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Finite group representations through 2-sheaves
Représentations des groupes finis et 2-faisceaux

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Résumé. La présente thèse expose une approche s'appuyant sur les 2-catégories pour l'étude des représentations modulaires des groupes finis et de la fusion dans les groupes finis. Le socle de cette approche est l'ubiquité des 2-faisceaux dans la théorie des représentations des groupes finis et leur bonne compatibilité avec diverses constructions catégoriques et bicatégoriques, telles que les produits et les adjonctions. Notamment, une généralisation du théorème de Bénabou-Roubaud permet d'établir une correspondance entre 2-foncteurs de Mackey cohomologiques et 2-faisceaux. Cette correspondance conduit à une formule analogue à celle des éléments stables de Cartan et Eilenberg pour de nombreuses catégories pertinentes pour la théorie des représentations des groupes finis, comme la catégorie stable des modules ou la catégorie dérivée de la catégorie des modules.

Abstract. This thesis exposes an approach based on 2-categories to the study of modular representations of finite groups and fusion in finite groups. The fundamental basis of this approach is the ubiquity of 2-sheaves in the theory of groups representations and their well-behaved compatibility with various categorical and bicategorical constructions, such as products and adjunctions. Most importantly, a generalization of the Bénabou-Roubaud theorem allows us to establish a correspondence between 2-sheaves and cohomological Mackey 2-functors. This correspondence leads to an analogous Cartan-Eilenberg stable elements formula for categories relevant to the representation of finite groups, such as the stable modules category and the derived category of the modules category.

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Introduction

Finite groups, despite being simultaneously one of the most basic and scrutinized algebraic structure, remain an important source of yet unsolved mathematical problems. A well-established method to study a mathematical object is to decompose it into simpler, more manageable parts. Finite groups are no exception, with well-known structures such as the composition series. Another strategy is to work one prime number p at a time. There is already a well-defined *topological* way to isolate the p-local information of a finite group G: the p-completion $BG_p^\wedge$ of the standard topological model of G, the classifying space BG. Our goal is to explore a well-behaved *algebraic* notion of the p-local part of a group or, dually, of a p-local invariant of a group. In particular, the mod p cohomology of a group should be a p-local invariant. We follow two well-known main tracks:

- *The p-fusion of a finite group:* a finite group G acts by conjugation on the subgroups of a p-Sylow subgroup. Aggregating these actions together allows one to define a category $\mathcal{F}_S(G)$, the fusion system associated to G. The mod p cohomology of G only depends on the fusion system $\mathcal{F}_S(G)$; and there is a partial converse result [Mis90]. Still, it is difficult to define a relevant notion of morphism of fusion systems and, by consequence, to compare fusion systems of two too different groups. A general perspective on fusion systems is given by [AKO11].
- *The p-modular representation theory of a finite group:* given a field $\Bbbk$ of characteristic $p > 0$, a p-modular representation of finite group G is a module for the group algebra $\Bbbk G$. These representations form an abelian category $\mathrm{Mod}(\Bbbk G)$, with an associated derived category $\mathrm{D}(\Bbbk G)$ and stable category $\mathrm{stMod}(\Bbbk G)$. In particular, the cohomology $H^*(G; \Bbbk)$, with its structure of $\Bbbk$-algebra, is given by a graded endomorphism set of $\mathrm{D}(\Bbbk G)$; similarly the Tate cohomology $\hat{H}^*(G; \Bbbk)$ shows up as a graded endomorphism set of $\mathrm{stMod}(\Bbbk G)$. Moreover the categories $\mathrm{Mod}(\Bbbk G)$, $\mathrm{D}(\Bbbk G)$ and $\mathrm{stMod}(\Bbbk G)$ are functorial in G: each of them defines a contravariant functor $\mathrm{gp}^{\mathrm{op}} \to \mathrm{Cat}$ from finite groups to categories.

The exact relation between the fusion system and the modular representation categories of a finite group is not straightforward. In this thesis, we aim at clarifying it. We chose the Cartan-Eilenberg stable elements formula as a starting point. This formula expresses the mod p cohomology of a finite group G in terms of the mod p cohomology of the p-subgroups of G and its fusion system $\mathcal{F}_S(G)$:

$$H^*(G; \mathbb{F}_p) \cong \lim_{P \in \mathcal{F}_S(G)^{\mathrm{op}}} H^*(P; \mathbb{F}_p)$$

An analogous formula exists for any *(global) cohomological Mackey functor* [Par17], a well-behaved generalization of the mod p cohomology functor of finite groups. Each of the three functors that the various categories of modular representations define is actually part of a richer structure of *cohomological Mackey 2-functor*. The latter is a categorification of cohomological Mackey functors, consisting of a 2-functor

$$\mathbb{M} \colon \mathbf{gpd}^{\mathrm{op}} \to \mathbf{Add}$$

between the opposite of the 2-category **gpd** of finite groupoids and the 2-category **Add** of additive categories, which preserves products, satisfies the (ambidextrous) Beck-Chevalley property (definition 3.1.2) and a cohomological identity (definition 4.1.8). This naturally leads us to look for a categorified version of the Cartan-Eilenberg formula for cohomological Mackey 2-functors.

We prove in this work a categorified Cartan-Eilenberg formula (theorem 4.2.12) for any *p-monadic Mackey 2-functor* $\mathbb{M}\colon \mathbf{gpd}^{\mathrm{op}} \to \mathbf{Add}$. In particular, a cohomological Mackey 2-functor taking values in $\mathbb{Z}_{(p)}$-linear and idempotent complete categories is p-monadic. Our formula states that, for any group G, there is an equivalence of categories:

$$\mathbb{M}(G) \simeq \operatorname*{bilim}_{P \in \hat{\mathcal{T}}_S(G)^{\mathrm{op}}} \mathbb{M}(P).$$

Compared to the classical formula, the isomorphism has been replaced by an equivalence and the limit by a pseudo bilimit, as we are now working in a bicategorical setting. Moreover the 2-category indexing the bilimit is now a newly introduced 2-category, the extended transporter category $\hat{\mathcal{T}}_S(G)$ (which is biequivalent to the classical orbit 1-category $\mathcal{O}_S(G)$), and not the fusion system $\mathcal{F}_S(G)$ anymore. While a direct proof of this formula is possible (as is done in [Mai21a]), we rather have chosen to make explicit the correspondence between p-monadic Mackey 2-functors and 2-sheaves on **gpd** for an adequate topology, the p-local topology. Then, by a 2-finality argument, we transform the descent condition of 2-sheaves into the categorified Cartan-Eilenberg formula.

The main original contributions of this thesis are the following.

- We prove a generalization of the Bénabou-Roubaud theorem (theorem 3.2.1) and an extension property for 2-sheaves of the Beck-Chevalley property (proposition 3.3.1). Combining these two results lets us deduce a correspondence between p-monadic Mackey 2-functors and 2-sheaves for the p-local topology (theorem 4.1.18).
- By the aforementioned correspondence, we obtain a biequivalence between p-monadic Mackey 2-functors on finite groupoids and Mackey 2-functors on finite p-groupoids (theorem 4.1.22).
- We prove a criterion characterizing 2-final 2-functors (theorem 1.3.13) and use it to compare descent bilimits with bilimits indexed by the extended transporter category $\hat{\mathcal{T}}_S(G)$ (proposition 4.2.10). This allows us to express the categorified Cartan-Eilenberg formula (theorem 4.2.12).

The thesis is structured as follows. In chapter 1, we recall usual bicategorical notions; we also introduce our notations for bipullbacks as they play a major role in the general theory of 2-sheaves. We give an original proof of a topological criterion characterizing 2-final 2-functors (this result appears in [Mai21b] as a standalone article). In chapter 2, we give a detailed introduction to the general theory of 2-sites and 2-sheaves; our framework mostly reflects the classical ones of sheaves or stacks. In chapter 3, we introduce the Beck-Chevalley property and prove two related fundamental results: the Bénabou-Roubaud theorem and the extension property for 2-sheaves of the Beck-Chevalley property. In chapter 4, we apply these results to the study of p-monadic Mackey 2-functors. We obtain our categorified Cartan-Eilenberg formula and show how it can be decategorified to recover the classical Cartan-Eilenberg formulas.

CHAPTER 1

2-Categories

The goal of this chapter is to summarize the basic bicategorical notions we will need throughout this thesis; a detailed presentation of bicategories can be found in [JY21]. Most of the constructions and results covered in this chapter are straightforward generalization of their 1-categorical counterparts, which can be found in classical references such as [Mac71].

We have chosen to work exclusively with strict 2-categories and strict 2-functors. This is a deliberate trade-off: while most of our intermediary results could certainly be generalized to a more relaxed setting, it would definitely increase the verbosity of our proofs, for little gain, as all our applications only involve strict 2-categories and strict 2-functors. Nevertheless, even in such a strict setting, the relevant constructions need some flexibility: we will use pseudo slice 2-category (example 1.1.20) and pseudo bilimits (definition 1.2.1). We avoid introducing variations over the same concept; specifically, we will not deal with lax or oplax notions.

We also inspect the concept of finality for a 2-functor and give a useful characterization of it (see theorem 1.3.13). This characterization is an original result of this thesis.

1.1. Preliminaries on 2-categories

General definitions. We define 2-categories and give the adequate notion of (higher) morphisms of 2-categories.

1.1.1. Definition. A *2-category* $\mathbf{C}$ is a category enriched over (1-)categories, that is the data of

- a class of objects $\operatorname{Ob}\mathbf{C}$,
- for each pair of objects $A, B \in \operatorname{Ob}\mathbf{C}$, a Hom-*category* $\mathbf{C}(A, B)$,
- for each object $A \in \operatorname{Ob}\mathbf{C}$, a distinguished identity $\mathrm{Id}_A \in \operatorname{Ob}\mathbf{C}(A, A)$ and
- for each objects $A, B, C \in \operatorname{Ob}\mathbf{C}$, a composition *functor*

$$\circ : \mathbf{C}(B, C) \times \mathbf{C}(A, B) \to \mathbf{C}(A, C)$$

satisfying the usual axioms of a category

$$f = f \circ \mathrm{Id}_A = \mathrm{Id}_B \circ f \tag{$2\mathrm{C}_{\mathrm{id}}$}$$

$$(f \circ g) \circ h = f \circ (g \circ h) \tag{$2\mathrm{C}_{\mathrm{asso}}$}$$

via the appropriate identities of composite functors.

In a 2-category $\mathbf{C}$, we call

- *Objects:* the elements of $\operatorname{Ob}\mathbf{C}$,
- *1-Morphisms* (or simply *morphisms*)*:* the objects of the hom-categories $\mathbf{C}(A, B)$, and
- *2-Morphisms:* the morphisms of the hom-categories $\mathbf{C}(A, B)$.

There is an internal notion of equivalence in a 2-category.

1.1.2. Definition. Let $\mathbf{C}$ be a 2-category and $f\colon A \to B$ be a 1-morphism of $\mathbf{C}$. The 1-morphism f is an *equivalence* if there exists a morphism $g\colon B \to A$ and two invertible 2-morphisms $fg \cong \mathrm{Id}_B$ and $gf \cong \mathrm{Id}_A$.

Two objects A and B of $\mathbf{C}$ are *equivalent* if there exists an equivalence $f\colon A \to B$; we write in this case $A \simeq B$.

1.1.3. Definition. A $(2,1)$-*category* $\mathbf{C}$ is a 2-category in which all the 2-morphisms are invertible.

1.1.4. Definition. Let $\mathbf{C}, \mathbf{D}$ be 2-categories. A *pseudofunctor* $\mathbb{F}$ from $\mathbf{C}$ and $\mathbf{D}$ is the data of

- for each object A of $\mathbf{C}$, an object $\mathbb{F}A$ of $\mathbf{D}$
- for each pair of objects A, B of $\mathbf{C}$, a functor
$$\mathbb{F}\colon \mathbf{C}(A,B) \to \mathbf{D}(\mathbb{F}A, \mathbb{F}B)$$
- for each object A of $\mathbf{C}$, a natural isomorphism $\mathbb{F}^0$
$$\begin{tikzcd} \mathbf{1} \arrow[r, "\mathrm{Id}"] \arrow[dr, bend right, "\mathrm{Id}"'] & \mathbf{C}(A,A) \arrow[d, "\mathbb{F}"] \\ & \mathbf{D}(\mathbb{F}A, \mathbb{F}A) \end{tikzcd} \quad (\mathbb{F}^0 \Rightarrow)$$
- for each objects A, B, C of $\mathbf{C}$, a natural isomorphism $\mathbb{F}^2$
$$\begin{tikzcd} \mathbf{C}(B,C) \times \mathbf{C}(A,B) \arrow[r, "\circ"] \arrow[d, "\mathbb{F}"'] & \mathbf{C}(A,C) \arrow[d, "\mathbb{F}"] \\ \mathbf{D}(\mathbb{F}B, \mathbb{F}C) \times \mathbf{D}(\mathbb{F}A, \mathbb{F}B) \arrow[r, "\circ"] \arrow[ur, Rightarrow, "\mathbb{F}^2"] & \mathbf{D}(\mathbb{F}A, \mathbb{F}C) \end{tikzcd}$$

subject to the relations given by the following commutative diagrams:

$$\begin{tikzcd} \mathbb{F}h \circ \mathbb{F}g \circ \mathbb{F}f \arrow[r, "\mathbb{F}h \circ \mathbb{F}^2_{g,f}"] \arrow[d, "\mathbb{F}^2_{h,g} \circ \mathbb{F}f"'] & \mathbb{F}h \circ \mathbb{F}(g \circ f) \arrow[d, "\mathbb{F}^2_{h, g\circ f}"] \\ \mathbb{F}(h \circ g) \circ \mathbb{F}f \arrow[r, "\mathbb{F}^2_{h\circ g, f}"'] & \mathbb{F}(h \circ g \circ f) \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{F}f \circ \mathbb{F}\,\mathrm{Id}_A \arrow[dr, "\mathbb{F}^2_{\mathrm{Id}_A, f}"'] & \mathbb{F}f \arrow[l, "\mathbb{F}f \circ \mathbb{F}^0_A"'] \arrow[r, "\mathbb{F}^0_B \circ \mathbb{F}f"] \arrow[d, equal] & \mathbb{F}\,\mathrm{Id}_B \circ \mathbb{F}f \arrow[dl, "\mathbb{F}^2_{f, \mathrm{Id}_B}"] \\ & \mathbb{F}f & \end{tikzcd}$$

1.1.5. Definition. A *2-functor* $\mathbb{F}\colon \mathbf{C} \to \mathbf{D}$ is a pseudofunctor where all the natural isomorphisms $\mathbb{F}^0$ and $\mathbb{F}^2$ are identities.

1.1.6. Example. Let $\mathbf{C}, \mathbf{D}$ be 2-categories and let D be an object of $\mathbf{D}$. The constant 2-functor at D is the 2-functor
$$\Delta D\colon \begin{cases} \mathbf{C} & \to & \mathbf{D} \\ C & \mapsto & D \\ f & \mapsto & \mathrm{Id}_D \\ \phi & \mapsto & \mathrm{Id}_{\mathrm{Id}_D} \end{cases}$$
We will regularly use the above notation to define 2-functors. The three mappings respectively refer to the mapping of objects, of 1-morphisms and of 2-morphisms by the 2-functor.

1.1.7. Remark. Whenever we work with a single 2-functor $\mathbb{F}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$, we will use the following shortened notations:

- For a 1-morphism $f\colon A \to B$ of $\mathbf{C}$, the symbol f^* denotes the 1-morphism $\mathbb{F}(f)\colon \mathbb{F}(B) \to \mathbb{F}(A)$.

- For a 2-morphism $\phi\colon f \to g$ of $\mathbf{C}$, the symbol ϕ^* denotes the 2-morphism $\mathbb{F}(\phi)\colon \mathbb{F}(f) \to \mathbb{F}(g)$.

1.1.8. Definition. A *pseudonatural transformation* ϕ between pseudofunctors $\mathbb{F}, \mathbb{G}\colon \mathbf{C} \to \mathbf{D}$ is the data of

- for each object A of $\mathbf{C}$, a 1-morphism $\phi_A\colon \mathbb{F}A \to \mathbb{G}A$
- for each 1-morphism $f\colon A \to B$ in $\mathbf{C}$, a 2-isomorphism ϕ_f

$$\begin{array}{ccc} \mathbb{F}A & \xrightarrow{\mathbb{F}f} & \mathbb{F}B \\ {\scriptstyle \phi_A}\downarrow & \overset{\phi_f}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle \phi_B} \\ \mathbb{G}A & \xrightarrow[\mathbb{G}f]{} & \mathbb{G}B \end{array}$$

making the following squares commute:

$$\begin{array}{ccc} \phi_A & = = = = = & \phi_A \\ {\scriptstyle \mathbb{G}^0_A\phi_A}\downarrow & & \downarrow{\scriptstyle \phi_A\mathbb{F}^0_A} \\ (\mathbb{G}\,\mathrm{Id}_A)\circ\phi_A & \xrightarrow[\phi_{\mathrm{Id}_A}]{} & \phi_A \circ (\mathbb{F}\,\mathrm{Id}_A) \end{array}$$

$$\begin{array}{ccccc} \mathbb{G}f\circ\mathbb{G}g\circ\phi_A & \xrightarrow{(\mathbb{G}f)\phi_g} & \mathbb{G}f\circ\phi_B\circ\mathbb{F}g & \xrightarrow{\phi_f(\mathbb{F}g)} & \phi_C\circ\mathbb{F}f\circ\mathbb{F}g \\ {\scriptstyle \mathbb{G}^2_{f,g}\phi_A}\downarrow & & & & \downarrow{\scriptstyle \phi_C\mathbb{F}^2_{f,g}} \\ \mathbb{G}(f\circ g)\circ\phi_A & & \xrightarrow[\phi_{f\circ g}]{\qquad\qquad\qquad} & & \phi_C\circ\mathbb{F}(f\circ g) \end{array}$$

1.1.9. Definition. A *modification* m between two pseudonatural transformations $\phi, \psi\colon \mathbb{F} \to \mathbb{G}\colon \mathbf{C} \to \mathbf{D}$ is the data, for each object A in $\mathbf{C}$, of a 2-morphism $m_A\colon \phi_A \to \psi_A$ subject to the relations

$$\begin{array}{ccc} \mathbb{G}f\circ\phi_A & \xrightarrow{(\mathbb{G}f)m_A} & \mathbb{G}f\circ\psi_A \\ {\scriptstyle \phi_f}\downarrow & & \downarrow{\scriptstyle \psi_f} \\ \phi_B\circ\mathbb{F}f & \xrightarrow[m_B(\mathbb{F}f)]{} & \psi_B\circ\mathbb{F}f \end{array}$$

1.1.10. Example. We now have several examples of 2-categories:

- the 2-category 1 with exactly one object and the identities as 1-morphisms and 2-morphisms.
- more generally, if C is a 1-category, it can be viewed as a 2-category with only identities as 2-morphisms.
- the 2-category **Cat** of small categories, with functors as 1-morphisms and natural transformations as 2-morphisms.
- the $(2,1)$-category **gpd** of finite groupoids, with functors as 1-morphisms and natural transformations as 2-morphisms. It is a full, 2-full sub-2-category of **Cat**.
- the $(2,1)$-category $\mathbf{gpd}^{\mathrm{f}}$ of finite groupoids, with *faithful* functors as 1-morphisms and natural transformations as 2-morphisms.
- Given two 2-categories $\mathbf{C}$ and $\mathbf{D}$, the 2-category $[\mathbf{C}, \mathbf{D}]$ of pseudofunctors between $\mathbf{C}$ and $\mathbf{D}$, with pseudonatural transformations as 1-morphisms and modifications as 2-morphisms.

1.1.11. Notation. Let $\mathbf{C}, \mathbf{D}$ be 2-categories and $\mathbb{F}, \mathbb{G}\colon \mathbf{C} \to \mathbf{D}$ be pseudofunctors. We use the notation $\mathrm{PsNat}(\mathbb{F}, \mathbb{G})$ as a short-hand for the category $[\mathbf{C}, \mathbf{D}](\mathbb{F}, \mathbb{G})$ of pseudonatural transformations between $\mathbb{F}$ and $\mathbb{G}$ and modifications.

1.1.12. Definition. Let $\mathbf{C}$, $\mathbf{D}$ be 2-categories and let $\mathbb{F}\colon \mathbf{C} \to \mathbf{D}$ be a pseudofunctor. The pseudofunctor $\mathbb{F}$ is a *biequivalence* if there exists a pseudofunctor $\mathbb{G}\colon \mathbf{D} \to \mathbf{C}$ and two equivalences $\mathbb{F}\mathbb{G} \simeq \mathrm{Id}_{\mathbf{D}}$ (in the 2-category $[\mathbf{D}, \mathbf{D}]$) and $\mathbb{G}\mathbb{F} \simeq \mathrm{Id}_{\mathbf{C}}$ (in the 2-category $[\mathbf{C}, \mathbf{C}]$).

We should cite the following results, which mostly allows us to only consider 2-functors.

1.1.13. Proposition. *Let* $\mathbf{C}$ *be a small* 2*-category. Note* $\mathbf{2Func}(\mathbf{C}, \mathbf{Cat})$ *the* 2*-category of* 2-functors *(with pseudonatural transformations and modifications), and recall that* $[\mathbf{C}, \mathbf{Cat}]$ *is the* 2*-category of* pseudofunctors *(with pseudonatural transformations and modifications). Then the inclusion* 2*-functor*

$$\mathbf{2Func}(\mathbf{C}, \mathbf{Cat}) \to [\mathbf{C}, \mathbf{Cat}]$$

is a biequivalence.

Proof. This is precisely [Pow89, §4.2] □

We have already seen that 1-categories are examples of 2-categories. Conversely, a 2-category canonically define two associated 1-categories.

1.1.14. Definition. Let $\mathbf{C}$ be a 2-category. The truncated 1-category $\tau_1\mathbf{C}$ of $\mathbf{C}$ is the 1-category with:

- *Objects:* the objects X of $\mathbf{C}$
- *Morphisms:* the equivalence class of 1-morphisms of $\mathbf{C}$ up to isomorphism.
- *Composition* is induced by the composition of 1-morphisms in $\mathbf{C}$.

Moreover there is a canonical projection 2-functor

$$\pi\colon \mathbf{C} \to \tau_1(\mathbf{C})$$

1.1.15. Definition. Let $\mathbf{C}$ be a 2-category. The underlying 1-category $\mathbf{C}^{(1)}$ of $\mathbf{C}$ is the 1-category with:

- *Objects:* the objects X of $\mathbf{C}$
- *Morphisms:* the 1-morphisms of $\mathbf{C}$.
- *Composition* is the composition of 1-morphisms in $\mathbf{C}$.

Moreover there is a canonical injection 2-functor

$$\iota\colon \mathbf{C}^{(1)} \to \mathbf{C}$$

1.1.16. Definition. Let $\mathbf{C}$ be a 2-category and $f\colon X \to Y$ a 1-morphism of $\mathbf{C}$.

- f is *faithful* if for any object W, the induced functor

  $$f_*\colon \mathbf{C}(W, X) \to \mathbf{C}(W, Y)$$

  is faithful.
- f is *full* if for any object W, the induced functor

  $$f_*\colon \mathbf{C}(W, X) \to \mathbf{C}(W, Y)$$

  is full.

1.1.17. Remark. In the 2-category $\mathbf{Cat}$ of small 1-categories, a 1-morphism is faithful (in the 2-categorical sense) if and only if it is a faithful functor (in the usual sense). The same is true for fullness.

Common constructions. We present the constructions of the opposite 2-category and the arrow 2-category.

The opposite 2-category is obtained by reversing the orientation of all 1-morphisms (but keeping the orientation of the 2-morphisms). Similarly to the 1-categorical construction, this is generally useful to define dual notions, to obtain dual results and to define contravariant 2-functors.

1.1.18. Definition. Let $\mathbf{C}$ be a 2-category. The *opposite* 2-*category* $\mathbf{C}^{\mathrm{op}}$ of $\mathbf{C}$ is the 2-category with:

- *Objects:* the objects C of $\mathbf{C}$.
- 1-*Morphisms* $C_1 \to C_2$*:* the 1-morphisms $f \colon C_2 \to C_1$ of $\mathbf{C}$.
- 2-*Morphisms* $f_1 \to f_2$*:* the 2-morphisms $\phi \colon f_1 \to f_2$.

The arrow 2-category is a (higher) pullback for 2-categories; we are mainly interested in specific instances of this construction: the slice 2-categories.

1.1.19. Definition. Let $\mathbb{R} \colon \mathbf{A} \to \mathbf{C}$ and $\mathbb{S} \colon \mathbf{B} \to \mathbf{C}$ be two 2-functors both landing in $\mathbf{C}$. The *(pseudo) arrow* 2-*category* $\mathbb{R} \downarrow \mathbb{S}$ is the 2-category with:

- *Objects:* the triples (A, B, f) consisting of an object A of $\mathbf{A}$, an object B of $\mathbf{B}$ and a 1-morphism $f \colon \mathbb{R}a \to \mathbb{S}b$ of $\mathbf{C}$.
- 1-*Morphisms* $(A_1, B_1, f_1) \to (A_2, B_2, f_2)$*:* the triples (r, s, ϕ) consisting of a 1-morphism $r \colon A_1 \to A_2$ of $\mathbf{A}$, a 1-morphism $s \colon B_1 \to B_2$ of $\mathbf{B}$ and an *invertible* 2-morphism ϕ:

$$\begin{array}{ccc} \mathbb{R}A_1 & \xrightarrow{f_1} & \mathbb{S}B_1 \\ {\scriptstyle \mathbb{R}r}\big\downarrow & \overset{\phi}{\underset{\sim}{\nearrow}} & \big\downarrow{\scriptstyle \mathbb{S}s} \\ \mathbb{R}A_2 & \xrightarrow[f_2]{} & \mathbb{S}B_2 \end{array}$$

- 2-*Morphisms* $(r_1, s_1, \phi_1) \to (r_2, s_2, \phi_2)$*:* the pairs (ρ, σ) consisting of a 2-morphism $\rho \colon r_1 \to r_2$ and a 2-morphism $\sigma \colon s_1 \to s_2$ such that:

$$\begin{array}{ccc} \mathbb{R}A_1 & \xrightarrow{f_1} & \mathbb{S}B_1 \\ {\scriptstyle \mathbb{R}r_1}\big\downarrow & \overset{\phi_1}{\underset{\sim}{\nearrow}} & {\scriptstyle \mathbb{S}s_1}\big\downarrow \overset{\mathbb{S}\sigma}{\Rightarrow} \big\downarrow{\scriptstyle \mathbb{S}s_2} \\ \mathbb{R}A_2 & \xrightarrow[f_2]{} & \mathbb{S}B_2 \end{array} \quad = \quad \begin{array}{ccc} \mathbb{R}A_1 & \xrightarrow{f_1} & \mathbb{S}B_1 \\ {\scriptstyle \mathbb{R}r_1}\big\downarrow \overset{\mathbb{R}\rho}{\Rightarrow} \big\downarrow{\scriptstyle \mathbb{R}r_2} & \overset{\phi_2}{\underset{\sim}{\nearrow}} & \big\downarrow{\scriptstyle \mathbb{S}s_2} \\ \mathbb{R}A_2 & \xrightarrow[f_2]{} & \mathbb{S}B_2 \end{array}$$

1.1.20. Example. We are specifically interested in a few instances of the arrow construction:

- Let $\mathbf{C}$ be a 2-category, C an object of $\mathbf{C}$. The *(pseudo) slice* 2-*category* $\mathbf{C}/C$ *over* C is the arrow 2-category $\mathrm{Id}_{\mathbf{C}} \downarrow \Delta C$ between the identity $\mathrm{Id}_{\mathbf{C}}$ of $\mathbf{C}$ and the constant 2-functor $\Delta C \colon \mathbf{1} \to \mathbf{C}$.
- Let $\mathbf{C}$ be a 2-category, C an object of $\mathbf{C}$. Conversely, the *(pseudo) slice* 2-*category* $C/\mathbf{C}$ *under* C is the arrow 2-category $\Delta C \downarrow \mathrm{Id}_{\mathbf{C}}$.
- Let $\mathbf{C}$,$\mathbf{D}$ be 2-categories, $\mathbb{I} \colon \mathbf{C} \to \mathbf{D}$ a 1-full, 2-full and faithful 2-functor and D an object of $\mathbf{D}$. By a slight abuse of notation, we denote by $D/\mathbf{C}$ the arrow 2-category $\Delta D \downarrow \mathbb{I}$.
- Let $\mathbf{C}, \mathbf{D}$ be a 2-categories, C an object of $\mathbf{C}$ and $\mathbb{F} \colon \mathbf{D} \to \mathbf{C}$. The *(pseudo) slice* 2-*category* $C/\mathbb{F}$ *of* $\mathbb{F}$ *under* C is the arrow 2-category $\Delta C \downarrow \mathbb{F}$.

Adjunctions and monads. We recall the definitions of adjunctions and monads, and some of their basic properties in **Cat**. We will not give any proof, further details can be found in [JY21, §6] for the general case, and in [Mac71, §IV and §VI] for the case of **Cat**.

1.1.21. Definition. Let $\mathbf{C}$ be a 2-category and C, D be two objects of $\mathbf{C}$. An adjunction between C and D is a quadruple $(\ell, r, \eta, \epsilon)$ of a 1-morphism $\ell\colon C \to D$, a 1-morphism $r\colon D \to C$, a 2-morphism $\eta\colon \mathrm{Id}_C \Rightarrow r\ell$ and a 2-morphism $\epsilon\colon \ell r \Rightarrow \mathrm{Id}_D$, such that:

$$\mathrm{Id}_\ell = \epsilon\ell \circ \ell\eta \tag{1.1.22}$$

$$\mathrm{Id}_r = r\epsilon \circ \eta r \tag{1.1.23}$$

We generally denote an adjunction $\ell \dashv r$, with the 2-morphisms η and ϵ omitted. The 2-morphism η is called the *unit* of the adjunction and the 2-morphism ϵ is called the *counit* of the adjunction.

1.1.24. Definition. Let $\mathbf{C}$ be a 2-category and C be an object of $\mathbf{C}$. A monad on C is a monoid object in the monoidal category $(\mathbf{C}(C,C), \circ, \mathrm{Id}_C)$ of endomorphisms of C, that is, a triple (T, η, μ) consisting of a 1-morphism $\mathsf{T}\colon C \to C$, a 2-morphism $\eta\colon \mathrm{Id}_C \to \mathsf{T}$ and a 2-morphism $\mu\colon \mathsf{T}\circ\mathsf{T} \Rightarrow \mathsf{T}$ such that the following diagrams commute:

$$\begin{array}{ccc} \mathsf{T}\circ\mathsf{T}\circ\mathsf{T} & \xRightarrow{\mathsf{T}\mu} & \mathsf{T}\circ\mathsf{T} \\ {\scriptstyle \mu\mathsf{T}}\Downarrow & & \Downarrow{\scriptstyle \mu} \\ \mathsf{T}\circ\mathsf{T} & \xRightarrow[\mu]{} & \mathsf{T} \end{array} \qquad \begin{array}{ccccc} \mathsf{T}\circ\mathsf{T} & \xLeftarrow{\mathsf{T}\eta} & \mathsf{T} & \xRightarrow{\eta\mathsf{T}} & \mathsf{T}\circ\mathsf{T} \\ & {\scriptstyle \mu}\searrow & \| & \swarrow{\scriptstyle \mu} & \\ & & \mathsf{T} & & \end{array}$$

1.1.25. Proposition. *Let $\mathbf{C}$ be a 2-category, C, D be two objects of $\mathbf{C}$ and $(\ell\colon C \to D, r\colon D \to C, \eta, \epsilon)$ be an adjunction $\ell \dashv r$ between C and D. Then the triple $(r\ell, \eta, r\epsilon\ell)$ defines a monad on C.*

1.1.26. Definition. Let $\mathbf{C}$ be a 2-category, C be an object of $\mathbf{C}$ and (T, η, μ) be a monad on C. A *left module* on T, or left T-module, is an object D of $\mathbf{C}$ endowed with a 1-morphism $v\colon D \to C$ and a 2-morphism $\nu\colon \mathsf{T}v \Rightarrow v$ such that the following diagrams commute

$$\begin{array}{ccc} \mathsf{T}\mathsf{T}v & \xRightarrow{\mu v} & \mathsf{T}v \\ {\scriptstyle \mathsf{T}\nu}\Downarrow & & \Downarrow{\scriptstyle \nu} \\ \mathsf{T}v & \xRightarrow[\nu]{} & v \end{array} \tag{1.1.27}$$

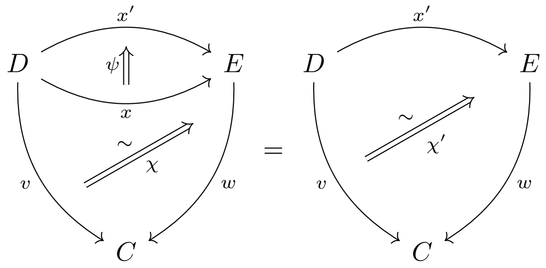
$$\begin{array}{ccc} v & \xRightarrow{\eta V} & \mathsf{T}v \\ \| & \swarrow{\scriptstyle \nu} & \\ v & & \end{array} \tag{1.1.28}$$

1.1.29. Definition. Let $\mathbf{C}$ be a 2-category, C be a category and (T, η, μ) be a monad on C. The *2-category of left T-modules* is the category with:

- *Objects:* the left T-modules.
- *1-morphisms $(D, v, \nu) \to (E, w, \omega)$:* the pairs (x, χ) consisting of a 1-morphism $x\colon D \to E$ and a 2-isomorphism $\chi\colon v \xRightarrow{\sim} wx$ such that:

$$\chi \circ \nu = \omega x \circ \mathsf{T}\chi$$

- *2-morphisms* $(x, \chi) \Rightarrow (x', \chi')$*:* the 2-morphisms $\psi : x \Rightarrow x'$ such that:



1.1.30. Definition. Let $\mathbf{C}$ be a 2-category, C be a category and (T, η, μ) be a monad on C. An *Eilenberg-Moore object* $(C^{\mathsf{T}}, U, \bar{\mu})$ for the monad T is a biterminal object in the 2-category of the left T-modules: for any other left T-module (D, v, ν), there is a 1-morphism $(k, \kappa) \colon (D, v, \nu) \to (C^{\mathsf{T}}, u, \bar{\mu})$, unique up to a unique 2-isomorphism.

1.1.31. Remark. Let $\mathbf{C}$ be a 2-category, C, D be two objects of $\mathbf{C}$ and $(\ell \colon C \to D, r \colon D \to C, \eta, \epsilon)$ be an adjunction $\ell \dashv r$ between C and D. Note $(\mathsf{T}, \mu, \epsilon)$ the associated monad on C, and assume it has an Eilenberg-Moore object $(C^{\mathsf{T}}, u, \bar{\mu})$. The object D endowed with the morphism r and natural transformation $r\epsilon \colon \mathsf{T}r \to r$ is a left T-module. Hence there is a canonical morphism in $\mathbf{C}$:

$$\tilde{r} \colon D \to C^{\mathsf{T}}$$

1.1.32. Definition. Let $\mathbf{C}$ be a 2-category, C, D be two objects of $\mathbf{C}$ and $(\ell \colon C \to D, r \colon D \to C, \eta, \epsilon)$ be an adjunction $\ell \dashv r$ between C and D. Denote $(\mathsf{T}, \mu, \epsilon)$ the associated monad on C. The adjunction $\ell \dashv r$ is *monadic* if the monad T has an Eilenberg-Moore object $(C^{\mathsf{T}}, u, \bar{\mu})$ and the canonical morphism

$$k \colon D \to C^{\mathsf{T}}$$

is an equivalence.

1.1.33. Definition. Let $\mathbf{C}, \mathbf{D}$ be 2-categories, $\mathcal{M}$ be a class of 1-morphisms of $\mathbf{C}$ and $\mathbb{F} \colon \mathbf{C} \to \mathbf{D}$ be a 2-functor. The 2-functor $\mathbb{F}$ is $\mathcal{M}$-monadic if for each 1-morphism $m \in \mathcal{M}$, its image $\mathbb{F}m$ has a left adjoint $m_!$ and the adjunction $m_! \dashv \mathbb{F}m$ is monadic.

Any monad in $\mathbf{C} = \mathbf{Cat}$ has an Eilenberg-Moore object, as follows.

1.1.34. Definition. Let C be a category and T be a monad on C. The *Eilenberg-Moore category* C^{T} of T is the category with:

- *Objects:* the pairs (c, f) with C an object of C and $f \colon \mathsf{T}c \to c$ a morphism of C.
- *Morphisms* $(c, f) \to (c', f')$*:* the morphisms $g \colon c \to c'$ such that the following square commutes:

$$\begin{array}{ccc} \mathsf{T}c & \xrightarrow{\mathsf{T}g} & \mathsf{T}c' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ c & \xrightarrow[g]{} & c' \end{array}$$

- *Composition* is induced by the composition of C.

There is a canonical functor

$$u \colon \begin{cases} C^{\mathsf{T}} & \to \quad C \\ (c, f) & \mapsto \quad c \\ g & \mapsto \quad g \end{cases}$$

and a natural transformation $\bar{\mu} \colon \mathsf{T}u \Rightarrow u$ with components

$$\bar{\mu}_{(c,f)} = f$$

1.1.35. Proposition. *Let C be a category and T be a monad on C. The Eilenberg-Moore category C^{T} endowed with U and $\bar{\mu}$ (definition 1.1.34) is a left module on T (definition 1.1.26). Moreover it is an Eilenberg-Moore object on T.*

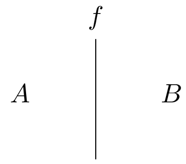
String diagrams. String diagrams are used to depict 2-morphisms in a 2-category. They are the dual diagrams of the "usual" pasting diagrams. We recall the general ideas of string diagrams, and introduce our notations. A fully detailed description of string diagrams can be found in [JY21, §3.7].

1.1.36. Notation. We will use the following notations for string diagrams:

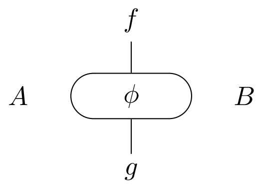
- An object A is represented by a labeled surface

$$A$$

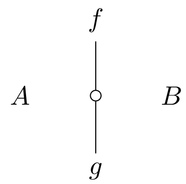
- A 1-morphism $f\colon A \to B$ is represented by a labeled vertical edge



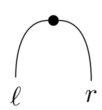
  with the source on the left and the target on the right.
- A 2-morphism $\phi\colon f \Rightarrow g\colon A \to B$ is represented by a labeled vertex



  with the source above and the target below. The identity 1-morphisms may be omitted.
- We will occasionally represent identity 2-morphisms by a white dot, that is, the string diagram



  is the identity between the 1-morphisms f and g.
- When dealing with an adjunction $\ell \dashv r$ (definition 1.1.21), units and counits will be represented by black dots. For instance, the string diagram



  is the unit of the adjunction $\ell \dashv r$.

1.1.37. Remark. The notation for the unit and the counit of an adjunction $\ell \dashv r$ gives a simple interpretation of the unit-counit laws (eq. (1.1.22) and eq. (1.1.23)):

1.2. Pseudo bilimits

Generalities. Pseudo bilimits are the natural adaptations of limits to 2-categories.

1.2.1. Definition. Let $\mathbf{I}$, $\mathbf{C}$ be 2-categories and $\mathbb{D}\colon \mathbf{I} \to \mathbf{C}$ a pseudofunctor. A *(pseudo) bilimit* of $\mathbb{D}$ is an object L of $\mathbf{C}$ together with a pseudonatural equivalence:

$$\Psi\colon \mathbf{C}(-, L) \simeq [\mathbf{I}, \mathbf{C}](\Delta -, \mathbb{D})$$

The pseudonatural transformation $\Psi_L(\mathrm{Id}_L)$ is the *standard cone* of L.

1.2.2. Remark. We call a pseudonatural transformation $\Delta X \to \mathbb{D}$ a cone over $\mathbb{D}$ with vertex X. The category $[\mathbf{I}, \mathbf{C}](\Delta X, \mathbb{D})$ of pseudonatural transformations and modifications is the category of cones over $\mathbb{D}$ with vertex X.

1.2.3. Remark. Any pseudonatural transformation $\Delta L \to \mathbb{D}$ induces (by applying Δ and whiskering) a pseudonatural transformation:

$$\mathbf{C}(-, L) \to [\mathbf{I}, \mathbf{C}](\Delta -, \mathbb{D})$$

In particular, for a bilimit (L, Ψ), the pseudonatural transformation induced by the standard cone $\Psi_L(\mathrm{Id}_L)$ is isomorphic to Ψ. We will generally describe a bilimit as a pair (L, ψ) with L an object of $\mathbf{C}$ and $\psi\colon \Delta L \to \mathbb{D}$ a cone over $\mathbb{D}$ with vertex L, which corresponds to the standard cone.

1.2.4. Remark. Let $\mathbf{I}$, $\mathbf{C}$, $\mathbf{D}$ be 2-categories, and $\mathbb{D}\colon \mathbf{I} \to \mathbf{C}$, $\mathbb{F}\colon \mathbf{C} \to \mathbf{D}$ be pseudofunctors. Let $\psi\colon \Delta X \to \mathbb{D}$ be a cone over D with vertex X. Then, the 2-functor $\mathbb{F}$ induces a cone

$$\mathbb{F}\psi\colon \Delta\mathbb{F}X \cong \mathbb{F}\Delta X \to \mathbb{F}\mathbb{D}$$

over $\mathbb{F}\mathbb{D}$ with vertex $\mathbb{F}X$.

1.2.5. Definition. Let $\mathbf{C}$, $\mathbf{D}$ be 2-categories and $\mathbb{F}\colon \mathbf{C} \to \mathbf{D}$ a pseudofunctor.

- The 2-functor $\mathbb{F}$ *preserves bilimits* if for any 2-functor $\mathbb{D}\colon \mathbf{I} \to \mathbf{C}$ which has a bilimit (L, ψ) in $\mathbf{C}$, the pair $(\mathbb{F}L, \mathbb{F}\psi)$ is a bilimit of the 2-functor $\mathbb{F}\mathbb{D}$ in $\mathbf{D}$.
- The 2-functor $\mathbb{F}$ *reflects bilimits* if, for any 2-functor $\mathbb{D}\colon \mathbf{I} \to \mathbf{C}$ and for any pair (L, ψ) of an object L of $\mathbf{C}$ and a cone $\psi\colon \Delta L \to \mathbb{D}$ such that $(\mathbb{F}L, \mathbb{F}\psi)$ is a bilimit of $\mathbb{F}\mathbb{D}$ in $\mathbf{D}$, the pair (L, ψ) is a bilimit of $\mathbb{D}$ in $\mathbf{C}$.

Biproducts. A biproduct is the counterpart of a product for bilimits. The terminology is slightly unfortunate, as the term biproduct is already used in the literature to denote an object of a category which is both a product and a coproduct. In this work, it will always denote the bilimit we present in this subsection.

1.2.6. Definition. Let n be a positive integer. The 2-category $\mathbf{n}$ is the 2-category with n objects $1, \ldots, n$ and only identities as 1-morphisms and 2-morphisms.

1.2.7. Definition. Let $\mathbf{C}$ be a 2-category and $A_1, \ldots, A_n$ be n objects of $\mathbf{C}$. The objects (A_i) define a diagram:

$$\begin{cases} \mathbf{n} & \to & \mathbf{C} \\ k & \mapsto & A_k \end{cases}$$

Its bilimit, if it exists, is called the biproduct of $A_1, \ldots, A_n$ and is noted $A_1 \times \ldots \times A_n$.

1.2.8. Remark. We have chosen to use the usual notation of products for biproduct. This is only slightly ambiguous, as a product of $A_1, \ldots, A_n$ is also a biproduct of $A_1, \ldots, A_n$. However, the converse is not true: a biproduct of $A_1, \ldots, A_n$ may exist even if there is no product of $A_1, \ldots, A_n$.

We will set up some general conventions when using biproducts.

1.2.9. Notation. The structural 1-morphisms are underlined and the structural 2-morphisms are overlined. By definition, the biproduct $A_1 \times \ldots \times A_n$ is endowed with n projection morphisms:

$$\underline{i} \colon A_1 \times \ldots \times A_n \to A_i \qquad 1 \leq i \leq n$$

For an ordered subset $\{i_1 < \ldots < i_k\} \subset \{1, \ldots, n\}$, there is a canonical 1-morphism induced by the projections $\underline{i_1}, \ldots \underline{i_k}$,

$$\underline{i_1 \ldots i_k} \colon A_1 \times \ldots \times A_n \to A_{i_1} \times \ldots \times A_{i_k}$$

with, for each $1 \leq j \leq k$, a structural 2-isomorphism:

$$(\overline{i_1 \ldots i_k})_j \colon \underline{j} \circ \underline{i_1 \ldots i_k} \Rightarrow \underline{i_j}$$

In the case $n = 3$, we can unfold these notations. There are three 1-morphisms:

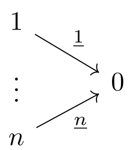
$$\underline{12} \colon A_1 \times A_2 \times A_3 \to A_1 \times A_2$$
$$\underline{13} \colon A_1 \times A_2 \times A_3 \to A_1 \times A_3$$
$$\underline{23} \colon A_1 \times A_2 \times A_3 \to A_2 \times A_3$$

The structural 2-isomorphisms can be combined to form:

$$\overline{1} = (\overline{13}_1)^{-1} \circ \overline{12}_1 \colon \underline{1} \circ \underline{12} \Rightarrow \underline{1} \circ \underline{13}$$
$$\overline{2} = (\overline{23}_1)^{-1} \circ \overline{12}_2 \colon \underline{2} \circ \underline{12} \Rightarrow \underline{1} \circ \underline{23}$$
$$\overline{3} = (\overline{23}_2)^{-1} \circ \overline{13}_2 \colon \underline{2} \circ \underline{13} \Rightarrow \underline{2} \circ \underline{23}$$

Bipullback. A bipullback is a straightforward analog of pullback for bilimits. They are usually only defined for a pair of 1-morphisms. Since they are associative (up-to equivalence), the bipullback of n 1-morphisms can be essentially defined by iterating the pullback of two 1-morphisms. We prefer giving a direct, unbiased definition of the pullback n 1-morphisms.

1.2.10. Definition. Let n be a positive integer. The (1-)category $\mathbf{Confl}_n$ is the category freely generated on the graph:



1.2.11. Definition. Let $\mathbf{C}$ be a 2-category, and $f_1 \colon X_1 \to X_0, \ldots, f_n \colon X_n \to X_0$ be n morphisms with the same target. The bipullback of $f_1, \ldots, f_n$ is the bilimit

$$(f_1 | \ldots | f_n) := \operatorname*{bilim}_{\mathbf{Confl}_n} F_{f_1, \ldots, f_n}$$

where $\mathbf{Confl}_n$ is seen as a locally discrete 2-category and

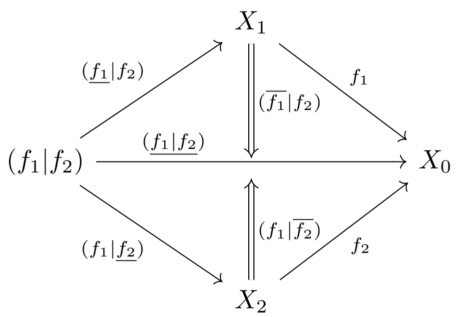
$$F_{f_1,\dots,f_n} : \begin{cases} \mathbf{Confl}_n & \to & \mathbf{C} \\ i & \mapsto & X_i & \text{for } 0 \le i \le n \\ \underline{i} & \mapsto & f_i & \text{for } 1 \le i \le n \end{cases}$$

1.2.12. Remark. Let $\mathbf{C}$ be a 2-category. A collection of morphisms $(f_i \colon X_i \to X)_i$ of $\mathbf{C}$ is precisely a collection of objects $(X_i, f_i)_i$ of the slice 2-category $\mathbf{C}/X$, and the bipullback $(f_1|\dots|f_n)$ in $\mathbf{C}$ corresponds to the biproduct $(X_1, f_1) \times \dots \times (X_n, f_n)$ in the slice.

1.2.13. Notation. We will take the following naming conventions for the components of the standard cone $\mathcal{C}$ of $(f_1|\dots|f_n)$:

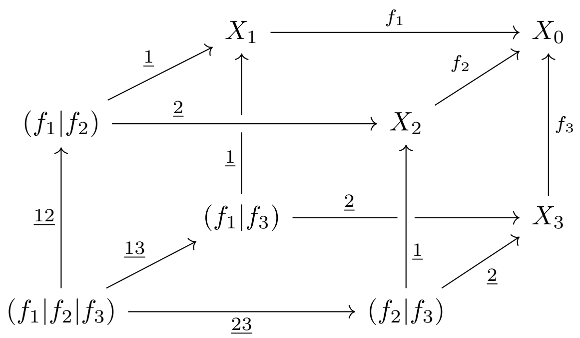
$$\underline{(f_1|\dots|f_n)} = \mathcal{C}_0 \colon (f_1|\dots|f_n) \to X_0$$
$$(f_1|\dots|\underline{f_i}|\dots|f_n) = \mathcal{C}_i \colon (f_1|\dots|f_n) \to X_i$$
$$(f_1|\dots|\overline{f_i}|\dots|f_n) = \mathcal{C}_{\underline{i}} \colon f_i \circ (f_1|\dots|\underline{f_i}|\dots|f_n) \to \underline{(f_1|\dots|f_n)}$$

We are mostly interested in the case where n is 2 or 3, since those are the one used to define descent for a 2-functor (definition 2.2.3). When $n = 2$, the standard cone can be represented as a square (with a diagonal):



Following remark 1.2.12, we will also use short notations of biproducts for the 1-morphisms $\underline{1} = (\underline{f_1}|f_2)$ and $\underline{2} = (f_1|\underline{f_2})$. When $n = 3$, we will use short notations of biproducts, which can be organized into a cube:

(1.2.14)



The faces adjacent to the bottom-left corner of the cube are:

$$\overline{1} \colon \underline{1} \circ \underline{12} \to \underline{1} \circ \underline{13}$$
$$\overline{2} \colon \underline{2} \circ \underline{12} \to \underline{1} \circ \underline{23}$$
$$\overline{3} \colon \underline{2} \circ \underline{13} \to \underline{2} \circ \underline{23}$$

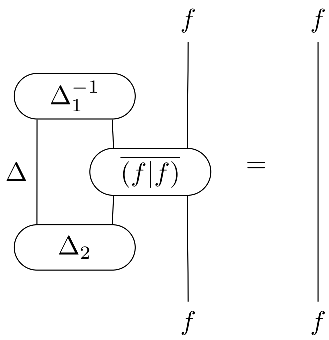
The faces adjacent to the top-right corner are the bipullbacks $\overline{(f_1|f_2)}$, $\overline{(f_1|f_3)}$ and $\overline{(f_2|f_3)}$.

1.2.15. Remark. The 2-morphisms $\overline{1}$, $\overline{2}$ and $\overline{3}$ are bipullback squares.

Properties of bipullbacks. We give an overview of properties of bipullback we will use throughout this thesis. These results are either direct application of the definition of bipullback or classical manipulations.

A bipullback of a 1-morphism with itself has an associated diagonal 1-morphism.

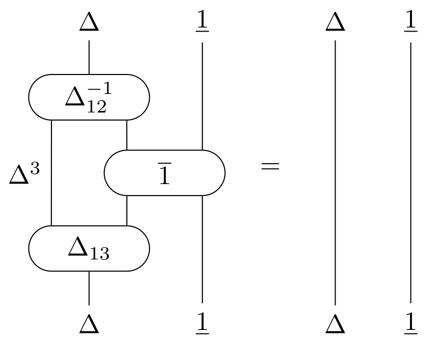
1.2.16. Lemma. *Let* $\mathbf{C}$ *be a* 2*-category and* $f\colon X \to Y$ *a morphism of* $\mathbf{C}$*. There is a unique (up to a unique* 2*-isomorphism)* 1*-morphism* $\Delta\colon X \to (f|f)$ *with, for each* $k = 1, 2$*, a structural* 2*-isomorphism* $\Delta_k\colon \underline{k} \circ \Delta \to \mathrm{Id}_X$ *such that:*



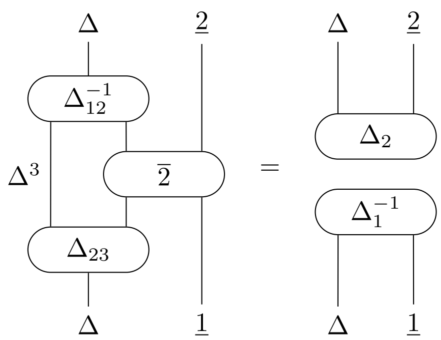
Similarly, a triple bipullback of a single 1-morphism has an associated diagonal 1-morphism.

1.2.17. Lemma. *Let* $\mathbf{C}$ *be a* 2*-category and* $f\colon X \to Y$ *a morphism of* $\mathbf{C}$*. There is a unique (up to a unique* 2*-isomorphism)* 1*-morphism* $\Delta^3\colon X \to (f|f|f)$ *with three* 2*-isomorphisms* $\Delta_{12}\colon \underline{12} \circ \Delta^3 \to \Delta$*,* $\Delta_{13}\colon \underline{13} \circ \Delta^3 \to \Delta$ *and* $\Delta_{23}\colon \underline{23} \circ \Delta^3 \to \Delta$*, satisfying the equations:*

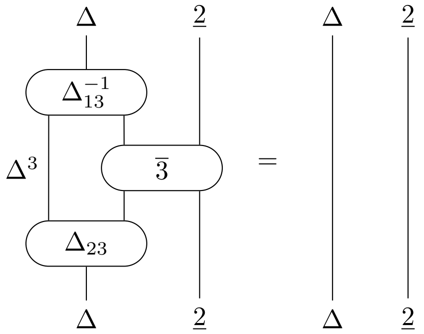
(1.2.18)



(1.2.19)



(1.2.20)

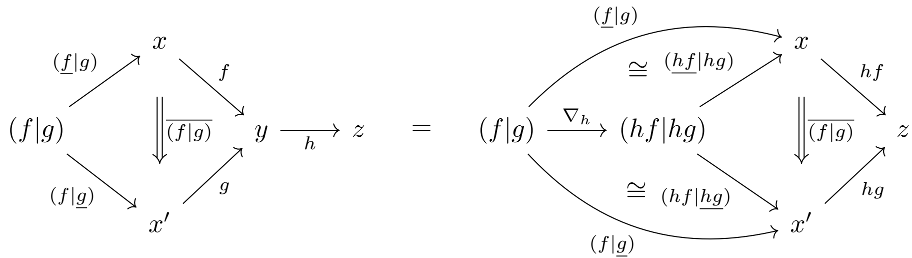


Applying the universal property of bipullbacks give simple results relating bipullbacks and composition of 1-morphisms.

1.2.21. Lemma. *Let* **C** *be a 2-category and* $f\colon x \to y$, $g\colon x' \to y$ *and* $h\colon y \to z$ *be morphisms of* **C**. *There is a unique (up to a unique isomorphism)* 1*-morphism*

$$\nabla_h\colon (f|g) \to (hf|hg)$$

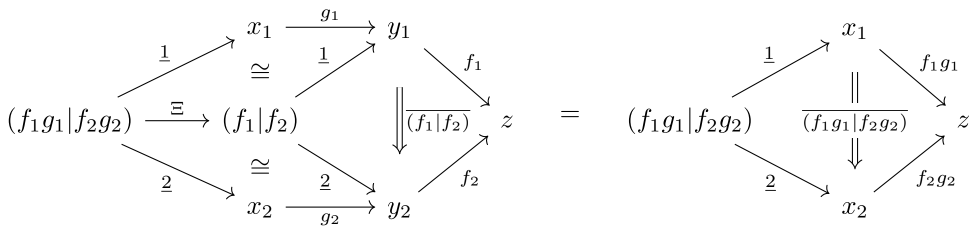
fitting in the equation:



1.2.22. Lemma. *Let* **C** *be a 2-category and* $f_1\colon y_1 \to z$, $f_2\colon y_2 \to z$, $g_1\colon x_1 \to y_1$ *and* $g_2\colon x_2 \to y_2$ *be morphisms of* **C**. *There is a unique (up to a unique isomorphism)* 1*-morphism*

$$\Xi\colon (f_1g_1|f_2g_2) \to (f_1|g_1)$$

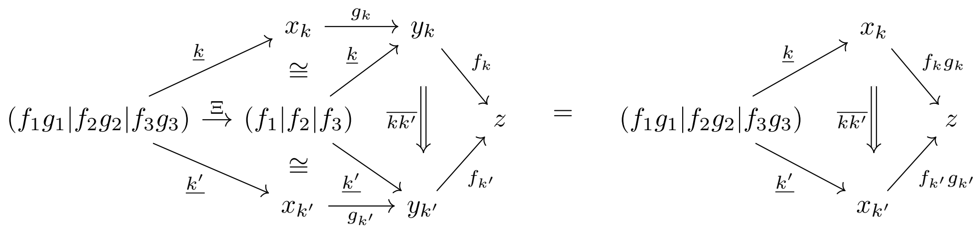
fitting in the equation:



1.2.23. Lemma. *Let* **C** *be a* 2*-category and* $f_1\colon y_1 \to z$, $f_2\colon y_2 \to z$, $f_3\colon y_3 \to z$, $g_1\colon x_1 \to y_1$ *and* $g_2\colon x_2 \to y_2$, $g_3\colon x_3 \to y_3$ *be morphisms of* **C**. *There is a unique (up to a unique isomorphism)* 1*-morphism*

$$\Xi\colon (f_1g_1|f_2g_2|f_3g_3) \to (f_1|f_2|f_3)$$

with three isomorphisms $\Xi^k : \underline{k}\Xi \to g_k\underline{k}$ *(for* $k = 1, 2, 3$*) such that for any* $k < k'$*,*



Similarly to (regular) pullbacks, a pasting of bipullbacks is a bipullback, and there is a partial converse result:

1.2.24. Lemma. *Let* $\mathbf{C}$ *a* 2*-category. Consider the diagram in* $\mathbf{C}$*:*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \uparrow & \cong & \uparrow{\scriptstyle h'} & \cong & \uparrow{\scriptstyle h} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

Assume that the rightmost inner square is a bipullback (of g *and* h*). Then the leftmost inner square is a bipullback (of* f *and* h'*) if and only the outer rectangle is a bipullback (of* gf *and* h*).*

An easy corollary is the following result, allowing to deal with bipullbacks of bipullbacks.

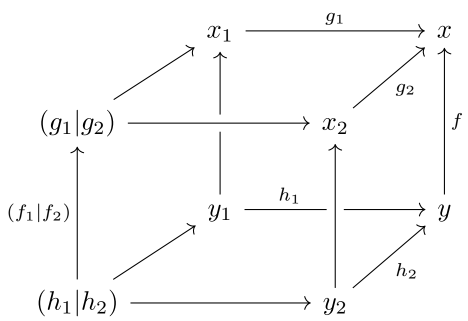
1.2.25. Lemma. *Let* $\mathbf{C}$ *be* 2*-category. Let* $f : y \to x$*,* $g_1 : x_1 \to x$ *and* $g_2 : x_2 \to y$*. We consider the bipullback, for* $l = 1, 2$*:*

$$\begin{array}{ccc} x_l & \xrightarrow{g_l} & x \\ {\scriptstyle f_l}\uparrow & & \uparrow{\scriptstyle f} \\ y_l & \xrightarrow[h_l]{} & y \end{array}$$

There is a unique (up to a unique isomorphism) 1*-morphism*

$$(f_1|f_2) : (h_1|h_2) \to (g_1|g_2)$$

fitting in the cube



Moreover all the faces of this cube are bipullbacks.

As a universal construction, bipullbacks are obviously pseudofunctorial.

1.2.26. Lemma. *Let* $\mathbf{C}$ *be a* 2*-category and* $f : y \to z$*,* $g : x \to z$ *and* $g' : x' \to z$ *be* 1*-morphisms of* $\mathbf{C}$*. Recall they define objects* $\mathbf{g} = (x, g)$ *and* $\mathbf{g}' = (x', g')$ *of the slice* 2*-category* $\mathbf{C}/z$*. Similarly their bipullbacks define objects* $(\underline{\mathbf{f}}|\mathbf{g}) = ((f|g), (\underline{f}|g))$ *and* $(\underline{\mathbf{f}}|\mathbf{g}') = ((f|g'), (\underline{f}|g'))$ *of the slice* 2*-category* $\mathbf{C}/y$*. There is a canonical functor:*

$$(f|-) : (\mathbf{C}/z)(\mathbf{g}, \mathbf{g}') \to (\mathbf{C}/y)((\underline{\mathbf{f}}|\mathbf{g}), (\underline{\mathbf{f}}|\mathbf{g}'))$$

1.2.27. Proposition. *Let* $\mathbf{C}$ *be* 2*-category and* $f\colon y \to z$ *be a* 1*-morphism of* $\mathbf{C}$*. Then the* 1*-morphism* f *defines a pseudofunctor*

$$(f|-)\colon \mathbf{C}/z \to \mathbf{C}/y$$

1.2.28. Remark. Strictly speaking, the 2-functor $(f|-)$ is only defined up to *equivalence* on objects, hence we actually need a choice of a 2-pullback $(f|g)$ for each 1-morphism $g\colon x \to z$.

Bipullbacks also preserves faithfulness and fullness of 1-morphisms.

1.2.29. Lemma. *Let* $\mathbf{C}$ *be a* 2*-category and* $f\colon x \to z$*,* $g\colon y \to z$ *be* 1*-morphisms of* $\mathbf{C}$*. Consider the bipullback:*

$$\begin{array}{ccc} x & \xrightarrow{f} & z \\ {\scriptstyle g'}\uparrow & \cong & \uparrow{\scriptstyle g} \\ (f|g) & \xrightarrow[f']{} & y \end{array}$$

Then:

- *if* f *is faithful,* f' *is also faithful*
- *if* f *is full,* f' *is also full*

1.3. 2-Final 2-functors

A 2-final 2-functor

$$\mathbb{F}\colon \mathbf{A} \to \mathbf{B}$$

is a functor that allows the restriction of 2-diagrams of shape $\mathbf{B}$ to $\mathbf{A}$ without changing their bicolimits. We present in this section a criterion characterizing 2-final 2-functors with homotopy invariants. This criterion can be seen as a 2-categorification of the one for final 1-functors [Mac71, §IX.3]. A related characterization for 2-filtered 2-functors is proved in [Des15, §1.3].

Combinatorial paths and homotopies in a 2-category. A 2-category $\mathbf{C}$ has an associated CW-complex $|\mathbf{C}|$, defined using the Duskin nerve [JY21, §5.4], which maps objects of C to vertices, 1-morphisms to 1-simplices and 2-morphisms to 2-simplices. There are thus notions of paths and homotopies of paths in $\mathbf{C}$. We give in this section a combinatorial approach to these.

We fix a 2-category $\mathbf{C}$.

1.3.1. Definition. A *path* (of 1-morphism) in $\mathbf{C}$ is a finite sequence of objects $(a_i)_{0\leq i\leq n}$ and a family of pairs $(\varepsilon_i, f_i)_{1\leq i\leq n}$ consisting of a sign $\varepsilon \in \{-1, 1\}$ and a morphism

$$f_i\colon \begin{cases} a_{i-1} & \to & a_i & \text{if } \varepsilon_i = 1 \\ a_i & \to & a_{i-1} & \text{if } \varepsilon_i = -1 \end{cases}$$

Such a path is said to have source a_0 and target a_n.

1.3.2. Notation. We write $p\colon a_0 \rightsquigarrow a_n$ to denote a path with source a_0 and target a_n.

A path can be pictured as a zig-zag of morphisms (potentially with consecutive morphisms in the same direction):

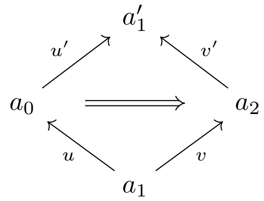
$$a_0 \xrightarrow{f_1} a_1 \xleftarrow{f_2} a_2 \xleftarrow{f_3} \dots \xrightarrow{f_n} a_n$$

Following the usual conventions, left-to-right arrows represents pairs with $\varepsilon = 1$ and right-to-left arrows pairs with $\varepsilon = -1$. The empty path (at an object a) should be represented by a.

There is an obvious notion of concatenation of paths with compatible target and source, given by the concatenation of the sequence of morphisms.

1.3.3. Definition. We say two paths p, p' of $\mathbf{C}$ are *elementary homotopic*, written $p \sim_{\text{elem}} p'$, in any of the following cases:

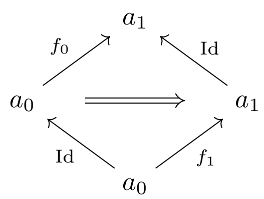
(1) $a \xrightarrow{\mathrm{Id}} a \sim_{\mathrm{elem}} a$, for any object a
(2) $a \xleftarrow{\mathrm{Id}} a \sim_{\mathrm{elem}} a$, for any object a
(3) $a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} a_2 \sim_{\mathrm{elem}} a_0 \xrightarrow{f_2 f_1} a_2$, for any composable pair f_1, f_2 of morphisms
(4) $a_0 \xleftarrow{f_1} a_1 \xleftarrow{f_2} a_2 \sim_{\mathrm{elem}} a_0 \xleftarrow{f_1 f_2} a_2$, for any composable pair f_1, f_2 of morphisms
(5) $a_0 \xleftarrow{u} a_1 \xrightarrow{v} a_2 \sim_{\mathrm{elem}} a_0 \xrightarrow{u'} a_1' \xleftarrow{v'} a_2$, for any 2-morphism



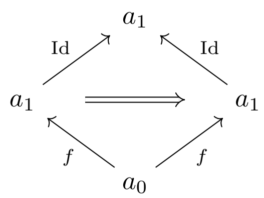
We then define a *homotopy* relation $\sim$ on paths as the smallest congruent (for the concatenation of paths), reflexive, symmetric and transitive relation encompassing the relation $\sim_{\mathrm{elem}}$.

1.3.4. Remark. We should pause to consider two consequences of (5):

- a 2-morphism $a_0 \underset{f_1}{\overset{f_0}{\Downarrow}} a_1$ can be arranged into the following square:



  This shows, together with (1) and (2), that $a_0 \xrightarrow{f_0} a_1 \sim a_0 \xrightarrow{f_1} a_1$, as one would expect.
- a 1-morphism $a_0 \xrightarrow{f} a_1$ can be used to form the square:



  Once again using (1) and (2), this proves that $(a_1 \xleftarrow{f} a_0 \xrightarrow{f} a_1) \sim a_1$. A similar argument (putting f on the upper side of the square) shows that $(a_0 \xrightarrow{f} a_1 \xleftarrow{f} a_0) \sim a_0$. Hence, up to homotopy, the paths $a_0 \xrightarrow{f} a_1$ and $a_1 \xleftarrow{f} a_0$ are mutual inverses.

Note that for two paths to be homotopic, they must have the same source and the same target.

It is natural to look for a category of paths up to homotopy:

1.3.5. Definition. The *(algebraic) fundamental groupoid* $\Pi_1(\mathbf{C})$ of $\mathbf{C}$ is the 1-category with:
- *Objects:* the objects of $\mathbf{C}$.
- *Morphisms:* the classes of paths between objects modulo the homotopy relation.

- *Composition* is induced by the concatenation of paths.

1.3.6. Definition. The 2-category $\mathbf{C}$ is said to be *connected* if for any pair of objects a, a' there is a path with source a and target a'.

The 2-category $\mathbf{C}$ is said to be *simply connected* if $p \sim p'$ for any pair of paths p, p' with same source and same target.

1.3.7. Remark. A 2-category $\mathbf{C}$ is nonempty, connected and simply connected if and only if its fundamental groupoid $\Pi_1(\mathbf{C})$ is equivalent to 1, the category with exactly one object and one morphism.

1.3.8. Remark. Given a 2-category $\mathbf{C}$ which is nonempty, connected and simply connected, its nerve $|\mathbf{C}|$ is not necessarily weakly contractible. Indeed higher homotopy groups may be nontrivial. For instance, one can realize the sphere S^2 as the nerve of the 2-category with two objects, two parallel 1-morphisms between these objects, and two parallel 2-morphisms between these 1-morphisms.

1.3.9. Remark. For any algebraic path p in $\mathbf{C}$, there is an associated topological path $|p|\colon I \to |\mathbf{C}|$. The following assertions, which should result from simplicial approximation, motivate the definitions of this section:

The 2-category $\mathbf{C}$ is connected (resp. simply connected) if and only if the CW-complex $|\mathbf{C}|$ is connected (resp. simply connected).

Two algebraic paths p, p' in $\mathbf{C}$ are homotopic if and only if the topological paths $|p|, |p'|$ are homotopic.

The categories $\Pi_1(\mathbf{C})$ and $\Pi_1(|\mathbf{C}|)$ are equivalent.

A criterion for 2-final 2-functors.

1.3.10. Definition. A 2-functor $\mathbb{F}\colon \mathbf{A} \to \mathbf{B}$ between 2-categories is 2*-final* if for any 2-diagram $\mathbb{D}\colon \mathbf{B} \to \mathbf{E}$, the pseudo bicolimits $\operatorname{bicolim}_{\mathbf{B}} \mathbb{D}$ and $\operatorname{bicolim}_{\mathbf{A}} \mathbb{D} \circ \mathbb{F}$ each exists if and only if the other one exists, and the canonical comparison morphism

$$\operatorname*{bicolim}_{\mathbf{A}} \mathbb{D} \circ \mathbb{F} \to \operatorname*{bicolim}_{\mathbf{B}} \mathbb{D}$$

is an equivalence.

1.3.11. Remark. In the above definition, $\mathbf{E}$ is only assumed to be a 2-category. However, if $\mathbf{B}$ is a $(2,1)$-category, the *pseudo* bicolimits can be equivalently computed in $\mathbf{E}_g$, the $(2,1)$-category with the objects of $\mathbf{E}$, the 1-morphisms of $\mathbf{E}$, and the *invertible* 2-morphisms of $\mathbf{E}$. Hence we could assume $\mathbf{E}$ to be a $(2,1)$-category, without changing the meaning of the definition.

1.3.12. Remark. A 1-final 1-functor $\mathbb{F}\colon \mathbf{A} \to \mathbf{B}$ between 1-categories is a functor such that, for any diagram $\mathbb{D}\colon \mathbf{B} \to \mathbf{E}$, the colimits $\operatorname{colim}_{\mathbf{B}} \mathbb{D}$ and $\operatorname{colim}_{\mathbf{A}} \mathbb{D} \circ \mathbb{F}$ each exists if and only if the other one exists, and the canonical comparison morphism

$$\operatorname*{colim}_{\mathbf{A}} \mathbb{D} \circ \mathbb{F} \to \operatorname*{colim}_{\mathbf{B}} \mathbb{D}$$

is an isomorphism.

A 2-final 1-functor $\mathbb{F}\colon \mathbf{A} \to \mathbf{B}$ between 1-categories (seen as 2-categories with only the identities as 2-morphisms) is 1-final, since any diagram is also a 2-diagram. The converse is not true, though: there are 1-final functors which are not 2-final.

1.3.13. Theorem. *Let $\mathbf{A}$, $\mathbf{B}$ be two 2-categories. A 2-functor $\mathbb{F}\colon \mathbf{A} \to \mathbf{B}$ is 2-final (definition 1.3.10) if and only if, for any object $b \in \mathbf{B}$, the slice 2-category $b/\mathbb{F}$ is nonempty, connected and simply connected (definition 1.3.6).*

We will first prove the backward implication.

Fix a 2-functor $\mathbb{D}\colon \mathbf{B} \to \mathbf{E}$. We will construct a pseudoinverse to the canonical comparison morphism

$$\operatorname*{bicolim}_{\mathbf{A}} \mathbb{D} \circ \mathbb{F} \to \operatorname*{bicolim}_{\mathbf{B}} \mathbb{D}$$

This morphism correspond to a family of functors, pseudonatural in e:

$$\mathcal{K}\colon [\mathbf{B}, \mathbf{E}](\mathbb{D}, \Delta e) \to [\mathbf{A}, \mathbf{E}](\mathbb{D} \circ \mathbb{F}, \Delta e).$$

We will construct a pseudoinverse $\mathbb{L}$ to $\mathbb{K}$. Given a cone $\phi\colon \mathbb{D} \circ \mathbb{F} \Rightarrow \Delta e$, we obtain a cone $\mathbb{L}(\phi)\colon \mathbb{D} \Rightarrow \Delta e$ as follows:

- objects in the slice 2-categories $b/\mathbb{F}$ define the 1-morphism components $\mathcal{L}(\phi)_b$ (see definition 1.3.14),
- paths in $b/\mathbb{F}$ define natural transformations between the components (see definition 1.3.16),
- homotopies between paths ensure the cohesion of these constructions (see proposition 1.3.17).

Consider an arbitrary cone under $\mathbb{D} \circ \mathbb{F}$ with vertex $e \in \mathbf{E}$, that is, a pseudonatural transformation $\phi\colon \mathbb{D} \circ \mathbb{F} \to \Delta(e)$. We first want to define a cone ψ under $\mathbb{D}$ with vertex e, using the cone ϕ.

As a first step, we fix an object b and we want to define the component at b $\psi_b\colon \mathbb{D}(b) \to e$. Since the slice 2-category $b/\mathbb{F}$ is nonempty, we consider the following candidate.

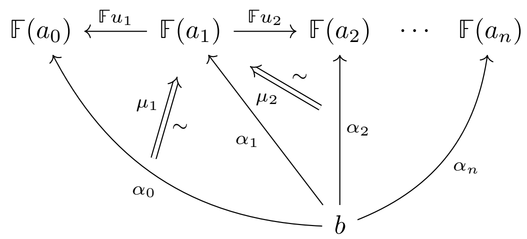
1.3.14. Definition. We fix an object in $b/\mathbb{F}$, that is an object $a(b) \in \mathbf{A}$ and a morphism $\alpha(b)\colon b \to \mathbb{F}(a(b))$. Define

$$\psi_{(a(b),\alpha(b))}\colon\ \mathbb{D}(b) \xrightarrow{\mathbb{D}(\alpha(b))} \mathbb{D}(\mathbb{F}(a(b))) \xrightarrow{\phi_{a(b)}} e$$

We then consider the dependence of $\psi_{(a(b),\alpha(b))}$ on $(a(b), \alpha(b))$. Fix another object $(a'(b), \alpha'(b)) \in b/\mathbb{F}$. Since $b/\mathbb{F}$ is connected there is a path

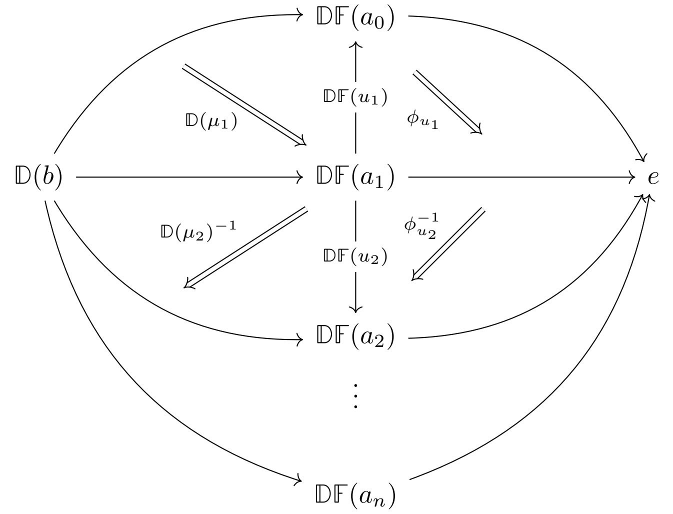
$$p\colon (a_0, \alpha_0) = (a(b), \alpha(b)) \rightsquigarrow (a_n, \alpha_n) = (a'(b), \alpha'(b))$$

which can be pictured as:



Applying the 2-functor $\mathbb{D}$ and using the cone ϕ, we obtain the pasting diagram:

(1.3.15)



We can thus define:

1.3.16. Definition. Any path $p\colon (a,\alpha) \rightsquigarrow (a',\alpha')$ in $b/\mathbb{F}$ defines a 2-isomorphism in $\mathbf{E}$

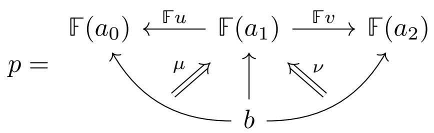
$$j(p)\colon \psi_{(a,\alpha)} \to \psi_{(a',\alpha')}$$

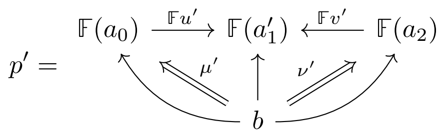
as given by the above pasting diagram 1.3.15.

1.3.17. Proposition. *For any paths $p, p'\colon (a,\alpha) \rightsquigarrow (a',\alpha')$ in $b/\mathbb{F}$ with same source and target,*

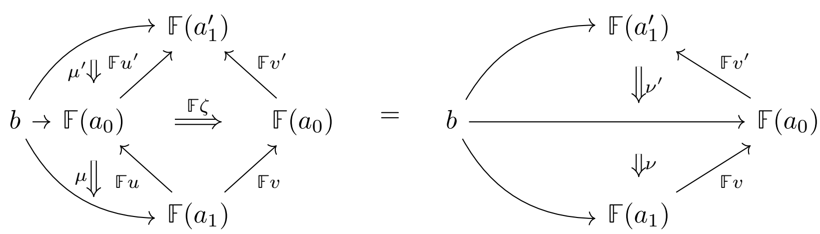
$$j(p) = j(p').$$

Proof. We first prove that two elementary homotopic paths $p \sim_{\mathrm{elem}} p'$ induce the same 2-isomorphism $j(p) = j(p')$. The four first cases are immediate consequences of the pseudonaturality of ϕ. We can thus assume that



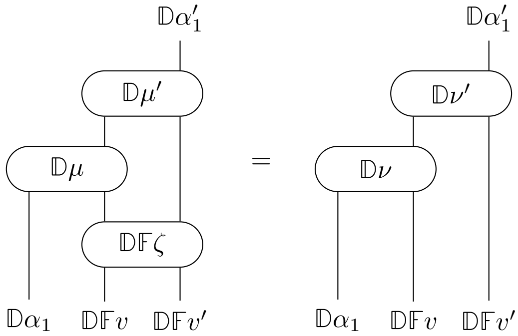


and that there is a 2-morphism $\zeta\colon u'u \Rightarrow v'v$ such that



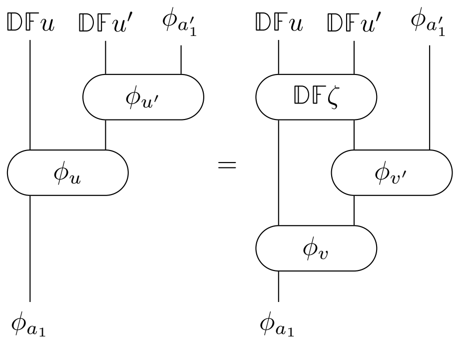
We can apply the 2-functor $\mathbb{D}$ and express this relation using string diagrams (see the subsection "String diagrams" of section 1.1):

(1.3.18)

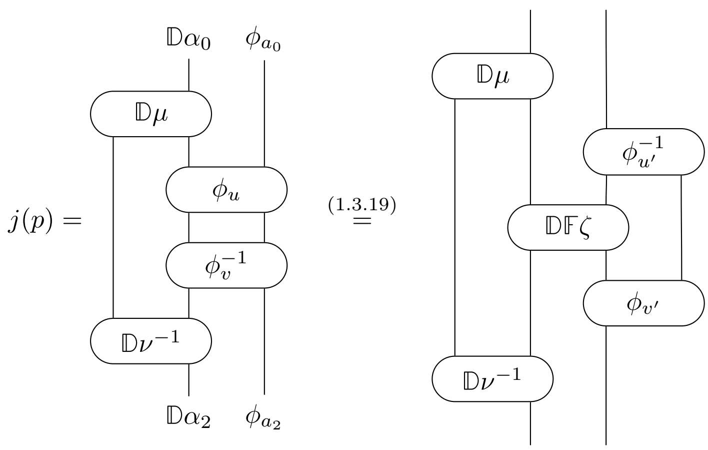


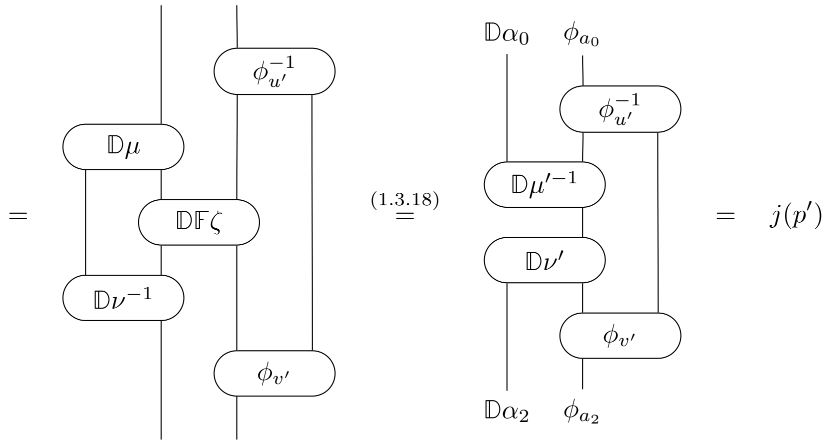
Similarly, the pseudonaturality of ϕ gives the relation:

(1.3.19)



We can now compute $j(p)$:





We now show that, for two homotopic paths $p \sim p'$, we have $j(p) = j(p')$. It suffices to show that the relation $\mathcal{R}$ on paths defined by

$$p\mathcal{R}p' \iff j(p) = j(p')$$

is reflexive, symmetric, transitive and congruent, since we have already proved that it contains $\sim_{\text{elem}}$. The three first properties are obviously satisfied. The last one is a direct consequence of the compatibility of j with the concatenation of paths: $j(p \cdot p') = j(p')j(p)$.

Since $b/\mathbb{F}$ is simply connected by hypothesis and j is homotopy invariant, the 2-isomorphism $j(p)$ only depends on the source and the target of p. Hence for any two objects (a, α) and (a', α') in $b/\mathbb{F}$, there is a unique 2-isomorphism $\psi_{(a,\alpha)} \Rightarrow \psi_{(a',\alpha')}$ in $\mathbf{E}$ induced by a path in $b/\mathbb{F}$. $\square$

1.3.20. Definition. Given a morphism $u \colon b \to b'$ in $\mathbf{B}$, there is a base change 2-functor:

$$u^* \colon \begin{cases} b'/\mathbb{F} & \to & b/\mathbb{F} \\ (x, \chi) & \mapsto & (x, \chi \circ u) \\ (v, \nu) & \mapsto & (v, \nu \cdot u) \\ \zeta & \mapsto & \zeta \end{cases}$$

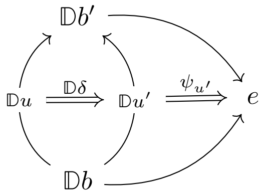
Note that this 2-functor also extends to a function between the respective sets of paths.

1.3.21. Proposition. *The application j maps base change to whiskering:*

$$j(u^*p) = j(p) \cdot \mathbb{D}u$$

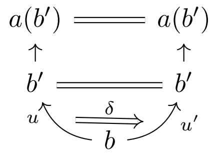
We can now use the above properties to construct a cone ψ under $\mathbb{D}$ with vertex e. For any $b \in \mathbf{B}$, fix an arbitrary object $(a(b), \alpha(b))$ in $b/\mathbb{F}$. This defines the components $\psi_b = \psi_{(a(b),\alpha(b))}$, as stated in definition 1.3.14. For a morphism $u \colon b \to b'$, note that $\psi_{(a(b'),\alpha(b'))} \circ \mathbb{D}u = \psi_{u^*(a(b'),\alpha(b'))}$; hence we can define ψ_u as the unique 2-isomorphism $j(p)$ induced by any path $p \colon u^*(a(b'), \alpha(b')) \rightsquigarrow (a(b), \alpha(b))$. We must check that ψ is indeed a pseudonatural transformation. The compatibility of j with the whiskering and the concatenation of paths implies the required compatibility of ψ with the composition of morphisms. It remains to check the compatibility with 2-morphisms. Let $u, u' \colon b \to b'$ be two parallel 1-morphisms and $\delta \colon u \Rightarrow u'$ be a 2-morphism in $\mathbf{B}$. By uniqueness of 2-morphisms induced by a path (proposition 1.3.17), it suffices to check that the pasting

(1.3.22)



is induced by a path. Indeed, fix a path $p\colon (u')^*(a(b'),\alpha(b')) \rightsquigarrow (a(b),\alpha(b))$ and recall that, by definition, $\psi_{u'} = j(p)$. We consider the path p' of length one:

$$p' \quad = \quad (a(b'),\alpha(b')u) \xleftarrow{(\mathrm{Id},\alpha(b')\delta)} (a(b'),\alpha(b')u') \quad =$$



The above pasting (1.3.22) is then induced by the concatenation $p' \cdot p$ of p' and p.

Through similar arguments, we can see that any other choice of the objects $(a(b),\alpha(b))_{b\in\mathbf{B}}$ leads to an isomorphic cone.

From now on, we assume that the objects $(a(b),\alpha(b))_{b\in\mathbf{B}}$ are fixed and we write $\mathcal{L}(\phi)$ for the cone ψ under $\mathbb{D}$ induced from the cone ϕ under $\mathbb{D}\circ\mathbb{F}$. Since we will not work with a single fixed cone ϕ anymore, we should write j_ϕ instead of j.

We would like to extend this mapping $\phi \mapsto \mathcal{L}\phi$ to a functor

$$\mathcal{L}\colon [\mathbf{A},\mathbf{E}](\mathbb{D}\circ\mathbb{F},\Delta e) \to [\mathbf{B},\mathbf{E}](\mathbb{D},\Delta e).$$

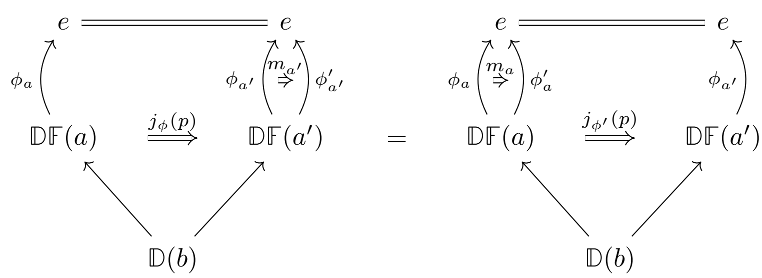
We use the proposition:

1.3.23. Proposition. *Let $m\colon \phi \to \phi'$ be a modification between two cones*

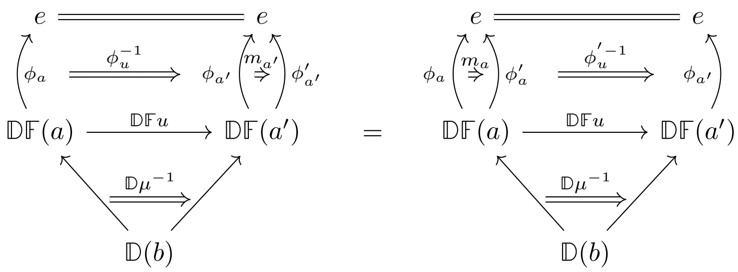
$$\phi,\phi'\colon \mathbb{D}\circ\mathbb{F} \Rightarrow \Delta e$$

For any path $p\colon (a,\alpha) \rightsquigarrow (a',\alpha')$ in $b/\mathbb{F}$, we have the following equality:

(1.3.24)



PROOF. For an empty path p, the proposition reduces to the tautology $m_a = m_a$. For a path $p = (a, \alpha) \xrightarrow{(u,\mu)} (a', \alpha')$ of length 1, we can decompose the equation as:



The lower parts of these diagrams are the same and the upper parts are equal, by the property of the modification m. Hence eq. (1.3.24) holds for a path $p = ((a, \alpha) \xrightarrow{(u,\mu)} (a', \alpha'))$. A similar decomposition of the diagrams shows that it also holds for a path $p = ((a, \alpha) \xleftarrow{(u,\mu)} (a', \alpha'))$ of length one in the reverse direction.

Since j_ϕ and $j_{\phi'}$ are compatible with paths concatenation, if eq. (1.3.24) holds for two composable paths p and p', it also holds for their concatenation pp'. We can thus conclude that it holds for any path p, as the path p is generated by paths of length 1. □

This property directly implies that the components

$$\mathcal{L}(m)_b \; = \; \mathbb{D}(b) \longrightarrow \mathbb{D}(a(b)) \underset{}{\overset{}{\Downarrow m_{a(b)}}} e$$

define a 2-morphism $\mathcal{L}(\phi) \to \mathcal{L}(\phi')$. The functoriality of $\mathcal{L}$ is straightforward.

Now consider the canonical functor

$$\mathcal{K}\colon \left\{ \begin{array}{lcl} [\mathbf{B}, \mathbf{E}](\mathbb{D}, \Delta e) & \to & [\mathbf{A}, \mathbf{E}](\mathbb{D} \circ \mathbb{F}, \Delta e) \\ \psi_\bullet & \mapsto & \psi_{\mathbb{F}(\bullet)} \\ m_\bullet & \mapsto & m_{\mathbb{F}(\bullet)} \end{array} \right.$$

sending cones under $\mathbb{D}$ with vertex e to cones under $\mathbb{D} \circ \mathbb{F}$ with vertex e. We are now ready to show that $\mathcal{L}$ and $\mathcal{K}$ are mutual pseudo-inverses.

1.3.25. Proposition. *There is a natural isomorphism* $\eta\colon \mathrm{Id} \Rightarrow \mathcal{KL}$.

PROOF. Let $\phi \in [\mathbf{A}, \mathbf{E}](\mathbb{D} \circ \mathbb{F}, \Delta e)$ be a cone under $\mathbb{D} \circ \mathbb{F}$ with vertex e. We will write $j = j_\phi$.

We want to define the component $\eta_\phi\colon \phi \to \mathcal{KL}\phi$ of η at ϕ. Since η_ϕ must be a modification, we have to define its components at each object $a_0 \in \mathbf{A}$:

$$\eta_{\phi,a_0}\colon \phi_{a_0} \Rightarrow \phi_{a(\mathbb{F}a_0)} \circ \mathbb{D}\alpha\mathbb{F}a_0.$$

Both $(a_0, \mathrm{Id}_{\mathbb{F}a_0})$ and $(a(\mathbb{F}a_0), \alpha(\mathbb{F}a_0))$ are objects of $\mathbb{F}a_0/\mathbb{F}$, which is connected. Hence, there is a path in $\mathbb{F}a_0/\mathbb{F}$:

$$p\colon (a_0, \mathrm{Id}_{\mathbb{F}a_0}) \rightsquigarrow (a(\mathbb{F}a_0), \alpha(\mathbb{F}a_0))$$

We can then define η_{ϕ,a_0} as:

$$\eta_{\phi,a_0} = j(p).$$

We have to check that η_ϕ is a modification. That is, for any morphism $f\colon a_0 \to a_1$ in $\mathbf{A}$, we have to check the commutativity of:

$$\begin{array}{ccc} \phi_a & \xRightarrow{\eta_{\phi,a_0}} & \phi_{a(\mathbb{F}a_0)} \circ \mathbb{D}\alpha(\mathbb{F}a_0) \\ \Big\Downarrow{\scriptstyle \phi_f} & & \Big\Downarrow{\scriptstyle \mathcal{L}(\phi)_{\mathbb{F}f}} \\ \phi_{a_1} \circ \mathbb{D}\mathbb{F}f & \xRightarrow{\eta_{\phi,a_1}\cdot \mathbb{D}\mathbb{F}f} & \phi_{a(\mathbb{F}a_1)} \circ \mathbb{D}\alpha(\mathbb{F}a_1) \circ \mathbb{D}\mathbb{F}f \end{array}$$

We first remark that there is a path $p_0 = ((a_0, \mathrm{Id}) \xrightarrow{(f,\mathrm{Id})} (a_1, \mathbb{F}f))$ in $\mathbb{F}a_0/\mathbb{F}$ and the induced 2-isomorphism is $\phi_f = j(p_0)$. Moreover, expanding the definitions, we have

$$\begin{array}{ll} \eta_{\phi,a_0} = j(p_1) & \text{for some } p_1\colon (a_0, \mathrm{Id}) \rightsquigarrow (a(\mathbb{F}a_0), \alpha(\mathbb{F}a_0)) \\ \eta_{\phi,a_1} = j(p_2) & \text{for some } p_2\colon (a_1, \mathrm{Id}) \rightsquigarrow (a(\mathbb{F}a_1), \alpha(\mathbb{F}a_1)) \\ \mathcal{L}(\phi)_{\mathbb{F}f} = j(p_3) & \text{for some } p_3\colon (a(\mathbb{F}a_0), \alpha(\mathbb{F}a_0)) \rightsquigarrow \mathbb{F}(f)^*(a(\mathbb{F}a_1), \alpha(\mathbb{F}a_1)) \end{array}$$

where p_1 and p_3 are paths in $\mathbb{F}a_0/\mathbb{F}$, and p_2 is a path in $\mathbb{F}a_1/\mathbb{F}$. We can check that $p_1 \cdot p_3$ and $p_0 \cdot \mathbb{F}(f)^* p_2$ are paths

$$(a_0, \mathrm{Id}) \rightsquigarrow \mathbb{F}(f)^*(a(\mathbb{F}a_1), \alpha(\mathbb{F}a_1)).$$

Hence

$$\begin{aligned} (\eta_{\phi,a_1} \cdot \mathbb{D}\mathbb{F}f) \circ \phi_f &= j(\mathbb{F}(f)^* p_2) \circ j(p_0) \\ &= j(p_0 \cdot \mathbb{F}(f)^* p_2) \\ &= j(p_1 \cdot p_3) \\ &= j(p_3) \circ j(p_1) \\ &= \mathcal{L}(\phi)_{\mathbb{F}f} \circ \eta_{\phi,a_0} \end{aligned}$$

We also have to check the naturality of η. For any modification $m\colon \phi \to \phi'$, we want the commutativity of the square:

$$\begin{array}{ccc} \phi & \xrightarrow{\eta_\phi} & \mathcal{K}\mathcal{L}\phi \\ \Big\downarrow{\scriptstyle m} & & \Big\downarrow{\scriptstyle \mathcal{K}\mathcal{L}m} \\ \phi' & \xrightarrow{\eta_{\phi'}} & \mathcal{K}\mathcal{L}\phi' \end{array}$$

That is, for any object $a_0 \in \mathbf{A}$:

$$\begin{array}{ccc} \phi_{a_0} & \xrightarrow{j_\phi(p)} & \phi_{a(\mathbb{F}a_0)} \circ \mathbb{D}\alpha(\mathbb{F}a_0) \\ \Big\downarrow{\scriptstyle m_{a_0}} & & \Big\downarrow{\scriptstyle m_{\mathbb{F}a_0}} \\ \phi'_{a_0} & \xrightarrow{j_{\phi'}(p)} & \mathcal{K}\mathcal{L}\phi'_{a_0} \end{array}$$

where $p\colon (a_0, \mathrm{Id}) \rightsquigarrow (a(\mathbb{F}a_0), \alpha(\mathbb{F}a_0))$ is a path in $\mathbb{F}a_0/\mathbb{F}$. This last square commutes by proposition 1.3.23. □

In the reverse direction we show:

1.3.26. Proposition. *There is a natural isomorphism* $\epsilon : \mathcal{L}\mathcal{K} \Rightarrow \mathrm{Id}$.

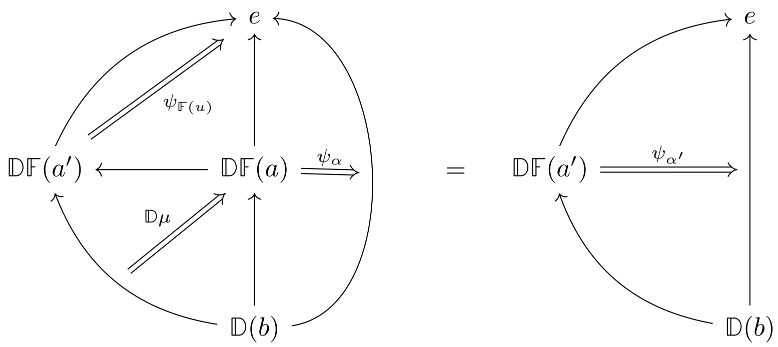
PROOF. Fix a cone $\psi\colon \mathbb{D} \Rightarrow \Delta e$ under $\mathbb{D}$. Write $\psi' = \mathcal{L}\mathcal{K}(\psi)$. For any $b \in \mathbf{B}$, we have:

$$\psi'_b = \mathcal{K}(\psi)_{a(b)} \circ \mathbb{D}(\alpha(b)) = \psi_{\mathbb{F}(a(b))} \circ \mathbb{D}(\alpha(b))$$

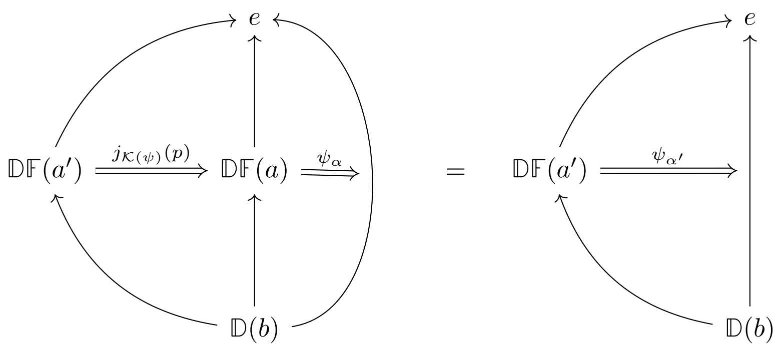
Hence we can define a 2-morphism $\epsilon_{\psi,b}\colon \psi'_b \Rightarrow \psi_b$ in $\mathbf{E}$ by:

$$\epsilon_{\psi,b} = \psi_{\alpha(b)}$$

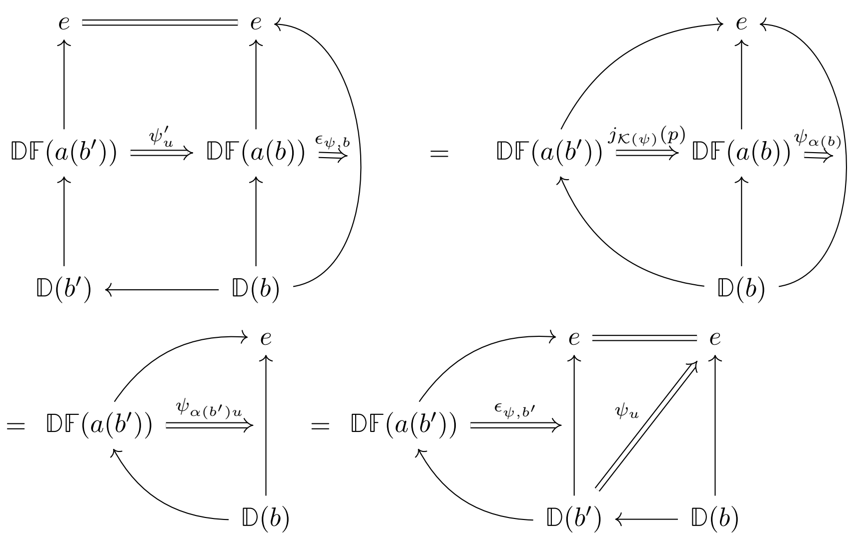
When b ranges over all objects of $\mathbf{B}$, these morphisms then form a modification $\epsilon_\psi\colon \psi' \to \psi$. Indeed for any morphism $(u,\mu)\colon (a,\alpha) \to (a',\alpha')$ in $b/\mathbb{F}$, we have:



which implies a similar formula for any path $p\colon (a',\alpha') \rightsquigarrow (a,\alpha)$ in $b/\mathbb{F}$:



This in turn implies that ϵ_ψ is a modification. Fix a morphism $u\colon b \to b'$ and consider a path $p\colon u^*(a(b'),\alpha(b')) \rightsquigarrow (a(b),\alpha(b))$ (hence we have $\psi'_u = j_{\mathcal{K}(\psi)}(p)$). We check the modification axiom at u:



Finally we have to check the naturality of $\epsilon\colon \mathcal{LK} \to \mathrm{Id}$, that is, for any modification of cones $m\colon \psi \to \psi'$, the commutativity of the square:

$$\begin{array}{ccc} \mathcal{LK}\psi & \xrightarrow{\epsilon_\psi} & \psi \\ \downarrow{\scriptstyle \mathcal{LK}m} & & \downarrow{\scriptstyle m} \\ \mathcal{LK}\psi' & \xrightarrow{\epsilon_{\psi'}} & \psi' \end{array}$$

Indeed, for any object $b \in \mathbf{B}$:

$$\epsilon_{\psi',b} \circ (\mathcal{LK}m)_b = \psi'_{\alpha(b)} \circ m_{a(b)}\alpha(b) = m_b \circ \psi_{\alpha(b)}.$$

□

Putting together proposition 1.3.25 and proposition 1.3.26, we deduce:

1.3.27. Proposition. *The canonical functor*

$$\mathcal{K}\colon [\mathbf{B},\mathbf{E}](\mathbb{D},\Delta e) \to [\mathbf{A},\mathbf{E}](\mathbb{D}\circ\mathbb{F},\Delta e)$$

is an equivalence.

Since this is true for any object e of $\mathbf{E}$, clearly $\operatorname{bicolim}\mathbb{D}$ exists if and only if $\operatorname{bicolim}\mathbb{D}\circ\mathbb{F}$ exists and, if it is the case, they are canonically equivalent.

We have thus proved one implication of theorem 1.3.13:

1.3.28. Proposition. *Let $\mathbb{F}\colon \mathbf{A} \to \mathbf{B}$ be a 2-functor. If for any object $b \in \mathbf{B}$, the slice 2-category $b/\mathbb{F}$ is nonempty, connected and simply connected, then the 2-functor $\mathbb{F}$ is 2-final.*

The reverse implication is proved by observing the following fact:

1.3.29. Proposition. *Let $\mathbb{F}\colon \mathbf{A} \to \mathbf{B}$ be a 2-functor. Then*

$$\Pi_1(b/\mathbb{F}) \simeq \operatorname*{bicolim}_{a\in\mathbf{A}} \mathbf{B}(b,\mathbb{F}a).$$

Proof. The wanted equivalence can be proved by constructing a family of equivalences, pseudonatural in the category T:

$$C_T\colon [\mathbf{A},\mathbf{Cat}](\mathbf{B}(b,\mathbb{F}-),\Delta T) \simeq [\Pi_1(b/\mathbb{F}),T].$$

Fix $\psi\colon \mathbf{B}(b,\mathbb{F}-) \Rightarrow \Delta T$ a pseudonatural transformation. We want to define a functor $C_T(\psi)\colon \Pi_1(b/\mathbb{F}) \to T$.

For any object $(a,\alpha\colon b \to \mathbb{F}(a))$ of $\Pi_1(b/\mathbb{F})$, set:

$$C_T(\psi)(a,\alpha) = \psi_a(\alpha)$$

For any morphism $(u,\mu\colon u\alpha \Rightarrow \alpha')\colon (a,\alpha) \to (a',\alpha')$ of $b/\mathbb{F}$, define the composite isomorphism:

$$C_T(\psi)(u,\mu):\ \psi_a(\alpha) \xrightarrow{(\psi_u)_\alpha} \psi_{a'}(u\circ\alpha) \xrightarrow{\psi_{a'}(\mu)} \psi_{a'}(\alpha')$$

This can be extended to paths using the relations:

$$C_T(\psi)((a',\alpha') \xleftarrow{(u,\mu)} (a,\alpha)) = C_T(\psi)((a,\alpha) \xrightarrow{(u,\mu)} (a',\alpha'))^{-1}$$
$$C_T(\psi)((a,\alpha)) = \mathrm{Id}_{\psi_a(\alpha)}$$
$$C_T(\psi)(p\cdot p') = C_T(\psi)(p') \circ C_T(\psi)(p)$$

One can check that such a definition is homotopy invariant, and gives a well-defined functor $C_T(\psi)\colon \Pi_1(b/\mathbb{F}) \to T$.

For a modification $m\colon \psi \to \psi'$, we define a natural transformation

$$C_T(m)\colon C_T(\psi) \Rightarrow C_T(\psi')$$

with components:

$$C_T(m)_{(a,\alpha)} = (m_a)_\alpha \tag{1.3.30}$$

To show that C_T is an equivalence, we show that it is a fully faithful and essentially surjective functor.

Indeed it is clear that (1.3.30) defines a bijection between modifications $\psi \to \psi'$ and natural transformations $C_T(\psi) \Rightarrow C_T(\psi')$. Hence C_T is fully faithful.

Moreover, given any functor $F\colon \Pi_1(b/\mathbb{F}) \to T$, one can define a pseudonatural transformation $\psi\colon \mathbf{B}(b, \mathbb{F}-) \to \Delta T$ by:

$$\psi_a : \begin{cases} \mathbf{B}(b, \mathbb{F}a) & \to & T \\ \alpha & \mapsto & F(a, \alpha) \\ \nu & \mapsto & F(\mathrm{Id}_a, \nu) \end{cases}$$

$$(\psi_u)_\alpha \colon F(a,\alpha) \xrightarrow{F(u,\mathrm{Id})} F(a', u \circ \alpha)$$

for any object a and morphism $u\colon a \to a'$ of $\mathbf{A}$. It is straightforward to check:

$$F = C_T(\psi)$$

Hence C_T is also essentially surjective. □

We can now prove:

1.3.31. Proposition. *Let $\mathbb{F}\colon \mathbf{A} \to \mathbf{B}$ be a 2-final 2-functor. Then, for any object b in $\mathbf{B}$:*

$$\Pi_1(b/\mathbb{F}) \simeq 1$$

Proof. We have a chain of equivalences:

$$\Pi_1(b/\mathbb{F}) \overset{1.3.29}{\simeq} \operatorname*{bicolim}_{a \in \mathbf{A}} \mathbf{B}(b, \mathbb{F}a) \overset{(1)}{\simeq} \operatorname*{bicolim}_{b' \in \mathbf{B}} \mathbf{B}(b, b') \overset{(2)}{\simeq} 1$$

The equivalence (1) is an application of the 2-finality of $\mathbb{F}$. The equivalence (2) is a consequence of the Yoneda lemma for 2-categories. Indeed we have the chain of equivalences, for any 1-category T, and pseudonatural in T:

$$[\mathbf{B}, \mathbf{Cat}](\mathbf{B}(b, -), \Delta T) \simeq \Delta T(b) \cong T \cong \mathbf{Cat}(1, T)$$

□

By combining proposition 1.3.31 and remark 1.3.7, we have:

1.3.32. Proposition. *Let $\mathbb{F}\colon \mathbf{A} \to \mathbf{B}$ be a 2-final 2-functor. Then, for any object b in $\mathbf{B}$, the 2-category $b/\mathbb{F}$ is nonempty, connected and simply connected.*

There is a dual notion of *2-initial 2-functor*, with a dual criterion, proven by a duality argument.

1.3.33. Definition. Let $\mathbb{F}\colon \mathbf{A} \to \mathbf{B}$ be a 2-functor between 2-categories. The 2-functor is said to be *2-initial* if, for any 2-diagram $\mathbb{D}\colon \mathbf{B} \to \mathbf{E}$, each of the bilimits $\mathrm{bilim}_{\mathbf{B}}\, \mathbb{D}$ and $\mathrm{bilim}_{\mathbf{A}}\, \mathbb{D} \circ \mathbb{F}$ exists whenever the other one exists, and the canonical comparison 1-morphism

$$\operatorname*{bilim}_{\mathbf{B}} \mathbb{D} \to \operatorname*{bilim}_{\mathbf{A}} \mathbb{D} \circ \mathbb{F}$$

is an equivalence.

1.3.34. Proposition. *Let $\mathbb{F}\colon \mathbf{A} \to \mathbf{B}$ be a 2-functor between 2-categories. The 2-functor $\mathbb{F}$ is initial if and only if the 2-functor $\mathbb{F}^{\mathrm{op}}\colon \mathbf{A}^{\mathrm{op}} \to \mathbf{B}^{\mathrm{op}}$ is final.*

1.3.35. Theorem. *Let $\mathbb{F}\colon \mathbf{A} \to \mathbf{B}$ be a 2-functor between 2-categories. The 2-functor $\mathbb{F}$ is initial if and only if, for any object $b \in B$, the slice 2-category $\mathbb{F}/b$ is nonempty, connected and simply connected.*

CHAPTER 2

2-Sheaves

We introduce in this chapter the notion of 2-sheaves. As suggested by their name, 2-sheaves are a categorification of sheaves; they are also a generalization of stacks. A 2-sheaf is a 2-functor between 2-categories

$$\mathbb{X}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$$

satisfying some gluing conditions. To make this a precise definition, it is first necessary to have an adequate notion of topology on a 2-category: this is the role of the Grothendieck coverages, defined with 2-sieves. In practice, it is often easier to generate a Grothendieck coverage using covering families of morphisms. In general, the pasting condition for 2-sheaves is defined using the Grothendieck coverage (definition 2.3.1). However, if the base 2-category $\mathbf{C}$ has finite bipullbacks, it can be reduced to a descent condition with respect to the covering families (definition 2.3.4). In practice, it is generally easier to check that a 2-functor satisfies the descent conditions for a set of covering families. The equivalence of the two notions (proposition 2.3.9) occupies a large part of this chapter. We prove along the way a useful criterion for checking the descent condition (proposition 2.2.4). Our main reference for 2-sheaves is [Str82]. However we took a slightly different approach to 2-sites, with a distinct terminology (borrowed from 1-sites). We will motivate and highlight those disparities as we introduce our notions.

Our major motivation for introducing and using 2-sheaves is their well-behaved extension properties, similar to those of sheaves and stacks. In section 2.4, we state that a 2-sheaf on a suitable 2-category has an essentially unique extension to a 2-sheaf on a larger 2-category (proposition 2.4.6). In the following chapter, we will see that, in a similar setting, we can also extend the existence of some adjoint functors (proposition 3.3.1).

Section 2.5 is a short aside discussing the relation between coproducts in 2-sites and product-preserving 2-sheaves.

As previously asserted, we would like to avoid considerations of size of our 2-categories. We will thus make the simplifying assumption that all the 2-categories we introduce are small, without stating it again.

2.1. 2-Sites

A 2-site is a 2-category with a prescribed topology. The appropriate notion of a topology for a 2-category is the data of a Grothendieck coverage, that is, a set of right-ideals of morphisms, called 2-sieves, closed under certain operations.

2-Sieves.

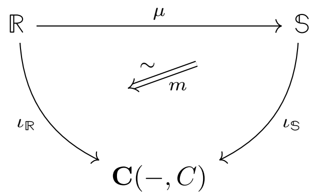
2.1.1. Definition. Let $\mathbf{C}$ be a 2-category and C an object of $\mathbf{C}$. A 2-sieve on C, or C-sieve, is a pair $(\mathbb{R}, \iota)$ consisting of a 2-functor $\mathbb{R}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ and a fully faithful pseudonatural transformation $\iota\colon \mathbb{R} \to \mathbf{C}(-, C)$.

2.1.2. Remark. It is interesting to spell out the definition of a sieve $(\mathbb{R}, \iota)$ on C. For each object D of $\mathbf{C}$, the component $\iota_D\colon \mathbb{R}D \to \mathbf{C}(D, C)$ is fully faithful, hence each $\mathbb{R}D$ should be seen as a full subcategory of $\mathbf{C}(D, C)$, which mostly amounts to a subset of the objects of $\mathbf{C}(D, C)$. The pseudonaturality of ι (essentially) states that theses subsets are closed under precomposition by 1-morphisms of $\mathbf{C}$ or, equivalently, that they form a right-ideal of morphisms of $\mathbf{C}$.

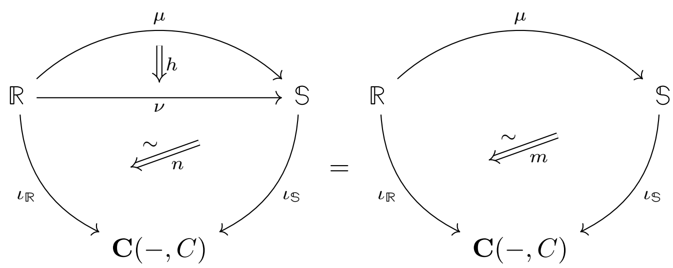
The 2-sieves over a fixed object C naturally form a 2-category.

2.1.3. Definition. Let $\mathbf{C}$ be a 2-category and C an object of $\mathbf{C}$. The 2-category $\mathbf{2Sieve}/C$ is the 2-category with:

- *Objects:* the 2-sieves $(\mathbb{R}, \iota)$ on C.
- 1-*Morphisms* $(\mathbb{R}, \iota_{\mathbb{R}}) \to (\mathbb{S}, \iota_{\mathbb{S}})$*:* the pairs (μ, m) where $\mu\colon \mathbb{R} \to \mathbb{S}$ is a pseudonatural transformation and $m\colon \iota_{\mathbb{S}}\mu \Rightarrow \iota_{\mathbb{R}}$ is an invertible modification:



- 2-*Morphisms* $(\mu, m) \Rightarrow (\nu, n)$: the modifications $h\colon \mu \Rightarrow \nu$ such that:



- Compositions are induced by the compositions of pseudonatural transformations and modifications.

2.1.4. Remark. The 2-category $\mathbf{2Sieve}/C$ is precisely the full, 2-full sub-2-category of the slice 2-category $[\mathbf{C}^{\mathrm{op}}, \mathbf{Cat}]/\mathbf{C}(-, C)$ whose objects are the 2-sieves on C.

2.1.5. Remark. In a morphism $(\mu, m)\colon (\mathbb{R}, \iota_{\mathbb{R}}) \to (\mathbb{S}, \iota_{\mathbb{S}})$ between 2-sieves, the natural transformation $\mu\colon \mathbb{R} \to \mathbb{S}$ is always fully faithful.

2.1.6. Remark. Two 2-sieves $(\mathbb{R}, \iota_{\mathbb{R}})$ and $(\mathbb{S}, \iota_{\mathbb{S}})$ are equivalent if and only if for every object D of $\mathbf{C}$, the following essential images are equal:

$$\iota_{\mathbb{R},D}(\mathbb{R}D) = \iota_{\mathbb{S},D}(\mathbb{S}D)$$

It is also possible to pull back 2-sieves along 1-morphisms of $\mathbf{C}$.

2.1.7. Lemma. *Let $\mathbf{C}$ be a 2-category and $f\colon D \to C$ a 1-morphism in $\mathbf{C}$. The 1-morphism f defines a 2-functor*

$$f^*\colon \mathbf{2Sieve}/C \to \mathbf{2Sieve}/D$$

Proof. Consider the pseudonatural transformation

$$f_*\colon \mathbf{D}(-, D) \to \mathbf{C}(-, C)$$

seen as a 1-morphism in $[\mathrm{C}^{\mathrm{op}}, \mathbf{Cat}]$. By proposition 1.2.27, it defines a 2-functor:

$$(f_*|-)\colon [\mathbf{C}^{\mathrm{op}}, \mathbf{Cat}]/\mathbf{C}(-, C) \to [\mathbf{C}^{\mathrm{op}}, \mathbf{Cat}]/\mathbf{C}(-, D).$$

In turn, by lemma 1.2.29 and remark 2.1.4, this 2-functor restricts to a 2-functor

$$f^* \colon \mathbf{2Sieve}/C \to \mathbf{2Sieve}/D$$

as claimed. □

2.1.8. Remark. To pull back 2-sieves along morphisms of $\mathbf{C}$, we do not need $\mathbf{C}$ to have bipullbacks, since the bipullback we consider is taken in $[\mathbf{C}^{\mathrm{op}}, \mathbf{Cat}]$, which has all (small) bilimits.

2-Sites.

2.1.9. Definition. Let $\mathbf{C}$ be a 2-category. A *Grothendieck* 2*-coverage* $\mathcal{R}$ on $\mathbf{C}$ is the data of a set of 2-sieves $\mathcal{R}_C$ on C, for each object C of $\mathbf{C}$, called *covering* 2*-sieves*, satisfying the following axioms:

- (GC1) For each object C of $\mathbf{C}$, the set $\mathcal{R}_C$ is closed under equivalence (in the 2-category of 2-sieves $\mathbf{2Sieve}/C$).
- (GC2) For each morphism $f \colon D \to C$ in $\mathbf{C}$ and covering 2-sieve $(\mathbb{R}, \iota) \in \mathcal{R}_C$, we have $f^*(\mathbb{R}, \iota) \in \mathcal{R}_D$.
- (GC3) For each object C of $\mathbf{C}$, the trivial 2-sieve is covering:

$$(\mathbf{C}(-, C), \mathrm{Id}) \in \mathcal{R}_C$$

- (GC4) Let C be an object of $\mathbf{C}$, $(\mathbb{S}, \iota_{\mathbb{S}})$ a 2-sieve on C and $(\mathbb{R}, \iota_{\mathbb{R}}) \in \mathcal{R}_C$ a covering 2-sieve on C. If for every object D of $\mathbf{C}$ and every object $r \in \mathbb{R}(D)$, the 2-sieve $(\iota_{\mathbb{R}} r)^*(\mathbb{S}, \iota_{\mathbb{S}})$ is in $\mathcal{R}_D$, then $(\mathbb{S}, \iota_{\mathbb{S}})$ is in $\mathcal{R}_C$.

2.1.10. Definition. A 2*-site* is a 2-category $\mathbf{C}$ endowed with a Grothendieck 2-coverage.

2.1.11. Remark. We will informally use the term *topology* as an equivalent for *Grothendieck* 2*-coverage*.

Covering families and 2-coverages. As in the classical case of a site over a 1-category, a Grothendieck 2-coverage can be presented by covering families of 1-morphisms. Compared to the pre-topologies defined in [Str82], we have chosen to take a much more flexible definition for our 2-coverages. This will require slightly more work from us in the following sections but will later greatly ease the definition of a specific topology on a 2-category. Note that a 2-coverage is a totally distinct notion from a Grothendieck 2-coverage; while there are related, we should never identify one with the other.

2.1.12. Definition. Let $\mathbf{C}$ be 2-category. A 2*-coverage* $\mathcal{C}$ on $\mathbf{C}$ is the choice, for each object C of $\mathbf{C}$, of a collection $\mathcal{C}_C$ of families of 1-morphisms $(f_i \colon C_i \to C)_i$ with the same codomain C, called *covering families of* C. The covering families must satisfy the following axioms:

- (Cov1) For each object C of $\mathbf{C}$, the collection $\mathcal{C}_C$ is nonempty.
- (Cov2) For any covering family $\mathcal{F} \in \mathcal{C}_C$ and morphism $h \colon D \to C$, there exists a covering family $\mathcal{G} \in \mathcal{C}_D$ such that, for any morphism $g \colon \tilde{D} \to D$ of $\mathcal{G}$, there is a morphism $f \colon \tilde{C} \to C$ of $\mathcal{F}$ and a factorization (up to an isomorphism):

$$\begin{array}{ccc} D & \xrightarrow{h} & C \\ {\scriptstyle g}\uparrow & \cong & \uparrow{\scriptstyle f} \\ \tilde{D} & \dashrightarrow & \tilde{C} \end{array}$$

2.1.13. Definition. Let $\mathbf{C}$ be 2-category. A *saturated* 2*-coverage* on $\mathbf{C}$ is a 2-coverage $\mathcal{C}$ satisfying the following additional axiom.

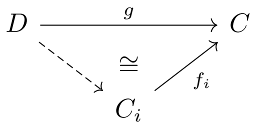
(Cov3) Let $\mathcal{F} \in \mathcal{C}_C$ be a covering family and, for each $f : C_f \to C \in \mathcal{F}$, let $\mathcal{G}_f \in \mathcal{C}_{C_f}$ be a family covering C_f. Then the family of morphisms

$$\{f \circ g \mid f \in \mathcal{F}, g \in \mathcal{G}_f\}$$

covers C.

2.1.14. Remark. Any coverage $\mathcal{C}$ induces a saturated coverage $\mathcal{C}^{\text{sat}}$.

2.1.15. Definition. Let $\mathbf{C}$ be a 2-category and C an object of $\mathbf{C}$. A family of 1-morphisms $\mathcal{F} = (f_i : C_i \to C)_i$ with the same codomain C defines a 2-sieve $\mathbb{R}_{\mathcal{F}}$ on C as follows. For any object D of $\mathbf{C}$, the category $\mathbb{R}_{\mathcal{F}}D$ is the full subcategory of $\mathbf{C}(D, C)$ whose objects are the 1-morphisms $g : D \to C$ which factor through some $f_i \in \mathcal{F}$, up to an isomorphism:



Endowed with the canonical inclusions, the categories $(\mathbb{R}_{\mathcal{F}}D)_D$ define a 2-sieve $\mathbb{R}_{\mathcal{F}}$ on C, called the 2-sieve generated by $\mathcal{F}$.

2.1.16. Remark. Any 2-sieve $(\mathbb{R}, \iota_{\mathbb{R}})$ is generated (up to equivalence) by a family of morphisms. Indeed consider the family of morphisms given by the reunion of the essential images of $\iota_{\mathbb{R}}$:

$$\mathcal{F} = \bigcup_{Y \in \mathbf{C}} \iota_{\mathbb{R},Y}(\mathbb{R}Y).$$

Then $(\mathbb{R}, \iota_{\mathbb{R}})$ is equivalent to the 2-sieve $\mathbb{R}_{\mathcal{F}}$ generated by $\mathcal{F}$.

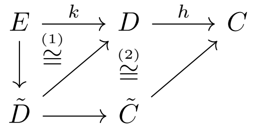
2.1.17. Proposition. *Let $\mathbf{C}$ be a 2-category endowed with a saturated 2-coverage $\mathcal{C}$. For each object C of $\mathbf{C}$, define the following set of 2-sieves on C:*

$$\mathcal{R}^{\mathcal{C}}_C = \{(\mathbb{R}, \iota_{\mathbb{R}}) \mid \exists \mathcal{F} \in \mathcal{C}_C \text{ and } \exists \mathbb{R}_{\mathcal{F}} \to (\mathbb{R}, \iota_{\mathbb{R}}) \text{ in } \mathbf{2Sieve}/C\}$$

Then, the collection of the sets $(\mathcal{R}^{\mathcal{C}}_C)_C$ is a Grothendieck coverage $\mathcal{R}^{\mathcal{C}}$ on $\mathbf{C}$.

PROOF. The axioms (GC1) and (GC3) are immediately satisfied by construction of $\mathcal{R}^{\mathcal{C}}$.

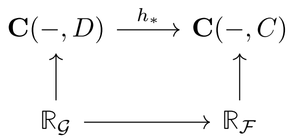
We first show that (GC2) is satisfied. Fix an object C of $\mathbf{C}$, a morphism $h : D \to C$ and a covering sieve $(\mathbb{R}, \iota) \in \mathcal{R}^{\mathcal{C}}_C$. By construction, there exists a covering family $\mathcal{F} \in \mathcal{C}_C$ and a 1-morphism $(\mu, m) : \mathbb{R}_{\mathcal{F}} \to (\mathbb{R}, \iota)$ in $\mathbf{2Sieve}/C$. Choose a covering family $\mathcal{G} \in \mathcal{C}_D$ as in (Cov2), with respect to the morphism $h : D \to C$. For any object E of $\mathbf{C}$, the functor $h_* : \mathbf{C}(E, D) \to \mathbf{C}(E, C)$ restricts to a functor $h_* : \mathbb{R}_{\mathcal{G}}E \to \mathbb{R}_{\mathcal{F}}E$. Indeed, given an object $k : E \to D$ of $\mathbb{R}_{\mathcal{G}}E$, we can form the following pasting diagram:



The isomorphism (1) is given by the definition of $\mathbb{R}_{\mathcal{G}}$ (definition 2.1.15) and the isomorphism (2) by the axiom (Cov2). Thus, the pseudonatural transformation

$$h_* : \mathbf{C}(-, D) \to \mathbf{C}(-, C)$$

fits in the commutative diagram:



By the universal property of bipullbacks, this diagram induces a 1-morphism in **2Sieve**$/D$:

$$\mathbb{R}_{\mathcal{G}} \to h^*(\mathbb{R}_{\mathcal{F}}).$$

Postcomposing by $h^*(\mu, m)\colon h^*(\mathbb{R}_{\mathcal{F}}) \to h^*(\mathbb{R}, \iota)$, we obtain the wanted 1-morphism $\mathbb{R}_{\mathcal{G}} \to h^*(\mathbb{R}, \iota)$.

We then show that (GC4) is satisfied. Fix an object C of $\mathbf{C}$, a 2-sieve $(\mathbb{S}, \iota_{\mathbb{S}})$ on C and a covering 2-sieve $(\mathbb{R}, \iota_{\mathbb{R}})$. Assume that for each object D of $\mathbf{C}$ and for each object r of $\mathbb{R}D$, the 2-sieve $(\iota_{\mathbb{R}} r)^*(\mathbb{S})$ is covering. There is a covering family $\mathcal{F} \in \mathcal{C}_C$ and a 1-morphism $\mu\colon \mathbb{R}_{\mathcal{F}} \to \mathbb{R}$ of 2-sieves. For any covering morphism $f\colon \tilde{C} \to C$ in $\mathcal{F}$, the sieve $f^*\mathbb{S}$ is covering, since it is equivalent to the covering sieve $(\iota_{\mathbb{R}} \mu f)^*(\mathbb{S})$. Hence there is a covering family $\mathcal{G}_f \in \mathcal{C}_{\tilde{C}}$ and a 1-morphism $\mu_f\colon \mathbb{R}_{\mathcal{G}_f} \to f^*\mathbb{S}$ of 2-sieves. By the axiom (Cov3), the family of morphisms

$$\tilde{\mathcal{F}} = \{ f \circ g \mid f \in \mathcal{F}, g \in \mathcal{G}_f \}$$

covers C. Moreover there is a 1-morphism of 2-sieves $\mathbb{R}_{\tilde{\mathcal{F}}} \to \mathbb{S}$ induced by the mapping $f \circ g \mapsto (f_* \circ \mu_f)(g)$, which means that the 2-sieve $\mathbb{S}$ is a covering 2-sieve. ☐

2.1.18. Definition. Let $\mathbf{C}$ be a 2-category endowed with a 2-coverage $\mathcal{C}$. The *Grothendieck coverage* $\mathcal{R}^{\mathcal{C}}$ *generated by* $\mathcal{C}$ is the Grothendieck coverage $\mathcal{R}^{\mathcal{C}^{\mathrm{sat}}}$ given by proposition 2.1.17.

2.2. Descent

For a 2-functor

$$\mathbb{F}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$$

descent describes a process for gluing together categories $\mathbb{F}X_i$ along a family of morphisms $(X_i \to X)_i$ with the same codomain. This section is vastly inspired by [Vis05], which explains in great details the notion of descent for stacks. Note that our descent diagrams are not the same as those described in [Str82] and [BL03]; nevertheless their bilimits are the same, as they have the same model (definition 2.2.7).

Descent diagrams.

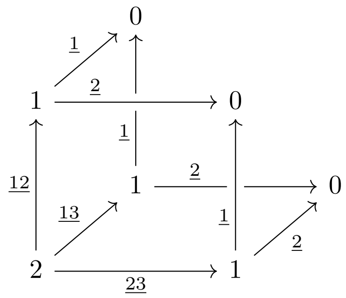
2.2.1. Definition. The *descent* 2-*category* **Desc** is the 2-category generated by the graph

$$0 \;\underset{\underline{2}}{\overset{\underline{1}}{\leftleftarrows}}\; 1 \;\overset{\underline{12},\ \underline{13},\ \underline{23}}{\Lleftarrow}\; 2$$

with three additional (non-trivial) 2-isomorphisms (and their inverses):

$$\bar{1}\colon \underline{1} \circ \underline{12} \Rightarrow \underline{1} \circ \underline{13}$$
$$\bar{2}\colon \underline{2} \circ \underline{12} \Rightarrow \underline{1} \circ \underline{23}$$
$$\bar{3}\colon \underline{2} \circ \underline{13} \Rightarrow \underline{2} \circ \underline{23}$$

2.2.2. Remark. The 2-category **Desc** can be pictured as one half of a cube



where the faces are precisely the 2-isomorphisms $\overline{1}$, $\overline{2}$ and $\overline{3}$. This diagram should be compared to the cube (1.2.14) naturally defined by a bipullback of three 1-morphisms.

2.2.3. Definition. Let $\mathbf{C}$ be a 2-category with finite bipullbacks and $\mathbb{X}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ a 2-functor. Let $(f_i\colon X_i \to X)_{i\in\mathcal{I}}$ be a family of morphisms in $\mathbf{C}$ with the same codomain X. The *descent diagram* for the family (f_i) is the 2-functor

$$\mathbb{D}^{\mathbb{X}}_{(f_i)} : \mathbf{Desc}^{\mathrm{op}} \to \mathbf{Cat}$$

defined as follows.

- *On objects,* $\mathbb{D}^{\mathbb{X}}_{(f_i)}$ is defined by the mapping

$$\begin{cases} 0 & \mapsto & \prod_{i\in\mathcal{I}} \mathbb{X}(X_i) \\ 1 & \mapsto & \prod_{i,j\in\mathcal{I}} \mathbb{X}(f_i|f_j) \\ 2 & \mapsto & \prod_{i,j,k\in\mathcal{I}} \mathbb{X}(f_i|f_j|f_k) \end{cases}$$

- *On* 1*-morphisms,* $\mathbb{D}^{\mathbb{X}}_{(f_i)}$ is defined using the corresponding structural 1-morphisms of the bipullbacks. For instance, for each $i, j \in \mathcal{I}$, there is a structural 1-morphism

$$\underline{1}\colon (f_i|f_j) \to X_i$$

which is mapped by $\mathbb{X}$ to

$$\mathbb{X}(\underline{1})\colon \mathbb{X}(X_i) \to \mathbb{X}(f_i|f_j).$$

Grouping together these 1-morphisms yields a 1-morphism:

$$\mathbb{D}^{\mathbb{X}}_{(f_i)}(\underline{1})\colon \prod_{i\in\mathcal{I}} \mathbb{X}(X_i) \to \prod_{i,j\in\mathcal{I}} \mathbb{X}(f_i|f_j)$$

- *On* 2*-morphisms,* $\mathbb{D}^{\mathbb{X}}_{(f_i)}$ is similarly defined using the corresponding structural 2-morphisms of the bipullbacks.

We are interested in the bilimits of the form

$$\operatorname*{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{(f_i)}.$$

However, directly manipulating descent diagrams turns out to be quite cumbersome. The following proposition provides an easier method to deal with bilimits of descent diagrams.

2.2.4. Proposition. *Let* $\mathbf{C}$ *be a* 2*-category with finite bipullbacks and* $\mathbb{X}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ *a* 2*-functor. Let* $(f_i\colon X_i \to X)_{i\in\mathcal{I}}$ *be a family of morphisms in* $\mathbf{C}$ *with the same codomain* X*. The data of a* 1*-morphism*

$$\tilde{u}\colon V \to \operatorname*{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{f_i}$$

is equivalent to the data of a family of 1*-morphisms* $u_i\colon V \to \mathbb{X}X_i$ *and for any* i, j*, a natural isomorphism* ζ_{ij}

$$\begin{tikzcd} & V \arrow[dl, "u_i"'] \arrow[dr, "u_j"] & \\ \mathbb{X}X_i \arrow[dr, "(\underline{f_i}|f_j)^*"'] & \overset{\zeta_{ij}}{\Longrightarrow} & \mathbb{X}X_j \arrow[dl, "(f_i|\underline{f_j})^*"] \\ & \mathbb{X}(f_i|f_j) & \end{tikzcd}$$

such that for any i, j, k*, the following cubical law holds:*

$$\begin{tikzcd}[column sep=tiny] & & V \arrow[dll, "u_i"'] \arrow[drr, "u_k"] & & \\ \mathbb{X}X_i \arrow[d] \arrow[dr] & & \overset{\zeta_{i,k}}{\Longrightarrow} & & \mathbb{X}X_k \arrow[d] \arrow[dl] \\ \mathbb{X}(f_i|f_j) \arrow[dr] & \cong \ \mathbb{X}(f_i|f_k) \arrow[d] & & \mathbb{X}(f_j|f_k) \ \cong \arrow[dl] & \\ & \mathbb{X}(f_i|f_j|f_k) & & & \end{tikzcd} = \begin{tikzcd}[column sep=tiny] & & V \arrow[dll, "u_i"'] \arrow[d, "u_j"] \arrow[drr, "u_k"] & & \\ \mathbb{X}X_i \arrow[d] & \overset{\zeta_{ij}}{\Longrightarrow} & \mathbb{X}X_j \arrow[dll] \arrow[drr] & \overset{\zeta_{jk}}{\Longrightarrow} & \mathbb{X}X_k \arrow[d] \\ \mathbb{X}(f_i|f_j) \arrow[dr] & & \cong & & \mathbb{X}(f_j|f_k) \arrow[dl] \\ & & \mathbb{X}(f_i|f_j|f_k) & & \end{tikzcd} \tag{2.2.5}$$

Under this correspondence:

- *The functor* $\tilde{u}$ *is faithful if and only if the morphism* $(u_i)_i\colon V \to \prod_i \mathbb{X}X_i$ *is faithful.*
- *The functor* $\tilde{u}$ *is full if and only if the following holds: for any objects* $a, b \in V$ *and any family of morphisms* $(g_i\colon u_i a \to u_i b)_i$ *such that for any* i, j *the following square commutes*

$$\begin{tikzcd} \underline{1}^* u_i a \arrow[r, "\zeta_{ij}"] \arrow[d, "\underline{1}^* g_i"'] & \underline{2}^* u_j a \arrow[d, "\underline{2}^* g_j"] \\ \underline{1}^* u_i b \arrow[r, "\zeta_{ij}"'] & \underline{2}^* u_j b \end{tikzcd}$$

  there is a morphism $g\colon a \to b$ *such that for all* i*,* $g_i = u_i g$*.*
- *The functor* $\tilde{u}$ *is essentially surjective if and only if the following holds: for any family of objects* $(a_i \in \mathbb{X}X_i)_{i\in\mathcal{I}}$ *and morphisms* $z_{ij}\colon \mathbb{X}\underline{1}a_i \to \mathbb{X}\underline{1}a_{j\,ij}$ *such that for any* $i, j, k \in \mathcal{I}$ *the following diagram in* $\mathbb{X}(f_i|f_j|f_k)$ *commutes*

$$\begin{tikzcd} (\underline{12}^*\underline{1}^*u_i)a_i \arrow[r, "(\overline{1}^*u_i)_{a_i}"] \arrow[d, "\underline{12}^*z_{ij}"'] & (\underline{13}^*\underline{1}^*u_i)a_i \arrow[r, "\underline{13}^*z_{ik}"] & (\underline{13}^*\underline{2}^*u_k)a_k \arrow[d, "(\overline{3}^*u_k)_{a_k}"] \\ (\underline{12}^*\underline{2}^*u_j)a_j \arrow[r, "(\overline{2}^*u_j)_{a_j}"'] & (\underline{23}^*\underline{1}^*u_j)a_j \arrow[r, "\underline{23}^*z_{jk}"'] & (\underline{23}^*\underline{2}^*u_k)a_k \end{tikzcd}$$

  there is an object $a \in \mathbb{X}X$ *and, for each* $i \in \mathcal{I}$*, an isomorphism* $\alpha_i\colon a \to f_i a$ *such that for any* $i, j \in \mathcal{I}$*, the following diagram in* $\mathbb{X}(f_i|f_j)$ *commutes*

$$\begin{tikzcd} \underline{1}^* u_i a \arrow[r, "\zeta_{ij}"] \arrow[d, "\underline{1}^* \alpha_i"'] & \underline{2}^* u_j a \arrow[d, "\underline{2}^* \alpha_j"] \\ \underline{1}^* a_i \arrow[r, "z_{ij}"'] & \underline{2}^* a_i \end{tikzcd}$$

Proof. These are direct consequences of proposition 2.2.12 below. □

Category of descent data. The goal of this subsection is to construct an explicit model of the bilimit of a descent diagram

$$\underset{\mathbf{Desc}^{\mathrm{op}}}{\mathrm{bilim}}\ \mathbb{D}^{\mathbb{X}}_{(f_i)}$$

for a 2-functor $\mathbb{X}$ and a family of morphisms $(f_i\colon X_i \to Y)_i$.

A *descent datum* for a family of morphisms $(f_i\colon X_i \to Y)_i$ is a pair consisting of a family of objects $(M_i \in \mathbb{X}X_i)_i$, seen as functors $M_i\colon 1 \to \mathbb{X}X_i$ and a family of natural transformations $(\zeta_{ij}\colon (\underline{f_i}|f_j)^*M_i \Rightarrow (f_i|\underline{f_j})^*M_j)_{ij}$ subject to the cubical relation:

$$(2.2.6)\qquad \begin{array}{ccccc} & & 1 & & \\ & \swarrow^{M_i} & \overset{\zeta_{i,k}}{\Longrightarrow} & \searrow^{M_k} & \\ \mathbb{X}X_i & & & & \mathbb{X}X_k \\ \downarrow & \searrow & & \swarrow & \downarrow \\ \mathbb{X}(f_i|f_j) & \cong & \mathbb{X}(f_i|f_k) & \cong & \mathbb{X}(f_j|f_k) \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbb{X}(f_i|f_j|f_k) & & \end{array} \quad = \quad \begin{array}{ccccc} & \swarrow^{M_i} & 1 \downarrow M_j & \searrow^{M_k} & \\ \mathbb{X}X_i & \overset{\zeta_{ij}}{\Longrightarrow} & \mathbb{X}X_j & \overset{\zeta_{jk}}{\Longrightarrow} & \mathbb{X}X_k \\ \downarrow & \swarrow & & \searrow & \downarrow \\ \mathbb{X}(f_i|f_j) & & \cong & & \mathbb{X}(f_j|f_k) \\ & \searrow & & \swarrow & \\ & & \mathbb{X}(f_i|f_j|f_k) & & \end{array}$$

The descent data for a family of morphisms (f_i) can be organized into a category:

2.2.7. Definition. The *category of descent data* $\mathcal{D}(\mathbb{X}, (f_i)_i)$ for the family of morphisms $(f_i\colon X_i \to Y)$ is the category with:

- *Objects:* the descent data $((M_i)_i, (\zeta_{ij})_{ij})$.
- *Morphisms* $((M_i), (\zeta_{ij})) \to ((M'_i), (\zeta'_{ij}))$*:* the families of natural transformations $(a_i\colon M_i \Rightarrow M'_i)_i$ such that for any i, j, the following square commutes:

$$\begin{array}{ccc} (\underline{f_i}|f_j)^*M_i & \xrightarrow{(\underline{f_i}|f_j)^*a_i} & (\underline{f_i}|f_j)^*M'_i \\ {\scriptstyle\zeta_{ij}}\downarrow & & \downarrow{\scriptstyle\zeta'_{ij}} \\ (f_i|\underline{f_j})^*M_j & \xrightarrow[(f_i|\underline{f_j})^*a_j]{} & (f_i|\underline{f_j})^*M_j \end{array}$$

This category gives an alternative description of the cones over $\mathbb{D}^{\mathbb{X}}_{f_i}$ with vertex 1:

2.2.8. Proposition. *There is an equivalence of categories:*

$$E\colon \mathcal{D}(\mathbb{X}, (f_i)) \simeq [\mathbf{Desc}^{\mathrm{op}}, \mathbf{Cat}](\Delta 1, \mathbb{D}^{\mathbb{X}}_{(f_i)})$$

Proof. We give an explicit construction of E and show that it is fully faithful and essentially surjective. To reduce the notations, we present a proof in the case where the family $(f_i)_i$ is reduced to one morphism $f : X \to Y$, but the general proof is analogous.

We associate to a descent datum (M, ζ) a pseudonatural transformation

$$E(M, \zeta) = C\colon \Delta 1 \to \mathbb{D}^{\mathbb{X}}_{f_i}$$

as follows:

$$\begin{aligned} C_0 &\colon 1 \to \mathbb{X}X = M \\ C_1 &\colon 1 \to \mathbb{X}(f|f) = \underline{1}^*M \\ C_2 &\colon 1 \to \mathbb{X}(f|f|f) = \underline{12}^*\underline{1}^*M \\ C_{\underline{1}} &\colon \underline{1}^*C_0 \Rightarrow C_1 = \mathrm{Id} \\ C_{\underline{2}} &\colon \underline{2}^*C_0 \Rightarrow C_1 = \zeta \\ C_{\underline{12}} &\colon \underline{12}^*C_1 \Rightarrow C_2 = \mathrm{Id} \end{aligned}$$

$$C_{\underline{13}}\colon \underline{13}^*C_1 \Rightarrow C_2 = (\overline{1}^{-1})^*M$$
$$C_{\underline{23}}\colon \underline{23}^*C_1 \Rightarrow C_2 = \underline{12}^*\zeta \circ (\overline{2}^{-1})^*M$$

The compatibility with the faces $\overline{1}^*$ and $\overline{2}^*$ can be read on the definition, and the compatibility with the face $\overline{3}^*$ is a direct consequence of eq. (2.2.6).

Given a morphism $a\colon (M,\zeta) \to (M',\zeta')$, we obtain a modification

$$E(a) = m\colon E(M,\zeta) \to E(M',\zeta')$$

as follows:

$$m_0 = a$$
$$m_1 = \underline{1}^*a$$
$$m_2 = \underline{12}^*\underline{1}^*a$$

It is clear that these mappings define a functor $E\colon \mathcal{D}(\mathbb{X}, f) \to [\mathbf{Desc}^{\mathrm{op}}, \mathbf{Cat}](\Delta 1, \mathbb{D}_f^{\mathbb{X}})$, which is moreover faithful. Furthermore, given any modification $n\colon E(M,\zeta) \Rrightarrow E(M',\zeta')$, we have:

$$(\underline{2}^*n_0)\zeta = (\underline{2}^*n_0)E(M,\zeta)_{\underline{2}} = E(M',\zeta')_{\underline{2}}(\underline{2}^*n_0) = \zeta'(\underline{2}^*n_0) \tag{2.2.9}$$
$$n_1 = n_1E(M,\zeta)_{\underline{1}} = E(M',\zeta')_{\underline{1}}(\underline{1}^*n_0) = \underline{1}^*n_0 \tag{2.2.10}$$
$$n_2 = n_2E(M,\zeta)_{\underline{12}} = E(M',\zeta')_{\underline{12}}(\underline{12}^*n_1) = \underline{12}^*n_1 \tag{2.2.11}$$

Using eq. (2.2.9), we check that $n_0\colon (M,\zeta) \to (M',\zeta')$ is a morphism of object of descent data; eq. (2.2.10) and eq. (2.2.9) imply that $n = E(n_0)$. Hence E is also full.

Finally we check the essential surjectivity of E. Given a cone $C\colon \Delta 1 \to \mathbb{D}_f^{\mathbb{X}}$, we consider the functor $M = C_0\colon 1 \to \mathbb{X}X$ and the natural transformation

$$\zeta\colon\ \underline{1}^*M \xRightarrow{C_{\underline{1}}} C_1 \xRightarrow{C_{\underline{2}}^{-1}} \underline{2}^*M$$

. The pair (M,ζ) is an object of descent data. Moreover there is an isomorphism $m\colon E(M,\zeta) \to C$ with components:

$$m_0 = \mathrm{Id}$$
$$m_1 = C_{\underline{1}}$$
$$m_2 = C_{\underline{12}}$$

Hence C is in the essential image of E.

Since E is essentially surjective and fully faithful, it is an equivalence. □

From the general description of bilimits in **Cat**, we deduce:

2.2.12. Proposition. *There is an equivalence*

$$\mathcal{D}(\mathbb{X}, (f_i)) \cong \operatorname*{bilim}_{\mathbf{Desc}} \mathbb{D}_{f_i}^{\mathbb{X}}$$

Descent condition. A 2-functor $\mathbb{X}$ satisfies the descent condition for a family of morphisms $(f_i\colon X_i \to X)$ if its value $\mathbb{X}(X)$ can be recovered by the process of descent previously described.

We first note that there is always a comparison 1-morphisms between the category $\mathbb{X}(X)$ and the descent bilimit of $\mathbb{X}$ along the $(f_i)_i$.

2.2.13. Definition. Let $\mathbf{C}$ be a 2-category with finite bipullbacks and $\mathbb{X}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ a 2-functor. Let $(f_i\colon X_i \to X)_{i\in\mathcal{I}}$ be a family of morphisms in $\mathbf{C}$ with the same codomain X. The functors $(\mathbb{X}f_i)$ and the natural transformations $\mathbb{X}\overline{(f_i|f_j)}$ define, by proposition 2.2.4, a canonical functor:

$$\mathbb{X}(X) \to \operatorname*{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}_{(f_i)}^{\mathbb{X}}$$

2.2.14. Definition. Let $\mathbf{C}$ be a 2-category with finite bipullbacks and $\mathbb{X}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ a 2-functor. Let $(f_i\colon X_i \to X)_{i\in\mathcal{I}}$ be a family of morphisms in $\mathbf{C}$ with the same codomain X. The 2-functor $\mathbb{X}$ *satisfies the descent condition* for the family (f_i) if the canonical comparison 1-morphism

$$\mathbb{X}(X) \to \operatorname*{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{(f_i)}$$

is an equivalence.

We show that the descent condition only depends on the isomorphism classes of the 1-morphisms.

2.2.15. Lemma. *Let $\mathbf{C}$ be a 2-category with finite bipullbacks and $\mathbb{X}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ a 2-functor. Let $(f_i\colon X_i \to X)_{i\in\mathcal{I}}$ and $(g_i\colon X_i \to X)_{i\in\mathcal{I}}$ be two families of morphisms with the same codomain X. Suppose that, for each $i \in \mathcal{I}$, there is an isomorphism*

$$f_i \cong g_i$$

Then $\mathbb{X}$ satisfies the descent condition for the (f_i) if and only if it satisfies the descent condition for the (g_i).

PROOF. For each $i \in \mathcal{I}$, there is an isomorphism $\alpha_i\colon f_i \to g_i$. By the universal property of bipullbacks, they induce a pseudonatural equivalence

$$\mathbb{D}^{\mathbb{X}}_{(g_i)} \to \mathbb{D}^{\mathbb{X}}_{(f_i)}$$

and thus an equivalence of categories fitting in the diagram:

$$\begin{array}{ccc} & \mathbb{X}(X) & \\ \swarrow & & \searrow \\ \mathrm{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{(g_i)} & \xrightarrow[\simeq]{} & \mathrm{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{(f_i)} \end{array}$$

Hence the 2-functor $\mathbb{X}$ satisfies the descent condition for the (f_i) if and only if it satisfies the descent condition for the (g_i). □

We would like the descent condition to remain true under composition of families of 1-morphisms; this is not true in general, but we have the following result with slightly stronger hypotheses.

2.2.16. Lemma. *Let $\mathbf{C}$ be a 2-category with finite bipullbacks and $\mathbb{X}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ a 2-functor. Let $(f_i\colon X_i \to X)_{i\in\mathcal{I}}$ be a family of morphisms in $\mathbf{C}$ with the same codomain X and, for each $i \in \mathcal{I}$, let $(f_{ij}\colon X_{ij} \to X_i)_{j\in\mathcal{J}_i}$ be a family of morphisms with the same codomain X_i. Consider the family of composite morphisms*

$$\mathcal{G} = \{g_{ij} = f_i \circ f_{ij} \mid i \in \mathcal{I}, j \in \mathcal{J}_i\}$$

and its canonical functor

$$G\colon \mathbb{X}(X) \to \operatorname*{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{\mathcal{G}}$$

Suppose that:

- *The 2-functor $\mathbb{X}$ satisfies the descent condition for the family $(f_i)_i$.*
- *For each $i \in \mathcal{I}$, the 2-functor $\mathbb{X}$ satisfies the descent condition for the family $(f_{ij})_j$.*

Under these common hypotheses, we have the following propositions:

(a) *G is faithful*

(b) *Assume that, for each $i, i' \in \mathcal{I}$, the comparison functor*

$$\mathbb{X}(f_i|f_{i'}) \to \operatorname*{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{(\Xi_{jj'})_{jj'}}$$

is faithful, where, for each $j \in \mathcal{J}_i$ and $j' \in \mathcal{J}_{i'}$, the 1*-morphism $\Xi_{jj'} : (f_i|f_{i'}) \to (g_{ij}|g_{i'j'})$ is given by lemma 1.2.22. Then G is full.*

(c) *Assume that:*

- *for each $i, i' \in \mathcal{I}$, the comparison functor*

$$\mathbb{X}(f_i|f_{i'}) \to \operatorname*{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{(\Xi_{jj'})_{jj'}}$$

is fully faithful, where, for each $j \in \mathcal{J}_i$ and $j' \in \mathcal{J}_{i'}$, the 1*-morphism $\Xi_{jj'} : (f_i|f_{i'}) \to (g_{ij}|g_{i'j'})$ is given by lemma 1.2.22.*

- *for each $i_1, i_2, i_3 \in \mathcal{I}$, the functor*

$$\mathbb{X}(f_{i_1}|f_{i_2}|f_{i_3}) \xrightarrow{(\Xi^*_{j_1j_2j_3})_{j_1j_2j_3}} \prod_{j_1,j_2,j_3} \mathbb{X}(g_{i_1j_1}|g_{i_2j_2}|g_{i_3j_3})$$

is faithful, where, for each $j_1 \in \mathcal{J}_{i_1}, j_2 \in \mathcal{J}_{i_2}, j_3 \in \mathcal{J}_{i_3}$, the 1*-morphism $\Xi_{j_1j_2j_3} : (f_{i_1}|f_{i_2})f_{i_3} \to (g_{i_1j_1}|g_{i_2j_2})g_{i_3j_3}$ is given by lemma 1.2.23.*

Then G is essentially surjective.

PROOF. The comparison functor

$$G : \mathbb{X}(X) \to \operatorname*{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{\mathcal{G}}$$

is induced by the functors g^*_{ij} and the natural transformations $\zeta_{iji'j'} = \overline{(g_{ij}|g_{i'j'})}^*$. We use the criteria of proposition 2.2.4 to show that G is fully faithful and essentially surjective.

(a) *Faithfulness.* First note that we have the decomposition:

$$(g^*_{ij})_{ij} \cong (f^*_{ij} \circ f^*_i)_{ij} = \left(\prod_i (f^*_{ij})_j\right) \circ ((f^*_i)_i)$$

Since, for each i, the functor

$$(f^*_{ij})_j : \mathbb{X}(X_i) \to \prod_j \mathbb{X}(X_{ij})$$

is faithful, so is the functor

$$\prod_i (f^*_{ij})_j : \prod_i \mathbb{X}(X_i) \to \prod_i \prod_j \mathbb{X}(X_{ij}).$$

As the functor

$$(f^*_i)_i : \mathbb{X}(X) \to \prod_i \mathbb{X}(X_i)$$

is also faithful, the composite functor $(g^*_{ij})_{ij}$ is faithful.

(b) *Fullness.* Fix two objects a, b of the category $\mathbb{X}X$ and a family of morphisms $(z_{ij} : g^*_{ij}a \to g^*_{ij}b)_{ij}$ such that for any i, j, i', j' the following square in $\mathbb{X}(g_{ij}|g_{i'j'})$ commutes:

$$\begin{array}{ccc} \underline{1}^* g^*_{ij} a & \xrightarrow{\zeta_{iji'j',a}} & \underline{2}^* g^*_{i'j'} a \\ {\scriptstyle \underline{1}^* z_{ij}} \downarrow & & \downarrow {\scriptstyle \underline{2}^* z_{i'j'}} \\ \underline{1}^* g^*_{ij} b & \xrightarrow[\zeta_{iji'j',b}]{} & \underline{2}^* g^*_{i'j'} b \end{array} \tag{2.2.17}$$

Fix $i \in \mathcal{I}$. We consider the objects $a' = f_i^* a$ and $b' = f_i^* b$ of $\mathbb{X}X_i$, and the family of morphisms $(z_{ij} \colon f_{ij}^* a' \to f_{ij}^* b')_j$. Fix $j, j' \in \mathcal{J}_i$. There is a 1-morphism

$$\nabla \colon (f_{ij}|f_{ij'}) \to (g_{ij}|g_{ij'})$$

given by lemma 1.2.21, with structural 2-isomorphisms $\nabla_k \colon \nabla \circ \underline{k} \to \underline{k}$ for $k = 1, 2$. Applying ∇^* to the square 2.2.17 and composing with the structural 2-isomorphisms gives the commutative diagram:

$$\begin{array}{ccccccc}
\underline{1}^* f_{ij}^* a' & \xrightarrow{(\nabla_1^*)^{-1}_{f_{ij}^* a'}} & \nabla^* \underline{1}^* f_{ij}^* a' & \xrightarrow{\nabla^* \varsigma_{ijij',a}} & \nabla^* \underline{1}^* f_{ij'}^* a' & \xrightarrow{(\nabla_2^*)_{f_{ij'}^* a'}} & \underline{2}^* f_{ij'}^* a' \\
\downarrow{\scriptstyle \underline{1}^* z_{ij}} & & \downarrow{\scriptstyle \nabla^* \underline{1}^* z_{ij}} & & \downarrow{\scriptstyle \nabla^* \underline{2}^* z_{ij}} & & \downarrow{\scriptstyle \underline{2}^* z_{ij'}} \\
\underline{1}^* f_{ij}^* b' & \xrightarrow[(\nabla_1^*)^{-1}_{f_{ij}^* b'}]{} & \nabla^* \underline{1}^* f_{ij}^* b' & \xrightarrow[\nabla^* \varsigma_{ijij',b}]{} & \nabla^* \underline{1}^* f_{ij'}^* b' & \xrightarrow[(\nabla_2^*)_{f_{ij'}^* b'}]{} & \underline{2}^* f_{ij'}^* b'
\end{array}$$

with the outer arrows $\overline{(f_{ij}|f_{ij'})}^*_{a'} \colon \underline{1}^* f_{ij}^* a' \to \underline{2}^* f_{ij'}^* a'$ (top) and $\overline{(f_{ij}|f_{ij'})}^*_{b'} \colon \underline{1}^* f_{ij}^* b' \to \underline{2}^* f_{ij'}^* b'$ (bottom).

Hence there is a morphism $z_i \colon f_i^* a \to f_i^* b$ such that $z_{ij} = (\mathbb{X} f_{ij}) z_i$ for all $j \in \mathcal{J}_i$. Since this is true for any $i \in \mathcal{I}$, we obtain a family of morphisms $(z_i \colon f_i^* a \to \mathbb{X} f_i^* b)_i$. We have to check that for any $i_1, i_2 \in \mathcal{I}$, the square

$$\begin{array}{ccc}
\underline{1}^* f_{i_1}^* a & \xrightarrow{\overline{(f_{i_1}|f_{i_2})}^*_a} & \underline{2}^* f_{i_2}^* a \\
\downarrow{\scriptstyle \underline{1}^* z_{i_1}} & & \downarrow{\scriptstyle \underline{2}^* z_{i_2}} \\
\underline{1}^* f_{i_1}^* b & \xrightarrow[\overline{(f_{i_1}|f_{i_2})}^*_b]{} & \underline{2}^* f_{i_2}^* b
\end{array}$$

in $\mathbb{X}(f_{i_1}|f_{i_2})$ commutes. Since, by hypothesis, the functor

$$\prod_{j_1, j_2} \Xi_{j_1 j_2}^* \colon \mathbb{X}(f_{i_1}|f_{i_2}) \to \mathbb{X}(g_{i_1 j_1}|g_{i_2 j_2})$$

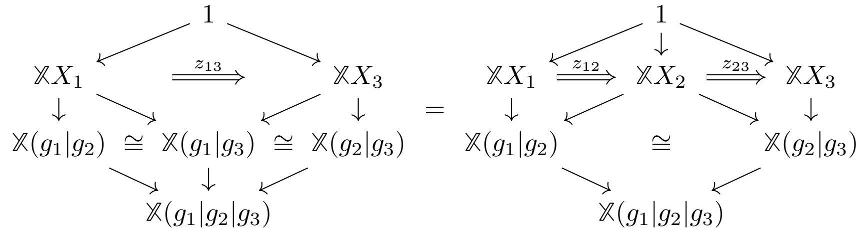
is faithful, it is sufficient to check that, for each $j_1 \in \mathcal{J}_{i_1}$ and $j_2 \in \mathcal{J}_{i_2}$, the square

$$\begin{array}{ccc}
\Xi_{j_1 j_2}^* \underline{1}^* f_{i_1}^* a & \xrightarrow{\Xi_{j_1 j_2}^* \overline{(f_{i_1}|f_{i_2})}^*_a} & \Xi_{j_1 j_2}^* \underline{2}^* f_{i_2}^* a \\
\downarrow{\scriptstyle \Xi_{j_1 j_2}^* \underline{1}^* z_{i_1}} & & \downarrow{\scriptstyle \Xi_{j_1 j_2}^* \underline{2}^* z_{i_2}} \\
\Xi_{j_1 j_2}^* \underline{1}^* f_{i_1}^* b & \xrightarrow[\Xi_{j_1 j_2}^* \overline{(f_{i_1}|f_{j_1})}^*_b]{} & \Xi_{j_1 j_2}^* \underline{2}^* f_{i_2}^* b
\end{array}$$

commutes. Up to the composition with the structural 2-morphisms of $\Xi_{j_1 j_2}$, this is precisely the square 2.2.17, hence it commutes. This guarantees the existence of a morphism $z \colon a \to b$ such that $f_i^* z = z_i$, for all $i \in \mathcal{I}$, hence $g_{ij}^* z = z_{ij}$ for all $i\mathcal{I}$ and $j \in \mathcal{J}_i$. Hence we have proved the fullness of the comparison functor G.

(c) *Essential surjectivity.* Let $(a_{ij}\colon 1 \to X_{ij})$ be a family of functors and $(z_{iji'j'}\colon \underline{1}^* a_{ij} \to \underline{2}^* a_{i'j'})$ a family of morphisms such that for any $i_1, i_2, i_3 \mathcal{I}, j_1 \in \mathcal{J}_{i_1}, j_2 \in \mathcal{J}_{i_2}, j_3 \in \mathcal{J}_3$, the following equation holds:

(2.2.18)



where g_k is a shorthand for $g_{i_k j_k}$, etc...

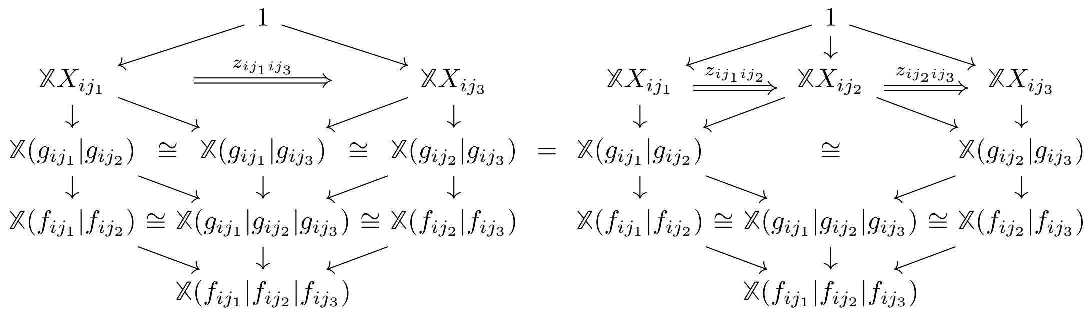
Fix $i \in \mathcal{I}$ and $j_1, j_2, j_3 \in \mathcal{J}_i$. We consider the 1-morphisms

$$
\begin{aligned}
\nabla_{12}&\colon (f_{ij_1}|f_{ij_2}) \to (g_{ij_1}|g_{ij_2})\\
\nabla_{13}&\colon (f_{ij_1}|f_{ij_3}) \to (g_{ij_1}|g_{ij_3})\\
\nabla_{23}&\colon (f_{ij_2}|f_{ij_3}) \to (g_{ij_2}|g_{ij_3})\\
\nabla_{123}&\colon (f_{ij_1}|f_{ij_2}|f_{ij_3}) \to (g_{ij_1}|g_{ij_2}|g_{ij_3})
\end{aligned}
$$

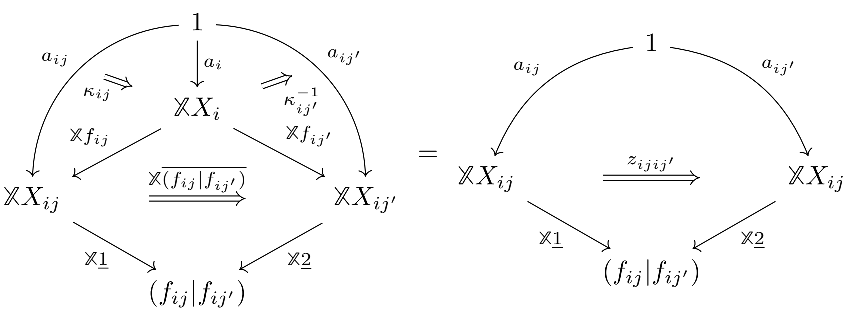
given by lemma 1.2.21.

Postcomposing eq. (2.2.18) with the induced 1-morphisms (and the structural 2-morphisms), yields the equation

(2.2.19)



which is precisely the cubical law for the $(z_{ij})_{j\in\mathcal{J}_i}$ with respect to the $(f_{ij})_j$. Hence there is a functor $a_i\colon 1 \to \mathbb{X}X_i$ and natural transformations $(\kappa_{ij}\colon a_{ij} \to \mathbb{X}f_{ij} \circ K_i)_j$ satisfying:



Fix $i, i' \in \mathcal{I}$. We want to define a morphism

$$z_{ii'}\colon \underline{1}^* a_i \to \underline{2}^* a_{i'}$$

in $\mathbb{X}(f_i|f_{i'})$. For each $j \in \mathcal{J}_i, j' \in \mathcal{J}_{i'}$ there is a morphism in $\mathbb{X}(g_{ij}|g_{i'j'})$:

$$\tilde{z}_{iji'j'} \quad : \quad \Xi^*_{jj'}\underline{1}^*a_i \xrightarrow{\sim} \underline{1}^*a_{ij} \xrightarrow{z_{iji'j'}} \underline{2}^*a_{i'j'} \xrightarrow{\sim} \Xi^*_{jj'}\underline{2}^*a_j$$

A careful expansion of the definition show that the following square commutes, for any $j_1, j_2 \in \mathcal{J}_i, j'_1, j'_2 \in \mathcal{J}_{i'}$:

$$\begin{array}{ccc}
\underline{1}^*\Xi^*_{j_1j'_1}\underline{1}^*a_i & \xrightarrow{\overline{(\Xi_{j_1j'_1}|\Xi_{j_2j'_2})}^*_{\underline{1}^*a_i}} & \underline{2}^*\Xi^*_{j_2j'_2}\underline{1}^*a_i \\
{\scriptstyle \underline{1}^*z_{ij_1i'j'_1}}\downarrow & & \downarrow{\scriptstyle \underline{2}^*z_{ij_2i'j'_2}} \\
\underline{1}^*\Xi^*_{j_1j'_1}\underline{2}^*a_{i'} & \xrightarrow[\overline{(\Xi_{j_1j'_1}|\Xi_{j_2j'_2})}^*_{\underline{2}^*a_{i'}}]{} & \underline{2}^*\Xi^*_{j_2j'_2}\underline{2}^*a_{i'}
\end{array}$$

Since the functor

$$\mathbb{X}(f_i|f_{i'}) \to \underset{\mathbf{Desc}^{\mathrm{op}}}{\mathrm{bilim}}\ \mathbb{D}^{\mathbb{X}}_{(\Xi_{jj'})_{jj'}}$$

is full, this show the existence of morphisms $z_{ii'} : \underline{1}^*a_i \to \underline{2}^*a_{i'}$ such that $\Xi^*_{jj'}z_{ii'} \cong \tilde{z}_{iji'j'}$.

Using the faithfulness of the functors

$$\mathbb{X}(f_{i_1}|f_{i_2}|f_{i_3}) \to \prod_{j_1,j_2,j_3} \mathbb{X}(g_{i_1j_1}|g_{i_2j_2}|g_{i_3j_3})$$

one can show that the $(z_i)_{i\in\mathcal{I}}$ satisfy the cubical law; hence there is a $a\colon 1 \to \mathbb{X}X$ such that $(a_i \cong f_i^*a)_i$, and these isomorphisms are coherent with the $(z_{ii'})$. Unrolling the definitions, we deduce a family of isomorphisms $(a_{ij} \cong g^*_{ij}a)_{ij}$, coherent with the $(z_{iji'j'})$. Hence the functor G is essentially surjective.

□

Refinement of covering families.

2.2.20. Definition. Let $\mathbf{C}$ be a 2-category. Let $\mathcal{F}$ and $\mathcal{G}$ be two families of 1-morphisms of $\mathbf{C}$ which all have the same codomain X. We say that $\mathcal{G}$ is *a refinement* of $\mathcal{F}$ if for any 1-morphism $g : X_g \to X$ in $\mathcal{G}$, there is a 1-morphism $f : X_f \to X$ in $\mathcal{F}$ and a 1-morphism $h\colon X_g \to X_f$ such that

$$fh \cong g.$$

2.2.21. Remark. If $\mathcal{G}$ is a refinement of $\mathcal{F}$, we may say that $\mathcal{G}$ is finer than $\mathcal{F}$ and $\mathcal{F}$ is coarser than $\mathcal{G}$.

We would hope the descent condition to hold for a family of morphisms $\mathcal{F}$, if it holds for a finer family $\mathcal{G}$. This is not true without stronger hypotheses (lemma 2.3.7), but we still have the following partial result.

2.2.22. Lemma. *Let $\mathbf{C}$ be a 2-category and $\mathbb{X}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ a 2-functor. Let $\mathcal{F} = (f_i\colon X_i \to X)_{i\in\mathcal{I}}$ and $\mathcal{G} = (g_j\colon Y_j \to X)_{j\in\mathcal{J}}$ be two families of 1-morphisms of $\mathbf{C}$ which all have the same codomain X. Suppose that $\mathcal{G}$ is a refinement of $\mathcal{F}$ and consider the canonical functors:*

$$F\colon \mathbb{X}(X) \to \underset{\mathbf{Desc}^{\mathrm{op}}}{\mathrm{bilim}}\ \mathbb{D}^{\mathbb{X}}_{\mathcal{F}}$$

$$G\colon \mathbb{X}(X) \to \underset{\mathbf{Desc}^{\mathrm{op}}}{\mathrm{bilim}}\ \mathbb{D}^{\mathbb{X}}_{\mathcal{G}}$$

Then:

(a) *If G is faithful, F is faithful.*

(b) If G is full, F is full.

Proof. First, note that G factors through F:

$$\begin{array}{ccc} \mathbb{X}(X) & \xrightarrow{\quad G \quad} & \\ \big\downarrow F & \cong & \searrow \\ \operatorname{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{\mathcal{F}} & \longrightarrow & \operatorname{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{\mathcal{G}} \end{array}$$

This factorization proves both implications. ☐

2.3. 2-Sheaves

In this section, we define 2-sheaves both for Grothendieck 2-coverages and for 2-coverages. We then show the equivalence of the two definitions.

2.3.1. Definition. Let $(\mathbf{C}, \mathcal{R})$ be a 2-site. A 2*-sheaf on the* 2*-site* $\mathbf{C}$ is a 2-functor $\mathbb{X} \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ such that for any object C of $\mathbf{C}$ and for any covering 2-sieve $(\mathbb{R}, \iota_{\mathbb{R}}) \in \mathcal{R}_C$, the canonical functor

$$\iota_{\mathbb{R}}^* \colon X(c) \simeq [\mathbf{C}^{\mathrm{op}}, \mathbf{Cat}](\mathbf{C}(-, C), X) \longrightarrow [\mathbf{C}^{\mathrm{op}}, \mathbf{Cat}](\mathbb{R}, X)$$

is an equivalence.

2.3.2. Remark. This definition is the same as the one in [Str82]. It guarantees that we are dealing with the same notion of 2-sheaf.

2.3.3. Definition. Let $\mathbf{C}$ be a 2-site. The 2*-categories* $\mathbf{Sh}(\mathbf{C})$ *of sheaves on* $\mathbf{C}$ is the full, 2-full sub-2-category of the 2-category of 2-functors $[\mathbf{C}^{\mathrm{op}}, \mathbf{Cat}]$ whose objects are the sheaves on $\mathbf{C}$.

2.3.4. Definition. Let $\mathbf{C}$ be a 2-category with finite bipullbacks and let $\mathcal{C}$ be a coverage on $\mathbf{C}$. A 2*-sheaf on* $\mathbf{C}$ *for the coverage* $\mathcal{C}$ is a 2-functor $\mathbb{X} \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ such that the following holds: for each object X of $\mathbf{C}$ and each covering family $\mathcal{G} \in \mathcal{C}_X$, the 2-functor $\mathbb{X}$ satisfies the descent condition for $\mathcal{G}$.

We are now able to complete the lemmas of the previous section: we will show that a 2-sheaf for a coverage also satisfies the descent condition for composite of covering families and for families coarser than covering families.

2.3.5. Lemma. *Let $\mathbf{C}$ be a 2-category with finite bipullbacks and $\mathcal{C}$ a coverage on $\mathbf{C}$. Let $\mathbb{X}$ be a sheaf on $\mathbf{C}$ for the coverage $\mathcal{C}$. Let $f_1 \colon X_1 \to X$ and $f_2 \colon X_2 \to X$ be morphisms of $\mathbf{C}$ with the same codomain X and $\mathcal{F}_\ell \in \mathcal{C}_{Y_\ell}$ be covering families of X_ℓ, for $\ell = 1, 2$. Write $\mathcal{F}_\ell = (f_{\ell i} \colon X_{\ell i} \to X_\ell)_{i \in \mathcal{I}_\ell}$. Then the functor*

$$\mathbb{X}(f_1 | f_2) \to \operatorname*{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{(\Xi_{i_1 i_2})_{i_1 i_2}}$$

is fully faithful, where the 1-morphisms $\Xi_{i_1 i_2} \colon (f_1 f_{1 i_1} | f_2 f_{2 i_2}) \to (f_1 | f_2)$ are given by lemma 1.2.22.

Proof. Since $\mathcal{C}$ is a coverage, we can find a covering family

$$\mathcal{G} = (g_j \colon Y_j \to (f_1 | f_2))_{j \in \mathcal{J}}$$

such that, for all $j \in \mathcal{J}$, there is an index $\phi(j) \in \mathcal{I}_1$ and an isomorphism

$$\begin{array}{ccc} (f_1 | f_2) & \xrightarrow{\ \ 1\ \ } & X_1 \\ {\scriptstyle g_j} \big\uparrow & \cong & \big\uparrow {\scriptstyle f_{1\phi(j)}} \\ Y_j & \longrightarrow & X_{1\phi(j)} \end{array}$$

Then, for each $j \in \mathcal{J}$, there is a covering family $\mathcal{H}_j = (h_{jk} \colon Z_{jk} \to Y_j)_{k \in \mathcal{K}_j}$ such that for all $k \in \mathcal{K}_j$, there is an index $\psi k \in \mathcal{I}_2$ and an isomorphism

$$\begin{array}{ccc} Y_j & \xrightarrow{2g_j} & X_2 \\ {\scriptstyle h_{jk}}\uparrow & \cong & \uparrow{\scriptstyle f_{2\psi(k)}} \\ Z_{jk} & \longrightarrow & X_{2\psi(k)} \end{array}$$

We consider the composite family of morphisms:

$$\mathcal{E} = \{g_j h_{jk} | j \in \mathcal{J}, k \in \mathcal{K}_j\}$$

Remark that $\mathcal{E}$ is a refinement of $(\Xi_{i_1 i_2})_{i_1 i_2}$, hence, by lemma 2.2.22, it is sufficient to show that the functor

$$E \colon \mathbb{X}(f_1|f_2) \to \underset{\mathbf{Desc}^{\mathrm{op}}}{\mathrm{bilim}}\, \mathbb{D}^{\mathbb{X}}_{\mathcal{E}}$$

is fully faithful. By the first part of lemma 2.2.16, the functor E is faithful. Hence the same argument can be used to show that, for any j, j' the functor

$$\mathbb{X}(g_j|g_{j'}) \to \underset{\mathbf{Desc}^{\mathrm{op}}}{\mathrm{bilim}}\, \mathbb{D}^{\mathbb{X}}_{(\tilde{\Xi}_{jkj'k'})}$$

is faithful, where the $\tilde{X}^{\mathbb{X}}_{jkj'k'} \colon (g_j h_{jk}|g_{j'} h_{j'k'})$ are given by lemma 1.2.22. Hence, we can apply the second part of lemma 2.2.16 to show that E is full. □

2.3.6. Lemma. *Let* $\mathbf{C}$ *be a* 2*-category with finite bipullbacks and* $\mathcal{C}$ *a coverage on* $\mathbf{C}$*. Let* $\mathbb{X}$ *be a sheaf on* $\mathbf{C}$ *for the coverage* $\mathcal{C}$*. Let* $f_\ell \colon X_\ell \to X$ *be morphisms of* $\mathbf{C}$ *with the same codomain* X *and* $\mathcal{F}_\ell \in \mathcal{C}_{Y_\ell}$ *be covering families of* X_ℓ*, for* $\ell = 1, 2, 3$*. Write* $\mathcal{F}_\ell = (f_{\ell i} \colon X_{\ell i} \to X_\ell)_{i \in \mathcal{I}_\ell}$*. Then the functor*

$$\mathbb{X}(f_1|f_2|f_3) \to \underset{\mathbf{Desc}^{\mathrm{op}}}{\mathrm{bilim}}\, \mathbb{D}^{\mathbb{X}}_{\Xi_{i_1 i_2 i_3}}$$

is faithful, where the 1*-morphisms* $\Xi_{i_1 i_2 i_3} \colon (f_1 f_{1i_1}|f_2 f_{2i_2}|f_3 f_{3i_3}) \to (f_1|f_2|f_3)$ *are given by lemma 1.2.22.*

Proof. This is an argument similar to the one of lemma 2.3.5 □

2.3.7. Lemma. *Let* $\mathbf{C}$ *be a* 2*-category with finite bipullbacks and* $\mathcal{C}$ *be a saturated coverage on* $\mathbf{C}$*. Let* $\mathbb{X}$ *be a sheaf on* $\mathbf{C}$ *for the coverage* $\mathcal{C}$*. Let* $\mathcal{F}$ *be a (not necessarily covering) family of morphisms of* $\mathbf{C}$ *with the same codomain* X *and* $\mathcal{G} \in \mathcal{C}_X$ *be a family covering* X*. Assume that* $\mathcal{G}$ *is refinement of* $\mathcal{F}$*. Then the* 2*-functor* $\mathbb{X}$ *satisfies the descent condition for* $\mathcal{F}$*.*

Proof. By lemma 2.2.22, it is sufficient to prove that the comparison functor

$$\mathbb{X}X \to \underset{\mathbf{Desc}^{\mathrm{op}}}{\mathrm{bilim}}\, \mathbb{D}^{\mathbb{X}}_{\mathcal{F}}$$

is essentially surjective.

We write $\mathcal{F} = (f_i \colon Y_i \to X)_{i \in \mathcal{I}}$ and $\mathcal{G} = (g_j \colon Z_j \to X)_{j \in \mathcal{J}}$. By hypothesis and lemma 2.2.15, we can assume that, for each $j \in \mathcal{J}$, there is an index $\phi(j) \in \mathcal{I}$ and a 1-morphisms $h_j \colon Z_j \to Y_{\phi(j)}$ such that

$$g_j = f_{\phi(j)} h_j.$$

Let $(a_i \colon 1 \to \mathbb{X}Y_i)_i$ be a family of functors in **Cat** and $(z_{ii'} \colon \underline{1}^* a_i \to \underline{2}^* a_{i'})_{ii'}$ a family of natural transformations which satisfy the cubical laws induced by the (f_i). We can form a family of functors $(a'_j = h_j^* a_{\phi(j)} \colon 1 \to \mathbb{X}Z_j)_j$ and natural transformations $(z'_{jj'} = \Xi^*_{jj'} z_{\phi(j)\phi(j')} \colon \underline{1}^* a'_j \to \underline{2} a'_{j'})_{jj'}$ where the

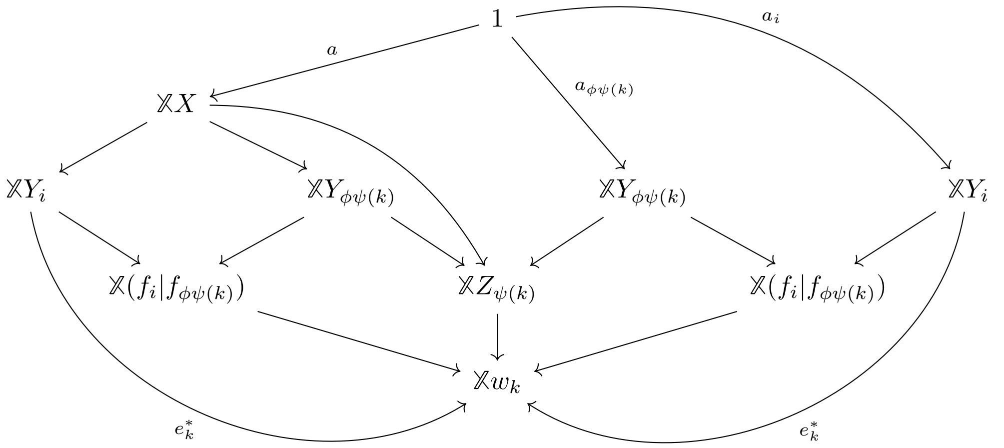
$$\Xi_{jj'} \colon (g_j|g_{j'}) \to (f_{\phi(j)}|f_{\phi(j')})$$

Clearly, the $(z'_{jj'})$ satisfy the cubical laws for the (g_j), hence there are a functor $a \colon 1 \to \mathbb{X}X$ and, for each $j \in \mathcal{J}$, an isomorphism $g_j^* a \cong a'_j$, which are compatible with the $(z'_{jj'})$.

We have to construct, for each $i \in \mathcal{I}$, an isomorphism $f_i^* a \cong a_i$. Fix $i \in \mathcal{I}$. There is a covering family of $(e_k \colon w_k \to Y_i)_{k \in \mathcal{K}} \in \mathcal{C}_{Y_i}$ of Y_i such that, for each $k \in \mathcal{K}$, there is an index $\psi(k) \in \mathcal{J}$ and an isomorphism

$$\begin{array}{ccc} Y_i & \xrightarrow{f_i} & x \\ {\scriptstyle e_k}\uparrow & \cong & \uparrow{\scriptstyle g_{\psi(k)}} \\ w_k & \longrightarrow & Z_{\psi(k)} \end{array}$$

which induce a 1-morphism $w_k \to (f_i | f_{\phi\psi(k)})$, and its structural 2-isomorphisms. Hence, for each k, we can construct the following natural isomorphism α_k (all faces being natural isomorphisms):



This should be read as a (convoluted) conjugation of the isomorphism $g_{\psi(k)} a \cong h_{\psi(k)} a_{\phi\psi(k)}$. We have thus defined a family of isomorphisms $(\alpha_k \colon e_k^* f_i^* a \to e_k^* a_i)_k$. One can check that for each k, k' the following square in $\mathbb{X}(e_k | e_{k'})$ commutes:

$$\begin{array}{ccc} \underline{1}^* e_k^* f_i^* a & \xrightarrow{\overline{(e_k|e_{k'})}^*_{f_i^* a}} & \underline{2}^* e_{k'}^* f_i^* a \\ {\scriptstyle \underline{1}^* \alpha_k}\downarrow & & \downarrow{\scriptstyle \underline{2}^* \alpha_{k'}} \\ e_k^* a_i & \xrightarrow[\overline{(e_k|e_{k'})}^*_{a_i}]{} & e_{k'}^* a_i \end{array}$$

Since (e_k) is a covering family, this shows that the family (α_k) defines an isomorphism $f_i^* a \cong a_i$. The same construction can be done for all $i \in \mathcal{I}$ and one can check that these isomorphisms are compatible with the given $(z_{ii'})$. Hence the functor

$$\mathbb{X}X \to \operatorname*{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}_{\mathcal{F}}^{\mathbb{X}}$$

is essentially surjective. $\square$

Finally we need a comparison between our gluing construction for 2-sieves and the gluing obtained by descent.

2.3.8. Proposition. *Let* $\mathbf{C}$ *be a* 2*-category with finite bipullbacks. Let*

$$\mathcal{F} = (f_i \colon X_i \to X)_{i \in \mathcal{I}}$$

be a family of morphisms of $\mathbf{C}$ *with the same codomain* X *and* $\mathbb{X}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ *a* 2*-functor. Then there is an canonical equivalence*

$$e : \mathrm{PsNat}(\mathbb{R}_{\mathcal{F}}, \mathbb{X}) \to \operatorname*{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{\mathcal{F}}$$

which fits in the diagram:

$$\begin{array}{ccc} \mathrm{PsNat}(\mathbf{C}(-,X),\mathbb{X}) & \xrightarrow{\simeq} & \mathbb{X}X \\ \downarrow & \cong & \downarrow \\ \mathrm{PsNat}(\mathbb{R}_{\mathcal{F}},\mathbb{X}) & \xrightarrow[e]{} & \mathrm{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{\mathcal{F}} \end{array}$$

PROOF. We use proposition 2.2.4 to define e and to show that it is an equivalence.
For each $i \in \mathcal{I}$ we define the component:

$$e_i \colon \begin{cases} \mathrm{PsNat}(\mathbb{R}_{\mathcal{F}},\mathbb{X}) & \to & \mathbb{X}(X_i) \\ (\alpha \colon \mathbb{R}_{\mathcal{F}} \to \mathbb{X}) & \mapsto & \alpha_{X_i}(f_i) \\ (m \colon \alpha \to \beta) & \mapsto & m_{X_i}(f_i) \end{cases}$$

For each $i, j \in \mathcal{I}$ we define the natural transformation

$$\zeta_{ij} : \underline{1}^* e_i \Rightarrow \underline{2}^* e_j$$

with components:

$$\zeta_{ij,\alpha} \colon \ \underline{1}^* \alpha_{X_i}(f_i) \xrightarrow{\sim} \alpha_{(f_i|f_j)}(f_i\underline{1}) \xrightarrow{\alpha_{(f_i|f_j)}(\overline{(f_i|f_j)})} \alpha_{(f_i|f_j)}(f_j\underline{2}) \xrightarrow{\sim} \underline{2}^* \alpha_{X_j}(f_j)$$

One can check that the $(\zeta_{ij})_{ij}$ satisfy the cubical law (with respect to the (f_i)), hence we have defined a functor

$$e : \mathrm{PsNat}(\mathbb{R}_{\mathcal{F}}, \mathbb{X}) \to \operatorname*{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{\mathcal{F}}.$$

We will show the faithfulness of e; fullness and essential surjectivity may be proved by similar arguments. Let $\alpha, \beta \colon \mathbb{R}_{\mathcal{F}} \to \mathbb{X}$ be pseudonatural transformations and $s, t \colon \alpha \to \beta$ parallel modifications. Assume that, for all $i \in \mathcal{I}$,

$$s_{X_i}(f_i) = t_{X_i}(f_i).$$

We want to prove that for any object Y of $\mathbf{C}$ and morphism $f \in \mathbb{R}_{\mathcal{F}}Y$:

$$s_Y(f) = t_Y(f).$$

Indeed, there is an $i \in \mathcal{I}$ and $g \colon Y \to X_i$ with an isomorphism

$$\gamma \colon f \to f_i g$$

Hence we have the four equations:

$$\begin{aligned} s_Y(f) &= \beta_Y(\gamma^{-1}) s_Y(g^* f_i) \alpha_Y(\gamma) \\ s_Y(g^* f_i) &= \beta_g(f_i)^{-1} [g^* s_{X_i}(f_i)] \alpha_g(f_i) \\ t_Y(g^* f_i) &= \beta_g(f_i)^{-1} [g^* t_{X_i}(f_i)] \alpha_g(f_i) \\ t_Y(f) &= \beta_Y(\gamma^{-1}) t_Y(g^* f_i) \alpha_Y(\gamma) \end{aligned}$$

Since $s_{X_i}(f_i) = t_{X_i}(f_i)$, we have $s_Y(f) = t_Y(f)$. This shows $s = t$, and thus proves the faithfulness of e. □

2.3.9. Proposition. *Let* $\mathbf{C}$ *be a* 2*-category with finite bipullbacks,* $\mathcal{C}$ *a coverage on* $\mathbf{C}$ *and* $\mathbb{X}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ *a* 2*-functor. The following propositions are equivalent:*

(i) $\mathbb{X}$ *is a sheaf for the coverage* $\mathcal{C}$*.*
(ii) $\mathbb{X}$ *is a sheaf for the saturated coverage* $\mathcal{C}^{\mathrm{sat}}$*.*
(iii) $\mathbb{X}$ *is a sheaf for the Grothendieck coverage* $\mathcal{R}^{\mathcal{C}}$ *generated by* $\mathcal{C}$*.*

PROOF. We prove $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$.

- $(i) \Rightarrow (ii)$*:* We apply lemma 2.2.16; by lemma 2.3.5 and lemma 2.3.6, the 2-functor $\mathbb{X}$ also satisfies the descent condition for the composite families.
- $(ii) \Rightarrow (ii)$*:* By proposition 2.3.8, for any family $\mathcal{F}$ covering an object X, the canonical functor
$$\mathbb{X}(X) \to \mathrm{PsNat}(\mathbb{R}_{\mathcal{F}}, \mathbb{X})$$
is an equivalence. Now consider a sieve $(\mathbb{R}, \iota_{\mathbb{R}})$ covering an object X. There exists a 1-morphism of sieves $\mathbb{R}_{\mathcal{G}} \to (\mathbb{R}, \iota_{\mathbb{R}})$ with $\mathcal{G}$ a covering family. Moreover, by remark 2.1.16, we can replace $(\mathbb{R}, \iota_{\mathbb{R}})$ by a sieve $\mathbb{R}_{\mathcal{F}}$ generated by a family of morphisms $\mathcal{F}$. Notice that $\mathcal{G}$ is refinement of $\mathcal{F}$, hence, by lemma 2.3.7 the canonical functor
$$\mathbb{X}(X) \to \mathrm{PsNat}(\mathbb{R}_{\mathcal{F}}, \mathbb{X})$$
is an equivalence.

The implication $(iii) \Rightarrow (i)$ is obvious. □

2.3.10. Example. A (usual) 1-sheaf $X\colon \mathrm{C}^{\mathrm{op}} \to \mathrm{Set}$ on a 1-site C is a 2-sheaf, viewing Set as a full, 2-full subcategory of **Cat**.

2.3.11. Example. A stack $X\colon \mathrm{C}^{\mathrm{op}} \to \mathbf{Cat}$ on a 1-site C is a 2-sheaf.

2.3.12. Remark. Let C be a 1-site. The 2-category $\mathbf{Sh}(\mathrm{C})$ of 2-sheaves on C is precisely the 2-category of stacks on C. In particular, it should not be mistaken with the category of 1-sheaves on C.

2.4. Morphisms of 2-sites

Given a 2-functor $\mathbb{W}\colon \mathbb{D} \to \mathbb{C}$ between 2-sites, we would like to be able to define a restriction 2-functor $\mathbb{W}^*\colon \mathbf{Sh}(\mathbf{C}) \to \mathbf{Sh}(\mathbf{D})$ and an extension 2-functor $\mathbb{W}_*\colon \mathbf{Sh}(\mathbf{D}) \to \mathbf{Sh}(\mathbf{D})$ on sheaves. The propositions of this section present sufficient conditions to define each.

2.4.1. Definition. Let $(\mathbf{C}, \mathcal{R})$ be a 2-site, $\mathbf{D}$ a 2-category and $\mathbb{W}\colon \mathbf{D} \to \mathbf{C}$ be a 2-functor. The *induced Grothendieck* 2*-coverage* $\mathbb{W}^*\mathcal{R}$ on $\mathbf{D}$ is the largest Grothendieck 2-coverage on $\mathbb{D}$ such that for any 2-sheaf $\mathbb{X}$ on $\mathbf{C}$, the composite 2-functor $\mathbb{X}\mathbb{W}$ is a sheaf on $\mathbf{D}$.

2.4.2. Proposition. *Let* $(\mathbf{C}, \mathcal{R})$ *and* $(\mathbf{D}, \mathcal{S})$ *be* 2*-sites,* $\mathbb{W}\colon \mathbf{D} \to \mathbf{C}$ *be a* 2*-functor and* $\mathbb{X}\colon \mathbf{C}^{\mathrm{op}} \to$ $\mathbf{Cat}$ *be a* 2*-sheaf on* $\mathbf{C}$*. Suppose that* $\mathcal{S} \subset \mathbb{W}^*\mathcal{R}$*. Then*
$$\mathbb{W}^*(\mathbb{X}) = \mathbb{X}\mathbb{W}\colon \mathbf{D}^{\mathrm{op}} \to \mathbf{Cat}$$
is a 2*-sheaf on* $\mathbf{D}$*.*

2.4.3. Proposition. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be* 2*-categories with bipullback, each endowed with a* 2*-coverage* $\mathcal{C}^{\mathbf{D}}$ *and* $\mathcal{C}^{\mathbf{D}}$*. Let* $\mathbb{W}\colon \mathbf{D} \to \mathbf{C}$ *be a* 2*-functor preserving* 2*-pullbacks, such that for any covering family* $\mathcal{F} \in \mathcal{C}^{\mathbf{D}}$*, the image by* $\mathbb{W}$
$$\mathbb{W}(\mathcal{F}) = \{\mathbb{W}(f) \mid f \in \mathcal{C}\}$$
is a covering family of $\mathcal{C}^{\mathbf{C}}$*. Then for any* 2*-sheaf* $\mathbb{X} \in \mathbf{Sh}(\mathbf{C})$*, the* 2*-functor*
$$\mathbb{W}^*(\mathbb{X}) = \mathbb{X}\mathbb{W}\colon \mathbf{D}^{\mathrm{op}} \to \mathbf{Cat}$$
is a 2*-sheaf on* $\mathbf{D}$*.*

Proof. Since the functor $\mathbb{W}$ preserves bipullbacks, for any covering family $\mathcal{F} \in \mathcal{C}_d^{\mathbf{D}}$, there is a pseudonatural equivalence:

$$\mathbb{D}^{\mathbb{X}}_{\mathbb{W}(\mathcal{F}} \simeq \mathbb{D}^{\mathbb{X}\mathbb{W}}_{\mathcal{F}}$$

Hence the equivalence:

$$\mathbb{X}\mathbb{W}(d) \simeq \underset{\mathbf{Desc}^{\mathrm{op}}}{\mathrm{bilim}}\, \mathbb{D}^{\mathbb{X}}_{\mathbb{W}(\mathcal{F}} \simeq \underset{\mathbf{Desc}^{\mathrm{op}}}{\mathrm{bilim}}\, \mathbb{D}^{\mathbb{X}\mathbb{W}}_{\mathcal{F}}$$

□

2.4.4. Definition. Let $\mathbf{C}$ be a 2-site whose Grothendieck 2-coverage is generated by a coverage $\mathcal{C}$. A *covering subsite* of $\mathbf{C}$ is a 2-site $\mathbf{D}$ and a 2-functor $\mathbb{I}\colon \mathbf{D} \to \mathbf{C}$ which is an equivalence on the Hom-categories and such that:

- the Grothendieck coverage on $\mathbf{D}$ is the one induced by $\mathbf{C}$
- for any object C of $\mathbf{C}$, there is a covering family $\mathcal{F} \in \mathcal{C}_C$ such that for each $f\colon c_f \to c \in \mathcal{F}$, the object c_f is in the essential image of $\mathbb{I}$.

2.4.5. Remark. We generally view a covering subsite as a full, 2-full sub-2-category and omit the inclusion 2-functor $\mathbb{I}$.

2.4.6. Proposition. *Let $\mathbf{C}$ be a 2-site, $(\mathbf{D}, \mathbb{I})$ a covering subsite of $\mathbf{C}$. Then the restriction along $\mathbb{I}$*

$$\mathbb{I}^*\colon \mathbf{Sh}(\mathbf{C}) \to \mathbf{Sh}(\mathbf{D})$$

is a biequivalence. That is, there is a (essentially unique) extension 2-functor along $\mathbb{I}$:

$$\mathbb{I}_*\colon \mathbf{Sh}(\mathbf{D}) \to \mathbf{Sh}(\mathbf{C})$$

Proof. This is precisely [Str82, Theorem 3.8]. □

2.5. Descent and coproduct

In this section, we will investigate the interactions between the descent condition and coproducts.

Consider a 2-category $\mathbf{C}$ with coproducts and finite bipullbacks. A product-preserving 2-functor

$$\mathbb{X}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$$

sends coproducts of $\mathbf{C}$ to products:

$$\mathbb{X}\left(\coprod_i X_i\right) \simeq \prod_i \mathbb{X}(X_i)$$

Hence, given a family of morphisms $(f_i\colon X_i \to X)_{i\in\mathcal{I}}$, writing

$$g = (f_i)\colon \coprod_i X_i \to X$$

a product-preserving 2-functor $\mathbb{X}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ induces a natural isomorphism

$$\mathbb{D}^{\mathbb{X}}_{(f_i)_i} \cong \mathbb{D}^{\mathbb{X}}_g$$

and $\mathbb{X}$ satisfies the descent condition for the family $(f_i)_i$ if and only it satisfies the descent condition for the 1-morphism g. This remark leads to the following proposition.

2.5.1. Proposition. *Let $\mathbf{C}$ be a 2-category with coproducts and $\mathcal{C}$ be a coverage on $\mathbf{C}$. Let $\mathbb{X}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ be a product-preserving 2-functor. Then $\mathbb{X}$ is a 2-sheaf on $\mathbf{C}$ if and only for any covering family $\mathcal{F} \in \mathcal{C}$, $\mathbb{X}$ satisfies the descent condition for the induced 1-morphism:*

$$(f)_{(f:X_f\to X)\in\mathcal{F}}\colon \coprod_f X_f \to X$$

2.5.2. Remark. For a single 1-morphism $g\colon Y \to X$, the descent diagram $\mathbb{D}_g^{\mathbb{X}}$ factors though $\mathbb{X}$:

$$\mathbb{D}_g^{\mathbb{X}}\colon \mathbf{Desc}^{\mathrm{op}} \xrightarrow{\tilde{\mathbb{D}}_g^{\mathrm{op}}} \mathbf{C}^{\mathrm{op}} \xrightarrow{\mathbb{X}} \mathbf{Cat}$$

The 2-functor $\tilde{\mathbb{D}}_g$ is defined on objects by

$$\begin{aligned}\tilde{\mathbb{D}}_g(0) &= X\\ \tilde{\mathbb{D}}_g(1) &= (g|g)\\ \tilde{\mathbb{D}}_g(2) &= (g|g|g)\end{aligned}$$

and in the most obvious way on 1-morphisms and 2-morphisms.

2.5.3. Proposition. *Let $\mathbf{C}$ be a 2-category with coproducts and finite bipullbacks and $g\colon Y \to X$ a morphism of $\mathbf{C}$. Let $\mathbb{X}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ be a bilimit-preserving functor. Assume that the canonical 1-morphism*

$$\operatorname*{bicolim}_{\mathbf{Desc}} \tilde{\mathbb{D}}_g \to X$$

is an equivalence. Then $\mathbb{X}$ satisfies the descent condition for g.

PROOF. There is a commutative diagram in $\mathbf{Cat}$

$$\begin{array}{ccc} \mathbb{X}(X) & \longrightarrow & \mathbb{X}(\mathrm{bicolim}_{\mathbf{Desc}}(\tilde{\mathbb{D}}_g)) \\ & \cong & \downarrow \\ & \searrow & \mathrm{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}_g^{\mathbb{X}} \end{array}$$

The rightmost 1-morphism is an equivalence since $\mathbb{X}$ preserves bilimits. The top 1-morphism is an equivalence, since it is the image by $\mathbb{X}$ of

$$\operatorname*{bicolim}_{\mathbf{Desc}} \tilde{\mathbb{D}}_g \to X$$

which is an equivalence by hypothesis. Hence the comparison 1-morphism

$$\mathbb{X}(X) \to \operatorname*{bilim}_{\mathbf{Desc}} \mathbb{D}_g^{\mathbb{X}}$$

is an equivalence. □

To have a productive discussion in the case where the coproduct is the *covered* object, we need to assume the *extensivity* of $\mathbf{C}$, which mostly amount to the disjointness of coproducts. We will use the characterization given by [BL03, Theorem 2.3] as our definition.

2.5.4. Definition. Let $\mathbf{C}$ be a 2-category with finite coproducts and finite bipullbacks. The 2-category $\mathbf{C}$ is *extensive* if, for all $j \in \mathcal{J}$, with $\mathcal{J}$ finite, the diagrams

$$\begin{array}{ccc} X_j & \xrightarrow{i_{X_j}} & \coprod_j X_j \\ \uparrow & \cong & \uparrow z \\ X'_j & \xrightarrow[x'_j]{} & Z' \end{array}$$

are bipullback squares if and only if the induced morphism

$$(x'_j)_j\colon \coprod_j X'_j \to Z'$$

is an equivalence.

2.5.5. Remark. Let $\mathbf{C}$ be an extensive 2-category with finite coproducts and finite bipullbacks. Let X and Y be objects of $\mathbf{C}$ and denote by 0 the initial object of $\mathbf{C}$. Then, the following commutative square

$$\begin{array}{ccc} X & \xrightarrow{i_X} & X \coprod Y \\ \uparrow & & \uparrow i_Y \\ 0 & \longrightarrow & Y \end{array}$$

is a bipullback square.

2.5.6. Lemma. *Let $\mathbf{C}$ be an extensive 2-category with finite coproducts and finite bipullbacks and $\mathbb{X}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ be a 2-functor such that $\mathbb{X}(0) = 1$. Let $(f_j\colon X_j \to Y_j)_j$ be a finite family of 1-morphisms in $\mathbf{C}$. For each j, write the composite*

$$\tilde{f}_j\colon X_j \to Y_j \to \coprod_k Y_k$$

Then, there is a pseudonatural equivalence

$$\mathbb{D}^{\mathbb{X}}_{(\tilde{f}_j)_j} \to \prod_j \mathbb{D}^{\mathbb{X}}_{f_j}$$

which induces an equivalence fitting in the commutative diagram:

$$\begin{array}{ccc} \mathbb{X}(\coprod_j Y_j) & \longrightarrow & \mathrm{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{(\tilde{f}_j)_j} \\ \downarrow & & \downarrow \simeq \\ \prod_j \mathbb{X}(Y_j) & \longrightarrow & \prod_j \mathrm{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{f_j} \end{array}$$

Proof. By remark 2.5.5, for any $j \neq k$, we have

$$(\tilde{f}_j|\tilde{f}_k) = 0.$$

Hence it suffices to check that for any j,

$$(\tilde{f}_j|\tilde{f}_j) \simeq (f_j|f_j)$$

and

$$(\tilde{f}_j|\tilde{f}_j|\tilde{f}_j) \simeq (f_j|f_j|f_j).$$

We show the first equivalence; the second is similar. Fix an index j. We can aggregate the following bipullback squares:

$$\begin{array}{ccccc} X_j & \xrightarrow{f_j} & Y_j & \xrightarrow{i_{Y_j}} & \coprod_k Y_k \\ \| & & \| & & \uparrow i_{Y_j} \\ X_j & \xrightarrow{f_j} & Y_j & = & Y_j \\ \uparrow & & f_j \uparrow & & \uparrow f_j \\ (f_j|f_j) & \longrightarrow & X_j & = & X_j \end{array}$$

Note that the top-right square is a bipullback by the extensivity of $\mathbf{C}$. Using lemma 1.2.24, the outer square is also a bipullback square, hence $(f_j|f_j)$ is a bipullback of $\tilde{f}_j$ with itself. □

2.5.7. Proposition. *Let* $\mathbf{C}$ *be an extensive* 2*-category with finite coproducts and finite bipullbacks. Let* $f : X \to \coprod_{j \in \mathcal{J}} Y_j$ *be a* 1*-morphism, with* $\mathcal{J}$ *finite, and, for each* $j \in \mathcal{J}$*, consider the bipullback:*

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & \coprod_i Y_j \\ \uparrow & & \uparrow{\scriptstyle i_{Y_j}} \\ X_j & \xrightarrow[f_j]{} & Y_j \end{array}$$

Let $\mathbb{X} \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ *be a* 2*-functor preserving finite products. Suppose that* $\mathbb{X}$ *satisfies the descent condition for each morphism* f_j*. Then* $\mathbb{X}$ *satisfies the descent condition for the morphism* f*.*

Proof. Consider the morphism:

$$g = \coprod_j f_j \colon \coprod_j X_j \to \coprod_j Y_j$$

By extensivity, we can identify (through an equivalence) $\coprod_j X_j$ and X and under this identification, we have $f \cong g$. Moreover since $\mathbb{X}$ preserves products, the equivalence given by lemma 2.5.6 yields a natural equivalence

$$\mathbb{D}^{\mathbb{X}}_g \to \prod_j \mathbb{D}^{\mathbb{X}}_{f_j}$$

which fits in the commutative diagram:

$$\begin{array}{ccc} \mathbb{X}(\coprod_j Y_j) & \longrightarrow & \mathrm{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_g \\ {\scriptstyle \simeq}\downarrow & & \downarrow{\scriptstyle \simeq} \\ \prod_j \mathbb{X}(Y_j) & \xrightarrow[\simeq]{} & \prod_j \mathrm{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{f_j} \end{array}$$

The uppermost morphism is an equivalence, since all the other morphisms are equivalences. □

2.5.8. Proposition. *Let* $\mathbf{C}$ *be an extensive* 2*-category with finite coproducts and finite bipullbacks and* $\mathcal{C}$ *be a* 2*-coverage on* $\mathbf{C}$*. Assume that for any finite coproduct* $\coprod_j X_j$ *the family of inclusions* $(i_{X_j})_j$ *is a covering family. Then, for any* 2*-sheaf* $\mathbb{X}$ *for the* 2*-coverage* $\mathcal{C}$ *such that* $\mathbb{X}(0) = 1$*,* $\mathbb{X}$ *preserves finite products.*

Proof. Consider a family of objects $(X_j)_j$ of $\mathbf{C}$. By lemma 2.5.6, there is a natural equivalence

$$\mathbb{D}^{\mathbb{X}}_{(i_{X_j})_j} \to \prod_j \mathbb{D}^{\mathbb{X}}_{\mathrm{Id}_{X_j}}$$

which fits in the commutative diagram:

$$\begin{array}{ccc} \mathbb{X}(\coprod_j Y_j) & \xrightarrow{\;\simeq\;} & \mathrm{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_g \\ \downarrow & & \downarrow{\scriptstyle \simeq} \\ \prod_j \mathbb{X}(Y_j) & \xrightarrow[\simeq]{} & \prod_j \mathrm{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}^{\mathbb{X}}_{f_j} \end{array}$$

The leftmost morphism is an equivalence, since all the other morphisms are equivalences. □

2.5.9. Remark. We restricted our attention to finite coproducts; all the propositions in this section hold if we bound coherently the size of the coproducts and the size of the covering families. For instance, we can have a similar discussion for a 2-category with *small coproducts* (and the corresponding notion of extensiveness), *small covering families* and 2-functors preserving *small products.*

CHAPTER 3

2-Sheaves and adjunctions

The goal of this chapter is to explore the interactions between 2-sheaves and adjunctions. In section 3.2, we prove an extension to 2-categories (theorem 3.2.1) of the Bénabou-Roubaud theorem [BR70], relating some descent bilimits to associated Eilenberg-Moore categories. It gives us a simple characterization of 2-sheaves over suitable topologies (proposition 3.2.13). In the reverse direction, we show that 2-sheaves are able to extend the existence of adjoints from suitable sub-2-sites (proposition 3.3.1).

We will heavily use string diagrams throughout this chapter (see notation 1.1.36).

3.1. The Beck-Chevalley property

The Beck-Chevalley (or base-change, BC for short) property is a coherence property between morphisms adjunct to images of the same 2-functor.

3.1.1. Definition. Let $\mathbf{C}$ be 2-category with finite bipullbacks. A class of morphisms $J \subset \mathbf{C}$ is stable under bipullbacks if for any morphism $j\colon X \to Z$ in J and any morphism $k\colon Y \to Z$ of $\mathbf{C}$, taking a bipullback

$$\begin{array}{ccc} X & \xrightarrow{\;j\;} & Z \\ \uparrow & \cong & \uparrow{\scriptstyle k} \\ (j|k) & \xrightarrow[(j|\underline{k})]{} & Y \end{array}$$

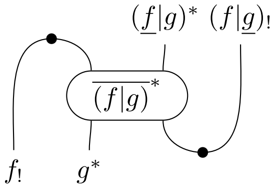
yields $(j|\underline{k}) \in J$.

3.1.2. Definition. Let $\mathbf{C}$ be a 2-category and $J \subset \mathbf{C}$ a class of morphisms stable under bipullbacks. A 2-functor $\mathbb{F}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ has the *left BC property* with respect to J if :

- For any morphism $f\colon X \to Y$ in J, the 1-morphism $f^* = \mathbb{F}(f)$ has a left adjoint $f_!$.
- For any bipullback square in $\mathbf{C}$

$$\begin{array}{ccc} x & \xrightarrow{\;f\;} & z \\ \uparrow & & \uparrow{\scriptstyle g} \\ (f|g) & \longrightarrow & y \end{array}$$

with $f \in J$, the mate $\overline{(f|g)}_!\colon (f|\underline{g})_!(\underline{f}|g)^* \Rightarrow g^* f_!$

$\overline{(f|g)}_! =$ 

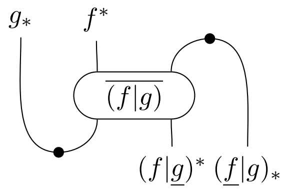
of $\overline{(f|g)}^*$ is an isomorphism.

A 2-functor $\mathbb{F}\colon \mathbf{C}^{op} \to \mathbf{Cat}$ has the *right BC property* with respect to J if :

- For any morphism $f\colon X \to Y$ in J, the functor $f^* = \mathbb{F}(f)$ has a right adjoint f_*.
- For any bipullback square in $\mathbf{C}$

$$\begin{array}{ccc} x & \xrightarrow{\ f\ } & z \\ \uparrow & & \uparrow{\scriptstyle g} \\ (f|g) & \longrightarrow & y \end{array}$$

with $g \in J$, the mate $\overline{(f|g)}_*\colon f^* g_* \Rightarrow (\underline{f}|g)_*(f|\underline{g})^*$

$\overline{(f|g)}_* =$ 

of $\overline{(f|g)}^*$ is an isomorphism.

A 2-functor $\mathbb{F}\colon \mathbf{C}^{op} \to \mathbf{Cat}$ has the *ambidextrous BC property* with respect to J to if it has both the left and right BC property, and the left and right adjoints are isomorphic, for any $f \in J$:

$$f_! \cong f_*$$

3.2. Bénabou-Roubaud for 2-functors

We will prove in this section an extension of Bénabou-Roubaud theorem [BR70]. For a pseudofunctor

$$\mathbb{X}\colon \mathrm{C}^{\mathrm{op}} \to \mathbf{Cat}$$

with domain a 1-*category* C and satisfying a left BC-property, the original theorem gives an equivalence between the Eilenberg-Moore category $\mathbb{X}(X)^{i^* i_!}$ and the category of descent data for i (definition 2.2.7). Our version allows the domain to be a 2-category. We have chosen to replace the category of descent data by the descent bilimit it models, but this is purely to better fit it in our framework of 2-sheaves. The proof is similar to the original one.

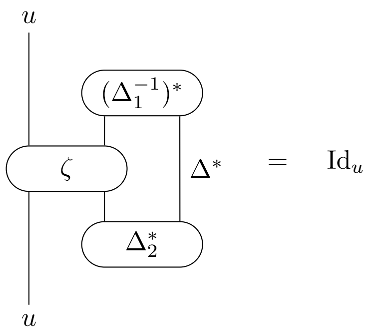
3.2.1. Theorem (Bénabou-Roubaud)**.** *Let* $\mathbf{C}$ *be a 2-category with finite bipullbacks and* $J \subset \mathbf{C}$ *a class of morphisms stable under bipullbacks. Let* $\mathbb{F}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ *be a 2-functor with the left BC property with respect to* J *(definition 3.1.2). Fix* $i\colon X \to Y$ *in* J*. The functor* $i^*\colon \mathbb{F}(Y) \to \mathbb{F}(X)$ *has a left adjoint* $i_!$*, hence defines a monad* $\mathsf{T} = i^* i_!$ *on* $\mathbb{F}(X)$ *(see proposition 1.1.25). Then the forgetful functor* $u\colon \mathbb{F}(X)^{\mathsf{T}} \to \mathbb{F}(X)$ *induces an equivalence*

$$\mathbb{F}(X)^{\mathsf{T}} \simeq \operatorname*{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}_i^{\mathbb{F}}$$

3.2.2. Remark. Requiring that the 2-functor $\mathbb{F}$ satisfies the left BC-property for a class of morphisms stable under bipullbacks is certainly an overly cautious assumption. A simple review of the proof shows that we only need a few mates associated to the bipullbacks $(i|i)$ and $(i|i|i)$ to be invertible.

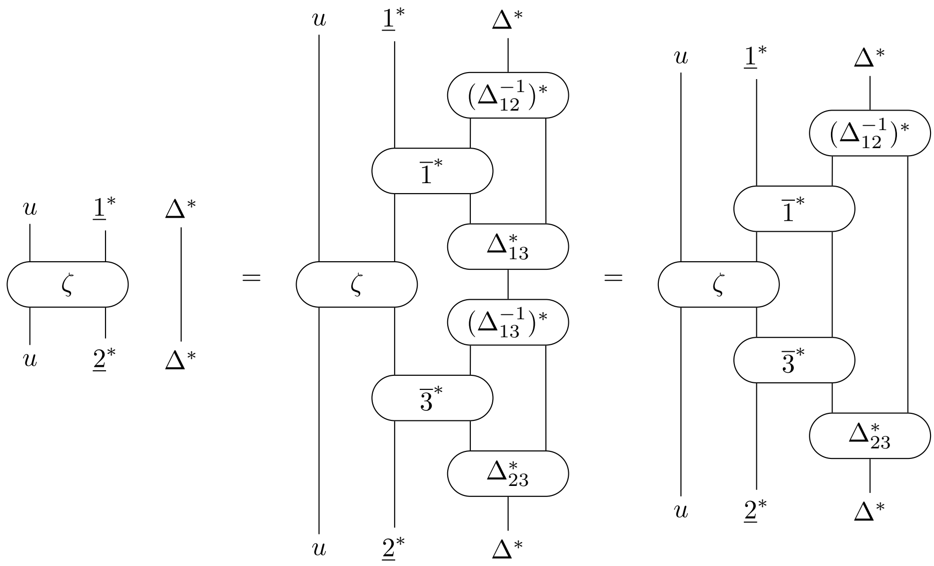
We will need a technical lemma relating descent and the diagonal morphism $\Delta\colon X \to (i|i)$:

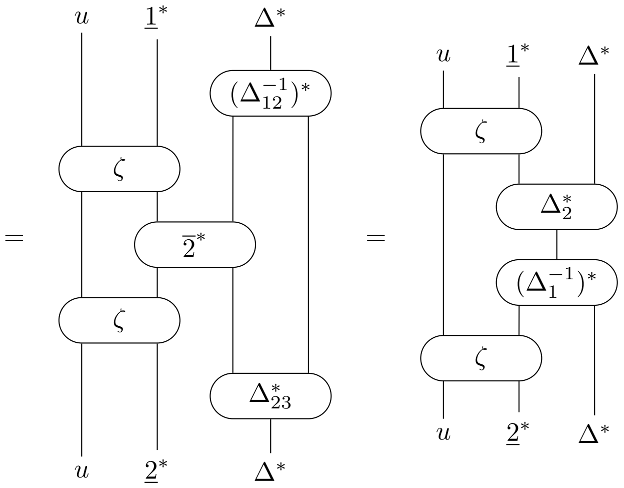
3.2.3. Lemma. *Let* $\mathbf{C}$ *be a 2-category with finite bipullbacks and* $\mathbb{F}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ *be a 2-functor. Fix a* 1*-morphism* $i\colon X \to Y$*. We consider the bipullback of* i *with itself* $(i|i)$*, with its two* *projections* $\underline{1}, \underline{2}\colon (i|i) \to X$. *Let* $u\colon V \to \mathbb{F}X$ *be a* 1*-morphism in* **Cat** *and* $\zeta\colon \underline{1}^*u \to \underline{2}^*u$ *be a* 2*-morphism, such that the pair* (u, ζ) *satisfies the cubical relation* (2.2.5) *with respect to* i. *Then*



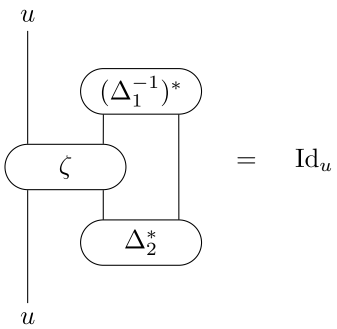
where Δ, Δ_1 *and* Δ_2 *are given by lemma 1.2.16.*

Proof. Let $\Delta^3\colon X \to (i|i|i)$ be the diagonal morphism given by lemma 1.2.17. We have the following computation, where the first and fourth equation use the relations of lemma 1.2.17, and the third uses the cubical law (2.2.5) for ζ:





Hence, we have the equation



which is precisely the identity we wanted to prove. ☐

We can now turn to the proof of theorem 3.2.1.

PROOF. Keeping the same notations as above, we fix $\lambda = \overline{(i|i)} \colon i(\underline{i}|i) \to i(i|\underline{i})$, $\beta = \lambda_!$ the invertible mate of λ^* and

$$L = \operatorname*{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}_i^{\mathbb{F}}$$

We note $\alpha \colon \mathsf{T}u \Rightarrow u$ the canonical natural transformation.

We will use proposition 2.2.4 to define a functor

$$\mathbb{F}(X)^{\mathsf{T}} \to L$$

and prove that it is an equivalence. Let

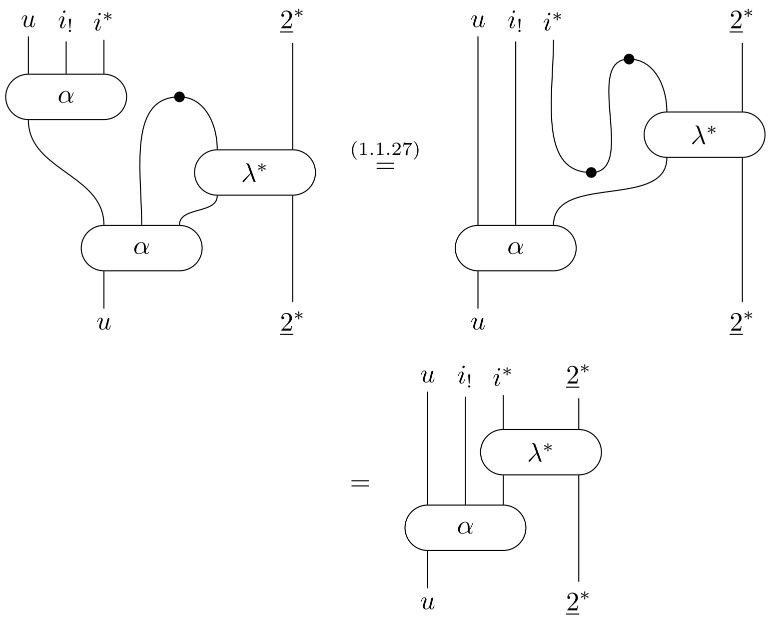
$$\zeta = \zeta_\lambda = \underline{1}^*u \xrightarrow{\underline{1}^*\eta u} \underline{1}^*i^*i_!u \xrightarrow{\lambda^* i_! u} \underline{2}^*i^*i_!u \xrightarrow{\underline{2}^*\alpha} \underline{2}^*u \tag{3.2.4}$$

Remark that we can express α using ζ, explicitly:

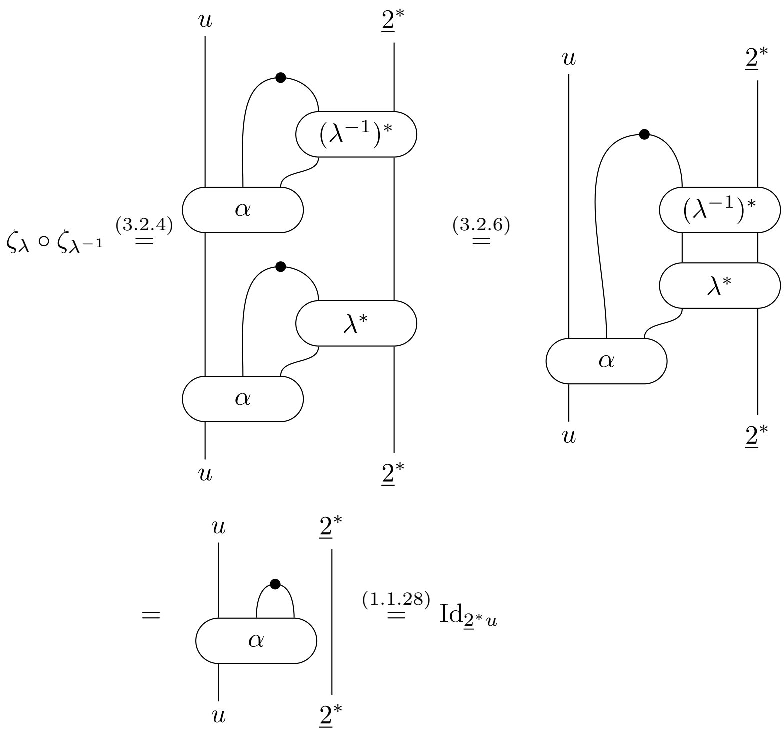
$$\alpha = \mathsf{T}u \xrightarrow{\beta^{-1}u} \underline{2}_!\underline{1}^*u \xrightarrow{\underline{2}_!\zeta} \underline{2}_!\underline{2}^*u \xrightarrow{\epsilon u} u \tag{3.2.5}$$

We first note that the following identity holds:

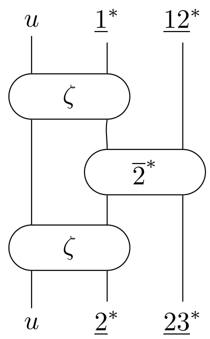
(3.2.6)



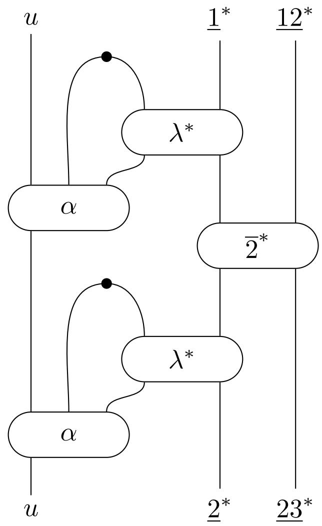
The natural transformation ζ_λ is inversible, with inverse $\zeta_{\lambda^{-1}}$:



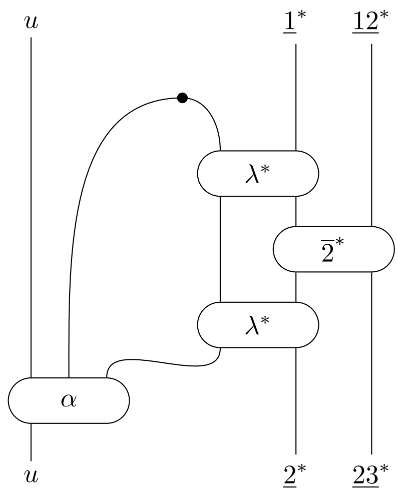
Symmetrically, $\zeta_{\lambda^{-1}} \circ \zeta_\lambda = \mathrm{Id}_{\underline{1}^* u}$. We have to check that the cubical law holds:

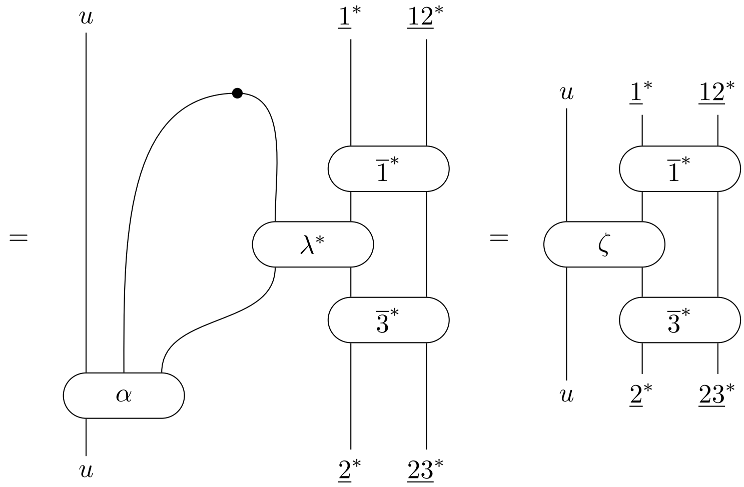


$\overset{(3.2.4)}{=}$



$\overset{(3.2.6)}{=}$





Hence, by proposition 2.2.4, u and ζ define a functor:

$$\tilde{u}\colon \mathbb{F}(X)^{\mathsf{T}} \to L$$

Since $u\colon \mathbb{F}(X)^{\mathsf{T}} \to \mathbb{F}(X)$ is faithful, so is $\tilde{u}$.

It remains to check the fullness of $\tilde{u}$. Let A, B be objects of $\mathbb{F}(X)^{\mathsf{T}}$ and $f\colon uA \to uB$ a morphism of $\mathbb{F}(X)^{\mathsf{T}}$ such that

$$\begin{array}{ccc} \underline{1}^*uA & \xrightarrow{\zeta_A} & \underline{2}^*uA \\ \downarrow{\scriptstyle \underline{1}^*f} & & \downarrow{\scriptstyle \underline{1}^*f} \\ \underline{1}^*uB & \xrightarrow{\zeta_B} & \underline{2}^*uB \end{array}$$

We have to check that f is a morphism of left T-module, that is that the following square commutes:

$$\begin{array}{ccc} \mathsf{T}uA & \xrightarrow{\alpha_A} & uA \\ {\scriptstyle \mathsf{T}f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathsf{T}uB & \xrightarrow[\alpha_B]{} & uB \end{array}$$

This last square can be decomposed into (using eq. (3.2.5)):

$$\begin{array}{ccccccc} \mathsf{T}uA & \xrightarrow{\beta_A^{-1}u} & \underline{2}_!\underline{1}^*uA & \xrightarrow{\underline{2}_!\zeta_A} & \underline{2}_!\underline{2}^*uA & \xrightarrow{\epsilon_A u} & uA \\ \downarrow{\scriptstyle \mathsf{T}f} & & \downarrow{\scriptstyle \underline{2}_!\underline{1}^*f} & & \downarrow{\scriptstyle \underline{2}_!\underline{2}^*f} & & \downarrow{\scriptstyle f} \\ \mathsf{T}uB & \xrightarrow{\beta_B^{-1}u} & \underline{2}_!\underline{1}^*uB & \xrightarrow{\underline{2}_!\zeta_B} & \underline{2}_!\underline{2}^*uB & \xrightarrow{\epsilon_B u} & uB \end{array}$$

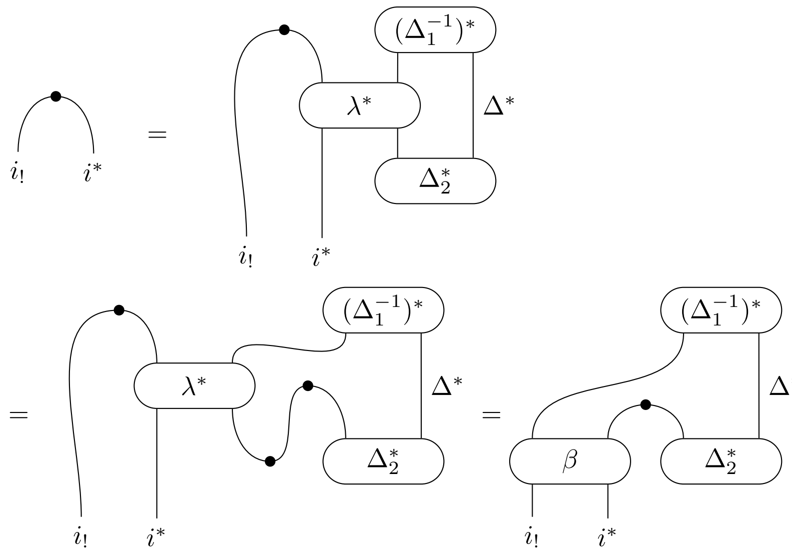
The middle square commutes by hypothesis, the leftmost and rightmost by naturality; hence the outter square commutes. This proves the fullness of $\tilde{u}$.

Finally, we prove the essential surjectivity of $\tilde{u}$. We fix a functor $\hat{u}\colon \mathbf{1} \to \mathbb{F}(X)$ and a natural transformation $\hat{\zeta}\colon \underline{1}^*\hat{u} \Rightarrow \underline{2}^*\hat{u}$, subject to the relation (2.2.5). We want to define a structure of left T-module on $\hat{u}$ by

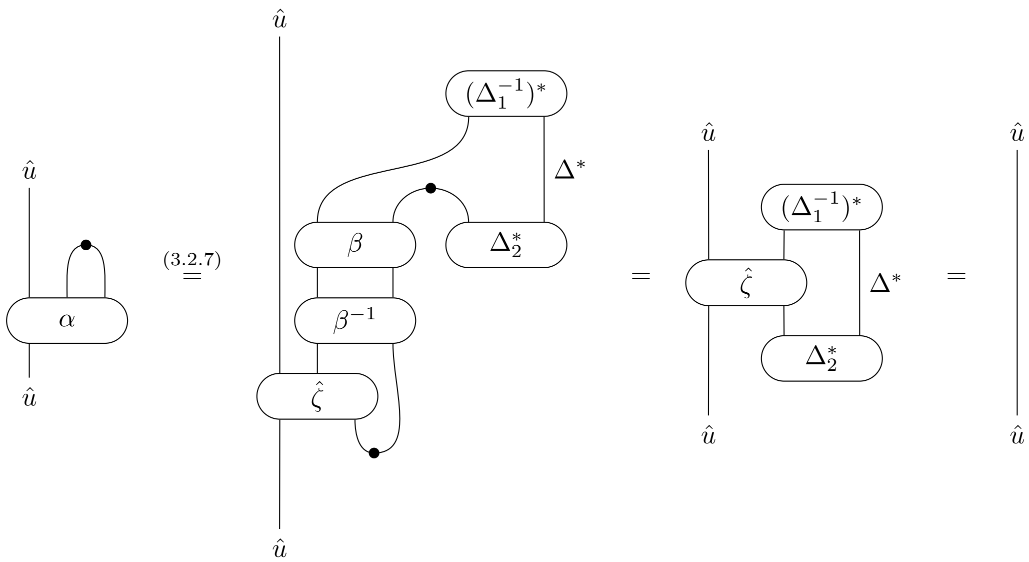
$$\hat{\alpha} = \mathsf{T}\hat{u} \xRightarrow{\beta^{-1}\hat{u}} \underline{2}_!\underline{1}^*\hat{u} \xRightarrow{\underline{2}_!\hat{\zeta}} \underline{2}_!\underline{2}^*\hat{u} \xRightarrow{\epsilon\hat{u}} \hat{u}$$

Indeed, we check that the morphism $\hat{\alpha}$ is unital. First note that we can derive from lemma 1.2.16 the following relation:

(3.2.7)

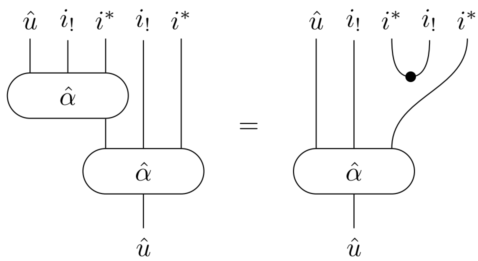


Hence, we have:



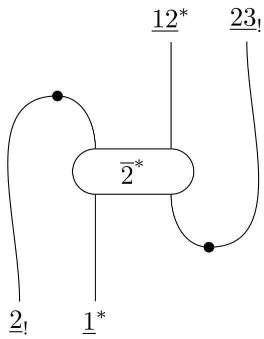
We also have to check that $\hat{\alpha}$ is associative:

(3.2.8)



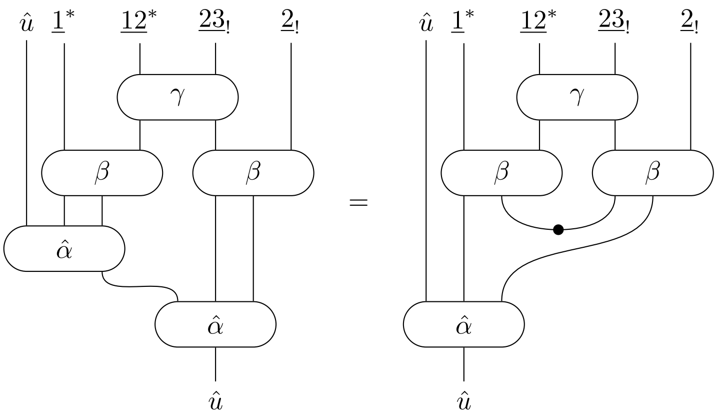
Note that the natural transformation $\overline{2}$ is a bipullback square of $\underline{1}$ and $\underline{2}$ (remark 1.2.15). Thus, we can introduce the invertible mate γ of $\overline{2}^*$:

(3.2.9) $\gamma =$

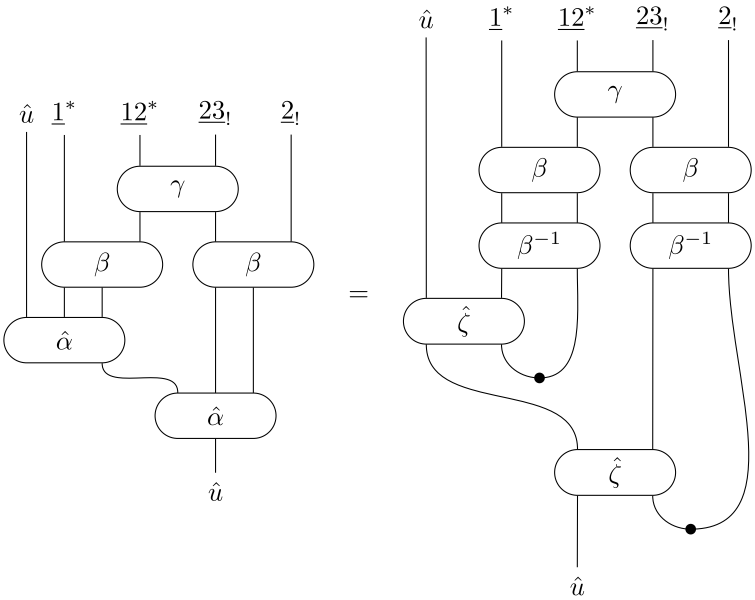


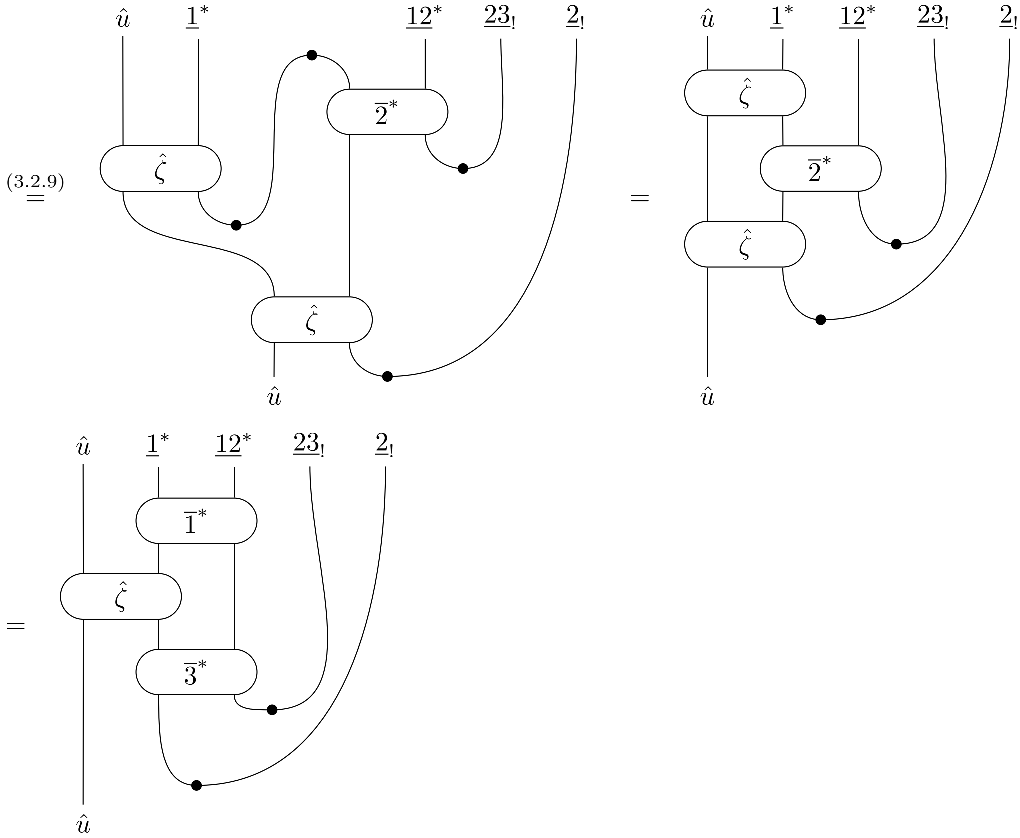
Hence, eq. (3.2.8) is equivalent to the equation:

(3.2.10)



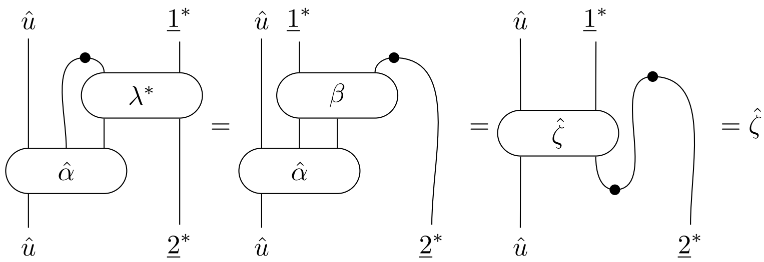
Expanding the definitions of $\hat{\alpha}$ and γ on the left hand side leads to the following computation:



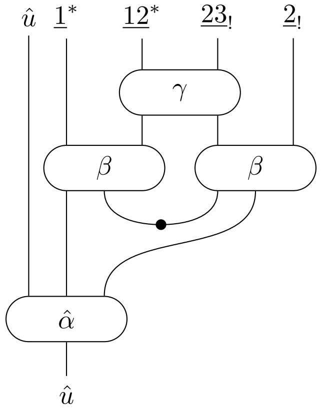


To simplify the right-hand side, first remark that we have the following relation:

(3.2.11)



Then, by expanding the definitions of β and γ in the right-hand side of eq. (3.2.10), we get:



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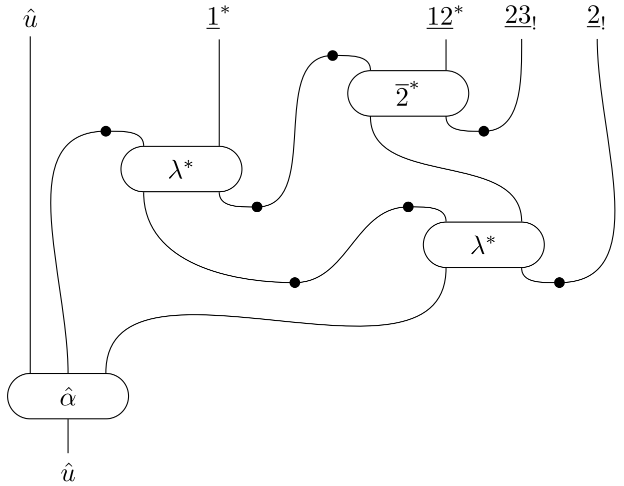

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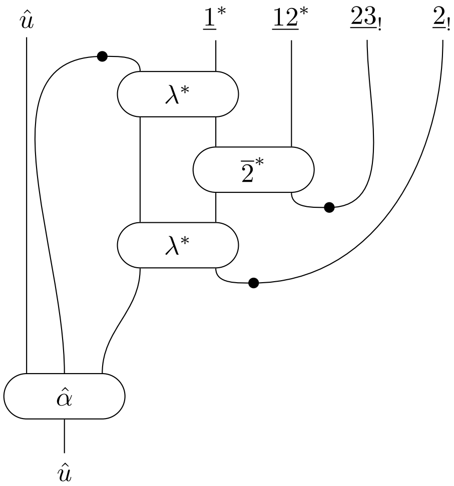

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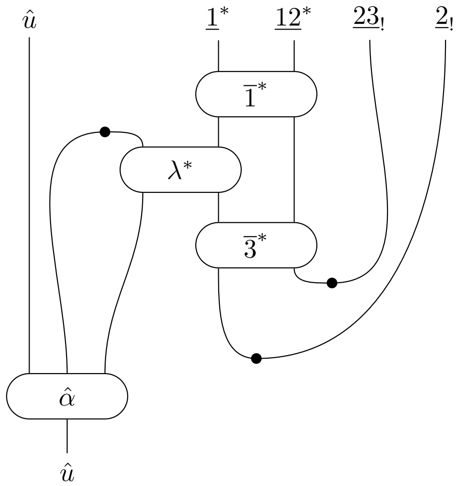
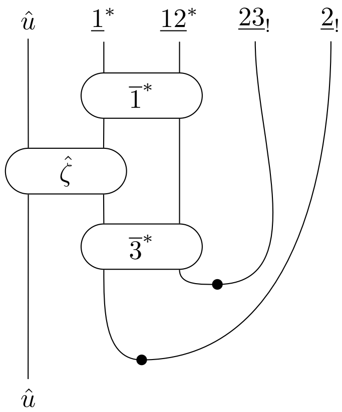


$\overset{(3.2.11)}{=}$



Thus eq. (3.2.10) holds, and so does eq. (3.2.8). Hence the couple $(\hat{u}, \alpha)$ defines a functor $K\colon \mathbf{1} \to \mathbf{C}^{\mathsf{T}}$ which moreover satisfy $\hat{u} = u \circ K$ and (by eq. (3.2.11)) $\hat{\zeta} = \zeta K$. This implies that

$\tilde{u}$ is essentially surjective. We have constructed a fully faithful and essentially surjective functor

$$\tilde{u}\colon \mathbb{F}(X)^{\mathsf{T}} \to \operatorname*{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}_i^{\mathbb{F}}$$

which is thus the wanted equivalence. □

3.2.12. Remark. Note that the functor $\tilde{u}\colon \mathbb{F}(X)^{\mathsf{T}} \to \mathrm{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}_i^{\mathbb{F}}$ fits into the following commutative triangle:

$$\begin{array}{ccc} & \mathbb{F}(Y) & \\ {}^{k^{\mathsf{T}}}\swarrow & & \searrow \\ \mathbb{F}(X)^{\mathsf{T}} & \longrightarrow & \mathrm{bilim}_{\mathbf{Desc}^{\mathrm{op}}} \mathbb{D}_i^{\mathbb{F}} \end{array}$$

where k^{T} is the canonical comparison functor.

An easy corollary of theorem 3.2.1 is the following characterization of 2-sheaves for specific topologies.

3.2.13. Proposition. *Let $\mathbf{C}$ be 2-category with finite bipullback and $\mathcal{I}$ a class of morphisms stable by bipullback. Let $\mathcal{P}$ be a 2-coverage of $\mathcal{C}$, with each covering family formed of a single 1-morphism. Hence we can view $\mathcal{P}$ as a class of 1-morphisms of $\mathbf{C}$. Moreover assume that $\mathcal{P} \subset \mathcal{I}$. Let $\mathbb{X}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ be 2-functor with the left BC property with respect to the family $\mathcal{I}$. Then the following assertions are equivalent:*

- *(i) $\mathbb{X}$ is a 2-sheaf for the coverage $\mathcal{P}$ (definition 2.3.4).*
- *(ii) $\mathbb{X}$ is $\mathcal{P}$-monadic (definition 1.1.33).*

3.2.14. Remark. The hypothesis that each covering family is formed of a single covering 1-morphism may seem restrictive. However, by proposition 2.5.1, as long as we are working with a 2-category $\mathbf{C}$ with coproducts, a product-preserving 2-functor is a sheaf if it satisfies the descent condition for such covering 1-morphisms.

3.3. Extension of the BC property

In this section we will prove a partial reciprocal result: any 2-sheaf that satisfies the BC property on a covering subsite, satisfies it globally.

3.3.1. Proposition. *Let $\mathbf{C}$ be 2-site with finite bipullbacks. Let $\mathcal{I}$ be a family of morphisms of $\mathbf{C}$ stable by bipullback. Let $\mathbf{D}$ be a covering subsite of $\mathbf{C}$ (definition 2.4.4) such that for any morphism $i\colon c \to c'$ of $\mathcal{I}$ and $f\colon d \to c'$ of $\mathcal{C}$ with domain d in $\mathbf{D}$, the bipullback $(i|f)$ is in $\mathbf{D}$. Let $\mathbb{F}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$ be a 2-sheaf on $\mathbf{C}$ such that its restriction $\mathbb{F}_{|\mathbf{D}}$ to $\mathbf{D}$ satisfies the left (respectively right, ambidextrous) BC property with respect to $\mathcal{I} \cap \mathbf{D}$. Then $\mathbb{F}$ satisfies the left (respectively right, ambidextrous) BC property with respect to $\mathcal{I}$.*

Proof. We present the proof for the left BC-property; the other cases are analogous.

We first show that for any morphism $i\colon c \to c'$, $i^* = \mathbb{F}i$ has a left adjoint. Fix $i\colon c \to c'$ in $\mathcal{I}$ and $(f_j\colon d'_j \to c')_{j\in\mathcal{J}}$ a covering family of c' with domain in $\mathbf{D}$. We consider, for each j, the bipullback square:

$$\begin{array}{ccc} c & \xrightarrow{\ i\ } & c' \\ {\scriptstyle f_j}\uparrow & & \uparrow{\scriptstyle f'_j} \\ d_j = (i|d'_j) & \xrightarrow[\ i_j\ ]{} & d'_j \end{array}$$

Since d is in $\mathcal{I}$, so is i_j and d is an object of $\mathbf{D}$. For each j_1, j_2, we define a morphism

$$i_{j_1,j_2} = (i_{j_1}|i_{j_2})\colon (f_{j_1}|f_{j_2}) \to (f'_{j_1}|f'_{j_2})$$

and, for each j_1, j_2, j_3, a morphism

$$i_{j_1,j_2,j_3} = (i_{j_1}|i_{j_2}|i_{j_3}) \colon (f_{j_1}|f_{j_2}|f_{j_3}) \to (f'_{j_1}|f'_{j_2})f'_{j_3}$$

All these morphisms are obtained as bipullback of morphisms of $\mathcal{I}$, hence are in $\mathcal{I}$. They induce (together with their coherence 2-morphisms) a pseudonatural transformation

$$\iota \colon \mathbb{D}^{\mathbb{F}}_{(f'_j)} \to \mathbb{D}^{\mathbb{F}}_{(f_j)}$$

such that the composite morphism

$$\tilde{i} \colon \ \mathbb{F}(c') \xrightarrow{\ \simeq\ } \mathrm{PsNat}(1, \mathbb{D}^{\mathbb{F}}_{f'_j}) \xrightarrow{\mathrm{PsNat}(1,\iota)} \mathrm{PsNat}(1, \mathbb{D}^{\mathbb{F}}_{f_j}) \xrightarrow{\ \simeq\ } \mathbb{F}(c)$$

is isomorphic to i^*. We will construct a left adjoint $\tilde{l}$ to $\tilde{i}$, which hence will be a left adjoint to i^*.

For any $j \in \mathcal{J}$, we note $l_j = (i_j)_!$, the left adjoint of i_j^*, which exists, since $i_j \in \mathcal{I}$. Similarly for any j_1, j_2, we note $l_{j_1,j_2} = (i_{j_1,j_2})_!$ and for any j_1, j_2, j_3, $l_{j_1,j_2,j_3} = (i_{j_1,j_2,j_3})_!$. These morphisms define the object-wise components of a pseudonatural transformation

$$\lambda \colon \mathbb{D}^{\mathbb{F}}_{(f_j)} \to \mathbb{D}^{\mathbb{F}}_{(f'_j)}$$

We detail how to construct the component of λ at the 1-morphism $\underline{1}$ of **Desc**; components at other morphisms are similarly defined. For any j_1, j_2, there is a bipullback in **D**:

$$\begin{array}{ccc} d_{j_1} & \xrightarrow{\ i_{j_1}\ } & d'_{j_1} \\ {\scriptstyle \underline{1}}\uparrow & \cong & \uparrow{\scriptstyle \underline{1}} \\ (f_{j_1}|f_{j_2}) & \xrightarrow[i_{j_1,j_2}]{} & (f'_{j_1}|f'_{j_2}) \end{array}$$

Since $\mathbb{F}_{|\mathbf{D}}$ has the left BC-property, it induces an isomorphism:

$$\begin{array}{ccc} \mathbb{F}d_{j_1} & \xrightarrow{\ l_{j_1}\ } & \mathbb{F}d'_{j_1} \\ {\scriptstyle \underline{1}^*}\downarrow & \cong & \downarrow{\scriptstyle \underline{1}^*} \\ \mathbb{F}(f_{j_1}|f_{j_2}) & \xrightarrow[l_{j_1,j_2}]{} & \mathbb{F}(f'_{j_1}|f'_{j_2}) \end{array}$$

Hence, letting j_1, j_2 varies over $\mathcal{J}$, these isomorphisms can be combined into a single isomorphism

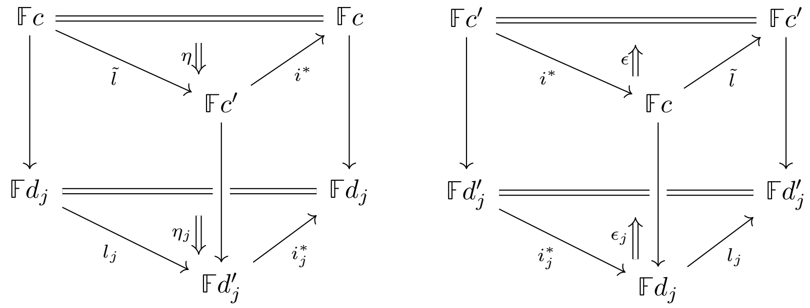
$$\lambda_{\underline{1}} \colon \lambda_1 \circ \mathbb{D}^{\mathbb{X}}_{(d_j}(\underline{1}) \to \mathbb{D}^{\mathbb{X}}_{(d'_j}(\underline{1}) \circ \lambda_0$$

A straightforward computation shows that λ is a pseudonatural transformation. As before, the pseudonatural transformation λ defines a 1-morphism:

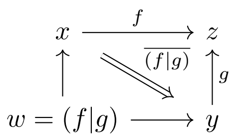
$$\tilde{l} \colon \ \mathbb{F}(c) \xrightarrow{\ \simeq\ } \mathrm{PsNat}(1, \mathbb{D}^{\mathbb{F}}_{f_j}) \xrightarrow{\mathrm{PsNat}(1,\lambda)} \mathrm{PsNat}(1, \mathbb{D}^{\mathbb{F}}_{f'_j}) \xrightarrow{\ \simeq\ } \mathbb{F}(c')$$

We will now show that we have an adjunction $\lambda \dashv \iota$ between pseudonatural transformations. For each $j \in \mathcal{J}$, we fix a unit $\eta_j \colon \mathrm{Id} \to i_j^* l_j$ and a counit $\epsilon_j \colon l_j i_j^* \to \mathrm{Id}$ satisfying the unit-counit laws. They induce, for each j_1, j_2, two natural transformations $\eta_{j_1,j_2} \colon \mathrm{Id} \to i^*_{j_1,j_2} l_{j_1,j_2}$ and $\epsilon_{j_1,j_2} \colon l_{j_1,j_2} i^*_{j_1,j_2} \to \mathrm{Id}$, which also satisfy the unit-counit laws. Similarly, they induce, for each j_1, j_2, j_3, two natural transformations $\eta_{j_1,j_2,j_3} \colon \mathrm{Id} \to i^*_{j_1,j_2,j_3} l_{j_1,j_2,j_3}$ and $\epsilon_{j_1,j_2,j_3} \colon l_{j_1,j_2,j_3} i^*_{j_1,j_2,j_3} \to \mathrm{Id}$, which also satisfy the unit-counit laws. All these natural transformations allow us to define two modifications $\eta \colon \mathrm{Id} \to \iota\lambda$ and $\epsilon \colon \iota\lambda \to \mathrm{Id}$ which satisfy the unit-counit laws, hence are the unit and counit of an adjunction $\lambda \dashv \iota$. This adjunction induces an adjunction $\tilde{l} \dashv \tilde{i} \cong i^*$.

Note that for all $j \in \mathcal{J}$, the unit $\eta\colon \mathrm{Id} \to \tilde{l}i^*$ and the counit $\epsilon\colon i^*\tilde{l} \to \mathrm{Id}$ of the adjunction $\tilde{l} \dashv i^*$ fit in the following commutative diagrams, where all vertical faces are isomorphisms and the one on the back is the identity:

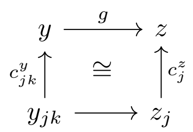


We now fix a bipullback in $\mathbf{C}$, with $f \in \mathcal{I}$

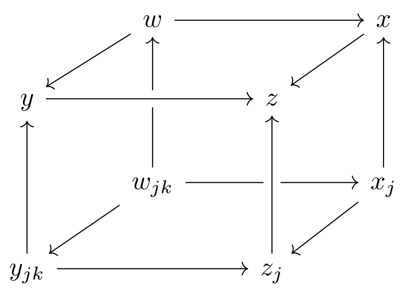


We want to show that the mate $\overline{(f|g)}_!$ of $\overline{(f|g)}^*$ is invertible. Fix a family of covering morphisms $(c_j^z\colon z_j \to z)_j$ with domain in $\mathbf{D}$. Taking the bipullbacks along f gives a covering family $(c_j^x\colon x_j \to x)$ of x with domain in $\mathbf{D}$.

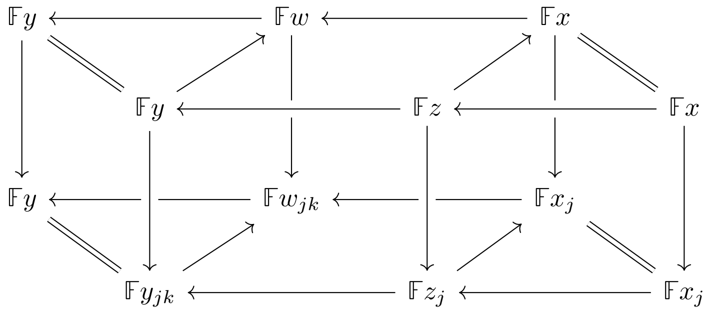
Taking the bipullbacks of the (c_j^z) along g yields a covering family $(c_j^y\colon y_j \to y)_j$ of y, but the y_j may not be in $\mathbf{D}$. Hence, for each $j \in \mathcal{J}$, we consider a covering family $(d_{jk}^y\colon y_{jk} \to y_j)_k$ of y_j with domain in $\mathbf{D}$. The composite family $(c_{jk}^y = c_j^y d_{jk}^y\colon y_{jk} \to y)_{j,k}$ is a covering family of y and for any j, k, there is a 2-isomorphism (which may not be a bipullback):



Taking the bipullbacks of the $(c_{jk}^y)_{j,k}$ along $(f|\underline{g})$, which is in $\mathcal{I}$, gives a covering family $(c_{jk}^w\colon w_{jk} \to w)$ of w with domain in $\mathbf{D}$. Notice that, for each j, k, there is a unique (up to isomorphism) 1-morphism $w_{jk} \to x_j$ making the following diagram (where all faces are 2-isomorphisms) commutes:



Note that all but the front and back faces are bipullbacks. By the previous discussion we can form the following commutative diagram, pasting the relevant cylinders on the left and on the right:



The upper face is precisely the mate $\overline{(x|y)_!}$ we are interested in. The lower pasted face is the mate associated to a bipullback lying in $\mathbf{D}$; it is thus invertible. All the vertical faces are invertible. Hence the remaining face $(c^y_{jk})^*\overline{(x|y)_!}$ is invertible. Since this is true for all j, k and $(c^y_{jk})_{j,k}$ is a covering family of y, this is sufficient to conclude that the 2-morphism $\overline{(x|y)_!}$ is invertible. $\square$

CHAPTER 4

Applications to modular representations of finite groups

A representation of a finite group G is a module over a group algebra $\Bbbk G$, for some commutative ring $\Bbbk$. The representations of a finite group G naturally form a category $\mathrm{Mod}(\Bbbk G)$. The study of this category and some related ones, such as the stable category of representations $\mathrm{stMod}(\Bbbk G)$ or the derived category $\mathrm{D}(\Bbbk G)$, occupy a large part of the theory of representations. Each of the mappings $G \mapsto \mathrm{Mod}(\Bbbk G)$, $G \mapsto \mathrm{stMod}(\Bbbk G)$ and $G \mapsto \mathrm{D}(\Bbbk G)$ extends to a pseudofunctor $\mathbf{gpd}^{\mathrm{op}} \to \mathbf{Add}$ with remarkable properties, abstracted into the notion of *Mackey 2-functor.*

When $\Bbbk$ is field of characteristic p, or more broadly, a commutative $\mathbb{Z}_{(p)}$-algebra, the representations are said to be p-modular. The p-modular representation categories, collected to form a Mackey 2-functor, are a *p-monadic* 2-functor (in the sense of definition 4.1.9). We explain how p-monadic Mackey 2-functors relate to sheaves for an adequate topology.

4.1. Mackey 2-functors

General Mackey 2-functors. We recall the notion of Mackey 2-functors introduced by Paul Balmer and Ivo Dell'Ambrogio in [BD20].

4.1.1. Definition. Let $\mathbf{G}$ be a $(2,1)$-category and $\mathcal{J}$ a class of 1-morphisms of $\mathbf{G}$. The pair $(\mathbf{G}, \mathcal{J})$ is *admissible* if it satisfies the following properties:

- The class $\mathcal{J}$ contains all equivalences and is closed by composition. Moreover if $fg \in \mathcal{J}$, $g \in \mathcal{J}$.
- For any 1-morphism $f \colon x \to y$ in $\mathcal{J}$, f is faithful.
- The 2-category $\mathbf{G}$ has all finite bipullbacks and $\mathcal{J}$ is closed under bipullbacks (definition 3.1.1).
- The 2-category $\mathbf{G}$ has all finite coproducts and $\mathcal{J}$ is closed under finite coproducts.

4.1.2. Example. We have several examples of admissible pairs $(\mathbf{G}, \mathcal{J})$:

(a) $\mathbf{G} = \mathbf{gpd}$, the 2-category of finite groupoids, and $\mathcal{J}$ is the class of the faithful functors between them.
(b) $\mathbf{G} = \mathbf{gpd}^{\mathrm{f}}$, the 2-category of finite groupoids, with faithful functors as 1-morphisms, and $\mathcal{J}$ is the class of all 1-morphisms of $\mathbf{gpd}^{\mathrm{f}}$.
(c) For a groupoid G, $\mathbf{G} = \mathbf{gpd}^{\mathrm{f}}/G$, the slice 2-category over G in $\mathbf{gpd}^{\mathrm{f}}$, and $\mathcal{J}$ is the class of all the 1-morphisms of $\mathbf{gpd}^{\mathrm{f}}/G$.
(d) For a prime p, $\mathbf{G} = p\text{-}\mathbf{gpd}$, the 2-category of finite p-groupoids (that is, finite groupoids in which each connected component is equivalent to a p-group) and $\mathcal{J}$ is the class of faithful functors between them.

We will generally keep the class $\mathcal{J}$ of 1-morphisms implicit.

4.1.3. Definition. Let $(\mathbf{G}, \mathcal{J})$ be an admissible pair. A Mackey 2-functor $\mathbb{M}$ on $(\mathbf{G}, \mathcal{J})$ is a 2-functor

$$\mathbb{M} \colon \mathbf{G}^{\mathrm{op}} \to \mathbf{Add}$$

satisfying the following axioms:

(Mack1) *Additivity:* For any objects x_1, x_2 of $\mathbf{G}$, the canonical functor induced by the 1-morphisms $i_k\colon x_k \to x_1 \amalg x_2$

$$\mathbb{M}(x_1 \amalg x_2) \xrightarrow{(i_1^*, i_2^*)} \mathbb{M}(x_1) \oplus \mathbb{M}(x_2)$$

is an equivalence.

(Mack2) For any morphism $j \in \mathcal{J}$, the functor j^* has a left adjoint $j_!$ and a right adjoint j_*:

$$j_! \dashv j^* \dashv j_*$$

(Mack3) *Beck-Chevalley:* For any morphism $f\colon x \to z$ in $\mathcal{J}$ and $g\colon y \to z$ of $\mathbf{G}$, consider the following bipullback square in $\mathbf{G}$:

$$\begin{array}{ccc} x & \xrightarrow{\;f\;} & z \\ {\scriptstyle \underline{1}}\uparrow & \overset{\lambda}{\Searrow} & \uparrow{\scriptstyle g} \\ (f|g) & \xrightarrow[\underline{2}]{} & y \end{array}$$

Then the mates $\lambda_!\colon \underline{2}_!\underline{1}^* \to g^*f_!$ and $(\lambda^{-1})_*\colon g^*f_* \to \underline{2}_*\underline{1}^*$ are invertible.

(Mack4) *Ambidexterity:* For any morphism $j \in \mathcal{J}$, the left adjoint $j_!$ and the right adjoint j_* of j^* are isomorphic:

$$j_! \cong j_*$$

4.1.4. Remark. To more precisely fit within the framework we developed in chapter 3, we should note that a Mackey 2-functor is precisely a 2-functor $\mathbb{M}\colon \mathbf{G}^{\mathrm{op}} \to \mathbf{Add}$ which preserves products and has the ambidextrous BC-property (definition 3.1.2).

4.1.5. Definition. Let $(\mathbf{G}, \mathcal{J})$ be an admissible pair. The 2-category $\mathbf{Mack}(\mathbf{G}, \mathcal{I})$ (or $\mathbf{Mack}(\mathbf{G})$) of Mackey 2-functors is the 2-category with:

- *Objects:* the Mackey 2-functors $\mathbb{M}$ on $(\mathbf{G}, \mathcal{J})$.
- 1-*Morphisms* $\mathbb{M} \to \mathbb{N}$*:* the pseudonatural transformations $\phi\colon \mathbb{M} \to \mathbb{N}$.
- 2-*Morphisms* $\phi \to \psi$*:* the modifications $m\colon \phi \to \psi$.

4.1.6. Remark. In this work, a 1-morphism $\phi\colon \mathbb{M} \to \mathbb{N}$ is simply a pseudonatural transformation, without any additionally requirement. In particular, for a morphism $j \in \mathcal{J}$, there are no specific compatibilities between the adjoints of $\mathbb{M}(j)$ and of $\mathbb{N}(j)$. This is a notable difference from the bicategory of Mackey 2-functors defined in [BD20, §6.3].

4.1.7. Example. Representation theory provides us with a large supply of Mackey 2-functors:

(a) For an additive category A, the representable 2-functor

$$\mathbf{Cat}(-, A)\colon \mathbf{gpd}^{\mathrm{op}} \to \mathbf{Add}$$

is a Mackey 2-functor. This example includes the linear representations $\mathrm{Mod}(\Bbbk-)$ over some commutative ring $\Bbbk$ with $A = \mathrm{Mod}(\Bbbk)$, the category of $\Bbbk$-modules.

(b) For a field $\Bbbk$, the stable modules categories

$$\mathrm{stMod}(\Bbbk-)\colon (\mathbf{gpd}^{\mathrm{f}})^{\mathrm{op}} \to \mathbf{Add}$$

and the derived categories

$$\mathrm{D}(\Bbbk-)\colon (\mathbf{gpd}^{\mathrm{f}})^{\mathrm{op}} \to \mathbf{Add}$$

form Mackey 2-functors.

(c) For a commutative ring $\Bbbk$, the permutation modules categories

$$\mathrm{perm}_{\Bbbk}(-)\colon \mathbf{gpd}^{\mathrm{op}} \to \mathbf{Add}$$

form a sub-Mackey 2-functor of $\mathrm{mod}(\Bbbk-)$.

(d) For a ring $\Bbbk$, the categories of cohomological Mackey 1-functors

$$\mathrm{coMack}_{\Bbbk}(-)\colon \mathbf{gpd}^{\mathrm{op}} \to \mathbf{Add}$$

form a Mackey 2-functor. This is a consequence of [BD20, Proposition 7.3.2] and the characterization of cohomological Mackey 1-functors given by Yoshida theorem [Web00, Theorem 7.1]:

$$\mathrm{coMack}_{\Bbbk}(G) = \mathrm{Fun}_{+}(\mathrm{perm}_{\Bbbk}(G)^{\mathrm{op}}, \mathrm{Ab})$$

Cohomological Mackey 2-functors. The Mackey 2-functors we are interested in, namely $\mathrm{stMod}(\Bbbk-)$ and $\mathrm{D}(\Bbbk-)$, both present the same additional property. For any injective group homomorphism $f\colon H \to G$, there is an isomorphism between the adjoint functors $f_!$ and f_* such that the composite natural transformation

$$\mathrm{Id} \xrightarrow{\eta} f_* f^* \cong f_! f^* \xrightarrow{\epsilon} \mathrm{Id}$$

acts by multiplication by the index $[G : H]$. In [BD21], Mackey 2-functors exhibiting this properties are called *cohomological* by analogy with the 1-dimensional case. We take a slightly different definition, as we are not willing to introduce the rectification of a Mackey 2-functor. Nevertheless both definitions have the same implication on the monadicity of the cohomological Mackey 2-functors (proposition 4.1.11).

We will assume in this subsection that the admissible pair $(\mathbf{G}, \mathcal{I})$ is composed of a sub-2-category of $\mathbf{gpd}$, such as $\mathbf{gpd}$, $\mathbf{gpd}^{\mathrm{f}}$ or p-$\mathbf{gpd}$, and its class of faithful 1-morphisms.

4.1.8. Definition. A Mackey 2-functor $\mathbb{M}\colon \mathbf{G}^{\mathrm{op}} \to \mathbf{Add}$ is *cohomological* if, for any injective morphism of groups $f\colon H \to G$ in $\mathbf{G}$, the composite natural transformation

$$\mathrm{Id}_{\mathbb{M}(G)} \xrightarrow{\eta_r} f_* f^* \cong f_! f^* \xrightarrow{\epsilon_l} \mathrm{Id}_{\mathbb{M}(G)}$$

is equal to $[G\colon H]\sigma$, where η_r is the unit of the adjunction $f^* \dashv f_*$, ϵ_l is the counit of the adjunction $f_! \dashv f^*$ and $\sigma\colon \mathrm{Id}_{\mathbb{M}(G)} \to \mathrm{Id}_{\mathbb{M}(G)}$ is some (arbitrary) invertible natural transformation.

We consider the set of injective group morphisms of index coprime to p:

$$\mathcal{P} = \{f\colon H \to G \mid p \nmid [G : H]\}$$

4.1.9. Definition. A 2-functor $\mathbb{M}\colon \mathbf{G}^{\mathrm{op}} \to \mathbf{Cat}$ is *p-monadic* if it is $\mathcal{P}$-monadic (definition 1.1.33).

4.1.10. Definition. We denote by $\mathbf{Mack}_p(\mathbf{G})$ the full, 2-full sub-2-category of $\mathbf{Mack}(\mathbf{G})$ whose objects are the p-monadic Mackey 2-functors.

4.1.11. Proposition. *Let $\mathbb{M}\colon \mathbf{G}^{\mathrm{op}} \to \mathbf{Add}_{\Bbbk}$ be a cohomological Mackey 2-functor taking values in the 2-category of $\Bbbk$-linear categories, for some commutative $\mathbb{Z}_{(p)}$-algebra $\Bbbk$. Assume moreover that, for every group G, the category $\mathbb{M}(G)$ is idempotent-complete. Then the 2-functor $\mathbb{M}$ is p-monadic.*

Proof. This is precisely [BD21, Theorem 3.10]. □

4.1.12. Example. We can apply proposition 4.1.11 to check that several of our previously defined Mackey 2-functors are actually p-monadic. For a commutative $\mathbb{Z}_{(p)}$-algebra $\Bbbk$:

(a) The categories of modules $\mathrm{Mod}(\Bbbk-)$ and finitely generated modules $\mathrm{mod}(\Bbbk-)$ over $\Bbbk$.

(b) The derived categories of modules $\mathrm{D}(\Bbbk-)$ over $\Bbbk$.

(c) When $\Bbbk$ is a field, the stable categories of modules $\mathrm{stMod}(\Bbbk-)$ and finitely generated modules $\mathrm{stmod}(\Bbbk-)$.
(d) The categories of cohomological Mackey 1-functors $\mathrm{coMack}_{\Bbbk}(-)$ over $\Bbbk$.

Sheaves and p-monadic Mackey 2-functors. We address in this subsection the central point of this thesis: the correspondence between 2-sheaves on finite groupoids **gpd**, with an adequate topology (definition 4.1.14), and p-monadic Mackey 2-functors (theorem 4.1.18).

4.1.13. Hypothesis. We will assume in this subsection that the admissible pair $(\mathbf{G}, \mathcal{I})$ is composed of an extensive sub-2-category of **gpd**, such as **gpd**, $\mathbf{gpd}^{\mathrm{f}}$ or p-**gpd**, and its class of faithful 1-morphisms. We will note $\mathbf{P}$ the full, 2-full sub-2-category of $\mathbf{G}$ on p-groupoids. Analogous results hold for slice 2-categories such as $\mathbf{G} = \mathbf{gpd}^{\mathrm{f}}/G$ or $\mathbf{P} = p\text{-}\mathbf{gpd}^{\mathrm{f}}/G$, where G is a fixed finite groupoid.

4.1.14. Definition. The *p-local topology* on $\mathbf{G}$ is the topology generated by the (finite) covering families of the form

$$(H_i \xrightarrow{f_i} G_i \to \coprod_{j\in\mathcal{I}} G_j)_{i\in\mathcal{I}}$$

where each $f_i \colon H_i \to G_i$ is an injective homomorphism between groups of index coprime to p.

4.1.15. Notation. The category of sheaves on $\mathbf{G}$ for the p-local topology is denoted by $\mathbf{Sh}(\mathbf{G}, p)$.

4.1.16. Remark. By proposition 2.5.7, since $\mathbf{G}$ is extensive, a product-preserving 2-functor $\mathbb{X} \colon \mathbf{G}^{\mathrm{op}} \to \mathbf{Cat}$ is a 2-sheaf for the p-local topology if and only if it satisfies the descent condition for all the injective group homomorphisms of index coprime to p.

4.1.17. Remark. We will see in proposition 4.2.13 that a bilimit-preserving 2-functor

$$\mathbb{X} \colon \mathbf{G}^{\mathrm{op}} \to \mathbf{Cat}$$

is a sheaf on $\mathbf{C}$ for the p-local topology.

4.1.18. Theorem. *Let $\mathbb{M} \colon \mathbf{G}^{\mathrm{op}} \to \mathbf{Cat}$. Assume that its restriction to $\mathbf{P}$ is a Mackey 2-functor. Then the following statements are equivalent:*

(i) The 2-functor $\mathbb{M} \colon \mathbf{G}^{\mathrm{op}} \to \mathbf{Add}$ is a p-monadic Mackey 2-functor.
(ii) The 2-functor $\mathbb{M} \colon \mathbf{G}^{\mathrm{op}} \to \mathbf{Cat}$ is a 2-sheaf for the p-local topology on $\mathbf{G}$.

Proof. We prove both implications of the equivalence:

(i) ⇒ (ii): Assume that $\mathbb{M}$ is a p-monadic Mackey 2-functor. In particular, it is product-preserving. By remark 4.1.16, it is sufficient to check that it satisfies the descent condition for an injective morphism $f \colon H \to G$ of index coprime to p. Since $\mathbb{M}$ is p-monadic, we can apply proposition 3.2.13, hence $\mathbb{M}$ is a 2-sheaf for the p-local topology on $\mathbf{G}$.

(ii) ⇒ (i): Assume that $\mathbb{M}$ is a 2-sheaf for the p-local topology. By proposition 3.3.1, since the restriction of $\mathbb{M}$ to $\mathbf{P}$ satisfies the ambidextrous BC-property, so does $\mathbb{M}$. Moreover, $\emptyset$, the initial groupoid, is a p-groupoid, hence $\mathbb{M}(\emptyset) = 1$ and, by proposition 2.5.8, the 2-functor $\mathbb{M}$ preserves products. Hence $\mathbb{M}$ is a Mackey 2-functor. □

4.1.19. Remark. The Mackey 2-functors listed in example 4.1.12 are p-monadic, hence they are 2-sheaves for the p-local topology. In particular, this includes the Mackey 2-functors $\mathrm{Mod}(\Bbbk-)$, $\mathrm{stMod}(\Bbbk-)$ and $\mathrm{D}(\Bbbk-)$.

4.1.20. Remark. For a group G, the 1-category G-set of finite G-sets is biequivalent to the slice 2-category $\mathbf{gpd}^{\mathrm{f}}/G$ [Del19, Proposition 5.5]. In [Bal15], the 1-category of finite G-sets is endowed with coverage, inducing the so-called *sipp topology.* Remark that the 2-functor

$$G\text{-set} \xrightarrow{\sim} \mathbf{gpd}^{\mathrm{f}}/G \to \mathbf{gpd}^{\mathrm{f}}$$

satisfies the hypotheses of proposition 2.4.3. Hence restricting a 2-sheaf on $\mathbf{gpd}^{\mathrm{f}}$ to G-set provides a stack on G-set. Applying this observation to $\mathrm{Mod}(\Bbbk-)$, $\mathrm{stMod}(\Bbbk-)$ and $\mathrm{D}(\Bbbk-)$, we recover the fundamental theorem of [Bal15].

4.1.21. Remark. One easily checks that a product-preserving 2-functor

$$\mathbb{M}\colon \mathbf{P}^{\mathrm{op}} \to \mathbf{Cat}$$

is a 2-sheaf for the p-local topology, since, by proposition 2.5.7, we only have to check the descent condition for isomorphisms, which holds trivially. Hence by theorem 4.1.18, a Mackey 2-functor on $\mathbf{P}$ is always p-monadic.

4.1.22. Theorem. *The restriction* 2*-functor*

$$\mathbf{Mack}_p(\mathbf{G}) \to \mathbf{Mack}(\mathbf{P})$$

is a biequivalence between the 2*-category of* p*-monadic Mackey* 2*-functors on* $\mathbf{G}$ *and the category of Mackey* 2*-functors on* $\mathbf{P}$*.*

Proof. By proposition 2.4.6, the restriction 2-functor

$$\mathbf{Sh}(\mathbf{G}, p) \to \mathbf{Sh}(\mathbf{P}, p)$$

is a biequivalence of 2-categories and by theorem 4.1.18, it induces a biequivalence

$$\mathbf{Mack}_p(\mathbf{G}) \to \mathbf{Mack}_p(\mathbf{P}) = \mathbf{Mack}(\mathbf{P})$$

as announced. □

4.1.23. Remark. Given an arbitrary Mackey 2-functor $\mathbb{M} \in \mathbf{Mack}(\mathbf{G})$, we can consider its restriction $\mathbb{M}_{|\mathbf{P}} \in \mathbf{Mack}(\mathbf{P})$ to $\mathbf{P}$, and then the (essentially) unique p-monadic extension $(\mathbb{M}_{|\mathbf{P}})^\sharp \in \mathbf{Mack}_p\mathbf{G}$. Notice that there is a canonical pseudonatural transformation $\alpha\colon \mathbb{M} \to (\mathbb{M}_{|\mathbf{P}})^\sharp$ (hence a 1-morphism in $\mathbf{Mack}(\mathbf{G})$), defined object-wise by the universal property of the descent bilimit. Moreover $((\mathbb{M}_{|\mathbf{P}})^\sharp, \alpha)$ is a biinitial object in $\mathbb{M}/\mathbf{Sh}(\mathbf{G}, p)$, what we call a 2-*sheafification* of $\mathbb{M}$.

4.1.24. Remark. The Mackey 2-functor $\mathbf{Mod}(\mathbb{Z}-) = \mathbf{Cat}(-, \mathrm{Ab})$ preserves the bilimits, as the inclusion $\mathbf{gpd} \to \mathbf{Cat}$ preserves the bicolimits and the representable 2-functor $\mathbf{Cat}(-, \mathrm{Ab})\colon \mathbf{Cat} \to \mathbf{Cat}$ preserves the bilimits. Hence, by remark 4.1.17, $\mathbf{Mod}(\mathbb{Z}-)$ is sheaf for the p-local topology, and thus by theorem 4.1.18, it is a p-monadic Mackey 2-functor. Nonetheless, for an injective group homomorphism $i\colon H \to G$ of non-trivial index coprime to p, the composite natural transformation

$$\mathrm{Id}_{\mathbf{Mod}(\mathbb{Z}G)} \xrightarrow{\eta_r} i_* i^* \cong i_! i^* \xrightarrow{\epsilon_l} \mathrm{Id}_{\mathbf{Mod}(\mathbb{Z}G)}$$

is equal to $[G\colon H]\,\mathrm{Id}$, which is not invertible.

4.1.25. Remark. Let

$$\mathbb{X}\colon \mathbf{G}^{\mathrm{op}} \to \mathbf{Cat}$$

be a 2-functor such that $\mathbb{X}(\emptyset) = 1$ and which is a sheaf for both the p-local topology and the q-local topological, with $p \neq q$. Then, it is clear that $\mathbb{X}$ is entirely defined by the category $\mathbb{X}(E)$, where E is the trivial group, and more precisely

$$\mathbb{X} = \mathbf{Cat}(-, \mathbb{X}(E))$$

In particular Mackey 2-functors which are p-monadic and q-monadic, for two distinct prime p, q, are of the form

$$\mathbb{X} = \mathbf{Cat}(-, A)$$

where A is an additive category.

4.2. Cartan-Eilenberg formulas

In this section, we present the link between the classical theory of fusion in a finite group and descent over the p-local topology. In particular, we show that the descent condition can be reinterpreted as a categorified version of the classical Cartan-Eilenberg stable elements formula. We also present a generic process to decategorify our categorified Cartan-Eilenberg formula and generate variations of the original Cartan-Eilenberg formula.

The classical Cartan-Eilenberg formula. We introduce the fusion system and the transporter category of a finite group, and use them to express the classical Cartan-Eilenberg formula.

Given a finite group G with a subgroup H, any element of g defines a group homomorphism:

$$c_g : \begin{cases} H & \to & gHg^{-1} \\ h & \mapsto & ghg^{-1} \end{cases}$$

This defines an action of G on its subgroups. The fusion system $\mathcal{F}_S(G)$ of G is a category that agglomerate all the information of the action by conjugation of G on the *p-subgroups* of a fixed p-Sylow subgroup S of G.

4.2.1. Definition. Let G be a finite group and p a prime integer. Fix a p-Sylow subgroup S of G. The *p-fusion system* $\mathcal{F}_S(G)$ of G is the category with:

- *Objects:* the subgroups P of S.
- *Morphisms $P \to Q$:* the conjugation homomorphisms

$$c_g : \begin{cases} P & \to & Q \\ p & \mapsto & gpg^{-1} \end{cases}$$

  induced by an element $g \in G$ such that $gPg^{-1} \subset Q$.
- *Composition* is induced by the composition of group homomorphisms.

The p-fusion system of G is naturally endowed with a forgetful functor to the category of finite groups:

$$U : \mathcal{F}_S(G) \to \mathrm{gp}$$

Several auxiliary categories are usually used in conjunction with the fusion system $\mathcal{F}_S(G)$; we will particularly be interested in the transporter category $\mathcal{T}_S(G)$.

4.2.2. Definition. Let G be a finite group and p a prime integer. Fix a p-Sylow subgroup S of G. The *p-transporter category* $\mathcal{T}_S(G)$ of G is the category with:

- *Objects:* the subgroups P of S.
- *Morphisms $P \to Q$:* the elements g of G such that $gPg^{-1} \subset Q$.
- *Composition* is induced by the multiplication in G.

The p-transporter category of G is naturally endowed with a forgetful functor to the category of finite groups

$$U : \mathcal{T}_S(G) \to \mathrm{gp}$$

4.2.3. Remark. We obviously have a factorization:

$$\begin{array}{ccc} \mathcal{T}_S(G) & \longrightarrow & \mathcal{F}_S(G) \\ & {\scriptstyle U} \searrow \quad \swarrow {\scriptstyle U} & \\ & \mathrm{gp} & \end{array}$$

The functor $\mathcal{T}_S(G) \to \mathcal{F}_S(G)$ is the identity on objects and full. In particular, it is final.

4.2.4. Remark. The transporter category $\mathcal{T}_S(G)$ entirely characterizes the group G, since G can be recovered as the group of automorphisms $\mathcal{T}_S(G)(e,e)$ of the trivial subgroup e. This is not the case for the fusion system associated to G.

The Cartan-Eilenberg formula expresses the mod p cohomology of a finite group G in term of the mod p cohomology of its p-subgroups and the conjugation action of G on these.

4.2.5. Proposition (Cartan-Eilenberg formula)**.** *Let G be finite group, p a prime integer and $\Bbbk$ a commutative $\mathbb{Z}_{(p)}$-algebra. Fix a p-Sylow S of G. Then the comparison morphism*

$$H^*(G;\Bbbk) \to \lim_{P \in \mathcal{F}_S(G)^{\mathrm{op}}} H^*(P;\Bbbk)$$

is an isomorphism.

4.2.6. Remark. The Cartan-Eilenberg formula can equivalently be stated using the transporter category for indexing

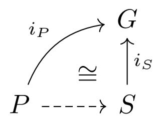
$$H^*(G;\Bbbk) \to \lim_{P \in \mathcal{T}_S(G)^{\mathrm{op}}} H^*(P;\Bbbk)$$

since the comparison functor $\mathcal{T}_S(G) \to \mathcal{F}_S(G)$ is final. Obviously, it is more interesting to state that the mod p cohomology is an invariant of fusion than an invariant of transport.

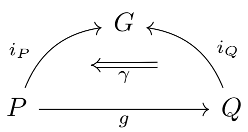
The categorification of the transporter category. For an object G of a $(2,1)$-category $\mathbf{G}$, we would like to define a 2-category analogous to the transporter category of a group.

4.2.7. Definition. Let $\mathbf{G}$ be a $(2,1)$-category. Let G be an object of $\mathbf{G}$ and $(S, i_S\colon S \to G)$ an object of the slice $(2,1)$-category $\mathbf{G}/G$. The *transporter 2-category* $\hat{\mathcal{T}}_{i_S}(G)$ of G (with respect to i_S) is full, 2-full sub-$(2,1)$-category of the slice $(2,1)$-category $\mathbf{G}/G$ whose object $(P, i_P\colon P \to G)$ factors through i_S. More explicitly $\hat{\mathcal{T}}_{i_S}(G)$ is the $(2,1)$-category with:

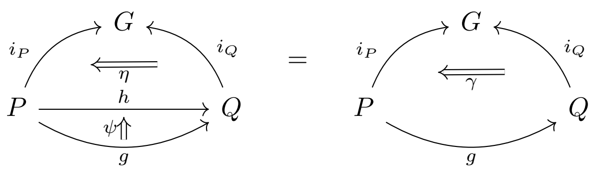
- *Objects:* the pairs (P, i_P) consisting of an object P and a 1-morphism $i_P\colon P \to G$ of $\mathbf{G}$ such that there is a factorization:



- 1-*Morphisms* $(P, i_P) \to (Q, i_Q)$*:* the pairs (g, γ) consisting of a 1-morphism $g\colon P \to Q$ in $\mathbf{G}$ and a 2-isomorphism γ:



- 2-*Morphisms* $(g, \gamma) \to (h, \eta)$*:* the 2-morphisms $\psi\colon g \to h$ such that:



The 2-category $\hat{\mathcal{T}}_{i_S}(G)$ is canonically endowed with a 2-functor

$$\mathbb{U}\colon \hat{\mathcal{T}}_{i_S}(G) \to \mathbf{G}$$

obtained as the restriction of the 2-functor $\mathbf{G}/G \to \mathbf{G}$.

4.2.8. Example. Suppose that $\mathbf{G} = \mathbf{gpd}^{\mathrm{f}}$, G is a finite group, seen as a one object groupoid, and S is a p-Sylow of G, with $i_S\colon S \to G$ denoting the inclusion. Then the 2-category $\hat{\mathcal{T}}_{i_S}(G)$ can be described as follows:

- Objects are essentially coproducts $\amalg_i P_i$ of subgroups P_i of S
- 1-Morphisms are entirely described by the 1-morphisms between connected components, that is between two subgroups P and Q of S. We can see that these morphisms are precisely elements g of G which induce a group homomorphism $c_g\colon P \to Q$.
- 2-Morphisms between two morphisms $g, g'\colon P \to Q$ can be identified with the elements q of Q such that $g' = qg$ (which is unique, if it exists)

Thus the underlying 1-category of $\hat{\mathcal{T}}_{i_S}(G)$ can be seen as an extension of the transporter category $\mathcal{T}_S(G)$ with coproducts. Hence, we will generally write $\hat{\mathcal{T}}_S(G)$ to denote $\hat{\mathcal{T}}_{i_S}(G)$.

Moreover, the truncated 1-category of $\hat{\mathcal{T}}_{i_S}(G)$ is precisely the orbit category $\mathcal{O}_S(G)$ of G, and the canonical projection 2-functor

$$\hat{\mathcal{T}}_{i_S}(G) \to \mathcal{O}_S(G)$$

is a biequivalence.

4.2.9. Proposition. *Suppose that* $\mathbf{G} = \mathbf{gpd}^{\mathrm{f}}$*. Let* H*,* G *be a finite groups and* $i\colon H \to G$ *an injective homomorphism. Then the canonical* 1*-morphism*

$$\operatorname*{bicolim}_{\hat{\mathcal{T}}_i(G)} \mathbb{U} \to G$$

is an equivalence

Proof. We note $\mathcal{C}$ the cone of G under $\mathbb{U}$. We want to show that for any groupoid Z, the functor

$$\mathbf{G}(G, Z) \to [\hat{\mathcal{T}}_i(G), \mathbf{G}](\mathbb{U}, \Delta Z)$$

is an equivalence. We note E the trivial group and, for any group K, $*_K$ its unique object.

- *Faithfulness:* Let $u, v\colon G \to Z$ and $\phi, \psi\colon u \to v$ such that $\phi\mathcal{C} = \psi\mathcal{C}$. In particular:
$$\phi_{*_G} = \phi_{\mathcal{C}_E(*_E)} = \psi_{\mathcal{C}_E(*_E)} = \psi_{*_G}$$
Hence $\phi = \psi$.
- *Fullness:* Let $u, v\colon G \to Z$ and $m\colon u\mathcal{C} \to v\mathcal{C}$ a modification. Define a transformation $\psi\colon u \to v$ by
$$\psi_{*_G} = m_{E,*_E}$$
For any $g \in G$, there is a 1-morphism:
$$(\mathrm{Id}, g)\colon (E, \mathcal{C}_E) \to (E, \mathcal{C}_E)$$
Hence the following square commutes:
$$\begin{array}{ccc} u\mathcal{C}_E & \xrightarrow{m_E} & v\mathcal{C}_E \\ {\scriptstyle u\mathcal{C}_{(\mathrm{Id},g)}}\downarrow & & \downarrow{\scriptstyle v\mathcal{C}_{(\mathrm{Id},g)}} \\ u\mathcal{C}_E & \xrightarrow[m_E]{} & v\mathcal{C}_E \end{array}$$
Then evaluating at $*_E$, we have the commutative square
$$\begin{array}{ccc} *_G & \xrightarrow{\phi_{*_G}} & *_G \\ {\scriptstyle u(g)}\downarrow & & \downarrow{\scriptstyle v(g)} \\ *_G & \xrightarrow[\phi_{*_G}]{} & *_G \end{array}$$
which shows the naturality of ψ. A straightforward check shows that $m = \psi\mathcal{C}$.

- *Essential surjectivity:* Let $\phi\colon \mathbb{U} \to \Delta Z$ a pseudonatural transformation. We consider the 1-morphism $u\colon G \to Z$ given by:
$$u\colon \begin{cases} *_G & \mapsto & \phi_E(*_E) \\ g & \mapsto & (\phi_{(\mathrm{Id}_E, g)})_{*_E} \end{cases}$$
We can check that ϕ is isomorphic to the cone $u\mathcal{C}$.

□

We will now relate our extended transporter 2-category to the 2-category of descent **Desc** (definition 2.2.1).

4.2.10. Proposition. *Let* $\mathbf{G}$ *a* $(2,1)$*-category with finite bipullback. Let* G *an object of* $\mathbf{G}$ *and* $(S, i_S\colon S \to G)$ *an object of the slice* $(2,1)$*-category. Then there is a* 2*-functor*
$$\mathbb{K}\colon \begin{cases} \mathbf{Desc} & \to & \hat{\mathcal{T}}_{i_S}(G) \\ 0 & \mapsto & (S, i_S) \\ 1 & \mapsto & (i_S|i_S) \\ 2 & \mapsto & (i_S|i_S|i_S) \\ \underline{x} & \mapsto & \underline{x} \\ \overline{x} & \mapsto & \overline{x} \end{cases}$$
Moreover the 2*-functor* $\mathbb{K}$ *is* 2*-final.*

Proof. We use the topological criterion for 2-final 2-functors; proposition 4.2.14, proposition 4.2.15 and proposition 4.2.16 guarantee that our 2-functor satisfy the hypothesis of theorem 1.3.13. □

4.2.11. Remark. Keeping the notations of proposition 4.2.10, let $\mathbb{X}\colon \mathbf{G} \to \mathbf{Cat}$ be a 2-functor. Then the following diagram commutes:
$$\begin{array}{ccc} \mathbf{Desc} & \xrightarrow{\quad\mathbb{K}\quad} & \hat{\mathcal{T}}_{i_S}(G) \\ & {\scriptstyle \mathbb{D}^{\mathbb{X}}_{i_S}} \searrow \qquad \swarrow {\scriptstyle \mathbb{X}\mathbb{U}} & \\ & \mathbf{Cat} & \end{array}$$

We defer the actual proof of proposition 4.2.10 to state a few (now) easy corollaries.

4.2.12. Theorem. *Let* $\mathbf{G}$ *a* $(2,1)$*-site with finite bipullback. Let* G *an object of* $\mathbf{G}$ *and* $(S, i_S\colon S \to G)$ *an object of the slice* $(2,1)$*-category. Assume that the family with only the* 1*-morphism* i_S *covers* G*. Then for any sheaf* $\mathbb{X}$ *on* $\mathbf{G}$*, we have the following categorified Cartan-Eilenberg formula:*
$$\mathbb{X}(G) \simeq \operatorname*{bilim}_{P \in \hat{\mathcal{T}}_{i_S}(G)} \mathbb{X}(P)$$

4.2.13. Proposition. *Let* $(\mathbf{G}, \mathcal{I})$ *satisfying hypothesis 4.1.13. Then a bilimit-preserving* 2*-functor*
$$\mathbb{X}\colon \mathbf{G}^{\mathrm{op}} \to \mathbf{Cat}$$
is sheaf on $\mathbf{G}$ *for the* p*-local topology.*

Proof. Indeed, since in particular it preserves products, we only have to check that it satisfies the descent condition for injective group homomorphisms $g\colon H \to G$ of index coprime to p. We can apply proposition 2.5.3 since we have the equivalence
$$G \overset{(1)}{\simeq} \operatorname*{bicolim}_{\hat{\mathcal{T}}_g(G)} \mathbb{U} \overset{(2)}{\simeq} \operatorname*{bicolim}_{\mathbf{Desc}} \tilde{\mathbb{D}}_g$$

where the equivalence (1) is precisely proposition 4.2.9 and the equivalence (2) is a consequence of proposition 4.2.10 and of the factorization

$$\begin{array}{ccc} \mathbf{Desc} & \longrightarrow & \hat{\mathcal{T}}_g(G) \\ & {\scriptstyle \tilde{\mathbb{D}}_g} \searrow \quad \swarrow {\scriptstyle \mathbb{U}} & \\ & \mathbf{G} & \end{array}$$

where $\tilde{\mathbb{D}}_g$ is defined in remark 2.5.2. □

We give in the remaining of this subsection the arguments justifying proposition 4.2.10. Recall that a bipullback over an object x in a 2-category $\mathbf{C}$ is precisely a biproduct in the slice 2-category $\mathbf{C}/x$. Hence we fix a $(2,1)$-category $\mathbf{S}$ and we assume that:

- all the binary biproducts exist in $\mathbf{S}$
- there is a weakly final object X, that is, for any object S in $\mathbf{S}$, there is a morphism $S \to X$.

As above we define a pseudofunctor

$$\mathbb{K} : \begin{cases} \mathbf{Desc} & \to & \mathbf{S} \\ 0 & \mapsto & X \\ 1 & \mapsto & X \times X \\ 2 & \mapsto & X \times X \times X \end{cases}$$

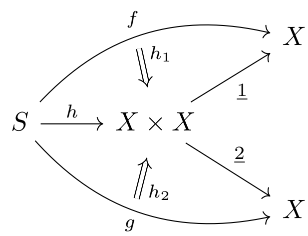
4.2.14. Proposition. *For any object S of $\mathbf{S}$, the slice $(2,1)$-category $S/\mathbb{K}$ is nonempty.*

Proof. By hypothesis, there is a morphism $S \to X = \mathbb{K}(0)$ in $\mathbf{S}$. Hence $S/\mathbb{K}$ is nonempty. □

4.2.15. Proposition. *For any object S of $\mathbf{S}$, the slice $(2,1)$-category $S/\mathbb{K}$ is connected.*

Proof. First, fix an object $(N, f\colon S \to \mathbb{K}(N))$ in $S/\mathbb{K}(N)$. There is a morphism $m\colon N \to 0$ in $\mathbf{Desc}$, hence a morphism $(m, \mathrm{Id})\colon (N,f) \to (0, \mathbb{F}m \circ f)$ in $S/\mathbb{K}$. It is thus sufficient to show that any two objects of the form $(0, f\colon S \to \mathbb{K}(0))$ are connected.

Now fix two such objects $(0,f)$ and $(0,g)$ in $S/\mathbb{F}$. By the biproduct universal property, they define an essentially unique morphism $h\colon S \to X \times X$ fitting in:



This last diagram can be rewritten as a path in $S/\mathbb{F}$:

$$(0,f) \xleftarrow{(\underline{1},h_1)} (1,h) \xrightarrow{(\underline{2},h_2)} (0,g)$$

□

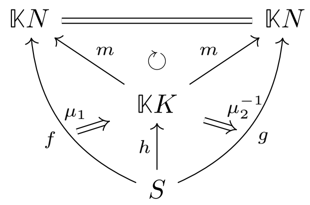
4.2.16. Proposition. *For any object S of $\mathbf{S}$, the slice $(2,1)$-category $S/\mathbb{K}$ is simply connected.*

Proof. We want to show that any two paths p, p' in $S/\mathbb{K}$. We will first reduce the problem gradually to paths with simpler shapes, by looking at projected path in $\mathbf{Desc}$.

(1) A path p with $\pi(p) = N \overset{m}{\longleftarrow} K \overset{m}{\longrightarrow} N$ is homotopic to a path p' with $\pi(p') = N \overset{\mathrm{Id}}{\longleftarrow} N$. Indeed, write p as:

$$(N, f) \xleftarrow{(m,\mu_1)} (K, h) \xrightarrow{(m,\mu_2)} (N, g)$$

We can define a 2-morphism ν in $\mathbf{S}$, by pasting:

$\nu =$ 

This defines a path p' in $S/\mathbb{K}$, homotopic to p:

$$p' = (N, f) \xleftarrow{(\mathrm{Id},\nu)} (N, g).$$

(2) Similarly, a path p with $\pi(p) = N \overset{m}{\longrightarrow} K \overset{m}{\longleftarrow} N$ is homotopic to a path p' with $\pi(p') = N \overset{id}{\longleftarrow} N$.

(3) A path p with

$$\pi(p) = 1 \overset{\underline{12}}{\longleftarrow} 2 \overset{\underline{13}}{\longrightarrow} 1$$

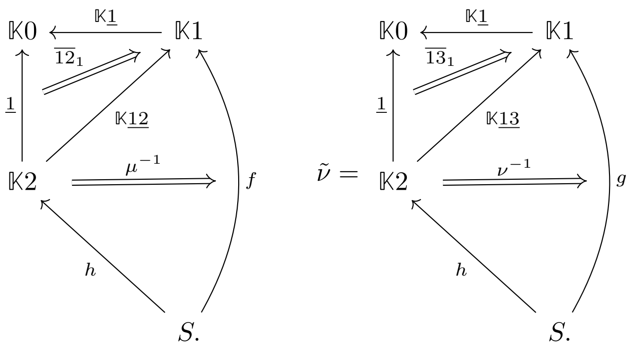
is homotopic to a path p' with

$$\pi(p') = 1 \overset{\underline{1}}{\to} 0 \overset{\underline{1}}{\leftarrow} 1.$$

Indeed, there is a 2-morphism $\overline{1}\colon \underline{1} \circ \underline{12} \Rightarrow \underline{1} \circ \underline{13}$ in **Desc**. Write p as:

$$(1, f) \xleftarrow{(\underline{12},\mu)} (2, h) \xrightarrow{(\underline{13},\nu)} (1, g)$$

We consider the 2-morphism in $\mathbf{S}$:

$\tilde{\mu} =$ $\tilde{\nu} =$ 

One can check that we have a 2-morphism in $S/\mathbb{K}$:

$$\mathbb{K}\overline{1}\colon (\mathbb{K}\underline{1}, \tilde{\mu}) \circ (\mathbb{K}\underline{12}, \mu) \Rightarrow (\mathbb{K}\underline{1}, \tilde{nu}) \circ (\mathbb{K}\underline{13}, \nu)$$

Writing $\tilde{h} = \underline{1} \circ h$, we deduce a path p' in $S/\mathbb{K}$, homotopic to p:

$$(1, f) \xrightarrow{(\mathbb{K}\underline{1},\tilde{\mu})} (0, \tilde{h}) \xleftarrow{(\mathbb{K}\underline{1},\tilde{\nu})} (1, g)$$

(4) By similar arguments, any path p with $\pi(p) = 1 \overset{\underline{12}}{\longleftarrow} 2 \overset{\underline{23}}{\longrightarrow} 1$ is homotopic to a path p' with $\pi(p') = 1 \overset{\underline{2}}{\to} 0 \overset{\underline{1}}{\leftarrow} 1$ and any path p with $\pi(p) = 1 \overset{\underline{13}}{\longleftarrow} 2 \overset{\underline{23}}{\longrightarrow} 1$ is homotopic to a path p' with $\pi(p') = 1 \overset{\underline{2}}{\to} 0 \overset{\underline{2}}{\leftarrow} 1$.

(5) Any path p with

$$\pi(p) = 0 \xrightarrow{1} 1 \xleftarrow{2} 0 \xrightarrow{1} 1 \xleftarrow{2} 0$$

is homotopic to *any* path p' with same source and target and with

$$\pi(p') = 0 \xrightarrow{1} 1 \xleftarrow{2} 0$$

Indeed, write p as:

$$(0, f_1) \xrightarrow{(\mathbb{K}\underline{1}, \mu_1)} (1, g_1) \xleftarrow{(\mathbb{K}\underline{2}, \nu_1)} (0, f_2) \xrightarrow{(\mathbb{K}\underline{1}, \mu_2)} (1, g_2) \xleftarrow{(\mathbb{K}\underline{1}, \nu_2)} (0, f_3).$$

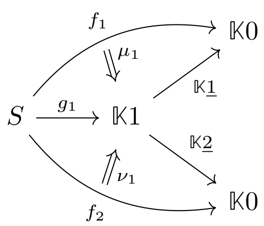
and p' as:

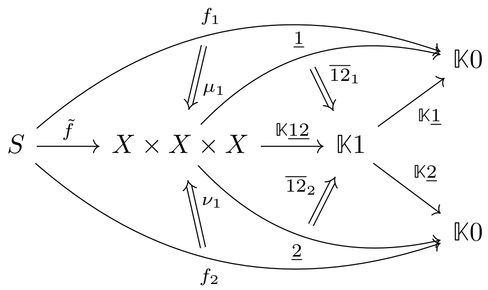
$$(0, f_1) \xrightarrow{(\mathbb{K}\underline{1}, \mu_3)} (1, g_3) \xleftarrow{(\mathbb{K}\underline{2}, \nu_3)} (0, f_3).$$

The morphisms f_1, f_2 and f_3 define an essentially unique morphism $\tilde{f} \colon S \to X \times X \times X$ in $\mathbf{S}$, with structural 2-isomorphisms, for $i = 1, 2, 3$:

$$\lambda_i \colon \underline{i} \circ \tilde{f} \Rightarrow f_i$$

Consider the diagrams:





Both the morphism g_1 and $\mathbb{K}\underline{12} \circ \tilde{f}$ satisfy the universal property of $\mathbb{K}1 = X \times X$ with respect to f_1 and f_2, thus there is a unique 2-isomorphism $\kappa_1 \colon g_1 \Rightarrow \mathbb{K}\underline{12} \circ \tilde{f}$ compatible with the structural 2-morphisms. Similarly, there are 2-isomorphisms $\kappa_2 \colon g_2 \to \mathbb{K}\underline{23} \circ \tilde{f}$ and $\kappa_3 \colon g_3 \to \mathbb{K}\underline{13} \circ \tilde{f}$, with some compatibilities. One can check that we have the following chain of homotopies:

$$
\begin{aligned}
& p = (0, f_1) \xrightarrow{(\mathbb{K}\underline{1}, \mu_1)} (1, g_1) \xleftarrow{(\mathbb{K}\underline{2}, \nu_1)} (0, f_2) \xrightarrow{(\mathbb{K}\underline{1}, \mu_2)} (1, g_2) \xleftarrow{(\mathbb{K}\underline{1}, \nu_2)} (0, f_3) \\
\text{(via } \overline{2}\text{)} \quad & \sim (0, f_1) \xrightarrow{(\mathbb{K}\underline{1}, \mu_1)} (1, g_1) \xrightarrow{(\mathbb{K}\underline{12}, \kappa_1)} (2, \tilde{h}) \xleftarrow{(\mathbb{K}\underline{23}, \kappa_2)} (1, g_2) \xleftarrow{(\mathbb{K}\underline{1}, \nu_2)} (0, f_3) \\
\text{(via } \overline{1} \text{ and } \overline{3}\text{)} \quad & \sim (0, f_1) \xrightarrow{(\mathbb{K}\underline{1}, \mu_3)} (1, g_3) \xrightarrow{(\mathbb{K}\underline{13}, \kappa_3)} (2, \tilde{h}) \xleftarrow{(\mathbb{K}\underline{13}, \kappa_3)} (1, g_3) \xleftarrow{(\mathbb{K}\underline{1}, \nu_3)} (0, f_3) \\
& \sim (0, f_1) \xrightarrow{(\mathbb{K}\underline{1}, \mu_3)} (1, g_3) \xleftarrow{(\mathbb{K}\underline{1}, \nu_3)} (0, f_3) = p'
\end{aligned}
$$

Since $S/\mathbb{K}$ is connected, to check that it is simply connected, it is sufficient to check that loops at an object $(0, f)$ are homotopic to the constant path. By the previous reduction, up to homotopy, it suffices to consider a loop p with $\pi(p) = 0 \xleftarrow{1} 1 \xrightarrow{2} 0$. But by the last case, we have $p^2 \sim p$, hence p is homotopic to the constant path. □

Cartan-Eilenberg formulas. In practice, the images of a Mackey 2-functor are generally richer than bare (additive) categories. We will present in this section one example exploiting this richer structure and to recover the classical Cartan-Eilenberg formula; the general process should be applicable to many more p-monadic Mackey 2-functors.

We consider the Mackey 2-functors of the derived categories

$$\mathbb{M} = \mathrm{D}(\Bbbk -) : \mathbf{gpd}^{\mathrm{f}} \to \mathbf{Cat}$$

where $\Bbbk$ is a field of characteristic p. Each $\mathrm{D}(\Bbbk kG)$ is a $\Bbbk$-linear category with $\mathbb{Z}$-graded Hom-sets and a distinguished object, the trivial module $\Bbbk$. They are the objects of the following 2-category.

4.2.17. Definition. The 2-category $\Bbbk\mathbf{Cat}_\bullet^{\mathrm{gr}}$ of $\Bbbk$-linear, graded and pointed categories is the 2-category with:

- *Objects:* The pairs $(C, \bullet_C)$ consisting of a $\Bbbk$-linear, $\mathbb{Z}$-graded category C and an object $\bullet_C$ of C.
- 1-*Morphisms* $(C, \bullet_C) \to (D, \bullet_D)$*:* the pairs (f, ϕ) consisting of a $\Bbbk$-linear functor F which preserves the graduation and an isomorphism $\phi\colon F(\bullet_C) \to \bullet_D$ of degree 0.
- 2-*Morphisms* $(F, \phi) \to (G, \gamma)$*:* the natural transformations $\alpha\colon F \to G$, with each component of degree 0, such that the following square commutes:

$$\begin{array}{ccc} F(\bullet_C) & \xrightarrow{\phi} & \bullet_D \\ \big\downarrow{\scriptstyle \alpha_{\bullet_C}} & & \big\| \\ G(\bullet_C) & \xrightarrow[\gamma]{} & \bullet_D \end{array}$$

The 2-category $\Bbbk\mathbf{Cat}_\bullet^{\mathrm{gr}}$ is endowed with a bilimit-reflecting 2-functor

$$\mathbb{W}\colon \Bbbk\mathbf{Cat}_\bullet^{\mathrm{gr}} \to \mathbf{Cat}$$

Our Mackey 2-functor $\mathbb{M} = \mathrm{D}(\Bbbk -)$ actually factors through $\mathbb{W}$:

$$\mathbb{M}\colon \mathbf{gpd}^{\mathrm{f}} \xrightarrow{\tilde{\mathbb{M}}} \Bbbk\mathbf{Cat}_\bullet^{\mathrm{gr}} \xrightarrow{\mathbb{W}} \mathbf{Cat}$$

Since $\mathbb{W}$ reflects bilimits, we can lift the categorified Cartan-Eilenberg formula (theorem 4.2.12) to $\mathbb{M}$. For any group G with a p-Sylow S,

$$\tilde{\mathbb{M}}(G) \simeq \operatorname*{bilim}_{\hat{\mathcal{T}}_S(G)^{\mathrm{op}}} \tilde{\mathbb{M}} \circ \mathbb{U} \tag{4.2.18}$$

Moreover there is a bilimit-preserving 2-functor to the 1-category of $\Bbbk$-linear and graded algebras:

$$\mathbb{H}\colon \begin{cases} \Bbbk\mathbf{Cat}_\bullet^{\mathrm{gr}} & \to & \Bbbk\mathrm{Alg}^{\mathrm{gr}} \\ (C, \bullet_C) & \mapsto & C^*(\bullet_C, \bullet_C) \\ (F, \phi) & \mapsto & C^*(\bullet_C, \bullet_C) \xrightarrow{F} D^*(F(\bullet_C), F(\bullet_C)) \xrightarrow{D^*(\phi^{-1}, \phi)} D^*(\bullet_D, \bullet_D) \\ \alpha & \mapsto & \mathrm{Id} \end{cases}$$

Hence applying $\mathbb{H}$ to eq. (4.2.18), we recover the classical Cartan-Eilenberg formula:

$$\begin{aligned} H^*(G; \Bbbk) &= (\mathbb{H} \circ \tilde{\mathbb{M}})(G) \\ &\simeq \mathbb{H}\big(\operatorname*{bilim}_{\hat{\mathcal{T}}_S(G)^{\mathrm{op}}} \tilde{\mathbb{M}}\mathbb{U}\big) \\ &\simeq \operatorname*{bilim}_{\hat{\mathcal{T}}_S(G)^{\mathrm{op}}} \mathbb{H}\tilde{\mathbb{M}}\mathbb{U} \\ &\simeq \lim_{P \in \mathcal{T}_S(G)^{\mathrm{op}}} H^*(P; \Bbbk) \end{aligned}$$

4.2.19. Remark. This example relies essentially on two properties:

- The 2-functor $\mathbb{W}\colon \Bbbk\mathbf{Cat}_\bullet^{\mathrm{gr}} \to \mathbf{Cat}$ reflects bilimits, allowing us to lift the categorified Cartan-Eilenberg formula.
- The 2-functor $\mathbb{H}\colon \Bbbk\mathbf{Cat}_\bullet^{\mathrm{gr}} \to \Bbbk\mathrm{Alg}^{\mathrm{gr}}$ preserves bilimits

These properties precisely outline the limitations of the process:

- It is quite common for Mackey 2-functors to land in categories with a triangulated structure. Unfortunately, the forgetful 2-functor $\mathbf{Cat}^{\mathrm{tr}} \to \mathbf{Cat}$, from the triangulated categories (with a reasonable structure of 2-category) to the categories, does not reflect bilimits. Hence we cannot apply the previous discussion to exploit the triangulated structure. A potential solution would to consider a more well-behaved (but still related) 2-category, such as the 2-category of derivators.
- Another common invariant is the Picard group associated to a symmetrical monoidal category. The 2-functor $\mathbf{Cat}^{\otimes,\mathrm{sym}} \to \mathbf{Cat}$, from symmetrical monoidal categories to categories, reflects bilimits, so there is a categorified Cartan-Eilenberg formula in $\mathbf{Cat}^{\otimes,\mathrm{sym}}$ for p-monadic Mackey 2-functors taking value in symmetrical monoidal categories ($\mathrm{Mod}(\Bbbk-)$, $\mathrm{stMod}(\Bbbk-)$). However, the Picard 2-functor, which maps a symmetrical monoidal category to the abelian group of its invertible objects modulo isomorphisms, does not preserve bilimits. Still, the 2-Picard 2-functor, mapping a symmetrical monoidal category to the 2-group of its invertible objects (without taking the quotient by the isomorphisms), does preserve bilimits.

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