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Hyperbolic surfaces with spikes and decorated Margulis space-times

Surfaces hyperboliques ciliées et espaces-temps de Margulis décorés

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"The opposite of depression is not happiness, but vitality".
— Andrew Solomon

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Introduction

Historical context

Bieberbach proved (1910-1912) that any group Γ of affine isometries of the n-dimensional Euclidean space $\mathbb{R}^n$ that acts properly discontinuously on $\mathbb{R}^n$ contains a finite-index subgroup isomorphic to $\mathbb{Z}^m$, $m \leq n$. Moreover, the quotient $\mathbb{R}^n/\Gamma$ is compact if and only if $m = n$. In 1964, Auslander proposed the following conjecture:

Conjecture 1 (Auslander)**.** If $\Gamma \subset \mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$ is a finitely generated group that acts on $\mathbb{R}^n$ properly discontinuously and cocompactly, then Γ is virtually solvable.

The conjecture has been proven to be true up to $n = 6$. In 1977, Milnor asked the following question [16]:

Q: Is the conjecture true if the cocompactness condition is dropped?

Meanwhile, in 1972, Tits proved that

Theorem (Tits alternative)**.** *Let* $\Gamma \subset \mathrm{GL}(n, \mathbb{F})$ *be a finitely generated group, where* $\mathbb{F}$ *is a field. Then* Γ *is either virtually solvable or it contains a free group of rank >1.*

Hence the Tits alternative implies that the answer to Milnor's question is negative if and only if there exists a properly discontinuous affine action of a free group (of rank >1).

Margulis spacetimes. In 1983, Margulis came up with examples for $n = 3$. These were complete non-compact Lorentzian manifolds, called *Margulis spacetimes*, obtained as a quotient of the (2,1)-Minkowski space $\mathbb{R}^{2,1}$ by a free group Γ, acting properly discontinuously by orientation-preserving affine isometries. The group of orientation-preserving affine isometries of $\mathbb{R}^{2,1}$ is given by $\mathrm{SO}(2,1) \ltimes \mathbb{R}^3$. The special linear group $\mathrm{SO}(2,1)$ is the isometry group of the two-dimensional hyperbolic space $\mathbb{H}^2$, which lies in the Minkowski space. Its Lie algebra $\mathfrak{so}_{2,1}$, equipped with its Killing form, is isomorphic to $\mathbb{R}^{2,1}$. The linear action of $\mathrm{SO}(2,1)$ on $\mathbb{R}^{2,1}$ coincides with its adjoint action on $\mathfrak{so}_{2,1}$. Consequently, the tangent bundle $\mathrm{T}(\mathrm{SO}(2,1))$ is isomorphic to $\mathrm{SO}(2,1) \ltimes \mathbb{R}^3$. We shall denote by G the isomorphic groups $\mathrm{SO}(2,1), \mathrm{PGL}(2,\mathbb{R})$ and by $\mathfrak{g}$, their Lie algebra.

Recent developments

Affine deformations. Consider the representation $\rho_0 : \Gamma \hookrightarrow G \ltimes \mathfrak{g} \simeq \mathrm{T}(G)$ of a discrete not virtually solvable Γ acting properly discontinuously on $\mathbb{R}^{2,1}$, like in the examples of Margulis. Fried and Goldman [8] proved that by projecting Γ onto its first coordinate, we virtually get the holonomy representation $\rho : \pi_1(S) \to G$ of a finite-type complete hyperbolic surface S. The projection onto the second coordinate $u : \Gamma \to \mathfrak{g}$ is a ρ-cocycle: for every $\gamma, \gamma' \in \Gamma$, u satisfies $u(\gamma\gamma') = u(\gamma) + \rho(\gamma) \cdot u(\gamma')$. It gives an infinitesimal deformation of ρ. The group Γ can thus be written as $\Gamma^{(\rho,u)} := \{(\rho(\gamma), u(\gamma)) \mid \gamma \in \pi_1(S)\}$, which gives an *affine deformation* of ρ. In the paper [9], the authors Goldman, Labourie and Margulis have studied affine deformations of free, discrete subgroups of G. An infinitesimal deformation u of ρ is said to be *proper* if $\Gamma^{(\rho,u)}$ acts properly discontinuously on $\mathbb{R}^{2,1}$. It has been proved in the paper [9] that for ρ convex cocompact, the corresponding u is proper if and only if u or $-u$ uniformly lengthens all closed geodesics:

$$\inf_{\gamma \in \Gamma \smallsetminus \{Id\}} \frac{\mathrm{d}l_\gamma(\rho)(u)}{l_\gamma(\rho)} > 0 \tag{1}$$

where l_γ is the length function. In both cases, the set of all such infinitesimal deformations forms an open, convex cone; the cone corresponding to the first case is called the *admissible cone*.

Strip deformations of compact surfaces. In the paper [5], the authors Danciger-Guéritaud-Kassel study admissible deformations of finite-type hyperbolic surfaces with non-empty boundary and without punctures, using *strip deformations*, first introduced by Thurston in [19]. A strip is the region in $\mathbb{H}^2$ bounded by two geodesics whose closures are disjoint. An arc on such a surface S is an embedding of $[0, 1]$ into S with its endpoints on its boundary ∂S such that it is not isotopic to a part of the boundary. Using the isotopy classes of these arcs, one can construct a simplicial complex called the *arc complex* which depends only on the topology of the surface. A k-simplex of this complex is generated by the isotopy classes of a family of $k + 1$ pairwise disjoint and distinct arcs. The pruned arc complex is a subspace formed by taking the union of the interiors of all those simplices σ such that the arcs corresponding to the 0-skeleton of σ decompose the surface into topological disks. A strip deformation is the process of cutting the surface along an embedded arc and gluing in a strip, without shearing. The authors uniquely realised (Theorem 4.4.1) an admissible deformation of the surface by performing strip deformations along positively weighted arcs, corresponding to a point in the pruned arc complex.

Drumm [6] constructed fundamental domains of some Margulis spacetimes with a convex cocompact linear part using specially crafted piecewise linear surfaces called *crooked planes*. The Crooked Plane Conjecture says that every Margulis spacetime is amenable to such a treatment. Charette, Drumm and Goldman proved this conjecture for rank two free groups in [3]. The general case (with convex cocompact linear part) follows from [5] which provides a dictionary between strip deformations and crooked planes.

Main results of the thesis

Surfaces with undecorated spikes. The main aim of this thesis is to generalise the parametrisation to include (possibly non-oriented) hyperbolic surfaces with *spikes* on their boundary. These complete non-compact surfaces are limits of a compact surface with convex polygonal boundary where the vertices become ideal. The admissible cone in this case is simply defined to be the set of all infinitesimal deformations that induce an admissible deformation of the convex core $S_\heartsuit$ of the spiked surface. This cone is an affine $\mathbb{R}^Q$-bundle over the admissible cone of $S_\heartsuit$, where Q is the total number of spikes.

The arc complex of a surface with spikes is spanned by the isotopy classes of arcs; this time we allow the arcs that separate off a disk from the surface, as long as the disk contains at least two spikes. In other words, we rule out the arcs that are isotopic to a horoball neighbourhood of a spike. Again, we define the pruned arc complex to be a subspace of the arc complex formed by taking the union of all those simplices σ such that the arcs corresponding to the 0-skeleton of σ decompose the surface into topological disks.

We parametrise (Theorem 6.1.1) the admissible cone using the pruned arc complex by performing strip deformations along a family of weighted embedded arcs that decompose the surface in topological disks, like in the compact case.

Theorem. *Let S_{sp} be a hyperbolic surface with spikes equipped with a metric $m \in \mathfrak{D}(S_{sp})$. Let $\widehat{\mathcal{A}}(S_{sp})$ be its pruned arc complex. Choose m-geodesic representatives from the isotopy classes of arcs. Then, the projectivised infinitesimal strip map $\mathbb{P}f : \widehat{\mathcal{A}}(S_{sp}) \longrightarrow \mathbb{P}^+(T_m\mathfrak{D}(S_{sp}))$ is a homeomorphism on its image $\mathbb{P}^+(\Lambda(m))$, where $\Lambda(m)$ denotes the admissible cone over m.*

Surfaces with decorated spikes. Next, we decorate all the spikes of such a surface with pairwise disjoint horoballs. A *horoball connection* is a geodesic arc on the surface that joins two decorated spikes. Its length is given by the geodesic segment intercepted by the two horoballs decorating its endpoints.

We define the *admissible cone* of a decorated surface to be the set of all infinitesimal deformations that uniformly lengthen every horoball connection. More precisely, every element (m, v) in the tangent space over a decorated metric m satisfies:

$$\inf_{\beta \in \mathcal{H}} \frac{\mathrm{d}l_\beta(m)(v)}{l_\beta(v)} > 0,$$

where $\mathcal{H}$ is the set of all horoball connections. Note that an admissible deformation also uniformly lengthens every closed loop of the surface, i.e., it satisfies (1), because we can always find a horoball connection that remains inside a very small neighbourhood of such a loop for arbitrarily long time and has bounded length outside. Thus the admissible cone in this case can be seen as a $\mathbb{R}^{2Q}$-bundle over the admissible cone of the convex core $S_\heartsuit$, whose fibres are open convex subsets, where Q is the total number of spikes.

On the decorated surface, we consider more arcs than in the undecorated case. In addition to the arcs already mentioned, we allow two new types: finite arcs that are isotopic to a horoball neighbourhood of a spike, and infinite arcs that are embeddings of $[0, \infty)$ such that the finite end is on the boundary and the infinite end converges to a spike. This time the pruned arc complex is defined to be the subspace of the arc complex formed by taking the union of all those simplices σ such that the arcs corresponding to the 0-skeleton of σ decompose the surface into topological disks with at most one spike.

The strip added along an infinite arc is the region in $\mathbb{H}^2$ bounded by two geodesics with the spike as the common endpoint. A strip deformation along a finite arc is defined as in the previous case. Again, we give a parametrisation (Theorem 6.1.2) of the admissible cone of a surface with decorated spikes, using its pruned arc complex.

Theorem. *Let S^h_{sp} be a hyperbolic surface with decorated spikes equipped with a decorated metric $m \in \mathfrak{D}(S^h_{sp})$. Let $\widehat{\mathcal{A}}(S^h_{sp})$ be its pruned arc complex. Choose m-geodesic representatives from the isotopy classes of arcs. Then, the projectivised infinitesimal strip map $\mathbb{P}f : \widehat{\mathcal{A}}(S^h_{sp}) \longrightarrow \mathbb{P}^+(T_m\mathfrak{D}(S^h_{sp}))$ is a homeomorphism onto its image $\mathbb{P}^+(\Lambda(m))$, where $\Lambda(m)$ denotes the admissible cone over m.*

Decorated Margulis Spacetimes. In the final chapter, we interpret admissible deformations of surfaces with decorated spikes as Margulis spacetimes with a certain type of decoration by lightlike lines (photons), one photon per spike. In this context, the above theorem provides fundamental domains of the Margulis spacetimes, adapted to the photons.

Full parametrisation. Ideal polygons and once-punctured polygons are two types of non-compact hyperbolic surfaces in which every simple closed curve is either homotopic to a point or to a puncture. Their arc complexes are known to be spheres of dimension one less than the dimension of their respective deformation spaces. We show (Theorems 5.0.1 and 5.0.2) that the arc complex parametrises the entire positively projectivised deformation space in these cases.

Theorem. *Let Π be the topological surface of an ideal polygon $\Pi_n^{\pentagon}$ $(n \geq 4)$ or a once punctured polygon $\Pi_n^{\odot}$ $(n \geq 2)$. Let $m \in \mathfrak{D}(\Pi)$ be a metric. Choose m-geodesic representatives from the isotopy classes of arcs. Then, the projectivised infinitesimal strip map $\mathbb{P}f : \mathcal{A}(\Pi) \longrightarrow \mathbb{P}^+(T_m\mathfrak{D}(\Pi))$ is a homeomorphism.*

Decorated Polygons A convex *decorated* polygon in $\mathbb{H}^2$ is a generalisation of a compact hyperbolic polygon whose vertices are allowed to be *hyperbolic* (truncations of hyper-ideal points) and *parabolic* (ideal points decorated with horoballs). We show the deformation space of such a polygon to be homeomorphic to an open ball. Finally, we prove (Theorem 5.0.3) that the subset of the space of all infinitesimal deformations, consisting of those which lengthen every diagonal and edge, is parametrised by the pruned arc complex of the surface.

Theorem. *Let $\Pi_n^{\circledwedge}$ $(n \geq 3)$ be a decorated polygon equipped with a hyperbolic metric $m \in \mathfrak{D}(\Pi_n^{\circledwedge})$. Choose m-geodesic representatives from the isotopy classes of arcs. Then the infinitesimal strip map $\mathbb{P}f$, when restricted to the pruned arc complex $\widehat{\mathcal{A}}(\Pi_n^{\circledwedge})$, is a homeomorphism onto its image $\mathbb{P}^+(\Lambda(m))$, where $\Lambda(m)$ denotes the admissible cone over m.*

As a consequence of this, we get that the pruned arc complex of such a polygon is homeomorphic to an open ball.

Theorem. *The pruned arc complex $\widehat{\mathcal{A}}(\Pi_n^{\circledwedge})$ of a decorated n-gon $\Pi_n^{\circledwedge}$ $(n \geq 3)$ is homeomorphic to an open ball of dimension $2n - 4$.*

Plan of the thesis

The thesis is divided into seven chapters. Chapter 1 recapitulates the necessary vocabulary from hyperbolic, Lorentzian and projective geometry. Chapter 2 introduces every type of surface mentioned above along with their deformation spaces and admissible cones. In Chapter 3, we discuss the arcs and the arc complexes of the different types of surfaces and study their topology. Chapter 4 gives the definitions of the various strip deformations along with examples. It also contains some estimations that will be required in the proofs. Chapter 5 contains the proofs of the parametrisation theorems for ideal, punctured, decorated polygons. In Chapter 6, we prove the parametrisation theorems for general surfaces with decorated and undecorated spikes. Finally, Chapter 7 talks about decorated Margulis spacetimes and how it is determined by an admissible deformation.

Chapter 1

Preliminaries

1.1 Pseudo-Riemannian manifolds

1.1.1 Scalar products

Let $\mathbb{R}^n$ be the usual n-dimensional real vector space and let $\langle \cdot, \cdot \rangle_{p,q}$ be the following non-degenerate symmetric bilinear form of signature $(p, q) \in \mathbb{N}^2$ with $p + q = n$:

for $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$,

$$\langle \mathbf{x}, \mathbf{y} \rangle_{p,q} = \sum_{i=1}^{p} x_i y_i - \sum_{j=p+1}^{p+q} x_j y_j.$$

The associated quadratic form is denoted by ${\| \cdot \|_{p,q}}^2$. The space $\mathbb{R}^n$ together with this quadratic form is denoted by $\mathbb{R}^{p,q}$. When $q = 1$, the bilinear form is called *Lorentzian scalar product* and the space $\mathbb{R}^{p,1}$ is known as the *Minkowski space* of dimension $p + 1$. We are primarily interested in the space $\mathbb{R}^{2,1}$ which we shall discuss further in the next section.

The isometry group of $\mathbb{R}^{p,q}$ is given by $\mathrm{O}(p, q) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A\mathbb{I}_{p,q}A^t = \mathbb{I}_{p,q}, \}$ where

$$\mathbb{I}_{p,q} = \begin{bmatrix} \mathbb{I}_p & 0 \\ 0 & -\mathbb{I}_q \end{bmatrix}.$$

A *totally positive* (resp. *negative*) subspace of $\mathbb{R}^{p,q}$ is a vector subspace of $\mathbb{R}^{p+q}$ on which the restriction of the quadratic form is positive (resp. negative) definite.

A *maximal* totally positive or negative subspace has the largest possible dimension, along all such subspaces. A maximal positive (resp. negative) subspace of $\mathbb{R}^{p,q}$ has dimension p (resp. q). The set of all maximal totally positive (resp. negative) subspaces is denoted by $\mathcal{P}$ (resp. $\mathcal{N}$).

In $\mathbb{R}^{p,q}$, it is possible to consistently orient all maximal totally positive subspaces. This is true for all maximal totally negative subspaces as well.

The group of orientation preserving isometries $\mathrm{Isom}^+(\mathbb{R}^{p,q})$ is given by the subgroup

$$\mathrm{SO}(p, q) := \mathrm{O}(p, q) \cap \mathrm{SL}(n, \mathbb{R}).$$

An element of this group either preserves or reverses the two consistent orientations on the two sets $\mathcal{P}, \mathcal{N}$. This group has two connected components — the one that preserves the orientations of these two sets individually is denoted as $\mathrm{SO}^+(p, q)$.

1.1.2 Manifolds of constant sectional curvature

A *pseudo-Riemannian* manifold M is a differentiable manifold equipped with a smooth metric tensor g such that $g : T_xM \times T_xM \longrightarrow \mathbb{R}$, is a non-degenerate scalar product, for every $x \in M$. The signature of this scalar product is the same for all tangent spaces and is called the *metric signature*. Clearly, all Riemannian manifolds are pseudo-Riemannian. A *Lorentzian* manifold is a pseudo-Riemannian manifold in which the metric tensor is a Lorentzian scalar product on every tangent space. Like in the case of Riemannian manifolds, a pseudo-Riemannian manifold comes with a Levi-Civita connection that lets us define the curvature tensor and geodesics.

Example 1.1.1.
- The space $\mathbb{R}^{p,q}$ is a complete flat pseudo-Riemannian manifold. In particular, the Minkowski space $\mathbb{R}^{p,1}$ is a Lorentzian manifold.
- Consider the space $\mathbb{R}^{p,q+1}$. Define the *generalised hyperbolic space* as the subspace

  $$\mathbb{H}^{p,q} := \{\mathbf{x} \in \mathbb{R}^{p,q+1} \mid \|\mathbf{x}\|^2_{p,q+1} = -1\}.$$

  The metric induced by $\|\cdot\|_{p,q}$ on this space is has signature (p, q). We have the following special cases:

  - $p = 0$: This is the unit sphere of dimension q, denoted by $\mathbb{S}^q$; the quadratic form induced by $\|\cdot\|_{0,q+1}$ is minus one times the usual metric on the sphere.
  - $q = 0$: This is a two-sheeted hyperboloid. The future-pointing sheet is the *classical hyperbolic space*, $\mathbb{H}^p$.
  - $p = 1$: This is a one-sheeted hyperboloid, which we projectivise, like in the previous case, to get the *de Sitter space*, dS^{q+1} with minus its usual Lorentz metric.
  - $q = 1$: This is a connected quadric, which we projectivise, like in the previous case, to get the *Anti-de Sitter space*, AdS^{p+1}.
- The de Sitter space dS^q (resp. the Anti-de Sitter space AdS^q) is a complete Lorentzian manifold with constant positive (resp. negative) sectional curvature.

The space $\mathbb{R}^{p,q}$ can be regarded as an affine space with its group of affine transformations

$$\mathrm{Aff}(\mathbb{R}^{p,q}) := \mathrm{O}(p, q) \ltimes \mathbb{R}^{p+q}.$$

An *affine* pseudo-Riemannian manifold is a differentiable manifold obtained by quotienting $\mathbb{R}^{p,q}$ with a discrete subgroup of $\mathrm{Aff}(\mathbb{R}^{p,q})$ that acts properly discontinuously on $\mathbb{R}^{p,q}$.

1.2 More on Minkowski space

In this section, we shall further study Minkowski space but shall restrict ourselves to the case where $p = 2$. In the rest of the thesis, we shall refer to its norm and scalar product as $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.

Vectors. There is the following classification of points in the Minkowski space: a non-zero vector $\mathbf{v} \in \mathbb{R}^{2,1}$ is said to be

- *space-like* if and only if $\|\mathbf{v}\|^2 > 0$,
- *light-like* if and only if $\|\mathbf{v}\|^2 = 0$,
- *time-like* if and only if $\|\mathbf{v}\|^2 < 0$.

A vector $\mathbf{v}$ is said to be *causal* if it is time-like or light-like. A causal vector $\mathbf{v} = (x, y, z)$ is called *positive* (resp. *negative*) if $z > 0$ (resp. $z < 0$). Note that by definition of the norm, every causal vector is either positive or negative. The set of all light-like points forms the *light-cone*, denoted by

$$L := \{\mathbf{v} = (x, y, z) \in \mathbb{R}^{2,1} \mid x^2 + y^2 - z^2 = 0\}.$$

The *positive* (resp. *negative*) cone is defined as the set of all positive (resp. *negative*) light-like vectors.

Subspaces. A vector subspace W of $\mathbb{R}^{2,1}$ is said to be

- *space-like* if $W \cap C = \{(0, 0, 0)\}$,
- *light-like* if $W \cap C = \operatorname{span}\{\mathbf{v}\}$ where $\mathbf{v}$ is light-like,
- *time-like* if W contains at least one time-like vector.

A subspace of dimension one is going to be called a line and a subspace of dimension two a plane. The adjective "affine" will be added before the words "line" and "plane" when we are referring to some affine subspace of the corresponding dimension.

Duals. Given a vector $\mathbf{v} \in \mathbb{R}^{2,1}$, its dual with respect to the bilinear form of $\mathbb{R}^{2,1}$ is denoted $\mathbf{v}^\perp$. For a light-like vector $\mathbf{v}$, the dual is given by the light-like hyperplane tangent to C along $\operatorname{span}\{\mathbf{v}\}$. For a space-like vector $\mathbf{v}$, the dual is given by the time-like plane that intersects C along two light-like lines, respectively generated by two light-like vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ such that $\operatorname{span}\{\mathbf{v}\} = \mathbf{v_1}^\perp \cap \mathbf{v_2}^\perp$. Finally, the dual of a time-like vector $\mathbf{v}$ is given by a space-like plane. One way to construct it is to take two time-like planes W_1, W_2 passing through $\mathbf{v}$. Then the space $\mathbf{v}^\perp$ is the vectorial plane containing the space-like lines $W_1^\perp$ and $W_2^\perp$.

1.3 The Hyperbolic Plane

In this section we will discuss the hyperbolic plane and notions related to it that will be used extensively in the following chapters.

1.3.1 The different models

There are different models for the hyperbolic plane. Each one has its own advantages and disadvantages. We shall be using the hyperboloid model inside the Minkowski space, two disk models and the upper half-plane model.

Hyperboloid model. The classical hyperbolic space of dimension two $\mathbb{H}^2$ can be identified with the upper sheet of the two-sheeted hyperboloid $\{\mathbf{v} = (x, y, z) \in \mathbb{R}^{2,1} \mid \|\mathbf{v}\|^2 = -1\}$, along with the restriction of the bilinear form. It is the unique (up to isometry) complete simply-connected Riemannian 2-manifold of constant curvature equal to -1. Its isometry group is isomorphic to $\mathrm{SO}(2,1)$ and the identity component $\mathrm{SO}^0(2,1)$ of this group forms the group of its orientation-preserving isometries; they preserve each of the two sheets of the hyperboloid individually. If the hyperbolic distance between two points $\mathbf{u}, \mathbf{v} \in \mathbb{H}^2$ is denoted by $d_{\mathbb{H}^2}(\mathbf{u}, \mathbf{v})$, then $\cosh d_{\mathbb{H}^2}(\mathbf{u}, \mathbf{v}) = -\langle \mathbf{u}, \mathbf{v} \rangle$. The geodesics of this model are given by the intersections of time-like hyperplanes with $\mathbb{H}^2$.

Klein's disk model. This model is the projectivisation of the hyperboloid model.

Let $\mathbb{P} : \mathbb{R}^{2,1} \smallsetminus \{\mathbf{0}\} \longrightarrow \mathbb{R}\mathrm{P}^2$ be the projectivisation of the Minkowski space. The projective plane $\mathbb{R}\mathrm{P}^2$ can be considered as the set $A \cup \mathbb{R}\mathbb{P}^1$, where $A := \{(x, y, 1) \,|\, x, y \in \mathbb{R}\}$ is an affine chart and the one-dimensional projective space represents the line at infinity, denoted by $\overleftrightarrow{l_\infty}$. The $\mathbb{P}$-image of a point $\mathbf{v} \in \mathbb{R}^{2,1}$ is denoted by $[\mathbf{v}]$. A line in A, denoted by $\overleftrightarrow{l}$, is defined as $A \cap V$ where V is a two-dimensional vector subspace of $\mathbb{R}^{2,1}$, not parallel to A.

In the affine chart A, the light cone is mapped to the unit circle and the hyperboloid is embedded onto its interior. This is the Klein model of the hyperbolic plane, denoted by $\mathbb{D}$ and whose boundary at infinity, denoted by $\partial_\infty \mathbb{D}$ is the unit circle. This model is non-conformal. The geodesics are given by open finite Euclidean straight line segments, denoted by l, lying inside $\mathbb{D}$, such that the endpoints of the closed segment $\bar{l}$ lie on $\partial_\infty \mathbb{D}$. The distance metric is given by the Hilbert metric $d_{\mathbb{D}}(w_1, w_2) = \frac{1}{2} \log[p, w_1; w_2, q]$, where p and q are the endpoints of $\bar{l}$, l being the unique hyperbolic geodesic passing through $w_1, w_2 \in \mathbb{D}$, and the cross-ratio $[a, b; c, d]$ is defined as $\frac{(c-a)(d-b)}{(b-a)(d-c)}$. The group of orientation-preserving isometries is identified with $PSU(1,1)$. A point p is called *real* (*ideal*, *hyperideal*) if $p \in \mathbb{D}$ (resp. $p \in \partial_\infty \mathbb{D}$, $p \in \overleftrightarrow{l} \cup A \backslash \overline{\mathbb{D}}$).

The dual of $\overleftrightarrow{l_\infty}$ is the point $(0, 0, 1)$ in A. The dual of any other projective line $\overleftrightarrow{l} = A \cap V$ is given by the point $A \cap V^\perp$. The dual $p^\perp$ of a point $p \in \mathbb{R}\mathrm{P}^2$ is the projective line $A \cap \operatorname{span}\{p\}^\perp$. If l is a hyperbolic geodesic, then $l^\perp$ is defined to be $\overleftrightarrow{l}^\perp$; it is given by the intersection point in $\mathbb{R}\mathrm{P}^2$ of the two tangents to $\partial_\infty \mathbb{D}$ at the endpoints of $\bar{l}$.

Notation: We shall use the symbol $\cdot^\perp$ for referring to the duals of both linear subspaces as well as their projectivisations.

Poincaré's unit disk model. Again, we start with the unit disk, but this time we endow it with the metric tensor $g = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$. This is a conformal model. Like in the previous case, the boundary at infinity $\partial_\infty \mathbb{D}$ is given by the unit circle.The geodesics are the diameters of the unit circle and arcs of circles that intersect the unit circle perpendicularly.

Upper Half-plane Model. The subset $\{z = x + iy \in \mathbb{C} \,|\, y > 0\}$ of the complex plane is the upper half-space model of the hyperbolic space of dimension 2, denoted by $\mathbb{U}$. The geodesics are given by semi-circles whose centres lie on $\mathbb{R}$ or straight lines that are perpendicular to $\mathbb{R}$. We shall call the former as *horizontal* and the latter as *vertical* geodesics. The boundary at infinity $\partial_\infty \mathbb{U}$ is given by $\mathbb{R} \cup \{\infty\}$. The orientation-preserving isometry group is given by $\mathrm{PSL}(2, \mathbb{R})$ that acts by Möbius transformations on $\mathbb{U}$.

Notation: We shall denote by G the isomorphic groups $\mathrm{Isom}(\mathbb{H}^2), \mathrm{SO}(2,1), \mathrm{PGL}(2, \mathbb{R})$ and by $\mathfrak{g}$ the Lie algebra of G.

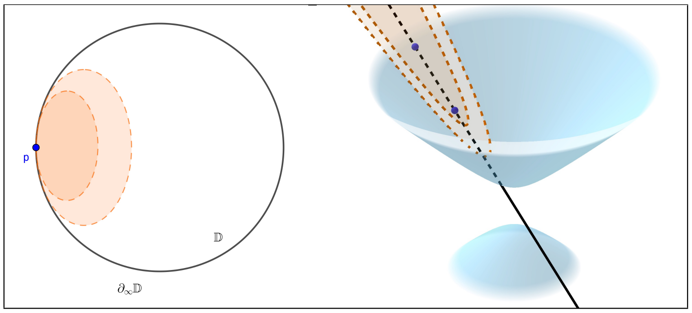


Figure 1.1: Concentric horoballs

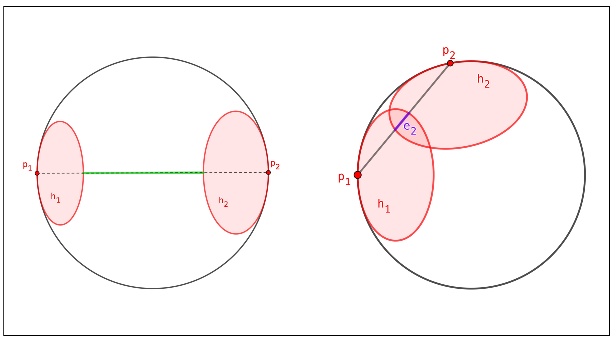


Figure 1.2: Length of horoball connections

1.4 Horoballs and decorated geodesics

An *open horoball* h based at $p \in \partial_\infty \mathbb{D}$ is the projective image of $H(\mathbf{v}) = \{\mathbf{w} \in \mathbb{H}^2 \mid \langle \mathbf{w}, \mathbf{v} \rangle > -1\}$ where $\mathbf{v}$ is a future-pointing light-like point in $\mathbb{P}^{-1}\{p\}$. If $k \geq k' > 0$, then $H(k\mathbf{v}_0) \subset H(k'\mathbf{v}_0)$. See Fig. 1.1.

The boundary of an open horoball $h(p) \subset \mathbb{D}$ based at $p \in \partial_\infty \mathbb{D}$ is called a *horocycle*. It is the projective image of the set

$$h(\mathbf{v}) := \{\mathbf{w} \in \mathbb{H}^2 \mid \langle \mathbf{w}, \mathbf{v} \rangle = -1\}.$$

In the projective disk model, it is a Euclidean ellipse inside $\mathbb{D}$, tangent to $\partial_\infty \mathbb{D}$ at p. In the upper half-plane model, horocycles are either Euclidean circles tangent to a point on the real line or horizontal lines which are horocycles based at ∞. In the Poincaré disk model, a horocycle is an Euclidean circle tangent to $\partial_\infty \mathbb{D}$ at $[p]$. A geodesic, one of whose endpoints is the centre of a horocycle, intersects the horocycles perpendicularly. Note that any horoball is completely determined by a future-pointing light-like vector in $\mathbb{R}^{2,1}$ and vice-versa. From now onwards, we shall use either of the notations introduced above to denote a horoball. Finally, the set of all horoballs of $\mathbb{H}^2$ forms an open cone (the positive light cone).

Given an ideal point $p \in \partial_\infty \mathbb{D}$, a *decoration* of p is the specification of an open horoball centred at p. A geodesic, whose endpoints are decorated, is called a *horoball connection*. The following definition is due to Penner [18].

The *length* of a horoball connection joining two horoballs $\mathbf{v}_1, \mathbf{v}_2$ is given by

$$l := \frac{1}{2}\ln(-\frac{\langle \mathbf{v}_1, \mathbf{v}_2 \rangle}{2}).$$

It is the signed length of the geodesic segment intercepted by the corresponding horocycles. In particular, is the horoballs are not disjoint, then the length of the horoball connection is negative.

1.5 Killing Vector Fields of $\mathbb{H}^2$

The Minkowski space $\mathbb{R}^{2,1}$ is isomorphic to $(\mathfrak{g}, \kappa)$ where $\mathfrak{g}$ is the Lie algebra of $G := \mathrm{PGL}(2, \mathbb{R})$ and κ is its Killing form, via the following map:

$$\mathbf{v} = (x, y, z) \mapsto V = \begin{pmatrix} y & x+z \\ x-z & -y \end{pmatrix}.$$

The Lie algebra $\mathfrak{g}$ is also isomorphic to the set $\mathscr{X}$ of all Killing vector fields of $\mathbb{H}^2$:

$$V \mapsto \left[\begin{array}{lll} X_v : & \mathbb{U} \longrightarrow & \mathrm{T}\mathbb{U} \\ & \mathbf{p} \mapsto & \frac{d}{dt}(e^{tV} \cdot \mathbf{p})|_{t=0} \end{array} \right]$$

Next, one can identify $\mathbb{R}^{2,1}$ with $\mathscr{X}$ via the map:

$$\mathbf{v} \mapsto \left[\begin{array}{lll} X_v : & \mathbb{H}^2 \longrightarrow & \mathrm{T}\mathbb{H}^2 \\ & p \mapsto & \mathbf{v} \wedge \mathbf{p} \end{array} \right]$$

where $\wedge$ is the Minkowski cross product:

$$(x_1, y_1, z_1) \wedge (x_2, y_2, z_2) := (-y_1 z_2 + z_1 y_2, -z_1 x_2 + x_1 z_2, x_1 y_2 - y_1 x_2).$$

Finally, in the upper half space model $\mathbb{U}$, one can identify $\mathscr{X}$ with the real vector space $\mathbb{R}_2[z]$ of polynomials of degree at most 2:

$$P(\cdot) \mapsto \left[z \mapsto P(z)\frac{\partial}{\partial z} \right]$$

The discriminant of a polynomial in $\mathbb{R}_2[z]$ corresponds to the quadratic form $\| \cdot \|$ in $\mathbb{R}^{2,1}$. So the nature of the roots of a polynomial determines the type of the Killing vector field. In particular, when

- $P(z) = 1$, the corresponding Killing vector field is parabolic, fixing ∞;
- $P(z) = z$, the corresponding Killing vector field is hyperbolic, fixing $0, \infty$;
- $P(z) = z^2$, the corresponding Killing vector field is parabolic, fixing 0.

Properties 1.5.1. Using these isomorphisms, we have that

- A spacelike vector $\mathbf{v}$ corresponds, in $\mathscr{X}$, to an infinitesimal hyperbolic translation whose axis is given by $\mathbf{v}^{\perp} \cap \mathbb{H}^2$. If $\mathbf{v}^+$ and $\mathbf{v}^-$ are respectively its attracting and repelling fixed points in C^+, then $(\mathbf{v}^-, \mathbf{v}, \mathbf{v}^+)$ are positively oriented in $\mathbb{R}^{2,1}$.

- A lightlike vector $\mathbf{v}$ corresponds, in $\mathscr{X}$, to an infinitesimal parabolic element that fixes the light-like line span $\{\mathbf{v}\}$.
- A timelike vector $\mathbf{v}$ corresponds, in $\mathscr{X}$, to an infinitesimal rotation of $\mathbb{H}^2$ that fixes the point $\frac{\mathbf{v}}{\sqrt{-\|\mathbf{v}\|}}$ in $\mathbb{H}^2$.

Properties 1.5.2.

1. Given a light-like vector $\mathbf{v} \in \mathbb{R}^{2,1}$, the set of all Killing vector fields that fix span $\{\mathbf{v}\}$ is given by its dual $\mathbf{v}^{\perp}$. In $\mathbb{R}\mathrm{P}^2$, the set of projectivised Killing vector fields that fix $[\mathbf{v}] \in \partial_\infty \mathbb{D}$ is given by the tangent line at $[\mathbf{v}]$.
2. The set of all Killing vector fields that fix a given ideal point $p \in \partial_\infty \mathbb{D}$ and a horocycle in $\mathbb{D}$ with centre at p is given by span $\{\mathbf{v}\}$, where $\mathbf{v} \in \mathbb{P}^{-1}(p)$ in $\mathbb{R}^{2,1}$.
3. The set of all Killing vector fields that fix a given hyperbolic geodesic l in $\mathbb{D}$ is given by $\mathbb{P}^{-1}(l^{\perp})$.

1.6 Some Useful Results

1.6.1 Projective Geometry

Definition 1.6.1. Let $\bar{l}$ be a projective line segment contained in $\overline{\mathbb{D}}$ with endpoints, denoted by A, B. Then the two projective triangles formed by $A^{\perp}, B^{\perp}$ and $\overleftrightarrow{l}$, with their disjoint interiors intersecting $\mathbb{D}$, are said to be *based* at $\bar{l}$.

Properties 1.6.2. Let $\bar{l}$ be a projective line segment contained in $\overline{\mathbb{D}}$. Then, any projective line $\overline{l'}$ that intersects $\mathbb{D}$, is disjoint from $\bar{l}$ if and only if its dual $l'^{\perp}$ is a space-like point contained in the interior of the bigon equal to the union of the two triangles based at $\bar{l}$.

Proof. Let the endpoints of $\bar{l}$ be denoted by A, B. There are three possibilities for l — either a geodesic segment (both $A, B \in \mathbb{D}$) or l is a geodesic ($A, B \in \partial_\infty \mathbb{D}$), or a geodesic ray (A or B on $\partial_\infty \mathbb{D}$, the other inside $\mathbb{D}$). It is enough prove the lemma for first case, the two others being limit cases of the first.

Let $\overleftrightarrow{l'}$ be another projective line that intersects $\mathbb{D}$. Let X, Y be the respective dual points $\bar{l}^{\perp}, \overline{l'}^{\perp}$. Since both the line segments intersect $\mathbb{D}$, neither X nor Y can line inside $\overline{\mathbb{D}}$. Then, $\bar{l}$ and $\overline{l'}$ intersect each other at a point $U \in \mathbb{D}$ if and only if $U = \overline{XY}^{\perp}$.

Using a hyperbolic isometry, we can assume that both the points A, B lie on the horizontal axis, on either side of the origin. Then the line segment $\bar{l}$ is given by the closed interval $[a, b] \times 0$, where $A = (a, 0), B = (b, 0)$, with $a < 0 < b$. Owing to this choice of A, B, the duals $A^{\perp}, B^{\perp}$ are vertical lines passing through $(\frac{1}{a}, 0), (\frac{1}{b}, 0)$, respectively, with their point of intersection X lying on the line at infinity $\overleftrightarrow{l_\infty}$. The union Δ of the two projective triangles based at $\bar{l}$ is given by the open vertical strip bounded by these two verticals, that contains $\mathbb{D}$. Now the line segment $\overline{XY}$ is a vertical line passing through $(y, 0)$, where y is the horizontal coordinate of Y. The coordinates of the dual point $\overline{XY}^{\perp}$ is given by $(\frac{1}{y}, 0)$. Then $\bar{l}$ and $\overline{l'}$ intersect each other if and only if

$$a \leq \frac{1}{y} \leq b \Leftrightarrow \frac{1}{a} \leq y \text{ or } y \geq \frac{1}{b}.$$

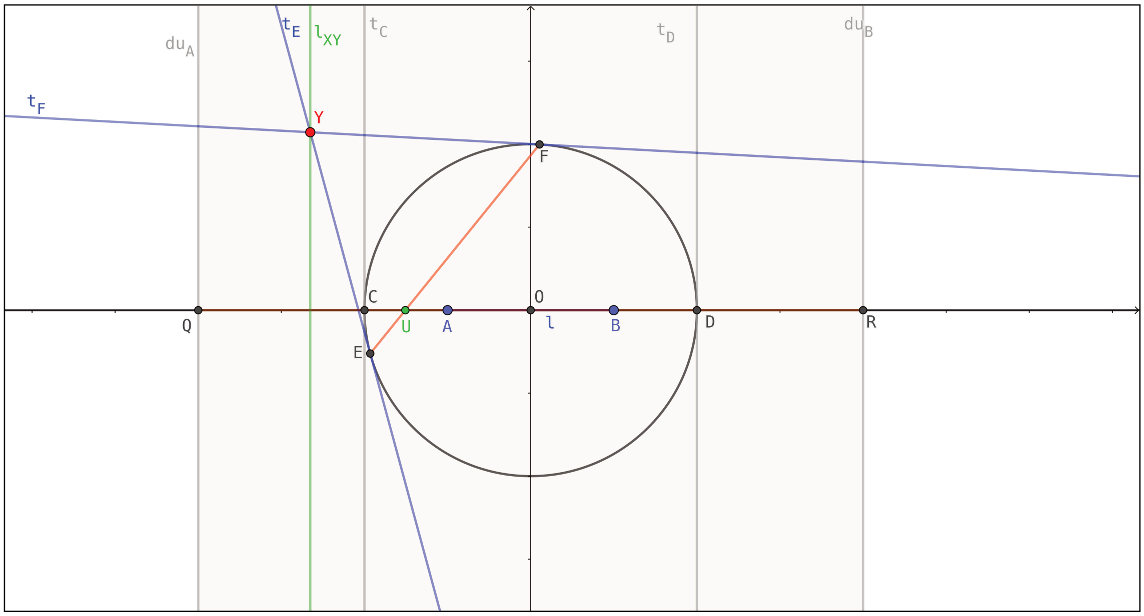


Figure 1.3: Properties 1.6.2

In other words, the line $\overleftrightarrow{l'}$ is disjoint from the segment $\bar{l}$ if and only if Y is a space-like point inside Δ. □

1.6.2 Calculations in $\mathbb{U}$

Lemma 1.6.3. *Let $\gamma_1 = (a_1, b_1)$ and $\gamma_2 = (a_2, b_2)$ be two geodesics in the upper half-plane model of $\mathbb{H}^2$ where a_1, b_1, a_2, b_2 are real numbers satisfying*

$$a_1 < b_1 < a_2 < b_2.$$

Let γ be the unique common perpendicular to γ_1 and γ_2. If x denotes the centre of the semi-circle containing γ, then

$$x = \frac{a_2 b_2 - a_1 b_1}{a_2 + b_2 - a_1 - b_1}.$$

Proof. Let $g = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{PGL}(2, \mathbb{R})$ be the inversion with respect to the semi-circle γ. Then, by definition of inversion, we have

$$x \mapsto \infty \Rightarrow rx + s = 0, \tag{1.1}$$
$$a \mapsto b \Rightarrow pa_1 + q = a_1 b_1 r + b_1 s, \tag{1.2}$$
$$b \mapsto a \Rightarrow pb_1 + q = a_1 b_1 r + a_1 s, \tag{1.3}$$
$$c \mapsto d \Rightarrow pa_2 + q = a_2 b_2 r + b_2 s. \tag{1.4}$$

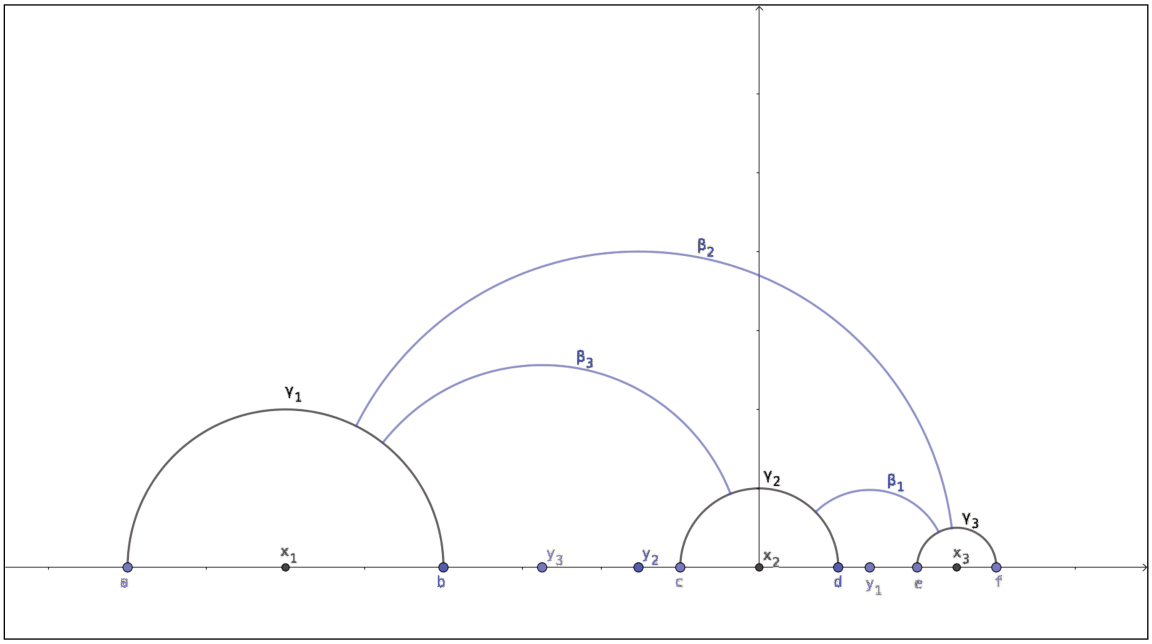


Figure 1.4: Common perpendiculars

, where "$\mapsto$" refers to the action of g. From eq.(1.2) and (1.3), we get that $p = -s$ and from the eqs.(1.1), (1.2) and (1.4), we get that

$$x = \frac{-s}{r} = \frac{a_2 b_2 - a_1 b_1}{a_2 + b_2 - a_1 - b_1}.$$

□

Lemma 1.6.4. *Let $\gamma_1, \gamma_2, \gamma_3$ be three pairwise disjoint semi-circular, possibly asymptotic geodesics in $\mathbb{U}$ such that none of them separates the remaining two geodesics from each other. For $i \in \mathbb{Z}_3$, let β_i be the common perpendicular to γ_{i-1} and γ_{i+1}, whenever possible. Let x_i be the centre of γ_i for $i = 1, 2, 3$. Let y_i be the centre of β_i or the common endpoint of $\gamma_{i-1}, \gamma_{i+1}$ for $i = 1, 2, 3$. Then the following equation holds:*

$$\frac{x_1 - x_2}{x_2 - x_3} = \frac{y_1 - y_2}{y_2 - y_3} \tag{1.5}$$

$$i.e., \ [\infty, x_1, x_2, x_3] = [\infty, y_1, y_2, y_3]\,. \tag{1.6}$$

Proof. Label the endpoints of $\gamma_1, \gamma_2, \gamma_3$ by $\{a, b\}, \{c, d\}, \{e, f\}$ such that

$$a < b \leq c < d \leq e < f.$$

Then from Lemma (1.6.3), we get that

$$x_1 = \frac{a+b}{2}, \qquad y_1 = \frac{ef-cd}{e+f-c-d}, \tag{1.7}$$

$$x_2 = \frac{c+d}{2}, \qquad y_2 = \frac{ef-ab}{e+f-a-b}, \tag{1.8}$$

$$x_3 = \frac{e+f}{2}, \qquad y_3 = \frac{cd-ab}{c+d-a-b}, \tag{1.9}$$

Using these coordinates, we calculate the right hand side of (1.5):

$$\begin{aligned} y_1 - y_2 &= \frac{ef-cd}{e+f-c-d} - \frac{ef-ab}{e+f-a-b} \\ &= \frac{ab(e+f-c-d)+cd(a+b-e-f)+ef(c+d-a-b)}{(e+f-c-d)(e+f-a-b)}, \\ y_2 - y_3 &= \frac{ef-ab}{e+f-a-b} - \frac{cd-ab}{c+d-a-b} \\ &= \frac{ab(e+f-c-d)+cd(a+b-e-f)+ef(c+d-a-b)}{(e+f-a-b)(c+d-a-b)}. \end{aligned}$$

Hence,

$$\frac{y_1-y_2}{y_2-y_3} = \frac{(c+d-a-b)}{(e+f-a-b)} = \frac{x_1-x_2}{x_2-x_3}. \tag{1.10}$$

□

Lemma 1.6.5. *Let y_1, y_2, y_3 be as in the hypothesis of the previous lemma. Then we have $y_3 < y_2 < y_1$.*

In order to prove this, we need the following lemma:

Lemma 1.6.6. *Let $\gamma_1 := (a,b)$ and $\gamma_2 := (b,c)$ be two asymptotic geodesics in $\mathbb{U}$. Let $\gamma_3 := (e,f)$ be another geodesic ultraparallel to γ_1, γ_2 such that*

$$a < b < c < e < f. \tag{1.11}$$

Let β_1, β_2 be the common perpendiculars to the pairs γ_2, γ_3 and γ_1, γ_3. Let y_i be the centre of the semi-circle β_i, for $i = 1, 2$. Then we have $y_1 > y_2$.

Proof. From Lemma (1.6.3), we know that,

$$y_1 = \frac{ef-bc}{e+f-b-c}, \qquad y_2 = \frac{ef-ab}{e+f-a-b}.$$

Calculating their difference, we get that,

$$\begin{aligned} y_1 - y_2 &= \frac{ef-bc}{e+f-b-c} - \frac{ef-ab}{e+f-a-b} \\ &= \frac{ef(c-a)+bc(a+b-e-f)+ab(e+f-b-c)}{(e+f-b-c)(e+f-a-b)}. \end{aligned}$$

Using the hypothesis (1.11), we know that the denominator of $y_1 - y_2$ is positive. So it suffices to check the sign of the numerator.

$$\begin{aligned}ef(c-a)+bc(a+b-e-f)+ab(e+f-b-c) &= ef(c-a)+b\{c(a+b-e-f)+a(e+f-b-c)\}\\ &= ef(c-a)+b\{c(b-e-f)-a(b-e-f)\}\\ &= (c-a)(ef+b(b-e-f))\\ &= (c-a)(e-b)(f-b).\end{aligned}$$

By eq(1.11), we have that the numerator is positive. Hence, $y_1 > y_2$. □

Proof of Lemma 1.6.5. Firstly, $x_1 < x_2 < x_3$. Then from (1.5) we get that, $y_1 - y_2$ and $y_2 - y_3$ have the same sign. So we shall calculate the sign of only one of them. Let β be the common perpendicular to γ_3 and $\gamma := (b, c)$. Let y' be the centre of the semi-circle β. Then using Lemma (1.6.6) for the geodesics $\gamma, \gamma_2, \gamma_3$, we get that $y' < y_1$. Again, by using the same Lemma for the geodesics $\gamma_1, \gamma, \gamma_3$, we get that $y_2 < y'$. Hence, $y_1 > y_2$. □

1.7 Simplicial Complex and PL-manifolds

The *link* of a simplex σ in a simplicial complex X, denoted by $\mathrm{Link}(\sigma, X)$, is the unique subcomplex of X such that the union of all simplices in X containing σ is given by $\sigma \bowtie \mathrm{Link}(\sigma, X)$. The codimension of a simplex σ of a simplicial complex X is defined as $\mathrm{codim}\,(\sigma) := \dim X - \dim \sigma$.

Definition 1.7.1. A simplicial complex is a d-manifold with boundary if the link of every 0-simplex is either $\mathbb{S}^{d-1}$ or $\mathbb{B}^{d-1}$ and there exists a 0-simplex whose link is $\mathbb{B}^{d-1}$.

Definition 1.7.2. A PL-manifold X is said to *PL-collapse* on to another PL-manifold Y if there exists a PL n-ball $\mathbb{B}^n$ and another PL $n-1$ ball $\mathbb{B}^{n-1} \subset \partial\mathbb{B}^n$ such that $X = Y \cup \mathbb{B}^n$ and $\mathbb{B}^{n-1} = Y \cap \mathbb{B}^n$.

Definition 1.7.3. (Simplicial) Collapsibility of a simplicial complex is defined recursively in the following way:

1. The void complex $\emptyset$ and any 0-simplex $\{\emptyset, v\}$ is collapsible.
2. If a simplicial complex contains a non-empty face σ such that its link and face-deletion are collapsible, then X is collapsible.

Definition 1.7.4. The *face-deletion* of simplex σ in a simplicial complex X, denoted by $\mathrm{fdel}(\sigma, X)$, is the subcomplex of X that contains all simplices that do not contain σ.

Lemma 1.7.5. *Let X be a simplicial complex. Let a be a 0-simplex such that* $\mathrm{Link}(a, X)$ *is closed ball. Then the X PL-collapses to the face-deletion of a.*

Proof. Take $Y = \mathrm{fdel}(a, X)$. The two balls required are $\mathrm{Star}(a, X)$ and $\mathrm{Link}(a, X)$. □

Properties 1.7.6. The following are true:

(a) Every simplex is simplicially collapsible.

(b) Every contractible simplicial complex of dimension one is collapsible.

(c) For every $d \geq 0$, a closed PL d-ball is PL-collapsible.

(d) $[Welker]$If at least one of two simplicial complexes X, Y is collapsible, then their join $X \bowtie Y$ is collapsible.

Definition 1.7.7. A subcomplex L of a simplicial complex K is called an *induced* subcomplex of K if whenever the 0-skeleton of a simplex σ of K is contained in the 0-skeleton of L, the entire simplex σ is also contained in L.

Theorem 1.7.8. *Let K be a combinatorial manifold with boundary. Suppose ∂K is an induced subcomplex of K. Let L be the simplicial complement of ∂K. Then, K collapses on to L.*

Theorem 1.7.9. *Any PL-collapsible d-manifold with boundary is a closed PL d-ball.*

Chapter 2

The Surfaces

In this chapter, we will introduce the different types of finite hyperbolic surfaces along with their deformation spaces. The first section gives a recap on compact orientable and non-orientable surfaces with non-empty boundary. In the second section, we shall construct hyperbolic surfaces with spikes. Finally, in the third section we will talk about hyperbolic polygons whose vertices are allowed to be real, ideal and hyperideal.

2.1 Compact surfaces with boundary

2.1.1 Orientable Surfaces

Any orientable compact surface is of the form $S_{g,n} := \mathbb{S}^2 \#((\mathbb{T}^2)^{\#g})\#((\mathbb{D}^2)^{\#n})$ where

- $\mathbb{S}^2$ is a sphere of dimension 2,
- $\mathbb{T}^2$ is the topological surface of $\mathbb{R}^2/\mathbb{Z}^2$,
- $\mathbb{D}$ is a closed 2-disk,
- the variable $g \in \mathbb{N}$ is called the genus of the surface and is additive under the connected sum, i.e, $S_g \# S_{g'} = S_{g+g'}$.
- the variable $n \in \mathbb{N}$ denotes that number of boundary components.

Next, we shall look at some examples and their common names.

Example 2.1.1. When $n = 0$, the surface is called *closed*.

Example 2.1.2. Suppose that $g = 0$.

1. When $n = 1$, we get back the disk $\mathbb{D}$.
2. When $n = 2$, we get an *annulus*.
3. When $n = 3$, the surface is called a *pair of pants*.

Example 2.1.3. When $g = 1, n = 1$, we shall call the surface a *one-holed* torus.

The Euler characteristic of such a surface is given by $\chi(S_{g,n}) = 2 - 2g - n$.

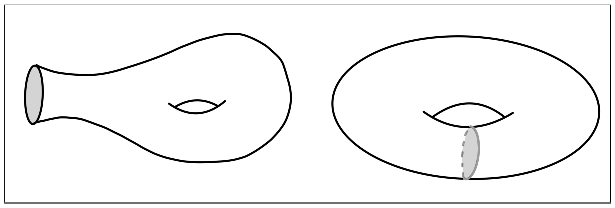


Figure 2.1: A one-holed torus and a torus

2.1.2 Non-orientable surfaces

Any closed non-orientable surface is of the form $T_{h,n} = (\mathbb{R}\mathrm{P}^2)^{\#h}\#((\mathbb{D}^2)^{\#n})$ where

- $\mathbb{R}\mathrm{P}^2$ is the projective plane,
- the variable $h \in \mathbb{N}$ here is again additive under the connected sum; the surface corresponding to $h = 0$ is the 2-sphere, which is orientable; so when we write $T_{h,n}$, we implicitly assume that $h > 0$. Also, we have the equality $T_h\#S_g = T_{h+2g}$, for any $h > 0$.

Example 2.1.4. When $h = 1, n = 1$, we get the *Möbius strip*.

Example 2.1.5. When $h = 2, n = 0$, we get the *Klein's bottle*.

2.1.3 Compact hyperbolic surfaces

We are primarily interested in those compact surfaces S which are hyperbolic and have non-empty boundary. From the Uniformisation Theorem, we know that the Euler characteristic of such a surface, denoted by $\chi(S)$, is negative. The following is the list of all the connected orientable and non-orientable surfaces that aren't hyperbolic, and hence excluded from the discussion:

$S_{0,0}$: a sphere $\mathbb{S}^2$,	$S_{0,2}$: annulus,
$S_{1,0}$: a torus $\mathbb{T}^2$,	$T_{1,1}$: a closed Möbius Strip
$T_{1,0}$: a projective plane $\mathbb{R}\mathrm{P}^2$,	$T_{2,0}$: Klein's bottle.

A complete finite-area hyperbolic metric with totally geodesic boundary on a compact hyperbolic surface $S = S_{g,n}$ or $T_{h,n}$ $(n > 0)$ is given by the following information:

- A discrete faithful representation, called a holonomy representation
$$\rho : \pi_1(S) \longrightarrow \mathrm{PGL}(2, \mathbb{R}),$$
that maps each b_i to a hyperbolic element. When $S = S_{g,n}$, the image $\rho(\pi_1(S))$ is a Fuchsian subgroup of $\mathrm{PSL}(2, \mathbb{R})$.
- A developing map $\mathrm{dev} : \widetilde{S} \longrightarrow \mathbb{H}^2$, such that the following diagram commutes: for all $\gamma \in \pi_1(S)$
$$\begin{tikzcd} \widetilde{S} \arrow[r, "\mathrm{dev}"] \arrow[d, "\gamma"'] & \mathbb{H}^2 \arrow[d, "\rho(\gamma)"] \\ \widetilde{S} \arrow[r, "\mathrm{dev}"'] & \mathbb{H}^2 \end{tikzcd}$$

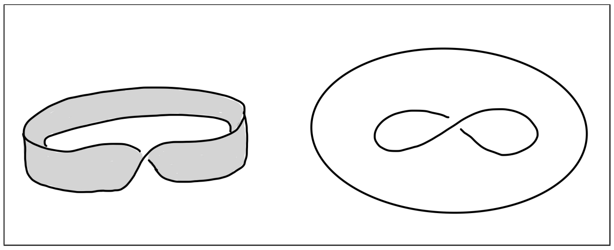


Figure 2.2: Möbius strip and a one-holed Möbius strip

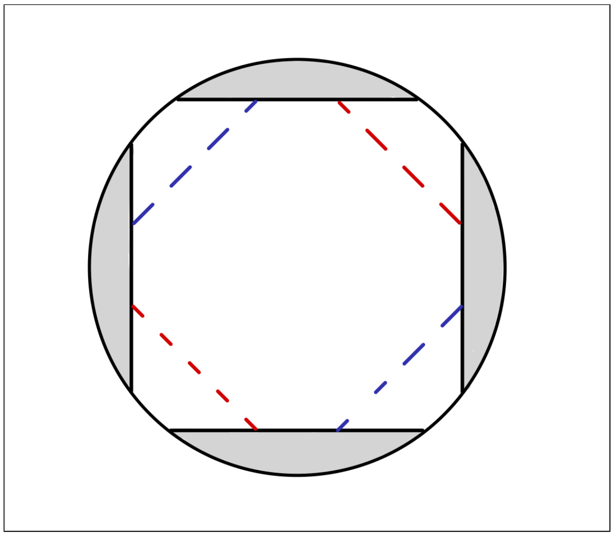


Figure 2.3: Universal cover of one-holed torus

for all $\gamma \in \pi_1(S)$. Here, $\widetilde{S}$ is the universal cover of S, on which an element $\gamma \in \pi_1(S)$ acts by deck transformations.

It follows from these conditions that the group $\Gamma := \rho(\pi_1(S))$ is a discrete finitely generated free group of $\mathrm{PGL}(2,\mathbb{R})$ containing only hyperbolic elements. The dev-image is a simply-connected region in $\mathbb{H}^2$ bounded by infinite geodesics corresponding to the lifts of its boundary components $\partial_i S$, for every $i = 1, \ldots, n$. These geodesics are pairwise disjoint in $\overline{\mathbb{H}^2}$. The deformation space $\mathfrak{D}(S)$ of the surface is the set of conjugacy classes of all possible holonomy representations. It is a connected component of the set

$$\{[\rho] : \rho \text{ is discrete, faithful}; \forall i, \rho(b_i) \text{ is hyperbolic}\} \subset \mathrm{Hom}(\pi_1(S), \mathrm{PGL}(2,\mathbb{R}))/\mathrm{PGL}(2,\mathbb{R}),$$

where the action of $\mathrm{PGL}(2,\mathbb{R})$ is by conjugation.

Let S be a compact hyperbolic surface endowed with a metric $m = [\rho] \in \mathfrak{D}(S)$. Given an element $[\gamma] \in \pi_1(S) \smallsetminus \{Id\}$, there exists a unique closed m-geodesic in this homotopy class, denoted by γ_g.

Definition 2.1.6. The *length function* is defined in the following way:

$$\begin{array}{rrcl} l_\gamma : & \mathfrak{D}(S) & \to & \mathbb{R}_{>0} \\ & [\rho] & \mapsto & 2\operatorname{arccosh}\left(\frac{tr(\rho(\gamma_g))}{2}\right). \end{array}$$

The following is a well-known result (for e.g. see [7]) which is usually proved using Fenchel-Nielson coordinates:

Theorem 2.1.7. *Let S_c be a compact hyperbolic surface with geodesic boundary.*

1. *If $S_c = S_{g,n}$, then its deformation space $\mathfrak{D}(S_{g,n})$ is homeomorphic to an open ball of dimension $6g - 6 + 3n$.*

2. *If $S_c = T_{h,n}$, then its deformation space $\mathfrak{D}(T_{h,n})$ is homeomorphic to an open ball of dimension $3h - 6 + 3n$.*

2.2 Surfaces with non-decorated spikes

These surfaces have been studied before in various contexts — Penner [18] gave a cell decomposition of their deformation spaces, Harer [11] has determined the topology of their arc complex (see Chapter 3 for more details), McShane [15] has determined the orthospectrum of a one-holed polygon, Parlier and Pournin [17] have studied the diameters of their flip graphs.

In the following we will start with the description of the simplest surface of this type and then gradually increase the topological complexity to obtain more generic examples.

The smaller surfaces

Ideal Polygons. An ideal $n(\geq 1)$-gon, denoted by $\Pi_n^{\pentagon}$, is the topological surface of a disk $\mathbb{B}^2$ with n points removed from its boundary. When $n(\geq 3)$, we can put a hyperbolic metric on it by taking the convex hull in $\mathbb{D}$ of n distinct points on $\partial_\infty\mathbb{D}$. The n ideal points are called *vertices* and they are marked as $x_1, \ldots, x_n$. The *edges* are the infinite geodesics of $\mathbb{D}$ joining two consecutive vertices. The restriction of the hyperbolic metric to an ideal polygon gives it a complete finite-area (equal to $\pi(n-2)$) hyperbolic metric with geodesic boundary. Its fundamental group is trivial. It is our first example of a hyperbolic surface with spikes. Fig. 2.4 shows an ideal pentagon in the projective model $\mathbb{D}$.

Ideal once-punctured polygons. For $n \geq 2$, an *ideal once-punctured n-gon*, denoted by $\Pi_n^{\odot}$, is another non-compact complete hyperbolic surface with geodesic boundary, obtained from an ideal $(n+2)$-gon, by identifying two consecutive edges using a parabolic element $T \in \mathrm{PSL}(2,\mathbb{R})$ that fixes the common vertex. The resulting surface has a missing point which we shall call a *puncture*. The fundamental group $\pi_1(\Pi_n^{\odot})$ of the surface is generated by the homotopy class of a simple closed loop that bounds a disk containing this puncture inside the surface. If $\rho : \pi_1(\Pi_n^{\odot}) \longrightarrow \mathrm{PSL}(2,\mathbb{R})$ is the holonomy representation, then $\rho(\pi_1(\Pi_n^{\odot})) \simeq \mathbb{Z}$, with $\rho(\gamma) = T$. The edges in this case are the connected components of the boundary of the surface. The vertices are the ideal points. Fig. 2.5 shows the construction of a punctured triangle $\Pi_3^{\odot}$ from an ideal pentagon $\Pi_5^{\pentagon}$, by identifying the two blue edges. The rightmost panel depicts the surface in a "polygonal" way.

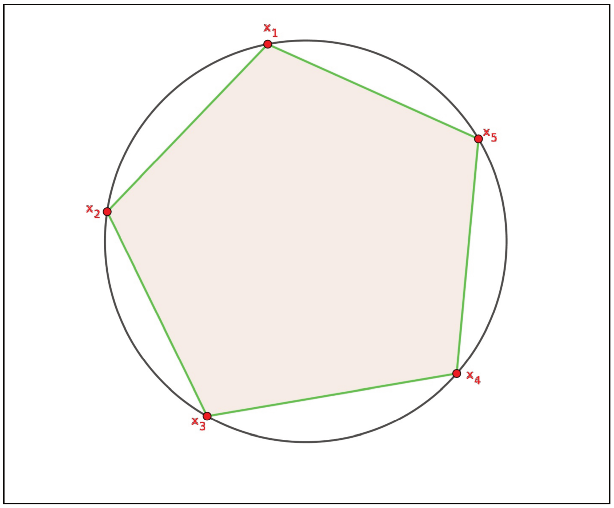


Figure 2.4: An ideal pentagon

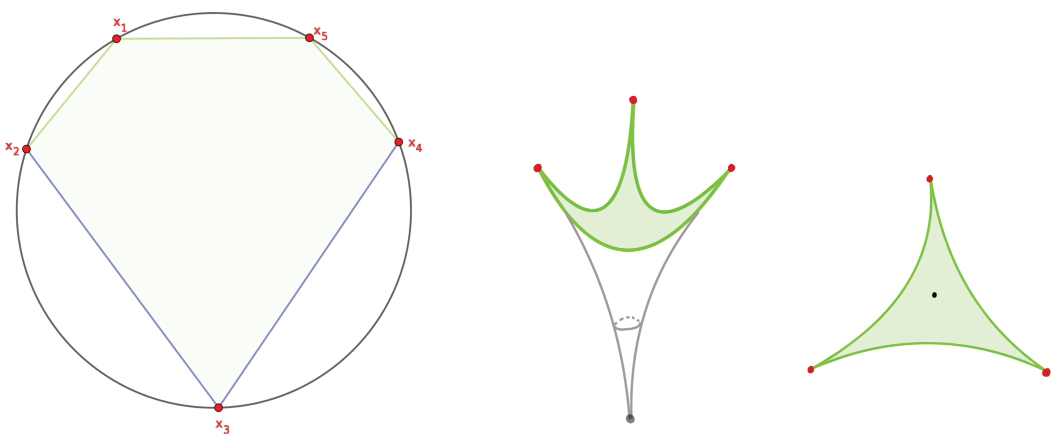


Figure 2.5: An ideal once-punctured triangle and its universal cover

General Surfaces

In this part, we shall construct a "big" hyperbolic surface with undecorated spikes from a compact surface S_c by gluing one-holed polygons along its boundary components.

Definition 2.2.1. Let S_c be $S_{g,m}$ or $T_{h,m}$, with $m \geq 0$. Consider $k(>0)$ ideal polygons $\Pi_{n_1}^{\pentagon}, \dots, \Pi_{n_k}^{\pentagon}$, with $n_j \geq 1$. Then, the surface S_{sp} obtained by taking the connected sum $S_c \# \Pi_{n_1}^{\pentagon} \# \dots \# \Pi_{n_k}^{\pentagon}$ is called a *surface with spikes*.

The vertices of the ideal polygons used in the construction are called *spikes*. The total number of spikes of such a surface is given by $Q := \sum_{i=1}^{k} n_i$. The connected components of the boundary of S_{sp} are either homeomorphic to the circle $\mathbb{S}^1$ (boundary component of S_c) or to open intervals (boundary of ideal polygons). The total number of connected components is $m + Q$.

Given an orientable (resp. non-orientable) surface with spikes such that $6g - 6 + 3n + Q > 0$ (resp. $3h - 6 + 3n + Q > 0$), we can put a complete finite-area hyperbolic metric on it.

The following are some examples of "small" hyperbolic surfaces:

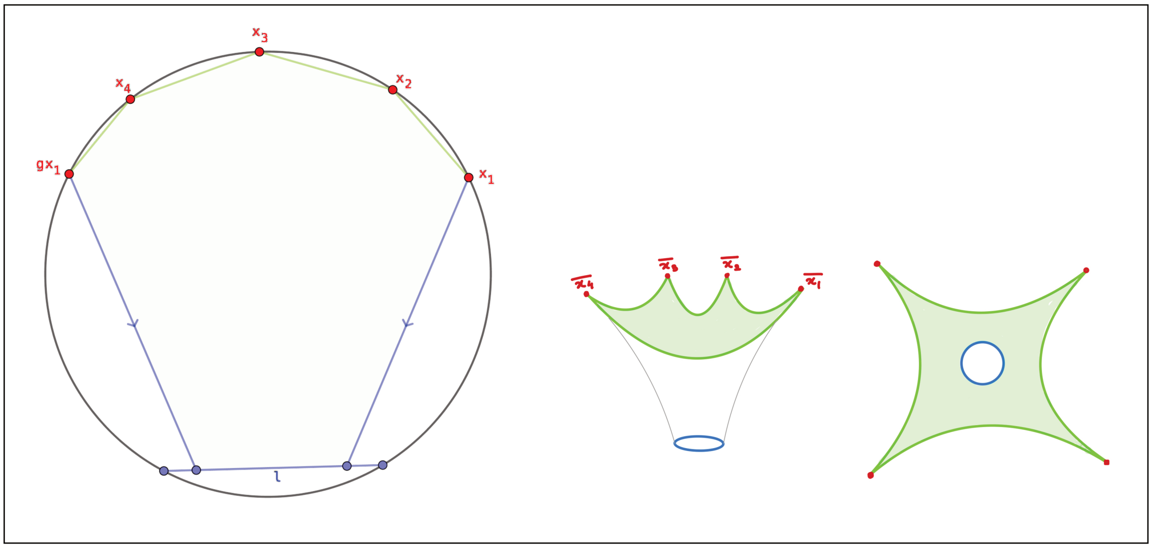


Figure 2.6: A fundamental domain for an ideal one-holed square

Example 2.2.2. The orientable surface $\mathbb{B}^2 \# \Pi_n^{\pentagon}$, for $n > 0$, is called an *ideal one-holed n-gon* and denoted by $\Pi_n^{\circledcirc}$. Its boundary consists of one simple closed curve, denoted by γ and n open intervals. Its fundamental group $\pi_1(\Pi_n^{\circledcirc})$ is generated by the homotopy class of γ and is isomorphic to $\mathbb{Z}$. Next, we put a hyperbolic structure on it in the following way:

Let $g \in \mathrm{PSL}(2, \mathbb{R})$ be a hyperbolic element with axis as a bi-infinite geodesic, denoted by l. See Fig. 2.6. It divides the boundary circle $\partial_\infty \mathbb{D}$ into two open intervals. Choose a point x_1 in any one of them and take $(n-1)$ distinct points $x_2, \ldots, x_n$ on the same interval between x_1 and its image $g \cdot x_1$. Mark all the points of their $\langle g \rangle$-orbit. All of them lie on the same side of l as the initial points. Join consecutive pairs using infinite geodesics. Drop two perpendiculars from x_1 and $g \cdot x_1$ and identify them using g. The quotient is a complete finite-area hyperbolic surface with geodesic boundary and the underlying topological surface is homeomorphic to that of an ideal once-punctured n-gon. If $\rho : \pi_1(\Pi_n^{\circledcirc}) \longrightarrow \mathrm{PSL}(2, \mathbb{R})$ is the holonomy representation, then $\rho(\gamma) = g$. The images of the ideal points $x_1, \ldots, x_n$ in the quotient are called vertices, and those of the bi-infinite geodesics as well as the closed boundary geodesic are called edges.

Example 2.2.3. The orientable surface $S := \mathbb{S}^2 \# \Pi_{n_1}^{\pentagon} \# \Pi_{n_2}^{\pentagon}$ for $n_1, n_2 > 0$ is called a (n_1, n_2)-*spiked annulus.* Any connected component of its boundary is homeomorphic to an open interval. It contains exactly one isotopy class of simple curves $[\gamma]$, where γ is a non-trivial simple closed curve. Its fundamental group is again isomorphic to $\mathbb{Z}$. Take an ideal n-gon in $\mathbb{D}$, where $n = n_1 + n_2 + 2$.

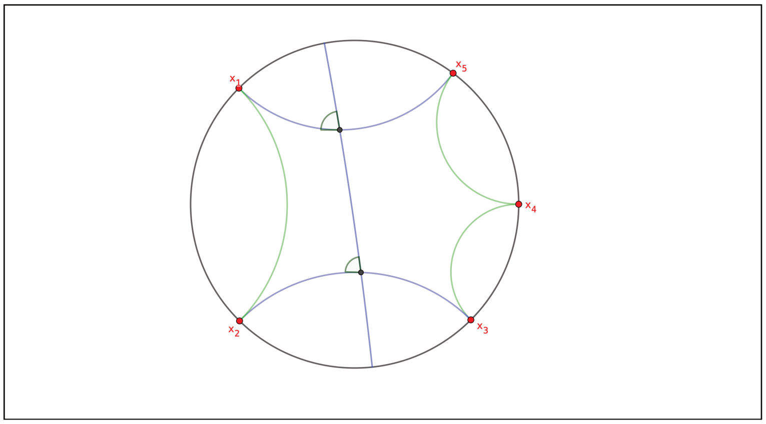


Figure 2.7: A fundamental domain for a (1,2)-spiked annulus

Its vertices are denoted by $x_1, \dots, x_n$ in the anti-clockwise direction. Let l_1, l_2 be the edges joining the pairs of vertices (x_{n_1+1}, x_{n_1+2}) and (x_n, x_1), respectively. Let $g \in \mathrm{PSL}(2,\mathbb{R})$ be a hyperbolic isometry whose axis intersects both l_1 and l_2 at the same angle, and whose translation length is given by the distance between the two points of intersection. Then the quotient surface $S = \Pi_n^{\pentagon} / \sim$, where for every $z \in l_1$, $z \sim g \cdot z$, is a complete finite-area hyperbolic surface with geodesic boundary, homeomorphic to a (n_1, n_2)-spiked annulus. Its holonomy representation $\rho : \pi_1(S) \longrightarrow \mathrm{PSL}(2,\mathbb{R})$ maps the generator $[\gamma]$ to g.

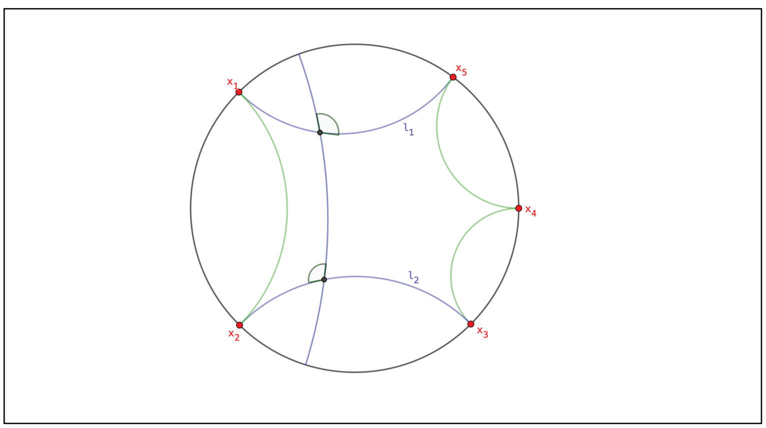


Figure 2.8: A fundamental domain for a 3-spiked Möbius strip

Example 2.2.4. The non-orientable surface $\mathbb{R}\mathrm{P}^2 \# \Pi_n^{\pentagon}$, for $n > 0$, is called a spiked Möbius strip. Its orientation double cover is a (n,n)-spiked annulus. Similar to the previous example, we consider an ideal $n+2$-gon with marked vertices $x_1, \dots, x_{n+2}$. See Fig. 2.8. Take any two edges l_1, l_2 of the polygon that don't have any common endpoint. Let $r \in \mathrm{PGL}(2,\mathbb{R})$ be the hyperbolic reflection along the common perpendicular to l_1, l_2 and $g \in \mathrm{PSL}(2,\mathbb{R})$ be a hyperbolic isometry whose axis intersects

l_1 and l_2 such that the angles of intersections are complementary. Let $h := rg \in \mathrm{PGL}(2,\mathbb{R})$. The quotient $\Pi_n^{\pentagon}/\sim$, where for every $z \in l_1$, $z \sim h \cdot z$, is a complete finite-area hyperbolic metric with geodesic boundary and is homeomorphic to a spiked Möbius strip with n marked spikes.

Definition 2.2.5. The smallest closed convex subset of a surface with spikes S which contains all closed geodesics of S is called the *convex core* of the surface and denoted by $S_{\heartsuit}$.

If S is a surface with spikes obtained from a compact surface S_c, with m boundary components, and k ideal polygons, then its convex core is usually a compact surface of the same genus with $(k+m)$ boundary components, each of which is homeomorphic to $\mathbb{S}^1$. The list of exceptions is given below.

Example 2.2.6. The following is a list of all hyperbolic surfaces whose convex cores are not hyperbolic surfaces:

- Ideal polygons $\Pi_n^{\pentagon}$ and ideal punctured polygons $\Pi_n^{\odot}$ have empty convex cores.
- The convex cores of a spiked Möbius strip, one-holed polygons and a spiked annulus are both homeomorphic to a circle.

Label the boundary components of the convex core as $\partial_1, \ldots, \partial_{m+k}$. These are called the *peripheral loops* of the surface S. Each peripheral loop is either isotopic to a boundary component of the compact surface S_c or it separates a one-holed m-gon (called a *crown*) from S, where $m \in \{n_1 \ldots, n_k\}$. There are k such crowns, which are labelled as $C_1, \ldots, C_k$.

Notation 2.2.1. For every $i = 1, \ldots, n$, define $q_i = 0$ if the i-th peripheral loop is isotopic to a boundary component of S_c and $q_i = n_j$, if the i-th peripheral loop separates a one-holed n_j-gon, for some $j \in \{1, \ldots, k\}$. Define the spike vector $\vec{q} := (q_1, \ldots, q_n)$. Finally, an orientable (resp. non-orientable) surface with genus g (resp. h), n peripheral loops and spike vector $\vec{q}$ is denoted by $S_{g,n}^{\vec{q}}$ (resp. $T_{h,n}^{\vec{q}}$).

Notation 2.2.2. Now we define the above notation for the exceptional cases that do not have hyperbolic convex cores. For ideal q-gons, ($q \geq 4$), we shall use the notation $S_{0,1}^{(q)}$. For (q_1, q_2)-spiked annulus, we shall use $S_{0,2}^{(q_1,q_2)}$. Finally, for a Mobïus strip with $q(\geq 1)$ spikes, we shall use $T_{1,1}^{(q)}$.

Next, we shall construct a hyperbolic metric on a general hyperbolic surface with spikes. We shall assume that the convex core of the surface is hyperbolic since we have already treated the cases where it is not hyperbolic. The surface with spikes and its convex core have the same homotopy type. In particular, $\pi_1(S_{\heartsuit}) = \pi_1(S_{sp})$. Let $([\rho], dev)$ be a hyperbolic structure on $S_{\heartsuit}$. The holonomy representation ρ of S_c gives a holonomy representation of S_{sp}. Next, we will construct the embedding R' of the universal cover of S_{sp} in $\mathbb{D}$. Start with the simply connected region $R := dev(\widetilde{S_{\heartsuit}})$ in $\mathbb{D}$ bounded by pairwise disjoint lifts of ∂_i, $i = 1, \ldots, n$. We choose Q distinct points on $\partial_\infty \mathbb{D}$ in the following way— whenever $q_i > 0$, take q_i ideal points $\underline{x^i} = (x_1^i, \ldots, x_{q_i}^i)$ on the same side of a lift of the peripheral loop ∂_i. Denote by $\mathbf{x} = (\underline{x^1}, \ldots, \underline{x^n}) \in (\partial_\infty \mathbb{D})^Q$ the n-tuple of vectors. Join consecutive pairs x_j, x_{j+1}, $j = 1, \ldots, q_i - 1$, by infinite geodesics. Then, R'' is the region bounded by the infinite geodesics corresponding to boundary components. It contains $dev(\widetilde{S_{\heartsuit}})$. See Fig. 2.10.

A metric on a surface with spikes S_{sp} can be seen as an ordered pair $(\rho, \mathbf{x})$. Two pairs $(\rho, \mathbf{x})$, $(\rho', \mathbf{x}')$ are said to be equivalent if there exists an element $g \in \mathrm{PGL}(2,\mathbb{R})$ such that for

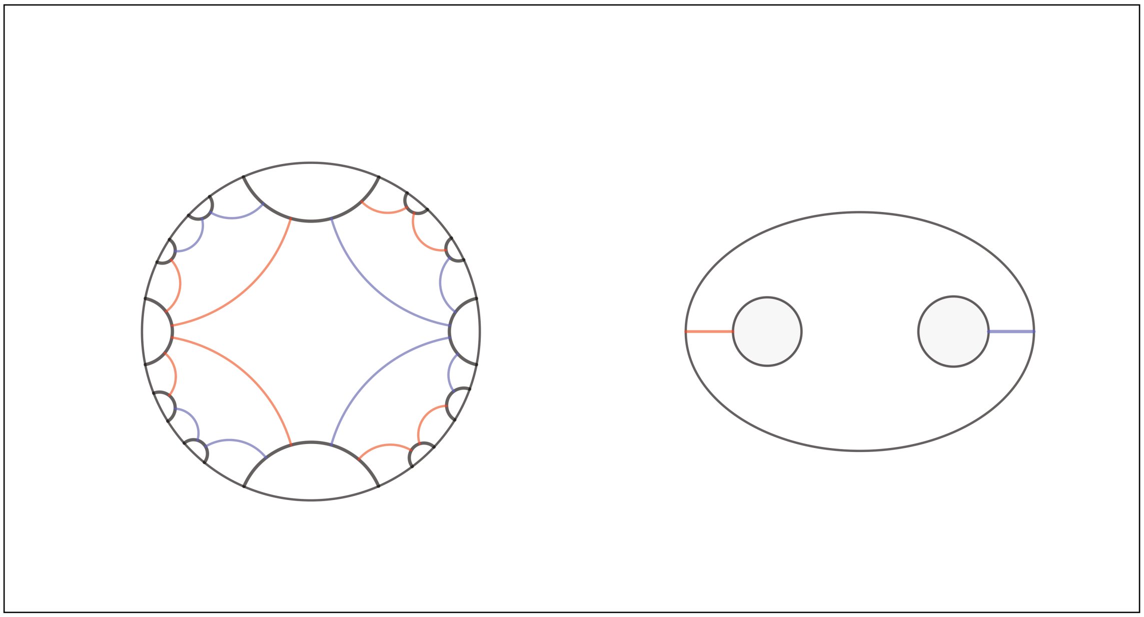


Figure 2.9: The universal cover of $S_{0,3}$

all $\gamma \in \pi_1(S)$, $\rho'(\gamma) = g\rho(\gamma)g^{-1}$ and $\mathbf{x}' = g \cdot \mathbf{x}$. Hence, elements in the deformation space are equivalence classes of such pairs. The following theorem about the dimension of the deformation space of a surface with non-decorated spikes is analogous to Theorem 2.1.7.

Theorem 2.2.7. *Let S_{sp} be a hyperbolic surface with Q non-decorated spikes.*

1. *If $S_{sp} = S_{g,n}^{\vec{q}}$, then its deformation space $\mathfrak{D}(S_{g,n}^{\vec{q}})$ is homeomorphic to an open ball of dimension $6g - 6 + 3n + Q$.*

2. *If $S_{sp} = T_{h,n}^{\vec{q}}$, then its deformation space $\mathfrak{D}(T_{h,n}^{\vec{q}})$ is homeomorphic to an open ball of dimension $3h - 6 + 3n + Q$.*

Deformation space of one-holed polygons We conclude this section by parametrising a subspace $\mathfrak{D}_0$ of the deformation space $\mathfrak{D}(\Pi_n^{\circledcirc})$ of one-holed polygons, which is defined in the following way: let $(\rho, \mathbf{x})$ be a metric on $\Pi_n^{\circledcirc}$ and let $\rho(\pi_1(\Pi_n^{\circledcirc})) = g \in \mathrm{PSL}(2, \mathbb{R})$, a hyperbolic element. Then two metrics $(\rho, \mathbf{x}), (\rho', \mathbf{x}')$ are said to be equivalent if there exists an element $h \in \mathrm{PGL}(2, \mathbb{R})$ such that for all $\gamma \in \pi_1(\Pi_n^{\circledcirc})$, $\rho'(\gamma) = h\rho(\gamma)h^{-1}$, $\mathbf{x}' = h \cdot \mathbf{x}$ and the axis of h is distinct from that of g. Then $\mathfrak{D}_0$ is defined to be the set of all such equivalence classes.

2.2.1 The hyperbolic surfaces with decorated spikes

Definition 2.2.8. A hyperbolic surface with decorated spikes is a surface obtained from a hyperbolic surface of type $S_{g,n}^{\vec{q}}$ or $T_{h,n}^{\vec{q}}$ by decorating each spike by a horoball. Such a surface is denoted by $S_{g,n}^{\vec{q},\vec{h}}$ (when orientable) and $T_{h,n}^{\vec{q},\vec{h}}$ (when non-orientable).

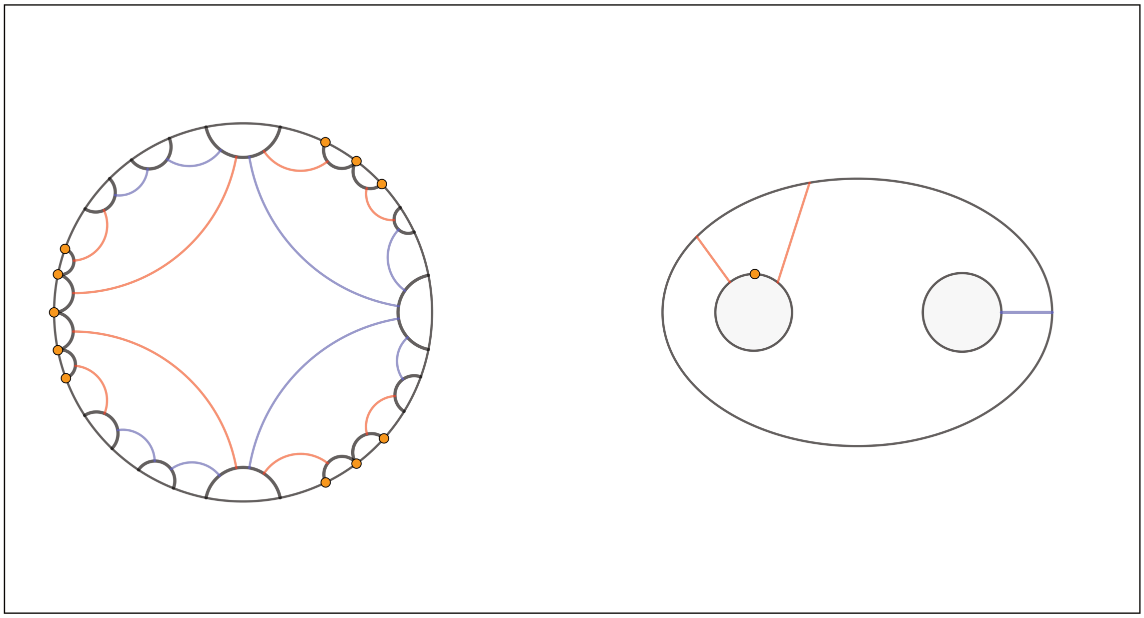


Figure 2.10: The universal cover of $S_{0,3}^{(0,1,0)}$

We shall use the symbol S_{sp}^h for referring to both cases at once.

The deformation space $\mathfrak{D}(S_{sp}^h)$ of such a surface is the trivial $\mathbb{R}_{>0}^Q$ bundle over $\mathfrak{D}(S_{sp})$ where the fibre over a point in the base space is given by the Busemann functions of the horoballs based at the Q spikes of the surface S_{sp}.

Definition 2.2.9. A *horoball connection* on a surface S_{sp}^h is a geodesic path joining two not necessarily distinct decorated spikes.

It is the image in the quotient of a bi-infinite geodesic in $\mathbb{D}$ joining a pair of decorated ideal vertices in $\overline{\mathbb{D}}$. Next we define its length.

Definition 2.2.10. Let β be a horoball connection with a lift $\widetilde{\beta}$ in the universal cover. The endpoints of the latter are two ideal points p_1, p_2 decorated with two horoballs h_1, h_2. Then the *length* of β is given by the lambda length of h_1, h_2. (See Definition (**??**)).

We have the following length function for horoball connections:

$$\begin{array}{rccc} l_\beta : & \mathfrak{D}(S_{sp}^h) & \longrightarrow & \mathbb{R} \\ & m & \mapsto & \lambda(h_1, h_2). \end{array}$$

In Chapter 6, we shall be interested in the deformations such that the lengths of all horoball connections increase uniformly. The set of all horoball connections is denoted by $\mathcal{H}$. Using Theorem 2.2.7, we have that

Theorem 2.2.11. *Let S_{sp}^h be a hyperbolic surface with Q decorated spikes.*

1. *If* $S^h_{sp} = S^{\vec{q},\vec{h}}_{g,n}$, *then its deformation space* $\mathfrak{D}(S^{\vec{q},\vec{h}}_{g,n})$ *is homeomorphic to an open ball of dimension* $6g - 6 + 3n + 2Q$.

2. *If* $S^h_{sp} = T^{\vec{q},\vec{h}}_{h,n}$, *then its deformation space* $\mathfrak{D}(T^{\vec{q},\vec{h}}_{h,n})$ *is homeomorphic to an open ball of dimension* $3h - 6 + 3n + 2Q$.

2.2.2 Trivial Bundles

Let S_{sp} be a hyperbolic surface with undecorated spikes with convex core $S_\heartsuit$. Let S^h_{sp} be the surface S_{sp} along with horoball decorations around every spike. Then we have that

$$\mathfrak{D}(S_\heartsuit) \xleftarrow{p_1} \mathfrak{D}(S_{sp}) \xleftarrow{p_2} \mathfrak{D}(S^h_{sp}),$$

where p_1 is trivial $\mathbb{R}^Q$ bundle map and the fibers of p_2 are convex subsets.

2.3 Hyperbolic polygons

In this section, we study the following objects:

- General compact polygons: These are hyperbolic polygons with real or truncated hyperideal vertices.

- Decorated polygons: These can have real, truncated hyperideal and ideal vertices that are decorated with horoballs.

All of these polygons are constructed, in the following section, from polygons in $\mathbb{R}\mathrm{P}^2$ satisfying certain properties.

2.3.1 Convex Polygons in $\mathbb{R}\mathrm{P}^2$

Start with a convex n-gon $\mathcal{P}_n$ in $\mathbb{R}\mathrm{P}^2$ such that each edge intersects $\mathbb{D}$. In the following we define the *truncation of a vertex* of $\mathcal{P}_n$ and a *truncated vertex*:

- Let p be a hyperideal vertex of the polygon $\mathcal{P}_n$ with the dual time-like line $p^\perp$ intersecting only the two edges l, l' adjacent to p in $\mathcal{P}_n$. The hyperbolic line segment ν supported on the line $p^\perp$ and intercepted between the two edges l and l' is called a *truncated hyperideal vertex*, and the removal of the triangle, formed by l, l' and ν, from the original polygon $\mathcal{P}_n$, is called the *truncation* of the hyperideal vertex p.

- Suppose that p is an ideal vertex of $\mathcal{P}_n$ and consider a horoball h based at p. Then the *truncation* of p with respect to the horoball h is the removal of the interior of h along with p.

An ideal vertex v is said to be *decorated* if a horoball, based at v is added. Among these polygons we shall consider only those which satisfy the following property:

$$\text{Property: The lengths of every truncated hyperideal vertex is positive.} \tag{2.1}$$

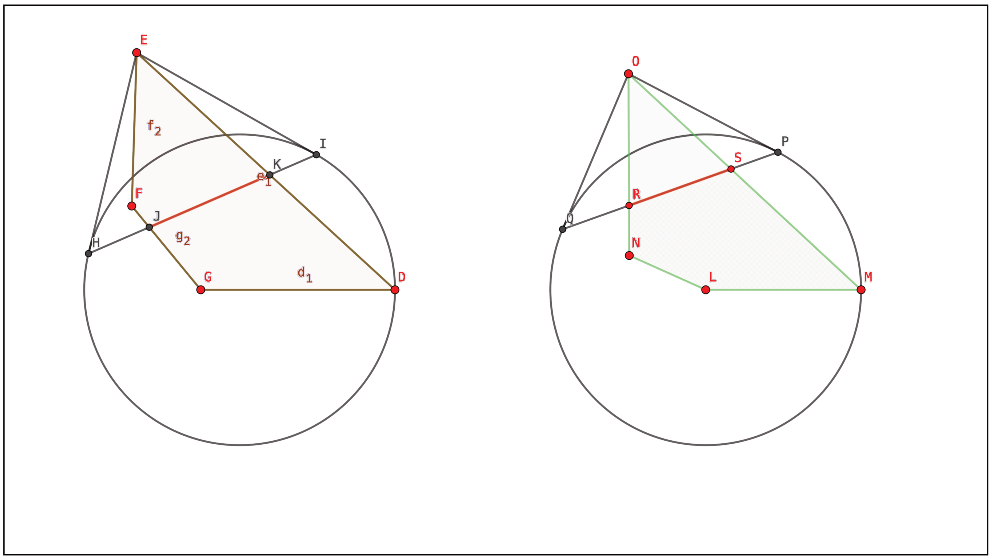


Figure 2.11: Non-example and example for Property 2.3.1

Remark 2.3.1. Property (2.1) implies that after every truncation of a hyperideal vertex, the remaining (truncated) vertices and edges of the polygon are entirely contained in one of the half spaces formed by the infinite geodesic carrying this segment or they are contained in complement of the horoball used for the truncation.

Fig. 2.11 shows one non-example (left) and one example (right) of convex polygons in $\mathbb{R}P^2$. In the non-example, after the truncation of the hyper-ideal vertex E, the length of the edge $\overline{EF}$ becomes negative.

Next, we shall define the different types of hyperbolic polygons whose deformation spaces we shall be parametrising using their respective arc complexes.

Compact Hyperbolic polygons. Starting from a convex polygon without any ideal vertices, one can construct a compact hyperbolic polygon by truncating hyperideal vertices whenever there are any. The topological surface of the resulting polygon shall be denoted by Π_n^c, where n is the total number of vertices of the original polygon. Fig. 2.12 shows a compact quadrilateral with two truncated hyperideal and two real vertices.

Decorated Polygons. The hyperbolic polygon obtained by truncating every hyperideal vertex and decorating every ideal vertex of a convex polygon satisfying Property (2.1) is called a decorated polygon and the underlying topological surface shall be denoted by $\Pi_n^{\oslash}$.

Note that a decorated polygon with no ideal vertices is just a compact polygon. A decorated polygon with only ideal vertices is called a *decorated* ideal polygon.

Given a decorated polygonal surface $\Pi_n^{\oslash}$, the term *generalised vertex* shall be used to refer to a real vertex or the truncation of a hyperideal vertex or a decorated ideal vertex. A generalised vertex is said to be of

- *hyperbolic type* if it is the truncation of a hyperideal point,

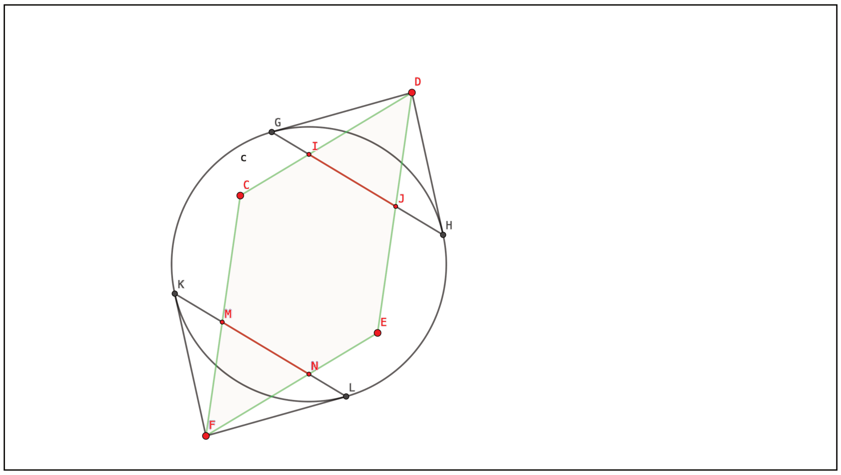


Figure 2.12: A compact quadrilateral

- *parabolic type* if it is a decorated ideal vertex,
- *elliptic type* if it is a real point.

Let $V = (\nu_1, \ldots, \nu_n)$ be the cyclically anti clock-wise ordered n-tuple of the generalised vertices of $\Pi_n^{\circledast}$. The type of V is defined to be the type of each vertex in the given order.

2.3.2 Deformation space of polygons

In this section, we shall prove that the deformation spaces of the different types of polygons, up to isometry, defined in the previous subsection are open balls.

For an ideal or punctured polygon, its *deformation space* is defined to be the set of all complete finite-area hyperbolic metrics with geodesic boundary, up to isometries that preserve the markings of the vertices.

Theorem 2.3.1. *The deformation space $\mathfrak{D}(\Pi)$ of an ideal polygon $\Pi = \Pi_n^{\pentagon}$, $n \geq 3$, is homeomorphic to an open ball $\mathbb{B}^{n-3}$.*

Proof. Let $x_1, \ldots, x_n \in \mathbb{R} \cup \{\infty\}$ denote the cyclically ordered vertices of an ideal polygon. Since $G = \mathrm{PGL}(2, \mathbb{R})$ acts triply transitively on $\partial_\infty \mathbb{D}$, there exists a unique $g \in G$ that maps (x_1, x_2, x_3) to $(\infty, 0, 1)$. Therefore, a metric on an ideal n-gon is determined by the real numbers $x_4, \ldots, x_n$. Hence, the deformation space $\mathfrak{D}(\Pi_n^{\pentagon}) = \left\{(x_4, \ldots, x_n) \in \mathbb{R}^{n-3} : 1 < x_4 < \ldots < x_n\right\}$ is homeomorphic to $\mathbb{D}^{n-3}$. $\square$

Theorem 2.3.2. *The deformation space $\mathfrak{D}(\Pi)$ of a punctured polygon $\Pi = \Pi_n^{\odot}$, $n \geq 1$, is homeomorphic to an open ball $\mathbb{B}^{n-1}$.*

Proof. From the construction of a punctured n-gon from an ideal $(n+2)$-gon, and the above discussion, we have that $\mathfrak{D}(\Pi_n^{\odot}) \simeq \mathbb{B}^{n-1}$. $\square$

Next we shall define the deformation space for decorated polygons, $\Pi_n^{\oslash}$ with $n \geq 3$.

Let P_1 and P_2 be two decorated polygons with generalised vertex set $V_1 = (\nu_1, \ldots, \nu_n)$ and $V_2 = (\mu_1, \ldots, \mu_n)$ such that for every $i = 1, \ldots, n$, ν_i and μ_i are of the same type. Then P_1 is said to be equivalent to P_2 if and only if there exists an isometry $\gamma \in Isom^+(\mathbb{D})$ such that for every i, $\gamma(\nu_i) = \mu_i$. In the case of parabolic vertices, the horoballs are mapped to each other via the isometry γ. The set of all equivalence classes is called the *deformation space* of the decorated polygon $\Pi_n^{\oslash}$ with the given type of vertices, and is denoted by $\mathfrak{D}(\Pi_n^{\oslash})$.

Theorem 2.3.3. *The deformation space $\mathfrak{D}(\Pi_n^{\oslash})$ of a decorated polygonal surface $\Pi_n^{\oslash}$ ($n \geq 3$), with a given type V of generalised vertices, is homeomorphic to an open ball of dimension $2n-3$.*

Proof. The theorem is proved by inducting on the total number of generalised vertices n.

For $n = 3$, there are six different combinations possible for the vertices of a decorated triangle. In each case we start with two generalised vertices ν_1 and ν_2 of specified types and then determine a region $K \subset \mathbb{R}\mathrm{P}^2$ such that a point p lies inside K if and only if p or its truncation can be the third vertex ν_3 of a decorated triangle whose other two vertices are ν_1 and ν_2. We show that the initial two points can be determined, up to isometries that preserve the markings, by using exactly one parameter that varies in an open interval. Finally, we show that K is homeomorphic to an open ball of dimension two, when ν_3 is hyperbolic or elliptic. If ν_3 is parabolic, we show that K is an open arc contained in $\partial_\infty \mathbb{D}$. A point in K is then decorated using a horocycle; this gives a trivial bundle K' of dimension two over a contractible space of dimension one.

Without loss of generality, we assume that the three vertices are cyclically ordered in the trigonometric sense as $\nu_1 - \nu_2 - \nu_3$. For $i = 1, 2, 3$, the edge opposite to the vertex ν_i is labeled as e_i.

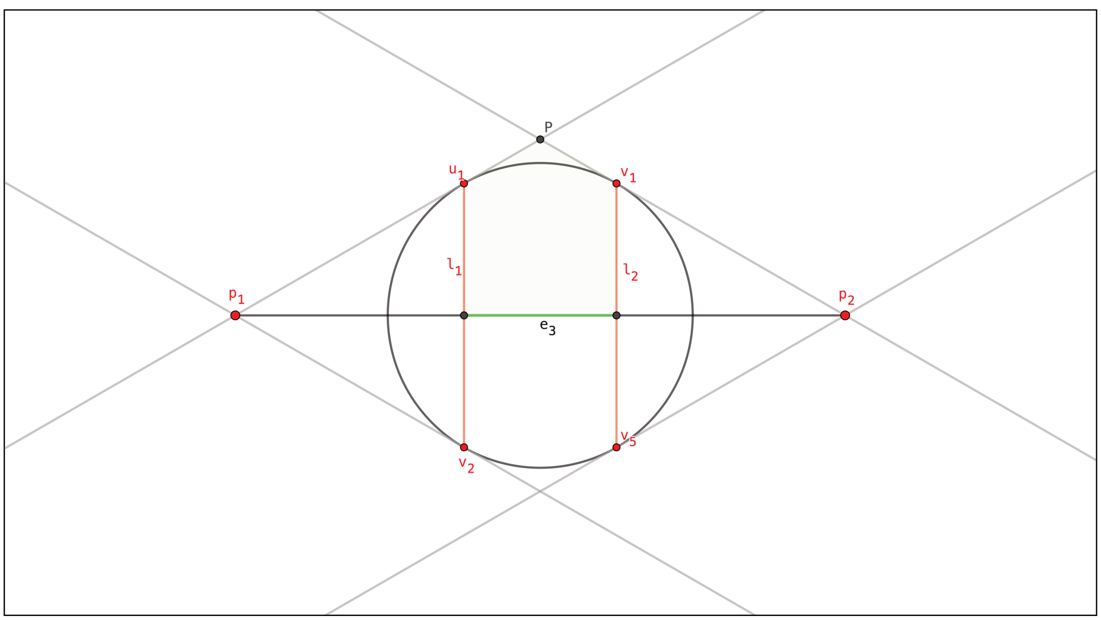


Figure 2.13: Base step Case (a)

(a) Suppose that ν_1 and ν_2 are both of hyperbolic type (Fig. 2.13(a)). Then they are the truncations of two hyperideal points p_1 and p_2 respectively. They are supported on the hyperbolic geodesics $l_1 := p_1{}^\perp \cap \mathbb{D}$ and $l_2 := p_1{}^\perp \cap \mathbb{D}$, respectively. The endpoints of l_1 are labeled as u_1, u_2 and those of l_2 as v_1, v_2 such that these four ideal points are ordered in the anti-clockwise manner, as u_1, u_2, v_2, v_1. Using the triple transitivity of the action of G on $\partial_\infty \mathbb{D}$, we can fix any three of these points and choose the fourth one from an open interval, contained in $\partial_\infty \mathbb{D}$. The line $\overleftrightarrow{e_3}$, supporting the edge e_3 passes through the hyperideal points p_1, p_2. Using Remark 2.3.1 for the geodesics l_1, l_2 we get that the third vertex ν_3 must be contained in one of the two disjoint open projective triangles bounded by $\overleftrightarrow{e_3}, \overleftrightarrow{l_1}, \overleftrightarrow{l_2}$, intersecting $\mathbb{D}$; we denote by Δ the one which is in accordance with the chosen ordering $\nu_1 - \nu_2 - \nu_3$. Let $\overleftrightarrow{t_1}, \overleftrightarrow{t_2}$ be the tangents at u_1 and v_1 from p_1 and p_2 respectively, to $\partial_\infty \mathbb{D}$. Then from Lemma 1.6.2, their point of intersection p lies inside Δ. Let P be the pentagon bounded by the lines $\overleftrightarrow{l_1}, \overleftrightarrow{l_2}, \overleftrightarrow{e_3}, \overleftrightarrow{t_1}, \overleftrightarrow{t_2}$ lying in Δ.

Now, by applying Lemma (1.6.2) to the geodesic with endpoints u_1, v_1, we get that the third vertex ν_3 is hyperbolic if and only if it is the truncation of a hyperideal point lying in the region P. Hence the region K is given by $P \smallsetminus \overline{\mathbb{D}}$. If ν_3 is of parabolic type, then K is given by the open contractible arc $u_1 v_1 := P \cap \partial_\infty \mathbb{D}$. The horoball decoration gives a trivial line bundle over $u_1 v_1$. Finally, any point in $K := P \cap \mathbb{D}$ can be a valid elliptic vertex and these are the only possibilities. So in all the three cases, the set K is homeomorphic to an open 2-ball.

(b) Suppose that ν_1 and ν_2 are both of parabolic type. Then these are two ideal points p_1 and p_2 decorated with horoballs h_1 and h_2, respectively. See Fig. (2.14). Using the triple-transitivity of the action of G on $\partial_\infty \mathbb{D}$, we can suppose that $p_1 = (-1, 0), p_2 = (1, 0)$. Again, e_3 denotes the common edge to the two vertices, whose length $l(e_3)$ is given by the lambda length between the horoballs h_1 and h_2 chosen to decorate the initial vertices. This accounts for one real parameter.

Let $\overleftrightarrow{t_1}$ and $\overleftrightarrow{t_2}$ be the tangents to the circle at p_1 and p_2 respectively. Again, the third vertex ν_3 is contained inside exactly one of the two disjoint triangles bounded by $\overleftrightarrow{t_1}, \overleftrightarrow{t_2}, \overleftrightarrow{e_3}$. Otherwise, a point outside these two triangles, intersects the geodesic carrying e_3, by Property (1.6.2). The triangle in accordance to the chosen ordering of the vertices, is denoted again by Δ and shaded green in the figure.

Now, from Property 1.6.2, the dual to any hyperideal point in Δ is disjoint from e_3. So we get that ν_3 is hyperbolic if and only if it is the truncation of a point in$K := \Delta \smallsetminus \overline{\mathbb{D}}$. In the figure, the triangle has been constructed using one such hyperbolic vertex, truncated from the hyperideal vertex p_3. In this case, all the edges e_1, e_2, e_3 have positive length which is given by the hyperbolic length of the green segments. If it is of parabolic type, then we first choose any point p_3 from the arc $p_1 p_2 := \Delta \cap \partial_\infty \mathbb{D}$ on the circle and then choose any horoball based at p. So the set K in this case is a trivial $\mathbb{R}$-bundle over the contractible one-dimensional space $p_1 p_2$. Finally, the third vertex ν_3 is elliptic if and only if it is a point in $K := \Delta \cap \mathbb{D}$. In the figure, such a point v has been used to complete the triangle. Once again, all the lengths are positive.

(c) Next, suppose that ν_1 and ν_2 are both of elliptic type. See Fig. (2.15). Using the transitive action of G on $\mathbb{D}$, we can assume that ν_1 is the origin $(0, 0)$. Up to a rotation around the origin, the vertex ν_2 can be considered to be on the open line segment with endpoints on $(0, 0)$ and $(1, 0)$ inside $\mathbb{D}$. So the two vertices are determined by the hyperbolic distance between

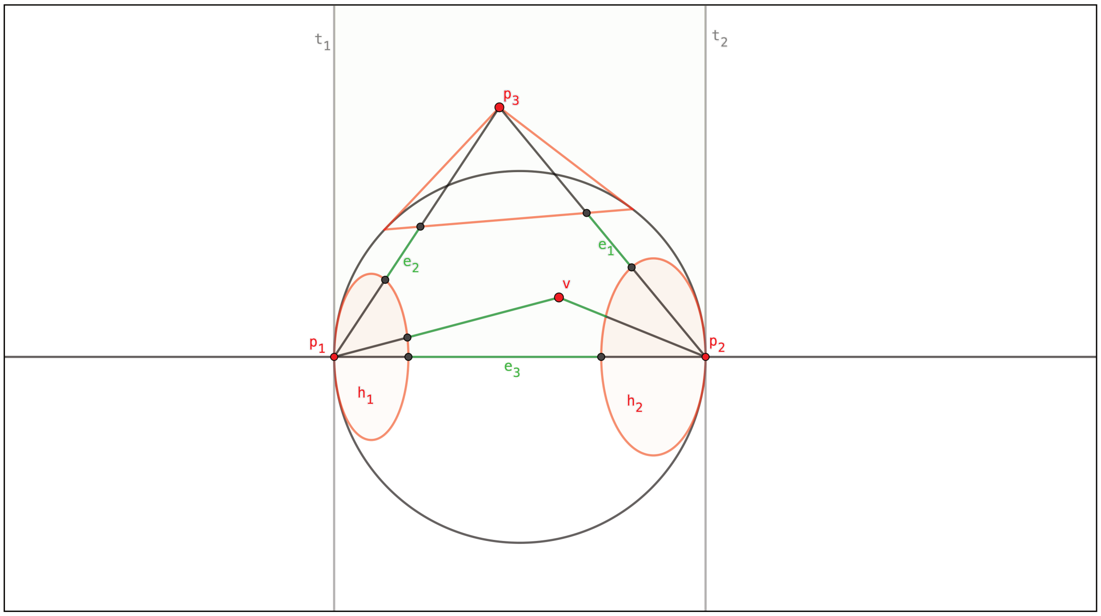


Figure 2.14: Base step Case (b)

ν_1 and ν_2, which varies in the interval $(0, \infty)$. The edge $\overline{e_3}$ joining the two real points ν_1 and ν_2 when extended, meets the dual lines ${\nu_1}^{\perp}, {\nu_2}^{\perp}$ at hyperideal points. Then the region where

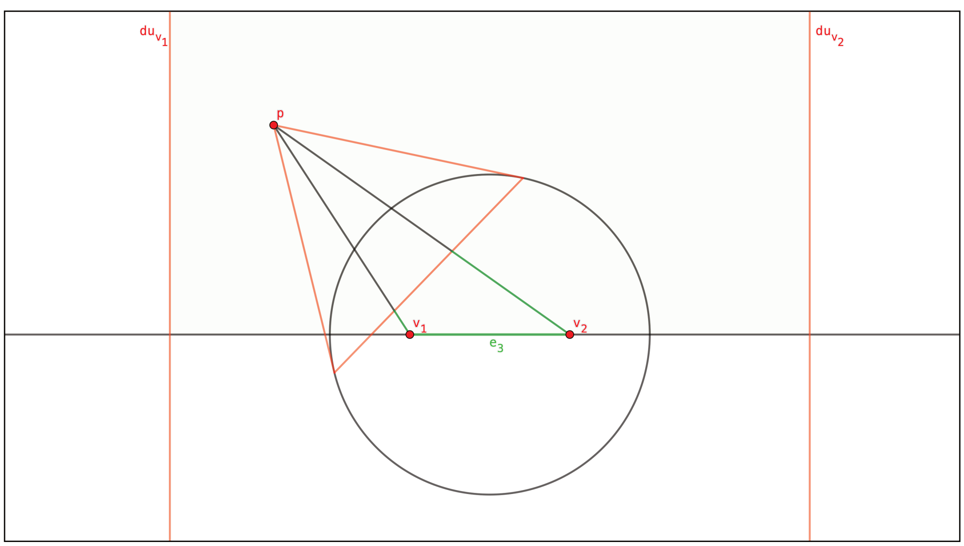


Figure 2.15: Base step Case (c)

the third vertex must lie is the projective triangle Δ bounded by $\nu_1{}^\perp, \nu_2{}^\perp$ and $\overleftrightarrow{e_3}$ which is in accordance with the ordering. Using Property 1.6.2, we get that the ν_3 can be hyperbolic if and only if its dual is a point in $K = \Delta \setminus \overline{\mathbb{D}}$. In the figure, one such hyperbolic ν_3 has been drawn. Next, the vertex ν_3 is parabolic if and only if its center is a point in $K = \Delta \cap \partial_\infty \mathbb{D}$. Finally, ν_3 is elliptic if and only if it is a point in $K = \Delta \cap \mathbb{D}$.

(d) Next, we suppose that ν_1 is of hyperbolic type and ν_2 is of parabolic type. See Figure (2.16). The vertex ν_1 is carried by a geodesic l whose endpoints are denoted by u, v; its dual is a space-like point p_1. The second vertex ν_2 an ideal point p_2, decorated with a horocycle h. Due to the given ordering of the initial vertices, the cyclic ordering in the trigonometric sense of these three ideal points is $u - v - p$. Using triple transitivity of the action of G on $\partial_\infty \mathbb{D}$, we can fix u, v and p_2. Then, the vertices are completely determined by the radius of h; this accounts for one real parameter. The edge $\overline{e_3}$, when extended, passes through $l^\perp$ and p.

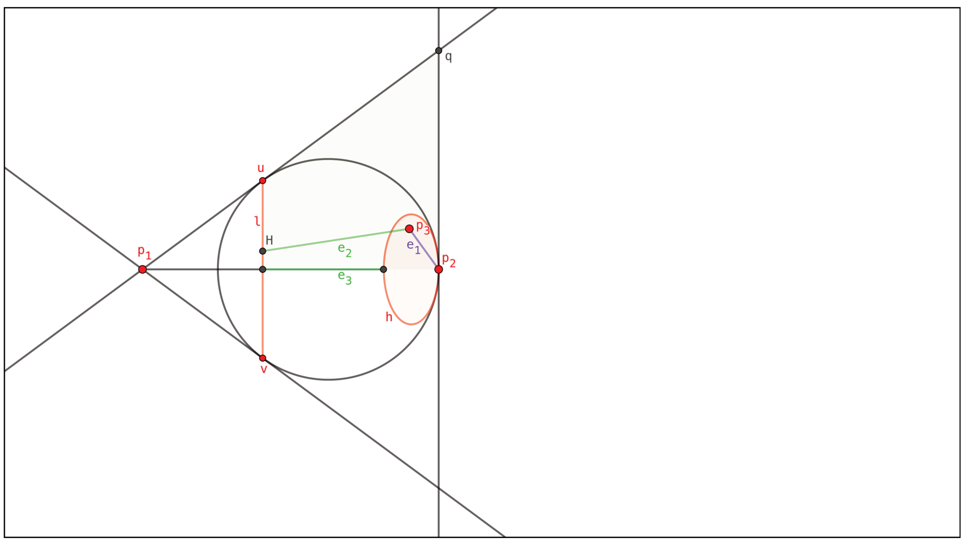


Figure 2.16: Base step Case (d)

The property (1.6.2) implies that the rest of the polygon is contained entirely in one of the projective triangles based at l. In this case, it is the projective triangle, denoted by $\Delta p_1 uv$ that contains p_2. Next we draw tangents $\overleftrightarrow{t_1}, \overleftrightarrow{t_2}, \overleftrightarrow{t_3}$ to $\partial_\infty \mathbb{D}$ at u, v and p_2, respectively. Let q_1, q_2 be the intersection points $\overleftrightarrow{t_1} \cap \overleftrightarrow{t_3}$, $\overleftrightarrow{t_2} \cap \overleftrightarrow{t_3}$, respectively. Then due to convexity, the third vertex ν_3 must lie inside exactly one of the two triangles $\Delta q_1 p_1 p_2$ and $\Delta q_2 p_1 p_2$ that intersects $\mathbb{D}$. The trigonometric ordering $\nu_1 - \nu_2 - \nu_3$ forces ν_3 to lie inside the former. Let $\Delta := \Delta p_1 uv \cap \Delta q_1 p_1 p_2$. Then Δ is a pentagon, which is shaded green in the figure.

Suppose that ν_3 is of hyperbolic type. Then it is carried by a geodesic which is disjoint from l and e_3. So the dual point $\nu_3{}^\perp$ is a hyperideal point in the triangle Δpqu. From above we also know that $\nu_3{}^\perp \in K := \Delta pqv \setminus \Delta_0$. Then ν_3 lies inside Δ if and only if $\nu_3{}^\perp$ is a hyperideal point inside K, which is homeomorphic to an open disk. If ν_3 is of parabolic type, then the centre of the horocycle carrying this vertex can be any point from the open arc $K = \overline{\Delta} \cap \partial_\infty \mathbb{D}$.

Like before, the radius of the horocycle based at such a point can vary in an open interval such that the horocycle does not intersect any other vertex or edge. Finally, the vertex ν_3 is elliptic if and only if it is a point inside $K = \Delta \setminus \Delta_0$, which is homeomorphic to an open disk.

(e) Next, we suppose that ν_1 is parabolic and ν_2 is elliptic. See Figure (2.17). Let h be the horocycle carrying ν_1 with centre at $p_1 \in \partial_\infty \mathbb{D}$. Since ν_2 is elliptic, it is a point p_2 in $\mathbb{D}$ and its dual ${\nu_2}^\perp$ is a space-like line. The edge $\overline{e_3}$ passes through p and ν_2. Using the transitive action of G on $\mathbb{D}$, we fix ν_2 to be the origin $(0,0)$ of the disk and rotate the closed disk $\overline{\mathbb{D}}$ to have $p = (1,0)$. Then the only parameter that determines the two generalised vertices is the horoball h which determines the length of the edge $\overline{e_3}$. Let $\overleftrightarrow{t}$ be the tangent to the unit circle at p_1. Then the third vertex is contained in the interior of exactly one of the two triangles

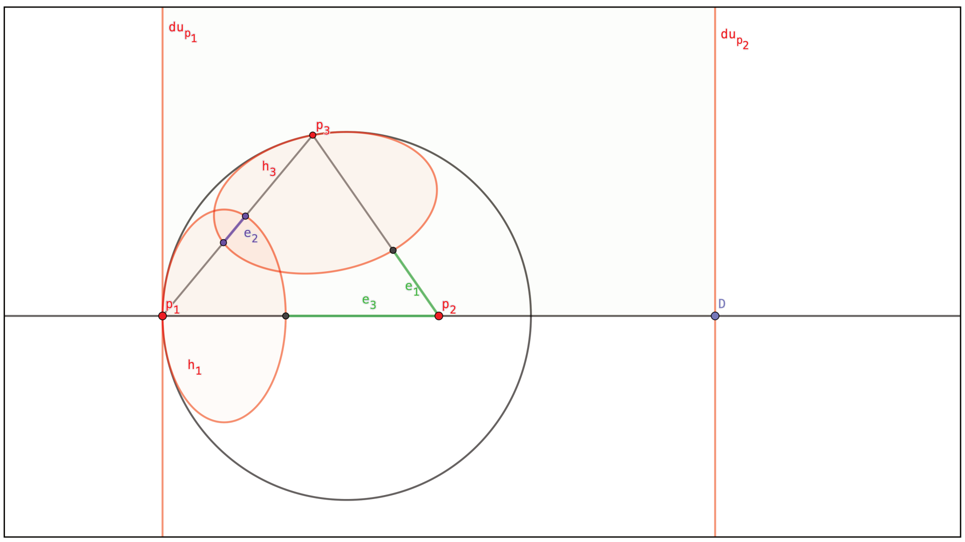


Figure 2.17: Base step Case (e)

bounded by $\overleftrightarrow{t}, \overleftrightarrow{e_3}, {\nu_2}^\perp$. This is because any point outside or on the boundary of these two triangles is either space-like or ideal. If it is hyperideal then its dual segment either intersects $\overline{e_3}$ or h. If it is ideal, then it is equal to p which makes the triangle degenerate. Denote by Δ that triangle which is in accordance with the ordering ν_1, ν_2, ν_3.Let $\overleftrightarrow{T}$ be the tangent to h from ν_2. Label the points $\overleftrightarrow{T} \cap h$, $\overleftrightarrow{T} \cap \partial_\infty \mathbb{D}$ and $\overleftrightarrow{T} \cap \overleftrightarrow{t_2}$ by a, b and c respectively. Then any line joining ν_2 and a point in the closed region Δ_0 bounded by $\overleftrightarrow{T}$, h and $\overleftrightarrow{t}$, contained in Δ, intersects h. So ν_3 is completely contained inside $\Delta \setminus \Delta_0$.

Suppose that ν_3 is of hyperbolic type. Then it is carried by a geodesic which is disjoint from h and e_3. So the dual ${\nu_3}^\perp$ is a hyperideal point in the triangle Δpqu. From above we also know that ${\nu_3}^\perp \in K := \Delta pqv \setminus \Delta_0$. Then ν_3 lies inside Δ if and only if ${\nu_3}^\perp$ is a hyperideal point inside K, which is homeomorphic to an open disk.

If ν_3 is of parabolic type, then the centre of the horocycle carrying this vertex can be any point from the open arc ub of $\partial_\infty \mathbb{D}$. Like before, the radius of the horocycle based at such a point can vary in an open interval such that it does not intersect any other vertex or edge. Finally, if ν_3 is of elliptic type then it can be any point inside $\Delta \smallsetminus \Delta_0$, which is homeomorphic to an open disk.

(f) Finally, we suppose that ν_1 is hyperbolic and ν_2 is elliptic. The vertex ν_1 is carried by a geodesic l whose endpoints are denoted by u, v. We fix ν_2 to be the origin of the disk and rotate $\overline{\mathbb{D}}$ to fix v. Then the two vertices are determined by the position of v which varies in an open interval in $\partial_\infty \mathbb{D} \smallsetminus \{u\}$. The third edge $\overline{e_3}$ when extended, passes through the hyperideal point ${\nu_1}^\perp$ and intersects the hyperideal line ${\nu_2}^\perp$ at a hyperideal point p. Let $\overleftrightarrow{t}$ be the tangent to $\partial_\infty \mathbb{D}$ at u. It intersects ${\nu_2}^\perp$ at a hyperideal point q. The third vertex ν_3 is contained in the open quadrilateral Q bounded by $\overleftrightarrow{t}, {\nu_2}^\perp, \overleftrightarrow{l}, \overleftrightarrow{e_3}$ such that its closure contains u. If ν_3 is of hyperbolic then K is given by the hyperideal points of Q. If ν_3 is of parabolic type, then K is given by the trivial bundle over the arc $K := Q \cap \partial_\infty \mathbb{D}$. Finally ν_3 is elliptic if and only if it is a point of $K := Q \cap \mathbb{D}$.

This concludes the base step.

Suppose that the statement is true for $n = 3, \dots, k$. In the induction step, there are three possibilities for the new vertex—elliptic, parabolic or hyperbolic. Without loss of generality, we assume that the elliptic vertices are added on at the end. So if the $(k+1)$-th vertex is non-elliptic then so are all of the previous ones.

Let $\Pi_{k+1}^{\oslash}$ be the polygon obtained from $\Pi_k^{\oslash}$ by adding a vertex ν between two vertices ν_i and ν_{i+1}; let $\overline{e}$ be the edge joining them. Let e be the infinite geodesic carrying $\overline{e}$. It divides $\mathbb{D}$ into two half spaces: the one containing $\Pi_k^{\oslash}$ is denoted by H_1, and the other half space, denoted by H_2, must contain the new vertex ν. Let e', e'' be the edges joining ν_{i+2}, ν_{i+1} and ν_i, ν_{i-1} respectively. Let Δ be the triangle bounded by the projective lines $\overleftrightarrow{e}, \overleftrightarrow{e'}, \overleftrightarrow{e''}$ that intersects H_2. Then Property 2.3.1 applied to the new polygon $\Pi_{k+1}^{\oslash}$ gives that the truncated vertex ν is entirely contained in the interior of $H_2 \cap \Delta$. Suppose that ν_i and ν_{i+1} are both elliptic. See Fig. (2.18). Then, these are also the intersection points $e \cap e''$ and $e \cap e'$, respectively. Also, owing to the order chosen, the new vertex ν is of elliptic type. In the figure, the region H_2 is shaded yellow and the triangle Δ is shaded blue.Then the polygon $\Pi_{k+1}^{\oslash}$ is convex if and only if ν is a point in $H_2 \cap \Delta$, which is homeomorphic to an open ball (shaded green in the figure). So using the induction hypothesis, we have that $\mathfrak{D}(\Pi_{k+1}^{\oslash}) \simeq \mathfrak{D}(\Pi_k^{\oslash}) \times \mathbb{D}^2 \simeq \mathbb{B}^{2k-1}$.

Next, we suppose that ν_i is parabolic and ν_{i+1} is hyperbolic. Then ν_i is an ideal point p decorated with a horoball h and ν_{i+1} is the truncation of a hyperideal point q, supported on the infinite geodesic l, whose endpoints are ideal points $u, v \in \partial_\infty \mathbb{D}$. The edge $\overline{e}$ joining these two vertices is a geodesic ray with the finite endpoint on l and the infinite end converging to p. It is carried by the straight line containing p, q. Let t_u, t_v are tangents to the boundary of the unit circle at u, v. Using the convexity condition (2.3.1) on ν_{i+1}, we have that the polygon $\Pi_k^{\oslash}$ as well as the $k+1$-th vertex ν are both contained in that associated triangle Δquv based at l that contains the point p. The projective line $\overleftrightarrow{e'}$, carrying the edge e' that joins the two vertices ν_{i+2}, ν_{i+1}, passes through q. Similarly, the projective line $\overleftrightarrow{e''}$ carrying the edge e'' that joins the two vertices ν_i, ν_{i-1}, passes through p. As in the previous case, we shall denote by Δ the projective triangle bounded by the three lines $\overleftrightarrow{e}, \overleftrightarrow{e'}, \overleftrightarrow{e''}$ that intersects the half plane H_2. So, $\nu \in \Delta$. Now, the new vertex ν is elliptic if and only if it is a real point in the interior of $R := H_2 \cap \Delta \cap \Delta quv$, which is a open disk of dimension 2. Next, we have that the new vertex can be parabolic if and only if it is an ideal point inside $R := \overline{H_2} \cap \Delta \cap \Delta quv$. Now, $\partial_\infty \mathbb{D} \cap \overline{H_2} = \partial_\infty \mathbb{D} \cap \Delta$ is the semicircular arc containing

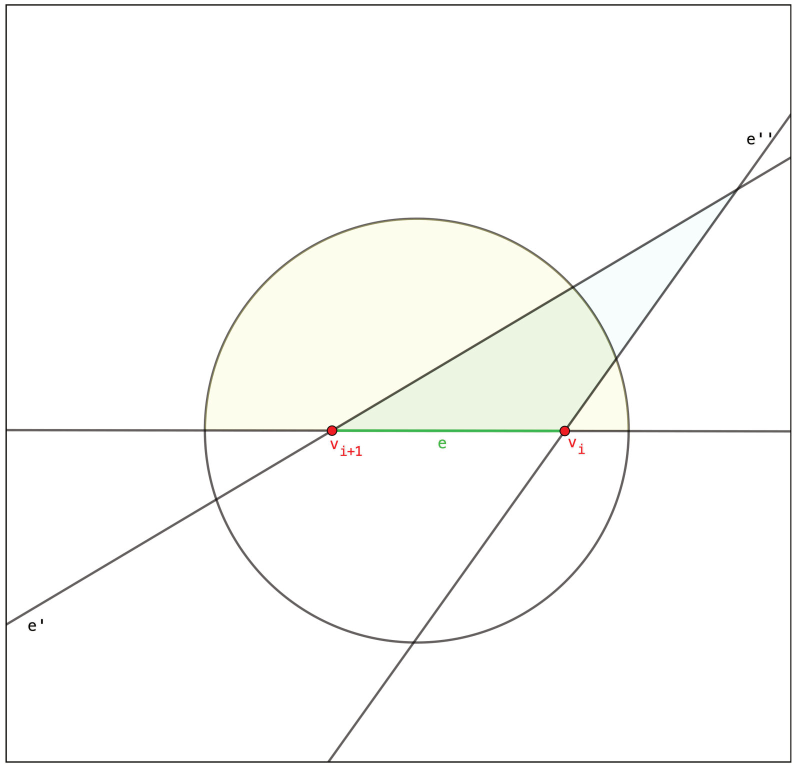


Figure 2.18: Induction step: Both ν_i, ν_{i+1} are elliptic

u, p and $\overleftrightarrow{e} \cap \partial_\infty \mathbb{D}$; the set $\partial_\infty \mathbb{D} \cap \Delta quv$ is the circular arc joining u, v and containing p. Since, $\overleftrightarrow{e}$ separated u, v, we have that $R \cap \partial_\infty \mathbb{D}$ is the arc joining u, p that is contained in the boundary of H_2. Hence we get that $\mathfrak{D}(\Pi^{\oslash}_{k+1}) \simeq \mathfrak{D}(\Pi^{\oslash}_k) \times \mathbb{D}^1 \times \mathbb{D}^1 \simeq \mathbb{D}^{2k-1}$. We are left with the case where ν is hyperbolic, i.e., it is a geodesic segment in $\mathbb{D}$ carried by an infinite geodesic l_1. Since l' must be disjoint from both l, e, Property 1.6.2, the dual ${l_1}^{\perp}$ is a hyperideal point inside $\Delta quv \cap \Delta_1$, where $\Delta_1, \Delta quv$ (shaded in blue, hatched in the figure) are the associated triangles based at e, l, respectively. Note that Δ contains the triangle Δ_1. Hence, we have that ν is hyperideal if and only if $\nu^{\perp}$ is a point in $\mathrm{int}\,(\Delta quv \cap \Delta_1) \setminus \overline{\mathbb{D}}$, which is a an open 2-ball (blue cross-hatched region). So using the induction hypothesis, we have that $\mathfrak{D}(\Pi^{\oslash}_{k+1}) \simeq \mathfrak{D}(\Pi^{\oslash}_k) \times \mathbb{D}^2 \simeq \mathbb{B}^{2k-1}$. Next suppose that ν_i is elliptic. See Fig. 2.20. Then the $(k+1)$-th vertex ν has to be elliptic as well. The required region is given by $R' := H_2 \cap \Delta \cap \Delta quv \cap \Delta_1$ (cross-hatched yellow region in the figure).

$\square$

2.4 Infinitesimal Deformations

In this section, we shall study infinitesimal deformations of hyperbolic surfaces with boundary.

Definition 2.4.1. An infinitesimal deformation of a point $m \in \mathfrak{D}(S)$ is a vector of $T_m\mathfrak{D}(S)$.

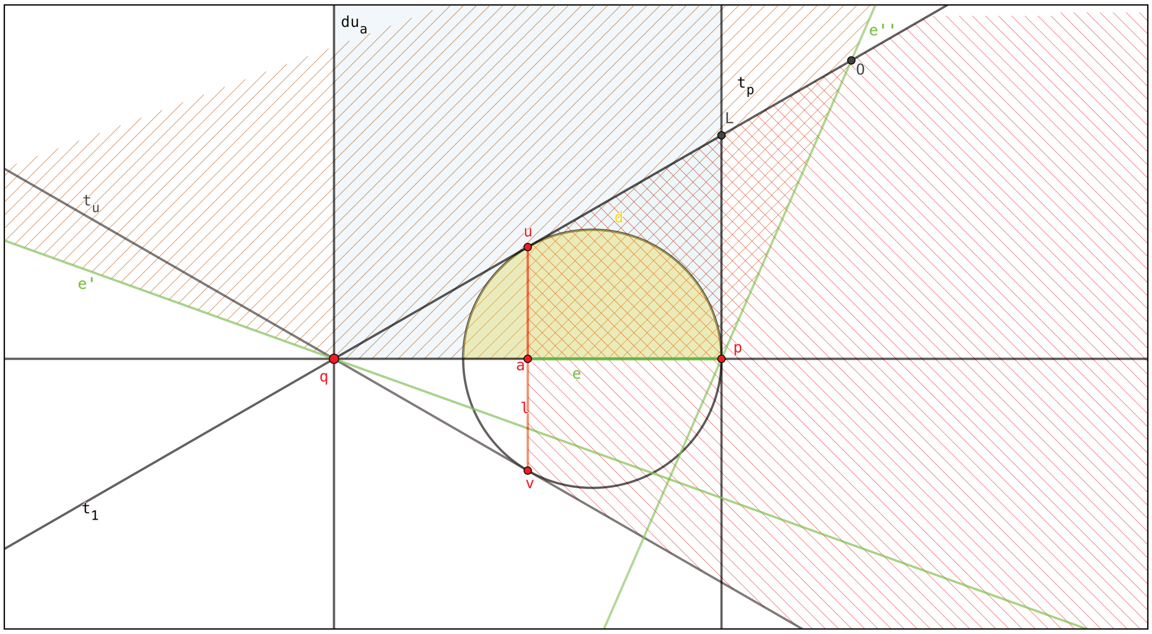


Figure 2.19: Induction step: ν_i is parabolic, ν_{i+1} is hyperbolic

The first subsection is dedicated to the compact surfaces, which is followed by non-compact surfaces with spikes, and finally hyperbolic polygons.

2.4.1 Compact Surfaces

Let $S_c = S_{g,n}$ or $T_{h,n}$ be a compact surface with non-empty boundary equipped with a hyperbolic structure as above. Let $G = \mathrm{PGL}(2,\mathbb{R}) \cong \mathrm{SO}(2,1)$. An *infinitesimal deformation* of its holonomy representation $\rho : \pi_1(S_c) \longrightarrow G$ is a vector of $T_\rho \mathrm{Hom}(\pi_1(S_c), G)$. It can be seen as an equivalence class of smooth paths $\{\rho_t\}_{t\in\mathbb{R}}$ with $\rho_0 = \rho$. Given such a path, we have that

$$\frac{\mathrm{d}\rho_t}{\mathrm{d}t}\Big|_{t=0}(\pi_1(S_c)) \subset TG \simeq G \ltimes \mathfrak{g},$$

in other words, for every $\gamma \in \pi_1(S)_c$, $\frac{\mathrm{d}\rho_t}{\mathrm{d}t}\Big|_{t=0}(\gamma) = (\rho(\gamma), u(\gamma))$,

Lemma 2.4.2. *The map $u : \pi_1(S_c) \longrightarrow \mathfrak{g}$ defined above satisfies the following equation:*

$$\textit{for every } \gamma_1, \gamma_2 \in \pi_1(S_c),\ u(\gamma_1\gamma_2) = u(\gamma_1) + \mathrm{Ad}(\rho(\gamma_1)) \cdot u(\gamma_2). \tag{2.2}$$

Proof. Rewriting the expression for ρ_t in the following way

$$\rho_t(\gamma) = \rho(\gamma)\mathrm{e}^{tu(\gamma)+o(t)}, \text{ for every } t \in \mathbb{R} \text{ and } \gamma \in \pi_1(S_c), \tag{2.3}$$

we get that

$$u(\gamma) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\rho_t(\gamma)\rho(\gamma)^{-1}.$$

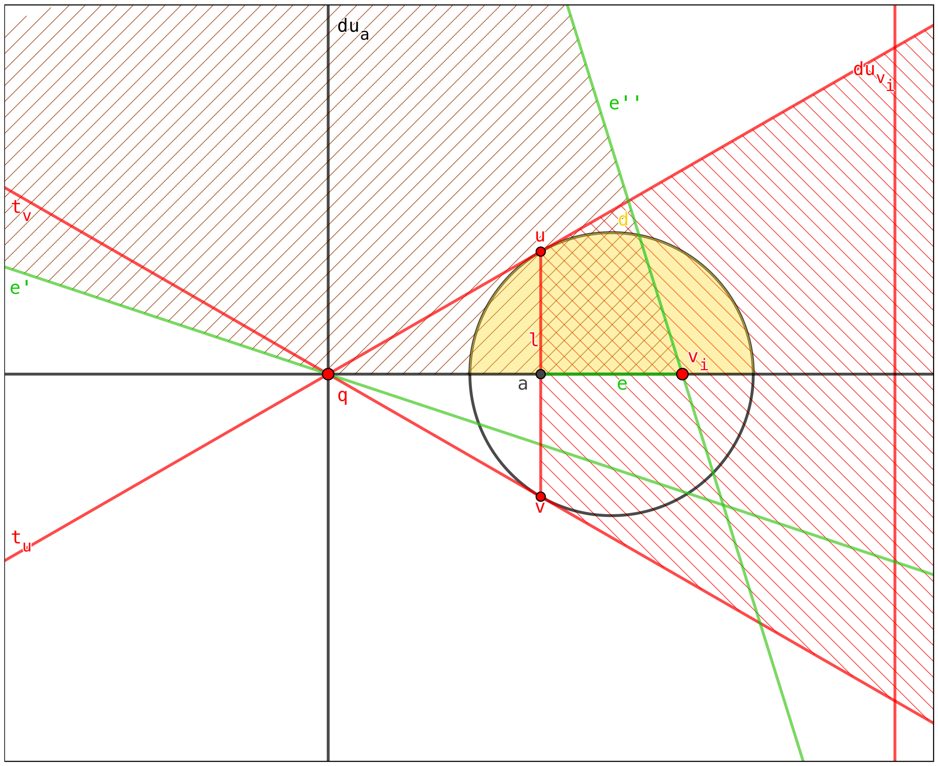


Figure 2.20: Induction step: ν_i is elliptic, ν_{i+1} is hyperbolic

Then for $\gamma_1, \gamma_2 \in \pi_1(S_c)$,

$$\begin{aligned}
u(\gamma_1\gamma_2) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \rho_t(\gamma_1)\rho_t(\gamma_2)\rho(\gamma_2)^{-1}\rho(\gamma_1)^{-1} \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \rho_t(\gamma_1)\rho(\gamma_2) + \rho(\gamma_1)\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\rho_t(\gamma_2)\right]\rho(\gamma_2)^{-1}\rho(\gamma_1)^{-1} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \rho_t(\gamma_1)\rho(\gamma_1)^{-1} + \rho(\gamma_1)\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\rho_t(\gamma_2)\rho(\gamma_2)^{-1}\right)\rho(\gamma_1)^{-1} \\
&= u(\gamma_1) + \mathrm{Ad}(\rho(\gamma_1)) \cdot u(\gamma_2).
\end{aligned}$$

□

A map $u : \pi_1(S_\heartsuit) \longrightarrow \mathfrak{so}_{2,1}$, satisfying (2.2) is called a ρ-cocycle.

Definition 2.4.3. A *ρ-coboundary* is a ρ-cocycle u such that for some $v_0 \in \mathfrak{g}$

$$u(\gamma) = \mathrm{Ad}(\rho(\gamma))v_0 - v_0, \text{ for every } \gamma \in \pi_1(S_c). \tag{2.4}$$

Two ρ-cocycles are equivalent if they differ by a coboundary. The set of equivalence classes of all ρ-cocycles forms the first cohomology group $\mathrm{H}^1_\rho(\pi_1(S_c), \mathfrak{g})$. An element $[u]$ of this group is an infinitesimal deformation of the metric $[\rho]$, i.e., $[u] \in T_m\mathfrak{D}(S_c)$.

Next, we will define a specific type of infinitesimal deformation of a compact surface, known as an admissible deformation.

Definition 2.4.4. Let S_c be a compact (possibly non-orientable) hyperbolic surface with non-empty boundary. Let $m \in \mathfrak{D}(S_c)$ and $v \in T_m\mathfrak{D}(S_c)$. Then v is said to be an *admissible* deformation of m if it satisfies:

$$\inf_{\gamma \in \Gamma \smallsetminus \{Id\}} \frac{\mathrm{d}l_\gamma(m)(v)}{l_\gamma(m)} > 0, \tag{2.5}$$

where l_γ is the length function as introduced in Definition (2.1.6).

In other words, an infinitesimal deformation is admissible if and only if the length of every non-trivial closed loop of S_c is uniformly lengthened. The following theorem was proved by Goldman-Labourie-Margulis in [9]

Theorem 2.4.5. *The set of all admissible deformations of a compact hyperbolic surface S_c with non-empty totally geodesic boundary forms an open convex cone of $T_m\mathfrak{D}(S_c)$.*

2.4.2 Surfaces with non-decorated and decorated spikes

Let S_{sp} be a hyperbolic surface with non-decorated spikes endowed with a metric $m \in \mathfrak{D}(S_{sp})$. The set of infinitesimal deformations of the metric is given by $T_m\mathfrak{D}(S_{sp})$. Let $[\rho]$ be the metric on the convex core and $V_0 \subset T_m\mathfrak{D}(S_{sp})$ be the set of infinitesimal deformations that do not deform S_c. Then the infinitesimal deformation space $T_{[\rho]}\mathfrak{D}(S_c) \simeq T_m\mathfrak{D}(S_{sp})/V_0$. Denote the quotient map $\pi : T_m\mathfrak{D}(S_{sp}) \longrightarrow T_m\mathfrak{D}(S_{sp})/V_0$.

Like in the case of compact surfaces, we will now discuss about admissible deformations. Let S_{sp} be a hyperbolic surface with non-decorated spikes equipped with a metric $m \in \mathfrak{D}(S_{sp})$. Let $[\rho]$ be the metric on its convex core S_c. Then the admissible cone $\Lambda(m)$ is defined as

$$\Lambda(m) := \pi^{-1}(\Lambda([\rho]))$$

It is the set of all infinitesimal deformations of m that lengthens every non-trivial closed loop. This is a vector bundle over the admissible cone of the convex core S_c of the surface. Every fiber is isomorphic to $V_0 \cong \mathbb{R}^Q$ containing the information of the movement of all the spikes.

Remark 2.4.1. In Chapter 5, we shall be parametrising the elements of $T_M(\mathfrak{D}_0)$ using the spinning arc complex, where $M \in \mathfrak{D}_0$. These are infinitesimal deformations of a one-holed polygon endowed with a metric M that fix the boundary loop pointwise.

Next we define admissible deformations for hyperbolic surfaces with decorated spikes.

Definition 2.4.6. Let S^h_{sp} be a hyperbolic surface with decorated spikes. The *admissible cone* for a given metric $m \in \mathfrak{D}(S^h_{sp})$, denoted by $\Lambda(m)$, is the set of all infinitesimal deformations of m that uniformly lengthens every horoball connection.

If the decoration on S^h_{sp} is such that the closures of the decorating horoballs are all pairwise disjoint, then an element $v \in T_m\mathfrak{D}(S^h_{sp})$ is admissible if and only if it satisfies the following condition:

$$\inf_{\beta \in \mathcal{H}} \frac{\mathrm{d}l_\beta(m)(v)}{l_\beta(m)} > 0, \tag{2.6}$$

where l_β is the length function as in Definition (2.2.10) and $\mathcal{H}$ is the set of all horoball connections.

Remark 2.4.2. Let S_{sp} be the underlying hyperbolic surface with undecorated spikes of S^h_{sp}. Let v be an admissible deformation of (S^h_{sp}, m). Then v is an admissible deformation for any decoration of S_{sp} such that closures of the decorating horoballs are pairwise disjoint.

If some of the decorating horoballs of the spikes of S^h_{sp} overlap then an admissible v satisfies

$$\inf_{\beta \in \mathcal{H}^-} \mathrm{d}l_\beta(m)(v) > 0,$$

where $\mathcal{H}^-$ is the set of horoball connections with non-positive length, and (2.6) for horoball connections with positive length.

Lemma 2.4.7. *Let $v \in \Lambda(m)$ be an admissible deformation of a surface S_{sp} endowed with a metric $m \in \mathfrak{D}(S_{sp})$. Then v satisfies* (2.5)*.*

Lemma 2.4.8. *The subspace $\Lambda(m)$ is an open convex cone of $T_m\mathfrak{D}(S)$.*

2.4.3 Generalised Polygons

Definition 2.4.9. Given a polygonal surface Π, a vector in the tangent space $T_m\mathfrak{D}(\Pi)$ is called an *infinitesimal deformation* of Π.

Definition 2.4.10. The *admissible cone* of a decorated polygonal surface $\Pi_n^{\circledcirc}$ is defined to be the set of all infinitesimal deformations of a metric $m \in \mathfrak{D}(\Pi_n^{\circledcirc})$, such that all the generalised vertices are moved away from each other. It is denoted by $\Lambda(m)$.

Lemma 2.4.11. *The admissible cone of a decorated polygon $\Pi_n^{\circledcirc}$, endowed with a metric m, is an open convex subset of $T_m\mathfrak{D}(\Pi_n^{\circledcirc})$.*

Proof. Two generalised vertices are moved away from each other if and only if the length of the edge joining them increases. Let $l_1, \ldots, l_N$ be the set of all edges and diagonals of the polygon. Then we can define the following smooth positive function for every $i = 1, \ldots, N$:

$$\begin{array}{rrl} l_i : & \mathfrak{D}(P) \longrightarrow & \mathbb{R}_{>0} \\ & m \mapsto & \text{length of } l_i \text{ w.r.t } m. \end{array}$$

An infinitesimal deformation v increases the length of l_i if and only if $dl_i(v) > 0$. So the admissible cone can be written as

$$\Lambda(m) = \bigcap_{i=1}^{N} \{dl_i > 0\},$$

which is open and convex in $T_m\mathfrak{D}(\Pi_n^{\circledcirc})$. $\square$

Chapter 3

Arcs and arc complexes

3.1 Arcs

An *arc* on a hyperbolic surface S with non-empty boundary, possibly with spikes, is an embedding α of a closed interval $I \subset \mathbb{R}$ into S. There are three possibilities depending on the nature of the interval:

1. $I = [a, b]$: In this case, the arc α is finite. We consider those finite arcs that verifiy: $\alpha(a), \alpha(b) \in \partial S$ and $\alpha(I) \cap S = \{\alpha(a), \alpha(b)\}$.

2. $I = [a, \infty)$: These are embeddings of hyperbolic geodesic rays in the interior of the surface such that $\alpha(a) \in \partial S$. There are two types:

   - The infinite end converges to a spike, i.e., $\alpha(t) \overset{t \to \infty}{\longrightarrow} x$, where x is a spike.
   - The infinite end spirals around a totally geodesic boundary component of the surface. They are called *spiraling* arcs.

3. $I = \mathbb{R}$: The infinite ends can either converge to a spike or spiral along a simple closed curve.

Definition 3.1.1. An arc α of a hyperbolic surface S with non-empty boundary is called *non-trivial* if each connected component of $S \smallsetminus \{\alpha\}$ has at least one spike or generalised vertex.

Let $\mathscr{A}$ be the set of all non-trivial arcs of the three types above. Two arcs $\alpha, \alpha' : I \longrightarrow S$ in $\mathscr{A}$ are said to be *isotopic* if there exists a homeomorphism $f : S \longrightarrow S$ that preserves the boundary and fixes all generalised vertices or (possibly decorated) spikes and a continuous function $H : S \times [0, 1] \longrightarrow S$ such that

1. $H(\cdot, 0) = \mathrm{Id}$ and $H(\cdot, 1) = f$,

2. for every $t \in [0, 1]$, the map $H(\cdot, t) : S \longrightarrow S$ is a homeomorphism,

3. for every $t \in I$, $f(\alpha(t)) = \alpha'(t)$.

We shall now give a formal definition of the arc complex.

Definition 3.1.2. The *arc complex* of a surface S, generated by a subset $\mathcal{K} \subset \mathscr{A}$, is a simplicial complex $\mathcal{A}(S)$ whose base set $\mathcal{A}(S)^{(0)}$ consists of the isotopy classes of arcs in $\mathcal{K}$, and there is an k-simplex for every $(k+1)$-tuple of pairwise disjoint and distinct isotopy classes.

The elements of $\mathcal{K}$ are called *permitted* arcs and the elements of $\mathscr{A} \smallsetminus \mathcal{K}$ are called *rejected* arcs. The permitted arcs are the building blocks of the different arc complexes. They are used to perform strip deformations of the surface. The way the surface is deformed depends on the nature of the arc used for the strip deformation. This shall be discussed in detail in Chapter 5.

Next we specify the elements of $\mathcal{K}$ for the different types of surfaces:

- In the case of a hyperbolic surface with non-decorated spikes, the set $\mathcal{K}$ of permitted arcs comprises of non-trivial finite arcs that separate at least two spikes from the surface.

- In the case of a hyperbolic surface with decorated spikes, the set $\mathcal{K}$ of permitted arcs comprises of non-trivial finite arcs and infinite arcs of type 2 whose infinite ends converge to spikes and whose finite ends lie on the boundary of the surface.

- In the case of generalised polygons, an arc is permitted if either both of its endpoints lie on two distinct edges of $\Pi_n^{\circledast}$ (*edge-to-edge* arc) or exactly one endpoint lies on a generalised vertex (*edge-to-vertex* arc).

- Finally, for a one holed-polygon $\Pi_n^{\circledcirc}$ with $n \geq 1$, we shall consider the *spinning* arc complex, denoted by $\mathcal{A}_{\circlearrowright}(\Pi_n^{\circledcirc})$, which is generated by non-trivial finite arcs that separate at least two spikes from the surface and the infinite arcs of the second type that spiral along its totally geodesic boundary.

Remark 3.1.1.

1. Two isotopy classes of arcs of S are said to be disjoint if it is possible to find a representative arc from each of the classes such that they are disjoint in S. Such a configuration can be realised by geodesic segments in the context of hyperbolic surfaces. In our discussion, we shall always choose such arcs as representatives of the isotopy classes.

2. In the cases of ideal and punctured polygons, we shall choose those geodesic arcs whose lifts are supported on projective lines that intersect outside $\mathbb{RP}^2 \smallsetminus \overline{\mathbb{D}}$.

3. The surfaces with non-decorated spikes that have finite arc complexes are ideal, punctured, one holed polygons and Möbius strip with spikes.

Definition 3.1.3. The 0-skeleton $\sigma^{(0)}$ of a top-dimensional simplex σ of the arc complex $\mathcal{A}(S)$ is called a *triangulation* of the surface S.

Definition 3.1.4. A finite arc of a one-holed ideal polygon or a once-punctured ideal polygon is called *maximal* if both its endpoints lie on the same connected component or edge.

Definition 3.1.5. A finite arc of a surface with non-decorated spikes is called *minimal* if it separates a quadrilateral with two ideal points from the surface.

3.1.1 Pruned arc complexes

Definition 3.1.6. We define a *big* simplex of the arc complex of the different types of surfaces:

- For a surface with non-decorated spikes, a simplex σ is said to be big if the arcs corresponding to $\sigma^{(0)}$ decompose the surface into topological disks.
- For a surface with decorated spikes, a simplex σ is said to be big if the arcs corresponding to $\sigma^{(0)}$ decompose the surface into topological disks with at most one spike.
- For a generalised polygon, a simplex σ is said to be big if the arcs corresponding to $\sigma^{(0)}$ decompose the surface into topological disks with at most one generalised vertex.

From the definition it follows that any simplex containing a big simplex is also big.

Definition 3.1.7. The *pruned arc complex* of a surface S is the union of the interiors of the big simplices of the arc complex $\mathcal{A}(S)$.

If S is a surface with arc complex $\mathcal{A}(S)$, then its pruned arc complex is denoted by $\widehat{\mathcal{A}}(S)$. Every point $x \in \widehat{\mathcal{A}}(S)$ is contained in the interior of a unique simplex, denoted by σ_x, i.e., there is a unique family of arcs $\{\alpha_1, \ldots, \alpha_p\}$, namely the 0-skeleton of σ_x, such that

$$x = \sum_{i=1}^{p} t_i \alpha_i, \sum_{i=1}^{p} t_i = 1, \text{ and } \forall i, t_i > 0.$$

Define the *support* of a point $x \in \widehat{\mathcal{A}}(S)$ as $\operatorname{supp}(x) := \sigma_x^{(0)}$.

3.2 Arc complexes of hyperbolic polygons

In this section, we shall discuss the topology of the arc complexes of hyperbolic polygons.

3.2.1 Ideal and Punctured Polygons

To every ideal polygon $\Pi_n^{\pentagon}$, one can associate a Euclidean regular polygon with n vertices, denoted by $\mathcal{P}_n$, in the following way:

- The vertices of $\mathcal{P}_n$ correspond to the infinite geodesics of the boundary of $\Pi_n^{\pentagon}$,
- Two vertices in $\mathcal{P}_n$ are consecutive if and only if the corresponding infinite geodesics have a common ideal endpoint.

Then we have the following bijection:

$$\left\{\text{Isotopy classes of permitted arcs of } \Pi_n^{\pentagon}\right\} \leftrightarrow \{\text{Diagonals of } \mathcal{P}_n\}$$

Two distinct isotopy classes are pairwise disjoint if and only if the corresponding diagonals in $\mathcal{P}_n$ don't intersect inside $\mathcal{P}_n$. However, the diagonals are allowed to intersect at vertices – this takes place whenever the arcs have exactly one endpoint on a common edge of the ideal polygon. One can construct the arc complex of $\mathcal{P}_n$ in the same way as before and one has $\mathcal{A}(\mathcal{P}_n) = \mathcal{A}\left(\Pi_n^{\pentagon}\right)$. Then a classical result from combinatorics states that

Theorem 3.2.1. *The arc complex $\mathcal{A}(\mathcal{P}_n)$ $(n \geq 4)$ is a sphere of dimension $n-4$. Its dual is an associahedron.*

See [18] for proof.

The following theorem about the arc complex of once-punctured polygons was proved by Penner in [18].

Theorem 3.2.2. *The arc complex $\mathcal{A}(\Pi_n^{\odot})$ of a punctured n-gon,($n \geq 2$), is homeomorphic to a sphere of dimension $n-2$.*

3.2.2 Decorated Hyperbolic Polygons

In this subsection, we shall prove that the pruned arc complex of a decorated hyperbolic polygon is an open manifold. Since the permitted arcs in this case are allowed to have one endpoint on a generalised vertex, we consider the following abstract set up to cover all the cases at the same time.

We start with the polygon $\mathcal{P}_{2n}$ (defined in the previous section))with $n \geq 2$ and partition its vertex set into two disjoint subsets G and R such that $|G| = |R| = n$ and for every pair of consecutive vertices, exactly one belongs to G and the other one belongs to R. Such a polygon is said to have an *alternate partitioning* and shall be denoted by $(\mathcal{P}_{2n}, C_{alt})$, where $C_{alt} := (G, R)$.

To every decorated polygon $\Pi_n^{\oslash}$, one can associate the polygon $(\mathcal{P}_{2n}, C_{alt})$ in the following way:

- a generalised vertex of $\Pi_n^{\oslash}$ corresponds to a vertex of $\mathcal{P}_{2n}$ in R,
- an edge of $\Pi_n^{\oslash}$ corresponds to a vertex of $\mathcal{P}_{2n}$ in G,

such that one R-vertex and one G-vertex are consecutive in $\mathcal{P}_{2n}$ if and only if the corresponding edge and generalised vertex of $\Pi_n^{\oslash}$ are consecutive. Again, we have the bijection:

$$\left\{\text{Isotopy classes of edge-to-edge arcs of } \Pi_n^{\oslash}\right\} \leftrightarrow \{G - G \text{ diagonals}\}$$

$$\left\{\text{Isotopy classes of edge-to-vertex arcs of } \Pi_n^{\oslash}\right\} \leftrightarrow \{G - R \text{ diagonals}\}$$

So the arc complex $\mathcal{A}(\Pi_n^{\oslash})$of a decorated n-gon is isomorphic to that of $\mathcal{P}_{2n}$ generated by the $G-G$ and $G-R$ diagonals. A 0-skeleton of a big simplex of $\mathcal{A}(\mathcal{P}_{2n})$ decomposes the surfaces into smaller polygons none of which has more than one R-vertex.

Lemma 3.2.3. *A simplex σ of the arc complex $\mathcal{A}(\mathcal{P}_{2n})$ is completely contained in ∂X if it is not a big simplex.*

Theorem 3.2.4. *The pruned arc complex $\widehat{\mathcal{A}}(\mathcal{P}_{2n})$, $n \geq 3$ of is an open manifold of dimension $2n-4$.*

Proof. Let $x \in \widehat{\mathcal{A}}(\mathcal{P}_{2n})$ be point which lies in the interior of unique simplex σ_x of dimension k. We need to show that there is a neighbourhood of x in $\widehat{\mathcal{A}}(\mathcal{P}_{2n})$ which is homeomorphic to an open ball of dimension $2n-4$. It suffices to prove that the link of σ_x in the arc complex is a sphere of dimension $2n-5-k$.The $k+1$ arcs of the 0-skeleton of σ_x divide the polygon $\mathcal{P}_{2n}$ into $k+2$ smaller polygons $\mathcal{P}_{n_1}, \ldots, \mathcal{P}_{n_{k+1}}$, with $3 \leq n_i \leq n_0 - 1$ for every $i = 1, \ldots, k+2$.

Lemma 3.2.5. *Let $s := \sum_{r=1}^{k+2} n_r$. Then we have,*

$$s = 2(k+1) + 2n_0.$$

Proof. In the sum $s := \sum_{r=1}^{k+2} n_r$ that counts the total number of edges of all the $k+2$ polygons, each edge e_p of $\mathcal{P}_{n_r}$ is counted $a(e_p)+1$ times, where $a(e_p)$ is the total number of arcs of σ_x that have an endpoint on e_p. □

Since σ_x is a big simplex, we have that none of the smaller polygons contain a $R-R$ diagonal. So each of their arc complexes is a sphere, from Theorem (3.2.1). The link is then given by

$$\begin{aligned}
\mathrm{Link}(\sigma_x, \mathcal{A}(\mathcal{P}_{2n})) &= \mathcal{A}(\mathcal{P}_{n_1}) \bowtie \ldots \mathcal{A}\left(\mathcal{P}_{n_{k+2}}\right) \\
&= \mathbb{S}^{n_1-4} \bowtie \ldots \mathbb{S}^{n_{k+2}-4} && \text{(from Theorem (3.2.1))} \\
&= \mathbb{S}^{s-4(k+2)+k+1} \\
&= \mathbb{S}^{2n_0-5-k} && \text{(from Lemma (3.2.5))}
\end{aligned}$$

□

3.2.3 Spinning arc complex

The main goal of this section is to show that the spinning arc complex generated by finite arcs and spiraling arcs is a sphere. In order to prove that, we shall compute two other arc complexes of an ideal one-holed polygon.

Recall that the finite arcs permitted for a surface with undecorated spikes separate at least two spikes from the surface.

The arc complex generated by the isotopy classes of finite arcs is denoted by $\mathcal{A}_f(\Pi_n^{\circledcirc})$. Its subcomplex generated by the finite arcs whose endpoints lie on the boundary geodesic γ forms a simplex, denoted by η. Also the 0-simplex corresponding to such an arc is big simplex (Definition (3.1.6)) of $\mathcal{A}_f(\Pi_n^{\circledcirc})$. This is because it decomposes the surface $\Pi_n^{\circledcirc}$ into a topological disk — an ideal polygon with $n+2$ vertices, whose arc complex is a sphere of dimension $n-2$.

Theorem 3.2.6. *The arc complex $\mathcal{A}_f(\Pi_n^{\circledcirc})$, $n \geq 1$, is a closed ball of dimension $n-1$.*

Proof. The proof is done by induction on the number of vertices n. For $n=1$, there is only one arc. It has one endpoint on γ. The arc complex $\mathcal{A}_f(\Pi_1^{\circledcirc})$ is then a 0-simplex, which is a closed ball of dimension 0. So the base case is verified.

Suppose that the statement holds for $n = 1, \ldots, n_0 - 1$. Consider the polygon $\Pi_{n_0}^{\circledcirc}$. The link of a 0-simplex corresponding to an arc one of whose endpoints lie on γ is a sphere of dimension n_0, from the above discussion. Next, let c be a finite arc with no endpoints on γ. It divides $\Pi_{n_0}^{\circledcirc}$ into an ideal k-gon and an ideal one-holed (n_0-k+2)-gon with boundary $\Pi_{n_0-k+2}^{\circledcirc}$. Then by induction hypothesis, we have that

$$\mathrm{Link}(c, \mathcal{A}_f(\Pi_{n_0}^{\circledcirc})) = \mathbb{S}^{k-4} \bowtie \mathbb{B}^{n_0-k+1} = \mathbb{B}^{n_0-2}.$$

So the complex $\mathcal{A}_f(\Pi_n^{\circledcirc})$ is a (n_0-1)-dimensional PL-manifold with boundary. The boundary of the complex consists of simplices spanned by arcs that do not intersect γ. This is the arc complex of a punctured n_0-gon. From Theorem 3.2.2, we have that $\partial\mathcal{A}_f(\Pi_{n_0}^{\circledcirc}) \simeq \mathbb{S}^{n_0-2}$. Also any simplex spanned by boundary 0-simplices lies in the boundary as well. This is because the corresponding arcs do not intersect the curve γ. It follows that the boundary is induced (see Definition 1.7.7). So by Theorem 1.7.8, the arc complex collapses onto the simplex generated by the finite arcs that intersect γ. Hence, $\mathcal{A}_\xi\left(\Pi_{n_0}^{\circledcirc}\right)$ is collapsible. Finally from Theorem (1.7.9), it follows that the arc complex is an (n_0-1)-dimensional ball.

□

The arc complex of $\Pi_n^{\circledcirc}$ generated by the isotopy classes of the infinite arcs and those finite arcs whose endpoints do not lie on the boundary loop γ, is called the *spinning* arc complex, and is denoted by $\mathcal{A}_{\circlearrowright}(\Pi_n^{\circledcirc})$. This is a finite simplicial complex whose dimension is $n-1$. We have the following theorem:

Theorem 3.2.7. *The spinning arc complex $\mathcal{A}_{\circlearrowright}(\Pi_n^{\circledcirc})$, $n \geq 1$, is homeomorphic to a sphere of dimension $n-1$.*

Proof. The subcomplex X_1 (resp. X_2) generated by the finite arcs and the spinning arcs, all of which spin in clock-wise (resp. anti-clockwise) direction, is isomorphic to $\mathcal{A}_f(\Pi_n^{\circledcirc})$, which is a ball of dimension $n-1$, from Theorem (3.2.6). Also, the boundaries $\partial X_1, \partial X_2$ of these two balls are identical because they are generated by all the finite arcs. So we have that,

$$\mathcal{A}_{\circlearrowright}\left(\Pi_n^{\circledcirc}\right) = \mathbb{B}^{n-1} \sqcup \mathbb{B}^{n-1}/\mathbb{S}^{n-2} \simeq \mathbb{S}^{n-1}.$$

□

3.3 Arc complex of general surfaces

In this section we shall be studying the topology of the pruned arc complexes of hyperbolic surfaces with undecorated spikes, followed by hyperbolic surfaces with decorated spikes.

3.3.1 Surfaces with undecorated spikes

Firstly, we shall discuss a theorem by Harer which proves that the pruned arc complex of an orientable surface is an open ball. In his paper, the terminology used is different from what we have seen up until now so we shall give a quick introduction to the objects involved in his result. Then by interpreting his result in the appropriate manner, we shall prove that the pruned arc complex in the case of orientable surfaces with spikes is an open ball. Finally, we shall derive the same result for non-orientable surfaces.

Harer's Terminology: Let $S_{g,r,s}$ be an orientable surface of genus g with r boundary components and s punctures:

$$S_{g,r,s} := S_{g,r} \smallsetminus \{y_1, \ldots, y_s\},$$

where $y_1, \ldots, y_s$ are points in the interior of $S_{g,r}$, that play the role of spikeless boundary components in our case. Mark Q distinct points $x_1, \ldots, x_Q$ on the boundary $\partial S_{g,r,s}$ such that each boundary component, denoted by $\partial_i S_{g,r,s}$ for $i = 1, \ldots, r$, contains at least one such point. These points shall play the role of spikes. Let $\Omega = \{x_1, \ldots, x_Q, y_1, \ldots y_s\}$.

The deformation space $\mathfrak{D}(S_{g,r,s})$ is an open ball of dimension $N_0 := 6g - 6 + 3r + 2s$. Define $\mathcal{T}(\Omega) := \{(m, \lambda)\}$, where $m \in \mathfrak{D}(S_{g,r,s})$ and λ is a positive projective weight on the points of Ω. Then,

$$\mathcal{T}(\Omega) = \mathfrak{D}(S_{g,r,s}) \times \mathbb{B}^{s+Q-1} \simeq \mathbb{B}^{6g-7+3r+3s+Q}.$$

Consider $\mathcal{K}$ to be the set of embedded arcs in $S_{g,r}$ whose endpoints belong to Ω. The arc complex spanned by the arcs in $\mathcal{K}$ is denoted by $\mathcal{A}(S_{g,r,s})$. A simplex σ of the arc complex is said to be "big" if the arcs corresponding to its 0-skeleton divide the surface $S_{g,r,s}$ into topological disks. The pruned arc complex $\widehat{\mathcal{A}}(S_{g,r,s})$ is defined to be the union of the interior of the big simplices.

Finally, let $\mathrm{MCG}(S_{g,r,s})$ be the mapping class group of the surface whose elements fix the points in Ω. Then, Harer proves the following theorem:

Theorem 3.3.1. *[11] There is a natural homeomorphism $\Phi : \mathcal{T}(\Omega) \longrightarrow \widehat{\mathcal{A}}(S_{g,r,s})$ that commutes with the action of the mapping class group $\mathrm{MCG}(S_{g,r,s})$.*

Interpretation: Let $S_{g,n}^{\vec{q}}$ be an orientable hyperbolic surface with undecorated spikes with k boundary components homeomorphic to a circle. Recall that in the case of hyperbolic surfaces with undecorated spikes, the permitted arcs are finite and have both their endpoints on two (not necessarily distinct) connected components of the boundary of the surface. By comparing these arcs with the permitted arcs used by Harer, we get that punctures and spikeless boundaries play the same role in the two surfaces; similarly, the points $y_1, \ldots, y_Q$ can be interpreted as spikes in our case. Then, $k = s$ and $n = r + s$. So, the arc complex $\mathcal{A}\left(S_{g,n}^{\vec{q}}\right)$ is isomorphic to the arc complex $\mathcal{A}\left(S_{g,r,s}\right)$. Also, the definition of a big simplex is the same in the two approaches. Hence, from Theorem (3.3.1) we have that

Corollary 3.3.2. *The pruned arc complex $\widehat{\mathcal{A}}(S_{g,n}^{\vec{q}})$ of a connected orientable surface $S_{g,n}^{\vec{q}}$ with non-decorated spikes is homeomorphic to an open ball of dimension $6g - 7 + 3n + Q$.*

Next, we prove a similar result for non-orientable surfaces with spikes using Harer's theorem.

The *orientation covering* of a surface S is defined as the pair $(\overline{S}, \pi)$, where,

$$\overline{S} := \{(p, o(p) \mid p \in S, o(p) \text{ is an orientation of } T_pS)\}$$

and

$$\pi : \begin{array}{ccc} \overline{T_{h,n}^{\vec{q}}} & \longrightarrow & T_{h,n}^{\vec{q}} \\ (p, o(p)) & \mapsto & p \end{array}.$$

Since the tangent space T_pS has exactly two orientations, π is a two-sheeted covering map. Since $T_{h,n}^{\vec{q}}$ is non-orientable, one has that $\overline{T_{h,n}^{\vec{q}}}$ is a orientable surface with

$$\chi(\overline{T_{h,n}^{\vec{q}}}) = 2\chi(T_{h,n}^{\vec{q}}) < 0.$$

So we get that $\overline{T_{h,n}^{\vec{q}}}$ is hyperbolic and that it is of the form $S_{h-1,2n}^{\vec{q}\sqcup\vec{q}}$, where $\vec{q}\sqcup\vec{q} := (q_1, \ldots, q_n, q_1, \ldots, q_n)$. Let $\Upsilon : S_{h-1,2n}^{\vec{q}\sqcup\vec{q}} \longrightarrow S_{h-1,2n}^{\vec{q}\sqcup\vec{q}}$ be the covering automorphism that exchanges the two points in every fibre of π.

We shall revert back to Harer's notation — suppose that $T_{h,n}^{\vec{q}}$ has k spikeless boundary components. Then, $S_{h-1,2n}^{2\vec{q}}$ has $s := 2k$ spikeless boundary components. Let $r := 2(n-k)$. We shall be working with $S_{h-1,r,s}$ which is an orientable surface of genus $h-1$ with r boundary components and s punctures. From Theorem (3.3.1), we know that $\widehat{\mathcal{A}}(S_{h-1,r,s})$ is an open ball of dimension

$$6(h-1) - 7 + 3(r+s) + 2Q = 6h - 13 + 6n + 2Q.$$

Let $\mathcal{I}(\mathcal{A}\left(S_{h-1,r,s}\right))$ be the subset of $\mathcal{A}\left(S_{h-1,r,s}\right)$ that is invariant under the action of Υ. Since the homeomorphism $\Phi : \mathcal{T}(\Omega) \longrightarrow \widehat{\mathcal{A}}(S_{h-1,r,s})$ commutes with the action of Υ, we get that $\Phi(\mathcal{I}(\mathcal{A}\left(S_{h-1,r,s}\right)))$ is the set of points $(m, \lambda) \in \mathcal{T}(\Omega)$ such that m and λ are invariant under Υ. Every permitted arc α of $T_{h,n}^{\vec{q}}$ lifts to two disjoint arcs α_1, α_2 in $S_{h-1,r,s}$ because Υ is a double cover and an arc is simply-connected. These two arcs are interchanged by the action of Υ. So, the isotopy classes $[\alpha^1], [\alpha^2]$ as well as the 1-simplex generated by them belong to $\mathcal{I}(\mathcal{A}(S_{h-1,r,s}))$. Consequently, we get the following map between the arc complexes:

$$\begin{array}{cccc} h: & \mathcal{A}\left(T^{\vec{q}}_{h,n}\right) & \longrightarrow & \mathcal{I}(\mathcal{A}(S_{h-1,r,s})) \\ (0-\text{skeleton}) & [\alpha] & \mapsto & \frac{[\alpha^1]+[\alpha^2]}{2}, \\ (k-\text{skeleton}) & \sum_{i=1}^{k+1} t_i[\alpha_i] & \mapsto & \sum_{i=1}^{k+1} t_i \frac{[\alpha_i^1]+[\alpha_i^2]}{2}, \end{array}$$

where $k \leq N_0$, $t_i \geq 0$ for $i = 1, \ldots, k+1$,with $\sum_{i=1}^{k+1} t_i = 1$.

Lemma 3.3.3. *The map $h : \widehat{\mathcal{A}}(T^{\vec{q}}_{h,n}) \longrightarrow \mathcal{I}(\widehat{\mathcal{A}}(S_{h-1,r,s}))$ is an isomorphism.*

Proof. Firstly, we show that this map is well-defined. A point $x \in \widehat{\mathcal{A}}(S)$ belongs to the interior of a unique big simplex σ_x of $\mathcal{A}(S)$:

$$x = \sum_{i=1}^{N_0} t_i\,[\alpha_i], \text{ with } t_i \in (0,1), [\alpha_i] \in \sigma_x^{(0)}, \text{ for every } i = 1, \ldots, N_0 \text{ and } \sum_i t_i = 1.$$

The union $\bigcup_i e_i$ of arcs decomposes the surface S into topological disks with at most two spikes. Being simply connected, they lift to twice as many disks partitioning the double cover. So the simplex formed by $\{[\alpha_i^1], [\alpha_i^2]\}_i$ is big. Hence we get that $h(x) \in \widehat{\mathcal{A}}(S_{h-1,r,s})$. Since Υ exchanges the two arcs $[\alpha_i^1], [\alpha_i^2]$ for every $i = 1, \ldots, N_0$, we get that $h(x) \in \mathcal{I}(\widehat{\mathcal{A}}(S_{h-1,r,s}))$.

Now we construct the inverse of h. Start with $y \in \mathcal{I}(\widehat{\mathcal{A}}(S_{h-1,r,s}))$. Since, $y \in \widehat{\mathcal{A}}(S_{h-1,r,s})$, there exists a unique big simplex σ_y such that $y \in \operatorname{int}(\sigma_y)$, i.e.,

$$y = \sum_{j=1}^{q} s_j\,\alpha_j, \text{ with } s_j \in (0,1), \alpha_j \in \sigma_y^{(0)}, \text{ for every } j = 1, \ldots, q \text{ and } \sum_j s_j = 1.$$

Since $y \in \mathcal{I}$, it is invariant under the action of Υ. The family of arcs in $\sigma_y^{(0)}$ project to equal or disjoint arcs in the quotient surface. Similarly, since σ_y is big the connected components of the complement of this family of arcs are disks and they project to equal or disjoint regions. If α, α' are two arcs in $\sigma_y^{(0)}$ that have equal weight t, then they project to the same arc β; so $h^{-1}(t([\alpha] + [\alpha']) := t\beta$. This concludes the proof of the lemma. □

Corollary 3.3.4. *The pruned arc complex $\widehat{\mathcal{A}}(T^{\vec{q}}_{h,n})$ of a non-orientable surface $T^{\vec{q}}_{h,n}$ with non-decorated spikes is homeomorphic to an open ball of dimension $3h - 7 + 3n + Q$.*

Proof. The subset $\Phi(\mathcal{I}(\mathcal{A}(S_{h-1,r,s})))$ is an open ball of dimension $3h - 7 + 3n + Q$ — it can be parametrised by the lengths of geodesic arcs of an Υ-invariant triangulation of $S_{h-1,r,s}$ and Υ-invariant projective weights on the set Ω. Using the isomorphism h, we get that

$$\widehat{\mathcal{A}}(T^{\vec{q}}_{h,n}) = h^{-1}(\mathcal{I}(\mathcal{A}(S_{h-1,r,s}))) = h^{-1}\Phi^{-1}(\mathbb{B}^{3h-7+3n+Q}).$$

□

3.3.2 Surfaces with decorated spikes

In this section, we shall prove that the pruned arc complex of a surface with decorated spikes is an open ball.

As mentioned previously, the arcs that are considered for spanning the arc complex for such a surface are either finite, separating at least one spike, or infinite with one endpoint exiting the surface through a spike. The former is referred to as an *edge-to-edge* arc, while the latter is called a *spike-to-edge* arc. Recall that every point $x \in \widehat{\mathcal{A}}(S_0)$ is contained in the interior of a unique big simplex denoted by σ_x, the arcs corresponding to whose 0-skeleton decomposes the surface into topological disks with at most one decorated spike.

Firstly, we will prove the following theorem for orientable surfaces $S_{g,n}^{\vec{q},\vec{h}}$:

Theorem 3.3.5. *The pruned arc complex $\widehat{\mathcal{A}}(S_{g,n}^{\vec{q},\vec{h}})$ of an orientable surface $S_{g,n}^{\vec{q},\vec{h}}$ with decorated spikes is an open ball of dimension $6g-7+3n+2Q$.*

We shall denote S_0 by the topological surface with genus g, n boundary components and $q_i \geq 0$ marked points on $\partial_i S_0$, $i=1,\dots,n$ such that $\chi(S_0) < 0$. Again, let Q be the total number of spikes. Let $\vec{\xi} = (\xi_1,\dots,\xi_Q)$ be the set of marked points on ∂S_0. Let $Q(i) = \sum_{j=1}^{i} q_i$. Then we see that $S_0 \smallsetminus \bigcup\{\xi_l\}_l$ is an orientable hyperbolic surface with spikes. The marked points are called *vertices* and the connected components of $\partial_i S \smallsetminus \{\xi_{Q(i-1)+1},\dots,\xi_{Q(i)}\}$ are called *edges*.

Firstly, we do a topological operation on S_0 called the *doubling* to obtain a "bigger" hyperbolic surface with boundary and without any marked points. This is done in two steps:

Step 1: We truncate small neighbourhoods of every marked point along embedded arcs, denoted by $V := \{r_l\}_{l=1}^{Q}$, that join the edges adjacent to the spikes. The elements of V are called *V-edges*. Let S be the resulting surface. For $i=1,\dots,n$, when $q_i > 0$, the i-th boundary of S, $\partial_i S$, is the union of $2q_i$ segments alternately partitioned into V-edges and the truncated boundary edges of S_0. When, $q_i = 0$, $\partial_i S = \partial_i S_0$. The truncated boundary edges along with any closed loop in ∂S_0 are called E-edges.

Step 2: Then we take a copy S' of S and glue it to S along the V-edges. The final surface, denoted as $\Sigma := S \sqcup S' / \sim$, has genus $2g$, with $2n+Q$ boundary components. If $\partial_i S_0$ had $q_i > 0$ E-edges, then after gluing we get q_i boundary components made out of two copies of every E-edge.

We get the Euler characteristic of the surface Σ, $\chi(\Sigma) = 2-4g-2n < 0$. So, it is hyperbolic. Since there are no spikes, we can consider all complete hyperbolic metrics with totally geodesic boundary. Its deformation space $\mathfrak{D}(\Sigma)$ is an open ball of dimension $12g-6+6n$. The surface Σ has a degree two symmetry $\iota \in \mathrm{MCG}(\Sigma)$ that exchanges the two surfaces S and S'.

Keeping this in mind, we construct an isomorphism, denoted by h, between the subcomplex $\mathrm{Fix}_\iota(\widehat{\mathcal{A}}(\Sigma))$ of the pruned arc complex of Σ, invariant under the involution ι, and the pruned arc complex $\widehat{\mathcal{A}}(S_0)$ of S_0 in the following way: At first we define it on the 0-skeleton of the arc complex and then we extend it linearly to a generic point on the pruned arc complex.

- Let e be an arc joining a spike ξ and an edge l of ∂S_0. Then the Step 1 above truncates e; in S it becomes an arc, again denoted by e, joining the corresponding R-edge and the initial G-edge l. Let $e' \in S'$ be the twin arc of e. Finally, after the Step 2, $e := e \sqcup e' / \sim$ becomes the arc that joins the two copies of l that form the totally geodesic boundary in Σ, and transverse to the R-edge. It is preserved as a set by the involution. Define $h(e) := e''$.

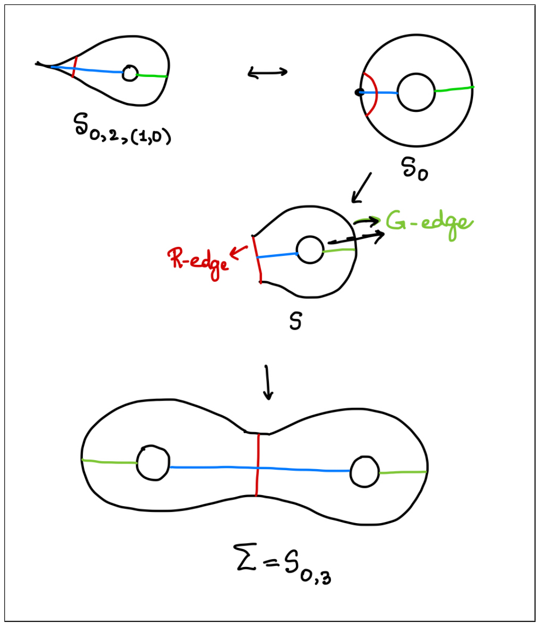


Figure 3.1: Doubling operation

- Let e be an edge-to-edge arc not in V. So e joins two distinct boundary edges of ∂S_0. The Step 1 doesn't change the arc e. It remains disjoint from its twin $e' \subset S'$ inside Σ. Define $h(e) := \frac{e+e'}{2}$.

The map is then extended linearly over any point $x \in \widehat{\mathcal{A}}(S)$.

Lemma 3.3.6. *The map* $h : \widehat{\mathcal{A}}(S_0) \longrightarrow \mathrm{Fix}_\iota(\widehat{\mathcal{A}}(\Sigma))$ *is an isomorphism.*

Proof. Firstly, we show that this map is well-defined. As discussed before, a point $x \in \widehat{\mathcal{A}}(S)$ belongs to the interior of a unique simplex σ_x of $\mathcal{A}(S)$. In other words,

$$x = \sum_{i=1}^{p} t_i \, e_i, \text{ with } t_i \in (0,1), e_i \in \sigma_x^{(0)}, \text{ for every } i = 1, \ldots, p \text{ and } \sum_i t_i = 1.$$

The union $\bigcup_i e_i$ of arcs decomposes the surface S into topological disks with at most one vertex.

Let $y := h(x) = \sum_{j=1}^{q} s_j \, \alpha_j$. Then from the definition of h, it follows that $s_j \in (0,1)$, and $\alpha_j \in \mathcal{A}_{\mathcal{K}}(\Sigma)^{(0)}$, for every $j = 1, \ldots, q$. The family of arcs $\{\alpha_j\}_j$ decomposes the surface Σ into topological disks. Otherwise there is a connected component K in the complement in Σ such that $\pi_1(K) \neq \{1\}$. So it is possible to find a non-trivial simple closed curve γ in K. Then either there was a curve in the complement of $\{e_i\}$ in S such that γ is one of its copies or the curve γ was created from a vertex-to-vertex arc by the doubling operation. None of these two cases is possible because $x \in \widehat{\mathcal{A}}(S_0)$ and by definition of the pruned arc complex of a surface with decorated spikes, the family of arcs $\{e_i\}_i$ decomposes the initial surface S_0 into disks with at most one vertex. So we have that y is a point of $\widehat{\mathcal{A}}(\Sigma)$. Finally, we verify that the point $h(x)$ is ι-invariant.

$$\begin{aligned}\iota(h(e)) &= \begin{cases} \frac{\iota(e)+\iota(e')}{2}, & \text{if } e \text{ is edge-to-edge} \\ \iota(e'') & \text{if } e \text{ is edge-to-vertex,} \end{cases} \\ &= \begin{cases} \frac{e'+e}{2}, \\ e'', \end{cases} \\ &= h(e). \end{aligned}$$

The inverse: Start with $y \in \mathrm{Fix}_\iota(\widehat{\mathcal{A}}(\Sigma))$. Then there exists a unique simplex σ_y such that $y \in \mathrm{int}\,(\sigma_y)$, i.e.,

$$y = \sum_{j=1}^{q} s_j\,\alpha_j, \text{ with } s_j \in (0,1), \alpha_j \in \sigma_y^{(0)}, \text{ for every } j = 1, \ldots, q \text{ and } \sum_j s_j = 1.$$

Since $\iota(y) = y$, for every $j \in \{1, \ldots, q\}$, either $\iota(a_j) = a_j$ or $\iota(a_j) = \alpha_k$ for some $k \in \{1, \ldots, q\} \smallsetminus \{i\}$. In the former case, there exist an edge-to-vertex arc e_j in S and its twin e'_j in S' such that $a_j = e_j \sqcup e'_j / \sim$. So we define $h^{-1}(s_j a_j) := s_j e_j$. In the latter case, we must also have $s_j = s_k =: s_{jk}$. Suppose that $a_j \in S$ and $a_j \in S'$. Then define $h^{-1}(s_{jk}(a_j + a_k)) := 2 s_{jk} a_j$. □

Now we shall prove Theorem 3.3.5.

Proof of Theorem 3.3.5. From Theorem 3.3.2, we get that there is a MCG(Σ)-invariant homeomorphism:

$$\mathrm{Fix}_\iota(\widehat{\mathcal{A}}(\Sigma)) \cong \mathrm{Fix}_\iota(T(\Omega)). \tag{3.1}$$

The subspace $\mathrm{Fix}_\iota(T(\Omega))$ is an open ball — it can be parametrised by the lengths of the geodesic arcs of an ι-invariant triangulation of Σ. Finally, using the isomorphism h from above we get that $\widehat{\mathcal{A}}(S_0)$ is an open ball of dimension $6g - 7 + 3n + 2Q$.

□

Using the same method as in the previous section, we can show that

Theorem 3.3.7. *The pruned arc complex $\widehat{\mathcal{A}}(T_{h,n}^{\vec{q},\vec{h}})$ of a non-orientable surface $T_{h,n}^{\vec{q},\vec{h}}$ with decorated spikes is an open ball of dimension $3h - 7 + 3n + 2Q$.*

3.4 Tiles

Let S be a hyperbolic surface endowed with a hyperbolic metric $m \in \mathfrak{D}(S)$. Let $\mathcal{K}$ be the set of permitted arcs for an arc complex $\mathcal{A}(S)$ of the surface. Given a simplex $\sigma \subset \mathcal{A}(S)$, the *edge set* is defined to be the set

$$\mathcal{E}_\sigma := \left\{ \alpha_g(m) \in \alpha | \alpha \in \sigma^{(0)} \right\},$$

where $\alpha_g(m)$ is a geodesic representative from its isotopy class. The set of all lifts of the arcs in the edge set in the universal cover $\widetilde{S} \subset \mathbb{D}$ is denoted by $\widetilde{\mathcal{E}_\sigma}$. The set of connected components of the surface S in the complement of the arcs of the edge set is denoted by $\mathcal{T}_\sigma$. The lifts of the elements in $\mathcal{T}_\sigma$ in $\mathbb{D}$ are called *tiles*; their collection is denoted by $\widetilde{\mathcal{T}_\sigma}$.

Remark 3.4.1. In the case of ideal polygons and decorated polygons, these components are homeomorphic to two-dimensional disks. In the case of punctured polygons, one of the components is a punctured disk.

The sides of a tile are either contained in the boundary of the original surface or they are the arcs of $\mathcal{E}_\sigma$. The former case is called a *boundary side* and the latter case is called an *internal side*. Two tiles d, d' are called *neighbours* if they have a common internal side. The tiles having finitely many edges are called *finite*.

If σ has maximal dimension in $\mathcal{A}(S)$, then the finite tiles can be of three types:

Type 1: The tile has only one internal side, i.e., it has only one neighbour.

Type 2: The tile has two internal sides, i.e., two neighbours.

Type 3: The tile has three internal sides, i.e., three neighbours.

Remark 3.4.2. Any tile, obtained from a triangulation using a simplex σ, must have at least one and at most three internal sides. Indeed, the only time a tile has no internal side is when the surface is an ideal triangle. Also, if a tile has four internal sides, then it must also have at least four distinct boundary sides to accommodate at least four endpoints of the arcs. The finite arc that joins one pair of non-consecutive boundary sides lies inside $\mathcal{K}$. This arc was not inside the original simplex, which implies that σ is not maximal. Hence a tile can have at most 3 internal sides.

Surfaces with non-decorated spikes: There are three types of tiles possible after triangulating (cf. first column of Fig 3.2):

- a hyperbolic quadrilateral with two ideal vertices and one permitted arc,
- a hyperbolic pentagon with one ideal vertex and two permitted arcs as alternating edges,
- a hyperbolic hexagon with three permitted arcs as alternating edges.

Decorated Polygons, Surfaces with decorated spikes: The different types of tiles possible in the case of a generalised polygon are shown in the last three columns of the table in Fig. 3.2.

- When there is only one internal side of the tile, that side is an edge-to-edge arc of the original surface. The tile contains exactly one generalised vertex ν and two boundary sides. The three cases corresponding to the three possible types of the vertex are given in the first row of the table in Fig. (3.2).
- When there are two internal sides (second row in Fig. (3.2)), one of them is an edge-to-vertex and the other one is of edge-to-edge type. So the tile contains a generalised vertex.
- There are two possibilities in this case: either all the three internal sides are of edge-to-edge type (fourth row in Fig. (3.2)) or two of them are edge-to-vertex arcs and one edge-to-edge arc (third row in Fig. (3.2)). In the former case, the tile does not contain any vertex whereas in the latter case it contains one.

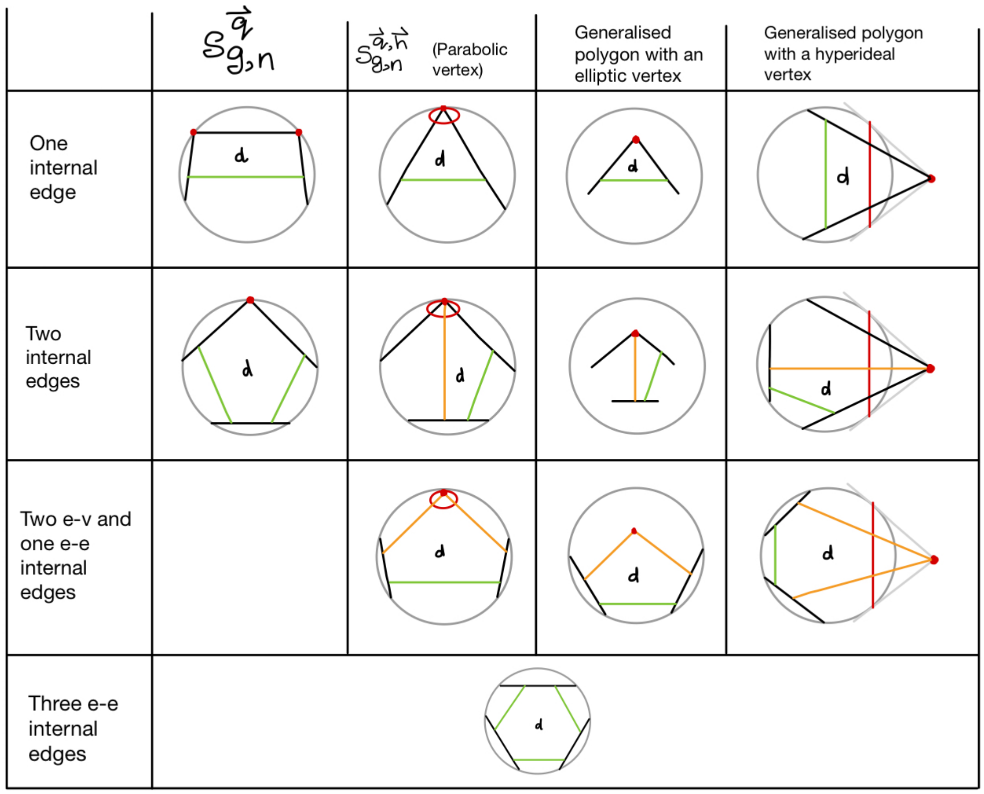


Figure 3.2: Tiles for different surfaces

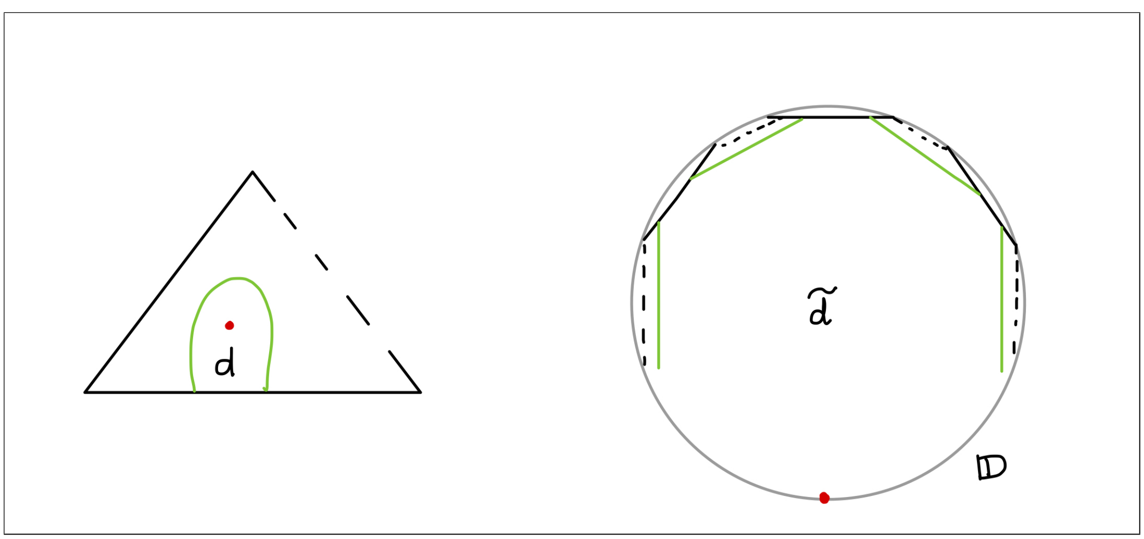


Figure 3.3: Infinite tile containing the puncture

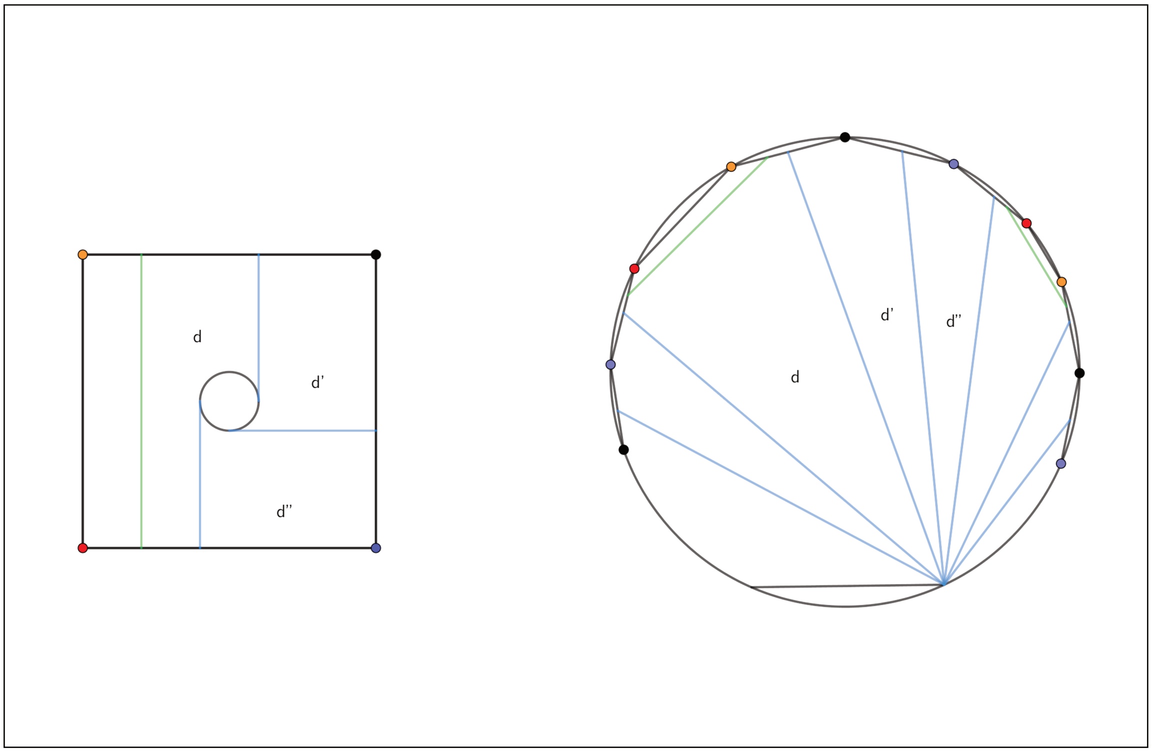


Figure 3.4: Tiles bounded by spiraling arcs

Punctured polygons: In the case of a punctured polygon, the possible connected components after cutting the surface along the arcs of the edgeset, can be of the three types as in the case of surfaces with non-decorated spikes and also a hyper-ideal punctured monogon. The lift of the latter in $\mathbb{H}^2$ is an infinite polygon with one ideal vertex, as in Fig. 3.3.

One-holed polygons. If $\mathcal{E}_\sigma$ is the edgeset of a top-dimensional simplex σ of the spinning arc complex $\mathcal{A}_{\circlearrowright}(\Pi_n^{\circledcirc})$ of a one holed polygon $\Pi_n^{\circledcirc}$, then the lifts of the possible tiles are those in the case of surfaces with non-decorated spikes and two types of spiraling. The first type of tile has an ideal point corresponding to one of the endpoints of the geodesic γ, two internal sides that are spiraling arcs and a third that is a finite arc. The finite arc is present only for $n \geq 2$. The second type of tile is a quadrilateral with one ideal vertex corresponding to a lift of a spike, another corresponding to one of the endpoints of the lift of the boundary loop and two spiraling internal edges. See Fig. 3.4.

Dual graph to a triangulation. Let $\sigma \in \mathcal{A}(\Pi)$ be a triangulation of a polygon Π. Then the corresponding *dual graph* is a graph embedded in the universal cover of the surface such that the vertex set is $\widetilde{\mathcal{T}_\sigma}$ and the edge set is given by unordered pairs of lifted tiles that share a lifted internal edge. A vertex of the graph has valency 1 (resp. 2, 3) if and only if the corresponding tile is of the type 1 (resp. 2, 3).

Refinement. Let σ be a top-dimensional simplex of an arc complex $\mathcal{A}(S)$ of a hyperbolic surface S. Let β be an arc such that $[\beta] \in \mathcal{A}(S)^{(0)} \smallsetminus \sigma^{(0)}$. So, β intersects every arc in the isotopy class of at least one arc in σ. The set $\sigma \cup [\beta]$ is called a *refinement* of the triangulation σ. Let $\mathcal{T}_{\sigma,r}$ be the set of connected components of $S \smallsetminus (\beta \bigcup_{\alpha \in \mathcal{E}_\sigma} \alpha)$ and $\widetilde{\mathcal{T}_{\sigma,r}}$ be the set of lifts of its elements. The elements of $\widetilde{\mathcal{T}_{\sigma,r}} \smallsetminus \widetilde{\mathcal{T}_\sigma}$ are called *small tiles*.

Chapter 4

Strip Deformations

In this chapter we shall introduce strip deformations, strip template, tiles and tile maps. We shall also recapitulate the main theorem proved by [5], and state the theorems that we shall prove in our context of more general surfaces.

Informally, a strip deformation of a hyperbolic surface is done by cutting it along a geodesic arc α_g in α and gluing a strip of the hyperbolic plane $\mathbb{H}^2$, without any shearing. The type of strip used depends on the type of arc and the surface being considered.

4.1 The different strips

Firstly, we define the different types of strips. Let l_1 and l_2 be any two geodesics in $\mathbb{D}$. Then there are three types of strips depending on the nature of their intersection:

- Suppose that l_1 and l_2 are disjoint in $\overline{\mathbb{D}}$. Then the region bounded by them in $\mathbb{D}$ is called a hyperbolic strip. The *width* of the strip is the length of the segment of the unique common perpendicular l to l_1 and l_2, contained in the strip. The *waist* of the strip is defined to be the set of points of intersection $l \cap l_1$ and $l \cap l_2$.

- Suppose that l_1 and l_2 intersect in $\partial_\infty \mathbb{D}$ at a point p. Let h be a horocycle based at p. Then the region bounded by them inside $\mathbb{D}$ is called a *parabolic strip.* The waist in this case is defined to be the ideal point p and the width (w.r.t h)is defined to be the length of the horocyclic arc of h subtended by l_1 and l_2.

- Suppose that l_1 and l_2 intersect inside $\mathbb{D}$ at a point v. Then any one of the regions bounded by v, l_1, l_2 inside $\mathbb{D}$ is called an *elliptic strip.* Again, the waist is defined to be the point v and the width is the Euclidean angle subtended at v by l_1 and l_2.

4.2 Strip template

Let S be a hyperbolic surface endowed with a metric $m \in \mathfrak{D}(S)$ on it. Let $\mathcal{K}$ be the set of permitted arcs (Definition (3.1.2)). A strip template is the following data:

- an m-geodesic representative α_g from every isotopy class α of arcs in $\mathcal{K}$, along which the strip deformation is performed,
- a point $p_\alpha \in \alpha_g$ where the waist of the strip being glued must lie.

A choice of strip template is the specification of this data. However, we shall see in the following section that even though we are allowed to choose the geodesic arcs in every case, the waists are sometimes fixed from beforehand by the nature of the arc being considered.

4.2.1 Finite arcs

Recall that finite arcs are embeddings of a closed and bounded interval into the surface with both the endpoints lying on the boundary of the surface. These arcs are present in the construction of every arc complex that we discuss. The strip glued along these arcs is of hyperbolic type except when the arc joins an elliptic vertex $v \in \mathbb{D}$ with an edge, in a decorated polygon. In this case, the strip glued is of elliptic type, with its waist at v. The representative α_g from the isotopy class of such an arc can be any geodesic segment from v to that edge.

In the case of a finite arc joining a truncated hyperideal vertex to an edge, the representative is chosen to be perpendicular to the vertex, and the waist is at the point of intersection of the arc with the truncated vertex.

In every other case, including edge-to-edge arcs in decorated polygons, we are free to chose the geodesic representative and the waist of the hyperbolic strip.

4.2.2 Infinite arcs

Let $[\alpha]$ be the isotopy class of a permitted infinite arc α of a hyperbolic surface S. Then α has one finite end lying on ∂S and one infinite end that either escapes the surface through a spike (decorated surfaces) or spirals about a simple closed geodesic curve (one holed-polygons). We can choose any geodesic arc α_g from $[\alpha]$ that does the same without any self intersection.

Consider the universal cover of the surface inside the disk model $\mathbb{D}$. The lift of such an arc is a geodesic ray in $\mathbb{D}$ whose infinite end is on an ideal point which is a lift of the spike (decorated surface) or one endpoint of the geodesic corresponding to the curve (spiraling arc). In both the cases, the strip added is of parabolic type, with its waist at this ideal point.

4.3 Strip deformations

4.3.1 Definitions

In this section we give a formal definition of a strip deformation and its infinitesimal version.

Definition 4.3.1. Given an isotopy class α of arcs and a choice of strip template $(\alpha_g, p_\alpha, w_\alpha)$, define the *strip deformation* along α to be a map

$$F_\alpha : \mathfrak{D}(S) \longrightarrow \mathfrak{D}(S)$$

where the image $F_\alpha(m)$ of a point $m \in \mathfrak{D}(S)$ is a new metric on the surface obtained by cutting it along the m-geodesic arc α_g in α chosen by the strip template and gluing a strip whose waist coincides with p_α. The type of strip used depends on the type of arc and the surface being considered.

Definition 4.3.2. Given an isotopy class of arcs α of a hyperbolic surface S and a strip template $\{(\alpha_g, p_\alpha, w_\alpha)\}_{\alpha\in\mathcal{K}}$ adapted to the nature of α for every $m \in \mathfrak{D}(S)$, define the *infinitesimal strip deformation*

$$\begin{aligned} f_\alpha : \quad \mathfrak{D}(S) &\longrightarrow T\mathfrak{D}(S) \\ m &\mapsto [m(t)] \end{aligned}$$

where the image $m(\cdot)$ is a path in $\mathfrak{D}(S)$ such that $m(0) = m$ and $m(t)$ is obtained from m by strip deforming along α with a fixed waist p_α and the width as tw_α.

Let $m = ([\rho, \vec{x}]) \in \mathfrak{D}(S)$ be a point in the deformation space of the surface, where ρ is the holonomy representation and denote $\Gamma = \rho(\pi_1(S))$. Fix a strip template $\{(\alpha_g, p_\alpha, w_\alpha)\}$ with respect to m. Let σ be a simplex of $\mathcal{A}(S)$. Given an arc α in the edgeset $\mathcal{E}_\sigma$, there exist tiles $\delta, \delta' \in \mathcal{T}_\sigma$ such that every lift $\widetilde{\alpha}$ of α in $\widetilde{S}$ is the common internal side of two lifts $\widetilde{\delta}, \widetilde{\delta}'$ of the tiles. Also, $p_{\gamma\cdot\widetilde{\alpha}} = \gamma \cdot p_{\widetilde{\alpha}}$, for every $\gamma \in \Gamma$. Then the infinitesimal deformation $f_\alpha(m)$ tends to pull the two tiles δ and δ' away from each other due to the addition of the infinitesimal strip. Let u be a infinitesimal strip deformation of ρ caused by $f_\alpha(m)$. Then we have a (ρ, u)-equivariant *tile* map $\phi : \widetilde{\mathcal{T}_\sigma} \to \mathfrak{g}$ such that for every $\gamma \in \Gamma$,

$$\phi(\rho(\gamma) \cdot \widetilde{\delta}) - \phi(\rho(\gamma) \cdot \widetilde{\delta}') = \rho(\gamma) \cdot v_{\widetilde{\alpha}}, \tag{4.1}$$

where $v_{\widetilde{\alpha}}$ is the Killing field in $\mathfrak{g} \simeq \mathscr{X}$ corresponding to the strip deformation $f_{\widetilde{\alpha}}(m)$ along a geodesic arc $\widetilde{\alpha}_g$, isotopic to $\widetilde{\alpha}$, adapted to the strip template chosen, and pointing towards $\widetilde{\delta}$:

- If $f_\alpha(m)$ is a hyperbolic strip deformation with strip template $(\alpha_g, p_\alpha, w_\alpha)$, then $v_{\widetilde{\alpha}}$ is defined to be the hyperbolic Killing vector field whose axis is perpendicular to $\widetilde{\alpha}_g$ at the point $\widetilde{p_\alpha}$, whose velocity is w_α.

- If α is an infinite arc joining a spike and a boundary component, then $f_\alpha(m)$ is a parabolic strip deformation with strip template $(\alpha_g, p_\alpha, w_\alpha)$, and $v_{\widetilde{\alpha}}$ is defined to be the parabolic Killing vector field whose fixed point is the ideal point where the infinite end of $\widetilde{\alpha}$ converges and whose velocity is .

- If α joins an elliptic vertex with an edge, then $f_\alpha(m)$ is an elliptic strip deformation with strip template $(\alpha_g, p_\alpha, w_\alpha)$, and $v_{\widetilde{\alpha}}$ is defined to be the elliptic Killing vector field whose fixed point is at the vertex and whose velocity is given by w_α.

Remark 4.3.1. Such a strip deformation $f_\alpha : \mathfrak{D}(S) \longrightarrow T_m\mathfrak{D}(S)$ does not deform the holonomy of a general surface with spikes (decorated or otherwise) if α is completely contained outside the convex core of the surface. However, it does provide infinitesimal motion to the spikes.

More generally, a linear combination of strip deformations $\sum_\alpha c_\alpha f_\alpha(m)$ along pairwise disjoint arcs $\{\alpha_i\} \subset \mathcal{E}_\sigma$ imparts motion to the tiles of the triangulation depending on the coefficient of each term in the linear combination. A tile map corresponding to it is a (ρ, u)-equivariant map $\phi : \widetilde{\mathcal{T}_\sigma} \to \mathfrak{g}$ such that for every pair $\delta, \delta' \in \mathcal{T}_\sigma$ which share an edge $\alpha \in \mathcal{E}_\sigma$, the equation 4.1 is satisfied by ϕ.

Definition 4.3.3. The *infinitesimal strip map* is defined as:

$$\begin{array}{cccc} \mathbb{P}f & : & \widehat{\mathcal{A}}(S) & \longrightarrow & \mathbb{P}^+(T_m\mathfrak{D}(S)) \\ & & \sum\limits_{i=1}^{\dim \mathfrak{D}(S)} c_i \alpha_i & \mapsto & \left[\sum\limits_{i=1}^{\dim \mathfrak{D}(S)} c_i f_{\alpha_i}(m)\right] \end{array}$$

where $\widehat{\mathcal{A}}(S)$ is the pruned arc complex of the surface (Definition (3.1.7)).

Two tile maps ϕ, ϕ' are said to be equivalent if for all $d \in \mathcal{T}_\sigma$,

$$\phi(d) - \phi'(d) = v_0 \in \mathfrak{g}.$$

The set of all equivalence classes of tile maps corresponding to a simplex $\sigma \subset \mathcal{A}(S)$ is denoted by Φ.

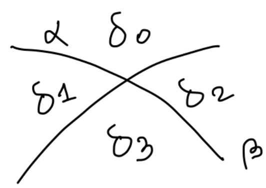


Figure 4.1: Refined tiles

Let $\sigma \cup [\beta]$ be a refinement of σ. A *consistent* tile map is a tile map $\phi : \mathcal{T}_{\sigma\cup[\beta]} \longrightarrow \mathfrak{g}$ that satisfies the consistency relation around every point of intersection: if the pairs $(\delta_1, \delta_0), (\delta_3, \delta_2)$ neighbour along α and the pairs $(\delta_1, \delta_3), (\delta_0, \delta_2)$ neighbour along β , then ϕ must satisfy

$$\phi(\delta_1) - \phi(\delta_0) = \phi(\delta_3) - \phi(\delta_2) = v_\alpha, \tag{4.2}$$
$$\phi(\delta_1) - \phi(\delta_3) = \phi(\delta_0) - \phi(\delta_2) = v_\beta, \tag{4.3}$$

where v_α and v_β are the Killing vector fields adapted to the strip templates and the nature of α and β. The set of all equivalence classes modulo $\mathfrak{g}$ of consistent tile maps is denoted by Φ^c. Then there is a natural inclusion

$$\Phi \subset \Phi^c.$$

Also, we have the bijection between formal expressions of the form $\sum\limits_{\alpha\in\mathcal{E}_\sigma\cup\{\beta\}} c_\alpha f_\alpha(m)$ and Φ^c.

Definition 4.3.4. A *neutral* tile map, denoted by ϕ_0, is a tile map that fixes the generalised vertex or a spike of a tile whenever it has one and satisfies the equation

$$\phi_0(\gamma \cdot \delta)) = \gamma \cdot \phi_0(\delta), \text{ for every } \gamma \in \Gamma. \tag{4.4}$$

Such a map belongs to the equivalence class corresponding to $0 \in T_m\mathfrak{D}(S)$.

4.3.2 Some Examples

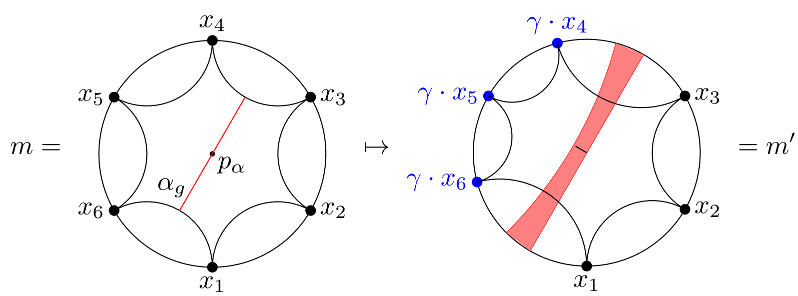
We will now illustrate this deformation using some simple examples. In each case, we take a strip map and compute one tile map that represents it as well as the infinitesimal deformation of the holonomy of the surface, caused by it.

Example 4.3.5. Firstly, we start with an undecorated ideal polygon $\Pi_n^{\pentagon}$, with $n \geq 4$. Recall that a point $m \in \mathfrak{D}(\Pi_n^{\pentagon})$ is determined by the equivalence class of the coordinates of its ideal vertices: $m = [x_1, \ldots, x_n]$.

Let α be a finite arc and α_g be a geodesic from its isotopy class. The geodesic carrying α_g divides the polygon into two regions, that are topological disks. Let d_1 be the region containing the vertices $x_1, \ldots, x_k$; the other is denoted by d_2. Let p_α be a point on α_g and $w_\alpha > 0$. Let $\gamma \in \mathrm{PSL}(2, \mathbb{R})$ be the hyperbolic element whose axis is perpendicular to α_g at p_α and whose translation length is w_α. Then, the hyperbolic strip deformation of m along α_g gives a new metric m' on S given by:

$$m' = [x_1, \ldots, x_k, \gamma \cdot x_{k+1}, \ldots, \gamma \cdot x_n] \in \mathfrak{D}(\Pi_n^{\pentagon}).$$

The edges joining the consecutive ideal points corresponding to the new vertices are redrawn. This is shown in the figure.



The infinitesimal version of this strip deformation is given by:

$$\begin{aligned} f_\alpha : \quad & \mathfrak{D}(S) \longrightarrow T\mathfrak{D}(S) \\ & [x_1, \ldots, x_n] \mapsto \left[\{[x_1, \ldots, x_k, \gamma_t \cdot x_{k+1}, \ldots, \gamma_t \cdot x_n]\}_{t>0}\right] \in T_m\mathfrak{D}(S), \end{aligned}$$

where $\gamma_t \in \mathrm{PSL}(2, \mathbb{R})$ is the hyperbolic element whose axis is perpendicular to α_g at p_α and whose translational length is tw_α.

Consider the universal cover (which is the surface itself) inside the upper half plane $\mathbb{U}$. Suppose that the arc α_g is carried by the geodesic $(-1, 1)$ and $p_\alpha = i$, so that $\gamma_t = \begin{bmatrix} e^{\frac{tw_\alpha}{2}} & 0 \\ 0 & e^{-\frac{tw_\alpha}{2}} \end{bmatrix}$, for $t \in \mathbb{R}$. Let $\exp : G \longrightarrow \mathfrak{g}$ be the usual exponential function. Then

$$\exp(tV_\alpha) = \{\gamma_t\}_{t \in \mathbb{R}}, \text{ where } V_\alpha = \begin{bmatrix} \frac{w_\alpha}{2} & 0 \\ 0 & -\frac{w_\alpha}{2} \end{bmatrix} \in \mathfrak{g}.$$

Then, infinitesimal strip deformation $f_\alpha(m)$ gives a vector field X on S, which is piecewise Killing:

$$\begin{aligned} X : \Pi_n^{\pentagon} \longrightarrow \quad & T\mathbb{H}^2 \\ p \mapsto \quad & \begin{cases} 0, & \text{if } p \in d_1, \\ v_\alpha \wedge p, & \text{if } p \in d_2 \cup \alpha_g, \end{cases} \end{aligned}$$

where $v_\alpha \in \mathbb{R}^{2,1}$ is the point corresponding to $V_\alpha \in \mathfrak{g}$.

We construct the following tile map:

$$\begin{array}{cccc} \phi & : & \widetilde{\mathcal{T}_\sigma} & \longrightarrow & \mathfrak{g} \\ & & d_1 & \mapsto & 0, \\ & & d_2 & \mapsto & v_\alpha, \end{array}$$

where σ is the 0-simplex given by the isotopy class of α and $\widetilde{\mathcal{T}_\sigma} = \{d_1, d_2\}$. The difference $\phi(d_2) - \phi(d_1) = v_\alpha$, so ϕ satisfies the equation (4.1).

If $\{g_t\} \subset \mathrm{PGL}(2,\mathbb{R})$ is a smooth one-parameter family generated by $V_0 \in \mathfrak{g}$, then

$$\left[\{[g_t \cdot x_1, \ldots, g_t \cdot x_k, \beta_t\gamma_t \cdot x_{k+1}, \ldots, g_t\gamma_t \cdot x_n]\}_{t>0}\right] = \left[\{[x_1, \ldots, x_k, \gamma_t \cdot x_{k+1}, \ldots, \gamma_t \cdot x_n]\}_{t>0}\right].$$

The corresponding tile map is then given by $\phi + V_0$. Hence for every strip deformation, we get a family of tile maps $\phi + \mathfrak{g}$ all equivalent modulo the additive action of $\mathfrak{g}$.

Now let β be another arc of the surface disjoint from α_g and let β_g be a geodesic representative from its isotopy class. Again, choose the waist $p_\beta \in \beta_g$ and the width $w_\beta > 0$. Let the Killing field corresponding to the hyperbolic strip deformation along β_g be given by an element $V_\beta \in \mathfrak{g}$, with $\exp(tV_\beta) = \{\gamma'_t\}_{t\in\mathbb{R}} \subset \mathrm{PSL}(2,\mathbb{R})$.

Then we can do infinitesimal strip deformations simultaneously along α and β in the following way:

The union of the two arcs $\alpha_g \cup \beta_g$ decomposes the surface into the regions d_1, d_2, d_3. In our example, we suppose that d_1 contains the first k_1 vertices, d_2 contains the next k_2 vertices and finally d_3 contains the remaining $n - (k_1 + k_2)$ vertices. Let β_g be the common boundary of d_2, d_3. Then the deformation done by fixing d_1, applying γ_t to the vertices of d_1, d_2 and finally by applying γ'_t on the vertices of d_3 is expressed in the following way:

$$\begin{aligned} f_{\alpha,\beta} : \mathfrak{D}(S) &\to \quad T\mathfrak{D}(S) \\ [x_1, \ldots, x_n] &\mapsto \quad \left[\{[x_1, \ldots, x_{k_1}, \gamma_t \cdot x_{k_1+1}, \ldots, \gamma_t \cdot x_{k_1+k_2}, \gamma_t\gamma'_t \cdot x_{n-k_1-k_2}, \ldots, \gamma_t\gamma'_t \cdot x_n]\}_{t>0}\right]. \end{aligned}$$

A piecewise Killing vector field X in this case is given by:

$$\begin{aligned} X : \Pi_n^{\pentagon} &\longrightarrow \quad T\mathbb{H}^2 \\ p &\mapsto \quad \begin{cases} v \wedge p, & \text{if } p \in d_1, \\ (v_\alpha + v) \wedge p, & \text{if } p \in d_2, \\ (v_\alpha + v_\beta + v) \wedge p, & \text{if } p \in d_3, \end{cases} \end{aligned}$$

where $v \in \mathfrak{g}$ is a fixed Killing field, v_α, v_β are Killing fields representing strip deformations along the arcs α, β, respectively.

The following is a tile map corresponding to the strip deformation $f_{\alpha,\beta}$

$$\begin{array}{cccl} \phi & : & \widetilde{\mathcal{T}_\sigma} & \longrightarrow & \mathfrak{g} \\ & & d_1 & \mapsto & v, \\ & & d_2 & \mapsto & v_\alpha + v, \\ & & d_3 & \mapsto & v_\alpha + v_\beta + v, \end{array}$$

where $\widetilde{\mathcal{T}_\sigma} = \{d_1, d_2, d_3\}$.

Remark 4.3.2. The strip deformation of a generalised polygon along an arc that joins a vertex ν to an edge is defined in a similar way. If ν is elliptic (resp. parabolic, hyperbolic), then the element $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ used above becomes elliptic (resp. parabolic, hyperbolic).

Remark 4.3.3. If d, d' are two tiles with a common internal arc α such that both the tiles contain the same elliptic (or parabolic, hyperbolic) vertex p, then $\phi(d) \wedge p = \phi(d') \wedge p = v_\alpha \wedge p$. The last term is zero because $p \in \operatorname{span}\{v_\alpha\}$. So the piecewise Killing vector field X is well-defined at the common vertex.

Next we move on to the polygons with non-trivial fundamental group - namely, punctured polygons $\Pi_n^{\odot}$ and one-holed polygons $\Pi_n^{\circledcirc}$.

Example 4.3.6. Let $S = \Pi_3^{\odot}$. Let $m = ([\rho, \vec{x}]) \in \mathfrak{D}(S)$ be a deformation of the surface, where ρ is the holonomy representation and $\vec{x} = (x_1, x_2, x_3) \in \partial_\infty \mathbb{D}^2$. The group $\Gamma := \rho(\pi_1(S))$ is generated by a parabolic element T in $\mathrm{PSL}(2, \mathbb{R})$, with fixed point corresponding to the puncture of the surface.

Fix a strip template. Let $x = t[\alpha] + (1-t)[\beta] \in \mathcal{A}(S)$, where α is a minimal arc (separates a tile containing two spikes) and β is a maximal arc in S (endpoints on the same edge). Define the simplex $\sigma := ([\alpha], [\beta])$. Then $f(x) = [\{m(t)\}]$, where $m(t) = ([\rho_t, \vec{x}(t)]])$, with $\rho_0 = \rho$ and $u := \rho_t'$ is the infinitesimal deformation of the holonomy. The surface is divided into three tiles, namely, d_1, d_2, d_3. Let d_1 be a punctured disk and the tiles d_2 be a pentagon with one spike and d_3 be a quadrilateral with two spikes. The tile d_1 (resp. d_3) has only one neighbour d_2, across the common internal arc β (resp. α). The lift of d_1 to the universal cover $\widetilde{\Pi_3^{\odot}} \subset \mathbb{D}$, is an infinite polygon with exactly one ideal vertex (at p), and is invariant under the action of T.

Again, we construct a piecewise Killing, (ρ, u)-equivariant vector field X on the universal cover that represents $f(x)$.

$$\begin{array}{rcl} X : \widetilde{\Pi_3^{\odot}} & \longrightarrow & T\mathbb{H}^2 \\ p \mapsto & & \begin{cases} v \wedge p, & \text{if } p \in \widetilde{d_1}, \\ T^i \cdot (v_\beta + v) \wedge p, & \text{if } p \in T^i \cdot \widetilde{d_2}, \\ T^i \cdot (v_\alpha + v_\beta + v) \wedge p, & \text{if } p \in T^i \cdot \widetilde{d_3}, \end{cases} \end{array}$$

where $v \in \mathfrak{g}$ is fixed and $i \in \mathbb{Z}$. The corresponding tile map is given by

$$\begin{array}{rccl} \phi & : & \widetilde{\mathcal{T}_\sigma} & \longrightarrow & \mathfrak{g} \\ & T^i \cdot d_1 & \mapsto & v, \\ & T^i \cdot d_2 & \mapsto & T^i \cdot (v_\beta + v), \\ & T^i \cdot d_3 & \mapsto & T^i \cdot (v_\alpha + v_\beta + v), \end{array}$$

where $\widetilde{\mathcal{T}_\sigma} = \{\widetilde{d_1}\} \cup \Gamma \cdot \widetilde{d_2} \cup \Gamma \cdot \widetilde{d_3}$. Next, we calculate the infinitesimal deformation u of ρ caused by $f(x)$. From the definition of X, we get that for $p \in \widetilde{d_1}$,

$$X(\rho(\gamma)p) = v \wedge (\rho(\gamma)p).$$

Using the equivariance of X, for every $\gamma \in \pi_1(S)$, we have that

$$\begin{array}{rll} & (\rho(\gamma) \cdot (v \wedge p) + u(\gamma)((\rho(\gamma)p) & = v \wedge (\rho(\gamma)p) \\ \Rightarrow & u(\gamma)((\rho(\gamma)p) & = v \wedge (\rho(\gamma)p) - \rho(\gamma) \cdot v \wedge (\rho(\gamma)p) \\ \Rightarrow & u(\gamma) & = (v - \rho(\gamma) \cdot v). \end{array}$$

So u is a coboundary.

Example 4.3.7. Let $S = \Pi_3^{\circledcirc}$ be a one-holed triangle. Let $m = ([\rho, \vec{x}]) \in \mathfrak{D}(S)$, where ρ is the holonomy representation and $\vec{x} = (x_1, x_2, x_3) \in \partial_\infty \mathbb{D}^2$. The group $\rho(\pi_1(S))$ is generated by a hyperbolic element $\rho(\partial S) = g$ in $\mathrm{PSL}(2, \mathbb{R})$. Fix a strip template. Let α be a minimal geodesic arc and β be a finite geodesic arc disjoint from α with one endpoint on the boundary. Define the simplex $\sigma := ([\alpha], [\beta])$. Let $x = t[\alpha] + (1-t)[\beta] \in \sigma$. Then $f(x) = [\{m(s)\}]$, where $m(s) = ([\rho_s, \vec{x}(s)]])$,

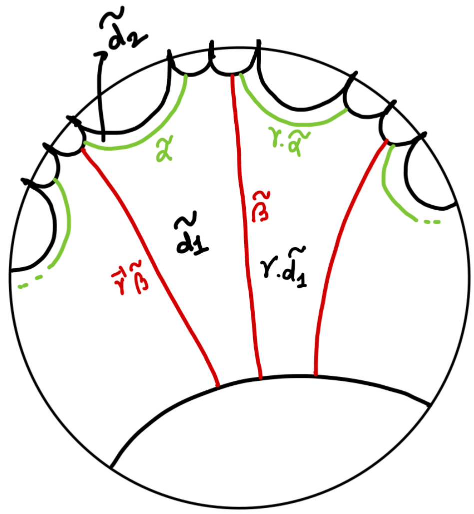


Figure 4.2: Universal cover of $\Pi_3^{\circledcirc}$

with $\rho_0 = \rho$ and $u := \frac{\mathrm{d}\rho_s}{\mathrm{d}s}\,|_{s=0}$ is the infinitesimal deformation of the holonomy. These two arcs divide the surface into two regions d_1, d_2, such that $d_2 x$ is the quadrilateral tile. Let $\widetilde{d_1}$ be a lift of d_1 whose internal edges are $\widetilde{\alpha}, \widetilde{\beta}, g^{-1}\widetilde{\beta}$. Let $\widetilde{d_2}$ be the lift d_2 neighbouring $\widetilde{d_1}$ along $\widetilde{\alpha}$. Let $v_{\widetilde{\alpha}} \in \mathfrak{g}$ (resp. $v_{\widetilde{\beta}}$) be the hyperbolic element whose axis is perpendicular to $\widetilde{\alpha}$ (resp. $\widetilde{\beta}$) at $\widetilde{p_\alpha}$ (resp. at $\widetilde{p_\alpha}$) and whose translational length is w_α (resp. w_β).

Then the associated piecewise Killing (ρ, u)-equivariant vector field of $\widetilde{S}$ is given by

$$\begin{aligned} X : \Pi_3^{\circledcirc} &\longrightarrow \quad T\mathbb{H}^2 \\ p &\mapsto \begin{cases} v \wedge p, & \text{if } p \in \widetilde{d_1}, \\ ((1-t)\sum_{i=1}^{n} \rho(g^i)\cdot v_{\widetilde{\beta}} + v) \wedge p, & \text{if } p \in \rho(g^{n+1})\cdot \widetilde{d_1} \text{ and for } n \geq 0 \\ (\rho(g^{n+1})\cdot t v_{\widetilde{\alpha}} + (1-t)\sum_{i=1}^{n} \rho(g^i)\cdot v_{\widetilde{\beta}} + v) \wedge p, & \text{if } p \in \rho(g^{n+1})\cdot \widetilde{d_2} \text{ and for } n \geq 0. \end{cases} \end{aligned}$$

Finally, we determine the ρ-cocycle u using the equivariance of X. Let $p \in \widetilde{d_1}$. Then for any $n \in \mathbb{N}$,

$$\begin{aligned} u(g^n)(\rho(g^n)p) &:= X(\rho(g^n)p) - \rho(g^n)\cdot(v \wedge p) \\ &= ((1-t)\sum_{i=1}^{n} \rho(g^i)\cdot v_{\widetilde{\beta}} + v) \wedge (\rho(g^n)p) - \rho(g^n)\cdot v \wedge (\rho(g^n)p) \\ &= (v - \rho(g^n)\cdot v + (1-t)\sum_{i=1}^{n} \rho(g^i)\cdot v_{\widetilde{\beta}}) \wedge (\rho(g^n)p). \end{aligned}$$

Similarly, for $p \in \widetilde{d_2}$ and any $n \in \mathbb{N}$, we get that

$$\begin{aligned} u(g^n)(\rho(g^n)p) &= X(\rho(g^n)p) - \rho(g^n) \cdot ((v + g \cdot tv_{\widetilde{\alpha}}) \wedge p) \\ &= (\rho(g^{n+1}) \cdot tv_{\widetilde{\alpha}} + (1-t)\sum_{i=1}^{n} \rho(g^i) \cdot v_{\widetilde{\beta}} + v) \wedge (\rho(g^n)p) - \rho(g^n) \cdot (v + g \cdot tv_{\widetilde{\alpha}}) \wedge (\rho(g^n)p) \\ &= (v - \rho(g^n) \cdot v + (1-t)\sum_{i=1}^{n} \rho(g^i) \cdot v_{\widetilde{\beta}}) \wedge (\rho(g^n)p). \end{aligned}$$

Hence, we have that for every $\gamma = g^n \in \pi_1(S)$, $u(\gamma) = v - \rho(\gamma) \cdot v + \rho(\gamma) \cdot (1-t)v_{\widetilde{\beta}}$.

4.3.3 Some useful estimates

Let S be a hyperbolic surface with (possibly decorated) spikes with a metric m. Consider a strip deformation $f_\alpha(m)$ along a finite arc α, with strip template $(\alpha_g, p_\alpha, w_\alpha)$. Then the strip added along α is hyperbolic. Let $w_\alpha(p)$ be the width of the strip at the point $p \in \alpha_g$. Let $\widetilde{\alpha_g}, \widetilde{p_\alpha}, \widetilde{p}$ be the lifts of α_g, p_α, p such that $\widetilde{p}, \widetilde{p_\alpha} \in \widetilde{\alpha_g}$. Suppose that $v_{\widetilde{\alpha}}$ is the Killing field acting across $\widetilde{\alpha_g}$ due to the strip deformation. Then, $\|v_{\widetilde{\alpha}}\| = w_\alpha$.

In the hyperboloid model $\mathbb{H}^2$, suppose that $v_{\widetilde{\alpha}} = (w_\alpha, 0, 0)$ and let the plane containing $\widetilde{\alpha_g}$ be $\{(x, y, z) \in \mathbb{R}^3 \mid y = 0\}$. So, $\widetilde{p_\alpha} = (0, 0, 1)$. A point p on the geodesic carrying $\widetilde{\alpha}$ is of the form $(x, 0, \sqrt{x^2+1})$, with $x \in \mathbb{R}$. Then we have

$$w_\alpha(p) = q(v_{\widetilde{\alpha}} \wedge p) = w_\alpha \sqrt{x^2+1} = -w_\alpha \langle p, \widetilde{p_\alpha} \rangle = w_\alpha \cosh d_{\mathbb{H}^2}(p, \widetilde{p_\alpha}). \tag{4.5}$$

Now suppose that the arc α is joining a decorated spike and a boundary component of a surface S_{sp}^h with decorated spikes. Then the infinitesimal strip added by $f_\alpha(m)$ is parabolic. Then, take $v_{\widetilde{\alpha}} = (w_\alpha, 0, w_\alpha)$. Then,

$$w_\alpha(p) = q(v_{\widetilde{\alpha}} \wedge p) = w_\alpha(\sqrt{x^2+1} - x).$$

Let L be the linear coordinate along the arc α such that $L < 0$ if p lies between $v_{\widetilde{\alpha}}$ and p_α and $L > 0$ if p_α lies between $v_{\widetilde{\alpha}}$ and p. Taking $x = \sinh L$ we get, $w_\alpha(p) = \mathrm{e}^L$.

The point p_α is called the point of *minimum impact* because $w_\alpha(p_\alpha) = w_\alpha$.

Definition 4.3.8. Let S be a hyperbolic surface with (possibly decorated) spikes with a metric m and corresponding strip template $\{(\alpha_g, p_\alpha, w_\alpha)\}$. Let $x = \sum_{i=1}^{N_0} c_i \alpha_i$ be a point in the pruned arc complex $\widehat{\mathcal{A}}(S)$. Then the *strip width function* is defined as:

$$\begin{array}{rccc} w_x : & \operatorname{supp}(x) & \longrightarrow & \mathbb{R}_{>0} \\ & p & \mapsto & c_i w_{\alpha_i}(p), \end{array}$$

Normalisation: Let S be a surface with (possibly decorated) spikes and $\mathcal{K}$ be the set of permitted arcs. Then for every $\alpha \in \mathcal{K}$, we choose $w_\alpha > 0$ such that the following equality holds for every $x \in \widehat{\mathcal{A}}(S)$:

$$\sum_{p \in \partial S \cap \operatorname{supp}(x)} w_x(p) = 1. \tag{4.6}$$

Lemma 4.3.9. *Let S^h_{sp} be a hyperbolic surface with decorated spikes endowed with a decorated metric m and a corresponding strip template $(\alpha_g, p_\alpha, w_\alpha)$. Let $x \in \widehat{\mathcal{A}}(S^h_{sp})$ and γ be a non-trivial closed geodesic in S^h_{sp}. Then,*

$$\mathrm{d}l_\gamma(f(x)) = \sum_{p\in\gamma\cap\mathrm{supp}(x)} w_x(p)\sin\angle_p(\gamma_g, \mathrm{supp}\,(x)) \geq 0. \tag{4.7}$$

Proof. We assume that $\mathrm{supp}\,(x)$ contains only one vertex $[\alpha]$. Suppose that α is an infinite arc joining a spike and a boundary component of S^h_{sp}. Consider the universal cover of the surface inside the upper half plane model $\mathbb{U}$. Using the transitivity of $\mathrm{PSL}(2,\mathbb{R})$, we can suppose that the lifts of γ and α are the geodesics $(-1,1), (a,\infty)$, for some $a \in (-1,1)$. Let ρ be the holonomy of the convex core. Then, $\rho(\gamma)$ is a hyperbolic element in $\mathrm{PSL}(2,\mathbb{R})$ of the form

$$\rho(\gamma) = \begin{bmatrix} \cosh\frac{l}{2} & \sinh\frac{l}{2} \\ \sinh\frac{l}{2} & \cosh\frac{l}{2}\end{bmatrix}, \text{ for some } l > 0.$$

Let $p := \alpha\cap\gamma$ be the intersection point. Then $p = (a, \sqrt{1-a^2})$. Suppose that the Killing field representing the infinitesimal parabolic strip deformation along α generates the following one parameter family with fixed point at ∞:

$$g_t = \begin{bmatrix} 1 & t \\ 0 & 1\end{bmatrix}, t \in \mathbb{R}_{>0}.$$

Now,

$$g_t\rho(\gamma) = \begin{bmatrix} \cosh\frac{l}{2} + t\sinh\frac{l}{2} & t\cosh\frac{l}{2} + \sinh\frac{l}{2} \\ \sinh\frac{l}{2} & \cosh\frac{l}{2}\end{bmatrix}$$

So the translational length is given by:

$$\begin{aligned} l_t &= 2\,\mathrm{arccosh}\left(\frac{tr(g_t\rho(\gamma))}{2}\right) \\ &= 2\,\mathrm{arccosh}\left(\cosh\frac{l}{2} + \frac{t\sinh\frac{l}{2}}{2}\right) \\ &= 2\ln\left(\cosh\frac{l}{2} + \frac{t\sinh\frac{l}{2}}{2} + \sqrt{(\cosh\frac{l}{2} + \frac{t\sinh\frac{l}{2}}{2})^2}\right) \\ &= 2\ln\left(\left(\cosh\frac{l}{2} + \sinh\frac{l}{2}\right) + \frac{t}{2}\left(\cosh\frac{l}{2} + o(t)\right)\right) \\ &= l + t + o(t). \end{aligned}$$

So we have that $\mathrm{d}l_\gamma(f(x)) = 1$.

Let $w(t)$ be the width of the strip added at the point p after a strip deformation γ_t. Then,

$$\begin{aligned}\cosh w(t) &= 1 + \frac{t^2}{2(1-a^2)},\\ \Rightarrow w(t) &= \frac{t}{\sqrt{1-a^2}} + o(t),\\ \Rightarrow w'(0) &= \frac{1}{\sqrt{1-a^2}},\\ &= w_x(p).\end{aligned}$$

Finally, by construction we have that $\sin \angle_p(\gamma_g, \operatorname{supp}(x)) = \sqrt{1-a^2}$. This concludes the proof for one intersection point. By linearity, we get the result for the general case with multiple intersection points.

□

Lemma 4.3.10. *Let x be a point of a hyperbolic surface S. Let B be a geodesic ball centered at x with radius r, where r is the injectivity radius of the surface. Then for every pair of distinct lifts B_1, B_2 of B in the universal cover of S, we have that $B_1 \cap B_2 = \emptyset$.*

Proof. Let x, B, B_1, B_2 be as in the hypothesis. Then, $B = \exp_x(B(0,r))$. For $i = 1,2$, let $x_i \in \mathbb{H}^2$ be the center of B_i. Then $x_2 = \gamma \cdot x_1$ for some $\gamma \in \pi_1(S)$.

If possible, let $\widetilde{z} \in B_1 \cap B_2$. Since the action of $\pi_1(S)$ is free, the point $z' := \gamma \cdot \widetilde{z}$ lies inside B_2. Join x_2 with z and z' by geodesic segments l_1, l_2, respectively. On the surface S, the path $l_1 \cup l_2$ is mapped to a loop based at z. Since B is the embedding of a ball by the exponential map, this loop is trivial. So $l_1 \cup l_2$ is a loop in $\widetilde{S}$ based at a lift $\widetilde{z}$, which is a contradiction. □

Lemma 4.3.11. *Let S be a hyperbolic surface with a deformation $m \in \mathfrak{D}(S)$. Then there exists $M > 0$ such that for every non-trivial closed geodesic γ and for every non-trivial geodesic arc α, the following inequality holds:*

$$\sum_{p \in \gamma \cap \alpha} w_\alpha(p) \leq M l_\gamma(m). \tag{4.8}$$

Proof. We prove the theorem for surfaces with decorated spikes. Let γ and α be as in the hypothesis. We further suppose that α is an infinite arc joining a decorated spike and a boundary component of S. We prove that there exists a positive constant M_0 such that for every unit segment η and for every arc α, the following inequality holds.

$$\sum_{p \in \eta \cap \alpha} w_\alpha(p) \leq M_0. \tag{4.9}$$

Given such a unit segment η, let $\eta \cap \alpha = \{p_1, \ldots, p_k\}$. We order them so that p_i lies closer to the horoball than p_j if and only if $i > j$. Take a lift $\widetilde{\alpha}$ of α in the universal cover inside $\mathbb{D}$. Since α is embedded, for every $i = 1, \ldots, k$, there is exactly one lift $\widetilde{p_i}$ of p_i that lies on $\widetilde{\alpha}$. Let r_0 be the injectivity radius of the surface. Cover the entire arc $\widetilde{\alpha}$ with balls $\{B_j\}_{j \in J}$ of radius $r := \frac{r_0}{2}$, such that two consecutive balls are tangent to each other.

We claim that each ball contains at most M_0 number of intersection points, where $M_0 := [\frac{1}{r_0}]+1$. Consider one lift of η and a ball B_r in the above covering. Then a reformulation of the claim is

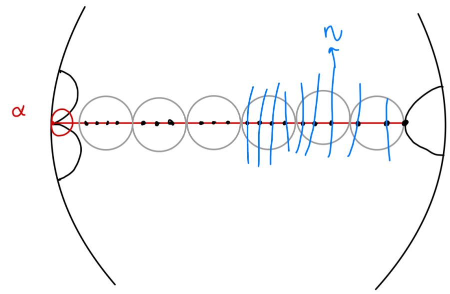


Figure 4.3: Intersection of the arc and the curve

that there are at most M_0-many balls that are in the same orbit as B. Now each of these balls are contained in the bigger ball B_{2r} with the same center and of radius r_0 and by Lemma 4.3.10, no two of them can intersect. Thus the maximum number of balls B_r in the same orbit intersecting η is M_0.

Next, we know that $w_{\widetilde{\alpha}}(\widetilde{p}_i) = \mathrm{e}^{L_i}$, where L_i is the negative arc coordinate of $\widetilde{p}_i$ along $\widetilde{\alpha}$. Then, $w_{\widetilde{\alpha}}(\widetilde{p}_i)$ decreases exponentially as i increases. In every ball, the maximum value of $w_{\widetilde{\alpha}}$ is attained at the rightmost point. Two such points in two consecutive balls are at most r_0 distance apart because the balls are of radius $\frac{r_0}{2}$ and tangent to each other. Inside the first ball, the maximum value of $w_{\widetilde{\alpha}}$ can be at most 1, if the point of unit impact is an intersection point.

So we have,

$$\sum_{p\in\eta\cap\alpha} w_\alpha(p) = \sum_{i=1}^{k} w_{\widetilde{\alpha}}(\widetilde{p}_i) = \sum_{i=1}^{k} \mathrm{e}^{L_i} \leq M_0(1+\mathrm{e}^{-r_0}+\ \mathrm{e}^{-2r_0}+\ldots) = \frac{M_0}{1-\mathrm{e}^{-r_0}}. \tag{4.10}$$

Finally, taking the sum over all the unit segments of γ, we get that

$$\sum_{p\in\gamma\cap\alpha} w_\alpha(p) \leq M l_\gamma(m),$$

where $M := \frac{M_0}{1-\mathrm{e}^{-r_0}}$. □

Lemma 4.3.12. *Let S_c be a compact hyperbolic surface equipped with a metric $m \in \mathfrak{D}(S_c)$. Then for every $\epsilon > 0$ there exists $M > 0$ such that whenever a geodesic arc α has m-length $l_\alpha(m) > M$, there exists a closed geodesic γ on the surface such that it intersects α as well as every geodesic arc, that is disjoint from α, at angle less than ϵ.*

Proof. Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $N\mathrm{diam}(S_c)\epsilon > 3\,\mathrm{area}(S_c)$. Take $M = N\mathrm{diam}(S_c)$. Consider an arc α of length M and its ϵ-neighbourhood $V_\epsilon(\alpha)$. The area of $V_\epsilon(\alpha)$ is at least $2M\epsilon$. So $V_\epsilon(\alpha)$ cannot be embedded inside the surface — it self-overlaps threefold. It follows that there exists a segment η of the arc α such that its length is $N'\mathrm{diam}(S_c)$ for $1 < N' < N$ and its endpoints lie at a distance 2ϵ from each other, with the velocities at those points being parallel. Join the two endpoints by a geodesic segment to get a closed loop. Then choose the unique closed geodesic γ in its homotopy class. We claim that γ satisfies the condition of the lemma. Firstly, we show that

$$\max_{p\in\gamma\cap\alpha} \angle_p(\gamma,\alpha) < \epsilon. \tag{4.11}$$

Let $\widetilde{\gamma}$ be a infinite geodesic lift of γ. Let $\widetilde{\eta}$ be a lift of the arc segment η and consider its $\rho(\gamma)$-orbit. For every $i \in \mathbb{Z}$, the two endpoints of $\rho(\gamma)^i \cdot \widetilde{\eta}$ are ϵ-close to one endpoint of $\rho(\gamma)^{i-1} \cdot \widetilde{\eta}$ and one endpoint of $\rho(\gamma)^{i+1} \cdot \widetilde{\eta}$. Let $p_i := \widetilde{\gamma} \cap \rho(\gamma^i) \cdot \widetilde{\eta}$. The entire geodesic $\widetilde{\gamma}$ is contained in the union $V := \bigcup_{i\in\mathbb{Z}} V_\epsilon(\rho(\gamma^i) \cdot \widetilde{\eta})$ of the ϵ-neighbourhoods of the lifts of η. So the angle of intersection at p_i satisfies

$$\angle_{p_i}(\widetilde{\gamma}, \rho(\gamma^i) \cdot \widetilde{\eta}) < \epsilon.$$

Now let α' be an arc disjoint from α that intersects γ. Then the point of intersection, denoted by p, lies inside $V_\epsilon(\alpha)$. Then from eq. (4.11) we have that α' intersects γ at an angle less than ϵ. □

The following lemma is an analogue of Proposition 2.3 in [5].

Lemma 4.3.13. *Let S_{sp}^h be a hyperbolic surface with decorated spikes endowed with metric $m \in \mathfrak{D}(S_{sp}^h)$. For any choice of minimally intersecting geodesic representatives $\{\alpha\}$ whose finite endpoints lie outside the horoball decoration of the spikes, there exists $\theta_0 \in (0, \frac{\pi}{2}]$ such that all the arcs intersect the boundary of the surface at an angle greater or equal to θ_0.*

4.4 Strip deformations of compact surfaces

In this section, we will recall the statement of the parametrisation theorem proved by Danciger-Guéritaud-Kassel in [5] for compact hyperbolic surfaces with totally geodesic boundary. We shall also give an idea of their proof, whose methods are going to be adapted to our case of surfaces with spikes.

Let S_c be a compact hyperbolic surface with totally geodesic boundary. Recall that when S_c is orientable (resp. non-orientable), it is of the form $S_{g,n}$ (resp. $T_{h,n}$), where g (resp. h) is the genus and n is the total number of boundary components. Its deformation space $\mathfrak{D}(S_c)$ is homeomorphic to an open ball of dimension $N_0 = 6g - 6 + 3n$ when S_c is orientable and $N_0 = 3h - 6 + 3n$ when S_c is non-orientable. A point m of the deformation space is expressed as $m = [\rho]$, where $\rho : \pi_1(S_c) \to \mathrm{PGL}(2,\mathbb{R})$ is a holonomy representation of the surface. Given such an element $m \in \mathfrak{D}(S_c)$, its admissible cone $\Lambda(m)$ is the set of all infinitesimal deformations that uniformly lengthen every non-trivial closed geodesic. It is an open convex cone of the vector space $T_m\mathfrak{D}(S_c)$.

The arcs that are used to span the arc complex $\mathcal{A}(S_c)$ of such a surface, are finite, non self-intersecting and their endpoints lie on boundary ∂S_c. The set of all such arcs is denoted by $\mathcal{K}$. A simplex of the arc complex is big if the arcs corresponding to its 0-skeleton decompose the surface into topological disks. The pruned arc complex $\widehat{\mathcal{A}}(S_c)$ of the surface S_c, given by the union of the interiors of all big simplices, is an open ball of dimension $N_0 - 1$. Any point x in $\widehat{\mathcal{A}}(S_c)$ belongs to the interior of a unique big simplex σ_x.

The strip deformations performed along the arcs are of hyperbolic type; their waists and widths are fixed by the choice of a strip template $\{(\alpha_g, p_\alpha, w_\alpha)\}_{\alpha\in\mathcal{K}}$. The infinitesimal strip map is given by

$$\begin{array}{rccc} f \quad : & \mathcal{A}(S_c) & \longrightarrow & T_m\mathfrak{D}(S_c) \\ & x = \sum_{i=1}^{N_0} c_i\alpha_i & \mapsto & \sum_{i=1}^{N_0} c_i f_{\alpha_i}(m), \end{array}$$

where $c_i \in [0,1]$ for every $i = 1, \dots, N_0$, and $\sum_{i=1}^{N_0} c_i = 1$. Then the following result was proved in [5]:

Theorem 4.4.1 (Danciger-Guéritaud-Kassel)**.** *Let $S_c = S_{g,n}$ or $T_{h,n}$ be a compact hyperbolic surface with totally geodesic boundary. Let $m = ([\rho]) \in \mathfrak{D}(S_c)$ be a metric. Fix a choice of strip template $\{(\alpha_g, p_\alpha, w_\alpha)\}_{\alpha \in \mathcal{K}}$ with respect to m. Then the restriction of the projectivised infinitesimal strip map $\mathbb{P}f : \widehat{\mathcal{A}}(S_c) \longrightarrow \mathbb{P}^+(T_m\mathfrak{D}(S_c))$ is a homeomorphism on its image $\mathbb{P}^+(\Lambda(m))$.*

Structure of the proof. Firstly, they show that the image of the map $\mathbb{P}f$ is given by the positively projectivised admissible cone. Since both the pruned arc complex and $\mathbb{P}^+(\Lambda(m))$ are homeomorphic to open balls of the same dimension, it is enough to show that $\mathbb{P}f$ is a covering map. A classical result from topology states that a continuous map between two manifolds is a covering map if the map is proper and also a local homeomorphism. So the authors prove that the projectivised strip map $\mathbb{P}f$ satisfies these two properties. Firstly they show that

Theorem 4.4.2. *The projectivised strip map $\mathbb{P}f : \widehat{\mathcal{A}}(S_c) \longrightarrow \mathbb{P}^+(\Lambda(m))$ is proper.*

Secondly, they show that the map $\mathbb{P}f$ is a local homeomorphism around points $x \in \widehat{\mathcal{A}}(S_c)$ such that $\operatorname{codim}(\sigma_x) \leq 2$, and then around points such that $\operatorname{codim}(\sigma_x) \geq 2$ by induction.

Theorem 4.4.3. *Let S_c be a compact hyperbolic surface with totally geodesic boundary, equipped with a metric $m \in \mathfrak{D}(S_c)$. Let $x \in \widehat{\mathcal{A}}(S_c)$ be a point contained in the interior of the simplex $\sigma_x \subset \mathcal{A}(S_c)$. Then, for every $i = 0, 1, 2$ and for every $x \in \widehat{\mathcal{A}}(S_c)$ with $\operatorname{codim}(\sigma_x) = i$, there exists an open neighbourhood U of x in $\widehat{\mathcal{A}}(S_c)$ such that $\mathbb{P}f|_U$ is a homeomorphism onto its image.*

For points belonging to the interior of simplices with codimension 0, it is enough to show that the f-images of the vertices of any top-dimensional simplex form a basis in the deformation space of the surface.

Theorem 4.4.4. *Let S_c be a compact hyperbolic surface with totally geodesic boundary, equipped with a metric $m \in \mathfrak{D}(S_c)$. Let σ be a codimension zero simplex and let $\mathcal{E}_\sigma$ be the corresponding edge set. Then the set of infinitesimal strip deformations $B = \{f_e(m) | e \in \mathcal{E}_\sigma\}$ forms a basis of $T_m\mathfrak{D}(S_c)$.*

Next let $x \in \widehat{\mathcal{A}}(S_{sp})$ such that $x \in \operatorname{int}(\sigma_x)$ where σ_x is a big simplex with $\operatorname{codim}(\sigma_x) = 1$. Since $\widehat{\mathcal{A}}(S_{sp})$ is an open ball, there exist two simplices σ_1, σ_2 such that

- $\operatorname{codim}(\sigma_1) = \operatorname{codim}(\sigma_2) = 0$,
- $\sigma_x = \sigma_1 \cap \sigma_2$.

The following theorem gives a sufficient condition for proving local homeomorphism of the projectivised strip map around points like in this case.

Theorem 4.4.5. *Let S_c be a compact hyperbolic surface with totally geodesic boundary, equipped with a metric $m \in \mathfrak{D}(S_c)$. Let $\sigma_1, \sigma_2 \in \mathcal{A}(S_c)$ be two top-dimensional simplices such that*

$$\operatorname{codim}(\sigma_1 \cap \sigma_2) = 1 \textit{ and } \operatorname{int}(\sigma_1 \cap \sigma_2) \subset \widehat{\mathcal{A}}(S_c).$$

Then we have that,

$$\operatorname{int}(\mathbb{P}f(\sigma_1)) \cap \operatorname{int}(\mathbb{P}f(\sigma_2)) = \varnothing. \tag{4.12}$$

The case $i = 2$ in Theorem 4.4.3 is a corollary of the following lemma:

Lemma 4.4.6. *Let S_c be a compact hyperbolic surface with totally geodesic boundary, equipped with a metric $m \in \mathfrak{D}(S_c)$. Let $x \in \mathcal{A}(S_c)$ such that* $\operatorname{codim}(\sigma_x) = 2$. *Then, $\mathbb{P}f|_{\mathrm{Link}(\sigma_p, \mathcal{A}(S_c))}$ is a homeomorphism.*

This lemma is a consequence of the next theorem.

Theorem 4.4.7. *Let S_c be a compact hyperbolic surface with totally geodesic boundary, equipped with a metric $m \in \mathfrak{D}(S_c)$. Let σ_1, σ_2 be two simplices of its arc complex $\mathcal{A}(S_c)$ satisfying the conditions of Theorem 4.4.5. Then there exists a choice of strip template such that $\mathbb{P}f(\sigma_1) \cup \mathbb{P}f(\sigma_2)$ is convex in $\mathbb{P}^+(T_m\mathfrak{D}(S_c))$.*

Idea of the proof of Lemma 4.4.6: Since $\operatorname{codim}(\sigma_x) = 2$, there is space to put two more arcs that are disjoint from all the arcs of σ_x. There are two possibilities:

- there exist exactly two disjoint regions (hyperideal quadrilaterals) in the complement of $\operatorname{supp}(x)$; every other connected component is a hyperideal triangle. Each of these regions can be decomposed into hyperideal triangles in two ways by a diagonal exchange such that the exchanges are independent of each other. So the sub-complex $\mathrm{Link}(\sigma_x, \mathcal{A}(S_c))$ of $\mathcal{A}(S_c)$ is a quadrilateral in this case.

- there exists exactly one region in the complement of $\operatorname{supp}(x)$, which is not a hyperideal triangle. This region can be decomposed into three hyperideal triangles by two additional arcs that are pairwise disjoint from the rest. These two arcs can be chosen in 5 ways, using the "pentagonal moves". As a result, the sub-complex $\mathrm{Link}(\sigma_x, \mathcal{A}(S_c))$ is a pentagon in this case.

So in both the cases, the restriction of the projectivised infinitesimal strip map to the link gives a P-L map

$$\mathbb{P}f|_{\mathrm{Link}(\sigma_p, \mathcal{A}(S_c))} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1.$$

Using Theorem 4.4.7, the authors prove that this map has degree one, which proves it to be a homemorphism. $\square$

Finally, the local homeomorphism of $\mathbb{P}f$ is obtained as a corollary:

Theorem 4.4.8. *Let $x \in \widehat{\mathcal{A}}(S_c)$ such that* $\operatorname{codim}(\sigma_x) \geq 2$. *Let $V \subset T_m\mathfrak{D}(S_c)$ be the vector subspace generated by the infinitesimal strip deformations $\{f_\alpha(m)\}_{\alpha \in \sigma_x^{(0)}}$. Then, the restriction map*

$$\mathbb{P}f : \mathrm{Link}(\sigma_x, \mathcal{A}(S_c)) \longrightarrow \mathbb{P}^+(T_m\mathfrak{D}(S_c)/V)$$

is a homeomorphism.

We recall the proof of the above theorem as done in [5]. We will use the same reasoning for our surfaces with spikes.

Proof. Firstly, we note that the V is a subspace of dimension $N_0 - 2$ because from Theorem 4.4.4 we get that $\{f_\alpha(m)\}_{\alpha \in \sigma_x^{(0)}}$ is linearly independent. So the space $\mathbb{P}^+(T_m\mathfrak{D}(S_c)/V)$ is homeomorphic to $\mathbb{S}^{\mathrm{codim}(\sigma_x)-1}$. The statement is verified for codim $(\sigma_x) = 2$. Suppose that the statement holds for $2, \ldots, d-1$. We need to show that

$$\mathbb{P}f|_{\mathrm{Link}(\sigma_x, \mathcal{A}(S_c))} : \mathbb{S}^{d-1} \longrightarrow \mathbb{S}^{d-1}$$

is a local homeomorphism. Let $x \in \mathrm{Link}(\sigma_x, \mathcal{A}(S_c))$. Then x is contained in the interior of a simplex σ_x whose codimension in $\mathrm{Link}(\sigma_x, \mathcal{A}(S_c))$ is $d - 1 - \dim \sigma_x$, which is less than d. So by induction hypothesis, the map $\mathbb{P}f|_{\mathrm{Link}(\sigma_x, \mathcal{A}(S_c))}$ restricted to $\mathrm{Link}(\sigma_x, \mathbb{P}f|_{\mathrm{Link}(\sigma_x, \mathcal{A}(S_c))})$ is a homeomorphism. This proves that $\mathbb{P}f|_{\mathrm{Link}(\sigma_x, \mathcal{A}(S_c))}$ is a local homeomorphism. Since $\mathbb{S}^{d-1}$ is compact and simply-connected for $d \geq 3$, it follows that $\mathbb{P}f|_{\mathrm{Link}(\sigma_x, \mathcal{A}(S_c))}$ is a homeomorphism. $\square$

Chapter 5

Strip deformations of Hyperbolic Polygons

The goal of this chapter is to prove the parametrisation theorems for four types of polygons — ideal polygons, ideal once-punctured polygons, decorated polygons and one-holed polygons. Let Π be the surface of any of these polygons and let $N_0 := \dim \mathfrak{D}(\Pi)$. Recall from Definition 4.3.3 that the projectivised infinitesimal strip map for a fixed $m \in \mathfrak{D}(\Pi)$ is defined as:

$$\begin{array}{rccc} \mathbb{P}f & : & \mathcal{A}(\Pi) & \longrightarrow & \mathbb{P}^+(T_m\mathfrak{D}(\Pi)) \\ & & \sum_{i=1}^{N_0} c_i\alpha_i & \mapsto & \left[\sum_{i=1}^{N_0} c_i f_{\alpha_i}(m)\right] \end{array}$$

where for every $i = 1, \ldots, N_0$, $c_i \in [0,1]$ and $\sum_{i=1}^{N_0} c_i = 1$.

Theorem 5.0.1. *Let $\Pi_n^{\pentagon}$ $(n \geq 4)$ be an ideal n-gon with a metric $m \in \mathfrak{D}(\Pi_n^{\pentagon})$. Fix a choice of strip template. Then, the infinitesimal strip map*

$$\mathbb{P}f : \mathcal{A}\left(\Pi_n^{\pentagon}\right) \longrightarrow \mathbb{P}^+(T_m\mathfrak{D}(\Pi_n^{\pentagon}))$$

is a homeomorphism.

Theorem 5.0.2. *Let $\Pi_n^{\odot}$ $(n \geq 2)$ be an ideal once-punctured n-gon with a metric $m \in \mathfrak{D}(\Pi_n^{\odot})$. Fix a choice of strip template. Then, the infinitesimal strip map*

$$\mathbb{P}f : \mathcal{A}\left(\Pi_n^{\odot}\right) \longrightarrow \mathbb{P}^+(T_m\mathfrak{D}(\Pi_n^{\odot}))$$

is a homeomorphism.

Theorem 5.0.3. *Let $\Pi_n^{\circledcirc}$ $(n \geq 3)$ be a decorated n-gon with a metric $m \in \mathfrak{D}(\Pi_n^{\circledcirc})$. Fix a choice of strip template. Then the infinitesimal strip map $\mathbb{P}f$, when restricted to the pruned arc complex $\widehat{\mathcal{A}}(\Pi)$, is a homeomorphism onto its image $\mathbb{P}^+(\Lambda(m))$, where $\Lambda(m)$ is the set of infinitesimal deformations that lengthens all edges and diagonals of the polygon.*

Theorem 5.0.4. *Let $\Pi_n^{\circledcirc}$ $(n \geq 1)$ be a one-holed n-gon with a metric $m \in \mathfrak{D}_0(\Pi_n^{\circledcirc})$. Let $\mathcal{A}_{\circlearrowright}(n)$ be its spinning arc complex. Then the infinitesimal strip map*

$$\mathbb{P}f : \mathcal{A}_{\circlearrowright}(n) \longrightarrow \mathbb{P}^{+}(T_m \mathfrak{D}_0(\Pi_n^{\circledcirc}))$$

is a homeomorphism.

The table in Fig. 5.1 recapitulates the topology of the deformation spaces and the arc complexes of the four types of polygons. We get that the projectivised strip maps in Theorems 5.0.1, 5.0.2 and 5.0.4, are defined from a sphere to another of the same dimension; in the case of decorated polygons, the projectivised strip map in Theorem 5.0.3 is defined from an open ball (pruned arc complex) to another of the same dimension (positively projectivised deformation space). See Fig. (**??**).

Π	$\Pi_n^{\pentagon}, n \geq 4$	$\Pi_n^{\odot}$, $n \geq 2$	$\Pi_n^{\oslash}, n \geq 3$	$\Pi_n^{\circledcirc}$ $n \geq 1$
$\mathfrak{D}(\Pi)$	$\mathbb{D}^{n-3}$	$\mathbb{D}^{n-1}$	$\mathbb{D}^{2n-3}$	$\mathbb{D}^{n}$
$\mathcal{A}(\Pi)$	$\mathbb{S}^{n-4}$	$\mathbb{S}^{n-2}$	–	$\mathbb{S}^{n-1}$
$\widehat{\mathcal{A}}(\Pi)$	idem	–	$\mathbb{D}^{2n-4}$	–

Figure 5.1: Arc complexes and deformation spaces of polygons

Ideal n-gon $\Pi_n^{\pentagon}$	$\mathbb{P}f : \mathbb{S}^{n-4} \longrightarrow \mathbb{S}^{n-4}$
Punctured n-gon $\Pi_n^{\odot}$	$\mathbb{P}f : \mathbb{S}^{n-2} \longrightarrow \mathbb{S}^{n-2}$
One-holed n-gon $\Pi_n^{\circledcirc}$	$\mathbb{P}f : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$
Decorated n-gon $\Pi_n^{\oslash}$	$\mathbb{P}f : \mathbb{D}^{2n-4} \longrightarrow \mathbb{D}^{2n-4}$

Figure 5.2: Projectivised strip map for the four types of polygons

Idea of the proofs. Each of the proofs of the four theorems follows the same strategy as discussed at the end of Chapter 4. Firstly, we show that the map $\mathbb{P}f$ is a local homeomorphism. Since the sphere is compact, we have that $\mathbb{P}f$ is a covering map for the first three cases — $\Pi_n^{\pentagon}$, $\Pi_n^{\odot}$, $\Pi_n^{\circledcirc}$. For the decorated polygons $\Pi_n^{\oslash}$, we show that it is a proper map in order to get a covering map. Finally, for $n \geq 6$ (ideal n-gon) and $n \geq 4$ (punctured n-gon), the spheres $\mathbb{S}^{n-4}$ and $\mathbb{S}^{n-2}$ are simply-connected, so the maps are homeomorphisms. The cases $\Pi_4^{\pentagon}, \Pi_5^{\pentagon}, \Pi_2^{\odot}, \Pi_3^{\odot}$ will be treated separately. Similarly, the open balls are simply-connected and hence we get a global homeomorphism for decorated polygons.

Let Π be the topological surface of any hyperbolic polygon. Every point $p \in \mathcal{A}(\Pi)$ belongs to a unique open simplex, denoted by σ_p. Like in [5], we prove that $\mathbb{P}f$ is a local homeomorphism for points p such that $\operatorname{codim}(\sigma_p) = 0, 1, 2$ and for p with $\operatorname{codim}(\sigma_p) \geq 3$, the proof is by induction on the codimension of the stratum.

5.1 Local homeomorphism: codimension 0 faces

In this section, for each of the four types of polygons, we shall prove the local homeomorphism of the projectivised strip maps around points that belong to the interior of codimension 0 simplices in their respective arc complexes.

5.1.1 Ideal polygons

Theorem 5.1.1. *Let $m \in \mathfrak{D}(\Pi_n^{\pentagon})$ be a metric on an ideal n-gon $\Pi_n^{\pentagon}$, with $n \geq 4$. Fix a choice of strip template. Let σ be a top-dimensional simplex of its arc complex $\mathcal{A}\left(\Pi_n^{\pentagon}\right)$ and let $\mathcal{E}_\sigma$ be the corresponding edge set. Then the set of infinitesimal strip deformations $B = \{f_e(m) | e \in \widetilde{\mathcal{E}_\sigma}\}$ forms a basis of the tangent space $T_m\mathfrak{D}(\Pi_n^{\pentagon})$.*

Proof. Since $\dim T_m\mathfrak{D}(\Pi) = \#\mathcal{E}_\sigma = n-3$, it is enough to show that the set B is linearly independent. We proceed by contradiction: suppose that there exists reals c_e, not all equal to 0, such that

$$\sum_{e\in\mathcal{E}_\sigma} c_e f_e(m) = 0. \tag{5.1}$$

Recall that $\widetilde{\mathcal{E}_\sigma}$ is the set of lifts of the arcs selected by the strip template for the metric $m \in \mathfrak{D}(\Pi_n^{\pentagon})$ and the set $\widetilde{\mathcal{T}_\sigma}$ consists of the lifts of the tiles formed by the arcs of σ. Since, an ideal polygon is simply-connected, we have that $\mathcal{E}_\sigma = \widetilde{\mathcal{E}_\sigma}, \mathcal{T}_\sigma = \widetilde{\mathcal{T}_\sigma}$. Then we get an equivalence class of tile maps, up to an additive constant in $\mathfrak{g}$, which do not deform the polygon. From this class, we can choose a neutral tile map $\phi_0 : \widetilde{\mathcal{T}_\sigma} \to \mathfrak{g}$ (see definition 4.3.4 in Chapter 4), which by definition, fixes all ideal vertices of the tiles in $\widetilde{\mathcal{T}_\sigma}$. The following lemma finds a permitted region for the $[\phi_0(d)]$ any type of tile d.

Lemma 5.1.2. *Let σ be a top-dimensional simplex of $\mathcal{A}\left(\Pi_n^{\pentagon}\right)$. Let $\phi_0 : \widetilde{\mathcal{T}_\sigma} \to \mathfrak{g}$ be a neutral tile map corresponding to the linear combination (5.1). Let $e \in \widetilde{\mathcal{E}_\sigma}$ be an internal edge of a tile $d \in \widetilde{\mathcal{T}_\sigma}$ such that $\phi_0(d) \neq 0$. Then the point $[\phi_0(d)] \in \mathbb{R}\mathrm{P}^2$ lies in the interior of the projective triangle, based at the geodesic $\overline{e}$ carrying e, that contains the tile d.*

Proof. Consider the dual graph of the triangulation of the surface by the top-dimensional simplex σ. It is a tree the valence of whose vertices is at most 3. Let τ be the sub-tree spanned by the tiles that are on the same side of e as d. Define $M(d)$ as the length of the longest path in τ joining d' and a leaf (quadrilateral). The lemma will be proved by induction on M. When $M(d) = 0$, the tile d is a quadrilateral. The neutral tile map ϕ_0 fixes the two ideal vertices of d. Applying Corollary 1.5.2 to these vertices, we get that $[\phi_0(d)]$ is the point of intersection of the tangents to $\partial_\infty\mathbb{D}$ at these ideal vertices. Lastly, the convexity of $\partial_\infty\mathbb{D}$ implies that $[\phi_0(d)]$ lies in the interior of Δ.

Next, we suppose the statement to be true for $M(d) = 0, \ldots, k$. Let $d \in \widetilde{\mathcal{T}_\sigma}$ be a tile such that $M(d) = k+1$. Then the tile d can be either a hexagon or a pentagon because a quadrilateral has only one neighbouring tile and it must lie outside the triangle Δ. We shall treat the two cases separately below:

- If d is a hexagon, then apart from e, it has two other internal edges $e', e'' \in \mathcal{E}_\sigma$ along which d neighbour two tiles d', d'', respectively. We note that both d', d'' lie inside Δ.

– Suppose that both $\phi_0(d'), \phi_0(d'')$ are non-zero. See Fig. 5.3. Denote by $\overleftrightarrow{t_1}, \overleftrightarrow{t_2}, \overleftrightarrow{t_3}, \overleftrightarrow{t_4}$, the tangents to $\partial_\infty \mathbb{D}$ at the endpoints P, Q and R, S of e', e'', respectively. Label the following points

$$X := \overleftrightarrow{t_1} \cap \overleftrightarrow{t_2}, \quad Y := \overleftrightarrow{t_3} \cap \overleftrightarrow{t_4}, \quad A := \overleftrightarrow{t_2} \cap \overleftrightarrow{t_3},$$
$$B := \overleftrightarrow{t_1} \cap \overleftrightarrow{t_3}, \quad C := \overleftrightarrow{t_1} \cap \overleftrightarrow{t_4}, \quad D := \overleftrightarrow{t_2} \cap \overleftrightarrow{t_4}.$$

By the induction hypothesis, the points $[\phi_0(d')]$ and $[\phi_0(d'')]$ lie inside the projective triangles ΔPQX and ΔRYS that contain d' and d'', respectively. Since these two triangles are disjoint, $\phi_0(d)$ cannot be equal to $\phi_0(d')$ as well as $\phi_0(d'')$. In other words, the coefficients $c_{e'}, c_{e''}$ in (5.1) cannot be simultaneously equal to zero. Without loss of generality, suppose that $c_{e'} = 0 \neq c_{e''}$. So, $\phi_0(d) = \phi_0(d') \neq \phi_0(d'')$. Consequently, $\phi_0(d)$ lies inside ΔPQX and $\phi_0(d) - \phi_0(d'')$ is a hyperbolic Killing vector field whose projective image lies on $\overleftrightarrow{e''} \setminus \overline{\mathbb{D}}$, i.e, the straight line joining the points $[\phi_0(d)]$ and $[\phi_0(d'')]$ intersects e'' outside $\overline{\mathbb{D}}$. Using Property 1.6.2 for e'', we know that the $[\phi_0(d)]$ must be contained in the region $K_1 := \mathbb{R}\mathrm{P}^2 \setminus \Delta''$ where Δ'' is the projective triangle based at e'' that does not contain d''. Since $\partial_\infty \mathbb{D}$ is convex, Δ'' is disjoint from K_1, which implies that $\phi_0(d) \neq \phi_0(d)'$, which is a contradiction. So we must have $c_{e'} \neq 0$. Using the same argument as in the case of e'', we get that $[\phi_0(d)]$ lies in the region $K_2 := \mathbb{R}\mathrm{P}^2 \setminus \Delta'$, where Δ' is the projective triangle based at $\overline{e'}$, not containing d'. Hence, the point $[\phi_0(d)]$ must lie inside the intersection $R_1 \cap R_2$, which is the quadrilateral $ABCD$ entirely contained in $\Delta \setminus e$, as required.

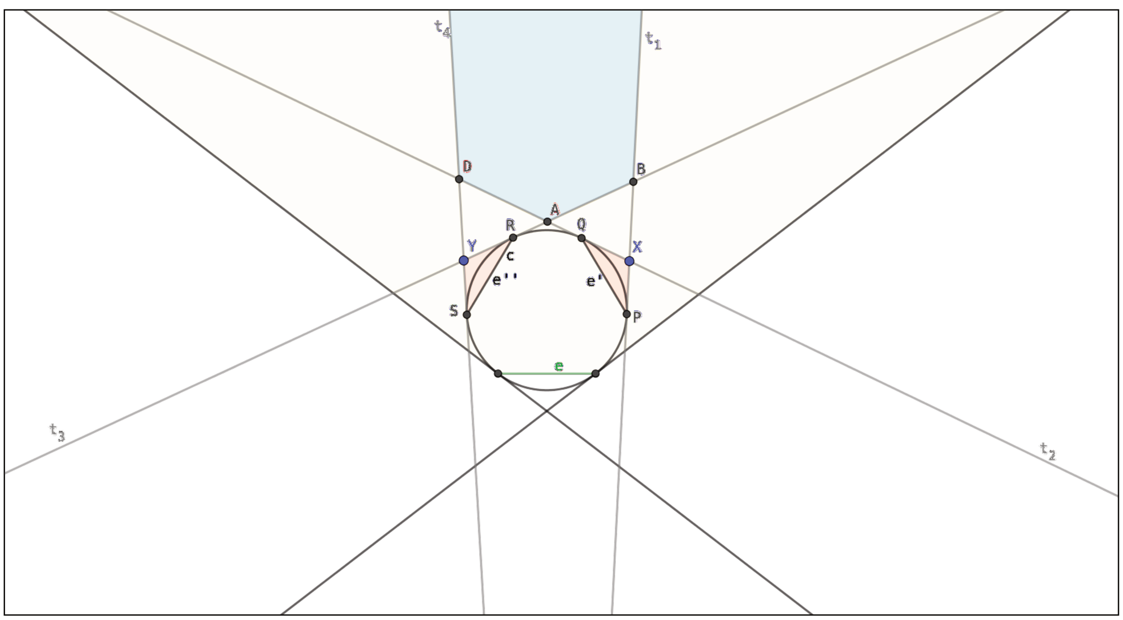


Figure 5.3: $\phi_0(d'), \phi_0(d'') \neq 0$

– Next we suppose that $\phi_0(d') = 0$ and $\phi_0(d'') \neq 0$. See Fig. 5.4. Again, by using the induction hypothesis on the tile d'' and the edge e'', we get that $\phi_0(d'')$ lies in the triangle ΔRYS, containing d''. Using the same argument and notation of the previous case, we have that the region where the point $[\phi_0(d)]$ must lie so that the straight line joining $[\phi_0(d)]$ and $[\phi_0(d'')]$ intersects $\overleftrightarrow{e''}$ outside $\overline{\mathbb{D}}$, is given by K_1. Label the points $\overleftrightarrow{e'} \cap \overleftrightarrow{t_3}$, $\overleftrightarrow{e'} \cap \overleftrightarrow{t_4}$ as T, O, respectively. Since $\phi_0(d) \neq 0$, the coefficient e' is non-zero. So, $[\phi_0(d)] \in \overleftrightarrow{e'} \setminus \overline{\mathbb{D}}$. Hence, the point $[\phi_0(d)]$ must lie in the intersection $(\overleftrightarrow{e'} \setminus \overline{\mathbb{D}}) \cap K_1$ which is a segment (coloured blue in the figure) completely contained inside Δ.

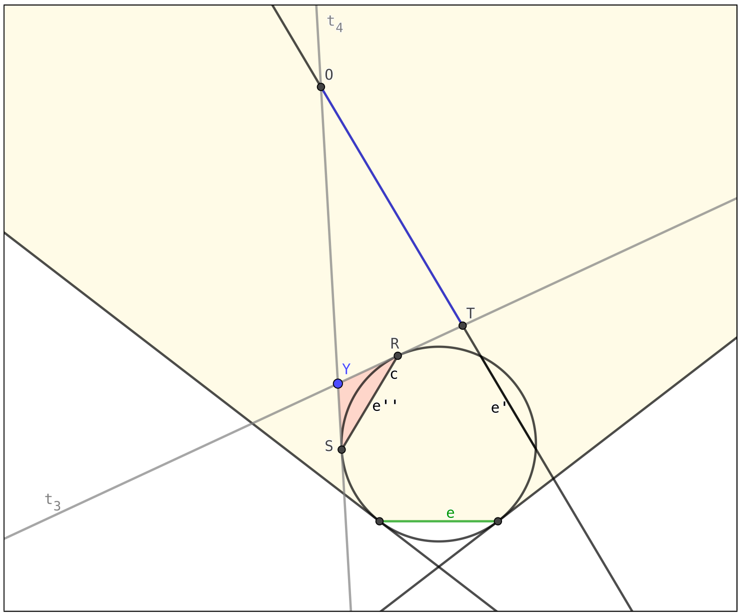


Figure 5.4: $\phi_0(d') = 0$

- Finally, we suppose that $\phi_0(d') = 0 = \phi_0(d'')$. Again, $\phi_0(d) \neq 0$ implies that $c_{e'}, c_{e''} \neq 0$. Then the point $[\phi_0(\delta)]$ is given by the intersection of the two straight lines $\overleftrightarrow{e'}, \overleftrightarrow{e''}$. Since e', e'' are disjoint, the intersection point is hyperideal and lies inside $\Delta \setminus e$.

- If d is a pentagon, then Corollary 1.5.2 implies that $\phi_0(d)$ must lie on the tangent $\overleftrightarrow{t}$ to the ideal vertex of d. Also, this tile has exactly one neighbour d' that is contained in Δ. Let $e' \in \mathcal{E}_\sigma$ be the common internal edge of d, d'.
  - If $\phi_0(d') = 0$, then $[\phi_0(d)] \in \overleftrightarrow{e'}$. So we have $[\phi_0(d)] = \overleftrightarrow{e'} \cap \overleftrightarrow{t}$, which lies inside Δ, by convexity of $\partial_\infty \mathbb{D}$.
  - If $\phi_0(d') \neq 0$, then by the induction hypothesis, $[\phi_0(d')]$ lies inside the projective triangle Δ' based at e' that doesn't contain d'. See Fig. 5.5 Again by Property 1.6.2, the point $[\phi_0(d)]$ is contained in the region $K := \mathbb{R}\mathrm{P}^2 \setminus \Delta'$. Let $\overleftrightarrow{t_1}, \overleftrightarrow{t_2}$ be the tangents to $\partial_\infty \mathbb{D}$ at the endpoints of $\overline{e'}$. Label the points $\overleftrightarrow{t_1} \cap \overleftrightarrow{t}$, $\overleftrightarrow{t_2} \cap \overleftrightarrow{t}$ by O_1, O_2 respectively. Then $\phi_0(d)$ is contained in the segment $\overline{O_1 O_2}$, which lies in the interior of Δ.

This proves the induction step and hence the lemma for ideal polygons. □

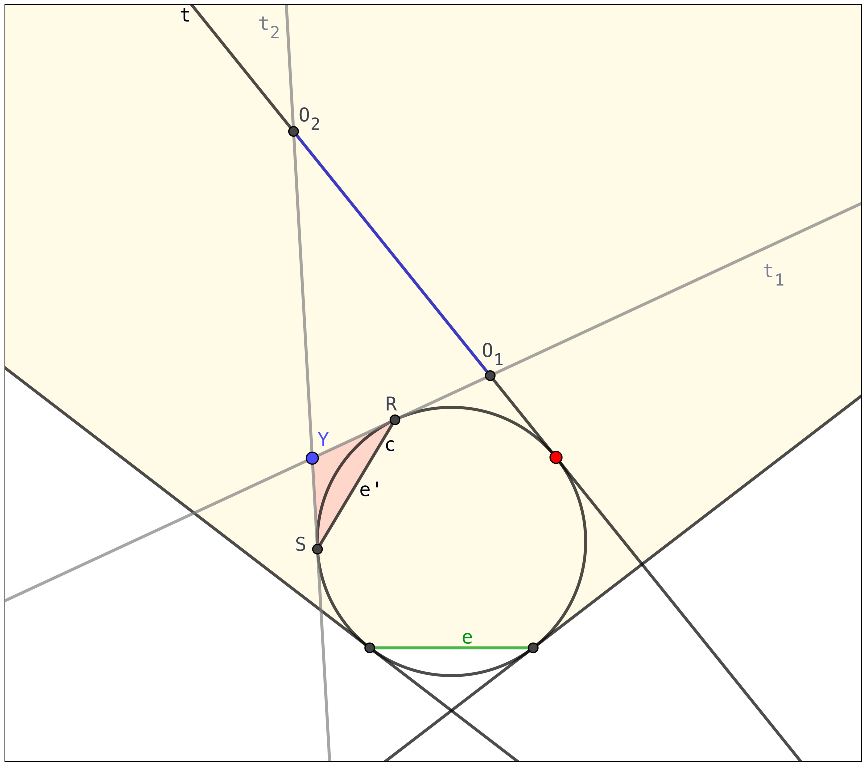


Figure 5.5: d is a pentagon, $\phi_0(d') \neq 0$

Now, we come back to the proof of the theorem. Let $e \in \mathcal{E}_\sigma$ be an arc such that $c_e \neq 0$. Let d, d' be the two tiles with common edge e. Then, $\phi_0(d) \neq \phi_0(d')$, and the point $[\phi_0(d) - \phi_0(d')]$ belongs to $\overleftrightarrow{e} \setminus \overline{\mathbb{D}}$. Let Δ, Δ' be the projective triangles based at $\bar{e}$. Let d, d' be the tiles in $\mathcal{T}_\sigma$ neighbouring along e such that $d \subset \Delta$ and $d' \subset \Delta'$.

If both $\phi_0(d), \phi_0(d')$ are non-zero, then the above lemma applied to the pairs d, e and d', e gives us that $[\phi_0(d)] \in \operatorname{int}(\Delta)$ and $[\phi_0(d')] \in \operatorname{int}(\Delta')$. Using 1.6.2, we get that the line joining $[\phi_0(d)]$ and $[\phi_0(d')]$ intersects $\overleftrightarrow{e}$ inside $\partial_\infty \mathbb{D}$, which is a contradiction.

If $\phi_0(d') = 0$, then $\phi_0(d) \in \overleftrightarrow{e} \setminus \overline{\mathbb{D}}$, which is disjoint from the interior of Δ. So we again reach a contradiction. Hence we must have $c_e = 0$ for every $e = 0$. This concludes the proof. $\square$

5.1.2 One-holed polygons

Now we shall prove Theorem 4.4.4 for one-holed polygons. We restate the theorem here.

Theorem 5.1.3. *Let $m = [(\rho, \mathbf{x})] \in \mathfrak{D}_0$ be a metric on a one-holed n-gon $\Pi_n^{\circledcirc}$, with $n \geq 1$. Let σ be a top-dimensional simplex of the spinning arc complex $\mathcal{A}_{\circlearrowright}(\Pi_n^{\circledcirc})$ and let $\mathcal{E}_\sigma$ be the corresponding edge set. Then the set of infinitesimal strip deformations $B = \{f_e(m) | e \in \mathcal{E}_\sigma\}$ forms a basis of $T_m \mathfrak{D}_0(\Pi_n^{\circledcirc})$.*

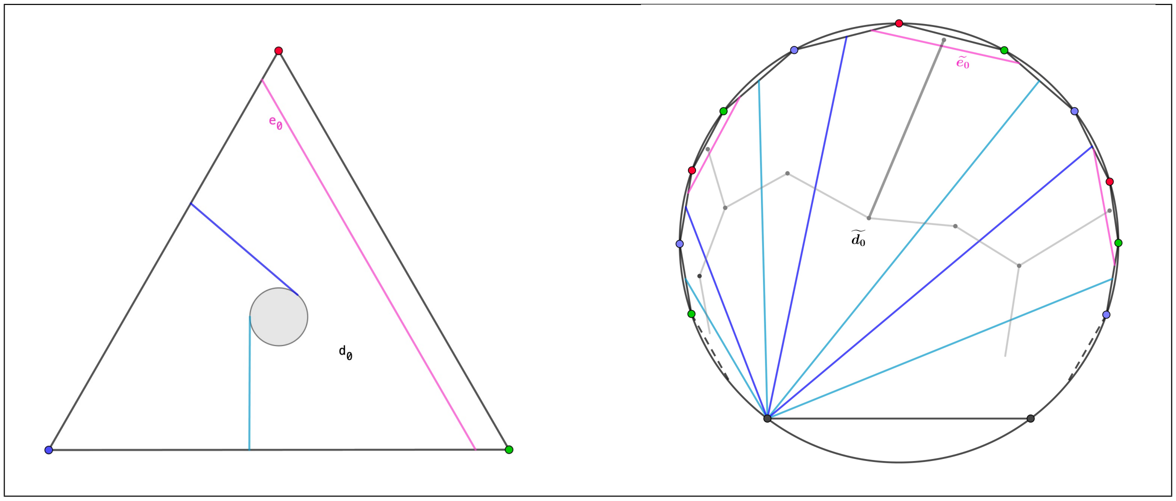


Figure 5.6: A triangulation of $\Pi_3^{\circledcirc}$ and the lift of the dual graph

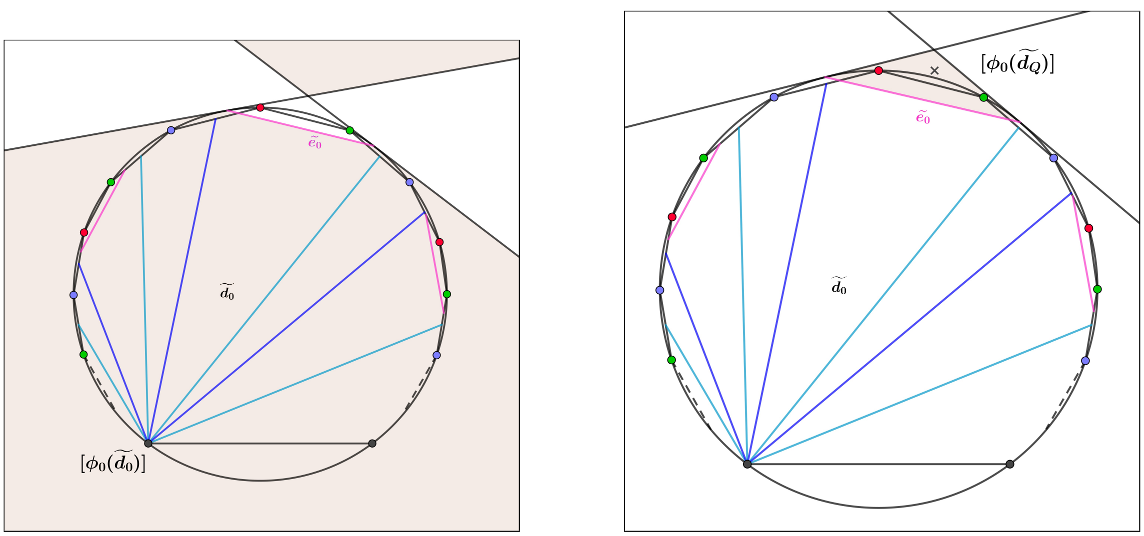


Figure 5.7: Base cases of Lemma 5.1.4

Proof. Let $\rho(\pi_1(\Pi_n^{\circledcirc})) = \langle g \rangle$, where g is a hyperbolic isometry in $\mathrm{PSL}(2,\mathbb{R})$. Let $\widetilde{\mathcal{E}_\sigma}$ and $\widetilde{\mathcal{T}_\sigma}$ be the sets of lifted arcs and tiles, respectively. Again, we have that $\dim T_m\mathfrak{D}(\Pi_n^{\circledcirc}) = \#\mathcal{E}_\sigma = n$. So, it is enough to show that the set B is linearly independent. We start with an equation $\sum_{e\in\mathcal{E}_\sigma} c_e f_e(m) = 0$ with a corresponding $\langle g \rangle$-invariant neutral map $\phi_0 : \widetilde{\mathcal{T}_\sigma} \longrightarrow \mathfrak{g}$.

Recall from Chapter 3 that the permitted arcs generating the spinning arc complex are of two types — finite arcs with their endpoints on the boundary and infinite arcs with one finite endpoint on some edge and another spiraling along the boundary loop. These arcs decompose the surface into two kinds of tiles: tiles with only finite internal edges (as in the case of ideal polygons) and tiles that spiral around the boundary loop. The latter case can be further subdivided into types: a quadrilateral tile with exactly one spike, two infinite internal edges spiraling in the same direction (clockwise or anti-clockwise) and a pentagonal tile with no spike, two spiraling internal edges and exactly one finite internal edge. Denote by p the endpoint of the axis of the isometry g such that every lift of every spiraling tile has one ideal vertex at p.

Let d be any spiraling tile and $\widetilde{d}$ be any lift. Since the linear combination represents zero, the boundary loop of the polygon is fixed point-wise. Hence there are two possibilities— either the Killing field $\phi_0(g^n \cdot \widetilde{d}) = 0$ for every $n \in \mathbb{Z}$, or $\phi_0(g^n \cdot \widetilde{d})$ is a parabolic element with fixed point as p, for every $n \in \mathbb{Z}$. In the latter case, for every $n \in \mathbb{Z}$, the point $[\phi_0(g^n \cdot \widetilde{d_0})] = p$.

Now if d contains a spike, then the Killing field $\phi_0(\widetilde{d})$ fixes the ideal vertex corresponding to the lift of the spike in $\widetilde{d}$. This is possible only if $\phi_0(\widetilde{d}) = 0$. Now, if every arc of $\mathcal{E}_\sigma$ is a spiraling arc, then every tile is a spiraling quadrilateral tile and so $\phi_0(d) = 0$ for every $d \in \widetilde{\mathcal{T}_\sigma}$. So the Theorem is proved in this case.

Next we assume that there exists at least one finite arc in the edgeset. As a result, there is at least one spiraling pentagonal tile. The following lemma is an analogous version of Lemma 5.1.2 adapted to this case.

Lemma 5.1.4. *Let σ be a top-dimensional simplex of $\mathcal{A}_{\circlearrowright}\left(\Pi_n^{\pentagon}\right)$. Let $\phi_0 : \widetilde{\mathcal{T}_\sigma} \to \mathfrak{g}$ be a neutral tile map corresponding to the linear combination* (5.1)*. Let $e \in \widetilde{\mathcal{E}_\sigma}$ be a finite internal edge of a tile $d \in \widetilde{\mathcal{T}_\sigma}$ such that $\phi_0(d) \neq 0$. Then the point $[\phi_0(d)] \in \mathbb{R}\mathrm{P}^2$ lies in the interior of the projective triangle, based at the geodesic $\overline{e}$ carrying e, that contains the tile d.*

Proof. Let $d \in \widetilde{\mathcal{T}_\sigma}$ be a tile with finite internal edge $e \in \widetilde{\mathcal{E}_\sigma}$. Now consider the dual graph of the triangulation of the universal cover of the surface by σ. It is an infinite tree invariant by the action of $\langle g \rangle$. It can be seen as the countable union of finite trees rooted at the tiles of the form $g^n \cdot d_0$, and the edges that join two adjacent lifts of two (not necessarily distinct) spiraling tiles. It is drawn in grey in the right panel of Fig. 5.6. There exists a unique lift d_0 of a pentagonal spinning tile with unique finite edge e_0 that lies nearest to d, with respect to the distance on the dual tree of σ. Let τ be the finite rooted sub-tree crossing the arc e_0 with root at the tile $\widetilde{d_0}$. Define $M(d)$ as the length of the longest path on τ joining d and a quadrilateral tile or the root tile $\widetilde{d_0}$ such that the path does not cross the internal edge e of d. Then the lemma is proved by induction on M.

If $M(d) = 0$, then the tile d is either a quadrilateral or the root tile $\widetilde{d_0}$. In the former case, we know that $\phi_0(d)$ is a hyperbolic Killing field with fixed points as the two ideal vertices of the quadrilateral; the point $[\phi_0(d)]$ is given by the intersection of the two tangents to the boundary circle $\partial_\infty\mathbb{D}$ at the ideal vertices. So the lemma is verified in this case. In the latter case, we know that $[\phi_0(\widetilde{d_0})] = p$. This point lies inside the desired triangle (see bottom left panel of Fig. 5.6). So the statement of the lemma is verified in this case.

Now suppose that the statement is true for $M = 0, \dots, k$. Consider a tile d inside τ such that $M(d) = k+1$. Then d is either a pentagon with one ideal vertex and two internal edges (both finite) or a hexagon with three internal edges and no spikes. Also, there exists a finite path of length $k+1$ in the tree τ starting from d and ending at a vertex which is either a quadrilateral or the root tile. By proceeding in the exact same way as in the induction step of Lemma 5.1.2 for ideal polygons, we get that the induction step is verified in this case as well.

□

Next we prove that the whenever $e \in \widetilde{\mathcal{E}_\sigma}$ is a finite arc, the coefficient c_e of $f_e(m)$ in the linear combination above is equal to 0. We will prove this by contradiction: suppose $c_e \neq 0$ for such a finite arc e. Let d, d' be the two tiles with common edge e. Then, $\phi_0(d) \neq \phi_0(d')$, and the point $[\phi_0(d) - \phi_0(d')]$ belongs to $\overleftrightarrow{e} \setminus \overline{\mathbb{D}}$. Let Δ, Δ' be the projective triangles based at the geodesic carrying the arc e such that $d \subset \Delta$ and $d' \subset \Delta'$.

If both $\phi_0(d), \phi_0(d')$ are non-zero, then the above lemma applied to the pairs d, e and d', e gives us that $[\phi_0(d)] \in \operatorname{int}(\Delta)$ and $[\phi_0(d')] \in \operatorname{int}(\Delta')$. Using 1.6.2, we get that the line joining $[\phi_0(d)]$ and $[\phi_0(d')]$ intersects the projective line $\overleftrightarrow{e}$ carrying the arc e inside $\partial_\infty \mathbb{D}$, which is a contradiction.

If $\phi_0(d') = 0$, then $\phi_0(d) \in \overleftrightarrow{e} \setminus \overline{\mathbb{D}}$, which is disjoint from the interior of Δ. So we again reach a contradiction.

Hence, we have $c_e = 0$ for every finite arc $e \in \widetilde{\mathcal{E}_\sigma}$. Now consider the path on the dual graph joining a lift $g^n \cdot \widetilde{d_0}$ of d_0 with a quadrilateral tile d_Q such that the path crosses only finite arcs. Such a path exists — for example on the sub-tree τ in the proof of the above lemma. For every adjacent pairs d, d' of tiles on this path, we have that $\phi_0(d) = \phi_0(d')$. Consequently, we get that $\phi_0(g^n \cdot \widetilde{d_0}) = \phi_0(d_Q)$, which implies that both these Killing fields must be trivial. Hence, we get that $\phi_0(d) = 0$ for every $d \in \widetilde{\mathcal{T}_\sigma}$. □

5.1.3 Punctured polygons

Theorem 5.1.5. *Let $m \in \mathfrak{D}(\Pi_n^\odot)$ be a metric on an ideal once-punctured n-gon $\Pi_n^\odot$, with $n \geq 2$. Fix a choice of strip template. Let σ be a top-dimensional simplex of its arc complex $\mathcal{A}(\Pi_n^\odot)$ and let $\mathcal{E}_\sigma$ be the corresponding edge set. Then the set of infinitesimal strip deformations $B = \{f_e(m) \mid e \in \mathcal{E}_\sigma\}$ forms a basis of the tangent space $T_m\mathfrak{D}(\Pi_n^\odot)$.*

Proof. Like in the case of ideal polygons, we have that $\dim T_m\mathfrak{D}(\Pi_n^\odot) = \#\mathcal{E}_\sigma = n-1$. So we only need to prove the linear independence of B. Again we start with an equation as in (5.1) with a corresponding neutral map $\phi_0 : \widetilde{\mathcal{T}_\sigma} \longrightarrow \mathfrak{g}$. This map is $\langle T \rangle$-invariant, where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is the generator of the fundamental group. So ϕ_0 satisfies the following equation:

$$T \cdot \phi_0(d) = \phi_0(T \cdot d), \text{ for every } d \in \widetilde{\mathcal{T}_\sigma}. \tag{5.2}$$

Recall from Chapter 3 that the permitted arcs generating the arc complex are finite arcs with their endpoints on the boundary. There is exactly one maximal arc e_M (separates the puncture from the spikes) in every triangulation. The surface is decomposed into four types of tiles. The first three types (quadrilateral, pentagon, hexagon) are finite hyperbolic polygons and the fourth one is a tile containing the puncture. It lifts to a tile,denoted by d_∞, with infinitely many edges, each given by a lift of the unique maximal arc $e_M \in \mathcal{E}_\sigma$ of the triangulation, and exactly one ideal vertex, denoted by p that corresponds to the puncture.

Now, we show that the Killing field $\phi_0(d_\infty)$ associate to the unique infinite tile $d_\infty \in \widetilde{\mathcal{T}_\sigma}$, is either zero or a parabolic element with fixed point $p \in \partial_\infty \mathbb{D}$ that corresponds to the puncture. We know that d is invariant under the action of the isometry T:

$$\phi_0(d_\infty) = \phi_0(T^n \cdot d_\infty) \text{ for every } n \in \mathbb{Z}. \tag{5.3}$$

Using the isomorphism between the Lie algebra $\mathfrak{g}$ and $\mathbb{R}^{2,1}$, we have that $\phi_0(d)$ is represented by the matrix $\begin{pmatrix} y & x+z \\ x-z & -y \end{pmatrix}$. The generator T acts on p by conjugation:

$$\begin{aligned} T \cdot \phi_0(d_\infty) &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & x+z \\ x-z & -y \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} y-x+z & 2(y+z) \\ x-z & x-z-y \end{pmatrix} \end{aligned}$$

From eqs. (5.2) and (5.3), we get that $y = 0, x = z$. Hence, $\phi_0(d_\infty)$ is either zero or a parabolic element, fixing the light-like line $\mathbb{R}p$ and $[\phi_0(d_\infty)] = p$.

We now prove an analogous version of Lemma 5.1.4 for a punctured polygon.

Lemma 5.1.6. *Let σ be a top-dimensional simplex of $\mathcal{A}(\Pi_n^{\odot})$. Let $\phi_0 : \widetilde{\mathcal{T}_\sigma} \to \mathfrak{g}$ be a neutral tile map corresponding to the linear combination* (5.1)*. Let $e \in \widetilde{\mathcal{E}_\sigma}$ be an internal edge of a tile $d \in \widetilde{\mathcal{T}_\sigma}$ such that $\phi_0(d) \neq 0$. Then the point $[\phi_0(d)] \in \mathbb{RP}^2$ lies in the interior of the projective triangle, based at the geodesic $\overline{e}$ carrying e, that contains the tile d.*

Proof. Let $d \in \widetilde{\mathcal{T}_\sigma}$ such that $\phi_0(d) \neq 0$ and let $e \in \widetilde{\mathcal{E}_\sigma}$ be an internal edge of d. Consider the dual graph of the triangulation of the universal cover of the surface by σ. It is an infinite tree invariant by the action of $\langle g \rangle$. It can be seen as the countable union of finite trees and rooted at the infinite tile d_∞. The latter has infinitely many edges, each given by a lift of the unique maximal arc $e_M \in \mathcal{E}_\sigma$ of the triangulation. There are two possibilities — either $d = d_\infty$ or there exists a unique lift $\widetilde{e_M}$ that separates d from $d\infty$. Let τ be the finite rooted sub-tree spanned by the tile d_∞ and all those tiles that are separated by $\widetilde{e_M}$ from d_∞. Define $M(d)$ as the length of the longest path on τ joining d and a quadrilateral tile or the root tile d_∞ such that the path does not cross the edge e of d. Then the lemma is proved by induction on M.

When $M(d) = 0$, d is either a quadrilateral or the tile d_∞. In the former case, we know that $\phi_0(d)$ is a hyperbolic Killing field with fixed points as the two ideal vertices of the quadrilateral; the point $[\phi_0(d)]$ is given by the intersection of the two tangents to the boundary circle $\partial_\infty \mathbb{D}$ at the ideal vertices. So the lemma is verified in this case. Next we suppose that $d = d_\infty$. Then from the discussion before the lemma, we have that $[\phi_0(d_\infty)] = p$ which lies inside the desired triangle. So the statement of the lemma is satisfied in this base case.

Now suppose that the statement is true for $M = 1, \dots, k$. Consider a tile d inside τ such that $M(d) = k+1$. Then d is either a pentagon with one ideal vertex and two internal edges (both finite) or a hexagon with three internal edges and no spikes. Also, there exists a finite path of length $k+1$ in the tree τ starting from d and ending at a vertex which is either a quadrilateral or the root tile. By proceeding in the exact same way as in the induction step of Lemma 5.1.2 for ideal polygons, we get that the induction step is verified in this case well. This finishes the proof of the lemma. ☐

Now suppose that the coefficient c_e of $f_e(m)$ is non-zero for some $e \in \widetilde{\mathcal{E}_\sigma}$. Let d, d' be the two tiles with common edge e. Then, $\phi_0(d) \neq \phi_0(d')$, and the point $[\phi_0(d) - \phi_0(d')]$ belongs to $\overleftrightarrow{e} \setminus \overline{\mathbb{D}}$.

Let Δ, Δ' be the projective triangles based at the geodesic carrying the arc e such that $d \subset \Delta$ and $d' \subset \Delta'$.

If both $\phi_0(d), \phi_0(d')$ are non-zero, then the above lemma applied to the pairs d, e and d', e gives us that $[\phi_0(d)] \in \operatorname{int}(\Delta)$ and $[\phi_0(d')] \in \operatorname{int}(\Delta')$. Using 1.6.2, we get that the line joining $[\phi_0(d)]$ and $[\phi_0(d')]$ intersects the projective line $\overleftrightarrow{e}$ carrying the arc e inside $\partial_\infty \mathbb{D}$, which is a contradiction.

If $\phi_0(d') = 0$, then $\phi_0(d) \in \overleftrightarrow{e} \setminus \overline{\mathbb{D}}$, which is disjoint from the interior of Δ. So we again reach a contradiction.

Hence, we have $c_e = 0$ for every arc $e \in \widetilde{\mathcal{E}_\sigma}$, which proves the theorem 5.1.5.

□

5.1.4 Decorated Polygons

Now we shall prove the linear independence in the case of decorated polygons.

Theorem 5.1.7. *Let $m \in \mathfrak{D}(\Pi_n^\diamond)$ be a metric on a decorated n-gon $\Pi_n^\diamond$, with $n \geq 3$. Fix a choice of strip template. Let σ be a top-dimensional simplex of its arc complex $\mathcal{A}(\Pi_n^\diamond)$ and let $\mathcal{E}_\sigma$ be the corresponding edge set. Then the set of infinitesimal strip deformations $B = \{f_e(m) \mid e \in \mathcal{E}_\sigma\}$ forms a basis of the tangent space $T_m\mathfrak{D}(\Pi_n^\diamond)$.*

Proof. Again, we have that $\dim T_m\mathfrak{D}(\Pi_n^\diamond) = \#\mathcal{E}_\sigma = 2n - 3$. So, it is enough to show that the above set is linearly independent. Since every decorated polygon is simply connected, we have that $\mathcal{E}_\sigma = \widetilde{\mathcal{E}_\sigma}$ and $\mathcal{T}_\sigma = \widetilde{\mathcal{T}_\sigma}$. Suppose that $\sum_{e \in \mathcal{E}_\sigma} c_e f_e(m) = 0$, with not all c_e's equal to 0. Let $\phi_0 : \widetilde{\mathcal{T}_\sigma} \to \mathfrak{g}$ be a neutral tile map; by definition, it fixes the generalised vertices of every tile. Suppose the tile d has a generalised vertex ν (Fig. 5.8, 5.9, 5.10). If ν is parabolic, the vector field $\phi_0(d)$ fixes the ideal point as well as the horoball decoration. If $\phi_0(d) \neq 0$, then the point $[\phi_0(d)]$ contained in the interior of the desired triangle, due to the convexity of $\partial_\infty \mathbb{D}$. Similarly, if ν is the truncation of a hyperideal point p or is an elliptic point v, then $[\phi_0(d)] = p$ and $[\phi_0(d)] = v$ respectively.

Lemma 5.1.8. *Let σ be a top-dimensional simplex of $\mathcal{A}(\Pi_n^\diamond)$. Let $\phi_0 : \widetilde{\mathcal{T}_\sigma} \to \mathfrak{g}$ be a neutral tile map corresponding to the linear combination* (5.1)*. Let e be an internal edge-to-edge arc of a tile $d \in \mathcal{T}_\sigma$ such that $\phi_0(d) \neq 0$. Then, $[\phi_0(d)]$ is contained in the interior of the projective triangle in $\mathbb{R}\mathrm{P}^2$, based at the geodesic $\overline{e}$ carrying e, that contains d.*

Proof. For every triangulation σ, there is at least one tile of type one and every tile has at least one internal edge-to-edge arc. Consider the dual graph of the triangulation of the decorated polygon by σ. It is a finite tree. Let τ be the finite rooted sub-tree crossing the arc e with root at the tile d. We will now prove that every tile on this sub-tree satisfies the lemma. Let $d \in \mathcal{T}_\sigma$ be any tile and e be an internal edge-to-edge arc. We define $M(d)$ to be the longest path in τ joining d and a tile containing one generalised vertex. The proof is done by induction on M.

When $M(d) = 0$, d is a tile of type one (one generalised vertex and one internal edge. From the discussion before the lemma, we get $[\phi_0(d)]$ lies in the desired triangle.

Now, let the statement be true for $M(d) = 0, \ldots, k$. Again, if d is a tile with a generalised vertex then we know already that the statement is verified. So we assume that d is a hexagon without any generalised vertex, such that $\phi_0(d) \neq 0$. Then it has two neighbouring tiles d', d'' contained in Δ, with common arcs e', e'' respectively. Both e', e'' are edge-to-edge arcs. The proof is then identical to that of Lemma 5.1.2. This proves the induction step. □

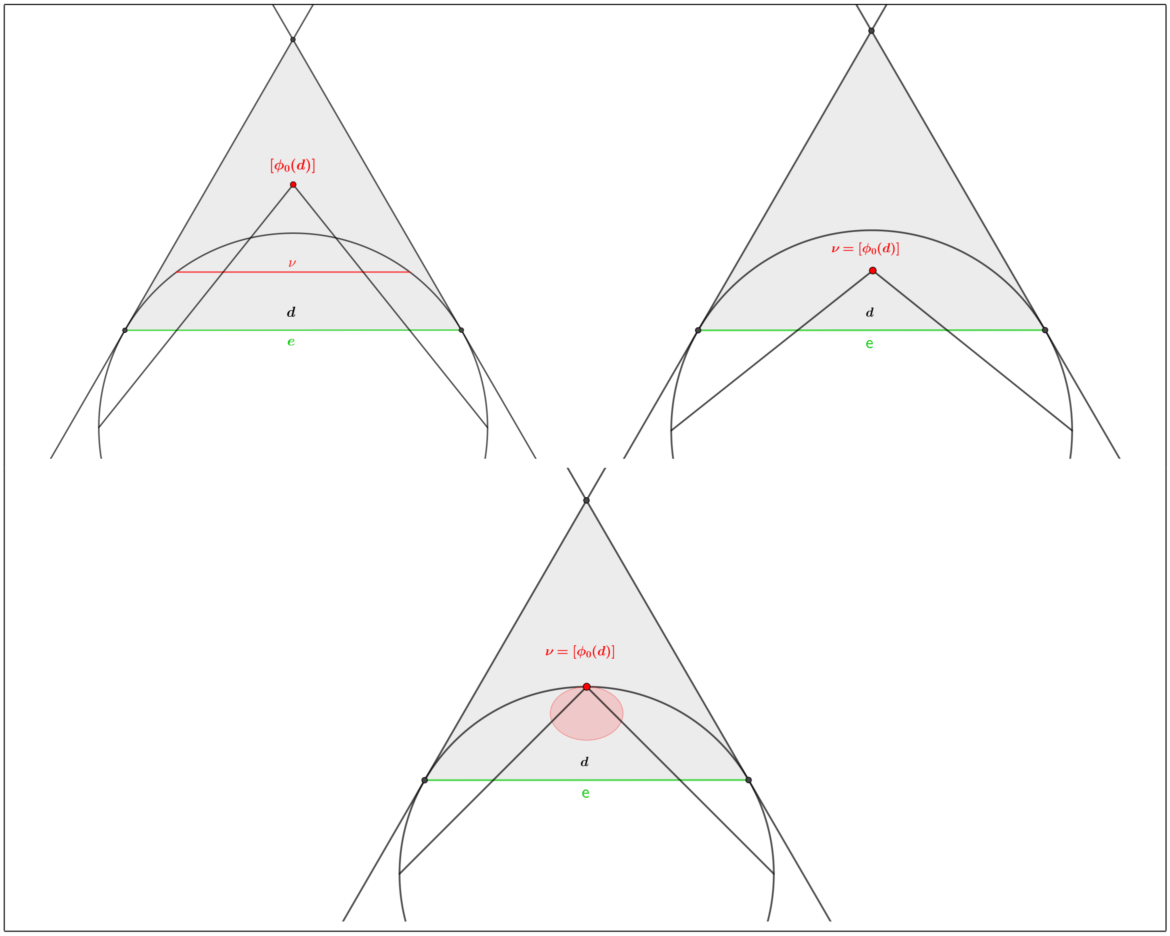


Figure 5.8: ϕ_0-images of tiles of type 1

Now we prove by contradiction that the coefficient c_e of any edge-to-edge arc e has to be zero. Let $e \in \mathcal{E}_\sigma$ be an edge-to-edge arc, that is common to the two neighbouring tiles d_1, d_2. Let $c_e \neq 0$. Then, $[\phi(d_1) - \phi(d_2)] \in \overleftrightarrow{e} \setminus \overline{\mathbb{D}}$. Since both $\phi_0(d_1)$ and $\phi_0(d_2)$ cannot be simultaneously equal to zero, we have two cases:

1. Let $\phi_0(d_1)$ and $\phi_0(d_2)$ be both non-zero. By the above lemma, $\phi_0(d_1)$ and $\phi_0(d_2)$ belong to two disjoint triangles associated to e. By Property 1.6.2, we have $[\phi_0(d_1) - \phi_0(d_2)]$ must intersect e inside $\mathbb{D}$, which is a contradiction.

2. Suppose that $\phi_0(d_1) = 0 \neq \phi_0(d_2)$. Then, the point $[\phi_0(d_1) - \phi_0(d_2)] = [\phi_0(d_2)]$ does not intersect $\overleftrightarrow{e}$, which is again a contradiction.

So, we have $\phi_0(d_1) = \phi_0(d_2)$, whenever two tiles $d_1, d_2 \in \mathcal{T}_\sigma$ have a common edge-to-edge arc.
Let $d, d' \in \mathcal{T}_\sigma$ be two tiles with different generalised vertices ν, ν' such that d and d' can be joined by a path in the dual tree that crosses only edge-to-edge arcs. Then, from the above discussion

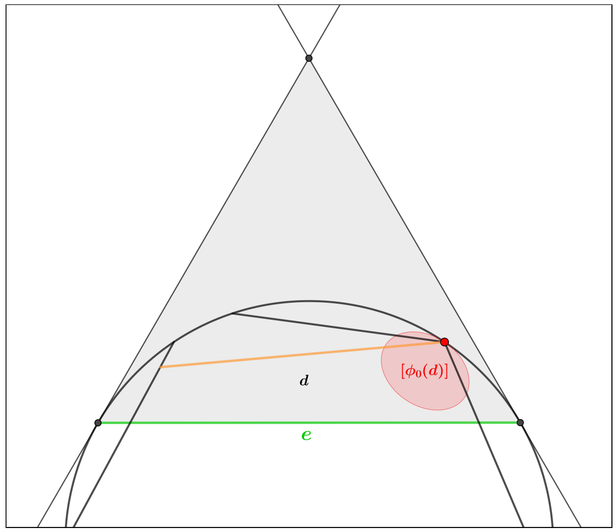


Figure 5.9: ϕ_0-images of tiles of type 2

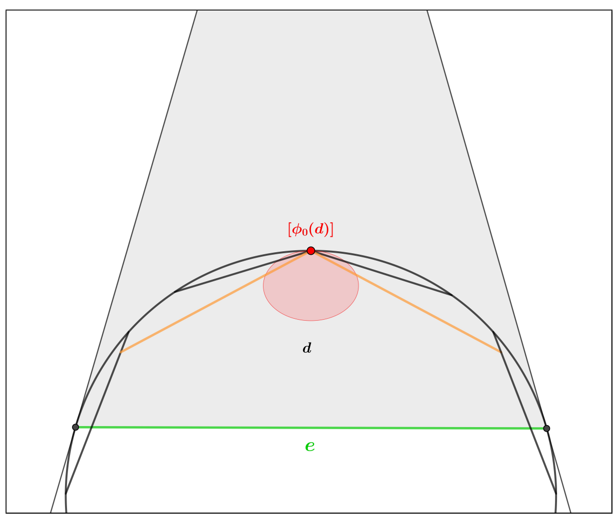


Figure 5.10: ϕ_0-image of tiles of type 3

we have that $[\phi_0(d)] = [\phi_0(d')]$. But $\phi_0(d')$ must fix ν' which is different from ν. So we get $\phi_0(d) = \phi_0(d') = 0$. Since every tile has an edge-to-edge arc and there is more than one generalised vertex, we get that $\phi_0(d) = 0$ for every $d \in \mathcal{T}_\sigma$. So we get that $c_e = 0$ for every $e \in \mathcal{E}_\sigma$, which proves the theorem. $\square$

5.2 Local homeomorphism: codimension 1 faces

In this section we show that the projectivised strip map $\mathbb{P}f : \widehat{\mathcal{A}}(\Pi) \longrightarrow \mathbb{P}^+(T_m\mathfrak{D}(\Pi))$ is a local homeomorphism around points belonging to the interiors of simplices of codimension 1.

Theorem 5.2.1. *Let Π be any one of the four types of polygons — ideal n-gons $\Pi_n^{\pentagon}$, once-punctured n-gons $\Pi_n^{\odot}$, one-holed n-gons $\Pi_n^{\circledcirc}$, decorated n-gons $\Pi_n^{\circledast}$. Let $m \in \mathfrak{D}(\Pi)$ be a metric. Let $\sigma_1, \sigma_2 \in \mathcal{A}(P)$ be two top-dimensional simplices such that*

$$\operatorname{codim}(\sigma_1 \cap \sigma_2) = 1 \text{ and } \operatorname{int}(\sigma_1 \cap \sigma_2) \subset \widehat{\mathcal{A}}(\Pi).$$

Then,

$$\operatorname{int}(\mathbb{P}f(\sigma_1)) \cap \operatorname{int}(\mathbb{P}f(\sigma_2)) = \varnothing. \tag{5.4}$$

Moreover, there exists a choice of strip template such that $\mathbb{P}f(\sigma_1) \cup \mathbb{P}f(\sigma_2)$ is convex in $\mathbb{P}^+(T_m\mathfrak{D}(\Pi))$.

Firstly, we will give a general idea of the proof for any type of polygon and then we will give the proof in each case in the subsequent sections 5.2.1-5.2.5.

Idea of the proof: Let $\mathcal{E}_{\sigma_1}$ and $\mathcal{E}_{\sigma_2}$ be the edge sets of σ_1 and σ_2 respectively. Since the simplex $\sigma_1 \cap \sigma_2$ has codimension one, we have that $\mathcal{E}_{\sigma_1} \backslash \mathcal{E}_{\sigma_2}$ (resp. $\mathcal{E}_{\sigma_2} \backslash \mathcal{E}_{\sigma_1}$) has exactly one arc, denoted by α_1 (resp. α_2). There are different possibilities for the pair $\{\alpha_1, \alpha_2\}$ in the case of every polygonal surface. Let $\widetilde{\mathcal{E}_{\sigma,r}}$ be the refined edgeset of $\widetilde{\mathcal{E}_{\sigma_1}}$ obtained by considering the refinement $\sigma := \sigma_1 \cup \{\alpha_2\}$. Let $\widetilde{\mathcal{T}_{\sigma,r}}$ be the refined tile set of $\widetilde{\mathcal{T}_{\sigma}}$.

In every case, we shall give a choice of strip template and then construct a tile map that represents the following linear combination for a chosen strip template and is coherent around every point of intersection:

$$c_{\alpha_1} f_{\alpha_1}(m) + c_{\alpha_2} f_{\alpha_2}(m) + \sum_{\beta \in \mathcal{E}_{\sigma_1} \cap \mathcal{E}_{\sigma_2}} c_\beta f_\beta(m) = 0, \tag{5.5}$$

where $c_\beta \leq 0$ for every $\beta \in \mathcal{E}_{\sigma_1} \cap \mathcal{E}_{\sigma_2}$ and $c_{\alpha_1}, c_{\alpha_2} > 0$. Drawing all arcs of $\sigma_1 \cup \sigma_2$ subdivides the surface into a system of tiles that refines both the triangulations. We will choose strip templates and assign Killing fields equivariantly to these tiles in a way that expresses this linear combination. The construction is done in the upper half plane model $\mathbb{U}$. We shall use the identification $\mathfrak{g} \simeq \mathbb{R}_2[z]$ from section 1.5.

5.2.1 Ideal Polygons

In this case $\Gamma = \{Id\}$. So $\widetilde{\mathcal{E}_{\sigma_1}} = \mathcal{E}_{\sigma_1}, \widetilde{\mathcal{E}_{\sigma_2}} = \mathcal{E}_{\sigma_2}, \widetilde{\mathcal{T}_{\sigma,r}} = \mathcal{T}_{\sigma,r}$. We choose an embedding of the ideal polygon into the upper half plane so that the point ∞ is distinct from all the vertices of the polygon, for $n \geq 5$. We shall consider the following strip template:

- For every isotopy class, choose the geodesic representative α_g which intersects the boundary of the polygon perpendicularly;
- For every isotopy class α, the waist p_α is given by the projection of ∞ on α_g.

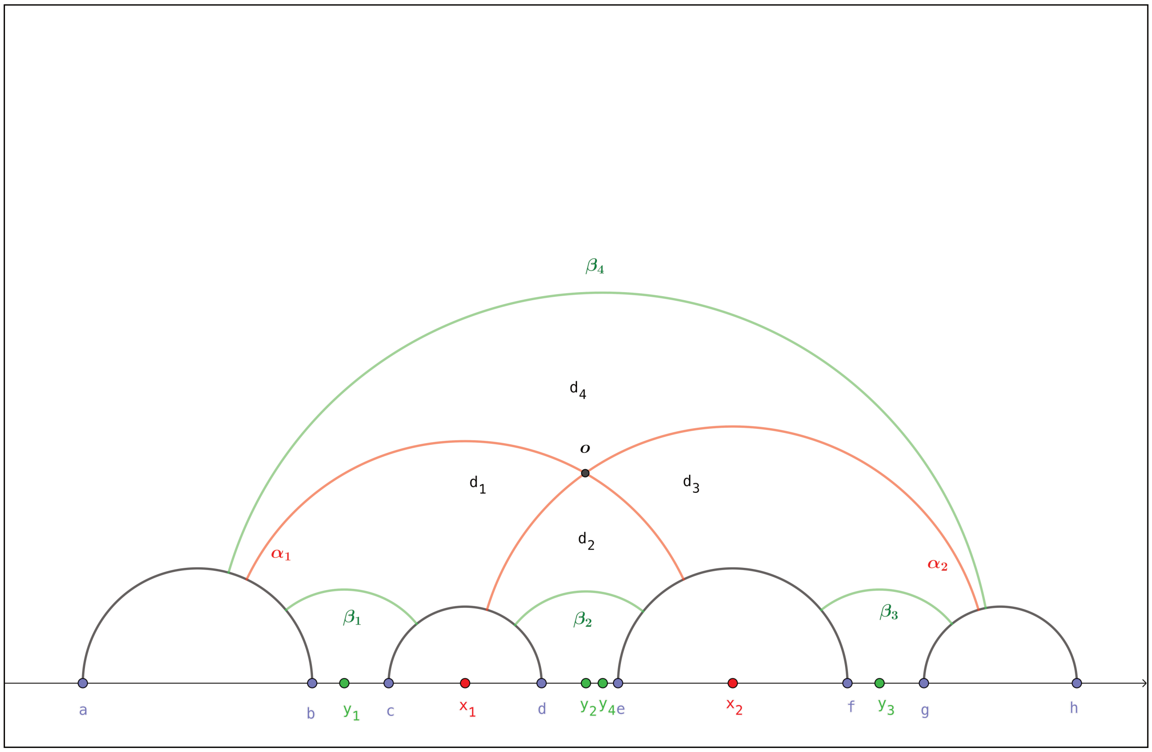


Figure 5.11: Codimension one: Ideal polygons

Then every geodesic arc used in the triangulation is carried by a semi-circle.

In an ideal polygon, any two arcs intersect at most once. The six situations possible for α_1, α_2 are given in Fig. 5.11. The geodesic arcs that are coloured green in the figure are common to both σ_1 and σ_2.

Let o be the point of intersection of α_1, α_2. In each of the six cases, there are four small tiles formed around o, namely d_j, $j = 1, 2, 3, 4$, labeled anti-clockwise such that d_1, d_2 lie below the semi-circle carrying α_1. For each j, d_j is either a quadrilateral with exactly one ideal vertex and two internal edges contained in α_1 and α_2, or it is a pentagon with exactly three internal edges: α_1, α_2 and a third arc $\beta_j \in \mathcal{E}_{\sigma_2} \cap \mathcal{E}_{\sigma_1}$. Let $\mathcal{J} \subset \{1, \ldots, 4\}$ be such that the tile d_j is pentagonal if and only if $j \in \mathcal{J}$. Note that the arc β_j intersects the boundary of the polygon perpendicularly, due to the choice of strip template. For $i = 1, 2$, let $x_i \in \mathbb{R}$ be the centre of the semi-circle carrying the geodesic arc α_i. For $j = 1, \ldots, 4$, let $y_j \in \mathbb{R}$ denote the ideal vertex of d_j or the centre of the semi-circle carrying the geodesic arc β_j.

We shall construct a tile map corresponding to the following linear combination:

$$c_{\alpha_1} f_{\alpha_1}(m) + c_{\alpha_2} f_{\alpha_2}(m) + \sum_{j \in \mathcal{J}} c_\beta f_\beta(m) = 0, \tag{5.6}$$

with $c_{\alpha_1}, c_{\alpha_2} > 0$ and $c_{\beta_j} < 0$ for every $j \in \mathcal{J}$.

Properties 5.2.2. A neutral tile map $\phi_0 : \mathcal{T}_{\sigma,r} \longrightarrow \mathbb{R}_2[z]$ represents the linear combination (5.6) if and only if it verifies the following properties:

1. The polynomial $\phi_0(\delta)$ vanishes at every ideal vertex of $\delta \in \mathcal{T}_{\sigma,r}$ whenever it has one.

2. The tile map is coherent around the intersection point o:
$$\phi_0(d_4) - \phi_0(d_1) = \phi_0(d_3) - \phi_0(d_2).$$

3. Let $\delta, \delta' \in \mathcal{T}_{\sigma,r}$ be two tiles with common internal edge contained in α_i for $i = 1, 2$ such that δ lies above and δ' lies below the semi-circle carrying the common internal edge. Then $\phi_0(\delta) - \phi_0(\delta')$ is a hyperbolic Killing field with attracting fixed point at ∞ and repelling fixed point at x_i. In particular, its axis intersects α_i at p_{α_i}. In terms of polynomials, we must have
$$\phi_0(\delta) - \phi_0(\delta') = (z \mapsto A(z - x_i)), \text{ for some } A > 0.$$

4. Suppose $\delta, \delta' \in \mathcal{T}_{\sigma,r}$ are two tiles with common internal edge β_j for $j \in \mathcal{J}$, such that δ lies above β_j. Then $\phi_0(\delta) - \phi_0(\delta')$ is a hyperbolic Killing field with attracting fixed point at y_j and repelling fixed point at ∞. In particular, its axis intersects β_j at p_{β_j}. In terms of polynomials, we must have
$$\phi_0(\delta) - \phi_0(\delta') = (z \mapsto B(z - y_j)), \text{ for some } B < 0.$$

Suppose that the endpoints of α_1 lie on the boundary geodesics (a, b) and (e, f) and those of α_2 lie on (c, d) and (g, h) such that the following inequalities hold for $n \geq 5$:
$$a < b \leq c < d \leq e < f \leq g < h. \tag{5.7}$$

We shall treat the case $n = 4$ separately.

Using Lemma 1.6.3, we get that
$$x_1 = \frac{ef - ab}{e + f - a - b}, \quad x_2 = \frac{gh - cd}{g + h - c - d}, \tag{5.8}$$
$$y_1 = \frac{cd - ab}{c + d - a - b}, \quad y_2 = \frac{ef - cd}{e + f - c - d}, \tag{5.9}$$
$$y_3 = \frac{gh - ef}{g + h - e - f}, \quad y_4 = \frac{gh - ab}{g + h - a - b}. \tag{5.10}$$

For $j = 1, \ldots, 4$, define
$$\begin{array}{rcl} \phi_0 & : \mathcal{T}_\sigma \longrightarrow & \mathbb{R}_2[z] \\ & \delta \longmapsto & \begin{cases} (z \mapsto a_j(z - y_j)), & \text{if } \delta = d_j \\ 0, & \text{otherwise,} \end{cases} \end{array}$$

where
$$a_1 = \tfrac{x_1 - y_4}{x_1 - y_1}, \quad a_2 = \tfrac{(x_1 - y_4)(x_2 - y_4) - (y_4 - y_1)(y_4 - y_3)}{(x_1 - y_1)(x_2 - y_3)} \quad a_3 = \tfrac{x_2 - y_4}{x_2 - y_3} \quad a_4 = 1.$$

The $a_i's$ as defined above are a nontrivial solution to the following system of linear equations in four unknowns:

$$a_1 - a_2 + a_3 - a_4 = 0, \tag{5.11}$$
$$a_1 y_1 - a_2 y_2 + a_3 y_3 - a_4 y_4 = 0, \tag{5.12}$$
$$a_1(x_1 - y_1) - a_4(x_1 - y_4) = 0 \tag{5.13}$$
$$a_3(x_2 - y_3) - a_4(x_2 - y_4) = 0. \tag{5.14}$$

Applying Lemma (1.6.6) to the geodesics $(a, b), (c, d), (e, f)$ and then to $(c, d), (e, f), (g, h)$ we get that for $j = 2, 4$,

$$y_1 < x_1 < y_j < x_2 < y_3, \tag{5.15}$$

So, $a_1, a_2, a_3 < 0$.

Remark 5.2.1. Note that for every $\delta \in \mathcal{T}_{\sigma,r}$, $\phi_0(\delta) \in \mathbb{R}_1[z]$. This is a consequence of our choice of normalisation and strip template. In fact, except for the case of one-holed polygons, we will use only Killing fields in $\mathbb{R}_1[z]$ in this chapter.

Verification of the properties:

1. Suppose that δ is a tile with an ideal vertex. If $\delta \in \operatorname{supp}(\phi_0)$, then $\delta = d_j$ for some $j \in \{1, \dots, 4\}$, so that ideal vertex is given by y_j. From the definition of the tile map we have that $\phi_0(\delta) = P_j$ which vanishes at y_j. If $\delta \notin \operatorname{supp}(\phi_0)$, then $\phi_0(\delta) = 0$, which automatically fixes its ideal vertex.

2. From eqs. (5.11) and (5.12) it follows that,
$$\begin{aligned}(\phi_0(d_4) - \phi_0(d_1))(z) &= (a_4 - a_1)z - a_4 y_4 + a_1 y_1 \\ &= (a_3 - a_2)z - a_3 y_3 + a_2 y_2 \\ &= (\phi_0(d_3) - \phi_0(d_2))(z).\end{aligned}$$

3. The tiles that share an edge carried by α_1 are the pairs $\{d_1, d_4\}$ and $\{d_2, d_3\}$. The tiles that share an edge carried by α_2 are the pairs $\{d_1, d_2\}$ and $\{d_3, d_4\}$. From the coherence property (2), it is enough to verify the property for $\{d_1, d_4\}$ and $\{d_3, d_4\}$. The tile d_4 lies above both the semicircles carrying the arcs α_1, α_2, respectively. From the definition of ϕ_0 we have that,
$$\begin{aligned}(\phi_0(d_4) - \phi_0(d_1))(z) &= (a_4 - a_1)z + a_1 y_1 - a_4 y_4, \\ (\phi_0(d_4) - \phi_0(d_3))(z) &= (a_0 - a_3)z + a_3 y_3 - a_4 y_4.\end{aligned}$$
Since $a_4 > 0$ and $a_1, a_3 < 0$, the leading coefficients $a_4 - a_1$ and $a_4 - a_3$ are both positive. The polynomials $\phi_0(d_4) - \phi_0(d_1)$ and $\phi_0(d_4) - \phi_0(d_3)$ vanish at x_1 and x_2 respectively, by eq.(5.13) and eq.(5.14) .

4. Suppose that the two tiles δ, δ' have a common edge of the form β_j for $j \in \mathcal{J}$, with δ lying above β_j. Then either $\delta' = d_4$ and $\delta \notin \operatorname{supp}(\phi_0)$ or $\delta = d_j$ for $1 \le j \le 3$ and $\delta' \notin \operatorname{supp}(\phi_0)$.
In the first case,
$$(\phi_0(\delta) - \phi_0(d_4))(z) = -a_4(z - y_0).$$
Since $-a_0 < 0$, the property is verified. In the second case, for $1 \le j \le 3$,
$$(\phi_0(d_j) - \phi_0(\delta'))(z) = a_j(z - y_j).$$
Since $a_j < 0$ for every $j = 1, 2, 3$, the leading coefficient is negative.

This finishes the proof for ideal polygons.

5.2.2 Punctured Polygons

Next, we shall prove Theorem 5.2.1 for punctured polygons.

Let σ_1 and σ_2 be as in the hypothesis with $\alpha_1 \in \mathcal{E}_{\sigma_1} \backslash \mathcal{E}_{\sigma_2}$ and $\alpha_2 \in \mathcal{E}_{\sigma_2} \backslash \mathcal{E}_{\sigma_1}$. These two arcs intersect either exactly once at a point o (when both are non-maximal) or twice at the points o_1, o_2 (when both are maximal). We suppose that the ideal point corresponding to the puncture is at ∞ in the universal cover $\mathbb{U}$ and that $\Gamma = \rho(\pi_1(\Pi_n^{\odot}))$ is generated by the parabolic element $T : z \mapsto z + 1$, after normalisation. Let $\widetilde{\mathcal{E}}_{\sigma,r}$ and $\widetilde{\mathcal{T}}_{\sigma,r}$ be the refined edge set and tile set respectively for the refinement $\sigma = \sigma_1 \cup \{\alpha_2\}$. We take the following strip template:

- From every isotopy class of arcs, we choose the geodesic arc which intersects the boundary of the surface perpendicularly.
- For every geodesic arc, the waist is chosen to be the point of projection of ∞. This choice of waist is Γ-equivariant because T fixes ∞.

We have the two following cases, depending on the maximality of α_1, α_2.

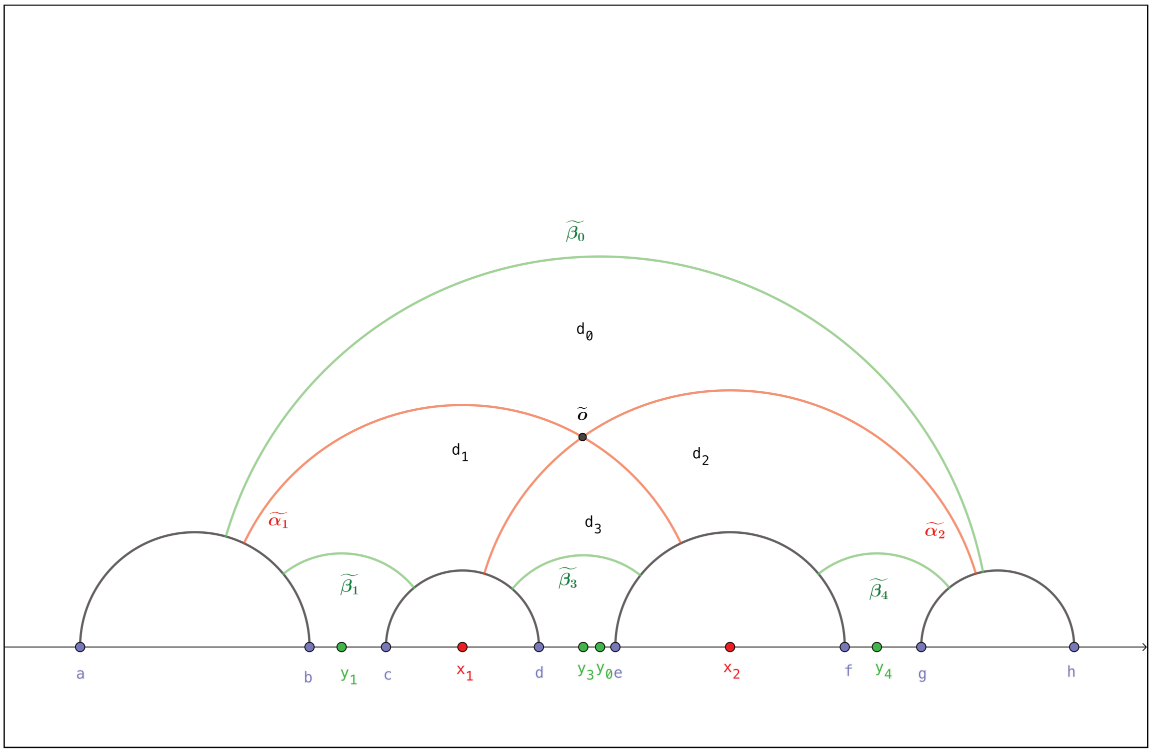


Figure 5.12: α_1 and α_2 are non-maximal arcs

1. See Fig. 5.12. When α_1, α_2 are both non-maximal, the construction is very similar to that in the case of the ideal polygons. Let $\widetilde{o}$ be a lift of the point o. Then $\widetilde{o} = \widetilde{\alpha_1} \cap \widetilde{\alpha_2}$ for two

lifts of α_1 and α_2 respectively. There are four finite tiles formed around $\widetilde{o}$, denoted by d_j, for $j = 0, \ldots, 3$. For each $j \in \{0, \ldots, 3\}$, the tile d_j is either a quadrilateral with an ideal vertex and exactly two arc edges carried by $\widetilde{\alpha_1}$ and $\widetilde{\alpha_2}$, or it is a pentagon with exactly three arc edges carried by $\widetilde{\alpha_1}, \widetilde{\alpha_2}$ and a third arc $\widetilde{\beta_j}$, which is a lift of an arc $\beta_j \in \mathcal{E}_{\sigma_2} \cap \mathcal{E}_{\sigma_1}$. Let $\mathcal{J} \subset \{1, \ldots, 4\}$ be such that the tile d_j is pentagonal if and only if $j \in \mathcal{J}$. For $i = 1, 2$, let x_i be the centre of the semi-circle containing $\widetilde{\alpha_i}$. For $j = 0, \ldots, 3$, let y_j denote the ideal vertex of d_j or the centre of the semi-circle containing $\widetilde{\beta_j}$. In this case, a tile map representing the linear combination (5.6) is a map $\phi_0 : \widetilde{\mathcal{T}_{\sigma,r}} \longrightarrow \mathbb{R}_2[z]$ that satisfies the following properties:

Properties 5.2.3. (a) ϕ_0 is Γ-equivariant: for every $m \in \mathbb{Z}$,

$$\phi_0(T^m \cdot \delta)(z) = \phi_0(\delta)(z - m).$$

(b) The polynomial $\phi_0(\delta)$ vanishes at every ideal vertex of $\delta \in \mathcal{T}_{\sigma,r}$ whenever it has one.

(c) The tile map is coherent around every point of intersection of the lifts of α_1 and α_2.

(d) Let $\delta, \delta' \in \widetilde{\mathcal{T}_{\sigma,r}}$ be two tiles neighbouring along an edge contained in a lift $T^m \cdot \widetilde{\alpha_i}$ of α_i, for some $i \in \{1, 2\}$, such that δ lies above $\widetilde{\alpha_i}$. Then the difference $\phi_0(\delta) - \phi_0(\delta')$ is a hyperbolic Killing vector field with attracting and repelling fixed points at ∞ and $x_i + m$, respectively. The axis intersects $\widetilde{\alpha_i}$ at $p_{\widetilde{\alpha_i}}$. In other words,

$$\phi_0(\delta) - \phi_0(\delta')(z) = A(z - x_i - m), \text{ for some } A > 0.$$

(e) Let $\delta, \delta' \in \mathcal{T}_{\sigma,r}$ be two tiles neighbouring along an edge $\widetilde{\beta} \in \widetilde{\mathcal{E}}_{\sigma_1} \cap \widetilde{\mathcal{E}}_{\sigma_2}$ such that δ lies above the edge. If $\widetilde{\beta} = T^m \cdot \widetilde{\beta}_j$ for some $j = 1, \ldots, 4$ then the difference $\phi_0(\delta) - \phi_0(\delta')$ is a hyperbolic Killing vector field with attracting and repelling fixed points at $x_j + m$ and ∞, respectively. The axis intersects $\widetilde{\beta}_j$ at $p_{\widetilde{\beta_j}}$. In other words,

$$\phi_0(\delta) - \phi_0(\delta')(z) = B(z - y_j - m), \text{ for some } B < 0.$$

Otherwise, $\phi_0(\delta) = \phi_0(\delta')$.

Let (a, b) and (e, f) be the two boundary geodesics that are joined by $\widetilde{\alpha_1}$. Similarly, let (c, d) and (g, h) be the two boundary geodesics joined by $\widetilde{\alpha_2}$ such that

$$a < b \leq c < d \leq e < f \leq g < h \leq a + 1. \tag{5.16}$$

We consider the non-trivial solution (a_0, a_1, a_2, a_3) of the system of equations (5.11)-(5.14) defined in the ideal polygon proof. For $j = 0, \ldots, 3$ and $m \in \mathbb{Z}$, define

$$\begin{aligned} \phi_0 \quad : \widetilde{\mathcal{T}_{\sigma,r}} &\longrightarrow \quad \mathfrak{g} \\ \delta &\longmapsto \quad \begin{cases} (z \mapsto a_j(z - y_j - m)), & \text{if } \delta = T^m \cdot d_j \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Verification of Properties 5.2.3:

(a) For every $m \in \mathbb{Z}$, $(T^m)'(z) = 1$. If $\delta = d_j$ for some $j \in \{1, \dots, 4\}$, then from the definition of ϕ_0 we have that for every $m \in \mathbb{Z}$,

$$\phi_0(T^m \cdot \delta)(z) = a_j(z - y_j - m) = \phi_0(\delta)(z - m).$$

So, the equivariance condition is satisfied in this case. For $\delta \notin \operatorname{supp}(\phi_0)$, the condition holds trivially.

Since the map has been proved to be Γ-equivariant, it suffices to verify the properties (1b)-(1e) around the point $\widetilde{o}$. This is identical to the proof in the case of ideal polygons.

This finishes the proof in the non-maximal case.

2. Let α_i be maximal for $i = 1, 2$. See Fig. 5.13. Let $\widetilde{o_1}, \widetilde{o_2}$ be two lifts of o_1, o_2 respectively, such that $\widetilde{o_1} = \widetilde{\alpha_1} \cap \widetilde{\alpha_2}$ and $\widetilde{o_2} = (T \cdot \widetilde{\alpha_1}) \cap \widetilde{\alpha_2}$ for two lifts $\widetilde{\alpha_1}, \widetilde{\alpha_2}$ of α_1, α_2, respectively. Let d_0 be the infinite tile around $\widetilde{o_1}$, and d_1 and d_2 be the tiles neighbouring d_0 along edges carried by $\widetilde{\alpha_1}$ and $\widetilde{\alpha_2}$, respectively. The fourth tile formed at the vertex $\widetilde{o_1}$ is denoted by d_3. The tiles around $\widetilde{o_2}$ are $d_0, d_2, T \cdot d_1$ and a fourth tile denoted by d_4. Again, for $j = 3, 4$, the tile d_i is either a quadrilateral with one ideal vertex, formed by two boundary edges of the punctured polygon or it is a pentagon with two arc edges respectively contained in $\widetilde{\alpha_1}, \widetilde{\alpha_2}$ and a third arc edge $\widetilde{\beta_j}$ which is a lift of an arc $\beta_j \in \widetilde{\mathcal{E}_{\sigma_1}} \cap \widetilde{\mathcal{E}_{\sigma_2}}$. For $i = 1, 2$, let x_i denote the centre of the semi-circle containing $\widetilde{\alpha}_i$. Let y_3 denote the ideal vertex of δ_3 or the centre of the semi-circle containing $\widetilde{\beta_3}$. Similarly, let y_4 denote the ideal vertex of d_4 or the centre of the semi-circle containing $\widetilde{\beta_4}$. Let $a, b, c, d \in \mathbb{R}$ be such that $\widetilde{\alpha_1}$ joins the two geodesics $(c - 1, d - 1)$ and (c, d), and $\widetilde{\alpha_2}$ joins (a, b) and $(a + 1, b + 1)$. Then a, b, c, d satisfies

$$a < b \le c < d \le a + 1.$$

Again, from Lemma 1.6.3, we have that

$$x_1 = \frac{c + d - 1}{2}, \quad x_2 = \frac{a + b + 1}{2} \tag{5.17}$$

$$y_3 = \frac{cd - ab}{c + d - a - b}, \quad y_4 = \frac{cd - (a + 1)(b + 1)}{c + d - a - b - 2} \tag{5.18}$$

Then ϕ_0 is defined as:

$$\begin{aligned} \phi_0 \quad : \widetilde{\mathcal{T}}_{\sigma,r} &\longrightarrow \quad \mathbb{R}_2[z] \\ \delta \mapsto & \quad \begin{cases} (z \mapsto a_i(z - x_i - m)), & \text{if } \delta = T^m \cdot d_i, \quad i = 1, 2, \\ (z \mapsto (a_1 + a_2)(z - m) - a_1 x_1 - a_2 x_2), & \text{if } \delta = T^m \cdot d_3 \\ (z \mapsto (a_1 + a_2)(z - m) - a_1 x_1 - a_2 x_2 - a_1), & \text{if } \delta = T^m \cdot d_4 \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where $a_1 = -1, a_2 = \frac{y_3 - x_1}{y_3 - x_2}$. In particular, $\phi_0(d_0) = 0$. We note that a_1, a_2 satisfy the following two equations

$$a_1(y_3 - x_1) + a_2(y_3 - x_2) = 0 \tag{5.19}$$

$$a_1(y_4 - x_1 - 1) + a_2(y_4 - x_2) = 0. \tag{5.20}$$

Applying Lemma (1.6.5) to the two triples of geodesics $(c-1, d-1), (a, b), (c, d)$ and $(a, b), (c, d), (a+1, b+1)$, we get that $x_1 < y_2 < x_2 < y_3$. As a result, $a_2 < 0$.

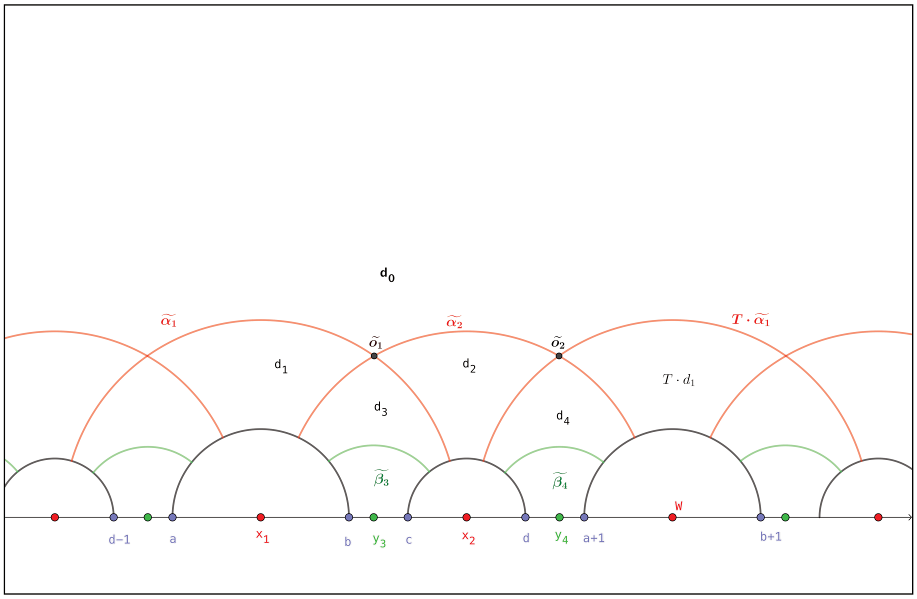


Figure 5.13: α_1, α_2 are maximal

Lemma 5.2.4. *The tile map defined above verifies the properties* (5.2.3)*.*

Proof. (a) The equivariance follows from the definition of ϕ_0.

(b) Suppose that δ is a tile with an ideal vertex. If $\delta \in \operatorname{supp}(\phi_0)$, then $\delta = d_j$ for some $j \in \{3, 4\}$, then that ideal vertex is given by y_j. From the equations (5.19) and (5.20), we get that $\phi_0(\delta)$ vanishes at y_j. If $\delta \notin \operatorname{supp}(\phi_0)$, then $\phi_0(\delta) = 0$, which automatically fixes its ideal vertex.

(c) We show that the map is coherent around $\widetilde{o_1}$:

$$\begin{aligned}(\phi_0(d_0) - \phi_0(d_1))(z) &= -a_1 z + a_1 x_1 \\ &= a_2 z - a_2 z - a_1 z + a_1 x_1 + a_2 x_2 - a_2 x_2 \\ &= a_2(z - x_2) - (a_1 + a_2)(z) - a_1 x_1 - a_2 x_3 \\ &= (\phi_0(d_2) - \phi_0(d_3))(z).\end{aligned}$$

Around the point $\widetilde{o_2}$:

$$\begin{aligned}(\phi_0(d_0) - \phi_0(d_2))(z) &= -a_2 z + a_2 x_2 \\ &= a_1 z - a_1 z - a_1 z + a_2 x_2 + a_1 x_1 - a_1 x_1 + a_1 - a_1 \\ &= a_1(z - x_1 - 1) - (a_1 + a_2)(z) - a_1 x_1 - a_2 x_2 - a_1 \\ &= (\phi_0(T \cdot d_1) - \phi_0(d_4))(z).\end{aligned}$$

Hence by equivariance, the map ϕ_0 is coherent around every intersection point.

(d) Let $\delta, \delta' \in \widetilde{\mathcal{T}}_{\sigma,r}$ be two tiles with a common internal edge of the form $T^m \cdot \widetilde{\alpha_i}$, for some $m \in \mathbb{Z}$ and $i = 1, 2$, such that δ lies above the edge. Using the equivariance and coherence of the map, it suffices to verify the property when $m = 0, \delta = d_0, \delta' = d_i$, $i = 1, 2$. From the calculations of the proof of the coherence property, we have that

$$(\phi_0(d_0) - \phi_0(d_1))(z) = -a_1(z - x_1)$$
$$(\phi_0(d_0) - \phi_0(d_2))(z) = -a_2(z - x_2).$$

Since $a_1, a_2 < 0$, the difference is of the desired form for every $i = 1, 2$.

(e) If the two tiles δ, δ' have a common internal edge of the form $\widetilde{\beta_j}$ for $m \in \mathbb{Z}$ and $j = 3, 4$ such that δ lies above the edge, then $\delta = \widetilde{\beta_j}$ and $\delta' \notin \operatorname{supp}(\phi_0)$.
For $j = 3$,

$$(\phi_0(d_3) - \phi_0(\delta'))(z) = (a_1 + a_2)z - a_1 x_1 - a_2 x_2.$$

For $j = 4$,

$$(\phi_0(d_4) - \phi_0(\delta'))(z) = (a_1 + a_2)z - a_1 x_1 - a_2 x_2 - a_1.$$

Since $a_1, a_2 < 0$, both of these polynomials have negative leading coefficient $a_1 + a_2$. By eqs. (5.19), (5.20), the polynomial $\phi_0(d_j) - \phi_0(\delta')$ vanishes at y_j, for $m \in \mathbb{Z}$ and $j = 3, 4$.
If two tiles $d_1, d_2 \in \widetilde{\mathcal{T}}_{\sigma,r}$ have a common internal edge $\beta \in \widetilde{\mathcal{E}}_{\sigma_1} \cap \widetilde{\mathcal{E}}_{\sigma_2}$, which is not of the above form, then $d_1, d_2 \notin \operatorname{supp}(\phi_0)$. So, $\phi_0(d_1) - \phi_0(d_2) = 0$.

□

So ϕ is a Γ-equivariant refined tile map that realises the required linear combination.

5.2.3 Decorated ideal polygons

In this section, we will prove Theorem 5.2.1 for polygons all of whose vertices are decorated ideal points.

The surface is contractible. So, $\widetilde{\mathcal{E}}_{\sigma,r} = \mathcal{E}_{\sigma,r}, \widetilde{\mathcal{T}}_{\sigma,r} = \mathcal{T}_{\sigma,r}$. Firstly, we remark that at most one of the two intersecting arcs α_1, α_2 can be of the edge-to-vertex type. Indeed, if for every $i = 1, 2$ the arc α_i joins the vertex v_i with the edge e_i then these four vertices must be cyclically ordered as e_1, v_2, v_1, e_2 and no two are consecutive. In particular, v_1, v_2 are not consecutive. So either there exists an arc β in $\mathcal{E}_{\sigma_1}$ that has one endpoint on e_1 and another on a vertex or an edge that lies between v_2 and v_1 or there exists an arc $\beta' \in \mathcal{E}_{\sigma_1} \cap \mathcal{E}_{\sigma_2}$ joining v_1, v_2. In the first case, the arc α_2 must intersect β, hence $\operatorname{codim}(\sigma_1 \cap \sigma_2) > 1$ which contradicts our hypothesis. The second case is not possible because there are no vertex-to-vertex arcs in this arc complex.

So we have the following two cases:

- The proof, in the case where both α_1 and α_2 are edge-to-edge arcs, is identical to that for ideal polygons.

- Let α_1 be an edge-to-edge arc joining two edges e_1, e_3 and let α_2 be an edge-to-vertex arc joining the edge e_2 and the decorated ideal vertex ν. Then there are three configurations as shown in the Fig.5.14. Since neither e_1, e_2 nor e_2, e_3 can be consecutive, there always exist two edge-to-edge arcs β_1 and β_2 in $\mathcal{E}_{\sigma_1} \cap \mathcal{E}_{\sigma_2}$ that respectively join these two pairs. Again, if e_1, ν or e_3, ν are not consecutive, there must exist two edge-to-vertex arcs β_4 and β_3 in

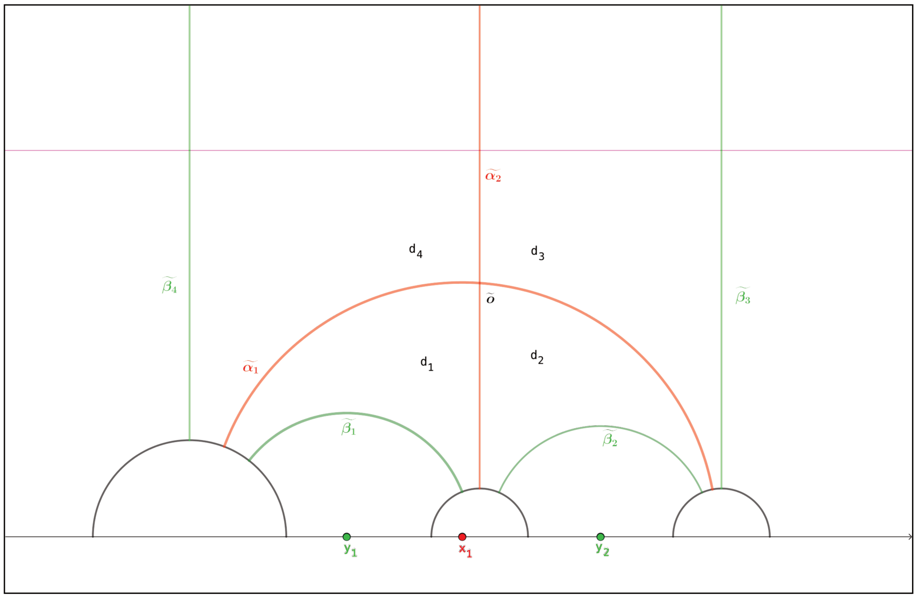


Figure 5.14: Codimension 1: Decorated ideal polygon

$\mathcal{E}_{\sigma_1} \cap \mathcal{E}_{\sigma_2}$, joining the pairs, respectively. Let $d_1, \ldots, d_4$ be the smaller tiles of the refinement $\sigma_1 \cup \{\alpha_2\}$, such that β_i is an internal edge of d_i whenever β_i exists.

We may suppose that $\nu = \infty$. We make the following choice of strip template:

- The arcs are chosen from the isotopy classes so that they intersect the boundary perpendicularly.
- For edge-to-edge arcs, the waists are chosen to the point of projection of the ideal point ∞. For edge-to-vertex arcs, the waist is always the point ∞.

The arcs β_1, β_2 are semi-circular with centres y_1, y_2 whereas β_3, β_4 are vertical lines. Let x_0 be the centre of the semi-circle carrying α_1. Using Lemma (1.6.3), we have that

$$y_1 < x_0 < y_2. \tag{5.21}$$

We shall construct a tile map corresponding to the linear combination (6.7).

Properties 5.2.5. A neutral tile map $\phi_0 : \mathcal{T}_{\sigma,r} \longrightarrow \mathbb{R}_2[z]$ represents the linear combination (6.7) if and only if it verifies the following properties:

1. The polynomial $\phi_0(d)$ vanishes at every ideal vertex of $\delta \in \mathcal{T}_{\sigma,r}$.
2. The tile map is coherent around the intersection point o.
3. Let $\delta, \delta' \in \widetilde{\mathcal{T}}_{\sigma,r}$ be two tiles with common internal edge contained in α_1 for such that δ lies above α_1. Then $\phi_0(\delta) - \phi_0(\delta')$ is a hyperbolic Killing field with attracting fixed

point at ∞ and repelling fixed point at x_1. In particular, its axis intersects α_1 at p_{α_1}. In terms of polynomials, we must have

$$\phi_0(\delta) - \phi_0(\delta') = (z \mapsto A(z - x_1)), \text{ for some } A > 0,$$

a linear polynomial with positive leading coefficient.

4. Let $\delta, \delta' \in \mathcal{T}_{\sigma,r}$ be two tiles with common internal edge contained in α_2 for $i = 1, 2$ such that δ lies to the left of the edge. Then $\phi_0(\delta) - \phi_0(\delta')$ is a parabolic Killing vector field with fixed point at ∞, pointing towards δ. In terms of polynomials, we must have

$$\phi_0(\delta) - \phi_0(\delta') = (z \mapsto B), \text{ for some } B < 0,$$

a constant polynomial.

5. Suppose $\delta, \delta' \in \mathcal{T}_{\sigma,r}$ are two tiles with common internal edge β_j for $j = 1, 2$, such that δ lies above β_j. Then $\phi_0(\delta) - \phi_0(\delta')$ is a hyperbolic Killing vector field with attracting fixed point at y_j and repelling fixed point at ∞. In particular, its axis intersects β_j at p_{β_j}. In terms of polynomials, we must have

$$\phi_0(\delta) - \phi_0(\delta') = (z \mapsto C(z - y_j)), \, C < 0.$$

6. Suppose $\delta, \delta' \in \mathcal{T}_{\sigma,r}$ are two tiles with common internal edge β_j for $j = 3, 4$, such that δ lies to the left β_j. Then $\phi_0(\delta) - \phi_0(\delta')$ is a parabolic Killing vector field with fixed point at ∞, pointing away from δ. In terms of polynomials, we must have

$$\phi_0(\delta) - \phi_0(\delta') = (z \mapsto D), \, D > 0.$$

We define the tile map

$$\begin{array}{rcl} \phi_0 & : \widetilde{\mathcal{T}}_{\sigma,r} \longrightarrow & \mathbb{R}_2[z] \\ \delta \mapsto & & \begin{cases} (z \mapsto a_j(z - y_j)), & \text{if } \delta = d_j, \quad i = 1, 2, \\ (z \mapsto a_3), & \text{if } \delta = d_3 \\ (z \mapsto a_4), & \text{if } \delta = d_4 \\ 0, & \text{otherwise,} \end{cases} \end{array}$$

where

$$a_1 = a_2 = -1, \quad a_3 = y_2 - x_0 > 0, \quad a_4 = y_1 - x_0 < 0.$$

Lemma 5.2.6. *The tile map defined above satisfies Properties* (5.2.5).

Proof. 1. Suppose that δ is a tile with a decorated ideal vertex. If $\delta \in \operatorname{supp}(\phi_0)$, then $\delta = d_j$ for some $j \in \{3, 4\}$ and that ideal vertex is given by ∞. From the definition of ϕ_0, we get that $\phi_0(d_j)$ is a constant polynomial. Hence, it fixes infinity and any horoball centered at ∞. If $\delta \notin \operatorname{supp}(\phi_0)$, then $\phi_0(\delta) = 0$, which automatically fixes its ideal vertex.

2. Consistency around $\widetilde{o}$:

$$\begin{aligned} \phi_0(d_1) - \phi_0(d_2)(z) &= y_1 - y_2 \\ &= y_1 - x_0 + x_0 - y_2 \\ &= \phi_0(d_4) - \phi_0(d_3)(z). \end{aligned}$$

3. Let $\delta, \delta' \in \mathcal{T}_{\sigma,r}$ be two tiles with a common internal edge of the form α_1 such that δ lies above the edge. Using the coherence of ϕ_0, it suffices to verify the property when $\delta = d_4, \delta' = d_1$. Substituting the values of a_1, a_4 and using the definition of ϕ_0, we get that
$$\phi_0(d_4) - \phi_0(d_1)(z) = z - x_0,$$
which is of the desired form.
4. Let $\delta, \delta' \in \mathcal{T}_{\sigma,r}$ be two tiles with a common internal edge of the form α_2 such that δ lies to the left of the edge. Using the coherence of ϕ_0, it suffices to verify the property when $\delta = d_4, \delta' = d_3$. From eq. (5.21),
$$\phi_0(d_4) - \phi_0(d_3)(z) = y_1 - y_2 < 0.$$
So the property is verified.
5. Let $\delta, \delta' \in \mathcal{T}_{\sigma,r}$ be two tiles with a common internal edge of the form β_j for $j = 1, 2$ such that δ lies above the edge. Then $\delta = d_j$ for $j = 1, 2$ and $\delta' \notin \operatorname{supp}(\phi_0)$. For $j = 1, 2$, we have that
$$\phi_0(d_j) - \phi_0(\delta')(z) = -z + y_j,$$
which is of the desired form.
6. Let $\delta, \delta' \in \mathcal{T}_{\sigma,r}$ be two tiles with a common internal edge of the form β_j for $j = 3, 4$ such that δ lies to the left of the edge. Then either $\delta \notin \operatorname{supp}(\phi_0), \delta' = d_4$ or $\delta = d_3, \delta' \notin \operatorname{supp}(\phi_0)$. In the first case, we have that
$$\phi_0(\delta) - \phi_0(d_4)(z) = x_0 - y_1 > 0.$$
In the second case, we have that
$$\phi_0(d_3) - \phi_0(\delta')(z) = y_2 - x_0 > 0.$$

So we have a neutral tile map representing the linear combination (6.7) for the chosen strip template. ☐

This concludes the proof of Theorem 5.2.1 for decorated ideal polygons.

5.2.4 Hyperideal polygons

In this section we will prove Theorem 5.2.1 for polygons with truncated hyperideal vertices.

Proof. Like in the previous case, denote by α_1, the edge-to-edge arc joining two edges e_1, e_3 and by α_2, the edge-to-vertex arc joining the edge e_2 and the truncated hyperideal vertex ν. The geodesic carrying the hyperideal vertex is taken to be the vertical line passing through the origin. There are three possible configurations. Fig.(5.15) shows the most general one. Again, let β_1 and β_2 in $\mathcal{E}_{\sigma_1} \cap \mathcal{E}_{\sigma_2}$ be the arcs joining the pairs e_1, e_2 and e_2, e_3 respectively. Finally, whenever e_1, ν or e_3, ν are not consecutive, let β_4 and β_3 in $\mathcal{E}_{\sigma_1} \cap \mathcal{E}_{\sigma_2}$ be the two edge-to-vertex arcs joining the pairs, respectively. Let $d_1, \ldots, d_4$ be the smaller tiles of the refinement $\sigma_1 \cup \{\alpha_2\}$, such that β_i is an internal edge of d_i whenever β_i exists.

We make the following choice of strip template :

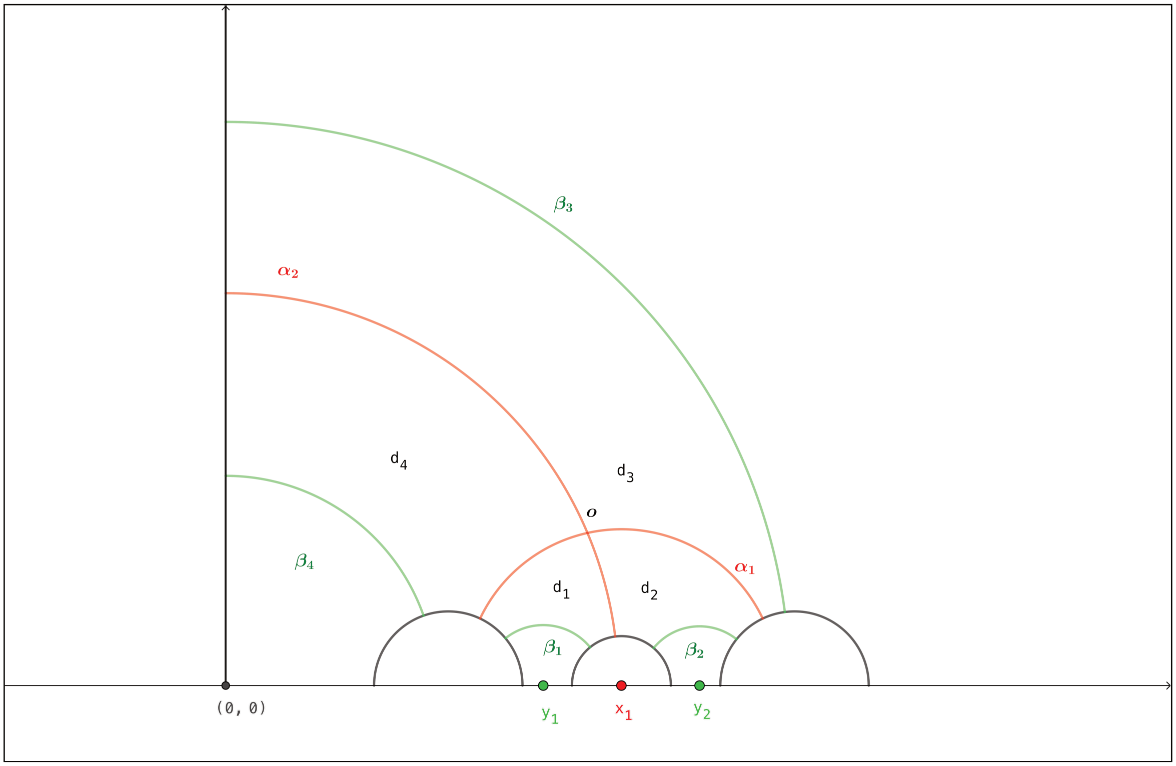


Figure 5.15: Codimension 1: Truncated hyper ideal polygon

- The arcs are chosen from the isotopy classes so that they intersect the boundary perpendicularly
- the waists for the edge-to-edge arcs are the points of projection of ∞.

Then β_1, β_2 are semi-circular whereas β_3, β_4 are vertical lines. Let x_0, y_1, y_2 be the respective centres of the semi-circles carrying $\alpha_1, \beta_1, \beta_2$. Using Lemma (1.6.3), we have that

$$0 < y_1 < x_0 < y_2. \tag{5.22}$$

We shall construct a tile map corresponding to the linear combination (6.7).

Properties 5.2.7. A neutral tile map $\phi_0 : \mathcal{T}_{\sigma,r} \longrightarrow \mathbb{R}_2[z]$ represents the linear combination (6.7) if and only if it verifies the following properties:

1. If a tile δ has ν as a truncated vertex, then the polynomial

$$\phi_0(\delta) = (z \mapsto kz), \text{ for some } k \in \mathbb{R}.$$

If δ contains any other vertex of the polygon, then $\phi_0(\delta)$ vanishes at the centre of the semi-circle carrying that vertex.

2. The tile map is coherent around the intersection point o.

3. Let $\delta, \delta' \in \mathcal{T}_{\sigma,r}$ be two tiles with common internal edge contained in α_1 such that δ lies above edge. Then $\phi_0(\delta) - \phi_0(\delta')$ is a hyperbolic Killing vector field with attracting fixed point at ∞ and repelling fixed point at x_0. In particular, for our choice of strip templates, its axis intersects α_1 at p_{α_1}. In terms of polynomials, we must have
$$\phi_0(\delta) - \phi_0(\delta') = (z \mapsto A(z - x_0)),\ A > 0.$$

4. Let $\delta, \delta' \in \mathcal{T}_{\sigma,r}$ be two tiles with common internal edge contained in α_2 for $i = 1, 2$ such that δ lies above the edge. Then $\phi_0(\delta) - \phi_0(\delta')$ is a hyperbolic Killing vector field with attracting fixed point at ∞ and repelling fixed point at 0. In terms of polynomials, we must have
$$\phi_0(\delta) - \phi_0(\delta') = (z \mapsto Bz),\ B > 0.$$

5. Suppose $\delta, \delta' \in \mathcal{T}_{\sigma,r}$ are two tiles with common internal edge β_j for $j = 1, 2$, such that δ lies above β_j. Then $\phi_0(\delta) - \phi_0(\delta')$ is a hyperbolic Killing vector field with attracting fixed point at y_j and repelling fixed point at ∞. In particular, its axis intersects α_i at p_{α_i}. In terms of polynomials, we must have
$$\phi_0(\delta) - \phi_0(\delta') = (z \mapsto C(z - y_j)),\ C < 0.$$

6. Suppose $\delta, \delta' \in \mathcal{T}_{\sigma,r}$ are two tiles with common internal edge β_j for $j = 3, 4$, such that δ lies to the left of β_j. Then $\phi_0(\delta) - \phi_0(\delta')$ is a Killing vector field with attracting fixed point at 0 and repelling fixed point at ∞. In terms of polynomials, we must have
$$\phi_0(\delta) - \phi_0(\delta') = (z \mapsto Dz),\ D < 0.$$

We define the tile map
$$\begin{aligned} \phi_0 \quad : \widetilde{\mathcal{T}}_{\sigma,r} &\longrightarrow \quad \mathbb{R}_2[z] \\ d \mapsto \quad & \begin{cases} (z \mapsto a_i(z - y_i)), & \text{if } \delta = d_i, \quad i = 1, 2, \\ (z \mapsto a_3 z), & \text{if } \delta = d_3 \\ (z \mapsto a_4 z), & \text{if } \delta = d_4 \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$
where
$$a_1 = \frac{x_0}{y_1(x_0 - y_2)} < 0, \quad a_2 = \frac{x_0}{(x_0 - y_2)} < 0,$$
$$a_3 = 1, \quad a_4 = \frac{y_2(x_0 - y_1)}{y_1(x_0 - y_2)} < 0.$$
These four numbers satisfy the following system of homogeneous equations:
$$a_1 - a_2 + a_3 - a_4 = 0, \tag{5.23}$$
$$a_1 y_1 - a_2 y_2 = 0, \tag{5.24}$$
$$a_2(y_2 - x_0) + a_3 x_0 = 0. \tag{5.25}$$

Lemma 5.2.8. *The tile map defined above satisfies Properties* (5.2.7).

Proof. 1. Suppose that δ is a tile with a truncated hyperideal vertex. If $\delta \in \operatorname{supp}(\phi_0)$, then $\delta = d_j$ for some $j \in \{3, 4\}$ and that ideal vertex is a segment on the vertical line given by $(0, \infty)$. From the definition of ϕ_0, we get that $\phi_0(d_j)(z)$ is a degree one polynomial vanishing at 0. If $\delta \notin \operatorname{supp}(\phi_0)$, then $\phi_0(\delta) = 0$, which automatically fixes its ideal vertex.

2. Around $\widetilde{o}$: from eqs. (5.23), (5.24)

$$\begin{aligned}(\phi_0(d_1) - \phi_0(d_2))(z) &= (a_1 - a_2)z + a_2 y_2 - a_1 y_1 \\ &= (a_4 - a_3)z \\ &= (\phi_0(d_4) - \phi_0(d_3))(z).\end{aligned}$$

3. Let $\delta, \delta' \in \mathcal{T}_{\sigma,r}$ be two tiles with a common internal edge of the form α_1 such that δ lies above the edge. Using the coherence of ϕ_0, it suffices to verify the property when $\delta = d_4, \delta' = d_1$. From eqs. (5.23),(5.24) we get that,

$$\begin{aligned}(\phi_0(d_4) - \phi_0(d_1))(z) &= (a_4 - a_1)z + a_1 y_1, \\ &= (a_3 - a_2)z + a_2 y_2.\end{aligned}$$

From eq. (5.25) we get that the Killing field $\phi_0(d_4) - \phi_0(d_1)$ vanishes at x_0. Also, the leading coefficient is positive because $a_3 > 0 > a_2$. Hence the property is verified.

4. Let $\delta, \delta' \in \mathcal{T}_{\sigma,r}$ be two tiles with a common internal edge of the form α_2 such that δ lies above the edge. Using the coherence of ϕ_0, it suffices to verify the property when $\delta = d_3, \delta' = d_4$.

$$(\phi_0(d_3) - \phi_0(d_4))(z) = (a_3 - a_4)z.$$

Since $a_3 > 0 > a_4$, the leading coefficient is positive. Hence, the property is verified.

5. Let $\delta, \delta' \in \mathcal{T}_{\sigma,r}$ be two tiles with a common internal edge of the form β_j for $j = 1, 2$ such that δ lies above the edge. Then $\delta = d_j$ for $j = 1, 2$ and $\delta' \notin \operatorname{supp}(\phi_0)$. For $j = 1, 2$, we have that

$$\phi_0(d_j) - \phi_0(\delta')(z) = a_j(z - y_j).$$

Since $a_1, a_2 < 0$, the difference has the desired form.

6. Let $\delta, \delta' \in \mathcal{T}_{\sigma,r}$ be two tiles with a common internal edge of the form β_j for $j = 3, 4$ such that δ lies above the edge. Then either $d \notin \operatorname{supp}(\phi_0)\, \delta' = d_3$ or $\delta = d_4, \delta' \notin \operatorname{supp}(\phi_0)$. In the first case, we have that

$$\phi_0(d) - \phi_0(d_3)(z) = -1.$$

In the second case, we have that

$$(\phi_0(d_4) - \phi_0(\delta'))(z) = a_4 < 0.$$

Hence, Lemma 5.2.8 is proved. □

So we have a neutral tile map representing the linear combination (6.7) for the chosen strip template. This concludes the proof of Theorem 5.2.1 in the case of hyperideal polygons. □

5.2.5 One-holed polygons and spun triangulations

In this section we will prove Theorem 5.2.1 for one-holed polygons, $\Pi_n^{\circledcirc}$ $(n \geq 1)$.

Let $\Gamma := \rho(\pi_1(\Pi_n^{\circledcirc})) \subset \mathrm{PSL}(2,\mathbb{R})$, where ρ is the holonomy representation of the surface. Then, Γ is an infinite cyclic group generated by some hyperbolic element γ of $\mathrm{PSL}(2,\mathbb{R})$. Using the transitivity of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{R} \cup \{\infty\}$, we assume that the attracting and repelling fixed points of γ are $\infty, 0$, respectively. So the element γ is of the form

$$\gamma = \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{R}),\ \lambda > 1.$$

Case 1: Suppose that both the arcs α_1, α_2 are finite. Then, they must be non-minimal. The proof in this case is identical to that of ideal polygons.

Case 2: The arc α_1 is finite whereas the arc α_2 is spiraling such that the finite endpoints of both the arcs lie on different edges. This case appears for $n > 1$. Again, we assume $\Gamma := \rho(\pi_1(\Pi_n^{\circledcirc}))$ is generated by γ and we consider its universal cover inside $\mathbb{U}$. From each isotopy class we choose a geodesic arc which meets the boundary perpendicularly. Then the lifts of α_2 are represented by vertical lines and those of α_1 are arcs of semi-circles, lying to the right of the axis of γ. Label their unique point of intersection as o. Let $\widetilde{o}$ be one of its lifts. Then it is the intersection of two lifts $\widetilde{\alpha_1}, \widetilde{\alpha_2}$ of α_1, α_2. Suppose that the endpoints of $\widetilde{\alpha_1}$ lie on the geodesics (a, b) and (e, f) and that of $\widetilde{\alpha_2}$ on (c, d). Then,

$$a < b \leq c < d \leq e < f. \tag{5.26}$$

If α_1 is a maximal arc, then $e = \lambda a, f = \lambda b$.

The four small tiles formed around $\widetilde{o}$ are labeled as $d_1, \ldots, d_4$ in the anti-clockwise manner so that the tiles d_1, d_2 lie below $\widetilde{\alpha_1}$. Also, for $j = 1, 2$, d_j either contains a spike or an internal edge $\widetilde{\beta_j}$ which is a lift of the arc $\beta_j \in \mathcal{E}_{\sigma_1} \cap \mathcal{E}_{\sigma_2}$. In the former case, the spike and in the latter case the centre of the semi-circle carrying β_j are denoted by y_j, for $j = 1, 2$. The centre of the semi-circle carrying $\widetilde{\alpha_1}$ is denoted by x_0. Each of the two tiles d_3, d_4 has an internal edge that are lifts of two spiraling arcs β_3, β_4; they are labeled as $\widetilde{\beta_3}$ and $\widetilde{\beta_4}$, respectively. When $\widetilde{\alpha_1}$ is a maximal arc, $\beta_3 = \beta_4$ and hence $\widetilde{\beta_4} = \gamma \cdot \widetilde{\beta_3}$. The arc $\widetilde{\beta_1}$ is the common perpendicular to the geodesics (a, b) and (c, d) and the arc $\widetilde{\beta_2}$ is the common perpendicular to (c, d) and (e, f). From Theorem 1.6.3, the coordinates for x_0, y_1, y_2 are given by

$$x_0 = \frac{ef - ab}{e + f - (a + b)}, \qquad y_1 = \frac{cd - ab}{c + d - a - b}, \qquad y_2 = \frac{ef - cd}{e + f - c - d}.$$

See Fig. 5.16. From Theorem 1.6.5, we get $y_1 < x_0 < y_2$. We shall construct a neutral tile map that represents the linear combination:

$$c_{\alpha_1} f_{\alpha_1}(m) + c_{\alpha_2} f_{\alpha_2}(m) + \sum_{j=1}^{n_0} c_{\beta_j} f_{\beta_j}(m) = 0, \tag{5.27}$$

where $c_{\alpha_1}, c_{\alpha_2} > 0$, $c_{\beta_j} < 0$ for $j = 1, \ldots, 4$ and

$$n_0 = \begin{cases} 3, & \text{if } \beta_3 = \beta_4, \\ 4, & \text{otherwise.} \end{cases}$$

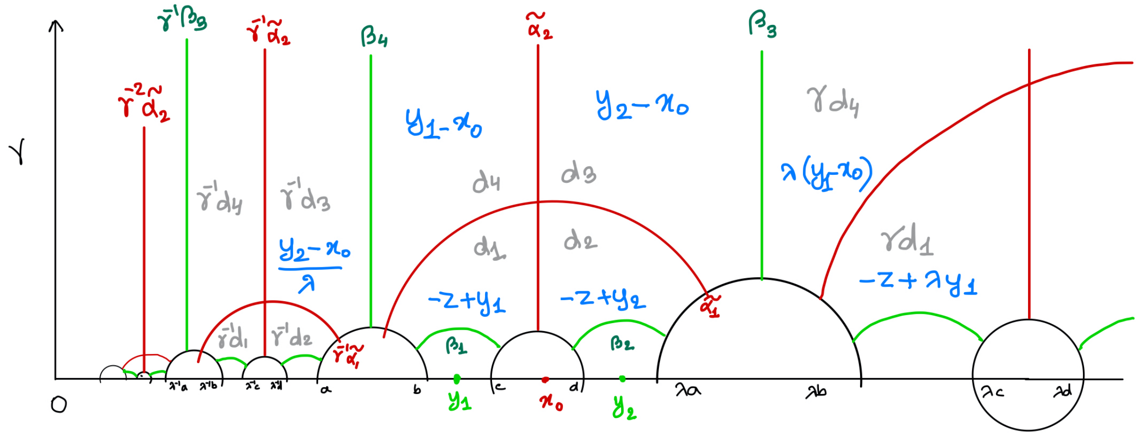


Figure 5.16: Codimension one: Case 2

Properties 5.2.9. A neutral tile map $\phi_0 : \widetilde{\mathcal{T}}_{\sigma,r} \longrightarrow \mathbb{R}_2[z]$ represents the equation (5.27) if and only if it verifies the following properties:

(a) If a tile $\delta \in \widetilde{\mathcal{T}}_{\sigma,r}$ contains a spike at a point $z_0 \in \mathbb{R}$, then $\phi_0(d)$ is a polynomial that vanishes at z_0. If a tile contains a spike at ∞, then $\phi_0(d)$ is a constant polynomial.

(b) ϕ_0 is Γ-equivariant: for all $\delta \in \widetilde{\mathcal{T}}_{\sigma,r}$, $\phi_0(\gamma \cdot d)z = \lambda\phi_0(d)(\frac{z}{\lambda})$.

(c) The map is consistent around every lift of every intersection point.

(d) If δ, δ' are two tiles that share an edge carried by a lift $\widetilde{\alpha_1}$ of α_1, then $\phi_0(\delta) - \phi_0(\delta')$ is a polynomial of the form $Az + B$. If δ lies above (resp. below) the lift, then $A > 0$ (resp. $A < 0$).

(e) If δ, δ' are two tiles that share an edge carried by a lift $\widetilde{\alpha_2}$ of α_2, then $\phi_0(\delta) - \phi_0(\delta')$ is a constant polynomial. If δ lies to the left of the lift, then this constant is negative.

(f) If δ, δ' are two tiles that share an edge carried by a lift of β_j for some $j = 1, \ldots, 4$, then $\phi_0(\delta) - \phi_0(\delta')$ is a degree one polynomial $az + b$ vanishing at the centre of the semi-circle carrying $\widetilde{\beta}$. If δ lies above the lift, then the leading coefficient a is negative.

(g) Let $(z_n)_n$ be a sequence of points in tiles $\delta_n \in \widetilde{\mathcal{T}_\sigma}$ that converges to a point z on the axis of the isometry γ, then $\phi_0(\delta_n)$ converges to 0.

Define $\phi_0 : \widetilde{\mathcal{T}}_{\sigma,r} \longrightarrow \mathbb{R}_2[z]$: for $q \in \mathbb{Z}$,

$$\delta \mapsto \begin{cases} (z \mapsto -z + \lambda^q y_1), & \text{if } \delta = \gamma^q \cdot d_1, \\ (z \mapsto -z + \lambda^q y_2), & \text{if } \delta = \gamma^q \cdot d_2, \\ (z \mapsto \lambda^q (y_2 - x_0)), & \text{if } \delta = \gamma^q \cdot d_3, \\ (z \mapsto \lambda^q (y_1 - x_0)), & \text{if } \delta = \gamma^q \cdot d_4, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 5.2.10. *The neutral map ϕ_0 defined above satisfies the properties 5.2.9 (a)-(g).*

Proof. (a) For $n = 2$, the only tiles that contain a real spike are of the form $\gamma^q \cdot \beta_j$, for $j = 1, 2$ and $q \in \mathbb{Z}$. The spikes are at $\gamma^q \cdot y_j$. From the definition of ϕ_0, it immediately follows that $\phi_0(d_j)(\gamma^q \cdot y_j) = 0$, for $j = 1, 2$. Hence the condition (b) is satisfied.

For $n \geq 3$, the tiles containing a real spike are not small. So their images by ϕ_0 are all equal to the zero polynomial.

The tiles that contain a spike at ∞ are in the orbit of d_3 or d_4. From the definition ϕ_0, the images of such tiles are constant polynomials. Hence the property is satisfied in this case as well.

(b) Suppose that $\delta = \gamma^q \cdot d_j$, for some $j = 1, 2$. Then,

$$\begin{aligned}
\phi_0(\gamma \cdot d) &= \phi_0(\gamma^{q+1} \cdot d_j)(z) \\
&= -z + \gamma^{q+1} y_j \\
&= \lambda \left(-\frac{z}{\lambda} + \gamma^q y_j \right) \\
&= \gamma \cdot \phi_0(\delta).
\end{aligned}$$

When $j = 3$,

$$\begin{aligned}
\phi_0(\gamma \cdot d) &= \phi_0(\gamma^{q+1} \cdot d_j)(z) \\
&= \lambda^{q+1}(y_2 - x_0) \\
&= \lambda \left(\lambda^q (y_2 - x_0)) \right) \\
&= \gamma \cdot \phi_0(\delta).
\end{aligned}$$

The calculations are identical for $j = 4$. Hence the map is Γ-equivariant.

(c) Any point of intersection is of the form $\gamma^q \cdot \widetilde{o} = \gamma^q \cdot \widetilde{\alpha_1} \cap \gamma^q \cdot \widetilde{\alpha_2}$, for some $q \in \mathbb{Z}$. It suffices to verify the property for $q = 0$. The tiles around this point are $d_1, \ldots, d_4$, in the anti-clock-wise manner so that the two tiles d_1, d_2 are below the finite arc $\widetilde{\alpha_1}$. Then,

$$\begin{aligned}
(\phi_0(d_4) - \phi_0(d_1))(z) &= y_1 - x_0 + z - y_1 \\
&= (z - x_0) \\
&= y_2 - x_0 + z - y_2 \\
&= (\phi_0(d_3) - \phi_0(d_2))(z).
\end{aligned}$$

So the property 5.2.9(c) is verified.

(d) Suppose δ, δ' are two tiles that share the edge carried by $\widetilde{\alpha_1}$. Then either $\delta = d_4, \delta' = d_1$ or $\delta = d_3, \delta' = d_2$. From calculations done above, we get that

$$(\phi_0(d_4) - \phi_0(d_1))(z) = z - x_0$$

which is a degree one polynomial with positive leading coefficient, as required by the property 5.2.9(d).

(e) Suppose δ, δ' are two tiles that share an edge carried by $\widetilde{\alpha_2}$. Then either $\delta = d_4, \delta' = d_3$ or $\delta = d_1, \delta' = d_2$. By property 5.2.9(c), it is enough to do the calculations for just one pair. The tile d_4 lies to the left of $\widetilde{\alpha_2}$. We have

$$\begin{aligned}(\phi_0(d_4) - \phi_0(d_3))(z) &= y_1 - x_0 - (y_2 - x_0) \\ &= y_1 - y_2 > 0.\end{aligned}$$

So the difference is a negative constant polynomial. Hence 5.2.9(e) is satisfied.

(f) Suppose that δ, δ' are two tiles that share an edge carried by the lift $\widetilde{\beta_j}$ of β_j for some $j = 1, \ldots, 4$. We assume that $\widetilde{\beta_j}$ is horizontal and that δ lies above δ'. Then, $\delta = d_1$ or d_2 and δ' is not a small tile. So

$$(\phi_0(d_j) - \phi_0(\delta'))(z) = -z + y_j.$$

So we have a degree one polynomial whose leading coefficient is negative and which vanishes at y_j, $j = 1, 2$.

If $\widetilde{\beta_j}$ is vertical and δ lies to its left, then $\delta = d_3$. If α_1 is maximal, then $\delta' = \gamma \cdot d_4$. Then,

$$\begin{aligned}(\phi_0(d_3) - \phi_0(\gamma \cdot d_4))(z) &= (y_2 - x_0) - \lambda(y_1 - x_0) \\ &= (y_2 - x_0 - \lambda y_1 + \lambda x_0) \\ &> (y_1 - x_0 - \lambda y_1 + \lambda x_0) \\ &= (1 - \lambda)(y_1 - x_0) > 0.\end{aligned}$$

So the difference between the two polynomials is a positive constant polynomial. If α_1 is not maximal, then δ' is not a small tile. Then,

$$(\phi_0(d_3) - \phi_0(\delta'))(z) = y_2 - x_0 > 0.$$

So we get a positive constant polynomial in this case as well.

Finally, suppose that $\widetilde{\beta_j}$ is vertical and δ lies to its right. Either α_1 is maximal, in which case $\delta = \beta_4$ and $\delta' = \gamma^{-1}\beta_3$ or α_1 is non-maximal with $\delta = \beta_4$ and δ' not a small tile. The former case is identical to the previous case, using the equivariance of ϕ_0. In the latter case,

$$(\phi_0(d_4) - \phi_0(\delta'))(z) = y_1 - x_0 < 0.$$

(g) Let $(z_n)_n$ be a sequence of points in the universal cover that converges to a point z on the axis of the isometry γ. Then the tiles $\delta_n \in \widetilde{\mathcal{T}_\sigma}$ containing them must be spiraling tiles of the form $d_n := \gamma^{-q_n} d_j$ for $j \in \{1, 2\}$ and a divergent monotonically increasing sequence $(q_n)_n \subset \mathbb{N}$. From the definition of ϕ_0, we get that $\phi_0(d_n)(z_n) = \gamma^{-q_n}(y_j - x_0)$, which converges to 0 as $n \to \infty$.

Thus all the properties have been verified by the tile map.

$\square$

This concludes the proof for the second case.

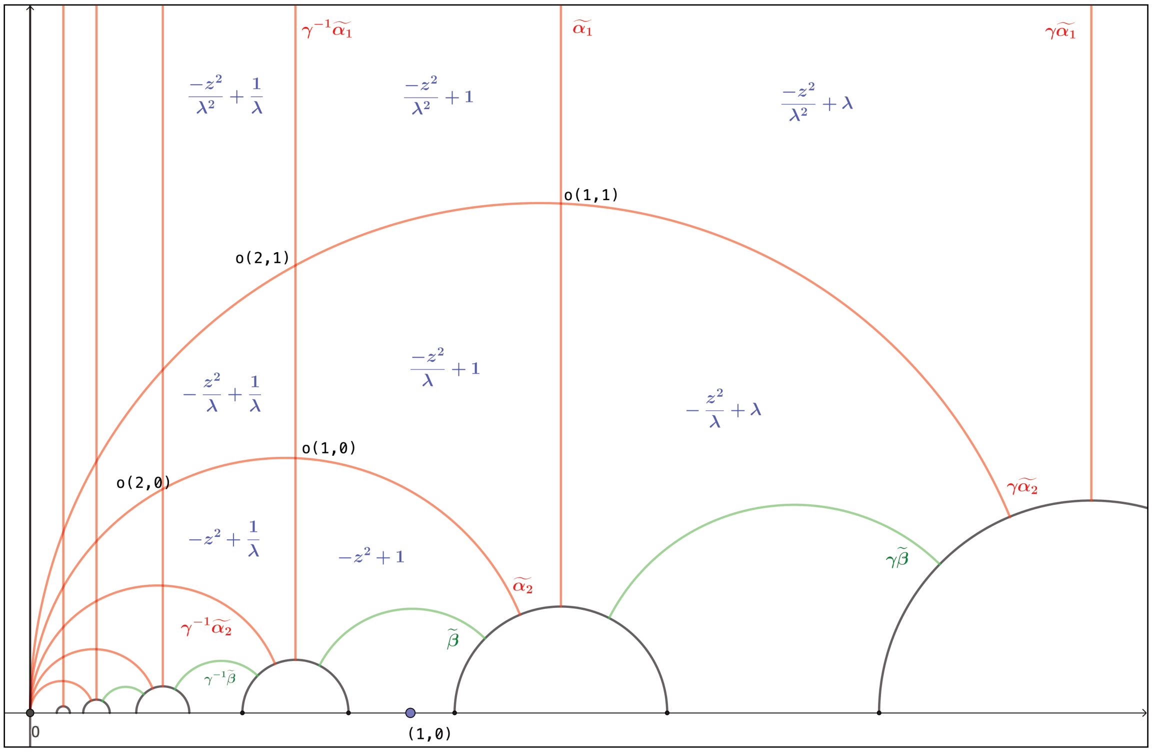


Figure 5.17: Case 3: The lifts of the two intersecting spiraling arcs α_1, α_2 are coloured red, the lifts of the finite arc β are coloured green, the Killing fields of the tiles are coloured in blue.

Case 3: Suppose that α_1, α_2 are infinite arcs spiraling in opposite directions with their endpoints lying on the same edge, denoted by l. When $n \geq 2$, there is a maximal arc β with both its endpoints lying on l such that $[\beta] \in \sigma_1^{(0)} \cap \sigma_2^{(0)}$.

The two spiraling arcs intersect each other inside the surface infinitely many times. So the refined tile set $\mathcal{T}_{\sigma,r}$ and the refined edgeset $\mathcal{E}_{\sigma,r}$, for the refinement $\sigma = \sigma_1 \cup \{\alpha_2\}$, are also countably infinite.

See Fig. 5.17. Then the lifts of l in $\mathbb{U}$ are geodesics of the form $(\lambda^q a, \lambda^q b)$, for some $a, b \in \mathbb{R}_{>0}, q \in \mathbb{Z}$, with $a < b < \lambda a$. We choose a, b such that $b \leq 1 < \lambda a$. When $n \geq 2$, the lifts of the arc β are incident on two consecutive lifts of l. Let $\widetilde{\beta}$ be the lift of β that joins (a, b) and $(\lambda a, \lambda b)$. The lifts of α_1 and α_2 are respectively the vertical geodesic rays, and the semi-circular geodesic rays passing through 0, with their finite endpoints on $(\lambda^q a, \lambda^q b)$.

We shall construct a refined tile map that represents the linear equation:

$$c_{\alpha_1} f_{\alpha_1}(m) + c_{\alpha_1} f_{\alpha_2}(m) + c_\beta f_\beta(m) = 0, \tag{5.28}$$

where $c_{\alpha_1}, c_{\alpha_1} > 0$ and $c_\beta < 0$. When $n = 1$, the arc β is replaced by a spike. In this case, we assume that a lift of the spike is at 1; then the rest of the ideal points corresponding to the spikes are elements of the sequence $\{\lambda^n\}_{n\in\mathbb{Z}}$. In this case we construct a refined tile map representing the equation:

$$c_{\alpha_1} f_{\alpha_1}(m) + c_{\alpha_1} f_{\alpha_2}(m) = 0, \tag{5.29}$$

Properties 5.2.11. A neutral tile map $\phi_0 : \widetilde{\mathcal{T}}_{\sigma,r} \longrightarrow \mathbb{R}_2[z]$ represents the equation (5.28) or (5.29) if and only if it verifies the following properties:

(a) ϕ_0 is Γ-equivariant: for all $\delta \in \widetilde{\mathcal{T}}_{\sigma,r}$, $\phi_0(\gamma \cdot d)(z) = \lambda \phi_0(\delta)(\frac{z}{\lambda})$.

(b) If a tile $\delta \in \widetilde{\mathcal{T}}_{\sigma,r}$ contains a spike at a point $z_0 \in \mathbb{R}$, then $\phi_0(d)$ is a polynomial that vanishes at z_0.

(c) The map is consistent around every lift of every intersection point.

(d) If δ, δ' are two tiles that share an edge carried by the lift $\widetilde{\alpha_1}$ of α_1, such that δ lies to the left of the lift, then $\phi_0(\delta) - \phi_0(\delta')$ is a parabolic Killing field with fixed point at ∞, pointing towards δ. In terms of polynomials, we must have

$$\phi_0(\delta) - \phi_0(\delta') = (z \mapsto A), \text{ for some } A < 0.$$

(e) If δ, δ' are two tiles that share an edge carried by the lift $\widetilde{\alpha_2}$ of α_2 such that δ lies above the semi-circle carrying $\widetilde{\alpha_2}$, then $\phi_0(\delta) - \phi_0(\delta')$ is a parabolic Killing field with fixed point at 0, pointing towards δ. In terms of polynomials, we must have

$$\phi_0(\delta) - \phi_0(\delta') = (z \mapsto Bz^2), \text{ for some } B > 0.$$

(f) If δ, δ' are two tiles that share an edge carried by the lift $\widetilde{\beta}$ of β such that δ lies above the lift, then $\phi_0(\delta) - \phi_0(\delta')$ is a hyperbolic Killing field with axis perpendicular to $\widetilde{\beta}$, directed away from δ.

(g) Let $(z_n)_n$ be a sequence of points in tiles $d_n \in \widetilde{\mathcal{T}_\sigma}$ that converges to a point z on the axis of the isometry γ, then $\phi_0(d_n)$ converges to 0.

Let $\widetilde{\alpha_1}, \widetilde{\alpha_2}$ be the lifts of α_1, α_2 such that their finite ends lie on $(\lambda a, \lambda b)$. Let

$$\widetilde{o}(p,q) := \gamma^q \cdot (\gamma^{-p} \cdot \widetilde{\alpha_1} \cap \widetilde{\alpha_2}), \text{ with } p \in \mathbb{N},\ q \in \mathbb{Z}.$$

Consider the intersection points $\{o(p,0)\}_{p\in\mathbb{N}} = \{\gamma^{-p} \cdot \widetilde{\alpha_1} \cap \widetilde{\alpha_2}\}$. For $p \in \mathbb{N}$, we label the small tiles lying above and below $\widetilde{\alpha_2}$ as d_{2p} and d_{2p+1}, respectively. The four tiles formed around this intersection point $o(p,0)$ are $d_{2p-1}, d_{2p}, d_{2p+1}, d_{2(p+1)}$. The pairs $\{d_{2p-1}, d_{2p+1}\}$ and $\{d_{2p}, d_{2(p+1)}\}$ share a common edge carried by $\gamma^{-p}\widetilde{\alpha_1}$; the pairs $\{d_{2p-1}, d_{2p}\}$ and $\{d_{2p+1}, d_{2(p+1)}\}$ share a common edge carried by $\widetilde{\alpha_2}$. Also, for every $p \in \mathbb{N}$, we have

$$d_{2p+1} = \gamma^{-1} d_{2p}. \tag{5.30}$$

Finally, any small tile is in the Γ-orbit of exactly one of the tiles $\{d_p\}_{p\in\mathbb{N}}$. For $n \geq 2$, let $\widetilde{\beta}$ be the lift of β that joins (a, b) and $(\lambda a, \lambda b)$. We choose the following strip template:

- From the isotopy classes of α_1 and α_2, we choose the geodesic representative whose finite end(s) intersect(s) the boundary perpendicularly.

- From the isotopy class of the lift $\gamma^q\widetilde{\beta}$ of β joining the boundary geodesics $(\lambda^{q-1}a, \lambda^{q-1}b)$ and $(\lambda^q a, \lambda^q b)$, we choose the geodesic arc that is the common perpendicular to these two geodesics. We scale a, b such that the endpoints of $\gamma^q\widetilde{\beta}$ are $(\lambda^q x, 0)$ and $(\frac{\lambda^q}{x}, 0)$, where $a \leq x \leq b$ and $\lambda a < \frac{1}{x} < \lambda b$.
- The waist of the finite arc $\widetilde{\beta}$ is chosen to be its point of intersection with the geodesic $(-1, 1)$ and the waists of $\gamma^n\widetilde{\beta}$ are chosen Γ-equivariantly.

Define $\phi_0 : \widetilde{\mathcal{T}_{\sigma,r}} \longrightarrow \mathbb{R}_2[z]$: for $q \in \mathbb{Z}$ and $p \in \mathbb{N}$,

$$\delta \mapsto \begin{cases} (z \mapsto -\frac{1}{\lambda^q} z^2 + \lambda^{q-p}), & \text{if } \delta = \gamma^q \cdot d_{2p+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Using eq.(5.30), we get that for every $q \in \mathbb{Z}$,

$$\phi_0(\gamma^q \cdot d_{2p}) = (z \mapsto -\frac{1}{\lambda^{q+1}} z^2 + \lambda^{q-p+1}).$$

Fig. 5.17 shows the ϕ_0-images of some tiles as degree two polynomials (coloured in blue).

Lemma 5.2.12. *The neutral map ϕ_0 defined above satisfies the properties 5.2.11 (b)-(g).*

Proof. (a) Let $\delta \in \widetilde{\mathcal{T}_{\sigma,r}}$ be a small tile. Then $\delta = \gamma^q \cdot d_{2p+1}$ for some $q \in \mathbb{Z}$. Then from the definition we have that,

$$\begin{aligned} \phi_0(\gamma \cdot d)(z) &= \phi_0(\gamma^{q+1} \cdot d_{2p+1})(z) \\ &= -\frac{1}{\lambda^{q+1}} z^2 + \lambda^{q+1-p} \\ &= \lambda \left(-\frac{1}{\lambda^q} (\frac{z}{\lambda})^2 + \lambda^{q-p} \right) \\ &= \lambda \phi_0(\gamma^q \cdot d_{2p+1})(\frac{z}{\lambda}) \\ &= (\gamma \cdot \phi_0(\delta))(z). \end{aligned}$$

When $\delta \in \widetilde{\mathcal{T}_{\sigma,r}}$ is not a small tile, the equivariance property holds trivially. Hence ϕ_0 is Γ-equivariant. So the rest of the properties shall be verified only for $q = 0$.

(b) For $n \geq 2$, the only tiles that contain a spike are not small. So the image by ϕ_0 of such a tile is the 0 polynomial. Hence (b) is satisfied.

For $n = 1$, only the tiles in the orbit of d_1 can contain a spike. The spike of the tile $\gamma^q \cdot d_1$ is at (λ^q,0). So $p = 0$ and the definition of ϕ_0 gives that $\phi_0(d_1)(1) = -1 + 1 = 0$. Since, the property is satisfied in this case as well.

(c) Any intersection point of two lifts of $\widetilde{\alpha_1}$ and $\widetilde{\alpha_2}$ is given by $\widetilde{o}(p, q) = \gamma^{-p+q} \cdot \widetilde{\alpha_1} \cap \gamma^q \cdot \widetilde{\alpha_2}$ for some $p \in \mathbb{N}$ and $q \in \mathbb{Z}$. We can assume that $q = 0$. The tiles around $\widetilde{o}(p, 0)$ are $d_{2p-1}, d_{2p}, d_{2p+1}, d_{2(p+1)}$. The pairs $\{d_{2p-1}, d_{2p+1}\}$ and $\{d_{2p}, d_{2(p+1)}\}$ share a common edge carried by $\gamma^{-p} \cdot \widetilde{\alpha_1}$, and the pairs $\{d_{2p-1}, d_{2p}\}$ and $\{d_{2p+1}, d_{2(p+1)}\}$ share a common edge carried by $\widetilde{\alpha_2}$. From the definition of ϕ_0 we get that

$$\begin{aligned}\phi_0(d_{2p+1})(z) - \phi_0(d_{2(p-1)+1})(z) &= \lambda^{-p} - \lambda^{-(p-1)} \\ &= \lambda^{-(p+1)+1} - \lambda^{-p+1} \\ &= \phi_0(d_{2(p+1)})(z) - \phi_0(d_{2p})(z) \\ &= \lambda^{-p}(1-\lambda).\end{aligned}$$

So the tile map is consistent around every intersection point.

(d) The tiles $d_{2p+1}, d_{2(p+1)}$ lie to the left of the arc $\gamma^{-p}\widetilde{\alpha_1}$. From the above calculations and using $\lambda > 1$, we have that $\phi_0(d_{2p+1}) - \phi_0(d_{2(p-1)+1})$ is the negative constant polynomial $z \mapsto \lambda^{-p}(1-\lambda)$.

(e) The tiles $d_{2p}, d_{2(p+1)}$ lie above the arc $\gamma^{-p}\widetilde{\alpha_1}$. Again, from the definition of ϕ_0,

$$\begin{aligned}\phi_0(d_{2p})(z) - \phi_0(d_{2p-1})(z) &= -\frac{1}{\lambda}z^2 + \lambda^{-p+1} + z^2 - \lambda^{-p+1} \\ &= (1-\frac{1}{\lambda})z^2.\end{aligned}$$

Since $\lambda > 1$, the leading coefficient is positive. Hence, the property (e) is verified.

(f) This case is possible only when $n \geq 2$ and $\delta = d_1$. The Killing field $(\phi_0(d_1))(z) = -z^2+1$ is a hyperbolic with attracting fixed point at 1 and repelling fixed point at -1. Since, the endpoints of $\widetilde{\beta}$ are $(x,0)$ and $(\frac{1}{x},0)$, the axis is perpendicular to $\widetilde{\beta}$. Hence the property (f) is satisfied.

(g) Let z_n be a sequence of points converging to a point z on the axis of γ. Then there are monotonically increasing divergent sequences $(p_n)_n, (q_n)_n \subset \mathbb{N}$ such that for all $n \in \mathbb{N}$, the tile $\gamma^{-q_n}d_{2(p_n)+1}$ contains at least one point of the sequence, $q_n \leq p_n$ and $\cup_n z_n \subset \cup_n \gamma^{-q_n} d_{2(p_n)+1}$. From the definition of the tile map ϕ_0 we get that $\phi_0(\gamma^{-q_n}d_{2(p_n)+1})$ tends to 0 as $n \to \infty$.

□

5.3 Local homeomorphsim: Codimension ≥ 2

Let $p \in \mathcal{A}(S)$ such that $\operatorname{codim}(\sigma_p) \geq 2$. In the cases of ideal polygons, punctured polygons and one-holed polygons with spiraling arcs, the arc complex is a sphere, we have that

$$\mathrm{Link}(\sigma_p, \widehat{\mathcal{A}}(\Pi_n^{\pentagon})) \simeq \mathbb{S}^{\operatorname{codim}(\sigma_p)-1.}$$

In order to prove that f is a local homeomorphism, it suffices to show that its restriction to the link of σ_p is a homeomorphism.

Theorem 5.3.1. *Let Π be a hyperbolic surface with boundary. Let $p \in \mathcal{A}(\Pi)$ such that $\operatorname{codim}(\sigma_p) = 2$. Then, $\mathbb{P}f|_{\mathrm{Link}(\sigma_p, \widehat{\mathcal{A}}(\Pi))}$ is a homeomorphism.*

Proof. We shall prove the theorem separately for the different types polygons.

- Ideal n-gons $\Pi = \Pi_n^{\pentagon}$, for $n \geq 6$: The complex $\mathrm{Link}(\sigma_p, \widehat{\mathcal{A}}(\Pi_n^{\pentagon}))$ is either a quadrilateral or a pentagon. So it is enough to show that the continuous map
$$\mathbb{P}f|_{\mathrm{Link}(\sigma_p, \widehat{\mathcal{A}}(\Pi_n^{\pentagon}))} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$$
has degree one.

  Suppose that the link is a pentagon. Let $\{\alpha_i\}_{i=1}^5$ be the five vertices of $\mathcal{A}\left(\Pi_5^{\pentagon}\right)$. Let θ_i, $i \in \mathbb{Z}_5$ be the angle formed at the origin by the vectors $f(\alpha_{i+1})$ and $f(\alpha_i)$, in $\mathfrak{D}(\Pi_5^{\pentagon})$. Then, for every $i = 1, \ldots, 5$, we have $\theta_i \in (0, \pi)$. By theorem 5.2.1, we know that there is a choice of strip template such that $\theta_1 + \theta_2 < \pi$. Also, the sum $\sum_{i=3}^5 \theta_i < 3\pi$. Since, f is a continuous map from the circle to itself, the quantity $\sum_{i=1}^5 \theta_i$ is always a multiple of 2π. Hence, we have $\sum_{i=1}^5 \theta_i = 2\pi$, which implies that the degree of f is 1. Hence, f is a homeomorphism for this choice of strip template. Since, the space of all strip templates is connected and since there is no continuous way of changing the sum of angles from 2π to 4π, we have that f is a homeomorphism for every choice of strip template. This also proves the homeomorphism of the projectivised strip map in the case of the ideal pentagon $\Pi_5^{\pentagon}$. The proof works similarly when the link is a quadrilateral.

- Punctured n-gons $\Pi = \Pi_n^{\odot}$, for $n \geq 5$ or one-holed n-gons $\Pi_n^{\circledcirc}$, for $n \geq 3$: Suppose that the complement $\Pi \setminus \bigcup_{\alpha \in \sigma^{(0)}} \alpha$ has one non-triangulated region. If this region contains the puncture or a hole, then it can be triangulated in six different ways using two disjoint arcs, exactly one of which is always a maximal arc. So we have that $\mathrm{Link}(\sigma, \mathcal{A}(\Pi))$ is a hexagon. Like in the case of ideal polygons, for $i = 1, \ldots, 6$, let θ_i be the angle subtended at the origin by the vectors $f(\alpha_{i+1})$ and $f(\alpha_i)$, in $\mathfrak{D}(\Pi)$. Let $\alpha_1, \alpha_3, \alpha_5$ be the vertices corresponding to the maximal arcs. By Theorem 5.2.1, there exists a strip template such that
$$\theta_i + \theta_{i+1} < \pi, i = 1, \ldots, 5.$$
So the degree of the map is 1. Since the arc complexes of a punctured triangle $\Pi_3^{\odot}$ and a one-holed bigon $\Pi_2^{\circledcirc}$ are also PL-homeomorphic to a hexagon, the homeomorphism of $\mathbb{P}f$ in these cases, is a consequence of the above proof.

  The two cases — exactly one non-triangulated region containing no puncture or hole, and two non-triangulated regions — are treated identically as in the case of ideal polygons.

□

Ideal Square, Punctured bigon, one-holed monogon: When $\Pi = \Pi_4^{\pentagon}$ or $\Pi_2^{\odot}$ or $\Pi_1^{\circledcirc}$, the arc complex $\mathcal{A}(\Pi)$ is a sphere of dimension 0. Let $[\alpha]$ and $[\beta]$ be the two isotopy classes of arcs. If we parametrise the deformation space using the length of α, we see that $f(\alpha)$ corresponds to the origin where as $f(\beta)$ is increases its length. So, we have $\mathbb{P}f(\alpha) \neq \mathbb{P}f(\beta)$, which proves the homeomorphism.

Theorem 5.3.2. *Let $p \in \widehat{\mathcal{A}}(\Pi)$ such that* $\mathrm{codim}(\sigma_p) \geq 2$. *Then, $\mathbb{P}f|_{\mathrm{Link}(\sigma_p, \mathcal{A}(\Pi))}$ is a homeomorphism.*

Proof. The statement is verified for codim $(\sigma_p) = 2$. Suppose that the statement holds for $2, \ldots, d-1$. We need to show that

$$\mathbb{P}f|_{\mathrm{Link}(\sigma_p, \mathcal{A}(\Pi))} : \mathbb{S}^{d-1} \longrightarrow \mathbb{S}^{d-1}$$

is a local homeomorphism. Let $x \in \mathrm{Link}(\sigma_p, \mathcal{A}\,(\Pi))$. Then x is contained in the interior of a simplex σ_x whose codimension in the link is $d - 1 - \dim \sigma_x$, which is less than d. So by the induction hypothesis, the map $\mathbb{P}f|_{\mathrm{Link}(\sigma_p, \mathcal{A}(\Pi))}$ restricted to $\mathrm{Link}(\sigma_x, \mathbb{P}f|_{\mathrm{Link}(\sigma_p, \mathcal{A}(\Pi))})$ is a homeomorphism. This proves that $\mathbb{P}f|_{\mathrm{Link}(\sigma_p, \mathcal{A}(\Pi))}$ is a local homeomorphism. Since $\mathbb{S}^{d-1}$ is compact and simply-connected for $d \geq 3$, it follows that $\mathbb{P}f|_{\mathrm{Link}(\sigma_p, \mathcal{A}(\Pi))}$ is a homeomorphism. $\square$

Thus, for the surfaces $\Pi = \Pi_n^{\pentagon}$ $(n \geq 6)$, $\Pi_n^{\odot}$ $(n \geq 3)$, $\Pi_n^{\circledcirc}$ $(n \geq 3)$, the $\mathbb{P}f : \widehat{\mathcal{A}}(\Pi) \longrightarrow \mathbb{P}^+(T_m\mathfrak{D}(\Pi))$ is a local homeomorphism and by compactness and the simply-connectedness of the sphere $\mathbb{S}^n$, we get that the map is a homeomorphism.

Chapter 6

Strip deformations of general surfaces

In this chapter, we shall parametrise the admissible cones of hyperbolic surfaces with decorated and undecorated spikes, using their pruned arc complexes.

6.1 Hyperbolic surfaces with spikes

Let S_{sp} be a hyperbolic surface with undecorated spikes. Recall that when S_{sp} is orientable (resp. non-orientable), it is of the form $S^{\vec{q}}_{g,n}$ (resp. $T^{\vec{q}}_{h,n}$). Its deformation space is homeomorphic to an open ball of dimension $N_0 = 6g - 6 + 3n + Q$, when S_{sp} is orientable and $N_0 = 3h - 6 + 3n + Q$, when S_{sp} is non-orientable, where Q is the total number of spikes. A point m of the deformation space is expressed as $m = [\rho, \mathbf{x}]$, where $\rho : \pi_1(S_c) \to \mathrm{PGL}(2, \mathbb{R})$ is a holonomy representation of the surface, $\mathbf{x}$ is a Q-tuple of distinct points in $\partial_\infty \mathbb{D}$ representing any one set of lifts of the spikes. Given a metric $m \in \mathfrak{D}(S_{sp})$, its admissible cone $\Lambda(m)$ is the set of all infinitesimal deformations that uniformly lengthen every non-trivial closed geodesic. It is an open convex cone of the vector space $T_m\mathfrak{D}(S_{sp})$. It is a bundle of affine spaces over the admissible cone of the convex core. The fibers correspond to the motion of the spikes. These motions don't affect the uniform lengthening property for geodesic curves.

Recall that the permitted arcs that span the arc complex $\mathcal{A}(S_{sp})$ of the surface are finite and their endpoints lie on its boundary ∂S_{sp}. The set of all permitted arcs is denoted by $\mathcal{K}$. A simplex of the arc complex is big if the arcs corresponding to its 0-skeleton decompose the surface into topological disks. In Chapter 3, the theorems 3.3.2,3.3.4 prove that the pruned arc complex $\widehat{\mathcal{A}}(S_{sp})$ of the surface, given by the union of the interiors of all big simplices, is an open ball of dimension $N_0 - 1$. Any point x in $\widehat{\mathcal{A}}(S_{sp})$ belongs to the interior of a unique big simplex σ_x. Finally, the strip deformations performed along the arcs are of hyperbolic type; their waists and widths are fixed by the choice of a strip template.

Theorem 6.1.1. *Let $S_{sp} = S^{\vec{q}}_{g,n}$ or $T^{\vec{q}}_{h,n}$ be a hyperbolic surface with undecorated spikes. Let $m \in \mathfrak{D}(S_{sp})$ be a metric. Fix a choice of strip template $\{(\alpha_g, p_\alpha, w_\alpha)\}_{\alpha \in \mathcal{K}}$ with respect to m. Then, the infinitesimal strip map $\mathbb{P}f : \widehat{\mathcal{A}}(S_{sp}) \longrightarrow \mathbb{P}^+(T_m\mathfrak{D}(S_{sp}))$ is a homeomorphism on its image $\mathbb{P}^+(\Lambda(m))$.*

Next, we will summarise the necessary facts, that we know from the previous chapters, about a hyperbolic surface with spikes.

Let S^h_{sp} be a hyperbolic surface with decorated spikes. It is obtained from a hyperbolic surface with spikes S_{sp} by decorating every spike with a horoball. Its deformation space is homeomorphic to an open ball of dimension $N_0 = 6g-6+3n+2Q$, when S_{sp} is orientable and $N_0 = 3h-6+3n+2Q$, when S_{sp} is non-orientable, where Q is the total number of spikes. Given a metric $m \in \mathfrak{D}(S^h_{sp})$, its admissible cone $\Lambda(m)$ is the set of all infinitesimal deformations that uniformly lengthen every horoball connection (and hence every closed geodesic). It is an open convex cone of the vector space $T_m\mathfrak{D}(S^h_{sp})$.

Recall that the permitted arcs that span the arc complex $\mathcal{A}\left(S^h_{sp}\right)$ of the surface are finite with their endpoints on the boundary ∂S^h_{sp} as well as infinite with one endpoint on the boundary and another end converging to a spike. Again, the set of all permitted arcs is denoted by $\mathcal{K}$. A simplex of the arc complex is big if the arcs corresponding to its 0-skeleton decompose the surface into topological disks with at most one spike. In Chapter 3, the theorems 3.3.5,3.3.7 prove that the pruned arc complex $\widehat{\mathcal{A}}(S^h_{sp})$ of the surface, given by the union of the interiors of all big simplices, is an open ball of dimension $N_0 - 1$. Any point x in $\widehat{\mathcal{A}}(S_c)$ belongs to the interior of a unique big simplex σ_x. Finally, the strip deformation performed along the finite arcs is of hyperbolic type; its waist and width are fixed by the choice of a strip template. The strip deformations performed along infinite arcs are of parabolic type with fixed point at the spike. We will prove the following parametrisation theorem:

Theorem 6.1.2. *Let $S^h_{sp} = S^{\vec{q},\vec{h}}_{g,n}$ or $T^{\vec{q},\vec{h}}_{h,n}$ be a hyperbolic surface with decorated spikes. Let $m \in \mathfrak{D}(S^h_{sp})$ be a decorated metric. Fix a choice of strip template $\{(\alpha_g, p_\alpha, w_\alpha)\}_{\alpha\in\mathcal{K}}$ with respect to m. Then, the infinitesimal strip map $\mathbb{P}f : \widehat{\mathcal{A}}(S^h_{sp}) \longrightarrow \mathbb{P}^+(T_m\mathfrak{D}(S^h_{sp}))$ is a homeomorphism on its image $\mathbb{P}^+(\Lambda(m))$.*

The rest of the chapter is dedicated to proving Theorem 6.1.1 and Theorem 6.1.2.

6.2 Local homeomorphism of strip maps

6.2.1 Codimension zero

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{2,1}$ be two linearly independent future-pointing light-like vectors whose projective images are denoted by A, B. Then A, B are ideal points; let AB be the unique hyperbolic geodesic joining these two points.

Definition 6.2.1. Given a Killing vector field $X \in \mathfrak{g}$, the *longitudinal motion* X_l imparted by X to the geodesic AB is defined as

$$X_l := \langle \mathbf{v}_X, \frac{\mathbf{a}\wedge\mathbf{b}}{\|(\mathbf{a}\wedge\mathbf{b})\|}\rangle, \tag{6.1}$$

where $\mathbf{v}_X$ is the vector in $\mathbb{R}^{2,1}$ corresponding to X.

The motion is called "longitudinal" because the vector X_l is equal to the component of $X(p)$ along the direction of the line AB, for every point $p \in \mathbb{H}^2$ lying on AB.

Lemma 6.2.2. *Let A, B, C, D be four ideal points ordered in anti-clockwise manner. Let AB and CD be two disjoint geodesics in $\mathbb{D}$. Let E, F, G be the intersection points $\overleftrightarrow{AD} \cap \overleftrightarrow{BC}$, $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ and $\overleftrightarrow{AC} \cap \overleftrightarrow{BD}$, respectively. Then the set of all Killing fields that impart at least the same amount*

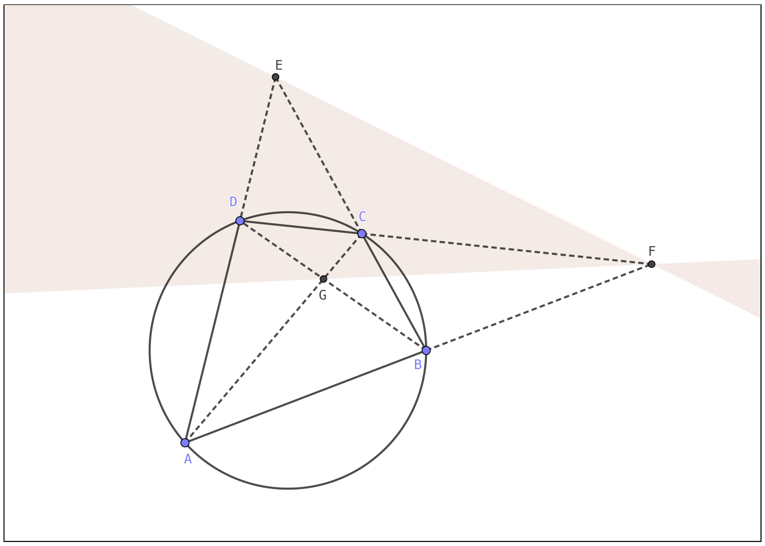


Figure 6.1: The shaded region is the bigon of Lemma 6.2.2

(absolute value) of longitudinal motion to AB as to CD, is given by the bigon bounded by $\overleftrightarrow{GF}$ and $\overleftrightarrow{EF}$, that contains the segment CD.

Proof. Firstly, we prove that the Killing vector fields who projective images are one of E, F, G, impart equal longitudinal motion to both AB and CD. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be future-pointing light-like vectors in the preimages of A, B, C, D such that

$$\begin{aligned} \mathbf{a} &= (-\cos\theta, -\sin\theta), 1), \quad & \mathbf{b} &= (\cos\theta, -\sin\theta, 1) \\ \mathbf{c} &= (\cos\theta, \sin\theta, 1), & \mathbf{d} &= (-\cos\theta, \sin\theta, 1), \end{aligned}$$

where $\theta \in [0, \frac{\pi}{2}]$. The existence of *theta* can be assumed up to applying a hyperbolic isometry to the quadruple (A, B, C, D) because any cross-ratio is realized by some rectangle. Then the points A, B, C, D form a rectangle in the projective plane. Also, we have that

$$E = [(\mathbf{a} \wedge \mathbf{d}) \wedge (\mathbf{c} \wedge \mathbf{b})], \quad F = [(\mathbf{a} \wedge \mathbf{b}) \wedge (\mathbf{c} \wedge \mathbf{d})], \tag{6.2}$$

$$G = [(\mathbf{a} \wedge \mathbf{c}) \wedge (\mathbf{b} \wedge \mathbf{d})]. \tag{6.3}$$

It follows directly from the definition of cross product that any Killing vector field that is a preimage of F imparts no longitudinal motion either to AB or to CD.

By inserting the coordinates of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ in the formulae (6.2),(6.3), we get that

$$\begin{aligned} &\mathbf{a} \wedge \mathbf{c} = 2(-\sin\theta, \cos\theta, 0), \quad \mathbf{b} \wedge \mathbf{d} = 2(-\sin\theta, -\cos\theta, 0), \\ &G = [(0, 0, -4\sin(2\theta))]. \end{aligned}$$

Furthermore, we have that

$$\begin{aligned} \mathbf{a} \wedge \mathbf{b} &= (0, 2\cos\theta, -\sin(2\theta)), \\ \mathbf{c} \wedge \mathbf{d} &= (0, -2\cos\theta, -\sin(2\theta)), \\ \|\mathbf{a} \wedge \mathbf{b}\|^2 = \|\mathbf{c} \wedge \mathbf{d}\|^2 &= (1 + \cos(2\theta))^2. \end{aligned}$$

If we take any Killing vector field $X_G = k((\mathbf{a}\wedge\mathbf{c})\wedge(\mathbf{b}\wedge\mathbf{d}))$ in the preimage of G, $k\in\mathbb{R}^*$, we get that
$$\langle X_G, \frac{\mathbf{a}\wedge\mathbf{b}}{\|\mathbf{a}\wedge\mathbf{b}\|}\rangle = \langle X_G, \frac{\mathbf{c}\wedge\mathbf{d}}{\|\mathbf{c}\wedge\mathbf{d}\|}\rangle = \frac{-4\sin^2(2\theta)}{(1+\cos(2\theta))} = -8\sin^2\theta.$$
Finally, we calculate the coordinates of E:
$$\begin{aligned}\mathbf{a}\wedge\mathbf{d} = r^2(-2\sin\theta, 0, \sin(2\theta)),\quad \mathbf{c}\wedge\mathbf{b} = r^2(2\sin\theta, 0, \sin(2\theta))\\ E = [4r^4(0, \sin\theta\sin(2\theta), 0)].\end{aligned}$$

If $X_E = k'(a\wedge d)\wedge(c\wedge b)$ for some $k'\in\mathbb{R}$, then we have
$$\langle X_E, \frac{\mathbf{a}\wedge\mathbf{b}}{\|\mathbf{a}\wedge\mathbf{b}\|}\rangle = -\langle X_E, \frac{\mathbf{c}\wedge\mathbf{d}}{\|\mathbf{c}\wedge\mathbf{d}\|}\rangle = 4\cos^2\theta.$$

By linearity, we get that the Killing vector fields whose projections lie on the straight line $\overleftrightarrow{EF}$ and $\overleftrightarrow{GF}$ impart equal or opposite longitudinal motions to AB and CD. Now, any Killing field, whose projective image lies on the straight line $c\wedge d$, imparts zero motion to CD and non-zero motion to AB. Since the longitudinal motion on a given line is a linear function of the Killing field, the bigon containing the segment CD is the desired one. □

Theorem 6.2.3. *Given a triangulation σ of a hyperbolic surface with undecorated spikes $S_{sp} = S^{\vec{q}}_{g,n}$ or $T^{\vec{q}}_{h,n}$ with corresponding edge set $\mathcal{E}_\sigma$, the set of infinitesimal strip deformations $B = \{f_e(m) | e \in \mathcal{E}_\sigma\}$ forms a basis of $T_m\mathfrak{D}(S_{sp})$.*

Proof. Let $\phi_0 : \widetilde{\mathcal{T}_\sigma} \longrightarrow \mathfrak{g}$ be a neutral tile map for the triangulation σ (Definition 4.3.4), representing the linear combination
$$\sum_{e\in\mathcal{E}_\sigma} c_e f_e(m) = 0.$$

Let e be a common internal edge of two tiles $d, d' \in \widetilde{\mathcal{T}_\sigma}$. Firstly we show that the longitudinal motions imparted to e by $\phi_0(d)$ and $\phi_0(d')$ are equal. We can decompose the Minkowski space as
$$\mathbb{R}^{2,1} = L_e \oplus {L_e}^\perp,$$
where L_e is the plane $\mathbb{P}^{-1}(\overleftrightarrow{e})$ and $L_e^\perp$ is the $\langle\cdot,\cdot\rangle$-dual of L_e. Then $\phi_0(d) = \mathbf{v}_t + \mathbf{v}_l$ and $\phi_0(d') = \mathbf{v}'_t + \mathbf{v}'_l$ with $\mathbf{v}_t, \mathbf{v}'_t \in L_e$ and $\mathbf{v}_l, \mathbf{v}'_l \in {L_e}^\perp$. Now from the definition of tile maps we have that the vector $\phi(d) - \phi(d')$ is a space-like point of L_e. Hence, $\mathbf{v}_l = \mathbf{v}'_l$.

Next we show that given a neutral tile map, the longitudinal motion along each arc in $\widetilde{\mathcal{E}_\sigma}$ is zero. We prove this by contradiction.

Lemma 6.2.4. *Suppose that e is an arc with maximal absolute value of longitudinal motion. Let $d \in \widetilde{\mathcal{T}_\sigma}$ be a tile with e as an internal edge. Then, the point $[\phi_0(d)]$ is contained in the interior of the projective triangle based at e, containing d.*

Proof. The tile d can be of type 1, 2 or 3 (see Section 3.4). We treat the cases separately:

1. Suppose that d is a tile of type 1, i.e., a quadrilateral with two ideal vertices. Since ϕ_0 is a neutral map, from Corollary 1.5.2 we know that the point $[\phi_0(d)]$ is given by the intersection point of the two tangents to the boundary circle at the two ideal vertices, which lies inside the desired triangle due to the convexity of $\partial_\infty\mathbb{D}$. So in this case the statement holds.

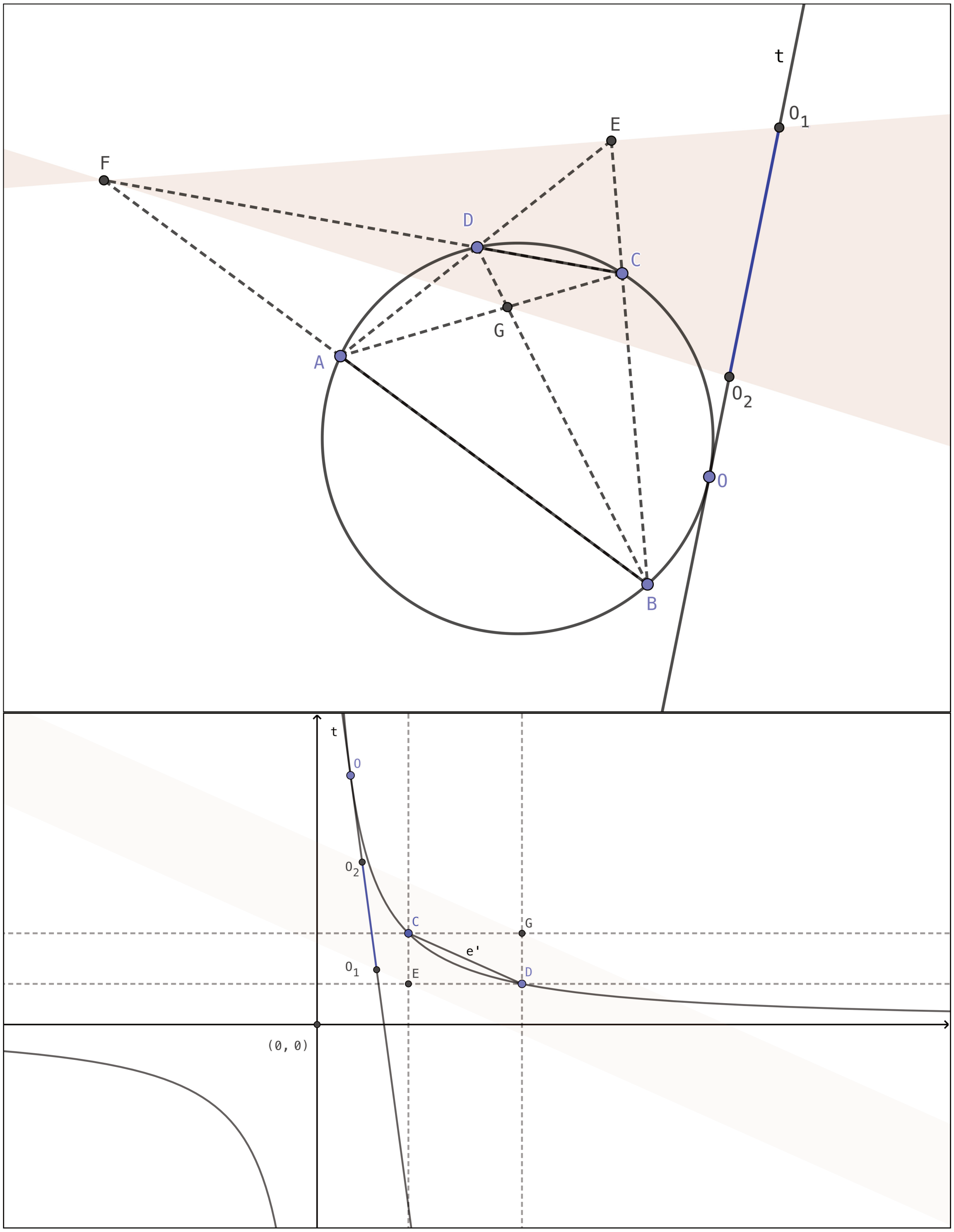


Figure 6.2: (Lemma 6.2.4) $[\phi_0(d)]$ lies on the blue segment $\overline{O_1O_2}$

2. Next we suppose that d is of type 2. Then d is a pentagon with one ideal vertex and exactly two internal edges — one of them is e and the other one is denoted by e'. Label the endpoints of e and e' as A, B, C, D in the trigonometric sense. We shall use the notations as in the hypothesis of Lemma 6.2.2. Then we get that $[\phi_0(d)]$ must lie in the bigon bounded by $\overleftrightarrow{EF}$ and $\overleftrightarrow{GF}$ containing e'. Also, since ϕ_0 is a neutral map, from Corollary 1.5.2 we know that $[\phi_0(d)]$ lies on the tangent, denoted by $\overleftrightarrow{t}$, to $\partial_\infty \mathbb{D}$ at the ideal vertex O of d. Let $O_1 := \overleftrightarrow{t} \cap \overleftrightarrow{EF}$ and $O_2 := \overleftrightarrow{t} \cap \overleftrightarrow{GF}$. Then, $[\phi_0(d)]$ is a point on the segment $\overline{O_1O_2}$ that intersects $\mathbb{D}$. We need to show that the segment $\overline{O_1O_2}$, that intersects $\mathbb{D}$, lies completely inside the projective triangle based at e, containing the tile d.

   Using a projective transformation, we map the line $\overleftrightarrow{AB}$ to $\overleftrightarrow{l_\infty}$ such that the points A, B are mapped to the points at infinity on the x and y axes, respectively. Then the boundary circle $\partial_\infty \mathbb{D}$ is sent to the hyperbola $\{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$ and the points C, D, O lie on the branch of the hyperbola that is contained in the first quadrant. The lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ are mapped to straight lines passing through C, parallel to the x-axis and y-axis, respectively. Similarly, the images of the lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BD}$ are straight lines passing through D, parallel to the x-axis and y-axis, respectively. So it is sufficient to show that the points O_1 and O_2 both lie in the first quadrant.

   Let the coordinates of the points C, D, E, G, O be given by:

$$\begin{array}{ll} C = (m_1, \frac{1}{m_1}) & D = (m_2, \frac{1}{m_2}) \\ E = (m_1, \frac{1}{m_2}) & G = (m_2, \frac{1}{m_1}) \\ O = (m, \frac{1}{m}). & \end{array}$$

   Then,

$$0 < m < m_1 < m_2 < \infty. \tag{6.4}$$

   The slope of $\overleftrightarrow{CD} = -\frac{1}{m_1 m_2}$. Since the point F now lies on $\overleftrightarrow{l_\infty}$, the line joining E and F is the straight line passing through $\overleftrightarrow{EF}$ that is parallel to $\overleftrightarrow{CD}$. Similarly, we have that $\overleftrightarrow{GF} \parallel \overleftrightarrow{CD}$. The equations of the straight lines $\overleftrightarrow{EF}$, $\overleftrightarrow{GF}$ and $\overleftrightarrow{t}$ are given by:

$$\begin{array}{ll} \overleftrightarrow{EF}: & y = -\frac{x}{m_1 m_2} + \frac{2}{m_2}; \\ \overleftrightarrow{GF}: & y = -\frac{x}{m_1 m_2} + \frac{2}{m_1}; \\ \overleftrightarrow{t}: & y = -\frac{x}{m^2} + \frac{2}{m} \end{array} .$$

   Then we get that the intersection points O_1, O_2 have the following coordinates:

$$\begin{array}{ll} O_1 & = (\frac{2(m_1 - m)m_2 m}{m_1 m_2 - m^2}, \frac{2(m_2 - m)}{(m_1 m_2 - m^2)}), \\ O_2 & = (\frac{2(m_2 - m)m_1 m}{m_1 m_2 - m^2}, \frac{2(m_1 - m)}{(m_1 m_2 - m^2)}). \end{array}$$

   It follows from (6.4) that O_1 and O_2 lie in the first quadrant, which proves the lemma in this case.

3. This case is identical to the proof of Claim 3.2(0) in [5].

This proves the lemma. □

Now let e be an internal edge of two neighbouring tiles $d, d' \in \widetilde{\mathcal{T}_\sigma}$, such that e has maximal longitudinal motion. Then, by Lemma 6.2.4, $[\phi_0(d)]$ and $[\phi_0(d')]$ belong to two projective triangles whose interiors are disjoint. If $c_e \neq 0$, then $[\phi_0(d) - \phi_0(d')]$ must be a point in $\overleftrightarrow{e} \setminus \overline{\mathbb{D}}$. But any line joining $[\phi_0(d)]$ and $[\phi_0(d')]$ intersects $\overleftrightarrow{e}$ inside $\mathbb{D}$. So we arrive at a contradiction. Hence the longitudinal motion along every arc is zero.

Finally, we prove that $\phi_0(d) = 0$ for every $d \in \widetilde{\mathcal{T}_\sigma}$. For every type of tile, we find three linearly independent vectors $\{\mathbf{v}_i\}_{i=1}^3 \subset \mathbb{R}^{2,1}$ such that

$$\langle \phi_0(d), \mathbf{v}_i \rangle = 0. \tag{6.5}$$

Consequently, $\phi_0(d)$ will be equal to zero.

- If d is a hexagon, then it has three internal edges, denoted by e_1, e_2, e_3. Choose space-like vectors $\mathbf{v}_i \in \mathbb{P}^{-1}\left\{{e_i}^{\perp}\right\}$ for $i = 1, 2, 3$. Then from above we know that the longitudinal motions along these edges are zero. So eq. (6.5) is satisfied for every i. These vectors are linearly independent because the projective lines carrying these three internal edges do not intersect in $\mathbb{R}\mathrm{P}^2$.

- Let d be a pentagon with two internal edges e_1, e_2 and one ideal vertex $p \in \partial_\infty \mathbb{D}$. Then, we choose space-like vectors $\mathbf{v}_i \in \mathbb{P}^{-1}\left\{{e_i}^{\perp}\right\}$ for $i = 1, 2$ and a future-pointing light-like vector $\mathbf{v}_3 \in \mathbb{P}^{-1}\{p\}$. Then using the longitudinal motion argument we get that the eq. (6.5) is satisfied for $i = 1, 2$. Since, $\phi_0(d)$ fixes the ideal vertex p, eq. (6.5) is satisfied by $\mathbf{v}_3$.

- Let d be a quadrilateral with internal edge e and ideal vertices $p_1, p_2 \in \partial_\infty \mathbb{D}$. Then choose $\mathbf{v_1} \in \mathbb{P}^{-1}\left\{{e_1}^{\perp}\right\}$, and $\mathbf{v_i} \in \mathbb{P}^{-1}\left\{{p_i}^{\perp}\right\}$ for $i = 2, 3$. Then, the eq. (6.5) is satisfied for every i.

Thus we get that $c_e = 0$ for every $e \in \widetilde{\mathcal{E}_\sigma}$, which finishes the proof of the theorem. □

Now we prove the same result for surfaces with decorated spikes.

6.2.2 Surfaces with decorated spikes

Theorem 6.2.5. *Given a triangulation σ of a hyperbolic surface with decorated spikes $S_{sp}^h = S_{g,n}^{\vec{q},\vec{h}}$ or $T_{h,n}^{\vec{q},\vec{h}}$ with corresponding edge set $\mathcal{E}_\sigma$, the set of infinitesimal strip deformations $B = \{f_e(m) | e \in \mathcal{E}_\sigma\}$ forms a basis of $T_m\mathfrak{D}(S_{sp}^h)$.*

Proof. We will follow the same strategy as in the case of hyperbolic surfaces with undecorated spikes: we start with a neutral tile map $\phi_0 : \widetilde{\mathcal{T}_\sigma} \longrightarrow \mathfrak{g}$ for the triangulation σ, representing the linear combination

$$\sum_{e \in \mathcal{E}_\sigma} c_e f_e(m) = 0,$$

and we show that the maximal longitudinal motion along any arc of the triangulation is zero.

Let e be a common internal edge of two tiles $d, d' \in \widetilde{\mathcal{T}_\sigma}$. From the definition of tile maps we know that when e is spike-to-edge, the difference $\phi_0(d) - \phi_0(d')$ is a light-like point in the plane L_e, and when e is an edge-to edge arc, the difference is a space-like point in L_e. Using the same argument as in the proof of Theorem 6.2.3, we get that the longitudinal motions imparted to e by $\phi_0(d)$ and $\phi_0(d')$ are equal. Moreover, when e is spike-to-edge, the Killing fields $\phi_0(d), \phi_0(d')$

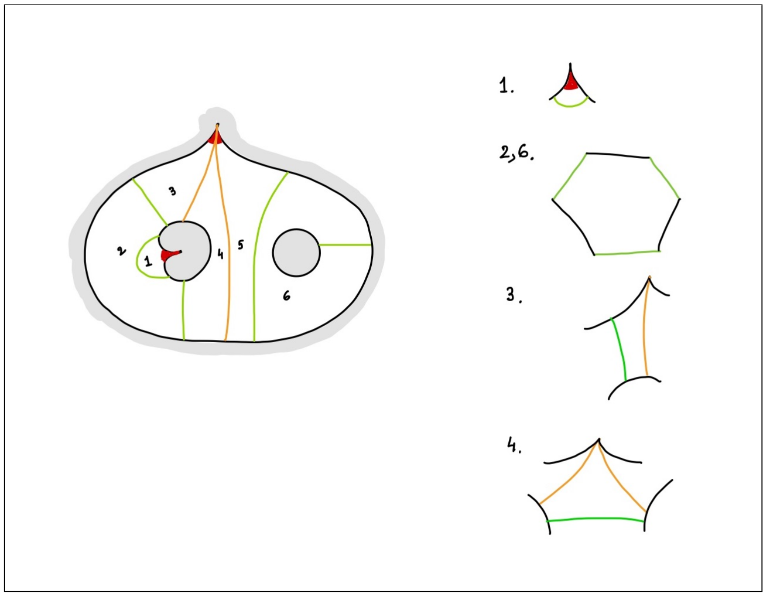


Figure 6.3: Tiles of a triangulation of $S_{0,3}^{(1,1,0)}$

are parabolic preserving the spike as well as the horoball decoration. So the longitudinal motion along e is zero in this case. It remains to show that the maximal longitudinal motion along any edge-to-edge arc is zero. Like in the case of surfaces with undecorated spikes, we shall now prove the following analogue of Lemma 6.2.4.

Lemma 6.2.6. *Suppose that e is an edge-to-edge arc with maximal longitudinal motion. Let $d \in \widetilde{\mathcal{T}_\sigma}$ be a tile with e as an internal edge. Then, the point $[\phi_0(d)]$ is contained in the interior of the projective triangle based at e, containing d.*

Proof. Once again there are three types of tiles. Figure 6.3 shows the different tiles formed after the triangulation of the surface.

- Suppose that d is of type one, i.e., it is a triangle with one decorated ideal vertex and one internal edge which is edge-to-edge. (Topmost tile in Fig. 6.3). The point $[\phi_0(d)]$ is given by the ideal vertex, which lies inside the desired triangle.
- Suppose that d is of type two, i.e., it is a quadrilateral with one decorated ideal vertex and two internal edges, one of which is edge-to-edge (Third tile from the top in Fig. 6.3). Once again, the point $[\phi_0(d)]$ is given by the ideal vertex, which lies inside the desired triangle.
- Suppose that d is of type three. Firstly, we suppose that it is a pentagon with one decorated ideal vertex and three internal edges, one of which is edge-to-edge (Fourth tile in Fig. 6.3). Once again, the point $[\phi_0(d)]$ is given by the ideal vertex, which lies inside the desired triangle. Secondly, if d is a hexagon with three edge-to-edge arcs (second tile from the top in Fig. 6.3), then the argument is exactly the same as in the proof for surfaces with undecorated spikes.

This finishes the proof of Lemma 6.2.6. $\square$

Now let e be an internal edge of two neighbouring tiles $d, d' \in \widetilde{\mathcal{T}_\sigma}$, such that e has maximal (non-zero) longitudinal motion. So e is an edge-to-edge arc. By Lemma 6.2.4, the points $[\phi_0(d)]$ and $[\phi_0(d')]$ belong to two projective triangles whose interiors are disjoint. If $c_e \neq 0$, then $[\phi_0(d) - \phi_0(d')]$ must be a point in $\overleftrightarrow{e} \setminus \overline{\mathbb{D}}$. But any line joining $[\phi_0(d)]$ and $[\phi_0(d')]$ intersects $\overleftrightarrow{e}$ inside $\mathbb{D}$. So we arrive at a contradiction. Hence, the longitudinal motion along every edge-to-edge arc is zero.

Now we prove that $\phi_0(d) = 0$ for every $d \in \widetilde{\mathcal{T}_\sigma}$. Every tile d of the triangulation has an internal edge-to-edge arc. Suppose that d has a decorated ideal vertex p. Then, either $\phi_0(d) = 0$ or $\phi_0(d) \in \mathbb{P}^{-1}\{p\}$. Let e be an internal edge-to-edge arc, with endpoints $A, B \in \partial_\infty \mathbb{D}$. Let $\mathbf{a} \in \mathbb{P}^{-1}\{A\}$ and $\mathbf{b} \in \mathbb{P}^{-1}\{B\}$ be future pointing light-like vectors. Since the longitudinal motion along e is zero we have that

$$\langle \phi_0(d), \frac{\mathbf{a} \wedge \mathbf{b}}{\|\mathbf{a} \wedge \mathbf{b}\|} \rangle = 0,$$

which is possible only if $\phi_0(d) = 0$. Finally suppose that d is a hexagon. Then it has three internal edges, denoted by e_1, e_2, e_3. Choose space-like vectors $\mathbf{v}_i \in \mathbb{P}^{-1}\left\{{e_i}^\perp\right\}$ for $i = 1, 2, 3$. Then from above we know that the longitudinal motions along its three internal (pairwise disjoint in $\mathbb{D}$) edges are zero. So $\phi_0(d) = 0$. $\square$

6.2.3 Codimension one and two

In this section we show that the projectivised strip map $\mathbb{P}f : \widehat{\mathcal{A}}(S_{sp}) \longrightarrow \mathbb{P}^+(T_m\mathfrak{D}(S_{sp}))$ is a local homeomorphism around points belonging to strata of codimension 1 and 2. Recall from Section 4.4 of Chapter 4 that we need to prove Theorems 4.4.5,4.4.7 which we restate here.

Theorem 6.2.7. *Let S_{sp} be a hyperbolic surface with spikes and $m \in \mathfrak{D}(S_{sp})$ be a metric. Let $\sigma_1, \sigma_2 \in \mathcal{A}(S_{sp})$ be two top-dimensional simplices such that*

$$\operatorname{codim}(\sigma_1 \cap \sigma_2) = 1 \text{ and } \operatorname{int}(\sigma_1 \cap \sigma_2) \subset \widehat{\mathcal{A}}(S_{sp}).$$

Then,

$$\operatorname{int}(\mathbb{P}f(\sigma_1)) \cap \operatorname{int}(\mathbb{P}f(\sigma_2)) = \varnothing. \tag{6.6}$$

Moreover, there exists a choice of strip template such that $f(\sigma_1) \cup f(\sigma_2)$ is convex in $\mathbb{P}^+(T_m\mathfrak{D}(S_{sp}))$.

Proof. Let $\mathcal{E}_{\sigma_1}$ and $\mathcal{E}_{\sigma_2}$ be the edge sets of σ_1 and σ_2 respectively. Since $\operatorname{codim}(\sigma_1 \cap \sigma_2) = 1$, we have that $\mathcal{E}_{\sigma_1} \setminus \mathcal{E}_{\sigma_2}$ (resp. $\mathcal{E}_{\sigma_2} \setminus \mathcal{E}_{\sigma_1}$) has exactly one arc, denoted by α_1 (resp. α_2). Let $\widetilde{\mathcal{E}}_{\sigma,r}$ be the refined edgeset of $\widetilde{\mathcal{E}}_{\sigma_1}$ obtained by considering the refinement $\sigma := \sigma_1 \cup \{\alpha_2\}$. Let $\widetilde{\mathcal{T}}_{\sigma,r}$ be the refined tile set of $\widetilde{\mathcal{T}_\sigma}$.

Firstly, we show that the arcs α_1 and α_2 intersect exactly once. From Chapter 3, we know that the arcs of σ_1 as well as σ_2 decompose the surface into topological disks. As a result the elements of the tile set corresponding to the refinement σ are also contractible. Now if the geodesic arcs α_1 and α_2 intersected non-trivially at least twice, there will be a tile in $\mathcal{T}_{\sigma,r}$ which is not homeomorphic to a disk. So the two arcs can intersect only once.

We choose an embedding of the universal cover of S_{sp} in the upper half plane so that the point ∞ is distinct from the endpoints of all the arcs in the lifted edgesets $\widetilde{\mathcal{E}}_{\sigma_1}, \widetilde{\mathcal{E}}_{\sigma_2}$. Then every geodesic arc used in the triangulation is carried by a semi-circle.

Let o be the point of intersection of α_1, α_2. Let $\widetilde{o}$ be a lift of the point o. Then $\widetilde{o} = \widetilde{\alpha_1} \cap \widetilde{\alpha_2}$ for two lifts of α_1 and α_2 respectively. There are four tiles formed around $\widetilde{o}$, denoted by d_j, for

$j = 1, \dots, 4$. For each $j \in \{1, \dots, 4\}$, the tile d_j is either a quadrilateral with an ideal vertex and exactly two arc edges carried by $\widetilde{\alpha_1}$ and $\widetilde{\alpha_2}$, or it is a pentagon with exactly three arc edges carried by $\widetilde{\alpha_1}, \widetilde{\alpha_2}$ and a third arc $\widetilde{\beta_j}$, which is a lift of an arc $\beta_j \in \mathcal{E}_{\sigma_1} \cap \mathcal{E}_{\sigma_2}$. Let $\mathcal{J} \subset \{1, \dots, 4\}$ be such that the tile d_j is a pentagon if and only if $j \in \mathcal{J}$. For $i = 1, 2$, let x_i be the centre of the semi-circle containing $\widetilde{\alpha_i}$. For $j = 1, \dots, 4$, let y_j denote the ideal vertex of d_j or the centre of the semi-circle containing $\widetilde{\beta_j}$. For each j, the tile d_j is either a quadrilateral with exactly one ideal vertex and two internal edges contained in α_1 and α_2, or it is a pentagon with exactly three internal edges: α_1, α_2 and a third arc $\beta_j \in \mathcal{E}_{\sigma_2} \cap \mathcal{E}_{\sigma_1}$. We shall construct a tile map corresponding to the following linear combination:

$$c_{\alpha_1} f_{\alpha_1}(m) + c_{\alpha_2} f_{\alpha_2}(m) + \sum_{j \in \mathcal{J}} c_{\beta_j} f_{\beta}(m) = 0, \tag{6.7}$$

with $c_{\alpha_1}, c_{\alpha_2} > 0$ and $c_{\beta_j} < 0$ for every $j \in \mathcal{J}$. In other words, we define a neutral tile map $\phi_0 : \widetilde{\mathcal{T}}_{\sigma,r} \longrightarrow \mathbb{R}_2[z]$ that is Γ-equivariant and satisfies Properties (1)-(4), as defined in Chapter 5.

Suppose that the endpoints of $\widetilde{\alpha_1}$ lie on the boundary geodesics (a, b) and (e, f) and those of $\widetilde{\alpha_2}$ lie on (c, d) and (g, h) such that the following inequalities hold:

$$a < b \leq c < d \leq e < f \leq g < h. \tag{6.8}$$

For $j = 1, \dots, 4$, define

$$\begin{aligned} \phi_0 \quad &: \widetilde{\mathcal{T}}_{\sigma,r} \longrightarrow \quad \mathbb{R}_2[z] \\ &d \longmapsto \quad \begin{cases} P_j := (z \mapsto a_j(z - y_j)), & \text{if } d = d_j \\ (\gamma \cdot z \mapsto \frac{\mathrm{d}\gamma(z)}{\mathrm{d}z} P_j(z))), & \text{if } d = \gamma \cdot d_j, \gamma \in \Gamma \smallsetminus \{Id\} \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where

$$a_1 = \tfrac{x_1 - y_4}{x_1 - y_1}, \quad a_2 = \tfrac{(x_1 - y_4)(x_2 - y_4) - (y_4 - y_1)(y_4 - y_3)}{(x_1 - y_1)(x_2 - y_3)} \quad a_3 = \tfrac{x_2 - y_4}{x_2 - y_3} \quad a_4 = 1.$$

So, $a_1, a_2, a_3 < 0$.

The tile map ϕ_0 is Γ-equivariant by definition. From the previous chapter we get that it verifies all the properties except possibly for a sub-case of (4) when there exist $j, j' \in \mathcal{J}$ and $\gamma \in \Gamma \smallsetminus \{Id\}$ such that $\widetilde{\beta'_j} = \gamma \cdot \widetilde{\beta_j}$. Without loss of generality, we suppose that $j < j'$. We shall now prove that ϕ_0 verifies this property for this case as well.

The geodesic arc $\gamma \cdot \beta_j$ is the common internal edge of $d_{j'}$ and $\gamma \cdot d_j$. Suppose that $d_{j'}$ lies above $\gamma \cdot d_j$. Since $j < j'$, we have that $j, j' \in \{1, 2, 3\}$. From the definition of ϕ_0, we get that $\phi_0(d_j), \phi_0(d_{j'})$ are hyperbolic Killing fields whose axes are perpendicular respectively to $\beta_j, \beta_{j'}$, with attracting fixed points at $y_j, y_{j'}$. Since both d_j and $d_{j'}$ lie above β_j and $\beta_{j'}$, this means that the two Killing fields are directed towards these edges. Now γ maps β_j to $\beta_{j'}$ and $\gamma \cdot d_j$ lies below it. Since ϕ_0 is Γ-equivariant, $\phi_0(\gamma \cdot d_j)$ is a hyperbolic Killing field directed towards $\gamma \cdot \beta_j$ and hence it points upwards. So the difference vector $\phi_0(d_{j'}) - \phi_0(\gamma \cdot d_j)$ is hyperbolic, normal to $\beta_{j'}$, directed downwards and towards $\gamma \cdot d_j$, as required. Finally, we suppose that $d_{j'}$ lies below $\gamma \cdot d_j$. So $j' = 4$ and $j \in \{1, 2, 3\}$. See Figure 6.4. Again from the definition of ϕ_0 we get that $\phi_0(d_4)$ is a hyperbolic Killing field with axis perpendicular to β_4 and attracting fixed point at ∞. So it is directed upwards. In this case, the tile $\gamma \cdot d_j$ lies above β_4, the Killing field $\phi_0(\gamma \cdot d_j)$ is directed downwards. Hence the difference vector field $\phi_0(d_{j'}) - \phi_0(\gamma \cdot d_j)$ is hyperbolic, directed upwards and towards $\gamma \cdot d_j$, as required. This finishes the proof of the theorem. $\square$

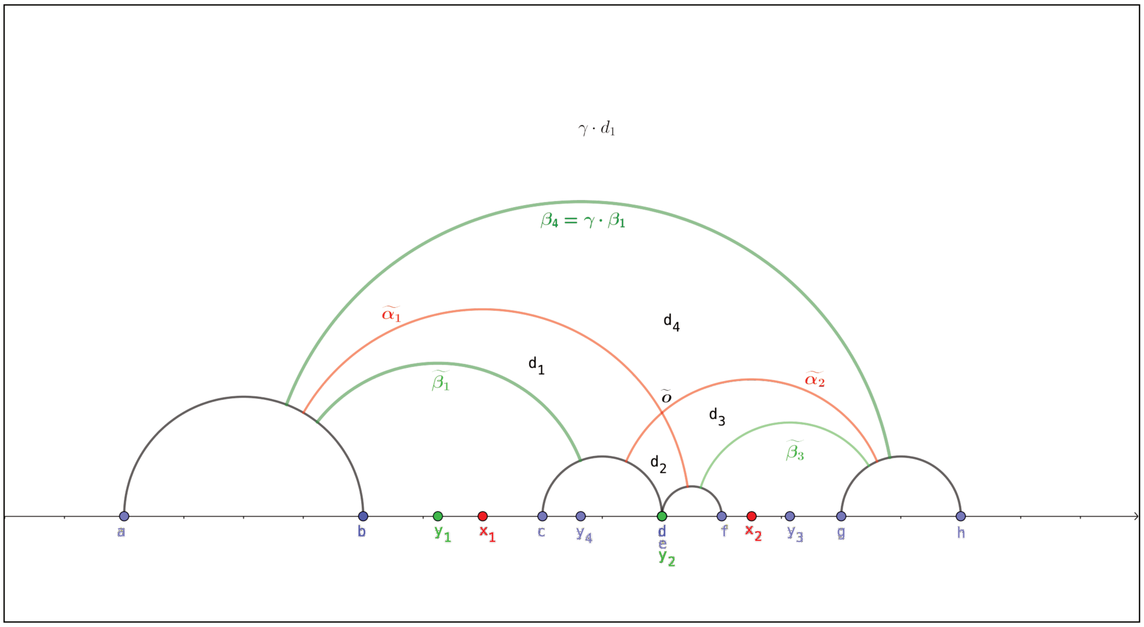


Figure 6.4: $d = e = y_2$ and $j = 1, j' = 4$

Theorem 6.2.8. *Let S^h_{sp} be a hyperbolic surface with decorated spikes and $m \in \mathfrak{D}(\Pi)$ be a metric. Let $\sigma_1, \sigma_2 \in \mathcal{A}\left(S^h_{sp}\right)$ be two top-dimensional simplices such that* $\operatorname{codim}(\sigma_1 \cap \sigma_2) = 1$. *Then,*

$$\operatorname{int}(\mathbb{P}f(\sigma_1)) \cap \operatorname{int}(\mathbb{P}f(\sigma_2)) = \varnothing \tag{6.9}$$

where int *denotes the interior of a simplex. Moreover, there exists a choice of strip template such that $f(\sigma_1) \cup f(\sigma_2)$ is convex in $\mathbb{P}^+(T_m\mathfrak{D}(S^h_{sp}))$.*

Proof. Let $\mathcal{E}_{\sigma_1}$ and $\mathcal{E}_{\sigma_2}$ be the edge sets of σ_1 and σ_2 respectively. Let $\mathcal{E}_{\sigma_1} \backslash \mathcal{E}_{\sigma_2} = \{\alpha_1\}$ and $\mathcal{E}_{\sigma_2} \backslash \mathcal{E}_{\sigma_1} = \{\alpha_2\}$. Let $\widetilde{\mathcal{E}}_{\sigma,r}$ be the refined edgeset of $\widetilde{\mathcal{E}}_{\sigma_1}$ obtained by considering the refinement $\sigma := \sigma_1 \cup \{\alpha_2\}$. Let $\widetilde{\mathcal{T}}_{\sigma,r}$ be the refined tile set of $\widetilde{\mathcal{T}_\sigma}$. We have two possibilities for α_1, α_2:

1. One of the two arcs, say α_1, is finite, while the other one, α_2, is an infinite spike-to-edge arc. There is only one point of intersection, denoted by o. We choose an embedding of the universal cover of the surface in the upper half plane such that one of the lifts of the spike to which α_2 converges, is at ∞. Let $\widetilde{\alpha_2}$ be the lift of α_2 that passes through ∞ and let $\widetilde{\alpha_1}$ be the lift of α_1 that intersects $\widetilde{\alpha_2}$ at a point $\widetilde{o}$, which is a lift of o in $\mathbb{U}$. See Figures 6.5, 6.6. The four tiles formed around $\widetilde{o}$ are labeled $d_1, \ldots, d_4$ in the trigonometric sense (as in the proof of Theorem **??**) such that the tiles d_3, d_4 lie above $\widetilde{\alpha_1}$, while d_1, d_2 lie below $\widetilde{\alpha_1}$. The horizontal line is the horoball decoration at ∞. For $j \in \{1, 2\}$, the tile d_j is a pentagon with exactly three arc edges carried by $\widetilde{\alpha_1}, \widetilde{\alpha_2}$ and a third arc $\widetilde{\beta_j}$, which is a lift of an arc $\beta_j \in \mathcal{E}_{\sigma_1} \cap \mathcal{E}_{\sigma_2}$. For $j \in \{3, 4\}$, the tile d_j is either

   - a quadrilateral with a decorated ideal vertex and exactly three internal edges carried by $\widetilde{\alpha_1}, \widetilde{\alpha_2}$ and a third arc $\widetilde{\beta_j}$, which is a lift of an arc $\beta_j \in \mathcal{E}_{\sigma_1} \cap \mathcal{E}_{\sigma_2}$ (Fig. 6.5),

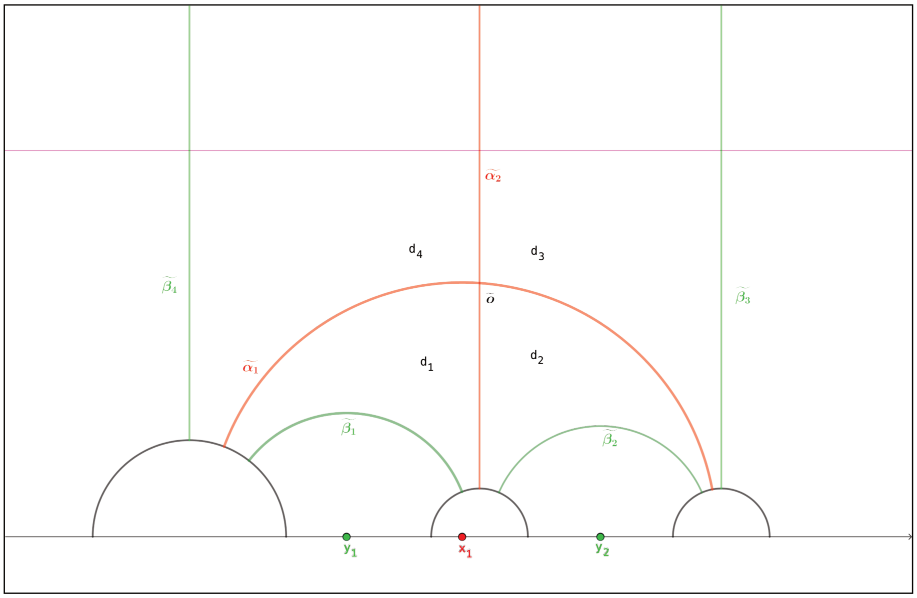


Figure 6.5

- or a triangle with exactly two arc edges carried by $\widetilde{\alpha_1}, \widetilde{\alpha_2}$ (Fig. 6.6).

For $j = 1, 2$, let y_j be the centre of the semi-circle carrying $\widetilde{\beta_j}$. Let x_0 be the centre of the semi-circle carrying $\widetilde{\alpha_1}$. We define the tile map

$$\begin{aligned}\phi_0 \quad &: \widetilde{\mathcal{T}}_{\sigma,r} \longrightarrow \quad \mathbb{R}_2[z] \\ d \mapsto &\quad \begin{cases} P_j := (z \mapsto a_i(z - y_i)), & \text{if } d = d_j, \qquad\qquad i = 1,2, \\ P_3 := (z \mapsto a_3), & \text{if } d = d_3 \\ P_4 := (z \mapsto a_4), & \text{if } d = d_4 \\ (\gamma(z) \mapsto \frac{\mathrm{d}\gamma(z)}{\mathrm{d}z} P_j(z))), & \text{if } d = \gamma \cdot d_j, \gamma \in \Gamma \smallsetminus \{Id\} \\ 0, & \text{otherwise,} \end{cases}\end{aligned}$$

where

$$a_1 = a_2 = -1, \quad a_3 = y_2 - x_0 > 0, \quad a_4 = y_1 - x_0 < 0.$$

This is a Γ-equivariant map by construction and satisfies the properties 5.2.5(1)-(4), as defined in Chapter 5. We shall now verify the sub-cases of (5) and (6) when there exist $j, j' \in \mathcal{J}$ and $\gamma \in \Gamma \smallsetminus \{Id\}$ such that $\widetilde{\beta'_j} = \gamma \cdot \widetilde{\beta_j}$. The only way to identify $\widetilde{\beta_3}$ and $\widetilde{\beta_4}$ is via a parabolic element in $\mathrm{PSL}(2, \mathbb{R})$ that has ∞ as fixed point. But such elements are not present in Γ. Hence, the verification of Property 5.2.5(6), as done in the proof of Theorem **??**, suffices for our general case as well. Thus we are left with the sub-case $\gamma \cdot \beta_1 = \beta_2$. Note that for this to happen, the endpoints of $\widetilde{\beta_1}$ must lie on two geodesics that do not intersect in $\overline{\mathbb{D}}$. Fig. 6.7 shows one such example. From the definition of ϕ_0, we get that $\phi_0(d_1), \phi_0(d_2)$ are hyperbolic Killing fields whose axes are perpendicular respectively to $\widetilde{\beta_1}, \widetilde{\beta_2}$, with attracting fixed points

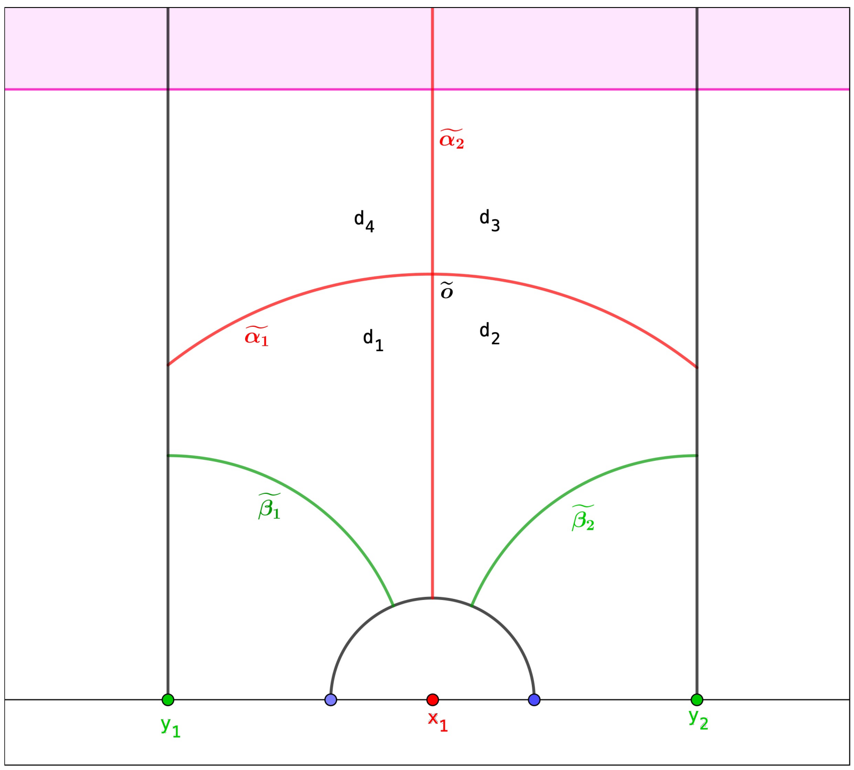


Figure 6.6: $\widetilde{\beta}_2 = \gamma \cdot \widetilde{\beta_1}$

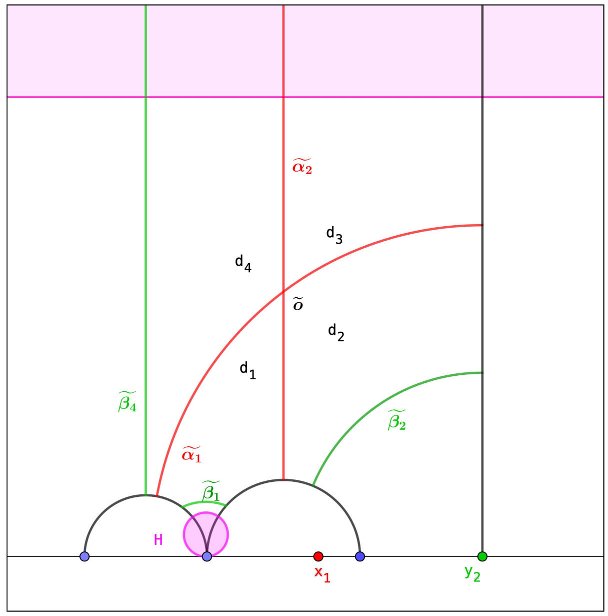


Figure 6.7

at y_1, y_2. Since both d_1 and d_2 lie above β_1 and β_2, this means that the two Killing fields are directed towards these edges. Now γ maps β_1 to β_2 and $\gamma \cdot d_1$ lies below β_2. Since ϕ_0 is Γ-equivariant, $\phi_0(\gamma \cdot d_1)$ is a hyperbolic Killing field directed towards $\gamma \cdot \beta_1$ and hence it points upwards. So the difference vector $\phi_0(d_2) - \phi_0(\gamma \cdot d_1)$ is hyperbolic, directed downwards and towards $\gamma \cdot d_1$, as required. This finishes the proof in this case.

2. Both the geodesic arcs α_1 and α_2 are finite. See Figure 6.8. There are six possibilities. We define the tile map as in the proof of Theorem **??**. It is possible to identify $\widetilde{\beta_j}$ with $\widetilde{\beta_{j'}}$ by an element $\gamma \in \Gamma \smallsetminus \{Id\}$ if the endpoints of $\widetilde{\beta_j}$ lie on two geodesics that do not intersect in $\overline{\mathbb{D}}$ (first column and bottom of the second column in Fig. 6.8.

□

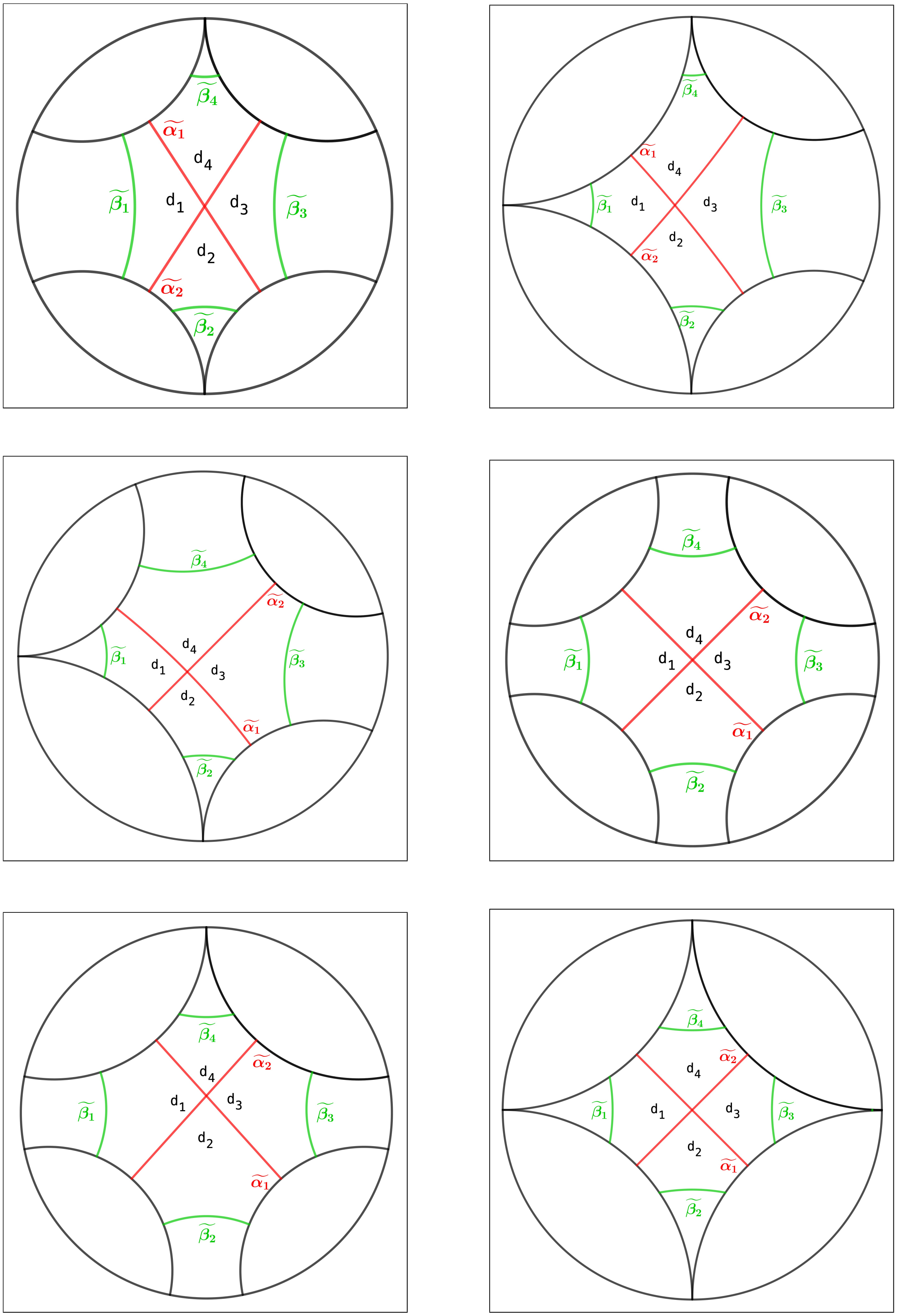
6.2.4 Codimension ≥ 2

The proofs for the local homeomorphism of $\mathbb{P}f$ around points belonging to the interiors of simplices in the pruned arc complex, in the cases of surfaces with decorated and undecorated spikes is identical to that in the cases of decorated and undecorated polygons in Chapter **??**.

6.2.5 Properness

Surfaces with decorated spikes

In this section we prove that the projectivised strip map $\mathbb{P}f$is proper in the case of surfaces with undecorated spikes.



$\widetilde{\beta_4}$
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d_1
d_3
$\widetilde{\beta_3}$
d_2
$\widetilde{\alpha_1}$
$\widetilde{\beta_2}$

Figure 6.8

Theorem 6.2.9. *Let S^h_{sp} be a hyperbolic surface with decorated spikes. Let $m \in \mathfrak{D}(S^h_{sp})$. Then the projectivised strip map $\mathbb{P}f : \widehat{\mathcal{A}}(S^h_{sp}) \longrightarrow \mathbb{P}^+(\Lambda(m))$ is proper.*

Proof. Let $(x_n)_n$ be a sequence in the pruned arc complex $\widehat{\mathcal{A}}(S^h_{sp})$ such that $x_n \to \infty$: for every compact K in $\widehat{\mathcal{A}}(S^h_{sp})$, there exists an integer $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $x_n \notin K$. We want to show that $\mathbb{P}f(x_n) \to \infty$ in the projectivised admissible cone $\mathbb{P}^+\Lambda(m)$. Recall that the admissible cone $\Lambda(m)$ is an open convex subset of $T_m\mathfrak{D}(S)$. Its boundary $\partial\Lambda(m)$ consists of $\vec{0} \in T_m\mathfrak{D}(S)$ and is supported by hyperplanes (and their limits) given by the kernels of linear functionals $\mathrm{d}l_\beta : T_m\mathfrak{D}(S) \longrightarrow \mathbb{R}$, where β is a horoball connection or a non-trivial closed geodesic of the surface. It suffices to show that $f(x_n)$ tends to infinity (in the sense of leaving every compact subset) inside $\Lambda(m)$ but stays bounded away from $\vec{0}$ so that $\mathbb{P}f(x_n)$ tends to infinity in $\mathbb{P}^+\Lambda(m)$.

From Lemma 4.3.11, we get that there exists a constant $M' > 0$ depending on the normalisation such that for every closed geodesic γ and every point $x \in \mathcal{A}\left(S^h_{sp}\right)$ the following inequality holds

$$\sum_{p \in \gamma \cap \mathrm{supp}(x)} w_x(p) \leq M' l_\gamma(m), \tag{6.10}$$

where $w_x : \mathrm{supp}\,(x) \to \mathbb{R}_{>0}$ is the strip width function. Let $K(S_\heartsuit)$ be a compact neighbourhood of the convex core $S_\heartsuit$ of the surface. Then every arc has bounded length outside $K(S_\heartsuit)$: there exists $C > 0$ such that for every geodesic arc α, $l_{\alpha \smallsetminus K(S_\heartsuit)} < C$. Given $\epsilon > 0$, we get a constant $M > 0$ from Lemma 4.3.12 applied to $\frac{\epsilon}{M'}$. Define

$$\mathcal{K}_M := \{\alpha \in \mathcal{K} \mid l_\alpha(m) \leq M + C\}.$$

Since there exist only finitely many geodesic arcs in S^h_{sp} (and hence finitely many permitted arcs) up to any given length, we have that $\mathcal{K}_M$ is finite. Consequently, the set Σ_M of simplices in $\mathcal{A}\left(S^h_{sp}\right)$ spanned by the arcs in $\mathcal{K}_M$ is also finite. We will show that there exists $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$, there exists a closed geodesic $\gamma(n)$ that satisfies:

$$\frac{\mathrm{d}l_{\gamma(n)}(f(x_n))}{l_{\gamma(n)}(m)} < \epsilon. \tag{6.11}$$

It is enough to prove the above inequality for two types of subsequences of $(x_n)_n$— a subsequence whose terms live in one of the finitely many simplices in Σ_M and a subsequence whose every term lies in simplices outside Σ_M. Finally, in both the cases we show that $f(x_n)$ does not converge to $\vec{0}$.

Case 1: Consider a subsequence $(y_n)_n$ of $(x_n)_n$ such that $y_n \in \sigma$ spanned by the arcs $\{\alpha_1, \dots, \alpha_N\} \subset \Sigma_M$, where $N \leq \dim \mathfrak{D}(S^h_{sp})$. Since $y_n \to \infty$, it has a subsequence that converges to a point $y \in \mathcal{A}\left(S^h_{sp}\right) \smallsetminus \widehat{\mathcal{A}}(S^h_{sp})$. So y_n is of the form:

$$y_n = \sum_{i=1}^N t_i(n)[\alpha_i], \text{with } t_i(n) \in (0,1] \text{ and } \sum_{i=1}^N t_i(n) = 1,$$

and the limit point y is then given by:

$$y = \sum_{i=1}^N t_i^\infty[\alpha_i],$$

where there exists $\mathcal{I} \subsetneq \{1, \ldots, N\}$ such that

$$\text{for } i \in \mathcal{I},\ t_i(n) \mapsto t_i^\infty \in (0,1], \text{ and } \sum_{i \in \mathcal{I}} t_i^\infty = 1,$$

$$\text{for } i \in \{1, \ldots, N\} \smallsetminus \mathcal{I},\ t_i(n) \to t_i^\infty = 0.$$

Since $y \in \mathcal{A}\left(S_{sp}^h\right) \smallsetminus \widehat{\mathcal{A}}(S_{sp}^h)$, in the complement of $\operatorname{supp}(y) = \bigcup_{i \in \mathcal{I}} \alpha_i$, there is either a loop or a horoball connection, denoted by β. By construction, β intersects only the arcs $\{\alpha_i\}_{i \notin \mathcal{I}}$. By continuity of the infinitesimal strip map f on σ, the sequence $(f(y_n))_n$ converges to $f(y) \in \partial\Lambda(m)$ and

$$\mathrm{d}l_\beta(f(y)) = \sum_{i \notin \mathcal{I}} t_i^\infty \mathrm{d}l_\beta(f_{\alpha_i}(m)) = 0.$$

Hence $f(y)$ fails to lengthen β.

Next we show that $f(y) \neq 0$. Let γ be the boundary component containing one endpoint of an arc α_i for $i \in \mathcal{I}$. Then we have

$$\begin{aligned}
\mathrm{d}l_\gamma(f(y)) &= \sum_{p \in \gamma \cap \operatorname{supp}(y)} w_y(p) \sin \angle_p(\gamma, \operatorname{supp}(y)) \\
&\geq t_i^\infty w_{\alpha_i}(p) \sin \angle_p(\gamma, \operatorname{supp}(y)) \\
&> 0.
\end{aligned}$$

Case 2: Consider a subsequence $(z_n)_n$ such that for every $n \in \mathbb{N}$ there exists an arc $\alpha_n \subset \operatorname{supp}(z_n)$ with $l_{\alpha_n}(m) > M + C$. So $l_{\alpha \cap K(S_\heartsuit)} > M$. From Lemma 4.3.12, there exists a geodesic, denoted by $\gamma(n)$, which satisfies

$$\theta_0 := \max_{p \in \gamma(n) \cap \operatorname{supp}(z_n)} \angle_p(\operatorname{supp}(z_n), \gamma(n)) < \frac{\epsilon}{M'}. \tag{6.12}$$

Thus we have

$$\begin{aligned}
\mathrm{d}l_{\gamma(n)}(f(z_n)) &= \sum_{p \in \gamma(n) \cap \operatorname{supp}(z_n)} w_{z_n}(p) \sin \angle_p(\gamma(n), \operatorname{supp}(z_n)) \\
&\leq \theta_0 \sum_{p \in \gamma(n) \cap \operatorname{supp}(z_n)} w_{z_n}(p) \\
&\leq l_{\gamma(n)}(m)\epsilon.
\end{aligned}$$

Hence we get that the closed geodesics $\{\gamma(n)\}_n$ do not get uniformly lengthened by the strip map. Hence, $f(z_n)$ converges to a point in $\partial\Lambda(m)$.

Now we show that $f(z_n) \not\to \vec{0}$. Let $\lambda := \lim_{n \to \infty} \operatorname{supp}(z_n)$ be the limit in Hausdorff topology. The normalisation condition states that for every $n \in \mathbb{N}$, we have

$$\sum_{p \in \partial S_{sp}^h \cap \operatorname{supp}(x_n)} w_{z_n}(p) = 1.$$

So for every $p \in \partial S_{sp}^h \cap \operatorname{supp}(x_n)$, we have $w_{z_n}(p) \geq \frac{1}{2N}$. Let b be a boundary component of the surface such that for every $n \in \mathbb{N}$ it contains an endpoint $p(n)$ of an arc α_n in z_n. Then

$$\mathrm{d}l_b(f(z_n)) \geq \frac{\sin \angle_{p(n)}(b, \operatorname{supp}(z_n))}{N} \geq \frac{\sin \theta_0}{N} > 0.$$

Thus we have that $f(z_n)$ is bounded away from $\vec{0}$.

□

Surfaces with undecorated spikes

Next we prove that the projectivised strip map is proper in the case of surfaces with undecorated spikes.

Theorem 6.2.10. *Let S_{sp} be a hyperbolic surface with undecorated spikes. Let $m \in \mathfrak{D}(S_{sp})$. Then the projectivised strip map $\mathbb{P}f : \widehat{\mathcal{A}}(S_{sp}) \longrightarrow \mathbb{P}^+(\Lambda(m))$ is proper.*

Proof. The proof is similar to that of Theorem 6.2.9. Let $(x_n)_n$ be a sequence in the pruned arc complex $\widehat{\mathcal{A}}(S_{sp})$ such that $x_n \to \infty$. We want to show that $\mathbb{P}f(x_n) \to \infty$ in the projectivised admissible cone $\mathbb{P}^+\Lambda(m)$. From Lemma 4.3.11, we get that there exists a constant $M' > 0$ depending on the normalisation such that for every closed geodesic γ and every point $x \in \mathcal{A}(S_{sp})$ the eq. 6.10 holds. Again let $K(S_\heartsuit)$ be a compact neighbourhood of the convex core $S_\heartsuit$ of the surface. Then there exists $C > 0$ such that the length of every geodesic arc α satisfies $l_{\alpha \smallsetminus K(S_\heartsuit)} < C$. Given $\epsilon > 0$, we get a constant $M > 0$ from Lemma 4.3.12 applied to $\frac{\epsilon}{M'}$. Define

$$\mathcal{K}_M := \{\alpha \in \mathcal{K} \mid l_\alpha(m) \leq M + C\}.$$

Note that $\mathcal{K}_M$ is finite. Consequently, the set Σ_M of simplices in $\mathcal{A}(S_{sp})$ spanned by the arcs in $\mathcal{K}_M$ is also finite. We show that there exists $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$, there exists a closed geodesic $\gamma(n)$ that satisfies eq. (6.11). Again we prove this inequality for two types of subsequences of $(x_n)_n$— a subsequence whose terms live in one of the finitely many simplices in Σ_M and a subsequence whose every term lies in simplices outside Σ_M. Finally, we prove that $f(x_n)$ does not converge to $\vec{0}$.

1. Consider a subsequence $(y_n)_n$ of $(x_n)_n$ such that $y_n \in \sigma$ spanned by the arcs $\{\alpha_1, \ldots, \alpha_N\} \subset \Sigma_M$, where $N \leq \dim \mathfrak{D}(S_{sp})$. So y_n is of the form:

$$y_n = \sum_{i=1}^{N} t_i(n)[\alpha_i], \text{with } t_i(n) \in (0, 1] \text{ and } \sum_{i=1}^{N} t_i(n) = 1.$$

   Then, $y_n \to y \in \mathcal{A}(S_{sp}) \smallsetminus \widehat{\mathcal{A}}(S_{sp})$ given by:

$$y = \sum_{i=1}^{N} t_i^\infty [\alpha_i],$$

   where there exists $\mathcal{I} \subsetneq \{1, \ldots, N\}$ such that

$$\text{for } i \in \mathcal{I},\ t_i(n) \mapsto t_i^\infty \in (0, 1],\ \text{and } \sum_{i \in \mathcal{I}} t_i^\infty = 1,$$

$$\text{for } i \in \{1, \dots, N\} \smallsetminus \mathcal{I},\ t_i(n) \to t_i^\infty = 0.$$

From the definition of pruned arc complex of a surface with undecorated spikes, it follows that in the complement of $\operatorname{supp}(y)$ there is a non-trivial closed curve, denoted by β. By construction, β intersects only the arcs $\{\alpha_i\}_{i \notin \mathcal{I}}$. By continuity of the infinitesimal strip map f on σ, the sequence $(f(y_n))_n$ converges to $f(y) \in \partial\Lambda(m)$ and

$$\mathrm{d}l_\beta(f(y)) = \sum_{i \notin \mathcal{I}} t_i^\infty \mathrm{d}l_\beta(f_{\alpha_i}(m)) = 0.$$

Hence $f(y)$ fails to lengthen β.

Using the same reasoning as in Case 1 of Theorem 6.2.9, we get that $f(y) \neq 0$.

2. Consider a subsequence $(z_n)_n$ such that for every $n \in \mathbb{N}$ there exists an arc $\alpha_n \subset \operatorname{supp}(z_n)$ with $l_{\alpha_n}(m) > M + C$. So $l_{\alpha \cap K(S_\heartsuit)} > M$. From Lemma 4.3.12, there exists a geodesic, denoted by $\gamma(n)$, which satisfies

$$\theta_0 := \max_{p \in \gamma(n) \cap \operatorname{supp}(z_n)} \angle_p(\operatorname{supp}(z_n), \gamma(n)) < \frac{\epsilon}{M'}. \tag{6.13}$$

Thus we have

$$\begin{aligned} \mathrm{d}l_{\gamma(n)}(f(z_n)) &= \sum_{p \in \gamma(n) \cap \operatorname{supp}(z_n)} w_{z_n}(p) \sin \angle_p(\gamma(n), \operatorname{supp}(z_n)) \\ &\leq \theta_0 \sum_{p \in \gamma(n) \cap \operatorname{supp}(z_n)} w_{z_n}(p) \\ &\leq l_{\gamma(n)}(m)\epsilon. \end{aligned}$$

Hence we get that the closed geodesics $\{\gamma(n)\}_n$ do not get uniformly lengthened by the strip map. Hence, $f(z_n)$ converges to a point in $\partial\Lambda(m)$.

Finally, we prove, by contradiction, that $f(z_n)$ does not converge to the trivial deformation. Suppose that $f(z_n) \to \vec{0}$. Let $\lambda := \lim_{n \to \infty} \operatorname{supp}(z_n)$ be the limit in Hausdorff topology. Since the vector $\vec{0}$ does not deform the convex core and that the lengths of the arcs of $\operatorname{supp}(z_n)$ are not bounded above, the set λ is the union of finite arcs and arcs that spiral along geodesic laminations. Up to extraction, we may assume that the weight on every given crown either goes to 0 or stays bounded below. There is at least one 1-crown, due to normalization. Let C be a 1-crown of the surface such that every arc in $\lambda \cap C$ spins along the peripheral loop ∂C of C only finitely many (possibly zero) times. Then $\mathrm{d}l_{\partial C}(f(z_n))$ is bounded from below in the limit because it intersects the arcs at an angle bounded from below, so we can conclude. Finally we suppose that whenever $\lambda \cap C' \neq \emptyset$ for a crown C', every arc in the intersection spins around $\partial C'$ infinitely many times. By identifying C' to a one-holed polygon $\Pi_{q_i}^{\circledcirc}$, we get that $\lambda \cap C'$ is the support of some simplex σ of the spinning arc complex $\mathcal{A}_{\circlearrowright}\left(\Pi_q^{\circledcirc}\right)$, where $q(>0)$ is the total number of spikes of C'. Let $\mathcal{T}_\sigma$ be the set of tiles and $\widetilde{\mathcal{T}}$ be its lift in the universal cover. Since $\vec{0}$ does not move the spikes, there exists a sequence of tiles maps $\phi_n : \widetilde{\mathcal{T}}_{\sigma_{z_n}} \to \mathfrak{g}$ converging to a neutral tile map $\phi_0 : \widetilde{\mathcal{T}} \to \mathfrak{g}$. But from Theorem 6.2.3, we know that such a neutral tile map cannot exist. Hence, the sequence $f(z_n)_n$ does not converge to $\vec{0}$.

$\square$

Chapter 7

Decorated Margulis Spacetimes

In this chapter, we give an application of the parametrisation of the admissible cone of a hyperbolic surface with decorated spikes as done in Chapter 6. It consists of parametrising Margulis spacetimes decorated with affine light-like lines that are pairwise disjoint and every pair has the same handedness and also finding crooked planes adapted to the photons. This is done by following Drumm's construction in [6] of fundamental domains of Margulis spacetimes using crooked planes and the parametrisation of (undecorated) Margulis spacetimes as done by Danciger-Guéritaud-Kassel in [5]. In the first section we shall recall the necessary vocabulary and give summaries of the proofs as done in the two references above.

7.1 Margulis Spacetimes

Recall from Chapter 1 that the three-dimensional Minkowski space, denoted by Min, is the vector space $\mathbb{R}^3$ equipped with the quadratic form $\|\cdot\|^2$ of signature (2,1), and corresponding bilinear form $\langle\cdot,\cdot\rangle$. It is a geodesically complete flat Lorentzian manifold with affine isometry group $\mathrm{Isom}(\mathbb{R}^{2,1})$ isomorphic to the semi-direct product $\mathrm{O}(2,1) \ltimes \mathbb{R}^3$. Its subgroup $\mathrm{Isom}^+(\mathbb{R}^{2,1})$, consisting of orientation-preserving affine isometries, is of the form $\mathrm{SO}(2,1) \ltimes \mathbb{R}^3$. We shall denote by G the isomorphic groups $\mathrm{SO}(2,1), \mathrm{PGL}(2,\mathbb{R})$ and by $\mathfrak{g}$, their isomorphic Lie algebras.

Definition 7.1.1. Let $\rho_0 : \Gamma \hookrightarrow G \ltimes \mathfrak{g}$ be the representation of a discrete not virtually solvable group Γ acting properly discontinuously and freely on $\mathbb{R}^{2,1}$. Then the quotient manifold $M := \mathbb{R}^{2,1}/\rho_0(\Gamma)$ is called a *Margulis spacetime*.

As mentioned in the introduction, Fried and Goldman [8] proved that by projecting Γ onto its first coordinate, we get the holonomy representation $\rho : \pi_1(S) \to G$ of a finite-type complete hyperbolic surface S. The projection onto the second coordinate $u : \Gamma \to \mathfrak{g}$ is a ρ-cocycle, which is also an infinitesimal deformation of ρ. The group Γ can thus be written as $\Gamma^{(\rho,u)} := \{(\rho(\gamma), u(\gamma)) \mid \gamma \in \pi_1(S)\}$, which gives an *affine deformation* of ρ. It has been proved in the paper [9] that for ρ convex cocompact, then the group $\Gamma^{(\rho,u)}$ acts properly if the ρ-cycle u or $-u$ uniformly lengthens all closed geodesics of the hyperbolic surface, i.e., the equivalence class $[u]$ of u, modulo coboundaries, lies in the admissible cone $\Lambda([\rho])$ of the hyperbolic surface $\mathbb{H}^2/\rho(\pi_1(S))$.

7.1.1 Margulis Invariant

Let ρ be a convex cocompact representation as before and u be a ρ-cocycle. Margulis defined an invariant that is used to detect the properness of such a cocycle. For every non-trivial $\gamma \in \pi_1(S)$, its image $\rho(\gamma)$ is a hyperbolic element of $\mathrm{SO}(2,1)$ with eigenvalues of the form $\lambda, 1, \lambda^{-1}$. Let v_1, v_2 be two future-pointing light-like eigenvectors corresponding to the eigenvalues λ, λ^{-1}, and let $v_0(\gamma)$ be the eigenvector of unit norm with eigenvalue 1 such that (v_1, v_0, v_2) is positively oriented. Then the *Margulis invariant* is defined as the map:

$$\begin{array}{rccc} \alpha_u : & \pi_1(S) & \longrightarrow & \mathbb{R} \\ & \gamma & \mapsto & \langle u(\gamma), v_0(\gamma) \rangle. \end{array}$$

The map α_u depends only on the cohomology class of u. Margulis showed the following lemma about the properness of a cocycle and the sign of the invariant:

Lemma 7.1.2 (Opposite sign lemma, Margulis [13])**.** *Suppose that* $\Gamma^{(\rho,u)} \subset \mathrm{Isom}^+(\mathbb{R}^{2,1})$ *acts properly on* $\mathbb{R}^{2,1}$*. Then either for every* $\gamma \in \pi_1(S)$*,* $\alpha_u(\gamma) > 0$ *or for every* $\gamma \in \pi_1(S)$*,* $\alpha_u(\gamma) < 0$*.*

7.2 Fundamental domains using Crooked Planes

In this section, we shall recall Drumm's construction [6] of fundamental domains for Margulis spacetimes using *crooked planes*.

7.2.1 Definition

Take any space-like vector $\mathbf{v} \in \mathbb{R}^{2,1}$. Then the associated Killing field is hyperbolic and has an attracting and a repelling fixed point at $p_+, p_- \in \partial_\infty \mathbb{D}$, respectively. Their preimages are light-like lines: $\mathbb{P}^{-1}p_+ = \mathbb{R}\mathbf{v}_+$, $\mathbb{P}^{-1}p_- = \mathbb{R}\mathbf{v}_-$, where $\mathbf{v}_+, \mathbf{v}_-$ are future-pointing light-like vectors. Denote by $l_\mathbf{v}$, the oriented hyperbolic geodesic from endpoints p_- to p_+. It divides $\mathbb{D}$ into two half-spaces — the one lying to the right of $l_\mathbf{v}$ is denoted by $H_+(\mathbf{v})$, the one to the left is denoted by $H_-(\mathbf{v})$. The geodesic $l_\mathbf{v}$ is transversely oriented in the following way: a directed geodesic l in $\mathbb{D}$ transverse to $l_\mathbf{v}$ is said to be pointing in the positive direction if the point $[p_+] \in \mathbb{D}$ lies to its left. When we refer to the geodesic $l_\mathbf{v}$ along with its transverse orientation, we shall denote it by $\vec{l_\mathbf{v}}$.

Definition 7.2.1. A *left crooked plane* $\mathcal{P}(\mathbf{v})$ *centered at 0, directed by a space-like vector* $\mathbf{v}$ is a subset of $\mathbb{R}^{2,1}$ that is the union of the following sets:

- A *stem*, defined as $\mathrm{St}(\mathcal{P}) := \{\mathbf{w} \in \mathbb{R}^{2,1} \mid \|\mathbf{w}\|^2 \leq 0\} \cap \mathbf{v}^\perp$. It meets the light-cone along the two light-like lines $\mathbb{R}\mathbf{v}_+$, $\mathbb{R}\mathbf{v}_-$.

- Two *wings*: The connected component of ${\mathbf{v}_+}^\perp \setminus \mathbb{R}\mathbf{v}_+$ (resp. of ${\mathbf{v}_-}^\perp \setminus \mathbb{R}\mathbf{v}_-$), that contains all the hyperbolic Killing fields whose attracting fixed point is given by p_+ (resp. p_-), is called a *positive wing* (resp. a *negative wing*). They are denoted by $\mathcal{W}^+(\mathbf{v})$ and $\mathcal{W}^-(\mathbf{v})$, respectively.

For any vector $\mathbf{v_0} \in \mathbb{R}^{2,1}$, the subset $\mathcal{P}(\mathbf{v_0}, \mathbf{v}) := \mathbf{v_0} + \mathcal{P}(\mathbf{v})$ is an *affine* left crooked plane *centered at* $\mathbf{v_0}$ and directed by a space-like vector $\mathbf{v}$. Then, $\mathcal{P}(\mathbf{0}, \mathbf{v}) = \mathcal{P}(\mathbf{v})$.

Crooked Halfspaces: The connected component of $\mathbb{R}^{2,1} \setminus \mathcal{P}(\mathbf{v_0}, \mathbf{v})$ containing the Killing fields whose non-repelling fixed points (space-like, time-like, light-like) lie in the half-plane $H_+(\mathbf{v}) \subset \mathbb{H}^2$ (resp. $H_-(\mathbf{v})$) is called the *positive crooked half-space* (resp. *negative crooked half-space)*, denoted by $\mathcal{H}^+(\mathbf{v})$ (resp. $\mathcal{H}^-(\mathbf{v})$).

Next we recall the definition of stem quadrant of a transversely oriented hyperbolic geodesic, as defined in [2].

Let $\mathbf{v} \in \mathbb{R}^{2,1}$ be a space-like vector, $\mathbf{v}_+, \mathbf{v}_-$ be future-pointing light-like vectors in $\mathbf{v}^\perp$ and $\overrightarrow{l_\mathbf{v}}$ be the hyperbolic geodesic in $\mathbb{D}$ with endpoints at $[\mathbf{v}_+], [\mathbf{v}_-]$ and oriented towards $[\mathbf{v}_+]$.

Definition 7.2.2. The set $\mathrm{SQ}(\overrightarrow{l_\mathbf{v}}) := \mathbb{R}_{>0}\mathbf{v}_+ - \mathbb{R}_{>0}\mathbf{v}_-$ is called the *stem quadrant* of the transversely oriented geodesic $l_\mathbf{v}$, associated to the positively oriented triplet $(\mathbf{v}_+, \mathbf{v}, \mathbf{v}_-)$.

In [6], Drumm gave a sufficient condition for two crooked planes to be disjoint.

Theorem 7.2.3 (Drumm)**.** *Let $\mathbf{v}, \mathbf{v}'$ be two space-like points in $\mathbb{R}^{2,1}$ such that their corresponding $\vec{l_\mathbf{v}}, \vec{l_{\mathbf{v}'}}$ geodesics are disjoint in $\overline{\mathbb{D}}$ and are transversely oriented away from each other. Then for every $\mathbf{w} \in \mathrm{SQ}(\overrightarrow{l_\mathbf{v}})$ and $\mathbf{w}' \in \mathrm{SQ}(\overrightarrow{l_{\mathbf{v}'}})$, we have $\overline{\mathcal{H}^+(\vec{l_\mathbf{v}})} + \mathbf{w} \subset \mathcal{H}^-(\vec{l_{\mathbf{v}'}}) + \mathbf{w}'$. In particular, the crooked planes $\mathcal{P}(\mathbf{w}, \mathbf{v})$ and $\mathcal{P}(\mathbf{w}', \mathbf{v}')$ are disjoint.*

7.2.2 Parametrisation of Margulis spacetimes

In this section we will recall the parametrisation of Margulis spacetimes using the pruned arc complex and the construction of the fundamental domain of a Margulis spacetime from an admissible deformation of a compact hyperbolic surface with boundary, as done in [5].

Drumm's construction of proper cocycles. Let $\rho : \Gamma \longrightarrow G$ be a convex cocompact representation. A fundamental domain for the action of $\rho(\Gamma)$ on the hyperbolic plane $\mathbb{D}$ is bounded by finitely many pairwise disjoint geodesics. These geodesics are used to construct the stems of pairwise disjoint crooked planes in $\mathbb{R}^{2,1}$. Then these planes are made disjoint from each other by adding points from their respective stem quadrants. The polyhedron bounded by these new crooked planes is a fundamental domain for the action of the group Γ and the resulting manifold X/Γ is complete. Finally, Drumm determined u.

From proper cocycles to Margulis spacetimes, [5]. Let S_c be a compact hyperbolic surface with totally geodesic boundary. Let $\rho : \pi_1(S_c) \longrightarrow \mathrm{PGL}(2, \mathbb{R})$ be a holonomy representation and $u : \pi_1(S_c) \longrightarrow \mathfrak{g}$ be a ρ-cocycle such that $[u] \in \Lambda([\rho])$. From Theorem 4.4.1 in [5], we know that the projectivised strip map when restricted to the pruned arc complex of the surface S_c is a homeomorphism onto its image $\Lambda([\rho])$. So there exists a point $x \in \widehat{\mathcal{A}}(S^h_{sp})$ and a unique simplex σ such that $\mathbb{P}f(x) = [u] \in \mathbb{P}^+\Lambda([\rho])$ and $x \in \mathrm{int}\,(\sigma)$. So $x = \sum_i t_i[\alpha_i]$ with $\sum_i t_i = 1$ and $f(x) = \sum_i t_i f_{\alpha_i}(m)$. Corresponding to this linear combination of strip maps, we get a class of tile maps $\phi : \widetilde{\mathcal{T}_\sigma} \longrightarrow \mathfrak{g}$ that are $(\rho(\pi_1(S_c)), u)$-equivariant. Let $\alpha \in \mathcal{E}_\sigma$ be any arc of σ and $\widetilde{\alpha}$ be any lift. There exists tiles $d_1, d_2 \in \widetilde{\mathcal{T}_\sigma}$ that have $\widetilde{\alpha}$ as their common internal edge. Suppose that the geodesic arc $\widetilde{\alpha}$ is positively transversely oriented from d_1 to d_2. Then the Killing field $\phi(d_2) - \phi(d_1)$ is hyperbolic and represents the term $t_\alpha f_\alpha(m)$ in $f(x)$. Let $\mathbf{v}_{\widetilde{\alpha}} \in \alpha^\perp$ be a hyperbolic Killing field with attracting and repelling fixed points given by $[\mathbf{v}^+_{\widetilde{\alpha}}], [\mathbf{v}^-_{\widetilde{\alpha}}]$ such that the triplet $(\mathbf{v}^+_{\widetilde{\alpha}}, \mathbf{v}_{\widetilde{\alpha}}, \mathbf{v}^-_{\widetilde{\alpha}})$ is positively oriented and the tile d_2 lies to the left of the axis when viewed from $[\mathbf{v}^-_{\widetilde{\alpha}}]$. Then the

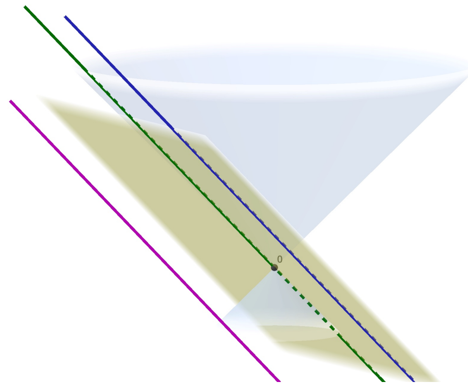

Figure 7.1: The different types of photons.

crooked plane associated to $\widetilde{\alpha}$ is given by $\mathcal{P}_{\widetilde{\alpha}} := \mathcal{P}(\mathbf{w}_{\widetilde{\alpha}}, \mathbf{v}_{\widetilde{\alpha}})$, where $\mathbf{w}_{\widetilde{\alpha}} := \frac{\phi(d_1)+\phi(d_2)}{2}$. For other arcs in the orbit of $\widetilde{\alpha}$, the crooked plane is defined as: for every $\gamma \in \pi_1(S_c)$, $\mathcal{P}_{\rho(\gamma)\cdot\widetilde{\alpha}} = \rho(\gamma)\cdot\mathcal{P}_{\widetilde{\alpha}} + u(\gamma)$.

Firstly, it is shown that for every two disjoint arcs $\widetilde{\alpha_1}, \widetilde{\alpha_2} \in \mathcal{E}_\sigma$ their associated crooked planes $\mathcal{P}_{\rho(\gamma)\cdot\widetilde{\alpha_1}}, \mathcal{P}_{\rho(\gamma)\cdot\widetilde{\alpha_2}}$ are disjoint by using Drumm's sufficient condition (Theorem 7.2.3). Then they consider a fundamental domain of the surface bounded by finitely many arcs in $\widetilde{\mathcal{E}_\sigma}$ and show that the associated crooked planes form a fundamental domain for the Margulis spacetime. We shall adapt this method to our surfaces with decorated spikes.

7.3 Decorating a Margulis spacetime

7.3.1 Photons and Killing fields

Consider the projective disk model $\mathbb{D}$ and a point $p \in \partial_\infty\mathbb{D}$. Recall that an open horoball h based at p is the projective image of the subset $H(\mathbf{v}) = \{\mathbf{w} \in \mathbb{H}^2 \mid \langle \mathbf{w}, \mathbf{v}\rangle > -1\}$ of the hyperboloid $\mathbb{H}^2$, where $\mathbf{v}$ is a future-pointing light-like point in $\mathbb{P}^{-1}\{p\}$. If $k > k' > 0$, then the horoball $h := \mathbb{P}H(k\mathbf{v}_0)$ is smaller than the horoball $h' := \mathbb{P}H(k'\mathbf{v}_0)$.

Definition 7.3.1. Let $\mathbf{v_0} \in \mathbb{R}^{2,1}$ be a future-pointing light-like vector and let $\mathbf{v} \in \mathbb{R}^{2,1}$ be any point. Then the affine line $\mathcal{L}(\mathbf{v}, \mathbf{v_0}) := \mathbf{v} + \mathbb{R}\mathbf{v_0}$ is called a *photon*.

A vector $\mathbf{u} \in \mathcal{L}(\mathbf{v}, \mathbf{v_0})$ corresponds to a Killing field that moves the vector $\mathbf{v_0}$ in the direction $\mathbf{u} \wedge \mathbf{v_0}$. A vector $\mathbf{w} \in H$ is moved in the direction $\mathbf{u} \wedge \mathbf{w}$.

- When $\mathbf{v} \in \mathbb{R}\mathbf{v_0}$, the photon $\mathcal{L}(\mathbf{v}, \mathbf{v_0})$ is the vectorial line $\mathbb{R}\mathbf{v_0}$, coloured green in Fig.7.1. Its non-zero points correspond to parabolic Killing fields that fix the ideal point $[\mathbf{v_0}]$ in the hyperbolic plane and preserve the horoballs based at this ideal point as sets: for $k \in \mathbb{R} \smallsetminus \{0\}$, $k\mathbf{v_0} \wedge \mathbf{v_0} = 0$. So the vector $\mathbf{v_0}$ and hence the set $H(\mathbf{v})$ is preserved by the flow of the Killing field associated to kv_0.

- When $\mathbf{v}$ is contained in the light-like plane $\mathbf{v_0}^\perp$, the photon also lies inside $\mathbf{v_0}^\perp$. Such a photon is coloured blue in Fig.7.1. Any vector $\mathbf{u}$ on such a photon, that is not contained in $\mathbb{R}\mathbf{v_0}$, is a hyperbolic Killing field with one of its fixed points at $[\mathbf{v_0}]$. We have that

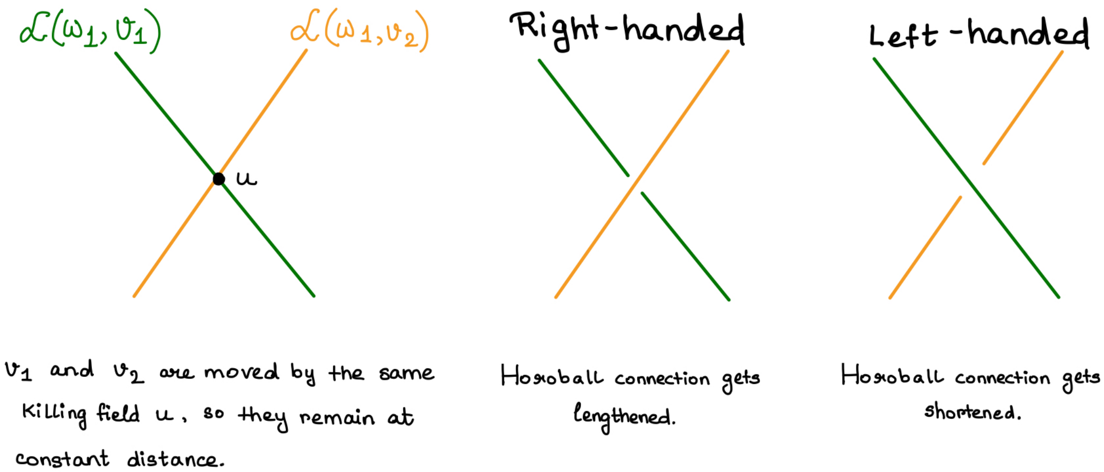

Figure 7.2: A pair of photons.

$\mathbf{u} \wedge \mathbf{v_0} \in \mathbf{u}^\perp \cap \mathbf{v_0}^\perp = \mathbb{R}\mathbf{v_0}$. So the vector $\mathbf{v_0}$ and the set H gets scaled by the flow of the Killing vector field $\mathbf{u}$. The connected component of the set $\mathbf{v_0}^\perp \smallsetminus \mathbb{R}\mathbf{v_0}$ that contains the hyperbolic Killing fields whose attracting (resp. repelling) fixed point is given by $[\mathbf{v_0}]$ shrinks (resp. enlarges) the horoballs centered at this point.

- When $\mathbf{v} \in \mathbb{R}^{2,1} \smallsetminus \mathbf{v_0}^\perp$, any vector $\mathbf{u} = \mathbf{v} + k\mathbf{v_0} \in \mathcal{L}(\mathbf{v}, \mathbf{v_0})$ moves the light-like vector away from $\mathbb{R}\mathbf{v_0}$ and in the direction given by $\mathbf{u} \wedge \mathbf{v_0}$. Such a photon is coloured in pink in Fig.7.1. When $\mathbf{v}$ lies above (resp. below), the point $[\mathbf{v_0}]$ is moved in the clockwise (resp. anticlockwise) direction on $\partial_\infty \mathbb{D}$.

The space of photons can be identified with the tangent bundle over the space of horoballs, modulo simultaneous scaling of all horoballs. This identification is equivariant for the actions of $G \ltimes \mathfrak{g} = T(G)$.

Handedness

Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{2,1}$ be two future-pointing light-like vectors. For $i = 1, 2$, let $\mathbf{w}_i \in \mathcal{W}^+(\mathbf{v}_i)$, where $\mathcal{W}^+(\mathbf{v}_i)$ is the positive wing of $\mathbf{v}_i$. Then the photon $\mathcal{L}(\mathbf{w}_i, \mathbf{v}_i)$ consists of hyperbolic Killing fields that have $[\mathbf{v}_i]$ as attracting fixed point. So for every $i = 1, 2$, the vector $\mathbf{v}_i$ gets infinitesimally deformed towards $k\mathbf{v}_i$ for $k > 1$ and the horoball $h_i := \mathbb{P}(H(\mathbf{v}_i))$ gets shrunken. Finally, consider the pair of photons $\{\mathcal{L}(\mathbf{w}_1, \mathbf{v}_1), \mathcal{L}(\mathbf{w}_2, \mathbf{v}_2)\}$. Any horoball connection joining the decorated spikes $([\mathbf{v}_1], h_1)$ and $([\mathbf{v}_2], h_2)$ gets lengthened.

Now let $\{\mathcal{L}(\mathbf{w}_1, \mathbf{v}_1), \mathcal{L}(\mathbf{w}_2, \mathbf{v}_2)\}$ be any pair of disjoint photons. They are contained in the two affine light-like planes $A_1 := \mathbf{w}_1 + \mathbf{v}_1{}^\perp, A_2 := \mathbf{w}_2 + \mathbf{v}_2{}^\perp$, respectively. Let $\mathbf{v} \in \mathcal{L}(\mathbf{w}_1, \mathbf{v}_1) \cap A_2$ and

$\mathbf{v}' \in \mathcal{L}(\mathbf{w}_2, \mathbf{v}_2) \cap A_1$. Then $\mathbf{v} = \mathbf{w}_1 + k_1\mathbf{v}_1$, $\mathbf{v}' = \mathbf{w}_2 + k_2\mathbf{v}_2$, where $k_1 = -\frac{\langle \mathbf{w}_1, \mathbf{v}_2\rangle}{\langle \mathbf{v}_1, \mathbf{v}_2\rangle}$ and $k_2 = -\frac{\langle \mathbf{w}_2, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_2\rangle}$. Then we have

$$\begin{aligned}\mathbf{v} - \mathbf{v}' &= \mathbf{w}_1 - \mathbf{w}_2 - \frac{\langle \mathbf{w}_1, \mathbf{v}_2\rangle}{\langle \mathbf{v}_1, \mathbf{v}_2\rangle}\mathbf{v}_1 + \frac{\langle \mathbf{w}_2, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_2\rangle}\mathbf{v}_2 \\ &= \frac{\langle \mathbf{w}_1 - \mathbf{w}_2, \mathbf{v}_1 \wedge \mathbf{v}_2\rangle}{\|\mathbf{v}_1 \wedge \mathbf{v}_2\|^2}(\mathbf{v}_1 \wedge \mathbf{v}_2).\end{aligned}$$

The sign of the real number $\langle \mathbf{w}_1 - \mathbf{w}_2, \mathbf{v}_1 \wedge \mathbf{v}_2\rangle$ gives the *handedness* of the pair $\{\mathcal{L}(\mathbf{w}_1, \mathbf{v}_1), \mathcal{L}(\mathbf{w}_2, \mathbf{v}_2)\}$.

7.3.2 From decorated surfaces to decorated Margulis spacetimes

In this section, we will adapt the parametrisation of Margulis spacetimes to our case of hyperbolic surfaces with decorated spikes. We start by defining decorated Margulis spacetimes.

Let S_{sp}^h be a hyperbolic surface with Q decorated spikes, endowed with a decorated metric $m = [\rho, \mathbf{x}, \mathbf{h}] \in \mathfrak{D}(S_{sp}^h)$. Then the metric on the convex core $S_\heartsuit$ is given by $[\rho]$. The admissible cone $\Lambda(m)$ is an affine bundle over the admissible cone $\Lambda([\rho])$ of the convex core; denote by π, the bundle projection $\pi : \Lambda(m) \longrightarrow \Lambda([\rho])$. The fibres are open subsets of $\mathbb{R}^{2Q}$ that are stable under the scaling of horoballs.

Let $[u] \in \Lambda(m)$ be an admissible deformation of the surface S_{sp}^h. Let $[u_0] := \pi([u])$. Then u_0 is a proper ρ-cocycle and the group of isometries $\Gamma^{(\rho,u_0)}$ acts properly discontinuously on $\mathbb{R}^{2,1}$. The quotient $M := \mathbb{R}^{2,1}/\Gamma^{(\rho,u_0)}$ is a Margulis spacetime, which we decorate with photons in the following way: the infinitesimal deformation $[u]$ imparts motion to every lift of each decorated spike of the surface. From the previous section, we know that set of Killing fields realising this particular variation to an ideal point decorated with a horoball, happens to be a photon. This collection of photons, denoted by $\mathscr{L}$, is $\Gamma^{(\rho,u_0)}$-equivariant and is the decoration of the underlying Margulis spacetime. The pair $(M, \mathscr{L})$ is called a *decorated* Margulis spacetime.

Next we will give another way of looking at this decoration using tile maps. We know that the projectivised strip map when restricted to the pruned arc complex is a homeomorphism onto its image $\Lambda(m)$. So there exists a point $x = \widehat{\mathcal{A}}(S_{sp}^h)$ and a unique big simplex σ such that $\mathbb{P}f(x) = [u] \in \mathbb{P}^+\Lambda(m)$ and $x \in \operatorname{int}(\sigma)$. So $x = \sum_i t_i[\alpha_i]$ with $\sum_i t_i = 1$, $t_i > 0$ for every i and $f(x) = \sum_i t_i f_{\alpha_i}(m)$. Corresponding to this linear combination of strip maps we get a class of tile maps $\phi : \widetilde{\mathcal{T}_\sigma} \longrightarrow \mathfrak{g}$. Now suppose that the surface has Q spikes and write the spike vector $\mathbf{x}$ as $(x_1, \ldots, x_Q)$. Since the arcs of σ decompose the surface into tiles with at most one spike, there exist exactly Q tiles $d_1, \ldots, d_Q$ such that $x_i \in d_i$ for every $i = 1, \ldots, Q$. Using the tile map, we get a collection of Q Killing fields $\phi(d_1), \ldots, \phi(d_Q)$ where $\phi(d_i)$ acts on the ideal point x_i. Now suppose that $\mathbf{h} = (h_1, \ldots, h_Q)$ is the horoball decoration given by the metric m. Then for each $i = 1, \ldots, Q$, there exists a future pointing light-like vector $\mathbf{v}_i$ and the set $H(\mathbf{v}_i)$ such that $x_i = [\mathbf{v}_i]$ and $h_i = [H(\mathbf{v}_i)]$. Then consider the collection of photons of the form $\phi(d_i) + \mathbb{R}\mathbf{v}_i$ for $i = 1, \ldots, Q$ and take their $\Gamma^{(\rho,u_0)}$-orbit.

Remark 7.3.1. Note that these photons are pairwise disjoint. If two photons intersect, their intersection point is a Killing field that realizes the motions of the two corresponding horoballs, hence the horoball connection has zero infinitesimal length variation. Hence the infinitesimal deformation $[u]$ fails to be admissible.

Remark 7.3.2. Every pair of photons has the same handedness, because every horoball connection is lengthened.

From decorated Margulis space-times to admissible deformations.

Let Γ be a finitely generated free discrete group acting properly discontinuously on $\mathbb{R}^{2,1}$ and its representation $\rho : \Gamma \longrightarrow G \ltimes \mathfrak{g}$. Let $(M := \mathbb{R}^{2,1}/\rho(\Gamma), \mathscr{L})$ be a decorated Margulis spacetime with convex cocompact linear part $\rho_0 : \Gamma \longrightarrow G$. Using Drumm's construction of proper cocycles, we have that $\Gamma = \Gamma^{(\rho_0,u_0)}$, where u_0 is a proper ρ_0-cocycle. The surface $S_c := \mathbb{D}/\rho_0(S_c)$ is compact with totally geodesic boundary. Denote its boundary components by $b_1, \ldots, b_n$.

The set $\mathscr{L}$ is $\Gamma^{(\rho_0,u_0)}$-equivariant; there exists finitely many pairs $(\mathbf{w}_1, \mathbf{v}_1), \ldots, (\mathbf{w}_Q, \mathbf{v}_Q)$ of points in $\mathbb{R}^{2,1}$ such that for every i, the vector $\mathbf{v}_i$ is future-pointing and light-like and $\mathscr{L}$ is generated by the photons $\mathcal{L}_i := \mathcal{L}(\mathbf{w}_i, \mathbf{v}_i)$, $i = 1, \ldots, Q$. This gives us Q ideal points $x_i = [\mathbf{v}_i] \in \partial_\infty \mathbb{D}$. Take the $\rho_0(\Gamma)$-orbit of this collection and join every consecutive pair, that lie on the same side of a lift of the boundary loop b_i, by a geodesic. Let R be the simply-connected region in $\mathbb{D}$ bounded these geodesics. Then we get a hyperbolic surface with decorated spikes $S^h_{sp} := R/\rho_0(\Gamma)$ with the metric $m = [\rho_0, \mathbf{x}, \mathbf{h}]$, where $\mathbf{x} = (x_1, \ldots, x_Q)$ and $\mathbf{h} = (h_1, \ldots, h_Q)$, $h_i = \mathbb{P}(H(\mathbf{v}_i))$. The surface S_c is the convex core of S^h_{sp}.

The admissible deformation of S^h_{sp} is determined in the following way: for every $i = 1, \ldots, Q$, the photon $\mathcal{L}_i$ imparts infinitesimal motion to the spike x_i as well as the horoball h_i in the sense that $\mathcal{L}_i$ is exactly the set of Killing fields all of whom cause h_i to vary in a certain infinitesimal way. Since no two photons intersect, every horoball connection is deformed and since every pair of photons has the same handedness, every horoball connection gets lengthened. Thus we get an admissible deformation $[u] \in \Lambda(m)$ with $\pi([u]) = [u_0]$.

From admissible deformations to decorated Margulis space-times.

Let S^h_{sp} be a hyperbolic surface with Q decorated spikes, endowed with a decorated metric m, which is of the form $m = [\rho, \mathbf{x}, \mathbf{h}] \in \mathfrak{D}(S^h_{sp})$. Let $[u] \in \Lambda(m)$ be an admissible deformation of the surface S^h_{sp}. Let $[u_0] := \pi([u])$. Then u_0 is a proper ρ-cocycle and the group of isometries $\Gamma^{(\rho,u_0)}$ acts properly discontinuously on $\mathbb{R}^{2,1}$. By Theorem 6.1.2 there exists a unique point $x = \widehat{\mathcal{A}}(S^h_{sp})$ and a unique big simplex σ such that $\mathbb{P}f(x) = [u] \in \mathbb{P}^+\Lambda(m)$ and $x \in \operatorname{int}(\sigma)$. So $x = \sum_i t_i[\alpha_i]$ with $\sum_i t_i = 1$, $t_i > 0$ for every i and $f(x) = \sum_i t_i f_{\alpha_i}(m)$. Corresponding to this linear combination of strip maps, we get a class of tile maps $\phi : \widetilde{\mathcal{T}_\sigma} \longrightarrow \mathfrak{g}$ that are $(\rho(\pi_1(S_c)), u)$-equivariant. Let $\alpha \in \mathcal{E}_\sigma$ be any arc of σ and $\widetilde{\alpha}$ be any lift. There exists tiles $d_1, d_2 \in \widetilde{\mathcal{T}_\sigma}$ that have $\widetilde{\alpha}$ as their common internal edge. The arc is either finite or joins a decorated spike with a bounary component. Let $\phi(d_2) - \phi(d_1)$ be the Killing field that represents the term $t_\alpha f_\alpha(m)$ in $f(x)$. When α is finite, the difference is a hyperbolic Killing field belonging to the stem quadrant $\mathrm{SQ}(\overrightarrow{l_{\mathbf{v}}})$. Otherwise, it is a parabolic Killing field fixing the ideal endpoint of $\widetilde{\alpha}$. Define the associated crooked plane as before: $\mathcal{P}_{\widetilde{\alpha}} := \mathcal{P}(\mathbf{w}_{\widetilde{\alpha}}, \mathbf{v}_{\widetilde{\alpha}})$, with $\mathbf{w}_{\widetilde{\alpha}} := \frac{\phi(d_1)+\phi(d_2)}{2}$. Let R be a fundamental domain of the surface S^h_{sp} bounded by some arcs $e_1, f_1, \ldots, e_k, f_k$ in $\widetilde{\mathcal{E}_\sigma}$ such that there exists $\gamma_1, \cdots, \gamma_k \in \pi_1(S^h_{sp})$ such that for $i = 1, \cdots, k$, $f_i = \rho(\gamma_i) \cdot e_i$. Since there are no parabolic elements in $\rho(\pi_1(S^h_{sp}))$, nor any spiralling arcs, for every pair (e_i, f_i) of spike-to-edge arcs, the spikes are distinct. So there exists an edge-to-edge arc $\alpha \in \mathcal{E}_\sigma$ whose lift $\widetilde{\alpha}$ separates e_i from f_i. Since the arc $\widetilde{\alpha}$ is disjoint to both e_i and f_i in $\overline{\mathbb{D}}$, its associated crooked plane $\mathcal{P}(\mathbf{w}_{\widetilde{\alpha}}, \mathbf{v}_{\widetilde{\alpha}})$ separates the crooked planes $\mathcal{P}_{e_i}, \mathcal{P}_{f_i}$. Hence we have that for every $i = 1, \ldots, k$, the crooked planes $\mathcal{P}_{e_i}, \mathcal{P}_{f_i}$ are disjoint and $(\rho(\gamma_i), u_0(\gamma_i))\mathcal{P}_{e_i} = \mathcal{P}_{f_i}$. The region $\mathcal{D}$ bounded by these crooked planes is a fundamental domain for the action of $\Gamma^{(\rho,u_0)}$ on $\mathbb{R}^{2,1}$.

Thus we have proved the following theorem:

Theorem 7.3.2. *Let S^h_{sp} be a hyperbolic surface with decorated spikes and let $\rho : \pi_1(S^h_{sp}) \to \mathrm{PGL}(2, \mathbb{R})$ be a holonomy representation. Let $\mathscr{M}^{\otimes}$ be the space of all Margulis spacetimes with convex cocompact linear part as ρ. Then there is a bijection $\Psi : \widehat{\mathcal{A}}(S^h_{sp}) \to \mathscr{M}^{\otimes}$.*

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