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Extension properties of holomorphic mappings with values in complex Hilbert manifolds

Extension de fonctions holomorphes à valeurs dans des variétés de Hilbert

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Abstract

Extension properties of holomorphic mappings with values in complex Hilbert manifolds

Our thesis contains the following two principal results. First is the existence of 1-complete neighborhoods. Namely:
Let $\Delta^k \subset \mathbb{C}^k$ be the unit polydisk and let $f : \Delta^k \to \mathcal{X}$ be an imbedding to a complex Hilbert manifold. We prove that for any $0 < r < 1$ there exists a fundamental system of 1-complete neighborhoods of $f(\bar{\Delta}^k_r)$. This result has several interesting consequences. Let's state one of them.

Let $H(r) = \{(z_1, z_2) \in \Delta^2 \mid |z_2| < r \text{ or } 1 - r < |z_1| < 1\}$ be the Hartogs figure of radius $r \in]0,1[$ in $\mathbb{C}^2$. We say that a finite dimensional complex manifold X is Hartogs if every holomorphic map $f : H(r) \to X$ extends holomorphically to Δ^2. If the same is true for a complex Hilbert manifold $\mathcal{X}$ then we say that $\mathcal{X}$ is Hilbert-Hartogs. We prove that a holomorphic map from a domain $\mathcal{D}$ in a Hilbert manifold $\mathcal{X}$ to a Hilbert-Hartogs manifold $\mathcal{Y}$ extends holomorphically to a neighborhood of a pseudoconcave boundary point of $\mathcal{D}$.

Our second main result concerns generalized loop spaces of finite dimensional complex manifolds. Remark that they naturally carry the structure of complex Hilbert manifolds. We denote the space of Sobolev maps $W^{k,2}$ from a compact real manifold S to a complex manifold X as $W^{k,2}(S, X)$.

We prove that the loop space $W^{k,2}(S, X)$ of a Hartogs manifold X is Hilbert-Hartogs.

Résumé

Extension de fonctions holomorphes à valeurs dans des variétés de Hilbert

Cette thèse contient les deux principaux résultats suivants. Le premier est l'existence d'un système fondamental de voisinages 1-complet:
Soit $\Delta^k \subset \mathbb{C}^k$ le polydisque unité et soit $f : \Delta^k \to X$ un plongement dans une variété de Hilbert complexe. Nous prouverons que pour tout $0 < r < 1$, il existe un système de voisinages 1-complet pour $f(\bar{\Delta}^k_r)$. Ce résultat a de nombreuses conséquences. Nous allons en énoncer un en particulier.

Soit $H(r) = \{(z_1, z_2) \in \Delta^2 \mid |z_2| < r \text{ ou } 1-r < |z_1|\}$ la figure de Hartogs de rayon r dans $\mathbb{C}^2$, $r \in]0, 1[$. Une variété complexe de dimension finie X est dite Hartogs si toute fonction holomorphe $f : H(r) \to X$ s'étend holomorphiquement à Δ^2. Si la même chose est vrai pour une variété de Hilbert $\mathcal{X}$, nous disons que $\mathcal{X}$ est Hilbert-Hartogs. Nous prouverons qu'une fonction holomorphe $f : \mathcal{D} \subset \mathcal{X} \to \mathcal{Y}$ sur un domaine $\mathcal{D}$, avec $\mathcal{X}$ une variété de Hilbert et $\mathcal{Y}$ une variété Hilbert-Hartogs, se prolonge holomorphiquement au voisinage d'un point pseudoconcave du bord du domaine $\mathcal{D}$.

Le second résultat principal concerne les espaces des lacets généralisés sur des variétés complexes de dimension finie. Nous pouvons d'abord remarquer qu'ils ont une stucture de variétés de Hilbert complexes. Notons l'espace des fonctions d'une variété réelle compacte S vers une variété complexe X qui sont de classe de Sobolev $W^{k,2}$ par $W^{k,2}(S, X)$.

Nous prouverons alors que l'espace des lacets généralisé $W^{k,2}(S, X)$ d'une variété de Hartogs X est une variété Hilbert-Hartogs.

Introduction

Several complex variables is a domain younger than complex analysis in one variable. One of the precursors is K. Weierstrass around 1879 with his work on "Some theorems on the theory of analytic functions of several variables". Another founder of several complex variables is H. Poincaré who proved that a meromorphic function on $\mathbb{C}^2$ wich is locally the quotient of two holomorphic functions of two complex variables is the quotient of two holomorphic functions defined on $\mathbb{C}^2$. P. Cousin generalized this result to several variables. Later in 1907, Poincaré wrote a paper that should be considered as the beginning of the study of biholomorphic maps. During the same period around 1906 F. Hartogs [H] worked on analytic continuation and proved in his thesis an analytic continuation of holomorphic functions from the Hartogs figure or Hartogs "pot" to the bidisk Δ^2. He used the Cauchy formula. Later E. Levi generalised this result to meromorphic functions. Complex analysis in several variables takes another direction due to the works of E. Cartan, K. Oka and others establishing a link between several complex variables and algebra, topology, algebraic geometry and some domains in physics. More recently in [Iv2] S. Ivashkovich studied extension properties of meromorphic maps with values in Kähler manifolds and proved the analytic continuation of such maps from the Hartogs figure to the unit polydisk. Following this idea one can ask the question about analytic continuation of holomorphic mappings from Hartogs figures with values in infinite dimensional complex manifolds.

This thesis focuses on the study of extension properties of holomorphic maps with values in complex Hilbert manifolds. The study is motivated by the fact that nowadays even if there are plenty of results on several complex variables the literature on infinite dimensional complex analysis is not rich. Many statements can be deduced from the finite dimensional case but there are some that do not generalize easily and others that stay unknown. An example of the difficulty to generalise is Royden's lemma. H. Royden proved in [Ro] the following statement:

Let X be a complex manifold of dimension n and let $f : \bar{\Delta}^q \to X$ be a holomorphic imbedding of a neighborhood of the closed unit polydisk in $\mathbb{C}^q$ to X for $q \in [\![1, n-1]\!]$. Then there exists a holomorphic imbedding $F : \bar{\Delta}^q \times \bar{\Delta}^{n-q} \to X$ extending f, i.e. $F|_{\bar{\Delta}^q \times \{0\}} = f$.

The proof crucially uses results on Stein manifolds. But there is no Stein theory in the infinite dimensional case yet. Even a local solvability of the $\bar{\partial}$-equation in Hilbert case is not clear. Nevertheless we can prove a weaker version of this result in the case of Hilbert manifolds. All Hilbert manifolds in this thesis are modeled over l^2 and are supposed to be second countable, i.e. their topology has a countable base. Here l^2 stands for the Hilbert space of square summable complex sequences with its standard Hermitian scalar product $< z, w >= \sum_k z_k \bar{w}_k$. The first result of our thesis is the following

Theorem 0.1. *Let $\phi : \bar{\Delta}^q \to \mathcal{X}$ be an imbedded analytic q-disk in a complex Hilbert manifold $\mathcal{X}$. Then $\phi(\bar{\Delta}^q)$ has a fundamental system of 1-complete neighborhoods.*

This theorem is interesting on its own and it also plays an improtant technical role in the proof of several results of our thesis. In particular it gives an opportunity to use the techniques of analytic continuation on Hilbert manifolds. Let recall first that the Hartogs figure in $\mathbb{C}^2$ is the following domain

$$H(r) = (\Delta \times \Delta_r) \cup (A_{1-r,1} \times \Delta). \tag{1}$$

Here Δ_r denotes the disk of radius r in $\mathbb{C}$ centered at zero, Δ the unit disk and $A_{1-r,1} = \Delta \setminus \bar{\Delta}_{1-r}$ the annulus for $r \in]0, 1[$.

Definition 0.1. *We say that a complex manifold X is Hartogs if every holomorphic mapping $f : H(r) \to X$ extends to a holomorphic mapping $\tilde{f} : \Delta^2 \to X$ from the unit bidisk to X. If the same is true for a complex Hilbert manifold $\mathcal{X}$ we say that $\mathcal{X}$ is Hilbert-Hartogs.*

Holomorphic mappings with values in Hilbert-Hartogs manifolds possess much stronger extension properties than what postulated in their definition. Consider an infinite dimensional analog of a Hartogs figure

$$H^\infty(r) := (\Delta \times B^\infty(r)) \cup (A_{1-r,1} \times B^\infty)$$

where B^∞ is the unit ball in l^2 and $B^\infty(r)$ the ball of radius r in l^2 centered at the origin.

Theorem 0.2. *Let $\mathcal{X}$ be a Hilbert-Hartogs manifold. Then for every $r > 0$ every holomorphic mapping $f : H^\infty(r) \to \mathcal{X}$ extends to a holomorphic mapping $\tilde{f} : \Delta \times B^\infty \to \mathcal{X}$.*

The proof of this theorem as well as that of theorem 3.1 in chapter 3 rely heavily on the existence of 1-complete neighborhoods as in theorem 0.1. There is a link between the Hartogs extension property of a Hilbert manifold $\mathcal{Y}$ and extension properties of mappings from a domain $\mathcal{D}$ in a complex Hilbert manifold $\mathcal{X}$, which is pseudoconcave in some of its boundary points, to $\mathcal{Y}$. Theorem 0.2 permits us to prove the following statement.

Corollary 0.1. *If a domain $\mathcal{D}$ in a complex Hilbert manifold $\mathcal{X}$ is pseudoconcave at a boundary point p then every holomorphic map $f : \mathcal{D} \to \mathcal{Y}$ to a Hilbert-Hartogs manifold $\mathcal{Y}$ extends holomorphically to a neighborhood of p.*

Now we are going to state the second main result of our thesis. Let S be a n-dimensional real compact manifold without boundary. Denote by $W^{k,2}(S,X)$ the space of maps from S to X of Sobolev class $W^{k,2}$. One usually calls $W^{k,2}(S,X)$ a generalized loop space. We take $k \geqslant \frac{n}{2} + \alpha$ with $\alpha \in]0,1[$ in order for maps from $W^{k,2}(S,X)$ to be at least continuous. L. Lempert showed that this mapping space inherits a natural complex structure from the ground manifold. More precisely it is shown in [L1] that the generalized loop space of Sobolev class $W^{k,2}(S,\mathcal{X})$ is a complex Hilbert manifold. In fact starting from a map $f \in W^{k,2}(S,X)$ we construct the pullback bundle $f^*TX \to S$. A neighborhood of the zero section of the pullback bundle $W^{k,2}(S, f^*TX)$ defines naturally a coordinate neighborhood of f in $W^{k,2}(S,X)$. Thus it provides the structure of a complex Hilbert manifold on $W^{k,2}(S,X)$. We prove that

Theorem 0.3. *A generalized loop space of a complex Hartogs manifold is a Hilbert-Hartogs manifold.*

One finds Hilbert-Hartogs manifolds more often than one could expect and we will end our exposition with a list of examples of Hibert-Hartogs manifolds. l^2 is obviously Hilbert-Hartogs. Let $\mathcal{X}$ be a complex submanifold of l^2. Then $\mathcal{X}$ is also Hilbert-Hartogs.

Furthermore we can prove that if the base space and the fiber of a Hilbert fibration are Hilbert-Hartogs then the total space is Hilbert-Hartogs, too. If $\mathcal{Y} \to \mathcal{X}$ is an unramified covering between Hilbert manifolds then $\mathcal{X}$ is Hilbert-Hartogs if and only if $\mathcal{Y}$ is. If for example $\Lambda = \mathrm{span}_{\mathbb{Z}}\{e_1, ie_1, e_2, ie_2, ...\}$ is the integer lattice in l^2, then $\mathbb{T}^\infty := l^2/\Lambda$ is Hartogs.

Remark that all connected Riemann surfaces except projective line are Hartogs. In fact consider the map

$$\begin{array}{rccc} f: & \mathbb{C}^2\backslash\{(0,\frac{1}{2})\} & \to & \mathbb{P}^1 \\ & (z_1, z_2) & \mapsto & [z_1 : z_2 - \frac{1}{2}]. \end{array}$$

One can see that the map f restricted to the Hartogs figure $H(r)$ with $r = 1/4$ does not extend to the whole bidisk Δ^2. Now take X a compact Riemann surface different from $\mathbb{P}^1$. By the uniformization theorem its universal cover Y is either Δ or $\mathbb{C}$. The result about coverings implies that X is Hartogs. If X is not compact then X is Stein and therefore Hartogs.

Now, Theorem 0.3 provides us a lot of interesting and non trivial examples of infinite dimensional Hilbert-Hartogs manifolds. For example let $S = \mathbb{S}^1$ be the circle and X be a connected Riemann surface. Theorem 0.3 implies that all loop spaces $W^{1,2}(\mathbb{S}^1, X)$ for $X \neq \mathbb{P}^1$ are Hilbert-Hartogs.

The results of this thesis are pulished in two papers "On Hilbert-Hartogs manifolds." [A-Z] and "Loop Spaces as Hilbert-Hartogs Manifolds" [A-I].

This thesis is organized as follows. In chapter one we recall briefly the analysis on complex Hilbert space and complex Hilbert manifolds and the basics on fiber bundles. In particular we recall the construction of the partition of unity and its consequences. In the second chapter we prove the existence of a fundamental system of 1-complete neighborhoods. In the third chapter we introduce the notion of Hartogs manifold and exploiting the existence of the fundamental system of 1-complete neighborhoods we prove several properties of these manifolds. In the last chapter we recall the complex Hilbert structure on loop spaces and we prove that loop spaces of Hartogs manifolds are Hilbert-Hartogs.

Introduction (French)

L'analyse complexe à plusieurs variables est un domaine beaucoup plus récent que son homologue à une variable. L'un des précurseurs est K. Weierstrass par ses travaux sur "Some theorems on the theory of analytic functions of several variables" aux alentours de 1879. Un autre des fondateurs de ce domaine est H. Poincaré qui prouva qu'une fonction, étant localement le rapport de deux fonctions holomorphes à deux variables, est le quotient de deux fonctions holomorphes entières, c'est-à-dire définies sur $\mathbb{C}^2$. P. Cousin généralisa ce dernier résultat aux cas des fonctions holomorphes à plusieurs variables. Plus tard, en 1907, Poincaré écrivit ce que l'on pourrait considérer comme le commencement de l'étude des biholomorphismes. Durant la même période, F. Hartogs [H] travailla sur le prolongement analytique et donna une preuve rigoureuse, dans sa thèse, du prolongement analytique sur le bidisque Δ^2 de fonctions holomorphes définies sur une figure de Hartogs ou autrement appelée "marmite" de Hartogs. Dans ce but, il utilisa essentiellement la formule de Cauchy. Ensuite, E. Levi généralisa ce résultat aux fonctions méromorphes. L'analyse complexe à plusieurs variables prend un nouveau tournant avec les travaux de E. Cartan, K. Oka et bien d'autres établissant un lien avec l'algèbre, la topologie, la géometrie algébrique et certains domaines de la physique. Plus récemment dans [Iv2], S.Ivashkovich étudia certaines propriétés d'extension d'application à valeur dans des variétés kählerienne et prouva le prolongement analytique de ces applications définies sur une figure de Hartogs vers le polydisque unité. En poursuivant cette idée, il est possible de réfléchir à la question du prolongement analytique d'une fonction holomorphe définie sur une figure de Hartogs et à valeurs dans une variété complexe de dimension infinie.

Cette thèse porte sur des propriétes d'extension de fonctions holomorphes dans des variétés de Hilbert complexes. L'étude est motivée par le fait qu'actuellement, comparée à la richesse de la littérature sur l'analyse complexe à plusieurs variables, il n'existe que très peu de résultats sur les variétés complexes de dimension infinie. Beaucoup de résultats se déduisent de la même manière que dans le cas d'un nombre fini de variables complexes. D'autres nécessitent un peu plus de travail et il existe encore d'autres dont la véracité reste inconnue. Le lemme de Royden est un exemple de la difficulté à généraliser. H. Royden prouva dans [Ro] l'énoncé suivant:

Soit X une variété complexe de dimension n et soit $f : \bar{\Delta}^q \to X$ un plongement holomorphe d'un voisinage du polydisque fermé de $\mathbb{C}^q$ dans X, avec $q \in [\![1, n-1]\!]$. Alors, il existe un plongement holomorphe $F : \bar{\Delta}^q \times \bar{\Delta}^{n-q} \to X$ qui étend f, c'est-à-

dire $F|_{\bar{\Delta}^q\times\{0\}} = f$.

La preuve utilise un résultat sur les variétés de Stein. Cependant, il n'y a pas encore d'analogue en dimension infinie des variétés de Stein. Même une résolution locale du problème de $\bar{\partial}$ en variété de Hilbert n'est pas claire. Néanmoins, nous pouvons prouver une version plus faible de ce résultat dans le cadre des variétés de Hilbert. Toutes les variétés de Hilbert dans cette thèse sont modelées sur l^2 et sont supposées avoir une topologie à base dénombrable. Ici l^2 correspond à l'espace de Hilbert des suites complexes de carrés sommables munit de sa structure hermitienne complexe, c'est-à-dire munit de son produit hermitien $< z, w >= \sum_k z_k\bar{w}_k$. Le premier résultat de cette thèse est le suivant:

Théorème 0.1. *Soit* $\phi : \bar{\Delta}^q \to \mathcal{X}$ *un q-disque analytique plongé dans une variété de Hilbert complexe* $\mathcal{X}$. *Alors* $\phi(\bar{\Delta}^q)$ *possède un système fondamental de voisinages 1-complet.*

Ce théorème est intéressant en soit et il joue aussi un important rôle technique dans plusieurs preuves des résultats de cette thèse. En particulier, il permet d'utiliser des techniques de prolongements analytiques dans des variétés de Hilbert. Dans la continuité des travaux de F. Hartogs, nous avons étudié les propriétés d'extension des variétés de Hartogs. Rappelons d'abord qu'une figure de Hartogs dans $\mathbb{C}^2$ est

$$H(r) := (\Delta \times \Delta_r) \cup (A_{1-r,1} \times \Delta).$$

Ici Δ_r correspond au disque de rayon r dans $\mathbb{C}$ centré en zéro, Δ le disque unité et $A_{1-r,1} = \Delta\backslash\bar{\Delta}_{1-r}$ à l'anneau pour $r \in]0, 1[$.

Définition 0.1. *Une variété complexe* X *est dite Hartogs si pour toute application holomorphe* $f : H(r) \to X$ *s'étend à une application holomorphe* $\tilde{f} : \Delta^2 \to X$ *définie sur le bidisque unité* Δ^2 *et à valeurs dans* X.

Si une variété $\mathcal{X}$ *de Hilbert vérife cette condition, nous dirons alors que c'est une varété Hilbert-Hartogs.*

Nous prouvons qu'une application holomorphe à valeurs dans une variété Hilbert-Hartogs possède des propriétés d'extension beaucoup plus fortes que ce qui est donné dans la définition. Considérons l'analogue d'une figure de Hartogs en dimension infinie

$$H^\infty(r) := (\Delta \times B^\infty(r)) \cup (A_{1-r,1} \times B^\infty)$$

où B^∞ est la boule unité de l^2 et $B^\infty(r)$ est la boule dans l^2 de rayon r centrée à l'origine.

Théorème 0.2. *Soit $\mathcal{X}$ une variété de Hilbert-Hartogs. Alors, pour tout $r > 0$, toute application holomorphe $f : H^{\infty}(r) \to \mathcal{X}$ s'étend en une application holomorphe $\tilde{f} : \Delta \times B^{\infty} \to \mathcal{X}$.*

La preuve de ce théorème ainsi que le théorème 3.1 dans le chapitre 3 sont fondées sur l'existence d'un système de voisinages 1-complet comme pour celle du théorème 0.1. Il y a une relation entre la propriété d'extension de Hartogs d'une variété de Hilbert $\mathcal{Y}$ et les propriétés d'extensions d'applications d'un domaine $\mathcal{D}$ d'une variété de Hilbert $\mathcal{X}$ qui est pseudoconcave en des points du bord et à valeurs dans $\mathcal{Y}$. Le théorème 0.2 permet de prouver le résultat suivant.

Corollaire 0.1. *Si un domaine $\mathcal{D}$ dans une variété de Hilbert complexe $\mathcal{X}$ est pseudoconcave en un point p de la frontière $\partial\mathcal{D}$ alors toute application holomorphe $f : \mathcal{D} \to \mathcal{Y}$ dans une variété Hilbert-Hartogs s'étend holomorphiquement à un voisinage de p.*

Nous allons maintenant énoncé le deuxième résultat principal de notre thèse. Soit S une variété réelle compacte de dimension n sans bord. Notons par $W^{k,2}(S, X)$ l'espace des applications de S vers X de classe de Sobolev $W^{k,2}$. L'espace $W^{k,2}(S, X)$ est habituellement appelé un espace des lacets généralisé. Nous choisissons $k \geqslant \frac{n}{2} + \alpha$ avec $\alpha \in]0, 1[$ afin que les applications de $W^{k,2}(S, X)$ soient au moins continues. L. Lempert a prouvé que des espaces de ce type héritent de la structure complexe de la variété d'arrivé. Plus précisement, dans [L1], il est prouvé qu'un espace des lacets généralisé de classe de Sobolev $W^{k,2}$ est une variété de Hilbert complexe. En effet, en partant d'une application $f \in W^{k,2}(S, X)$, nous pouvons construire le tiré en arrière du fibré tangent au dessus de S, c'est-à-dire $f^*TX \to S$. Un voisinage de la section nulle dans l'espace des sections du tiré en arrière du fibré tangeant $W^{k,2}(S, f^*TX)$ définie un voisinage des coordonnées de $f \in W^{k,2}(S, X)$. Ainsi on obtient une structure de variété complexe sur $W^{k,2}(S, X)$. Nous démontrons que

Théorème 0.3. *L'espace des lacets généralisé d'une variété de Hartogs complexe est une variété Hilbert-Hartogs.*

Les variétés Hilbert-Hartogs apparaissent beaucoup plus souvent que ce que nous pouvons attendre et nous allons finir notre exposé par des exemples de variétés Hilbert-Hartogs. l^2 est évidemment Hilbert-Hartogs. Soit $\mathcal{X}$ une sous-variété complexe de l^2. Alors $\mathcal{X}$ est aussi Hilbert-Hartogs.

Plus encore, nous pouvons démontrer que que si l'espace de base et la fibre d'une fibration de Hilbert est Hilbert-Hartogs alors l'espace total de la fibration est aussi Hilbert-Hartogs. Si $\mathcal{Y} \to \mathcal{X}$ est un revêtement non-ramifié entre des variétés de Hilbert, alors $\mathcal{X}$ est Hilbert-Hartogs si et seulement si $\mathcal{Y}$ l'est aussi. Si Λ est le réseau d'entier dans l^2 défini par $\Lambda := \mathrm{span}_{\mathbb{Z}}\{e_1, ie_1, e_2, ie_2, ...\}$, alors $\mathbb{T}^{\infty} := l^2/\Lambda$ est Hilbert-Hartogs.

En outre, remarquons que toutes les surfaces de Riemann connexe à l'exception des droites projectives sont Hartogs. Effectivement, considérons l'application

$$\begin{array}{rccc} f: & \mathbb{C}^2\backslash\{(0,\frac{1}{2})\} & \to & \mathbb{P}^1 \\ & (z_1,z_2) & \mapsto & [z_1, z_2-\frac{1}{2}]. \end{array}$$

Nous pouvons observer que l'application restreinte à la figure de Hartogs $H(r)$ avec $r = 1/4$ ne s'étend pas à tout le bidisque Δ^2. Désormais, soit X une surface de Riemann compacte connexe différente de $\mathbb{P}^1$. Son recouvrement universel Y, par le théorème d'uniformisation, est Δ ou $\mathbb{C}$. L'énoncé sur les recouvrements, cité précedemment, permet de déduire que X est Hartogs. Si X n'est pas compacte, alors X est Stein et donc est Hartogs.

Le théorème 0.3 donne aussi de nombreux exemples intéressants et non triviaux de variétés de dimension infinie qui sont Hilbert-Hartogs. Par exemple, soit $S = \mathbb{S}^1$ le cercle et soit X une surface de Riemann connexe différente de $\mathbb{P}^1$. L'espace $W^{1,2}(\mathbb{S}^1, X)$ est Hilbert-Hartogs.

Les résultats de cette thèse sont publiés dans l'article "On Hilbert-Hartogs manifolds" [A-Z] et dans l'article "Loop spaces as Hilbert-Hartogs" [A-I].

Cette thèse est organisée comme suit. Le premier chapitre est un bref rappel des résultats d'analyse complexe dans des espaces de Hilbert, des variétés de Hilbert et quelques notions de base sur les fibrés. En particulier, nous rappelons la construction d'une partition de l'unité et ses conséquences. Dans le second chapitre, nous prouvons l'existence d'un système fondamental de voisinages 1-complet. Dans le troisième chapitre, nous introduisons la notion de variété de Hartogs et en exploitant l'existence d'un système fondamental de voisinages 1-complet, nous prouvons plusieurs propriétés de ces variétés. Dans le dernier chapitre, nous rappelons la structure de variété de Hilbert complexe des espaces des lacets et nous expliquons comment l'espace des lacets d'une variété de Hartogs est Hilbert-Hatogs.

Chapter 1

Preliminaries

1.1 Complex analysis in Hilbert spaces

In this chapter we recall the notion of holomorphic maps in Hilbert spaces. We will state only results that will be used in this thesis. For more details we refer to [Mu]. To avoid any specific consideration we restrict ourself to the separable Hilbert space l^2.

The notation $\mathbb{N}$ corresponds to the positive integers $\{1, 2, ...\}$ and $\mathbb{N}_0$ to the nonnegative intergers $\{0, 1, 2, ...\}$. The notation l^2 stands for the Hilbert space of sequences of complex numbers $z = \{z_k\}_{k=1}^{\infty}$ such that $||z||_2^2 := \sum_k |z_k|^2 < \infty$ with the standard Hermitian scalar product $< z, w >= \sum_k z_k \bar{w}_k$ and standard basis $\{e_1, e_2, ...\}$. We take also a Hilbert space F with its norm $|| \cdot ||_F$. We write the norms $|| \cdot ||_2$ or $|| \cdot ||_F$ simply by $|| \cdot ||$ when it is clear from the context.

Definition 1.1. *Let U be an open subset of l^2. We say that a map $f : U \to F$ is $\mathbb{C}$-differentiable ($\mathbb{R}$-differentiable) at $z \in U$ if there exists a continuous $\mathbb{C}$-linear ($\mathbb{R}$-linear) map $T : l^2 \to F$ such that*

$$f(z + h) = f(z) + T(h) + o(||h||)$$

where $\frac{o(||h||)}{||h||} \to 0$ as $h \to 0$. We write $T = df_z$ and call df_z the differential of f at z.

Definition 1.2. *We say that a map $f : U \to F$ is holomorphic on U if f is $\mathbb{C}$-differentiable at every point $z \in U$.*

Remark 1.1. We denote by $\mathcal{O}(U, F)$ the set of all holomorphic maps from U to F. In the case of functions we denote $\mathcal{O}(U, \mathbb{C})$ simply by $\mathcal{O}(U)$.

Analogously to the finite dimensional case we can separate the differential in a $\mathbb{C}$-linear part ∂f_z and $\mathbb{C}$-antilinear part $\bar{\partial} f_z$. For $z \in l^2$ we can write $z = x + iy$ with x, y real sequences in l^2 and the differential of f at the point z can be written for $h = u + iv \in l^2$ with $u = \mathrm{Re}(h)$ and $v = \mathrm{Im}(h)$ as:

$$df_z(h) = \sum_{k=1}^{\infty} \frac{\partial f}{\partial x_k}(z) u_k + \sum_{k=1}^{\infty} \frac{\partial f}{\partial y_k}(z) v_k.$$

Taking the complex coordianate i.e in terms of h and $\bar{h}$, we can write it as

$$df_z(h) = \sum_{k=1}^{\infty} \frac{\partial f}{\partial z_k}(z) h_k + \sum_{k=1}^{\infty} \frac{\partial f}{\partial \bar{z}_k}(z) \bar{h}_k.$$

where $\frac{\partial f}{\partial z_k} = \frac{1}{2}\left(\frac{\partial f}{\partial x_k} - i \frac{\partial f}{\partial y_k}\right)$ and $\frac{\partial f}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial f}{\partial x_k} + i \frac{\partial f}{\partial y_k}\right)$. The first sum is denoted by $\partial f_z(h)$ and corresponds to the $\mathbb{C}$-linear part of the differential and the second one is denoted by $\partial f_z(h)$ and corresponds to the $\mathbb{C}$-antilinear part. From this last observation one can compute easily the equalities

$$\partial f_z(h) = \frac{1}{2}\left(df_z(h) - i\ df_z(ih)\right) \quad \text{and} \quad \bar{\partial} f_z(h) = \frac{1}{2}\left(df_z(h) + i\ df_z(ih)\right). \tag{1.1}$$

Those equalities are also convenient because one can express the linear part and the antilinear part without fixing coordinates on a basis.

Clearly we have $df_z(h) = \partial f_z(h) + \bar{\partial} f_z(h)$. We can characterize holomorphic maps as follows:

Theorem 1.1. *Let $f : U \to F$ be a $\mathbb{R}$-differentiable map from an open set $U \subset E$. Then f is holomorphic in U if and only if $\bar{\partial} f_z = 0$ for all $z \in U$.*

This theorem is an analogon of the Cauchy-Riemann conditions for holomorphic maps where the antilinear part vanishes.

Morever, in the finite dimensional case, we have that an analytic map is the same as a holomorphic map. We will see further that the equivalence is preserved in the infinite dimensional case. Recall the following

Definition 1.3. *A homogeneous continuous polynomial of degree m is a map $P_m : l^2 \to F$ such that there exists a continuous multilinear map $B \in \mathcal{L}_c(\underbrace{l^2 \times \ldots \times l^2}_{m}, F)$ satisfying for all $x \in E$ $P_m(x) = Bx^m$, here $Bx^m := B(\underbrace{x, \ldots, x}_{m})$.*

Definition 1.4. *Let U be an open subset of l^2. A mapping $f : U \to F$ is said to be analytic if for each $a \in U$ there exist a ball $B(a, r) \subset U$ and a sequence of continuous m-homogeneous polynomials P_m such that*

$$f(x) = \sum_{m=0}^{\infty} P_m(x - a) \tag{1.2}$$

uniformly for $x \in B(a, r)$.

Remark 1.2. The sequence P_m which appears above is uniquely determined by f and

a. That is why we shall write it as $P_m = P^m f(a)$ for every $m \in \mathbb{N}_0$. Then the series

$$\sum_{m=0}^{\infty} P^m f(a)(x-a)$$

is called the Taylor series of f at a.

As it was already mentioned holomorphy and analyticity are equivalent also in infinite dimension. The first implication "analyticity" $\implies$ "holomorphy" is almost clear since the homogeneous continuous polynomial $P^1 f(a)$ is $\mathbb{C}$-linear by definition.

Analogously to the notion of separate holomorphic maps in $\mathbb{C}^n$ one defines the notion of Gâteaux-holomorphy as follows.

Definition 1.5. *Let U be an open subset of l^2. A mapping $f : U \to F$ is said to be G-holomorphic (G for Gâteaux) if for all $a \in U$ and $b \in l^2$ the mapping $\lambda \mapsto f(a+\lambda b)$ is holomorphic on the open set $\{\lambda \in \mathbb{C} \mid a + \lambda b \in U\}$. We shall denote by $\mathcal{O}_G(U, F)$ the vector spaces of all G-holomorphic mappings from U into F.*

It is clear that a holomorphic map is continuous and is G-holomorphic. To obtain the equivalence between analyticity and holomorphy, we shall show that a continuous and G-holomorphic map is analytic. This result is deduced from the Cauchy integral formula.

Theorem 1.2. *Let U be an open subset of l^2 and let f be a G-holomorphic map in $\mathcal{O}_G(U, F)$. Let $a \in U$, $t \in l^2$ and $r > 0$ such that $a + \xi t \in U$ for all $\xi \in \mathbb{C}$ such that $|\xi| \leqslant r$. Then for each $\lambda \in \Delta(0, r)$ we have the Cauchy integral formula*

$$f(a+\lambda t) = \frac{1}{2\pi i}\int_{|\xi|=r} \frac{f(a+\xi t)}{\xi - \lambda} d\xi.$$

Proof. Take $\psi \in \mathcal{L}_c(F, \mathbb{C})$ a continuous linear form on F and consider the function $g(\xi) = \psi \circ f(a + \xi t)$. Then g is holomorphic on a neighborhood of the closed disc $\bar{\Delta}(0, r)$. From the standard Cauchy formula we obtain that

$$\psi \circ f(a+\lambda t) = g(\lambda) = \frac{1}{2\pi i}\int_{|\xi|=r} \frac{g(\xi)}{\xi-\lambda} = \frac{1}{2\pi i}\int_{|\xi|=r} \frac{\psi \circ f(a+t\xi)}{\xi-\lambda} d\xi$$

for each $\lambda \in \Delta(0, r)$. It gives the result since $\mathcal{L}_c(F, \mathbb{C})$ separates the points of F, i.e. if $x \neq y$ in F then there exists $\psi \in \mathcal{L}_c(F, \mathbb{C})$ such that $\psi(x) \neq \psi(y)$.

□

From theorem 1.2 we compute that for $\lambda \in \Delta(0, r)$ and $\forall \xi \in \partial\Delta(0, r)$

$$\frac{f(a+\xi t)}{\xi-\lambda} = \frac{f(a+\xi t)/\xi}{1-\lambda/\xi} = \sum_{m=0}^{\infty} \lambda^m \frac{f(a+\xi t)}{\xi^{m+1}}.$$

Then the integration over $|\xi| = r$ and the Cauchy formula give

$$f(a+\lambda t) = \sum_{m=0}^{\infty} \lambda^m \left(\frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(a+t\xi)}{\xi^{m+1}} d\xi \right).$$

By comparing with the Taylor expansion formula we deduce the identity

$$P^m f(a)(t) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(a+t\xi)}{\xi^{m+1}} d\xi$$

for $r > 0$ such that $a + \xi t \in U$ for all $\xi \in \bar{\Delta}(0, r)$.

One can remark that the coefficients of the series on the right hand side given by the integral is a homogeneous polynomial that is obviously continuous since f is continuous.

Proposition 1.1. *Let U be an open subset of l^2 and let $f : U \to F$ be G-holomorphic. For $a \in U$ and $m \in \mathbb{N}$, let $P_a^m : l^2 \to F$ be defined by*

$$P_a^m(t) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(a+\xi t)}{\xi^{m+1}} d\xi$$

where $r > 0$ is chosen so that $a + \xi t \in U$ for all $\xi \in \bar{\Delta}(0, r)$. Then:

i) $P_a^m(t)$ is independent from the choice of r,

ii) P_a^m is m-homogeneous, i.e

$$\forall t \in l^2, \forall \mu \in \mathbb{C}, \quad P_a^m(\mu t) = \mu^m P_a^m(t).$$

Proof.

i) Let $t \in l^2$ and $0 < s < r$ be such that $a + \xi t \in U$ for all $\xi \in \Delta(0, r)$. We have that

$$\sum_{m=0}^{\infty} \lambda^m \int_{|\xi|=r} \frac{f(a+\xi t)}{\xi^{m+1}} d\xi = 2\pi i f(a+\lambda t) = \sum_{m=0}^{\infty} \lambda^m \int_{|\xi|=s} \frac{f(a+\xi t)}{\xi^{m+1}} d\xi$$

for every $\lambda \in \Delta(0, s)$. By identification we conclude that

$$\int_{|\xi|=r} \frac{f(a+\xi t)}{\xi^{m+1}} d\xi = \int_{|\xi|=s} \frac{f(a+\xi t)}{\xi^{m+1}} d\xi$$

and then P_a^m is independent of r.

ii) For $t \in l^2$ and $\mu \in \mathbb{C}$ consider for r small enough the series expansion for every $\lambda \in \Delta(0, r)$

$$\sum_{m=0}^{\infty} \lambda^m \int_{|\xi|=s} \frac{f(a+\xi\mu t)}{\xi^{m+1}} d\xi = 2\pi i f(a+\lambda\mu t) = \sum_{m=0}^{\infty} \lambda^m \mu^m \int_{|\xi|=s} \frac{f(a+\xi t)}{\xi^{m+1}} d\xi.$$

Then by identification we see that P_a^m is m-homogeneous.

□

Theorem 1.3. *Let U be a open subset of l^2. Then for each mapping $f : U \to F$ the following conditions are equivalent :*

1) f is analytic,

2) f is holomorphic,

3) f is continuous and G-holomorphic.

Proof. 1) $\implies$ 2) is clear from the definition.
2) $\implies$ 3) is also clear.
3) $\implies$ 1): Let $f : U \to F$ be a G-holomorphic and continous map. Let $a \in U$ and $r > 0$ such that $B(a, r) \subset U$. Take $t \in E$ with $||t|| \leqslant 1$. From the G-holomorphy one can write the expansion

$$f(a + \lambda t) = \sum_{m=0}^{\infty} \lambda^m \left(\frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(a + \xi t)}{\xi^{m+1}} d\xi \right).$$

Since f is continuous there is a ball $\bar{B}(a, s) \subset B(a, r)$ and a constant $c > 0$ such that for every $x \in B(a, s)$ the map is bounded by c i.e. $||f(x)|| < c$. Then the homogeneous continuous polynomials are uniformly bounded

$$\left\| \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(a + \xi t)}{\xi^{m+1}} d\xi \right\| \leqslant \frac{1}{2\pi i} \int_{|\xi|=r} \frac{||f(a + \xi t)||}{r^{m+1}} d\xi \leqslant \frac{c}{r^{m+1}}.$$

This last means that for $x \in B(a, s)$ the map f is analytic.

□

Generally in this text we prove that a map is holomorphic by proving the continuity and the G-holomorphicity i.e. using 3) $\implies$ 2).

Remark 1.3. The continuity of a G-holomorphic map to be holomorphic is important. Consider the Hilbert space l^2 and the subspace $\mathbb{C}_0^{\mathbb{N}}$ of finite sequences, i.e $(a_k)_k$ such that $a_k = 0$ for $k \geqslant k_0$. Here k_0 depends on the sequence. Define the linear form $\phi : \mathbb{C}_0^{\mathbb{N}} \to \mathbb{C}$ with $\phi(a) = \sum_k k a_k$. We can extend it by zero on l^2, i.e. we take an algebraic complement M to $\mathbb{C}_0^{\mathbb{N}}$ and set $\phi|_M = 0$. Then the sequence $a^{(p)} \in (l^2)^{\mathbb{N}}$ defined by $a^{(p)} = \{1, \frac{1}{2}, ..., \frac{1}{p}, 0, 0, ...\}$ converges to the harmonic sequence $a = \{1, \frac{1}{2}, \frac{1}{3}, ...\} \in l^2$ but for all $p \in \mathbb{N}$

$$\phi(a^{(p)}) = p \underset{p \to +\infty}{\longrightarrow} +\infty.$$

Therefore ϕ is not bounded on l^2. But for any $a, b \in l^2$ and $\lambda \in \mathbb{C}$ such that $a + \lambda b \in U$ we have $\lambda \mapsto \phi(a + \lambda b) = \phi(a) + \lambda\phi(b)$ and so the map is G-holomorphic but not holomorphic.

Many properties of holomorphic mappings can be derived from the corresponding properties of holomorphic functions of finitely many complex variables. Let us give some of them. The following weak uniqueness theorem can be derived from the finite dimensional case.

Proposition 1.2. *Let U be a connected open subset of l^2 and let $f \in \mathcal{O}(U, F)$. If f is identically zero on a non-empty open set $V \subset U$ then f is identically zero on all of U.*

Proof. Let A the set of all points $a \in U$ such that f is identically zero on a neighborhood of a. Clearly A is nonempty because it contains V and A is obviously open. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in A wich converges to a point $b \in U$. Choose $r > 0$ such that $B(b, r) \subset U$ and choose $N \in \mathbb{N}$ such that $a_N \in B(b, r)$. Let denote $r_N < r$ the radius of the ball $B(a_N, r_N)$ where f is identically zero. Let $\psi \in \mathcal{L}_c(F, \mathbb{C})$ be a continuous linear form, $x \in B(b, r_N)$ and consider the function of one complex variable

$$g(\lambda) = \psi \circ f(x + \lambda(a_N - b)).$$

The function g is holomorphic and is identically zero on a neighborood of 1 due to the definition of a_N and $g(1) = \psi \circ f(a_N + x - b)$. By the identity principle of one complex variable the function g is identically zero. Since $\mathcal{L}_c(F, \mathbb{C})$ separates the points of F we conclude that $f(x) = 0$ for all $x \in B(b, r_N)$ and hence $b \in A$. A is open and closed in U connected then $A = U$.

□

Proposition 1.3. *Let U be a connected open subset of l^2 and let $f \in \mathcal{O}(U)$. If there exists $a \in U$ such that $|f(x)| \leqslant |f(a)|$ for every $x \in U$ then f is constant on U.*

Proof. One can find a proof in [Mu]. We will give the proof later because it can be deduced from the Maximum Principle for plurisubharmonic functions stated in the next chapter.

□

Lemma 1.1. *(Schwarz) Let $U = B(a, r) \subset l^2$ and let $f \in \mathcal{O}(U, F)$. Suppose that $||f(z)|| \leqslant c$ for every $x \in B(a, r)$ and suppose there exists $m \in \mathbb{N}$ such that $P^j f(a) = 0$ for every $j < m$. Then*

$$||f(x)|| \leqslant c \left(\frac{||x - a||}{r} \right)^m \text{ for every } x \in B(a, r).$$

Proof. Let $x \in B(a, r)$ and $\psi \in \mathcal{L}_c(F, \mathbb{C})$ be a continuous linear form such that $||\psi|| = 1$. Consider then the function g of one complex variable given by:

$$\text{for } 0 < |\lambda| < \frac{r}{||x - a||}, \qquad g(\lambda) = \frac{\psi \circ f(a + \lambda(x - a))}{\lambda^m}$$

and $g(0) = \psi \circ P^m f(a)(x-a)$. The Taylor series of f at a converges to f on the ball $B(a,r)$. So one can write g as

$$g(\lambda) = \sum_{j=m}^{\infty} \lambda^{j-m} \psi \circ P^j f(a)(x-a)$$

on the disc $\Delta\left(0, \frac{r}{||x-a||}\right)$. It means that g is holomorphic on that disc. Let s be a real number such that $||x-a|| < s < r$. Since $||f|| \leqslant c$ on $B(a,r)$ it follows that

$$\text{for } |\lambda| = \frac{s}{||x-a||}, \qquad |g(\lambda)| \leqslant c\left(\frac{||x-a||}{s}\right)^m.$$

By the maximum modulus principle this inequality is true also for $|\lambda| \leqslant s/||x-a||$. Then taking $\lambda = 1$ we get that

$$|\psi \circ f(x)| \leqslant c\left(\frac{||x-a||}{s}\right)^m.$$

To conclude we let $s \to r$ and we use a consequence of the Hahn-Banach Theorem:

if $x_0 \neq 0$ then it exists $\psi \in F'$ such that $||\psi|| = 1$ and $\psi(x_0) = ||x_0||$.

□

Proposition 1.4. *Let U be an open subset of l^2 and let $f \in \mathcal{O}_G(U,F)$. Then f is continuous if and only if f is locally bounded.*

Proof. Let $f : U \to F$ be Gâteaux-holomorphic and locally bounded. Let $a \in U$ choose $r > 0$ and $c > 0$. Such that $||f(x)|| \leqslant c$ for all $x \in B(a,r)$. By applying the Schwarz lemma to the mapping $f(x) - f(a)$ we obtain that

$$\forall x \in B(a,r),\ ||f(x) - f(a)|| \leqslant 2c\frac{||x-a||}{r}.$$

Then f is continuous on a.

□

1.2 Complex Hilbert manifolds

In this section we denote by $\mathcal{X}$ a Hausdorff topological space which is second countable. The latter means that the topology of $\mathcal{X}$ has a countable base.

Definition 1.6. *A Hausdorff topological space $\mathcal{X}$ is called a complex Hilbert manifold modeled over l^2 if it carries an atlas $\{(U_\alpha, \phi_\alpha, V_\alpha)\}_{\alpha \in A}$ that satisfies the following:*

- *$\{U_\alpha\}_{\alpha \in A}$ is an open covering of $\mathcal{X}$ and $\{V_\alpha\}$ are open subsets of l^2,*

- *For $\alpha \in A$, $\phi_\alpha : U_\alpha \to V_\alpha$ is a homeomorphism from U_α to V_α,*
- *For $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$ the transition maps $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ are holomorphic.*

U_α is called a coordinate chart and $\phi_\alpha(x)$ a local coordinate of x.

Since we suppose $\mathcal{X}$ second countable we can prove following [Mu] the existence of a partition of unity subordinated to an open covering of $\{\Omega_\alpha\}_\alpha$ of an open set $\Omega \subset \mathcal{X}$. The second countability allows us to suppose that A is countable. Consider the subspace $\mathbb{Q}_0[i]^{\mathbb{N}}$ of l^2 which consists of finite sequences of complex numbers with rational real and imaginary part

$$\mathbb{Q}_0[i]^{\mathbb{N}} := \{(z_n = x_n + iy_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid (x_n), (y_n) \in \mathbb{Q}^{\mathbb{N}} \;\&\; \exists n_0 \in \mathbb{N}, \forall n \geqslant n_0, x_n = y_n = 0\}.$$

For every $x \in \mathbb{Q}_0[i]^{\mathbb{N}} \cap \phi_\alpha(\Omega_\alpha)$ there exists ϵ_x such that $B(x, 2\epsilon_x) \subset \phi_\alpha(\Omega_\alpha)$. Then

$$\{\phi_\alpha^{-1}(B(x, \epsilon_x)) \mid x \in \mathbb{Q}_0[i]^{\mathbb{N}} \cap \phi_\alpha(\Omega_\alpha)\}$$

is a countable covering of Ω_α and since A is countable one obtains a countable covering $\{B_n\}_{n\in\mathbb{N}}$ of $\mathcal{X}$ subordinated to $\{\Omega_\alpha\}_{\alpha\in A}$. By the axiom of choice there is a function $\tau : \mathbb{N} \to A$ such that $B_n = \phi_{\tau(n)}^{-1}(B(x_n, \epsilon_n)) \subset \Omega_{\tau(n)}$. Construct a sequence of smooth functions $0 \leqslant f_n \leqslant 1$ defined on $\Omega_{\tau(n)}$ such that

$$\text{for } x \in \mathcal{X}, \quad f_n(x) = \begin{cases} 1 & \text{if } ||\phi_{\tau(n)}(x) - x_n|| < \epsilon_{x_n} \\ 0 & \text{if } ||\phi_{\tau(n)}(x) - x_n|| > 2\epsilon_{x_n}. \end{cases} \tag{1.3}$$

We can extend the functions f_n by zero to $\mathcal{X}$. Then define another sequence $(\psi_n)_n$ by

$$\begin{cases} \psi_1 = & f_1, \\ \psi_n = & f_n \; \prod_{j=1}^{n-1}(1 - f_j) \quad \text{if } n \geqslant 2. \end{cases} \tag{1.4}$$

Clearly $0 \leqslant \psi_n \leqslant 1$ on $\mathcal{X}$ and $\mathrm{supp}(\psi_n) \subset \Omega_{\tau(n)}$ for every n. Furthermore one can prove by induction that

$$\psi_1 + ... + \psi_n = 1 - \prod_{j=1}^{n}(1 - f_j). \tag{1.5}$$

Notice that

- From (1.5) it follows that $\psi_1 + ... + \psi_n = 1$ on B_n because $f_n = 1$ on B_n.
- From the definition of (ψ_k) we see that $\psi_k = 0$ on B_n for every $k > n$.

The first item guarantees that $\sum_{n\in\mathbb{N}} \psi_n(x) = 1$ for every $x \in \mathcal{X}$ and the second one gives that (ψ_n) is locally finite on $\mathcal{X}$. Construct then the partition of unity $\{\eta_\alpha\}_\alpha$ subordinated to $\{\Omega_\alpha\}_{\alpha\in A}$ by:

$$\eta_\alpha = \begin{cases} \sum_{\tau(n)=\alpha} \psi_n & \text{if } \alpha \in \tau(\mathbb{N}), \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

Remark 1.4. From the partition of unity subordinated to an atlas of a complex Hilbert manifold $\mathcal{X}$ one can construct an inner product $g_x(\cdot,\cdot)$ on the tangent spaces $T_x\mathcal{X}$ smoothly depending on x as follows

$$\forall x \in \mathcal{X} \ \forall v, w \in T_x\mathcal{X} \ g_x(v,w) = \sum_{\alpha\in A} \eta_\alpha(x) < (d\phi_\alpha)_x(u), (d\phi_\alpha)_x(v) >.$$

This inner product defines on every $T_x\mathcal{X} \cong l^2$ a norm equivalent to the standard one on l^2, providing thus $\mathcal{X}$ with the structure of a Riemann Hilbert manifold. For a peicewise smooth path $\gamma : [a,b] \to \mathcal{X}$ its length is defined as

$$L(\gamma) = \int_a^b \|\gamma'(t)\| \, dt.$$

Then, supposing that $\mathcal{X}$ is connected, one defines the distance function on $\mathcal{X}$:

$$d(x,y) := \inf\{L(\gamma) : \gamma \text{ is a peicewise smooth path joining } p \text{ with } q\}.$$

We refer to [Lg], Proposition 6.1, for the proof that d is in fact a distance which induces the original topology of $\mathcal{X}$.

Example 1.1. An example of a complex Hilbert manifold is the projective space $\mathbb{P}(l^2)$. Define the equivalence relation $\mathcal{R}$ on $l^2\backslash\{0\}$ by

$$\forall x, y \in l^2\backslash\{0\}, \ x\mathcal{R}y \iff \exists\lambda \in \mathbb{C}, \ y = \lambda x.$$

Then we define the projective space by $\mathbb{P}(l^2) := (l^2\backslash\{0\})/\mathcal{R}$. For $z \in l^2\backslash\{0\}$ we denote by $[z]$ the corresponding point in $\mathbb{P}(l^2)$. To show that this space is a complex Hilbert manifold we need to construct an atlas.

Consider $\Omega_j := \{[z] \in \mathbb{P}(l^2) \mid z_j \neq 0\}$ and let $\phi_j : \Omega_j \to l^2$ be defined by $\phi_j([z]) = \{\frac{z_k}{z_j}\}_{k\in\mathbb{N}\backslash\{j\}}$. Then $\{\Omega_j\}_{j\in\mathbb{N}}$ is an open covering of $\mathbb{P}(l^2)$, $\{\phi_j\}_{j\in\mathbb{N}}$ are homeomorphisms, and the transition maps $\phi_l \circ \phi_j^{-1} : \phi_j(\Omega_j) \to \phi_l(\Omega_l)$ given by

$$\phi_l \circ \phi_j^{-1}(z) = \left(\frac{z_0}{z_l}, \frac{z_1}{z_l}, ..., \frac{z_{j-1}}{z_l}, \frac{1}{z_l}, \frac{z_j}{z_l}, ..., \frac{z_{l-1}}{z_l}, \frac{z_{l+1}}{z_l}, ...\right)$$

are holomorphic on $\phi_j(\Omega_j)$.

In order to describe all the tools used in this thesis we give the definition of Hilbert bundles.

Definition 1.7. *A fiber bundle $(\mathcal{E}, \mathcal{B}, \pi, \mathcal{F})$ consist of*

- *a topological space $\mathcal{E}$ called the total space,*
- *a topological space $\mathcal{B}$ called the base space,*
- *a continuous surjective map $\pi : \mathcal{E} \to \mathcal{B}$ called the projection map,*
- *a topological space $\mathcal{F}$ called the fiber.*

The map π is asked to satisfy the following constraint:

there exists a collection $\{(U_\alpha, h_\alpha)\}_{\alpha \in A}$ called a trivialisation cover, where the collection $\{U_\alpha\}_{\alpha \in A}$ is an open cover of $\mathcal{B}$ and for each $\alpha \in A$ a homeomorphism

$$h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathcal{F}$$

such that the following diagram is commutative:

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[r, "h_\alpha"] \arrow[dr, "\pi"'] & U_\alpha \times \mathcal{F} \arrow[d, "pr_1"] \\
& U_\alpha
\end{tikzcd}$$

$$\forall (b, v) \in U_\alpha \times \mathcal{F}, \;\; \pi \circ h_\alpha^{-1}(b, v) = b.$$

In the diagram pr_1 denotes the projection on the first factor. We call the fiber over $b \in \mathcal{B}$ the set $\mathcal{F}_b := \pi^{-1}(\{b\}) \approx \mathcal{F}$. When $\mathcal{E}, \mathcal{B}$ and $\mathcal{F}$ are Hilbert manifold and π is a smooth surjective submersion we call $(\mathcal{E}, \mathcal{B}, \pi, \mathcal{F})$ a Hilbert fiber bundle. When $\mathcal{F}$ is discrete we say that $\mathcal{E}$ is an unramified cover of $\mathcal{B}$.

One can add a special structure of the bundle that we call $\mathcal{S}$-bundle for $\mathcal{S}$ a structure like smooth, holomorphic or Sobolev structures. As in this thesis we work with complex manifolds and holomorphic maps we will be more specific by defining an holomorphic Hilbert bundle.

Definition 1.8. *$(\mathcal{E}, \mathcal{B}, \pi, \mathcal{F})$ is called a holomorphic Hilbert fiber bundle if $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{B}$ are complex Hilbert manifolds and $\pi : \mathcal{E} \to \mathcal{B}$ is a holomorphic surjective submersion map satisfying a local trivialisation condition i.e.*

there exists a collection $\{(U_\alpha, h_\alpha)\}_{\alpha\in A}$ called a trivialisation cover where the collection $\{U_\alpha\}_{\alpha\in A}$ is an open cover of $\mathcal{B}$ and for each $\alpha \in A$, a biholomorphism

$$h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathcal{F}$$

such that the following diagram is commutative:

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "h_\alpha"] \arrow[dr, "\pi"'] & U_\alpha \times \mathcal{F} \arrow[d, "pr_1"] \\ & U_\alpha \end{tikzcd}$$

$$\forall (b, v) \in U_\alpha \times \mathcal{F},\ \pi \circ h_\alpha^{-1}(b, v) = b.$$

Example 1.2. The first example one can give of a Hilbert bundle is the trivial bundle. Let $\mathcal{B}$ and $\mathcal{F}$ be Hilbert manifolds and consider the total space $\mathcal{E} := \mathcal{B} \times \mathcal{F}$ and the projection map given by the natural projection on the first factor.

When we take two local trivialisations (U_α, h_α) and (U_β, h_β) we have that

$$h_{\alpha\beta} := h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathcal{F} \to (U_\alpha \cap U_\beta) \times \mathcal{F}$$

is a biholomorphism that acts on the fiber $\mathcal{F}$. Then the collection of $\{h_{\alpha\beta}\}$ satisfies a cocyle condition:

- for any $\alpha \in A$, $h_{\alpha\alpha} = \mathrm{Id}_{U_\alpha\times\mathcal{F}}$ on $U_\alpha \times \mathcal{F}$,
- for any $\alpha, \beta \in A$, $h_{\alpha\beta}^{-1} = h_{\beta\alpha}$ on $(U_\alpha \cap U_\beta) \times \mathcal{F}$,
- for any $\alpha, \beta, \gamma \in A$, $h_{\alpha\beta} \circ h_{\beta\gamma} \circ h_{\gamma\alpha} = \mathrm{Id}_{U_{\alpha\beta\gamma}\times\mathcal{F}}$ on $U_{\alpha\beta\gamma} \times \mathcal{F}$ where $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\beta$.

All these biholomorphisms are called transition maps and form the structure group of the fiber $\mathcal{F}$. Given a collection of transition maps $\{h_{\alpha\beta}\}$ one can retrieve a bundle with these transition maps. An outline of this construction is as follows: Let $\mathcal{E}$ be defined by

$$\mathcal{E} := \bigcup_{\alpha\in A} U_\alpha \times \mathcal{F}$$

equipped with the natural product topology. Define an equivalence relation in $\mathcal{E}$ by setting

$$(b, v) \sim (c, w) \text{ for } (b, v) \in U_\beta \times \mathcal{F} \text{ and } (c, w) \in U_\alpha \times \mathcal{F}$$

if and only if

$$c = b \qquad \text{and} \qquad w = h_{\alpha\beta}(b, v).$$

The fact that this is a well-defined equivalence relation is a consequence of the cocyle condition given previously. Then consider $\tilde{\mathcal{E}} := \mathcal{E}/\sim$ equipped with the quotient topology. Let $\pi : \tilde{\mathcal{E}} \to \mathcal{B}$ be the mapping which sends a representative (b, v) of a point of $\tilde{\mathcal{E}}$ into the first coordinate. Then $\tilde{\mathcal{E}}$ is a fiber bundle.

From the definition of a fiber bundle one can see that a group appears describing the matching conditions between overlapping local trivialization charts. In fact the mapping $h_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \mathcal{F} \to (U_\alpha \cap U_\beta) \times \mathcal{F}$ can be written by

$$h_{\alpha\beta}(x, v) = (x, t_{\alpha\beta}(x)v)$$

where $x \mapsto t_{\alpha\beta}(x)$ is a continuous map. For any $x \in X$, the set $\{t_{\alpha\beta}(x)\}$ forms a group G that acts continuously on the fiber $\pi^{-1}(\{x\}) \approx \mathcal{F}$ and is called the structure group. When the structure group have specific properties, one can define the following.

Definition 1.9. *Let G be a topological group. A principal G-bundle is a fiber bundle $(\mathcal{E}, \mathcal{B}, \pi, \mathcal{F})$ together with a continuous action $G \times \mathcal{E} \to \mathcal{E}$ that preserves the fibers and such that is free and transitive on the fibers $\mathcal{F}_b$, i.e every $b \in \mathcal{B}$ has a neighborhood $U \subset \mathcal{B}$ and there is a trivialization $U \times G \to \mathcal{E}|U$ under wich the action of $g \in G$ on $U \times G$,*

$$U \times G \ni (\beta, h) \mapsto (\beta, gh) \in U \times G$$

and on $\mathcal{E}|U$ correspond.

Since the group action is free and transitive we can identify the fiber $\mathcal{F}$ with the group G. From the definition, the elements $\{t_{\alpha\beta}\}$ satisfy the same cocycle conditions as $\{h_{\alpha\beta}\}$.

Definition 1.10. *Let $(\mathcal{E}, \mathcal{B}, \pi, \mathcal{F})$ and $(\mathcal{E}', \mathcal{B}, \pi', \mathcal{F}')$ be two Hilbert bundles. A bundle homomorphism is a morphism $\Phi : \mathcal{E} \to \mathcal{E}'$ such that it preserves the fibers, i.e. for any $b \in \mathcal{B}$, Φ maps the fiber $\mathcal{F}_b$ to $\mathcal{F}'_b$.*

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "\Phi"] \arrow[dr, "\pi"] & & \mathcal{E}' \arrow[dl, "\pi'"] \\
& \mathcal{B} &
\end{tikzcd}$$

Then a bundle isomorphism is a bundle homomorphism which is an isomorphism between the total spaces. By abuse of language a fiber bundle is said to be trivial if there is a bundle isomomorphism between the fiber bundle and the trivial bundle. In the context of holomorphic Hilbert bundles we require that every isomorphism is a biholomophism to say that a bundle is holomorphically trivial.

Definition 1.11. *Let (E, F, π, B') be a fiber bundle and $f : B \to B'$ a map. The "pullback" bundle of E is defined by the set*

$$f^*E := \{(b, e) \in B' \times E \mid f(b) = \pi(e)\}$$

equipped with the projection $pr : f^*E \to B$ given by $pr(b, e) = b$. One obtains that (f^*E, F, pr, B) is a fiber bundle called the pullback bundle.

$$\begin{tikzcd} f^*E \arrow[d, "\mathrm{pr}"] & E \arrow[d, "\pi"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

When the fiber $\mathcal{F}$ is a vector space, we call E a vector bundle.

A section of a fiber bundle $(\mathcal{E}, \mathcal{B}, \pi, \mathcal{F})$ is a continuous map $\sigma : \mathcal{B} \to \mathcal{E}$ that satisfies the condition

$$\pi \circ s = \mathrm{Id}_{\mathcal{B}}.$$

It means that σ sends a point b in the base space into the fiber over that point, i.e $\pi^{-1}(\{b\})$. In the literature the space of sections is denoted oftenly by $\Gamma(\mathcal{B}, \mathcal{E})$ and when the sections are holomorphic we denote it by $\mathcal{O}(\mathcal{B}, \mathcal{E})$. Since fiber bundles do not have in general globally defined sections we deal with local sections over an open set $U \subset \mathcal{B}$.

Definition 1.12. *A local section of a fiber bundle $(\mathcal{E}, \mathcal{B}, \pi, \mathcal{F})$ over an open set $U \subset \mathcal{B}$ is a continuous map $\sigma : U \to \mathcal{E}$ satisfying the condition*

$$\pi(\sigma(x))) = x \qquad \text{for all } x \in U.$$

Definition 1.13. *Let X and Y be two topological spaces. Let $f : X \to Y$ and $g : X \to Y$ be two continuous functions. We say that f and g are homotopic if there exists a continuous function $H : X \times [0, 1] \to Y$ such that $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$.*

We say that a topological space X is contractible if the identity map $Id : X \to X$ is homotopic to some constant map.

Theorem 1.4. *(Kuiper cf [Ku]) For any infinite dimensional separable Hilbert space H the group $GL(H)$ of invertible linear operators on H is contractible.*

The idea of the proof is to squeeze the space to a finite dimensional simplex and to glue carefully using a partition of unity, see [Ku].

This theorem implies that any vector bundle over a compact space X with $GL(H)$ as structure group is topologically trivial. The idea is to use the following lemma.

Lemma 1.2. *A G-principal bundle (E, π, F, B) is trivial if and only if it admits a global section.*

Proof. If $E \cong B \times F$ then any constant section $s : B \to B \times F$ can do it. Reciprocally, let $s : B \to E$ be a global section. Fix some $y \in F$ and define $\phi : E \to B \times F$ by $\phi(e) = (\pi(e), g_e(y))$ where g_e is the element of the structure group G that acts on F such that $g_e(s \circ \pi(e)) = e$. This g_e continuously depend on e since in local trivialization equation $g_e(s \circ \pi(e)) = e$ becomes $g_e \cdot (s \circ \pi(e)) = e$ and can be solved as $g_e = (s \circ \pi(e))^{-1}e$. Since G acts continuously and freely ϕ is an isomorphism.

□

Let (E, π, F, B) be a fiber bundle with contractible fiber F. We have a homotopy $H : F \times [0,1] \to F$ such that $H(\cdot, 0) = \mathrm{Id}_F$ and $H(\cdot, 1) = f$ an element of F. Let us consider the fiber bundle $E \times [0,1]$. Then we can define a section s_α on the chart $U_\alpha \times [0,1]$ by

$$s_\alpha(b,t) = h_\alpha^{-1}\Big(b, H\Big(h_\alpha \circ s(b), t\Big)\Big).$$

Since on $U_\alpha \cap U_\beta$ the sections $s_\alpha(\cdot, 1)$ coincide with $s_\beta(\cdot, 1)$ one can define a global section on B. Then $E \times \{1\}$ is trivial. Considering now the map $p_1 : B \times [0,1] \to B \times \{1\}$ and $p_0 : B \times [0,1] \to B \times \{0\}$ we have that

$$E \times \{0\} \approx p_0^*(E) \approx p_1^*(E) \approx B \times F \times \{1\}.$$

Then the fiber bundle is trivial.

□

1.3 Trivialization of a 1-cochain

To finish this chapter the last tool we need is the notion of a cochain. Let D be a complex manifold and $\mathcal{U} := \{U_j\}_{j \in J}$ be a covering of D by open subsets. There is a notion of p-cochain for any integer p but as in the latter we just need the case $p = 0$ and 1 we just give it for those special cases.

Remark 1.5. In this thesis we will essentially work with holomorphic G-valued maps with $G = (\mathcal{L}(l^2), +)$ the group of operators in l^2 with the addition operations or with $G = (GL(l^2), \circ)$ the group of invertible operators of l^2 with the composition.

Definition 1.14. *Let $\mathcal{C}^0(\mathcal{U}, \mathcal{G})$ be the multiplicative group of all collections $\{f_j\}_{j \in J}$ of holomorphic functions $f_j \in \mathcal{G}(U_j)$. The collection $\{f_j\}_{j \in J}$ is called* 0*-cochain on* $\mathcal{U}$ *with coefficent in the sheaf $\mathcal{G}$ of $GL(l^2)$-valued holomorphic maps.*

Definition 1.15. *We define a* 1*-cochain* $\mathcal{C}^1(\mathcal{U},\mathcal{G})$ *as a collection* $\{f_{ij}\}$ *where* $f_{ij} \in \mathcal{G}(U_i \cap U_j)$. *We denote by* $\mathcal{Z}^1(\mathcal{U},\mathcal{G})$ *the group of* 1*-cochains satisfying the cocycle condition:*

- *for any* $i,j \in J$ *such that* $U_i \cap U_j \neq \emptyset$ *we have* $f_{ji} = f_{ij}^{-1}$ *on* $U_i \cap U_j$,
- *for any* $i,j,k \in J$ *such that* $U_i \cap U_j \cap U_k \neq \emptyset$ *we have* $f_{ij} \circ f_{jk} = f_{ik}$.

where f_{ij}^{-1} *corresponds to the inverse of* f_{ij} *in the group* $(GL(l^2),\circ)$, *i.e.* $f_{ij} \circ f_{ij}^{-1} = Id$ *with* $Id : U_i \cap U_j \to GL(l^2)$ *the constant map equal to* $Id_{GL(l^2)}$. *We call an element of* $\mathcal{Z}^1(\mathcal{U},\mathcal{G})$ *a* 1*-cocycle or simply a cocycle.*

Definition 1.16. *The coboundary homomorphism* $\delta : \mathcal{C}^0(\mathcal{U},\mathcal{G}) \to \mathcal{C}^1(\mathcal{U},\mathcal{G})$ *is defined by*

$$\delta(\{f_j\}_{j\in J}) := \{f_i \circ f_j^{-1}\}.$$

Remark 1.6. For this last case we can give the same previous definitions by replacing the group $(GL(l^2),\circ)$ by the group $(\mathcal{L}(l^2),+)$.

One can remark that $\delta(\mathcal{C}^0(\mathcal{U},\mathcal{G})) \subset \mathcal{Z}^1(\mathcal{U},\mathcal{G})$. Then we can define the first Cech cohomology group $\mathcal{H}^1(\mathcal{U},\mathcal{G})$ of the covering $\mathcal{U}$ with coefficients in the sheaf $\mathcal{G}$ by the quotient

$$\mathcal{H}^1(\mathcal{U},\mathcal{G}) := \mathcal{Z}^1(\mathcal{U},\mathcal{G})/\delta(\mathcal{C}^0(\mathcal{U},\mathcal{G})).$$

Definition 1.17. *A finite dimensional complex manifold* X *is called a Stein manifold if it satisfies the following conditions:*

i) X *is holomorphically convex i.e for every compact subset* $K \subset X$, $\hat{K}$ *is also compact where* $\hat{K} := \{x \in K \mid \forall f \in \mathcal{O}(X), |f(x)| \leqslant \sup_K |f|\}$,

ii) X *is holomorphically separable, i.e for* $x, y \in X$ *if* $x \neq y$ *then there exists* $f \in \mathcal{O}(X)$ *such that* $f(x) \neq f(y)$,

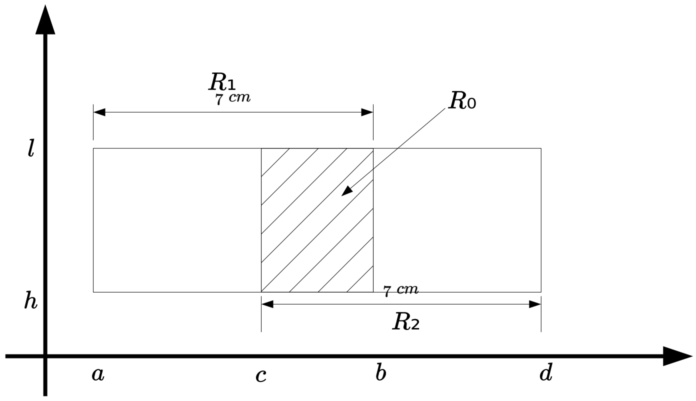
iii) For any $x \in X$ *there exist function* $f_1, ..., f_n \in \mathcal{O}(X)$, $n = dim(X)$, *whose differentials* df_j *are* $\mathbb{C}$*-linearly independant at* p.

Roughly speaking Stein manifolds generalize the notion of domain of holomorphy that corresponds to the biggest domain where one can extend a holomorphic map. Stein manifolds have some good properties and generalise the euclidian case $\mathbb{C}^n$.

In the following chapters, we will need at some point to trivialize cocycles. This is a result due to Grauert [G]. The first stage is the one dimensional case known as Cartan's lemma.

Theorem 1.5. *(Cartan's lemma [Bu]) Let $a < c < b < d$ and $h < l$ be real numbers. Consider the closed rectangles $R_1 = [a, b] \times [h, l]$ and $R_2 = [c, d] \times [h, l]$ and let $R_0 := R_1 \cap R_2 = [c, b] \times [h, l]$. Let f be a $GL(l^2)$-valued holomorphic map on a neighborhood of R_0. Then there exist f_1 and f_2 $GL(l^2)$-valued holomorphic maps defined respectively on R_1 and R_2 such that*

$$f(z) = f_1(z) \circ f_2(z)^{-1} \text{ for } z \in R_0.$$



One can extend this result to cubes in $\mathbb{C}^n$ and now from the contractibility of the structure group $GL(l^2)$, see theorem 1.4, and the result of Grauert [G] stating the correspondance between holomorphic and topological vector bundles we can state the following theorem that reflects a specifically infinite dimensional feature of Hilbert bundles, see [Bu].

Theorem 1.6. *Let D be a Stein manifold and $\mathcal{E}$ a holomorphic $GL(l^2)$-principal bundle over D. Then $\mathcal{E}$ is holomorphically trivial and moreover if $\mathcal{U} = \{U_\alpha\}$ is a locally finite Stein covering of D then for every cocyle $f \in Z^1(\mathcal{U}, \mathcal{G})$ there exists a cochain $c \in C^0(\mathcal{U}, \mathcal{G})$ such that $\delta(c) = f$.*

Here $\mathcal{G}$ is the sheaf of holomorphic mappings with values in the group of invertible operators on l^2. For the proof we refer to [Bu].

Chapter 2

Existence of 1-complete neighborhoods

2.1 Plurisubharmonic functions

Definition 2.1. *Let $\mathcal{D}$ be a topological space. A function $f : \mathcal{D} \to [-\infty, \infty[$ is said to be upper semicontinuous if the set $\{m \in \mathcal{D} \mid f(x) < c\}$ is open for each $c \in \mathbb{R}$.*

Definition 2.2. *Let U be an open subset of l^2. A function $f : U \to [-\infty, \infty[$ is said to be plurisubharmonic if f is upper semicontinuous and*

$$f(a) \leqslant \frac{1}{2\pi i} \int_0^{2\pi} f(a + e^{i\theta} b) d\theta$$

for each $a \in U$ and $b \in l^2$ such that $a + \bar{\Delta} b \subset U$.

We shall denote by $PSH(U)$ the set of plurisubhamonic functions and by $PSH_c(U)$ the set of plurisubharmonic continuous functions on U. We simply say subharmonic functions for the one dimensional case.

Example 2.1. For $f \in \mathcal{O}(U)$ we have that $\Re e(f), \Im m(f), |f|$ belong to $PSH(U)$.

Proposition 2.1. *Let U be an open subset of l^2.*

i) If $f, g \in PSH(U)$ and α, β are non-negative constants then $\alpha f + \beta g \in PSH(U)$.

ii) If $f_j \in PSH(U)$ for every $j \in J$ such that $\sup_{j \in J} f_j < \infty$ on U and is upper semicontinuous on U then $\sup_{j \in J} f_j \in PSH(U)$.

iii) Let $f : U \to [-\infty, \infty[$ be upper semicontinuous. Then f is plurisubharmonic if and only if the restriction of f to $U \cap H$ is plurisubharmonic for each finite dimensional subspace H of l^2.

Proof. see [Mu]

□

Theorem 2.1. *(Maximum principle for plurisubharmonic function)*
Let U be a connected open set in l^2 and let $f \in PSH(U)$.

i) If there exists $a \in U$ such that $f(x) \leqslant f(a)$ for every $x \in U$ then $f(x) = f(a)$ for every $x \in U$.

ii) If $f \equiv -\infty$ on a non empty open susbet of U then $f \equiv -\infty$ on U.

Proof.

i) Let us consider the set $A := \{x \in U \mid f(x) = f(a)\}$. The set A coincide $U \backslash \{x \in U \mid f(x) < f(a)\}$. Since f is upper semicontinuous it is closed.

Let $b \in A$ and let $r > 0$ such that $B(b,r) \subset U$. Take x in $B(b,r)$ such that $f(x) < f(a)$. The upper semicontinuity of f allows to find $\epsilon > 0$ such that $B(x,\epsilon) \subset B(a,r)$ and $f(y) < f(a)$ for every $y \in B(x,\epsilon)$. Then

$$\frac{1}{2\pi i}\int_0^{2\pi} f\Big(b + e^{i\theta}(x-b)\Big)d\theta < \frac{1}{2\pi}\int_0^{2\pi} f(a)d\theta = f(a) = f(b)$$

This is a contradiction. Then $B(b,r) \subset A$ and A is open, hence $A = U$.

ii) Let A be the set of all $a \in U$ such that $f \equiv -\infty$ on a neighborhood of a. By hypothesis A is a non empty open set. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in A which converges to a point $a \in U$. Take $r > 0$ such that $B(a,r) \subset U$ and then take $N \in \mathbb{N}$ such that $a_N \in B(a,r)$ and choose $\epsilon > 0$ such that $B(a_N,\epsilon) \subset B(a,r)$ and $f \equiv -\infty$ on $B(a_N,\epsilon)$. Since $x + (a_N - a) \in B(a_N,\epsilon)$ for $x \in B(a,\epsilon)$, it follows that

$$f(x) \leqslant \frac{1}{2\pi}\int_0^{2\pi} f\Big(x + e^{i\theta}(a_N - a)\Big)d\theta = -\infty.$$

Therefore $f(x) = -\infty$ for $x \in B(a,\epsilon)$. So $a \in A$ and A is closed. Therefore $A = U$.

□

Let $f : U \to F$ be a holomorphic map from an open connected subset U of l^2 to a complex Hilbert space F. By corollary 31.6 in [Mu], $||f||$ is PSH on U. This implies the maximum principle for Hilbert valued holomorphic maps. Let V be an open bounded subset of l^2 and f be an F-valued holomorphic map defined in a neighborhood of $\bar{V}$, then

$$\sup_{z\in\overline{V}} ||f(z)||_F = \sup_{z\in\partial V} ||f(z)||_F. \tag{2.1}$$

When the plurisubharmonic functions are of class $\mathcal{C}^2$, one can characterize them with the second differential. Let recall first what is a function of class $\mathcal{C}^2$.

Definition 2.3. *Let U be an open set of l^2. A function $f : U \to \mathbb{R}$ is said to be twice $\mathbb{R}$-differentiable if f is $\mathbb{R}$-differentiable and the differential $df : U \to \mathcal{L}_c(l^2,\mathbb{R})$ is differentiable as well. Here $\mathcal{L}_c(l^2,\mathbb{R})$ denotes the space of continuous $\mathbb{R}$-linear forms on l^2.*

Then the differential of the mapping df at point $a \in U$ is called the second differential of f at a and will be denoted by $d^2 f_a$. A function f is said to be in $\mathcal{C}^2(U, \mathbb{R})$ if the map $a \mapsto d^2 f_a$ is continuous. Thus $d^2 f_a$ can be regarded as an element of $\mathcal{L}_c(l^2, \mathcal{L}_c(l^2, \mathbb{R}))$ which is isometric to $\mathcal{L}_c(l^2 \times l^2, \mathbb{R})$ provided the spaces in question equipped with the natural norms. More precisely, for $A \in \mathcal{L}_c(l^2, \mathcal{L}_c(l^2, \mathbb{R}))$ set

$$||A||_{op} = \sup_{\|x\|_{l^2}=1} \|Ax\|_{op} = \sup_{\|x\|_{l^2}=1} \sup_{\|y\|_{l^2}=1} | < y, Ax > | = \sup_{\|x\|, \|y\|_{l^2}=1} | < y, Ax > |, \quad (2.2)$$

and for $A \in \mathcal{L}_c(l^2 \times l^2, \mathbb{R})$ the standard norm is, see [Mu]

$$\|A\| = \sup_{\|x\|_{l^2}, \|y\|_{l^2}=1} |Axy|. \quad (2.3)$$

We see that (2.2) and (2.3) are equal because we view an operator A as a bilinear form $Axy =< y, Ax >$. Moreover an analog of the Schwarz theorem gives that $d^2 f_a$ is symmetric.

Theorem 2.2. *Let U be an open set of l^2 and $f : U \to \mathbb{R}$ be a twice differentiable function. Then the bilinear function $d^2 f_a$ is symmetric for each $a \in U$, i.e*

$$\forall v, w \in l^2, \ d^2 f_a(v, w) = d^2 f_a(w, v).$$

Proof. see [Mu].

□

Then one can write the expansion of order two for a function $f \in \mathcal{C}^2(U, \mathbb{R})$

$$\forall a \in U, \forall v \in l^2, \ f(a + v) = f(a) + df_a(v) + \frac{1}{2} \, d^2 f_a(v, v) + o(\|v\|^2).$$

The characterization of subharmonicity in the one dimensional case is given by the following.

Proposition 2.2. *Let U be an open subset of $\mathbb{C}$. Then for each function $f \in \mathcal{C}^2(U, \mathbb{R})$ the following condition are equivalent:*

i) *f is subharmonic on U,*

ii) $4\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \geqslant 0$ *on U,*

iii) *For each $a \in U$ the integral*

$$M(r) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta \quad (r \geqslant 0)$$

is an increasing function of r.

Proof.

i) $\Longrightarrow$ *ii)*: Let $\bar{\Delta}(a,r) \subset U$ be the complex disk centered in a with radius r. We define a function $u : [0,r] \to \mathbb{R}$ by

$$u(\epsilon) = \int_0^{2\pi} \Big(f(a+\epsilon e^{i\theta}) - f(a)\Big) d\theta.$$

Since f is subharmonic $u(\epsilon) \geqslant 0$ for every $\epsilon \in [0,r]$. We compute the expansion of u at 0 :

$$u(\epsilon) = \textstyle\int_0^{2\pi} 2\Re e\left(\frac{\partial f}{\partial z}(a)\epsilon e^{i\theta}\right) d\theta + \int_0^{2\pi} \Re e\left(\frac{\partial^2 f}{\partial z^2}(a)\epsilon^2 e^{i2\theta}\right) d\theta + \int_0^{2\pi} \frac{\partial^2 f}{\partial z \partial \bar{z}}(a)\epsilon^2 d\theta + o(\epsilon^2)$$

$= 2\pi \frac{\partial^2 f}{\partial z \partial \bar{z}}(a)\epsilon^2 + o(\epsilon^2)$. Then $\frac{\partial^2 f}{\partial z \partial \bar{z}}(a) \geqslant 0$ for every $a \in U$.

ii) $\Longrightarrow$ *iii)* : By deriving under the integral, one can compute M' and M'' and see that

$$M''(r) + \frac{1}{r}M'(r) = 4\int_0^{2\pi} \frac{\partial^2 f}{\partial z \partial \bar{z}}(a + re^{i\theta}) d\theta \geqslant 0.$$

Thus $(rM'(r))' \geqslant 0$ and $r \mapsto rM'(r)$ is an increasing function. Since $M'(0) \geqslant 0$ we can conclude that $M'(r) \geqslant 0$ for every $r \geqslant 0$.

iii) $\Longrightarrow$ *i)*: It is clear.

□

The item *ii)* points out the term $\frac{\partial^2 f}{\partial z \partial \bar{z}}$. In order to generalize this to several variables one can take the complex Hessian of f in $a \in U \subset l^2$ which is given in the standard basis of l^2 by the Hermitian matrix

$$\mathcal{H}_f(a) := \left(\frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}(a)\right)_{j,k\in\mathbb{N}}.$$

One can define a Hermitian quadratic form associated to this Hermitian matrix called the Levi form.

Definition 2.4. *Let U be an open set of l^2 and $f \in \mathcal{C}^2(U,\mathbb{C})$. The Levi form of f at point $a \in U$ is defined by:*

$$\forall v \in l^2,\ \mathcal{L}_{f,a}(v) = (\partial\bar{\partial} f)_a(v,v).$$

Moreover one can compute for $f \in \mathcal{C}^2(U,\mathbb{R})$ and $g \in \mathcal{O}(V,l^2)$ with $g(V) \subset U \subset l^2$ that

$$\mathcal{L}_{f\circ g,a}(v) = \partial\bar{\partial}(f\circ g)_a(v,v) = (\partial\bar{\partial} f)_{g(a)}\Big(\partial g_a v, \partial g_a v\Big) = \mathcal{L}_{f,g(a)}(\partial g_a v). \tag{2.4}$$

Let U be an open subset of a Hilbert manifold $\mathcal{X}$ and $(\Omega_\alpha, \phi_\alpha)$ be an atlas of $\mathcal{X}$. Let f be in $\mathcal{C}^2(U,\mathbb{R})$ and $(\Omega_\alpha, \phi_\alpha)$ be a chart containing a point a. Since $\phi_\beta \circ \phi_\alpha^{-1}$ is holomorphic for any chart Ω_β containing a, we can choose f, g, v and a replaced respectively by $f \circ \phi_\alpha^{-1}$, $\phi_\beta \circ \phi_\alpha^{-1}$,$(d\phi_\alpha)_a v$ and $\phi_\alpha(a)$ in equation 2.4 to obtain:

$$\begin{aligned}\mathcal{L}_{f\circ\phi_\alpha^{-1},\phi_\alpha(a)}\Big((d\phi_\alpha)_a v\Big) &= \mathcal{L}_{f\circ\phi_\beta^{-1}\circ\phi_\beta\circ\phi_\alpha^{-1},\phi_\alpha(a)}\Big((d\phi_\alpha)_a v\Big)\\ &= \mathcal{L}_{f\circ\phi_\beta^{-1},\phi_\beta(a)}\Big(d(\phi_\beta\circ\phi_\alpha^{-1})_{\phi_\alpha(a)}(d\phi_\alpha)_a v\Big)\\ &= \mathcal{L}_{f\circ\phi_\beta^{-1},\phi_\beta(a)}\Big((d\phi_\beta)_a v\Big).\end{aligned}$$

Remark 2.1. For U an open subset of a Hilbert manifold $\mathcal{X}$ and $f \in \mathcal{C}^2(U,\mathbb{R})$ the Levi form is given by

$$\forall a \in U\ \forall v \in T_a\mathcal{X}\ \mathcal{L}_{f,a}(v) = (\partial\bar\partial f)_a(v,v).$$

Proposition 2.3. *Let U be an open subset of l^2 and let $f \in \mathcal{C}^2(U,\mathbb{R})$. Then f is plurisubharmonic on U if and only if the Levi form $\mathcal{L}_{f,a}$ is positive for each $a \in U$, i.e*

$$\mathcal{L}_{f,a}(v) \geqslant 0 \quad \textit{for each} \quad a \in U \quad \textit{and} \quad v \in l^2.$$

Proof. Given $a \in U$ and $v \in l^2$ we define the function $u(\xi) = f(a+\xi v)$ for $\xi \in \Delta(0,r)$ a suitable disk in $\mathbb{C}$. Then

$$\frac{\partial^2 u}{\partial\xi\partial\bar\xi}(\xi) = (\partial\bar\partial f)_{a+\xi v}(v,v) = \mathcal{L}_{f,a+\xi v}(v).$$

We can conclude by theorem 2.2 that if $\mathcal{L}_{f,a+\xi v}$ is positive then f is subharmonic for any direction and the plurisubharmonic. Conversly the plurisubharmonicity of f implies that for all a and v the term $\frac{\partial^2 u}{\partial\xi\partial\bar\xi}$ is positive so the Levi form is positive.

□

Definition 2.5. *Let U be an open subset of l^2. A function $f \in \mathcal{C}^2(U,\mathbb{R})$ is said to be strictly plurisubharmonic on U if the Levi form $\mathcal{L}_{f,a}$ is positive definite for every $a \in U$, i.e*

$$\mathcal{L}_{f,a}(v) > 0 \quad \textit{for each} \quad a \in U \quad \textit{and} \quad v \in l^2\backslash\{0\}.$$

We have the following lemma in the finite dimensional case.

Lemma 2.1. *Let U be an open subset of $\mathbb{C}^n$, and let f be a strictly plurisubharmonic function on U of class $\mathcal{C}^2$. Then there exists a positive function $c \in \mathcal{C}^0(U,\mathbb{R})$ such that*

$$\mathcal{L}_{f,a}(v) \geqslant c(a)||v||^2 \quad \forall v \in l^2.$$

Proof. Let $(U_j)_{j\in\mathbb{N}}$ be an increasing sequence of relatively compact open sets whose union is U. Since f is strictly plurisubharmonic on each $U_{j+1} \supset \overline{U_j}$, it follows that for every $j \in \mathbb{N}$

$$c_j := \inf\{\mathcal{L}_{f,a}(v) \mid a \in \bar{U}_j,\ v \in l^2 \text{ with } \|v\| = 1\} > 0.$$

Now one can construct continuous functions $(\eta_j)_{j\in\mathbb{N}}$ such that

$$\forall j \in \mathbb{N},\ \eta_j \equiv 1 \text{ on } U_j \text{ and } \overline{supp(\eta_j)} \subset U_{j+1}.$$

Define functions $(\psi_j)_{j\in\mathbb{N}}$ by

$$\begin{cases} \psi_1 = & \eta_1, \\ \psi_n = & \eta_n \ \prod_{j=1}^{n-1}(1-\eta_j) \quad \text{ if } n \geqslant 2. \end{cases}$$

and consider the function

$$c := \frac{1}{\sum_{j\in\mathbb{N}} \frac{1}{c_j}\psi_j}.$$

Then the function c is continuous and

$$\forall a \in U_{j+1}\backslash U_j, \forall v \in l^2\backslash\{0\},\ \mathcal{L}_{f,a}\left(\frac{v}{\|v\|}\right) \geqslant c_j \geqslant c(a).$$

□

Unfortunately, this lemma is not true on l^2. Take for example the function f defined by $f(z) = \sum_{j=1}^{\infty} \frac{|z_j|^2}{j}$ for $z \in l^2$. The Levi form on a point $a \in l^2$ is

$$\forall v \in l^2, \quad \mathcal{L}_{f,a}(v) = \sum_{j=1}^{\infty} \frac{1}{j}|v_j|^2,$$

and this cannot be bound from below by $c||v||^2$.

To finish this section we recall that a function $u : U \to \mathbb{R}$ with $U \subset \mathcal{X}$ an open subset of a complex Hilbert manifold $\mathcal{X}$ is called strictly plurisubharmonic if for every coordinate chart $\phi_j : \Omega_j \to U_j \subset l^2$ the composition $u \circ \phi_j^{-1}$ is strictly plurisubharmonic in U_j.

2.2 Pseudoconvexity and pseudoconcavity

In the theory of functions of several complex variables pseudoconvex sets are important as they help to characterize domains of holomorphy.

Definition 2.6. *Let $\mathcal{D}$ be a domain with $\mathcal{C}^2$ boundary in a complex Hilbert manifold $\mathcal{X}$. Let $a \in \partial\mathcal{D}$ and $U \subset \mathcal{X}$ be a neighborhood of a. A function $f : U \to \mathbb{C}$ of class $\mathcal{C}^2$*

with $\nabla_a f \neq 0$ *is called a local defining function if the set* $\mathcal{D}$ *near* a *is given by:*

$$\mathcal{D} \cap U = \{z \in U \mid f(z) < 0\}.$$

Definition 2.7. *$\mathcal{D}$ is called pseudoconvex at a boundary point a if there exists a local defining function near a such that the Levi form is non negative defnite. $\mathcal{D}$ is called strongly pseudoconvex if the Levi form for some local defining function at a boundary point a is positive definite.*

This notion does not depend on local coordinates.

Definition 2.8. *$\mathcal{D}$ is called pseudoconvex if it is pseudoconvex at all boundary points.*

Definition 2.9. *$\mathcal{D}$ is called q-pseudoconcave at a boundary point a if there exists a q-dimensional subspace $V \subset T_a^c\partial\mathcal{D}$ on which the Levi form of some defining function is negative definite.*

In order to define 1-complete neighborhoods, we introduce first some terminology. By a closed coordinate ball we shall mean a closed subset $\bar{\mathcal{B}}$ of $\mathcal{X}$ such that there exists coordinate chart $(U_\alpha, \phi_\alpha, V_\alpha)$ with

1) $\bar{\mathcal{B}} \subset U_\alpha$ and $\bar{B}^\infty \subset V_\alpha$ where $\bar{B}^\infty$ is the closed unit ball in l^2,

2) $\bar{\mathcal{B}} = \phi_\alpha^{-1}(\bar{B}^\infty)$.

In this situation $\mathcal{B} := \phi_\alpha^{-1}(B^\infty)$ will be called an open coordinate ball or simply a coordinate ball.

Definition 2.10. *Let $\mathcal{K}$ be a compact in $\mathcal{X}$. By a 1-complete neighborhood of $\mathcal{K}$ we understand an open set $\mathcal{U} \supset \mathcal{K}$ such that*

i) $\mathcal{U}$ is contained in a finite union of open coordinate balls centered at points of $\mathcal{K}$, i.e.

$$\mathcal{U} \subset \bigcup_{\alpha=1}^{n} \mathcal{B}_\alpha \text{ with } \mathcal{B}_\alpha = \phi_\alpha^{-1}(B^\infty) \text{ and } \phi_\alpha^{-1}(0) = k_\alpha \in \mathcal{K},$$

ii) $\mathcal{U}$ possesses a strictly plurisubharmonic exhaustion function $\psi : \mathcal{U} \to [0, t_0)$, i.e.

– *for every $t < t_0$ one has that $\overline{\psi^{-1}([0,t))} \subset \mathcal{U}$.*

We highlight the fact that all closures are taken in the topology of $\mathcal{X}$.

The usual definition of a 1-complete manifold V in finite dimension is that there should exist a strictly psh. function ψ on V such that for every $c \in \mathbb{R}$ the set $\psi^{-1}((-\infty, c))$ is relatively compact in U. Equivalently one can say that there should exist a str.psh. $\psi : U \to [0, +\infty)$ such that for every $c \in \mathbb{R}^+$ the set $\psi^{-1}([0, c))$ is relatively compact in U. If ψ takes values in $[0, t_0)$ one can compose it with $\frac{1}{t_0 - t}$ to get a exhaustion function from $U \to [0,+\infty)$. As for relative compactness condition it can be replaced by the following condition in the case when we are dealing with neighboorhoods of a compact set, say K, in some finite dimensional X. Take first a relatively compact neighborhood U of K and then say that U is 1-complete if there exist a str.psh. $\psi : U \to [0,+\infty)$ such that for every $c \in \mathbb{R}^+$ the closure in X of the set $\psi^{-1}([0,c))$ is contained in U. In Hilbert case "relative compactness" U is replaced by the condition that $\mathcal{U}$ is contained in a finite union of interiors of closed coordinate balls.

Therefore for the neighborhoods of compacts our definition agrees with the standard one in finite dimensional case.

2.3 Main theorem

The first main theorem of this thesis is the existence of one complete neighborhoods, i.e Theorem 2.3 from the introduction. This will be done following the original idea of H. Royden from [Ro].

By an analytic q-disk in a complex Hilbert manifold $\mathcal{X}$ we understand a holomorphic mapping ϕ of a neighborhood of a closure of a relatively compact strongly pseudoconvex domain $D \Subset \mathbb{C}^q$ into $\mathcal{X}$. We shall denote the image $\phi(\bar{D})$ by Φ. We say that Φ is imbedded if ϕ is an imbedding in a neighborhood of $\bar{D}$.

We choose $r > 0$ such that ϕ extends as an imbedding to a $2r$-neighborhood of D, i.e to $D^{2r} := \{z \in \mathbb{C}^q : d(z,D) < 2r\}$.

Theorem 2.3. *Let $\phi : \bar{D} \to \mathcal{X}$ be an imbedded analytic q-disk in a complex Hilbert manifold $\mathcal{X}$. Then $\phi(\bar{D})$ has a fundamental system of 1-complete neighborhoods.*

Remark 2.2. This theorem holds true for polydiscs as well. The point is that we suppose that ϕ is holomorphic in a neighborhood of the closure of a relatively compact domain D. Now suppose that the closure of a relatively compact pseudoconvex (but not necessarily strongly) domain $B \Subset \mathbb{C}^q$ possesses a fundamental system of str. ps. convex neighborhoods, say $\{D_k\}$ (a polydisc is certainly such). If $\phi : \bar{B} \to \mathcal{X}$ is a holomorphic imbedding in a neighborhood of $\bar{B}$ then it will be such in a neighborhood of all $\bar{D}_k$ for $k >> 1$. Now given a neighborhood U of $\phi(\bar{B})$ take k such that $\phi(\bar{D}_k) \subset U$ and find by Theorem 2.3 a 1-complete neighborhood V_k of $\phi(\bar{D}_k)$ such that $V_k \subset U$. Then $\{V_k\}$ will be a fundamental system of 1-complete neighborhoods of $\phi(\bar{B})$.

Step 1: Redressing of the coordinates

Lemma 2.2. *Let $\phi : B^q(a,\epsilon) \to \mathcal{X}$ be a holomorphic map of a ball centered at $a \in \mathbb{C}^q$ into a complex Hilbert manifold $\mathcal{X}$ such that $d_a\phi : \mathbb{C}^q \to T_b\mathcal{X}$ is injective, here $b = \phi(a)$. Then one can find a coordinate chart (V,h) in a neighborhood of b such that:*

i) V is mapped by h onto a neighborhood V' of the point $(a,0) \in l^2$ with $h(b) = (a,0)$. If $z = (z_1,...,z_q), w = (w_{q+1},,,,)$ are standard coordinates in l^2 then $V' = \{(z,w) \in l^2 : ||z-a|| < \delta, ||w|| < \delta\}$ for an appropriate $\delta > 0$.

ii) The map $h \circ \phi : \phi^{-1}(V) \to V'$ is given by $(h\circ\phi)(z_1,...,z_q) = (z_1,...,z_q,0,...)$.

Proof. Take some coordinate chart (V_1,h_1) in a neighborhood of b in $\mathcal{X}$. Choose a frame $\{v_i\}_{i=1}^{\infty}$ in l^2 in such a way that:

(a) $d_a\phi(e_i) = v_i, i = 1,...q$, for the standard basis $e_1,...,e_q$ of $\mathbb{C}^q$,

(b) span $\{v_1,...,v_q\}$ is orthogonal to span $\{v_{q+1},...\}$.

Let $z' = (z_1',...,z_q'), w' = (w_{q+1}',...)$ be affine coordinates in l^2 which correspond to the frame $\{v_i\}$ and such that $h_1(b) = (a,0)$ in these coordinates. Write $h_1\circ\phi = (\phi_1,\phi_2)$ in coordinates (z',w'). One has $\frac{d(z_1',...,z_q')}{d(z_1,...,z_q)}(0) = 1_q$ due to (a). By the implicit function theorem there exist neighborhoods $U \ni a$ in $\mathbb{C}^q$ and $V_2 \ni a$ in $L = \text{span}\{v_1,...,v_q\}$ such that $\phi_1 : U \to V_2$ is a biholomorphism. U can be taken of the form $\{||z-a|| < \delta\}$ for an appropriate $\delta > 0$.

Therefore $\phi(U)$ is a graph $w' = \psi(z')$ over V_2. Make a coordinate change $h_2 : (z^{'},w^{'}) \to (z^{''} = z^{'}, w^{''} = w^{'} - \psi(z^{'}))$ to get that $(h_2\circ h_1)\circ\phi$ has the form $z \to (\phi_1(z),0)$. Finally make one more coordinate change $(z^{''},w^{''}) \to (z = \phi_1^{-1}(z^{''}), w = w^{''})$ to get the final chart (V,h) with $(h\circ\phi)(z) = (z,0)$. V' can be taken to have the form $U\times V_3$ where $V_3 = \{w, ||w|| < \delta\}$ for an appropriate $\delta > 0$ (for that one might need to shrink V_2 and therefore U). Lemma 2.2 is proved.

□

Remark 2.3. U was chosen to be a ball. Note that it can be chosen to be a cube as well. Remark also that V' can be taken in the form of a product $V' = U \times V''$, where V'' is a Hilbert ball ($V^{''} = V_3$ in the notations of the proof of Lemma 2.2).

Step 2: Trivializaion of the infinitesimal neighborhood.

Denote by $D^r := \{z \in \mathbb{C}^q \mid d(z,D) < r\}$ the r-neighborhood of D. Cover $\phi(\overline{D^r})$ with a finite collection of coordinate neighborhoods $\{(V_\alpha,h_\alpha)\}_{\alpha=1}^N$ with centers at a_α as in Lemma 2.2. Denote by z_α, w_α the corresponding coordinates in $V_\alpha' = U_\alpha \times V_\alpha^{''} \subset l^2$. Note that z_α glues to a global coordinate z on D. Denote by $J_{\alpha,\beta}$ the Jacobian matrix of the coordinate change $(z_\alpha, w_\alpha) = (h_\alpha \circ h_\beta^{-1})(z_\beta, w_\beta)$. Since $h_\alpha \circ h_\beta^{-1}(z, 0) = (z, 0)$ we see that the jacobian $J_{\alpha,\beta}$ of $\phi_\alpha \circ \phi_\beta^{-1}$ has the form

$$J_{\alpha,\beta}(z,0) = \begin{pmatrix} I_q & A_{\alpha,\beta}(z) \\ 0 & B_{\alpha,\beta}(z) \end{pmatrix}. \tag{2.5}$$

By construction the operator valued functions $B_{\alpha,\beta}$ form a multiplicative cocycle, i.e they are the transition functions of an appropriate vector l^2-bundle over $\overline{D^r}$ (the normal bundle to $f(\overline{D^r})$). Indeed

$$\begin{pmatrix} I_q & A_{\alpha,\beta}(z) \\ 0 & B_{\alpha,\beta}(z) \end{pmatrix} \cdot \begin{pmatrix} I_q & A_{\beta,\gamma}(z) \\ 0 & B_{\beta,\gamma}(z) \end{pmatrix} = \begin{pmatrix} I_q & A_{\beta,\gamma}(z) + A_{\alpha,\beta}(z)B_{\beta,\gamma}(z) \\ 0 & B_{\alpha,\beta}(z)B_{\beta,\gamma}(z) \end{pmatrix} = \begin{pmatrix} I_q & A_{\alpha,\gamma}(z) \\ 0 & B_{\alpha,\gamma}(z) \end{pmatrix} \tag{2.6}$$

This bundle is trivial by Theorem 1.6 and therefore one can find holomorphic operator valued functions $B_\alpha : U_\alpha \to End(l^2)$ such that $B_{\alpha,\beta} = B_\alpha \circ B_\beta^{-1}$ on $U_\alpha \cap U_\beta$. Make a coordinate change in V_α as follows: $\tilde{z}_\alpha = z_\alpha$ and $\tilde{w}_\alpha = B_\alpha(z_\alpha)^{-1} w_\alpha$. Then in the new coordinates the Jacobian matrix of the coordinate change (when restricted to $w_\beta = 0$) will have the form

$$J_{\alpha,\beta}(z;0) = \begin{pmatrix} I_q & A_{\alpha,\beta}(z) \\ 0 & I_\infty \end{pmatrix} \tag{2.7}$$

for some $A_{\alpha,\beta}(z)$. Notations for coordinates (z_α, w_α) and charts $(V'_\alpha = U_\alpha \times V''_\alpha, h_\alpha)$ will be not change at this stage.

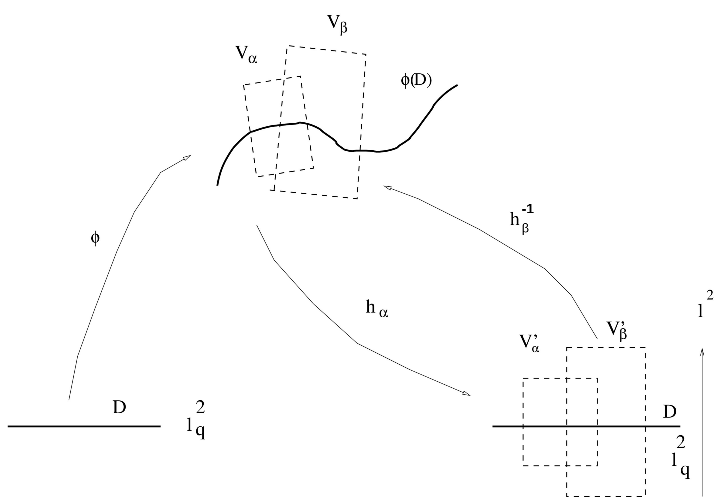


Figure 2.1: $z = (z_1, ..., z_q)$ serves as common coordinate for all charts in this step.

Transition mappings between new coordinate charts have the form

$$\begin{cases} z_\alpha = z_\beta + \sum\limits_{n=1}^{\infty} A^n_{\alpha,\beta}(w_\beta), \\ w_\alpha = w_\beta + \sum\limits_{n=2}^{\infty} B^n_{\alpha,\beta}(w_\beta), \end{cases} \tag{2.8}$$

where $A^n_{\alpha,\beta}$ (resp. $B^n_{\alpha,\beta}$) are holomorphic functions of $z_\beta = z \in U_\alpha \cap U_\beta$ with values in the spaces of continuous n-homogeneous vector valued polynomials on w_β. We shall use the standard notations as follows, see [Mu]. For a n-homogeneous polynomial B^n we denote as $\hat{B}^n$ the (unique) symmetric n-linear form defining B^n, i.e. such that $B^n(w) = \hat{B}^n(\underbrace{w, ..., w}_{n-times}) =: \hat{B}w^n$. According to the same standard notations when writing $\hat{B}^n u^l v^k$ one means $\hat{B}^n(\underbrace{u, ..., u}_{l-times}, \underbrace{v, ..., v}_{k-times})$, $n = l + k$. Our goal in what follows is to eliminate the terms $B^n_{\alpha,\beta}$. Note that on $U_\alpha \cap U_\beta \cap U_\gamma$ one has

$$w_\alpha = w_\beta + \sum_{n=2}^{\infty} B^n_{\alpha,\beta}(w_\beta) = w_\gamma + \sum_{n=2}^{\infty} B^n_{\beta,\gamma}(w_\gamma) +$$

$$+ \sum_{n=2}^{\infty} B^n_{\alpha,\beta}\left(w_\gamma + \sum_{m=2}^{\infty} B^m_{\beta,\gamma}(w_\gamma)\right) = w_\gamma + \sum_{n=2}^{\infty} B^n_{\alpha,\gamma}(w_\gamma).$$

Therefore

$$\sum_{n=2}^{\infty} B^n_{\alpha,\gamma}(w_\gamma) = \sum_{n=2}^{\infty} B^n_{\beta,\gamma}(w_\gamma) + \sum_{n=2}^{\infty} B^n_{\alpha,\beta}\left(w_\gamma + \sum_{m=2}^{\infty} B^m_{\beta,\gamma}(w_\gamma)\right). \tag{2.9}$$

This gives for every degree $N \geq 2$ of homogenuity the following *finite* relation between homogeneous polynomials of degree N

$$B^N_{\alpha,\gamma}(w_\gamma) = B^N_{\beta,\gamma}(w_\gamma) + \sum_{(m-1)k+n=N} C^k_{n-k} \hat{B}^n_{\alpha,\beta} w_\gamma^{n-k} B^m_{\beta,\gamma}(w_\gamma)^k. \tag{2.10}$$

In the right hand side of 2.10 we have the term $B^N_{\beta,\gamma}(w_\gamma)$ and terms in the sum with $k \geq 0$. Each of this terms should have the right degree of homogenuity equal to N, which gives only a finite number of possibilities for k, m and n. Note that if $N = 2$ then, since $n, m \geq 2$, the only possibility in 2.10 is $k = 0$ and therefore we get a cocyle condition

$$B^2_{\alpha,\gamma}(w_\gamma) = B^2_{\beta,\gamma}(w_\gamma) + B^2_{\alpha,\beta}(w_\gamma). \tag{2.11}$$

Remark 2.4. It is worth to point out at this stage that we do not have the cocycle condition for higher N-s. For them we have only relation 2.10.

Now solve the additive Cousin problem $B^2_{\alpha,\beta} = B^2_\beta - B^2_\alpha$ for the acyclic covering

$\{U_\alpha\}$ of D and make the following (quadratic in w) change of variables

$$\begin{cases} \tilde{z}_\alpha = z_\alpha \\ \tilde{w}_\alpha = w_\alpha + B^2_\alpha(w_\alpha). \end{cases} \tag{2.12}$$

It is not difficult to check that relation 2.8 becomes

$$\begin{cases} \tilde{z}_\alpha = \tilde{z}_\beta + \sum_{n=1}^{\infty} \tilde{A}^n_{\alpha,\beta}(\tilde{w}_\beta), \\ \tilde{w}_\alpha = \tilde{w}_\beta + \sum_{n=3}^{\infty} \tilde{B}^n_{\alpha,\beta}(\tilde{w}_\beta) \end{cases} \tag{2.13}$$

with appropriate $\tilde{A}^n_{\alpha,\beta}$ and $\tilde{B}^n_{\alpha,\beta}$.

Now we can trivialize the infinitesimal neighborhood to all orders. From 2.10 we see that if $B^n_{\alpha,\beta} = 0$ for all $n < N$ and all α, β then $\{B^N_{\alpha,\beta}\}$ is an additive cocyle, which can be resolved as $B^N_{\alpha,\beta} = B^N_\beta - B^N_\alpha$ and then the coordinate change

$$\begin{cases} z_\alpha = z_\beta \\ \tilde{w}_\alpha = w_\alpha + B^N_\alpha(z)(w_\alpha). \end{cases} \tag{2.14}$$

will give us coordinates in which the transition functions will take the form

$$\begin{cases} z_\alpha = z_\beta + \sum_{n=1}^{\infty} A^n_{\alpha,\beta}(w_\beta), \\ w_\alpha = w_\beta + \sum_{n=N+1}^{\infty} B^n_{\alpha,\beta}(w_\beta). \end{cases} \tag{2.15}$$

$\mathbb{C}^q$-valued functions $A^n_{\alpha,\beta}(z_\beta)$ are holomorphic in $z \in U_\alpha \cap U_\beta$ with values in the space of continuous homogeneous polynomials of degree n in w-s. They satisfy the additive cocycle condition for every fixed $n \leq N$.

For every fixed $n \leq N$ we solve $A^n_{\alpha,\beta} = A^n_\beta - A^n_\alpha$ and in every chart V'_α make the coordinate change:

$$\begin{cases} \tilde{z}_\alpha = z_\alpha - \sum_{n=1}^{N} A^n_\alpha(z_\alpha)(w_\alpha) \\ \tilde{w}_\alpha = w_\alpha. \end{cases} \tag{2.16}$$

We get that in new coordinates the coordinate changes vanish to order N, i.e. have the form

$$\begin{cases} z_\alpha = z_\beta + \sum_{n=N+1}^{\infty} A^n_{\alpha,\beta}(w_\beta), \\ w_\alpha = w_\beta + \sum_{n=N+1}^{\infty} B^n_{\alpha,\beta}(w_\beta). \end{cases} \tag{2.17}$$

In steps 1 and 2 we constructed a covering $\{V_\alpha\}$ of a neighborhood V of $\phi(\bar{D})$ by coordinate charts with coordinate mappings $h_\alpha : V_\alpha \to V'_\alpha$ such that $V'_\alpha = U_\alpha \times V''_\alpha$, where U_α is a covering of $\bar{D}$ with coordinates z_α and V''_α are open sets in l^2 with coordinates w_α. Finally the coordinate changes have the form 2.17. We take $N = 3$ in the sequel, i.e. coordinates match to order three.

Let ρ_α be nonnegative $\mathcal{C}^\infty$ functions with support contained in V_α such that $\sum_\alpha \rho_\alpha = 1$ in a neighborhood of $\phi(\bar{D})$. Define the following (vector valued) functions on the manifold $\mathcal{X}$

$$\begin{cases} u = \sum\limits_\alpha \rho_\alpha z_\alpha, \\ w = \sum\limits_\alpha \rho_\alpha w_\alpha. \end{cases} \tag{2.18}$$

In V_α we have

$$\begin{cases} u - z_\alpha = \sum\limits_\beta \rho_\beta (z_\beta - z_\alpha) = O(w_\alpha^4), \\ w - w_\alpha = \sum\limits_\beta \rho_\beta (w_\beta - w_\alpha) = O(w_\alpha^4). \end{cases} \tag{2.19}$$

Since D is a strongly pseudoconvex domain there exists a strictly plurisubharmonic exhaustion function θ of class $\mathcal{C}^2$ defined in a neighborhood of $\bar{D}$ such that

$$D = \{z \in \mathbb{C}^q \mid \theta(z) < 1\} \quad \text{and} \quad \theta^{-1}(\{0\}) = \{0\} \subset D.$$

Since u and w are differentiable coordinates in a neighborhood $\mathcal{V}$ of $\phi(\bar{D})$ we can shrink $\mathcal{V}$ in such a way that the image of $\mathcal{V}$ under (u, w) is $W \times B^\infty(\delta)$ with W a neighborhood of $\bar{D}$. Set $\xi^2 = \|w\|^2 = \sum |w_k|^2$. Then in $V_\alpha \cap \mathcal{V}$ we have due to 2.19

$$|u_k - z_{\alpha,k}| = O(\xi^4), \qquad |w_k|^2 - |w_{\alpha,k}|^2 = O(\xi^5).$$

Indeed, the first one is clear and the second one comes from

$$|w_k|^2 = |w_{\alpha,k} + O(w_{\alpha,k}^4)|^2 = |w_{\alpha,k}|^2 + |O(w_{\alpha,k}^5)| + |O(w_{\alpha,k}^8)| = |w_{\alpha,k}|^2 + O(\xi^5). \tag{2.20}$$

For a given $0 < \lambda < 1$ consider the following function

$$\begin{array}{cccc} \psi_\lambda : & W \times B^\infty(\delta) & \longrightarrow & \mathbb{R} \\ & (x, y) & \mapsto & \theta(x) + \lambda^{-2}||y||^2 \end{array} \tag{2.21}$$

We can observe that since θ is smooth then ψ_λ is smooth too. On $V_\alpha \cap \mathcal{V}$ we have

$$\begin{aligned} g(z_\alpha, w_\alpha) &:= \psi_\lambda(u, w) - \psi_\lambda(z_\alpha, w_\alpha) = \theta(u) - \theta(z_\alpha) + \lambda^{-2}\big(||w||^2 - ||w_\alpha||^2\big) \\ &= d\theta_{z_\alpha}(u - z_\alpha) + o(\|u - z_\alpha\|) + \lambda^{-2} \textstyle\sum_k |w_k|^2 - |w_{\alpha,k}|^2 \\ &= O(\textstyle\sum_k |u_k - z_{\alpha,k}|) + \lambda^{-2} O(\textstyle\sum_k |w_k|^2 - |w_{\alpha,k}|^2) = O(w_\alpha^4) + \lambda^{-2} O(w_\alpha^5). \end{aligned}$$

Therefore the Hessian of g, i.e. the difference with respect to the coordinates

(z_α, w_α) corresponds to

$$\frac{\partial^2 g}{\partial z_{\alpha,j}\partial \bar{z}_{\alpha,k}} = O(w_\alpha^4) + \lambda^{-2}O(w_\alpha^5), \qquad \frac{\partial^2 g}{\partial z_{\alpha,j}\partial \bar{w}_{\alpha,k}} = O(w_\alpha^3) + \lambda^{-2}O(w_\alpha^4),$$

$$\frac{\partial^2 g}{\partial w_{\alpha,j}\partial \bar{z}_{\alpha,k}} = O(w_\alpha^3) + \lambda^{-2}O(w_\alpha^4), \qquad \frac{\partial^2 g}{\partial w_{\alpha,j}\partial \bar{w}_{\alpha,k}} = O(w_\alpha^2) + \lambda^{-2}O(w_\alpha^3).$$

So the Hessian of g consists of terms $O(\xi^2) + \lambda^{-2}O(\xi^3)$. The Hessian $\mathcal{H}_{\psi_\lambda(z_\alpha,w_\alpha)}$ of $\psi_\lambda(z_\alpha, w_\alpha)$ is an infinite matrix of the form $\begin{pmatrix} \mathcal{H}_\theta & 0 \\ 0 & \lambda^{-2}I_\infty \end{pmatrix}$ where $\mathcal{H}_\theta$ is the Hessian of θ and I_∞ is the identity matrix.

The Hessian $\mathcal{H}_{\psi_\lambda(u,w)}$ of $\psi_\lambda(u, w)$ satisfies

$$\mathcal{H}_{\psi_\lambda(u,w)} = \begin{pmatrix} \mathcal{H}_\theta & 0 \\ 0 & \lambda^{-2}I_\infty \end{pmatrix} + O(\xi^2) + \lambda^{-2}O(\xi^3) = \mathcal{H}_{\psi_\lambda(z_\alpha,w_\alpha)} + O(\xi^2) + \lambda^{-2}O(\xi^3).$$

Let $\mu > 0$ be such that $\mathcal{H}_\theta(u, 0)(v', v') \geqslant \mu||v'||^2$ for all $u \in W$ and all $v' \in \mathbb{C}^q$, see lemma 2.1. This gives us the bound $\mathcal{H}_{\psi_\lambda(u,w)}(v, v) \geqslant \mu||v'||^2 + \lambda^{-2}||v''||^2$ for all $u, w \in W \times B^\infty(\delta)$ provided $\delta > 0$ small enough and all $(v', v'') \in \mathbb{C}^q \oplus l^2$. Therefore $\psi_\lambda(u, w)$ is strictly plurisubharmonic in $\mathcal{V}$.

For $\lambda > 0$ small enough set

$$\mathcal{V}_\lambda := \{(u, w) \mid \psi_\lambda(u, w) < 1 + \lambda\}.$$

Notice that $\bar{\mathcal{V}}_\lambda = \{(u, w) \mid \psi_\lambda(u, w) \leqslant 1+\lambda\}$ and this set is contained in a finite union of coordinate balls by construction. One has obviously that $\mathcal{V}_\lambda \supset \phi(\bar{D})$ and $\bigcap_{\lambda>0} \mathcal{V}_\lambda = \phi(\bar{D})$. But this is not enough for $\mathcal{V}_\lambda$ be a fundamental system of neighborhoods of the compact $\phi(\bar{D})$.

For example, $N_k = \{z \in l^2 : |z_1|, ..., |z_k| < 1/k\}$ satisfy $\bigcap_1^\infty N_k = 0$, but they do not form a fundamental system of neighborhoods for the norm topology of l^2 .

Let us prove that nevertheless in our case $\mathcal{V}_\lambda$ is a fundamental system of neighborhood of $\phi(\bar{D})$. Indeed, the str. psh function has the form

$$\psi_\lambda(u, w) = \theta(u) + \lambda^{-2} \|w\|^2 , \tag{2.22}$$

where u is the coordinate on D and w is the (non-holomorphic) normal coordinate. Therefore

$$\mathcal{V}_\lambda := \{(u, w) : \psi_\lambda(u, w) < 1+\lambda\} \subset \{(u, w) : \theta(u) < 1+\lambda,\ \|w\|^2 < \lambda^2(1+\lambda)\} \tag{2.23}$$

is contained in the tubular $\max\{\lambda, \lambda^2(1+\lambda)\}$-neighborhood of $\phi(\bar{D})$, i.e. is contained in a given neighborhood of $\phi(\bar{D})$ provided $\lambda > 0$ is small enough. The latter is true because $\phi(\bar{D})$ is compact.

What is left to prove is that $\psi_\lambda(u, w)$ is a strictly plurisubharmonic exhausting function on $\mathcal{V}_\lambda$, i.e. each $\mathcal{V}_\lambda$ is 1-complete. More precisely, we need to prove that $\psi = \psi_\lambda$ and $\mathcal{U} = \mathcal{V}_\lambda$ and $\mathcal{K} = \phi(\bar{D})$ satify the definition 2.10. Indeed, for every $t < 1+\lambda$, $\overline{\psi^{-1}([0,t))} \subset \mathcal{V}_\lambda = \psi^{-1}([0, 1+\lambda))$. Theorem is proved. $\square$

Chapter 3

Hartogs manifolds and Hilbert-Hartogs manifolds

3.1 The notion of a Hilbert-Hartogs manifold

We denote as $H^1_q(r)$ the q-concave Hartogs figure in $\mathbb{C}^{q+1}$, i.e. the following domain

$$H^1_q(r) = (\Delta^q \times \Delta_r) \cup (A^q_{1-r,1} \times \Delta). \tag{3.1}$$

Here Δ_r denotes the disk of radius r in $\mathbb{C}$ centered at zero, Δ the unit disk, Δ^q_r the polydisk in $\mathbb{C}^q$ and $A^q_{1-r,1} = (\Delta \setminus \bar{\Delta}_{1-r})^q$ the ring domain. The precise value of $r \in]0,1[$ is usually irrelevant. If something holds for some $0 < r < 1$ then the same usually holds for all other $0 < r' < 1$. The envelope of holomorphy of $H^1_q(r)$ is the unit polydisk Δ^{q+1}.

Definition 3.1. *We say that a complex manifold X is q-Hartogs if every holomorphic mapping $f : H^1_q(r) \to X$ extends to a holomorphic mapping $\tilde{f} : \Delta^{q+1} \to X$.*

If the same holds for a complex Hilbert manifold $\mathcal{X}$ we say that $\mathcal{X}$ is q-Hilbert-Hartogs. 1-Hartogs manifolds are called simply Hartogs.

First, we can observe that Hartogs manifolds have better extension properties on Hartogs figures than it is postulated in their definition. Let us make a first step in studying this.

For positive integers q, n and real $r \in]0,1[$ we call a Hartogs figure of bidimension (q,n) or a q-concave Hartogs figure in $\mathbb{C}^{q+n}$ the domain

$$H^n_q(r) := (\Delta^q \times \Delta^n(r)) \cup \left(A^q_{1-r,1} \times \Delta^n \right). \tag{3.2}$$

The envelope of holomorphy of $H^n_q(r)$ is Δ^{q+n}.

If holomorphic maps with values in Hilbert manifold $\mathcal{X}$ holomorphically extend from $H^n_q(r)$ to Δ^{q+n} we shall say that $\mathcal{X}$ possesses a holomorphic extension property in bidimension (q,n).

We shall prove that the holomorphic extendability in bidimension $(q,1)$, i.e. being q-Hartogs, implies the holomorphic extendability in all bidimensions (q,n) for $n \geq 1$

and moreover holomorphic extendability in bidimension (q, n) implies holomorphic extendability in all bidimensions (p, m) with $p \geq q$ and $m \geq n$. The proof follows the lines of Lemmas 2.2.1 and 2.2.2 in [Iv4].

Theorem 3.1. *If a complex Hilbert manifold $\mathcal{X}$ possesses a holomorphic extension property in bidimension (q, n) for some $q, n \geq 1$ then $\mathcal{X}$ possesses this property in every bidimension (p, m) with $p \geq q, m \geq n$.*

Proof. Let us denote by Z the coordinates $Z := (z, z_{q+1}, w)$ with $z \in \mathbb{C}^q$, $z_{q+1} \in \mathbb{C}$ and $w \in \mathbb{C}^n$. We shall prove our theorem by induction. Let us increase q first. Notice that

$$H_{q+1}^n(r) \supset \bigcup_{z_{q+1} \in \Delta} H_q^n(r) \times \{z_{q+1}\},$$

here $H_q^n(r) := \{(z, w) : \|z\| < 1, \|w\| < r \text{ or } 1 - r < \|z\| < 1, \|w\| < 1\}$ with $||z|| = \max_{j=1,\dots,n} |z_j|$ and $||w|| = \max_{j=1,\dots,n} |w_j|$.

Therefore a given holomorphic mapping $f : H_{q+1}^n(r) \to \mathcal{X}$ to a q-Hartogs manifold extends along these slices to a map $\tilde{f} : \Delta^q \times \Delta \times \Delta^n \to \mathcal{X}$. We need to prove that $\tilde{f}$ is holomorphic as function of the couple (z, z_{q+1}, w). This will prove that our $\mathcal{X}$ is $(q + 1)$-Hartogs.

Continuity of $\tilde{f}$. First we need to prove that this extension $\tilde{f}$ is continuous. Let a sequence $Z_k = (z_1^k, \dots, z_q^k, z_{q+1}^k, w_1^k, \dots, w_n^k)$ converge to $Z_0 = (z_1^0, \dots, z_q^0, z_{q+1}^0, w_1^0, \dots, w_n^0) \in \Delta^{q+1+n}$. Take $R \in]0, 1[$ such that $\|z^0\|, |z_{q+1}^0|, \|w^0\| < R$ and the same for $||z^k||, |z_{q+1}^k|$ and $||w^k||$ with k big enough. Consider the following imbedded $(q + n)$-disk ϕ_0 in the Hilbert manifold $\mathcal{Y} := \mathbb{C}^{q+1+n} \times \mathcal{X}$:

$$\phi_0(z, w) = \Big\{ \big(z, z_{q+1}^0, w, \tilde{f}(z, z_{q+1}^0, w)\big) : z \in \bar{\Delta}^q(R), w \in \bar{\Delta}^n(R) \Big\},$$

i.e. ϕ_0 is the map to the graph of the restriction of $\tilde{f}$ to the $(q+n)$-disk $\bar{\Delta}^q(R) \times \{z_{q+1}^0\} \times \bar{\Delta}^n(R) \subset \Delta^{q+1+n}$. Denote by Φ_0 the image of ϕ_0, i.e. the graph of $\tilde{f}|_{\bar{\Delta}^q(R) \times \{z_{q+1}^0\} \times \bar{\Delta}^n(R)}$. Furthermore, set

$$\phi_k(z, w) = \Big\{ \big(z, z_{q+1}^k, w, \tilde{f}(z, z_{q+1}^k, w)\big) : z \in \bar{\Delta}^q(R), w \in \bar{\Delta}^n(R) \Big\},$$

and denote by Φ_k its image, i.e. the graph of $\tilde{f}|_{\bar{\Delta}^q(R) \times \{z_{q+1}^k\} \times \bar{\Delta}^n(R)}$.

It will be convenient for the future references to formulate the next step of the proof in the form of a lemma. In the proof of this lemma we shall use the Riemann Hilbert structure on $\mathcal{X}$, see Remark 1.4.

Lemma 3.1. *Let $\phi_n : \bar{D} \to \mathcal{X}$ be a sequence of analytic q-disks in a Hilbert manifold $\mathcal{X}$ and let Φ_n be their graphs, here $D \Subset \mathbb{C}^q$. Suppose that there exists an analytic disk*

$\phi_0 : \bar{D} \to \mathcal{X}$ with the graph Φ_0 such that for any neighborhood $\mathcal{V} \supset \Phi_0$ one has $\Phi_n \subset \mathcal{V}$ for $n >> 1$. Then ϕ_n converges uniformly on $\bar{D}$ to ϕ_0.

Proof. Since ϕ_0 is uniformly continuous on $\bar{D}$ for a given $\epsilon > 0$ we can find $\delta > 0$ such that if $||u - v|| < \delta$ then $d(\phi_0(u), \phi_0(v)) < \epsilon$. Here d is some Riemannian metric on $\mathcal{X}$. In addition we can assume that $\delta < \epsilon$. Take N such that for $\forall n \geq N$ one has $\Phi_n \subset \Phi_0^\delta$, where Φ_0^δ is a δ-neighborhood of Φ_0 with respect to the product metric on $\mathcal{Z} := \mathbb{C}^q \times \mathcal{X}$. Fix $u \in \bar{D}$. For every $n \geq N$ there exists $v_n \in \bar{D}$ such that the distance between $(u, \phi_n(u))$ and $(v_n, \phi_0(v_n))$ is less than δ. But then the distance between u and v_n is less than δ, i.e. $||u - v_n|| < \delta$. Therefore $d(\phi_0(u), \phi_0(v_n)) < \epsilon$. This proves that the distance between $(u, \phi_n(u))$ and $(u, \phi_0(u))$ is not more than $\epsilon + \delta < 2\epsilon$.

The same holds for $u_1 \in \bar{D}$ in a neighborhood of u with the same N. Using compactness of $\bar{D}$ we find N such that $d(\phi_n(u), \phi_0(u)) < \epsilon$ for all $u \in \bar{D}$ and all $n \geqslant N$.

□

To apply this lemma in our setting notice that by Theorem 2.3 for a given neighborhood $\mathcal{V}$ of Φ_0 there exists a 1-complete neighborhood $\mathcal{V}_\lambda$ with $\Phi_0 \subset \mathcal{V}_\lambda \subset \mathcal{V}$. Denote by $\psi_\lambda : \mathcal{V}_\lambda \to [0, 1 + \lambda)$ the corresponding strictly plurisubharmonic exhaustion function. Notice furthermore that for z_{q+1}^k close enough to z_{q+1}^0 the graph of $\tilde{f}$ over $H_q^n(r) \times \{z_{q+1}^k\}$ belongs to $\mathcal{V}_\lambda$. Therefore its graph over the whole polydisk $\bar{\Delta}^{n+q}(R) \times \{z_{q+1}^k\} \times \bar{\Delta}^n(R)$ belongs to $\mathcal{V}_\lambda$ by the maximum principle. Indeed, fix $\tau < 1 + \lambda$ such that $\mathcal{V}_\lambda(\tau) := \{\psi_\lambda < \tau\}$ is still a neighborhood of Φ_0. We know that $\overline{\mathcal{V}_\lambda(\tau)} \subset \mathcal{V}_\lambda$. Consider the family of polydiscs $\Delta_t^q(R) := \Delta^q(R) \times \{z_{q+1}^k\} \times \{w(t)\}$ where $w(t)$ is a curve in $\Delta^n(R)$ starting at 0 and going to the boundary. Note that for t close to 0, $\Delta_t^q(R) \subset H_n^q(r) \times \{z_{q+1}^k\}$ and therefore $\Gamma_{\tilde{f}}|_{\Delta_t^q(R)} \subset \mathcal{V}(\tau)$. The same is true for all t for the graph of $\tilde{f}$ restricted to a neighborhood of the boundary of $\Delta_t^q(R)$.

Let t_1 be the maximal number in $[0, 1]$ such that $\tilde{f}|_{\Delta^q(R) \times \{z_{q+1}^k\} \times \{w(t_1)\}} \subset \bar{\mathcal{V}}_\lambda(\tau)$, i.e. $(\psi_\lambda \circ \tilde{f})|_{\Delta^q(R) \times \{z_{q+1}^k\} \times \{w(t)\}} \leq \tau$. Suppose $t_1 < 1$, then we can find $t_1 < t_2 < 1$ as close to t_1 as we wish such that

$$\max(\psi_\lambda \circ \tilde{f})|_{\Delta^q(R) \times \{z_{q+1}^k\} \times \{w(t_2)\}} > \tau. \tag{3.3}$$

But $\tilde{f}_{\Delta^q(R) \times \{z_{q+1}^k\} \times \{w(t_2)\}}$ is close to $\tilde{f}_{\Delta^q(R) \times \{z_{q+1}^k\} \times \{w(t_1)\}}$ when t_2 is close to t_1, because $\tilde{f}$ is holomorphic (therefore continuous) when restricted to $\Delta^q \times \{z_{q+1}^k\} \times \Delta^n$. Therefore it stays in $\mathcal{V}_\lambda$. So the restriction $(\psi_\lambda \circ \tilde{f})|_{\Delta^q(R) \times \{z_{q+1}^k\} \times \{w(t_2)\}}$ is well defined. And now 3.3 contradicts to the maximum principle for the psh function $(\psi_\lambda \circ \tilde{f})|_{\Delta^q(R) \times \{z_{q+1}^k\} \times \{w(t_2)\}}$ since

$$\max(\psi_\lambda \circ \tilde{f})|_{\partial\Delta^q(R) \times \{z_{q+1}^k\} \times \{w(t_2)\}} \leq \tau. \tag{3.4}$$

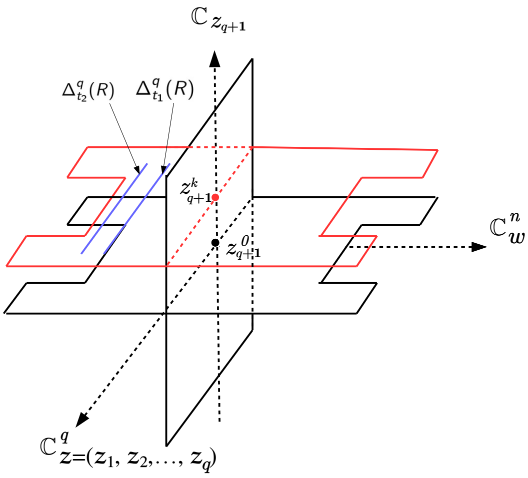


Figure 3.1: Extension on the slice z^k_{q+1}.

Therefore $t_1 = 1$ and consequently $\Phi_k \subset \mathcal{V}_\lambda$. Therefore the question of continuity is reduced to Lemma 3.1 just proved. Remark that we even do not need the strict plurisubharmonicity of ψ_λ.

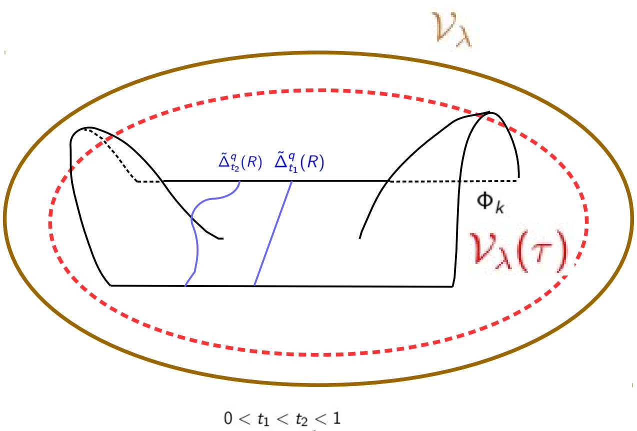


Figure 3.2: Analytic $(q+n)$-disc Φ_0 is not shown here, only its 1-complete neighborhoods $\mathcal{V}_\lambda(\tau) \subset \mathcal{V}_\lambda$. It is shown however how analytic $(q+n)$-disc Φ_k, the graph of the restriction $\tilde{f}|_{\Delta^q(R)\times\{z^k_{q+1}\}\times\Delta^n(R)\}}$, may behave when $z^k_{q+1} \to z^0_{q+1}$. But if so, then analytic q-discs $\tilde{f}|_{\Delta^q_t(R)}$ drawn on this picture as $\tilde{\Delta}^q_t(R)$ slide out of $\mathcal{V}_\lambda(\tau)$. And this is not possible, because they "slide continuously" (not leaving the bigger 1-complete neighborhood $\mathcal{V}_\lambda$ at once) and their boundaries stay in $\mathcal{V}_\lambda(\tau)$.

Holomorphy of $\tilde{f}$. Holomorphy becomes now a local question. Let denote by $Z^0 = (z^0, z^0_{q+1}, w^0, 0, ..., 0)$ a point in $\Delta^{q+2} \times \{0\}$. We shall prove that $\tilde{f}$ is holomorphic in a neighborhood of Z^0. The rest can be done by an obvious induction. Let $Z^1 = (z^0, z^0_{q+1}, 0)$ be the projection of Z^0 to $\mathbb{C}^{q+1}$. Since $Z^1 \in H^{q+1}_n$ our mapping $\tilde{f}$ is holomorphic in a neighborhood of Z^1. Denote by $I = [0, w^0_1]$ the closed interval in the

$(q+2)$-th coordinate plane from 0 to the point w_1^0 - the first coordinate of Z^0. By W denote the set of $t \in I$ such that $\tilde{f}$ is holomorphic in a neighboroohd of t. $Z^1 \in I$, i.e. is non-empty and is obviously open. All we need to prove is that I is closed. Let t_1 be a cluster point of W. By continuity of $\tilde{f}$ we can find a poydisk neighborhood U containing t_1 and a coordinate chart (Ω, ψ) such that $\tilde{f}(U) \subset \Omega$. Since t_1 is a cluster point of W we can find $t_0 \in W \cap U$. After shrinking if necessary we find ourselves in the following situation. For $\epsilon > 0$ small enough and appropriate simply connected neighborhoods $V_0 \ni t_0$ and $V_0 \subset V_1 \supset [t_0, t_1]$ we have the following:

i) $(\psi \circ \tilde{f})|_{V_0 \times \Delta^{q+n}(\epsilon)} \to \Omega$ is holomorphic,

ii) for every fixed $p = (z, z_{q+1}, w_2, ..., w_n)$ the restriction $(\psi \circ \tilde{f})|_{V_0 \times \{p\}}$ is holomorphic on V_1.

By the classical Hartogs theorem $\tilde{f}$ is holomorphic on $\Delta^{q+n} \times V_1$. Therefore $W = I$. This proves the holomorphy of $\tilde{f}$ in a neighborhood of $\Delta^{q+1} \times \Delta \times \{(w_2^0, ..., w_n^0)\}$ for every $(w_2^0, ..., w_n^0) \in \Delta^{n+1}$.

The extension on the Hartogs figure $H_q^{n+1}(r)$ follows the same lines and can be done also in two steps. Set

$$E_q^{n+1}(r) = H_q^n(r) \times \Delta(r),$$

and remark that $E_q^{n+1}(r) \subset H_q^{n+1}(r)$. Extend f from $E_q^{n+1}(r)$ to $\Delta^{q+n} \times \Delta(r)$ exactly as above. Then extend f from $\Delta^{q+n} \times \Delta(r)$ to Δ^{q+n+1} again in the same way.

□

3.2 Hartogs property and pseudoconcavity

There is a link between the Hartogs phenomena and concavity.

A complex Hilbert manifold $\mathcal{X}$ satisfies the q-Levi extension condition if for any domain $D \subset \mathbb{C}^n$ with $\mathcal{C}^2$ boundary such that ∂D is q-pseudoconcave at p any holomorphic map $f : D \to \mathcal{X}$ extends holomorphically to a neighborhood of p. We shall prove that $\mathcal{X}$ being q-Hartogs is equivalent to having q-Levi extension property. But first let us make the following observation.

Remark 3.1. We can replace the polydisks in the definitions of Hartogs figures by the balls of the corresponding dimension, i.e. we can take

$$H_q^n(r) = (B^q \times B^n(r)) \ \cup \ ((B^q \setminus B^q(r)) \times B^n).$$

Now we can state the corresponding theorem.

Theorem 3.2. *A complex Hilbert manifold $\mathcal{X}$ satisfies the q-Levi extension condition if and only if $\mathcal{X}$ is q-Hartogs.*

Proof. q-Hartogs $\Rightarrow$ q-Levi extension.

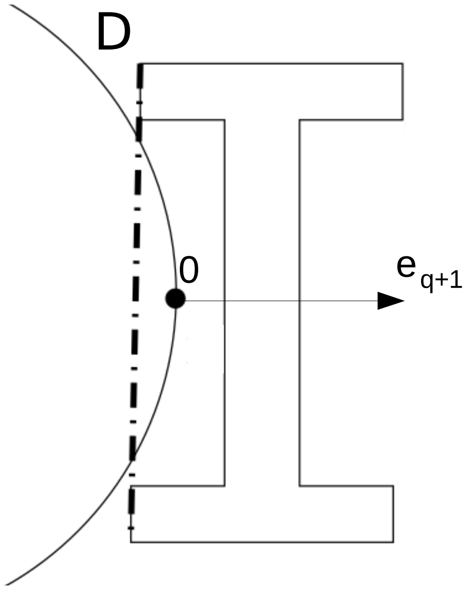


Figure 3.3: q-Hartogs figure in a neighborhood of p.

Let $\mathcal{X}$ be q-Hartogs and consider a domain $D \subset \mathbb{C}^n$. Let U an open set in $\mathbb{C}^n$ such that $U \cap D = \{z \in U \mid u(z) < 0\}$ with $\nabla u \neq 0$. Then there exist $v_j \in T_0 \partial D \subset \mathbb{C}^n$ such that $\mathcal{L}_{u,p}(v_j) < 0$ for $j = 1, ..., q$. After a complex linear change of coordinate we can suppose that $p = 0$ and

$$v_1 = e_1,\ v_2 = e_2,\ \dots\ ,\ v_q = e_q, \nabla u_p = e_{q+1}.$$

We can define the following figure: $\phi : H^1_q(r) \times \Delta^{n-q-1} \to \mathbb{C}^n$

$$\phi : z \mapsto \eta z_1 e_1 + ... + \eta z_q e_q + (\eta\epsilon z_{q+1} - r')e_{q+1} + \delta z_{q+2} e_{q+2} + ... + \delta z_n e_n. \tag{3.5}$$

We can choose ϵ, r', δ and η such that $\phi(H^1_q(r) \times \Delta^{n-q-1}) \subset U \cap D$ and $\eta\epsilon > r'$. The latter insures that the image of the polydisk $\phi(\Delta^n)$ contains the origin.

Now for any $f : D \to \mathcal{X}$ the map $f|_{\phi(H^1_q(r) \times \Delta^{n-q-1})}$ extends to $\phi(\Delta^{q+1} \times \Delta^{n-q-1})$ because $\mathcal{X}$ is q-Hartogs. So it extends to a neighborhood of $p = 0$. Therefore $\mathcal{X}$ satifies the q-Levi extension condititon.

q-Levi extension $\Rightarrow$ q-Hartogs. This direction can be proved by representing $B^q \times \Delta$ as an increasing union of pseudoconcave domains starting from $H^q_1(r)$.

Suppose $\mathcal{X}$ satisfies the q-Levi extension condition and let $f : H^1_q(r) \to X$ be a

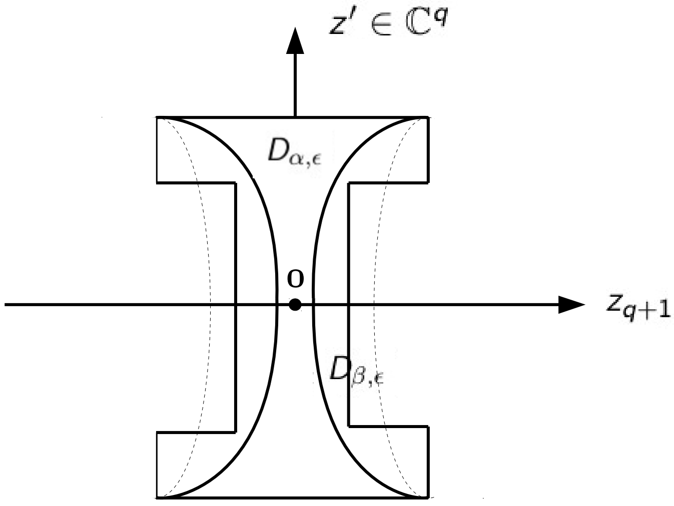


Figure 3.4: Exhaustion of the polydisk by a family of pseudoconcave $D_{\alpha,\epsilon}$ starting from the Hartogs figure. The “flat” part of the boundary $G_1 := \partial B^q \times \bar{\Delta}$ doesn't play any role, only the concave one, i.e., $G_2 = \partial D_{\alpha,\epsilon} \cap \left(B^q \times \bar{\Delta}\right)$. Through this concave part we extend mappings from $D_{\alpha,\epsilon}$ to $D_{\beta,\epsilon}$ for $\beta < \alpha$.

holomorphic map. Consider for $\alpha \in [1, \infty[$ and $\epsilon \in]0, 1[$ the following function

$$u_{\alpha,\epsilon}(z) = |z_{q+1}|^2 - (1-\epsilon)||z'||^{2\alpha} - \epsilon \tag{3.6}$$

where $z' = (z_1, ..., z_q)$ and let $D_{\alpha,\epsilon} = \{z \in B^q \times \Delta \mid u_{\alpha,\epsilon}(z) < 0\}$. One can see that

- For a fixed $\epsilon_0 \in]0, 1[$ small enough there exists $\alpha_0 \geq 1$ such that $D_{\alpha_0,\epsilon_0} \subset H^1_q(r)$,
- $D_{\alpha,\epsilon}$ is strictly pseudoconcave for all $\epsilon \in]0, 1[$ and $\alpha \in [1, \infty[$ at all points of $G_2 = \partial D_{\alpha,\epsilon} \cap \left(B^q \times \bar{\Delta}\right)$.

Let $\Gamma_{\epsilon_0} = \{\alpha \geq 1 \mid f \text{ extends to } D_{\alpha,\epsilon_0}\}$. Then Γ_{ϵ_0} is nonempty (because $\alpha_0 \in \Gamma_{\epsilon_0}$) and closed. As $\mathcal{X}$ satisfies the q-Levi extension condition Γ_{ϵ_0} is open. Therefore f extends to D_{1,ϵ_0}. Now let $\Gamma = \{\epsilon < 1 \mid f \text{ extends to } D_{1,\epsilon}\}$. For the same reasons as above Γ is nonempty and closed. It is also open because $\mathcal{X}$ satisfies the q-Levi extension condition. Hence f extends holomorphically to $D_{1,1}$:

$$D_{1,1} = \{z \in B^q \times \Delta \mid |z_{q+1}|^2 - 1 < 0\} = B^q \times \Delta. \tag{3.7}$$

We proved that f extends from $H^1_q(r)$ to $B^q \times \Delta$. Therefore X is q-Hartogs.

□

3.3 Extension from infinite dimensional domains

Let us define the infinite dimensional q-concave Hartogs figure $H_q^\infty(r) \subset l^2$ for $r \in [0,1]$ by

$$H_q^\infty(r) = (\Delta^q \times B^\infty(r)) \cup (A^q_{1-r,1} \times B^\infty)$$

Theorem 3.3. *Let $\mathcal{X}$ be a q-Hartogs Hilbert manifold. Then for every $r > 0$ every holomorphic mapping $f : H_q^\infty(r) \to \mathcal{X}$ extends to a holomorphic mapping $\tilde{f} : \Delta^q \times B^\infty \to \mathcal{X}$.*

Proof. We identify $\mathbb{C}^q$ with $l^2_q = \operatorname{span}\{e_1, ..., e_q\} \subset l^2$. For a unit vector $v \in l^2$ orthogonal to $\mathbb{C}^q$ set $L_v := \operatorname{span}\{e_1, ..., e_q, v\}$. Remark that $L_v \cap H_q^\infty(r) = H_q^1(r)$ and therefore given a holomorphic mapping $f : H_q^\infty(r) \to \mathcal{X}$ its restriction to $L_v \cap H_q^\infty(r)$ holomorphically extends to $L_v \cap (\Delta^q \times B^\infty)$. We conclude that for every line $< v > \perp \mathbb{C}^q$ the restriction $f|_{L_v}$ holomorphically extends onto $L_v \cap (\Delta^q \times B^\infty)$, giving an extension $\tilde{f}$ of f onto $\Delta^q \times B^\infty$. This extension is correctly defined because for unit vectors $v \neq w$ orthogonal to $\mathbb{C}^q$ the spaces L_v and L_w intersect only by $\mathbb{C}^q$.

Let us prove the continuity of $\tilde{f}$. Consider the sequence $(Z_n)_{n\geqslant 1}$ defined by $Z_n = (z^n, w^n)$ such that $Z_n \to Z_0 = (z^0, w^0)$. Here $z^0, z^n \in \Delta^q$ and $w^0, w^n \in B^\infty$. Take R such that $1 - r < R < 1$ and $||z^n||, ||w^n|| < R$ for all $n \in \mathbb{N}$ as well as $||z^0||, ||w^0|| < R$. Let $\phi_n : \bar{\Delta}^q(R) \times \bar{\Delta} \to \mathcal{X}$ be the analytic disk defined by $\phi_n(z, \eta) = \tilde{f}(z, \eta w^n)$ and ϕ_0 be defined by $\phi_0(z, \eta) = \tilde{f}(z, \eta w^0)$.

Theorem 2.3 gives a 1-complete neighborhood V of the graph of Φ_0. For w^n close enough to w^0 the graph Φ_n of ϕ_n over $L_{w_n} \cap H_q^\infty(r)$ is contained in V because $L_{w_n} \cap H_q^\infty(r) \subset H_q^\infty(r)$ where $\tilde{f}$ is holomorphic. In exactly the same manner as in the proof of Theorem 3.1 we have that by the maximum principle the graph Φ_n of ϕ_n over the whole set $\bar{\Delta}^q(R) \times \bar{\Delta}$ is contained in V. Then by Lemma 3.1 $(\phi_n)_n$ converges uniformly to ϕ_0, i.e. we have $\tilde{f}(z, \eta w^n) \underset{n\to\infty}{\rightrightarrows} \tilde{f}(z, \eta w^0)$ on $\bar{\Delta}^q(R) \times \bar{\Delta}$. So taking $z = z^n$ and $\eta = 1$ gives $\tilde{f}(z^n, w^n) \underset{n\to\infty}{\longrightarrow} \tilde{f}(z^0, w^0)$. Therefore $\tilde{f}$ is continuous.

What is left to prove is that this extension is Gâteaux differentiable. Take some $z^0 \in \Delta^q \times B^\infty$ and fix some direction v at z^0. Let $l := \{z^0 + tv : t \in \mathbb{C}\}$ be the line through z^0 in the direction v. Find (at most) two vectors w_1, w_2 such that $e_1, ..., e_q, w_1, w_2$ is the orthonormal basis of the subspace L containing $\mathbb{C}^q$, z^0 and l.

Let $L_{w_1,w_2} = \mathbb{C}^q \bigoplus \operatorname{span}\{w_1, w_2\}$. Remark that $L_{w_1,w_2} \cap H_q^\infty(r) = H_q^2(r)$ and therefore the theorem 3.1 is applicaple. It gives us the holomorphy of $\tilde{f}|_{(\Delta^q \times \Delta^2) \cap L_{w_1,w_2}}$ and therefore the differentiability of $\tilde{f}$ in the direction v.

Every continuous Gâteaux differentiable map is holomorphic, see Theorem 1.3 and therefore Theorem 3.3 is proved.

$\square$

As an important consequence we obtain the following statement.

Corollary 3.1. *If a domain $\mathcal{D}$ in a complex Hilbert manifold $\mathcal{X}$ is q-pseudoconcave at a boundary point p then every holomorphic map $f : \mathcal{D} \to \mathcal{Y}$ to a q-Hartogs Hilbert manifold $\mathcal{Y}$ extends holomorphically to a neighborhood of p.*

Proof. The problem is local in $\mathcal{X}$ and therefore we can assume that $\mathcal{X} = l^2$. The q-pseudoconcavity of the point $p \in \partial\mathcal{D}$ implies that there exists $U \subset l^2$ a neighborhood of p such that $U \cap \mathcal{D} = \{z \in U \mid u(z) < 0\}$ with $\nabla u \neq 0$. Take $v_j \in T_p\partial\mathcal{D} \subset l^2$ such that $\mathcal{L}_{u,p}(v_j) < 0$ for $j = 1, ..., q$. After a complex linear change of coordinates we can suppose that $p = 0$ and $v_1 = e_1,\ v_2 = e_2,\ \dots\ ,\ v_q = e_q, \nabla u_p = e_{q+1}$. We define the following figure $\phi : H_q^\infty(r) \to l^2$ by

$$z \mapsto \eta z_1 e_1 + ... + \eta z_q e_q + (\eta\epsilon z_{q+1} - r')e_{q+1} + \delta \sum_{s=q+2}^{\infty} z_s e_s. \tag{3.8}$$

We choose ϵ, r', δ and η such that $\phi(H_q^\infty(r)) \subset U \cap \mathcal{D}$ and $\eta\epsilon > r'$ to insure that the image of the polydisk $\phi(\Delta^q \times B^\infty)$ contains the origin.

Consider now the map $f \circ \phi : H_q^\infty(r) \to \mathcal{Y}$. By theorem 3.3 the map $f \circ \phi$ extends to $\tilde{f} : \Delta^q \times B^\infty \to \mathcal{Y}$ and then $\tilde{f} \circ \phi^{-1}$ gives the desired extension on a neighborhood of p.

□

3.4 Mappings to Hilbert fiber bundles

Proposition 3.1. *a) If $\mathcal{X}$ is a Hilbert manifold and $\mathcal{Y}$ is some unramified cover of $\mathcal{X}$ then $\mathcal{X}$ and $\mathcal{Y}$ are q-Hartogs or not simultaneously.*

b) If the fiber $\mathcal{F}$ and the base $\mathcal{B}$ of a complex Hilbert fiber bundle $(\mathcal{E}, \mathcal{F}, \pi, \mathcal{B})$ are q-Hartogs then the total space $\mathcal{E}$ is also q-Hartogs.

Proof. a) Suppose that $\mathcal{X}$ is q-Hartogs. We shall show that $\mathcal{Y}$ satisfies the Levi extension condition: for any domain D in $\mathbb{C}^{q+1}$ with $\mathcal{C}^2$ boundary such that ∂D is q-Levi pseudoconcave at $p \in \partial D$. Let $f : D \to \mathcal{Y}$ be holomorphic. As $\mathcal{X}$ is q-Hartogs we can extend $\pi \circ f$ to an open neighborhood V of p with π the projection map. Take a neighborhood $U \subset V$ of p such that there is a trivialisation:

$$h = (\pi, b) : \pi^{-1}(U) \tilde{\to} U \times F.$$

As F is a discrete set we can extend f on a neighborhood of $h^{-1}(U \times \{b \circ f(p)\}) \subset \mathcal{Y}$ Hence f satisfies the q-Levi extension condition and is q-Hartogs.

Reciprocally, suppose that $\mathcal{Y}$ is q-Hartogs. Let D be a domain in $\mathbb{C}^{q+1}$ with $\mathcal{C}^2$ boundary such that ∂D is q-Levi pseudoconcave at p and $f : D \to \mathcal{X}$ be holomorphic. For $p \in \partial D$, it exists $\mathfrak{p} \in \mathcal{Y}$ such that $\pi(\mathfrak{p}) = f(p)$. We take a neighborhood V of $\mathfrak{p}$ that is biholomorphic to $\pi(V)$ because π is locally a biholomorphism. Then, $(\pi|_V)^{-1} \circ f$ extends to a neighborhood of p. Hence we compose the last extension by π and it gives an extension of f to a neighborhood of p. $\mathcal{X}$ satisfies the q-Levi extension condition so is q-Hartogs.

b) We shall show that $\mathcal{E}$ satisfies the Levi extension condition. Let D be a domain in $\mathbb{C}^{q+1}$ with $\mathcal{C}^2$ boundary such that ∂D is q-Levi pseudoconcave at p. Let $f : D \to \mathcal{E}$ be holomorphic. As $\mathcal{B}$ is q-Hartogs, we can extend $\pi \circ f$ to an open neighborhood V of p. Take a neighborhood $U \subset V$ of p such that there is a trivialisation:

$$h = (\pi, pr) : \pi^{-1}(U) \tilde{\to} U \times \mathcal{F}.$$

Then $pr \circ f$ extends to a neighborhood of p as $\mathcal{F}$ is q-Hartogs. So $\mathcal{E}$ satisfies the q-Levi extension condition and is q-Hartogs.

□

Example 3.1. From this proposition one can derive several examples of Hilbert-Hartogs manifolds. If for example $\Lambda = \mathrm{span}_{\mathbb{Z}}\{e_1, ie_1, e_2, ie_2, ...\}$ is the integer lattice in l^2 then $\mathbb{T}^\infty := l^2/\Lambda$ is Hartogs. In fact consider the map $f : H(r) \to l^2$. Since f is holomorphic the Cauchy integral gives an extension to the $\Delta^2_{1-\epsilon}$ for all $\epsilon > 0$:

$$f(z_1, z_2) := \frac{1}{2\pi i} \int_{|\xi_1|=|\xi_2|=1-\epsilon} \frac{f(\xi_1, \xi_2)}{(\xi_1 - z_1)(\xi_2 - z_2)} d\xi_1 d\xi_2.$$

This extension is clearly defined by the fact that $(\partial\Delta_{1-\epsilon})^2 \subset H(r)$. Therefore l^2 is Hartogs. Since l^2 is an unramified cover of $\mathbb{T}^\infty$ the latter is Hartogs as well by Proposition 3.1. From the last proposition 3.1 $\mathbb{T}^\infty$ is also Hartogs.

Example 3.2. Hilbert fiber bundles with Hartogs fibers over connected Riemann surfaces different from $\mathbb{P}^1$ are also Hilbert-Hartogs. Indeed as it was explained in the introduction such Riemann surfaces are Hartogs. Therefore the complex Hilbert fiber bundles with Hartogs fibers over them are Hilbert-Hartogs by the proposition just proved. As for Riemann sphere $\mathbb{P}^1$ it was explained that it is not Hartogs, see introduction. Now take the Hopf surface $H = \mathbb{C}^2\backslash\{0\}/\{z \sim 2z\}$. It is naturally fibered over $\mathbb{P}^1$ with fiber being a torus and the latter is Hartogs. But H is not Hartogs because the natural projection $\pi : \mathbb{C}^2\backslash\{0\} \to H$ does not extend to the origin, its limit set at zero is the whole H.

Chapter 4

Loop spaces as complex Hilbert manifolds

4.1 Weak derivative

Hilbert-Hartogs manifolds appear more often than one may expect. We will prove a theorem that gives us a way to construct loop spaces that are Hilbert-Hartogs. Working with mapping spaces we need to choose a class of smoothness that we will use.

For an open $\Omega \subset \mathbb{R}^n$ and for $p \in [\![1, \infty[\![$ let $L^p(\Omega)$ be the standard Lebesgue space with the norm $||u||_{L^p} = \left(\int_\Omega |u(x)|^p d\mu(x)\right)^{1/p}$. In the special case of $p = 2$ $L^2(\Omega)$ has the structure of a Hilbert space with the scalar product: $\forall u, v \in L^2(\Omega)$ $< u, v >_{L^2} := \int_{\mathbb{R}^n} u(x)\overline{v(x)} d\mu(x)$. Now we introduce the concept of weak derivative which allows us to define properly Sobolev spaces.

Definition 4.1. *Let $\Omega \subset \mathbb{R}^n$ be open. We denote by $\mathcal{C}_0^\infty(\Omega, \mathbb{C})$ the set of test functions (or smooth functions with compact support) on Ω as*

$$\mathcal{C}_0^\infty(\Omega, \mathbb{C}) := \left\{ u \in \mathcal{C}^\infty(\Omega, \mathbb{C}) \mid supp(u) \subset \Omega \textit{ is compact} \right\}.$$

An example of a test function on $\mathbb{R}^n$ is the function

$$u(x) = \begin{cases} \exp\left(\frac{1}{||x||^2 - 1}\right) & \text{for } ||x|| < 1 \\ 0 & \text{for } ||x|| > 1. \end{cases}$$

The idea of considering this space lies in the following lemma.

Lemma 4.1. *The space of test functions is dense in L^p, i.e. $\overline{\mathcal{C}_0^\infty(\Omega, \mathbb{C})} = L^p(\Omega)$.*

This can be proved as follows. Let v be in $L^p(\Omega)$. We consider the convolution of v with the function $u_\epsilon(x) := \frac{1}{\epsilon^n} u\left(\frac{x}{\epsilon}\right)$ provided u was taken such that $\int_{\mathbb{R}^n} u(x)dx = 1$

$$v_\epsilon(x) := \int_\Omega v(x - y) u_\epsilon(y) d\mu(y).$$

So we have $v_\epsilon \in \mathcal{C}_0^\infty(\Omega, \mathbb{C})$ and $v_\epsilon \to v$ in $L^p(\Omega)$.

From the integrability and using the integration by part, it allows us to define a notion of derivation called weak derivative.

Working on L^p spaces many functions are not differentiable Functions of L^p spaces are not all differentiable. Although, the integrability allows us to use integration by parts (the boundary values vanish since the support of u is a compact subset of Ω) to have the concept of weak derivatives. Let $\Omega \subset \mathbb{R}^n$ be an open set, $f \in \mathcal{C}^1(\bar{\Omega})$ and $u \in \mathcal{C}_0^\infty(\Omega, \mathbb{C})$. Then

$$\int_\Omega \frac{\partial f}{\partial x_i}(x)\overline{u(x)}d\mu(x) = -\int_\Omega f(x)\overline{\frac{\partial u}{\partial x_i}(x)}d\mu(x).$$

Similarly we obtain for m-times conitnuously differentiable functions $f \in \mathcal{C}^m(\bar{\Omega})$:

$$\int_\Omega D^\alpha f(x)\overline{u(x)}d\mu(x) = (-1)^{|\alpha|}\int_\Omega f(x)\overline{D^\alpha u(x)}d\mu(x)$$

where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}}$ with $\alpha \in \mathbb{N}^n$ a multi-index and $|\alpha| := \alpha_1 + ... + \alpha_n \leqslant k$.

We can now drop the assumption $f \in \mathcal{C}^m$ to state:

Definition 4.2. *Let $\Omega \subset \mathbb{R}^n$ be open, $\alpha \in \mathbb{N}^n$ and $f \in L^2(\Omega)$. Then $g \in L^2(\Omega)$ is called the weak α-th derivative of f if*

$$\forall u \in \mathcal{C}_0^\infty(\Omega, \mathbb{C}) \quad \int_\Omega g(x)\overline{u(x)}d\mu(x) = (-1)^{|\alpha|}\int_\Omega f(x)\overline{D^\alpha u(x)}d\mu(x).$$

Remark 4.1. Using the scalar product of the Hilbert space $L^2(\Omega)$ we can write the equality simply by

$$< g \mid u >_{L^2} = (-1)^{|\alpha|} < f \mid D^\alpha u >_{L^2} .$$

Remark 4.2. The weak derivative coincides with the classical derivatives for differentiable functions. Moreover the weak derivative g of a function f is unique. Suppose h is another weak derivative of f. Then $< g - h \mid u >= 0$ for all $u \in \mathcal{C}_0^\infty(\Omega, \mathbb{C})$ and by lemma 4.1 we obtain that $g - h = 0$, almost everywhere.

From now on we denote the weak derivative of f by $D^{(\alpha)}f$. We may define a space of weakly differentiable functions.

Definition 4.3. *Let $\Omega \subset \mathbb{R}^n$ be open. The Sobolev class $W^{k,2}(\Omega)$ or $W^{k,2}(\Omega, \mathbb{C})$ is defined to be*

$$W^{k,2}(\Omega) := \left\{ f \in L^2(\Omega) \mid \forall \alpha \in \mathbb{N}^n,\ |\alpha| \leqslant k,\ D^{(\alpha)}f \in L^2(\Omega) \right\}.$$

Proposition 4.1. *The space $W^{k,2}(\Omega)$ endowed with the scalar product*

$$\forall f, g \in W^{k,2}(\Omega),\ < f \mid g >_{W^{k,2}} = \sum_{|\alpha| \leqslant k} < D^{(\alpha)} f \mid D^{(\alpha)} g >_{L^2}$$

is a Hilbert space.

Proof. Obviously $< \cdot \mid \cdot >_{W^{k,2}}$ is a scalar product. The difficulty is to prove that $W^{k,2}(\Omega)$ is complete for this norm. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to the norm $||\cdot||^2_{W^{k,2}} := \sum_{|\alpha|\leqslant k} ||D^\alpha \cdot||^2_{L^2}$. The sequences $D^{(\alpha)} f$ are Cauchy sequences in the Hilbert spaces $L^2(\Omega)$ so there is f_α such that $D^{(\alpha)} f_n \to f_\alpha$. Let denote by f_0 the function f_α for $|\alpha| = 0$. For every $u \in \mathcal{C}_0^\infty(\Omega, \mathbb{C})$ we have

$$\begin{aligned} < f_\alpha \mid u > &= < \lim_{n\to\infty} D^{(\alpha)} f_n \mid u > = \lim_{n\to\infty} < D^{(\alpha)} f_n \mid u > \\ &= \lim_{n\to\infty} (-1)^{|\alpha|} < f_n \mid D^\alpha u > = (-1)^{|\alpha|} < f_0 \mid D^\alpha u > \\ &= < D^{(\alpha)} f_0 \mid u > . \end{aligned}$$

So $f_0 \in W^{k,2}(\Omega)$ because $D^{(\alpha)} f_0 = f_\alpha \in L^2(\Omega)$ and $f_n \to f_0$ with respect to $||\cdot||_{W^{k,2}}$. □

A function $f \in L^2$ has a certain decrease at infinity. For the Sobolev space $W^{k,2}(\mathbb{R}^n)$, we also have that weak derivatives have a certain decrease at infinity. The Fourier transform allows us to translate derivatives into multiplication with polynomials. That is why we can characterize the class $W^{k,2}$ by the Fourier transform.

4.2 The Fourier transform

Definition 4.4. *The Fourier transform of a function $u \in L^1(\mathbb{R}^n)$ is defined by*

$$\mathcal{F}u(\xi) = \hat{u}(\xi) := \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} u(x) e^{-i<x,\xi>} d\mu(x).$$

In order to obtain relavant properties on the Fourier transform, we need some better hypothesis on the function u. That is why when one works with this transform it is more appropriate to define it on the Schwarz space:

Definition 4.5. *The Schwarz space denoted by $\mathcal{S}(\mathbb{R}^n)$ is defined as following.*

$$\mathcal{S}(\mathbb{R}^n) := \left\{ u \in \mathcal{C}^\infty(\mathbb{R}^n) \mid \sup_{\mathbb{R}^n} |x^\beta D^\alpha u| < +\infty \text{ for all } \alpha, \beta \in \mathbb{N}^n \right\}.$$

In the literature a function from the Schwarz space is sometimes called smooth rapidly decreasing functions or simply Schwarz function.

Example 4.1. The function $\gamma(x) = e^{-x^2}$ and every test function are Schwarz functions.

Remark 4.3. Clearly $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for $p \in [1, \infty[$. Since $\mathcal{C}_0^\infty(\mathbb{R}^n, \mathbb{C})$ is a subset of $\mathcal{S}(\mathbb{R}^n)$ we have that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

One important property of $\mathcal{S}(\mathbb{R}^n)$ is the fact that the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an isomorphism.

Lemma 4.2. *Let $u \in \mathcal{S}(\mathbb{R}^n)$ and α a multi-index. Then*

a) $\mathcal{F}u \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $D^\alpha(\mathcal{F}u) = (-i)^{|\alpha|}\mathcal{F}(x^\alpha u)$,

b) $\mathcal{F}(D^\alpha u) = i^{|\alpha|}\xi^\alpha \mathcal{F}u$.

Proof. a) We have

$$D^\alpha(\mathcal{F}u)(\xi) = \frac{\partial^{|\alpha|}}{\partial \xi^\alpha}\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} u(x)e^{-i<x,\xi>}dx \overset{(*)}{=} \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} u(x)\frac{\partial^{|\alpha|}}{\partial \xi^\alpha}e^{-i<x,\xi>}dx$$

$$= (-i)^{|\alpha|}\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} u(x)x^\alpha e^{-i<x,\xi>}dx = (-i)^{|\alpha|}\mathcal{F}(x^\alpha u)(\xi).$$

The equality $(*)$ is ensured by the dominated convergence theorem. As $\mathcal{F}(x^\alpha u)$ is continuous for every $\alpha \in \mathbb{N}^n$ $\mathcal{F}u$ is $\mathcal{C}^\infty$.

(b) By integration by parts and the fact that u rapidly decreases one obtains

$$\mathcal{F}(D^\alpha u)(\xi) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}(D^\alpha u)(x)e^{-i<x,\xi>}dx = \frac{(-1)^{|\alpha|}}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} u(x)\frac{\partial}{\partial x^\alpha}e^{-i<x,\xi>}dx$$

$$= (-1)^{|\alpha|}(-i)^{|\alpha|}\xi^\alpha(\mathcal{F}u)(\xi) = i^{|\alpha|}\xi^\alpha(\mathcal{F}u)(\xi).$$

□

Proposition 4.2. $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$, *i.e. if $u \in \mathcal{S}(\mathbb{R}^n)$ then $\mathcal{F}u \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. See [T].

□

We can compose the Fourier transform and show that for $u \in \mathcal{S}(\mathbb{R}^n)$

$$\forall x \in \mathbb{R}^n \ (\mathcal{F}\mathcal{F}u)(x) = u(-x).$$

It means that $\mathcal{F}^2$ is a reflection. Moreover $\mathcal{F}^4 = \mathrm{Id}_{\mathcal{S}(\mathbb{R}^n)}$. This gives the following proposition.

Proposition 4.3. *We can define the inverse Fourier transform by*

$$\mathcal{F}^* u(\xi) := \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} u(x)e^{i<x,\xi>}d\mu(x).$$

Then $\mathcal{F}$ is an isomorphism on the Schwarz space $\mathcal{S}(\mathbb{R}^n)$ and we have the identity

$$\mathcal{F}\mathcal{F}^* = \mathcal{F}^*\mathcal{F} = Id \quad on\ \mathcal{S}(\mathbb{R}^n).$$

Furthermore we have the so-called Plancherel equation

$$\forall\ u, v \in \mathcal{S}(\mathbb{R}^n), \quad < \mathcal{F}u \mid \mathcal{F}v >_{L^2} = < u \mid v >_{L^2} . \tag{4.1}$$

Remark 4.4. By the density of the space C_0^∞ of compact support smooth functions on the Schwarz space we can extend the Fourier transform to the Hilbert space $L^2(\mathbb{R}^n)$. Then $\mathcal{F}$ extends to a linear operator $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

The Plancherel equation implies that $\mathcal{F}$ is an isometric isomorphism: $\|\mathcal{F}u\|_{L^2} = \|u\|_{L^2}$ for $u \in \mathcal{S}(\mathbb{R}^n)$ and by density for $u \in L^2(\mathbb{R}^n)$.

Lemma 4.3. *(Formula for $W^{k,2}$). Let u be in $W^{k,2}(\mathbb{R}^n)$. Then for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqslant k$*

$$\mathcal{F}(D^{(\alpha)}u) = i^{|\alpha|}\xi^\alpha \mathcal{F}u.$$

Proof. For $v \in \mathcal{S}(\mathbb{R}^n)$ and using (4.1) we have

$$\begin{aligned} < \mathcal{F}(D^\alpha u) \mid \mathcal{F}v > = < D^\alpha u \mid v > = (-1)^{|\alpha|} < u \mid D^\alpha v > &= (-1)^{|\alpha|} < \mathcal{F}u \mid \mathcal{F}(D^\alpha v) > \\ &= (-1)^{|\alpha|} < \mathcal{F}u \mid i^{|\alpha|}\xi^\alpha \mathcal{F}v > \\ &= < i^{|\alpha|}\xi^\alpha \mathcal{F}u \mid \mathcal{F}v >. \end{aligned}$$

Since the Fourier transform is a bijection on $\mathcal{S}(\mathbb{R}^n)$ we have that

$$\forall v \in \mathcal{S}(\mathbb{R}^n),\ < w \mid v > = < \mathcal{F}(D^{(\alpha)}u) - i^{|\alpha|}\xi^\alpha \mathcal{F}u \mid v > = 0.$$

where $w = \mathcal{F}(D^{(\alpha)}u) - i^{|\alpha|}\xi^\alpha \mathcal{F}u$. As $\mathcal{C}_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ the previous inequality is true also for all $v \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. We still do not know if the function w is in $L^2(\mathbb{R}^n)$. So we consider w in a ball $B_r \subset \mathbb{R}^n$ of radius r.

$$\forall v \in \mathcal{C}_0^\infty(B_r),\ < w \mid v >_{L^2(B_r)} = 0.$$

Then $w = 0$ almost everywhere in every ball B_r of radius $r > 0$.

$\square$

Remark 4.5. Compared to lemma 4.2, we only showed an analog of the second formula. In fact, for a function $u \in W^{k,2}(\mathbb{R}^n)$ we only know that $\mathcal{F}u \in L^2(\mathbb{R}^n)$ and the derivative of $\mathcal{F}u$ does not need to exist.

From lemma 4.3 one can deduce the characterisation of Sobolev map by the Fourier transform.

$$\begin{aligned} u \in W^{k,2}(\mathbb{R}^n) &\Longleftrightarrow \forall \alpha \in \mathbb{N}^n, |\alpha| \leqslant k, \quad D^{(\alpha)}u \in L^2(\mathbb{R}^n) \\ &\Longleftrightarrow \forall \alpha \in \mathbb{N}^n, |\alpha| \leqslant k, \quad \xi^\alpha \mathcal{F}u \in L^2(\mathbb{R}^n) \\ &\Longleftrightarrow \ (1+\|\xi\|)^k \mathcal{F}u \in L^2(\mathbb{R}^n) \end{aligned}$$

It allows to extend the definition of Sobolev spaces for any positive $k \in \mathbb{R}^+$

$$W^{k,2}(\mathbb{R}^n) = \{u \mid (1+\|\xi\|)^k \mathcal{F}u \in L^2(\mathbb{R}^n)\}. \tag{4.2}$$

A standard result about Sobolev spaces is the Sobolev embedding theorem.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be open and let $k, m \in \mathbb{N}$ be such that $k > \frac{n}{2} + m$. If $u \in W^{k,2}(\Omega)$ then there exists a function $v \in \mathcal{C}^m(\Omega)$ that is equal to u almost everywhere.*

Roughly speaking for $k > \frac{n}{2} + m$ we have $W^{k,2}(\Omega) \subset \mathcal{C}^m(\Omega)$.

Many other results are considered under the name "Sobolev embedding theorem" and for completeness we just state them:
i) If $k \geq \frac{n}{2} + \alpha$ with $0 < \alpha < 1$ then $W^{k,2}(\mathbb{R}^n) \subset \mathcal{C}^\alpha(\mathbb{R}^n)$ (α-Hölder continuous) and this inclusion is a compact operator. In particular $W^{n,2}(\mathbb{R}^n) \subset \mathcal{C}^0(\mathbb{R}^n)$.
ii) If $0 \leq k < \frac{n}{2}$ then $W^{k,2}(\mathbb{R}^n) \subset L^{\frac{2n}{n-2k}}(\mathbb{R}^n)$.

From (4.2) one can derive that if $f, g \in W^{k,2}(\mathbb{R}^n)$ then $fg \in W^{k,2}(\mathbb{R}^n)$ provided $k > \frac{n}{2}$. This enable us to define correctly a $W^{k,2}$-vector bundle over an n-dimensional real manifold provided $k > \frac{n}{2}$. By that we mean that the transition functions of the bundle are in $W^{k,2}$. Moreover this implies that for a polynomial $P(t_1, .., t_l)$ and $f_1, ..., f_l \in W^{k,2}$ one has $P(f_1, ..., f_l) \in W^{k,2}$. And this implies via the Weierstrass approximation theorem that for $F \in \mathcal{C}^k$ one has $F(f_1, .., f_l) \in W^{k,2}$. Therefore we can speak about Sobolev $W^{k,2}$ mappings between smooth manifolds and pullback bundles under such mappings. Condition $k > \frac{n}{2}$ will be always assumed from now on.

For the proof of the Theorem 4.1 and more details on i) and ii) we refer to [T].

4.3 Generalized loop spaces

Let S be a compact, connected, n-dimensional real manifold without boundary. Let X be a finite dimensional complex manifold. In order to speak about the Sobolev space $W^{k,2}(S, X)$ it is convenient to suppose that X is embedded smoothly to some $\mathbb{R}^N$. If X is not compact we suppose that this imbedding is proper.

We define $W^{k,2}(S, X)$ as the Sobolev space of maps $g : S \to X$ of class $W^{k,2}$. Following [L1], we endow this space with the Sobolev topology and complex structure as follows. A neighborhood of $g \in W^{k,2}(S, X)$ is obtained by the following construction. Cover $g(S) \subset X$ by finitely many coordinate charts (X_i, α_i) of X with $\alpha_i : X_i \to \mathbb{C}^n$

and similarly cover S by coordinate charts (V_i, β_i) making sure that $g(\bar{V}_i) \subset X_i$. In Hilbert spaces $W^{k,2}(V_i, \mathbb{C}^n)$ choose neighborhoods $\mathcal{U}_i$ of $\alpha_i \circ g \circ \beta_i^{-1}$.

Definition 4.6. *A neighborhood of $g \in W^{k,2}(S, X)$ is defined as*

$$\{h \in W^{k,2}(S, X) \mid \alpha_i \circ h \circ \beta_i^{-1} \in \mathcal{U}_i \text{ for all } i\}. \tag{4.3}$$

This construction endows $W^{k,2}(S, X)$ with the $W^{k,2}$-topology and makes it a topological space. Now in order to obtain a complex manifold structure we need to construct charts on $W^{k,2}(S, X)$. For a set $U \subset S \times X$ and $s \in S$ we write

$$U^s := \{x \in X \mid (s, x) \in U\} \qquad \text{and} \qquad \begin{array}{cccc} \epsilon^s & : & U^s & \to & U \\ & & x & \mapsto & (s, x). \end{array}$$

Lemma 4.4. *Given $g \in W^{k,2}(S, X)$, there is a $W^{k,2}$-diffeomorphism G between a neighborhood $U \subset S \times X$ of the graph of g and a neighborhood of the zero section in g^*TX such that*

i) $G(., g(.))$ is the zero section of g^*TX;

ii) $G^s = G \circ \epsilon^s$ maps U^s biholomorphically on a convex neighborhood of $0 \in T_{g(s)}X$;

iii) $dF^s_{g(s)}$ is the identity map.

Proof. We recall the argument from [L1] just pointing out the smoothness of G. Let (Ω_j, ϕ_j) be an atlas of the complex manifold X. Then the sets $S_j = g^{-1}(\Omega_j) \subset S$ form an open covering of S. Consider $U_j \subset S_j \times X$ a neighborhood of the graph of $g|_{S_j}$. We can construct locally the diffeomorphism G_j by

$$\begin{array}{cccc} G_j : & U_j & \longrightarrow & g^*TX \\ & (s, x) & \mapsto & \Big(s, (d\phi_j^{-1})_{\phi_j(g(s))}\big(\phi_j(x) - \phi_j(g(s))\big)\Big). \end{array}$$

Notice that G_j is of class $W^{k,2}$ because such is g. It remains to glue all the G_j. Take $\{\eta_j\}$ a $\mathcal{C}^k$-partition of unity subordinated to the covering $\{S_j\}$ and define $G(s, x) = \sum_j \eta_j(s) G_j(s, x)$ and choose the restriction of G to a suitable $U \subset \bigcup_j U_j$.

$\square$

For $g \in W^{k,2}(S, X)$ choose U and G as in the previous lemma. Those $h \in W^{k,2}(S, X)$ whose graph $\Gamma_h := \{(s, h(s)) \mid s \in S\}$ is contained in U form a neighborhood $\mathcal{U} \subset W^{k,2}(S, X)$.

$$\mathcal{U} := \left\{h \in W^{k,2}(S, X) \mid \Gamma_h \subset U\right\}.$$

For $h \in \mathcal{U}$ associate a section $\psi(h) \in W^{k,2}(g^*TX)$ by putting $\psi(h)(s) = G(s, h(s))$. Thus $\psi_{h,G}$ is a homeomorphism between $\mathcal{U}$ and an open set of $W^{k,2}(g^*TX)$. The link to another homeomorphism $\psi' = \psi'_{g',G'}$ associated with $g' \in W^{k,2}(S,X)$ and corresponding G' is by the fact that $G' \circ G^{-1}$ defines a bundle morphism between open fiber subundles of g^*TX and g'^*TX holomorphic on the fibers. Thus it induces the map $\psi_{g',G'} \circ \psi_{g,G}^{-1}$ between open subsets of $W^{k,2}(g^*TX)$ and $W^{k,2}(g'^*TX)$. So we can observe that $\psi_{g',G'} \circ \psi_{g,G}^{-1}$ is holomorphic. Thus all the charts $(\mathcal{U}, \psi)$ associated with different g, G are compatible and define a complex manifold structure on $W^{k,2}(S,X)$. Thus $W^{k,2}(S,X)$ has the structure of a complex Hilbert manifold.

Another way to understand this complex structure on $W^{k,2}(S,X)$ is to describe analytic disks in $W^{k,2}(S,X)$.

Lemma 4.5. *Let D and X be finite dimensional complex manifolds and let S be an n-dimensional compact real manifold without boundary. A mapping $F : D \times S \to X$ represents a holomorphic map from D to $W^{k,2}(S,X)$ (denoted by F_*) if and only if the following holds:*

i) for every $s \in S$ the map $F(\cdot, s) : D \to X$ is holomorphic;

ii) for every $z \in D$ one has $F(z, \cdot) \in W^{k,2}(S,X)$ and the

correspondence $D \ni z \to F(z,\cdot) \in W^{k,2}(S,X)$ is continuous with respect to the Sobolev topology on $W^{k,2}(S,X)$ and the standard topology on D.

Proof. The problem is local in D and therefore F can be supposed to be defined for z close to $z_0 \in D$ and $s \in S$ mapping every $\{z\} \times S$ to an neighborhood of 0 in a complex Hilbert space and $W^{k,2}(S, \big(F(z_0, \bullet)\big)^* TX)$. After this reduction the statement of this lemma becomes obvious. □

For more details we refer to [L1].

4.4 Loop spaces of Hartogs manifolds are Hilbert-Hartogs

In this chapter we shall prove the second main result of this thesis, namely the following theorem.

Theorem 4.2. *A generalized loop space of a q-Hartogs complex manifold is a q-Hartogs Hilbert manifold.*

Proof. The proof will be achieved in a number of steps.

Step 1. Localisation in holomorphic variable.

Since q-Hartogs extension property of a Hilbert manifold is equivalent to that of q-Levi it is enough to prove the latter. Let D be a domain in $\mathbb{C}^{q+1}$ with smooth

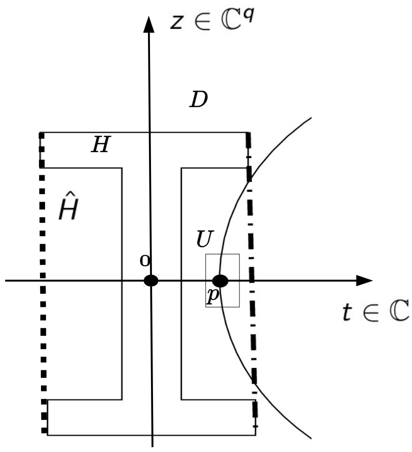


Figure 4.1: Hartogs figure near a pseudoconcave point

boundary, which is strongly pseudoconcave at its boundary point p. Let furthermore $F : D \times S \to X$ be a mapping to a q-Hartogs manifold such as in Lemma 4.5.

All we need is to find an extension of F to a mapping $\tilde{F} : U \times S \to X$, where U is a neighborhood of p, such that it obeys the conditions of Lemma 4.5 as well.

Take a q-Hartogs figure H in D near p such that its associated polydic $\hat{H}$ contains a neighborhood V of p and we take as U a polydisc $\Delta_R^{q+1}(p)$ such that its closure is contained in this V, see the picture 4.1. The natural coordinates on $\hat{H}$ we denote by (z, t). Since X is q-Hartogs we have that for every $s \in S$ mapping $F(\cdot, s) : D \to X$ holomorphically extends to $\hat{H}$, i.e. to a fixed neighborhood V of $\bar{U}$. Denote by $\tilde{F}$: $V \times S \to X$ this extension.

Step 2. Continuity in classical topology.

We shall prove that mapping $\tilde{F}$ is continuous with respect to the natural topologies on V, S and X. For that it is sufficient to prove that for any sequence $s_n \to s_0 \in S$ $F(*, s_n)$ converges uniformly on $\bar{U}$ to $F(*, s_0)$. Consider the analytic $(q+1)$-disc $\tilde{F}(\cdot, s_0) : \bar{U} \to X$ in X. Denote by $G_0 := \Gamma_{F(\cdot, s_0)}$ its graph in $V \times X$. Due to Lemma 3.1 all we need to prove is that for any sequence $s_n \in S$ converging to s_0 the graphs $G_n := \Gamma_{F(\cdot, s_n)}$ over $\bar{U}$ enter to a given neighborhood of G_0. By Royden's lemma there exists a polydisc neighborhood, say W, of the graph of $F(\cdot, s_0)$ over the whole polydisc $\hat{H}$. In fact over an any given relatively compact subpolydisc of $\hat{H}$ and so one may need to shrink H and therefore U on this step. For every $s \in S$ denote by $g(z, t, s) = (z, t, F(z, t, s))$ the mapping to the graph of $F(\cdot, s)$. We have a sequence of holomorphic mappings

$$g(\cdot, s_n)|_{\hat{H}\cap D} : \hat{H} \cap D \to W \cong \Delta^N \quad \text{for} \quad n >> 1,$$

uniformly converging to $g(\cdot, s_0)$ on compacts of $\hat{H} \cap D$. But then their holomorphic extensions to $\hat{H}$ will still take values in W and uniformly converge on compacts of $\hat{H}$ by maximum principle. Therefore graphs G_n converge over $\bar{U}$ to the graph G_0. The step is proved.

Step 3. Continuity in Sobolev topology.

Again fix some $s_0 \in S$ and take some $\epsilon > \epsilon_1 > 0$. Furthermore take a cut-off function ρ_0 equal to 1 in $B(s_0, \epsilon_1)$, non-negative everywhere and supported in $B(s_0, \epsilon)$. Here by $B(s_0, \epsilon)$ and $B(s_0, \epsilon_1)$ we denote the coordinate balls centered at $s_0 \in S$. Take $\epsilon > 0$ small enough to ensure that $\tilde{F}|_{\hat{H}\times B(s_0,\epsilon)}$ is contained in some coordinate chart X_0 of X and denote by α_0 the corresponding coordinate map to $\mathbb{C}^m$ where $m = \dim X$. Set $F_{\rho_0} = \rho_0 \left(\alpha_0 \circ \tilde{F}\right)$. Again in order to achieve that one should shrink H and therefore U if necessary.

Consider our map $F_{\rho_0} : \hat{H} \times B(s_0, \epsilon) \to \mathbb{C}^m$ and notice that:

i) for every $(z, t) \in H$ the map $F_{\rho_0}(z, t, *)$ is in $W^{k,2}(B(s_0, \epsilon), \mathbb{C}^m)$ and has compact support.

ii) $(z, t) \mapsto F_{\rho_0}(z, t, *)$ is a continuous map from H to $W^{k,2}(B(s_0, \epsilon), \mathbb{C}^m)$ with respect to the standard topology on H and Sobolev topology on $W^{k,2}(B(s_0, \epsilon), \mathbb{C}^m)$,

iii) moreover, for every $s \in B(s_0, \epsilon)$ mapping $F_{\rho_0}(z, t, s)$ is holomorphic on the associated polydisc $\hat{H}$ (not only on the Hartogs figure H).

These items hold true because a multiplication by a smooth cut-off function preserves the Sobolev class and continuity in Sobolev topology on H, where we had this continuity for F. Also, since this cut-off function depends only on the space variable s it doesn't spoils the holomorphicity in (z, t). According to Lemma 4.5 items *i)* and *ii)* mean that $F_{\rho_0}(z, t, *)$ is a holomorphic mapping from H to $W^{k,2}(B(s_0, \epsilon), \mathbb{C}^m)$, the latter being a complex Hibert space. Holomorphic mappings from the Hartogs figure $H = H_q^1(r)$ to a complex Hilbert space holomorphically extend to the associated polydisc $\hat{H}$. Denote for the moment by $\hat{F}_{\rho_0}$ this extension. And now remark that for every s our extension $\hat{F}_{\rho_0}(*, s)$ coincides on $\hat{H}$ with $F_{\rho_0}(*, s)$ as it is given in *iii)* . This is obvious since both $\hat{F}_{\rho_0}(*, s)$ and $F_{\rho_0}(*, s)$ are holomorphic on $\hat{H}$ for a fixed s and coincide on H. Therefore they coincide everywhere by the uniqueness theorem.

Remark 4.6. The fact that a holomorphic map $F : H(r) \to F$ with values in a complex Banach space F holomorphically extends to Δ^2 can be found in [Ra]. But in the case of a l^2 it is particularly simple. Take a composition $w_j \circ F$ of F with any coordinate function and extend it to Δ^2 by the classical Hartogs theorem. Then use the plurisubharmonicity of $\rho(w) := \sum_j |w_j|^2$ to prove that the extension

$$z \to \{(w_j \circ F)(z)\}_j$$

thus obtained is locally bounded in Δ^2.

Since ρ_0 is equal to 1 on $B(s_0, \epsilon_1)$ we proved that for $(z,t) \in \hat{H}$ the restriction $F(z,t,*)|_{B(s_0,\epsilon_1)} \in W^{k,2}(B(s_0,\epsilon_1), X_0)$ and it continuously depends on (z,t). Since S is compact one can cover it by a finite number of balls $\{B(s_j,\epsilon_1)\}_{s\in J}$ and take $\epsilon > \epsilon_1$ such that $\tilde{F}|_{\hat{H}\cap B(s_j,\epsilon)}$ takes its values in some chart X_j of X for any $j \in J$. Take a cut-off function ρ_j equal to 1 in $B(s_j,\epsilon_1)$ and supported in $B(s_j,\epsilon)$ and set $F_{\rho_j} = \rho_j.(\alpha_j \circ \tilde{F})$, where $\alpha_j : X \to \mathbb{C}^m$ is a coordinate map. As above we obtain continuity of every F_{ρ_j} and therefore of $F(z,t,*)|_{B(s_j,\epsilon_1)}$ together with their holomorphicity and therefore continuity on $\hat{H}$.

By the defintion of Sobolev topology, see Definition 4.6, we get that for all $(z,t) \in \Delta_R^q \times \Delta_R$, $F(z,t,*)$ is in $W^{k,2}(S,X)$ and the correspondance $(z,t) \to F(z,t,*)$ is continuous in Sobolev topology. Theorem is proved.

□

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