infty} \sqrt{n}\mathbb{P}(\sigma_n(1)=1)\mathbb{P}(\rho_n(1)=1) \geq \alpha.$$

In this case,

$$\begin{aligned}\liminf_{n\to\infty} \frac{\mathbb{E}(\mathrm{LCS}(\sigma_n,\rho_n))}{\sqrt{n}} &\geq \liminf_{n\to\infty} \frac{\mathbb{E}(\#_1(\sigma_n\circ\rho_n^{-1})))}{\sqrt{n}}\\ &\geq \liminf_{n\to\infty} \sqrt{n}\mathbb{P}(\sigma_n(1)=1)\mathbb{P}(\rho_n(1)=1) \geq \alpha.\end{aligned}$$

- Assume that

$$\lim_{n\to\infty} \max(\mathbb{P}(\sigma_n(1)=1),\mathbb{P}(\rho_n(1)=1))=0.$$

In this case,

$$\begin{aligned}\mathbb{P}\left(\sigma_n^{-1}\circ\rho_n(1)=1\right) &= \sum_{i=1}^n \mathbb{P}(\sigma_n(1)=i)\mathbb{P}(\rho_n(1)=i)\\ &= \mathbb{P}(\sigma_n(1)=1)\mathbb{P}(\rho_n(1)=1)\\ &+\frac{(1-\mathbb{P}(\sigma_n(1)=1))(1-\mathbb{P}(\rho_n(1)=1))}{n-1}\\ &= o(1).\end{aligned}$$

For any conjugation invariant random permutation σ_n on $\mathfrak{S}_n$

$$\mathbb{E}(\#(\sigma_n)) = \mathbb{E}\left(\sum_{i=1}^{n} \frac{1}{c_i(\sigma_n)}\right) = \sum_{i=1}^{n} \mathbb{E}\left(\frac{1}{c_i(\sigma_n)}\right) = n\mathbb{E}\left(\frac{1}{c_1(\sigma_n)}\right),$$

and for $n_\beta := \lfloor \frac{1}{\beta} \rfloor + 1$, with the same β as in Lemma 7.12,

$$\begin{aligned}\frac{\mathbb{E}(\#(\sigma_n))}{n} &= \sum_{k=1}^{\infty} \frac{1}{k}\mathbb{P}(c_1(\sigma_n) = k) \\ &\leq \mathbb{P}(c_1(\sigma_n) = 1) + \sum_{k=2}^{n_\beta} \mathbb{P}(c_1(\sigma_n) = k) + \frac{1}{n_\beta + 1} \sum_{k=n_\beta+1}^{\infty} \mathbb{P}(c_1(\sigma_n) = k) \\ &\leq \mathbb{P}(\sigma_n(1) = 1) + \sum_{k=2}^{n_\beta} \mathbb{P}(c_1(\sigma_n) = k) + \frac{1}{n_\beta + 1}.\end{aligned}$$

Consequently, under (7.2), by Lemma 7.13, we have

$$\frac{\mathbb{E}(\#(\sigma_n \circ \rho_n^{-1}))}{n} \leq \frac{1}{n_\beta + 1} + o(1) < \beta + o(1).$$

Hence, we obtain Proposition 1.8 thanks to Lemma 7.12.

□

Proof of Corollary 1.9. This is a direct application of Proposition 1.8. In fact, if

$$\mathbb{P}(\sigma_n(1) = 1) \geq \frac{\sqrt{2}}{\sqrt[4]{n}},$$

then

$$\liminf_{n\to\infty} \sqrt{n}\mathbb{P}(\sigma_n(1) = 1)\mathbb{P}(\rho_n(1) = 1) \geq 2.$$

Otherwise,

$$\lim_{n\to\infty} \max(\mathbb{P}(\sigma_n(1) = 1), \mathbb{P}(\rho_n(1) = 1)) = 0.$$

□

7.4.3 Proof of (1.4), Theorem 1.10 and Proposition 1.11.

By observing that if σ_n and ρ_n are independent random permutations with conjugation invariant distributions then $\sigma_n^{-1} \circ \rho_n$ is conjugation invariant, proving (1.4) is equivalent to prove the following.

Theorem 7.14. For any sequence of conjugation invariant random permutations $\{\sigma_n\}_{n\geq 1}$,

$$\liminf_{n\to\infty} \frac{\mathbb{E}(\mathrm{LIS}(\sigma_n))}{\sqrt{n}} \geq 2\sqrt{\theta}.$$

The argument will be by comparison with the uniform measure on $\mathfrak{S}_n$ and the uniform measure on the set of involutions. We will use the uniform permutation on $\mathfrak{S}_n$ if we have few cycles. Otherwise, we will use the uniform measure on the set of involutions since it has approximately $\frac{n}{2}$ cycles with high probability. In this section, we denote by $\mathfrak{S}_n^2 := \{\sigma \in \mathfrak{S}_n, \sigma \circ \sigma = Id_n\}$ the set of involutions of $\mathfrak{S}_n$ and $\sigma_{inv,n}$ a uniform random permutation on $\mathfrak{S}_n^2$. The distribution of $\lambda(\sigma_{inv,n})$ on the set of Young diagrams $\mathbb{Y}_n$ is known as the Gelfand distribution. For our purpose we recall that (Méliot, 2011, Theorem 1) guarantees that

$$\sup_{s\in\mathbb{R}} \left| \frac{1}{\sqrt{2n}} L_{\lambda(\sigma_{inv,n})}\left(s\sqrt{2n}\right) - \Omega(s) \right| \xrightarrow[n\to\infty]{\mathbb{P}} 0$$

and one can find of prove that

$$\lim_{n\to\infty} \frac{\mathbb{E}(\#_1(\sigma_n))}{\sqrt{n}} = 1$$

in (Flajolet and Sedgewick, 2009, Page 692, Proposition IX.19) which yields the following result.

Lemma 7.15. If σ_n is conjugation invariant and supported on $\mathfrak{S}_n^2$ then

$$\liminf_{n\to\infty} \frac{\mathbb{E}(\mathrm{LIS}(\sigma_n))}{\sqrt{n}} \geq 2.$$

Idea of the proof. If $\frac{\mathbb{E}(\#_1(\sigma_n))}{\sqrt{n}} \geq 2$ the result is trivial. Otherwise, the technique of proof is identical to that of Lemma 7.12. Going back to Lemma 7.10, we replace A_σ by

$$A'_\sigma := \{\rho \in \mathfrak{S}_n; \sigma = \rho \circ (i_1, i_2) \circ \cdots \circ (i_{\#_1(\sigma))-1}, i_{\#_1(\sigma))}), \#_1(\rho) = 0\}$$

if n is even and by

$$A'_\sigma := \{\rho \in \mathfrak{S}_n; \sigma = \rho \circ (i_1, i_2) \circ \cdots \circ (i_{\#_1(\sigma))-2}, i_{\#_1(\sigma))-1}), \#_1(\rho) = 1\}$$

if n is odd. We denote by T'' the Markov operator on $\mathfrak{S}_n^2$ associated to the stochastic matrix $\left[\frac{1_{A'_\sigma}(\rho)}{\mathrm{card}(A_\sigma)}\right]_{\sigma,\rho\in\mathfrak{S}_n^2}$. It means that we merge couples of fixed points to obtain the uniform distribution on permutations having only cycles of length 2 when n is even and having a unique fixed point when n is odd. Similarly to that we did in Lemma 7.10, for any permutation σ, we have the following.

- Almost surely,
$$|\mathrm{LIS}(T''(\sigma)) - \mathrm{LIS}(\sigma)| \leq \#_1(\sigma).$$

- More generally, almost surely,
$$\max_{i\geq 1} \left| \sum_{k=1}^{i} (\lambda_k(\sigma) - \lambda_k(T''(\sigma))) \right| \leq \#_1(\sigma).$$

Moreover, if σ_n is conjugation invariant, the law of $T''(\sigma_n)$ does not depend on the law of σ_n. Consequently, Lemma 7.15 follows using the same techniques as in the proof of Lemma 7.12. □

For our purpose, one can obtain then a lower asymptotic bound.

Proposition 7.16. Let $\{\sigma_n\}_{n\geq 1}$ be a sequence of random permutations each one being conjugation invariant. Assume that there exists a sequence $(\beta_n)_{n\geq 1}$ such that

$$\lim_{n\to\infty} \beta_n = +\infty,$$

and for any $n \geq 1$,

$$\mathbb{P}(\#_1(\sigma_n^2) > \beta_n) = 1.$$

Then

$$\liminf_{n\to\infty} \frac{\mathbb{E}\left(\mathrm{LIS}(\sigma_n)\right)}{\sqrt{\beta_n}} \geq 2.$$

Proof. Giving $A \subset \mathbb{N}$ finite, we denote by $\mathfrak{S}_A$ (resp. $\mathfrak{S}_A^2$) the set of permutations (resp. involutions) of A. A random permutation σ_A supported on $\mathfrak{S}_A$ is called *conjugation invariant* if for any $\sigma \in \mathfrak{S}_A$, $\sigma \circ \sigma_A \circ \sigma^{-1}$ is equal in distribution to σ_A.
Fix $\varepsilon > 0$. By Lemma 7.15, there exists n_0 such that for any $A \subset \mathbb{N}$ with $n_0 < \mathrm{card}(A) < +\infty$, for any random permutation $\hat{\sigma}_A$ supported on $\mathfrak{S}_A^2$ conjugation invariant,

$$\frac{\mathbb{E}(\mathrm{LIS}(\hat{\sigma}_A))}{\sqrt{\mathrm{card}(A)}} \geq 2 - \varepsilon.$$

Let σ_n be a conjugation invariant random permutation and σ'_n be the restriction of σ_n on $fix(\sigma_n^2)$. In particular, almost surely $\mathrm{LIS}(\sigma'_n) \leq \mathrm{LIS}(\sigma_n)$. One can see that for any $A \subset [n]$ such that $\mathbb{P}(fix(\sigma_n^2) = A) > 0$, for any $\hat{\sigma}_1, \hat{\sigma}_2 \in \mathfrak{S}_A$,

$$\mathbb{P}(\sigma'_n = \hat{\sigma}_1 | fix(\sigma_n^2) = A) = \mathbb{P}(\sigma'_n = \hat{\sigma}_2 \circ \hat{\sigma}_1 \circ \hat{\sigma}_2^{-1} | fix(\sigma_n^2) = A). \tag{7.20}$$

Consequently, if $\beta_n > n_0$,

$$\begin{aligned}
\frac{\mathbb{E}\left(\mathrm{LIS}(\sigma_n)\right)}{\sqrt{\beta_n}} &= \sum_{\substack{|A|>\beta_n \\ \mathbb{P}(fix(\sigma_n^2)=A)>0}} \frac{\mathbb{E}\left(\mathrm{LIS}(\sigma_n) | fix(\sigma_n^2) = A\right)}{\sqrt{\beta_n}} \mathbb{P}(fix(\sigma_n^2) = A) \\
&\geq \sum_{\substack{|A|>\beta_n \\ \mathbb{P}(fix(\sigma_n^2)=A)>0}} (2-\varepsilon)\sqrt{\frac{\mathrm{card}(A)}{\beta_n}} \mathbb{P}(fix(\sigma_n^2) = A) \\
&\geq \sum_{\substack{|A|>\beta_n \\ \mathbb{P}(fix(\sigma_n^2)=A)>0}} (2-\varepsilon)\mathbb{P}(fix(\sigma_n^2) = A) = 2 - \varepsilon.
\end{aligned}$$

This yields Proposition 7.16. □

We will now prove Theorem 7.14.

Proof. In this proof, we use the following convention. Let $A, B \subset \mathfrak{S}_n$ and $f : \mathfrak{S}_n \to \mathbb{R}$. If $\mathbb{P}(\sigma_n \in A) = 0$, we assign $\mathbb{P}(\sigma_n \in B|\sigma_n \in A) = 0$ and $\mathbb{E}(f(\sigma_n)|\sigma_n \in A) = 0$.

We have[2]

$$\begin{aligned}\mathbb{E}(\mathrm{LIS}(\sigma_n)) &= \mathbb{E}\left(\mathrm{LIS}(\sigma_n)\middle|\#(\sigma_n) \leq \frac{(2+\theta)n}{6}\right)\mathbb{P}\left(\#(\sigma_n) \leq \frac{(2+\theta)n}{6}\right)\\ &+ \mathbb{E}\left(\mathrm{LIS}(\sigma_n)\middle|\#(\sigma_n) > \frac{(2+\theta)n}{6}\right)\mathbb{P}\left(\#(\sigma_n) > \frac{(2+\theta)n}{6}\right).\end{aligned}$$

Since the condition on the number of cycles is conjugation invariant, it is sufficient to prove Theorem 7.14 in the two particular cases.

- Assume that almost surely $\#(\sigma_n) \leq \frac{(2+\theta)n}{6}$. By Lemma 7.11, for any $0 < \gamma < 2$,

  $$\mathbb{P}\left(\frac{\mathrm{LIS}(\sigma_n)}{\sqrt{n}} > \gamma\right) \geq \mathbb{P}\left(\frac{\sum_{i=1}^n(\lambda_i(T(\sigma_n)) - \gamma\sqrt{n})_+}{n} > \frac{2+\theta}{6}\right).$$

  As $T(\sigma_n)$ is distributed according to the $Ew(0)$, by choosing $\gamma = 2\sqrt{\theta} - \varepsilon$ for some $\varepsilon > 0$ in Lemma 7.12, we can conclude that the right-hand side goes to 1 as n goes to infinity.

- Assume that almost surely $\#(\sigma_n) > \frac{(2+\theta)n}{6}$. We can write,

  $$\begin{aligned}\mathbb{E}(\mathrm{LIS}(\sigma_n)) &= \mathbb{E}(\mathrm{LIS}(\sigma_n)|\#_1(\sigma_n) \geq 2\sqrt{n\theta})\mathbb{P}(\#_1(\sigma_n) \geq 2\sqrt{n\theta})\\ &+ \mathbb{E}(\mathrm{LIS}(\sigma_n)|\#_1(\sigma_n) < 2\sqrt{n\theta})\mathbb{P}(\#_1(\sigma_n) < 2\sqrt{n\theta}).\end{aligned}$$

  Clearly, if $\mathbb{P}(\#_1(\sigma_n) \geq 2\sqrt{n\theta}) > 0$, then

  $$\mathbb{E}(\mathrm{LIS}(\sigma_n)|\#_1(\sigma_n) \geq 2\sqrt{n\theta}) > 2\sqrt{n\theta}.$$

  One can check easily that for any $\sigma \in \mathfrak{S}_n$

  $$\#_1(\sigma^2) \geq 6\#(\sigma) - 3\#_1(\sigma)) - 2n.$$

  Consequently, under the condition $\#_1(\sigma_n) < 2\sqrt{n\theta}$,, almost surely,

  $$\#_1(\sigma_n^2) > \theta n - 6\sqrt{\theta n}.$$

[2] We recall that θ is the same as in (1.4).

We can then conclude by Proposition 7.16 that if $\mathbb{P}(\#_1(\sigma_n) < 2\sqrt{n\theta}) > 0$, then

$$\liminf_{n\to\infty} \frac{\mathbb{E}\left(\mathrm{LIS}(\sigma_n)\middle|\#_1(\sigma_n) < 2\sqrt{\theta n}\right)}{\sqrt{n\theta - 6\sqrt{n\theta}}} \geq 2.$$

Thus, if $\mathbb{P}(\#_1(\sigma_n) < 2\sqrt{n\theta}) > 0$, then

$$\liminf_{n\to\infty} \frac{\mathbb{E}\left(\mathrm{LIS}(\sigma_n)\middle|\#_1(\sigma_n) < 2\sqrt{n\theta}\right)}{\sqrt{n}} \geq 2\sqrt{\theta}.$$

□

The proofs of Theorem 1.10 and Proposition 1.11 are based on the following observation.

Lemma 7.17. For any permutations σ, ρ, almost surely,

$$|\mathrm{LCS}(\sigma,\rho) - \mathrm{LCS}(T'(\sigma),\rho)| \leq \#(\sigma).$$

The proof is identical to that of Lemma 7.10. This guarantees the convergence when one of the permutations follows the $Ew(0)$ distribution.

Lemma 7.18. Assume that $\sigma_{Ew,0,n}$ and ρ_n are independent. Then

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\mathrm{LCS}(\sigma_{Ew,0,n},\rho_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) = F_2(s),$$

$$\lim_{n\to\infty} \frac{\mathbb{E}\left(\mathrm{LCS}(\sigma_{Ew,0,n},\rho_n)\right)}{\sqrt{n}} = 2 \quad \text{and} \quad \frac{\mathrm{LCS}(\sigma_{Ew,0,n},\rho_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} 2.$$

Proof. Note that if σ_n is distributed according the uniform distribution, one can see that the independence between σ_n and ρ_n implies that $\sigma_n^{-1} \circ \rho_n$ follows also the uniform distribution. In this case,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\mathrm{LCS}(\sigma_n,\rho_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) = \lim_{n\to\infty} \mathbb{P}\left(\frac{\mathrm{LIS}(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) = F_2(s), \tag{7.21}$$

$$\lim_{n\to\infty} \frac{\mathbb{E}\left(\mathrm{LCS}(\sigma_n,\rho_n)\right)}{\sqrt{n}} = \lim_{n\to\infty} \frac{\mathbb{E}\left(\mathrm{LIS}(\sigma_n)\right)}{\sqrt{n}} = 2, \tag{7.22}$$

and

$$\frac{\mathrm{LCS}(\sigma_n,\rho_n)}{\sqrt{n}} \stackrel{d}{=} \frac{\mathrm{LIS}(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} 2. \tag{7.23}$$

The second equality of (7.21) is due to Baik, Deift, and Johansson, 1999 and the second equality of (7.22) and the convergence of (7.23) are due to (Vershik and Kerov, 1977). Hence, one can conclude by Lemma 7.17 since

$$\mathbb{E}(\#(\sigma_n)) = log(n) + O(1)$$

and $\mathrm{LCS}(\sigma_{Ew,0,n}, \rho_n)$ is equal in distribution to $\mathrm{LCS}(T'(\sigma_n), \rho_n)$. □

Using again Lemma 7.17, Lemma 7.18 implies Proposition 1.11 since $T'(\sigma_n)$ is distributed according to $Ew(0)$. Finally we give a sketch of proof of Theorem 1.10.

Sketch of proof of Theorem 1.10. Using the same technique as in Lemma 7.12, we can prove that for any $\varepsilon > 0$,

$$\mathbb{P}\left(\frac{\mathrm{LCS}(\sigma_n, \rho_n)}{\sqrt{n}} > G^{-1}\left(\frac{\#(\sigma_n)}{2n} + \varepsilon\right) - \varepsilon\right) \xrightarrow[n\to\infty]{} 1.$$

Consequently,

$$\liminf_{n\to\infty} \frac{\mathbb{E}(\mathrm{LCS}(\sigma_n, \rho_n))}{\sqrt{n}} \geq \mathbb{E}\left(G^{-1}\left(\liminf_{n\to\infty} \frac{\#(\sigma_n)}{2n}\right)\right).$$

Since G^{-1} is convex, we can conclude using Jensen's inequality. □

7.5 Proof of Theorem 1.7

Theorem 1.7 is clearly a direct application of Proposition 2.24. Moreover, since this proof combines the techniques we already used we will give only a sketch of the proof.

Sketch of the proof of Proposition 2.24. Before we start, let

$$\theta' := 4 - \sqrt{13} \text{ and } \theta'' := 2(1-\theta') = 2\sqrt{6\theta' - 2} = 2\sqrt{13} - 6 = 1.21\ldots.$$

As in the previous subsection one can examine only the case where almost surely $\#_1(\sigma_n) < \theta''\sqrt{n}$. We distinguish two cases.

- If almost surely $\#(\sigma_n) > n\theta'$.
  We recall that
  $$\#_1(\sigma^2) \geq 6\#(\sigma) - 3\#_1(\sigma) - 2n.$$
  Consequently, under the condition $\#_1(\sigma_n) < \theta''\sqrt{n}$, almost surely,
  $$\#_1(\sigma_n^2) > n(6\theta' - 2) - 3\theta'\sqrt{n}.$$

We can then conclude by Proposition 7.16 that

$$\liminf_{n\to\infty} \frac{\mathbb{E}\left(\mathrm{LIS}(\sigma_n)\right)}{\sqrt{n(6\theta'-2)-3\theta'\sqrt{n}}} \geq 2.$$

Thus,

$$\liminf_{n\to\infty} \frac{\mathbb{E}\left(\mathrm{LIS}(\sigma_n)\right)}{\sqrt{n}} \geq 2\sqrt{6\theta'-2} = \theta''.$$

- If almost surely $\#(\sigma_n) \leq n\theta'$. Using Lemma 5.13 for $k=1$, we obtain that for any $\varepsilon, \varepsilon' > 0$ there exists n_0 such that for any $n > n_0$, for any conjugation invariant random permutation σ_n such that almost surely $\#(\sigma_n) \leq n\theta'$,

$$\mathbb{P}(\mathrm{LIS}(\sigma_n) > 2\sqrt{n}(1-\theta'-\varepsilon)) > 1-\varepsilon'.$$

Consequently,

$$\liminf_{n\to\infty} \frac{\mathbb{E}\left(\mathrm{LIS}(\sigma_n)\right)}{\sqrt{n}} \geq 2(1-\theta') = \theta''.$$

This concludes the proof.

□

8 Some conjugation invariant random permutations

"En mathématiques, on ne comprend pas les choses, on s'habitue seulement à elles."

John von Neumann

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We give in this chapter some examples of conjugation invariant random permutation that satisfies $(\mathcal{H}^{\mathbb{P}}_{inv,\alpha})$ for some choices of parameters. In particular, one can apply Corollary 2.5.

8.1 Ewens

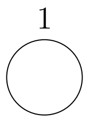
Definition 8.1. Let θ be a non-negative real number. We say that a random permutation $\sigma_{Ew,\theta,n}$ follows the Ewens distribution with parameter θ if for all $\sigma \in \mathfrak{S}_n$,

$$\mathbb{P}(\sigma_{Ew,\theta,n} = \sigma) = \frac{\theta^{\#(\sigma)-1}}{\prod_{i=1}^{n-1}(\theta + i)}. \tag{8.1}$$

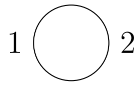
Note that when $\theta = 1$, the Ewens distribution is just the uniform distribution on $\mathfrak{S}_n$, whereas when $\theta = 0$ we have the uniform distribution on permutations having a unique cycle. For general θ, the Ewens distribution is clearly conjugation invariant since it only involves the cycle structure of θ.

We want to recall the interpretation of the Ewens distribution via a nice stochastic process known as "the Chinese restaurant process". Suppose that there are an infinite number of circular tables with infinite capacity.

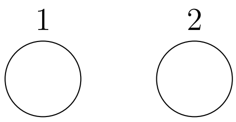
- At $t = 0$, all tables are empty.
- At $t = 1$, the person "1" comes and sits in the first table.



- At $t = 2$, the person "2" comes and sits in the table near person 1 with probability $\frac{1}{1+\theta}$

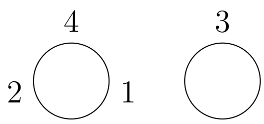


  and sits alone in a new table with probability $\frac{\theta}{1+\theta}$.

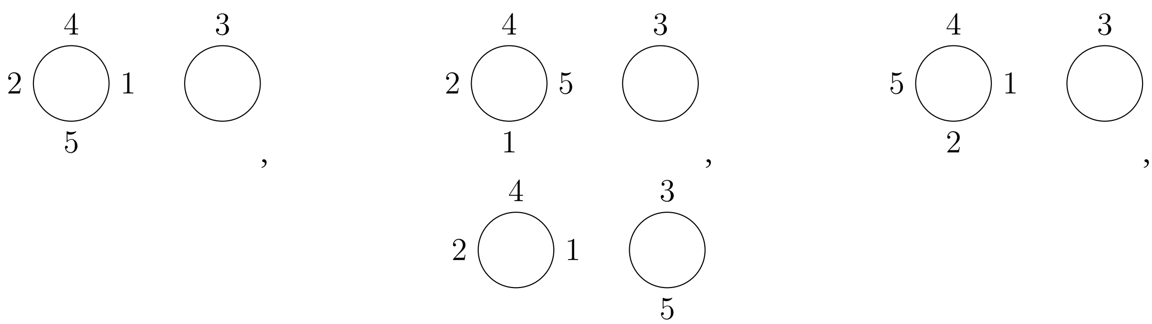


- At $t = n$, the person "n" comes, she/he chooses to sit alone in a new table with probability $\frac{\theta}{\theta+n-1}$ and in an occupied table i with probability $\frac{|B_i|}{\theta+n-1}$, where B_i is the number of persons at the table i. In this case, she/he chooses her/his position uniformly in gaps between two persons.

For example, if we have the following configuration[1],

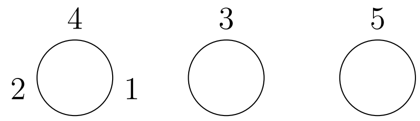


at $t = 5$, the probability to switch to each of the following configurations



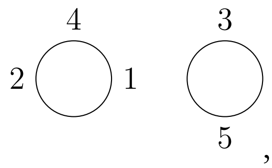
is $\frac{1}{\theta+4}$ and the probability to switch to

[1] we omitted empty tables



is equal to $\frac{\theta}{4+\theta}$.

To obtain the associated permutation to a configuration one reads the elements on each non-empty circle counterclockwise to get a cycle. For example, to the configuration



,

we associate the permutation $(1,4,2)(3,5)$.

Using the Chinese restaurant process description of the Ewens distribution, it is obvious to see that the number of cycles $\#(\sigma_{Ew,\theta,n})$ is the sum of n independent Bernoulli random variables with parameters $\left\{\frac{\theta}{\theta+i}\right\}_{0\leq i\leq n-1}$. For further reading, we recommend (Aldous, 1985; McCullagh, 2011; Chafaï, Doumerc, and Malrieu, 2013). In particular, we have the following classic result.

Proposition 8.2.

$$\mathbb{E}(\#_1(\sigma_{Ew,\theta,n})) = \frac{n\theta}{n-1+\theta} \quad \text{and} \quad \mathbb{E}(\#(\sigma_{Ew,\theta,n})) = 1 + \sum_{i=2}^{n} \frac{\theta}{i-1+\theta} \leq 2 + \theta log(n).$$

In particular, for uniform distribution, we have

$$\frac{\#(\sigma_{unif,n})}{log(n)} \xrightarrow[n\to\infty]{\mathbb{P}} 1. \tag{8.2}$$

Proof. Since the number of cycles of the uniform law is the sum of n independent random Bernoulli variables of parameters $1, \frac{1}{2}, \ldots, \frac{1}{n}$ and using Chebyshev's inequality, we obtain

$$\mathbb{P}\left(\left|\frac{\#(\sigma_{unif,n})}{\log(n)} - 1\right| > \alpha\right) \leq \frac{\frac{\sum_{i=1}^{n}\frac{i-1}{i^2}}{\log(n)^2}}{\left(\alpha + 1 - \frac{\sum_{i=1}^{n}\frac{1}{i}}{\log n}\right)^2} = O\left(\frac{1}{\log(n)}\right).$$

□

Remark 8.3. This convergence holds almost surely. The proof uses martingale techniques. One can find a proof of this result in (Chafaï, Doumerc, and Malrieu, 2013).

One can now apply our results using the following two results.

Corollary 8.4. Let $(\theta_n)_{n\geq 1}$ be a sequence of non-negative real numbers such that:

$$\lim_{n\to\infty} \frac{\theta_n \log(n)}{n^{\frac{1}{\alpha}}} = 0. \tag{8.3}$$

Then $(\sigma_{Ew,\theta_n,n})_{n\geq 1}$ satisfies $(\mathcal{H}^{\mathbb{P}}_{inv,\alpha})$.

Proposition 8.5. For any $\theta \geq 0, \alpha > 0$ and $p \geq [1, \infty[$, $(\sigma_{Ew,\theta,n})_{n\geq 1}$ satisfies $(\mathcal{H}^{\mathbb{L}^p}_{inv,\alpha})$.

Proof. Using Bernstein inequality, if $\theta \geq 1$,

$$\begin{aligned}
\mathbb{P}(\#(\sigma_{Ew,\theta,n}) > (3p+1)\theta\log(n)+2) &\leq \mathbb{P}(\#(\sigma_{Ew,\theta,n}) > \mathbb{E}(\ \#(\sigma_{Ew,\theta,n})) + 3p\theta\log(n)) \\
&\leq \exp\left(\frac{-\frac{9}{2}\theta^2p^2\log(n)^2}{\mathrm{var}(\#(\sigma_{Ew,\theta,n})) + \frac{3p}{3}\theta\log(n)}\right) \\
&\leq \exp\left(\frac{-\frac{9}{2}\theta^2p^2\log(n)^2}{(p+1)\theta\log(n)+2}\right) = O\left(n^{-\frac{9}{4}\theta\frac{p^2}{p+1}}\right) = O\left(n^{-\frac{9p}{8}}\right).
\end{aligned}$$

Consequently,

$$\mathbb{E}(\#(\sigma_{Ew,\theta,n})^p) \leq ((3p+1)\theta\log(n)+2)^p + n^pO(n^{-\frac{9p}{8}}) = O(\log^p(n)).$$

When $\theta < 1$, one can conclude since $\mathbb{E}(\#(\sigma_{Ew,\theta,n})^p) < \mathbb{E}(\#(\sigma_{Ew,1,n})^p)$. □

8.2 Generalized Ewens

Definition 8.6. Let $\hat{\theta} = (\hat{\theta}_i)_{i\geq 1}$ be a sequence of positive real numbers, we say that $\sigma_{GEw,\hat{\theta},n}$ follows the generalized Ewens distribution on $\mathfrak{S}_n$ with parameter $\hat{\theta}$ if for all $\sigma \in \mathfrak{S}_n$,

$$\mathbb{P}(\sigma_{GEw,\hat{\theta},n} = \sigma) = \frac{\prod_{i\geq 1} \hat{\theta}_i^{\#_i(\sigma)}}{\sum_{\sigma\in\mathfrak{S}_n}\prod_{i\geq 1}\hat{\theta}_i^{\#_i(\sigma)}}.$$

This generalization was studied in some cases in details by Ercolani and Ueltschi (2014). In the general case, it is not obvious to have a good control on the number of cycles. Nevertheless, by using some results of Ercolani and Ueltschi, we can conclude in some cases. In particular, we use the following results:

Lemma 8.7. Let $\hat{\theta} = (\hat{\theta}_i)_{i\geq 1}$.

- If $\hat{\theta}_i = e^{i^\gamma}$ with $\gamma > 1$, then $\#(\sigma_{GEw,\hat{\theta},n}) \xrightarrow[n\to\infty]{\mathbb{P}} 1$ (Ercolani and Ueltschi, 2014, Theorem 3.1).
- If $\hat{\theta}_i \to \theta$, then $\frac{1}{\theta\log(n)}\mathbb{E}(\#(\sigma_{GEw,\hat{\theta},n})) \to 1$ (Ercolani and Ueltschi, 2014, Theorem 6.1).
- If $\hat{\theta}_i = i^{-\gamma}$ with $\gamma > 1$, then $\#(\sigma_{GEw,\hat{\theta},n}) \xrightarrow{d} 1 + \sum_i Poisson\{\theta_i\}$ (Ercolani and Ueltschi, 2014, Theorem 7.1).

- If $\sum_{k=1}^{n-1} \frac{\hat{\theta}_k \hat{\theta}_{n-k}}{\hat{\theta}_n} \to 0$, then $\#(\sigma_{GEw,\hat{\theta},n}) \xrightarrow[n\to\infty]{\mathbb{P}} 1$ (Ercolani and Ueltschi, 2014, Theorem 3.1).
- If $\frac{\hat{\theta}_i}{i^\gamma} \to 1$ with $\gamma > 0$, then $\lim_{n\to\infty} n^{\frac{-\gamma}{\gamma+1}} \mathbb{E}(\#(\sigma_{GEw,\hat{\theta},n})) = \left(\frac{\Gamma(\gamma)}{\gamma^\gamma}\right)^{\frac{1}{\gamma+1}}$ (Ercolani and Ueltschi, 2014, Theorem 5.1).

8.3 Pitman-Ewens

Pitman and Yor (1997) introduced a two-parameters generalization of the Ewens distribution. We will not detail here the definition but one can see Pitman and Yor (1997) for more details. We chose to give this exemple since it is an example of conjugation invariant random permutations where the expectation of the nombre of cycles is polynomial. Using the same notations as in (Pitman and Yor, 1997), for fixed $0 < \alpha < 1$ and $\theta \geq 0$, the Pitman-Ewens random permutations satisfies ($\mathcal{H}^{\mathbb{P}}_{inv,\beta}$) for any $\beta < \frac{1}{\alpha}$.

8.4 Kingman Virtual permutations

Virtual permutations are introduced by Kerov, Olshanski, and Vershik (1993) as the projective limit of $\mathfrak{S}_n$. We are interested in this chapter only in conjugation invariant random virtual permutations also known as central measures as defined and totally characterized by Tsilevich (1998). Those measures are the counterpart for random permutations of the Kingman exchangeable random partitions (Kingman et al., 1975; Kingman, 1978).

Let n be a positive integer and π_n be the projection of $\mathfrak{S}_{n+1}$ on $\mathfrak{S}_n$ obtained by removing $n+1$ from the cycles' structure of the permutation. For example,

$$\pi_3((1,\,3)\,(2,\,4)) = \pi_3((1,\,4,\,3)\,(2)) = \pi_3((1,\,3)\,(2)\,(4)) = (1,\,3)\,(2).$$

We define the space of virtual permutations $\mathfrak{S}^\infty$ as the projective limit of $\mathfrak{S}_n$ as n goes to infinity:

$$\mathfrak{S}^\infty := \{(\hat{\sigma}_n)_{n\geq 1};\ \forall n \geq 1,\ \pi_n(\hat{\sigma}_{n+1}) = \hat{\sigma}_n\} = \varprojlim \mathfrak{S}_n.$$

Therefore, a random virtual permutation [2] is a sequence $(\sigma_n)_{n\geq 1}$ of random permutations such that $\pi_n(\sigma_{n+1}) \overset{a.s}{=} \sigma_n$. We say that it is conjugation invariant if for all positive integer n, σ_n is conjugation invariant.

According to (Tsilevich, 1998, Section 2) there exists a one-to-one correspondence between the set of conjugation invariant probability distributions on $\mathfrak{S}^\infty$ and the set of Borel probability distributions on

$$\Sigma := \left\{(x_i)_{i\geq 1};\ x_1 \geq x_2 \geq \cdots \geq 0,\ \sum_i x_i \leq 1\right\}.$$

[2] We will use the same $\sigma-$algebra as in (Tsilevich, 1998). For more details, one strongly recommend (Kerov, 2003). We will not assume that the reader is familiar with this discussion : we give a detailed description of those measures.

Let $0 \leq a \leq 1$. We denote

$$\Sigma_a := \left\{ (x_i)_{i\geq 1};\ x_1 \geq x_2 \geq \cdots \geq 0,\ \sum_i x_i = a \right\}.$$

Let ν be a probability measure on Σ. We denote by $(\sigma_{Kig,\nu,n})_{n\geq 1}$ a Kingman random virtual permutation such that the associated distribution on Σ is ν. We will study this correspondence in three parts:

- Let $x = (x_i)_{i\geq 1} \in \Sigma_1$. If $\nu = \delta_x$, then for all positive integer n, for all $\sigma \in \mathfrak{S}_n$,

  (8.4) $$f(n,x,\sigma) := \mathbb{P}(\sigma_{Kig,\delta_x,n} = \sigma) = \prod_{j\geq 1} \frac{\#_j(\sigma)!}{((j-1)!)^{\#_j(\sigma)}} \sum_m \prod_{i\geq 1} x_i^{m_i}.$$

  Here, the sum is over all sequences of non-negative integers $m = (m_i)_{i\geq 1}$ such that $\forall j \geq 1, \mathrm{card}(\{i; m_i = j\}) = \#_j(\sigma)$. For more details, see (Tsilevich, 1998, Section 2).

- If $\nu(\Sigma_1) = 1$, ν is called a 1-measure. In this case, the distribution of $(\sigma_{Kig,\nu,n})_{n\geq 1}$ is a mixture of the previous distributions i.e. for all positive integer n, for all $\sigma \in \mathfrak{S}_n$,

  (8.5) $$\mathbb{P}(\sigma_{Kig,\nu,n} = \sigma) = \int_{x\in\Sigma_1} f(n,x,\sigma) d\nu(x).$$

  Corollary 8.8. Assume that ν is a 1-measure then $(\sigma_{Kig,\nu,n})_{n\geq 1}$ satisfies ($\mathcal{H}^{\mathbb{P}}_{inv,1}$). Moreover, for any $0 < \alpha < 1$, if

  $$\int_{x\in\Sigma_1} \sum_{i=1}^{\infty} \left(1 - (1-x_i)^n\right) d\nu(x) = o\left(n^{\frac{1}{\alpha}}\right),$$

  then $(\sigma_{Kig,\nu,n})_{n\geq 1}$ satisfies ($\mathcal{H}^{\mathbb{P}}_{inv,\alpha}$).

  Corollary 8.9. If $x_n = o(n^{-\beta})$ with $\beta > \alpha$, then $(\sigma_{Kig,\nu,n})_{n\geq 1}$ satisfies ($\mathcal{H}^{\mathbb{P}}_{inv,\alpha}$).

  To explain the relation with the Ewens distributions, we need first to introduce the Poisson-Dirichlet distributions. Let $\theta > 0$ and let $1 \geq x_1 \geq x_2 \geq \cdots \geq 0$ be a Poisson point process on $(0,1]$ with intensity $\lambda(t) = \frac{\theta \exp(-t)}{t}$. We define the random variable $S := \sum_{i\geq 1} x_i$. It is proved that the sum S is almost surely finite. We can find a proof for example in Holst (2001). The point process $\hat{x} := \left(\frac{x_i}{S}\right)_{i\geq 1}$ defines a measure on Σ_1 known as the Poisson-Dirichlet distribution with parameter θ. It was introduced by Kingman et al. (1975) and it is a useful tool to study some problems of combinatorics, analytic number theory, statistics and population genetics. See (Kingman, 1980; Donnelly and Grimmett, 1993; Arratia, Barbour, and Tavaré, 2003; Tenenbaum, 1995).

  The Poisson-Dirichlet distribution with parameter $\theta > 0$ represents also the limiting distribution of normalized cycles' lengths of the Ewens distribution with the same parameter, see

Arratia, Barbour, and Tavaré (2003). As a consequence, using the description of these measures in (Tsilevich, 1998, Section 2), if ν follows the Poisson-Dirichlet distribution with parameter θ, $\sigma_{Kig,\nu,n}$ follows the Ewens measure with same parameter θ.

- In the general case, the correspondence is given by the formula:

$$\mathbb{P}(\sigma_{Kig,\nu,n} = \sigma) = \int_{x\in\Sigma} f(n,x,\sigma)d\nu(x),$$

where

$$f(n,x,\sigma) := \begin{cases} \prod_{j\geq 1} \frac{\#_j(\sigma)!}{((j-1)!)^{\#_j(\sigma)}} \sum_m \prod_{i\geq 1} x_i^{m_i} & \text{if } \sum_{i=1}^{\infty}(x_i) = 1 \\ \sum_{j=0}^{l} \binom{l}{j} x_0^j (1-x_0)^{n-j} f(n-j,\, y,\, \sigma^j) & \text{if } 0 < \sum_{i=1}^{\infty}(x_i) < 1 \\ \mathbb{1}_{\sigma = Id_n} & \text{if } \sum_{i=1}^{\infty}(x_i) = 0 \end{cases}. \tag{8.6}$$

Here, again the sum is over all sequences of non-negative integers $m = (m_i)_{i\geq 1}$ such that $\forall j \geq 1, \text{card}\{i; m_i = j\} = \#_j(\sigma)$, $y := \frac{x}{\sum_i x_i}$, $x_0 := 1 - \sum_{i=1}^{\infty} x_i$, l is the number of fixed points of σ, σ^j is the permutation obtained by removing j fixed points of σ and Id_n is the identity of $\mathfrak{S}_n$. For more details, we recommend (Tsilevich, 1998, Section 2). We can have a combinatorial interpretation of (8.6). Let $x = (x_i)_{i\geq 1} \in \Sigma$. At the beginning, we have an infinite number of circles $\{C_n\}_{n\in\mathbb{Z}}$. At each step $n \geq 1$ we choose an integer pos_n with probability distribution $\sum_{j\geq 1} x_j\delta_j + (1 - \sum_{i\geq 1} x_i)\delta_0$ independently from the past. We insert then the number n uniformly on the circle C_{pos_n} if $pos_n > 0$ and on the circle C_{-n} if $pos_n = 0$. At each step, one reads the elements on each non-empty circle counterclockwise to get a cycle. For example, if $pos_1 = 4$, $pos_2 = 1$, $pos_3 = 4$, $pos_4 = 0$ and $pos_5 = 0$, we obtain the permutation $(1,3)(2)(4)(5)$. We use the notation $(\sigma_n)_{n\geq 1} = (A_n, \tau_n)_{n\geq 1}$ with $A_n = \{1 \leq i \leq n; \forall i \geq 1, \sigma_n(i) = i\}$ with the previous description, almost surely, $A := \cup_{n\geq 1} A_n \overset{a.s}{=} \{i \geq 1; pos_i = 0\}$ and $(\tau_n)_{n\notin A}$ is a virtual permutation over $\mathbb{N}^* \setminus A$.

With this description, we have

$$\mathbb{E}\left(\#\left(\sigma_{Kig,\delta_x,n}\right)\right) = n\left(1 - \sum_{i\geq 1} x_i\right) + \sum_{i=1}^{\infty}(1-(1-x_i)^n).$$

In the general case, we do not expect the Tracy-Widom fluctuations neither for LIS nor for LDS. We limit then our study to the case where there exists $0 < x_0 < 1$ such that $\nu(\Sigma_{1-x_0}) = 1$. Unlike all previous examples when LIS(σ_n) and LDS(σ_n) have the same asymptotic fluctuations, in this case, the expected length of the longest increasing subsequence is larger than $(1-x_0)n$ and we will show that there exist some cases where the expected length of the longest decreasing subsequence is asymptotically proportional to $\sqrt{n}$ with Tracy-Widom fluctuations.

Corollary 8.10. Let $0 < x_0 < 1$ and ν be a probability measure on Σ satisfying $\nu\left(\Sigma_{1-x_0}\right) = 1$. Let $\hat{\nu}$ be the 1-measure such that $d\hat{\nu}(x) = d\nu\left(\frac{x}{1-x_0}\right)$. If there exists a positive integer k such that for all real numbers $s_1, s_2, \ldots, s_k$,

$$\lim_{n\to\infty} \mathbb{P}\left(\forall 1 \le i \le k,\ \frac{\lambda_i'(\sigma_{Kig,\hat{\nu},n}) - 2\sqrt{n}}{n^{\frac{1}{6}}} \le s_i\right) = F_{2,k}(s_1,\ \ldots,\ s_k),$$

then for all real numbers $s_1, s_2, \ldots, s_k$,

$$\lim_{n\to\infty} \mathbb{P}\left(\forall 1 \le i \le k,\ \frac{\lambda_i'(\sigma_{Kig,\nu,n}) - 2\sqrt{(1-x_0)n}}{((1-x_0)n)^{\frac{1}{6}}} \le s_i\right) = F_{2,k}(s_1,\ \ldots,\ s_k).$$

In particular, for all real s,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\mathrm{LDS}(\sigma_{Kig,\nu,n}) - 2\sqrt{(1-x_0)n}}{((1-x_0)n)^{\frac{1}{6}}} \le s\right) = F_2(s).$$

This corollary is a direct application of Proposition 5.10. Here are some examples of measures ν that meet the assumptions of the previous corollary:

- When $\nu = \delta_x$ and $x_i = o(\frac{1}{i^{6+\varepsilon}})$.
- When $d\nu(x) = dPD(\beta)(\frac{x}{\alpha}), \beta \ge 0, 0 < \alpha \le 1$ and $PD(\beta)$ is Poisson-Dirichlet distribution with parameter β.

As a consequence, recalling (8.6), if there exists $0 < x_0 < 1$ such that $\nu\left(\Sigma_{1-x_0}\right) = 1$, then the number of fixed points of $\sigma_{Kig,\nu,n}$ is larger than a binomial random variable with parameters x_0 and n. Consequently,

$$\mathbb{E}(\mathrm{LIS}(\sigma_{Kig,\nu,n})) \ge \mathbb{E}(\#_1(\sigma_{Kig,\nu,n})) \ge nx_0.$$

In this case, we conjecture that the fluctuations are Gaussian.

Conjecture 8.11. Let $0 < x_0 < 1$, ν be a probability measure on Σ satisfying $\nu(\Sigma_{1-x_0}) = 1$. Then $\forall s \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\mathrm{LIS}(\sigma_{Kig,\nu,n}) - x_0 n}{\sqrt{x_0(1-x_0)n}} \le s\right) = \int_{-\infty}^{s} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx.$$

One bound is simple to prove by the remark above.

For the descent process, we have the following result:

Theorem 8.12. If there exists $0 \le x_0 \le 1$ such that $\nu(\Sigma_{1-x_0}) = 1$, then for all finite set $A \subset \mathbb{N}^*$,

$$\lim_{n\to\infty} \mathbb{P}(A \subset D\left(\sigma_{Kig,\nu,n}\right)) = \det([k_{x_0}(j-i)]_{i,j\in A}),$$

with

$$\sum_{l\in\mathbb{Z}} k_{x_0}(l)z^l = \frac{1}{1-(1+x_0z)e^{(1-x_0)z}} = \frac{-1}{z+\sum_{l=1}^{\infty}\hat{a}_l(x_0)z^{l+1}}, \tag{8.7}$$

where

$$\hat{a}_l(x_0) := \frac{(1-x_0)^{l+1}}{(l+1)!} + \frac{x_0(1-x_0)^l}{l!}, \tag{8.8}$$

and D is defined by (2.14).

The proof of this result we give in Section 9.1 consists in studying in a first step the case where the corresponding measure ν is concentrated on Σ_1. We prove that the limiting point process is determinantal with kernel $(i,j) \mapsto k_0(j-i)$. In a second step, we prove that the kernel depends only on $\sum_{i\geq 1} x_i$.

Theorem 8.12 implies that for a general random virtual permutation stable under conjugation, we have the following result.

Corollary 8.13. For any probability measure ν on Σ,

$$\lim_{n\to\infty} \mathbb{P}(A \subset D(\sigma_{Kig,\nu,n})) = \int_\Sigma \det\left(\left[k_{1-\sum_i x_i}(j-i)\right]_{i,j\in A}\right) d\nu(x). \tag{8.9}$$

For the total number of descents we have

Proposition 8.14. For any probability measure ν on Σ,

$$\lim_{n\to\infty} \frac{\mathbb{E}(\mathcal{N}_D(\sigma_{Kig,\nu,n}))}{n} = \frac{1}{2}\left(1 - \int_\Sigma \left(1-\sum_i x_i\right)^2 d\nu(x)\right).$$

We will prove Corollary 8.13 and Proposition 8.14 in Section 9.1.

8.5 Shape of RSK for Kingman Virtual permutations

Corollary 8.15. Let $(\sigma_n)_{n\geq 1}$ be a Kingman virtual permutation. Then for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}}\left|\frac{1}{2\sqrt{n}} L_{\lambda(\sigma_n)}\left(2s\sqrt{n-\#_1(\sigma_n)}\right) - \sqrt{1-\frac{\#_1(\sigma_n)}{n}}\,\Omega(s)\right| < \varepsilon\right) = 1. \tag{8.10}$$

We will now prove those results. Note that for 1-measures i.e. when $\nu(\Sigma_1) = 1$, we have the following.

Lemma 8.16. (Kammoun, 2018) Assume that $\nu(\Sigma_1) = 1$. Then, for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}}\left|\frac{1}{\sqrt{2n}}L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right) - \Omega(s)\right| < \varepsilon\right) = 1. \tag{8.11}$$

Corollary 8.17. If $\nu(\Sigma_1) = 1$, then for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}}\left|\frac{1}{2\sqrt{n}}L_{\lambda(\sigma_n)}\left(2s\sqrt{n-\#_1(\sigma_n)}\right) - \sqrt{1-\frac{\#_1(\sigma_n)}{n}}\Omega\left(s\right)\right| < \varepsilon\right) = 1. \tag{8.12}$$

Proof. Fix $0 < \varepsilon < 1$. By Markov's inequality, we obtain

$$\lim_{n\to\infty} \mathbb{P}\left(\sqrt{1-\frac{\#_1(\sigma_n)}{n}} > \frac{1}{1+\varepsilon}\right) = 1.$$

Let n_0 such that $\forall n > n_0$,

$$\mathbb{P}\left(\sup_{s\in\mathbb{R}}\left|\frac{1}{\sqrt{2n}}L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right) - \Omega(s)\right| < \frac{\varepsilon}{2}\right) > 1-\frac{\varepsilon}{2} \tag{8.13}$$

and

$$\mathbb{P}\left(\sqrt{1-\frac{\#_1(\sigma_n)}{n}} > \max\left(\frac{1}{1+\varepsilon}, 1-\frac{\varepsilon}{4+2\varepsilon}\right)\right) > 1-\frac{\varepsilon}{2}. \tag{8.14}$$

If $\sqrt{1-\frac{\#_1(\sigma_n)}{n}} > \frac{1}{1+\varepsilon}$ and $\sup_{s\in\mathbb{R}}\left|\frac{1}{\sqrt{2n}}L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right) - \Omega(s)\right| < \frac{\varepsilon}{2}$ then for any $|s| \geq 1+\varepsilon$,

$$\sqrt{1-\frac{\#_1(\sigma_n)}{n}}\Omega\left(s\right) = \sqrt{1-\frac{\#_1(\sigma_n)}{n}}|s| = \Omega\left(\sqrt{1-\frac{\#_1(\sigma_n)}{n}}s\right),$$

and consequently by replacing s by $\sqrt{1-\frac{\#_1(\sigma_n)}{n}}s$ we obtain, for any $|s| \geq 1+\varepsilon$,

$$\left|\frac{1}{2\sqrt{n}}L_{\lambda(\sigma_n)}\left(2s\sqrt{n-\#_1(\sigma_n)}\right) - \sqrt{1-\frac{\#_1(\sigma_n)}{n}}\Omega\left(s\right)\right| < \frac{\varepsilon}{2} < \varepsilon.$$

Consequently for all $\varepsilon > 0$, for all $n > n_0$,

$$\mathbb{P}\left(\sup_{|s|\geq 1+\varepsilon}\left|\frac{1}{2\sqrt{n}}L_{\lambda(\sigma_n)}\left(2s\sqrt{n-\#_1(\sigma_n)}\right) - \sqrt{1-\frac{\#_1(\sigma_n)}{n}}\Omega\left(s\right)\right| < \varepsilon\right) > 1-\varepsilon. \tag{8.15}$$

To conclude we will now prove that $\forall n > n_0$,

$$\mathbb{P}\left(\sup_{|s|< 1+\varepsilon}\left|\frac{1}{2\sqrt{n}}L_{\lambda(\sigma_n)}\left(2s\sqrt{n-\#_1(\sigma_n)}\right) - \sqrt{1-\frac{\#_1(\sigma_n)}{n}}\Omega\left(s\right)\right| < \varepsilon\right) > 1-\varepsilon. \tag{8.16}$$

By triangle inequality, we have

$$\left|\frac{1}{2\sqrt{n}}L_{\lambda(\sigma_n)}\left(2s\sqrt{n-\#_1(\sigma_n)}\right)-\sqrt{1-\frac{\#_1(\sigma_n)}{n}}\Omega\left(s\right)\right|$$
$$\leq \frac{1}{2\sqrt{n}}\left|L_{\lambda(\sigma_n)}\left(2s\sqrt{n-\#_1(\sigma_n)}\right)-L_{\lambda(\sigma_n)}\left(2s\sqrt{n}\right)\right|+\left|\frac{1}{\sqrt{2n}}L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right)-\Omega(s)\right|$$
$$+\left(1-\sqrt{1-\frac{\#_1(\sigma_n)}{n}}\right)\Omega\left(s\right).$$

Since the Lipschitz constant for $L_{\lambda(\sigma_n)}$ is 1 then

$$\frac{1}{2\sqrt{n}}\left|L_{\lambda(\sigma_n)}\left(2s\sqrt{n-\#_1(\sigma_n)}\right)-L_{\lambda(\sigma_n)}\left(2s\sqrt{n}\right)\right|\leq 1-\sqrt{1-\frac{\#_1(\sigma_n)}{n}}.$$

If

$$\sqrt{1-\frac{\#_1(\sigma_n)}{n}}>1-\frac{\varepsilon}{4+2\varepsilon}$$

and $\sup_{s\in\mathbb{R}}\left|\frac{1}{\sqrt{2n}}L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right)-\Omega(s)\right|<\varepsilon$ then for any $|s|<1+\varepsilon$,

$$\left|\frac{1}{2\sqrt{n}}L_{\lambda(\sigma_n)}\left(2s\sqrt{n-\#_1(\sigma_n)}\right)-\sqrt{1-\frac{\#_1(\sigma_n)}{n}}\Omega\left(s\right)\right|<\frac{\varepsilon}{4+2\varepsilon}+\frac{\varepsilon}{2}+\frac{\varepsilon}{4+2\varepsilon}(1+\varepsilon)=\varepsilon.$$

We have than (8.16) which concludes the proof. □

We will now prove the result for the probability measures ν on Σ such that ν almost surely $x_2=0$. Using the previous description of those measure $\sigma_{Kig,\nu,n}$ has almost surely at most one cycle having more than 1 element. This yields that if $(\sigma_{Kig,\nu,n})_{n\geq 1}=(A_n,(\tau_n))_{n\geq 1}$ then $\text{card}(A_n)\leq \#_1(\sigma_n^\nu)\leq \text{card}(A_n)+1$ and for any $A\subset\{1,2,\ldots,n\}$ conditionally to that $A_n=A$, τ_n is distributed according to the uniform measure on permutations with one cycle on $\{1,2,\ldots,n\}\setminus A_n$.

Lemma 8.18. Let $(\sigma_n)_{n\geq 1}=(A_n,\tau_n)_{n\geq 1}$ a virtual permutation. We have

- For all $n\geq 1$

$$\text{card}(A_n)\leq\lambda_1(\sigma_n)\leq\text{card}(A_n)+\lambda_1(\tau_n). \tag{8.17}$$

- For all $n\geq 1$, for all $i\geq 2$,

$$\text{card}(A_n)+\sum_{j=1}^{i-1}\lambda_j(\tau_n)\leq\sum_{j=1}^{i}\lambda_j(\sigma_n)\leq\text{card}(A_n)+\sum_{j=1}^{i}\lambda_j(\tau_n). \tag{8.18}$$

- For all $n \geq 1$

$$\max_{i\geq 2}\left|\sum_{k=2}^{m}(\lambda_k(\sigma_n)-\lambda_k(\tau(n))\right|\leq \lambda_1(\tau(n)). \tag{8.19}$$

Proof. Those inequalities are an application of Greene's theorem.

- The points of A_n are fixed points of σ_n and then $A_n \subset \mathfrak{I}_1(\sigma_n)$ which implies the first inequality of (8.17).

- Let $i_1 < i_2 < \cdots < i_{\mathrm{LIS}(\sigma_n)}$ is an increasing subsequence of σ_n of maximal length. We have

$$\{i_1, i_2, \dots i_{\mathrm{LIS}(\sigma_n)}\} = (\{i_1, i_2, \dots i_{\mathrm{LIS}(\sigma_n)}\} \cap A_n) \cup (\{i_1, i_2, \dots i_{\mathrm{LIS}(\sigma_n)}\} \cap (\{1,2,\dots,n\} \setminus A_n)),$$
$$(\{i_1, i_2, \dots i_{\mathrm{LIS}(\sigma_n)}\} \cap A_n) \subset A_n,$$
$$(\{i_1, i_2, \dots i_{\mathrm{LIS}(\sigma_n)}\} \cap (\{1,2,\dots,n\} \setminus A_n)) \subset \mathfrak{I}_1(\tau_n).$$

  This yields the second inequality of (8.17)

- Assume $I \subset \mathfrak{I}_{i-1}(\tau_n) \subset \mathfrak{I}_{i-1}(\sigma_n)$. Then $(I \cup A_n) \subset \mathfrak{I}_i(\sigma_n)$. This yields the first inequality of (8.18).

- Let $I \subset \mathfrak{I}_i(\sigma_n)$ of maximal length. Then

$$I = (I \cap A_n) \cup (I \cap (\{1,2,\dots,n\} \setminus A_n)),$$
$$(I \cap A_n) \subset A_n,$$
$$(I \cap (\{1,2,\dots,n\} \setminus A_n)) \subset \mathfrak{I}_i(\tau_n).$$

  This yields the second inequality of (8.18).

- By subtracting (8.17) from (8.18). We obtain

$$-\lambda_1(\tau(n)) \leq -\lambda_i(\tau(n)) \leq \sum_{j=2}^{i}(\lambda_j(\sigma_n) - \lambda_j(\tau(n)) \leq \lambda_1(\tau(n)).$$

  This yields (8.18).

□

We have the following.

Lemma 8.19. Assume that $\nu(\{x \in \Sigma; x_2 = 0\}) = 1$. Then for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}} \frac{1}{\sqrt{2n}}\left|L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right) - L_{\lambda(\tau_n)}\left(s\sqrt{2n}\right)\right| < \varepsilon\right) = 1.$$

Proof. Using Lemma 7.9 we obtain

$$\sup_{s\in\mathbb{R}} \left|L_{\lambda(\sigma_n)}(s) - L_{\lambda(\tau_n)}\right| \leq 2\sqrt{\max_{m\geq 2}\left|\sum_{k=l+1}^{m}(\lambda_k(\sigma_n)-\lambda_k(\tau_n))\right|} + \sqrt{2}. \tag{8.20}$$

By Lemma 8.18, we obtain

$$\sup_{s\in\mathbb{R}} \left|L_{\lambda(\sigma_n)}(s) - L_{\lambda(\tau_n)}\right| \leq 2\sqrt{\lambda_1(\tau_n)} + \sqrt{2}. \tag{8.21}$$

Since $\lambda_1(\tau_n)$ is equal in distribution to $\mathrm{LIS}(\tilde{\sigma}_{n-\mathrm{card}(A_n)})$, where $\tilde{\sigma}_n$ is distributed according to the uniform distribution of Ewens of parameter $\theta = 0$, then for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\lambda_1(\tau_n)}{n} > \varepsilon\right) = 0. \tag{8.22}$$

This yields Lemma 8.19. □

We can then conclude if $\nu(\{x \in \Sigma; x_2 = 0\}) = 1$.

Proposition 8.20. If $\nu(\{x \in \Sigma; x_2 = 0\}) = 1$, then for all $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}}\left|\frac{1}{2\sqrt{n}}L_{\lambda(\sigma_n)}\left(2s\sqrt{n-\#_1(\sigma_n)}\right) - \sqrt{1-\frac{\#_1(\sigma_n)}{n}}\Omega(s)\right| < \varepsilon\right) = 1. \tag{8.23}$$

Proof. Using Lemma 8.19. This result is equivalent to that for all $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}}\left|\frac{1}{2\sqrt{n}}L_{\lambda(\tau_n)}\left(2s\sqrt{n-\mathrm{card}(A_n)}\right) - \sqrt{1-\frac{\mathrm{card}(A_n)}{n}}\Omega(s)\right| < \varepsilon\right) = 1.$$

Fix $\varepsilon, \varepsilon' > 0$,

$$\begin{aligned}\mathbb{P}&\left(\sup_{s\in\mathbb{R}}\left|\frac{1}{2\sqrt{n}}L_{\lambda(\tau_n)}\left(2s\sqrt{n-\mathrm{card}(A_n)}\right) - \sqrt{1-\frac{\mathrm{card}(A_n)}{n}}\Omega(s)\right| < \varepsilon\right)\\ &= \sum_{k=0}^{n} p_{k,n,\varepsilon}\mathbb{P}(\mathrm{card}(A_n) = n-k),\end{aligned}$$

with

$$p_{k,n,\varepsilon} := \mathbb{P}\left(\sup_{s\in\mathbb{R}}\left|\frac{1}{2\sqrt{n}}L_{\lambda(\tau_n)}\left(2s\sqrt{n-\mathrm{card}(A_n)}\right) - \sqrt{1-\frac{\mathrm{card}(A_n)}{n}}\Omega(s)\right| < \varepsilon \middle| \mathrm{card}(A_n) = n-k\right).$$

Remark that

$$\begin{aligned} p_{k,n,\varepsilon} &:= \mathbb{P}\left(\sup_{s\in\mathbb{R}}\left|\frac{1}{2\sqrt{n}}L_{\lambda(\tilde{\sigma}_k)}\left(2s\sqrt{k}\right)-\sqrt{\frac{k}{n}}\Omega\left(s\right)\right|<\varepsilon\right) \\ &= \mathbb{P}\left(\sup_{s\in\mathbb{R}}\sqrt{\frac{k}{n}}\left|\frac{1}{2\sqrt{k}}L_{\lambda(\tilde{\sigma}_k)}\left(2s\sqrt{k}\right)-\Omega\left(s\right)\right|<\varepsilon\right) \\ &\geq \mathbb{P}\left(\sup_{s\in\mathbb{R}}\left|\frac{1}{2\sqrt{k}}L_{\lambda(\tilde{\sigma}_k)}\left(2s\sqrt{k}\right)-\Omega\left(s\right)\right|<\varepsilon\right) \end{aligned}$$

where the law of $\tilde{\sigma}_k$ is $Ew(0)$. Lemma 7.4 applied to $(\tilde{\sigma}_k)_{k\geq 1}$ guarantee the existence of n_0 such that for all $k \geq \frac{\varepsilon^2 n_0}{1+\frac{2}{\pi}}$, for all $n \geq 1$

$$p_{k,n,\varepsilon} \geq 1-\varepsilon'.$$

Moreover,

$$\sup_{s\in\mathbb{R}}\left|\frac{1}{2\sqrt{k}}L_{\lambda(\tilde{\sigma}_k)}\left(2s\sqrt{k}\right)-\Omega\left(s\right)\right| \leq \sup_{s\in\mathbb{R}}\left|\frac{1}{2\sqrt{k}}L_{\lambda(\tilde{\sigma}_k)}\left(2s\sqrt{k}\right)-|s|\right| + \sup_{s\in\mathbb{R}}\left|\Omega(s)-|s|\right|.$$

Since $|s| = L_{\varnothing}(s) = \frac{1}{2\sqrt{k}}L_{\varnothing}(2s\sqrt{k})$, by 7.9, we obtain

$$\sup_{s\in\mathbb{R}}\left|\frac{1}{2\sqrt{k}}L_{\lambda(\tilde{\sigma}_k)}\left(2s\sqrt{k}\right)-|s|\right| < 1$$

and consequently

$$\sup_{s\in\mathbb{R}}\left|\frac{1}{2\sqrt{k}}L_{\lambda(\tilde{\sigma}_k)}\left(2s\sqrt{k}\right)-\Omega\left(s\right)\right| \leq 1+\frac{2}{\pi}.$$

Hence, for all $n > n_0$, for all $k < \frac{\varepsilon^2 n_0}{1+\frac{2}{\pi}}$,

$$p_{k,n,\varepsilon} = 1 \geq 1-\varepsilon'.$$

This yields Proposition 8.20

□

For the general case. Let $(\sigma_n)_{n\geq 1} = (A_n, \tau_n)_{n\geq 1}$ be a virtual permutation. For fixed n, we define the following Markov operator T. Let $\sigma = (a, \tau)$ be a realization of σ_n. If τ has one cycle, σ remains unchanged ($T(\sigma) = \sigma$). Otherwise, we choose with uniform probability C_1 and C_2 two different cycles of τ, and then independently two elements $i \in C_1$ and $j \in C_2$ uniformly within each cycle. In this case, $T(\sigma) = \sigma \circ (i, j)$. In the reminder of the proof we denote by τ_n^k the restriction of $T^k(\sigma_n)$ on $\{1, 2, \ldots, n\} \setminus A_n$.

Lemma 8.21.

(8.24)
$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}}\left|\frac{1}{2\sqrt{n}}L_{\lambda(\sigma_n)}\left(2s\sqrt{n-\#_1(\sigma_n)}\right)-\frac{1}{2\sqrt{n}}L_{\lambda(T^{n-1}(\sigma_n))}\left(2s\sqrt{n-\#_1(\sigma_n)}\right)\right|<\varepsilon\right)=1,$$

Proof. Similarly to Lemma 5.11, we obtain the following. For any positive integers i and n, we have almost surely

$$\left|\sum_{k=1}^{i}\lambda_k(\sigma_n)-\lambda_k\left(T^{n-1}(\sigma_n)\right)\right|\leq 2\#(\tau_n). \tag{8.25}$$

Moreover,

$$\lim_{n\to\infty}\frac{\mathbb{E}(\#(\tau_n))}{n}=\lim_{n\to\infty}\frac{\int_\Sigma\sum x_i(1-x_i)^{n-1}d\nu(x)}{n}=0. \tag{8.26}$$

Theorem 7.4 yields Lemma 8.21. □

Lemma 8.22. For any integer n, for any subset B_n of $\{1,2,\dots,n\}$ satisfying $|B_n|<n$ and $\mathbb{P}(A_n=B_n)\neq 0$, we have

$$\mathbb{P}\left(\tau_n^{n-1}=\tau|A_n=B_n\right)=\frac{1}{(n-\mathrm{card}(A_n)-1)!}\mathbb{1}_{\#(\tau)=1}\mathbb{1}_{supp(\tau)=\{1,2,\dots,n\}\setminus B_n}.$$

Sketch of proof. Note that by construction for all positive integer $i<n$,

$$\#(\tau_n^i)\overset{a.s}{=}\max(\#(\tau_n)-i,1). \tag{8.27}$$

In particular,

$$\#(\tau_n^{n-1})\overset{a.s}{=}1. \tag{8.28}$$

Since for a fixed value of A_n the modification of the cycles of σ_n does not depend on the "label" of the elements, the distribution of $T^{n-1}(\sigma_n)$ is stable under conjugation, the same result holds for τ_n. We recall that the only distribution of probability on permutations verifying (8.28) and stable under conjugation is the uniform distribution for the permutations with a unique cycle. We can give a rigorous proof of the stability under conjugation by a direct computation using the same technique as in the proof of (Kammoun, 2018, Lemma 29). □

Lemma 8.23. For any $\varepsilon>0$,

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{s\in\mathbb{R}}\left|\frac{1}{2\sqrt{n}}L_{\lambda(T^{n-1}(\sigma_n))}\left(2s\sqrt{n-\#_1(\sigma_n)}\right)-\sqrt{1-\frac{\#_1(\sigma_n)}{n}}\Omega(s)\right|<\varepsilon\right)=1. \tag{8.29}$$

Proof. Lemma 8.22 implies in particular that $T^{n-1}(\sigma_{Kig,\nu,n})$ is equal in distribution to $\sigma_{Kig,\hat{\nu},n}$, where

$\hat{\nu}$ is given by the formula $\hat{\nu}(A) = \nu(\Pi^{-1}(A))$ and

$$\Pi : \Sigma \mapsto \Sigma$$
$$(x_1, x_2, x_3, \dots) \to (\sum_{i\geq 1} x_i, 0, 0, \dots).$$

By Proposition 8.20, we have for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}} \left| \frac{1}{2\sqrt{n}} L_{\lambda(T^{n-1}(\sigma_n))}\left(2s\sqrt{n - \#_1(T^{n-1}(\sigma_n))}\right) - \sqrt{1 - \frac{\#_1(T^{n-1}(\sigma_n))}{n}}\Omega\left(s\right)\right| < \varepsilon\right) = 1.$$

We replace s by sh, where

$$h = \begin{cases} \sqrt{\frac{n-\#_1(\sigma(n))}{n-\#_1(T^{n-1}(\sigma_n))}} & \text{if } \#_1(T^{n-1}(\sigma_n)) < n \\ 1 & \text{if } \#_1(T^{n-1}(\sigma_n)) = n \end{cases},$$

we obtain the following convergence for any $\varepsilon > 0$.

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}} \left| \frac{1}{2\sqrt{n}} L_{\lambda(T^{n-1}(\sigma_n))}\left(2s\sqrt{n - \#_1(\sigma_n)}\right) - \sqrt{1 - \frac{\#_1(T^{n-1}(\sigma_n))}{n}}\Omega\left(sh\right)\right| < \varepsilon\right) = 1.$$

Consequently we need to prove that for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}} \left| \sqrt{1 - \frac{\#_1(T^{n-1}(\sigma_n))}{n}}\Omega\left(s\right) - \sqrt{1 - \frac{\#_1(\sigma_n)}{n}}\Omega(sh)\right| < \varepsilon\right) = 1.$$

Let

$$E_{n,\varepsilon} := \sup_{s\in\mathbb{R}} \left| \sqrt{1 - \frac{\#_1(T^{n-1}(\sigma_{Kig,\nu,n}))}{n}}\Omega\left(s\right) - \sqrt{1 - \frac{\#_1(\sigma_{Kig,\nu,n})}{n}}\Omega(sh)\right| < \varepsilon.$$

Fix $\varepsilon, \varepsilon' > 0$. We will prove now that there exists $n_0 > 0$, such that for all $n > n_0, \mathbb{P}(E_{n,\varepsilon}) < (1-\varepsilon')^2$. Since $\lim_{\alpha\to 1^-} \nu(\alpha < x_0 < 1) = 0$ there exists $0 < \alpha < 1$ such that $\nu(x_0 \in [0, \alpha) \cup \{1\}) > 1 - \varepsilon'$. Under the condition $x_0 = 1$ (when it is defined), we have almost surely $\sigma_n = Id_n$ and then $\mathbb{P}(E_n | x_0 = 1) = 1$ and consequently :

- $\mathbb{P}(x_0 < \alpha) = 0$ and $\mathbb{P}(x_0 = 1) > 0$ then

$$\begin{aligned}\mathbb{P}\left(E_{n,\varepsilon}\right) &\geq \mathbb{P}(E_{n,\varepsilon} | x_0 = 1)\mathbb{P}(x_0 = 1) \\ &\geq \mathbb{P}(E_{n,\varepsilon} | x_0 = 1) \geq 1 - \varepsilon' \geq (1 - \varepsilon')^2.\end{aligned}$$

- $\mathbb{P}(x_0 < \alpha) > 0$ and $\mathbb{P}(x_0 = 1) = 0$ then

$$\begin{aligned}\mathbb{P}\left(E_{n,\varepsilon}\right) &\geq \mathbb{P}(E_{n,\varepsilon}|x_0 < \alpha)\mathbb{P}(x_0 < \alpha)\\ &\geq \mathbb{P}(E_{n,\varepsilon}|x_0 < \alpha)(1-\varepsilon').\end{aligned}$$

- If $\mathbb{P}(x_0 < \alpha) > 0$ and $\mathbb{P}(x_0 = 1) > 0$, then

$$\begin{aligned}\mathbb{P}\left(E_{n,\varepsilon}\right) &\geq \mathbb{P}(E_{n,\varepsilon}|x_0 = 1)\mathbb{P}(x_0 = 1) + \mathbb{P}(E_{n,\varepsilon}|x_0 < \alpha)\mathbb{P}(x_0 < \alpha)\\ &\geq \mathbb{P}(E_{n,\varepsilon}|x_0 = 1) + \mathbb{P}(E_{n,\varepsilon}|x_0 < \alpha)\mathbb{P}(x_0 < \alpha) \geq \mathbb{P}(E_{n,\varepsilon}|x_0 < \alpha)(1-\varepsilon').\end{aligned}$$

Since $\mathbb{P}\left(\#_1\left(\sigma_{Kig,\nu,n}\right) < n^{\frac{\alpha+1}{2}}\right) \to 1$, there exists n_0 such that for all $n > n_0$, $\mathbb{P}\left(\#_1\left(\sigma_{Kig,\nu,n}\right) < n^{\frac{\alpha+1}{2}}\right) > 1-\varepsilon'$. Which concludes the proof. □

9 Further discussion

"The art of doing mathematics consists in finding that special case which contains all the germs of generality."
David Hilbert

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In this chapter, we give some additional results and we give some possible ways to continue this work. In Section 9.1, we improve Corollary 2.22. We chose to present the result here as it does not use Markov techniques. We give in Section 9.2 a different point of view on our Markov processes using an optimal transport formulation. We chose not to detail this section since we did not find any nice application. In Section 9.3, we present a possible extension to the study of cycle structure of product of permutations. In Section 9.4, we suggest a way to improve the results we presented in Chapter 4. Finally, in Section 9.5, we give a generalization of Theorem 4.5.

9.1 Descent process

We now give an improvement of Proposition 2.20; If $f \in \widetilde{\mathcal{L}oc}$[1], one can replace $(\mathcal{H}^{\mathbb{P}}_{inv,1})$ by $(\mathcal{H}^{\text{tr},1}_{inv,1})$ in Proposition 2.20. We will prove this result only for the decent process but the generalization to $\widetilde{\mathcal{L}oc}$ is immediate.

Theorem 9.1. Assume that the sequence of random permutations $(\sigma_n)_{n\geq 1}$ satisfies $(\mathcal{H}^{\text{tr},1}_{inv,1})$. Then for all finite set $A \subset \mathbb{N}^* := \{1, 2, \dots\}$,

$$\text{(DPP)} \qquad \lim_{n\to\infty} \mathbb{P}(A \subset D(\sigma_n)) = \det([k_0(j-i)]_{i,j\in A}).$$

We prove also results of convergence for virtual permutations (Theorem 8.12, Corollary 8.13 and Proposition 8.14).

Let A be a finite subset of $\mathbb{N}^*$ and $m := \max(A)$. The idea of the proof of Theorem 9.1 is to study the descent process under the condition $\{\sigma_n(\{1, 2, \dots, m+1\}) \cap \{1, 2, \dots, m+1\} = \varnothing\}$ and to show that it does not depend on the law of σ_n.

Lemma 9.2. Let $E_{n,m} := \{\sigma \in \mathfrak{S}_n, \sigma(\{1, 2, \dots, m+1\}) \cap \{1, 2, \dots, m+1\} = \varnothing\}$. Assume that the law of σ_n is conjugation invariant and $\mathbb{P}(\sigma_n \in E_{n,m}) > 0$. Then for any $b_1, b_2, \dots, b_{m+1}$ distinct elements of $\{1, \dots, n\}$,

$$\mathbb{P}(\sigma_n(1) = b_1, \dots, \sigma_n(m+1) = b_{m+1} | E_{n,m}) = \frac{\mathbb{1}_{\min_i(b_i) > m+1}}{\binom{n-m-1}{m+1}}.$$

Proof. The event $E_{n,m}$ reads as the disjoint union of the events $\{\sigma(1) = b_1, \dots, \sigma(m+1) = b_{m+1}\}$ where $b_1, b_2, \dots, b_{m+1}$ are distinct elements of $\{m+2, m+3, \dots, n\}$. Let $b_1, b_2, \dots, b_{m+1}$ and $c_1, c_2, \dots, c_{m+1}$ verify the previous condition. Let $\hat{\sigma} \in \mathfrak{S}_n$ be a permutation such that for any $1 \leq i \leq m+1, \hat{\sigma}(c_i) = b_i$ and $\hat{\sigma}(j) = j$ if $j \notin (\{b_i\}_{i\leq m+1} \cup \{c_i\}_{i\leq m+1})$. By conjugation invariance, we have

$$\begin{aligned}\mathbb{P}(\sigma_n(1) = b_1, \dots, \sigma_n(m+1) = b_{m+1}) &= \mathbb{P}(\hat{\sigma} \circ \sigma_n \circ \hat{\sigma}^{-1}(1) = b_1, \dots, \hat{\sigma} \circ \sigma_n \circ \hat{\sigma}^{-1}(m+1) = b_{m+1}) \\ &= \mathbb{P}(\sigma_n(1) = c_1, \dots, \sigma_n(m+1) = c_{m+1})\end{aligned}$$

and thus

$$\mathbb{P}(\sigma_n(1) = b_1, \dots, \sigma_n(m+1) = b_{m+1} | E_{n,m}) = \mathbb{P}(\sigma_n(1) = c_1, \dots, \sigma_n(m+1) = c_{m+1} | E_{n,m})$$

and the lemma follows. □

[1] We recall that $\widetilde{\mathcal{L}oc}$ is defined in 2.4

Proof of Theorem 9.1. Under ($\mathcal{H}^{\mathrm{tr},1}_{inv,1}$),

$$\begin{aligned}\mathbb{P}(\sigma_n \in E_{n,m}) &\geq 1 - \sum_{i=1}^{m+1} \mathbb{P}(\sigma_n(i) \leq m+1) \\ &= 1 - (m+1)\left(\mathbb{P}(\sigma_n(1) = 1) + \frac{m(1 - \mathbb{P}(\sigma_n(1) = 1))}{n-1}\right) \xrightarrow[n\to\infty]{} 1.\end{aligned}$$

Similarly, $\mathbb{P}(\sigma_{unif,n} \in E_{n,m}) \xrightarrow[n\to\infty]{} 1$. Therefore, since the law of σ_n is conjugation invariant we can use Lemma 9.2 for n large enough to get

$$\mathbb{P}(A \subset D(\sigma_n)|E_{n,m}) = \mathbb{P}(A \subset D(\sigma_{unif,n})|E_{n,m}).$$

Thus,

$$\lim_{n\to\infty} \left(\mathbb{P}(A \subset D(\sigma_n)) - \mathbb{P}(A \subset D(\sigma_{unif,n}))\right) = 0.$$

Since $\sigma_{unif,n}$ satisfies (DPP) by Theorem 2.21, this concludes the proof. □

Before proving Theorem 8.12, we need to recall that a point process X on a discrete space $\mathfrak{X}$ is fully characterized by its correlation function ρ. Moreover, a point process defined on $\mathbb{N}^*$ is 1-dependent if for all A and B finite subsets of $\mathbb{N}^*$ such that the distance between A and B is larger than 1, $\rho(A \cap B) = \rho(A)\rho(B)$. It is called stationary on $\mathbb{N}^*$ if for all positive integer k, for all finite subset $A \subset \mathbb{N}^*$, $\rho(A) = \rho(A + k)$.

To prove Theorem 8.12, we will use the following result.

Theorem 9.3. Borodin, Diaconis, and Fulman (2010) A stationary 1-dependent simple point process X on $\mathbb{N}^*$ is determinantal with kernel K given by $K(i,j) = k(j-i)$ and

$$\sum_{i\in\mathbb{Z}} k(i) z^i = \frac{-1}{z + \sum_{i\geq 1} a_i z^{i+1}},$$

where $a_i := \mathbb{P}(\{1, 2, \dots, i\} \subset X)$.

Proof of Theorem 8.12. If $x_0 = 1$, the theorem is obvious since $D(\sigma_{Kig,\nu,n}) = \delta_\varnothing$. Next we split the proof into two steps depending on whether $x_0 = 0$ or not.

Step 1 : We assume $x_0 = 0$ so that $\nu(\Sigma_1) = 1$. Using equalities (8.4) and (8.5) we obtain:

$$\begin{aligned}\mathbb{P}(\sigma_{Kig,\nu,n}(1) = 1) = \sum_{\sigma\in\mathfrak{S}_n, \sigma(1)=1} \mathbb{P}(\sigma_{Kig,\nu,n} = \sigma) &= \sum_{\sigma\in\mathfrak{S}_n, \sigma(1)=1} \int_{\Sigma_1} f(n, x, \sigma) d\nu(x) \\ &= \int_{\Sigma_1} \sum_{\sigma\in\mathfrak{S}_n, \sigma(1)=1} f(n, x, \sigma) d\nu(x) \\ &= \int_{\Sigma_1} \mathbb{P}\left(\sigma_{Kig,\delta_x,n}(1) = 1\right) d\nu(x).\end{aligned}$$

Using Beppo Levi theorem, it is thus enough to prove

$$\mathbb{P}\left(\sigma_{Kig,\delta_x,n}(1)=1\right) \xrightarrow[n\to\infty]{} 0.$$

Using the same combinatorial interpretation as in the beginning of Subsection 5.3.2, we have for any $x \in \Sigma_1$,

$$\mathbb{P}\left(\sigma_{Kig,\delta_x,n}(1)=1\right) = \sum_{i\geq 1} \mathbb{P}(\sigma_{Kig,\delta_x,n}(1)=1|pos_1=i)\mathbb{P}(pos_1=i) = \sum_{i\geq 1} x_i(1-x_i)^{n-1}.$$

Let $\varepsilon > 0$. Since $\sum_i x_i = 1$, there exists n_0 such that $(\sum_{i>n_0} x_i) < \frac{\varepsilon}{2}$ and

$$\mathbb{P}\left(\sigma_{Kig,\delta_x,n}(1)=1\right) = \sum_{i\geq 1} x_i(1-x_i)^{n-1} \leq \sum_{i=1}^{n_0} x_i(1-x_i)^{n-1} + \frac{\varepsilon}{2}.$$

As for all $i \leq n_0$, $x_i(1-x_i)^{n-1}$ converges to 0 when n goes to infinity, there exists n_1 such that for $n > n_1$ $\sum_{i=1}^{n_0} x_i(1-x_i)^{n-1} < \frac{\varepsilon}{2}$ and therefore

$$\mathbb{P}(\sigma_{Kig,\delta_x,n}(1)=1) \to 0.$$

Theorem 8.12 follows from Theorem 9.1 when $x_0 = 0$.

Step 2: we now assume that $0 < x_0 < 1$ and $\nu(\Sigma_{1-x_0}) = 1$. We have

$$\mathbb{P}(\sigma_{Kig,\nu,n}(1)=1) = x_0 + \int_\Sigma \sum_{i\geq 1} x_i(1-x_i)^{n-1} d\nu(x) \geq x_0 > 0,$$

which prevents the use of Theorem 9.1. The strategy is instead to use Theorem 9.3, namely to prove that the limiting process is stationary, 1-dependent and its correlation function is such that $\forall k \geq 1$,

$$\rho(\{1,2,\dots,k\}) = \frac{(1-x_0)^{k+1}}{(k+1)!} + \frac{x_0(1-x_0)^k}{k!}.$$

To do so we need to prove this result in the particular case $d\nu_1(x) := dPD(1)(\frac{x}{1-x_0})$ since for any finite subset B,

$$\lim_{n\to\infty} \left(\mathbb{P}(B \subset D\left(\sigma_{Kig,\nu,n}\right)) - \mathbb{P}(B \subset D(\sigma_{Kig,\nu_1}, n))\right) = 0. \tag{9.1}$$

Indeed, let B be a finite subset of $\mathbb{N}^*$ and $B' := B \cup (B+1)$. We use the same interpretation of the random virtual permutations in this case as in the proof of Proposition 5.10. We choose a random subset A_n of $\{1,2,\dots,n\}$ of fixed points where each point belongs to A_n with probability x_0 independently from the others. After that, we permute the elements according to $\mathbb{P}_{n-|A_n|}$, where $(\mathbb{P}_n)_{n\geq 1}$ is the probability distribution on $\mathfrak{S}^\infty$ associated to $\hat{\nu}$ where $d\hat{\nu}(x) = d\nu(\frac{x}{1-x_0})$. Let $C_n := A_n \cap B'$

and

$$E_n := \{\sigma \in \mathfrak{S}_n, \forall i \in B' \setminus C_n,\ \sigma(i) > \max(B')\}.$$

We have

$$\mathbb{P}\left(B \subset D\left(\sigma_{Kig,\nu,n}\right)|E_n\right) = \sum_{X \subset B'} \mathbb{P}\left(B \subset D\left(\sigma_{Kig,\nu,n}\right)|E_n, C_n = X\right)\mathbb{P}(C_n = X).$$

With similar arguments as in the proof of Lemma 9.2, it is not difficult to show that the quantity $\mathbb{P}(B \subset D\left(\sigma_{Kig,\nu,n}\right)|E_n, C_n = X)$ is defined for $n > |B'| + \max(B')$ and does not depend on ν. Moreover, $\mathbb{P}(C_n = X) = x_0^{|X|}(1-x_0)^{|B'|-|X|}$. Thus $\mathbb{P}\left(B \subset D\left(\sigma_{Kig,\nu,n}\right)|E_n\right)$ does not depend on ν. We have

$$\begin{aligned}\mathbb{P}(\sigma_{Kig,\nu,n} \in E_n) &= \sum_{X \subset B'} \mathbb{P}(\sigma_{Kig,\nu,n} \in E_n | C_n = X)\mathbb{P}(C_n = X) \\ &\geq 1 - \sum_{X \subset B'} \sum_{j \in B' \setminus X} \mathbb{P}(\sigma_{Kig,\nu,n}(j) \leq \max(B') | C_n = X)\mathbb{P}(C_n = X).\end{aligned}$$

Moreover, using the notation $p_k := \mathbb{P}(\sigma_{Kig,\hat{\nu},k}(1) = 1)$ and observing that $p_k \to 0$ as $k \to \infty$ thanks to Step 1, we have

$$\begin{aligned}&\mathbb{P}(\sigma_{Kig,\nu,n}(j) \leq \max(B')|C_n = X) \\ =& \sum_{k=0}^{n-|B'|} \mathbb{P}(\sigma_{Kig,\nu,n}(j) \leq \max(B')|C_n = X, |A_n| = |X| + n - |B'| - k) \\ &\mathbb{P}(|A_n| = |X| + n - |B'| - k|C_n = X) \\ =& \sum_{k=0}^{n-|B'|} x_0^{n-|B'|-k}(1-x_0)^k \binom{n-|B'|}{k} \\ &\mathbb{P}(\sigma_{Kig,\nu,n}(j) \leq \max(B')|C_n = X, |A_n| = |X| + n - |B'| - k) \\ \leq& x_0^{n-|B'|} + x_0^{n-|B'|-1}(1-x_0)(n-|B'|) \\ &+ \sum_{k=2}^{n-|B'|} x_0^{n-|B'|-k}(1-x_0)^k \binom{n-|B'|}{k} \left(p_{k+|B'|-|X|} + \frac{\max(B')}{|B'|-|X|+k-1}\right) \\ &\xrightarrow[n\to\infty]{} 0.\end{aligned}$$

This yields

$$\lim_{n\to\infty} \mathbb{P}(\sigma_{Kig,\nu,n} \in E_n) = 1$$

and therefore the convergence (9.1) is proven.

We compute now

$$\lim_{n\to\infty} \mathbb{P}(B \subset D(\sigma_n^{\nu_1})).$$

The finite subset B can be decomposed as $B = \bigcup_{i=1}^{l} B_i$ where each B_i consists in consecutive elements of $\mathbb{N}^*$ and the distance between B_i and B_j is larger than one if $i \neq j$. For example,

$$B = \{1, 2, 3, 5, 6, 8, 11, 12\} = \{1, 2, 3\} \cup \{5, 6\} \cup \{8\} \cup \{11, 12\}.$$

Note that every finite subset has a such decomposition. Let $B'_i := B_i \cup (B_i + 1)$. We have $B' := B \cup (B+1) = \bigcup_{i=1}^{l} B'_i$ and if $i \neq j$, then $B'_i \cap B'_j = \varnothing$. From now we assume that $n > |B'| + \max(B')$. We have

$$\mathbb{P}(B \subset D(\sigma_{Kig,\nu_1,n})|E_n) = \sum_{X \subset B'} \mathbb{P}(B \subset D(\sigma_{Kig,\nu_1,n})|C_n = X, E_n)\mathbb{P}(C_n = X). \tag{9.2}$$

If $B \cap X \neq \varnothing$, then $\mathbb{P}(B \subset D(\sigma_{Kig,\nu_1,n})|C_n = X, E_n) = 0$. Indeed, conditionally on E_n, if $i \in B \cap X$, then $\sigma_{Kig,\nu_1,n}(i) = i$ and $\sigma_{Kig,\nu_1,n}(i+1)$ is either equal to $i+1$ or larger than $\max(B')$ and in both cases, there is no descent on i. Consequently, (9.2) becomes

$$\begin{aligned}
\mathbb{P}(B \subset D(\sigma_{Kig,\nu_1,n})|E_n) &= \sum_{X \subset B' \setminus B} \mathbb{P}(B \subset D(\sigma_{Kig,\nu_1,n})|C_n = X, E_n)\mathbb{P}(C_n = X) \\
&= \sum_{U \subset \{1,2,\dots,l\}} \mathbb{P}\left(B \subset D(\sigma_{Kig,\nu_1,n}) \middle| C_n = \bigcup_{i \in U} (B'_i \setminus B_i), E_n\right) \\
&\quad \times \mathbb{P}\left(C_n = \bigcup_{i \in U} (B'_i \setminus B_i), E_n\right).
\end{aligned}$$

The second equality comes from the fact that $B'_i \setminus B_i$ contains exactly one element. We denote by $U^c := \{1, 2, \dots, l\} \setminus U$ and by $W(U) := \bigcup(\bigcup_{i \in U} B_i \bigcup_{i \in U^c} B'_i)$. We have

$$\mathbb{P}\left(B \subset D(\sigma_{Kig,\nu_1,n}) \middle| C_n = \bigcup_{i \in U} (B'_i \setminus B_i), E_n\right) = \frac{|\mathfrak{E}_2|}{|\mathfrak{E}_1|},$$

where $$\mathfrak{E}_1 = \left\{(e_k)_{k \in W(U)}, \forall k \in W(U),\ \max(B') < e_k \leq n, i \neq j \Rightarrow e_i \neq e_j\right\}$$

and $$\mathfrak{E}_2 := \left\{(e_k)_{k \in W(U)} \in \mathfrak{E}_1, \forall k \in \bigcup_{i=1}^{l} B_i \setminus \bigcup_{i \in U} \{\max(B_i)\},\ e_{k+1} < e_k\right\}.$$

Therefore,

$$|\mathfrak{E}_1| := \frac{(n - \max(B'))!}{(n - \max(B)' - |W(U)|)!}$$

and

$$|\mathfrak{E}_2| = \frac{(n - \max(B'))!}{(n - \max(B') - \sum_{i\in U}|B_i|)! \prod_{i\in U}|B_i|!} \\ \times \frac{(n - \max(B') - \sum_{i\in U}|B_i|)!}{(n - \max(B') - \sum_{i\in U}|B_i| - \sum_{i\in U^c}|B_i'|)! \prod_{i\in U^c}|B_i'|!} \\ = \frac{(n - \max(B'))!}{(n - \max(B') - |W(U)|)! \prod_{i\in U}|B_i|! \prod_{i\in U^c}|B_i'|!}.$$

As a consequence,

$$\mathbb{P}\left(B \subset D(\sigma_{Kig,\nu_1,n}) \middle| C_n = \bigcup_{i\in U}(B_i' \setminus B_i), E_n\right) = \frac{|\mathfrak{E}_2|}{|\mathfrak{E}_1|} = \frac{1}{\prod_{i\in U}|B_i|! \prod_{i\in U^c}|B_i'|!}.$$

Then

$$\mathbb{P}(B \subset D(\sigma_{Kig,\nu_1,n})|E_n) = \sum_{U\subset\{1,2,\dots,l\}} \frac{x_0^{|U|}(1-x_0)^{|B|+l-|U|}}{\prod_{i\in U}|B_i|! \prod_{i\in U^c}|B_i'|!} \\ = \prod_{i=1}^{l} \frac{(1-x_0)^{|B_i|}}{|B_i|!}\left(x_0 + \frac{1-x_0}{|B_i|+1}\right) \\ = \prod_{i=1}^{l} \hat{a}_{|B_i|}(x_0),$$

where we recall that

$$\hat{a}_k(x_0) = \frac{(1-x_0)^{k+1}}{(k+1)!} + \frac{x_0(1-x_0)^k}{k!}.$$

This implies that the limiting process is stationary and 1-dependent. Consequently, by Theorem 9.3 it is determinantal and the kernel satisfies (8.7). □

Corollary 8.13 is at the same time a generalization and a direct application of Theorem 8.12.

Proof of Corollary 8.13. We denote by $f(n, x, \sigma) := \mathbb{P}(\sigma_{Kig,\delta_x,n} = \sigma)$ (see (8.6)), by $\rho(n, x, .)$ the correlation function of the descent process of $\sigma_{Kig,\delta_x,n}$ and by $\rho_{lim}(x_0, .)$ the correlation function of the determinantal process with kernel $K_{x_0}(i,j) := k_{x_0}(j-i)$. Let A be a finite subset of $\mathbb{N}^*$. We have

$$\mathbb{P}(A \subset D(\sigma_{Kig,\nu,n})) = \sum_{\sigma\in\mathfrak{S}_n, A\subset D(\sigma)} \mathbb{P}(\sigma_{Kig,\nu,n} = \sigma) \\ = \sum_{\sigma\in\mathfrak{S}_n, A\subset D(\sigma)} \int_\Sigma f(n,x,\sigma)d\nu(x) = \int_\Sigma \sum_{\sigma\in\mathfrak{S}_n, A\subset D(\sigma)} f(n,x,\sigma)d\nu(x) \\ = \int_\Sigma \rho(n,x,A)d\nu(x).$$

Using the convergence of $\rho(n, x, A)$ to $\rho_{lim}(1 - \sum_{i\geq1} x_i, A)$ and the dominated convergence theorem, we obtain:

$$\mathbb{P}(A \subset D(\sigma_{Kig,\nu,n})) \xrightarrow[n\to\infty]{} \int_{\Sigma} \rho_{lim}\left(1 - \sum_{i\geq 1} x_i, A\right) d\nu(x).$$

□

Using this corollary, we can now prove Proposition 8.14.

Lemma 9.4. For any random permutation σ_n stable under conjugation, $\mathbb{P}(i \in D(\sigma_n))$ does not depend on i.

Proof. Let $1 \leq i < n$. We have

$$\begin{aligned}
\mathbb{P}(i \in D(\sigma_n)) &= \mathbb{P}(i \in D(\sigma_n)|\sigma_n(i) = i, \sigma_n(i+1) = i+1)\mathbb{P}(\sigma_n(i) = i, \sigma_n(i+1) = i+1) \\
&+ \mathbb{P}(i \in D(\sigma_n)|\sigma_n(i) = i, \sigma_n(i+1) \neq i+1)\mathbb{P}(\sigma_n(i) = i, \sigma_n(i+1) \neq i+1) \\
&+ \mathbb{P}(i \in D(\sigma_n)|\sigma_n(i) \neq i, \sigma_n(i+1) = i+1)\mathbb{P}(\sigma_n(i) \neq i, \sigma_n(i+1) = i+1) \\
&+ \mathbb{P}(i \in D(\sigma_n)|\sigma_n(i) \notin \{i, i+1\}, \sigma_n(i+1) \notin \{i, i+1\}) \\
&\times \mathbb{P}(\sigma_n(i) \notin \{i, i+1\}, \sigma_n(i+1) \notin \{i, i+1\}) \\
&+ \mathbb{P}(i \in D(\sigma_n)|\sigma_n(i) = i+1, \sigma_n(i+1) \neq i) \\
&\times \mathbb{P}(\sigma_n(i) = i+1, \sigma_n(i+1) \neq i) \\
&+ \mathbb{P}(i \in D(\sigma_n)|\sigma_n(i) \neq i+1, \sigma_n(i+1) = i) \\
&\times \mathbb{P}(\sigma_n(i) \neq i+1, \sigma_n(i+1) = i) \\
&+ \mathbb{P}(i \in D(\sigma_n)|\sigma_n(i) = i+1, \sigma_n(i+1) = i)\mathbb{P}(\sigma_n(i) = i+1, \sigma_n(i+1) = i).
\end{aligned}$$

Using the conjugation invariance, we obtain,

$$\begin{aligned}
&\mathbb{P}(i \in D(\sigma_n)|\sigma_n(i) = i, \sigma_n(i+1) = i+1) = 0 \\
&\mathbb{P}(i \in D(\sigma_n)|\sigma_n(i) = i, \sigma_n(i+1) \neq i+1) = \frac{i-1}{n-2} \\
&\mathbb{P}(i \in D(\sigma_n)|\sigma_n(i) \neq i, \sigma_n(i+1) = i+1) = \frac{n-i-1}{n-2} \\
&\mathbb{P}(i \in D(\sigma_n)|\sigma_n(i) \notin \{i, i+1\}, \sigma_n(i+1) \notin \{i, i+1\}) = \frac{1}{2} \\
&\mathbb{P}(i \in D(\sigma_n)|\sigma_n(i) = i+1, \sigma_n(i+1) \neq i) = \frac{i-1}{n-2} \\
&\mathbb{P}(i \in D(\sigma_n)|\sigma_n(i) \neq i+1, \sigma_n(i+1) = i) = \frac{n-i-1}{n-2} \\
&\mathbb{P}(i \in D(\sigma_n)|\sigma_n(i) = i+1, \sigma_n(i+1) = i) = 1.
\end{aligned}$$

We have then, using again the conjugation invariance,

$$\begin{aligned}\mathbb{P}(i \in D(\sigma_n)) &= \mathbb{P}(\sigma_n(1) = 1, \sigma_n(2) \neq 2) \\ &+ \mathbb{P}(\sigma_n(1) = 2, \sigma_n(2) = 1) \\ &+ \mathbb{P}(\sigma_n(1) \neq 2, \sigma_n(2) = 1) \\ &+ \frac{1}{2}\mathbb{P}(\sigma_n(1) \notin \{1,2\}, \sigma_n(2) \notin \{1,2\})\end{aligned}$$

and the lemma follows. □

Proof of Proposition 8.14. Let ν be a probability measure on Σ. By Lemma 9.4 and by using (8.8) and (8.9) for $A = \{1\}$, we obtain

$$\begin{aligned}\frac{\mathbb{E}(\mathcal{N}_D(\sigma_{Kig,\nu,n}))}{n} = \frac{n-1}{n}\mathbb{P}\left(1 \in D(\sigma_{Kig,\nu,n})\right) &\to \int_\Sigma \hat{a}_1 \left(1 - \sum_{i\geq 1} x_i\right) d\nu(x) \\ &= \frac{1}{2}\left(1 - \int_\Sigma \left(1 - \sum_i x_i\right)^2 d\nu(x)\right).\end{aligned}$$

□

9.2 Optimal transport formulation

9.2.1 General Case

Given a positive integer n, we denote by $M_{n!}(\mathbb{R}_+)$ the set of square matrices of size $n!$ indexed by the symmetric group $\mathfrak{S}_n$ and having values on the set of non-negative real numbers. For $f : \sigma_n \to \mathbb{R}_+$ Let $C_n(f) \in M_{n!}(\mathbb{R}_+)$ be the cost matrix defined by the formula

$$C_n(f) = [|f(\sigma) - f(\rho)|]_{\sigma,\rho\in\mathfrak{S}_n}.$$

Given $\pi_1 = (\pi_{1,\sigma})_{\sigma\in\mathfrak{S}_n}$ and $\pi_2 = (\pi_{2,\sigma})_{\sigma\in\mathfrak{S}_n}$, we denote by

$$C_{opt,n}(f, \pi_1, \pi_2) := \min_{P\text{stochastic };\pi_1 P = \pi_2} ||\pi_1(P \odot C_n(f))||_1 = \min_{P\text{stochastic };\pi_1 P = \pi_2} \sum_{\sigma,\rho} \pi_{1,\sigma}, P_{\sigma,\rho} C_n(f)_{\sigma,\rho}.$$

Here, $\odot$ is the Hadamard product for matrices. It is not difficult to see that if

$$\lim_{n\to\infty} C_{opt,n}(f, (\mathbb{P}(\sigma_n = \sigma))_{\sigma\in\mathfrak{S}_n}, (\mathbb{P}(\sigma_{ref,n} = \sigma))_{\sigma\in\mathfrak{S}_n}) = 0,$$

then the exists $\rho_n \overset{d}{=} \sigma_{ref,n}$ such that

$$\mathbb{E}(|f(\sigma_n) - f(\rho_n)|) \xrightarrow[n\to\infty]{} 0.$$

In particular, if
$$f(\sigma_{ref,n}) \xrightarrow[n\to\infty]{ConvType} X,$$
then there exists $Y \overset{d}{=} X$ such that
$$f(\sigma_n) \xrightarrow[n\to\infty]{ConvType} Y,$$
where $ConvType \in \{d, \mathbb{P}, \mathbb{L}^1\}$.

9.2.2 Back to the conjugation invariant case

If σ_n is conjugation invariant then

$$\begin{aligned} C_{opt,n}(f, (\mathbb{P}(\sigma_n = \sigma))_{\sigma\in\mathfrak{S}_n}, (\mathbb{P}(\sigma_{Ew,0,n} = \sigma))_{\sigma\in\mathfrak{S}_n}) &\le \mathbb{E}(|f(T(\sigma_n)) - f(\sigma_n)|) \\ &\le \mathbb{E}\left(\varepsilon_{n,\#(\sigma_n)}(f)\right) \end{aligned}$$

and then we obtain a weaker version of our universality results. But in general, this upper bound is much larger than the optimal cost. For example, for the longest increasing subsequence, the typical entries of the matrix $C_n(LIS)$ are of order $\sqrt[6]{n}$ but this upper bound is equivalent to study the case where the entries are equal to $2(n - \#(\sigma \circ \rho^{-1}))$ which is typically of order $n - \log(n)$.

9.3 Cycle structure

9.3.1 General discussion

We make a few remarks on the optimality of the assumptions H_3 and H_4 in Theorem 1.13. We assume hereafter that H_1 and H_2 hold true and consider for the sake of clarity the case $m = 2$.

- The assumption H_3 is optimal in the sense that if
$$\liminf_{n\to\infty} n^{-\frac{k}{2}} \min(\mathbb{E}((\#_1 \sigma_n)^k), \mathbb{E}((\#_1 \rho_n)^k)) = \varepsilon_k > 0,$$
then
$$\liminf_{n\to\infty} \mathbb{E}((\#_1(\sigma_n\rho_n))^k) \ge \mathbb{E}(\xi_1^k) + \varepsilon_k^2.$$
Indeed, going back to the equation (*), one can see that in the case $v_1 = v_2 = \cdots = v_k = 1$, if $\hat{g}$ is the class of the graph with adjacency matrix Id_k the event
$$\{(\hat{\mathcal{G}}_1^{1,2,\ldots,k}(\sigma_n, \rho_n), \hat{\mathcal{G}}_2^{1,2,\ldots,k}(\sigma_n, \rho_n)) = (\hat{g}, \hat{g})\}$$
will contribute to the limit, leading to the term ε_k^2.

- Similarly H_4 is optimal in the sense that

$$\text{if } \liminf_{n\to\infty}\left(\frac{\min(\mathbb{E}(\#_2\,\sigma_n),\mathbb{E}(\#_2\,\rho_n))}{n}\right)=\varepsilon'>0,\ \text{then } \liminf_{n\to\infty}\mathbb{E}\left((\#_1(\sigma_n\rho_n))^2\right)\geq 2+\varepsilon'^2.$$

  Indeed, as above, in the case $v_1=v_2=1$, if $\hat{g}'$ is the class of the graph with adjacency matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, the event $\{(\hat{\mathcal{G}}_1^{1,2,\dots,k}(\sigma_n,\rho_n),\hat{\mathcal{G}}_2^{1,2,\dots,k}(\sigma_n,\rho_n))=(\hat{g}',\hat{g}')\}$ will contribute to the limit.

- Assume now that one of the bounds in H_3 is not satisfied. More precisely, assume that there exists $k\geq 1$ such that

$$\liminf_{n\to\infty} n^{-\frac{k}{2}}\mathbb{E}((\#_1\sigma_n)^k)=\varepsilon_k>0,\ \text{or}\ \liminf_{n\to\infty}\frac{\mathbb{E}(\#_2\,\sigma_n)}{n}=\varepsilon'>0.$$

  Then, by similar arguments, one can check that the convergences

$$\forall k\geq 1,\ \lim_{n\to\infty} n^{-\frac{k}{2}}\mathbb{E}((\#_1\,\rho_n)^k)=0 \quad \text{and} \quad \lim_{n\to\infty}\frac{\mathbb{E}(\#_2\,\rho_n)}{n}=0$$

  are a necessary condition to obtain (1.10) and that the convergences

$$\forall k\geq 1,\ \lim_{n\to\infty} n^{-\frac{k}{2}}\mathbb{E}((\#_1\,\rho_n)^k)=0,\ \limsup_{n\to\infty} n^{-\frac{k}{2}}\mathbb{E}((\#_1\,\sigma_n)^k)<\infty$$

  and $\lim_{n\to\infty}\frac{\mathbb{E}(\#_2\,\rho_n)}{n}=0$ are a sufficient condition to obtain (1.10).

One can see also that the condition H_1 can be replaced by the following condition.
$\hat{H}_1$: For any $k_1,k_2\geq 1$, for any $\varepsilon>0$, there exists n_0 such that for any $n>n_0$, for any $g_1\in\mathbb{G}^n_{k_1}$, $g_2\in\mathbb{G}^n_{k_2}$,

$$\begin{aligned}(1-\varepsilon)\mathbb{P}(\sigma_n\in\mathfrak{S}_{n,g_1})\mathbb{P}(\rho_n\in\mathfrak{S}_{n,g_2}) &\leq \mathbb{P}(\sigma_n\in\mathfrak{S}_{n,g_1},\rho_n\in\mathfrak{S}_{n,g_2})\\ &\leq (1+\varepsilon)\mathbb{P}(\sigma_n\in\mathfrak{S}_{n,g_1})\mathbb{P}(\rho_n\in\mathfrak{S}_{n,g_2}).\end{aligned}$$

When both permutations are conjugation invariant, we don't need a uniform bound.

9.3.2 ω-random non-uniform permutations

A possible extension to our work is to study ω-random permutation. One can see (Puder, 2014; Puder and Parzanchevski, 2015) for rigorous definitions and results in the uniform case . Let F_k be the free group of rank k and let $x_1,x_2,\dots,x_k$ be a basis of F_k.

Definition 9.5. $\omega\in F_k$ is primitive if there exist $y_2,y_3,\dots,y_k\in F_k$ such that

$$<\omega,y_2,y_3,\dots,y_k>=F_k.$$

Here, $<y_1,y_2,\dots,y_k>$ is the smallest group in the sense of inclusion containing $\{y_i,1\leq i\leq k\}$.

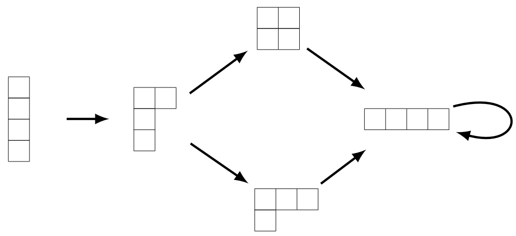


Figure 9.1: The directed classes graph for $\mathfrak{S}_4$

Definition 9.6. Given $\omega \in F_k$, let $\pi(\omega) \in \mathbb{N} \cup \{+\infty\}$ be the minimal rank of a subgroup of F_k containing ω as a non primitive word.

The function π plays a central role to understand the expectation of fixed points of an ω-random uniform permutations. Given a word $\omega \in F_m =< x_1, x_2 \dots, x_m >$ and m random permutations $\sigma_{1,n}, \sigma_{2,n}, \dots, \sigma_{m,n}$ on $\mathfrak{S}_n$, We define σ_n as the random permutation obtained by replacing x_i by $\sigma_{i,n}$ in ω. For example, if $w = x_1^2 x_3^{-2} x_2^3$ then $\sigma_n = \sigma_{1,n}^2 \sigma_{3,n}^{-2} \sigma_{2,n}^3$.

Theorem 9.7. (Puder, 2014; Puder and Parzanchevski, 2015) For any $\omega \in F_k$, if $\sigma_{1,n}, \sigma_{2,n}, \dots, \sigma_{m,n}$ are i.i.d. uniform permutations than

$$\mathbb{E}_\omega(\mathrm{tr}(\sigma_n)) = 1 + \Theta\left(n^{1-\pi(\omega)}\right).$$

We conjecture the following

Conjecture 9.8. Let $\omega \in F_m$ be a non primitive word. Assume that

(H_1) For any $n \geq 1$, $(\sigma_{1,n}, \dots, \sigma_{m,n})$ are independent.

(H_2) For any $n \geq 1$ and $1 \leq \ell \leq m$, $\sigma_{\ell,n}$ is conjugation invariant except maybe for one $\ell \in \{1, \dots, m\}$.

- There exists $1 \leq i \leq m$ such that for any $k \geq 1$,

$$(H_3) \qquad \sigma_{i,n} \text{ satisfies } (\mathcal{H}_{inv,2}^{\mathrm{tr},k}),$$
$$(H_4) \qquad \forall \ell \leq \pi(\omega), \sigma_{i,n}^{\ell} \text{ satisfies } (\mathcal{H}_{inv,1}^{\mathrm{tr},1}).$$

We have then

$$\mathbb{E}_\omega(\mathrm{tr}(\sigma)) = 1 + \Theta\left(n^{1-\pi(\omega)}\right).$$

9.4 A different walk

When the graph satisfies the hypothesis presented in Chapter 4, the walk can be seen as a walk on the a directed version of the classes graph. Let denote it by $\tilde{G}_n$. For example, for $\mathfrak{S}_4$ we obtain

Figure 9.1. Any walk on this graph will arrive to i_n^*. And from any random (or deterministic) walk on $\tilde{G}_n$ and any initial distribution constant on the classes of G_n we can define a random walk that arrives to the uniform distribution on $V_n^{i_n^*}$ after a finite time and by staying G_n-invariant on every step. It is then possible to choose the deterministic walk that minimize the error in every step. For sake of simplicity, we will give only an example for to the symmetric group and the convergence in $\mathbb{L}^1$. The other groups and the other types of convergence can be studied in a similar way. Let f be a statistic as in Chapter 4 and let $\widetilde{\varepsilon_{n,i,i'}}(f) := \max_{\sigma \in V_n^i} \max_{\rho \in V_n^{i'} \cap \text{next}(\sigma)} d_F(f(\sigma), f(\rho))$ (with the convention $\max(\varnothing) = \infty$).

Let $G'_{n,f} = (V_{G'_{n,f}}, E_{G'_{n,f}})$ the graph such that $V_{G'_{n,f}} = V_{G_n}$ and

$$E_{G'_{n,f}} = E_{G_{n,f}} \cap (\cup_{i \in I_n} \{(\sigma, \rho); \sigma \in V_n^i, \rho \in V^{argmin(i' \to \widetilde{\varepsilon_{n,i,i'}})}(f)\}).$$

One can consider the uniform random walk on that graph. In this case one can replace $\varepsilon_{n,k}$ by the new control.

$$\widetilde{\varepsilon_{n,i}}(f) := \min_{i' \in \text{next}(i)} \widetilde{\varepsilon_{n,i,i'}}(f) \tag{9.3}$$

For the example of $\mathfrak{S}_4$, depending on the statistic f the graph of classes will be one of those of Figure 9.2.

9.5 Colored permutations

A possible improvement of Theorem 4.5 is the following.

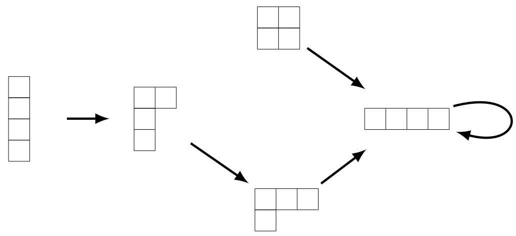
Proposition 9.9. Let $(\pi_n = (\sigma_n, f_n))_{n \geq 1}$ be a sequence of random colored permutations and assume that:

- σ_n is independent of f_n.
- For all $1 \leq p \leq m$,
$$\frac{\mathbb{E}(\text{card}\{i, f_n(i) = j\})}{n} \to \gamma_j$$
and
$$\max_j \mathbb{V}\text{ar}(\text{card}(\{i, f_n(i) = j\})) = O(n).$$
- σ_n is conjugation invariant.
- $\frac{\#\sigma_n}{n^{\frac{1}{6}}} \xrightarrow{\mathbb{P}} 0,$

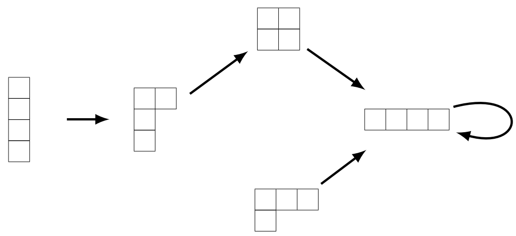
then

$$\mathbb{P}\left(\frac{\text{LIS}(\pi_n) - 2m\sqrt{\hat{\gamma}n}}{m\sqrt[6]{n\hat{\gamma}}} < s\right) \to F_2^{\tilde{\gamma}}(s), \tag{9.4}$$

If $\widetilde{\varepsilon}_{n,(2,1,1,\underline{0}),(3,1,\underline{0})}(f) < \widetilde{\varepsilon}_{n,(2,1,1,\underline{0}),(2,2,\underline{0})}(f)$, we obtain



If $\widetilde{\varepsilon}_{n,(2,1,1,\underline{0}),(3,1,\underline{0})}(f) > \widetilde{\varepsilon}_{n,(2,1,1,\underline{0}),(2,2,\underline{0})}(f)$, we obtain



If $\widetilde{\varepsilon}_{n,(2,1,1,\underline{0}),(3,1,\underline{0})}(f) = \widetilde{\varepsilon}_{n,(2,1,1,\underline{0}),(2,2,\underline{0})}(f)$, we obtain

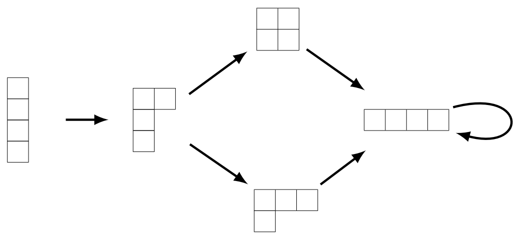


Figure 9.2: The possible new graph classes of $\mathfrak{S}_4$

where $\hat{\gamma} = \max_i\{m_i, 1 \leq i, \leq m\}$ and $\tilde{\gamma} = \mathrm{card}(\{i, \gamma_i = \hat{\gamma}\})$.

The idea of the proof is that as soon as $\max_j \mathbb{V}\mathrm{ar}(\mathrm{card}(\{i, f_n(i) = j\})) = O(n)$, one can prove that the error between considering the real random proportions and considering fixed proportions is $o(n^{\frac{1}{6}})$ in probability. When $\mathbb{V}\mathrm{ar}(\mathrm{card}(\{i, f_n(i) = j\})) = 0$, and with high probability, only colors i such that $\gamma_i = \hat{\gamma}$ can maximize the longest increasing subsequence and the fluctuations of LIS behave asymptotically as the maximum of Tracy-Widom distributions.

$f(\sigma)$	X	Hypotheses	Theorem
$\frac{\mathrm{LIS}(\sigma)}{\sqrt{n}}, \frac{\mathrm{LDS}(\sigma)}{\sqrt{n}}$	2	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$ $(\mathcal{H}^{L^p}_{inv,2})$	Theorem 1.4
$\frac{\mathrm{LISC}(\sigma)}{\sqrt{n}}, \frac{\mathrm{LDSC}(\sigma)}{\sqrt{n}}$	2	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$	Corollary 2.9
$\frac{\mathrm{LIS}(\sigma)-2\sqrt{n}}{n^{\frac{1}{6}}}, \frac{\mathrm{LDS}(\sigma)-2\sqrt{n}}{n^{\frac{1}{6}}}$	Tracy-Widom	(1.2)	Theorem 1.5
$\frac{\lambda_i(\sigma)}{\sqrt{n}}$	2	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$ $(\mathcal{H}^{L^p}_{inv,2})$	Proposition 5.9
$\left(\frac{\lambda_i(\sigma)-2\sqrt{n}}{n^{\frac{1}{6}}}\right)_{1\leq i\leq d}$	Airy ensemble	(1.2)	Theorem 5.7
$s \to \frac{L_{\lambda(\sigma)}(s\sqrt{2n})}{\sqrt{2n}}$	Ω	$(\mathcal{H}^{\mathbb{P}}_{inv,1})$	Theorem 7.4
$\frac{\mathrm{LCS}(\sigma,\rho)}{\sqrt{n}}$	2	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$	Proposition 1.11
$\frac{\mathrm{LCS}(\sigma,\rho)-\sqrt{n}}{n^{\frac{1}{6}}}$	Tracy-Widom	(1.2)	Theorem 1.5
$\frac{\mathcal{K}_j(\sigma)}{n^j}$	$\frac{1}{j!^2}$	$(\mathcal{H}^{\mathbb{P}}_{inv,1})$	Corollary 2.19
$\frac{\mathcal{K}_j(\sigma)-\frac{n^j}{(j!)^2}}{\sqrt{n}}$	$\mathcal{N}\left(0, \frac{\binom{4j-2}{2j-1}-2\binom{2j-1}{j}^2}{2((2m-1)!)^2}\right)$	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$	Corollary 2.19
$\frac{\mathcal{N}_{exc}(\sigma)}{n}$	$\frac{1}{2}$	$(\mathcal{H}^{\mathbb{P}}_{inv,1})$	Corollary 2.19
$\frac{\mathcal{N}_{exc}(\sigma)-\frac{n}{2}}{\sqrt{n}}$	$\mathcal{N}(0, \frac{1}{12})$	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$	Corollary 2.19
$\mathbb{1}_{D(\sigma)\subset A}$	$Ber(\det([k_0(j-i)]_A))$	$(\mathcal{H}^{\mathrm{tr},1}_{inv,1})$	Theorem 9.1
$\frac{\mathcal{N}_D(\sigma)}{n}$	$\frac{1}{2}$	$(\mathcal{H}^{\mathbb{P}}_{inv,1})$	Corollary 2.19
$\frac{\mathcal{N}_D(\sigma)-\frac{n}{2}}{\sqrt{n}}$	$\mathcal{N}(0, \frac{1}{12})$	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$	Corollary 2.19
$\frac{\mathcal{N}_{peak(\sigma)}}{n}$	$\frac{1}{3}$	$(\mathcal{H}^{\mathbb{P}}_{inv,1})$	Corollary 2.19
$\frac{\mathcal{N}_{peak}(\sigma)-\frac{n}{2}}{\sqrt{n}}$	$\mathcal{N}(0, \frac{2}{45})$	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$	Corollary 2.19
$\frac{\mathrm{LAS}(\sigma)}{n}$	$\frac{2}{3}$	$(\mathcal{H}^{\mathbb{P}}_{inv,1})$	Corollary 2.15
$\frac{\mathrm{LAS}(\sigma)-\frac{2n}{3}}{\sqrt{n}}$	$\mathcal{N}(0, \frac{8}{45})$	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$	Corollary 2.15

Table 9.1: Recap of universality results

$f(\sigma)$	X	Hypotheses
$\frac{\mathrm{LIS}(\sigma)}{\sqrt{n}}, \frac{\mathrm{LDS}(\sigma)}{\sqrt{n}}$	2	$(\mathcal{H}^{\mathrm{tr},1}_{inv,2})$
$\frac{\mathrm{LIS}(\sigma)-2\sqrt{n}}{n^{\frac{1}{6}}}, \frac{\mathrm{LDS}(\sigma)-2\sqrt{n}}{n^{\frac{1}{6}}}$	Tracy-Widom	$(\mathcal{H}^{\mathbb{P}}_{inv,2})$
$s \to \frac{L_{\lambda(\sigma)}(s\sqrt{2n})}{\sqrt{2n}}$	Ω	$(\mathcal{H}^{\mathrm{tr},1}_{inv,1})$

Table 9.2: Our conjectures.

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