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Autour des équations stochastiques fractionnaires : Variations et estimation

Présentée par

Zeina MAHDI-KHALIL

Sous la direction de

Ciprian TUDOR

Soutenu le 16 Juin 2020 devant le jury d'examen composé de

M. Jean-Christophe BRETON,	Rapporteur	Université de Rennes 1
M. Serguei DACHIAN,	Président du Jury	Université de Lille
Mme Marina KLEPTSYNA,	Examinateur	Université du Mans
M. Paul RAYNAUD DE FITTE,	Rapporteur	Université de Rouen
M. Ciprian TUDOR,	Directeur de thèse	Université de Lille

À

mon père et ma mère,

Hassan, Ahmad et Jawad,

mon frère Hssein,

ma soeur Amira

et aussi à mes soeurs,

modeste témoignage de mon infinie amour et tendresse...

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Soon, when all is well, you're going to look back on this period of your life and be so glad that you never gave up. [1]

1. Brittany Burgunder.

Autour des équations stochastiques fractionnaires : Variations et estimation

Résumé :

Cette thèse est consacrée à l'étude de certaines classes d'équations aux dérivées partielles stochastiques de type fractionnaire dirigées par un bruit gaussien additif. Le caractère fractionnaire de ces équations est donné soit par l'opérateur différentiel qui intervient (le laplacien fractionnaire) ou bien par le bruit aléatoire. La perturbation aléatoire qui dirige ces équations peut avoir une corrélation en temps ou en espace.

Dans un premier temps, on analyse l'équation de la chaleur stochastique avec un opérateur différentiel donné par le laplacien fractionnaire d'ordre $\alpha \in (1,2)$. Le bruit aléatoire qui intervient dans cette équation est additif et il se comporte comme un processus de Wiener par rapport à la variable temporelle et comme un bruit blanc ou coloré par rapport à la variable spatiale. Nous obtenons des résultats concernant l'existence de la solution, la régularité de ses trajectoires ainsi que sa loi de probabilité. Nous remarquons un lien étroit entre la solution de l'équation fractionnaire de la chaleur et certains processus stochastiques fractionnaires (mouvement brownien fractionnaire ou bifractionnaire). En utilisant ce lien, nous étudions le comportement asymptotique des variations généralisées de la solution, en temps et en espace. Nous proposons également, dans la situation où l'équation initiale dépend d'un paramètre de dérive (ou de drift), des estimateurs pour ce paramètre. Les estimateurs s'expriment en fonction des variations généralisées de la solution et nous utilisons les comportements de celles-ci pour obtenir les propriétés asymptotiques (consistance, normalité asymptotique) de nos estimateurs.
Dans un deuxième temps, on analyse l'équation stochastique des ondes sur un intervalle fini en espace. Ici le caractère fractionnaire est donné par le bruit gaussien qui se comporte comme un mouvement brownien fractionnaire avec un indice de Hurst $H \in \left(\frac{1}{2}, 1\right)$ par rapport à la variable temporelle et comme un mouvement brownien standard en espace. Notre analyse est basée sur l'écriture différente sous la forme d'une série trigonométrique du noyau associé à l'équation des ondes. Des différentes propriétés de la solution sont ainsi obtenues, parmi lesquelles l'existence, la continuité höldérienne de ses trajectoires, la propriété dite de scaling et le comportement par rapport à l'indice de Hurst.

Mots-clefs : équation stochastique fractionnaire, auto-similarité, variation généralisée, estimation du paramètre, équation d'onde.

On fractional stochastic heat equation : Variation and estimation

Abstract : This doctoral thesis is devoted to the study of the fractional stochastic heat equation driven by additive Gaussian noises. The word "fractional" concerns the appearance of the fractional Laplacian operator or it refers to the driven fractional noise. The Gaussian random may have a non trivial correlation in time and/or in space.

First, we analyze the stochastic differential heat equation with a fractional Laplacian operator with exponent $\alpha \in (1, 2)$. The random noise is considered to be white in time and white or colored with respect to the space variable. We obtain several results concerning the existence of the solution, the regularity of its paths and its law. We noticed a link between the solution of fractional heat equation and some fractional stochastic processes (Fractional Brownian motion or bi-Fractional Brownian motion). Using this link, we study the asymptotic behavior of the generalized variations of the solution, in time and in space. We also propose, in the situation where the initial equation depends on a drift parameter, estimators for this parameter. The estimators are expressed as a function of the generalized variations of the mild solution. We use the behavior of these variations to prove some asymptotic properties (the consistency, asymptotic normality) of our estimators.

In a second part, we analyze the wave stochastic equation on a finite interval in space. In this case, the character "fractional" is given by the Gaussian noise which behaves in time as a Fractional Brownian motion with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$ with respect to the variable of time and as a standard Brownian motion in space. Our analysis is based on the expression of the Green kernel associated to the wave equation, which can be written as a trigonometric series. We establish various properties for the solution, including the scaling property, the pathwise regularity or the asymptotic behavior with respect to the Hurst parameter.

Key-words : fractional stochastic heat, self-similarity, generalized variation, parameter estimator, wave equation.

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Chapitre 1

Introduction

Sommaire

L'équation de la chaleur se situe parmi les équations aux dérivées partielles les plus utilisées. Elle modélise le phénomène physique de conductivité thermique. L'équation de la chaleur classique, écrite à l'aide du laplacien ordinaire, est utilisée surtout pour modéliser les situations standards, qui se passent dans un environnement normal. D'une autre part, les phénomènes physiques arrivent parfois dans un environnement différent, anormal (matériaux hétérogènes, tissus organiques etc) et cela peut influencer l'évolution du flux d'énergie. Dans ces situations, les chercheurs en physique ou les mathématiciens utilisent souvent comme modèle l'équation fractionnaire de la chaleur, en remplaçant le laplacien standard par le laplacien fractionnaire. En outre, le bruit ou la perturbation stochastique est inévitable et omni-présente dans la nature ainsi que dans les systèmes créés par l'homme. Il est donc important d'analyser les effets stochastiques dans l'étude des systèmes différentiels fractionnaires, en particulier d'étudier l'influence de l'exposant du laplacien fractionnaire sur la solution de ce type d'équation, sur la continuité de ses trajectoires, sur sa loi de probabilité ou sur ses variations, parmi d'autres.

Nous présentons dans cette partie introductive l'organisation du manuscrit, les objets de base utilisés dans notre étude et nous faisons également une description succincte des principaux résultats obtenus.

1.1 Organisation de la thèse

Le corps de cette thèse se présente en deux parties. Dans la première partie (Chapitres 2 et 3), on travaille sur l'équation de la chaleur stochastique fractionnaire où le laplacien ordinaire est remplacé par le laplacien fractionnaire, noté $(-\Delta)^{\frac{\alpha}{2}}$, d'ordre $\alpha \in (1, 2]$. Le cas $\alpha = 2$ correspond à l'équation de la chaleur classique. La deuxième partie du manuscrit (Chapitre 4) est consacrée à l'étude de la solution de l'équation stochastique des ondes dirigée par un bruit aléatoire qui se comporte comme un mouvement brownien fractionnaire en temps.

Dans le Chapitre 2, on introduit tout d'abord l'équation de la chaleur stochastique fractionnaire avec un bruit additif blanc en temps et en espace. On étudie l'existence de la solution et sa relation avec le mouvement brownien bi-fractionnaire. On démontre que notre solution existe seulement quand la dimension spatiale est égale à 1 (comme dans le cas du laplacien ordinaire pour $\alpha = 2$).

La solution de l'équation de la chaleur est un processus stochastique gaussien qui dépend de deux paramètres : l'espace et le temps. Nous analysons ce processus par rapport à ces deux paramètres. Quand la variable spatiale est fixée, en regardant la solution comme un processus en fonction du temps, on démontre que cette solution est égale en distribution à un mouvement bi-fractionnaire d'un paramètre de Hurst explicite (qui dépend de l'ordre α du laplacien en particulier). Ce fait nous aide à déduire des différentes propriétés de notre solution : auto-similarité, continuité höldérienne, variations généralisées, etc. Concernant l'analyse par rapport à la variable d'espace, on utilise le fait que notre solution est un mouvement brownien perturbé (c'est à dire la somme d'un mouvement brownien fractionnaire et d'un processus gaussien régulier), fait démontré dans [27]. En analysant les propriétés du mouvement brownien fractionnaire perturbé, on obtient certaines propriétés importantes de la solution par rapport à la variable spatiale. On remarque que la régularité en espace est α multipliée par la régularité en temps (le même résultat ayant été obtenu pour le cas laplacien ordinaire $\alpha = 2$ dans [54]).

Dans la deuxième partie de ce chapitre, on étudie aussi l'équation de la chaleur fractionnaire mais cette fois dirigée par un bruit additif blanc en temps et coloré en espace (c.à.d ayant une corrélation spatiale non triviale). La covariance de notre bruit est donnée par le noyau de Riesz d'où l'apparition d'un nouveau paramètre $\gamma \in (0, d)$ dans notre étude. On démontre que la solution existe pour $d < \alpha+\gamma$ (d étant la dimension spatiale), ce qui donne un domaine beaucoup plus vaste que le cas du bruit blanc espace-temps (où la solution n'existe que pour $d = 1$). Pour le cas temporel (c'est à dire, l'analyse de la solution comme fonction de temps, la variable spatiale étant fixée), notre solution est aussi égale en distribution à un mouvement bi-fractionnaire d'un paramètre de Hurst déterminé (en fonction de d, α, γ) d'où la facilité de déduire la limite de la q-variation de notre solution. Concernant le comportement en espace, la situation est plus complexe. La solution de l'équation fractionnaire de la chaleur est dans ce cas liée au drap brownien fractionnaire isotropique (ou mouvement brownien fractionnaire de Lévy isotropique), en fait sa loi coïncide avec la loi d'un "smooth" drap brownien fractionnaire perturbé (c. à. d., en lui ajoutant un processus "smooth"). En étudiant (ou en utilisant) les propriétés du drap fractionnaire perturbé, on en déduit la variation spatiale de notre solution.

Ce chapitre fait l'objet d'un article publié dans la revue *Probability Theory and Mathematical Statistics* ([39]).

Le Chapitre 3 fait aussi l'objet d'un article publié [40] dans la revue *Modern Stochastics :Theory and Applications,* en collaboration avec le directeur de thèse. Il s'agit toujours de l'équation fractionnaire de la chaleur, mais cette fois on suppose que cette équation dépend d'un paramètre de dérive (ou de drift). On vise l'estimation de ce paramètre de drift en supposant que la solution est observée à des temps discrets et à un point $x \in \mathbf{R}$ fixé. Cette estimation est faite via la méthode des variations généralisées (ou "power-variation"). Nous donnons aussi la vitesse de convergence de nos estimateurs sous la distance de Wasserstein. La construction des estimateurs est basée sur l'observation du comportement asymptotique des variations généralisées (en espace ou en temps) de la solution de notre équation paramétrique. Ces variations sont obtenues d'une façon similaire que dans notre premier article en observant la relation entre notre solution et le mouvement brownien bi-fractionnaire (cas temporel) ou avec le mouvement brownien fractionnaire perturbé (cas spatial). En utilisant la limite de ces variations, on démontre que nos estimateurs sont consistants et asymptotiquement normaux et nous donnons leur vitesse de convergence sous la distance de Wasserstein en appliquant certains résultats de la théorie récente de Stein-Malliavin.

Ce chapitre contient également deux cas : le cas du bruit blanc espace-temps et le cas du bruit blanc en temps et coloré en espace. Dans les deux situations, on obtient deux estimateurs du paramètre de drift : un basé sur la variation temporelle de la solution de l'EDS stochastique considérée et l'autre basé sur sa variation en espace. Malgré le fait que l'ordre de variation apparu dans la construction de nos quatre estimateurs est différent (il dépend de l'ordre α du laplacien fractionnaire, et de γ qui représente l'indice de correlation de l'espace), ils sont tous les quatre asymptotiquement normales. En plus, ils ont tous le même ordre de convergence $n^{-\frac{1}{2}}$ et la même distance par rapport à la distribution gaussienne. Notons que le fait de savoir l'ordre de convergence sous la distance de Wasserstein des q-variations d'un mouvement brownien fractionnaire a facilité l'obtention de l'ordre de convergence de nos estimateurs. Le cas du laplacien ordinaire ($\alpha = 2$) a été étudié en [49].

Dans la deuxième partie du manuscrit (composée du Chapitre 4), qui présente un article dans la revue [41], on introduit l'équation stochastique des ondes avec un bruit additif W sur un intervalle spatial fini. Dans ce chapitre, l'équation des ondes modélise les vibrations d'une corde perturbée par une force aléatoire. Notre bruit est fractionnaire en temps i.e. il se comporte en temps comme un mouvement brownien fractionnaire d'indice de Hurst $H \in \left(\frac{1}{2}, 1\right)$ et il est blanc en espace, c'est à dire il a la covariance d'un mouvement brownien standard. Ce qui est nouveau dans notre étude, c'est l'analyse de l'équation des ondes sur un intervalle fini en espace, c.à.d. la variable de l'espace appartient à un intervalle fini $[0, L]$ avec des conditions de Dirichlet sur les bornes de l'intervalle. Le noyau associé à notre équation peut-être écrit comme une série trigonométrique ce qui rend le calcul différent du cas de la corde infinie (déjà analysée dans la littérature) et cela a un impact sur les propriétés de notre solution. En utilisant la représentation trigonométrique de noyau de Green de notre équation, on obtiendra l'existence de la solution, la propriété dite "scaling", l'analyse de l'incrément temporel et spatial, le comportement asymptotique de la solution par rapport au paramètre de Hurst H.

Ces deux parties sont composées des trois articles suivants :

Articles :

— (Zeina Mahdi Khalil and Ciprian Tudor)(2018) "On the distribution and q-variation of the solution to the heat equation with fractional Laplacian", **Probability Theory and Mathematical Statistics,** 39(2), 315-335.
— (Zeina Mahdi Khalil and Ciprian Tudor)(2019) "Estimation of the drift parameter for the fractional stochastic heat equation via power variation", **Modern Stochastics : Theory and Applications,** 6(4), 397-417.

— (Zeina Khalil Mahdi and Ciprian Tudor)(2020) "Vibrations of a finite string under a fractional Gaussian random noise", accepté dans **Revue Roumaine de Mathématiques Pures et Appliquées,**

Avant de commencer l'énoncé des articles, on présente dans cette introduction quelques notions de base utilisées dans l'ensemble du manuscrit ainsi qu'un résumé des principaux résultats obtenus.

1.2 Préliminaires

Dans cette partie préliminaire, nous allons introduire quelques notions ou résultats de base utilisés souvent dans cette thèse.

1.2.1 Rappels

Nous allons rappeler les concepts d'auto-similarité, accroissements stationnaires et variations généralisées d'un processus stochastique.

Les processus auto-similaires sont des processus qui possèdent la propriété dite de scaling, c.à.d ils préservent leur loi de probabilité après un changement d'échelle. On va voir plus tard que la plupart des processus qui apparaissent dans ce travail (les processus liés au mouvement brownien fractionnaire, les solutions des équations stochastiques) possèdent cette propriété de scaling.

Definition 1. *Un processus stochastique $(X_t)_{t\geq 0}$ est auto-similaire d'indice $H > 0$ si, pour toute constante $c > 0$, les deux processus $(X_{ct})_{t\geq 0}$ et $(c^H X_t)_{t\geq 0}$ ont les mêmes lois fini-dimensionnelles.*

Les notions de stationnarité ou accroissements stationnaires vont aussi être utilisées plus tard. Une classe importante de processus auto-similaires est celle des processus à accroissements stationnaires.

Definition 2.
- *Un processus est dit stationnaire strict si pour tout $h \geq 0$, $(X_{t+h})_{t\geq 0} \equiv^{(d)} (X_t)_{t\geq 0}$ (i.e. la loi de $(X_{t+h})_{t\geq 0}$ ne dépend pas de $h > 0$).*
- *Un processus est dit à accroissements stationnaires si la loi des accroissements $(X_{t+h} - X_h)_{t\geq 0}$ ne dépend pas de $h \geq 0$, i.e. $(X_{t+h} - X_h)_{t\geq 0} \equiv^{(d)} (X_t)_{t\geq 0}$ (comme le (fBm).)*

où $\equiv^{(d)}$ désigne l'égalité au sens des lois fini-dimensionnelles.

1.2.2 Variation d'un processus stochastique

Les variations généralisées des processus stochastiques constituent un élément central dans notre travail. Dans ce manuscrit, on étudie deux types de variations (voir [36]) :

— La variation exacte d'ordre q (ou la q-variation) avec $q \in [0, \infty[$ sur un intervalle $[A_1, A_2]$, représente la limite en probabilité (si cette limite existe), lorsque $n \to \infty$, de la suite aléatoire
$$V^{q,n}(X) = \sum_{i=0}^{n-1} \left| X_{t_{i+1}} - X_{t_i} \right|^q, \tag{1.1}$$
où $t_i = A_1 + \frac{i}{n}(A_2 - A_1), i = 0, .., n$ constitue une partition de l'intervalle $[A_1, A_2]$.

— La variation renormalisée d'ordre q (qui est souvent utilisée pour les processus auto-similaires pour $q \geq 1$ entier) représente la limite en distribution quand $n \to \infty$ de la suite de variables aléatoires
$$V_{q,n}(X) = \sum_{i=0}^{n-1} \left[\frac{(X_{t_{i+1}} - X_{t_i})^q}{\mathbf{E}(X_{t_{i+1}} - X_{t_i})^q} - \mu_q \right], \tag{1.2}$$
où $\mu_q = \mathbf{E} Z^q$ avec $Z \sim N(0, 1)$.

1.2.3 Le mouvement brownien fractionnaire et ses extensions

Plusieurs processus de type fractionnaire vont intervenir dans notre travail. Nous rappelons leurs définitions et quelques propriétés de base.

Le mouvement brownien fractionnaire

Dans tous les travaux qui constituent cette thèse de doctorat, les processus stochastiques qui interviennent sont liés au mouvement brownien fractionnaire (noté (fBm)). Ce processus stochastique a fait l'objet d'études intensives dans les dernières décades. Ses propriétés bien-connues (auto-similarité, mémoire longue, stationnarité des accroissements etc) font de lui un candidat naturel pour modéliser différents phénomènes physiques. L'analyse du mouvement brownien fractionnaire et des processus liés à celui-ci a été faite à la fois d'un point de vue probabiliste ou bien statistique.

Rappelons qu'on peut définir le (fBm) de plusieurs manières :

— $(B_t^H)_{t \in [0,T]}$ est un processus gaussien centré de covariance
$$R^H(t, s) := R(t, s) = \frac{1}{2} \left(t^{2H} + s^{2H} - |t - s|^{2H} \right), \quad s, t \in [0, T]. \tag{1.3}$$

En particulier, pour $H = \frac{1}{2}$, $B^{\frac{1}{2}} := B$ est le mouvement brownien standard.

— $(B_t^H)_{t\in[0,T]}$ est une intégrale de Wiener par rapport au processus de Wiener, i.e.

$$B_t^H = \int_{-\infty}^{0} [(t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}]W(du) + \int_0^t (t-u)^{H-\frac{1}{2}} W(du),$$

avec W un mouvement brownien standard.

— $(B_t^H)_{t\in[0,T]}$ est l'unique processus gaussien auto-similaire d'indice H avec des accroissements stationnaires.

Le (fBm) n'est pas une semi-martingale (sauf pour $H = \frac{1}{2}$), ni un processus de Markov (voir [56]). Parmi les propriétés du (fBm) qui seront utilisées dans la suite nous rappelons également que

— Les trajectoires du (fBm) sont δ-Hölder continues $\forall \delta < H$.

— Le (fBm) admet une variation exacte (i.e. dans le sens de (1.1)) d'ordre $\frac{1}{H}$ et cette variation est égale à $\mathbf{E}(|Z|^{\frac{1}{H}})$ où Z est une variable normale centrée et réduite.

Le mouvement brownien bi-fractionnaire

Le mouvement brownien bi-fractionnaire représente une extension du mouvement brownien fractionnaire. Il a été introduit dans [30]. Le bi-(fBm) $(B_t^{H,K})_{t\in[0,T]}$ est un processus gaussien centré de covariance

$$R^{H,K}(t,s) := R(t,s) = \frac{1}{2^K}\left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right), \quad s,t \in [0,T] \tag{1.4}$$

avec $H \in (0,1)$ et $K \in (0,1]$. En particulier, pour $K = 1$, $B^{H,1} := B^H$ est le (fBm) de paramètre de Hurst $H \in (0,1)$.

Parmi ses propriétés, nous citons :

— Le bi-(fBm) est HK-auto-similaire, ce qui est équivalent à dire que pour tout $c > 0$, les processus gaussiens $(B_{ct}^{H,K})_{t\geq 0}$ et $c^{HK}(B_t^{H,K})_{t\geq 0}$ ont la même covariance.

— $\mathbf{E}|B_t^{H,K} - B_s^{H,K}|^2 \leq 2^{1-K}|t-s|^{2HK}$. Il découle par le critère de Kolmogorov que le bi-(fBm) est δ-Hölder continu $\forall \delta < HK$.

— Le bi-(fBm) admet une variation exacte (i.e. dans le sens de (1.1)) d'ordre $\frac{1}{HK}$ et celle-ci est égale à $(C_{H,K})(A_2 - A_1)$ avec $C_{H,K} = (2^{\frac{1-K}{2}})^{\frac{1}{HK}}\mathbf{E}(|Z|^{\frac{1}{HK}})$.

Relation entre le (fBm) et le bi-(fBm) Il a été démontré dans [36] la relation suivante entre un (fBm) et un bi-(fBm) : Soit $H \in (0,1), K \in (0,1]$. Si $(B_t^{HK})_{t\geq 0}$ est un (fBm) avec un paramètre de Hurst HK et $(B_t^{H,K})_{t\geq 0}$ est un bi-(fBm), alors

$$\left(C_1 X_t^{H,K} + B_t^{H,K}, t \geq 0\right) \equiv^{(d)} \left(C_2 B_t^{HK}, t \geq 0\right), \tag{1.5}$$

avec $C_1 > 0$ et $C_2 = 2^{\frac{1-K}{2}}$. $X^{H,K}$ est un processus gaussien régulier C^∞, en particulier $X^{H,K}$ satisfait (1.21), et il est indépendant de $B^{H,K}$.

1.2.4 Le drap brownien fractionnaire isotropique

Nous allons utiliser également une version multi-paramétrique du mouvemement brownien fractionnaire. Il y a plusieurs variantes du (fBm) à plusieurs paramètres, dont le drap fractionnaire isotropique (introduit ci-dessous) et le drap fractionnaire anisotropique (dont la covariance est définie comme produit de covariances du (fBm) standard).

Le drap fractionnaire isotropique $(B^H(\mathbf{t}), \mathbf{t} \in \mathbf{R}^d)$ avec l'indice de Hurst $H \in (0,1)$ est défini comme un processus gaussien centré de covariance :

$$\mathbf{E}(B^H(\mathbf{t})B^H(\mathbf{s})) = \frac{1}{2}\left(\|\mathbf{s}\|^{2H} + \|\mathbf{t}\|^{2H} - \|\mathbf{t}-\mathbf{s}\|^{2H}\right) \text{ pour tout } \mathbf{s}, \mathbf{t} \in \mathbf{R}^d. \tag{1.6}$$

Nous avons noté par $\|\cdot\|$ la norme euclidienne de l'espace $\mathbf{R}^d$.

Le drap isotropique multiparamètre (fBm) est auto-similaire et ses accroissements sont stationnaires dans le sens suivant : pour tout $\mathbf{h} \in \mathbf{R}^d_+$

$$\left(B^H(\mathbf{x}+\mathbf{h}) - B^H(\mathbf{h})\right)_{\mathbf{x}\in\mathbf{R}^d_+} \equiv^{(d)} \left(B^H(\mathbf{x})\right)_{\mathbf{x}\in\mathbf{R}^d_+} \tag{1.7}$$

et pour tout $a > 0$,

$$(B^H(a\mathbf{x})_{\mathbf{x}\in\mathbf{R}^d_+} \equiv^{(d)} a^H \left(B^H(\mathbf{x})\right)_{\mathbf{x}\in\mathbf{R}^d_+}. \tag{1.8}$$

Rappelons qu'on note $\equiv^{(d)}$ l'équivalence dans le sens de distribution. C'est possible aussi de démontrer que les accroissements du multi-paramètre (fBm) sont stationnaires (voir Proposition 6 dans [29]). Une propriété importante qui rend ce drap isotropique différent du drap anisotropique est que pour tout $x, y \in \mathbf{R}^d$

$$\mathbf{E}\left(B^H(\mathbf{x}) - B^H(\mathbf{y})\right)^2 = \|\mathbf{x}-\mathbf{y}\|^{2H}$$

ce qui implique, en raison du caractère gaussien, et pour tout $n \geq 1$

$$\mathbf{E}\left(B^H(\mathbf{x}) - B^H(\mathbf{y})\right)^n = \mathbf{E}|Z|^n \|\mathbf{x}-\mathbf{y}\|^{nH} \tag{1.9}$$

où Z est une variable normale standard. De (1.9) on peut déduire, par un argument de type Kolmogorov, l'existence d'une version continue de B^H isotropique, voir [35].

On peut également montrer une représentation intégrale du drap fractionnaire isotropique, i.e. si W est un drap brownien standard

$$B_t^H = \int_{R^d} [\|t-u\|^{H-\frac{1}{2}} - \|u\|^{H-\frac{1}{2}}] W(du) + \int_{R^d} ||t-u||^{H-\frac{d}{2}} W(du)$$

est un drap fractionnaire isotropique à une constante multiplicative près.

Variation d'un drap isotropique (fBm) :

Le drap isotropique (fBm) $\left((B^H(\mathbf{x}), \mathbf{x} \in \mathbf{R}^d)\right)$ admet une variation exacte (i.e. dans le sens de (1.1)) d'ordre $\frac{1}{H}$ sur $[A_1, A_2]$ qui est égale à :

$$(A_2 - A_1)\mathbf{E}|B_{\mathbf{1}}|^{1/H} = (A_2 - A_1)\sqrt{d}\mathbf{E}|Z|^{1/H}.$$

1.2.5 La méthode Delta

Soit T_n un estimateur de γ, on désire estimer le paramètre $\phi(\gamma)$, où ϕ est une fonction connue. Il est naturel d'estimer $\phi(\gamma)$ par $\phi(T_n)$. On peut alors se demander comment les propriétés asymptotiques de T_n se transfèrent à $\phi(T_n)$.
Si $\sqrt{n}(T_n - \gamma) \to^L X$ lorsque $n \to \infty$, a-t-on $\sqrt{n}(\phi(T_n) - \phi(\gamma)) \to^L Y$? La méthode delta offre une réponse à cette question.

Théorème 1.2.1. *(Méthode Delta) Soit ϕ une application de $\mathbf{R}^k$ dans $\mathbf{R}^m$, différentiable en γ. Soit T_n des vecteurs aléatoires de $\mathbf{R}^k$ (à valeurs dans le domaine de définition de ϕ) et $(r_n)_n$ une suite de nombres réels tendant vers ∞. Alors, lorsque $n \to \infty$*

$$r_n(\phi(T_n) - \phi(\gamma)) \to^L D\phi(\gamma)(T)$$

dès que $r_n(T_n - \gamma) \to^L T$ où $D\phi$ est le gradient de ϕ.

Nous notons par $\to^L$ la convergence en loi.
Rappelons aussi qu'une suite d'estimateurs $(T_n)_{n\geq 1}$ de γ est dite consistante si $T_n \to \gamma$ en probabilité (pour simplifier on dit souvent estimateur au lieu de suite d'estimateurs). La suite T_n est dite asymptotiquement normale s'il existe une suite de réels strictement positifs r_n telle que $r_n \to \infty$, et

$$r_n(T_n - \gamma) \to^L N(0, V(\gamma)).$$

$V(\gamma)$ est appelée la matrice de variance-covariance asymptotique de la suite $r_n(T_n - \gamma)$.
Notons qu'un estimateur asymptotiquement normal est consistant.

Dans ce dernier point, on rappelle la définition de la distance de Wasserstein qui sera utilisée dans le Chapitre 3.

Definition 3. *La distance Wasserstein entre deux variables aléatoires sur* $\mathbf{R}^d$, F *et* G, *est définie par*

$$d_W(F,G) = \sup_{h\in\mathcal{A}} |\mathbf{E}h(F) - \mathbf{E}h(G)|$$

où $\mathcal{A}$ *est la classe des fonctions lipchitiziennes* $h : \mathbf{R}^d \to \mathbf{R}$ *tels que* $\|h\|_{Lip} \leq 1$, *avec*

$$\|h\|_{Lip} = \sup_{x,y\in\mathbf{R}^d, x\neq y} \frac{|h(x)-h(y)|}{\|x-y\|_{\mathbf{R}^d}}.$$

1.3 Résultats principaux

Nous allons passer en revue le contenu des chapitres suivants.

1.3.1 Résumé du Chapitre 2

Le Chapitre 2 correspond à la publication [39], en collaboration avec C.A. Tudor. Dans ce chapitre, on étudie la solution de l'équation de la chaleur stochastique avec un laplacien fractionnaire et avec un bruit gaussien additif.

Il y a plusieurs définitions possibles du laplacien fractionnaire. Nous avons choisi ci-dessous sa définition donnée en fonction de sa transformée de Fourier. Rappelons que la transformée de Fourier $\mathcal{F}(f)$ d'une fonction f convenable $f : \mathbf{R}^d \to \mathbf{R}$ est donnée par $\mathcal{F}(f)(\xi) = \int_{\mathbf{R}^d} f(x)e^{-ix\cdot\xi}\,dx$.

Pour $1 < \alpha \leq 2$, le laplacien fractionnaire d'ordre α (noté $(-\Delta)^{\frac{\alpha}{2}}$) peut-être défini comme un opérateur sur des fonctions $f : \mathbf{R}^d \to \mathbf{R}$ par la formule :

$$\mathcal{F}((-\Delta)^{\frac{\alpha}{2}} f)(\xi) = |\xi|^\alpha \mathcal{F}(f)(\xi).$$

On considère l'équation différentielle stochastique :

$$\frac{\partial}{\partial t}u(t,x) = -(-\Delta)^{\frac{\alpha}{2}}u(t,x) + \dot{W}(t,x), t \geq 0, x \in \mathbf{R}^d \tag{1.10}$$

$$u(0,x) = 0 \quad pour \quad tout \quad x \in \mathbf{R}^d. \tag{1.11}$$

Le noyau de Green G_α (i.e. la solution fondamentale de (1.10)) de l'équation de la chaleur stochastique fractionnaire est défini par sa transformée de Fourier :

$$\mathcal{F}G_\alpha(t,\cdot)(\xi) = e^{-t\|\xi\|^\alpha}, \quad t > 0, \xi \in \mathbf{R}^d. \tag{1.12}$$

Dans un premier temps, on regarde la solution de l'équation fractionnaire de la chaleur dans le cas du bruit additif blanc en temps et en espace. Dans ce cas le bruit W est un champ gaussien centré $(W(t,A), t \geq 0, A \in \mathcal{B}_b(\mathbf{R}^d))$ de covariance

$$\mathbf{E}W(t,A)W(s,B) = (t \wedge s)\lambda(A \cap B), \tag{1.13}$$

λ est la mesure de Lebesgue sur $\mathbf{R}^d$.
La solution mild de l'équation (1.10) $\left\{u = u(t,x); t \geq, x \in \mathbf{R^d}\right\}$ s'exprime sous la forme d'une intégrale de Wiener par rapport au bruit gaussien W, i.e. pout tout $t \geq 0, x \in \mathbf{R}^d$

$$u(t,x) = \int_0^t \int_{\mathbf{R}^d} G_\alpha(t-s, x-y)W(ds,dy).$$

Un premier résultat donne les conditions pour l'existence de la solution mild.

Proposition 1.3.1. *Soit le processus* $\left(u(t,x), t \geq 0, x \in \mathbf{R}^d\right)$ *donné par (1.10). La solution est bien définie seulement dans le cas unidimensionel* $d = 1$.

La condition $d = 1$ pour l'existence de la solution représente une restriction importante pour les applications pratiques du modèle (1.10). Une approche possible pour dépasser cette barrière est de remplacer le bruit blanc espace-temps par un bruit aléatoire plus régulier que le bruit blanc en lui imposant une corrélation spatiale. Un tel exemple de corrélation est celui donné par le noyau de Riesz. On considère l'équation de la chaleur stochastique :

$$\frac{\partial}{\partial t}u(t,x) = -(-\Delta)^{\frac{\alpha}{2}}u(t,x) + \dot{W}^\gamma(t,x), \quad t \geq 0, x \in \mathbf{R}^d, \tag{1.14}$$

avec $u(0,x) = 0$ pour tout $x \in \mathbf{R}^d$. $\dot{W}^\gamma$ est le bruit blanc-coloré en espace, i.e. $W^\gamma(t,A), t \geq 0, A \in \mathcal{B}(\mathbf{R}^d)$ est un champ gaussien centré de covariance

$$\mathbf{E}W^\gamma(t,A)W^\gamma(s,B) = (t \wedge s)\int_A \int_B f(x-y)dxdy, \tag{1.15}$$

où f est la fonction Riesz kernel d'ordre γ définie par

$$f(x) = R_\gamma(x) := g_{\gamma,d}\|x\|^{-d+\gamma}, \quad 0 < \gamma < d, \tag{1.16}$$

avec $g_{\gamma,d} = 2^{d-\gamma}\pi^{d/2}\Gamma((d-\gamma)/2)/\Gamma(\gamma/2)$.
Dans ce cas, on a $\mu(d\xi) = \|\xi\|^{-\gamma}d\xi$ et on a la relation de Parceval :

$$\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\varphi(x)f(x-y)\psi(y)dxdy = (2\pi)^{-d}\int_{\mathbf{R}^d}\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(d\xi) \tag{1.17}$$

pour tout $\varphi, \psi \in \mathcal{S}(\mathbf{R}^d)$ (l'espace de Schwartz sur $\mathbf{R^d}$).

Proposition 1.3.2.

$$u(t,x) = \int_0^t\int_{\mathbf{R}^d} G_\alpha(t-u, x-z)W^\gamma(du,dz), \tag{1.18}$$

la solution de (1.14) existe pour $d < \alpha + \gamma$.

Un premier pas dans l'analyse des solutions de (1.10) et (1.14) est le calcul de sa covariance. Nous commençons par la covariance par rapport au temps, en supposant que la variable spatiale est fixée.

Proposition 1.3.3.

1. *Si $x \in \mathbf{R}$ est fixé, la solution $(u(t,x))_{t\geq 0}$ de l'équation stochastique (1.10) avec bruit blanc en espace et en temps, a comme covariance : pour tout $s,t \geq 0$ et $x \in \mathbf{R}$*

$$\mathbf{E}u(t,x)u(s,x) = c_{1,\alpha}\left[(t+s)^{1-\frac{1}{\alpha}} - |t-s|^{1-\frac{1}{\alpha}}\right],$$

avec une constante $c_{1,\alpha}$ (sa formule explicite sera donnée dans le chapitre suivant).

2. *La solution $(u(t,x))_{t\geq 0}$ de l'équation stochastique (1.14) avec bruit blanc en temps et coloré en espace a comme covariance : pour tout $s,t \geq 0$, et pour tout $x \in \mathbf{R}^d$*

$$\mathbf{E}u(t,x)u(s,x) = c_{1,\alpha,\gamma}\left[(t+s)^{1-\frac{d-\gamma}{\alpha}} - |t-s|^{1-\frac{d-\gamma}{\alpha}}\right],$$

avec une certaine constante $c_{1,\alpha,\gamma}$.

La formule explicite de la covariance nous aide à conclure que notre solution dans les deux cas d'équations (bruit blanc en espace ou colorié en espace) est égale en distribution à un mouvement brownien bi-fractionnaire avec les indices H, K bien définis.

Corollaire 1. *1. La solution $(u(t,x))_{t\geq 0}$ de l'équation stochastique (1.10) avec bruit blanc en temps et en espace a la même loi en distribution que $c_{2,\alpha}B^{\frac{1}{2},1-\frac{1}{\alpha}}$, où $B^{\frac{1}{2},1-\frac{1}{\alpha}}$ est un bi-(fBm) avec paramètre de Hurst $H=\frac{1}{2}$, $K=1-\frac{1}{\alpha}$ et*

$$c_{2,\alpha}^2 = c_{1,\alpha}2^{1-\frac{1}{\alpha}}. \tag{1.19}$$

2. La solution $(u(t,x))_{t\geq 0}$ de l'équation stochastique (1.14) avec bruit blanc en temps et coloré en espace a la même loi que $c_{2,\alpha,\gamma}(B_t^{\frac{1}{2},1-\frac{d-\gamma}{\alpha}})_{t\geq 0}$ avec $B^{\frac{1}{2},1-\frac{d-\gamma}{\alpha}}$ est un bi-(fBm) avec $H=\frac{1}{2}$,$K=1-\frac{d-\gamma}{\alpha}$ et

$$c_{2,\alpha,\gamma}^2 = c_{1,\alpha,\gamma}2^{1-\frac{d-\gamma}{\alpha}}. \tag{1.20}$$

Vu qu'on connaît les propriétés du bi-(fBm) d'indice (H,K), sa Hölder-continuité (pour tout $\delta < HK$), sa variation (il admet une $1/HK$-variation), autosimilarité, on va en déduire des propriétés concernant la solution de l'équation fractionnaire de la chaleur. Parmi celles-ci, on mentionne :

1. La solution $(u(t,x))_{t\geq 0}$ de l'équation stochastique (1.10) avec bruit blanc en espace et en temps, est auto-similaire d'ordre $\frac{1}{2}(1-\frac{1}{\alpha})$, et Hölder-continue en temps d'ordre δ, pour tout $\delta \in (0, \frac{1}{2}(1-\frac{1}{\alpha}))$.

2. La solution $(u(t,x))_{t\geq 0}$ de l'équation stochastique (1.14) avec bruit blanc en temps et coloré en espace est auto-similaire d'ordre $\frac{1}{2}(1-\frac{d-\gamma}{\alpha})$, et Hölder-continue en temps d'ordre δ, pour tout $\delta \in (0, \frac{1}{2}(1-\frac{d-\gamma}{\alpha}))$.

De plus, le résultat (1.5) montre que le mouvement brownien bifractionnaire est la somme d'un mouvement brownien fractionnaire et d'un processus gaussier régulier. On appelle cela un mouvement brownien perturbé. On peut démontrer que les variations d'un (fBm) perturbé se comportent d'une manière similaire que les variations d'un mouvement brownien fractionnaire.

lemme 1. *Soit $(B_t^H)_{t\geq 0}$ un (fBm) avec $H \in (0, \frac{1}{2}]$ et $(X_t)_{t\geq 0}$ un processus gaussien centré tels que*

$$\mathbf{E}\,|X_t - X_s|^2 \leq C|t-s|^2 \text{ pour tout } s,t \geq 0. \tag{1.21}$$

Soit

$$Y_t^H = B_t^H + X_t \text{ pour tout } t \geq 0.$$

1. *Le processus* Y *a* $\frac{1}{H}$*-variation exacte sur l'intervalle* $[A_1, A_2]$ *, elle est égale à*

$$\mathbf{E}|Z|^{1/H}(A_2 - A_1).$$

2. *Soit*

$$V_{q,n}(Y^H) := \sum_{i=0}^{n-1} \left[\frac{n^{Hq}}{(A_2 - A_1)^{qH}} (Y^H_{t_{i+1}} - Y^H_{t_i})^q - \mu_q \right].$$

Si $H \in (0, \frac{1}{2})$ *et* $q \geq 2$*, alors on a*

$$\frac{1}{\sqrt{n}} V_{q,n}(Y^H) = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\frac{n^{Hq}}{(A_2 - A_1)^{qH}} (Y^H_{t_{i+1}} - Y^H_{t_i})^q - \mu_q \right] \to N(0, \sigma^2_{H,q}). \quad (1.22)$$

Remark 1.3.1. *On définit* $\sigma^2_{H,q} = q! \sum_{v \in \mathbf{Z}} \rho_H(v)^q$*, avec* $\rho_H(v) = \frac{1}{2}\left(|v+1|^{2H} + |v-1|^{2H} - 2|v|^{2H}\right)$ *for* $v \in \mathbf{Z}$*.*

On conclut, par le Lemme 1 et le Corollaire 1, le comportement asymptotique de la variation temporelle de la solution de (1.10) et (1.14).

Proposition 1.3.4. ***(Variation temporelle)***

1. *On fixe* $A_1 < A_2$ *et* $x \in \mathbf{R}$*. Soit* $t_j = A_1 + \frac{j}{n}(A_2 - A_1), n \geq 1, j = 0, 1, .., n$ *une partition de l'intervalle* $[A_1, A_2]$*. Alors le processus* $(u(t,x), t \geq 0)$ *admet une variation d'ordre* $\frac{2\alpha}{\alpha - 1}$ *qui est égale à*

$$c_{2,\alpha}^{\frac{2\alpha}{\alpha-1}} C_{1/2, 1-\frac{1}{\alpha}}(A_2 - A_1).$$

$(u(t,x))_{t \geq 0}$ *est la solution de l'équation stochastique (1.10) avec bruit blanc en espace et en temps.*

2. $(u(t,x))_{t \geq 0}$ *solution de l'équation stochastique (1.14) avec bruit blanc en temps et coloré en espace. On fixe* $A_1 < A_2$ *et* $x \in \mathbf{R}$*. Soit* $t_j = A_1 + \frac{j}{n}(A_2 - A_1), n \geq 1, j = 0, 1, .., n$ *une partition de l' intervalle* $[A_1, A_2]$*. Alors* $(u(t,x), t \geq 0)$ *admet une variation d'ordre* $\frac{2\alpha}{\alpha + \gamma - d}$ *égale à*

$$c_{2,\alpha,\gamma}^{\frac{2\alpha}{\alpha+\gamma-d}} C_{\frac{1}{2}, 1-\frac{d-\gamma}{\alpha}}(A_2 - A_1),$$

avec $C_{H,K} = (2^{\frac{1-K}{2}})^{\frac{1}{HK}} \mathbf{E}(|Z|^{\frac{1}{HK}})$*.*

Puisque la solution de (1.10) et la solution de (1.14) sont tous les deux un (fBm) perturbé, on en déduit du Lemme 1 leur variation renormalisée :

Proposition 1.3.5. *1. On fixe, $A_1 < A_2$ et $x \in \mathbf{R}$. Soit $t_j = A_1 + \frac{j}{n}(A_2 - A_1), n \geq 1, j = 0, 1, .., n$ une partition de l'intervalle $[A_1, A_2]$, alors*

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\left(\frac{n^{\frac{\alpha-1}{2\alpha}}}{c_{2,\alpha} 2^{\frac{1}{2\alpha}} (A_2 - A_1)^{\frac{\alpha-1}{2\alpha}}} \right)^q (u(t_{i+1}, x) - u(t_i, x))^q - \mu_q \right] \to N(0, \sigma^2_{\frac{1}{2}(1-\frac{1}{\alpha}),q}).$$

$(u(t, x))_{t \geq 0}$ est la solution de l'équation stochastique(1.10) avec bruit blanc en espace et en temps.

2. $(u(t, x))_{t \geq 0}$ solution de l'équation stochastique (1.14) avec bruit blanc en temps et coloré en espace. On fixe $A_1 < A_2$ et $x \in \mathbf{R}$. Soit $t_j = A_1 + \frac{j}{n}(A_2 - A_1), n \geq 1, j = 0, 1, .., n$ une partition de l'intervalle $[A_1, A_2]$, alors

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\left(\frac{n^{\frac{\alpha+\gamma-d}{2\alpha}}}{c_{2,\alpha,\gamma} 2^{\frac{d-\gamma}{2\alpha}} (A_2 - A_1)^{\frac{\alpha+\gamma-d}{2\alpha}}} \right)^q (u(t_{i+1}, x) - u(t_i, x))^q - \mu_q \right] \to N(0, \sigma^2_{\frac{1}{2}(1-\frac{d-\gamma}{\alpha}),q}).$$

Pour étudier les variations en espace, on sait de l'article [27] que $(u(t, x))_{t \geq 0}$, la solution de l'équation stochastique (1.10) avec bruit blanc en espace et en temps, peut-être écrite comme un mouvement brownien fractionnaire perturbé :

$$u(t, x) \equiv^{(d)} m_\alpha B^{\frac{\alpha-1}{2}}(x) + S(x),$$

avec $B^{\frac{\alpha-1}{2}}$ est un mouvement brownien fractionnaire de paramètre de Hurst $\frac{\alpha-1}{2} \in [0, \frac{1}{2}]$, $(S(x))_{x \in \mathbf{R}}$ est un processus gaussien C^∞, et $m_\alpha = (2\Gamma(\alpha)|\cos(\alpha\pi/2)|)^{-\frac{1}{2}}$.
De cette décomposition, on remarque que la régularité de la solution est celle du (fBm) $B^{\frac{\alpha-1}{2}}$. Ce qui implique en particulier que la fonction $u(t, x)$ est höldérienne d'ordre $\delta \in (0, \frac{\alpha-1}{2})$.
Puisque la solution est un (fBm) perturbé, on déduit la variation en espace de la solution de (1.10) ainsi que celle renormalisée en appliquant le Lemme 1.

Proposition 1.3.6. *1. On fixe $A_1 < A_2$ et $t > 0$. Soit $x_j = A_1 + \frac{j}{n}(A_2 - A_1)$ pour $j = 0, .., n$ et $n \geq 1$. Donc le processus $(u(t, x), x \in \mathbf{R})$, solution de (1.10), admet par rapport à la variable d'espace $\frac{2}{\alpha-1}$- variation, i.e. on a la limite en probabilité suivante :*

$$\lim_{n\to\infty}\sum_{j=0}^{n-1}|u(t,x_{j+1})-u(t,x_j)|^{\frac{2}{\alpha-1}} = m_\alpha^{\frac{2}{\alpha-1}}\mathbf{E}|Z|^{\frac{2}{\alpha-1}}(A_2-A_1).$$

2. *On fixe $A_1 < A_2$ et $t > 0$. Soit $x_j = A_1 + \frac{j}{n}(A_2 - A_1)$ pour $j = 0, .., n$ et $n \geq 1$. Alors pour $\alpha \in (1, 2)$, (ce qui entraîne $H = \frac{\alpha-1}{2} \in (0, 1/2)$)*

$$\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\left[\left(\frac{n^{\frac{\alpha-1}{2}}}{m_\alpha}\right)^q (u(t,x_{i+1})-u(t,x_i))^q - \mu_q\right] \to N(0,\sigma^2_{\frac{\alpha-1}{2},q}).$$

Quand le bruit n'est plus blanc en espace et il a une corrélation spatiale donnée par le noyau de Riesz, la solution existe en dimension $d \geq 1$ (en fait, pour $d < \alpha + \gamma$). Cela nous amène à analyser les draps browniens fractionnaires de type isotropique, voir le paragraphe 1.2.4. Nous prouvons que la loi du processus gaussien $(u(t,x), x \in \mathbf{R}^d)$ défini par (1.18) coïncide en loi avec un drap fractionnaire isotropique plus un processus à trajectoires régulières.

Proposition 1.3.7. *On fixe $t > 0$, $(u(t,x))_{x\in\mathbf{R}^d}$, solution de l'équation stochastique (1.14) avec bruit blanc en temps et coloré en espace, est égale en distribution à*

$$c_{3,\alpha,\gamma}B^{\frac{\alpha+\gamma-d}{2}}(x) + S(x), x \in \mathbf{R}^d$$

où $B^{\frac{\alpha+\gamma-d}{2}}$ est un drap fractionnaire isotropique avec un paramètre de Hurst $\frac{\alpha+\gamma-d}{2}$, $S(x), x \in \mathbf{R}^d$, est un processus gaussien C^∞ et $c_{3,\alpha,\gamma}$ une constante explicite.

En étudiant les q-variations d'un drap fractionnaire perturbé (un résultat similaire comme celui obtenu dans le Lemme 1), on va déduire la forme exacte des q-variations en espace de la solution ainsi que sa variation renormalisée.

Proposition 1.3.8. *1. Le processus $(u(t,x))_{x\in\mathbf{R}^d}$, solution de l'équation stochastique (1.14) avec bruit blanc en temps et coloré en espace, admet une $\frac{2}{\alpha+\gamma-d}$- variation donnée par $c_{3,\alpha,\gamma}^{\frac{2}{\alpha+\gamma-d}}(A_2-A_1)\sqrt{d}\mathbf{E}|Z|^{\frac{2}{\alpha+\gamma-d}}$.*

2. *On fixe* $A_1 < A_2$ *et* $t > 0$. *Soit* $x_j^k = A_1 + \frac{j}{n}(A_2 - A_1)$ *pour* $j = 0, .., n$, $n \geq 1$ *et pour tout* $k = 1, .., d$. *Soit* $\mathbf{x}_j = (x_j^1, .., x_j^d)$. *Donc si* $\alpha + \gamma - d < 1$ *(ce qui entraîne* $H = \frac{\alpha+\gamma-d}{2} \in (0, 1/2)$*)*

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\left(\frac{n^{\frac{\alpha+\gamma-d}{2}} d^{-H/2}}{c_{3,\alpha,\gamma}} \right)^q (u(t, \mathbf{x}_{i+1}) - u(t, \mathbf{x}_i))^q - \mu_q \right] \to N(0, \sigma^2_{\frac{\alpha+\gamma-d}{2}, q}).$$

1.3.2 Résumé du Chapitre 3

Le Chapitre 3 correspond à la publication [40] en collaboration avec C.A. Tudor. Dans ce chapitre, on étudie l'équation de la chaleur stochastique fractionnaire paramétrique (c. à. d., dépendant d'un paramètre de dérive ou de drift) avec un bruit gaussien additif. Ce bruit aléatoire est défini comme dans le chapitre précédent, il est soit un bruit blanc espace-temps avec la covariance (1.13), soit un bruit blanc-coloré avec la covariance donnée par (1.15).

D'abord, on considère le cas où le bruit est blanc en temps et en espace, soit

$$\frac{\partial u_\theta}{\partial t}(t, x) = -\theta(-\Delta)^{\frac{\alpha}{2}} u_\theta(t, x) + \dot{W}(t, x), \quad t \geq 0, x \in \mathbf{R} \tag{1.23}$$

avec $u(0, x) = 0$, avec $(-\Delta)^{\frac{\alpha}{2}}$ le laplacien fractionnaire d'ordre $\alpha \in (1, 2]$, $\theta > 0$ et W processus gaussien défini par sa covariance (1.13).

Le but est de donner une estimation du paramètre $\theta > 0$ basée sur l'observation de la solution à des temps discrets ou à des points discrets en espace (en supposant que soit la variable temps, soit la variable espace est connue). Pour cela on va constuire deux estimateurs basés sur l'analyse du comportement asymptotique de la variation généralisée d'ordre q (en espace et en temps) de la solution.

Le noyau associé à l'opérateur $-\theta(-\Delta)^{\frac{\alpha}{2}}$ est $G_\alpha(\theta t, x)$ avec

$$G_\alpha(t, x) = \int_{\mathbf{R}} e^{it\xi - t|\xi|^\alpha} d\xi. \tag{1.24}$$

Une première observation est que le paramètre de drift peut être transformé en un paramètre de diffusion via un changement de temps. Ce fait jouera un rôle important dans l'estimation de θ.

lemme 2. *Supposons que le processus* $(u_\theta(t,x), t \geq 0, x \in \mathbf{R})$ *satisfait (1.23). On va définir*

$$v_\theta(t,x) := u_\theta\left(\frac{t}{\theta}, x\right), \quad t \geq 0, x \in \mathbf{R}. \tag{1.25}$$

Alors le processus $(v_\theta(t,x), t \geq 0, x \in \mathbf{R})$ *satisfait*

$$\frac{\partial v_\theta}{\partial t}(t,x) = -(-\Delta)^{\frac{\alpha}{2}} v_\theta(t,x) + (\theta)^{-\frac{1}{2}} \dot{\widetilde{W}}(t,x), \quad t \geq 0, x \in \mathbf{R} \tag{1.26}$$

avec $v_\theta(0,x) = 0$ *pour tout* $x \in \mathbf{R}$*, et* $\dot{\widetilde{W}}$ *est un bruit blanc en espace et en temps.*

En utilisant le résultat dans le Corollaire 1, on en déduit facilement les lois de u_θ en espace et en temps.

Proposition 1.3.9. *1.* ***cas temporel :*** *Pour tout* $x \in \mathbf{R}$ *et* $\theta > 0$*, on a*

$$(u_\theta(t,x), t \geq 0) \equiv^{(d)} \left(\theta^{-\frac{1}{2\alpha}} c_{2,\alpha} B_t^{\frac{1}{2}, 1-\frac{1}{\alpha}}, t \geq 0\right),$$

avec $B^{\frac{1}{2}, 1-\frac{1}{\alpha}}$ *est un bi-(fBm) de paramètre de Hurst* $H = \frac{1}{2}$ *et* $K = 1 - \frac{1}{\alpha}$*.*

2. ***cas spatial :*** *Pour tout* $t \geq 0, \theta > 0$*, on a l'égalité en distribution*

$$(u_\theta(t,x), x \in \mathbf{R}) \equiv^{(d)} \left(\theta^{-\frac{1}{2}} m_\alpha B^{\frac{\alpha-1}{2}}(x) + S_{\theta t}(x), x \in \mathbf{R}\right),$$

avec $B^{\frac{\alpha-1}{2}}$ *est un (fBm) avec* $H = \frac{\alpha-1}{2} \in (0, \frac{1}{2}]$*,* $(S_{\theta t}(x))_{x \in \mathbf{R}}$ *est un processus gaussien centré* C^∞ *.*

Cette proposition nous montre que notre solution est dans tous les cas (spatial ou temporel) un (fBm) perturbé ou un drap fractionnaire isotropique perturbé. Par conséquent, on obtient le résultat suivant.

Proposition 1.3.10. *1.* ***Variation temporelle :*** *Soit* $t_j = A_1 + \frac{j}{n}(A_2 - A_1), j = 0, .., n$ *une partition de l'intervalle* $[A_1, A_2]$*.*

$$S^{n, \frac{2\alpha}{\alpha-1}}_{[A_1,A_2]} := \sum_{i=0}^{n-1} |u_\theta(t_{j+1}, x) - u_\theta(t_j, x)|^{\frac{2\alpha}{\alpha-1}} \to_{n\to\infty} c_{2,\alpha}^{\frac{2\alpha}{\alpha-1}} 2^{\frac{1}{\alpha-1}} \mu_{\frac{2\alpha}{\alpha-1}} (A_2 - A_1) |(\theta)|^{\frac{-1}{\alpha-1}} \tag{1.27}$$

en probabilité.

2. ***Variation spatiale :*** *Soit* $x_j = A_1 + \frac{j}{n}(A_2 - A_1), j = 0, .., n$ *une partition de l'intervalle* $[A_1, A_2]$.

$$\sum_{i=0}^{n-1} |u_\theta(t, x_{j+1}) - u_\theta(t, x_j)|^{\frac{2}{\alpha-1}} \to_{n\to\infty} m_\alpha^{\frac{2}{\alpha-1}} \mu_{\frac{2}{\alpha-1}} (A_2 - A_1) |\theta|^{\frac{-1}{\alpha-1}},$$

et si $q := \frac{2}{\alpha-1}$ *est un entier*

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\left(\frac{n}{m_\alpha^{\frac{2}{\alpha-1}} (A_2 - A_1)} \right) \theta^{\frac{1}{\alpha-1}} \left(u_\theta(t, x_{i+1}) - u_\theta(t, x_i)\right)^{\frac{2}{\alpha-1}} - \mu_{\frac{2}{\alpha-1}} \right] \to^{(d)} N(0, \sigma^2_{\frac{\alpha-1}{2}, \frac{2}{\alpha-1}}).$$

Cette proposition nous aide à définir deux estimateurs, l'un basé sur la variation en espace de notre solution, l'autre sur la variation en temps.

1. **estimateur basé sur la variation temporelle**

$$\widehat{\theta}_{n,1} = \left(\left(c_{2,\alpha}^{\frac{2\alpha}{\alpha-1}} 2^{\frac{1}{\alpha-1}} \mu_{\frac{2\alpha}{\alpha-1}} (A_2 - A_1) \right)^{-1} \sum_{i=0}^{n-1} |u_\theta(t_{j+1}, x) - u_\theta(t_j, x)|^{\frac{2\alpha}{\alpha-1}} \right)^{1-\alpha}$$

et

$$\widehat{\theta_{n,1}^{\frac{1}{1-\alpha}}} = \frac{1}{c_{2,\alpha}^{\frac{2\alpha}{\alpha-1}} 2^{\frac{1}{\alpha-1}} \mu_{\frac{2\alpha}{\alpha-1}} (A_2 - A_1)} \sum_{i=0}^{n-1} |u_\theta(t_{j+1}, x) - u_\theta(t_j, x)|^{\frac{2\alpha}{\alpha-1}}. \tag{1.28}$$

2. **estimateur basé sur la variation spatiale**

$$\widehat{\theta}_{n,2} = \left[(m_\alpha^{\frac{2}{\alpha-1}} \mu_{\frac{2}{\alpha-1}} (A_2 - A_1))^{-1} \sum_{i=0}^{n-1} |u_\theta(t, x_{j+1}) - u_\theta(t, x_j)|^{\frac{2}{\alpha-1}} \right]^{1-\alpha}. \tag{1.29}$$

Avant de démontrer que nos estimateurs associés sont consistants et asymptotiquement normaux sous la distance de Wasserstein, on donne l'énoncé du lemme suivant.

lemme 3. *Soit* $(B_t^H)_{t\geq 0}$ *un (fBm) avec* $H \in (0, \frac{1}{2}]$ *et* $(X_t)_{t\geq 0}$ *un processus gaussien centré tels que*

$$\mathbf{E} |X_t - X_s|^2 \leq C|t - s|^2 \textit{pour tout } s,t \geq 0. \tag{1.30}$$

Soit

$$Y_t^H = B_t^H + X_t \textit{ pour tout } t \geq 0.$$

$$V_{q,n}(Y^H) := \sum_{i=0}^{n-1} \left[\frac{n^{Hq}}{(A_2 - A_1)^{qH}} (Y^H_{t_{i+1}} - Y^H_{t_i})^q - \mu_q \right],$$

alors pour $H \in (0, 1/2)$, et $q \geq 2$ on a

$$\frac{1}{\sqrt{n}} V_{q,n}(Y^H) = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\frac{n^{Hq}}{(A_2 - A_1)^{qH} a^q} (Y^H_{t_{i+1}} - Y^H_{t_i})^q - \mu_q \right] \to^{(d)} N(0, \sigma^2_{H,q}),$$

et

$$d_W \left(\frac{1}{\sqrt{n}} V_{q,n}(Y^H), N(0, \sigma^2_{H,q}) \right) \leq C \frac{1}{\sqrt{n}}.$$

Comme la construction de nos deux estimateurs est basée sur le comportement asymptotique de la variation temporelle, respectivement spatiale de notre solution ; et comme notre solution est un (fBm) perturbé dans les deux cas spatial et temporel ; on en déduit d'après la Lemme 3 le comportement de nos estimateurs.

Proposition 1.3.11. *1. Soit $q := \frac{2\alpha}{\alpha-1}$ un entier pair et considérons l'estimateur (basé sur la **variation temporelle**) $\widehat{\theta}_{n,1}$. Donc $\widehat{\theta}_{n,1} \to_{n\to\infty} \theta$ en probabilité et*

$$\sqrt{n} \left[\widehat{\theta}_{n,1}^{\frac{1}{1-\alpha}} - \theta^{\frac{1}{1-\alpha}} \right] \to^{(d)} N(0, s^2_{1,\theta,\alpha}) \text{ avec } s^2_{1,\theta,\alpha} = \sigma^2_{\frac{1}{q},q} \theta^{\frac{2}{1-\alpha}} \mu^{-2}_{\frac{2\alpha}{\alpha-1}}. \tag{1.31}$$

Et pour n grand

$$d_W \left(\sqrt{n} \left[\widehat{\theta}_{n,1}^{\frac{1}{1-\alpha}} - \theta^{\frac{1}{1-\alpha}} \right], N(0, s^2_{1,\theta,\alpha}) \right) \leq c \frac{1}{\sqrt{n}}.$$

*2. L'estimateur (basé sur la **variation spatiale**) $\widehat{\theta}_{n,2}$ converge en probabilité quand n $n \to \infty$ vers le paramètre θ. En plus, si $q := \frac{2}{\alpha-1}$ est un entier pair*

$$\sqrt{n} \left[\widehat{\theta}_{n,2}^{\frac{1}{1-\alpha}} - \theta^{\frac{1}{1-\alpha}} \right] \to^{(d)} N(0, s^2_{2,\theta,\alpha}) \text{ avec } s^2_{2,\theta,\alpha} = \sigma^2_{\frac{\alpha-1}{2}, \frac{2}{\alpha-1}} \mu^{-2}_{\frac{2}{\alpha-1}} \theta^{\frac{2}{1-\alpha}}. \tag{1.32}$$

En plus, pour n grand on a,

$$d_W \left(\sqrt{n} \left[\widehat{\theta}_{n,2}^{\frac{1}{1-\alpha}} - \theta^{\frac{1}{1-\alpha}} \right], N(0, s^2_{2,\theta,\alpha}) \right) \leq c \frac{1}{\sqrt{n}}.$$

En utilisant la methode Delta (voir le Théorème 1.2.1), en prenant $T_n = \widehat{\theta_n^{\frac{1}{1-\alpha}}}$, $\phi(x) = x^{1-\alpha}$, $T = \theta^{\frac{1}{1-\alpha}}$, on démontre finalement que nos estimateurs associés sont asymptotiquement normaux sous la distance de Wasserstein, ils ont la même distance à la distribution gaussienne et le même ordre de convergence.

Proposition 1.3.12. *1.* ***estimateur basé sur la variation temporelle*** $\widehat{\theta}_{n,1}$ *:*

$$\sqrt{n}(\widehat{\theta}_{n,1} - \theta) \to^{(d)} N(0, s_{1,\theta,\alpha}^2 (1-\alpha)^2 \theta^{\frac{2\alpha}{\alpha-1}}) \tag{1.33}$$

et pour n grand

$$d_W\left(\sqrt{n}(\widehat{\theta}_{n,1} - \theta), N(0, s_{1,\theta,\alpha}^2 (1-\alpha)^2 \theta^{\frac{2\alpha}{\alpha-1}})\right) \leq c\frac{1}{\sqrt{n}}.$$

2. ***estimateur basé sur la variation spatiale*** $\widehat{\theta}_{n,2}$ *: Quand* $n \to \infty$

$$\sqrt{n}(\widehat{\theta}_{n,2} - \theta) \to^{(d)} N\left(0, s_{2,\theta,\alpha}^2 (1-\alpha)^2 \theta^{\frac{2\alpha}{\alpha-1}}\right),$$

et pour n grand

$$d_W\left(\sqrt{n}(\widehat{\theta}_{n,2} - \theta), N(0, s_{2,\theta,\alpha}^2 (1-\alpha)^2 \theta^{\frac{2\alpha}{\alpha-1}})\right) \leq c\frac{1}{\sqrt{n}}.$$

Nous analysons également la situation quand le bruit possède une corrélation en espace donnée par le noyau de Riesz (1.16), i.e., il est de la forme (1.15). L'approche proposée pour estimer le paramètre de drift θ suit le même chemin avec la différence que dans cette situation la solution de l'équation fractionnaire de la chaleur et par conséquent ses variations et les estimateurs qui en découlent vont dépendre du paramètre γ dans (1.16). Nous donnons également deux estimateurs construits en fonction des observations discrètes de la solution (soit à des temps discrets soit à des points discrets en espace). On obtient la consistance, la normalité asymptotique et la distance par rapport à la loi normale de nos estimateurs.

1.3.3 Résumé du Chapitre 4

Le dernier chapitre de la thèse porte sur l'équation stochastique des ondes dirigée par un bruit ayant une corrélation en temps cette fois-ci. Le caractère fractionnaire de l'équation

est maintenant donné par ce bruit (et non plus par l'opérateur différentiel) qui se comporte comme un mouvement brownien fractionnaire par rapport à la variable temporelle. Le Chapitre 4 correspond à la publication [41] en collaboration avec C.A. Tudor.
On se concentre sur l'équation stochastique des ondes, avec un bruit additif gaussien, sur un segment fini c.à.d. la variable en espace appartient à un interval fini $[0, L]$. On suppose des conditions de Dirichlet sur les bornes de l'intervalle $[0, L]$. Cette équation modélise les vibrations d'une corde perturbée par une force aléatoire. Le bruit est blanc en espace et fractionnaire en temps i.e. il se comporte comme un (fBm) par rapport au temps. Il existe beaucoup d'études sur l'équation des ondes avec un bruit stochastique (voir [26], [19], [12], [14], [20], [47], [48]) et en particulier sur l'équation des ondes avec un bruit fractionnaire (en temps ou en espace), voir [7], [13], [23], [34], [50], [56], mais tous ces travaux traitent en général le cas où la variable spatiale appartient à $\mathbf{R}$.

L'EDPS considérée est la suivante

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) &= c^2 \Delta u(t,x) + \dot{W}^H(t,x), \ t \in [0,T], \ x \in [0,L], \\ u(0,x) &= 0, \quad x \in [0,L], \\ \frac{\partial u}{\partial t}(0,x) &= 0, \quad x \in [0,L], \\ u(t,0) &= u(t,L) = 0, \quad t \in [0,T], \end{cases} \tag{1.34}$$

avec $c, L > 0$. L représente la longueur de la corde et c est relié à la tension dans la corde. La perturbation W^H est un champ gaussien centré $W^H = \{W_t^H(A); t \in [0,T], A \in B_b([0,L])\}$, sur l'espace probabilisé $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbf{P})$, de covariance

$$\mathbf{E}\left(W_t^H(A) W_s^H(B)\right) = R_H(t,s)\lambda(A \cap B), \forall A, B \in \mathcal{B}_b([0,L]), \tag{1.35}$$

où R_H est la covariance du (fBm)

$$R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s,t \in [0,T]. \tag{1.36}$$

La solution mild du problème $\{u = (u(t,x); t \geq, x \in [0,L])\}$ s'exprime sous la forme d'une intégrale de Wiener par rapport au bruit gaussien W^H

$$u(t,x) = \int_0^t \int_0^L G_{t,x}(s,y) W^H(ds,dy) \text{ pour tout } t \in [0,T], x \in [0,L] \tag{1.37}$$

avec le noyau de Green $G_{t,x}$ donné par, pour $0 \leq s \leq t \leq T$ et $x, y \in [0, L]$

$$G_{t,x}(s,y) = \sum_{n=1}^{\infty} \frac{2}{L w_n} \sin(w_n(t-s)) \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \text{ avec } w_n = \frac{n\pi c}{L}. \tag{1.38}$$

Un élément central dans l'étude de la solution (1.37) est l'espace de Hilbert $\mathcal{H}$ associé au (fBm) B^H avec le paramètre de Hurst $H \in \left(\frac{1}{2}, 1\right)$ et les propriétés de l'intégrale de Wiener par rapport au mouvement brownien fractionnaire. En particulier, cette intégrale de Wiener vérifie l'isométrie

$$\mathbf{E} \int_0^T f(u) dB_u^H \int_0^T g(u) dB_u^H = \alpha_H \int_0^T \int_0^T f(u) g(u) |u-v|^{2H-2} := \langle f, g \rangle_{\mathcal{H}} \tag{1.39}$$

pour tout $f, g \in |\mathcal{H}|$ où $|\mathcal{H}|$ l'espace des fonctions mesurables $f : [0,T] \to \mathbf{R}$ telles que

$$\int_0^T \int_0^T |f(u)g(u)||u-v|^{2H-2} < \infty.$$

La solution (1.37) existe si l'intégrale de Wiener est bien définie, i.e.

$$\sup_{t \in [0,T]} \mathbf{E} u(t,x)^2 < \infty, \tag{1.40}$$

pour tout $x \in [0, L)$.
En se basant sur la représentation trigonométrique du noyau (1.38) de notre équation, on obtiendra l'existence de la solution, la propriété dite "scaling", l'estimation de l'incrément temporel et spatial, le comportement asymptotique de la solution par rapport au paramètre de Hurst H.

Le point de départ est l'observation que la solution (1.37) peut s'exprimer sous la forme d'une série trigonométrique à coefficients aléatoires, i.e.

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi x}{L}\right) \tag{1.41}$$

où

$$T_n(t) = \frac{2}{L w_n} \int_0^t \int_0^L \sin(w_n(t-u)) \sin\left(\frac{n\pi y}{L}\right) W^H(du, dy). \tag{1.42}$$

Un rôle clé est joué par les variables aléatoires T_n qui ont une certaine propriété d'orthogonalité.

lemme 4. *Pour tout $n \geq 1$, $(T_n(t))_{t \in [0,T]}$ est un processus gaussien centré dont la covariance vérifie :*

$$|\mathbf{E} T_n(t) T_n(s)| \leq \frac{2L}{\pi^2 c^2} R_H(t,s) \frac{1}{n^2} \text{ pour tout } s,t \in [0,T]$$

où R_H est donnée par (1.36).

Si $n \neq m$, alors $T_n(t)$ et $T_m(s)$ sont des variables aléatoires indépendantes, pour tout $s,t \in [0,T]$.

Une conséquence du résultat précédent est que pour tout $n \geq 1$, le processus $(T_n(t))_{t\in[0,T]}$ a les mêmes lois fini-dimensionnelles que le processus

$$\left(\frac{\sqrt{2}}{\sqrt{L}w_n}\int_0^t \sin(w_n(t-u))dB_t^H\right), \tag{1.43}$$

où $(B_t^H)_{t\in[0,T]}$ est un (fBm) avec un indice de Hurst H.

Un calcul immédiat donne alors l'existence de la solution de (1.34) quelque soit le paramètre de Hurst dans l'intervalle $\left(\frac{1}{2},1\right)$.

Proposition 1.3.13. *Pour $H \in \left(\frac{1}{2},1\right)$, l'intégrale stochastique dans (1.41) est bien-définie et on a*

$$\sup_{t\in[0,T],x\in[0,L]} \mathbf{E}u(t,x)^2 < \infty.$$

En utilisant les propriétés des coefficients T_n qui apparaissent dans (1.41) et des identités trigonométriques, on arrive à obtenir certains résultats satisfaits par la solution mild (1.41). Nous prouvons :

— **Une propriété de changement d'échelle** : Pour tout $a > 0$, le processus $(u_c(at,x), t \geq 0)$ est égal en distribution au processus $\left(a^{H+1}u_{ac}(t,x), t \geq 0\right)$.

— **Le comportement par rapport aux temps nodaux** : Soit $T_k = \frac{k\pi}{w_1} = \frac{kL}{c}$, $k = 1, 2, ...$ série de temps. Ces temps sont nommés *temps nodaux*. Soient T_k, T_l deux temps nodaux $k > l$ et supposons que k, l ont la même parité. Donc

$$u(T_k,x) - u(T_l,x) =^{(d)} u(T_k - T_l, x).$$

Cela veut dire que la position de x sur la corde au temps T_k est obtenu en loi en ajoutant la position du même point au temps T_l et $T_k - T_l$.

— **La relation avec la solution faible** : Un autre concept de la solution du problème (1.34) est la solution faible. On dit qu'un processus stochastique $(u(t,x), t \in [0,T], x \in [0,L])$ est une solution faible de (1.34) si pour toute fonction test $\varphi \in C^\infty([0,T]\times[0,L])$ avec $\varphi(T,x) = \frac{\partial\varphi}{\partial t}(T,x) = 0$ pour tout $x \in [0,L]$ et $\varphi(t,0) = \varphi(t,L) = 0$ pour tout $t \in [0,T]$, on a

$$\int_0^T dt \int_0^L dx \quad u(t,x)\left(\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2}\right) = \int_0^T\int_0^L \varphi(s,y)W^H(ds,dy). \tag{1.44}$$

On démontre que notre solution (1.37) satisfait (1.34) dans le sens faible.

Une partie importante du chapitre est consacrée à l'analyse des accroissements de la solution. Comme cette solution est un processus gaussien, beaucoup de propriétés de ses trajectoires découlent de l'analyse de ses accroissements.

Commençons par l'accroissement en temps du processus (1.41), fixons la variable d'espace $x \in [0, L]$ et étudions le comportement du processus $(u(t, x), t \in [0, T])$.

Nous démontrons le résultat suivant.

Proposition 1.3.14. *Soit $x \in [0, L]$. Alors*

$$\mathbf{E}\left|u(t,x)-u(s,x)\right|^2 \leq C(\varepsilon)|t-s|^{2H-\varepsilon},$$

pour tout $\varepsilon \in (0, 2H)$ et pour tout $0 \leq s \leq t \leq T$.

La preuve, assez longue, demande une analyse assez exacte de la norme dans l'espace $\mathcal{H}$ des fonctions trigonométriques. La Proposition 1.3.14 précise la continuité höldérienne en temps des trajectoires de la solution. En effet, pour tout $x \in [0, L]$, le processus $(u(t, x), t \in [0, T])$ est Hölder continu d'ordre δ, pour tout $\delta \in (0, H)$.

Le comportement en espace est basé également sur certaines propriétés des séries trigonométriques de fonctions, en particulier

$$\sum_{n\geq 1} \frac{\sin^2(nx)}{n^2} = \frac{\pi}{2}x - \frac{1}{2}x^2. \tag{1.45}$$

La régularité spatiale de la solution est la suivante :

Proposition 1.3.15. *Pour tout $t \in [0, T]$ et pour tout $x, y \in (0, L)$ avec $|x - y|$ petit,*

$$\mathbf{E}(u(t,x)-u(t,y))^2 \leq C|x-y|.$$

Par conséquent, pour tout $t \in [0, T]$, le processus $(u(t, x), x \in [0, L])$ est Hölder continu d'ordre δ, pour tout $\delta \in \left(0, \frac{1}{2}\right)$.

On s'intéresse aussi, dans ce chapitre, au comportement de la solution de (1.34) lorsque l'indice de Hurst converge vers les valeurs extrêmes de son intervalle de définition, c.à.d. quand $H \to 1$ et $H \to \frac{1}{2}$. Le résultat obtenu, énoncé ci-dessous, montre que la solution converge en distribution vers la solution d'une équation stochastique des ondes avec un bruit qui est la "limite" (quand $H \to 1$ et $H \to \frac{1}{2}$) du bruit (1.35).

Proposition 1.3.16. *Soit $(u(t, x), t \in [0, T], x \in [0, L])$ solution de (1.41).*

1. *Quand* $H \to \frac{1}{2}$*, pour tout* $x \in [0, L]$*, le processus* $(u(t, x), t \in [0, T])$ *converge faiblement dans l'espace* $C[0, T]$ *vers le processus* $(u_0(t, x), t \in [0, T])$ *définit par*

$$u_0(t, x) = \int_0^t \int_0^L G_{t,x}(s, y)W(ds, dy)$$

où W *est un bruit blanc en temps et en espace et* G *est donné par (1.38).*

2. *Quand* $H \to 1$*, pour tout* $x \in [0, L]$*, le processus* $(u(t, x), t \in [0, T])$ *converge faiblement dans l'espace* $C[0, T]$ *vers le processus* $(u_1(t, x), t \in [0, T])$ *définie par*

$$u_1(t, x) = \sum_{n \geq 1} \frac{\sqrt{2}}{\sqrt{L} w_n} \left(\int_0^t \sin(w_n(t - u))du \right) Z_n$$

où $(Z_n)_{n \geq 1}$ *suite de variables aléatoires indépendantes standards.*

Première partie

Fractional Stochastic heat equation : estimation and variation

Chapitre 2

On the distribution and q-variation of the solution to the heat equation with fractional Laplacian

Sommaire

We study the probability distribution of the solution to the linear stochastic heat equation with fractional Laplacian and white noise in time and white or correlated noise in space. As an application, we deduce the behavior of the q-variations of the solution in time and in space.

Key Words and Phrases : Stochastic heat equation, Fractional Brownian motion, Fractional Laplacian, Quadratic variation.

2.1 Introduction

In this work we study the probability distribution and other properties of the solution to the fractional stochastic heat equation driven by an additive Gaussian noise which is white in time and white or correlated in space. Here, the word "fractional" concerns the appearance of the fractional Laplacian operator in the equation and it does not refers to the noise.

Several recent and less recent works showed an interesting connection between the solution to the classical stochastic linear heat equation and some stochastic processes related to the (bi)fractional Brownian motion. Consider the stochastic partial differential equation

$$\frac{\partial}{\partial t}u(t,x) = \Delta u(t,x) + \dot{W}(t,x), \quad t \geq 0, x \in \mathbf{R}^d \tag{2.1}$$

with vanishing initial condition $u(0,x) = 0$ for every $x \in \mathbf{R}^d$. In (2.1), we denoted by Δ the standard Laplacian on $\mathbf{R}^d$ and by W the random noise which is defined as a centered Gaussian process $(W(t,A), t \geq 0, A \in \mathcal{B}_b(\mathbf{R}^d))$ with covariance

$$\mathbf{E}W(t,A)W(s,B) = (t \wedge s) \int_A \int_B \|x-y\|^{-\gamma} dxdy, \text{ if } \gamma \in (0,d)$$

and

$$\mathbf{E}W(t,A)W(s,B) = (t \wedge s)\lambda(A \cap B), \text{ when } \gamma = 0.$$

We denoted by $\|\cdot\|$ the Euclidean norm in $\mathbf{R}^d$ and by $\mathcal{B}_b(\mathbf{R}^d)$ the class of bounded Borel sets in $\mathbf{R}^d$.

In the first case, the noise is said to be white in time and correlated in space with spatial correlation given by the Riesz kernel. In the second case, we have a time-space white noise, i.e. the noise behaves as a Wiener process both in time and in space.

The solution to (2.1) is usually defined in the mild sense, i.e. as the Wiener integral with respect to the noise W by

$$u(t,x) = \int_0^t \int_{\mathbf{R}^d} G(t-s, x-y) W(ds, dy) \tag{2.2}$$

where G is the fundamental solution of the heat equation, i.e. the deterministic function which solves $\frac{\partial}{\partial t}u(t,x) = \Delta u(t,x)$.

We say that the solution to (2.1) exists if the Wiener integral in (2.2) is well-defined in $L^2(\Omega)$. We know (see e.g. [56]) that the necessary and sufficient condition for the existence of the solution is

$$d < 2 + \gamma$$

which means $d = 1$ in the case of the time-space white noise ($\gamma = 0$).

Also, the solution is connected to the bifractional Brownian motion. Recall that (see [30]), given constants $H \in (0,1)$ and $K \in (0,1]$, the bifractional Brownian motion (bi-fBm for short) $(B_t^{H,K})_{t\in[0,T]}$ is a centered Gaussian process with covariance

$$R^{H,K}(t,s) := R(t,s) = \frac{1}{2^K}\left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right), \quad s,t \in [0,T]. \tag{2.3}$$

In particular, for $K = 1$, $B^{H,1} := B^H$ is the fractional Brownian motion with Hurst parameter $H \in (0,1)$.

It is known that for every fixed $x \in \mathbf{R}^d$, the process $(u(t,x), t \geq 0)$ given by (2.2) coincides in distribution, modulo a constant, to the bifractional Brownian motion with parameters $H = \frac{1}{2}$ and $K = 1 - \frac{d-\gamma}{2}$ (so $H = K = \frac{1}{2}$ if $\gamma = 0$ and $d = 1$).

Concerning the behavior with respect to the space variable, we know from [27] that for fixed $t > 0$ and for $d = 1$, the process $(u(t,x), x \in \mathbf{R})$ has the same law as a Brownian motion plus an independent Gaussian process with C^∞ sample paths.

When the noise is correlated in time, for example when the noise behaves as a fractional Brownian motion with respect to the time variable, there are also links between the law of the solution to the heat equation and the fractional processes, see e.g. [28] or [57].

All these connections are very useful to deduce various properties of the solution to the heat equation by using known results for the fractional Brownian motion.

In this work, our purpose is to do a similar analysis for the solution to the fractional stochastic heat equation, i.e. when the Laplacian is replaced by the fractional Laplacian of order $\alpha \in (1,2]$, denoted $-(-\Delta)^{\frac{\alpha}{2}}$, in the equation (2.1). We want to understand the influence of the parameter α on the law and on the sample paths regularity of the solution. We give a necessary and sufficient condition for the existence of the solution for the fractional heat equation and we study the connection with fractional Brownian motion and related processes. We prove the following facts : while with respect to the time variable the solution still remains a bifractional Brownian motion (whose Hurst parameters will be explicitly given), the behaviour of the solution in space will be related to the isotropic fractional Brownian sheet. The result was known in dimension $d = 1$ from [27], but in dimension $d \geq 2$, we notice the appearance, for the first time in the literature, of the multiparameter fractional Brownian motion of isotropic type.

We apply these findings to the study of the q-variations of the solution to the fractional heat equation. For a stochastic process $(X_t)_{t\geq 0}$, we will consider two types of variations :

— the exact q-variations ($q \in (0,\infty)$) over an interval $[A_1, A_2]$, meaning the limit in probability of the sequence

$$V^{q,n}(X) = \sum_{i=0}^{n-1} \left| X_{t_{i+1}} - X_{t_i} \right|^q$$

if $t_i = A_1 + \frac{i}{n}(A_2 - A_1), i = 0, .., n$ constitutes a partition of the interval $[A_1, A_2]$.

— the renormalized $q-$variations (which are usually defined for self-similar stochastic processes and for $q \geq 1$ integer) as the limit in distribution, as $n \to \infty$, of the sequence

$$V_{q,n}(X) = \sum_{i=0}^{n-1} \left[\frac{(X_{t_{i+1}} - X_{t_i})^q}{\mathbf{E}(X_{t_{i+1}} - X_{t_i})^q} - \mu_q \right]$$

where $\mu_q = \mathbf{E} Z^q$ with $Z \sim N(0,1)$ (we will use this notation througout our work).

More details on these notions will be given below in Section 2. In the same Section 2, we also prove some new facts concerning the variations of the perturbed fractional Brownian motion. In Section 4 we study the solution to the fractional heat equation with time-space white noise while Section 4 is devoted to the analysis of the white-colored noise case.

2.2 Variations of the perturbed fractional Brownian motion

In this section we introduce the notion of exact q-variation and of renormalized q-variation for stochastic processes. We will also recall some known results for the fractional Brownian motion and bifractional Brownian motion. In the last part, we obtain the q-variation for a perturbed fBm, i.e. the sum of a fBm and of a smooth process. This result for the perturbed fBm will be applied several times in this work.

2.2.1 Exact q-variations and renormalized q-variations for stochastic processes

We first define the concept of *exact q-variation* for stochastic processes.

Definition 4. *Let $A_1 < A_2$ and for $n \geq 1$, let $t_i = A_1 + \frac{i}{n}(A_2 - A_1)$ for $i = 0, .., n$. A continuous stochastic process $(X_t)_{t\geq 0}$ admits a q-variation (or a variation of order q) over the interval $[A_1, A_2]$ if the sequence*

$$V^{n,q}_{[A_1,A_2]}(X) = \sum_{i=0}^{n-1} \left| X_{t_{i+1}} - X_{t_i} \right|^q$$

converges in probability as $n \to \infty$. The limit, when it exists, is called the exact q-variation of X over the interval $[A_1, A_2]$.

If $[A_1, A_2] = [0, t]$, we will simply denote $V^{n,q}_{[A_1,A_2]}(X) := V^{n,q}_t(X)$. Moreover, if $t = 1$, we denote $V^{n,q}_t(X) := V^{q,n}$. In the case $q = 2$ the limit of $V^{2,n}$ is called the quadratic variation, while for $q = 3$ we have the cubic variation.

Let us recall the following result concerning the q-variation of the bifractional Brownian motion (see Proposition 1 in [36]).

Proposition 2.2.1. *If $(B^{H,K}_t)_{t\geq 0}$ is a bi-fBm with Hurst parameters $H \in (0, 1), K \in (0, 1]$ then $B^{H,K}$ admits a variation of order $\frac{1}{HK}$ over any interval $[A_1, A_2]$ which is equal to*

$$C_{H,K}(A_2 - A_1)$$

where Z is a standard normal random variable and

$$C_{H,K} = (2^{\frac{1-K}{2}})^{\frac{1}{HK}} \mathbf{E}|Z|^{\frac{1}{HK}}. \tag{2.4}$$

By taking $K = 1$, we notice that the fractional Brownian motion has $\frac{1}{H}$-variation over the interval $[A_1, A_2]$ given by $\mathbf{E}|Z|^{\frac{1}{H}}(A_2 - A_1)$.

We will also study the asymptotic behavior of the *normalized q-variation* of the solution to the fractional heat equation. Even if this notion is usually studied for self-similar stochastic processes, one can discuss it for more general stochastic processes. In order to define this object, let us recall the case of the fractional Brownian motion. Let B^H be a fBm with Hurst parameter $H \in (0, 1)$ and define

$$V_{q,n}(B^H) := \sum_{i=0}^{n-1} \left[\frac{n^{Hq}}{(A_2 - A_1)^{Hq}} (B^H_{t_{i+1}} - B^H_{t_i})^q - \mu_q \right].$$

This is called the (centered) renormalized q-variation because the random variable $(B^H_{t_{i+1}} - B^H_{t_i})^q$ is normalized, i.e. the expectation of $\frac{n^{Hq}}{(A_2-A_1)^{Hq}}(B^H_{t_{i+1}} - B^H_{t_i})^q$ is $\mu_q = \mathbf{E}Z^q$. Many works treated recently the limit behavior in distribution of the renormalized q-variations for various stochastic processes, see e.g. [56] and the references therein.

We recall the following result concerning the variations of the fractional Brownian motion (see [11], [24], [25] or [55] ; the reader may also consult Section 1 in [44] for a survey of these results). We will restrict below to the case when the Hurst parameter H is less or equal than $\frac{1}{2}$ since only this case will be needed in the sequel. That is, if $q \geq 2$ is an integer and $H \in (0, \frac{1}{2}]$, then

$$\frac{1}{\sqrt{n}} V_{q,n}(B) := \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\frac{n^{Hq}}{(A_2 - A_1)^{Hq}} (B_{t_{i+1}} - B_{t_i})^q - \mu_q \right] \to N(0, \sigma^2_{H,q}). \tag{2.5}$$

Above, $\sigma^2_{H,q}$ denote a strictly positive constant depending on q and H.

In this work, we will analyze the asymptotic behavior of the renormalized q-variation for various stochastic processes, not necessarily self-similar. For a general process $(X_t)_{t\geq 0}$, by studying the asymptotic behavior of the renormalized q-variation we will generally mean to find a constant $\mu \in \mathbf{R}$ and two deterministic sequence $f(n), g(n)$ which converge to zero as $n \to \infty$ such that $\mathbf{E}\left[f(n)^{-1}\left(X_{t_{i+1}} - X_{t_i}\right)\right]^2$ is close to 1 and

$$g(n) \sum_{i=0}^{n-1} \left[\frac{\left(X_{t_{i+1}} - X_{t_i}\right)^q}{f(n)} - \mu_q \right]$$

converges in distribution to a non-trivial limit as $n \to \infty$. We illustrate below the case of the perturbed fBm.

2.2.2 q-variation of the perturbed fractional Brownian motion

In the next sections, we will see that the solution to the fractional heat equation can be decomposed into the sum of a fBm (with Hurst parameter less or equal than one half) and a smooth process. Therefore, we need to understand the variations of such stochastic processes. This can be relatively easily obtained from the results known for the fBm and recalled above.

Concerning the asymptotic behavior of the exact and renormalized q variation of the sum of a fBm and a smooth process, we have the following lemma.

Lemma 1. *Let $(B^H_t)_{t\geq 0}$ be a fBm with $H \in (0, \frac{1}{2}]$ and consider a centered Gaussian process $(X_t)_{t\geq 0}$ such that*

$$\mathbf{E}\left|X_t - X_s\right|^2 \leq C|t-s|^2 \text{ for every } s, t \geq 0. \tag{2.6}$$

Define

$$Y^H_t = B^H_t + X_t \text{ for every } t \geq 0.$$

1. *The process Y has $\frac{1}{H}$-variation over the interval $[A_1, A_2]$ which is equal to*

$$\mathbf{E}|Z|^{1/H}(A_2 - A_1).$$

2. *Let*

$$V_{q,n}(Y^H) := \sum_{i=0}^{n-1} \left[\frac{n^{Hq}}{(A_2 - A_1)^{qH}} (Y^H_{t_{i+1}} - Y^H_{t_i})^q - \mu_q \right].$$

Then if $H \in (0, \frac{1}{2})$ and $q \geq 2$

$$\frac{1}{\sqrt{n}} V_{q,n}(Y^H) = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\frac{n^{Hq}}{(A_2 - A_1)^{qH}} (Y^H_{t_{i+1}} - Y^H_{t_i})^q - \mu_q \right] \to N(0, \sigma^2_{H,q}). \tag{2.7}$$

If $H = \frac{1}{2}$, $q = 2$ and the process $(X_t)_{t\geq 0}$ is independent of B, then

$$\frac{1}{\sqrt{n}} V_{2,n}(Y^H) = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\frac{n}{(A_2 - A_1)} (Y^H_{t_{i+1}} - Y^H_{t_i})^2 - 1 \right] \to N(0, \sigma^2_{\frac{1}{2},2}). \tag{2.8}$$

Proof : Concerning point 1., we use Minkowski inequality to write

$$\begin{aligned}
&\left(\sum_{i=0}^{n-1} \left| B^H_{t_{i+1}} - B^H_{t_i} \right|^{\frac{1}{H}} \right)^H - \left(\sum_{i=0}^{n-1} \left| X_{t_{i+1}} - X_{t_i} \right|^{\frac{1}{H}} \right)^H \\
&\leq \left(\sum_{i=0}^{n-1} \left| Y^H_{t_{i+1}} - Y^H_{t_i} \right|^{\frac{1}{H}} \right)^H \\
&\leq \left(\sum_{i=0}^{n-1} \left| B^H_{t_{i+1}} - B^H_{t_i} \right|^{\frac{1}{H}} \right)^H + \left(\sum_{i=0}^{n-1} \left| X_{t_{i+1}} - X_{t_i} \right|^{\frac{1}{H}} \right)^H .
\end{aligned} \tag{2.9}$$

Since by Proposition 3.2.1, the sequence

$$\sum_{i=0}^{n-1} \left| B^H_{t_{i+1}} - B^H_{t_i} \right|^{\frac{1}{H}}$$

converges in probability, as $n \to \infty$, to the desired limit $\mathbf{E}|Z|^{1/H}(A_2 - A_1)$, it suffices to show that $\sum_{i=0}^{n-1} \left| X_{t_{i+1}} - X_{t_i} \right|^{\frac{1}{H}}$ converges to zero. We have, via (2.6) and the fact that for $s < t$, $X_t - X_s \sim N(0, \sigma^2)$ with $\sigma^2 \leq C(t - s)$

$$\mathbf{E}\sum_{i=0}^{n-1}\left|X_{t_{i+1}}-X_{t_i}\right|^{\frac{1}{H}}\leq C\sum_{i=0}^{n-1}|t_{i+1}-t_i|^{\frac{1}{H}}\leq Cn^{1-\frac{1}{H}}\to_{n\to\infty}0.$$

Let us prove point 2. Consider first the situation $H\in(0,\frac{1}{2})$. By using Newton's formula we can write

$$\frac{1}{\sqrt{n}}V_{q,n}(Y^H)=\frac{1}{\sqrt{n}}(V_{q,n}(B)+R_n)$$

where

$$\frac{R_{n,r}}{\sqrt{n}}=(A_2-A_1)^{qH}\frac{n^{Hq}}{\sqrt{n}}\sum_{r=0}^{q-1}C_q^r\sum_{i=0}^{n-1}(B_{t_{i+1}}-B_{t_i})^r(X_{t_{i+1}}-X_{t_i})^{q-r}:=\sum_{r=0}^{q-1}R_{n,r}. \tag{2.10}$$

It suffices to show that $\frac{R_{n,r}}{\sqrt{n}}$ converges to zero in $L^1(\Omega)$ for every $r=0,..,q-1$. Using (2.6) we have for every $s,t\geq 0$ and for $r=0,..,q-1$

$$\mathbf{E}|X_t-X_s|^{2(q-r)}\leq C|t-s|^{2(q-r)}$$

and then we can write, for $r=0,..,q-1$

$$\begin{aligned}n^{Hq}\mathbf{E}\frac{|R_{n,r}|}{\sqrt{n}} &\leq n^{Hq}\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\left(\mathbf{E}(B_{t_{i+1}}-B_{t_i})^{2r}\right)^{\frac{1}{2}}\left(\mathbf{E}(X_{t_{i+1}}-X_{t_i})^{2(q-r)}\right)^{\frac{1}{2}}\\ &\leq Cn^{(H-1)(q-r)+\frac{1}{2}}\leq Cn^{H-\frac{1}{2}}\end{aligned}$$

and this converges to zero as $n\to\infty$ since $H<\frac{1}{2}$.

If $H=\frac{1}{2}$ and $q=2$, we have

$$\begin{aligned}\frac{1}{\sqrt{n}}V_{2,n}(Y^{\frac{1}{2}}) &= \frac{1}{\sqrt{n}}V_{2,n}(B)+\frac{2}{\sqrt{n}}\sum_{i=0}^{n-1}\frac{n}{A_2-A_1}(B_{t_{i+1}}-B_{t_i})(X_{t_{i+1}}-X_{t_i})\\ &\quad+\frac{1}{\sqrt{n}}\frac{n}{A_2-A_1}\sum_{i=0}^{n-1}(X_{t_{i+1}}-X_{t_i})^2.\end{aligned}$$

Clearly, by (2.6)

$$\sqrt{n}\mathbf{E}\sum_{i=0}^{n-1}(X_{t_{i+1}}-X_{t_i})^2\leq cn^{-\frac{1}{2}}\to_{n\to\infty}0$$

and since X is independent of B,

$$\mathbf{E}\left(\sqrt{n}\sum_{i=0}^{n-1}(B_{t_{i+1}}-B_{t_i})(X_{t_{i+1}}-X_{t_i})\right)^2 = n\mathbf{E}\sum_{i=0}^{n-1}(B_{t_{i+1}}-B_{t_i})^2(X_{t_{i+1}}-X_{t_i})^2$$

$$\leq n\sum_{i=0}^{n-1}\left(\mathbf{E}(B_{t_{i+1}}-B_{t_i})^4\right)^{\frac{1}{2}}\left(\mathbf{E}(X_{t_{i+1}}-X_{t_i})^4\right)^{\frac{1}{2}} \leq Cn\sum_{i=0}^{n-1}\frac{1}{n}\frac{1}{n^2} \leq C\frac{1}{n} \to_{n\to\infty} 0.$$

■

Remark 2.2.1. *The condition (2.6) can be replaced by the following condition :*

$$\textit{The process } X \textit{ has absolute continuous path on } [0,\infty). \tag{2.11}$$

Indeed in this case,

$$\mathbf{E}\sum_{i=0}^{n-1}\left|X_{t_{i+1}}-X_{t_i}\right|^{\frac{1}{H}} \leq \sup_i\left(\left|X_{t_{i+1}}-X_{t_i}\right|^{\frac{1}{H}-1}\right)\sum_{i=0}^{n-1}\left|X_{t_{i+1}}-X_{t_i}\right|.$$

The quantity $\sup_i\left|X_{t_{i+1}}-X_{t_i}\right|^{\frac{1}{H}-1}$ *converges to zero almost surely as* $n\to\infty$ *due to the continuity of* X *while the sum is bounded by the total variation of* X *on the interval* $[0,1]$. *Also, with* $R_{n,r}$ *given by (2.10)*

$$\mathbf{E}\frac{1}{\sqrt{n}}|R_{n,r}| \leq \left(\sup_i\left|X_{t_{i+1}}-X_{t_i}\right|^{q-r-1}\left|B^H_{t_{i+1}}-B^H_{t_i}\right|^r\right)\sum_{i=0}^{n-1}\left|X_{t_{i+1}}-X_{t_i}\right|$$

and by the continuity of X *and* B^H *for* $r=0,1,\dots,q-1$ *the factor* $\sup_i\left|X_{t_{i+1}}-X_{t_i}\right|^{q-r-1}\left|B^H_{t_{i+1}}-B^H_{t_i}\right|^r$ *converges to zero almost surely as* $n\to\infty$.

2.3 Heat equation with fractional Laplacian driven by space-time white noise

We first consider the fractional heat equation driven by a time-space white noise. We study the existence, the probability distribution and the variations of the solution both in time and in space.

2.3.1 The equation and its solution

Consider the stochastic partial differential equation

$$\frac{\partial}{\partial t}u(t,x) = -(-\Delta)^{\frac{\alpha}{2}}u(t,x) + \dot{W}(t,x), \quad t \geq 0, x \in \mathbf{R}^d \tag{2.12}$$

with vanishing initial condition $u(0,x) = 0$ for every $x \in \mathbf{R}^d$. In (2.12), $-(-\Delta)^{\frac{\alpha}{2}}$ denotes the fractional Laplacian with exponent $\frac{\alpha}{2}$, $\alpha \in (1,2]$ and W is space-time white noise, i.e. $\big(W(t,A), t \geq 0, A \in \mathcal{B}(\mathbf{R}^d)\big)$ is a centered Gaussian field with covariance

$$\mathbf{E}W(t,A)W(s,B) = (t \wedge s)\lambda(A \cap B)$$

where λ denotes the Lebesgue measure on $\mathbf{R}^d$. We refer to [31], [32], [33] for the precise definition and other properties of the fractional Laplacian operator. We will here use only the expression of the Green kernel G_α (or the fundamental solution) associated to the fractional Laplacian, i.e. the deterministic kernel that solves the heat equation without noise $\frac{\partial}{\partial t}u(t,x) = -(-\Delta)^{\frac{\alpha}{2}}u(t,x)$. This Green kernel is defined through its Fourier transform

$$\mathcal{F}G_\alpha(t,\cdot)(\xi) = e^{-t\|\xi\|^\alpha}, \quad t > 0, \xi \in \mathbf{R}^d \tag{2.13}$$

where $\mathcal{F}G_\alpha(t,\cdot)$ is the Fourier transform of the function $y \to G_\alpha(t,y)$.

The mild solution to (2.12) is understood in the mild sense, i.e.

$$u(t,x) = \int_0^t \int_{\mathbf{R}^d} G_\alpha(t-u, x-z)W(du,dz) \tag{2.14}$$

where the above integral $W(du,dz)$ is a Wiener integral with respect to the Gaussian noise W.

First, we notice that the solution exists only in spatial dimension $d = 1$.

Proposition 2.3.1. *Let $\big(u(t,x), t \geq 0, x \in \mathbf{R}^d\big)$ be given by (3.5). Then the solution is well-defined if and only if $d = 1$. Moreover, in this case, for every $T > 0$*

$$\sup_{t \in [0,T], x \in \mathbf{R}} \mathbf{E}|u(t,x)|^2 < \infty.$$

Proof : From the Wiener isometry, the Plancherel identity and the expression of the Fourier transform (2.13), we have for every $t > 0, x \in \mathbf{R}^d$

$$\begin{aligned}\mathbf{E}u(t,x)^2 &= \int_0^t du \int_{\mathbf{R}^d} dz\, |G_\alpha(t-u,x-z)|^2 \\ &= (2\pi)^{-d} \int_0^t du \int_{\mathbf{R}^d} d\xi\, |\mathcal{F}G_\alpha(u,\cdot)(\xi)|^2 \\ &= (2\pi)^{-d} \int_0^t du \int_{\mathbf{R}^d} d\xi e^{-2u\|\xi\|^\alpha} = C_{d,\alpha} \int_0^t duu^{-\frac{d}{\alpha}}\end{aligned}$$

with $C_{d,\alpha} = (2\pi)^{-d} \int_{\mathbf{R}^d} d\xi e^{-2\|\xi\|^\alpha} < \infty$. The integral $\int_0^t u^{-\frac{d}{\alpha}} du$ is finite if and only if $1-\frac{d}{\alpha} > 0$ which means $d < \alpha$ or $d = 1$ since $\alpha \in (1,2]$. Moreover, for every $t \in [0,T], x \in \mathbf{R}^d$,

$$\mathbf{E}u(t,x)^2 \leq C_{1,\alpha} \frac{1}{1-\frac{1}{\alpha}} T^{1-\frac{1}{\alpha}} < \infty.$$

■

Next, we will focus on the probability distribution of the solution in spatial dimension $d = 1$. We will treat separately the behavior in time and in space.

2.3.2 Behavior in time

We consider here the process $(u(t,x), t \geq 0)$ with $x \in \mathbf{R}$ fixed. The distribution and the properties of this Gaussian process will follow easily from the computation of its covariance.

Proposition 2.3.2. *For every $s,t \geq 0$ and $x \in \mathbf{R}$ we have*

$$\mathbf{E}u(t,x)u(s,x) = c_{1,\alpha} \left[(t+s)^{1-\frac{1}{\alpha}} - |t-s|^{1-\frac{1}{\alpha}} \right]$$

where

$$c_{1,\alpha} = \frac{1}{2\pi(\alpha-1)} \Gamma\left(\frac{1}{\alpha}\right).$$

Consequently, the process $(u(t,x))_{t\geq 0}$ has the same law as the process $c_{2,\alpha} B^{\frac{1}{2},1-\frac{1}{\alpha}}$, where $B^{\frac{1}{2},1-\frac{1}{\alpha}}$ is a bifractional Brownian motion with Hurst parameters $H = \frac{1}{2}$ and $K = 1-\frac{1}{\alpha}$ and

$$c_{2,\alpha}^2 = c_{1,\alpha} 2^{1-\frac{1}{\alpha}}. \tag{2.15}$$

Proof : We follow the lines of the proof of Proposition 2.3.1. Assuming that $0 \leq s \leq t$, we have from (2.13)

$$\begin{aligned}
\mathbf{E}u(t,x)u(s,x) &= \int_0^{t\wedge s} du \int_{\mathbf{R}} dz G_\alpha(t-u,x-z)G_\alpha(s-u,x-z) \\
&= \int_0^{t\wedge s} du \int_{\mathbf{R}} d\xi \mathcal{F}G_\alpha(t-u,\cdot)(\xi)\overline{\mathcal{F}G_\alpha(s-u,\cdot)(\xi)} \\
&= (\pi)^{-1} \int_0^s du \int_0^\infty d\xi e^{-(t+s-2u)|\xi|^\alpha} \\
&= (\pi)^{-1}\frac{2}{\alpha} \int_0^\infty d\xi \mid \xi \mid^{\frac{2}{\alpha}-1} e^{-|\xi|^2} \int_0^s du(t+s-2u)^{-1/\alpha}
\end{aligned}$$

and the conclusion is obtained since

$$\int_0^\infty d\xi \mid \xi \mid^{\frac{2}{\alpha}-1} e^{-|\xi|^2} = \frac{1}{2}\Gamma\left(\frac{1}{\alpha}\right).$$

Then

$$\mathbf{E}u(t,x)u(s,x) = \frac{1}{2}(\pi)^{-1}\frac{2}{\alpha}\frac{1}{2}\Gamma\left(\frac{1}{\alpha}\right)\frac{1}{1-\frac{1}{\alpha}}\left[(t+s)^{1-\frac{1}{\alpha}} - |t-s|^{1-\frac{1}{\alpha}}\right].$$

Remark 2.3.1. *If $\alpha = 2$, then $c_{1,\alpha} = \frac{1}{2\sqrt{\pi}}$ since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. We retrieve a well-known formula (see [54] or [56]).*

From Proposition 2.3.2, we can deduce many properties of the process $t \to u(t,x)$. In particular, for every $x \in \mathbf{R}$,

- the process $(u(t,x))_{t\geq 0}$ is self-similar of order $\frac{1}{2}\left(1-\frac{1}{\alpha}\right)$ and it is Hölder continuous of order δ, for any $\delta \in \left(0, \frac{1}{2}\left(1-\frac{1}{\alpha}\right)\right)$.
- We have the following decomposition in law : $u(t,x) + Y_t = CB_t^{\frac{1}{2}(1-\frac{1}{\alpha})}$ where Y is a Gaussian process with absolute continuous paths and $B^{\frac{1}{2}(1-\frac{1}{\alpha})}$ denotes a fBm with Hurst index $\frac{1}{2}(1-\frac{1}{\alpha})$ and $C>0$ (see [36]).

Let us end this paragraph by stating the result on the behavior of the variation of the solution in time.
In the sequel we used the fact that two processes with the same finite dimensional distributions have the same variations (see e.g. Proposition 4 in [36]).

Proposition 2.3.3. *Fix $A_1 < A_2$ and $x \in \mathbf{R}$. Let $t_j = A_1 + \frac{j}{n}(A_2 - A_1), n \geq 1, j = 0, 1, .., n$ be a partition of the interval $[A_1, A_2]$. Then the process $(u(t,x), t \geq 0)$ admits variation of order $\frac{2\alpha}{\alpha-1}$ which is equal to*

$$c_{2,\alpha}^{\frac{2\alpha}{\alpha-1}} C_{1/2, 1-\frac{1}{\alpha}}(A_2 - A_1)$$

with $C_{1/2,1-\frac{1}{\alpha}}(A_2 - A_1)$ from (2.4) and $c_{2,\alpha}$ from (3.7).

Proof : This is an immediate consequence of Proposition 3.2.1 and Proposition 2.3.2. ■

Remark 2.3.2. *For $\alpha = 2$, we retrieve a result from [54] : the solution to the standard heat equation with time-space white noise has non-trivial quartic variation.*

Concerning the renormalized q-variations, we have the following.

Proposition 2.3.4. *Fix $A_1 < A_2$ and $x \in \mathbf{R}$. Let $t_j = A_1 + \frac{j}{n}(A_2 - A_1), n \geq 1, j = 0, 1, .., n$ be a partition of the interval $[A_1, A_2]$. Then*

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\left(\frac{n^{\frac{\alpha-1}{2\alpha}}}{c_{2,\alpha} b 2^{\frac{1}{2\alpha}} (A_2 - A_1)^{\frac{\alpha-1}{2\alpha}}} \right)^q (u(t_{i+1}, x) - u(t_i, x))^q - \mu_q \right] \to N(0, \sigma^2_{\frac{1}{2}(1-\frac{1}{\alpha}), q})$$

with $\sigma^2_{\frac{1}{2}(1-\frac{1}{\alpha}),q}$ a strictly positive constant depending on q and H..

Proof : From [36], we know that

$$(u(t,x) + C_1 X_t)_{t \geq 0} \equiv^{(d)} c_{2,\alpha} 2^{\frac{1}{2\alpha}} B_t^{\frac{1}{2}(1-\frac{1}{\alpha})}$$

where " $\equiv^{(d)}$ " means the equivalence of finite dimensional distributions, $B^{\frac{1}{2}(1-\frac{1}{\alpha})}$ is a fBm with Hurst parameter $\frac{1}{2}(1 - \frac{1}{\alpha})$, $C_1 > 0$ and $(X_t)_{t \geq 0}$ is a Gaussian process which satisfies (2.11) from Remark 2.2.1. Therefore, $(2^{-\frac{1}{2\alpha}} u(t,x), t \geq 0)$ is a perturbed fBm in the sense of Lemma 1. Also note that its Hurst parameter is strictly less than $\frac{1}{2}$. We can then apply Lemma 1 to obtain the conclusion. ■

2.3.3 Behavior in space

An analysis of the process (3.5) with respect to its space variable has been done in [27]. Let us recall the main facts.

From Proposition 3.1 in [27] we know that for every $t > 0$ the process $(u(t,x), x \in \mathbf{R})$ can be decomposed as

$$u(t,x) \equiv^{(d)} m_\alpha B^{\frac{\alpha-1}{2}}(x) + S(x) \tag{2.16}$$

where $B^{\frac{\alpha-1}{2}}$ is a fractional Brownian motion with Hurst parameter $\frac{\alpha-1}{2} \in [0, \frac{1}{2}]$, $(S(x))_{x\in\mathbf{R}}$ is a centered Gaussian process with C^∞ sample paths and m_α is the following numerical constant

$$m_\alpha = \left(2\Gamma(\alpha)|\cos(\alpha\pi/2)|\right)^{-\frac{1}{2}}.$$

From the decomposition (2.16) we notice that the regularity of $x \to u(t,x)$ is given by the fractional Brownian motion $B^{\frac{\alpha-1}{2}}$. In particular for every $x, y \in \mathbf{R}$ and $t > 0$

$$\mathbf{E}\,|u(t,x) - u(t,y)|^2 \leq C|x-y|^{\alpha-1}$$

which implies that the function $x \to u(t,x)$ is Hölder continuous of order $\delta \in (0, \frac{\alpha-1}{2})$. Actually, we have a more precise result in Lemma 2.1 in [27], i.e. for every $x, y \in \mathbf{R}$ and $t > 0$ and for δ close to zero,

$$\mathbf{E}\,|u(t,x+\delta) - u(t,x)|^2 = m_\alpha^2 \delta^{\alpha-1} + O(\delta^2).$$

From the decomposition (2.16) we can deduce the q- variation of the solution u with respect to the space variable.

Proposition 2.3.5. *Fix $A_1 < A_2$ and $t > 0$. Let $x_j = A_1 + \frac{j}{n}(A_2 - A_1)$ for $j = 0,..,n$ and $n \geq 1$. Then the process $(u(t,x), x \in \mathbf{R})$ has $\frac{2}{\alpha-1}$- variation, i.e. we have the following limit in probability*

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} |u(t,x_{j+1}) - u(t,x_j)|^{\frac{2}{\alpha-1}} = m_\alpha^{\frac{2}{\alpha-1}} \mathbf{E}|Z|^{\frac{2}{\alpha-1}} (A_2 - A_1).$$

Proof : Notice that the process S satisfies condition (2.6). This follows from [27], but also from the proof of Proposition 2.4.6 below. We can then apply Lemma 1, point 1. ■.

From Lemma 1 and (2.16) we have the following result.

Proposition 2.3.6. *Fix $A_1 < A_2$ and $t > 0$. Let $x_j = A_1 + \frac{j}{n}(A_2 - A_1)$ for $j = 0,..,n$ and $n \geq 1$. Then if $\alpha \in (1,2)$*

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\left(\frac{n^{\frac{\alpha-1}{2}}}{m_\alpha} \right)^q (u(t,x_{i+1}) - u(t,x_i))^q - \mu_q \right] \to N(0, \sigma^2_{\frac{\alpha-1}{2},q}). \tag{2.17}$$

If $\alpha = 2$ *(i.e.* $\frac{\alpha-1}{2} = \frac{1}{2}$*), then*

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\frac{n^{\alpha-1}}{m_2^2} \left(u(t, x_{i+1}) - u(t, x_i)\right)^2 - 1 \right] \to N(0, \sigma^2_{\frac{1}{2},2})$$

with the constant $\sigma_{H,q}$ *from (3.11).*

Proof : It suffices to note that from [27], $\mathbf{E}|S(x) - S(y)|^2 \leq c|t-s|^2$ and that the Hurst index $\frac{\alpha-1}{2}$ is less than $\frac{1}{2}$. For $\alpha = 2$ the result is known from [49].

■

Let us make some comments :

Remark 2.3.3.
1. *When* $\alpha = 2$*, we have (exact) quadratic variation in space for the solution. We retrieve again a known result from [54].*
2. *We notice that the regularity in space is* α *times the regularity in time (i.e.* $\frac{\alpha-1}{2\alpha}$*-Hölder continuity in time and* $\frac{\alpha-1}{2}$*-Hölder regularity in space). The phenomenon was known for* $\alpha = 2$*.*

2.4 Heat equation with fractional Laplacian and white-colored noise

In this section, we will add a new parameter to the heat equation (2.12), by considering a Gaussian noise which behaves as a fractional Brownian motion in space, i.e. its spatial covariance is given by the so-called Riesz kernel. More precisely, we will consider the stochastic heat equation

$$\frac{\partial}{\partial t} u(t, x) = -(-\Delta)^{\frac{\alpha}{2}} u(t, x) + \dot{W}^{\gamma}(t, x), \quad t \geq 0, x \in \mathbf{R}^d \tag{2.18}$$

with $u(0, x) = 0$ for every $x \in \mathbf{R}^d$. In (2.12), $-(-\Delta)^{\frac{\alpha}{2}}$ denotes the fractional Laplacian with exponent $\frac{\alpha}{2}$, $\alpha \in (1, 2]$ and W^{γ} is the so-called white-colored noise, i.e. $W^{\gamma}(t, A), t \geq 0, A \in \mathcal{B}(\mathbf{R}^d)$ is a centered Gaussian field with covariance

$$\mathbf{E} W^{\gamma}(t, A) W^{\gamma}(s, B) = (t \wedge s) \int_A \int_B f(x-y) dx dy$$

where f is the so-called Riesz kernel of order γ given by

$$f(x) = R_{\gamma}(x) := g_{\gamma,d} \|x\|^{-d+\gamma}, \quad 0 < \gamma < d, \tag{2.19}$$

where $g_{\gamma,d} = 2^{d-\gamma}\pi^{d/2}\Gamma((d-\gamma)/2)/\Gamma(\gamma/2)$. In this case, if we consider the measure $\mu(d\xi) = \|\xi\|^{-\gamma}d\xi$ we have the identity

$$\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\varphi(x)f(x-y)\psi(y)dxdy = (2\pi)^{-d}\int_{\mathbf{R}^d}\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(d\xi). \tag{2.20}$$

for any $\varphi, \psi \in \mathcal{S}(\mathbf{R}^d)$ (the Schwartz space on $\mathbf{R}^d$).

As usual, the mild solution to (2.12) is given by

$$u(t,x) = \int_0^t \int_{\mathbf{R}^d} G_\alpha(t-u, x-z)W^\gamma(du,dz) \tag{2.21}$$

where the above integral $W^\gamma(du,dz)$ is a Wiener integral with respect to the Gaussian noise W^γ.

Let us first give the necessary and sufficient condition for the existence of the mild solution.

Proposition 2.4.1. *The mild solution (2.21) to the heat equation (2.18) is well-defined if and only if*

$$d < \alpha + \gamma. \tag{2.22}$$

Moreover, in this case, for every $T > 0$

$$\sup_{t\in[0,T], x\in\mathbf{R}} \mathbf{E}|u(t,x)|^2 < \infty.$$

Proof : Using the identity (2.20), we have for $t \geq 0, x \in \mathbf{R}^d$,

$$\begin{aligned}
\mathbf{E}u(t,x)^2 &= g_{\gamma,d}\int_0^t du \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} dydz G_\alpha(t-u, x-z)G_\alpha(t-u,x-y)|y-z|^{-(d-\gamma)} \\
&= g_{\gamma,d}(2\pi)^{-d}\int_0^t du \int_{\mathbf{R}^d} d\xi e^{-2\|\xi\|^\alpha u}\|\xi\|^{-\gamma} \\
&= C_{1,\alpha,\gamma}\int_0^t du u^{-\frac{d-\gamma}{\alpha}}
\end{aligned}$$

with $C_{1,\alpha,\gamma} = g_{\gamma,d}(2\pi)^{-d}\int_{\mathbf{R}^d} d\xi e^{-2\|\xi\|^\alpha}\|\xi\|^{-\gamma} < \infty$. The integral du is finite if and only if $1 - \frac{d-\gamma}{\alpha} >$ which implies (2.22). The last bound in the statement is also trivial from the above computation. ■

2.4.1 Behavior in time

In the next result we deduce the law of the Gaussian process $u(t,x), t \geq 0$ with $x \in \mathbf{R}^d$ fixed.

Proposition 2.4.2. *For every $s,t \geq 0$, and for every $x \in \mathbf{R}^d$, we have*

$$\mathbf{E}u(t,x)u(s,x) = c_{1,\alpha,\gamma}\left[(t+s)^{1-\frac{d-\gamma}{\alpha}} - |t-s|^{1-\frac{d-\gamma}{\alpha}}\right]$$

where

$$c_{1,\alpha,\gamma} = g_{\gamma,d}(2\pi)^{-d}\int_{\mathbf{R}^d} d\xi \|\xi\|^{-\gamma} e^{-\|\xi\|^{\alpha}} \frac{1}{2(1-\frac{d-\gamma}{\alpha})}. \tag{2.23}$$

Consequently, the process $(u(t,x))_{t\geq 0}$ has the same law as $c_{2,\alpha,\gamma}(B_t^{\frac{1}{2},1-\frac{d-\gamma}{\alpha}})_{t\geq 0}$ where $B^{\frac{1}{2},1-\frac{d-\gamma}{\alpha}}$ is a bi-fBm with $H=\frac{1}{2}$ and $K = 1-\frac{d-\gamma}{\alpha}$ and

$$c_{2,\alpha,\gamma}^2 = c_{1,\alpha,\gamma} 2^{1-\frac{d-\gamma}{\alpha}}. \tag{2.24}$$

Proof : As in the proof of Proposition 2.3.2 we have for $0 \leq s \leq t$ and for $x \in \mathbf{R}^d$,

$$\begin{aligned}\mathbf{E}u(t,x)u(s,x) &= g_{\gamma,d}(2\pi)^{-d}\int_0^{t\wedge s} du \int_{\mathbf{R}^d} d\xi \|\xi\|^{-\gamma} e^{-(t-u)\|\xi\|^{\alpha}} e^{-(s-u)\|\xi\|^{\alpha}} \\ &= g_{\gamma,d}(2\pi)^{-d}\int_{\mathbf{R}^d} d\xi \|\xi\|^{-\gamma} e^{-\|\xi\|^{\alpha}} \int_0^s du (t+s-2u)^{-\frac{d-\gamma}{\alpha}} \\ &= c_{1,\alpha,\gamma}\left((t+s)^{1-\frac{d-\gamma}{\alpha}} - \mid t-s \mid^{1-\frac{d-\gamma}{\alpha}}\right)\end{aligned}$$

with $c_{1,\alpha,\gamma}$ given by (2.23). ∎

As a consequence, the process $t \to u(t,x)$ is Hölder continuous of order δ for every $\delta \in (0, 1-\frac{d-\gamma}{\alpha})$ and it is self-similar of order $\frac{1}{2}\left(1-\frac{d-\gamma}{\alpha}\right)$.

Now, it is immediate to obtain the q-variations of the process u in time. The proof is similar to the proof of Proposition 2.3.3.

Proposition 2.4.3. *Fix $A_1 < A_2$ and $x \in \mathbf{R}$. Let $t_j = A_1 + \frac{j}{n}(A_2 - A_1), n \geq 1, j = 0,1,..,n$ be a partition of the interval $[A_1, A_2]$. Then the process $(u(t,x), t \geq 0)$ admits a variation of order $\frac{2\alpha}{\alpha+\gamma-d}$ which is equal to*

$$c_{2,\alpha,\gamma}^{\frac{2\alpha}{\alpha+\gamma-d}} C_{\frac{1}{2},1-\frac{d-\gamma}{\alpha}}(A_2 - A_1)$$

with $C_{\frac{1}{2},1-\frac{d-\gamma}{\alpha}}$ from (2.4) and $c_{2,\alpha,\gamma}$ from (2.24).

And from the proof of Proposition 2.3.4 and Remark 2.2.1, we have

Proposition 2.4.4. *Fix $A_1 < A_2$ and $x \in \mathbf{R}$. Let $t_j = A_1 + \frac{j}{n}(A_2 - A_1), n \geq 1, j = 0, 1, .., n$ be a partition of the interval $[A_1, A_2]$. Then*

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\left(\frac{n^{\frac{\alpha+\gamma-d}{2\alpha}}}{c_{2,\alpha,\gamma} 2^{\frac{d-\gamma}{2\alpha}} (A_2 - A_1)^{\frac{\alpha+\gamma-d}{2\alpha}}} \right)^q (u(t_{i+1}, x) - u(t_i, x))^q - \mu_q \right] \to N(0, \sigma^2_{\frac{1}{2}(1-\frac{d-\gamma}{\alpha}),q})$$

with $\sigma^2_{\frac{1}{2}(1-\frac{1}{\alpha}),q}$ is a strictly positive constant depending on q and H..

When $\gamma = 0$ and $d = 1$, we retrieve the result in the case of the white noise in space.

The next step is to study the behavior in space of (2.21). Now, we work in spatial dimension $d \geq 1$. In this case the solution will be related to the multiparameter isotropic fractional Brownian motion. For this reason, let us present below the definition and the basic properties of this process.

2.4.2 Isotropic fractional Brownian motion

In this paragraph we will use bold notation to indicate vectors in $\mathbf{R}^d$ to distinguish them from real numbers in order to avoid confusion.

The isotropic multiparameter fBm (also known as the Lévy fBm) $(B^H(\mathbf{x}), \mathbf{x} \in \mathbf{R}^d)$ with Hurst parameter $H \in (0, 1)$ is defined as a centered Gaussian process, starting from zero, with covariance function

$$\mathbf{E}(B^H(\mathbf{x})B^H(\mathbf{y})) = \frac{1}{2} \left(\|\mathbf{x}\|^{2H} + \|\mathbf{y}\|^{2H} - \|\mathbf{x} - \mathbf{y}\|^{2H} \right) \text{ for every } \mathbf{x}, \mathbf{y} \in \mathbf{R}^d \tag{2.25}$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbf{R}^d$. It can be also represented as a Wiener integral with respect to the Wiener sheet, see [29], [37].

The isotropic multiparameter fBm is self-similar and it has stationary increments in the following sense : for every $\mathbf{h} \in \mathbf{R}^d_+$

$$\left(B^H(\mathbf{x} + \mathbf{h}) - B^H(\mathbf{h}) \right)_{\mathbf{x} \in \mathbf{R}^d_+} \equiv^{(d)} \left(B^H(\mathbf{x}) \right)_{\mathbf{x} \in \mathbf{R}^d_+} \tag{2.26}$$

and for every $a > 0$,

$$(B^H(a\mathbf{x})_{\mathbf{x} \in \mathbf{R}^d_+} \equiv^{(d)} \left(B^H(\mathbf{x}) \right)_{\mathbf{x} \in \mathbf{R}^d_+} \tag{2.27}$$

Recall that we denoted by $\equiv^{(d)}$ the equivalence in the sense of finite dimensional distributions. It is also possible to prove the stationarity of the increments in some generalized sense, by using higher order difference (see Proposition 6 in [29]). An important property, which makes this Gaussian sheet different from the anisotropic fractional Brownian motion is that for every $x, y \in \mathbf{R}^d$

$$\mathbf{E}\left(B^H(\mathbf{x}) - B^H(\mathbf{y})\right)^2 = \|\mathbf{x}-\mathbf{y}\|^{2H}$$

which implies, due to the Gaussianity, that for every $n \geq 1$

$$\mathbf{E}\left(B^H(\mathbf{x}) - B^H(\mathbf{y})\right)^n = \mathbf{E}|Z|^n \|\mathbf{x}-\mathbf{y}\|^{nH} \tag{2.28}$$

where Z is a standard normal random variable. From (2.28) one can deduce, via a standard argument, the existence of a continuous version for B^H, see e.g. [35].

Following the one-parameter case, we define the q-variation of the isotropic fBm as the limit in probability as $n \to \infty$, of the sequence

$$V^{n,q}_{[A_1,A_2]}(B^H) = \sum_{i=0}^{n-1} \left|B^H(\mathbf{x}_{i+1}) - B^H(\mathbf{x}_i)\right|^q$$

where $\mathbf{x}_i = (x_i^{(1)}, .., x_i^{(d)})$ with $x_i^{(j)} = A_1 + \frac{i}{n}(A_2 - A_1)$ for $i = 0, .., n$ and $j = 1, .., d$.

Let us state the result on the variation of the isotropic fractional Brownian sheet. Even if its proof follows easily from the one-parameter case, it has not been stated before, as far as we know.

Proposition 2.4.5. *The isotropic fBm $\left((B^H(\mathbf{x}), \mathbf{x} \in \mathbf{R}^d)\right)$ has $\frac{1}{H}$-variation over $[A_1, A_2]$ which is equal to*

$$(A_2 - A_1)\mathbf{E}|B_{\mathbf{1}}|^{1/H} = (A_2 - A_1)\sqrt{d}\mathbf{E}|Z|^{1/H}$$

Proof : Consider the sequence

$$Y_{n,q} = n^{qH-1} \sum_{i=0}^{n-1} \left|B^H(\mathbf{x}_{i+1}) - B^H(\mathbf{x}_i)\right|^q.$$

From (2.27) and (2.26), it has the same law as

$$Y'_{n,q} = (A_2 - A_1)^{qH} \frac{1}{n} \sum_{i=0}^{n-1} \left|B^H(\mathbf{j}+\mathbf{1}) - B^H(\mathbf{j})\right|^q$$

with $\mathbf{j} = (j, .., j) \in \mathbf{R}^d$. The sequence $\left(B^H(\mathbf{j}+\mathbf{1}) - B^H(\mathbf{j})\right)_{\mathbf{j}\in\mathbf{Z}^d}$ is stationary and it has the same law as $d^{H/2}(B_{j+1} - B_j)$ where B is a one-parameter fBm with Hurst parameter H.

By the ergodic theorem, $Y'_{n,q}$ converges to $(A_2 - A_1)^{qH}\mathbf{E}|B_{\mathbf{1}}|^q$. Taking $q = \frac{1}{H}$, we obtain the conclusion. ■

Following the proof of Lemma 3, can obtain the q-variation of the isotropic fBm perturbed by a regular multiparameter process.

Lemma 2. *Let $(B^H(\mathbf{x}))_{\mathbf{x}\in\mathbf{R}^d}$ be a d-parameter isotrpic fBm and consider $(X(x))_{\mathbf{x}\in\mathbf{R}^d}$ a d-parameter stochastic process that satisfies*

$$\mathbf{E}\left(X(\mathbf{x}) - X(\mathbf{y})\right|^2 \leq C\|\mathbf{x}-\mathbf{y}\|^2, \text{ for every } \mathbf{x},\mathbf{y}\in\mathbf{R}^d. \tag{2.29}$$

Define

$$Y(\mathbf{x}) = B^H(\mathbf{x}) + X(\mathbf{x}) \text{ for every } \mathbf{x}\in\mathbf{R}^d.$$

Then

1. *the process $(Y(\mathbf{x}))_{\mathbf{x}\in\mathbf{R}^d}$ has $\frac{1}{H}$-variation which is equal to*

$$(A_2 - A_1)\sqrt{d}\mathbf{E}|Z|^{1/H}.$$

2. *Then if $H\in(0,\frac{1}{2})$ and $q\geq 2$*

$$\frac{1}{\sqrt{n}}V_{q,n}(Y^H) = \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\left[\frac{n^{Hq}d^{-Hq/2}}{(A_2-A_1)^{qH}}(Y^H(\mathbf{x}_{i+1}) - Y^H(\mathbf{x}_i))^q - \mu_q\right] \to N(0,\sigma^2_{H,q}). \tag{2.30}$$

Proof : As in the proof of Lemma 3, we have the double inequality (2.9) due to Minkowski inequality. The sequence

$$\sum_{i=0}^{n-1}\left|B^H(\mathbf{x}_{i+1}) - B^H(\mathbf{x}_i)\right|^{\frac{1}{H}}$$

converges again almost surely and in L^1 to the desired limit, by Proposition 2.4.5. It remains to show that $\sum_{i=0}^{n-1}|X(\mathbf{x}_{i+1}) - X(\mathbf{x}_i)|^{\frac{1}{H}}$ converges to zero in probability and this is an easy consequence of the hypothesis (2.29).

For point 2. it suffices to observe that the vector $\left(B^H(\mathbf{x}_{i+1}) - B^H(\mathbf{x}_i)\right)_{0,1,..,n-1}$ has the same law as $d^{H/2}(B^H(x_{j+1}) - B(x_j))_{0,1,..,n-1}$ where B is a one-parameter fBm with Hurst parameter H and to apply Lemma 3. ■

2.4.3 Behavior in space

We go back to the process (2.21) and we analyze its behavior in space. We prove the following result.

Proposition 2.4.6. *Fix $t > 0$. Then the process $(u(t,x))_{x\in\mathbf{R}^d}$ has the same finite dimensional distribution as*

$$c_{3,\alpha,\gamma}B^{\frac{\alpha+\gamma-d}{2}}(x) + S(x), x \in \mathbf{R}^d$$

where $B^{\frac{\alpha+\gamma-d}{2}}$ is an isotropic multiparameter fBm with Hurst index $\frac{\alpha+\gamma-d}{2}$, $S(x), x \in \mathbf{R}^d$ is a Gaussian process with C^∞ paths which satisfies (2.29) and

$$c^2_{3,\alpha,\gamma} = (2\pi)^{-d}\int_{\mathbb{R}^d} dw\|w\|^{-(\alpha+\gamma)}(1-\cos(w\cdot e_1)) \text{ with } e_1 = (1, 0..., 0). \tag{2.31}$$

Proof : Define, for $x \in \mathbf{R}^d$,

$$S(x) = \int_t^\infty \int_{\mathbf{R}^d} \left[G_\alpha(s,z) - G_\alpha(u, x-z)\right] W^\gamma(ds, dz). \tag{2.32}$$

Then we have

$$\begin{aligned}
\mathbf{E}|S(x)|^2 &= \int_t^\infty ds \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} dzdz' \left(G_\alpha(s,z) - G_\alpha(s,x-z)\right)\left(G_\alpha(s,z') - G_\alpha(s,x-z')\right) f(z-z') \\
&= (2\pi)^d \int_t^\infty du \int_{\mathbf{R}^d} d\xi\|\xi\|^{-\gamma}\left(\mathcal{F}G_\alpha(s,\cdot)(\xi) - \mathcal{F}G_\alpha(s,x-\cdot)(\xi)\right)\overline{\left(\mathcal{F}G_\alpha(s,\cdot)(\xi) - \mathcal{F}G_\alpha(s,x-\cdot)(\xi\right.} \\
&= (2\pi)^{-d}\int_t^\infty du \int_{\mathbf{R}^d} d\xi\|\xi\|^{-\gamma}e^{-2u\|\xi\|^\alpha}\left|1-e^{-i\xi\cdot x}\right|^2
\end{aligned}$$

where we used the Parseval formula (2.20). Now, by using Fubini and computing the integral du, we get

$$\mathbf{E}|S(x)|^2 = (2\pi)^{-d}\int_{\mathbf{R}^d} d\xi\|\xi\|^{-\gamma-\alpha}(1-\cos(\xi\cdot x))e^{-2t\|\xi\|^\alpha} < \infty.$$

The function under the integral $d\xi$ is integrable at infinity because of the presence of the exponential function, while in the vicinity of zero we use $|1-\cos(\xi\cdot x)| \leq c\|\xi\|^2$ and then

$$\|\xi\|^{-\gamma-\alpha}(1-\cos(\xi\cdot x))e^{-2t\|\xi\|^\alpha} \leq C\|\xi\|^{-\alpha-\gamma+2}$$

which is integrable for ξ close to zero since $-\alpha-\gamma+2+d>0$. (since $\alpha<2$ et $\gamma<d$) In the same we can control the increments of S to prove (2.29). Indeed, for $x_1, x_2 \in \mathbf{R}^d$,

$$\begin{aligned}
\mathbf{E}\left|S(x_2)-S(x_1)\right|^2 &= (2\pi)^{-d}\int_t^\infty du\int_{\mathbf{R}^d} d\xi\|\xi\|^{-\gamma}e^{-2u\|\xi\|^\alpha}\left|1-e^{-i\xi\cdot(x_2-x_1)}\right|^2\\
&= 2(2\pi)^{-d}\int_t^\infty du\int_{\mathbf{R}^d} d\xi\|\xi\|^{-\gamma}e^{-2u\|\xi\|^\alpha}(1-\cos(\xi\cdot(x_2-x_1))\\
&= (2\pi)^{-d}\int_{\mathbf{R}^d} d\xi\|\xi\|^{-\gamma-\alpha}(1-\cos(\xi\cdot(x_2-x_1)))e^{-2t\|\xi\|^\alpha}\\
&\leq (2\pi)^{d}\|x_2-x_1\|^2\int_{\mathbf{R}^d} d\xi\|\xi\|^{-\gamma-\alpha+2}e^{-2t\|\xi\|^\alpha}
\end{aligned}$$

since $1-\cos(\xi\cdot(x_2-x_1))\leq\|x_2-x_1\|^2\|\xi\|^2$.

Let us put $U(x)=u(t,x)-S(x)$ for every $x\in\mathbf{R}^d$. We will show that $(U(x))_{x\in\mathbf{R}^d}$ is modulo a constant, an isotropic fBm. We can write, for $x,y\in\mathbf{R^d}$, by using the independence of u and S (because the noise W^γ is white in time)

$$\begin{aligned}
\mathbf{E}\left|U(x)-U(y)\right|^2 &= \mathbf{E}\left[\int_0^t\int_{\mathbf{R}^d}\left(G_\alpha(u,x-z)-G_\alpha(u,y-z)\right)W^\gamma(du,dz)\right]^2\\
&\quad+\mathbf{E}\left[\int_t^\infty\int_{\mathbf{R}^d}\left(G_\alpha(u,x-z)-G_\alpha(u,y-z)\right)W^\gamma(du,dz)\right]^2\\
&= \mathbf{E}\left[\int_0^\infty\int_{\mathbf{R}^d}\left(G_\alpha(u,x-z)-G_\alpha(u,y-z)\right)W^\gamma(du,dz)\right]^2\\
&= 2(2\pi)^{-d}\int_0^\infty du\int_{\mathbf{R}^d} d\xi\|\xi\|^{-\gamma}e^{-2u\|\xi\|^\alpha}(1-\cos(\xi\cdot(x-y)))\\
&= (2\pi)^{-d}\int_{\mathbf{R}^d} d\xi\|\xi\|^{-\gamma-\alpha}(1-\cos(\xi\cdot(x-y)))
\end{aligned}$$

where we computed the integral du. We will have

$$\mathbf{E}\left|U(x)-U(y)\right|^2=(2\pi)^{-d}\|y-x\|^{\alpha+\gamma-d}\int_{\mathbb{R}^d} dw\|w\|^{-(\alpha+\gamma)}(1-\cos(w\cdot e_1))$$

with $e_1=(1,0,...,0)$. The last identity follows from Proposition 2 in [29].

The last relation implies that U coincides in law with $c_{3,\alpha,\gamma}B^{\frac{\alpha+\gamma-d}{2}}$ where $B^{\frac{\alpha+\gamma-d}{2}}$ is an isotropic fBm and $c^2_{3,\alpha,\gamma}$ given by (2.31). ■

From Lemma 2 and Proposition 2.4.6, we have

Proposition 2.4.7. *The process $x \to u(t,x)$ has $\frac{2}{\alpha+\gamma-d}$- variation given by $c_{3,\alpha,\gamma}^{\frac{2}{\alpha+\gamma-d}}(A_2 - A_1)\sqrt{d}\mathbf{E}|Z|^{\frac{2}{\alpha+\gamma-d}}$.*

and

Proposition 2.4.8. *Fix $A_1 < A_2$ and $t > 0$. Let $x_j^k = A_1 + \frac{j}{n}(A_2 - A_1)$ for $j = 0, .., n$, $n \geq 1$ and for every $k = 1, .., d$. Also let $\mathbf{x}_j = (x_j^1, .., x_j^d)$. Then if $\alpha + \gamma - d < 1$*

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\left(\frac{n^{\frac{\alpha+\gamma-d}{2}} d^{-H/2}}{c_{3,\alpha,\gamma}} \right)^q (u(t, \mathbf{x}_{i+1}) - u(t, \mathbf{x}_i))^q - \mu_q \right] \to N(0, \sigma^2_{\frac{\alpha+\gamma-d}{2},q})$$

with $c_{3,\alpha,\gamma}$ from (2.31) and $\sigma_{\frac{\alpha+\gamma-d}{2},q}$ from (2.17).

Notice that we restricted to the situation $\alpha + \gamma - d < 1$ which means that the parameter of the isotropic fBm associated to the solution is strictly less than one-half and we can apply Lemma 2.

Chapitre 3

Estimation of the drift parameter for the fractional stochastic heat equation via power variation

Sommaire

We define power variation estimators for the drift parameter of the stochastic heat equation with fractional Laplacian and with additive Gaussian noise which is white in time and white or correlated in space. We prove that these estimators are consistent and asymptotically normal and we derive their rate of convergence under the Wasserstein metric.

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Key Words and Phrases : Stochastic heat equation, Fractional Brownian motion, Fractional Laplacian, q variation ; drift parameter estimation.

3.1 Introduction

The purpose of this work is to estimate the drift parameter $\theta > 0$ of the fractional stochastic heat equation

$$\frac{\partial u_\theta}{\partial t}(t,x) = -\theta(-\Delta)^{\frac{\alpha}{2}} u_\theta(t,x) + \dot{W}(t,x), \quad t \geq 0, x \in \mathbf{R} \tag{3.1}$$

with vanishing initial conditions, where $(-\Delta)^{\frac{\alpha}{2}}$ denotes the fractional Laplacian of order $\alpha \in (1,2]$, $\theta > 0$ and W is a Gaussian noise which is white in time and white or correlated in space.

The parameter estimation for stochastic partial differential equations (SPDEs in the sequel) constitutes a research direction of wide interest in probability theory and mathematical statistics. We refer, among many others, to the recent surveys [38] and [15]. On the other side, there are relatively few works that consider the solution to a SPDE observed at discrete points in time and/or in space. Among the first works in this direction, we refer to [43] and [42] for the maximum likelihood and least square estimators for parabolic, respectively elliptic-type SPDEs driven by a space-time white noise. The study in [42] has been then extended by [6], by adding a time-varying volatility in the noise term and by using power variation techniques to estimate the parameter of the model. Other recent works on parameter estimates for discretely sampled SPDEs via power variations are [16], [18], [7], [49] and [60].

In our work, we extend the above results into two directions. Firstly, we replace the standard Laplacian operator used in all the above references by a fractional Laplacian. On the other hand, we consider a simpler form, comparing to [6], [42], of the differential operator. Secondly, we also consider a noise term which is correlated in space. Our purpose is to propose power variation-type estimators for the drift parameter in the stochastic model (3.1), based on discrete observation of the solution in time or in space, and to analyze the consistency and the limit distribution of the estimators by taking advantage of the link between the solution and the fractional Brownian motion. Our approach to construct and analyze the estimators for the drift parameter is based on the asymptotic behavior of the q-variations of the mild solution to (3.1). It is well-known (see e.g. [27], [39], [56]) that there exists a strong link between the law of this mild solution with $\theta = 1$ and the fractional

Brownian motion and related processes. We will use this connection in order to deduce the behavior of the q-variations (of suitable order q) of the solutions to (3.1) and to prove the consistency, asymptotic normality and Berry-Esséen bounds under the Wasserstein distance for the associated estimators. For the situation when W is a space-time white noise, we will obtain two estimators for the drift parameter : one based on the temporal variations and one based on the spatial variations of the mild solution u_θ. Similarly, two estimators are defined when the Gaussian noise W is white in time and colored in space (with the spatial covariance given by the Riesz kernel). Even if the order of the variations which appears in the definition of the estimator is different in the four cases (this order may depend on the parameter α of the fractional Laplacian and/or on the spatial correlation), all the estimators are asymptotically normal, they have the same rate of convergence of order $n^{-\frac{1}{2}}$ and they have the same distance to the Gaussian distribution. The case of the standard Laplacian (i.e. $\alpha = 2$) has been studied in [49].

We organize the paper as follows. In Section 2 we present general facts on the stochastic heat equation with fractional Laplacian and the behavior of the variations of the perturbed fractional Brownian motion. In Section 3 we discuss the drift parameter estimation for the fractional heat equation with space-time white noise while in Section 4 we treat the case when the noise is correlated in space.

We will denote by c, C a generic positive constant that may change from line to line (or even inside of the the same line). By $\to^{(d)}$ we denote the convergence in distribution while $\equiv^{(d)}$ stands for the equivalence of finite dimensional distributions.

3.2 The fractional heat equation driven by space-time white noise

We start by treating the fractional stochastic heat equation with space-time white noise. We recall the basic properties of the solution, its relation with fractional Brownian motion and then we discuss the estimation of the drift parameter θ via the q-variations.

3.2.1 General properties of the solution

On the standard probability space $(\Omega, \mathcal{F}, P)$, we consider a centered Gaussian field $(W(t, A), t \geq 0, A \in \mathcal{B}_b(\mathbf{R}))$ with covariance

$$\mathbf{E}W(t, A)W(s, B) = (s \wedge t)\lambda(A \cap B) \text{ for every } s, t \geq 0, A, B \in \mathcal{B}_b(\mathbf{R}) \tag{3.2}$$

where λ denotes the Lebesgue measure on $\mathbf{R}$ and $\mathcal{B}_b(\mathbf{R})$ is the class of bounded Borel subsets of $\mathbf{R}$. The Gaussian field W is usually called the space-time white noise.

We will consider the stochastic heat equation

$$\frac{\partial u_\theta}{\partial t}(t,x) = -\theta(-\Delta)^{\frac{\alpha}{2}} u_\theta(t,x) + \dot{W}(t,x), \quad t \geq 0, x \in \mathbf{R} \tag{3.3}$$

with vanishing initial condition $u(0,x) = 0$ for every $x \in \mathbf{R}$. In the above equation, $(-\Delta)^{\frac{\alpha}{2}}$ represents the fractional Laplacian of order α. We will assume in the sequel that $\alpha \in (1,2]$. We refer to [22], [31], [32], [33] for the precise definition and other properties of the fractional Laplacian operator. We will denote its Green kernel (or the fundamental solution) by G_α, which represents the deterministic kernel that solves the heat equation without noise $\frac{\partial}{\partial t} u(t,x) = -(-\Delta)^{\frac{\alpha}{2}} u(t,x)$. It is known from the above references that for $t > 0, x \in \mathbf{R}$

$$G_\alpha(t,x) = \int_{\mathbf{R}} e^{it\xi - t|\xi|^\alpha} d\xi. \tag{3.4}$$

It is immediate to see that the fundamental solution associated to the operator $-\theta(-\Delta)^{\frac{\alpha}{2}} u_\theta(t,x)$ is $G_\alpha(\theta t, x)$.

The solution to (4.1) is understood in the mild sense, i.e.

$$u_\theta(t,x) = \int_0^t \int_{\mathbf{R}} G_\alpha(\theta(t-s), x-y) W(ds,dy) \tag{3.5}$$

where the stochastic integral $W(ds,dy)$ is the usual Wiener integral with respect to the space-time white noise, which satisfies the isometry

$$\mathbf{E}\left(\int_0^T \int_{\mathbf{R}} H(s,y) W(ds,dy) \right)^2 = \int_0^T \int_{\mathbf{R}} H(s,y)^2 dy ds$$

for every $T > 0$ and for every measurable square integrable function H.

For $\theta = 1$, the solution to the heat equation (4.1) has been studied in [39]. This solution exists only if the spatial dimension is $d = 1$ and it is connected to the bifractional Brownian motion. Recall that (see [30]), given constants $H \in (0,1)$ and $K \in (0,1]$, the bifractional Brownian motion (bi-fBm for short) $(B_t^{H,K})_{t\geq 0}$ is a centered Gaussian process with covariance

$$R^{H,K}(t,s) := R(t,s) = \frac{1}{2^K}\left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK} \right), \quad s,t \geq 0. \tag{3.6}$$

In particular, for $K = 1$, $B^H := B^{H,1}$ is the fractional Brownian motion (fBm in the sequel) with Hurst parameter $H \in (0,1)$.

Let us recall some of the results in [39] which will be needed in the sequel.

— The mild solution (3.5) is well-defined. For every $x \in \mathbf{R}$, the process $(u_1(t,x), t \geq 0)$ coincides in distribution, modulo a multiplcative constant, with the bifractional Brownian motion, i.e.

$$(u_1(t,x), t \geq 0) \equiv^{(d)} \left(c_{2,\alpha} B_t^{\frac{1}{2}, 1-\frac{1}{\alpha}}, t \geq 0 \right)$$

where $B^{\frac{1}{2},1-\frac{1}{\alpha}}$ is a bifractional Brownian motion with Hurst parameters $H = \frac{1}{2}$, $K = 1 - \frac{1}{\alpha}$ and

$$c_{2,\alpha}^2 = c_{1,\alpha} 2^{1-\frac{1}{\alpha}} \text{ with } c_{1,\alpha} = \frac{1}{2\pi(\alpha-1)} \Gamma\left(\frac{1}{\alpha}\right). \tag{3.7}$$

— For every $t \geq 0$, we have (see Proposition 3.1 in [27])

$$(u_1(t,x), x \in \mathbf{R}) \equiv^{(d)} \left(m_\alpha B^{\frac{\alpha-1}{2}}(x) + S_t(x), x \in \mathbf{R} \right) \tag{3.8}$$

where $B^{\frac{\alpha-1}{2}}$ is a fractional Brownian motion with Hurst parameter $\frac{\alpha-1}{2} \in]0, \frac{1}{2}]$, $(S_t(x))_{x \in \mathbf{R}}$ is a centered Gaussian process with C^∞ sample paths and m_α is an explicit numerical constant.

The above facts, combined with the decomposition (3.18) of the bifractional Brownian motion, show that the solution to the heat equation, can be expressed as the sum of a fBm and a smooth process (we will call this sum a perturbed fractional Brownian motion).

3.2.2 Variations of the perturbed fractional Brownian motion

Since the process (3.5) is connected to the perturbed fBm (i.e., the sum of a fBm and of a smooth Gaussian process), let us recall some facts concerning the asymptotic behavior of the variation of the perturbed fBm. Some of the results below are directly taken from [39] while those concerning the rate of convergence under the Wasserstein distance are deduced from [45].

We first define the notion of *(exact) q-variation* for stochastic processes.

Definition 5. *Let $A_1 < A_2$ and for $n \geq 1$, let $t_i = A_1 + \frac{i}{n}(A_2 - A_1)$ for $i = 0,..,n$. A continuous stochastic process $(X_t)_{t \geq 0}$ admits a q-variation (or a variation of order q) over the interval $[A_1, A_2]$ if the sequence*

$$S_{[A_1,A_2]}^{n,q}(X) := \sum_{i=0}^{n-1} \left| X_{t_{i+1}} - X_{t_i} \right|^q$$

converges in probability as $n \to \infty$. The limit, when it exists, is called the exact q-variation of X over the interval $[A_1, A_2]$.

If $[A_1, A_2] = [0, t]$, we will simply denote $S_t^{n,q}(X) := S_{[0,t]}^{n,q}(X)$. Moreover, if $t = 1$, we denote $S^{q,n}(X) := S_t^{n,q}(X)$. In the case $q = 2$ the limit of $S^{2,n}$ is called the quadratic variation, while for $q = 3$ we have the cubic variation.

Let us recall the following result (see [39]) concerning the exact variation of the perturbed fractional Brownian motion, i.e. the sum of a fBm and a smooth Gaussian process. In the rest of this section, we will fix an interval $[A_1, A_2]$ with $A_1 < A_2$ and a partition $t_j = A_1 + \frac{j}{n}(A_2 - A_1)$, $n \geq 1, j = 0, .., n$ of this interval. Also, we denote by Z a standard normal random variable, and set $\mu_q = \mathbf{E} Z^q$ for $q \geq 1$. Define $\sigma_{H,q}^2 = q! \sum_{v \in \mathbf{Z}} \rho_H(v)^q$, with $\rho_H(v) = \frac{1}{2}\left(|v+1|^{2H} + |v-1|^{2H} - 2|v|^{2H}\right)$ for $v \in \mathbf{Z}$.

Lemma 3. *Let $(B_t^H)_{t \geq 0}$ be a fBm with $H \in (0, \frac{1}{2}]$ and consider a centered Gaussian process $(X_t)_{t \geq 0}$ such that*

$$\mathbf{E}\left|X_t - X_s\right|^2 \leq C|t-s|^2 \text{ for every } s, t \geq 0. \tag{3.9}$$

Define

$$Y_t^H = aB_t^H + X_t \text{ for every } t \geq 0$$

with $a \neq 0$.

1. *The process Y^H has $\frac{1}{H}$-variation over the interval $[A_1, A_2]$ which is equal to*

$$a^{-\frac{1}{H}} \mathbf{E}|Z|^{1/H}(A_2 - A_1).$$

2. *Let*

$$V_{q,n}(Y^H) := \sum_{i=0}^{n-1} \left[\frac{n^{Hq}}{(A_2 - A_1)^{qH} a^q} (Y_{t_{i+1}}^H - Y_{t_i}^H)^q - \mu_q \right]. \tag{3.10}$$

Then, if $H \in (0, \frac{1}{2})$ and $q \geq 2$ is an integer,

$$\frac{1}{\sqrt{n}} V_{q,n}(Y^H) = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\frac{n^{Hq}}{(A_2 - A_1)^{qH} a^q} (Y_{t_{i+1}}^H - Y_{t_i}^H)^q - \mu_q \right] \to^{(d)} N(0, \sigma_{H,q}^2). \tag{3.11}$$

If $H = \frac{1}{2}$, $q = 2$ and the process $(X_t)_{t \geq 0}$ is independent by B, then

$$\frac{1}{\sqrt{n}} V_{2,n}(Y^H) = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\frac{n}{(A_2 - A_1) a^2} (Y_{t_{i+1}}^{\frac{1}{2}} - Y_{t_i}^{\frac{1}{2}})^2 - 1 \right] \to^{(d)} N(0, \sigma_{\frac{1}{2},2}^2). \tag{3.12}$$

Using the recent Stein-Malliavin theory, it is also possible to deduce the rate of convergence in the above Central Limit Theorem (CLT in the sequel) under the Wasserstein distance. Before stating and proving the result, let us briefly recall the definition of this distance.

The Wasserstein distance between the laws of two $\mathbf{R}^d$-valued random variables F and G is defined as

$$d_W(F,G) = \sup_{h\in\mathcal{A}} |\mathbf{E}h(F) - \mathbf{E}h(G)| \tag{3.13}$$

where $\mathcal{A}$ is the class of Lipschitz continuous function $h : \mathbf{R}^d \to \mathbf{R}$ such that $\|h\|_{Lip} \leq 1$, where

$$\|h\|_{Lip} = \sup_{x,y\in\mathbf{R}^d, x\neq y} \frac{|h(x)-h(y)|}{\|x-y\|_{\mathbf{R}^d}}.$$

Proposition 3.2.1. *Assume $H \leq \frac{1}{2}$. Let Y^H as in Lemma 3 and let $V_{q,n}(Y^H)$ be given by (3.10). Then for n large and with $\sigma_{H,q}$ from (3.11)*

$$d_W\left(\frac{1}{\sqrt{n}}V_{q,n}(Y^H), N(0,\sigma^2_{H,q})\right) \leq C\frac{1}{\sqrt{n}}.$$

Proof : From the proof of Lemma 2.1 in [39], we can express the variation of Y^H as the variation of the fBm B^H plus a rest term, i.e.

$$\frac{1}{\sqrt{n}}V_{q,n}(Y^H) = \frac{1}{\sqrt{n}}V_{q,n}(B^H) + R_n$$

where R_n satisfies, for every $n \geq 1$,

$$\mathbf{E}|R_n| \leq cn^{H-1}. \tag{3.14}$$

By the definition of the Wasserstein distance, we can write

$$\begin{aligned} & d_W\left(\frac{1}{\sqrt{n}}V_{q,n}(Y^H), N(0,\sigma^2_{H,q})\right) \\ \leq \;& d_W\left(\frac{1}{\sqrt{n}}V_{q,n}(B^H), N(0,\sigma^2_{H,q})\right) + d_W\left(\frac{1}{\sqrt{n}}V_{q,n}(Y^H), \frac{1}{\sqrt{n}}V_{q,n}(B^H)\right) \\ \leq \;& d_W\left(\frac{1}{\sqrt{n}}V_{q,n}(B^H), N(0,\sigma^2_{H,q})\right) + \mathbf{E}|R_n|. \end{aligned}$$

In order to estimate $d_W(\frac{1}{\sqrt{n}}V_{q,n}(B^H), N(0,\sigma^2_{H,q}))$, we will use the chaos expansion of the random variable $V_{q,n}(B^H)$ and several results from [45]. Notice that (see e.g. the proof of Corollary 3 in [44]),

$$V_{q,n}(B^H) = \sum_{k=1}^{q} k!C_q^k \mu_{q-k} \sum_{i=0}^{n-1} H_k\left(\frac{n^{HK}}{(A_2-A_1)^{HK}}\left(B^H_{t_{i+1}} - B^H_{t_i}\right)\right),$$

where H_k is the kth probabilistic Hermite polynomial $H_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right)$ for $k \geq 1$ with $H_0(x) = 1$. We know from [45] that the vector

$$(F_{1,n}, F_{2,n}, ..., F_{q,n}) := \left(\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} H_k \left(\frac{n^{HK}}{(A_2 - A_1)^{HK}} \left(B^H_{t_{i+1}} - B^H_{t_i} \right) \right) \right)_{k=1,..,q}$$

converges in distribution to a centered Gaussian vector with diagonal covariance matrix C (the explicit expression of C can be found in [45], it is not needed in our work). Moreover, Proposition 6.2.2 and Corollary 7.4.3 in [45] implies that

$$d_W\left((F_{k,n})_{k=1,..,q}, N(0,C)\right) \leq c\frac{1}{\sqrt{n}}.$$

This will easily lead to

$$d_W\left(\frac{1}{\sqrt{n}} V_{q,n}(B^H), N(0, \sigma^2_{H,q})\right) \leq c\frac{1}{\sqrt{n}}. \tag{3.15}$$

Since $H \leq \frac{1}{2}$, we obtain the conclusion via (3.14) and (3.15). ■

3.2.3 Estimators for the drift parameter

Our purpose is to estimate the parameter $\theta > 0$ based on the observations of the process u_θ. We will define two estimators : the first is based on the temporal variations of the process u_θ while the second is constructed via its variation in space. Their behavior is strongly related to the law of the process u_θ, therefore we start by analyzing the distribution of this Gaussian process.

The law of the solution

Let $G_\alpha(t,x)$ the Green kernel associated to the operator $-(-\Delta)^{\frac{\alpha}{2}}$. Then the Green kernel associated to the operator $-\theta(-\Delta)^{\frac{\alpha}{2}}$ is

$$G_\alpha(\theta t, x).$$

Lemma 4. *Suppose that the process $(u_\theta(t,x), t \geq 0, x \in \mathbf{R})$ satisfies (4.1). Define*

$$v_\theta(t,x) := u_\theta\left(\frac{t}{\theta}, x\right), \quad t \geq 0, x \in \mathbf{R}. \tag{3.16}$$

Then the process $(v_\theta(t,x), t \geq 0, x \in \mathbf{R})$ *satisfies*

$$\frac{\partial v_\theta}{\partial t}(t,x) = -(-\Delta)^{\frac{\alpha}{2}} v_\theta(t,x) + (\theta)^{-\frac{1}{2}} \dot{\widetilde{W}}(t,x), \quad t \geq 0, x \in \mathbf{R} \tag{3.17}$$

with $v_\theta(0,x) = 0$ *for every* $x \in \mathbf{R}$*, where* $\dot{\widetilde{W}}$ *is a space-time white noise, i.e. a centered Gaussian random field with covariance (3.2).*

Proof : From (3.5), we have for every $t \geq 0, x \in \mathbf{R}$,

$$\begin{aligned} v_\theta(t,x) = u_\theta\left(\frac{t}{\theta}, x\right) &= \int_0^{\frac{t}{\theta}} \int_{\mathbf{R}} G_\alpha(t-\theta s, x-y) W(ds, dy) \\ &= \int_0^t \int_{\mathbf{R}} G_\alpha(t-s, x-y) W(d\frac{s}{\theta}, dy) \\ &= \theta^{-\frac{1}{2}} \int_0^t \int_{\mathbf{R}} G_\alpha(t-s, x-y) \tilde{W}(ds, dy) \end{aligned}$$

where, for $t \geq 0, A \in \mathcal{B}(\mathbf{R})$, we denoted $\tilde{W}(t,A) := \theta^{\frac{1}{2}} W\left(\frac{t}{\theta}, A\right)$. Notice that $\tilde{W}$ has the same finite dimensional distributions as W, due to the scaling property of the white noise. ■

We can deduce the law of the process u_θ in time and in space.

Proposition 3.2.2. *For every* $x \in \mathbf{R}$ *and* $\theta > 0$*, we have*

$$(u_\theta(t,x), t \geq 0) \equiv^{(d)} \left(\theta^{-\frac{1}{2\alpha}} c_{2,\alpha} B_t^{\frac{1}{2}, 1-\frac{1}{\alpha}}, t \geq 0\right),$$

where $B^{\frac{1}{2}, 1-\frac{1}{\alpha}}$ *is a bifractional Brownian motion with parameters* $H = \frac{1}{2}$ *and* $K = 1 - \frac{1}{\alpha}$ *and* $c_{2,\alpha}$ *is given by (3.7).*

Proof : Fix $x \in \mathbf{R}$ and $\theta > 0$. Then for every $s, t \geq 0$, we have

$$\begin{aligned} \mathbf{E} u_\theta(t,x) u_\theta(s,x) &= \mathbf{E} v_\theta(\theta t, x) v_\theta(\theta s, x) \\ &= \theta^{-1} \mathbf{E} u_1(\theta t, x) u_1(\theta s, x) = \theta^{-1} c_{1,\alpha} \left[(\theta t + \theta s)^{1-\frac{1}{\alpha}} - |\theta t - \theta s|^{1-\frac{1}{\alpha}}\right] \\ &= \theta^{-\frac{1}{\alpha}} c_{2,\alpha}^2 \mathbf{E} B_t^{\frac{1}{2}, 1-\frac{1}{\alpha}} B_s^{\frac{1}{2}, 1-\frac{1}{\alpha}}. \end{aligned}$$

■

Proposition 3.2.3. *For every $t \geq 0, \theta > 0$, we have the following equality in distribution*

$$(u_\theta(t,x), x \in \mathbf{R}) \equiv^{(d)} \left(\theta^{-\frac{1}{2}} m_\alpha B^{\frac{\alpha-1}{2}}(x) + \theta^{-\frac{1}{2}} S_{\theta t}(x), x \in \mathbf{R}\right),$$

where $B^{\frac{\alpha-1}{2}}$ is a fractional Brownian motion with Hurst parameter $\frac{\alpha-1}{2} \in (0, \frac{1}{2}]$, $(S_{\theta t}(x))_{x\in\mathbf{R}}$ is a centered Gaussian process with C^∞ sample paths and m_α from (3.8).

Proof : The result is immediate since for every $t > 0, \theta > 0$

$$(u_\theta(t,x), x \in \mathbf{R}) = (v_\theta(\theta t, x), x \in \mathbf{R}) \equiv^{(d)} \theta^{-\frac{1}{2}} (u_1(\theta t, x), x \in \mathbf{R})$$

$$\equiv^{(d)} \left(\theta^{-\frac{1}{2}} m_\alpha B^{\frac{\alpha-1}{2}}(x) + \theta^{-\frac{1}{2}} S_{\theta t}(x), x \in \mathbf{R}\right),$$

where we used (3.8). ■

Notice that the Hurst parameter of the fBm in Proposition 3.2.3 may be $\frac{1}{2}$ if $\alpha = 2$.

Estimators based on the temporal variation

Proposition 3.2.2 indicates that the process u_θ behaves as a bi-fBm in time. Recall the following connection between the fBm and the bi-fBm (see [36]) : Let $H \in (0,1), K \in (0,1]$. If $(B_t^{HK})_{t\geq 0}$ is a fBm with Hurst parameter HK and $(B_t^{H,K})_{t\geq 0}$ is a bi-fBm, then

$$\left(C_1 X_t^{H,K} + B_t^{H,K}, t \geq 0\right) \equiv^{(d)} \left(C_2 B_t^{HK}, t \geq 0\right), \tag{3.18}$$

with $C_1 > 0$ and $C_2 = 2^{\frac{1-K}{2}}$. In (3.18), $X^{H,K}$ is a Gaussian process, independent of $B^{H,K}$ with C^∞-sample paths. (see Remark 2.2.1). Therefore, the bi-fBm is a perturbed fBm and the same holds true for the solution $(u_\theta(t,x), t \geq 0)$, by Proposition 3.2.2. Therefore, we obtain, by using the notation $t_j = A_1 + \frac{j}{n}(A_2 - A_1)$, $n \geq 1, j = 0, .., n$

Lemma 5. *Let u_θ be the solution to (4.1). Then for every $x \in \mathbf{R}$*

$$S^{n,\frac{2\alpha}{\alpha-1}}_{[A_1,A_2]} := \sum_{i=0}^{n-1} |u_\theta(t_{j+1},x) - u_\theta(t_j,x)|^{\frac{2\alpha}{\alpha-1}} \to_{n\to\infty} c_{2,\alpha}^{\frac{2\alpha}{\alpha-1}} 2^{\frac{1}{\alpha-1}} E|Z|^{\frac{2\alpha}{\alpha-1}} (A_2 - A_1)|\theta|^{\frac{-1}{\alpha-1}} \tag{3.19}$$

in probability.

Relation (3.19) motivates the definition of the following estimator for the parameter $\theta > 0$ of the model (4.1)

$$\widehat{\theta}_{n,1} = \left(\left(c_{2,\alpha}^{\frac{2\alpha}{\alpha-1}} 2^{\frac{1}{\alpha-1}} \mu_{\frac{2\alpha}{\alpha-1}} (A_2 - A_1) \right)^{-1} \sum_{i=0}^{n-1} |u_\theta(t_{j+1}, x) - u_\theta(t_j, x)|^{\frac{2\alpha}{\alpha-1}} \right)^{1-\alpha} \tag{3.20}$$

$$= \left(c_{2,\alpha}^{\frac{2\alpha}{\alpha-1}} 2^{\frac{1}{\alpha-1}} \mu_{\frac{2\alpha}{\alpha-1}} (A_2 - A_1) \right)^{\alpha-1} \left(S^{n, \frac{2\alpha}{\alpha-1}} (u_\theta(\cdot, x)) \right)^{1-\alpha}, \tag{3.21}$$

and so

$$\widehat{\theta_{n,1}^{\frac{1}{1-\alpha}}} = \frac{1}{c_{2,\alpha}^{\frac{2\alpha}{\alpha-1}} 2^{\frac{1}{\alpha-1}} \mu_{\frac{2\alpha}{\alpha-1}} (A_2 - A_1)} \sum_{i=0}^{n-1} |u_\theta(t_{j+1}, x) - u_\theta(t_j, x)|^{\frac{2\alpha}{\alpha-1}}. \tag{3.22}$$

We obtain the consistency and the asymptotic normality of the above estimator. In order to have the above quantity well-defined we need to assume that $\frac{2\alpha}{\alpha-1}$ is an even integer.

Proposition 3.2.4. *Assume $q := \frac{2\alpha}{\alpha-1}$ is an even integer and consider the estimator $\widehat{\theta}_{n,1}$ defined by (3.20). Then $\widehat{\theta}_{n,1} \to_{n\to\infty} \theta$ in probability and*

$$\sqrt{n} \left[\widehat{\theta_{n,1}^{\frac{1}{1-\alpha}}} - \theta^{\frac{1}{1-\alpha}} \right] \to^{(d)} N(0, s_{1,\theta,\alpha}^2) \text{ with } s_{1,\theta,\alpha}^2 = \sigma_{\frac{1}{q},q}^2 \theta^{\frac{2}{1-\alpha}} \mu_{\frac{2\alpha}{\alpha-1}}^{-2}. \tag{3.23}$$

Moreover, for n large enough

$$d_W \left(\sqrt{n} \left[\widehat{\theta_{n,1}^{\frac{1}{1-\alpha}}} - \theta^{\frac{1}{1-\alpha}} \right], N(0, s_{\theta,\alpha}^2) \right) \leq c \frac{1}{\sqrt{n}}.$$

Proof : From Proposition 3.2.2 and the relation between the fBm and the bi-fBm (3.18), we obtain that

$$\left(u_\theta(t, x) + c_{2,\alpha} \theta^{-\frac{1}{2\alpha}} X_t \right) \equiv^{(d)} c_{2,\alpha} \theta^{-\frac{1}{2\alpha}} 2^{\frac{1}{2\alpha}} B^{\frac{\alpha-1}{2\alpha}},$$

where $B^{\frac{\alpha-1}{2\alpha}}$ is a fBm with Hurst parameter $\frac{\alpha-1}{2\alpha} \in (0, \frac{1}{2})$. Therefore, u_θ is a perturbed fBm and we obtain, by taking $H = \frac{\alpha-1}{2\alpha}$ and $q = \frac{1}{H} = \frac{2\alpha}{\alpha-1}$

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\frac{n\theta^{\frac{1}{\alpha-1}}}{c_{2,\alpha}^{\frac{2\alpha}{\alpha-1}} 2^{\frac{1}{\alpha-1}} (A_2 - A_1)} (u_\theta(t_{j+1}, x) - u_\theta(t_j, x))^{\frac{2\alpha}{\alpha-1}} - \mu_{\frac{2\alpha}{\alpha-1}} \right] \to^{(d)} N(0, \sigma_{\frac{1}{q},q}^2).$$

This means

$$\sqrt{n}\mu_{\frac{2\alpha}{\alpha-1}}\theta^{\frac{1}{\alpha-1}}\left[\widehat{\theta}_{n,1}^{\frac{1}{1-\alpha}}-\theta^{\frac{1}{1-\alpha}}\right]\rightarrow^{(d)} N(0,\sigma^2_{\frac{1}{q},q})$$

which is equivalent to (3.23). ■

Using the so-called delta-method, we can get the asymptotic behavior of the estimator $\widehat{\theta}_n$. Recall that if $(X_n)_{n\geq 1}$ is a sequence of random variables such that

$$\sqrt{n}(X_n-\gamma_0)\rightarrow^{(d)} N(0,\sigma^2)$$

and g is a function such that $g'(\gamma_0)$ exists and does not vanish, then

$$\sqrt{n}(g(X_n)-g(\gamma_0))\rightarrow^{(d)} N(0,\sigma^2 g'(\gamma_0)^2). \tag{3.24}$$

Proposition 3.2.5. *Consider the estimator (3.20) and let $s_{1,\theta,\alpha}$ be given by (3.23). Then as $n\to\infty$*

$$\sqrt{n}(\widehat{\theta}_{n,1}-\theta)\rightarrow N(0,s^2_{1,\theta,\alpha}(1-\alpha)^2\theta^{\frac{2\alpha}{\alpha-1}}) \tag{3.25}$$

and for n large enough

$$d_W\left(\sqrt{n}(\widehat{\theta}_{n,1}-\theta),N(0,s^2_{1,\theta,\alpha}(1-\alpha)^2\theta^{\frac{2\alpha}{\alpha-1}})\right)\leq c\frac{1}{\sqrt{n}}.$$

Proof : By applying the delta-method for the function $g(x)=x^{1-\alpha}, X_n=\widehat{\theta}_{n,1}^{\frac{1}{1-\alpha}}$ and $\gamma_0=\theta^{\frac{1}{1-\alpha}}$, we immediately obtain the convergence (3.25). Concerning the rate of convergence, we can write, with $\widetilde{\gamma}_0$ a random point located between X_n and γ_0

$$\begin{aligned}\sqrt{n}(g(X_n)-g(\gamma_0)) &= \sqrt{n}g'(\widetilde{\gamma}_0)(X_n-\gamma_0)\\ &= g'(\gamma_0)\sqrt{n}(X_n-\gamma_0)+\sqrt{n}(X_n-\gamma_0)(g'(\widetilde{\gamma}_0)-g'(\gamma_0))\\ &=: g'(\gamma_0)\sqrt{n}(X_n-\gamma_0)+T_n.\end{aligned}$$

We have, for n large

$$\begin{aligned}\mathbf{E}|T_n| &= \mathbf{E}\left|\sqrt{n}(X_n-\gamma_0)(g'(\widetilde{\gamma}_0)-g'(\gamma_0))\right|\leq\left(\mathbf{E}\left(\sqrt{n}(X_n-\gamma_0)\right)^2\right)^{\frac{1}{2}}\left(\mathbf{E}(g'(\widetilde{\gamma}_0)-g'(\gamma_0))^2\right)^{\frac{1}{2}}\\ &\leq c\left(\mathbf{E}(g'(\widetilde{\gamma}_0)-g'(\gamma_0))^2\right)^{\frac{1}{2}}\leq c\left(\mathbf{E}\left(\widehat{\theta}_{n,1}^{\frac{\alpha}{\alpha-1}}-\theta^{\frac{\alpha}{\alpha-1}}\right)^2\right)^{\frac{1}{2}}\\ &\leq c\left(\mathbf{E}\left(\widehat{\theta}_{n,1}^{\frac{1}{\alpha-1}}-\theta^{\frac{1}{\alpha-1}}\right)^2\right)^{\frac{1}{2}}\leq c\frac{1}{\sqrt{n}}\end{aligned}$$

where we used the assumption $\alpha > 1$ and the fact that for every $p \geq 1$, (see e.g. [45], page 94)

$$\sup_{n\geq 1} \mathbf{E}|X_n|^p < \infty.$$

Therefore, by the triangle inequality and Proposition 4.4.1, for n large enough

$$\begin{aligned} &d_W\left(\sqrt{n}(\widehat{\theta}_{n,1}-\theta), N(0, s^2_{1,\theta,\alpha}(1-\alpha)^2\theta^{\frac{\alpha}{\alpha-1}})\right) \\ &\leq c d_W\left(\sqrt{n}(X_n-\gamma_0), N(0, s^2_{1,\theta,\alpha})\right) + \mathbf{E}|T_n| \leq c\frac{1}{\sqrt{n}}. \end{aligned}$$

■

Estimators based on the spatial variation

It is possible to define an estimator for the parameter θ based on the spatial variations of the solution (3.5). The result in Proposition 3.2.3 says that the process $(u_\theta(t,x), x \in \mathbf{R})$ is a perturbed fBm, so we know its exact variation in space. Below $x_j = A_1 + \frac{j}{n}(A_2 - A_1), j = 0, .., n$ will denote a partition of the interval $[A_1, A_2]$.

Proposition 3.2.6. *Let u_θ be given by (4.1). Then we have the probability convergence,*

$$\sum_{i=0}^{n-1} |u_\theta(t,x_{j+1}) - u_\theta(t,x_j)|^{\frac{2}{\alpha-1}} \to_{n\to\infty} m_\alpha^{\frac{2}{\alpha-1}} E|Z|^{\frac{2}{\alpha-1}}(A_2-A_1)|\theta|^{\frac{-1}{\alpha-1}}$$

and if $q := \frac{2}{\alpha-1}$ is an integer

$$\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\left[\left(\frac{n}{m_\alpha^{\frac{2}{\alpha-1}}(A_2-A_1)}\right)\theta^{\frac{1}{\alpha-1}}\left(u_\theta(t,x_{i+1})-u_\theta(t,x_i)\right)^{\frac{2}{\alpha-1}} - \mu_{\frac{2}{\alpha-1}}\right] \to^{(d)} N(0, \sigma^2_{\frac{\alpha-1}{2},\frac{2}{\alpha-1}}).$$

The above Proposition 3.2.6 leads to the definition of the estimator

$$\widehat{\theta}_{n,2} = \left[(m_\alpha^{\frac{2}{\alpha-1}}\mu_{\frac{2}{\alpha-1}}(A_2-A_1))^{-1}\sum_{i=0}^{n-1}|u_\theta(t,x_{j+1})-u_\theta(t,x_j)|^{\frac{2}{\alpha-1}}\right]^{1-\alpha}, \tag{3.26}$$

and we can immediately deduce from Proposition 3.2.3 its asymptotic properties.

Proposition 3.2.7. *The estimator (3.26) converges in probability as $n \to \infty$ to the parameter θ. Moreover, if $q := \frac{2}{\alpha-1}$ is an even integer*

$$\sqrt{n}\left[\widehat{\theta}_{n,2}^{\frac{1}{1-\alpha}} - \theta^{\frac{1}{1-\alpha}}\right] \to^{(d)} N(0, s^2_{2,\theta,\alpha}) \text{ with } s^2_{2,\theta,\alpha} = \sigma^2_{\frac{\alpha-1}{2},\frac{2}{\alpha-1}}\mu^{-2}_{\frac{2}{\alpha-1}}\theta^{\frac{2}{1-\alpha}}. \tag{3.27}$$

Moreover, for n large

$$d_W\left(\sqrt{n}\left[\widehat{\theta}_{n,2}^{\frac{1}{1-\alpha}} - \theta^{\frac{1}{1-\alpha}}\right], N(0, s^2_{2,\theta,\alpha})\right) \leq c\frac{1}{\sqrt{n}}.$$

Proof : Using the law of the process $(u_\theta(t,x), x \in \mathbf{R})$ obtained in Proposition 3.2.3, we deduce that the Gaussian process $\left(\theta^{\frac{1}{2}} m_\alpha^{-1} u_\theta(t,x), x \in \mathbf{R}\right)$ is a perturbed fractional Brownian motion. Therefore, by relation (3.11) in Lemma 3,

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\left(\frac{n\theta^{\frac{1}{\alpha-1}}}{(A_2-A_1)m_\alpha^{\frac{2}{\alpha-1}}}\left(u_\theta(t,x_{j+1})-u_\theta(t,x_j)\right)^{\frac{2}{\alpha-1}} - \mu_{\frac{2}{\alpha-1}}\right)\\
= \quad & \sqrt{n}\mu_{\frac{2}{\alpha-1}}\theta^{\frac{1}{\alpha-1}}\left[\widehat{\theta}_{n,2}^{\frac{1}{1-\alpha}} - \theta^{\frac{1}{1-\alpha}}\right] \to^{(d)}_{n\to\infty} N\left(0, \sigma^2_{\frac{\alpha-1}{2},\frac{2}{\alpha-1}}\right).
\end{aligned}$$

Moreover, Proposition 3.2.1 implies that

$$d_W\left(\sqrt{n}\mu_{\frac{2}{\alpha-1}}\theta^{\frac{1}{\alpha-1}}\left[\widehat{\theta}_{n,2}^{\frac{1}{1-\alpha}} - \theta^{\frac{1}{1-\alpha}}\right], N(0, \sigma^2_{\frac{\alpha-1}{2},\frac{2}{\alpha-1}})\right) \leq c\frac{1}{\sqrt{n}}$$

and this obviously leads to the desired conclusion. ■

By using the delta-method, we can obtain the asymptotic distribution of $\widehat{\theta}_{n,2}$.

Proposition 3.2.8. *Let $\widehat{\theta}_{n,2}$ be given by (3.26). Then, with $s_{2,\theta,\alpha}$ from (3.27), as $n \to \infty$*

$$\sqrt{n}(\widehat{\theta}_{n,2} - \theta) \to^{(d)} N\left(0, s^2_{2,\theta,\alpha}(1-\alpha)^2\theta^{\frac{2\alpha}{\alpha-1}}\right),$$

and for n large enough

$$d_W\left(\sqrt{n}(\widehat{\theta}_{n,2} - \theta), N(0, s^2_{2,\theta,\alpha}(1-\alpha)^2\theta^{\frac{2\alpha}{\alpha-1}})\right) \leq c\frac{1}{\sqrt{n}}.$$

Proof : It suffices to apply (3.24) to the function $g(x) = x^{1-\alpha}$ and $\gamma_0 = \theta^{\frac{1}{1-\alpha}}$ and to follow the proof of Proposition 4.4.2. ■

Remark 3.2.1.
- *The estimators (3.20) and (3.26) coincide with the estimators in [49] in the case of the standard Laplacian $\alpha = 2$.*
- *The distance of the estimators (3.20) and (3.26) to their limit distribution is of the same order, although they involve q-variations with different q.*

3.3 Heat equation with fractional Laplacian and white-colored noise

In this section, we will consider the stochastic heat equation with an additive Gaussian noise which behaves as a Wiener process in time and as a fractional Brownian motion in space, i.e. its spatial covariance is given by the so-called Riesz kernel. We will again study the distribution of the solution, its connection with fractional and bifractional Brownian motion and we apply the q-variation method to obtain an asymptotically normal estimator for the drift parameter.

3.3.1 General properties of the solution

We will consider the stochastic heat equation

$$\frac{\partial}{\partial t}u_\theta(t,\mathbf{x}) = -\theta(-\Delta)^{\frac{\alpha}{2}}u_\theta(t,\mathbf{x}) + \dot{W}^\gamma(t,\mathbf{x}), \quad t \geq 0, \mathbf{x} \in \mathbf{R}^d. \tag{3.28}$$

with $u_\theta(0,\mathbf{x}) = 0$ for every $\mathbf{x} \in \mathbf{R}^d$. In (3.28), $-(-\Delta)^{\frac{\alpha}{2}}$ denotes the fractional Laplacian with exponent $\frac{\alpha}{2}$, $\alpha \in (1,2]$ and W^γ is the so-called white-colored noise, i.e. $W^\gamma(t,A), t \geq 0, A \in \mathcal{B}(\mathbf{R}^d)$ is a centered Gaussian field with covariance

$$\mathbf{E}W^\gamma(t,A)W^\gamma(s,B) = (t \wedge s)\int_A \int_B f(\mathbf{x}-\mathbf{y})d\mathbf{x}d\mathbf{y}, \tag{3.29}$$

where f is the so-called Riesz kernel of order γ given by

$$f(\mathbf{x}) = R_\gamma(\mathbf{x}) := g_{\gamma,d}\|\mathbf{x}\|^{-d+\gamma}, \quad 0 < \gamma < d, \tag{3.30}$$

where $g_{\gamma,d} = 2^{d-\gamma}\pi^{d/2}\Gamma((d-\gamma)/2)/\Gamma(\gamma/2)$. As usual, the mild solution to (3.28) is given by

$$u_\theta(t,\mathbf{x}) = \int_0^t \int_{\mathbf{R}^d} G_\alpha(\theta(t-s), \mathbf{x}-\mathbf{z})W^\gamma(ds, d\mathbf{z}), \tag{3.31}$$

where the above integral $W^\gamma(ds,d\mathbf{z})$ is a Wiener integral with respect to the Gaussian noise W^γ.

We know the following facts concerning the mild solution (3.31) when $\theta = 1$.

— The mild solution (3.28) is well-defined as a square integrable process satisfying

$$\sup_{t\in[0,T],\mathbf{x}\in\mathbf{R}^d} \mathbf{E}|u_1(t,\mathbf{x})|^2 < \infty$$

if and only if

$$d < \gamma + \alpha. \tag{3.32}$$

In particular, condition (3.32) shows that the solution exists in any spatial dimension d, via suitable choice of the parameter γ.

— Assume (3.32) is satisfied. Then for every $\mathbf{x} \in \mathbf{R}^d$, we have the following equivalence in distribution

$$(u_1(t,\mathbf{x}), t \geq 0) \equiv^{(d)} \left(c_{2,\alpha,\gamma} B_t^{\frac{1}{2}, 1-\frac{d-\gamma}{\alpha}}, t \geq 0 \right), \tag{3.33}$$

where $B^{\frac{1}{2}, 1-\frac{d-\gamma}{\alpha}}$ is a bifractional Brownian motion with Hurst parameters $H = \frac{1}{2}$ and $K = 1 - \frac{d-\gamma}{\alpha}$ and

$$c^2_{2,\alpha,\gamma} = c_{1,\alpha,\gamma} 2^{1-\frac{d-\gamma}{\alpha}} \tag{3.34}$$

with

$$c_{1,\alpha,\gamma} = (2\pi)^{-d} \int_{\mathbf{R}^d} d\xi \|\xi\|^{-\gamma} e^{-\|\xi\|^\alpha} \frac{1}{2(1-\frac{d-\gamma}{\alpha})}.$$

— For every $t \geq 0$, we have (see Proposition 4.6 in [39])

$$\left(u_1(t,\mathbf{x}), \mathbf{x} \in \mathbf{R}^d\right) \equiv^{(d)} \left(m_{\alpha,\gamma} B^{\frac{\alpha+\gamma-d}{2}}(\mathbf{x}) + S_t(\mathbf{x}), \mathbf{x} \in \mathbf{R}^d \right) \tag{3.35}$$

where $B^{\frac{\alpha+\gamma-d}{2}}$ is an isotropic d-dimensional fractional Brownian motion (see the next section) with Hurst parameter $\frac{\alpha+\gamma-d}{2}$, $(S_t(\mathbf{x}))_{\mathbf{x}\in\mathbf{R}^d}$ is a centered Gaussian process with C^∞ sample paths and $m^2_{\alpha,\gamma}$ is an explicit numerical constant.

3.3.2 Perturbed isotropic fractional Brownian motion

Since the law of the solution (3.31) is related to the isotropic fBm, let us recall the definition of this process. The isotropic d-parameter fBm (also known as the Lévy fBm) $(B_d^H(\mathbf{x}), \mathbf{x} \in \mathbf{R}^d)$ with Hurst parameter $H \in (0,1)$ is defined as a centered Gaussian process, starting from zero, with covariance function

$$\mathbf{E}(B_d^H(\mathbf{x}) B_d^H(\mathbf{y})) = \frac{1}{2} \left(\|\mathbf{x}\|^{2H} + \|\mathbf{y}\|^{2H} - \|\mathbf{x}-\mathbf{y}\|^{2H} \right) \text{ for every } \mathbf{x}, \mathbf{y} \in \mathbf{R}^d \tag{3.36}$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbf{R}^d$. It can be also represented as a Wiener integral with respect to the Wiener sheet, see [29], [37].

As in the one-parameter case, we define the q-variation of the isotropic fBm as the limit in probability when $n \to \infty$, of the sequence

$$S^{n,q}_{[A_1,A_2]}(B^H) = \sum_{i=0}^{n-1} \left| B^H_d(\mathbf{x}_{i+1}) - B^H_d(\mathbf{x}_i) \right|^q,$$

where $\mathbf{x}_i = (x_i^{(1)}, .., x_i^{(d)})$ with $x_i^{(j)} = A_1 + \frac{i}{n}(A_2 - A_1)$ for $i = 0, .., n$ and $j = 1, .., d$. And from [39] we know that the isotropic fbm$\big((B^H_d(\mathbf{x}), \mathbf{x} \in \mathbf{R}^d)\big)$ has $\frac{1}{H}$-variation over $[A_1, A_2]$ which is equal to

$$(A_2 - A_1)\mathbf{E}|B^H_d(\mathbf{1})|^{1/H} = (A_2 - A_1)\sqrt{d}\mathbf{E}|Z|^{1/H}.$$

The q-variation of the isotropic fBm perturbed by a regular multiparameter process has been obtained in [39], Lemma 4.1.

Lemma 6. *Let $(B^H(\mathbf{x}))_{\mathbf{x}\in\mathbf{R}^d}$ be a d-parameter isotropic fBm and consider $(X(\mathbf{x}))_{\mathbf{x}\in\mathbf{R}^d}$ a d-parameter stochastic process, independent of B^H, that satisfies*

$$\mathbf{E}\left|X(\mathbf{x}) - X(\mathbf{y})\right|^2 \leq C\|\mathbf{x} - \mathbf{y}\|^2, \text{ for every } \mathbf{x}, \mathbf{y} \in \mathbf{R}^d. \tag{3.37}$$

Define

$$Y(\mathbf{x}) = B^H_d(\mathbf{x}) + X(\mathbf{x}) \text{ for every } \mathbf{x} \in \mathbf{R}^d.$$

Then

1. *the process $(Y(\mathbf{x}))_{\mathbf{x}\in\mathbf{R}^d}$ has $\frac{1}{H}-$variation which is equal to*
$$(A_2 - A_1)\sqrt{d}\mathbf{E}|Z|^{1/H}.$$
2. *Then if $H \in (0, \frac{1}{2})$ and $q \geq 2$*

$$\frac{1}{\sqrt{n}} V_{q,n}(Y^H) := \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\frac{n^{Hq} d^{-Hq/2}}{(A_2 - A_1)^{qH}} (Y^H(\mathbf{x}_{i+1}) - Y^H(\mathbf{x}_i))^q - \mu_q \right] \to^{(d)} N(0, \sigma^2_{H,q}). \tag{3.38}$$

It is immediate to deduce the rate of convergence in the above central limit theorem. Recall that we denoted by d_W the Wasserstein distance.

Proposition 3.3.1. *Let Y^H be as in the statement of Lemma 6. Then for n large*

$$d_W\left(\frac{1}{\sqrt{n}} V_{q,n}(Y^H), N(0, \sigma^2_{H,q})\right) \leq C\frac{1}{\sqrt{n}}.$$

Proof : We notice that the Gaussian vector $\big(B^H_d(\mathbf{x}_{i+1}) - B^H_d(\mathbf{x}_i)\big)_{0,1,..,n-1}$ has the same law as $d^{H/2}(B^H(x_{j+1}) - B^H(x_j))_{0,1,..,n-1}$ where B is a one-parameter fBm with Hurst parameter H and we then apply Lemma 3. Therefore, the distribution of the sequence $\frac{1}{\sqrt{n}} V_{q,n}(B^H_d)$ is independent of $d \geq 1$ and we can use the same argument as in the proof of Proposition 3.2.1 above. ■

3.3.3 Estimators for the drift parameter

Throughout this section we will assume (3.32). As in the previous section, we will construct and analyze estimators for the drift parameter θ by using the limit behavior of the variations (in time and in space) of the process (3.31).

The law of the solution

Let us start by analyzing the distribution of the solution to (3.28) and its link with the (bi)fractional Brownian motion.

Proposition 3.3.2. *For every $\mathbf{x} \in \mathbf{R}^d$ and $\theta > 0$, we have*

$$(u_\theta(t,\mathbf{x}), t \geq 0) \equiv^{(d)} \left(\theta^{-\frac{d-\gamma}{2\alpha}} c_{2,\alpha,\gamma} B_t^{\frac{1}{2}, 1-\frac{d-\gamma}{\alpha}}, t \geq 0 \right),$$

where $B^{\frac{1}{2}, 1-\frac{d-\gamma}{\alpha}}$ is a bifractional Brownian motion with parameters $H = \frac{1}{2}$ and $K = 1 - \frac{d-\gamma}{\alpha}$ and the constant $c_{2,\alpha,\gamma}$ is defined by (3.34).

Proof : Denote

$$v_\theta(t,\mathbf{x}) = u_\theta\left(\frac{t}{\theta}, \mathbf{x}\right) \text{ for every } t \geq 0, \mathbf{x} \in \mathbf{R}^d.$$

Then, as in Lemma 8, v_θ solves the equation

$$\frac{\partial v_\theta}{\partial t}(t,\mathbf{x}) = -(-\Delta)^{\frac{\alpha}{2}} v_\theta(t,\mathbf{x}) + (\theta)^{-\frac{1}{2}} \dot{\widetilde{W}}^\gamma(t,\mathbf{x}), \quad t \geq 0, \mathbf{x} \in \mathbf{R}^d \tag{3.39}$$

with $v_\theta(0,\mathbf{x}) = 0$ for every $\mathbf{x} \in \mathbf{R}^d$, where $\dot{\widetilde{W}}^\gamma$ is a white colored Gaussian noise (i.e. a Gaussian process with zero mean and covariance (3.29)).

Fix $\mathbf{x} \in \mathbf{R}^d$ and $\theta > 0$. We have, for every $s, t \geq 0$,

$$\begin{aligned} \mathbf{E} u_\theta(t,\mathbf{x}) u_\theta(s,\mathbf{x}) &= \mathbf{E} v_\theta(\theta t,\mathbf{x}) v_\theta(\theta s,\mathbf{x}) \\ &= \theta^{-1} \mathbf{E} u_1(\theta t,\mathbf{x}) u_1(\theta s,\mathbf{x}) = \theta^{-1} c_{1,\alpha,\gamma} \left[(\theta t + \theta s)^{1-\frac{d-\gamma}{\alpha}} - |\theta t - \theta s|^{1-\frac{d-\gamma}{\alpha}} \right] \\ &= \theta^{-\frac{d-\gamma}{\alpha}} c_{2,\alpha,\gamma}^2 \mathbf{E} B_t^{\frac{1}{2}, 1-\frac{d-\gamma}{\alpha}} B_s^{\frac{1}{2}, 1-\frac{d-\gamma}{\alpha}}. \end{aligned}$$

■

For the behavior with respect to the space variable, we obtain the following result.

Proposition 3.3.3. *For every $t \geq 0, \theta > 0$, we have the following equality in distribution*

$$\left(u_\theta(t,\mathbf{x}), \mathbf{x} \in \mathbf{R}^d\right) \equiv^{(d)} \left(\theta^{-\frac{1}{2}} m_{\alpha,\gamma} B^{\frac{\alpha+\gamma-d}{2}}(\mathbf{x}) + \theta^{-\frac{1}{2}} S_{\theta t}(\mathbf{x}), \mathbf{x} \in \mathbf{R}^d\right)$$

where $B^{\frac{\alpha+\gamma-d}{2}}$ is a fractional Brownian motion with Hurst parameter $\frac{\alpha+\gamma-d}{2} \in (0, \frac{1}{2}]$, $(S_{\theta t}(\mathbf{x}))_{\mathbf{x}\in\mathbf{R}^d}$ is a centered Gaussian process with C^∞ sample paths and $m_{\alpha,\gamma}$ from (3.35).

Proof : The result is immediate since for fixed time $t > 0$

$$\left(u_\theta(t,\mathbf{x}), \mathbf{x} \in \mathbf{R}^d\right) = \left(v_\theta(\theta t,\mathbf{x}), \mathbf{x} \in \mathbf{R}^d\right) \equiv^{(d)} \theta^{-\frac{1}{2}} \left(u_1(\theta t,\mathbf{x}), \mathbf{x} \in \mathbf{R}^d\right)$$

$$\equiv^{(d)} \left(\theta^{-\frac{1}{2}} m_{\alpha,\gamma} B^{\frac{\alpha+\gamma-d}{2}}(\mathbf{x}) + \theta^{-\frac{1}{2}} S_{\theta t}(\mathbf{x}), \mathbf{x} \in \mathbf{R}^d\right).$$

■

Estimators based on the temporal variation

Again $t_j = A_1 + \frac{j}{n}(A_2 - A_1), j = 0, .., n$ will denote a partition of the interval $[A_1, A_2]$.

Lemma 7. *Assume (3.32). Let u_θ be the solution to (3.28). Then for every $\mathbf{x} \in \mathbf{R}^d$, the process $(u_\theta(t,\mathbf{x}), t \geq 0)$ admits $\frac{2\alpha}{\alpha+\gamma-d}$-variation over the interval $[A_1, A_2]$, i.e.*

$$S^{n,\frac{2\alpha}{\alpha+\gamma-d}}_{[A_1,A_2]} := \sum_{i=0}^{n-1} |u_\theta(t_{j+1},\mathbf{x}) - u_\theta(t_j,\mathbf{x})|^{\frac{2\alpha}{\alpha+\gamma-d}} \to_{n\to\infty} c_{2,\alpha,\gamma}^{\frac{2\alpha}{\alpha+\gamma-d}} 2^{\frac{d-\gamma}{\alpha+\gamma-d}} E(|Z|^{\frac{2\alpha}{\alpha+\gamma-d}})(A_2-A_1)|\theta|^{\frac{\gamma-d}{\alpha+\gamma-d}} \tag{3.40}$$

in probability.

Proof : Clearly, for fixed $\mathbf{x} \in \mathbf{R}^d$,

$$\sum_{i=0}^{n-1} |u_\theta(t_{j+1},\mathbf{x}) - u_\theta(t_j,\mathbf{x})|^{\frac{2\alpha}{\alpha+\gamma-d}} = \sum_{i=0}^{n-1} |v(\theta t_{j+1},\mathbf{x}) - v(\theta t_j,\mathbf{x})|^{\frac{2\alpha}{\alpha+\gamma-d}},$$

where $(v_\theta(t,\mathbf{x}), t \geq 0) \equiv^{(d)} (\theta^{-\frac{1}{2}} u_1(t,\mathbf{x}), t \geq 0)$. And from Proposition 4.3 in [39] we know that u_1 admits a variation of order $\frac{2\alpha}{\alpha+\gamma-d}$ which equal $c_{2,\alpha,\gamma}^{\frac{2\alpha}{\alpha+\gamma-d}} C_{\frac{1}{2},1-\frac{d-\gamma}{\alpha}}(A_2 - A_1)$ with $C_{\frac{1}{2},1-\frac{d-\gamma}{\alpha}} = 2^{\frac{d-\gamma}{\alpha+\gamma-d}} \mu_{\frac{2\alpha}{\alpha+\gamma-d}}$ and means that

$$\begin{aligned}\sum_{i=0}^{n-1} |u_\theta(t_{j+1},\mathbf{x}) - u_\theta(t_j,\mathbf{x})|^{\frac{2\alpha}{\alpha+\gamma-d}} &\to_{n\to\infty} c_{2,\alpha,\gamma}^{\frac{2\alpha}{\alpha+\gamma-d}} 2^{\frac{d-\gamma}{\alpha+\gamma-d}} \mu_{\frac{2\alpha}{\alpha+\gamma-d}} (\theta A_2 - \theta A_1)|\theta^{-\frac{1}{2}}|^{\frac{2\alpha}{\alpha+\gamma-d}} \\ &= c_{2,\alpha,\gamma}^{\frac{2\alpha}{\alpha+\gamma-d}} 2^{\frac{d-\gamma}{\alpha+\gamma-d}} \mu_{\frac{2\alpha}{\alpha+\gamma-d}} (A_2 - A_1)|\theta|^{\frac{\gamma-d}{\alpha+\gamma-d}}.\end{aligned}$$

■

From relation (3.40) we can naturally define the following estimator for the parameter $\theta > 0$ of the stochastic partial differential equation (3.28)

$$\begin{aligned}\widehat{\theta}_{n,3} &= \left(\left(c_{2,\alpha,\gamma}^{\frac{2\alpha}{\alpha+\gamma-d}} 2^{\frac{d-\gamma}{\alpha+\gamma-d}} \mu_{\frac{2\alpha}{\alpha+\gamma-d}} (A_2 - A_1)\right)^{-1} \sum_{i=0}^{n-1} |u_\theta(t_{j+1},\mathbf{x}) - u_\theta(t_j,\mathbf{x})|^{\frac{2\alpha}{\alpha+\gamma-d}}\right)^{\frac{\alpha+\gamma-d}{\gamma-d}} \\ &= \left(c_{2,\alpha,\gamma}^{\frac{2\alpha}{\alpha+\gamma-d}} 2^{\frac{d-\gamma}{\alpha+\gamma-d}} \mu_{\frac{2\alpha}{\alpha+\gamma-d}} (A_2 - A_1)\right)^{\frac{d-\gamma}{\alpha+\gamma-d}} \left(S^{n,\frac{2\alpha}{\alpha+\gamma-d}}(u_\theta(\cdot,\mathbf{x}))\right)^{\frac{\alpha+\gamma-d}{\gamma-d}},\end{aligned} \tag{3.41}$$

and so

$$\widehat{\theta}_{n,3}^{\frac{\gamma-d}{\alpha+\gamma-d}} = \frac{1}{c_{2,\alpha,\gamma}^{\frac{2\alpha}{\alpha+\gamma-d}} 2^{\frac{d-\gamma}{\alpha+\gamma-d}} \mu_{\frac{2\alpha}{\alpha+\gamma-d}} (A_2 - A_1)} \sum_{i=0}^{n-1} |u_\theta(t_{j+1},\mathbf{x}) - u_\theta(t_j,\mathbf{x})|^{\frac{2\alpha}{\alpha+\gamma-d}}. \tag{3.42}$$

We have the following asymptotic behavior.

Proposition 3.3.4. *Assume $\frac{2\alpha}{\alpha+\gamma-d} := q$ is an even integer and consider the estimator $\widehat{\theta}_{n,3}$ in (3.41). Then $\widehat{\theta}_{n,3} \to_{n\to+\infty} \theta$ in probability and*

$$\sqrt{n}\left[\widehat{\theta}_{n,3}^{\frac{\gamma-d}{\alpha+\gamma-d}} - \theta^{\frac{\gamma-d}{\alpha+\gamma-d}}\right] \to^{(d)} N(0, s^2_{3,\theta,\alpha,\gamma}) \text{ with } s^2_{3,\theta,\alpha,\gamma} = \sigma^2_{\frac{1}{q},q} \theta^{\frac{2(\gamma-d)}{\alpha+\gamma-d}} \mu^{-2}_{\frac{2\alpha}{\alpha+\gamma-d}}, \tag{3.43}$$

and for n large enough

$$d_W\left(\sqrt{n}\left[\widehat{\theta}_{n,3}^{\frac{\gamma-d}{\alpha+\gamma-d}} - \theta^{\frac{\gamma-d}{\alpha+\gamma-d}}\right], N(0, s^2_{\theta,\alpha})\right) \le c\frac{1}{\sqrt{n}}. \tag{3.44}$$

Proof : From Proposition 3.3.2 and the relation between the fractional and bifractional Brownian motion (see (3.18)), we can see that

$$\left(c_{2,\alpha,\gamma}^{-1} 2^{\frac{d-\gamma}{2\alpha}} \theta^{\frac{d-\gamma}{2\alpha}} u_\theta(t,\mathbf{x}), t \ge 0\right)$$

is a perturbed fBm with Hurst parameter $H = \frac{\alpha-d+\gamma}{2\alpha}$. By taking $H = \frac{\alpha+\gamma-d}{2\alpha}$ and $q = \frac{1}{H} = \frac{2\alpha}{\alpha+\gamma-d}$ in Lemma 3, we get

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\frac{n\theta^{\frac{d-\gamma}{\alpha+\gamma-d}}}{c_{2,\alpha,\gamma}^{\frac{2\alpha}{\alpha+\gamma-d}} 2^{\frac{d-\gamma}{\alpha+\gamma-d}} (A_2 - A_1)} (u_\theta(t_{j+1}, \mathbf{x}) - u_\theta(t_j, \mathbf{x}))^{\frac{2\alpha}{\alpha+\gamma-d}} - \mu_{\frac{2\alpha}{\alpha+\gamma-d}} \right] \to N(0, \sigma^2_{\frac{1}{q},q}),$$

or, equivalently

$$\sqrt{n}\mu_{\frac{2\alpha}{\alpha+\gamma-d}} \theta^{\frac{d-\gamma}{\alpha+\gamma-d}} \left[\widehat{\theta}_{n,3}^{\frac{\gamma-d}{\alpha+\gamma-d}} - \theta^{\frac{\gamma-d}{\alpha+\gamma-d}} \right] \to N(0, \sigma^2_{\frac{1}{q},q}),$$

which is equivalent to (3.43). The bound (3.44) follows easily from Proposition 3.2.1. ■

We finally obtain the asymptotic normality and the rate of convergence for the estimator $\widehat{\theta}_{n,3}$.

Proposition 3.3.5. *Let $\widehat{\theta}_{n,3}$ be given by (3.41) and $s_{3,\theta,\alpha,\gamma}$ be given by (3.43). Then as $n \to \infty$,*

$$\sqrt{n} \left(\widehat{\theta}_{n,3} - \theta \right) \to^{(d)} N \left(0, s_{3,\theta,\alpha,\gamma} \left(\frac{\alpha+\gamma-d}{\gamma-d} \right)^2 \theta^{\frac{2\alpha}{\alpha+\gamma-d}} \right)$$

and

$$d_W \left(\sqrt{n} \left(\widehat{\theta}_{n,3} - \theta \right), N \left(0, s_{3,\theta,\alpha,\gamma} \left(\frac{\alpha+\gamma-d}{\gamma-d} \right)^2 \theta^{\frac{2\alpha}{\alpha+\gamma-d}} \right) \right) \leq c \frac{1}{\sqrt{n}}.$$

Proof : It suffices to apply (3.24) with $g(x) = x^{\frac{\alpha+\gamma-d}{\gamma-d}}$ and $\gamma_0 = \theta^{\frac{\gamma-d}{\gamma+\alpha-d}}$ and to follow the proof of Proposition 4.4.2. ■

3.3.4 Estimators based on the spatial variation

We will repeat the method employed in the previous parts of our work in order to define an estimator expressed in terms of the variations in space of the process (3.31) for the parameter θ in (3.28).

Recall that we show in Proposition 3.3.3 that for every fixed time $t > 0$,

$$\left(\theta^{\frac{1}{2}} m_{\alpha,\gamma}^{-1} u_\theta(t, \mathbf{x}), \mathbf{x} \in \mathbf{R}^d \right)$$

is a perturbed multiparameter isotropic fractional Brownian motion as defined in Lemma 6. Then we can deduce the variation in space of u_θ, by recalling that $\mathbf{x}_i = (x_i^{(1)}, .., x_i^{(d)})$ with $x_i^{(j)} = A_1 + \frac{i}{n}(A_2 - A_1)$ for $i = 0, .., n$ and $j = 1, .., d$.

Proposition 3.3.6. *Let u_θ be given by (3.31). Then*

$$\sum_{i=0}^{n-1} |u_\theta(t, \mathbf{x}_{j+1}) - u(_\theta t, \mathbf{x}_j)|^{\frac{2}{\alpha+\gamma-d}} \to_{n\to\infty} m_{\alpha,\gamma}^{\frac{2}{\alpha+\gamma-d}} (A_2 - A_1)\sqrt{d} E(|Z|^{\frac{2}{\alpha+\gamma-d}})\, |\theta|^{\frac{-1}{\alpha+\gamma-d}}.$$

Proof : We use Lemma 6, point 1 ■

Define, for every $n \geq 1$

$$\widehat{\theta}_{n,4} = \left[(m_{\alpha,\gamma}^{\frac{2}{\alpha+\gamma-d}} \mu_{\frac{2}{\alpha+\gamma-d}} \sqrt{d}(A_2 - A_1))^{-1} \sum_{i=0}^{n-1} |u_\theta(t, \mathbf{x}_{j+1}) - u_\theta(t, \mathbf{x}_j)|^{\frac{2}{\alpha+\gamma-d}} \right]^{-(\alpha+\gamma-d)}, \quad (3.45)$$

and so

$$\widehat{\theta}_{n,4}^{\frac{-1}{\alpha+\gamma-d}} = \frac{1}{m_{\alpha,\gamma}^{\frac{2}{\alpha+\gamma-d}} \mu_{\frac{2}{\alpha+\gamma-d}} \sqrt{d}(A_2 - A_1)} \sum_{i=0}^{n-1} |u_\theta(t, \mathbf{x}_{j+1}) - u_\theta(t, \mathbf{x}_j)|^{\frac{2}{\alpha+\gamma-d}}. \quad (3.46)$$

We can deduce the asymptotic properties of the estimator by using Lemma 6 and Proposition 3.3.1.

Proposition 3.3.7. *The estimator (3.45) converges in probability as $n \to \infty$ to the parameter θ. Moreover, if $\frac{2}{\alpha+\gamma-d}$ is an even integer, then*

$$\sqrt{n}\left[\widehat{\theta}_{n,4}^{\frac{-1}{\alpha+\gamma-d}} - \theta^{\frac{-1}{\alpha+\gamma-d}}\right] \to N(0, s_{4,\theta,\alpha,\gamma}^2) \text{ with } s_{4,\theta,\alpha,\gamma}^2 = \sigma^2_{\frac{\alpha+\gamma-1}{2}, \frac{2}{\alpha+\gamma-d}} \mu^{-2}_{\frac{2}{\alpha+\gamma-d}} \theta^{\frac{-2}{\alpha+\gamma-d}}.$$

We also have, for n large enough,

$$d_W\left(\sqrt{n}\left[\widehat{\theta}_{n,4}^{\frac{-1}{\alpha+\gamma-d}} - \theta^{\frac{-1}{\alpha+\gamma-d}}\right], N(0, s_{4,\theta,\alpha,\gamma}^2)\right) \leq c\frac{1}{\sqrt{n}}.$$

Finally,

Proposition 3.3.8. *With $\widehat{\theta}_{n,4}$ from (3.45), as $n \to \infty$,*

$$\sqrt{n}\left(\widehat{\theta}_{n,4} - \theta\right) \to^{(d)} N\left(0, s_{4,\theta,\alpha,\gamma}\left(\frac{\alpha+\gamma-d}{\gamma-d}\right)^2 \theta^{\frac{2\alpha}{\alpha+\gamma-d}}\right)$$

and

$$d_W\left(\sqrt{n,4}\left(\widehat{\theta}_n - \theta\right), N\left(0, s_{4,\theta,\alpha,\gamma}\left(\frac{\alpha+\gamma-d}{\gamma-d}\right)^2 \theta^{\frac{2\alpha}{\alpha+\gamma-d}}\right)\right) \leq c\frac{1}{\sqrt{n}}.$$

Proof : Apply again (3.24) with $g(x) = x^{\frac{\alpha+\gamma-d}{\gamma-d}}$ and $\gamma_0 = \theta^{\frac{\gamma-d}{\gamma+\alpha-d}}$. ■

Remark 3.3.1. *Notice that in the case $\gamma = 1$ (i.e. there is no spatial correlation and in this case d has to be 1), we retrieve the results in Section 3.2. Observe, as in Section 3.2, that the distance of the estimators (3.41) and (3.45) to their limit distributions is of the same order, although they involve q-variations of different orders.*

3.4 Conclusion

To conclude, in our paper we provide estimators based on power variation for the drift parameter θ of the solution to the fractional stochastic heat equation (4.1). The novelty of our approach is that it allows, comparing with the literature on statistical inference for SPDEs (see [15], [42], [6] etc), to consider the case of a Gaussian noise with non-trivial spatial correlation and to treat the situation when the differential operator in the heat equation (4.1) is the fractional Laplacian instead of the standard Laplacian. The proofs of the asymptotic behavior of the estimators are relatively simple and they are based on the link between the law of the solution and the fractional Brownian motion, using known results on the behavior of the power variations of the fBm. Our approach also gives the rate of convergence of the estimators under the Wasserstein distance via some recent results in Stein-Malliavin calculus (see [45]). We assumed for simplicity a vanishing initial condition in (4.1) but the case of a non-trivial initial value, whose power variations are dominated by those of the fBm, can be also treated by our approach. Another open problem of interest that could be treated via our techniques is adding an unknown volatility parameter in the disturbance term and jointly estimating the drift and the volatility parameters. The case of the fractional heat equation on bounded domains is also interesting but in this case the fundamental solution and implicitly the law of the mild solution changes. Consequently, the relation between the law of the solution and the fBm is not obvious and therefore new techniques are needed.

Deuxième partie

Wave stochastic heat equation

Chapitre 4

Vibrations of a finite string under a fractional random noise

Sommaire

We study the solution to the stochastic wave equation with Dirichlet boundary conditions driven by a Gaussian noise which behaves as a fractional Brownian motion in time and as a Wiener process in space. We obtain the existence of the solution as well as various other properties (pathwise regularity, scaling properties, or the behavior with respect to the Hurst parameter).

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4.1 Introduction

The wave equation represents a mathematical model for the vibrations of a perfectly flexible string. The stochastic wave equation models the vibrations of a string perturbed by a random force. Traditionally, the random noise is considered to be white in time, i.e. it behaves as a standard Brownian motion with respect to its time variable. There exists a vast literature on stochastic partial differential equations in general, and on the wave equation in particular (see e.g. [26], [19], [12], [14], [20], [47], [48] and the references therein). More recently, due to the development of the stochastic calculus with respect to the fractional Brownian motion (fBm in the sequel) and related processes, several authors considered the stochastic wave equation with fractional noise (in time and/or in space), i. e. which behaves as a fBm both in time and space. Among others, we mention the works [7], [13], [23], [34], [50], [56]. The model considered in these references assume that the string is infinite, i.e. the spatial variable belongs to the whole real line or to an interval with infinite Lebesgue measure. As far as we know, the case of a finite string, i.e. when the spatial variable belongs to a finite interval $[0, L]$, with Dirichlet boundary conditions at the endpoints of the interval has not been yet treated.

Our purpose is to analyze the vibrations of a finite string forced by a random process that behaves as a fBm in time and which is white in space. We will analyze the existence of the mild solution to the wave equation, its relation with the weak solution and other properties of the solution, including the pathwise regularity, the scaling properties or the behavior with respect to the Hurst parameter. The main difference with respect to the case of the infinite string is the fact the Green kernel associated to the wave equation is different, it can be written as a trigonometric series. This makes the calculation different and lead to a different behavior of the solution. In particular, the solution is not anymore self-similar in time or stationary in space and its pathwise regularity is not the same as for the infinite random string.

We organized the paper as follows. Section 2 contains some preliminaries on the wave equation and on the calculus related to fBm. In Section 3 we discuss the existence and various distributional and trajectorial properties of the solution to (4.1) while in Section 4 we analyze the behavior of the solution when the Hurst parameter approaches its critical values.

4.2 Preliminaries

We consider the boundary-value problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) &= c^2 \Delta u(t,x) + \dot{W}^H(t,x),\ t \in [0,T],\ x \in [0,L], \\ u(0,x) &= 0, \quad x \in [0,L], \\ \frac{\partial u}{\partial t}(0,x) &= 0, \quad x \in [0,L], \\ u(t,0) &= u(t,L) = 0, \quad t \in [0,T] \end{cases} \tag{4.1}$$

with $c, L > 0$. The constant L represents the length of the string while c is related to the tension. The random perturbation W^H is a *fractional-white Gaussian noise* which is defined as a real valued centered Gaussian field $W^H = \{W_t^H(A); t \in [0,T], A \in B_b([0,L])\}$, over a given complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, with covariance function given by

$$\mathbf{E}\left(W_t^H(A) W_s^H(B)\right) = R_H(t,s)\lambda(A\cap B), \forall A, B \in \mathcal{B}_b([0,L]), \tag{4.2}$$

where R_H is the covariance of the fractional Brownian motion

$$R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s,t \in [0,T]. \tag{4.3}$$

We denoted by $B_b(I)$ the class of bounded Borel subsets of $I \subset \mathbb{R}$ and by λ the Lebesgue measure on $\mathbb{R}$. We will assume throughout this work $H \in \left(\frac{1}{2}, 1\right)$.

We will consider the mild formulation for the solution to the wave equation (4.1). That is, the mild solution to (4.1) is defined as the Green kernel (or the fundamental solution) associated to the wave equation on $(t,x) \in [0,T] \times [0,L]$ integrated in the Wiener sense with respect to the centered Gaussian process with covariance (4.3), i.e.

$$u(t,x) = \int_0^t \int_0^L G_{t,x}(s,y) W^H(ds,dy) \text{ for every } t \in [0,T], x \in [0,L] \tag{4.4}$$

where the Green kernel $G_{t,x}$ is given by, for $0 \le s \le t \le T$ and $x,y \in [0,L]$

$$G_{t,x}(s,y) = \sum_{n=1}^{\infty} \frac{2}{L w_n} \sin(w_n(t-s)) \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \text{ with } w_n = \frac{n\pi c}{L}. \tag{4.5}$$

We will say that the solution to (4.1) exists if the Wiener integral in the righ-hand side of (4.4) is well-defined and

$$\sup_{t\in[0,T]} \mathbf{E} u(t,x)^2 < \infty, x \in [0,L]. \tag{4.6}$$

In order to check the square integrability of (4.4), let us denote by $\mathcal{P}$ the Hilbert space associated with the Gaussian field with covariance (4.23). The inner product in the space $\mathcal{P}$ is given by (see e.g. [6])

$$\begin{aligned}\langle f, g\rangle_{\mathcal{P}} &= \mathbf{E}\int_0^T\int_0^L f(s,y)W^H(ds,dy)\int_0^T\int_0^L g(s,y)W^H(ds,dy) \\ &= \alpha_H\int_0^T\int_0^T dudv|u-v|^{2H-2}\int_0^L dy f(u,y)g(v,y)\end{aligned} \tag{4.7}$$

for any measurable functions $f, g : [0,T]\times[0,L]\to\mathbb{R}$ such that

$$\int_0^T\int_0^T dudv|u-v|^{2H-2}\int_0^L dy|f(u,y)g(v,y)| < \infty.$$

We will also need to introduce the space of integrands with respect to the fractional Brownian motion B^H with Hurst parameter $H\in\left(\frac{1}{2},1\right)$. Denote by $\mathcal{H}$ the Hilbert space associated with the fBm (see e.g. [46]) and recall that

$$\mathbf{E}\int_0^T f(u)dB_u^H\int_0^T g(v)dB_u^H = \alpha_H\int_0^T\int_0^T f(u)g(v)|u-v|^{2H-2}dudv := \langle f,g\rangle_{\mathcal{H}} \tag{4.8}$$

for any $f, g\in|\mathcal{H}|$ where $|\mathcal{H}|$ the space of measurable function $f:[0,T]\to\mathbb{R}$ such that

$$\int_0^T\int_0^T |f(u)f(v)||u-v|^{2H-2}dudv < \infty.$$

4.3 Existence and basic properties of the solution

We start by showing the existence of the solution to the wave equation (4.1), i.e. we prove that the process $(u(t,x), t\in[0,T], x\in[0,L])$ given by (4.4) is well-defined and (4.6) holds true. Then we deduce other properties related to the law of the solution to (4.1).

4.3.1 Existence of the solution

We will use the series representation of the Green kernel (4.5). Notice that (see e.g. [47]) the Wiener integral (4.4) can be also written as

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t)\sin\left(\frac{n\pi x}{L}\right) \tag{4.9}$$

with

$$T_n(t) = \frac{2}{Lw_n} \int_0^t \int_0^L \sin(w_n(t-u)) \sin\left(\frac{n\pi y}{L}\right) W^H(du, dy). \tag{4.10}$$

We will start by giving some useful properties of the family $(T_n, n \geq 1)$.

Lemma 8. *For every $n \geq 1$, $(T_n(t))_{t\in[0,T]}$ is a centered Gaussian process and its covariance function satisfies*

$$|\mathbf{E}T_n(t)T_n(s)| \leq \frac{2L}{\pi^2 c^2} R_H(t,s) \frac{1}{n^2} \text{ for every } s,t \in [0,T]$$

where R_H is given by (4.3).

If $n \neq m$, then $T_n(t)$ and $T_m(s)$ are independent random variables, for every $s,t \in [0,T]$.

Proof : For every $s,t \in [0,T]$, we have, with $\alpha_H = H(2H-1)$, by (4.7)

$$\begin{aligned} \mathbf{E}T_n(t)T_m(s) &= \frac{4}{L^2 w_n w_m} \alpha_H \int_0^t du \int_0^s dv |u-v|^{2H-2} \sin(w_n(t-u)) \sin(w_m(s-v)) \\ &\quad \times \int_0^L dy \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{m\pi y}{L}\right) \end{aligned}$$

and since

$$\int_0^L dy \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{m\pi y}{L}\right) = \begin{cases} 0, & \text{if } n \neq m \\ \frac{L}{2}, & \text{if } m = n \end{cases}$$

we get

$$\mathbf{E}T_n(t)T_m(s) = 0 \text{ if } n \neq m \tag{4.11}$$

and for $n = m$, by (4.8),

$$\begin{aligned} \mathbf{E}T_n(t)T_n(s) &= \frac{4}{L^2 w_n^2} \frac{L}{2} \alpha_H \int_0^t du \int_0^s dv |u-v|^{2H-2} \sin(w_n(t-u)) \sin(w_m(s-v)) \\ &= \frac{2}{L w_n^2} \langle \sin(w_n(t-\cdot)) 1_{[0,t]})(\cdot), \sin(w_n(s-\cdot) 1_{[0,s]}(\cdot)) \rangle_{\mathcal{H}}, \end{aligned} \tag{4.12}$$

so

$$\begin{aligned} |\mathbf{E}T_n(t)T_n(s)| &\leq \frac{4}{L^2 w_n^2} \frac{L}{2} \alpha_H \left| \int_0^t du \int_0^s dv |u-v|^{2H-2} \sin(w_n(t-u)) \sin(w_m(s-v)) \right| \\ &\leq \frac{4}{L^2 w_n^2} \frac{L}{2} \alpha_H \int_0^t du \int_0^s dv |u-v|^{2H-2} = \frac{2L}{\pi^2 c^2} R_H(t,s) \frac{1}{n^2}. \end{aligned}$$

Relation (4.11) gives the independence of the Gaussian random variables $T_m(t)$ and $T_n(s)$ for every $n \neq m$ and for every $s, t \in [0, T]$. ■

An easy and useful consequence of the covariance expression of T_n is the following.

Lemma 9. *For every $n \geq 1$, the process $(T_n(t))_{t\in[0,T]}$ has the same finite dimensional distributions as the process*

$$\left(\frac{\sqrt{2}}{\sqrt{L} w_n} \int_0^t \sin(w_n(t-u)) dB_u^H \right)_{t\in[0,T]}, \tag{4.13}$$

with $(B_t^H)_{t\in[0,T]}$ a fractional Brownian motion with Hurst parameter.

Proof : Both processes are centered Gaussian processes with covariance given by (4.12). ■

Let us show that the mild solution (4.9) is well-defined. By C we denote a generic strictly positive constant that may change from line to line.

Proposition 4.3.1. *For every $H \in \left(\frac{1}{2}, 1\right)$, the stochastic integral in (4.9) is well-defined and it holds that*

$$\sup_{t\in[0,T], x\in[0,L]} \mathbf{E} u(t,x)^2 < \infty.$$

Proof : We have from Lemma 8 and (4.12), for every $t \in [0, T]$ and $x \in [0, L]$

$$\mathbf{E} u(t,x)^2 = \sum_{n=1}^{\infty} \mathbf{E} T_n(t)^2 \left(\sin\left(\frac{n\pi x}{L}\right)\right)^2 \leq C t^{2H} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

so

$$\sup_{t\in[0,T], x\in[0,L]} \mathbf{E} u(t,x)^2 \leq C.$$

■

Notice that the solution existence for every $H \in \left(\frac{1}{2}, 1\right)$. The same holds in the case of the wave equation with fractional-white noise with space variable in $\mathbb{R}$ (see e.g. [7]).

Remark 4.3.1. *It is possible to give an alternative representation of the Green kernel G (4.5) and implicitly of the solution (4.9) by calculating the sum of the series in (4.5). By applying the trigonometric identities*

$$\sin(x)\sin(y) = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right) \text{ and } \sin(x)\cos(y) = \frac{1}{2}\left(\sin(x+y) - \sin(x-y)\right)$$

we obtain

$$\begin{aligned} G_{t,x}(u,z) &= -\sum_{n\geq 1}\frac{1}{2Lw_n}\sin\Big(\frac{n\pi}{L}(c(t-u)+z-x)\Big)+\sum_{n\geq 1}\frac{1}{2Lw_n}\sin\Big(\frac{n\pi}{L}(c(t-u)-(z-x))\Big) \\ &\quad -\sum_{n\geq 1}\frac{1}{2Lw_n}\sin\Big(\frac{n\pi}{L}(c(t-u)+x+z)\Big)+\sum_{n\geq 1}\frac{1}{2Lw_n}\sin\Big(\frac{n\pi}{L}(c(t-u)-(z+x)\Big) \\ &:= S_1+S_2+S_3+S_4. \end{aligned}$$

and by using the formula

$$f(x)=\sum_{n\geq 1}\frac{\sin(nx)}{n}=\begin{cases}\frac{\pi-x}{2}, & \textit{if } x\in(0,2\pi)\\ f(x+2\pi), & \textit{if } x\in\mathbb{R}\end{cases} \tag{4.14}$$

we can compute the sum of the four series above. By assuming $cT<L$, we find that G is not zero only on the sets $(x-z\leq c(t-u)<x+z)\cap(x+z\leq L)$ (and its value is $\frac{L}{2\pi}$) and on the set $(x-z\leq c(t-u)<2L-(x+z))\cap(x+z\geq L)$ (and its value is $-\frac{L}{2\pi}$). This corresponds with the formula given in [14]. In this work, we will not use this expression of the fundamental solution G.

4.3.2 Scaling property

The process $(u(t,x),t\geq 0)$ is not self-similar. On the other hand, it verifies some scaling properties that depend on the parameter c in (4.1). In the following result (and only here), let us use the notation $u(t,x)=u_c(t,x)$ in order to express the dependence on the parameter c. We will denote by "$\equiv^{(d)}$" the equivalence of finite-dimensional distributions. We have :

Proposition 4.3.2. *Fix $x\in(0,L)$. For every $a>0$, the process $(u_c(at,x),t\geq 0)$ has the same finite-dimensional distributions as the process $\big(a^{H+1}u_{ac}(t,x),t\geq 0\big)$.*

Proof : We can write, for every $a>0$,

$$\begin{aligned} u_c(at,x) &= \sum_{n\geq 1}\frac{2}{n\pi c}\int_0^{at}\int_0^L\sin\Big(\frac{n\pi c}{L}(ta-u)\Big)\sin\Big(\frac{n\pi z}{L}\Big)W^H(du,dz)\times\sin\Big(\frac{n\pi x}{L}\Big) \\ &= \sum_{n\geq 1}\frac{2}{n\pi c}\int_0^{t}\int_0^L\sin\Big(\frac{n\pi ca}{L}(t-u)\Big)\sin\Big(\frac{n\pi z}{L}\Big)W^H(d(au),dz)\times\sin\Big(\frac{n\pi x}{L}\Big) \end{aligned}$$

and by using the scaling property in time of the random noise W^H,

$$\begin{aligned} u_c(at,x) &\equiv^{(d)} \sum_{n\geq 1} \frac{2}{n\pi c} a^H \int_0^t \int_0^L \sin\left(\frac{n\pi ca}{L}(t-u)\right) \sin\left(\frac{n\pi z}{L}\right) W^H(du,dz) \times \sin\left(\frac{n\pi x}{L}\right) \\ &= a^{H+1} \sum_{n\geq 1} \frac{2}{n\pi ca} \int_0^t \int_0^L \sin\left(\frac{n\pi ca}{L}(t-u)\right) \sin\left(\frac{n\pi z}{L}\right) W^H(du,dz) \times \sin\left(\frac{n\pi x}{L}\right) \\ &= a^{H+1} u_{ac}(t,x). \end{aligned}$$

■

Recall that (see e.g. [56]), when $x \in \mathbb{R}$, the process given by (4.4) is self-similar in time and stationary in space. When we work on a finite interval in space, these properties are not true. Instead, we have the scaling property from Proposition 4.3.2, which say that, the dilation of time is similar, modulo scaling, with a dilation of the tension of the string.

4.3.3 Behavior at nodal times

Let $T_k = \frac{k\pi}{w_1} = \frac{kL}{c}$, $k = 1, 2, ...$ be a sequence of times. These times are usually called *nodal times of vibrations.*

We have, for $k \geq 1$ integer and for every $x \in [0, L]$

$$u(T_k,x) = \sum_{n\geq 1} \frac{2}{n\pi c} \int_0^{T_k} \int_0^L \sin\left(\frac{n\pi c}{L}(T_k-u)\right) \sin\left(\frac{n\pi z}{L}\right) W^H(du,dz) \times \sin\left(\frac{n\pi x}{L}\right).$$

Since

$$\sin\left(\frac{n\pi c}{L}(T_k-u)\right) = \sin\left(nk\pi - w_n u\right) = (-1)^{kn+1} \sin(w_n u) \tag{4.15}$$

we obtain

$$u(T_k,x) = \sum_{n\geq 1} \frac{2}{n\pi c} \int_0^{T_k} \int_0^L (-1)^{kn+1} \sin\left(w_n u\right) \sin\left(\frac{n\pi z}{L}\right) W^H(du,dz) \times \sin\left(\frac{n\pi x}{L}\right). \tag{4.16}$$

The increment of the solution between two nodal times satisfies the following interesting property. We denote by "$=^{(d)}$" the equality in distribution of two random variables.

Proposition 4.3.3. *Let T_k, T_l be two nodal times with $k > l$ and assume that k, l have the same parity. Then*

$$u(T_k,x) - u(T_l,x) =^{(d)} u(T_k - T_l, x).$$

Proof : For every k, l integers with $k > l$ and with the same parity (both are even or both are odd), we have from (4.16)

$$\begin{aligned}
& u(T_k, x) - u(T_l, x) \\
= & \sum_{n\geq 1} \frac{2}{Lw_n} \int_{T_l}^{T_k} (-1)^{kn+1} \sin(w_u u) \sin\left(\frac{n\pi z}{L}\right) W^H(du, dz) \times \sin\left(\frac{n\pi x}{L}\right) \\
& + \sum_{n\geq 1} \frac{2}{Lw_n} \int_0^{T_l} \left((-1)^{kn+1} - (-1)^{ln+1}\right) \sin(w_u u) \sin\left(\frac{n\pi z}{L}\right) W^H(du, dz) \times \sin\left(\frac{n\pi x}{L}\right) \\
= & \sum_{n\geq 1} \frac{2}{Lw_n} \sin\left(\frac{n\pi x}{L}\right) \int_{T_l}^{T_k} (-1)^{kn+1} \sin(w_u u) \sin\left(\frac{n\pi z}{L}\right) W^H(du, dz)
\end{aligned}$$

because $(-1)^{kn+1} - (-1)^{ln+1} = 0$ when k and l have the same parity. Thus $u(T_k, x) - u(T_l, x)$ is a centered Gaussian random variable and, from (4.7) and (4.15), its variance can be computed as follows

$$\begin{aligned}
& \mathbf{E}\left(u(T_k, x) - u(T_l, x)\right)^2 \\
= & \sum_{n\geq 1} \frac{4}{L^2 w_n^2} \frac{L}{2} \left(\sin\left(\frac{n\pi x}{L}\right)\right)^2 \alpha_H \int_{T_l}^{T_k} \int_{T_l}^{T_k} \sin(w_n u) \sin(w_n v) |u - v|^{2H-2} du dv \\
= & \sum_{n\geq 1} \frac{4}{L w_n^2} \frac{L}{2} \left(\sin\left(\frac{n\pi x}{L}\right)\right)^2 \alpha_H \int_0^{T_k - T_l} \int_0^{T_k - T_l} \sin(w_n (T_l - u)) \sin(w_n (T_l - v)) |u - v|^{2H-2} du dv \\
= & \sum_{n\geq 1} \frac{4}{L w_n^2} \frac{L}{2} \left(\sin\left(\frac{n\pi x}{L}\right)\right)^2 \alpha_H \int_0^{T_k - T_l} \int_0^{T_k - T_l} \sin(w_n u) \sin(w_n v) |u - v|^{2H-2} du dv \\
= & \mathbf{E}\left(u(T_k - T_l, x)\right)^2.
\end{aligned}$$

■

This suggests that the position of the point x on the random string at time T_k is obtained in law by adding the position of the same point at times T_l and $T_k - T_l$.

4.3.4 Relation with the weak solution

Another concept of solution to the boundary value problem (4.1) is the weak solution. We will say that a stochastic process $(u(t, x), t \in [0, T], x \in [0, L])$ is a weak-solution to

(4.1) if for every test function $\varphi \in C^{\infty}([0,T]\times[0,L])$ with $\varphi(T,x) = \frac{\partial\varphi}{\partial t}(T,x) = 0$ for every $x \in [0,L]$ and $\varphi(t,0) = \varphi(t,L) = 0$ for every $t \in [0,T]$, we have

$$\int_0^T dt \int_0^L dx \quad u(t,x)\left(\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2}\right) = \int_0^T \int_0^L \varphi(s,y) W^H(ds,dy). \tag{4.17}$$

We can show that our solution (4.4) also satisfies (4.1) in the weak sense.

Proposition 4.3.4. *The mild solution (4.4) is also a weak solution for (4.1),*

Proof : Let $\varphi \in C^{\infty}([0,T]\times[0,L])$ as above. Then

$$\begin{aligned}&\int_0^T dt \int_0^L dx u(t,x)\left(\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2}\right)\\ =& \int_0^T\int_0^L \left(\int_u^T dt \int_0^L dx G_{t,x}(u,y)\left(\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2}\right)\right) W^H(du,dy).\end{aligned}$$

By integrating twice by parts and using the assumptions on φ,

$$\begin{aligned}&\int_u^T dt \int_0^L dx G_{t,x}(u,y)\frac{\partial^2\varphi}{\partial t^2} = -\int_u^T dt \int_0^L dx \frac{\partial G_{t,x}(u,y)}{\partial t}\frac{\partial\varphi}{\partial t}(t,x)\\ =& \int_0^L dx \frac{\partial G_{t,x}(u,y)}{\partial t}\Big|_{t=u}\varphi(u,y) + \int_u^T dt\int_0^L dx \frac{\partial^2 G_{t,x}(u,y)}{\partial t^2}\varphi(t,x),\end{aligned}$$

and

$$\int_u^T dt \int_0^L dx G_{t,x}(u,y)\frac{\partial^2\varphi}{\partial x^2} = \int_u^T dt \int_0^L dx \frac{\partial^2 G_{t,x}(u,y)}{\partial x^2}\varphi(t,x).$$

We used the fact that G satisfies (4.1) when there is no noise. Notice that

$$\begin{aligned}\int_0^L dx \frac{\partial G_{t,x}(u,y)}{\partial t}\Big|_{t=u} &= \frac{2}{L}\sum_{n\geq 1}\left(\int_0^L dx \sin\left(\frac{n\pi x}{L}\right)\right)\sin\left(\frac{n\pi y}{L}\right)\\ &= \frac{2}{\pi}\sum_{n\geq 1}\frac{1}{n}\left[1-(-1)^n\right]\sin\left(\frac{n\pi y}{L}\right)\end{aligned}$$

we obtain the conclusion since by (4.14)

$$\sum_{n\geq 1}\frac{1}{n}\sin\left(\frac{n\pi z}{L}\right) = \frac{\pi}{2} - \frac{\pi z}{2L}$$

and for a similar formula for the sawtooth wave function (see e.g. [58])

$$\sum_{n\geq 1} \frac{1}{n}(-1)^n \sin\left(\frac{n\pi z}{L}\right) = -\frac{\pi z}{2L}.$$

■

4.4 Behavior of the increments of the solution

In this part, we study the regularity of the sample paths of the solution (4.9) with respect to its time and space variables.

4.4.1 The temporal increment

Let us fix the space variable $x \in [0, L]$ and study the behavior of the process $(u(t, x), t \in [0, T])$. We will need the following auxiliary lemma from [23]. Recall that $\mathcal{H}$ is the Hilbert space associated to the fractional Brownian motion.

Lemma 10. *Let $f(x) = \cos(x)$ for $x \in \mathbb{R}$. Then for every $a, b \in \mathbb{R}$ with $a < b$ we have*

$$\|f 1_{[a,b]}(\cdot)\|^2_{\mathcal{H}} \leq 2\alpha_H \int_0^{b-a} dv \cos(v) v^{2H-2}(b - a - v).$$

We have the following result.

Proposition 4.4.1. *Let $x \in [0, L]$. Then*

$$\mathbf{E}\,|u(t, x) - u(s, x)|^2 \leq C(\varepsilon)|t - s|^{2H-\varepsilon}$$

for every $\varepsilon \in (0, 2H)$ and for every $0 \leq s \leq t \leq T$. C is a constant that depends on ε.

Proof : Let $s, t \in [0, T]$ with $s \leq t$. Recall that C denotes a generic strictly positive constant (that may change from line to line). We have

$$\begin{aligned}
&\mathbf{E}\,|u(t, x) - u(s, x)|^2 \\
= \;& C \sum_{n\geq 1} \frac{1}{n^2} \sin^2\left(\frac{n\pi x}{L}\right) \int_s^t \int_s^t dudv \sin(w_n(t - u)) \sin(w_n(t - v))|u - v|^{2H-2} \\
&+ C \sum_{n\geq 1} \frac{1}{n^2} \int_0^s \int_0^s dudv |u - v|^{2H-2} \\
&\times \left(\sin(w_n(t - u)) - \sin(w_n(s - u))\right) \left(\sin(w_n(t - v)) - \sin(w_n(s - v))\right) \\
:= \;& C(T_1 + T_2).
\end{aligned}$$

We show that $T_i \leq C|t-s|^{2H}$ for $i = 1, 2$. For T_1, this is trivial since, by majorizing the sinus function by 1,

$$T_1 \leq \sum_{n\geq 1} \frac{1}{n^2} \int_s^t \int_s^t dudv|u-v|^{2H-2} = C|t-s|^{2H} \sum_{n\geq 1} \frac{1}{n^2} = C|t-s|^{2H}.$$

Let us focus on the term T_2. Using the trigonometric identity

$$\sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

we can write

$$\begin{aligned} T_2 &\leq C\sum_{n\geq 1} \frac{1}{n^2} \sin^2\left(\frac{w_n(t-s)}{2}\right) \\ &\times \int_0^s \int_0^s dudv \cos\left(\frac{w_n(t+s-2u)}{2}\right) \cos\left(\frac{w_n(t+s-2v)}{2}\right) |u-v|^{2H-2} \\ &= C\sum_{n\geq 1} \frac{1}{n^2} \sin^2\left(\frac{w_n(t-s)}{2}\right) \\ &\times \int_{\frac{t-s}{2}}^{\frac{t+s}{2}} \int_{\frac{t-s}{2}}^{\frac{t+s}{2}} dudv \cos(w_n u)\cos(w_n v)|u-v|^{2H-2} \\ &= C\sum_{n\geq 1} \frac{1}{n^{2+2H}} \sin^2\left(\frac{w_n(t-s)}{2}\right) \int_{w_n\frac{t-s}{2}}^{w_n\frac{t+s}{2}} \int_{w_n\frac{t-s}{2}}^{w_n\frac{t+s}{2}} dudv \cos(u)\cos(v)|u-v|^{2H-2} \\ &= C\sum_{n\geq 1} \frac{1}{n^{2+2H}} \sin^2\left(\frac{w_n(t-s)}{2}\right) \|\cos(\cdot)1_{[w_n\frac{t-s}{2}, w_n\frac{t+s}{2}]}\|_{\mathcal{H}}^2 \end{aligned}$$

by using (4.8). Now, via Lemma 10,

$$\begin{aligned} T_2 &\leq C\sum_{n\geq 1} \frac{1}{n^{2+2H}} \sin^2\left(\frac{w_n(t-s)}{2}\right) \int_0^{w_n s} \cos(v)v^{2H-2}(w_n s - v)dv \\ &= Cs\sum_{n\geq 1} \frac{1}{n^{1+2H}} \sin^2\left(\frac{w_n(t-s)}{2}\right) \int_0^{w_n s} \cos(v)v^{2H-2}dv \\ &\quad -C\sum_{n\geq 1} \frac{1}{n^{2+2H}} \sin^2\left(\frac{w_n(t-s)}{2}\right) \int_0^{w_n s} \cos(v)v^{2H-1}dv. \end{aligned}$$

Via an integration by parts

$$\int_0^{w_n s} \cos(v) v^{2H-1} dv = \sin(w_n s)(w_n s)^{2H-1} - (2H-1) \int_0^{w_n s} \sin(v) v^{2H-2} dv$$

and this implies

$$\begin{aligned} T_2 &\leq Cs \sum_{n\geq 1} \frac{1}{n^{1+2H}} \sin^2\left(\frac{w_n(t-s)}{2}\right) \int_0^{w_n s} v^{2H-2} \cos(v) dv \\ &\quad -C \sum_{n\geq 1} \frac{1}{n^{2+2H}} \sin^2\left(\frac{w_n(t-s)}{2}\right) \sin(w_n s)(w_n s)^{2H-1} \\ &\quad -C \sum_{n\geq 1} \frac{1}{n^{2+2H}} \sin^2\left(\frac{w_n(t-s)}{2}\right) \int_0^{w_n s} \sin(v) v^{2H-2} dv \\ &:= +T_{2,1} - T_{2,2} + T_{2,3}. \end{aligned} \tag{4.18}$$

We will bound separately the three summands $T_{2,1}, T_{2,2}$ and $T_{2,3}$. First, since the integral $\int_0^\infty v^{2H-2} \cos(v) dv$ is convergent,

$$\begin{aligned} T_{2,1} &\leq C \sum_{n\geq 1} \frac{1}{n^{1+2H}} \sin^2\left(\frac{w_n(t-s)}{2}\right) \left| \int_0^{w_n s} v^{2H-2} \cos(v) dv \right| \\ &\leq C \sum_{n\geq 1} \frac{1}{n^{1+2H}} \sin^2\left(\frac{w_n(t-s)}{2}\right) \end{aligned}$$

and writting, for every $\varepsilon \in (0, 2H)$

$$\sin^2\left(\frac{w_n(t-s)}{2}\right) = \left|\sin\left(\frac{w_n(t-s)}{2}\right)\right|^{2H-\varepsilon} \left|\sin\left(\frac{w_n(t-s)}{2}\right)\right|^{2-2H+\varepsilon} \leq C(\varepsilon)(n(t-s))^{2H-\varepsilon}$$

we get

$$T_{2,1} \leq C(\varepsilon)|t-s|^{2H-\varepsilon} \sum_{n\geq 1} \frac{1}{n^{1+\varepsilon}} \leq C(\varepsilon)|t-s|^{2H-\varepsilon}. \tag{4.19}$$

For $T_{2,2}$, we have

$$\begin{aligned} T_{2,2} &\leq C \sum_{n\geq 1} \frac{1}{n^3} \sin^2\left(\frac{w_n(t-s)}{2}\right) \leq C(\varepsilon)|t-s|^{2H-\varepsilon} \sum_{n\geq 1} \frac{1}{n^{3-2H+\varepsilon}} \\ &\leq C|t-s|^{2H-\varepsilon}. \end{aligned} \tag{4.20}$$

Finally, using the convergence of the integral $\int_0^\infty v^{2H-2}\sin(v)dv$

$$T_{2,3} \leq C\sum_{n\geq 1}\frac{1}{n^{2+2H}}\sin^2\left(\frac{w_n(t-s)}{2}\right)\leq C(\varepsilon)|t-s|^{2H-\varepsilon}. \tag{4.21}$$

By plugging (4.19), (4.20) and (4.21) in (4.18), we obtain the conclusion. ■

An immediate consequence is the following.

Corollary 4.4.1. *For every $x\in[0,L]$, the process $(u(t,x),t\in[0,T])$ is Hölder continuous of order δ, for every $\delta\in(0,H)$.*

Proof : This follows from the above Proposition 4.4.1 and the Kolmogorov continuity criterion. ■

4.4.2 Spatial increment

For the study of the spatial increment, we recall the following formula (see e.g. [58]) : for every $x\in(0,\frac{\pi}{2})$,

$$\sum_{n\geq 1}\frac{\sin^2(nx)}{n^2}=\frac{\pi}{2}x-\frac{1}{2}x^2. \tag{4.22}$$

The spatial regularity of the process (4.4) states as follows.

Proposition 4.4.2. *For every $t\in[0,T]$ and for every $x,y\in(0,L)$ with $|x-y|$ small enough,*

$$\mathbf{E}(u(t,x)-u(t,y))^2\leq C|x-y|.$$

Proof : First, for $t\in[0,T]$ and $x,y\in(0,L)$,

$$\begin{aligned} u(t,x)-u(t,y) &= \sum_{n\geq 1}T_n(t)\left(\sin\left(\frac{n\pi x}{L}\right)-\sin\left(\frac{n\pi y}{L}\right)\right)\\ &= 2\sum_{n\geq 1}T_n(t)\sin\left(\frac{n\pi(x-y)}{2L}\right)\cos\left(\frac{n\pi(x+y)}{2L}\right)\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}\left(u(t,x)-u(t,y)\right)^2 &= 4\sum_{n\geq 1}\mathbf{E}T_n(t)^2\sin^2\left(\frac{n\pi(x-y)}{2L}\right)\cos^2\left(\frac{n\pi(x+y)}{2L}\right)\\ &\leq C\sum_{n\geq 1}\frac{1}{n^2}\sin^2\left(\frac{n\pi(x-y)}{2L}\right).\end{aligned}$$

By using the above formula (4.22), for $x, y \in (0, L)$,

$$\begin{aligned}\mathbf{E}\left(u(t,x)-u(t,y)\right)^2 &\leq C\left(\frac{\pi}{2}(x-y)-\frac{1}{2}|x-y|^2\right)\\ &\leq C|x-y|\end{aligned}$$

when $|x-y|$ is small enough. ■

The Hölder regularity in space is obtained via Proposition 4.4.2 and the Kolmogorov criterion.

Corollary 4.4.2. *For every $t \in [0,T]$, the process $(u(t,x), x \in [0,L])$ is Hölder continuous of order δ, for every $\delta \in \left(0, \frac{1}{2}\right)$.*

Notice that the regularity of the solution (4.9), both in time and in space, is different with respect to the case of the wave equation with fractional-colored noise with space variable in the whole real line. Recall (see [23], see also [20] for the white-noise case), that when $x \in \mathbb{R}$, then the corresponding solution is Hölder continuous of order $\delta \in (0, \frac{H}{2})$ in time and of order $\delta \in (0, H)$ in space. In the case of the finite string, while some regularity is gained in time, the solution seems to be less regular in space since $H > \frac{1}{2}$.

4.5 Behavior with respect to the Hurst parameter

Now, we analyze the behavior of the solution to (4.1) with respect to the Hurst parameter. Recall that $H \in \left(\frac{1}{2}, 1\right)$ and the covariance of the solution is not defined for $H = \frac{1}{2}$ and $H = 1$ (see the covariance formula (4.12)). We will see what happens when H converges to its extreme values, i.e. when $H \to \frac{1}{2}$ and $H \to 1$. The behavior of several fractional processes with respect to the Hurst parameter has been studied in [1], [2], [5] while the particular case of solutions to SPDEs can be found in [52], [53].

Recall that a space-time white noise is a real valued centered Gaussian field $W = \{W_t(A); t \in [0,T], A \in B_b([0,L])\}$, over a given complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, with covariance function given by

$$\mathbf{E}\left(W_t(A)W_s(B)\right) = (t \wedge s)\lambda(A \cap B), \forall A, B \in \mathcal{B}_b([0,L]). \tag{4.23}$$

By $C([0,T])$ we denote the set of continuous functions on $[0,T]$.

Proposition 4.5.1. *Let $(u(t,x), t \in [0,T], x \in [0,L])$ be given by (4.9). Then, as $H \to \frac{1}{2}$, for every $x \in [0,L]$, the process $(u(t,x), t \in [0,T])$ converges in the space $C[0,T]$ to the process*

$(u_0(t,x), t\in[0,T])$ defined by

$$u_0(t,x)=\int_0^t\int_0^L G_{t,x}(s,y)W(ds,dy) \tag{4.24}$$

where W is a space-time white noise and G is given by (4.5).

As $H\to 1$, for every $x\in[0,L]$, the process $(u(t,x), t\in[0,T])$ converges in the space $C[0,T]$ to the process $(u_1(t,x), t\in[0,T])$ defined by

$$u_1(t,x)=\sum_{n\geq 1}\frac{\sqrt{2}}{\sqrt{L}w_n}\left(\int_0^t \sin(w_n(t-u))du\right)Z_n \tag{4.25}$$

with $(Z_n)_{n\geq 1}$ independent standard normal random variables.

Proof : Recall that for every $s,t\in[0,T]$,

$$\mathbf{E}u(t,x)u(s,x)=\sum_{n\geq 1}\frac{4}{L^2w_n^2}\frac{L}{2}H(2H-1)\int_0^t\int_0^s dudv\sin(w_n(t-u))\sin(w_n(s-v))|u-v|^{2H-2}. \tag{4.26}$$

Let us look to the limit as $H\to\frac{1}{2}$ and $H\to 1$ of the quantity

$$I(H)=H(2H-1)\int_0^t du\int_0^s dv\sin(w_n(t-u))\sin(w_n(s-v))|u-v|^{2H-2}.$$

Assume $s\leq t$. We can express $I(H)$ as follows

$$\begin{aligned}I(H) &= H(2H-1)\int_s^t du\int_0^s dv\sin(w_n(t-u))\sin(w_n(s-v))(u-v)^{2H-2}\\ &\quad +H(2H-1)\int_0^s du\int_0^u dv\sin(w_n(t-u))\sin(w_n(s-v))(u-v)^{2H-2}\\ &\quad +H(2H-1)\int_0^s du\int_u^s dv\sin(w_n(t-u))\sin(w_n(s-v))(v-u)^{2H-2}\\ &:= I_1(H)+I_2(H)+I_3(H).\end{aligned}$$

Let us calculate first $I_1(H)$. By integrating by parts in the integral dv,

$$\begin{aligned}I_1(H) &= H\int_s^t du\sin(w_n(t-u))\left[-\sin(w_n(s-v))(u-v)^{2H-1}\Big|_{v=0}^{v=s}\right.\\ &\quad \left.-\int_0^s w_n\cos(w_n(s-v))(u-v)^{2H-1}\right]dv\\ &= H\int_s^t du\sin(w_n(t-u))\left[\sin(w_ns)u^{2H-1}-\int_0^s dvw_n\cos(w_n(s-v))(u-v)^{2H-1}\right].\end{aligned}$$

We take the limit of the above quantity when $H \to \frac{1}{2}$. Clearly

$$H \sin(w_n(t-u)) \sin(w_n s) u^{2H-1} \to_{H\to\frac{1}{2}} \frac{1}{2} \sin(w_n(t-u)) \sin(w_n s)$$

and

$$H \sin(w_n(t-u)) \cos(w_n(s-v))(u-v)^{2H-1} \to_{H\to\frac{1}{2}} \frac{1}{2} \sin(w_n(t-u)) \cos(w_n(s-v)).$$

The first term of $I_1(H)$ is dominated by $\min(1,u)$ which is integrable over $[s,t]$. The second term is dominated by $min(1,u-v)$. By applying the dominated convergence theorem,

$$I_1(H) \quad \to_{H\to\frac{1}{2}} \quad \frac{1}{2} \int_s^t du \sin(w_n(t-u)) \left(\sin(w_n(s) - \int_0^s dv w_n \cos(w_n(s-u)) \right) = 0.$$

For $I_2(H)$, we similarly get

$$\begin{aligned} I_2(H) \quad &= \quad H \int_0^s du \sin(w_n(t-u)) \left[\sin(w_n s) u^{2H-1} - \int_0^u dv \quad w_n \cos(w_n(s-v))(u-v)^{2H-1} \right] \\ &\to_{H\to\frac{1}{2}} \quad \frac{1}{2} \int_0^s du \sin(w_n(t-u)) \left[\sin(w_n s) + \sin(w_n(s-v)) \big|_{v=0}^{v=u} \right] \\ &= \quad \frac{1}{2} \int_0^s du \sin(w_n(t-u)) \sin(w_n(s-u)). \end{aligned}$$

Finally,

$$\begin{aligned} I_3(H) \quad &= \quad H \int_0^s du \sin(w_n(t-u)) \int_u^s dv \quad w_n \cos(w_n(s-v))(v-u)^{2H-1} \\ &\to \quad \frac{1}{2} \int_0^s du \sin(w_n(t-u)) \sin(w_n(s-u)). \end{aligned}$$

We obtained

$$I(H) \to_{H\to\frac{1}{2}} \int_0^s du \sin(w_n(t-u)) \sin(w_n(s-u)). \tag{4.27}$$

Since the terms of the series are dominated by Cn^{-2} which ensures the normal convergence, from (4.27) and (4.26) we have

$$\begin{aligned} \mathbf{E}u(t,x)u(s,x) \quad &\to_{H\to\frac{1}{2}} \sum_{n\geq 1} \frac{4}{L^2 w_n^2} \frac{L}{2} \int_0^s du \sin(w_n(t-u)) \sin(w_n(s-u)) \\ &= \quad \mathbf{E}u_0(t,x)u_0(s,x) \end{aligned}$$

with u_0 given by (4.24). This gives the convergence of finite-dimensional distributions of u to those of u_0, since both processes are Gaussian. The tightness is obtained from Proposition 4.4.1 and the Billingsley criterion (see [8, Theorem 12.3] or [9]).

If $H \to 1$, it is easy to see that

$$I(H) \to \int_0^t \sin(w_n(t-u))du \int_0^s \sin(w_n(s-v))dv$$

and thus

$$\begin{aligned}\mathbf{E}u(t,x)u(s,x) \quad &\to_{H\to 1} \sum_{n\geq 1} \frac{4}{L^2 w_n^2}\frac{2}{L} \\ &\int_0^t \sin(w_n(t-u))du \int_0^s \sin(w_n(s-v))dv = \mathbf{E}u_1(t,x)u_1(s,x)\end{aligned}$$

with u_1 given by (4.25). So we have the convergence of finite dimensional distributions of u to those of u_1 and the tightness is obtained as above. ■

Let us remark that the above result shows that the solution (4.4) converges, when H approaches its extreme values, to the solution to the wave equation driven by the "limit of the noise". Indeed, when $H \to \frac{1}{2}$, it is clear that the fractional-white noise (4.2) converges to the white noise (4.23) while when H is close to 1 the solution (4.4) has the same law as the process (4.13) and we use that fact that $(B_t^1)_{t\in[0,T]} =^{(d)} (tZ, t \in [0,1])$ with Z a standard normal random variable.

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