Université des Sciences et Technologies de Lille
Laboratoire de Mécanique de Lille - UMR CNRS 8107

MEMOIRE D'HABILITATION A DIRIGER DES RECHERCHES
Spécialité: MECANIQUE

Présenté par **MARIUS BULIGA**

Outils géométriques dans l'étude des grandes déformations, de l'endommagement et de la mécanique non régulière

Rapport d'activités

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Pour Claudia, Matei, Ioan

Remerciements

Je voudrais commencer par rendre hommage à mes professeurs roumains qui, par leurs exemples personnels et par leurs cours de mathématiques et de mécanique, m'ont donné envie de faire ce métier, notamment à Eugen Sóos et Alexandru Popescu-Zorica.

Je dois sans doute l'entrée dans le monde de la recherche actuelle à la coïncidence de l'accès à la bibliothèque de l'Ecole Polytechnique de Palaiseau et du début de la grande aventure de l'Internet.

Je remercie ensuite Tudor Raţiu pour l'opportunité qu'il m'a donné de passer une période de travail longue et intense au Département de Mathématiques de l'Ecole Polytechnique Fédérale de Lausanne. Beaucoup des résultats présentés dans ce mémoire ont germé au bord du Leman.

Ce mémoire a été écrit suite à l'insistance de Géry de Saxcé. Je le remercie pour son soutien et j'espère que notre collaboration continuera longtemps. Je tiens également à remercier Claude Vallée pour les nombreuses discussions fertiles et aussi pour l'occasion qu'il m'a accordée de participer à plusieurs éditions du Colloque International de Théories Variationnelles qui sont chères à ma mémoire.

Je suis très honoré que Pierre Alart, Ioan Ionescu et Tudor Raţiu aient accepté de rapporter mon travail. Je remercie également Olivier Allix, Djimedo Kondo, Claude Vallée et Géry de Saxcé pour avoir accepté de faire partie de mon jury.

Ma plus grande reconnaissance va à ma famille – Claudia, Matei et Ioan – pour les nombreuses façons par lesquelles ils ont changé ma vie pour toujours.

1 Introduction

La synthèse des travaux de recherche qui suit est basée sur une sélection d'articles groupés en quatre thèmes de recherche:

1. Mécanique de la rupture fragile (3 articles)
2. Bipotentiels (2 articles)
3. Calcul des variations, élasticité, quasiconvexité (2 articles)
4. Géométrie sub-riemannienne et structures de dilatation (5 articles)

Des douze articles présentés ici,à cet instant 4 sont publiés, 1 accepté pour publication et 7 sont soumis à publication. Parmi les articles soumis à publication se trouve un qui a été cité plusieurs fois.

J'ai choisi ces articles à partir d'une liste de 32 articles: 11 publiés, 8 soumis à publication, 14 e-prints arXiv non publiés. Cette distribution inhabituelle vient de mon choix de mettre mon travail sur les arXiv, la plus grande collection des article en ligne, qui représente un futur probable pour la communication des recherches en mathématiques. 3 des articles publies, et tous les articles soumis à publications sont des e-prints arXiv. Parmi les 23 e-prints il y a 4 qui sont cités plusieurs fois.

Toutefois, à partir de 2006 j'ai décide de retourner à la façon habituelle de publier, motivé principalement par des raisons pratiques. L'année dernière, j'ai soumis 9 articles pour publication (de ces 9 j'ai soumis 8 aux arXiv, donc il s'agit des articles nouveaux); 3 ont été acceptés pour publication et pour les 6 autres j'attends le résultat final. J'envisage dans le futur proche de soumettre les autres articles qui se trouvent dans les arXiv, sous la forme présente ou sous une forme nouvelle, en accord avec mes points de vue actuels.

Chaque des quatre thèmes de recherche commence par un article publié et continue par des article nouveaux.

La liste des articles publiés qui ne seront pas présentés ici est la suivante:

[19] M. Buliga, Topological Substratum of the Derivative, *Stud. Cerc. Mat. (Mathematical Reports)*, 45, **6**, 453-465, (1993)

[9] P. Ballard, M. Buliga, A. Constantinescu, Reconstruction d'un champ de contraintes résiduelles à partir des contraintes mesurées sur des surfaces successives. Existence et unicité. *C. R. Acad. Sci., Paris*, Sér. II 319, No.10, 1117-1122 (1994)

[20] M. Buliga, On Special Relativistic Approach to Large Deformations in Continuous Media, *Rev. Roum. de Math. Pures et Appl.*, t. XLI, **1-2**, 5-15, (1996)

[22] M. Buliga, Geometric evolution problem and action-measures, in: Proceedings of PAMM Conference PC 122, Constanta 1998, Tech. Univ. Budapest, (1998)

[23] M. Buliga, Brittle crack propagation based on an optimal energy balance, *Rev. Roum. des Math. Pures et Appl.*, **45**, no. 2, 201–209 (2001)

[97] G. de Saxcé, M Buliga, C. Vallée, C. Lerintiu, Construction of a bipotential for a multivalued constitutive law, PAMM, **6** , 1 (December 2006), Special Issue: GAMM Annual Meeting 2006 - Berlin

[28] M. Buliga, Vrănceanu' nonholonomic spaces from the viewpoint of distance geometry, (in romanian, original title: Spaţiile neolonome ale lui Vrănceanu din punctul de vedere al geometriei distanţei), to appear in *Revista Fundaţiei Acad. Prof. Gh. Vrănceanu*, (2007)

2 Curriculum vitae

2.1 État civil

Nom patronymique: Buliga

Prenoms: Marius, Luchian.

Date et lieu de naissance: 22 novembre 1967, Bucarest, Roumanie.

Nationalité: Roumaine.

Situation de famille: marié, deux enfants.

Adresse personnelle: Str. Antiaeriana 115, bl. A1, sc. 11, ap. 139, sector 5, Bucarest, Roumanie

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2.2 Fonction

Chargé de Recherche, Département de Mécanique de Milieux Continus, Institut de Mathématiques de l'Académie Roumaine, Bucarest (Roumanie)

Établissement actuel: Institut de Mathématiques de l'Académie Roumaine,

Adresse PO-BOX 1-764, 014700, Bucarest, Roumanie

mél: Marius.Buliga@imar.ro

Page web: (vivement conseillé pour une image détaillé de mes travaux)

`http://www.imar.ro/~mbuliga/`

2.3 Diplômes, qualifications, titres étrangers

1997 – Thèse de Doctorat en Mathématiques de l'Institut de Mathématiques, Académie Roumaine, intitulée: "Formulations variationnelles en mécanique de la fissuration fragile", Directeur de thèse Eugen Sóos.

1995 – Diplôme de master (DEA) (mécanique non linéaire) de l' École Polytechnique, Paris, dissertation intitulée: "Modélisation de la décohésion d'interface fibres-matrice dans les matériaux composites".

1994 – Diplôme d'auditeur, majeure Science de l'Ingénieur, École Polytechnique, Paris. Titre de la dissertation: "Reconstruction d'un champ de contraintes résiduelles à partir des contraintes mesurées sur des surfaces successives".

1992 – Diplôme de mathématicien , Faculté de Mathématiques, Université de Bucarest. Titre de la dissertation: ”Le contenu topologique de la différentiabilité”.

2.4 Vie professionnelle

2000 – *Chargé de Recherche*, Département de Mécanique de Milieux Continus, Institut de Mathématiques de l'Académie Roumaine, Bucarest (Roumanie)

2001 – 2006 *Chercheur invité, post-doctorant*, Chaire d'Analyse Géométrique, Institut de Mathématiques B, École Polytechnique Fédérale de Lausanne (Suisse).

1998 – *Chercheur*, Institut de Mathématiques, Département de Mécanique des Milieux Continus, Académie Roumaine

1997-2000 – *Professeur Associé*, Faculté de Mathématiques, Université Hyperion, Bucarest

1995 – *Assistant de Recherche*, Institut de Mathématiques, Département de Mécanique des Milieux Continus, Académie Roumaine

1993 – *Assistant de Recherche*, Faculté de Mathématiques, Université de Bucarest, et l'Institut de Génie Civil

1992-1993 – *Professeur d'Informatique*, Lycée Petru Poni , Bucarest (Sept. 1992 - Feb. 1993)

Chercheur invité

2000 Département de Mathématiques, École Polytechnique Fédérale de Lausanne, pour 3 mois

2004 Institut des Hautes Études Scientifiques, Nov. 2004

2005 Laboratoire de Mécanique de Lille - UMR CNRS 8107, pour un mois

2.5 Activités de recherche

3 contrats de recherche sur fonds nationaux et internationaux

Grandes orientations: Mes recherches sont consacrées à la Mécanique des Solides dans ses aspects théoriques et numériques, et à l'Analyse Géométrique des espaces métriques. Les thèmes abordés concernent:

La Mécanique de la rupture fragile: une formulation rigoureuse de la *rupture fragile* basée sur la *fonctionnelle de Mumford-Shah* et sur des techniques d'analyse géométrique.

Des fissures avec des formes complexes peuvent apparaître et se propager sans prescription sur leur géométries. J'ai montré que une théorie basée seulement sur la fonctionnelle Mumford-Shah conduit à des résultats non raisonnables du point de vue mécanique.

En conséquence j'ai proposé des critères de propagation fragile des fissures qui généralisent les critères bien connues de Griffith et Irwin. Le travail est illustré par des résultats numériques.

Élasticité non linéaire, Convexité et Calcul de Variations: des études sur les propriétés de *quasiconvexité* d'un potentiel élastique non linéaire.

Par un résultat classique de Morrey, les propriétés de continuité de certaines fonctionnelles sur des espaces de Sobolev sont en relation avec la quasiconvexité de la fonction potentiel w. Le cas des espaces de Sobolev des fonctions $u : \Omega \subset R^n \to R^n$ est très intéressant pour la mécanique des milieux continus.

En élasticité le potentiel w n'est pas en général défini sur un espace vectoriel de matrices $n \times n$ (l'algèbre de Lie $gl(n, R)$) mais sur l'ensemble des matrices $n \times n$ avec déterminant positif (le groupe de Lie $GL(n, R)$). Parfois la fonction w est définie sur un sous-groupe, comme c'est le cas de l'élasticité incompressible, ou il faut considérer le groupe des matrices de déterminant 1, i.e. le groupe $SL(n, R)$. Pour n pair , un autre groupe qui attire l'attention est $Sp(n, R)$, le groupe des matrices symplectiques. Il est donc intéressant de trouver des conditions nécessaires et/ou suffisantes pour l'inférieure semi-continuité de la fonctionnelle et de notions (qui restent à trouver) de convexité du $w : G \to \mathbb{R}$, avec G un groupe de matrices.

Très peu est connu sur la continuité des intégrales associées aux potentiels diff-quasiconvex. Il est cependant facile à prouver que la diff-quasiconvexité, introduite par Giaquinta, Modica, Soucek est une condition nécessaire.

J'ai introduit la notion de *quasiconvexité multiplicative* qui est mieux adaptée au cas de *grandes déformations*, quand il est important de prendre en compte le fait que les déformations sont *invertibles*.

J'ai également étudié l'importance de la convexité de Schur pour le comportement des matériaux élastiques isotropes, notamment les élastomères nematiques.

Bipotentiels: Les lois constitutives des materiaux standard sont en général multivaluées et également associées. Le graphe d'une telle loi constitutive est inclus dans le graphe de la sous-differentiale d'un super-potentiel ϕ (qui est aussi inférieur semi-continu). La loi constitutive prend alors la forme d'une inclusion différentielle, $y \in \partial\phi(x)$.

Cependant, certaines des lois constitutives sont non-associées. Elles ne peuvent pas être traités dans le cadre des matériaux standard. Pour contourner ce problème, une réponse possible, proposée d'abord par Gery de Saxcé, consiste en construisant une fonction b (le bipotential) avec deux variables, bi-convexe, qui satisfait une inégalité généralisant celle de Fenchel. Physiquement, le bipotential représente la dissipation.

En collaboration avec G. de Saxcé et C. Vallée, nous sommes en train de donner une base rigureuse du point de vue mathématique à la théorie du bipotentiel. Nous utilisons pour cela des outils de l'analyse convexe et de géométrie symplectique.

Géométrie sub-riemannienne: La géométrie sub-riemannienne, ou de Carnot-Carathéodory, ou encore géométrie non-holonome, est un sujet de recherche en contact avec plusieurs domaines, notamment: l'analyse des opérateurs hypoelliptiques, théorie du contrôle, l'analyse dans les espaces métriques mesurés. Parmi les principaux contributeurs à ce sujet on compte Hörmander, Gromov, Cheeger , Folland, Stein, Margulis, Mostow.

La géométrie intrinsèque des liaisons non holonomes n'est pas riemannienne, ce qui engendre le besoin d'adapter les outils d'analyse non linéaire. Cela est possible grâce à des résultats nouveaux d'analyse géométrique dans espaces métriques plus générales que les espaces riemanniens.

Mon interet pour le sujet de recherche de la géométrie sub-riemannienne a commencé pendant le temps passé au département de mathématiques de l'EPFL. J'ai proposé et étudié la notion de structure de dilatations, les proprietés de courbure d'une varieté sub-riemannienne et quelques applications dans la mécanique hamiltonienne.

Il semble maintenant que les structures de dilatation sont intéressantes par elles-mêmes, avec un champ d'applications possibles contenant strictement la géométrie sub-riemannienne, mais aussi les espaces ultramétriques ou les groupes contractibles.

Autres: un problème de reconstruction de contraintes résiduelles dans un solide élastique, mesurées par diffraction des rayons X, après un enlèvement de matière qui induit la redistribution des contraintes, et donc le tenseur mesuré est différent du tenseur original.

Un étude de la méthode de Rougée pour la description des propriétés locales d'un milieu continu, appliquée pour un milieu continu en relativité restreinte.

2.6 Activités pédagogiques

6 enseignements de cours et/ou travaux dirigés

2 fascicules de cours polycopiés rédigés

1 support de cours en forme électronique (page web)

Description de charges

1993-1994- cours de Théorie de la Relativité , 4^{eme} semestre, cours, 28 h/semestre, pendant deux ans, suite à l'initiative de Prof. L. Beju qui a proposé d'expérimenter l'enseignement de ce cours en deuxième année d'études, Faculté de Mathématiques, Université de Bucarest.

1993-1994- travaux dirigés de l'Introduction à l'Informatique 1^{er}semestre, t.d. 56 h/semestre, l'Institut de Génie Civil, Bucarest

1995-1999- cours et t.d. de Mécanique des Milieux Continus pour mathématiciens, 7^{eme} semestre, cours 28 h/semestre, t.d. 140 h/semestre (2h de t.d./semaine pour 5 groupes), à la Faculté de Mathématiques, Université de Bucarest.

1997-1999- cours et t.d. de Géométrie linéaire, deux premiers semestrs, cours 84 h/an, t.d. 168 h/an, à la Faculté de Mathématiques, Université Hyperion, Bucarest

1997-1999- cours et t.d. de Géométrie Différentielle, les 2 semestres suivants, cours 84 h/an, t.d. 168 h/an, à la Faculté de Mathématiques, Université Hyperion, Bucarest.

2001-2006 t.d. Analyse I et II, t.d. 108 h/an, pour le cours de T. Ratiu à l'EPFL

2001-2006 Collabore occasionnellement à l'enseignement du cours d' Analyse I et II à l'EPFL.

Créations d'enseignement

1998- Cours avancé "Problèmes à discontinuité libre", Institut de Mathématiques de l'Académie Roumaine

1999- Cours avancé "Méthodes énergétiques en mécanique de la fissure fragile", Institut de Mathématiques de l'Académie Roumaine

2001-2002- avec T. Ratiu, séminaire de travail "Sub-Riemannian geometry and Lie groups", EPFL

2003- un des organisateurs du Séminaire Borel 2003, dans le cadre du 3ème Cycle Romand de Mathématiques, intitulé "Tangent spaces of metric spaces", Université de Berne (Suisse), (2 exposés en 12 semaines)

Cours polycopiés et support de cours

- Rédaction de notes du cours "Théorie de la Relativité restreinte" (1993).
- Rédaction de notes du cours et t.d. de "Géométrie linéaire" (1998).
- Rédaction de notes du cours et t.d. de "Géométrie Différentielle" (1998).
- support informatique pour les t.d. de Analyse I, II, EPFL (2003-2006)

2.7 Travaux. Articles. Réalisations

Publications dans des revues avec comité de lecture

1. M. Buliga, Topological Substratum of the Derivative, *Stud. Cerc. Mat. (Mathematical Reports)*, 45, **6**, 453-465, (1993)

2. P. Ballard, M. Buliga, A. Constantinescu, Reconstruction d'un champ de contraintes résiduelles à partir des contraintes mesurées sur des surfaces successives. Existence et unicité. *C. R. Acad. Sci., Paris*, Sér. II 319, No.10, 1117-1122 (1994)

3. M. Buliga, On Special Relativistic Approach to Large Deformations in Continuous Media, *Rev. Roum. de Math. Pures et Appl.*, t. XLI, **1-2**, 5-15, (1996)

9 cit. - 4. M. Buliga, Energy Minimizing Brittle Crack Propagation, *J. of Elasticity*, **52**, 3, 201-238, (1999)

3 cit. - 5. M. Buliga, Brittle crack propagation based on an optimal energy balance, *Rev. Roum. des Math. Pures et Appl.*, **45**, no. 2, 201–209 (2001)

1 cit. - 6. M. Buliga, Lower semi-continuity of integrals with G-quasiconvex potential, *Z. Angew. Math. Phys.*,bf 53, 6, 949-961, (2002)

7. M. Buliga, G. de Saxcé, C. Vallée, Existence and construction of bipotentials for graphs of multivalued laws, *J. of Convex Analysis*, **15**, 1, (2008)

8. M. Buliga, Dilatation structures I. Fundamentals, *J. Gen. Lie Theory Appl.*, Vol **1** (2007), No. 2, 65-95

Publications acceptées dans une revue à comité de lecture

9. M. Buliga, Equilibrium and absolute minimal solutions of brittle fracture models based on energy-minimization methods, *Int. Journal of Fracture*, (2007)

10. M. Buliga, Vrănceanu' nonholonomic spaces from the viewpoint of distance geometry, (en roumain, titre original: Spaţiile neolonome ale lui Vrănceanu din punctul de vedere al geometriei distanţei), *Revista Fundaţiei Acad. Prof. Gh. Vrănceanu*, (2007)

Communication à des congrès et colloques avec actes publiés

11. G. de Saxcé, M Buliga, C. Vallée, C. Lerintiu, Construction of a bipotential for a multivalued constitutive law, *Proc. Appl. Math. Mech.*, vol. **6** , no. 1 (2006), 153-154

12. M. Buliga, Geometric evolution problems and action-measures, *PAMM Appl. Math. Bull.*, vol. **LXXXVI** (1998), T. U. Budapest, 53-58

Articles soumises au publication

13. M. Buliga, G. de Saxcé, C. Vallée, Construction of bipotentials and a minimax theorem of Fan, (submitted), http://arxiv.org/abs/math.FA/0610136, (2006)

14. M. Buliga, Contractible groups and linear dilatation structures, (submitted) http://xxx.arxiv.org/abs/0705.1440, (2007)

15. M. Buliga, Linear dilatation structures and inverse semigroups, (submitted) http://xxx.arxiv.org/abs/0705.4009, (2007)

16. M. Buliga, Microfractured media with a scale and Mumford-Shah energies, (submitted)
http://xxx.arxiv.org/abs/0704.3791, (2007)

17. M. Buliga, Four applications of majorization to convexity in the calculus of variations, (submitted) (2007)

18. M. Buliga, Dilatation structures in sub-riemannian geometry, (submitted) http://arxiv.org/abs/0708.4298, (2007)

19. M. Buliga, Self-similar dilatation structures and automata, (submitted), (2007),

20. M. Buliga, Dilatation structures with the Radon-Nikodym property, (submitted) http://arxiv.org/abs/0706.3644, (2007)

Publications électroniques

2 cit. - 21. M. Buliga, Majorisation with applications to the calculus of variations, http://arxiv.org/abs/math.FA/0105044, (2001), see also the updated version "Four applications of majorization to convexity in the calculus of variations"

1 cit. - 22. M. Buliga, Sub-Riemannian geometry and Lie groups. Part I, http://arxiv.org/abs/math.MG/0210189, (2002)

1 cit. - 23. M. Buliga, Symplectic, Hofer and sub-Riemannian geometry, http://arxiv.org/abs/math.SG/0201107, (2002)

24. M. Buliga, Volume preserving bi-Lipschitz homeomorphisms on the Heisenberg group, http://arxiv.org/abs/math.SG/0205039, (2002)

25. M. Buliga, Tangent bundles to sub-Riemannian groups, http://arxiv.org/abs/math.MG/0307342, (2003)

26. M. Buliga, Curvature of sub-Riemannian spaces, http://arxiv.org/abs/math.MG/0311482, (2003)

27. M. Buliga, Sub-Riemannian geometry and Lie groups. Part II. Curvature of metric spaces, coadjoint orbits and associated representations, http://arxiv.org/abs/math.MG/0407099, (2004)

28. M. Buliga, Energy concentration and brittle crack propagation, http://arxiv.org/abs/math.AP/0510225, (2005)

29. M. Buliga, Quasiconvexity versus group invariance, http://arxiv.org/abs/math.AP/0511235, (2005)

30. M. Buliga, Perturbed area functionals and brittle damage mechanics, http://arxiv.org/abs/math.AP/0511240, (2005)

31. M. Buliga, Energy minimizing brittle crack propagation II, http://arxiv.org/abs/math.AP/0511301, (2005)

32. M. Buliga, The variational complex of a diffeomorphisms group, http://arxiv.org/abs/math.AP/0511302, (2005)

33. M. Buliga, Dilatation structures II. Linearity, self-similarity and the Cantor set, (2006), http://xxx.arxiv.org/abs/math.MG/0612509

34. M. Buliga, On the Kirchheim-Magnani counterexample to metric differentiability, http://arxiv.org/abs/0710.1350, (2007)

Preprints, mémoires, dissertations, notes

1. *Lower semicontinuity of variational integrals defined on groups of diffeomorphisms*, IMAR preprint 17/1998

2. *Variational Formulations in brittle fracture mechanics* (in Romanian), PhD Thesis, Institute of Mathematics of the Romanian Academy, 1997

3. *Modélisation de la décohesion d'interface fibres-matrice dans les matériaux composites*, mémoire de D.E.A., École Polytechnique, 1995

4. *Energetic criteria in fracture mechanics*, scientific report, grant MCT-ANSTI 627/1998-1999

2.8 Communications à des congrès et colloques

1992- Lie-Lobacevski Symposium, Université de Bucharest,

1996- Conférences Nationales de Mécanique des Solides (Roumanie), Constanta 1996, Iasi 1997

1996- Differential Equations and Calculus of Variations, summer school and workshop, Pisa, 1996

1999- M. Buliga, Energetic criterions in brittle fracture mechanics, The Fourth International Congress on Industrial and Applied Mathematics (ICIAM 99), 1999

1999- Applied Analysis and Mechanics Seminars, Hilary Term 1999, Mathematical Institute, Oxford, "Quasiconvexity versus group invariance", invited by J.M. Ball

1999- Scuola Internazionale Superiore di Studi Avanzati, Trieste, "The variational complex of a diffeomophisms group", invited by A. Braides

2000- Département de Mathématiques, École Polytechnique Fédérale de Lausanne, "Variational rigidity", invited by T. Ratiu

2002- Mathematical Institute, University of Bern, "Towards rectifiability in Carnot groups: a theory of irreducible representations of volume preserving bi-Lipschitz homeomorphisms", invited by M. Reimann

2004- Centre Bernoulli, École Polytechnique Fédérale de Lausanne, "A claim about Hamiltonian mechanics", invited by T. Ratiu

2004- Mathematical Institute, University of Bern, "Metric profiles and Mitchell theorem 1", invited by M. Reimann

2004- IMA - EPFL , "Majorisation and multiplicative quasiconvexity", invited by B. Dacorogna

2004- Journées d'automne de la Société Mathématique Suisse, "Curvature of metric profiles"

2004- Universität Stuttgart, Fakultät Mathematik und Physik, "Convexity notions, groups and nonlinear elasticity" invited by A. Mielke.

2005- Mathematical Institute, University of Bern, "Differential structures for sub-Riemannian spaces", invited by M. Reimann.

2005- Centre de Mathématiques et d'Informatique, Université de Provence, Séminaire de Géométrie et Singularités, "Flots hamiltoniens d'isométries" invited by B. Kolev.

2005- Laboratoire de Mécanique de Lille, "Un test pour les critères énérgetiques de rupture", invited by G. De Saxcé.

2006- GAMM 2006, Berlin, Germany, G. De Saxcé, M. Buliga, C. Vallée, C. Lerinţiu, Construction of a bipotential for a multivalued constitutive law

2006- 8-ème Colloque Franco-Roumain de Mathématiques Appliquées, Chambéry, "Convexité de Schur et élastomères nématiques"

2006- Geometric and Asymptotic Group Theory with Applications, Manresa,Spain, "Dilatation Structures"

2006- December Monthly Conference of the Institute of Mathemathics of the Romanian Academy, Bucharest, "Travelling salesman through fractals"

2007- Geometric linearization of graphs and groups, January 22-26, 2007, Centre Interfacultaire Bernoulli, EPFL, Lausanne, Switzerland, "Dilatation structures and linearization of self-similar actions"

2007- International Symposium on Defect and Material Mechanics, March 25-29, 2007 - Aussois, France, "Fracture fattening and energy release rates"

2007- 6-th Congress of Romanian Mathematicians, June 28 - July 4, 2007 - Bucharest, Romania, Section: Theoretical Computer Science, Operations Research and Mathematical Programming, "Self-similar dilatation structures and automata"

2007- Viertes Deutsch-Rumänisches Seminar über Geometrie Dortmund, 15-18 July 2007, "Linear dilatation structures and conical groups"

2007- The eight international workshop on differential geometry and its applications, August 19-25, 2007, "Babeş-Bolyai" University, Cluj-Napoca, Romania, "nonholonomic spaces and geometric group theory"

2.9 Rapports de contrats

- M. Buliga, Energetic criteria in fracture mechanics, rapport scientifique, contrat de recherche du Ministère de la Recherche et des Technologies (Roumanie), MCT-ANSTI 627/1998-1999

2.10 Participation à des comités scientifiques des congrès et colloques

- j'ai fait partie des organisateurs du Séminaire Borel 2003, 3ème Cycle Romand de Mathématiques, intitulé "Tangent spaces of metric spaces", Université de Berne (Suisse).

2.11 Activités administratives et autres responsabilités collectives

- grant owner MCT-ANSTI 627/1998-1999 "Energetic Criteria in Fracture Mechanics"

- grant member CEEX06-11-12/2006

- représentant de la partie roumaine, dans le programme SCOPES, financé par le Fond National Suisse de Recherche, "Lausanne-Bucharest common project on topology, geometry and mechanics", (2001-2005).

2.12 Citations par les auteurs d'autres publications

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3 Résumé des activités

Mes recherches sont consacrées à la Mécanique des Solides dans ses aspects théoriques et numériques, et à l'Analyse Géométrique des espaces métriques. J'ai donc une double spécialisation, comme mécanicien et mathématicien, qui s'explique par mon parcours et par l'organisation des études universitaires en Roumanie pendant les années '80 et '90.

a. Études en Roumanie (1987-1994)

En Roumanie j'ai suivi des études de la section de Mécanique des Solides de la Faculté de Mathématiques, Université de Bucarest. Après 2 années d' études générales de mathématique, les membres de la section de Mécanique ont suivi pendant 3 ans des cours approfondis en e.d.p, calcul de variations, analyse fonctionnelle, mais aussi des cours d'élasticité, plasticité, rhéologie, fluides, ... La période totale des études était de 5 ans (comparée à 4 ans pour les autres sections de la Faculté de Mathématique) et les étudiants de cette section étaient préparés pour faire ensuite de la recherche.

J'ai suivi ce double enseignement, mathématique et mécanique, par passion de la mathématique mais aussi parce que j'ai été toujours fasciné par la physique. J'ai eu la chance de faire partie d'un petit groupe des étudiants sélectionnés par le professeur Eugen Sóos depuis le commencement des études. En plus de cet enseignement non standard, nous avons bénéficié des cours supplémentaires, donnés par les meilleurs rechercheurs roumains, ex-membres de l'Institut de Mathématiques de l'Académie Roumaine, détruit par les autorités communistes. *Pour nous, la distinction entre la mécanique et les mathématiques est artificielle.*

Après la révolution roumaine, toutes les institutions ont subi des grandes transformations. La deuxième partie de mes études en Roumanie est faite dans des conditions meilleures (l'Institut a été refait), mais dans une atmosphère de transition et de confusion du point de vue de l'organisation. Suite à 5 années d'études j'ai reçu une diplôme de mathématicien. En présent, pour les mêmes études l'Université de Bucarest offre une diplôme de master. Je sais maintenant qu'à cette époque, les connaissances des étudiants roumains sur le système d'enseignement dans l'Union Européenne étaient très minces. D'ailleurs, il me semble que la situation était symétrique de l'autre côté du rideau de fer.

Après la fin des études universitaires, j'ai travaillé une première période comme professeur d'informatique dans un lycée de Bucarest. Peu après j'ai eu la chance d'enseigner le cours de relativité restreinte à la Faculté de Mathématique, suite à la proposition du professeur Iulian Beju.

Mes domaines de recherche du début étaient les milieux continus en relativité restreinte et la topologie générale.

À la fin ce cette période, tous les membres du groupe selectioné par Sóos ont continué les études ailleurs, notamment en France et les États Unis.

b. Période parisienne (1994-1995)

Mon arrivée à l'École Polytechnique, comme élève auditeur (et après comme étudiant en D.E.A. de Mécanique non linéaire) m'a permis d'élargir mes horizons, en particulier à cause de l'accès à une grande bibliothèque de spécialité.

En collaboration avec Patrick Ballard et Andrei Constantinescu, nous avons résolu un problème inverse d'élasticité (détermination des contraintes résiduelles par mesure de contraintes sur de surfaces successives).

Pendant le stage de D.E.A. fait au LPMTM, Université Paris 13, j'ai eu l'occasion de m'initier au domaine naissant des formulations énergétiques en la mécanique de la rupture fragile. Le sujet de stage était de contribuer à la formulation d'un modèle de rupture fragile qui utilise la fonctionnelle Mumford-Shah. J'ai vite prouvé que un modèle basé exclusivement sur cette fonctionnelle (et sur une simple discrétisation temporelle) ne donne pas de bons résultats dans le cas d'apparition d'une fissure. J'ai proposé donc un modèle modifié, en utilisant des idées des problèmes inverses d'élasticité (que j'avais appris auparavant, pendant les études a l'École Polytechnique), notamment la fonction de Dirichlet-Neumann.

c. Deuxième période roumaine (1995-1999)

Pour des raisons personnelles, je retourne en Roumanie. Je m'inscrit en thèse, sur le sujet *"Formulations variationnelles en mécanique de la rupture fragile"*, Directeur de thèse: Eugen Sóos. Décidé d'apprendre plus sur le sujet de la fonctionnelle Mumford-Shah, je contacte par mél. le principal spécialiste, Luigi Ambrosio (ENS Pisa), qui a la gentillesse de m'envoyer ses articles et avec qui je noue pour une période une collaboration. À ma déception je ne trouve pas la même ouverture auprès de l'équipe française qui poursuit le même but: un modèle rigoureux de rupture fragile basé sur la fonctionnelle Mumford-Shah. Néanmoins, au but d'un an et demi, j'arrive à formuler un tel modèle, en tenant compte aussi des faiblesses que j'ai découvert auparavant. Trois publications et cinq publications électroniques sont ensuite dédiées à ce sujet, dont le premier est mentionné dans la monographie de référence de L. Ambrosio, N. Fusco, L. Pallara. J'ai obtenu un contrat de recherche de la part du Ministère de la Recherche et des Technologies Roumain sur le sujet des critères énergétiques en mécanique de la rupture fragile. Plus tard je commence à étudier avec E. Sóos et M. Craciun le critère de fissuration de Sih.

Après la thèse de doctorat (en mathématiques!), je deviens intéressé par l'élasticité non linéaire. Le raison est simple: trouver une extension du modèle de fissuration fragile pour des matériaux élastiques non linéaires. Je découvre vite que l'élasticité est un domaine intéressant en soi et j'essaye de trouver des conditions nécessaires et/ou suffisantes pour la semi-continuité inférieure de la fonctionnelle énergie élastique et des notions de convexité du potentiel élastique défini sur un groupe de matrices, au lieu d'un espace vectoriel. En deux articles j'ai introduit la bonne notion de quasiconvexité dans le sens variationnel, associée à un groupe d'homéomorphismes bi-Lipschitziens.

Suite à une invitation de la part de J.M. Ball j'ai eu l'occasion de présenter la notion de quasiconvexité multiplicative dans un exposé à l'Oxford Mathematical Institute. Au même temps je fais une série de visites à l'ENS (Pise) et SISSA (Trieste), ou j'ai l'occasion d'échanger des idées avec L. Ambrosio et A. Braides, sur le sujet de formulations énergétiques de la rupture fragile.

Par l'intermédiaire de E. Sóos je commence une collaboration avec C. Vallée (Poitiers) qui continue jusqu'à présent.

c. Période suisse (2000-2006)

Suite à une visite de T. Ratiu en Roumanie, j'ai l'occasion de discuter avec lui de l'analyse sur de groupes de difféomorphismes, domaine dans lequel Ratiu a démontré des résultats fondamentaux. Il est intéressé par les utilisations que j'ai donné en mécanique de la rupture et l'élasticité et il m'invite pour 3 mois dans sa chaire à l'EPFL. Après cette visite il m'offre la possibilité de passer quelques années à l'EPFL. D'une position d'invité, je suis devenu ensuite premier assistant.

À l'EPFL j'ai découvert les relations entre la convexité de Schur et les notions de convexité en élasticité, suite à une coïncidence : j'ai eu l'occasion d'apprendre plus, en même temps, sur la convexité de Schur (de la part de Ratiu) et sur sur la convexité de rang un, de la part de Dacorogna.

Ensuite, pendant 4 ans, je travaille sur le sujet de la géométrie sub-riemannienne, qui reste parmi mes principales thèmes de recherche. J'ai collaboré dans ce sujet avec l'équipe de M. Reimann (Berne), avec S. Vodop'yanov (Novosibirsk) et j'ai fait une visite à l'IHES pour discuter ce sujet.

J'ai organise avec Tudor Ratiu le séminaire de travail "Sub-Riemannian geometry and Lie groups", en 2001-2002, à l'EPFL. J'ai été parmi les organisateurs du séminaire Borel 2003, "Tangent spaces of metric spaces", ou j'ai présenté des constructions de fibrés tangents d'un group sub-riemannien.

La période de travail à l'EPFL signifie pour moi la liberté totale de recherche, sans soucis matériels. J'en ai profité pleinement.

Depuis 2004 j'ai recommence à étudier des sujets de mécanique de la rupture et d'élasticité, suite à la collaboration avec C. Vallée. La façon de modéliser l'évolution quasistatique d'une fissure a attiré l'attention de A. Mielke (WIAS), et j'ai eu l'occasion de collaborer avec lui à l'EPFL et en Allemagne. Enfin, une collaboration a été établie avec G. Oleaga (Madrid) sur le sujet de la mécanique de la rupture.

Plus tard j'ai rencontré G. De Saxcé (Lille-1), avec lequel nous avons travaillé sur une méthode de construction des bipotentiels.

d. Le présent (2006-2007)

Je suis retourné en Roumanie à partir de juillet 2006 et j'ai repris mon poste de chargé de recherche à l'Institut de Mathématiques de l'Académie Roumaine. Dès mon arrivée, je commencé une période de travail intense, en essayant de mettre sur le papier toute une suite d'idées plus ou moins développées pendant le stage en Suisse.

Depuis mon rencontre avec G. De Saxcé, le sujet des bipotentiels est parmi mes préoccupations. Trois articles ont été écrits en collaboration avec G. De Saxcé et C. Vallée. Deux sont publiés et le troisième attends le verdict final.

Deux autre articles soumis à publication portent sur la fissuration fragile. J'ai la satisfaction de voir que mes premiers essais en la matière semblent pointer dans une bonne direction. Dans le premier de ces deux articles, je propose une analyse rigoureuse des modèles de fissuration fragile basés sur la fonctionnelle Mumford-Shah. L'analyse porte sur des diffèrent façons de définir l'évolution quasistatique d'un corps élastique fissuré et sur les implications physiques des hypothèses mathématiques assez techniques. Dans le deuxième article je traite sur une méthode d'homogeneization non standard de la fonctionnelle Mumford-Shah, qui semble prédire correctement la concentration d'endommagement dans un milieu élastique périodique microfissuré. Les deux articles sont soumis à publication.

Enfin, j'ai commencé à jeter les bases de la géométrie sub-riemannienne à partir de la notion de structure de dilatation. Cette notion était deja esquissée dans des preprints arXiv de la période suisse. Maintenant je commence un étude poussé dans cette direction, et je produis 6 articles (un publié et 5 soumises à publication) sur le sujet. Je considère ce sujet de travail comme le plus important dans ma carrière jusqu'à cette date.

L'étendue de la notion de structure de dilatation est plus générale que la géométrie sub-riemannienne. J'ai établi aussi des connections avec l'analyse dans les corps ultramétriques et avec la théorie des automates. Dans le dernier an j'ai présenté ce travail lors de cinq exposés, en Espagne (dans le cadre d'une conférence satellite de l'ICM2006), en Suisse, en Allemagne et deux fois en Roumanie.

4 Mécanique de la rupture fragile et la fonctionnelle Mumford-Shah

4.1 Description du sujet

Une grande partie de la difficulté des problèmes de fissuration fragile consiste en la nature géométrique de la fissure. Les premiers contributions en ce domaine concernent surtout le comportement d'un materiau fragile. Parmi les références fondamentales se trouvent: Eshelby [54], Griffith [68], Irwin [60], Gurtin [72] [73], Rice [94].

Dans presque toutes les études, la géométrie de la fissures est a priori fixée. Parmi les peu nombreuses exceptions, on trouve les articles de Stumpf, Le [110], ou Ohtsuka [89] [90] [91] [92].

La géométrie de la fissure peut être prescrite de manière forte, comme dans le cas des fissures rectangulaires ou elliptiques, qui préservent leur forme le long de l'évolution. Une prescription faible de la géométrie de la fissure apparaît, par exemple, dans le cas d'une fissure unidimensionnelle qui évolue dans une configuration de référence bidimensionnelle, à la condition que la fissure soit à tout instant une courbe simple. Dans ce cas, l'évolution de la fissure est réduite au mouvement d'un point. Dans tous ces cas, la nature géométrique de la principale inconnue du problème – la fissure – n'est pas prise en compte.

Une nouvelle direction de recherche dans ce domaine commence avec l'article de Mumford et Shah [88], sur un problème de traitement d'image. Ce problème de segmentation de l'image, c.a.d. trouver l'ensemble des contours d'une image et construire une version plus simple de l'image en tenant compte de l'ensemble des contours, est similaire du point de vue mathématique au problème d'évolution quasistatique des fissures fragiles.

Dans l'article [88] Mumford et Shah proposent l'approche variationnel suivant pour le problème de segmentation d'image: soit $g : \Omega \subset \mathbb{R}^2 \rightarrow [0, 1]$ l'image initiale, comprise comme une distribution des niveaux de gris (1 c'est blanc et 0 c'est noir) sur le support Ω de l'image. Soit $g : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ l'image cherchée et $K \subset \Omega$ l'ensemble des contours. K est (contenu dans) l'ensemble des points ou la fonction u n'est pas continue, c.a.d. $u \in C^1(\Omega \setminus K, \mathbb{R})$. On cherche une paire (u, K) qui minimise la fonctionnelle

$$I(u, K) \ = \ \int_\Omega \alpha \mid \nabla u \mid^2 \ \mathrm{d}x + \int_\Omega \beta \mid u - g \mid^2 \ \mathrm{d}x + \gamma \mathcal{H}^1(K) \quad . \qquad (4.1.1)$$

Le paramètre α contrôle la régularité de la fonction u, le paramètre β contrôle la distance L^2 entre l'image u et l'image initiale g et le paramètre γ contrôle la longueur (mesure de Hausdorff 1D) de l'ensemble K des contours. Les auteurs remarquent que pour $\beta = 0$ la fonctionnelle I peut être utilisée pour un traitement énergétique du problème de fissuration fragile.

4.2 Contributions

Mes contributions dans ce sujet,

[21] M. Buliga, Energy Minimizing Brittle Crack Propagation, *J. of Elasticity*, **52**, 3, 201-238, (1999)

[22] M. Buliga, Geometric evolution problems and action-measures, *PAMM Appl. Math. Bull.*, vol. **LXXXVI** (1998), T. U. Budapest, 53-58

[23] M. Buliga, Brittle crack propagation based on an optimal energy balance, *Rev. Roum. des Math. Pures et Appl.*, **45**, no. 2, 201–209 (2001)

[29] M. Buliga, Equilibrium and absolute minimal solutions of brittle fracture models based on energy-minimization methods, (submitted), 2007

[33] M. Buliga, Microfractured media with a scale and Mumford-Shah energies, (submitted), http://xxx.arxiv.org/abs/0704.3791, (2007)

[41] M. Buliga, Energy concentration and brittle crack propagation, http://arxiv.org/abs/math.AP/0510225, (2005)

[43] M. Buliga, Perturbed area functionals and brittle damage mechanics, http://arxiv.org/abs/math.AP/0511240, (2005)

[44] M. Buliga, Energy minimizing brittle crack propagation II, http://arxiv.org/abs/math.AP/0511301, (2005)

sont liées à la propagation des fissures fragiles dans un milieu élastique.

Ma dissertation de D.E.A., "Modélisation de la décohésion d'interface fibre-matrice dans les matériaux composites", a été faite après un stage à LPMTM, Université Paris 13, 1995. À la fin du D.E.A., j'ai décidé de retourner en Roumaine, ou plus tard j'ai soutenu un doctorat en mathématiques (1997), à l'Institut de Mathématiques de l'Academie Roumaine, avec le titre "Thermo-mécanique de la rupture. Formulations variationnelles en mécanique de la rupture fragile".

4.3 Résumés des articles

4.3.1 Energy minimizing brittle crack propagation

Cet article porte sur la modélisation de la fissuration quasistatique d'un solide élastique fragile. Les hypothèses du travail sont les suivantes : le corps élastique linéaire, avec ou sans fissures initiales, évolue d'une manière quasistatique suite à l'action des déplacements imposés sur la frontière. Au cours de son évolution, des fissures de géométrie arbitraire peuvent apparaître et/ou se propager.

Dans ce qui suit, je présente un modèle d'apparition d'une fissure fragile suite au déplacement imposé sur la frontière d'un corps élastique.

L'état d'un corps fragile avec la configuration de référence Ω est décrit par une paire déplacement-fissure. $(\mathbf{u}, K)$ est une telle paire si:

(1) K est une fissure dans le corps, vue comme une surface quelconque,

(2) $\mathbf{u}$ est un déplacement du corps fissuré avec la fissure $K \subset \overline{\Omega}$, compatible avec le déplacement imposé sur la frontière $\mathbf{u}_0$, c.a.d. $\mathbf{u} \in C^1(\Omega \setminus K)$ et $\mathbf{u} = \mathbf{u}_0$ sur $\partial\Omega$.

L'énergie totale du corps dans l'état $(\mathbf{u}, K)$ a la forme d'une fonctionnelle Mumford-Shah:

$$E(\mathbf{u}, K; \mathbf{u}_0) = \int_\Omega w(\nabla \mathbf{u}) \, \mathrm{d}x + F(\mathbf{u}_0, K) \quad .$$

Le premier terme de la fonctionnelle E représente l'énergie élastique du corps soumis au déplacement $\mathbf{u}$. Le terme suivant représente l'énergie utilisée pour produire la fissure K dans le corps, avec le déplacement imposé $\mathbf{u}_0$ comme paramètre. Dans ce modèle l'apparition de la fissure est vue comme un problème d'équilibre.

Quand le déplacement $\mathbf{u}_0$ est imposé sur la frontière (extérieure) $\partial\Omega$, l'état $(\mathbf{v}, S)$ du corps fragile minimise l'énergie totale $E(\cdot, \cdot; \mathbf{u}_0)$. La fissure qui apparaît est S. Remarquez que S peut être aussi l'ensemble vide; dans ce cas le modèle nous dit que le corps soumis au déplacement $\mathbf{u}_0$ ne se fissure pas.

L'apparition des fissures fragiles et la segmentation de l'image sont deux problèmes à discontinuité libre. Les inconnus, la fissure ou la collection des contours, sont des surfaces (lignes) de discontinuité pour le déplacement ou pour l'image finale; leur position ou géométrie sont complètement libres.

Nous allons utiliser un approche énergétique du problème de l'évolution de la fissuration fragile, quasistatique. Nous allons discrétiser le temps et transformer le problème en une suite des problèmes de minimisation d'énergie. Francfort et Marigo [59] procèdent de la même manière dans le cas de l'endommagement fragile brutal. Pour bien formuler ce passage du discret au continu (par rapport au temps), nous allons utiliser le cadre des mouvements minimisants généralisés, introduit par De Giorgi [64]. Pour cela nous introduisons dans la Section 2 la notion de mouvement minimisant de l'énergie, comme un cas particulier d'un mouvement minimisant généralisé.

Dans la Section 3, après les préliminaires concernant la statique d'un corps fragile, nous présentons une variante du critère de Griffith pour la propagation d'une fissure fragile en Sous-section 3.3, comme un critère de sélection parmi les possibles évolutions de la fissure. A la fin de cette section nous formulons le problème d'évolution quasistatique d'une fissure fragile sous la forme (14).

Dans Subsection 4.1 est donnée une formulation de ce problème, en termes de mouvements minimisants d'énergie, en utilisant une fonctionnelle Mumford-Shah (Définition 4.1). Dans ce modèle, il y a une seule constante de materiau reliée à la fissuration: la constante G de Griffith. Quelques propriétés du modèle sont explorés dans la Sous-section 4.2, dans les cas antiplan et unidimensionnel. Nous prouvons que dans ce modèle l'apparition des fissures peut se produire. Dans la relation (23) se trouve l'expression du σ_c, la tension critique qui conduit à la rupture, d'après le modèle, dans le cas d'un expérience de traction unidimensionnelle. On déduit d'ici que σ_c et G ne peuvent pas être tous les deux des constantes de materiau dans ce modèle.

Dans la Section 5 est contenue la formulation faible du modèle introduit dans la Définition 4.1, discrétisé par rapport au temps. La Sous-section 5.1 traite des fonctions spéciales à variation ou déformation bornée. L'existence des solutions du modèle discrétisé (Définition 5.1, Théorème 5.3) est une conséquence des résultats dus aux De Giorgi et Ambrosio [65], Ambrosio [1] [2], Ambrosio, Coscia et Dal Maso [4]. Le cas antiplan est discuté en Sous-section 5.3. Nous comparons les solutions faibles (Définition 5.1) et fortes (Définition 4.1) dans la Sous-section 5.4.

Dans la Section 6 nous comparons notre modèle avec le modèle de Ambrosio et Braides [3], qui est aussi basé sur des mouvements minimisants généralisés. Dans cet modèle sont introduites des forces de viscosité et la propagation des fissures pendant un déplacement imposé constant en temps est permise; par contre, l'apparition des fissures ne peut pas se produire d'une manière physiquement acceptable.

Dans la Section 7 nous prouvons un résultat partiel d'existence des solutions du modèle en variable temporelle continue, sous l'hypothèse d'une borne supérieure uniforme (par rapport au temps) de la puissance communiquée par le reste de l'univers au corps fissuré.

La Section 8 est dédiée à l'approche numérique du modèle. Nous utilisons des résultats de convergence variationnelle de Ambrosio-Tortorelli [5] et la méthode numérique de Richardson-Mitter [95].

4.3.2 Equilibrium and absolute minimal solutions of brittle fracture models based on energy-minimization methods

Nous pouvons distinguer quatre directions de recherche liées aux modèles de fissuration fragile, vus comme des problèmes à discontinuité libre, et basés sur la minimisation d'une fonctionnelle énergie de Mumford-Shah. Ces directions sont : (i) l'étude qualitative du modèle, à supposer que les solutions du modèle existent, (ii) la comparaison avec l'expérience et les autres modèles classiques, (iii) la formulation faible du modèle, l'étude de la régularité des solutions faibles, (iv) la recherche des résultats d'approximation qui peuvent mener aux algorithmes numériques. Pour un chercheur intéressé à la mécanique, les directions (i), (ii) et (iv) sont plus intéressant que (iii). Pour le chercheur orienté plus vers la mathématique les points d'intérêt sont complémentaires.

Dans cet article, nous sommes intéressés par les deux premières directions mentionnées auparavant. Nous formulons un modèle général de fissuration fragile quasistatique, puis nous définissons des états d'équilibre et des états minimaux absolus et nous explorons leurs propriétés fondamentales. Dans le cas de la rupture fragile 3D nous prouvons une relation entre une généralisation de l'intégrale de Rice et la concentration d'énergie élastique, toutes les deux vues dans le sens de la théorie de la mesure.

4.3.3 Microfractured media with a scale and Mumford-Shah energies

Nous voulons comprendre la concentration d'endommagement observée dans les milieux élastiques micro-fissurés. En raison du comportement différent par rapport au changement d'échelle de l'aire et du volume (ou de longueur et de l'aire en 2D) la méthode traditionnelle d'homogénéisation qui emploie des tableaux périodiques des cellules semble échouér, une fois appliqué à la fonctionnelle Mumford-Shah et aux domaines périodiquement micro-fissurés.

Dans cet article nous nous écartons de l'homogénéisation traditionnelle. Le principal résultat concerne l'utilisation des énergies de Mumford-Shah et mène à une explication de la concentration observée de l'endommagement dans les corps élastiques micro-fissurés.

Le premier résultat d'homogénéisation, au sujet de la fonctionnelle Mumford-Shah semble être dû à Braides, Defranceschi, Vitali [18]. L'article de Focardi , Gelli [58] (voir aussi les références là-dedans) fait partie d'une autre piste de recherche qui pourrait être pertinente pour cet article: homogénéisation des domaines perforés.

Le résultat principal de cet article concerne l'utilisation de l'énergie Mumford-Shah pour donner une explication de la concentration d'endommagement observée dans les corps élastiques micro-fissurés.

Au lieu d'effectuer une homogénéisation de l'énergie du corps micro-fissuré et d'étudier alors les minima de l'énergie homogénéisée, nous procédons d'une manière différente. Nous étudions une suite des problèmes sur des corps élastiques contenant une distribution périodique des fissures, avec la configuration de référence Ω_ε, indicée par un paramètre d'échelle ε. Pour chaque ε la configuration Ω_ε est composée d'un nombre $M(\varepsilon) \approx \varepsilon^{-3}$ des cellules fissurées de dimension ε. Pour chaque ε on a un problème de minimisation d'une fonctionnelle Mumford-Shah pour lequel on peut prouver l'existence d'une solution. Notons avec $N(\varepsilon)$ le nombre de cellules endommagées de la configuration Ω_ε. Nous prouvons une estimation de la grandeur de $N(\varepsilon)$ qui montre que $N(\varepsilon) \approx \varepsilon^{-2}$. Cela veut dire que l'endommagement a tendance à se concentrer en bandes de petit volume, ce qui est conforme aux resultats expérimentaux.

5 Bipotentiels

5.1 Description du sujet

Les outils de base de la mécanique de milieux continus sont les équation de compatibilité cinématique et d'équilibre. De l'information additionnelle doit être fournie par les lois constitutives traduisant le comportement matériel. Sous sa forme la plus simple, une loi de comportement est donnée par un graphe rassemblant des couples des variables duales; souvent ce graphe résulte de l'essai expérimental.

Pour beaucoup de situations physiquement pertinentes, les lois de comportement sont multivoques et également associées. Le graphe d'une telle loi constitutive est inclu dans le graphe du sous-differentiel d'un surpotentiel ϕ (qui est aussi semi-continu inférieurement). La loi de comportement prend la forme d'une inclusion différentielle, $y \in \partial\phi(x)$. Tout surpotentiel ϕ a une fonction polaire ϕ^* qui satisfait une relation fondamentale, l'inégalité de Fenchel, $\forall x, y \; \phi(x) + \phi^*(y) \geq \langle x, y\rangle$. La loi de comportement peut être également écrite comme $x \in \partial\phi^*(y)$. Dans la littérature, ce genre de matériaux s'appellent souvent des matériaux standard ou des matériaux standard généralisés [74].

Du point de vue des applications, il est important de savoir si un surpotentiel existe pour un graphe donné, et de le construire. La réponse à ce problème est fournie par un théorème célèbre dû à Rockafellar [96] qui assure qu'un graphe admet un surpotentiel si et seulement si le graphe est cycliquement monotone maximal.

Cependant, certaines lois de comportement sont non-associées. Elles ne peuvent pas être traités dans le cadre des matériaux standard. Pour contourner ce problème, une réponse possible, proposée d'abord dans [98], consiste à construire une fonction b à deux variables, bi-convexe, qui satisfait une inégalité généralisant celle de Fenchel, c.a.d. $\forall x, y \; b(x, y) \geq \langle x, y\rangle$. G. de Saxcé appelle une telle fonction bipotentiel. Physiquement, le bipotentiel représente la dissipation. Dans le cas des lois de comportement associées, le bipotentiel est séparé : $b(x, y) = \phi(x) + \phi^*(y)$.

Quant aux lois de comportement non associées qui peuvent être exprimées avec l'aide des bipotentiels, elles ont la forme d'une relation implicite entre les variables duales, $y \in \partial b(\cdot, y)(x)$. En mécanique, nous dirons que ces lois sont des lois de normalité implicites ou faibles. Les applications des bipotentiels à la mécanique des solides sont diverses: la loi du frottement du Coulomb [99], le modèle non-associé de Drücker-Prager [100] et le modèle Cam-Clay [101] en mécanique des sols, la plasticité cyclique ([99], [13]) et la viscoplasticité [77] des métaux avec une loi cinématique non linéaire d'écrouissage, la loi d'endommagement de Lemaitre [12], les lois coaxiales ([53], [113]). De tels matériaux s'appellent des matériaux standard implicites. Un synthèse concernant ces lois exprimées en termes de bipotentiels peut être trouvé dans [53] et [113].

L'utilisation des bipotentiels dans les applications est particulièrement attrayante dans des simulations numériques par la méthode des éléments finis, mais l'intérêt n'est pas limité à ces aspects. Par exemple, les théorèmes de borne de l'analyse limite ([103],

[16]) et de la théorie de l'adaptation plastique ([105], [53], [17], [14]) peuvent être reformulés dans le cadre plus large des lois faibles de normalité. D'un point de vue numérique appliqué, la méthode du bipotentiel suggère de nouveaux algorithmes, rapides mais robustes, comme les estimateurs variationnels d'erreurs évaluant la précision du maillage en éléments finis ([75], [76], [102], [104], [15], [78], [79]). Les applications à la mécanique de contact [55], à la dynamique des matériaux granulaires ((([56], [57], [62][106]), à la plasticité cyclique des métaux [102] et à la plasticité de sols ([11], [78]) illustrent la pertinence de cette approche.

5.2 Contributions

Mes contributions à ce sujet sont due à une collaboration avec G. de Saxcé (qui a introduit les bipotentiels) et C. Vallée:

> [97] G. de Saxcé, M Buliga, C. Vallée, C. Lerintiu, Construction of a bipotential for a multivalued constitutive law, PAMM, **6** , 1 (December 2006), Special Issue: GAMM Annual Meeting 2006 - Berlin
>
> [25] M. Buliga, G. de Saxcé, C. Vallée, Existence and construction of bipotentials for graphs of multivalued laws, *J. of Convex Analysis*, **15**, 1, (2008)
>
> [26] M. Buliga, G. de Saxcé, C. Vallée, Construction of bipotentials and a minimax theorem of Fan, (submitted),
> http://arxiv.org/abs/math.FA/0610136, (2006)

5.3 Résumés des articles

5.3.1 Existence and construction of bipotentials for graphs of multivalued laws

Dans tous les articles déjà mentionnés au sujet des applications mécaniques, des bipotentiels ont été construits pour certaines lois de comportement multivoques. Néanmoins, afin de comprendre mieux l'approche du bipotentiel, on doit résoudre les problèmes suivants :

1) (existence) quels sont les conditions à satisfaire par une loi multivoque telle qu'elle peut être exprimée avec l'aide d'un bipotentiel ?

2) y a-t-il un procédé pour construire une classe des bipotentiels pour une loi multivoque ? On s'attend à ce que génériquement la loi ne détermine pas uniquement le bipotentiel.

Nous donnons un premier traitement mathématique de ces problèmes et nous prouvons des résultats d'existence (théorème 3.2) et de construction (théorème 6.7) des bipotentiels pour une classe des lois multivoques.

Une des idées principales est de construire le bipotentiel comme une enveloppe inférieure. Cela pourrait être considéré comme paradoxal parce que, généralement, il est fortement improbable qu'une enveloppe inférieure, même de fonctions convexes, soit convexe. Néanmoins, nous avons été convaincus de la pertinence de cette approche par des exemples inspiré de la mécanique et nous avons souhaité en comprendre la raison. Cela nous a menés à introduire l'outil principal des recouvrements lagrangiens convexes (définition 4.1) satisfaisant une condition implicite de convexité.

La méthode que nous donnons dans cet article s'applique seulement aux BB-graphes (Définition 3.1) admettant au moins un recouvrement lagrangien convexe par des graphes cycliquement monotones maximaux. C'est une classe intéressante de graphes des lois multivoques pour les raisons suivants:

(a) elle contient la classe des graphes des sous-differentiels des surpotentiels convexes et semicontinus inférieurement ,

(b) toute loi non associée provenant des applications mécaniques mentionnées auparavant est un BB-graphe, qui admet un recouvrement lagrangien convexe, pertinent du point de vue physique, par des graphes cycliquement monotones.

Concernant le point (b), il est important de savoir que les résultats de cet article ne s'appliquent pas à quelques BB-graphes d'intérêt mécanique, comme le bipotentiel associé au contact avec frottement de Coulomb [98]. C'est parce que nous n'employons dans cet article que des recouvrements lagrangiens convexes avec des graphes cycliquement monotones *maximaux* (voir également la Remarque 5.1).

5.3.2 Construction of bipotentials and a minimax theorem of Fan

Du point de vue mathématique, une loi constitutive non associée est un opérateur à valeurs multiples $T : X \to 2^Y$ *qui n'est pas censé être monotone*. Ici X, Y sont des espaces localement convexes duaux, avec le produit de dualité $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$.

Afin de comprendre mieux l'approche du bipotentiel, dans l'article [25] nous avons résolu deux problèmes principaux : (a) quand le graphe d'un opérateur multivoque donné, en général non monotone, peut être exprimé comme l'ensemble de points critiques d'un bipotential, et (b) une méthode de construction d'un bipotentiel associé (dans le sens du point (a)) à l' opérateur donné. Notre outil principal était la notion du recouvrement lagrangien convexe du graphe de l'opérateur, et une notion de convexité implicite de ce recouvrement.

Dans cet article nous prouvons un autre théorème de reconstruction pour un bipotential à partir d'une couverture lagrangienne convexe, cette fois en utilisant une notion de convexité reliée à un théorème de minimax de Fan.

6 Calcul des variations, quasiconvexité, élasticité

6.1 Description du sujet

Par un résultat classique de Morrey, les propriétés de continuité des fonctionnelles du type

$$u \mapsto I(u) = \int_\Omega w(Du(x))\, \mathrm{d}x$$

sur des espaces de Sobolev sont en relation avec la quasiconvexité de la fonction potentiel w. Le cas des espaces de Sobolev des fonctions $u : \Omega \subset R^n \to R^n$ est très intéressant pour la mécanique des milieux continus.

En élasticité le potentiel w n'est pas en général défini sur un espace vectoriel de matrices $n \times n$ (l'algèbre de Lie $gl(n,R)$) mais sur l'ensemble des matrices $n \times n$ avec déterminant positif (le groupe de Lie $GL(n,R)$). Parfois la fonction w est définie sur un sous-groupe, comme dans le cas de l'élasticité incompressible où il faut considérer le groupe des matrices de déterminant 1, i.e. le groupe $SL(n,R)$. Pour n pair, un autre groupe qui attire l'attention est $Sp(n,R)$, le groupe des matrices symplectiques. Il est donc intéressant de trouver des conditions nécessaires et/ou suffisantes pour la semi-continuité 'inférieure de la fonctionnelle et de notions (qui restent à trouver) de convexité du $w : G \to \mathbb{R}$, où G est un groupe de matrices.

Ce problème est ouvert depuis quelque temps. Deux définitions de la quasi-convexité sont pertinentes: la première se trouve dans l'article de Ball [7], et la deuxième, nommée "Diff-quasiconvexity", est introduite dans Giaquinta, Modica, Soucek in [63], page 174, définition 3. Ces deux définitions sont en vérité équivalentes.

Très peu de choses sont connues sur la continuité des intégrales associées aux potentiels diff-quasiconvexes. Il est cependant facile à prouver que la diff-quasiconvexité est une condition nécessaire (voire [63] proposition 2, page 174).

6.2 Contributions

La liste des contributions à ce sujet est la suivante:

[24] M. Buliga, Lower semi-continuity of integrals with G-quasiconvex potential, *Z. Angew. Math. Phys.*, bf 53, 6, 949-961, (2002)

[32]M. Buliga, Four applications of majorization to convexity in the calculus of variations, (submitted) (2007), updated version of the paper "Majorisation with applications to the calculus of variations"
http://arxiv.org/abs/math.FA/0105044

[42] M. Buliga, Quasiconvexity versus group invariance,
http://arxiv.org/abs/math.AP/0511235, (2005)

[45] M. Buliga, The variational complex of a diffeomorphisms group, http://arxiv.org/abs/math.AP/0511302, (2005)

6.3 Résumés des articles

6.3.1 Lower semi-continuity of integrals with G-quasiconvex potential

Dans cet article [24] je propose une notion pertinente de quasiconvexité dans le sens variationnel, associée à un groupe d'homéomorphismes bi-Lipschitziens. Soit G un groupe de matrices inversibles. Conformément au théorème de Rademacher tout fonction de Lipschitz de $\mathbb{R}^n$ à $\mathbb{R}^n$ est dérivable p.p.t. Le groupe $[G]$ associé à G est alors le groupe des homéomorphismes bi-Lipschitziens $\phi : \mathbb{R}^n \to \mathbb{R}^n$ tels que $D\phi(x) \in G$ p.p.t. par rapport à $x \in \mathbb{R}^n$.

La question est de savoir dans quelles conditions une fonction $w : G \to \mathbb{R}$ donne une fonctionnelle qui est semicontinue inférieurement sur $[G]$ par rapport à la convergence uniforme (soit la convergence *-faible dans un espace de Sobolev).

Le résultat principal de l'article est un théorème qui répond à cette question. La preuve nécessite en quelques points des techniques d'intégration géométrique introduites par Gromov et étudiées par Dacorogna.

La version plus longue, math.FA/0105097, discute aussi des notions pertinentes pour la convexité de rang un. Le problème des G-lagrangiens nuls n'avait pas été posé auparavant. Un G-lagrangien nul est un potentiel w tel que l'intégrale associée est continue. Dans le cas $G = GL(n, R)$, on retrouve les lagrangiens nuls classiques, qui sont identifiables à des formes différentielles (dans un espace de jets; cela conduit à un bi-complexe variationnel).

6.3.2 Four applications of majorization to convexity in the calculus of variations

Il y a une ressemblance forte entre les deux théorèmes suivants. Le premier théorème est Horn [80] (1954), Thompson [111] (1971), le théorème 1.) :

Theorem 6.1 *Soit X, Y soit deux matrices définies positives quelconques $n \times n$ et soit $x_1 \geq x_2 \geq ... \geq x_n$ et $y_1 \geq y_2 \geq ... \geq y_n$ les ensembles respectifs de valeurs propres. Alors il existe une matrice unitaire U telle que XU et Y ont le même spectre si et seulement si :*

$$\prod_{i=1}^{k} x_i \geq \prod_{i=1}^{k} y_i \quad , \ k = 1, ..., n-1$$

$$\prod_{i=1}^{n} x_i \ = \ \prod_{i=1}^{n} y_i$$

Le deuxième théorème est (Dacorogna-Marcellini-Tanteri [51] (2000), théorème 20, voir également Dacorogna, Marcellini [50]):

Theorem 6.2 *Soit* $0 \leq \lambda_1(A) \leq ... \leq \lambda_n(A)$ *les valeurs singulières d'une matrice* $A \in \mathbb{R}^{n\times n}$ *et*

$$E(a) = \left\{ A \in \mathbb{R}^{n\times n} \ : \ \lambda_i(A) = a_i \ , \ i = 1, ..., n \ , \ \det A = \prod_{i=1}^{n} a_i \right\}$$

Alors

$$Pco\ E = \ Rco\ E(a) = \left\{ A \in \mathbb{R}^{n\times n} \ : \ \prod_{i=\nu}^{n} \lambda_i(A) \leq \prod_{i=\nu}^{n} a_i \ , \ \nu = 2, ..., n \ , \right.$$
$$\left. \det A = \prod_{i=1}^{n} a_i \right\}$$

où PCo*,* Rco *sont les notations pour les enveloppes polyconvexe, convexe de rang un respectivement.*

Dans ces deux théorèmes est décrit un ensemble $\{y \ : \ y \prec\prec x\}$ où $\prec\prec$ est une relation de préordre définie avec l'aide de certaines inégalités entre les produits qui apparaissent dans les formulations des deux théorèmes précédentes.

La bonne relation de préordre à considérer est reliée à la relation classique de majoration. La majoration est une notion familière en analyse stochastique, en algèbre linéaire et en théorie des groupes de Lie. Dans cet article, une première tentative est faite pour appliquer la majoration à l'élasticité et au calcul des variations. Nous obtiendrons des preuves plus simples de résultats connus mais aussi de nouveaux résultats.

Il est significatif de noter que la majeure partie des résultats concernant la majoration employée dans cet article précèdent l'ouvrage fondamental de Morrey (1952) [87] sur la quasiconvexité. Cependant, il semble que il n'y avait pas jusqu'à maintenant beaucoup d'interaction entre ces champs de recherche.

Dans la section 3 on donne synthèse des propriétés de la relation de majoration. Dans la section 4, on énumère des propriétés des valeurs singulières et des valeurs propres des matrices reliées à la majoration.

L'article continu avec quatre applications.

La première application est dans le domaine de l'élasticité non-linéaire. Le théorème 5.6 donne des conditions simples, nécessaires et suffisantes, pour qu'une énergie objective et isotrope soit convexe de rang un sur l'ensemble de matrices avec déterminant positif. Le sujet a une longue histoire, commençant par 1954 Baker, Ericksen (1954) [8].

Comme deuxième application, nous employons le majoration afin de fournir une preuve très courte d'un théorème de Thompson et Freede [112], Ball [7], ou Le Dret [82] (dans cet article théorème 6.2).

Ensuite, nous prouvons (théorème 6.3) un résultat de semi-continuité inférieure pour des fonctionnelles de la forme $\int_{\Omega} w(D\phi(x))\,\mathrm{d}x$, avec $w(F) = h(\ln V_F)$. Ici $F = R_F U_F = V_F R_F$ est la décomposition polaire de $F \in gl(n, \mathbb{R})$ et $\ln V_F$ est la contrainte logarithmique de Hencky.

Nous clôturons l'article par une preuve du théorème de Dacorogna-Marcellini-Tanteri basée seulement sur des résultats classiques de majoration. Ceci explique la ressemblance entre les théorèmes 6.1 et 6.2. Des résultats proches peuvent être trouvés dans [108] où Silhavy exprime les inégalités de Baker-Ericksen en utilisant aussi la multiplication au lieu de la division.

7 Géométrie sub-riemannienne et structures de dilatations

7.1 Description du sujet

Mon intérêt pour le sujet de recherche de la géométrie sub-riemannienne a commencé pendant le temps passé au département de mathématiques de l'EPFL. J'ai eu l'occasion de collaborer avec une partie des collègues de l'Institut de Mathématiques de l'Université de Berne, aussi bien qu'avec certains invités à EPFL.

La géométrie sub-riemannienne, ou de Carnot-Carathéodory, ou encore géométrie non-holonome, est un sujet de recherche en contact avec plusieurs domaines, notamment : l'analyse des opérateurs hypoelliptiques, la théorie du contrôle, l'analyse dans les espaces métriques mesurés. Parmi les principaux contributeurs à ce sujet on compte Hörmander [81], Gromov [70] [69] [71], Cheeger [48], Folland, Stein [61], Margulis, Mostow [83] [84].

L'intérêt pour ces espaces vient de plusieurs propriétés intéressantes du point de vue métrique : ce sont des fractales (la dimension de Hausdorff par rapport à la distance de Carnot-Carathéodory est strictement plus grande que la dimension topologique, cf. Mitchell [85]); l'espace métrique tangent en un point d'une variété sub-riemannienne régulière est un groupe de Carnot; l'espace asymptotique (dans le sens de la distance de Gromov-Hausdorff [70]) d'un groupe fini généré avec une croissance polynômiale est également un groupe de Carnot, par un théorème célèbre de Gromov [69]; enfin, sur de tels espaces nous avons assez de structure pour développer un calcul différentiel ressemblant à celui proposé par Cheeger [48] et pour prouver des théorèmes comme la version de Pansu du théorème de Rademacher [93], en menant à une preuve ingénieuse d'un résultat de rigidité appartenant à Margulis.

Une variété sub-riemannienne est un triple (M, D, g), ou M est une variété connexe de dimension n, D est un sous-fibré de TM, nommé distribution horizontale, et g est un produit scalaire euclidien défini seulement sur D. De plus D est totalement non-integrable, c.a.d. que pour toute paire de points $x, y \in M$, il existe une courbe $c : [0, 1] \to M$ telle que $c(0) = x$, $c(1) = y$, et c est une courbe qui est presque partout tangente à la distribution D. On peut alors définir la distance Carnot-Carathéodory $d(x, y)$ comme l'infimum des longueurs des courbes c avec les propriétés énumérées.

La notion de structure de dilatation vient de mes efforts pour comprendre les résultats de base de la géométrie sub-riemannienne, particulièrement la dernière section de l'article de Bellaïche [10] et le point de vue intrinsèque de Gromov [71].

Dans ces articles, comme dans d'autres sur la géométrie sub-riemannienne, même si les résultats fondamentaux admettent une formulation intrinsèque (en termes de la distance Carnot-Carathéodory), leur preuves emploient les outils de la géométrie différentielle.

À mon avis ces outils ne sont pas intrinsèques à la géométrie sub-riemannienne. Par conséquent j'ai essayé de trouver un cadre dans lequel la géométrie sub-riemannienne serait un modèle, si nous employons la même façon de parler que dans le cas de la géométrie hyperbolique (avec son ensemble d'axiomes) et du disque de Poincaré comme modèle de la géométrie hyperbolique.

Les résultats de cet effort sont les notions de structure de dilatation et de paire des structures de dilatation, l'une regardant l'autre vers le bas. À la première notion sont consacrés les articles [27], [30] (le deuxième article traitant le sujet d'une version "linéaire" d'une structure de dilatation, en correspondance avec les groupes de Carnot ou avec les groupes contractibles, plus généraux).

Aujourd'hui, il semble que les structures de dilatation sont intéressantes par elles-mêmes, avec un champ d'applications possibles contenant strictement la géométrie sub-riemannienne, mais aussi les espaces ultramétriques ou les groupes contractibles. Une structure de dilatation code la similitude approximative d'un espace métrique et induit des opérations non associatives, mais approximativement associatives, sur l'espace métrique, aussi bien que sur un fibré tangent (dans le sens métrique) avec des opérations de groupe dans chaque fibre (l'espace tangent à un point).

Structures de dilatation: un exemple très connu Pour se faire une idée sur les structures de dilatations, je vais presenter l'exemple le plus trivial.

Soit $(\mathbb{V}, \| \cdot \|)$ un espace vectoriel de dimension finie, normé et réel. Par définition la dilatation basée en x, de coefficient $\varepsilon > 0$, est la fonction

$$\delta_{\varepsilon}^{x} : \mathbb{V} \rightarrow \mathbb{V} \quad , \quad \delta_{\varepsilon}^{x} y = x + \varepsilon(-x + y) \quad .$$

Pour x fixé, les dilatations basées en x forment un groupe à un paramètre qui contracte tout voisinage borné de x en un point, uniformément par rapport à x dans un ensemble compact.

La distance d étant induite par la norme, l'espace métrique $(\mathbb{V}, d)$ est complet et localement compact. Pour tout $x \in \mathbb{V}$ et tout $\varepsilon > 0$ la distance d se comporte bien par rapport aux dilatations δ_{ε}^{x} dans le sens: pour tout $u, v \in \mathbb{V}$ nous avons

$$\frac{1}{\varepsilon} d(\delta_{\varepsilon}^{x} u, \delta_{\varepsilon}^{x} v) \ = \ d(u, v) \quad . \tag{7.1.1}$$

À partir des dilatations, nous pouvons reconstruire la structure algébrique de l'espace vectoriel $\mathbb{V}$. Par exemple définissons pour $x, u, v \in \mathbb{V}$ et $\varepsilon > 0$:

$$\Sigma_{\varepsilon}^{x}(u, v) = \delta_{\varepsilon^{-1}}^{x} \delta_{\varepsilon}^{\delta_{\varepsilon}^{x} u}(v) \quad .$$

Un calcul simple montre que $\Sigma_{\varepsilon}^{x}(u, v) = u + \varepsilon(-u + x) + (-x + v)$, donc nous pouvons récupérer l'opération d'addition en $\mathbb{V}$ par la limite:

$$\lim_{\varepsilon \rightarrow 0} \Sigma_{\varepsilon}^{x}(u, v) = u + (-x + v) \quad . \tag{7.1.2}$$

C'est une translation de l'opération d'addition telle que l'élément neutre est x. Ainsi, pour $x = 0$, nous récupérons l'opération habituelle d'addition.

Essayons de prendre les dilatations en tant que données de base pour l'exemple ci-dessus. À savoir, au lieu de donner à l'espace $\mathbb{V}$ une structure d'espace vectoriel normé, nous donnons seulement la distance d et les dilatations δ_ε^x pour tout $x \in X$ et $\varepsilon > 0$. Nous devrions ajouter quelques relations qui prescrivent:

- le comportement de la distance par rapport aux dilatations, par exemple une certaine forme de la relation (7.1.1),
- l'interaction entre les dilatations, par exemple l'existence de la limite du côté gauche de la relation (7.1.2).

Nous dénotons une telle collection des données par $(\mathbb{V}, d, \delta)$ et nous appelons cela une structure de dilatation.

7.2 Contributions

J'ai organise avec Tudor Ratiu le séminaire de travail "Sub-Riemannian geometry and Lie groups", en 2001-2002. Dans ce séminaire j'ai présenté une suite de 12 lectures. J'ai été parmi les organisateurs du séminaire Borel 2003, "Tangent spaces of metric spaces", où j'ai présenté 2 lectures. Depuis, j'ai donné plusieurs communications à des conférences sur les structures de dilatations, voir la notice de travaux. Les articles suivants sont dédiés à ce sujet:

[27] M. Buliga, Dilatation structures I. Fundamentals, J. Gen. Lie Theory Appl. Vol 1 (2007), No 2, 65-95

[28] M. Buliga, Vrănceanu' nonholonomic spaces from the viewpoint of distance geometry, (in romanian, original title: Spaţiile neolonome ale lui Vrănceanu din punctul de vedere al geometriei distanţei), to appear in *Revista Fundaţiei Acad. Prof. Gh. Vrănceanu*, (2007)

[30] M. Buliga, Contractible groups and linear dilatation structures, (submitted) http://xxx.arxiv.org/abs/0705.1440, (2007)

[31] M. Buliga, Linear dilatation structures and inverse semigroups, (submitted) http://xxx.arxiv.org/abs/0705.4009, (2007)

[34] M. Buliga, Dilatation structures in sub-riemannian geometry, (submitted) http://arxiv.org/abs/0708.4298, (2007)

[47] M. Buliga, Dilatation structures with the Radon-Nikodym property, (submitted)
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7.3 Résumés des articles

7.3.1 Dilatation structures I. Fundamentals

Une structure de dilatation est un concept entre une structure différentielle et un groupe. Dans cet article, nous étudions les propriétés fondamentales des structures de dilatation dans les espaces métriques. C'est une partie d'une série des articles qui prouvent qu'une telle structure induit une analyse "non commutative", dans le sens du calcul différentiel, sur une grande classe d'espaces métriques, certains d'entre eux fractaux. Nous décrivons également un calcul formel et universel avec des arbres planaires décorés binaires, qui est à la base de toute structure de dilatation.

À tout espace métrique (X, d) doté d'une structure de dilatation est associé un fibré tangent. L'espace tangent (au sens métrique) en un point est un groupe conique. Les groupes coniques généralisent les groupes de Carnot, qui sont des groupes nilpotents avec une graduation positive. Chaque structure de dilatation mène à un calcul différentiel non commutatif sur l'espace métrique (X, d).

Plusieurs articles importants sont consacrés à l'étude des structures sur un espace métrique qui induisent une analyse raisonnable, comme par exemple Cheeger [48] ou Margulis-Mostow [83] [84].

Les constructions proposées en cet article sont apparues suites aux essais de comprendre certains problèmes de l'analyse dans les varietes sub-riemanniennes. Des parties de cet article peuvent être vues comme une formulation rigoureuse des considérations qui se trouvent dans la dernière section de Bellaïche [10].

Une structure de dilatation est simplement un fibré de semigroupes de (quasi) contractions sur l'espace métrique (X, d), satisfaisant un certain nombre d'axiomes. La structure de fibré tangent, associée à une structure donnée de dilatations sur l'espace métrique (X, d), est obtenue par un passage à la limite, à partir d'une structure algébrique qui vit sur l'espace métrique.

Avec l'aide de la structure de dilatation nous construisons un fibré (au-dessus de l'espace métrique) des opérations: à chaque $x \in X$ et paramètre ε, pour la simplicité ici $\varepsilon \in (0, +\infty)$, il y a une opération non-associative definie par:

$$\Sigma_{\varepsilon}^{x} : U(x) \times U(x) \rightarrow U(x)$$

où $U(x)$ est un voisinage de x. La non-associativité de cette opération est contrôlée par le paramètre ε. Quand ε tend vers 0, l'opération Σ_{ε}^{x} converge vers une opération de groupe sur l'espace tangent de (X, d) en x.

Soit δ_{ε}^{x} la dilatation basée en $x \in X$, de paramètre ε. Le fibré d'opérations satisfait un genre d'associativité faible, même si pour tout $y \in X$ l'opération Σ_{ε}^{y} est non-associative. L'associativité faible est décrite par la relation:

$$\Sigma_{\varepsilon}^{x}(u, \Sigma_{\varepsilon}^{\delta_{\varepsilon}^{x}}(v, w)) = \Sigma_{\varepsilon}^{x}(\Sigma_{\varepsilon}^{x}(u, v), w)$$

pour tout $x \in X$ et tout $u, v, w \in X$ suffisamment près de x.

Nous décrivons brièvement plus loin le contenu de l'article. Dans la section 2 nous donnons des premiers exemples des structures de dilatation. Les notions et résultats de base de la géométrie métrique et des groupes dotés de dilatations sont mentionnés dans la section 3.

Dans la section 4 nous présentons un formalisme basé sur les arbres binaires planaires décorés. Ce formalisme sera employé pour prouver les résultats principaux du article. Nous montrons que, d'un point de vue algébrique, les structures de dilatation (avec plus de précision le formalisme de la section 4) induisent un paquet de déformations à un paramètre des opérations binaires, qui ne sont pas associatives, mais faiblement associatives. C'est une structure qui ressemble au fibré tangent à un groupe de Lie.

Les sections 5, 6 et 7 sont consacrées aux structures de dilatation et comprennent les résultats principaux de l'article. Après que nous ayons présenté et expliqué les axiomes des structures de dilatation, nous décrivons dans la section 5 plusieurs des propriétés métriques principales d'une telle structure.

Une partie des résultats principaux de cet article peut être synthétisée en deux théorèmes:

Theorem 7.1 *Soit (X, d, δ) une structure de dilatation. Alors l'espace métrique (X, d) admet un espace métrique tangente à tout $x \in X$. Plus précisément, nous avons la limite suivante:*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \sup \left\{ \mid d(u, v) - d^{x}(u, v) \mid \ : d(x, u) \leq \varepsilon \ , \ d(x, v) \leq \varepsilon \right\} \ = \ 0 \ .$$

Theorem 7.2 *Soit (X, d, δ) une structure forte de dilatation. Alors pour tout $x \in X$ le triple $(U(x), \Sigma^x, \delta^x, d^x)$ est un un groupe local, plus précisément:*

(a) Σ^x est une opération locale de groupe sur $U(x)$, avec x comme élément neutre et inv^x comme fonction élément inverse,

(b) la distance d^x est invariante à gauche par rapport à l'opération de groupe du point (a),

(c) pour tout $\varepsilon \in (0, 1)$, la dilatation δ^x_ε est un automorphisme de l'opération de groupe du point (a),

(b) pour tout $u, v \in X$ tels que $d(x, u) \leq 1$ et $d(x, v) \leq 1$, pour tout $\mu \in (0, A)$ nous avons:

$$d^x(u, v) \ = \ \frac{1}{\mu} d^x(\delta^x_\mu u, \delta^x_\mu v) \quad .$$

Le groupe conique $(U(x), \Sigma^x, \delta^x)$ peut être considéré comme l'espace tangent à (X, d, δ) en x.

7.3.2 Contractible groups and linear dilatation structures

Un espace métrique (X, d) qui admet une structure forte de dilatation possède un espace métrique tangent en tout point $x \in X$, et chacun de ces espaces métriques tangents possède la structure algébrique d'un groupe conique normé. Les groupes coniques sont des exemples particuliers de groupes contractibles. La structure des groupes contractible est connue de manière assez détaillée, dû à Siebert [107], Wang [115], Glöckner et Willis [67], Glöckner [66].

Par un résultat classique de Siebert [107] proposition 5.4, nous pouvons caractériser la structure algébrique des espaces métriques tangents qui sont associés aux structures de dilatation d'une certain type: ce sont des groupes de Carnot, c. à d. des groupes de Lie simplement connexes dont l'algèbre de Lie admet un graduation entière positive.

Les groupes de Carnot apparaissent dans beaucoup de situations, en particulier en relation avec la géométrie sub-riemannienne cf. Bellaïche [10], des groupes avec croissance polynômiale cf. Gromov [69], ou de la rigidité de Margulis cf. Pansu [93]. Une partie du mon programme de recherche se propose de prouver que les structures de dilatation sont des objets naturels sur ces sujets mathématiques. À cet égard le corollaire 4.7 représente une généralisation de certains résultats difficiles en géométrie sub-riemannienne, concernant la structure du l'espace métrique tangent en un point d'une variété sub-riemannienne régulière.

La linéarité est également une propriété qui peut être expliquée avec l'aide des structures de dilatation. Dans la deuxième section de l'article, nous expliquons pourquoi la linéarité peut être formulée en termes de dilatations. Il y a en fait deux sortes de linéarités : la linéarité d'une structure de dilatation et celle d'une fonction entre deux structures de dilatation.

Notre résultat principal est une caractérisation des groupes contractibles en termes de structures de dilatation. À chaque groupe conique normé (ou groupe contractible normé), nous pouvons naturellement associer une structure linéaire de dilatation. Réciproquement, toute structure de dilatation linéaire et forte vient d'une structure de dilatation d'un groupe contractible normé.

7.3.3 Linear dilatation structures and inverse semigroups

Dans cet article, nous nous demandons s'il y a un rapport entre les structures de dilatation et les semigroupes inverses.

Dans un espace vectoriel normé, les transformations affines admettent une description en termes de dilatations: $A : \mathbb{V} \rightarrow \mathbb{V}$ est affine si et seulement si pour tout $\varepsilon \in (0,1)$, $x, y \in \mathbb{V}$ nous avons

$$A \, \delta_{\varepsilon}^{x} y \; = \; \delta_{\varepsilon}^{Ax} A y \quad . \tag{7.3.3}$$

Toute dilatation associée à l'espace vectoriel $\mathbb{V}$ est une transformation affine, par conséquent pour tout $x, y \in \mathbb{V}$ et $\varepsilon, \mu > 0$ nous avons

$$\delta_{\mu}^{y} \, \delta_{\varepsilon}^{x} \; = \; \delta_{\varepsilon}^{\delta_{\mu}^{y} x} \delta_{\mu}^{y} \quad . \tag{7.3.4}$$

Parfois, les compositions de dilatations sont des dilatations. C'est le sujet du prochain théorème, qui est équivalent au théorème de Menelaos en géométrie euclidienne.

Theorem 7.3 *Pour tous $x, y \in \mathbb{V}$ et $\varepsilon, \mu > 0$ tels que $\varepsilon \mu \neq 1$, il existe un et un seul $w \in \mathbb{V}$ tel que*

$$\delta_{\mu}^{y} \, \delta_{\varepsilon}^{x} \; = \; \delta_{\varepsilon \mu}^{w} \quad .$$

Pour la preuve voir Artin [6]. Une conséquence directe de ce théorème est le résultat suivant.

Corollary 7.4 *Le semigroupe inverse généré par les dilatations de l'espace vectoriel $\mathbb{V}$ est fait de toutes les dilatations et toutes les translations de $\mathbb{V}$.*

Dans cet article nous prouvons que pour les structures de dilatation linéaires (qui satisfont une forme générale de la relation (7.3.4)) il existe une généralisation du corollaire 7.4. Le résultat est nouveau pour des groupes de Carnot et la preuve semble être nouvelle, même pour les espaces vectoriels.

Definition 7.5 *Une structure de dilatation (X, d, δ) a la propriété de Menelaos si pour tous $x, y \in X$ suffisamment proches et pour tout $\varepsilon, \mu \in \Gamma$ tels que $\nu(\varepsilon), \nu(\mu) \in (0,1)$ nous avons*

$$\delta_\varepsilon^x \, \delta_\mu^y \;=\; \delta_{\varepsilon\mu}^w \quad ,$$

où $w \in X$ est le point fixe de la contraction $\delta_\varepsilon^x \delta_\mu^y$.

Theorem 7.6 *Une structure linéaire de dilatation a la propriété de Menelaos.*

7.3.4 Dilatation structures with the Radon-Nikodym property

Dans cet article, j'explique ce qu'est une paire de structures de dilatation, l'une regardant l'autre vers le bas. Une telle paire de structures de dilatation va nous mener à une définition intrinsèque d'une distribution comme un champ de filtres topologiques.

Pour toute structure de dilatation, il y a une notion associée de différentiabilité qui généralise le différentiabilité de Pansu [93]. Ceci permet l'introduction de la propriété de Radon-Nikodym pour les structures de dilatation, qui est la généralisation de la propriété de Radon-Nikodym pour les espaces de Banach.

Après une section consacrée aux espaces métriques et aux dérivés métriques, on prouve que pour une structure de dilatation avec la propriété de Radon-Nikodym la longueur des courbes absolument continues s'exprime comme une intégrale de la norme de la tangente à la courbe, comme en géométrie riemannienne.

Plus loin on montre que la propriété de Radon-Nikodym se transfère du haut vers le bas, c. à d. à partir d'une structure de dilatation "en haut" qui regarde vers une autre structure de dilatation "en bas". À mon avis, ce résultat explique intrinsèquement le fait que les courbes absolument continues dans des variétés sub-riemanniennes régulières sont dérivable presque partout, comme prouvé par Margulis, Mostow [83], Pansu [93] (pour des groupes de Carnot) ou Vodopyanov [114].

7.3.5 Dilatation structures in sub-riemannian geometry

Plusieurs articles sont consacrés à l'établissement des fondements de la géométrie sub-riemannienne, comme Mitchell [85], Bellaïche [10], un grand article de Gromov [71] demandant un point de vue intrinsèque pour la géométrie sub-riemannienne, Margulis, Mostow [83] [84], consacrés au théorème de Rademacher pour les variétés sub-riemanniennes et à la construction intrinsèque d'un fibré tangent, et Vodopyanov [114] (entre d'autres articles).

Il y a une raison pour l'existence de tant d'articles sur le même sujet : les propriétés géométriques fondamentales des espaces sub-riemanniennes sont très difficiles à prouver. Probablement, le plus difficile est de fournir une construction rigoureuse du fibré tangent d'un tel espace à partir des propriétés de la distance Carnot-Carathéodory et ensuite de généraliser d'un façon ou d'une autre le calcul différentiel proposé par Pansu.

Basé sur la notion de structure de dilatation [27], j'ai essayé de donner un traitement intrinsèque ła géométrie sub-riemannienne dans l'article [47].

Dans cet article, je prouve que les espaces sub-riemanniens réguliers admettent des structures de dilatation (théorèmes 6.3, 6.4). À partir de l'existence des repères normaux prouvé par Bellaïche, nous déduisons le reste de propriétés des espaces sub-riemanniens en employant le formalisme des structures de dilatation.

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Energy Minimizing Brittle Crack Propagation

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Abstract. We propose a minimizing movement model for quasi-static brittle crack evolution. Cracks (fissures) appear and/or grow without any prescription of their shape or location when time-dependent displacements are imposed on the exterior boundary of the body. We use an energetic approach based on Mumford–Shah type functionals. By the discretization of the time variable we obtain a sequence of free discontinuity problems.

We find exact solutions and estimations which lead us to the conclusion that in this model crack appearance is allowed but the constant of Griffith G and the critical stress which causes the fracture in an uni-dimensional traction experiment cannot be both constants of material.

A weak formulation of the model is given in the frame of special functions with bounded deformation. We prove the existence of weak constrained incremental solutions of the model. A partial existence result for the minimizing movement model is obtained under the assumption of uniformly bounded (in time) power communicated to the body by the rest of the universe.

The model is of applicative interest. A numerical approach and examples, using an Ambrosio–Tortorelli variational approximation of the energy functional, are given in the last section.

Mathematics Subject Classifications (1991): 73M25, 58E30, 49M10.

Key words: brittle fracture propagation, free discontinuity problems, minimizing movements, variational approximation, functions with bounded deformations.

1. Introduction

This paper concerns the study of quasi-static brittle crack evolution. We work under the following assumptions: a linear elastic body, with or without initial cracks inside, evolves in a quasi-static manner under an imposed path of boundary displacements. During its evolution cracks with unprescribed geometry may appear and/or grow.

The difficulty of brittle crack propagation problems consists in the nature of the main unknown: the crack itself, at various moments in time. The research in this field concerns mainly the constitutive behavior of a brittle material, like the basic paper of Griffith [27]. Amongst the basic references we can quote: Eshelby [24], Irwin [30], Gurtin [28], [29], Rice [38].

In almost all the studies the geometry of the crack is prescribed. There are few exceptions, as the papers of Ohtsuka [34–37] or Stumpf and Le [39]. The geometry of the crack can be prescribed in a strong form, like in the case of a plane rectangu-

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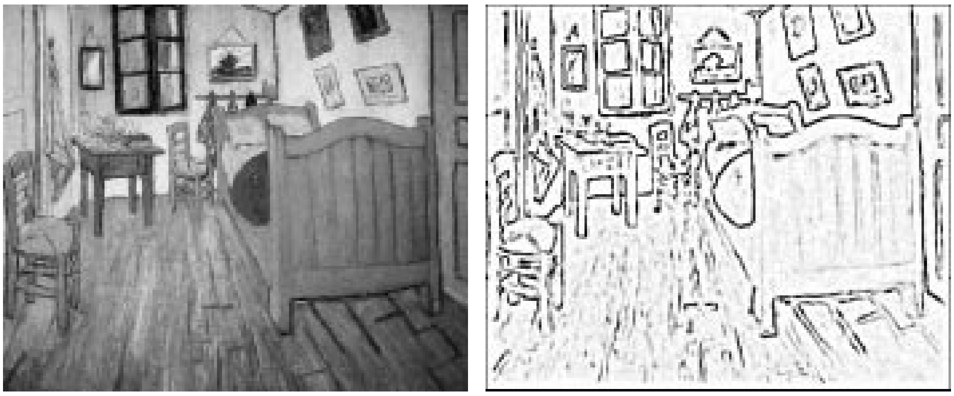

Figure 1. Example of image segmentation with the Mumford–Shah functional. The left figure is a black-and-white copy of a Van Gogh's painting; in the right figure we see the set of edges.

lar or elliptic crack which is supposed to remain plane rectangular or elliptic during its growth. We find a weak prescription upon the evolution of the crack in the case of a body with two-dimensional configuration, when the crack is supposed to have only an edge, which is a point. Therefore, in this case, the evolution of the crack is conveniently reduced to the movement of a point. Under these assumptions the geometrical nature of the main unknown is obscured.

A new direction of research in brittle fracture mechanics begins with the article of Mumford and Shah [33] regarding the problem of image segmentation. This problem, which consists in finding the set of edges of an image and constructing a smoothed (away from the edges) version of that image, turns out to be intimately related to the problem of brittle crack evolution.

In the article mentioned above Mumford and Shah propose the following variational approach to the problem of image segmentation: let $g\colon \Omega \subset R^2 \to [0, 1]$ be the original picture, given as a distribution of grey levels (1 is white and 0 is black). Let $u\colon \Omega \to R$ be the output picture and let K be the set of edges of the objects in the picture. K is (contained in) the set where u has jumps, i.e. $u \in C^1(\Omega \setminus K, R)$. The pair formed by the smoothed picture u and the set of edges K minimizes then the functional

$$I(u, K) = \int_\Omega \alpha |\nabla u|^2 \,\mathrm{d}x + \int_\Omega \beta |u - g|^2 \,\mathrm{d}x + \gamma \mathcal{H}^1(K). \qquad (1)$$

The parameter α controls the smoothness away from the edges of the new picture u, β controls the L^2 distance between the smoothed picture and the original one and γ controls the total length of the edges given by this variational method. The authors remark that for $\beta = 0$ the functional I might be useful for an energetic treatment of fracture mechanics. In the followings is presented a model of brittle crack appearance in the case of imposed boundary displacements.

The state of a brittle body with reference configuration Ω is described by a pair displacement-crack. $(\mathbf{u}, K)$ is such a pair if:

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(1) K is a crack in the body, seen as a surface,
(2) $\mathbf{u}$ is a displacement of the body with the crack $K \subset \overline{\Omega}$, compatible with the imposed boundary displacement $\mathbf{u}_0$, i.e. $\mathbf{u} \in C^1(\Omega \setminus K)$ and $\mathbf{u} = \mathbf{u}_0$ on $\partial\Omega$.

The total energy of the body in the state $(\mathbf{u}, K)$ is a Mumford–Shah functional of the form

$$E(\mathbf{u}, K; \mathbf{u}_0) = \int_\Omega w(\nabla \mathbf{u})\,\mathrm{d}x + F(\mathbf{u}_0, K).$$

The first term of the functional E represents the elastic energy of the body with the displacement $\mathbf{u}$. The second term represents the energy consumed to produce the crack K in the body, with the boundary displacement $\mathbf{u}_0$ as parameter.

In this model the brittle crack appearance is seen as an equilibrium problem. When the displacement $\mathbf{u}_0$ is imposed on the (exterior) boundary $\partial\Omega$ the state $(\mathbf{v}, S)$ of the brittle body is a minimizer of the total energy $E(\cdot, \cdot, ; \mathbf{u}_0)$. The crack predicted by the model is S. Notice that S may be the empty-set; in this case the model predicts that no crack appears when $\mathbf{u}_0$ is imposed.

Brittle crack appearance and image segmentation are free discontinuity problems. The unknowns, the crack or the collection of edges, are discontinuity surfaces for the displacement field or for the smoothed image; their location is entirely unprescribed.

We shall use an energetic approach to quasi-static brittle crack evolution. Therefore, we proceed to a time discretization which transforms the problem of crack evolution into a sequence of energy minimization problems. Francfort and Marigo [26] proceed in the same way in the case of brittle brutal damage evolution. However, it is only a belief that when the time step goes to zero, the discretized evolution converges to an almost continuous (in time) evolution. We have found in the frame of generalized minimizing movements, introduced by De Giorgi [20], stronger mathematical reasons to support this belief. That is why we introduce in Section 2 the notion of energy minimizing movement as a particular case of a generalized minimizing movement.

In Section 3, after the preliminaries concerning the statics of a brittle body, the Griffith criterion of brittle crack propagation is presented in Subsection 3.3, as a selection criterion amongst all possible crack evolutions. At the end of this section we formulate the problem of quasi-static brittle crack evolution in the form (14).

In Subsection 4.1 we give an energy minimizing movement formulation to this problem using a Mumford–Shah energy functional (Definition 4.1). In this model we have only one material constant connected to fracture, namely the constant of Griffith G. Some features of the model are explored in Subsection 4.2 in the anti-plane and uni-dimensional cases. We prove that crack appearance is allowed (we refer to [18] for more information, especially concerning fiber-matrix debonding in composites). The relation (23) contains the expression of σ_c, the critical stress which lead to fracture in an uni-dimensional traction experiment. We infer from this relation that σ_c and G cannot be both constants of material in this model.

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Section 5 concerns the weak formulation of the incremental (that is discretized in time) model of crack evolution introduced in Definition 4.1. Subsection 5.1 deals with special functions with bounded variation or deformation. The existence of weak constrained incremental solutions of the model (Definition 5.1, Theorem 5.3) is a consequence of more general results due to De Giorgi and Ambrosio [21], Ambrosio [1–2], Bellettini, Coscia and Dal Maso, [15], Ambrosio, Coscia and Dal Maso [6]. The anti-plane case is discussed in Subsection 5.3. We compare the notions of weak (according to Definition 5.1) and strong (Definition 4.1) solution in Subsection 5.4.

In Section 6 a comparison is made with the model of Ambrosio and Braides [4], also based on generalized minimizing movements. In this model viscosity forces are introduced and crack propagation under imposed constant boundary displacement is allowed; on the contrary, crack appearance can not occur in a physically acceptable way.

In Section 7 we prove a partial existence result of the energy minimizing movement described in the model, under the assumption of uniformly bounded power communicated by the rest of the universe to the body during its evolution.

Section 8 is devoted to the numerical approach to the model. We use here functional convergence results of Ambrosio and Tortorelli [11–12] and the numerical method of Richardson and Mitter [32].

This paper continues a part of the work [17].

2. General Energy Minimizing Movements

An energy minimizing movement is a particular case of a generalized minimizing movement. The latter notion has been introduced by De Giorgi in [20], inspired by the paper [13] of Almgren, Taylor and Wang. The definition of a generalized minimizing movement (according to Ambrosio [3]) is presented below

DEFINITION 2.1. Let S be a topological space and

$$F\colon (1, +\infty) \times N \times S \times S \to R \cup \{+\infty\}$$

be a function. For any $u_0 \in S$, a function $u\colon [0, +\infty) \to S$ is a generalized minimizing movement associated to F with initial datum u_0, and we write $u \in GMM(F, u_0)$, if there exists a diverging sequence $(s_i)_{i\in N}$, $s_i > 1$, and there are functions $u_i\colon N \to S$ such that:

(i) $u_i(0) = u_0$;

(ii) for any $k \in N$ and any i, $u_i(k+1)$ minimizes the functional

$$v \mapsto F(s_i, k, v, u_i(k))$$

over S;

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(iii) for any $t \geqslant 0$, $u_i([s_i t]) \to u(t)$ in S as $i \to +\infty$.

As the name tells, the notion of a generalized minimizing movement extends the notion of minimizing movement. With S, F and $u_0 \in S$ as in Definition 2.1, $u\colon [0, +\infty) \to S$ is a minimizing movement associated to F with initial datum u_0, and we write $u \in MM(F, u_0)$, if there are functions $u_s(k)$, for any $s > 1$ and $k \in N$, such that:

(i) $u_s(0) = u_0$;
(ii) for any $k \in N$ and any $s \in (0, +\infty)$, $u_s(k+1)$ minimizes the functional

$$v \mapsto F(s, k, v, u_s(k))$$

over S;
(iii) for any $t \geqslant 0$, $u_s([st]) \to u(t)$ in S as $s \to +\infty$.

The canonical example of (generalized) minimizing movement is given by the choice: $S = R^n$, $f\colon R^n \to R$ Lipschitz continuous and C^2 and

$$F(s, k, u, v) = f(u) + \frac{s}{2}|u - v|^2.$$

In this case, for any $u_0 \in R^n$ there is only one minimizing movement, namely the unique solution of the Cauchy problem

$$u'(t) = -\nabla f(u(t)), \qquad u(0) = u_0.$$

Notice that the minimizing movement associated to F and u_0 might not be unique, mainly because the functional $v \mapsto F(s, k, u_s(k), v)$ may have more than one minimizer. The nonuniqueness of a generalized minimizing movement is of higher order, because there might be different generalized minimizing movements depending on the choice of the diverging sequence s_i. For examples and techniques of investigation of the sets $MM(F, u_0)$ and $GMM(F, u_0)$ we refer to Ambrosio [3].

An energy minimizing movement is a generalized minimizing movement associated to a particular function F. It is designed to be a 'weak stable' solution of an evolution problem of the following type

$$\begin{cases} \mathbf{A}(u(t), \alpha(t), t) = 0, & \forall t \geqslant 0 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\alpha(t) \leqslant \mathbf{L}(\alpha(t), u(t)), & \forall t \geqslant 0 \\ u(0) = u_0, \quad \alpha(0) = \alpha_0. \end{cases} \tag{2}$$

There are two unknowns in this problem: u and α. The evolution of the unknown u is quasi-static. Suppose that we don't have a proper law of evolution of α, or that the law of evolution that we have gives too many solutions. We may assume that we have the expression of the total energy $f(u, \alpha)$ of the system in the state (u, α) and

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a set of constraints, not in a differential form, upon the evolution of α. We make then a time discretization with time step δ and recursively find $(u_{k+1}^{\delta}, \alpha_{k+1}^{\delta})$ from $(u_k^{\delta}, \alpha_k^{\delta})$, by a minimization process of the total energy f under some constraints. A weak stable solution of the previous problem is a limit of sequences $(u_k^{\delta}, \alpha_k^{\delta})_k$ when the time step δ converges to 0.

In the next definition S may be seen as the space of all pairs $x = (u, \alpha)$, endowed with a topology.

DEFINITION 2.2. Let S be a topological space and

$$F: (1, +\infty) \times N \times S \times S \to R \cup \{+\infty\},$$

$$F(s, k, x, y) = f(s, x, y) + \psi(k/s, y)$$

be a function, with $f: N \times S \times S \to R$ and $\psi: [0, \infty) \times S \to \{0, +\infty\}$. For any $x_0 \in S$, an energy minimizing movement associated to the energy f with the constraints ψ and initial datum x_0 is any generalized minimizing movement $x: [0, +\infty) \to S$, $x \in GMM(F, x_0)$.

Let us denote by $S(\lambda)$ the following set

$$S(\lambda) = \{y \in S: \psi(\lambda, y) = 0\}.$$

From Definition 2.2 we notice that $x: [0, +\infty) \to S$ is an energy minimizing evolution associated to f, with the constraints ψ and initial datum x_0 if there exists a diverging sequence $(s_i)_{i\in N}$, $s_i > 1$, and there are functions $x_i: N \to S$ such that:

(i) $x_i(0) = x_0$;
(ii) for any $k \in N$ and any $i \in N$, $x_i(k+1)$ minimizes the functional f over the set $S(k/s_i)$ (in particular $x_i(k+1)$ belongs to $S(k/s_i)$);
(iii) for any $t > 0$, $x_i([s_i t]) \to x(t)$ in S as $i \to +\infty$.

3. Notations and Preliminaries

3.1. Notations and constitutive assumptions

The open bounded set $\Omega \subset R^3$ represents the reference configuration of an elastic body and $\mathbf{u}: \Omega \to R^3$ is the displacement field of the body. We shall always suppose, without mentioning further, that the open set Ω and its closure have the same topological boundary.

The expression of the elastic (or free) energy of the body is

$$\int_{\Omega} w(\nabla \mathbf{u})\, \mathrm{d}x.$$

The first Piola-Kirchhoff stress tensor $\boldsymbol{S}$ is

$$\boldsymbol{S}(\mathbf{u}) = \frac{\mathrm{d}w}{\mathrm{d}\nabla}(\nabla \mathbf{u})$$

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and the equilibrium equation of the body in the absence of volumic forces is

$$\operatorname{div} \boldsymbol{S}(\mathbf{u}) = 0 \quad \text{in } \Omega .$$

In this paper we suppose that the body is linear elastic and homogeneous, i.e. the function $w(\nabla \mathbf{u})$ has the form

$$w(\nabla \mathbf{u}) = \tfrac{1}{2} \boldsymbol{C} \nabla \mathbf{u} : \nabla \mathbf{u},$$

with the elasticity 4-tensor $\boldsymbol{C}$ having the symmetries

$$\boldsymbol{C}_{ijkl} = \boldsymbol{C}_{jikl} = \boldsymbol{C}_{klij}.$$

Under these assumptions the stress tensor $\boldsymbol{S}$ becomes the Cauchy stress tensor

$$\sigma = \sigma(\mathbf{u}) = \boldsymbol{C} \nabla \mathbf{u} = \boldsymbol{C} \varepsilon(\mathbf{u}),$$

where $\varepsilon(\mathbf{u})$ is the symmetric part of $\nabla \mathbf{u}$, i.e.

$$\varepsilon(\mathbf{u}) = \tfrac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T).$$

We suppose moreover that w satisfies the growth conditions

$$\forall \ \mathbf{F} \in R^9, \qquad \mathbf{F} = \mathbf{F}^T, \qquad c|\mathbf{F}|^2 \leqslant w(\mathbf{F}) \leqslant C|\mathbf{F}|^2,$$

where c and C belong to $(0, +\infty)$.

In the cases of plane or anti-plane displacements the domain $\Omega \subset R^2$ represents a section in the cylindrical reference configuration of the body $\Omega \times R$ and the body is supposed to be isotropic.

If $\mathbf{u}: \Omega \to R^2$ is a plane displacement then the displacement relative to the three-dimensional configuration of the body has the following expression

$$(x_1, x_2, x_3) \in \Omega \times R \mapsto (u_1(x_1, x_2), u_2(x_1, x_2), 0) \in R^3.$$

The anti-plane displacement is a function $u: \Omega \to R$. The three-dimensional displacement has the following form

$$(x_1, x_2, x_3) \in \Omega \times R \mapsto (0, 0, u(x_1, x_2)) \in R^3.$$

In this case the elastic energy takes the form

$$\int_\Omega \mu |\nabla u|^2 \, \mathrm{d}x,$$

where μ is one of the two Lamé's constants.

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3.2. Statics of a fractured elastic body

For any measurable set $B \subset R^n$, $|B| = \mathcal{L}^n(B)$ denotes the Lebesgue measure of B and $\mathcal{H}^k(B)$ the k-dimensional Hausdorff measure of B.

By a crack set in the body Ω we mean (according to Ball [14]) a topologically closed countably rectifiable set, generically denoted by K. We shall always suppose that K is a subset of $\overline{\Omega}$.

Given the function f, a point $x \in \Omega \subset R^n$ and an unit vector (or direction) $\mathbf{n} \in R^n$, the approximate limit of f in x associated to the direction $\mathbf{n}$ is denoted by $\tilde{f}(x,\mathbf{n})$ and it is defined by the following expression

$$\lim_{\rho\to 0_+} \frac{\int_{B_\rho(x)\cap\{y:(y-x)\cdot\mathbf{n}\geqslant 0\}} |f(y)-\tilde{f}(x,\mathbf{n})|\,\mathrm{d}y}{|B_\rho(x)\cap\{y:(y-x)\cdot\mathbf{n}\geqslant 0\}|} = 0. \tag{3}$$

Given a field of unit vectors $x \in K \mapsto \mathbf{n}(x)$ normal to K, the lateral limits f^+ and f^- of any function $f\colon \Omega\setminus K \to R^n$ are $f^+\colon K \to R$ and $f^-\colon K \to R$, defined by

$$f^+(x) = \tilde{f}(x,\mathbf{n}(x)), \qquad f^-(x) = \tilde{f}(x,-\mathbf{n}(x)).$$

This means that f^+ and f^- satisfy the equalities

$$\forall x \in K, \quad \lim_{\rho\to 0_+} \frac{\int_{B_\rho(x)\cap\{y:(y-x)\cdot\mathbf{n}\geqslant 0\}} |f(y)-f^+(x)|\,\mathrm{d}y}{|B_\rho(x)\cap\{y:(y-x)\cdot\mathbf{n}\geqslant 0\}|} = 0,$$

$$\forall x \in K, \quad \lim_{\rho\to 0_+} \frac{\int_{B_\rho(x)\cap\{y:(y-x)\cdot\mathbf{n}\leqslant 0\}} |f(y)-f^-(x)|\,\mathrm{d}y}{|B_\rho(x)\cap\{y:(y-x)\cdot\mathbf{n}\leqslant 0\}|} = 0.$$

Note that for any $x \in K$ the triplet $(f^+(x), f^-(x), \mathbf{n}(x))$ is unique up to a change of sign of $\mathbf{n}$ and a permutation of f^+, f^-, i.e.

$$(f^+(x), f^-(x), \mathbf{n}(x)) \sim (f^-(x), f^+(x), -\mathbf{n}(x)).$$

We denote by $[f] = f^+ - f^-$ the jump of f over K. Notice that the tensor field over K defined by $[f]\otimes\mathbf{n}$ is uniquely determined by f and K. If f takes values in R^n then the same is true for the symmetric part of the tensor field defined above, namely

$$\{[f]\odot\mathbf{n}\}_{ij} = \tfrac{1}{2}([f]_i\mathbf{n}_j + [f]_j\mathbf{n}_i).$$

The jump of f over the crack set K can be described by the following measure

$$\mathrm{j}(f,K) = [f]\odot\mathbf{n}\,\mathrm{d}\mathcal{H}^{n-1}_K, \qquad \mathrm{j}(f,K)(B) = \int_{B\cap K} [f]\odot\mathbf{n}\,\mathrm{d}\mathcal{H}^{n-1}. \tag{4}$$

Consider a crack set $K \subset \Omega$ formed by a finite collection of smooth surfaces. By a displacement compatible with K we mean a function $\mathbf{u}\colon \overline{\Omega}\setminus K \to R^k$ (where

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k might be 1, 2 or 3) which is C^1 and has continuous lateral limits on K. In this section we shall consider the space $W^{1,2}(\Omega \setminus K)$ as the set of weak displacements compatible with the crack set K.

Let n be the dimension of the reference configuration Ω. For any $u_0 \in H^{1/2}(\partial\Omega, R^n)$ and for any crack set K, such that $\mathcal{H}^{n-1}(\partial\Omega \setminus K) > 0$, a solution (if any) of the following problem

$$\begin{cases} \operatorname{div} \sigma(\mathbf{u}) = 0, & \text{in } \Omega \setminus K \\ \sigma^+(\mathbf{u})\mathbf{n} = \sigma^-(\mathbf{u})\mathbf{n} = 0, & \text{on } K \\ \mathbf{u} = \mathbf{u}_0, & \text{on } \partial\Omega \setminus K \end{cases} \tag{5}$$

will be denoted by $\mathbf{u} = \mathbf{u}(\mathbf{u}_0, K)$. The solution is unique up to rigid displacements of $\Omega \setminus K$ equal to 0 on $\partial\Omega$. If K and $\partial\Omega$ are such that a Korn inequality holds on the space $W^{1,2}(\Omega \setminus K)$, then the problem (5) has a solution. For this paper the fact that $\mathbf{u}(\mathbf{u}_0, K)$ is unique up to a class of rigid displacements is irrelevant, therefore $\mathbf{u}(\mathbf{u}_0, K)$ will be called ‘the solution’ of the problem (5).

We use the same notation – $u = u(u_0, K)$ – in the anti-plane case, when $n = 2$, $k = 1$ and the problem (5) becomes

$$\begin{cases} \mu \operatorname{div} \nabla u = 0, & \text{in } \Omega \setminus K, \\ (\nabla u)^+\mathbf{n} = (\nabla u)^-\mathbf{n} = 0, & \text{on } K, \\ u = u_0, & \text{on } \partial\Omega \setminus K. \end{cases} \tag{6}$$

The solution $\mathbf{u}(\mathbf{u}_0, K)$ of the problem (5) minimizes the functional

$$E(\mathbf{v}) = \int_\Omega w(\nabla \mathbf{v})\,\mathrm{d}x$$

over the following set of weak displacements compatible with the crack set K and the boundary displacement $\mathbf{u}_0$

$$\{\mathbf{v} \in W^{1,2}(\Omega \setminus K, R^n) : \mathbf{v} = \mathbf{u}_0 \quad \text{on } \partial\Omega \setminus K\}.$$

By standard arguments the functional

$$\mathbf{v} \in W^{1,2}(\Omega, R^n) \mapsto \int_\Omega \sigma(\mathbf{u}(\mathbf{u}_0, K)) : \nabla \mathbf{v}\,\mathrm{d}x$$

depends only on the trace of $\mathbf{v}$ on $\partial\Omega$, hence it gives raise to the linear continuous function

$$\mathbf{T}(K) \colon H^{1/2}(\partial\Omega, R^n) \to H^{-(1/2)}(\partial\Omega, R^n),$$

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$$\langle \mathbf{T}(K)\mathbf{u}_0, \mathbf{v}\rangle = \int_\Omega \sigma(\mathbf{u}(\mathbf{u}_0, K)) : \nabla \mathbf{v}' \, \mathrm{d}x \quad \text{for any } \mathbf{v}' = \mathbf{v} \text{ on } \partial\Omega. \tag{7}$$

In the latter definition $\langle \cdot, \cdot \rangle$ is the duality product of the pair of spaces $H^{1/2}(\partial\Omega, R^n)$ and $H^{-(1/2)}(\partial\Omega, R^n)$.

The function $\mathbf{T}(K)$ is called the Dirichlet-to-Neumann map of the elastic body Ω with the crack set K. Under the assumptions concerning the symmetries of the elasticity tensor $\boldsymbol{C}$, the function $\mathbf{T}(K)$ is self-adjoint, that is for any $\mathbf{u}$, $\mathbf{v}$ we have

$$\langle \mathbf{T}(K)\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{T}(K)\mathbf{v}, \mathbf{u}\rangle. \tag{8}$$

In the same way the Dirichlet-to Neumann map associated to the problem (6) is defined.

Finally, we remark that the elastic energy of the body can be expressed using the Dirichlet-to-Neumann map. Indeed, we have

$$\int_\Omega w(\nabla \mathbf{u}(\mathbf{u}_0, K)) \, \mathrm{d}x = \tfrac{1}{2}\langle \mathbf{T}(K)\mathbf{u}_0, \mathbf{u}_0\rangle. \tag{9}$$

3.3. The Griffith criterion of brittle crack propagation

Let us consider in the elastic body Ω an initial crack set K_0 which evolves and becomes at the moment t the crack set K_t. We assume that the crack set always increases in time, i.e.,

$$\forall 0 < t < t', \qquad K_t \subset K_{t'}. \tag{10}$$

We suppose that the evolution of the body is quasi-static. At the moment t the state of the body is characterized by the displacement-crack pair $(\mathbf{u}(t), K_t)$, where $\mathbf{u}(t)$ is the displacement of the body, compatible with the crack set K_t. Let us denote by $\mathbf{u}_0(t)$ the trace of $\mathbf{u}(t)$ on $\partial\Omega$. We have then the equality $\mathbf{u}(t) = \mathbf{u}(\mathbf{u}_0(t), K_t)$. We make the assumption that the function $t \mapsto \mathbf{u}_0(t)$ is sufficiently regular in time.

The power given to the body by the rest of the universe at the moment t has the following expression

$$P(t) = \int_{\partial\Omega} \boldsymbol{S}(\mathbf{u}(t))\mathbf{n} \cdot \dot{\mathbf{u}}_0(t) \, \mathrm{d}x = \langle \mathbf{T}(K_t)\mathbf{u}_0(t), \dot{\mathbf{u}}_0(t)\rangle.$$

Let us consider a given curve $t \mapsto (\mathbf{u}(t), K_t)$, such that for any t we have $\mathbf{u}(t) = \mathbf{u}(\mathbf{u}_0(t), K_t)$. For a given t we introduce the following curve of displacements

$$\forall \tau \geqslant 0, \qquad \tilde{\mathbf{u}}(\tau) = \mathbf{u}(\mathbf{u}_0(t + \tau), K_t).$$

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$\tilde{\mathbf{u}}(\tau)$ represents the displacement of the body at the moment $t+\tau$ in the presence of the crack K_t. An easy calculation leads us to the equality

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_\Omega w(\nabla \tilde{\mathbf{u}}(\tau))\, \mathrm{d}x_{|_{\tau=0}} = P(t). \tag{11}$$

Therefore $P(t)$ represents the power consumed at the moment t by the body in order to modify its displacement, constrained to follow the path of imposed boundary displacements $t \mapsto \mathbf{u}_0(t)$, without any modification of the actual crack set K_t.

The Griffith criterion of brittle crack propagation asserts that during the propagation of the crack K_t the following inequality is true at any moment t

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\{ \int_\Omega w(\nabla \mathbf{u}(t))\, \mathrm{d}x + G\mathcal{H}^{n-1}(K_t) \right\} \leqslant P(t). \tag{12}$$

Here G is the constant of Griffith, supposed to be a material constant.

The relation (12) can be written in a different form using the map $\mathbf{T}(K_t)$. Let us assume that the crack evolution is smooth in the sense that the function $t \mapsto \mathbf{T}(K_t)$ is differentiable, i.e., the Dirichlet-to-Neumann map varies smoothly in time. The Griffith criterion takes the following form

$$\begin{aligned} &\frac{1}{2}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{T}(K_t)]\mathbf{u}_0(t), \mathbf{u}_0(t)\right\rangle + \tfrac{1}{2}\langle \mathbf{T}(K_t)\dot{\mathbf{u}}_0(t), \mathbf{u}_0(t)\rangle \\ &\quad + \tfrac{1}{2}\langle \mathbf{T}(K_t)\mathbf{u}_0(t), \dot{\mathbf{u}}_0(t)\rangle + G\frac{\mathrm{d}}{\mathrm{d}t}\{\mathcal{H}^{n-1}(K_t)\} \\ &\qquad \leqslant \langle \mathbf{T}(K_t)\mathbf{u}_0(t), \dot{\mathbf{u}}_0(t)\rangle. \end{aligned}$$

The function $\mathbf{T}(K_t)$ is self-adjoint, therefore we obtain the following expression of the Griffith criterion

$$\frac{1}{2}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{T}(K_t)]\mathbf{u}_0(t), \mathbf{u}_0(t)\right\rangle + G\frac{\mathrm{d}}{\mathrm{d}t}\{\mathcal{H}^{n-1}(K_t)\} \leqslant 0. \tag{13}$$

Notice that we have the following equality

$$P(t) - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w(\nabla \mathbf{u}(t))\, \mathrm{d}x = -\frac{1}{2}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{T}(K_t)]\mathbf{u}_0(t), \mathbf{u}_0(t)\right\rangle.$$

The left-hand member of the previous equality is usually called the energy release rate due only to the crack propagation.

$\mathbf{u}_0(t)$ plays the role of a time-dependent parameter, since in the last inequality $\dot{\mathbf{u}}_0(t)$ does not appear. As we have seen, this is a consequence of relations (8), (9) and (12).

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The problem of quasi-static brittle propagation of an initial crack in an elastic body under a time-dependent imposed displacement $\mathbf{u}_0(t)$ is of the type (2). If we put apart the constraint (10), we have the following formulation

$$\begin{cases} \mathbf{u}(t) - \mathbf{u}(\mathbf{u}_0(t), K_t) = 0, & \forall t \geqslant 0, \\ \frac{1}{2}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{T}(K_t)]\mathbf{u}_0(t), \mathbf{u}_0(t)\right\rangle + G\frac{\mathrm{d}}{\mathrm{d}t}\{\mathcal{H}^{n-1}(K_t)\} \leqslant 0, & \forall t \geqslant 0, \\ \mathbf{u}(0) = \mathbf{u}_0, \quad K_0 = K. \end{cases} \tag{14}$$

4. The Model

In the left term of the Griffith criterion (12) there appears the time-derivative of an energetic functional. Let us consider the set M of all admissible displacement-crack pairs $(\mathbf{u}, K)$ with the following properties

(1) $K \subset \overline{\Omega}$ is a crack set;
(2) $\mathbf{u} \in C^1(\overline{\Omega} \setminus K, R^n)$;
(3) for $\mathcal{H}^{n-1}$-almost any $x \in K$ there exist the normal $\mathbf{n}(x)$ at K in x and $\mathbf{u}^+(x)$, $\mathbf{u}^-(x)$.

Notice that the field $\mathbf{n}$ of normals induces an orientation in the neighborhood of K. The item (3) in the definition of M can be replaced by imposing the existence of traces $\mathbf{u}^+$ and $\mathbf{u}^-$ of $\mathbf{u}$ on K with respect to this orientation.

The Mumford–Shah energy functional over M has the following expression

$$I: M \to R \cup \{+\infty\}, \qquad I(u, K) = \int_\Omega w(\nabla \mathbf{u})\,\mathrm{d}x + G\mathcal{H}^{n-1}(K). \tag{15}$$

4.1. Introduction of the Model

According to Definition 2.2 and the constraint (10) we give an energy minimizing movement formulation to the problem (14) using the functional defined in (15).

DEFINITION 4.1. Let us define the functions

$$J: M \times M \to R,$$

$$J((\mathbf{u}, K), (\mathbf{v}, L)) = \int_\Omega w(\nabla \mathbf{v})\,\mathrm{d}x + G\mathcal{H}^{n-1}(L \setminus K),$$

$$\Psi: [0, \infty) \times M \to \{0, +\infty\},$$

$$\Psi(\lambda, (\mathbf{v}, K)) = \begin{cases} 0, & \text{if } \mathbf{v} = \mathbf{u}_0(\lambda) \text{ on } \partial\Omega \setminus K \\ +\infty, & \text{otherwise.} \end{cases}$$

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We consider the initial data $(\mathbf{u}_0, K) \in M$ such that $\mathbf{u}_0 = \mathbf{u}(\mathbf{u}_0(0), K)$. For any $s \geqslant 1$ we define the sequences

$$k \in N \mapsto \mathbf{u}^s(k), \qquad L^s(k), \qquad K^s(k),$$

$(\mathbf{u}^s(k), L^s(k)) \in M$ and $(\mathbf{u}^s(k), K^s(k)) \in M$, recursively:

(i) $(\mathbf{u}^s, K^s)(0) = (\mathbf{u}_0, K)$, $L^s(0) = K$,
(ii) for any $k \in N$ $(\mathbf{u}^s, L^s)(k+1) \in M$ minimizes the functional

$$(\mathbf{v}, L) \in M \mapsto J((\mathbf{u}^s, K^s)(k), (\mathbf{v}, L)) + \Psi((k+1)/s, (\mathbf{v}, L))$$

over M. In order to verify the constraint (10), $K^s(k+1)$ is defined by the formula:

$$K^s(k+1) = K^s(k) \cup L^s(k+1).$$

$(\mathbf{u}, L)\colon [0, +\infty) \to M$ is an energy minimizing movement associated to J with the constraints (10), Ψ and initial data $(\mathbf{u}_0, K)$, and we write $(\mathbf{u}, L) \in GMM(\mathbf{u}_0, K, \Psi)$, if there is a diverging sequence (s_i) such that for any $t > 0$ we have

$$\begin{cases} \mathbf{u}^{s_i}([s_i t]) \to \mathbf{u}(t) \quad \text{in } L^2(\Omega, R^n), \\ j(\mathbf{u}^{s_i}, L^{s_i})([s_i t]) \to j(\mathbf{u}, L)(t) \quad \text{weakly as Radon measures} \end{cases} \tag{16}$$

as $i \to \infty$ and

$$\mathcal{H}^{n-1}(L(t)) \leqslant \liminf_{i \to \infty} \mathcal{H}(L^{s_i}([s_i t])). \tag{17}$$

In the previous definition $1/s$ is the step of the discretization of the time variable. The approximate displacement of the body at the moment k/s is $\mathbf{u}^s(k)$. The *active crack* at the same moment is $L^s(k)$ and the *total crack* is $K^s(k)$. The state of the brittle body is $(\mathbf{u}^s(k), L^s(k))$ while $K^s(k)$ takes account of the history of fissuration. Any sequence $k \mapsto (\mathbf{u}^s, L^s, K^s)(k)$ constructed using the rules (i) and (ii) from the Definition 4.1 is called an incremental solution. We use the same name for a sequence of displacement-crack pairs $k \mapsto (\mathbf{u}^s, L^s)(k)$. Notice that in rule (ii) the triplet $(\mathbf{u}^s, L^s, K^s)(k)$ appears in the expression of the functional j only through $K^s(k)$.

The time step goes to 0 as i converges to ∞ and the incremental solution $(\mathbf{u}^{s_i}, L^{s_i})([s_i t])$ converges to $(\mathbf{u}, L)(t)$, for any $t > 0$. $L(t)$ is called the *active crack* at the moment t and

$$K(t) = \cup_{s \in [0,t]} L(s)$$

is called the *damaged region* of the body at the same moment. Notice that the damaged region $K(t)$ might not be a crack set, because it is *a priori* a noncountable union of surfaces.

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The convergence of the incremental solution to the energy minimizing movement deserves a discussion. The measure $\mathrm{j}(\mathbf{u}, L)$ associated to a displacement-crack pair contains information about the placement and the opening of the crack L under the displacement $\mathbf{u}$. The weak convergence of $\mathrm{j}(\mathbf{u}^{s_i}, L^{s_i})([s_i t])$ to $\mathrm{j}(\mathbf{u}, L)(t)$ as Radon measures means that for any $\phi \in C_0(\Omega, M^{n\times n})$ we have

$$\lim_{i\to\infty} \int_{L^{s_i}([s_i t])} ([\mathbf{u}^{s_i}([s_i t])] \odot \mathbf{n}) : \phi \, \mathrm{d}\mathcal{H}^{n-1}$$
$$= \int_{L(t)} ([\mathbf{u}(t)] \odot \mathbf{n}) : \phi \, \mathrm{d}\mathcal{H}^{n-1}.$$

Therefore (16) asserts that the incremental displacement converges to the displacement at the moment t and (in a weak sense) the placement and the opening of the incremental crack set converges to the placement and the opening of the active crack set at the same moment. Generally $\mathrm{j}(\mathbf{u}, L)$ is not null on the part of L where the jump of $\mathbf{u}$ is not null, therefore this measure gives information only about the opened crack. The role of the condition (17) is to control the area of the crack $L(t)$, in order to eliminate the parts of the active crack which are not opened.

4.2. Features of the model

We investigate further the behavior of the model proposed in Definition 4.1 in the particular case of anti-plane displacements. There are some obvious adjustments to be made. Ω is now a bounded domain in R^2 and the displacement is a scalar function u. The functional J will take the form

$$J((u, K), (v, L)) = \int_\Omega \mu |\nabla v|^2 \, \mathrm{d}x + G\mathcal{H}^1(L \setminus K). \tag{18}$$

For a displacement-crack pair (u, L) we introduce the notation

$$\mathrm{j}(u, L) = [u] \, \mathrm{d}\mathcal{H}^1_{|L}.$$

Let us consider a particular type of imposed displacement on $\partial\Omega$. We split the boundary of the body into three parts

$$\partial\Omega = \overline{\Gamma^1_u} \cup \overline{\Gamma^2_u} \cup \overline{\Gamma_f},$$

$$\Gamma^i_u \cap \Gamma_f = \emptyset, \qquad \overline{\Gamma^1_u} \cap \overline{\Gamma^2_u} = \emptyset, \qquad \mathcal{H}^1(\Gamma^1_u) \cdot \mathcal{H}^1(\Gamma^2_u) \cdot \mathcal{H}^1(\Gamma_f) > 0.$$

At any moment $t \geqslant 0$, Γ_f is force free, i.e. the displacement is not prescribed on this part of the boundary. On Γ^1_u and Γ^2_u the imposed displacement is defined by

$$u_0(t)(x) = \begin{cases} 0 & \text{on } \Gamma^1_u \\ t\delta & \text{on } \Gamma^2_u \end{cases},$$

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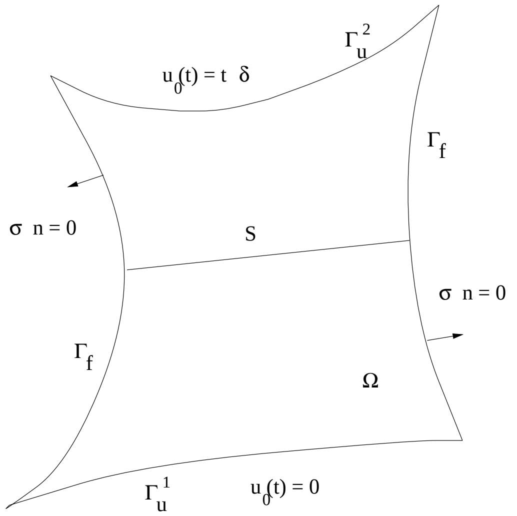

Figure 2. The geometry of the body and imposed displacement.

where δ is a positive constant with dimension of speed. This displacement is homogeneous in the time variable

$$\forall t > 0, \quad u_0(t) = t u_0(1).$$

We suppose that at the moment $t = 0$ there are no cracks in the body. This assumption takes the form $K = \emptyset$. At $t = 0$ we have $u_0(0) = 0$, hence the initial data are $(u_0 = 0, K = \emptyset)$.

Let us consider a time discretization given by the parameter $1/s$ and the incremental solution $k \in N \mapsto (u^s, L^s)(k)$ introduced in Definition 4.1 for the initial data and the imposed boundary conditions described above. In order to shorten the notations we shall omit for the moment the superscript s.

The incremental solution $(u, L)\colon N \to M$ is recursively defined by the following two rules:

(i) $u(0) = 0$ and $K(0) = \emptyset$;

(ii) for any $k \in N$ we seek to determine the crack set $L(k+1)$ and the displacement $u(k+1)$ such that $(u(k+1), L(k+1)) \in M$, $u(k+1) = (k+1)/s u_0(1)$ on $(\Gamma_u^1 \cup \Gamma_u^2) \setminus L(k+1)$ and $(u(k+1), L(k+1))$ is a minimizer of the functional

$$(v, L) \mapsto J((u(k), K(k)), (v, L)),$$

where $(v, L) \in M$, $v = (k+1)/s u_0(1)$ on $(\Gamma_u^1 \cup \Gamma_u^2) \setminus L$. The set $K(k+1)$ is given by the formula

$$K(k+1) = K(k) \cup L(k+1).$$

Let $u_\emptyset$ denote the displacement of the body Ω, without cracks, under the prescribed displacement on the boundary $u_0(1)$. With the use of a notation made

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earlier, $u_\emptyset$ is defined by $u_\emptyset = u(u_0(1), \emptyset)$. For any $k \in N$ we have $(k/su_\emptyset, \emptyset) \in M$ and $k/su_\emptyset = k/su_0(1)$ on $\Gamma_u^1 \cup \Gamma_u^2$. Therefore, with the notation

$$J_k = J((u(k), K(k)), (u(k+1), L(k+1)))$$

for any $k \in N$ we have

$$J_k \leqslant J((u(k), K(k)), ((k+1)/su_\emptyset, \emptyset)).$$

The last inequality may be written as

$$J_k \leqslant \left(\frac{k}{s}\right)^2 \int_\Omega \mu |\nabla u_\emptyset|^2 \, \mathrm{d}x, \tag{19}$$

$$J_k = \int_\Omega \mu |\nabla u(k+1)|^2 \, \mathrm{d}x + G\mathcal{H}^1(L(k+1) \setminus K(k)). \tag{20}$$

We can always find a curve in $\overline{\Omega}$ which is a length minimizer in the family of all curves in $\overline{\Omega}$ separating Γ_u^1 from Γ_u^2. Let us denote such a curve by S (which exists but it might not be unique). The curve S splits the domain $\overline{\Omega}$

$$\overline{\Omega} = \Omega^1 \cup \Omega^2, \qquad \Gamma_u^1 \subset \Omega^1, \qquad \Gamma_u^2 \subset \Omega^2,$$

$$\Omega^1 \cap \Omega^2 = \emptyset, \qquad \overline{\Omega^1} \cap \overline{\Omega^2} = S.$$

We define the following displacement

$$u_S(x) = \begin{cases} 0 & x \in \Omega^1 \\ \delta & x \in \Omega^2. \end{cases}$$

It is easy to see that for any $k \in N$ the pair $(k/su_S, S)$ belongs to M and $k/su_S = k/su_0(1)$ on $(\Gamma_u^1 \cup \Gamma_u^2) \setminus S$. We have therefore the inequality

$$J_k \leqslant G\mathcal{H}^1(S \setminus K(k)), \tag{21}$$

with J_k given by (20). From (21) we derive the following conclusion: *for large time k/s the crack set $K(k)$ is not void.* Indeed, suppose that the function $k \in N \mapsto (k/su_\emptyset, \emptyset)$ is an incremental solution constructed by the rules (i) and (ii) above. Then for any $k \in N$ the inequality (21) becomes an equality and the inequality (21) takes the form

$$\left(\frac{k}{s}\right)^2 \int_\Omega \mu |\nabla u_\emptyset|^2 \, \mathrm{d}x \leqslant G\mathcal{H}^1(S), \tag{22}$$

which lead to a contradiction. Therefore this model can predict crack appearance.

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We get more information about the behavior of the model if we use it in the case of an uni-axial traction experiment. The body with modulus of elasticity E has the configuration $\Omega = (0, L) \subset R$ and any crack set is a finite collection of points in the interval Ω, so the body is either undamaged or totally broken. The imposed displacement at the time t is

$$u_0(t) = t u_0(1),$$

where $u_0(1) = 0$ at $x = 0$ and $u_0(1) = D$ at $x = L$. The function $J((u, K), (v, L))$ takes the expression

$$J((u, K), (v, L)) = \int_0^L \tfrac{1}{2} E(v'(x))^2 \, \mathrm{d}x + G\#(L \setminus K),$$

where $\#(M)$ is the number of elements of the set M.

At the time k/s we have only two kinds of displacement-crack pairs which compete. These are:

(1) $(k/s u_\emptyset, \emptyset)$, where $u_\emptyset(x) = xD/L$;
(2) $(k/s u_S, S)$, where $S = \{x_1, \ldots, x_N\}$ is a crack set and u_S is a piecewise constant function on $[0, L] \setminus S$ such that $u_S(0) = 0$ and $u_S(1) = D$.

For any displacement-crack pair (u, K) we have

$$J((u, K), (k/s u_S, S)) = \begin{cases} (k/s)^2 \int_0^L \frac{1}{2} E(u'_\emptyset)^2 \, \mathrm{d}x & \text{if } S = \emptyset, \\ G\#(S \setminus K) & \text{if } S \neq \emptyset, \end{cases}$$

therefore among all pairs $(k/s u_S, S)$ it is sufficient to consider only the pairs with $\#(S) = 1$ or $S = \emptyset$.

For small time k/s the body remains uncracked and for large time k/s a crack appears in the body. Precisely, for small k/s we have

$$(u(k), K(k)) = (k/s u_\emptyset, \emptyset)$$

and for large k/s we have

$$(u(k), K(k)) = (k/s u_S, S),$$

with $\#(S) = 1$. An inequality similar to (22) leads us to an equation for the critical time t_c when the crack appears

$$t_c^2 \int_0^1 \tfrac{1}{2} E(u'_\emptyset)^2 \, \mathrm{d}x = G.$$

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We obtain the following expression of the uni-axial stress $\sigma_c = t_c E u'_\emptyset$, existing in the uncracked body when the model predicts its fracture:

$$\sigma_c = \left(\frac{2EG}{L}\right)^{1/2} . \tag{23}$$

We see that the stress σ_c and the quantity G cannot be both constants of material in this model.

5. Existence of weak incremental solutions

5.1. The spaces SBV and SBD

This section is dedicated to a brief voyage through the spaces **SBV** and **SBD**.

We use the notation $\mu \ll \lambda$ if the measure μ is absolutely continuous with respect to the measure λ. For any measure μ we denote by $|\mu|(B)$ the variation of μ over the Borel set $B \subset \Omega$, defined by the relation

$$|\mu|(B) = \sup\left\{\sum_{i=1}^{\infty} |\mu(A_i)| : \cup_{i=1}^{\infty} A_i \subset B,\, A_i \cap A_j = \emptyset \ \forall i \neq j\right\} .$$

The measure μ has finite total variation (over Ω) if $|\mu|(\Omega) < +\infty$.

$\mathbf{BV}(\Omega, R^n)$ is the space of functions $\mathbf{u} \in L^1(\Omega, R^n)$ with the distributional derivative $D\mathbf{u}$ representable as a vector measure with finite total variation. We refer to the book of Evans and Gariepy [25] for the main properties of such functions. The approximate limit of $\mathbf{u}$ at the point $x \in \Omega$ is that $\tilde{\mathbf{u}}(x)$ defined by the equality

$$\lim_{\rho \to 0_+} \frac{\int_{B_\rho(x)} |\mathbf{u}(y) - \tilde{\mathbf{u}}(x)| \, \mathrm{d}y}{|B_\rho(x)|} = 0.$$

The Lebesgue set of $\mathbf{u}$ is the set of points where $\mathbf{u}$ has an approximate limit. The complementary set is a $\mathcal{L}^n$ negligible set denoted by $\mathbf{S_u}$. De Giorgi proved in [23] that for any $\mathbf{u} \in \mathbf{BV}(\Omega, R^n)$ the set $\mathbf{S_u}$ is countably rectifiable. Moreover, for $\mathcal{H}^{n-1}$ almost every $x \in \mathbf{S_u}$ there is a triplet $(\mathbf{u}^+(x), \mathbf{u}^-(x), \mathbf{n}(x))$ such that

(1) $\mathbf{n}(x)$ is a unit vector normal to $\mathbf{S_u}$ at x;
(2) $(\mathbf{u}^+(x), \mathbf{u}^-(x))$ are the approximate limits of $\mathbf{u}$ in x associated with the direction $\mathbf{n}(x)$ (for the definition see (3)).

This triplet is uniquely determined up to a change of sign of $\mathbf{n}$ and an interchange of $\mathbf{u}^+, \mathbf{u}^-$. The jump of $\mathbf{u}$ across $\mathbf{S_u}$ is $[\mathbf{u}] = \mathbf{u}^+ - \mathbf{u}^-$; notice that the tensor field $[\mathbf{u}] \otimes \mathbf{n}$ over $\mathbf{S_u}$ is independent of the choice of the field of normals $\mathbf{n}$.

For any $\mathbf{u} \in \mathbf{BV}(\Omega, R^n)$ the measure $D\mathbf{u}$ admits the decomposition into absolute continuous and singular parts with respect to the Lebesgue measure dx: $D\mathbf{u} =$

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$D^a\mathbf{u} + D^s\mathbf{u}$. Calderon and Zygmund [19] theorem gives the following decomposition of the measure $D\mathbf{u}$ into three mutually singular parts

$$D\mathbf{u} = \nabla\mathbf{u}(x)\,\mathrm{d}x + [\mathbf{u}] \otimes \mathbf{n}\,\mathrm{d}\mathcal{H}^{n-1}_{|\mathbf{S_u}} + C(\mathbf{u}).$$

$\nabla\mathbf{u}$ is the approximate gradient of $\mathbf{u}$ defined for almost every $x \in \Omega$ by the equality

$$\lim_{\rho\to 0_+} \frac{\int_{B_\rho(x)} |\mathbf{u}(y) - \mathbf{u}(x) - \nabla\mathbf{u}(x)\cdot(y-x)|\,\mathrm{d}y}{|B_\rho(x)\|y-x|} = 0.$$

The jump part of $D\mathbf{u}$ is

$$D^j\mathbf{u} = [\mathbf{u}] \otimes \mathbf{n}\,\mathrm{d}\mathcal{H}^{n-1}_{|\mathbf{S_u}}.$$

$C(\mathbf{u})$ is called the Cantor part of $D\mathbf{u}$; for any Borel set $B \subset \Omega$ the quantity $C(\mathbf{u})(B)$ is defined by $C(\mathbf{u})(B) = D^s\mathbf{u}(B \setminus \mathbf{S_u})$. We have therefore

$$D^a\mathbf{u} = \nabla\mathbf{u}\,\mathrm{d}x, \qquad D^s\mathbf{u} = [\mathbf{u}] \otimes \mathbf{n}\,\mathrm{d}\mathcal{H}_{|K} + C(\mathbf{u}).$$

The space $\mathbf{SBV}(\Omega, R^n)$ of special functions with bounded variation was introduced by De Giorgi and Ambrosio in the study of a class of free discontinuity problems ([21], [1], [2]). A general reference to **SBV** and free-discontinuity problems is Ambrosio, Fusco and Pallara [10]. This space is defined as follows:

$$\mathbf{SBV}(\Omega, R^n) = \{\mathbf{u} \in \mathbf{BV}(\Omega, R^n) : |D^s\mathbf{u}|(\Omega \setminus \mathbf{S_u}) = 0\}.$$

For any $\mathbf{u} \in \mathbf{BV}(\Omega, R^n)$, $\mathbf{u}$ is a special function with bounded variation if and only if the Cantor part of $D\mathbf{u}$ is null.

For several versions of the compactness theorem in **SBV** we refer to the aforementioned papers of De Giorgi and Ambrosio. We shall use this theorem in the following form:

THEOREM 5.1. *Let $(\mathbf{u}_h)_h$ be a sequence in $\mathbf{SBV}(\Omega, R^k)$ and C be a constant such that for any h*

$$\int_\Omega |\nabla\mathbf{u}_h|^2\,\mathrm{d}x + \mathcal{H}(\mathbf{S}_{\mathbf{u}_h}) + \|\mathbf{u}_h\|_{L^\infty} \leqslant C.$$

Then there exist $\mathbf{u} \in \mathbf{SBV}(\Omega, R^k)$ and a subsequence, still denoted by $(\mathbf{u}_h)_h$, such that

$$\begin{cases} \mathbf{u}_h \to \mathbf{u} \quad \textit{in } L^2(\Omega, R^k), \\ \nabla\mathbf{u}_h \to \nabla\mathbf{u} \quad \textit{weakly in } L^2(\Omega, M^{n\times k}), \\ D^j\mathbf{u}_h \to D^j\mathbf{u} \quad \textit{weakly as Radon measures}, \end{cases}$$

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and

$$\mathcal{H}^{n-1}(\mathbf{S_u}) \leqslant \liminf_{h\to\infty} \mathcal{H}^{n-1}(\mathbf{S}_{\mathbf{u}_h}).$$

A description of the space of special functions with bounded deformation $\mathbf{SBD}(\Omega)$, can be found in Ambrosio, Coscia and Dal Maso [6]. Any function $\mathbf{u} \in L^1(\Omega, R^n)$ belongs to $\mathbf{BD}(\Omega)$ if $E\mathbf{u}$, the symmetric part of the distributional derivative of $\mathbf{u}$, is representable as a vector measure with finite total variation.

For any $\mathbf{u} \in \mathbf{BD}(\Omega)$ the measure $E\mathbf{u}$ decomposes with respect to the Lebesgue measure into absolute continuous and singular parts

$$E\mathbf{u} = E^a\mathbf{u} + E^s\mathbf{u}.$$

We denote by $|E\mathbf{u}|$ the variation of the measure $E\mathbf{u}$. Kohn introduced in [31] the set $\Theta_\mathbf{u}$

$$\Theta_\mathbf{u} = \left\{ x \in \Omega\colon \limsup_{\rho\to 0^+} \frac{|E\mathbf{u}|(B_\rho(x))}{\rho^{n-1}} > 0 \right\}$$

and proved that it is countably rectifiable. Let $\mathbf{J_u}$ be the subset of Ω of all points $x \in \Omega$ such that there is a unit vector $\nu(x)$ with the property that $\mathbf{u}$ has different approximate limits $\mathbf{u}^+(x) = \tilde{\mathbf{u}}(x, \nu(x))$, $\mathbf{u}^-(x) = \tilde{\mathbf{u}}(x, -\nu(x))$ defined by the relation (3). It is straightforward that $\mathbf{J_u} \subset \mathbf{S_u}$. However, $\mathbf{S_u}$ may not be countably rectifiable. In [6] it is proved that $\Theta_\mathbf{u}$ coincides with $\mathbf{J_u}$ up to a $\mathcal{H}^{n-1}$ negligible set, therefore $\mathbf{J_u}$ is countably rectifiable. The triplet $(\mathbf{u}^+(x), \mathbf{u}^-(x), \mathbf{n}(x))$ exists for $\mathcal{H}^{n-1}$ almost every $x \in J_\mathbf{u}$, where $\mathbf{n}(x)$ is the normal unit vector to $\Theta_\mathbf{u}$ at x; as previously the tensor field over $\mathbf{J_u}$ defined by $[\mathbf{u}] \otimes \mathbf{n}$ is uniquely determined. We denote by $[\mathbf{u}] \odot \mathbf{n}$ its symmetric part.

The difference between $\mathbf{S_u}$ and $\mathbf{J_u}$ is subtle. Let us quote only the fact that for a function $\mathbf{u} \in \mathbf{SBV}(\Omega, R^n)$ these sets coincide up to a $\mathcal{H}$-negligible set.

The following decomposition theorem is due to Ambrosio, Coscia and Dal Maso [6] and asserts that

$$E\mathbf{u} = \varepsilon(\mathbf{u})(x)\,\mathrm{d}x + [\mathbf{u}] \odot \mathbf{n}\,\mathrm{d}\mathcal{H}^{n-1}_{|\mathbf{J_u}} + E^c(\mathbf{u}).$$

Here $\varepsilon(\mathbf{u})$ is the approximate symmetric gradient, defined for almost every $x \in \Omega$ by

$$\lim_{\rho\to 0^+} \frac{1}{\rho^n} \int_{B_\rho(x)} \frac{(\mathbf{u}(y) - \mathbf{u}(x) - \varepsilon(\mathbf{u})(x)(y-x)) \cdot (y-x)}{|y-x|^2}\,\mathrm{d}y = 0.$$

The jump part of $E\mathbf{u}$ is

$$E^j\mathbf{u} = [\mathbf{u}] \odot \mathbf{n}\,\mathrm{d}\mathcal{H}^{n-1}_{|\mathbf{J_u}}.$$

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$E^c\mathbf{u}$ is the Cantor part of $E\mathbf{u}$, that is the part of $E^s\mathbf{u}$ not concentrated on $\mathbf{J_u}$. Therefore we have

$$E^a\mathbf{u} = \varepsilon(\mathbf{u})\,\mathrm{d}x, \qquad E^s\mathbf{u} = E^j\mathbf{u} + E^c\mathbf{u}.$$

The definition of $\mathbf{SBD}(\Omega)$ is the following:

$$\mathbf{SBD}(\Omega, R^n) = \{\mathbf{u} \in \mathbf{BD}(\Omega) : |E^s\mathbf{u}|(\Omega \setminus \mathbf{J_u}) = 0\}.$$

We have the inclusion

$$\mathbf{SBV}(\Omega, R^n) \subset \mathbf{SBD}(\Omega).$$

For the compactness theorem in **SBD** we refer to Bellettini, Coscia and Dal Maso [15]. We shall use this theorem in the following form:

THEOREM 5.2. *Let us consider the function*

$$\mathbf{F} \in M^{n\times n}_{\mathrm{sym}} \mapsto w(\mathbf{F}) = (1/2)\mathbf{CF}:\mathbf{F},$$

with $\boldsymbol{C}$ *a positive definite symmetric 4-order tensor. Let* $(\mathbf{u}_h)_h$ *be a sequence in* $\mathbf{SBD}(\Omega)$ *and* C *a constant such that for any* h

$$\int_\Omega w(\varepsilon(\mathbf{u}_h))\,\mathrm{d}x + \mathcal{H}^{n-1}(\mathbf{J}_{\mathbf{u}_h}) + \|\mathbf{u}_h\|_{L^\infty} \leqslant C.$$

Then there exist $\mathbf{u} \in \mathbf{SBV}(\Omega, R^k)$ *and a subsequence, still denoted by* $(\mathbf{u}_h)_h$*, such that*

$$\begin{cases} \mathbf{u}_h \to \mathbf{u} \quad \text{in } L^2(\Omega, R^k), \\ \varepsilon(\mathbf{u}_h) \to \varepsilon(\mathbf{u}) \quad \text{weakly in } L^2(\Omega, M^{n\times n}_{\mathrm{sym}}), \\ E^j\mathbf{u}_h \to E^j\mathbf{u} \quad \text{weakly as Radon measures}, \end{cases}$$

and

$$\mathcal{H}^{n-1}(\mathbf{J_u}) \leqslant \liminf_{h\to\infty} \mathcal{H}^{n-1}(\mathbf{J}_{\mathbf{u}_h}).$$

5.2. Existence of weak constrained incremental solutions

In order to give a weak formulation of the model described in Definition 4.1 let us weaken first the space M of displacement-crack pairs. We introduce the new set of weak displacement-crack pairs $\mathcal{M}$

$$\mathcal{M} = \{(\mathbf{u}, K) : K \text{ is } \sigma\text{-rectifiable}, \mathbf{u} \in \mathbf{SBD}(\Omega) \text{ and } |E^s\mathbf{u}|(\Omega \setminus K) = 0\}. \qquad (24)$$

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Given $(\mathbf{u}, K) \in \mathcal{M}$, the set K is countably rectifiable but it is not necessarily closed; we have also weaker conditions on the regularity of the displacement $\mathbf{u}$. A direct consequence of (29) is that any (strong) displacement-crack pair $(\mathbf{u}, K)$ such that $\mathbf{u} \in L^\infty(\Omega, R^n)$ belongs to the set $\mathcal{M}$.

Let us define the functional $\mathcal{J}$, the weak version of the functional J introduced at Definition 4.1: $\mathcal{J}: \mathcal{M} \times \mathcal{M} \to R$,

$$\mathcal{J}((\mathbf{u}, K), (\mathbf{v}, L)) = \int_\Omega w(\varepsilon(\mathbf{v}))\, dx + G\mathcal{H}^{n-1}(L \setminus K). \tag{25}$$

Before the introduction of the weak form of the function Ψ from the same definition, let us explain what we mean by $\mathbf{u} = \mathbf{u}_0$ on the boundary of Ω. We consider, for technical reasons, that there is an open bounded set Λ with piecewise Lipschitz boundary such that $\overline{\Omega} \subset \Lambda$. The imposed boundary displacement is $\mathbf{u}_0 \in \mathbf{SBD}(\Lambda)$ such that $\mathbf{J}_{\mathbf{u}_0} \cap \overline{\Omega} = \emptyset$. Then, for any $\mathbf{u} \in \mathbf{SBD}(\lambda)$, $\mathbf{u} = \mathbf{u}_0$ on $\partial\Omega$ means that $\mathbf{u} = \mathbf{u}_0$ in $\Lambda \setminus \Omega$. We denote the set of all such functions $\mathbf{u}$ by $\mathbf{SBD}(\Omega, \mathbf{u}_0)$. The reason for this choice of defining boundary conditions is that the space $\mathbf{SBD}(\Omega, \mathbf{u}_0)$ is closed in $\mathbf{SBD}(\Lambda)$ in the L^2 convergence. Note that $\mathbf{SBD}(\Omega, \mathbf{u}_0)$ can be identified with a subspace of $\mathbf{SBD}(\Omega)$ by the inclusion map $\mathbf{u} \mapsto \mathbf{u}_{|\Omega}$.

Let us consider a curve of imposed displacements $\lambda \mapsto \mathbf{u}_0(\lambda)$ such that $\|\mathbf{u}_0(\lambda)\|_{L^\infty(\Lambda)} < +\infty$. We impose a supplementary condition for a displacement field $\mathbf{u}$ to be admissible at the time λ, namely

$$\|\mathbf{u}\|_{L^\infty(\Lambda)} \leqslant \|\mathbf{u}_0(\lambda)\|_{L^\infty(\Lambda)}. \tag{26}$$

The space of all $\mathbf{u} \in \mathbf{SBD}(\Omega, \mathbf{u}_0(\lambda))$ such that the constraint (26) holds will be denoted by $\mathbf{SBD}^\infty(\Omega, \mathbf{u}_0(\lambda))$.

The function $\tilde{\Psi}$, introduced instead of Ψ, is defined as follows

$$\tilde{\Psi}: [0, +\infty) \times \mathcal{M} \to \{0, +\infty\},$$

$$\tilde{\Psi}(\lambda, (\mathbf{u}, K)) = \begin{cases} 0 & \text{if } \mathbf{u} \in \mathbf{SBD}^\infty(\Omega, \mathbf{u}_0(\lambda)) \text{ and} \\ & \mathcal{H}^{n-1}(K \setminus \mathbf{J}_\mathbf{u}) = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

DEFINITION 5.1 (weak version of Definition 4.1). Let us consider the space $\mathcal{M}$ endowed with the topology given by the convergence

$$(\mathbf{u}_h, K_h) \to (\mathbf{u}, K) \quad \text{if } \mathbf{u}_h L^2 \to \mathbf{u}. \tag{27}$$

Let us consider also the function $\mathcal{J}$, the curve of imposed displacements $t \mapsto \mathbf{u}_0(t)$ with the associated function $\tilde{\Psi}$ and the initial data $(\mathbf{u}_0, K) \in M$ such that $\mathbf{u}_0 = \mathbf{u}(\mathbf{u}_0(0), K)$.

For any $s \geqslant 1$ we recursively define $(\mathbf{u}^s, K^s): N \to \mathcal{M}$ as:

(i) $(\mathbf{u}^s, K^s)(0) = (\mathbf{u}_0, K)$;

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(ii) for any $k \in N$ $(\mathbf{u}^s, \mathbf{J}_{\mathbf{u}^s})(k+1) \in \mathcal{M}$ minimizes the functional

$$(\mathbf{v}, L) \in \mathcal{M} \mapsto \mathcal{J}((\mathbf{u}^s, K^s)(k), (\mathbf{v}, L)) + \tilde{\Psi}((k+1)/s, (\mathbf{v}, L))$$

over $\mathcal{M}$. In order to verify the constraint (10), $K^s(k+1)$ is defined by the formula

$$K^s(k+1) = K^s(k) \cup \mathbf{J}_{\mathbf{u}^s(k+1)}. \tag{28}$$

An energy minimizing movement associated to $\mathcal{J}$ with the constraints (10), $\tilde{\Psi}$ and initial data $(\mathbf{u}_0, K)$ is any $(\mathbf{u}, \mathbf{J}_\mathbf{u})\colon [0, +\infty) \to \mathcal{M}$ having the property: there is a diverging sequence (s_i) such that for any $t > 0$ $\mathbf{u}^{s_i}([s_i t]) \to \mathbf{u}(t) \in \mathbf{SBD}^\infty(\Omega, \mathbf{u}_0(t))$ in $L^2(\Omega, R^n)$ as $i \to \infty$. The active crack at the time t is $\mathbf{J}_{\mathbf{u}(t)}$ and the damaged region at the same instant is

$$K(t) = \cup_{s\in[0,t]} \mathbf{J}_{\mathbf{u}(s)}.$$

Let us remark that the disappearance of the set $L^s(k+1)$ from the definition of the incremental solution (28) is only apparent, because if $(\mathbf{u}^s, L^s)(k+1)$ minimizes the functional

$$(\mathbf{v}, L) \in \mathcal{M} \mapsto \mathcal{J}((\mathbf{u}^s, K^s)(k), (\mathbf{v}, L)) + \tilde{\Psi}((k+1)/s, (\mathbf{v}, L))$$

then $\tilde{\Psi}((k+1)/s, (\mathbf{u}^s, L^s)(k+1)) = 0$, hence

$$\mathcal{H}^{n-1}(K \setminus \mathbf{J}_\mathbf{u}) = 0.$$

From Theorem 5.2 we notice that functionals like $\mathcal{J}$ are L^2 sequential lower semi-continuous and coercive on closed sets $\mathbf{V} \subset \mathbf{SBD}(\Omega)$ of functions equally bounded in L^∞ norm. If we consider in particular the functional

$$\mathbf{v} \in \mathbf{V} \mapsto \mathcal{J}((\mathbf{u}^s, K^s)(k), (\mathbf{v}, \mathbf{J}_\mathbf{v}))$$

the following theorem is true by a trivial induction:

THEOREM 5.3 (existence of weak incremental constrained solutions). *Let $\Omega, \Lambda \subset R^n$ be bounded open sets with piecewise smooth boundary such that $\overline{\Omega} \subset \Lambda$. Let*

$$\mathbf{u}_0\colon N \to \mathbf{SBD}(\Lambda) \cap L^\infty(\Lambda)$$

be a given sequence of imposed displacements such that $\mathbf{J}_{\mathbf{u}_0(\lambda)} \cap \overline{\Omega} = \emptyset$ and let $(\mathbf{u}_0, K)$ be a given admissible displacement-crack pair in Ω such that $\mathbf{u}_0 = \mathbf{u}(\mathbf{u}_0(0), K)$ on $\partial\Omega$.

Then there exists a sequence $(\mathbf{u}, K)\colon N \to \mathcal{M}$ such that:

(i) $\mathbf{u}(0) = \mathbf{u}_0$ *and* $K(0) = K$;

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(ii) *for any* $k \in N$ *there is* $(\mathbf{u}(k+1), \mathbf{J}_{\mathbf{u}(k+1)}) \in \mathcal{M}$*, such that* $\mathbf{u}(k+1) = \mathbf{u}_0(k+1)$ *on* $\partial\Omega$ *and* $(\mathbf{u}(k+1), \mathbf{J}_{\mathbf{u}(k+1)})$ *is a minimizer of the functional*

$$(\mathbf{v}, L) \in \mathcal{M}, \mathbf{v} = \mathbf{u}_0(k+1) \quad \text{on } \partial\Omega \mapsto \mathcal{J}((\mathbf{u}(k), K(k)), \mathbf{v}, L)).$$

The set $K(k+1)$ *is given by the formula*

$$K(k+1) = K(k) \cup \mathbf{J}_{\mathbf{u}(k+1)}.$$

5.3. The anti-plane case

In the anti-plane case we have to replace $\mathbf{SBD}(\Omega)$ by $\mathbf{SBV}(\Omega, R)$. Let us consider a larger domain $\overline{\Omega} \subset \Lambda \subset R^2$, a boundary condition $\mathbf{u}_0 \in \mathbf{SBV}(\Lambda, R) \cap L^\infty(\Lambda)$ and $u \in \mathbf{SBV}(\Lambda, R)$ such that $\mathbf{u} = \mathbf{u}_0$ in $\Lambda \setminus \overline{\Omega}$. We don't need the constraint (26) because in this case we have a maximum principle. Indeed, with the notations

$$I(\mathbf{u}) = \int_\Omega \mu |\nabla \mathbf{u}|^2 \, \mathrm{d}x + G\mathcal{H}^1(\mathbf{S}_\mathbf{u}),$$

$$\overline{\mathbf{u}}(x) = \begin{cases} \mathbf{u}(x) & \text{if } |\mathbf{u}(x)| \leqslant \|\mathbf{u}_0\|_{L^\infty(\Lambda)}, \\ \|\mathbf{u}_0\|_{L^\infty(\Lambda)} & \text{otherwise,} \end{cases}$$

we have the inequality $I(\overline{\mathbf{u}}) \leqslant I(\mathbf{u})$ and we notice that $\overline{\mathbf{u}} = \mathbf{u}_0$ on $\Lambda \setminus \overline{\Omega}$.

The set of **SBV** displacements compatible with the boundary displacement $\mathbf{u}_0$ is denoted by $\mathbf{SBV}(\Omega, \mathbf{u}_0)$.

The set of weak displacement-crack pairs will be

$$\begin{aligned} \mathcal{N} \ = \ &\{(\mathbf{u}, K) \colon K \text{ is } \sigma\text{-rectifiable}, \mathbf{u} \in \mathbf{SBV}(\Omega) \text{ and} \\ &|D^s\mathbf{u}|(\Omega \setminus K) = 0\}. \end{aligned}$$

For a given path of imposed boundary displacements $\lambda \mapsto \mathbf{u}_0(\lambda) \in \mathbf{SBV}(\Lambda, R) \cap L^\infty(\Lambda, R)$ we define

$$\tilde{\Phi} \colon [0, +\infty) \times \mathcal{N} \to \{0, +\infty\},$$

$$\tilde{\Phi}(\lambda, (\mathbf{u}, K)) = \begin{cases} 0 & \text{if } \mathbf{u} \in \mathbf{SBV}(\Omega, \mathbf{u}_0(\lambda)) \text{ and} \\ & \mathcal{H}^{n-1}(K \setminus \mathbf{S}_\mathbf{u}) = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

With this setting we obtain the notion of a weak incremental solution in the case of anti-plane displacements as in Definition 5.1. All we have to do is to replace the space $\mathcal{M}$ by $\mathcal{N}$, the function $\tilde{\Psi}$ by $\tilde{\Phi}$ and $\mathbf{J}_\mathbf{u}$ by $\mathbf{S}_\mathbf{u}$. The existence of weak incremental solutions is a consequence of Theorem 5.1.

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The partial regularity results of De Giorgi, Carriero and Leaci [22] tell us that weak incremental solutions give raise to strong incremental solutions. The existence of incremental solutions is therefore true in the anti-plane case. For $\mathcal{H}^{n-1}$-smoothness of $\mathbf{S_u}$, where $\mathbf{u}$ is a minimizer of the Mumford–Shah functional, we refer to Ambrosio and Pallara [7], Ambrosio, Fusco and Pallara [8], [9].

5.4. Justification of the weak formulation

Let us compare the Definitions 4.1 and 5.1, where strong, respectively weak (constrained) energy minimizing movements were introduced.

We consider the Sobolev space associated to the crack set K (see [5])

$$W_K^{1,2} = \left\{ \mathbf{u} \in \mathbf{SBV}(\Omega, R^n) \colon \int_\Omega |\nabla \mathbf{u}|^2 \, \mathrm{d}x + \int_K [\mathbf{u}]^2 \, \mathrm{d}\mathcal{H}^{n-1} < +\infty, \right.$$
$$\left. |D^s \mathbf{u}| \ll \mathcal{H}^{n-1}_{|K} \right\}.$$

The following equality has been proved in [22]

$$W^{1,2}(\Omega \setminus K, R^n) \cap L^\infty(\Omega, R^n) = W_K^{1,2}(\Omega, R^n) \cap L^\infty(\Omega, R^n). \tag{29}$$

Therefore if $\mathbf{u} = \mathbf{u}(\mathbf{u}_0, K)$ and $\mathbf{u} \in L^\infty(\Omega, R^n)$ then $\mathbf{u}$ is a special function with bounded variation. Also, if $(\mathbf{u}, K) \in M$ is a displacement-crack pair and $\mathbf{u}$ is essentially bounded, then $\mathbf{u} \in \mathbf{SBV}(\Omega, R^n)$ and $\overline{\mathbf{S_u}} \subset K$. These inclusions may lead to the introduction of the following space of weak displacement-crack pairs

$$\mathcal{M}' = \{(\mathbf{u}, K) \colon K \text{ is } \sigma\text{-rectifiable}, \mathbf{u} \in \mathbf{SBV}(\Omega, R^n) \text{ and }$$
$$|D^s \mathbf{u}|(\Omega \setminus K) = 0\}.$$

However the bulk part of the functional J (in weak form $\mathcal{J}$) controls only the symmetric part of the gradient of the displacement. This is the reason of considering the larger space $\mathcal{M}$ defined at (24). In conclusion, the pair $(\mathbf{u}, L)$ is replaced by the pair $(\mathbf{u}, \mathbf{J_u})$ (or, in the anti-plane case, by $(u, \mathbf{S}_u)$). The weak version of the measure $\mathrm{j}(\mathbf{u}, L)$ is then $E^j \mathbf{u}$ (or $D^j u$ in the anti-plane case).

The following proposition is a direct consequence of Theorem 5.2.

PROPOSITION 5.1. *Let $\mathbf{u}_h$ be a sequence in $\mathbf{SBD}(\Omega)$ which converges in $L^2(\Omega, R^n)$ to $\mathbf{u} \in \mathbf{SBD}(\Omega)$ such that*

$$\int_\Omega w(\varepsilon(\mathbf{u}_h)) \, \mathrm{d}x + \mathcal{H}^{n-1}(\mathbf{J}_{\mathbf{u}_h}) + \|\mathbf{u}_h\|_{L^\infty} \leqslant C \tag{30}$$

for some constant C independent of h. Then there exists a subsequence, still denoted by $\mathbf{u}_h$, such that

$$\begin{cases} \varepsilon(\mathbf{u}_h) \to \varepsilon(\mathbf{u}) & \text{weakly in } L^2(\Omega, M^{n\times n}_{\mathrm{sym}}), \\ E^j \mathbf{u}_h \to E^j \mathbf{u} & \text{as Radon measures} \end{cases}$$

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and

$$\mathcal{H}^{n-1}(\mathbf{J_u}) \leqslant \liminf_{h\to\infty} \mathcal{H}^{n-1}(\mathbf{J}_{\mathbf{u}_h}).$$

From this proposition we infer the following corollary

COROLLARY 5.1. *Let us consider* $(\mathbf{u}, \mathbf{J_u}) : [0, +\infty) \to \mathcal{M}$ *an energy minimizing movement,* (s_i) *a diverging sequence and* $(\mathbf{u}^s(k), \mathbf{J}_{\mathbf{u}^s(k)})$ *an incremental solution as in Definition* 5.1*, such that*

$$\mathbf{u}^{s_i}([s_i t]) \to \mathbf{u}(t) \tag{31}$$

in $L^2(\Omega, R^n)$ *as* $i \to \infty$*, for any* $t > 0$*. We have then*

$$\begin{cases} E^j \mathbf{u}^{s_i}([s_i t]) \to E^j \mathbf{u}(t) \quad \textit{as Radon measures} \\ \mathcal{H}^{n-1}(\mathbf{J}_{\mathbf{u}(t)}) \leqslant \liminf_{i\to\infty} \mathcal{H}^{n-1}(\mathbf{J}_{\mathbf{u}^{s_i}([s_i t])}). \end{cases} \tag{32}$$

Therefore the relations (31), (32) are the weak version of (16). Moreover, we notice that (32) is a consequence of (31). However, this is a priori true only in the case of weak *constrained* energy minimizing movements.

6. Introduction of Small Viscosity

In the paper [4] Ambrosio and Braides introduce a generalized minimizing movement based model for the propagation of a crack in the presence of viscous forces in the body. They give as initial datum at $t = 0$ the anti-plane displacement $u_0 \in \mathbf{SBV}(\Omega, R) \cap L^\infty(\Omega, R)$. For a given s they recursively define a sequence $(u_k^s)_k$ in $\mathbf{SBV}(\Omega, R)$ and an increasing sequence of closed rectifiable sets $(K_k^s)_k$ as follows: $u_0^s = u_0$, $K_0^s = \emptyset$ and $u_{k+1}^s = w$, $K_{k+1}^s = \overline{\mathbf{S}_w} \cup K_k^s$, where w is a minimizer of the functional

$$v \mapsto \int_\Omega |\nabla v|^2 \,\mathrm{d}x + \mathcal{H}^1(\mathbf{S}_v \setminus K_k^s) + s \int_\Omega |v - u_k^s|^2 \,\mathrm{d}x \tag{33}$$

over the set of all v such that

$$v \in \mathbf{SBV}(\Omega, R), \qquad \|v\|_\infty \leqslant \|u_0\|_\infty.$$

The generalized minimizing movements obtained as limits of such incremental solutions, when s diverges, correspond to the following situation: a body evolves from the initial state u_0, with the initial crack $\mathbf{S}_{u_0}$, under a constant imposed boundary displacement. The equation of evolution for the displacement is

$$\mathrm{div}\, \nabla u(t) + \dot{u}(t) = 0.$$

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The authors obtain an existence result for the generalized minimizing movement introduced by them. After the introduction of the piecewise constant function

$$u^s(t) = u^s_{[st]},$$

they find the following uniform estimate

$$\|u^s(t') - u^s(t)\|_{L^2} \leqslant M\sqrt{t' - t + \frac{1}{s}} \quad \text{if } t' \geqslant t. \tag{34}$$

Therefore there exists a diverging sequence $(s_i)_i$ such that u^{s_i} converges to u uniformly in $L^\infty([0, T], L^2(\Omega, R))$, for all $T > 0$ and

$$u \in C^{0,1/2}([0, +\infty); L^2(\Omega, R)). \tag{35}$$

This result is obtained under the assumption of constant imposed boundary displacement, equal to the trace on the boundary of the initial datum u_0.

It is natural to introduce the Lamé constant μ and the viscosity λ in the expression of the functional (33) and modify it as follows

$$v \mapsto \int_\Omega \mu|\nabla v|^2 \,\mathrm{d}x + \mathcal{H}^{n-1}(\mathbf{S}_v \setminus K^s_k) + \lambda s \int_\Omega |v - u^s_k|^2 \,\mathrm{d}x.$$

We obtain the more physical case of an anti-plane displacement satisfying at any moment t the equation

$$\operatorname{div} \mu\nabla u(t) + \lambda \dot{u}(t) = 0.$$

The estimate (34) becomes

$$\|u^s(t') - u^s(t)\|_{L^2} \leqslant M\sqrt{t' - t + \frac{1}{\lambda s}} \quad \text{if } t' \geqslant t.$$

We expect to obtain our model, in the case of anti-plane displacements, when the viscosity λ converges to 0. It is easy to see that if λ converges to 0 then the uniform estimate from above is lost.

We notice that the crack appearance can not occur in this model in a physically acceptable way.

Indeed suppose that for any $t > t' > 0$ we have

$$\mathbf{S}_{u(t')} \subset \mathbf{S}_{u(t)}.$$

This hypothesis means that the damaged region

$$K(t) = \cup_{s\in[0,t]}\mathbf{S}_{u(s)}$$

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is the active crack $\mathbf{S}_{u(t)}$. We suppose moreover that the energy

$$E(t) = \int_\Omega |\nabla u(t)|^2 \, \mathrm{d}x + \mathcal{H}^1(K(t))$$

is a decreasing function.

From the above suppositions, the compactness theorem in **SBV** and (35) we infer that:

(a) the function $t \mapsto E(t)$ is decreasing lower semicontinuous,
(b) the function $t \mapsto \mathcal{H}^1(K(t))$ is increasing lower semicontinuous,
(c) the elastic energy

$$t \mapsto \int_\Omega |\nabla u(t)|^2 \, \mathrm{d}x$$

is a decreasing function (from (a) and (b)) and it is lower semicontinuous.

A straightforward consequence of items (a), (b), (c) is that for any t the lateral limits of the function $s \mapsto \mathcal{H}^1(K(s))$ at the moment t are both equal to the value $\mathcal{H}^1(K(t))$, that is the length of the crack grows continuously with time.

We mention however that we don't know if for any minimizing movement $u(t)$ the energy $E(t)$ decreases with time. Again from the compactness theorem in **SBV** all we can prove is that for any $t < t'$ we have the inequalities

$$\liminf_{i\to\infty} E(u^{s_i}(t)) \geqslant \liminf_{i\to\infty} E(u^{s_i}(t')),$$

$$E(u(t)) \leqslant \liminf_{i\to\infty} E(u^{s_i}(t)), \qquad E(u(t')) \leqslant \liminf_{i\to\infty} E(u^{s_i}(t')),$$

where

$$E(u) = \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \mathcal{H}^1(\mathbf{S}_u).$$

7. A Partial Existence Result

The main open theoretical problem is the general existence of an energy minimizing movement according to our definitions. Below is described a partial existence result based on a sound physical assumption (36). Nevertheless, we do not know if (36) can be proved from the basic assumptions of the model.

THEOREM 7.1. *Let us consider for any given s an incremental solution $k \mapsto (\mathbf{u}^s(k), K^s(k)) \in M$, according to Definition* 4.1, *such that $\mathbf{u}^s(k)$ are equally bounded in L^∞. Let us suppose that the power communicated by the rest of the universe to the body is uniformly bounded at any time t. The incremental form of*

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this assumption consists in the existence of a constant P such that for any k and s we have

$$\langle \mathbf{T}_k^s \tfrac{1}{2}(\mathbf{u}_0((k+1)/s) + \mathbf{u}_0(k/s)), \Delta \mathbf{u}_0(k,s)\rangle \leqslant P/s, \tag{36}$$

where $\mathbf{T}_k^s = \mathbf{T}(K^s(k))$ *and* $\Delta \mathbf{u}_0(k,s) = \mathbf{u}_0((k+1)/s) - \mathbf{u}_0(k/s)$. *Then for any* $t > 0$ *there exist diverging sequences* $(s_i)_i$ *and* $(k_i)_i$ *such that* k_i/s_i *converges to* t,

$$\begin{cases} \mathbf{u}^{s_i}(k_i) \to \mathbf{u}(t) \quad \text{in } L^2(\Omega, R^n), \\ j(\mathbf{u}^{s_i}, L^{s_i})(k_i) \to j(\mathbf{u}, L)(t) \quad \text{weakly as Radon measures} \end{cases} \tag{37}$$

as $i \to \infty$ *and*

$$\mathcal{H}^{n-1}(L(t)) \leqslant \liminf_{i\to\infty} \mathcal{H}(L^{s_i}(k_i)). \tag{38}$$

Proof. For any $k \in N$ we introduce the displacement

$$\mathbf{v}^s(k+1) = \mathbf{u}(\mathbf{u}_0((k+1)/s), K^s(k)).$$

From the minimality assumption on the incremental solution we have for any $k \in N$ the inequality

$$J((\mathbf{u}^s(k), K^s(k)), (\mathbf{v}^s(k+1), K^s(k))) \geqslant J(k,s),$$

$$J(k,s) = J((\mathbf{u}^s(k), K^s(k)), (\mathbf{u}^s(k+1), K^s(k+1))).$$

Also, because $\mathbf{u}^s(k)$ is uniformly (with respect to k and s) essentially bounded, from the relation (29) and the minimality of the incremental solution we have $\mathcal{H}^{n-1}(L^s(k) \setminus \mathbf{J}_{\mathbf{u}^s(k)}) = 0$ for any s, k. The latter inequality means that

$$\int_\Omega w(\nabla \mathbf{v}^s(k+1))\,\mathrm{d}x \geqslant \int_\Omega w(\nabla \mathbf{u}^s(k+1))\,\mathrm{d}x + G\mathcal{H}^{n-1}(K^s(k+1) \setminus K^s(k)).$$

The crack growth condition $K^s(k) \subset K^s(k+1)$ implies that the latter relation can be written as

$$\left(\int_\Omega w(\nabla \mathbf{v}^s(k+1))\,\mathrm{d}x - \int_\Omega w(\nabla \mathbf{u}^s(k))\,\mathrm{d}x\right) + \int_\Omega w(\nabla \mathbf{u}^s(k))\,\mathrm{d}x + G\mathcal{H}^{n-1}(K^s(k)) \geqslant \int_\Omega w(\nabla \mathbf{u}^s(k+1))\,\mathrm{d}x + G\mathcal{H}^{n-1}(K^s(k+1)).$$

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This is the incremental form of the Griffith criterion of crack propagation (12). Indeed, we have

$$\int_\Omega w(\nabla \mathbf{v}^s(k+1))\,\mathrm{d}x = \tfrac{1}{2}\langle \mathbf{T}(K^s(k))\mathbf{u}_0((k+1)/s), \mathbf{u}_0((k+1)/s)\rangle,$$

$$\int_\Omega w(\nabla \mathbf{u}^s(k))\,\mathrm{d}x = \tfrac{1}{2}\langle \mathbf{T}(K^s(k))\mathbf{u}_0(k/s), \mathbf{u}_0(k/s)\rangle,$$

therefore

$$\int_\Omega w(\nabla \mathbf{v}^s(k+1))\,\mathrm{d}x - \int_\Omega w(\nabla \mathbf{u}^s(k))\,\mathrm{d}x$$
$$= \langle \mathbf{T}(K^s(k))\tfrac{1}{2}(\mathbf{u}_0((k+1)/s) + \mathbf{u}_0(k/s)),$$
$$\mathbf{u}_0((k+1)/s) - \mathbf{u}_0(k/s)\rangle.$$

$\mathbf{v}^s(k+1)$ represents the displacement of the body with the boundary displacement $\mathbf{u}_0(k/s+1/s)$ in the presence of the crack $K^s(k)$. $\mathbf{u}^s(k)$ represents the displacement of the body with the boundary displacement $\mathbf{u}_0(k/s)$ in the presence of the same crack $K^s(k)$. According to (11), the quantity

$$\left(\int_\Omega w(\nabla \mathbf{v}^s(k+1))\,\mathrm{d}x - \int_\Omega w(\nabla \mathbf{u}^s(k))\,\mathrm{d}x\right)\Big/\left(\frac{1}{s}\right)$$

is the discretized expression of the power communicated by the rest of the universe to the body at the time k/s, when a time discretization with step $1/s$ is considered.

We deduce from the inequality (39) that

$$P/s + \int_\Omega w(\nabla \mathbf{u}^s(k))\,\mathrm{d}x + G\mathcal{H}^{n-1}(K^s(k))$$
$$\geqslant \int_\Omega w(\nabla \mathbf{u}^s(k+1))\,\mathrm{d}x + G\mathcal{H}^{n-1}(K^s(k+1)).$$

We have therefore

$$Pk/s \geqslant \int_\Omega w(\nabla \mathbf{u}^s(k+1))\,\mathrm{d}x + G\mathcal{H}^{n-1}(K^s(k+1)).$$

From $L^s(k+1) \subset K^s(k+1)$ we infer that

$$Pk/s \geqslant \int_\Omega w(\nabla \mathbf{u}^s(k+1))\,\mathrm{d}x + G\mathcal{H}^{n-1}(L^s(k+1)).$$

The latter inequality and the equally boundedness of $\mathbf{u}^s(k)$ allow us to apply the compactness Theorem for **SBD** 5.2. We deduce that for any $t > 0$ there exist

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diverging sequences $(s_i)_i$ and $(k_i)_i$ such that k_i/s_i converges to t and $(\mathbf{u}^{s_i}, L^{s_i})(k_i)$ converges to an element of $M\ (\mathbf{u}, L)(t)$ in the sense of the relations (37), (38).

8. Numerical Approach to the Model

The models presented in this paper are of applicative interest. In order to use them we have to know how to minimize a Mumford–Shah functional. This can be done by approximating, in the sense of variational convergence, the original functional by a volume integral. There are several ways to approximate the Mumford–Shah functional by volume integrals (for a general reference we quote Braides [16]). One idea is to replace the displacement-crack pair $(\mathbf{u}, K)$ with the pair $(\mathbf{u}, f)$, where f is a smoothed version of the characteristic function of the crack set K, taking values in the interval $[0, 1]$. The original functional may be replaced by an Ambrosio–Tortorelli approximation, introduced in [11], [12].

Let us consider, for given $g: \Omega \to R$ and $c > 0$, functionals of the form

$$I_c(u, f) = \int_\Omega \left\{ \alpha\phi(f)|\nabla u|^2 + \beta(u-g)^2 + \right. \\ \left. + \gamma \left[c\psi(f)|\nabla f|^2 + \frac{f^2}{4c} \right] \right\} \mathrm{d}x. \tag{39}$$

We suppose that the functions ϕ, ψ have the following properties:

(a) $\psi(x) > 0$ for any $x \in (0, 1]$;
(b) $\int_0^1 2x\psi^{1/2}(x)\, \mathrm{d}x = 1$;
(c) $\phi(0) = 1$, $\phi(1) = 0$ and $\phi(x) \in (0, 1)$ for any $x \in (0, 1)$.

Under these assumptions it is known that when c converges to 0 then I_c converges in the variational sense (or Γ-convergence) to the Mumford–Shah functional

$$I(u) = \alpha \int_\Omega |\nabla u|^2\, \mathrm{d}x + \beta \int_\Omega |u-g|^2\, \mathrm{d}x + \gamma \mathcal{H}^1(\mathbf{S}_u). \tag{40}$$

This result, due to Ambrosio and Tortorelli, tells us that for any $u \in \mathbf{SBV}(\Omega, R)$ the followings are true:

(i) for any sequence (u_h, f_h, c_h) such that $u_h \to u$ and $f_h \to 0$ in L^2, $c_h \to 0$, we have

$$\liminf_{h\to\infty} I_{c_h}(u_h, f_h) \geqslant I(u);$$

(ii) there is a sequence (u_h, f_h, c_h) such that $u_h \to u$ and $f_h \to 0$ in L^2, $c_h \to 0$, and

$$\limsup_{h\to\infty} I_{c_h}(u_h, f_h) \leqslant I(u).$$

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A consequence of this result is that if:

(i) (u_h, f_h) is a minimizer of the functional I_{c_h} and $c_h \to 0$; and
(ii) there is a function $u \in \mathbf{SBV}(\Omega, R)$ such that $u_h \to u$ and $f_h \to 0$ in L^2,

then u is a minimizer of the Mumford–Shah functional I.

The numerical approach to the problem of minimizing the Mumford–Shah functional consists in the replacement of this functional with an approximate functional I_c. After a numerical minimization of I_c over a conveniently chosen set we obtain a minimizing pair (u^c, f^c). The function f^c represents an approximation of the characteristic function of the set $\mathbf{S}_u$, where u is a minimizer of I.

We shall use this idea for the model presented here, in the anti-plane case. Instead of a sequence of incremental solutions (u_h, K_h) we shall consider a sequence of pairs (u_h^c, f_h^c). The crack-growth condition $K_h \subset K_{h+1}$ will be replaced by: $f_h^c(x) \leqslant f_{h+1}^c(x)$ for any $x \in \Omega$. Notice that f_h^c is an approximation of the characteristic function of the damaged region.

We shall not be concerned further with the regularity of the functions that we are dealing with. We set M to be the space of all pairs of smooth enough functions $u: \Omega \subset R^2 \to R$, $f: \Omega \to [0, 1]$. The number c and functions ϕ, ψ are given, as well as a sequence of imposed boundary displacements $u_0^n: \Gamma_u \subset \partial\Omega \to R$. As for the material constants, we set $\gamma = G/\mu$, which has the dimension of a length.

DEFINITION 8.1. Let us define the functions

$$J_c: M \times M \to R,$$

$$F(g) = \int_\Omega \left\{ \Phi(g)|\nabla v|^2 + \gamma \left[c\psi(g)|\nabla g|^2 + \frac{g^2}{4c} \right] \right\} \, \mathrm{d}x,$$

$$J_c((u, f), (v, g)) = \begin{cases} F(g) & \text{if } g \geqslant f, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\Psi: N \times M \to \{0, +\infty\},$$

$$\Psi(n, (v, g)) = \begin{cases} 0 & \text{if } (1-g)(v - u_0^n) = 0 \text{ on } \Gamma_u, \\ +\infty & \text{otherwise.} \end{cases}$$

We consider the initial data (u_0, f_0) such that $u_0 = u(u_0^0, K)$ and f_0 satisfies

$$\sup\{|f(x) - \chi_K(x)|: x \in \Omega\} \leqslant c,$$

where χ_K is the characteristic function of the set K.

We recursively define the sequence (u_h^c, f_h^c) as follows:

(i) $(u_0^c, f_0^c) = (u_0, f_0)$;

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(ii) for any $k \in N$ the pair (u^c_{k+1}, f^c_{k+1}) minimizes over M the functional

$$(v, g) \mapsto J_c((u^c_k, f^c_k), (v, g)) + \Psi(k+1, (v, g)).$$

For the approximate model described in Definition 8.1 we shall use the gradient descent method described in Richardson and Mitter [32]. The domain Ω is discretized in pixels and the various partial derivatives of functions u^c and f^c are replaced by finite differences. With the notation

$$J^k_c(u, f) = J_c((u^c_k, f^c_k), (u, f))$$

the gradient descent of the functional J^k_c has the form

$$\dot{u} = -C_u \partial_u J^k_c(u, f),$$

$$\dot{f} = -C_f \partial_f J^k_c(u, f)$$

with variable controls C_u and C_f. In order to respect the constraint Ψ, after each step of the descent a projection of f on the convex set

$$\{g: \Omega \to [0, 1]: g(x) \geqslant f^c_k(x) \ \ \forall x \in \Omega\}$$

is performed. The boundary condition for the displacement u is satisfied in the usual way by setting the value of u on the pixels of $\partial\Omega$ equal to the value of u^{k+1}_0.

The simplest choice for the functions ϕ and ψ is

$$\phi(x) = (1-x)^2, \qquad \psi(x) = 1.$$

Richardson and Mitter remark in [32] that the parameter β (see (1)), which is equal to 0 in Definitions 4.1 and 8.1, has a strong influence on the speed of the gradient descent method they propose: small β causes low speed of the gradient descent. In our problem β is null and this causes a very slow rate of convergence. There is an empirical reason for which the Mumford–Shah functional behaves badly when β is zero, in the problem of crack evolution: unlike the case of image segmentation, where the information is scattered all over Ω, in the problem of crack evolution the displacement that causes the growth of the crack is a datum concentrated on the boundary of Ω. The viscous force induced by β should serve to transport this information inside Ω.

For numerical reasons we shall mix our model with an Ambrosio and Braides model with small, but not zero, viscosity. We replace the functional J_c by

$$J^*_c((u, f), (v, g)) = J_c((u, f), (v, g)) + \beta s \int_\Omega |v-u|^2 \, \mathrm{d}x.$$

The sequence of imposed boundary displacements (u^n_0) is the discretized in time version of a path of displacements $u_0(t)$. For a fixed step of discretization $1/s$ we have

$$u^n_0 = u_0(n/s).$$

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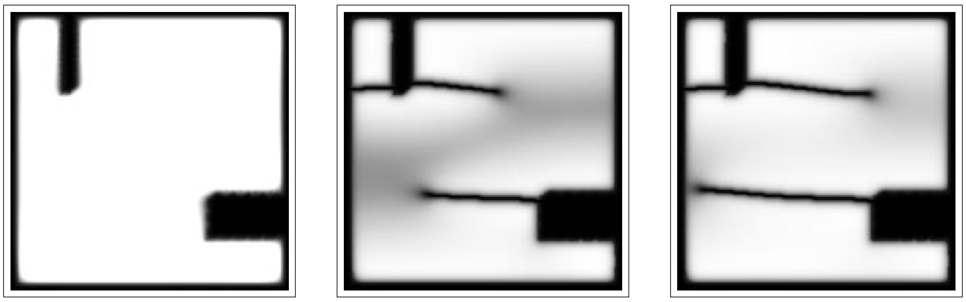
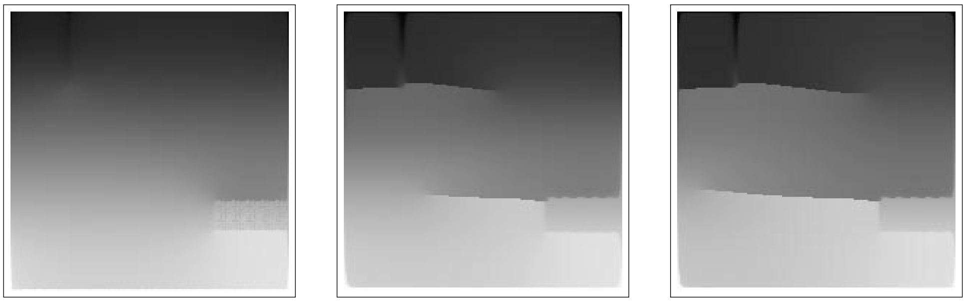


(a) (b) (c)

Figure 3. (a) The initial geometry of the body; (b) and (c) the aspect of the evolving cracks.



(a) (b) (c)

Figure 4. (a) The initial displacement of the body; (b) and (c) represent the displacement of the body fractured as in (b) and (c) previous pictures.

In order to eliminate the effects of the viscosity we replace also the sequence (u_0^n) with the following one, for a given natural P

$$\forall n \in N, \quad k \in \{0, 1, \ldots P - 1\} U_0^{nP+k} = u_0^n.$$

Therefore at any time n/s, the boundary displacement becomes $u_0(n/s)$ and after that it remains constant in the interval $[n/s, (n + P)/s]$, in order to let the influence of the viscosity to become negligible.

In Figure 1 we see how the Richardson and Mitter method works for the image segmentation problem. Recall that the Mumford–Shah functional (1) is used. The parameters α, β and γ have been left to our choice, in order to get a good result.

The results of the numerical method for a cylinder with a rectangular cross-section of 0.1 m × 0.1 m are shown in the next four figures. We remove from this cross-section small rectangles (Figures 3 and 4) or parts of ellipsis (Figures 5 and 6) and study what happened with the body obtained in this way during an imposed path of boundary displacements. The material (carbon steel) has the constant $\gamma = G/\mu = 0.0000025$ m and it has a pure elastic behavior. The boundary conditions are described further. The rectangular section is a square $[0, 0.1] \times [0, 0.1]$. The

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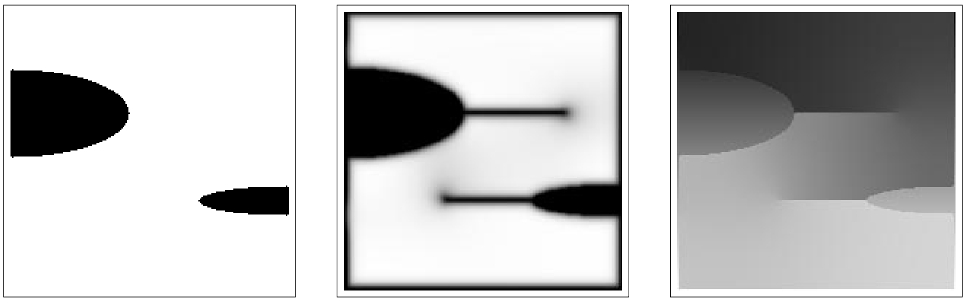
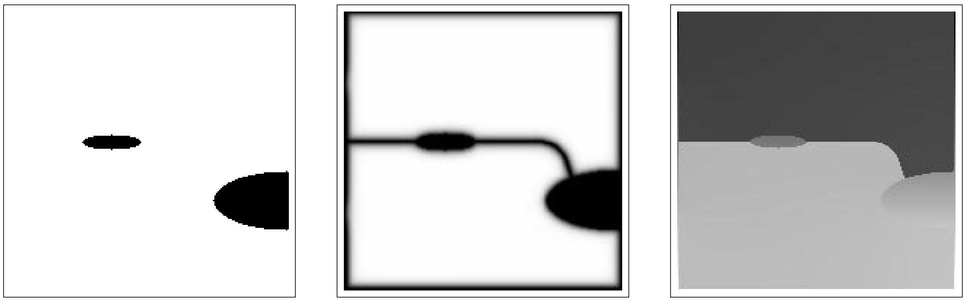


(a) (b) (c)

Figure 5. (a) The initial geometry; (b) final aspect of the cracks; (c) final displacement.



(a) (b) (c)

Figure 6. (a) The initial geometry of the body; (b) final aspect of the cracks (c) final displacement.

displacement u_0^n is imposed on the faces $[0, 0.1] \times \{0\}$, where u_0^n is constant and equal to 0, and $[0, 0.1] \times \{0.1\}$, where the displacement u_0^n is constant and grows slowly with n, from the value 0 m to the value 0.0041 m. The other two faces are force free.

The approximate characteristic function of the crack, i.e., the function $f : \Omega \to [0, 1]$ is represented with the following convention: there are 256 grey levels, numbered from 0 (black) to 255 (white); the number 0 (no crack there) corresponds to the level 255 and the number 1 (certainly a crack there) corresponds to the level 0. We have a linear correspondence between the numbers from $(0, 1)$ and the intermediary grey level. In this way we obtain a kind of picture of the shape of the crack in the cross-section of the body. Therefore a pixel is black either if there was no material there from the start, or if it belongs to the actual crack. Irrelevant black pixels appear on the boundary of the picture, maybe as an effect of error accumulation during the minimization process.

The displacement function u is represented in the complete square cross-section, but is irrelevant in the portions removed from the section. The representation was made with the following convention: the 255 level (white) correspond to the max-

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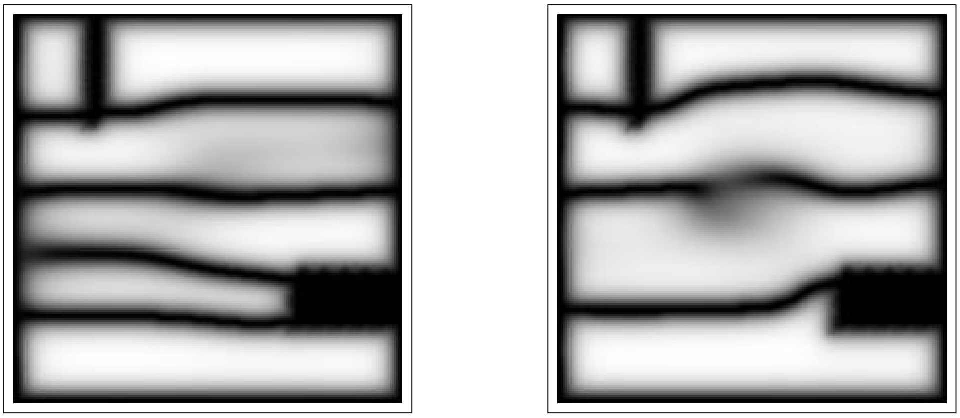

(a) (b)

Figure 7. Examples of local minima.

imum value of u and the 0 level (black) correspond to the minimum value of u; all the intermediary values of u are represented as grey levels, with a linear law of correspondence.

9. Final Remarks

This energetic approach to quasi-static brittle fracture propagation has the quality that it does not contain any prescription of the shape or location of the cracks. We have seen that the model provides a way of working with cracks which suddenly appear in the body. We have partially investigated this feature of the model and we have concluded that the model is not compatible with a critical stress based model of damage of an elastic body.

In this paper we did not study the bifurcation of an existing crack. A crack bifurcates when its shape suffers a change of topology. The most common example is a crack in a two-dimensional configuration, initially with only one edge in the body, which develops in time new branches. During this phenomenon the number of edges of the crack increases.

The numerical results presented in the last section have the following feature: during the evolution of the crack new concentrations of the elastic energy density do not appear *in the interior of the body*. It may seem that we have an example of crack bifurcation in Figures 3(b) and (c), but the two branches from the top of the Figure 3(b) do not grow simultaneously. We have noticed that a first crack grows to the left until its edge reaches the boundary of the rectangle and, after that, a second crack grows to the right.

There is no method to find the global minimum of a functional like the Ambrosio–Tortorelli approximation. We have experimented with our programs for a large variety of data. We have obtained from time to time solutions which were

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obviously local but not global minima. We have found that some of these local minima loose old edges (Figure 7(a)), eventually developing instead new ones (Figure 7(b)).

Our numerical results indicate that there is a sort of conservation law of 'edges' (i.e. maxima or singularities of the elastic energy density) of the solutions of the model, asserting that during the evolution of the crack the number of these 'edges' can only decrease. If such a conservation law is true, it may be a consequence of the fact that in the Mumford–Shah functional there is no term which controls the creation of a new 'edge'.

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Equilibrium and absolute minimal states of Mumford-Shah functionals and brittle fracture propagation

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Abstract

By a combination of geometrical and configurational analysis we study the properties of absolute minimal and equilibrium states of general Mumford-Shah functionals, with applications to models of quasistatic brittle fracture propagation. The main results concern the mathematical relations between physical quantities as energy release rate and energy concentration for 3D cracks with complex shapes, seen as outer measures living on the crack edge.

Keywords: 3D brittle fracture; energy methods; Mumford-Shah functional

1 Introduction

A new direction of research in brittle fracture mechanics begins with the article of Mumford & Shah [12] regarding the problem of image segmentation. This problem, which consists in finding the set of edges of a picture and constructing a smoothed version of that picture, it turns to be intimately related to the problem of brittle crack evolution. In the before mentioned article Mumford and Shah propose the following variational approach to the problem of image segmentation: let $g : \Omega \subset \mathbb{R}^2 \rightarrow [0,1]$ be the original picture, given as a distribution of grey levels (1 is white and 0 is black), let $u : \Omega \rightarrow R$ be the smoothed picture and K be the set of edges. K represents the set where u has jumps, i.e. $u \in C^1(\Omega \setminus K, R)$. The pair formed by the smoothed picture u and the set of edges K minimizes then the functional:

$$I(u,K) \ = \ \int_{\Omega} \alpha \ \mid \nabla u \mid^2 \ \mathrm{d}x \ + \ \int_{\Omega} \beta \ \mid u - g \mid^2 \ \mathrm{d}x \ + \ \gamma \mathcal{H}^1(K) \ \ .$$

The parameter α controls the smoothness of the new picture u, β controls the L^2 distance between the smoothed picture and the original one and γ controls the total length of the edges given by this variational method. The authors remark that for $\beta = 0$ the functional I might be useful for an energetic treatment of fracture mechanics.

An energetic approach to fracture mechanics is naturally suited to explain brittle crack appearance under imposed boundary displacements. The idea is presented in the followings.

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The state of a brittle body is described by a pair displacement-crack. $(\mathbf{u}, K)$ is such a pair if K is a crack — seen as a surface — which appears in the body and $\mathbf{u}$ is a displacement of the broken body under the imposed boundary displacement, i.e. $\mathbf{u}$ is continuous in the exterior of the surface K and $\mathbf{u}$ equals the imposed displacement $\mathbf{u}_0$ on the exterior boundary of the body.

Let us suppose that the total energy of the body is a Mumford-Shah functional of the form:

$$E(\mathbf{u}, K) \ = \ \int_{\Omega} w(\nabla \mathbf{u}) \ \mathrm{d}x \ + \ F(\mathbf{u}_0, K) \quad .$$

The first term of the functional E represents the elastic energy of the body with the displacement $\mathbf{u}$. The second term represents the energy consumed to produce the crack K in the body, with the boundary displacement $\mathbf{u}_0$ as parameter. Then the crack that appears is supposed to be the second term of the pair $(\mathbf{u}, K)$ which minimizes the total energy E.

After the rapid establishment of mathematical foundations, starting with De Giorgi, Ambrosio [8], Ambrosio [1], [2], the development of such models continues with Francfort, Marigo [9], [10], Mielke [11], Dal Maso, Francfort, Toader, [7], Buliga [4], [5], [6].

In this paper we introduce and study equilibrium and absolute minimal states of Mumford-Shah functionals, in relation with a general model of quasistatic brittle crack propagation.

On the space of the states of a brittle body, which are admissible with respect to an imposed Dirichlet condition, we introduce a partial order relation. Namely the state $(\mathbf{u}, K)$ is "smaller than" $(\mathbf{v}, L)$ if $L \subset K$ and $E(\mathbf{u}, K) \leq E(\mathbf{v}, L)$. Equilibrium states for the Mumford-Shah energy E are then minimal elements of this partial order relation. Absolute minimal states are just minimizers of the energy E.

Both equilibrium states and absolute minimal ones are good candidates for solutions of models for quasistatic brittle crack propagation. Usually such models, based on Mumford-Shah energies, take into consideration only absolute minimal states. However, it seems to me that equilibrium states are better, because it is physically sound to define a state of equilibrium $(\mathbf{u}, K)$ of a brittle body as one with the property that its total energy $E(\mathbf{u}, K)$ cannot be lowered by increasing the crack further.

For this reason we study here properties of equilibrium and absolutely minimal states of general Mumford-Shah energies. This study culminates with an inequality between the energy release rate and elastic energy concentration, both defined as outer measures living on the edge of the crack. This result generalizes for tri-dimensional cracks with complex geometries what is known about brittle cracks with simple geometry in two dimensions. In the two dimensional case, for cracks with simple geometry, classical use of complex analysis lead us to an equality between the energy release rate and elastic energy concentration at the tip of the crack. We prove that for absolute minimal states (corresponding to cracks with complex geometry) such an equality still holds, but for general equilibrium states we only have an inequality. Roughly stated, such a difference in properties of equilibrium and absolute minimal states comes from the mathematical fact that the class of first variations around an equilibrium state is only a semigroups.

This research might be relevant for 3D brittle fracture criteria applied for cracks with complex geometries. Indeed, it is very difficult even to formulate 3D fracture criteria, because in three dimensions a crack of arbitrary shape does not have a finite number of "crack tips" (as in 2D classical theory), but an "edge" which is a collection of piecewise smooth curves in the 3D space.

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2 Notations

Partial derivatives of a function f with respect to coordinate x_j are denoted by $f_{,j}$. We use the convention of summation over the repeating indices. The open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$ is denoted by $B(x,r)$.

We assume that the body under study has an open, bounded, with locally Lipschitz boundary, reference configuration $\Omega \subset \mathbb{R}^n$, with $n = 1, 2$ or 3. In the paper we shall use Hausdorff measures $\mathcal{H}^k$ in $\mathbb{R}^n$. For example, if $n = 3$ then $\mathcal{H}^n$ is the volume measure, $\mathcal{H}^{n-1}$ is the area measure, $\mathcal{H}^{n-2}$ is the length measure. If $n = 2$ then $\mathcal{H}^n$ is the area measure, $\mathcal{H}^{n-1}$ is the length measure, $\mathcal{H}^{n-2}$ is the counting measure.

Definition 2.1 *A smooth diffeomorphism with compact support in Ω is a function $\phi : \Omega \to \Omega$ with the following properties:*

i) ϕ is bijective;

ii) ϕ and ϕ^{-1} are C^∞ functions;

iii) ϕ equals the identity map of Ω near the boundary $\partial\Omega$:

$$supp\ (id_\Omega - \phi) \ \subset\subset \Omega \quad .$$

The set of all diffeomorphisms with compact support in Ω is denoted by $\mathcal{D}$ or $\mathcal{D}(\Omega)$.

The set $\mathcal{D}(\Omega)$ it is obviously non void because it contains at least the identity map id_Ω. Remark also that it is a group with respect to function composition.

For any C^∞ vector field η on Ω there is an unique associated one parameter flow, which is a function $\phi : I \times \Omega \to \Omega$, where $I \subset \mathbb{R}$ is an open interval around $0 \in \mathbb{R}$, with the properties:

f1) $\forall t \in I$ the function $\phi(t, \cdot) \ = \ \phi_t(\cdot)$ satisfies i) and ii) from definition 2.1,

f2) $\forall t, t' \in I$, if $t - t' \in I$ then we have $\phi_{t'} \circ \phi_t^{-1} \ = \ \phi_{t-t'}$,

f3) $\forall\ t \in I$ we have $\eta \ = \ \dot{\phi}_t \circ \phi_t^{-1}$, where $\dot{\phi}_t$ means the derivative of $t \mapsto \phi_t$.

The vector field $\eta = 0$ generates the constant flow $\phi_t = id_\Omega$. If η has compact support in Ω then the associated flow $t \mapsto \phi_t$ is a curve in $\mathcal{D}$.

A crack set K is a piecewise Lipschitz surface with a boundary. This means that exists bi-Lipschitz functions $(f_\alpha)_{\alpha \in 1...M}$, each of them defined over a relatively open subset D_α of $\mathbb{R}^{n-1}_+ = \left\{ y \in \mathbb{R}^{n-1} : y_{n-1} \geq 0 \right\}$, with ranges in $\mathbb{R}^n$, such that:

$$K = \cup_{\alpha=1}^{M} f_\alpha(D_\alpha) \quad ,$$

$$if \quad \alpha \neq \beta \quad then \quad f_\alpha(D_\alpha \setminus \partial \mathbb{R}^{n-1}_+) \cap f_\beta(D_\beta \setminus \partial \mathbb{R}^{n-1}_+) = \emptyset \quad .$$

The edge of the crack K is defined by

$$dK = \cup_{\alpha=1}^{M} f_\alpha(D_\alpha \cap \partial \mathbb{R}^{n-1}_+) \quad .$$

We shall denote further by $B_r(dK)$ the tubular neighborhood of radius r of dK, given by the formula:

$$B_r(dK) = \cup_{x \in dK} B(x, r) \quad .$$

We denote by $[f] = f^+ - f^-$ the jump of the function f over the surface K with respect to the field of normals $\mathbf{n}$.

3 Mumford-Shah type energies

Definition 3.1 *We describe the state of a brittle body by a pair $(\mathbf{v}, S)$. The crack is seen as a piecewise Lipschitz surface S in the topological closure $\overline{\Omega}$ of the reference configuration Ω of the body and $\mathbf{v}$ represents the displacement of the body from the reference configuration. The displacement $\mathbf{v}$ has to be compatible with the crack , i.e. $\mathbf{v}$ has the regularity C^1 outside the surface S.*

The space of states of the brittle body with reference configuration Ω is denoted by $Stat(\Omega)$.

The main hypothesis in models of brittle crack propagation based on Mumford-Shah type energies is the following.

Brittle fracture hypothesis. *The total energy of the body subject to the boundary displacement $\mathbf{u}_0$ depends only on the state of the body $(\mathbf{v}, S)$ and it has the expression:*

$$E(\mathbf{v}, S) = \int_\Omega w(\nabla \mathbf{v}) \, dx + F(S; \mathbf{u}_0) \quad . \tag{3.0.1}$$

The first term of this functional is the elastic energy associated to the displacement $\mathbf{v}$; the second term represents the energy needed to produce the crack S, with the boundary displacement $\mathbf{u}_0$ as parameter.

We suppose that the elastic energy potential w is a smooth, non negative function.

The most simple form of the function F is the Griffith type energy:

$$F(S; \mathbf{u}_0) = Const. \cdot Area\ (S) \quad ,$$

that is the energy consumed to create the crack S is proportional, through a material constant, to the area of S.

One may consider expressions of the surface energy F, different from (3.0.1), for example:

$$F(\mathbf{v}, S) \;=\; \int_S \phi(\mathbf{v}^+, \mathbf{v}^-, \mathbf{n}) \; \mathrm{d}s \quad ,$$

where $\mathbf{n}$ is a field of normals over S , $\mathbf{v}^+$, $\mathbf{v}^-$ are the lateral limits of $\mathbf{v}$ on S with respect to directions $\mathbf{n}$, respective $-\mathbf{n}$ and ϕ has the property:

$$\phi(\mathbf{v}^+, \mathbf{v}^-, \mathbf{n}) \;=\; \phi(\mathbf{v}^-, \mathbf{v}^+, -\mathbf{n}) \quad .$$

The function ϕ, depending on the displacement of the "lips" of the crack, is a potential for surface forces acting on the crack. The expression (3.0.1) does not lead to such forces.

In general we shall suppose that the function F has the properties:

h1) is sub-additive: for any two crack sets A, B we have

$$F(A \cup B; \mathbf{u}_0) \;\leq\; F(A; \mathbf{u}_0) \;+\; F(B; \mathbf{u}_0) \quad ,$$

h2) for any $x \in \Omega$ and $r > 0$, let us denote by δ_r^x the dilatation of center x and coefficient r:

$$\delta_r^x(y) = x + r(y - x) \quad .$$

Then, there is a constant $C \geq 1$ such that for any $A \subset \Omega$ with $F(A; \mathbf{u}_0) < +\infty$ we have:

$$F(\delta_r^x(A) \cap \Omega; \mathbf{u}_0) \;\leq\; C r^{n-1} F(A; \mathbf{u}_0) \quad .$$

The particular case $F(A; \mathbf{u}_0) = G\mathcal{H}^{n-1}(A)$ satisfies these two assumptions. In general these assumptions are satisfied for functions $F(\cdot; \mathbf{u}_0)$ which are measures absolutely continuous with respect to the area measure $\mathcal{H}^{n-1}$.

A weaker property than h2), is the property h3) below. We don't explain here why h3) is weaker than h2), but remark that h3) is satisfied by the same class of examples given for h2).

For any $A \subset \Omega$, let us denote by $B(A, r)$ the tubular neighborhood of A:

$$B(A, r) \;=\; \cup_{x \in A} B(x, r) \quad .$$

We shall suppose that F satisfies:

h3) for any $A \subset \Omega$ such that $F(A; \mathbf{u}_0) < +\infty$, we have

$$\limsup_{r \to 0} \frac{F(\partial B(A, r) \;\cap\; \Omega; \mathbf{u}_0)}{r} \;<\; +\infty \quad .$$

4 The space of admissible states of a brittle body

Definition 4.1 *The class of admissible states of a brittle body with respect to the crack F and with respect to the imposed displacement $\mathbf{u}_0$ is defined as the collection of all states $(\mathbf{v}, S)$ such that*

(a) $\mathbf{u} = \mathbf{u}_0$ *on* $\partial\Omega \setminus S$,

(b) $F \subset S_u$.

This class of admissible states is denoted by $Adm(F, \mathbf{u}_0)$.

An admissible displacement $\mathbf{u}$ is a function which has to be equal to the imposed displacement on the boundary of Ω (condition (a)). Any such function $\mathbf{u}$ is reasonably smooth in the set $\Omega \setminus S_u$ and the function $\mathbf{u}$ is allowed to have jumps along the set S. Physically the set represents the collection of all cracks in the body under the displacement $\mathbf{u}$. The condition (b) tells us that the collection of all cracks associated to an admissible displacement $\mathbf{u}$ contains F, at least.

For some states $(\mathbf{u}, S)$, the crack set S may have parts lying on the boundary of Ω, that is $S \cap \partial\Omega$ is a surface with positive area. In such cases we think about $S \cap \partial\Omega$ as a region where the body has been detached from the machine which imposed upon the body the displacement $\mathbf{u}_0$.

In a weak sense the whole space of states of a brittle body may be identified with the space of special functions with bounded deformation $\mathbf{SBD}(\Omega)$, see [3]. Indeed, to every displacement field $\mathbf{u}$ which is a special function with bounded deformation we associate the state of the brittle body described by $(\mathbf{u}, \overline{\mathbf{S}}_u)$, where generally for any set A we denote by $\overline{A}$ the topological closure of A. (Note that, technically, the crack set $\overline{\mathbf{S}}_u$ may not be a collection of surfaces with Lipschitz regularity.)

On the space of states of a brittle body we introduce a partial order relation. The definition is connected to definition 4.1 and the brittle fracture hypothesis.

Definition 4.2 *Let* $(\mathbf{u}, S), (\mathbf{v}, L) \in Stat(\Omega)$ *be two states of a brittle body with reference configuration* Ω. *If*

(a) $S \subset L$,

(b) $\mathbf{u} = \mathbf{v}$ *on* $\partial\Omega \setminus L$,

(c) $E(\mathbf{v}, L) \leq E(\mathbf{u}, S)$,

then we write $(\mathbf{v}, L) \leq (\mathbf{u}, S)$. *This is a partial order relation.*

There are many pairs $(\mathbf{u}, S), (\mathbf{v}, L) \in Stat(\Omega)$ such that $(\mathbf{v}, L) \leq (\mathbf{u}, S)$ and $(\mathbf{u}, S) \leq (\mathbf{v}, L)$, but $\mathbf{u} \neq \mathbf{v}$. Nevertheless such pairs have the same total energy E, the same crack set $S = L$, and $\mathbf{u} = \mathbf{v}$ on $\partial\Omega \setminus L$.

For a given boundary displacement $\mathbf{u}_0$ and for given initial crack set K, on the set of admissible states $Adm(\mathbf{u}_0, K)$ we have the same partial order relation.

Definition 4.3 *An element* $(\mathbf{u}, S) \in Adm(\mathbf{u}_0, K)$ *is minimal with respect to the partial order relation* $\leq$ *if for any* $(\mathbf{v}, L) \in Adm(\mathbf{u}_0, K)$ *the relation* $(\mathbf{v}, L) \leq (\mathbf{u}, S)$ *implies* $(eu, S) \leq (\mathbf{v}, L)$.

The set of equilibrium states with respect to given crack K *and imposed boundary displacement* $\mathbf{u}_0$ *is denoted by* $Eq(\mathbf{u}_0, K)$ *ant it consists of all minimal elements of* $Adm(\mathbf{u}_0, K)$ *with respect to the partial order relation* $\leq$.

An element $(\mathbf{u}, S) \in Adm(\mathbf{u}_0, K)$ *with the property that for any* $(\mathbf{v}, L) \in Adm(\mathbf{u}_0, K)$ *we have* $E(\mathbf{u}, S) \leq E(\mathbf{v}, L)$*, is called an absolute minimal state. The set of absolute minimal states is denoted by* $Absmin(\mathbf{u}_0, K)$.

The physical interpretation of equilibrium states is the following. An equilibrium state $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$ is one such that any other state $(\mathbf{v}, L) \in Adm(\mathbf{u}_0, K)$, which is comparable to $(\mathbf{u}, S)$ with respect to the relation $\leq$, has the property $(\mathbf{u}, S) \leq (\mathbf{v}, L)$. In other words, equilibrium states are those with the property: the total energy E cannot be made smaller by prolongating the crack set S or by modifying the displacement $\mathbf{u}$ compatible with the crack set S and imposed boundary displacement $\mathbf{u}_0$.

Absolute minimal states are just equilibrium states with minimal energy.

Remark 4.4 *There might exist several minimal elements of of* $Adm(\mathbf{u}_0, K)$*, such that any two of them are not comparable with respect to the partial order relation* $\leq$.

For given expressions of the functions w and F, we formulate the following

Equilibrium hypothesis (EH). *For any piecewise* C^1 *imposed boundary displacement* $\mathbf{u}_0$ *and any crack* K *the set of equilibrium states* $Eq(\mathbf{u}_0, K)$ *is not empty.*

Without supplementary hypothesis on the total energy E, the EH does not imply that the set of absolute minimal states $Absmin(\mathbf{u}_0, K)$ is non empty. Therefore the following hypothesis is stronger than EH.

Strong equilibrium hypothesis (SEH). *For any piecewise* C^1 *imposed boundary displacement* $\mathbf{u}_0$ *and any crack* K *the set of equilibrium states* $Absmin(\mathbf{u}_0, K)$ *is not empty.*

5 Models of quasistatic evolution of brittle cracks

We shall describe here two models of quasistatic brittle crack propagation, according to Francfort, Marigo [9], [10], Mielke [11], section 7.6, or Buliga [6], [5]. At a first sight the models seem to be identical, but subtle differences exist. Further, instead of referring to a particular different model, we shall write about a general model of brittle crack propagation based on energy functionals, as if there is only one, general model, with different variants, according to the choice among axioms listed further. Whenever necessary, the exposition will contain variants of statements or assumptions which specializes the general model to one of the actual models in use.

As an input of the model we have an initial crack set $K \subset \overline{\Omega}$ and a curve of imposed displacements $t \in [0, T] \mapsto \mathbf{u}_0(t)$ on the boundary of Ω, the initial configuration of the body.

We like to think about the configuration Ω as being an open, bounded subset of $\mathbb{R}^n$, $n = 1, 2, 3$, with sufficiently regular boundary (that is: piecewise Lipschitz boundary).

The initial crack set K has the status of an initial condition. Thus, we suppose that $\partial(\mathbb{R}^n \setminus \Omega) = \partial\Omega$. For the same configuration Ω we may consider any crack set $K \subset \overline{\Omega}$ as an initial crack. The crack set K may be empty.

Remark 5.1 *Models suitable for the evolution of brittle cracks under applied forces would be of great interest. Present formulations of the models of brittle crack propagation allows only the introduction of conservative force fields, as it is done in [11] or [10]. The reason is that models based on energy minimization cannot deal with arbitrary force fields. In the case of a conservative force field it is enough to introduce the potential of the force field inside the expression of the total energy of the fractured body. Thus, in this particular case we do not have to change substantially the formulation of the model presented here, but only to slightly modify the expression of the energy functional.*

In order to simplify the model presented here, we suppose that no conservative force fields are imposed on Ω or parts of $\partial\Omega$. In the models described in [11] or [10] such forces may be imposed.

Definition 5.2 *A solution of the model is a curve of states of the brittle body* $t \in [0, T] \mapsto (\mathbf{u}(t), S_t)$ *such that:*

(A1) (initial condition) $K \subset S_0$,

(A2) (boundary condition) for any $t \in [0, T]$ *we have* $\mathbf{u}(t) = \mathbf{u}_0(t)$ *on* $\partial\Omega \setminus S_t$,

(A3) (quasistatic evolution) for any $t \in [0, T]$ *we have* $(\mathbf{u}(t), S_t) \in Eq(\mathbf{u}_0(t), S_t)$,

(A4) (irreversible fracture process) for any $t \leq t'$ *we have* $S_t \subset S_{t'}$,

(A5) (selection principle) for any $t \leq t'$ *and for any state* $(\mathbf{v}, S_t) \in Adm(\mathbf{u}_0(t'), S_t)$ *we have* $E(\mathbf{v}, S_t) \geq E(\mathbf{u}(t'), S_{t'})$.

From definition 4.3 we see that (A2) is just a part of (A3). The axiom (A2) is present in the previous definition only for expository reasons.

The selection principle (A5) enforces the irreversible fracture process axiom (A4). Indeed, we may have severe non-uniqueness of solutions of the model. The axiom (A5) selects among all solutions satisfying (A1), ..., (A4), the ones which are energetically economical. The crack set S_t does not grow too fast, according to (A5). For imposed displacement $\mathbf{u}_0(t')$, the body with crack set $S_{t'}$ is softer than the same body with the crack set S_t, for any $t \leq t'$.

As presented in definition 5.2, the model has been proposed in Buliga [6].

In the models described in [11], [9], [10] we don't need the selection principle (A5) and the axiom (A3) takes the stronger form:

(A3') (quasistatic evolution) for any $t \in [0, T]$ *we have* $(\mathbf{u}(t), S_t) \in Absmin(\mathbf{u}_0(t), S_t)$.

6 The existence problem

The existence of equilibrium, or absolutely minimal states clearly depends on the ellipticity properties of the elastic energy potential w (as shown for example in [2], [3] or [9]). This is related to the existence of minimizers of the elastic energy functional, as shown by relation (7.0.1) further on. Some form of ellipticity of the function w is sufficient, but it is not clear if such conditions are also necessary. Much effort, especially of a mathematical nature, has been spent on this problem.

In this paper we are not concerned with the existence problem, however. Our purpose is to find general properties of solutions of brittle fracture propagation models based on Mumford-Shah functionals. These properties do not depend on particular forms of the elastic energy potential w, but on the hypothesis made in the general model. As any other model, the one studied in this paper is better fitted to some physical situations than others. If some property of solutions of this model are incompatible with a particular physical case, then we must deduce that the model is not fitted for this particular case (meaning that at least one of the hypothesis of the model is not suitable to this physical case). We are thus able to provide a complementary information to the one provided by the existence problem. See further the Conclusions section for more on the subject.

7 Absolute minimal states versus equilibrium states

The differences between the models come from the difference between equilibrium states and absolute minimal states.

Absolute minimal states are equilibrium states, but not any equilibrium state is an absolute minimal state.

Let us denote by $(\mathbf{u}, S)$ an equilibrium state of the body, with respect to the imposed displacement $\mathbf{u}_0$ and initial crack set K.

Consider first the class of all admissible pairs $(\mathbf{v}, S')$ such that $S = S$. We have, as an application of definition 4.3, then:

$$\int_\Omega w(\nabla \mathbf{u}) \, \mathrm{d}x \ \leq \ \int_\Omega w(\nabla \mathbf{v}) \, \mathrm{d}x \quad \forall \ \mathbf{v}, \ \mathbf{v} \ = \ \mathbf{u}_0 \text{ on } \partial\Omega \setminus K \ , \ \mathbf{v} \in C^1(\Omega \setminus K) \ . \quad (7.0.1)$$

Thus any equilibrium state minimizes the elastic energy functional (in the class of admissible pairs with the same associated crack set). A sufficient condition for the existence of such minimizers is the polyconvexity of the elastic energy potential w.

The elastic energy potential function $w : M^{n\times n}(\mathbb{R}) \rightarrow \mathbb{R}$ associates to any strain $\mathbf{F} \in M^{n\times n}(\mathbb{R})$ (here $n = 2$ or 3) the real value $w(\mathbf{F}) \in \mathbb{R}$. If this function is smooth enough then we can define the (Cauchy) stress tensor as coming from the elastic energy potential:

$$\sigma(\mathbf{u}) \ = \ \frac{\partial w(\mathbf{F})}{\partial \mathbf{F}}(\nabla \mathbf{u}) \ .$$

The variational inequality (7.0.1) implies that in the sense of distributions we have:

$$div \ \sigma(\mathbf{u}) \ = \ 0$$

and that on the crack set S we have

$$\sigma(\mathbf{u})^{+}\mathbf{n} \ = \ \sigma(\mathbf{u})^{-}\mathbf{n} \ = 0 \quad ,$$

where the signs + and − denotes the lateral limits of $\sigma(\mathbf{u})$ with respect to the field of normals $\mathbf{n}$.

7.1 Configurational relations for absolute minimal states

We can also make smooth variations of the pair $(\mathbf{u}, S)$. Here appears the first difference between absolute minimal and equilibrium states. We suppose further that $S \setminus K \neq \emptyset$, in fact we suppose that $S \setminus K$ is a surface with positive area.

If $(\mathbf{v}, L) \in Adm(\mathbf{u}_0, K)$ is an admissible state and $\phi \in \mathcal{D}$ is a diffeomorphism of Ω with compact support, such that $K \subset \phi(K)$, then $(\mathbf{v} \circ \phi^{-1}, \phi(S))$ is admissible too.

If $(\mathbf{u}, S)$ is an absolute minimal state then, as an application of definition 4.3, we have:

$$E(\mathbf{u}, S) \ \leq \ E(\mathbf{u} \circ \phi^{-1}, \phi(S)) \quad \forall \phi \in \mathcal{D} \, , \, K \subset \phi(K) \quad . \tag{7.1.2}$$

We may use (7.1.2) in order to derive a first variation equality.

We shall restrict further to the group $\mathcal{D}(K)$ of diffeomorphisms $\phi \in \mathcal{D}$ such that $supp\ (\phi - id) \cap K = \emptyset$. Vector fields η which generate one-parameter flows in $\mathcal{D}(K)$ are those with the property $supp\, \eta \cap K = \emptyset$. Further we shall work only with such vector fields.

We shall admit further that for any smooth vector field η there exist the derivatives at $t = 0$ of the functions:

$$t \ \mapsto \ \int_{\Omega} w(\nabla(\mathbf{u} \circ \phi_t^{-1})) \ \mathrm{d}x \quad , \quad t \ \mapsto \ F(\phi_t(K); \mathbf{u}_0) \quad ,$$

where ϕ_t is the one parameter flow generated by the vector field η. The relation (7.1.2) implies then:

$$\frac{d}{dt}_{|t=0} F(\phi_t(S); \mathbf{u}_0) \ = \ - \frac{d}{dt}_{|t=0} \int_{\Omega} w(\nabla(\mathbf{u} \circ \phi_t^{-1})) \ \mathrm{d}x \quad . \tag{7.1.3}$$

Let us compute the right hand side of (7.1.3). We have

$$- \frac{d}{dt}_{|t=0} \int_{\Omega} w(\nabla(\mathbf{u} \circ \phi_t^{-1})) \ \mathrm{d}x \ = \ \int_{\Omega} \{ -w(\nabla \mathbf{u}) \ div \ \eta \ + \ \sigma(\mathbf{u})_{ij} (\nabla \mathbf{u})_{ik} (\nabla \eta)_{kj} \} \ \mathrm{d}x \quad .$$

For any vector field η, let us define, for any $x \in S$, $\lambda(x) = \eta(x) \cdot \mathbf{n}(x)$, $\eta^T(x) = \eta(x) - \lambda)(x)\mathbf{n}(x)$, where $\mathbf{n}$ is a fixed field of normals over S.

With these notations, and recalling that the divergence of the stress field equals 0, we have:

$$- \frac{d}{dt}_{|t=0} \int_{\Omega} w(\nabla(\mathbf{u} \circ \phi_t^{-1})) \ \mathrm{d}x \ = \ \int_{S} [w(\nabla \mathbf{u})] \lambda \ \mathrm{d} \ \mathcal{H}^{n-1} \ +$$

$$+ \ \lim_{r \to 0} \int_{\partial B_r(dS)} \{ [w(\nabla \mathbf{u})] \lambda \ - \ [\sigma(\mathbf{u})_{ij} (\nabla \mathbf{u})_{ik}] \eta_k \mathbf{n}_j \} \ \mathrm{d} \ \mathcal{H}^{n-1} \quad . \tag{7.1.4}$$

Definition 7.1 *We introduce three kind of variations in terms of a vector field η which generates an one parameter flow $\phi_t \in \mathcal{D}(K)$:*

(a) (crack neutral variations) for $\eta = 0$ on S; in this case we have $\phi_t(S) = S$ for any t,

(b) (crack normal variations) for $\eta = \lambda \mathbf{n}$ on $S \setminus K$, with $\lambda : S \rightarrow \mathbb{R}$ a scalar, smooth function, such that $\lambda(x) = 0$ for any $x \in K \cup dS$,

(c) (crack tangential variations) for $\eta \cdot \mathbf{n} = 0$ on S.

For the case (a) of crack neutral variations the relation (7.1.4) gives no new information, when compared with (7.0.1).

In the case (b) of crack normal variations, the relation (7.1.4) implies

$$\frac{d}{dt}_{|t=0} F(\phi_t(K); \mathbf{u}_0) \ = \ \int_S [w(\nabla \mathbf{u})] \lambda \ \mathrm{d} \ \mathcal{H}^{n-1} \quad .$$

In the particular case $F(S; \mathbf{u}_0) \ = \ \mathcal{H}^{n-1}(S)$ we obtain:

$$\int_S \left\{ [w(\nabla \mathbf{u})] \ + H \right\} \lambda \ \mathrm{d}\mathcal{H}^{n-1} \ = \ 0 \quad ,$$

where $H \ = \ -div_s \mathbf{n} \ = \ - \ div \ \mathbf{n} \ + \mathbf{n}_{i,j} \mathbf{n}_i \mathbf{n}_j$ is the mean curvature of the surface S. Therefore we have

$$[w(\nabla \mathbf{u})(x)] \ + H(x) = 0 \tag{7.1.5}$$

for any $x \in S \setminus K$.

In the case (c) of crack tangential variations, the relation (7.1.4) implies

$$\frac{d}{dt}_{|t=0} F(\phi_t(S); \mathbf{u}_0) \ =$$

$$= \ \lim_{r \rightarrow 0} \int_{\partial B_r(dS)} \left\{ [w(\nabla \mathbf{u})] \lambda \ - \ [\sigma(\mathbf{u})_{ij} (\nabla \mathbf{u})_{ik}] \eta_k \mathbf{n}_j \right\} \ \mathrm{d}\mathcal{H}^{n-1} \ . \tag{7.1.6}$$

This last relation admits an well known interpretation, briefly explained in the next subsection.

7.2 Absolute minimal states for $n = 2$

Let us consider the case $n = 2$ and the function

$$F(S; \mathbf{u}_0) \ = \ G \ \mathcal{H}^1(S) \quad ,$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure, i.e. the length measure. Let us suppose, for simplicity, that the initial crack set K is empty and the crack set S of the absolute minimal state $(\mathbf{u}, S)$ has only one edge, i.e. $dS \ = \ \{x_0\}$. Let us choose a vector field η with compact support in Ω such that η is tangent to S. The equality (7.1.6) becomes then

$$G\ \eta(x_0) \cdot \tau(x_0) \ = \ \lim_{r \rightarrow 0} \int_{\partial B_r(x_0)} \{[w(\nabla \mathbf{u})] \eta \cdot \mathbf{n} \ - \ [\sigma(\mathbf{u})_{ij} (\nabla \mathbf{u})_{ik}] \eta_k \mathbf{n}_j\} \ \mathrm{d}\mathcal{H}^{n-1} \ ,$$

where $\tau(x)$ is the unitary tangent in $x \in K$ at K. If we suppose moreover that the crack S is straight near x_0, and the material coordinates are chosen such that near x_0 we have $\eta(x) \ = \ \tau(x) \ = \ (1,0)$, then the equality (7.1.6) takes the form:

$$G \ = \ \lim_{r \rightarrow 0} \int_{\partial B_r(x_0)} \{[w(\nabla \mathbf{u})] \mathbf{n}_1 \ - \ [\sigma(\mathbf{u})_{ij} (\nabla \mathbf{u})_{i1}] \mathbf{n}_j\} \ \mathrm{d}\mathcal{H}^{n-1} \ . \qquad (7.2.7)$$

We recognize in the right term of (7.2.7) the integral J of Rice; therefore at the edge of the crack the integral J has to be equal to the constant G, interpreted as the constant of Griffith.

The equality (7.2.7) tells us that at the edge of a crack set belonging to an absolute minimal state the Griffith criterion is fulfilled with equality.

7.3 Configurational inequalities

For equilibrium states which are not absolute minimal states we obtain just an inequality, instead of the equality from relation (7.1.6). Also, for such equilibrium states there is no relation like (7.1.5) between the mean curvature of the crack set and the jump of elastic energy potential. We explain this further.

The reason lies in the fact that if $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$ is an equilibrium state with $S \setminus K$ having positive area, and $\phi \in \mathcal{D}(K)$ is a diffeomorphism preserving the initial crack set K, then we don't generally have the relation (7.1.2).

Indeed, in order to be able to compare $(\mathbf{u}, S)$ with $(\mathbf{u} \circ \phi^{-1}, \phi(S))$, we have to impose $S \subset \phi(S)$. Only for these diffeomorphisms $\phi \in \mathcal{D}(K)$ the relation (7.1.2) is true. The class of these diffeomorphisms is not a group, like $\mathcal{D}(K)$, but only a semigroup. Technically, this is the reason for having only an inequality replacing (7.1.6), and for the disappearance of relation (7.1.5).

There is a necessary condition on the edge dS of the crack set S, in order to have a trivial vector field η which generates a one parameter flow $\phi_t \in \mathcal{D}(K)$ with $S \subset \phi_t(S)$ for any $t \in [0, T]$ (with $T > 0$ sufficiently small). This condition is $dS \setminus K \neq \emptyset$.

Thus, for $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$ with $S \setminus K$ with positive area, and $dS \setminus K \neq \emptyset$, we have

$$E(\mathbf{u}, S) \ \leq \ E(\mathbf{u} \circ \phi_t^{-1}, \phi_t(S)) \quad \forall t \in [0, T] \quad , \qquad (7.3.8)$$

for any one parameter flow $\phi_t \in \mathcal{D}(K)$ with $S \subset \phi_t(S)$ for any $t \in [0, T]$.

In relation (7.3.8) crack normal variations (case (b) of definition 7.1) are prohibited. But these type of variations led us to the relation (7.1.5). We deduce that for an equilibrium state $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$, such that $S \setminus K$ has positive area, and $dS \setminus K \neq \emptyset$, the relation (7.1.5) does not necessarily hold.

The crack tangential variations (case (c) of definition 7.1) are allowed in relation (7.3.8) only for $t \geq 0$. That is why we get only a first variation inequality:

$$\frac{d}{dt}_{|t=0} F(\phi_t(S); \mathbf{u}_0) \ \geq$$

$$\geq \ \lim_{r \rightarrow 0} \int_{\partial B_r(dK)} \left\{ [w(\nabla \mathbf{u})]\lambda \ - \ [\sigma(\mathbf{u})_{ij} (\nabla \mathbf{u})_{ik}] \eta_k \mathbf{n}_j \right\} \ \mathrm{d}\mathcal{H}^{n-1} \ , \qquad (7.3.9)$$

for any vector field η which generates one parameter flow $\phi_t \in \mathcal{D}(K)$ with $S \subset \phi_t(S)$ for any $t \in [0, T]$.

The physical interpretation of relation (7.3.9) is the following: the crack set S of an equilibrium state satisfies the Griffith criterion of fracture, but, in distinction with the case of an absolute minimal state, there is an inequality instead of the previous equality. We are aware of at least one example where this inequality is strict. This case concerns a crack set in 3D formed by a pair of intersecting, transversal planar cracks. Such a crack set has an edge (in form of a cross), but also a "tip" (at the intersection of the edges of the planar cracks. The physical implications of the inequality (7.3.9) are that such a 3D crack may propagate in different ways, either along a crack tangential variation, or along a more topologically complex shape, by loosing its "tip". An article in preparation is dedicated to this subject.

We may interpret the Griffith criterion of fracture, in the form given by relation (7.3.9), as a first order stability condition for the crack S associated to the state of a brittle body. Surprisingly then, absolute minimal states are first order neutral (stable and unstable), even if globally stable (as global minima of the total energy). There might exist equilibrium states for which we have strict inequality in relation (7.3.9). Such states are surely not absolute minimal, but they seem to be first order stable, if our interpretation of (7.3.9) is physically sound.

7.4 Concentration of energy from comparison with admissible states

We can obtain energy concentration estimates from comparison of the energy of the equilibrium state $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$ with other particular admissible pairs.

Let $x_0 \in \Omega$ be a fixed point and $r > 0$ such that $B(x_0, r) \subset \Omega$. We construct the following admissible pair $(\mathbf{v}_r, S_r)$:

$$\mathbf{v}_r(x) \ = \ \begin{cases} \mathbf{u}(x) & \text{if } x \in \Omega \setminus B(x_0, r) \\ 0 & \text{if } x \in \Omega \cap B(x_0, r) \end{cases} \ ,$$

$$S_r \ = \ S \cup \partial B(x_0, r) \ .$$

We have then the inequality $E(\mathbf{u}, S) \ \leq \ E(\mathbf{v}_r, S_r)$, for any $r > 0$ sufficiently small. We use the properties h1), h2) of F to deduce that for any $x_0 \in \Omega$ and $r > 0$ we have :

$$\int_{B(x_0,r)} w(\nabla \mathbf{u}) \ \mathrm{d}x \ \leq \ C \Omega_n(x; \mathbf{u}_0) \ r^{n-1} \quad , \qquad (7.4.10)$$

where $\Omega_n(x_0; \mathbf{u}_0)$ is a number defined by

$$\Omega_n(x_0; \mathbf{u}_0) = F(\partial B(x_0, 1); \mathbf{u}_0) \quad .$$

In the case of Griffith type surface energy $F(S;\mathbf{u}_0) = G\mathcal{H}^{n-1}(S)$ we have

$$\Omega_n(x_0;\mathbf{u}_0) = G\omega_n \quad ,$$

with ω_n the area of the boundary of the unit ball in n dimensions, that is $\omega_1 = 2$, $\omega_2 = 2\pi$, $\omega_3 = 4\pi^2$.

This inequality lead us to the following energy concentration property for $\mathbf{u}$:

$$\limsup_{r\to 0} \frac{\int_{B(x_0,r)} w(\nabla \mathbf{u}) \, \mathrm{d}x}{r^{n-1}} \ \leq \ C\Omega_n(x_0;\mathbf{u}_0) \quad . \tag{7.4.11}$$

The term from the left hand side of the relation (7.4.11) is the concentration factor of the elastic energy around the point x_0.

The relation (7.4.11) shows that the distribution of elastic energy of the body in the state $(\mathbf{u}, S)$ is what we expect it to be, from the physical viewpoint. Indeed, let us go back to the case $n = 2$. It is well known that in the case of linear elasticity in two dimensions, if $(\mathbf{v}, S)$ is a pair displacement-crack such that $div \ \sigma(\mathbf{v}) = 0$ outside S and $\sigma(\mathbf{v})^+\mathbf{n} = \sigma(\mathbf{v})^-\mathbf{n} = 0$ on S then $\mathbf{v}$ behaves like $\sqrt{r}$ near the edge of the crack, hence the elastic energy behaves like r^{-1}. We recover then the relation (7.4.11) for $n = 2$.

The relation (7.4.11) does imply that elastic energy concentration has an upper bound, but it does not imply that the energy concentration is positive at the tip of the crack. In the case $n = 2$, for example, and for general form of the elastic energy density, the relation (7.4.11) tells us that if there is a concentration of energy (that is if the density of elastic energy goes to infinity around the point x in the reference configuration) then the elastic energy density behaves like r^{-1}. But it might happen that the elastic energy density is nowhere infinite. In this case we simply have

$$\limsup_{r\to 0} \frac{\int_{B(x_0,r)} w(\nabla \mathbf{u}) \, \mathrm{d}x}{r^{n-1}} \ = \ 0$$

which is not in contradiction with (7.4.11).

From the hypothesis h3) upon the surface energy F we get a slightly different estimate. We need first a definition.

Definition 7.2 *For the equilibrium state $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$ and for any open set $A \subset \Omega$ we define:*

$$CE(\mathbf{u}, S)(A) \ = \ \limsup_{r\to 0} \frac{\int_{B((dS\cap A,r)\cap\Omega} w(\nabla \mathbf{u}) \ dx}{r} \quad ,$$

$$CF(S;\mathbf{u}_0)(A) \ = \ \limsup_{r\to 0} \frac{F(\partial B(dS \cap A, r); \mathbf{u}_0)}{r} \quad .$$

The functions $CE(\mathbf{u}, S)(\cdot)$, $CF(S;\mathbf{u}_0)(\cdot)$ are sub-additive functions which by well-known techniques induce outer measures over the σ-algebra of borelian sets in Ω.

The function $CE(\mathbf{u}, S)(\cdot)$ is called the elastic energy concentration measure associated to the equilibrium state $(\mathbf{u}, S)$. Likewise, the function $CF(S;\mathbf{u}_0)(\cdot)$ is called the surface energy concentration measure associated to $(\mathbf{u}, S)$.

Theorem 7.3 *Let $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$ be an equilibrium state. Then for any open set $A \subset \Omega$ we have*

$$CE(\mathbf{u}, S)(A) \leq CF(S; \mathbf{u}_0)(A) \quad .$$

Proof. We consider, for any closed subset A of Ω the following admissible state $(\mathbf{u}_{r,A}, S_{r,A})$ given by:

$$\mathbf{u}_{r,A}(x) = \begin{cases} \mathbf{u}(x) & \text{if } x \in \Omega \setminus B(dS \cap A, r) \\ 0 & \text{if } x \in \Omega \cap B(dS \cap A, r) \end{cases} ,$$

$$S_{r,A} = S \cup \partial B(dS \cap A, r) \quad .$$

The state $(\mathbf{u}, S)$ is an equilibrium state and $(\mathbf{u}_{r,A}, S_{r,A})$ is a comparable state, therefore we obtain:

$$\int_{B(dS \cap A, r) \cap \Omega} w(\nabla \mathbf{u}) \, \mathrm{d}x \leq F(\partial B(dS \cap A, r); \mathbf{u}_0) \quad .$$

We get eventually:

$$\limsup_{r \to 0} \frac{\int_{B(dS \cap A, r) \cap \Omega} w(\nabla \mathbf{u}) \, \mathrm{d}x}{r} \leq \limsup_{r \to 0} \frac{F(\partial B(dS \cap A, r); \mathbf{u}_0)}{r} \quad . \qquad \square$$

Theorem 7.3 shows that an equilibrium state satisfies a kind of Irwin type criterion. Indeed, Irwin criterion is formulated in terms of stress intensity factors. Closer inspection reveals that really it is formulated in terms of elastic energy concentration factor, and that for special geometries of the crack set, and for linear elastic materials, we are able to compute the energy concentration factor as a combination of stress intensity factors.

8 Energy release rate and energy concentration

From relations (7.1.3), (7.1.6), we deduce that a good generalization of the J integral of Rice (which is classically a number) might a functional :

$$\eta \ , \ supp \ \eta \subset\subset \Omega \ \mapsto \ - \frac{d}{dt}_{|t=0} \int_{\Omega} w(\nabla(\mathbf{u}.\phi_t^{-1})) \, \mathrm{d}x \quad ,$$

where ϕ_t is the flow generated by η.

Definition 8.1 *For any equilibrium state $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$ and for any vector field η which generates a one parameter flow $\phi_t \in \mathcal{D}(K)$, such that (there is a $T > 0$ with) $S \subset \phi_t(S)$ for all $t \in [0, T]$, we define the energy release rate along the vector field η by:*

$$ER(\mathbf{u}, S)(\eta) = - \frac{d}{dt}_{|t=0} \int_{\Omega} w(\nabla(\mathbf{u} \circ \phi_t^{-1})) \, dx \qquad (8.0.1)$$

Denote by $\mathcal{V}(K,S)$ the family of all vector fields η generating a one parameter flow $\phi_t \in \mathcal{D}(K)$, such that there is a $T > 0$ with $S \subset \phi_t(S)$ for all $t \in [0,T]$. Formally this set plays the role of the tangent space at the identity for the (infinite dimensional) semigroup of all $\phi \in \mathcal{D}(K)$ such that $S \subset \phi(S)$.

Remark that $ER(\mathbf{u},S)(\eta)$ is a linear expression in the variable η. Indeed, we have

$$ER(\mathbf{u},S)(\eta) \ = \int_\Omega \{\sigma(\nabla \mathbf{u})_{ij}\mathbf{u}_{i,k}\eta_{k,j} \ - \ w(\nabla \mathbf{u}) \ div \ \eta\} \ \mathrm{d}x \quad .$$

Nevertheless, the set $\mathcal{V}(K,S)$ is not a vector space (mainly because the class of all $\phi \in \mathcal{D}(K)$ such that $S \subset \phi(S)$ is only a semigroup, and not a group). Therefore, the energy release rate is not a linear functional in a classical sense.

Definition 8.2 *With the notations from definition 8.1, the total variation of the energy release rate in a open set $D \subset \Omega$ is defined by:*

$$\mid ER \mid (\mathbf{u},S)(D) \ = \ \sup \, ER(\mathbf{u},S)(\eta) \quad , \tag{8.0.2}$$

over all vector fields $\eta \in \mathcal{V}(K,S)$, with support in D, supp $\eta \subset D$, such that for all $x \in \Omega$ we have $\|\eta(x)\| \leq 1$.

The function $\mid ER \mid (\mathbf{u},S)(\cdot)$ is positive and sub-additive, therefore induces an outer measures over the σ-algebra of borelian sets in Ω.

We call this function the energy release rate associated to $(\mathbf{u},S) \in Eq(\mathbf{u}_0,K)$.

The number $\mid ER(\mathbf{u},S) \mid (D)$ measures the maximal elastic energy release rate that can be obtained by propagating the crack set S inside the the set D, with sub-unitary speed, *by preserving it's shape topologically.*

In the case $n = 2$, as explained in subsection 7.2, let x_0 be the crack tip of the crack set S, and J the Rice integral. Then for an open set $D \subset \Omega$ we have:

- $\mid ER(\mathbf{u},S) \mid (D) = J$ if the crack tip belongs to D, that is $x_0 \in D$,
- $\mid ER(\mathbf{u},S) \mid (D) = 0$ if the crack tip does not belong to D.

For short, if we denote by δx_0 the Dirac measure centered at the crack tip x_0, we can write:

$$\mid ER(\mathbf{u},S) \mid = J \ \delta x_0 \quad .$$

It is therefore the appropriate generalization of the Rice integral in three dimensions.

Suppose that for any crack set L and boundary displacement $\mathbf{u}_0$ the surface energy has the expression:

$$F(S;\mathbf{u}_0) \ = \ G\mathcal{H}^{n-1}(S) \quad .$$

Then $CF(S,\mathbf{u}_0)(\Omega)$ is just G times the perimeter (length if $n = 3$) of the edge of the crack S which is not contained in K (technically, it is the Hausdorff measure $\mathcal{H}^{n-2}$ of $dS \setminus K$).

There is a mathematical formula which expresses the perimeter of the edge of an arbitrary crack set L as an "area release rate". Indeed, it is well known that the variation of the area of the crack set $\phi_t(L)$, along a one parameter flow generated by the vector field $\eta \in \mathcal{V}(K, L)$, has the expression:

$$\frac{d}{dt}_{|t=0} \mathcal{H}^{n-1}(\phi_t(S)) \ = \ \int_S div_{tan}\eta \ \mathrm{d}\mathcal{H}^{n-1}(x) \quad ,$$

where the operator div_{tan} is the tangential divergence with respect to the surface S. If we denote by $\mathbf{n}$ the field of normals to the crack set S, then the expression of div_{tan} operator is:

$$div_{tan}\eta \ = \ \eta_{i,i} \ - \ \eta_{i,j}\mathbf{n}_i\mathbf{n}_j \quad .$$

Further, the perimeter of $dS \setminus K$, the edge of the crack set S outside K, admits the following description, similar in principle to the expression of the elastic energy release rate given in definition 8.2:

$$\mathcal{H}^{n-2}(dS \setminus K) \ = \ \sup \left\{ \int_S div_{tan}\eta \ \mathrm{d}\mathcal{H}^{n-1}(x) : \eta \in \mathcal{V}(K, S), \forall x \in X \ \ \|\eta(x)\| \leq 1 \right\} \quad .$$

By putting together this expression of the perimeter, with relation (7.1.6), we obtain therefore the following proposition.

Proposition 8.3 *If for any crack set L we have $F(L; \mathbf{u}_0) = G\mathcal{H}^{n-1}(L)$ then for any absolute minimal state $(\mathbf{u}, S) \in Absmin(\mathbf{u}_0, K)$ such that $S \setminus K \neq \emptyset$ we have*

$$\mid ER(\mathbf{u}, S) \mid (\Omega) \ = \ CF(\mathbf{u}, S)(\Omega) \quad .$$

At this point let us remark that for a general equilibrium state in three dimensions $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$ there is no obvious connection between the energy release rate $\mid ER(\mathbf{u}, S) \mid$, as in definition 8.2, ant the elastic energy concentration $CE(\mathbf{u}, S)$, as in definition 7.2.

The following theorem gives a relation between these two quantities.

Theorem 8.4 *Let $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$ be an equilibrium state of the brittle body with reference configuration Ω, and $D \subset \Omega$ an arbitrary open set. Then we have the following inequality:*

$$\mid ER(\mathbf{u}, S) \mid (D) \ \leq \ CE(\mathbf{u}, S)(D) \quad . \tag{8.0.3}$$

Remark 8.5 *For an arbitrary crack set L, we can't a priori deduce from the EH the existence of a displacement $\mathbf{u}'$ with $(\mathbf{u}', L) \in Adm(\mathbf{u}_0, K)$ and such that for any other state $(\mathbf{v}, L) \in Adm(\mathbf{u}_0, K)$ we have*

$$\int_\Omega w(\nabla \mathbf{u}') \ dx \ \leq \int_\Omega w(\nabla \mathbf{v}) \ dx \quad .$$

From the mechanical point of view such an assumption is natural. There are mathematical results which supports this hypothesis, but as far as I know, not with the regularity needed in this paper. Fortunately, we shall not need to make such an assumption in order to prove theorem 8.4.

Proof. (First part) Let us consider an arbitrary vector field $\eta \in \mathcal{V}(K, S)$, with compact support in D, such that for any $x \in \Omega$ we have $\|\eta(x)\| \leq 1$.

In order to prove the theorem it is enough to show that

$$ER(\mathbf{u}, S)(\eta) \ \leq \ CE(\mathbf{u}, S)(D) \quad . \tag{8.0.4}$$

Indeed, suppose (8.0.4) is true for any vector field $\eta \in \mathcal{V}(K, S)$, with compact support in D, such that for any $x \in \Omega$ we have $\|\eta(x)\| \leq 1$. Then, by taking the supremum with respect to all such vector fields η and using definition 8.2, we get the desired relation (8.0.3).

The inequality (8.0.4) is a consequence of proposition 8.6 and relation (8.0.9), which are of independent interest. We shall resume the proof of theorem 8.4, by giving the proof of the inequality (8.0.4), after we prove the before mentioned results. □

Let ϕ_t be the one parameter flow generated by the vector field η. We can always find a curvilinear coordinate system $(\alpha_1, ..., \alpha_{n-1}, \gamma)$ in the open set D such that:

- on the part of the edge $dS \cap supp\, \eta$ of the crack set S we have $\gamma = 0$,
- the surface $\gamma = t$ (constant) is the boundary of an open set B_t such that

$$\phi_t(S) \setminus S \subset B_t \subset supp\, \eta \subset D \quad ,$$

- there exists $T > 0$ such that for all $t \in [0, T]$ we have

$$B_t \subset B(dS \cap D, t) \cap D \quad , \tag{8.0.5}$$

where $B(dS \cap D, t)$ is the tubular neighbourhood of $dS \cap D$, of radius t.

Consider also the one parameter flow ψ_t, $t \geq 0$, which is equal to identity outside the open set D and, in curvilinear coordinates just introduced, it has the expression

$$\psi_t(x(\alpha_i, \gamma)) = x(\alpha_i, t + \gamma) \quad .$$

Notice that $\psi_t(\Omega) = \Omega \setminus B_t$. We shall use these notations for proving that the elastic energy concentration is a kind of energy release rate, after the following result.

Proposition 8.6 *With the notations made before, we have:*

$$\lim_{t \to 0} \frac{1}{t} \left(\int_{\Omega \setminus B_t} w(\nabla \mathbf{u}) \ dx - \int_{\Omega \setminus B_t} w(\nabla(\mathbf{u} \circ \psi_t^{-1})) \ dx \right) \ = 0 \quad . \tag{8.0.6}$$

Proof. Recalling that $\psi_t(\Omega) = \Omega \setminus B_t$, we use the change of variables $x = \psi_t(y)$ to prove that (8.0.6) is equivalent with

$$\lim_{t \to 0} \frac{1}{t} \left(\int_\Omega \left(w(\nabla \mathbf{u}(y)(\nabla \psi_t)^{-1}(y)) - w((\nabla \mathbf{u})(\psi_t(y)) \right) \det \nabla \psi_t(y) \, \mathrm{d}y \right) = 0 \quad .$$

The previous relation is just

$$\frac{d}{dt}_{|t=0} \int_\Omega \left(w(\nabla \mathbf{u}(y)(\nabla \psi_t)^{-1}(y)) - w((\nabla \mathbf{u})(\psi_t(y)) \right) \det \nabla \psi_t(y) \, \mathrm{d}y = 0 \quad . \tag{8.0.7}$$

We shall prove this from $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$ and from an approximation argument. Notations from subsection 7.1 will be in use.

Denote by ω the vector field which generates the one parameter flow ψ_t. Let us compute, using integration by parts:

$$\frac{d}{dt}_{|t=0} \int_\Omega \left(w(\nabla \mathbf{u}(y)(\nabla \psi_t)^{-1}(y)) - w((\nabla \mathbf{u})(\psi_t(y))) \right) \det \nabla \psi_t(y) \, \mathrm{d}y =$$

$$= \int_\Omega \left(\sigma_{ij} \mathbf{u}_{i,jk} \omega_k + \sigma_{ij} \mathbf{u}_{i,k} \omega_{k,j} \right) \, \mathrm{d}y \quad . \tag{8.0.8}$$

For any $\gamma > 0$, sufficiently small, choose a smooth scalar function $f^\gamma : \Omega \to [0, 1]$, such that:

(a) $f^\gamma(x) = 0$ for all $x \in B_\gamma$, $f^\gamma(x) = 1$ for all $x \in \Omega \setminus B_{2\gamma}$,

(b) as γ goes to 0 we have:

$$\lim_{\gamma \to 0} \int_\Omega f^\gamma \left(\sigma_{ij} \mathbf{u}_{i,jk} \omega_k + \sigma_{ij} \mathbf{u}_{i,k} \omega_{k,j} \right) \, \mathrm{d}y = \int_\Omega \left(\sigma_{ij} \mathbf{u}_{i,jk} \omega_k + \sigma_{ij} \mathbf{u}_{i,k} \omega_{k,j} \right) \, \mathrm{d}y \quad ,$$

$$\lim_{\gamma \to 0} \int_\Omega f^\gamma_{,j} \sigma_{ij} \mathbf{u}_{i,k} \omega_k \, \mathrm{d}y = 0 \quad .$$

For all sufficiently small $\gamma > 0$ it is true that:

$$\int_\Omega \left(\sigma_{ij} \mathbf{u}_{i,jk} \omega^\gamma_k + \sigma_{ij} \mathbf{u}_{i,k} \omega^\gamma_{k,j} \right) \, \mathrm{d}y =$$

$$= \int_\Omega \left(f^\gamma \left(\sigma_{ij} \mathbf{u}_{i,jk} \omega_k + \sigma_{ij} \mathbf{u}_{i,k} \omega_{k,j} \right) + f^\gamma_{,j} \sigma_{ij} \mathbf{u}_{i,k} \omega_k \right) \, \mathrm{d}y \quad .$$

Thus, from (a), (b) above we get the equality:

$$\lim_{\gamma \to 0} \int_\Omega \left(\sigma_{ij} \mathbf{u}_{i,jk} \omega^\gamma_k + \sigma_{ij} \mathbf{u}_{i,k} \omega^\gamma_{k,j} \right) \, \mathrm{d}y = \int_\Omega \left(\sigma_{ij} \mathbf{u}_{i,jk} \omega_k + \sigma_{ij} \mathbf{u}_{i,k} \omega_{k,j} \right) \, \mathrm{d}y \quad .$$

Recall that $(\mathbf{u}, S)$ is an equilibrium state, therefore the stress field $\sigma = \sigma(\nabla \mathbf{u})$ has divergence equal to 0. Integration by parts shows that for any sufficiently small $\gamma > 0$ we have:

$$\int_\Omega \left(\sigma_{ij} \mathbf{u}_{i,jk} \omega^\gamma_k + \sigma_{ij} \mathbf{u}_{i,k} \omega^\gamma_{k,j} \right) \, \mathrm{d}y = \int_\Omega -\sigma_{ij,j} \left(\mathbf{u}_{i,k} \omega^\gamma_k \right) \, \mathrm{d}y = 0 \quad .$$

We obtained therefore the relation:

$$\int_{\Omega} \left(\sigma_{ij} \mathbf{u}_{i,jk} \omega_k + \sigma_{ij} \mathbf{u}_{i,k} \omega_{k,j}\right) \ \mathrm{d}y \ = 0 \quad .$$

This is equivalent to relation (8.0.7), by computation (8.0.8). □

A straightforward consequence of (8.0.6) is that the elastic energy concentration is related to a kind of configurational energy release rate. Namely, we see that

$$\limsup_{t \rightarrow 0} \frac{1}{t} \int_{B_t} w(\nabla \mathbf{u}) \ \mathrm{d}x \ =$$

$$= \ \limsup_{t \rightarrow 0} \frac{1}{t} \left(\int_{\Omega} w(\nabla \mathbf{u}) \ \mathrm{d}x - \int_{\Omega \setminus B_t} w(\nabla(\mathbf{u} \circ \psi_t^{-1})) \ \mathrm{d}x \right) \quad . \tag{8.0.9}$$

We turn back to the proof of theorem 8.0.3. Recall that what it is left to prove is relation (8.0.4).

Proof of (8.0.4). By construction, for all sufficiently small $t > 0$ we have:

$$\frac{1}{t} \int_{B(dS,t) \cap D} w(\nabla \mathbf{u}) \ \mathrm{d}x \ \geq \ \frac{1}{t} \int_{B_t} w(\nabla \mathbf{u}) \ \mathrm{d}x \quad .$$

because $B_t \subset B(dS, t) \cap D$. We write the right hand side member of this inequality as a sum of three terms:

$$\frac{1}{t} \int_{B_t} w(\nabla \mathbf{u}) \ \mathrm{d}x \ =$$

$$= \ \frac{1}{t} \left(\int_{\Omega} w(\nabla \mathbf{u}) \ \mathrm{d}x - \int_{\Omega} w(\nabla(\mathbf{u} \circ \phi_t^{-1})) \ \mathrm{d}x \right) \ +$$

$$+ \ \frac{1}{t} \left(\int_{\Omega} w(\nabla(\mathbf{u} \circ \phi_t^{-1})) \ \mathrm{d}x - \int_{\Omega \setminus B_t} w(\nabla(\mathbf{u} \circ \psi_t^{-1})) \ \mathrm{d}x \right) \ +$$

$$+ \ \frac{1}{t} \left(\int_{\Omega \setminus B_t} w(\nabla(\mathbf{u} \circ \psi_t^{-1})) \ \mathrm{d}x - \int_{\Omega \setminus B_t} w(\nabla(\mathbf{u})) \ \mathrm{d}x \right) \quad .$$

As t goes to 0, the first term converges to $EC(\mathbf{u}, S)(\eta)$ and the third term converges to 0 by proposition 8.6. We want to show that

$$\lim_{t \rightarrow 0} \frac{1}{t} \left(\int_{\Omega} w(\nabla(\mathbf{u} \circ \phi_t^{-1})) \ \mathrm{d}x - \int_{\Omega \setminus B_t} w(\nabla(\mathbf{u} \circ \psi_t^{-1})) \ \mathrm{d}x \right) \ = 0 \quad . \tag{8.0.10}$$

The proof of this limit is identical with the proof of proposition 8.6. Indeed, in that proof we worked with the one parameter flow ψ_t generated by the vector field ω. This one parameter flow is a semigroup (with respect to composition of functions), but after inspection of the proof it can be seen that we only used the following: for any $x \in \Omega \setminus S$

$$\lim_{t \rightarrow 0} \psi_t(x) \ = \ x \quad \text{and} \quad \frac{d}{dt}_{|t=0} \psi_t(x) \ = \ \omega(x) \quad .$$

Therefore we can modify the proof of proposition 8.6 by considering, instead of ψ_t, the diffeomorphisms λ_t defined by:

$$\lambda_t \ = \ \psi_t \circ \phi_t^{-1} \quad .$$

The rest of the proof goes exactly as before, thus leading us to relation (8.0.10).

Eventually, we have:

$$CE(\mathbf{u}, S)(D) \ = \ \limsup_{t \to 0} \frac{1}{t} \int_{B(dS,t)\cap D} w(\nabla \mathbf{u}) \ \mathrm{d}x \ \geq$$

$$\geq \ \limsup_{t \to 0} \frac{1}{t} \int_{B(dS,t)\cap D} w(\nabla \mathbf{u}) \ \mathrm{d}x \ = \ ES(\mathbf{u}, S)(\eta) \ +$$

$$+ \ \lim_{t \to 0} \frac{1}{t} \left(\int_{\Omega} w(\nabla(\mathbf{u} \circ \phi_t^{-1})) \ \mathrm{d}x - \int_{\Omega \setminus B_t} w(\nabla(\mathbf{u} \circ \psi_t^{-1})) \ \mathrm{d}x \right) \ +$$

$$+ \ \lim_{t \to 0} \frac{1}{t} \left(\int_{\Omega \setminus B_t} w(\nabla(\mathbf{u} \circ \psi_t^{-1})) \ \mathrm{d}x - \int_{\Omega \setminus B_t} w(\nabla(\mathbf{u})) \ \mathrm{d}x \right) \ = \ ES(\mathbf{u}, S)(\eta)$$

and (8.0.4) is therefore proven. □

9 A constraint on some minimal solutions

Let us consider now a solution of the model of brittle crack propagation described in section 5. More precisely, for given boundary conditions $\mathbf{u}_0(t)$ and initial crack set K, we shall call a solution $(\mathbf{u}(t), S_t) \in Eq(\mathbf{u}_0(t), S_t)$ of the model described by axioms (A1),..., (A5), by the name "equilibrium solution". Likewise, a solution $(\mathbf{u}(t), S_t) \in Absmin(\mathbf{u}_0(t), S_t)$ of the model described by axioms (A1),(A2),(A3'),(A4), will be called a "minimal solution".

We shall deal with a minimal solution $(\mathbf{u}(t), S_t) \in Absmin(\mathbf{u}_0(t), S_t)$ for which the crack set S_t propagates smoothly, *without topological changes*. Namely we shall suppose that there exists a vector field η with compact support in Ω, such that for all $t \in [0, T]$ we have $S_t = \phi_t(K)$, where ϕ_t is the one parameter flow generated by η.

Because the problem is quasistatic, time enters only as a parameter, therefore we may suppose moreover that for all $x \in \Omega$ we have $\eta(x) \leq 1$.

At each moment $t \in [0, T]$ we shall have $\eta \circ \phi_t \in \mathcal{V}(K, S_t)$.

Theorem 9.1 *Suppose that for any crack set L and boundary displacement $\mathbf{u}_0$ the surface energy has the expression:*

$$F(S; \mathbf{u}_0) \ = \ G\mathcal{H}^{n-1}(S) \quad .$$

Let $(\mathbf{u}(t), S_t) \in Absmin(\mathbf{u}_0(t), S_t)$ be a minimal solution, with $S_0 = K$, such that exists a vector field η with $\|\eta(x)\| \leq 1$ for all $x \in \Omega$ and for all $t \in [0, T]$ we have $S_t = \phi_t(K)$, where ϕ_t is the one parameter flow generated by η.

Then for any $t \in [0,T]$ and any open set $D \subset \Omega$ we have the equalities:

$$| ER(\mathbf{u}(t), \phi_t(S)) | (D) = EC(\mathbf{u}(t), \phi_t(S))(D) =$$
$$= CF(\phi_t(S); \mathbf{u}_0(t))(D) = G\mathcal{H}^{n-2}(dS \setminus K) \quad . \qquad (9.0.1)$$

Proof. Theorems 8.4 and 7.3 tell us that for any open set $D \subset \Omega$, and for any $t \in [0,T]$ we have

$$| ER(\mathbf{u}(t), \phi_t(S)) | (D) \leq EC(\mathbf{u}(t), \phi_t(S))(D) \leq CF(\phi_t(S); \mathbf{u}_0(t))(D) \quad .$$

Proposition 8.3 tells that

$$CF(\phi_t(S); \mathbf{u}_0(t))(\Omega) = | ER(\mathbf{u}(t), \phi_t(S)) | (\Omega) \quad .$$

We deduce that for any open set $D \subset \Omega$, and for any $t \in [0,T]$ the string of equalities (9.0.1) is true. $\square$

This result is natural in two dimensional linear elasticity. Nevertheless, in the case of three dimensional elasticity, the constraints on the elastic energy concentration provided by theorem 9.1 might be too hard to satisfy.

Indeed, from (9.0.1) we deduce that in particular the elastic energy concentration has to be absolutely continuous with respect to the perimeter measure of the edge of the crack.

10 Conclusions

We have proposed a general model of brittle crack propagation based on Mumford-Shah functionals. We have defined equilibrium and absolute minimal solutions of the model.

By a combination of analytical and configurational analysis, we defined measures of energy release rate and energy concentrations for equilibrium and absolute minimal solutions and we have shown that there is a difference between such solutions, as shown mainly by theorems 7.3, 8.4 and 9.1.

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Microfractured media with a scale and Mumford-Shah energies

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Abstract

We want to understand he concentration of damage in microfractured elastic media. Due to the different scallings of the volume and area (or area and length in two dimensions) the traditional method of homogenization using periodic arrays of cells seems to fail when applied to the Mumford-Shah functional and to periodically fractured domains.

In the present paper we are departing from traditional homogenization. The main result implies the use of Mumford-Shah energies and leads to an explanation of the observed concentration of damage in microfractured elastic bodies.

1 Introduction

A new direction of research in brittle fracture mechanics begins with the article of Mumford & Shah [15] regarding the problem of image segmentation. This problem, which consists in finding the set of edges of a picture and constructing a smoothed version of that picture, it turns to be intimately related to the problem of brittle crack evolution. In the before mentioned article Mumford and Shah propose the following variational approach to the problem of image segmentation: let $g : \Omega \subset \mathbb{R}^2 \rightarrow [0,1]$ be the original picture, given as a distribution of grey levels (1 is white and 0 is black), let $u : \Omega \rightarrow R$ be the smoothed picture and K be the set of edges. K represents the set where u has jumps, i.e. $u \in C^1(\Omega \setminus K, R)$. The pair formed by the smoothed picture u and the set of edges K minimizes then the functional:

$$I(u,K) \ = \ \int_{\Omega} \alpha \ \mid \nabla u \mid^2 \ \mathrm{d}x \ + \ \int_{\Omega} \beta \ \mid u - g \mid^2 \ \mathrm{d}x \ + \ \gamma \mathcal{H}^1(K) \ \ .$$

The parameter α controls the smoothness of the new picture u, β controls the L^2 distance between the smoothed picture and the original one and γ controls the total length of the edges given by this variational method. The authors remark that for $\beta = 0$ the functional I might be useful for an energetic treatment of fracture mechanics.

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An energetic approach to fracture mechanics is naturally suited to explain brittle crack appearance under imposed boundary displacements. The idea is presented in the followings.

The state of a brittle body is described by a pair displacement-crack. $(\mathbf{u}, K)$ is such a pair if K is a crack — seen as a surface — which appears in the body and $\mathbf{u}$ is a displacement of the broken body under the imposed boundary displacement, i.e. $\mathbf{u}$ is continuous in the exterior of the surface K and $\mathbf{u}$ equals the imposed displacement $\mathbf{u}_0$ on the exterior boundary of the body.

Let us suppose that the total energy of the body is a Mumford-Shah functional of the form:

$$E(\mathbf{u}, K) \ = \ \int_\Omega w(\nabla \mathbf{u}) \ \mathrm{d}x \ + \ F(\mathbf{u}_0, K) \quad .$$

The first term of the functional E represents the elastic energy of the body with the displacement $\mathbf{u}$. The second term represents the energy consumed to produce the crack K in the body, with the boundary displacement $\mathbf{u}_0$ as parameter. Then the crack that appears is supposed to be the second term of the pair $(\mathbf{u}, K)$ which minimizes the total energy E.

Models for brittle damage, based on functionals of the Mumford-Shah type have have been proposed by Francfort-Marigo [11], Buliga [6], among others. Such models have been studied intensively from the mathematical point of view, espacially by the Italian school of geometric measure theory, to name a few: De Giorgi, Ambrosio, Dal Maso, Buttazzo.

The first homogenization result, concerning the Mumford-Shah functional, seems to be Braides, Defranceschi, Vitali [5]. In this paper it is done the homogenization of a Mumford-Shah functional of the form:

$$\int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u\right) \ \mathrm{d} + \int_{S_u} g\left(\frac{x}{\varepsilon}, (u^+ - u^- \otimes \nu_u\right) \ \mathrm{d}\mathcal{H}^{n-1} \quad .$$

The paper Focardi, Gelli [14] (and the references therein) are part of another line of research which might be relevant for this paper: homogenization of perforated domains.

In the present paper we are departing from traditional homogenization. The line of research concerning perforated domains is close to our problem, but for various reasons the results from perforated domains don't apply here.

We want to understand he concentration of damage in microfractured elastic media. Due to the different scallings of the volume and area (or area and length in two dimensions) the traditional method of homogenization using periodic arrays of cells seems to fail when applied to the Mumford-Shah functional and to periodically fractured domains.

The main result, theorem 4.2, implies the use of Mumford-Shah energies and leads to an explanation of the observed concentration of damage in microfractured elastic bodies.

Instead of performing a homogenization of the total energy of the microfractured body and then study the minimizers of the homogenized energy, we proceed along a different path. We study sequences of problems on fractured elastic bodies, indexed by a scale parameter ε. Each such problem has (at least approximative) solutions. We find estimates of the area of the damaged region in terms of the scale ε.

2 Notations

Let Ω be a bounded, open subset of $\mathbb{R}^2$, with locally Lipschitz boundary. We denote by $Y = [0,1]^2$ the unit closed square in $\mathbb{R}^2$.

For a given $\varepsilon > 0$ let $\mathbb{Z}_\varepsilon \subset \mathbb{R}^2$ be the lattice of points in $\mathbb{R}^2$ with coordinates of the form $(\varepsilon m, \varepsilon n)$, for all $m, n \in \mathbb{Z}$.

We denote by $\mathbb{Z}(\varepsilon, \Omega) \subset \mathbb{Z}_\varepsilon$ the set of all $z \in \mathbb{Z}_\varepsilon$ such that

$$z + \varepsilon Y \subset \Omega \quad .$$

To any $z \in \mathbb{Z}(\varepsilon, \Omega)$ we associate the cell

$$D_z = z + \varepsilon Y \ \subset \ \Omega \quad .$$

The set $\mathbb{Z}(\varepsilon, \Omega)$ is finite for any $\varepsilon > 0$. We denote the cardinal of this set by $N(\varepsilon)$ and we notice that as ε goes to 0 we have

$$\lim_{\varepsilon \to 0} \frac{N(\varepsilon)\varepsilon^2}{A(\Omega)} = 1 \quad ,$$

where $A(\Omega)$ denotes the area of Ω. Thus for small ε the number of cells $N(\varepsilon)$ is approximately equal to $A(\Omega)/\varepsilon^2$.

3 The model

We take Ω to be the configuration set of a microfractured linear elastic body. We explain further what we mean by this.

The elastic properties of the body are described by an elastic potential

$$w : M^{2\times 2}_{sym}(\mathbb{R}) \to \mathbb{R} \quad .$$

We suppose that the function w is quadratic and strictly positive definite.

For a given displacement $\mathbf{u} : \Omega \to \mathbb{R}^2$, the elastic energy of the body is given by

$$\int_\Omega w(e(\mathbf{u})) \, \mathrm{d}x \quad ,$$

where $e(\mathbf{u})$ is the deformation of the displacement $\mathbf{u}$, that is the symmetric part of the gradient of $\mathbf{u}$: for any $x \in \Omega$

$$e(\mathbf{u})(x) \ = \ \frac{1}{2}\left(\nabla \mathbf{u}(x) + (\nabla \mathbf{u})^T(x)\right) \quad .$$

For a fixed $\varepsilon > 0$ we suppose that the body contains a distribution of micro-fractures at the scale ε, seen as a union of (Lipschitz) curves

$$F_\varepsilon = \bigcup_{z \in \mathbb{Z}(\varepsilon,\Omega)} (z + \varepsilon F_z) \quad ,$$

where for each $z \in \mathbb{Z}(\varepsilon, \Omega)$ the (Lipschitz) curve F_z lies inside the unit cell Y:

$$F_z \subset (0,1)^2 \quad .$$

We explain further what we mean by an imposed boundary displacement $\mathbf{u}_0$, and what we mean by $\mathbf{u} = \mathbf{u}_0$ on the boundary of Ω.

We consider, for simplicity, that $\mathbf{u}_0 : \partial\Omega \rightarrow \mathbb{R}^n$ is a continuous and therefore bounded function. Then, for any $\mathbf{u} \in \mathbf{SBD}(\Omega)$, $\mathbf{u} = \mathbf{u}_0$ if the approximate limit of $\mathbf{u}$ equals $\mathbf{u}_0$ in any point of $\partial\Omega$ where the first exists, i.e.: for all $x \in \partial\Omega$, if there exists $\mathbf{v}(x)$ such that

$$\lim_{\rho \rightarrow O_+} \frac{\int_{B_\rho(x) \cap \Omega} \mid \mathbf{u}(y) - \mathbf{v}(x) \mid \ \mathrm{d}y}{\mid B_\rho(x) \cap \Omega \mid} = 0$$

then $\mathbf{v}(x) \ = \mathbf{u}_0(x)$.

Definition 3.1 *The class of admissible displacements with respect to the distribution of cracks F_ε and with respect to the imposed displacement $\mathbf{u}_0$ is defined as the collection of all $\mathbf{u} \in SBD(\Omega)$ such that*

(a) $\mathbf{u} = \mathbf{u}_0$ *on* $\partial\Omega$*,*

(b) $F_\varepsilon \subset S_u$.

This class of admissible displacements is denoted by $Adm(F_\varepsilon, \mathbf{u}_0)$.

This definition deserves an explanation. An admissible displacement $\mathbf{u}$ is a function which has to be equal to the imposed displacement on the boundary of Ω (condition (a)). Any such function $\mathbf{u}$ is a special function with bounded deformation, that is a reasonably smooth function on the set $\Omega \backslash S_u$ and the function $\mathbf{u}$ is allowed to have jumps along the set S_u. For the technical details see the Appendix. We have to think about S_u as being a collection of curves, with finite length. Physically the set S_u represents the collection of all cracks in the body under the displacement $\mathbf{u}$. The condition (b) tells us that the collection of all cracks associated to an admissible displacement $\mathbf{u}$ contains F_ε, at least.

Definition 3.2 *With the notations from definition 3.1, the total energy of an admissible displacement $\mathbf{u} \in Adm(F_\varepsilon, \mathbf{u}_0)$ is given by*

$$E_\varepsilon(\mathbf{u}) = \int_\Omega w(e(\mathbf{u})) \ dx \ + \ G\mathcal{H}^1(S_u \setminus F_\varepsilon) \quad .$$

The energy of an admissible displacement is of Mumford-Shah type. It contains two terms.

The first term measures the elastic energy of the body under the displacement u. Notice that in the expression of the elastic energy we have integrated over the whole domain Ω. This is simply because the collection of cracks associated to u (that is the set S_u) has Lebesque measure 0, therefore we have

$$\int_{\Omega} w(e(\mathbf{u}))\ \mathrm{d}x \ = \ \int_{\Omega \setminus S_u} w(e(\mathbf{u}))\ \mathrm{d}x \quad .$$

In physical terms, the right hand side expression would make more sense than the left hand side, but from the mathematical point of view they are the same. This is not meaning that the elastic energy neglects the fractures. Indeed, further we shall infimize the energy E_ε over the whole set of admissible displacements. According to condition (b) of definition 3.1, this set is defined with respect to the collection of cracks F_ε, therefore the infimum of the energy E_ε depends on the set of cracks F_ε.

The second term of the Mumford-Shah energy measures the surface energy caused by the apparition of new cracks. The collection of new cracks is the set $S_u \setminus F_\varepsilon$. The constant G has the dimension of energy per unit area, and it is physically related to the Griffith constant.

In [4] has been proven that functionals like E_ε are L^1 inferior semi-continuous and coercive, hence on closed subspaces $\mathbf{V}$ of $\mathbf{SBD}(\Omega)$ the functional E_ε has a minimizer. Such a closed subspace of $\mathbf{SBD}(\Omega)$ is the space of all admissible displacements $Adm(F_\varepsilon, \mathbf{u}_0)$. Therefore we have:

Theorem 3.3 *On the space $Adm(F_\varepsilon, \mathbf{u}_0)$ we consider the topology given by the convergence: $\mathbf{u}_h \to \mathbf{u}$ if*

$$\left\{ \begin{array}{l} \mathbf{u}_h \ \ L^2 \ \to \ \ \mathbf{u} \ \ , \\ \mathcal{H}^{n-1}(S_{u_h} \Delta S_u) \to 0 \quad . \end{array} \right.$$

Then there exists a minimizer of the functional E_ε over the set $Adm(F_\varepsilon, \mathbf{u}_0)$.

In the following section we shall use approximate minimizers.

Definition 3.4 *For a given $\delta > 0$, a function $\mathbf{u} \in Adm(F_\varepsilon, \mathbf{u}_0)$ is a δ-approximate minimizer if*

$$E_\varepsilon(\mathbf{u}) \ \leq \ \delta + \inf \left\{ E_\varepsilon(\mathbf{v}) \ : \ \mathbf{v} \in Adm(F_\varepsilon, \mathbf{u}_0) \right\} \quad .$$

For fixed $\delta > 0$, we model an approximate displacement of a microfractured body as a sequence of displacements $\mathbf{u}_\varepsilon$, with ε converging to 0, such that for each $\varepsilon > 0$ the displacement $\mathbf{u}_\varepsilon \in Adm(F_\varepsilon, \mathbf{u}_0)$ is a δ-approximate minimizer of the Mumford-Shah energy E_ε, over the set $Adm(F_\varepsilon, \mathbf{u}_0)$.

Notice that in the model, at this stage, there is no relation between the crack sets $F_\varepsilon, F_{\varepsilon'}$, for two different scales $\varepsilon, \varepsilon'$.

4 An estimate related to damage concentration

For fixed $\varepsilon, \delta > 0$, given F_ε and imposed boundary displacement $\mathbf{u}_0$, let $\mathbf{u} \in Adm(F_\varepsilon, \mathbf{u}_0)$ be a δ-approximate minimizer of the Mumford-Shah energy E_ε.

In this section we want to estimate the number of ε-cells $z + \varepsilon Y$, $z \in \mathbb{Z}(\varepsilon, \Omega)$, where the initial cracks $z + \varepsilon F_z$ propagated.

Let $l > 0$ be a given length.

Definition 4.1 *For any cell $D_z = z + \varepsilon Y$, $z \in \mathbb{Z}(\varepsilon, \Omega)$, and any δ-approximate minimizer $\mathbf{u}$ we define the emergent crack in the cell D_z by*

$$S_u(z) = (z + \varepsilon Y) \cap (S_u \setminus (z + \varepsilon F_z)) \quad .$$

A cell D_z is called active if the length of the emergent crack is greater than εl, that is:

$$\mathcal{H}^1(S_u(z)) \geq \varepsilon l \quad .$$

We denote by $M(\varepsilon, l)$ the number of active cells. (In this notation we don't mention the dependence of $M(\varepsilon, l)$ on the δ-approximate minimizer $\mathbf{u}$.)

Theorem 4.2 *Suppose that for fixed $\delta > 0$, the crack sets F_ε are chosen so that there exists an approximate displacement of a microfractured body $\mathbf{u}_\varepsilon$, with ε converging to 0, with the property that the sequence*

$$\inf \{ E_\varepsilon(\mathbf{v}) \, : \, \mathbf{v} \in Adm(F_\varepsilon, \mathbf{u}_0) \}$$

is bounded.

Then the number of active cells $M(\varepsilon, l)$ is of order $1/\varepsilon$ and the area of the damaged region of the body

$$Damaged(\varepsilon, \Omega) \; = \; \bigcup_{D_z \; active} D_z$$

is of order ε.

Proof. Let $M > 0$ such that for all $\varepsilon > 0$ we have

$$\inf \{ E_\varepsilon(\mathbf{v}) : \, \mathbf{v} \in Adm(F_\varepsilon, \mathbf{u}_0) \} \leq M \quad .$$

According to definition 3.4, for any $\varepsilon > 0$ we have

$$E_\varepsilon(\mathbf{u}_\varepsilon) = \int_\Omega w(e(\mathbf{u}_\varepsilon)) \, \mathrm{d}x + G \mathcal{H}^1 \left(S_{u_\varepsilon} \setminus F_\varepsilon \right) \leq$$

$$\leq \delta + \inf \{ E_\varepsilon(\mathbf{v}) : \, \mathbf{v} \in Adm(F_\varepsilon, \mathbf{u}_0) \} \leq \delta + M \quad .$$

From definition 4.1 we get the following estimate:

$$\mathcal{H}^1 \left(S_{u_\varepsilon} \setminus F_\varepsilon \right) = \sum_{z \in \mathbb{Z}(\varepsilon, \Omega)} \mathcal{H}^1 \left(S_u(z) \right)) \geq M(\varepsilon, l) \; l \; \varepsilon \quad .$$

We have therefore

$$G \; M(\varepsilon, l) \; l \; \varepsilon \leq G \mathcal{H}^1 \left(S_{u_\varepsilon} \setminus F_\varepsilon \right) \leq E_\varepsilon(\mathbf{u}_\varepsilon) \leq M + \delta \quad .$$

All in all we have obtained the estimate:

$$M(\varepsilon, l) \leq \frac{1}{\varepsilon} \frac{M + \delta}{Gl} \quad .$$

The area of the damaged region of the body is

$$Area(Damaged(\varepsilon, \Omega)) = \sum_{D_z \text{ active}} Area(D_z) = \varepsilon^2 M(\varepsilon, l) \leq \varepsilon \frac{M + \delta}{Gl} \quad .$$

The proof is done. □

5 Conclusions

The theorem implies that the area of the damaged region is much smaller than the total area of the body, as ε goes to zero. In this model the use of Mumford-Shah energies leads to an explanation of the observed concentration of damage in microfractured elastic bodies.

Notice that we need more precise estimates in order to prove that the damaged region (at the scale ε) converges, as ε goes to zero, to a curve with finite length. All we know at this moment is that the area of the damaged region goes to zero as the scale parameter ε.

In experiments it has been observed that the damaged region is approximately straight. It is possible that Mumford-Shah energies might explain this, since geometries of the active crack set, that is $S_u \setminus F_\varepsilon$, close to a straight line would be preferred by the energy E_ε. See [7] for examples that in some situations the leading term of a Mumford-Shah energy is the one accounting for the length of the crack, and not the elastic energy part.

Finally, in theorem 4.2 we obtained an estimate of the number of cells where cracks of length at least $\varepsilon \; l$ appear. It would be interested to study the interplay between ε and l in this estimate.

6 Appendix.Functions with bounded variation or deformation

This section is dedicated to a brief voyage trough the spaces **SBV** and **SBD**.

The space $\mathbf{SBV}(\Omega, R^n)$ of special functions with bounded variation was introduced by De Giorgi and Ambrosio in the study of a class of free discontinuity problems ([9], [1], [2]). For any function $\mathbf{u} \in L^1(\Omega, R^n)$ let us denote by $D\mathbf{u}$ the distributional derivative of $\mathbf{u}$ seen as a vector measure. The variation of $D\mathbf{u}$ is a scalar measure defined like this: for any Borel measurable subset B of Ω the variation of $D\mathbf{u}$ over B is

$$\mid D\mathbf{u} \mid (B) \; = \; sup \; \left\{ \sum_{i=1}^{\infty} \mid D\mathbf{u}(A_i) \mid \; : \; \cup_{i=1}^{\infty} A_i \subset B \; , \; A_i \cap A_j = \emptyset \;\; \forall i \neq j \right\} \quad .$$

A function $\mathbf{u}$ has bounded variation if the total variation of $D\mathbf{u}$ is finite. We send the reader to the book of Evans & Gariepy [13] for basic properties of such functions.

The space $\mathbf{SBV}(\Omega, R^n)$ is defined as follows:

$$\mathbf{SBV}(\Omega, R^n) \ = \ \left\{ \mathbf{u} \in L^1(\Omega, R^n) : \ \mid D\mathbf{u} \mid (\Omega) < +\infty \ , \ \mid D^s\mathbf{u} \mid (\Omega \setminus \mathbf{S_u}) = 0 \right\} \ .$$

The Lebesgue set of $\mathbf{u}$ is the set of points where $\mathbf{u}$ has approximate limit. The complementary set is a $\mathcal{L}^n$ negligible set denoted by $\mathbf{S_u}$. If $\mathbf{u}$ is a special function with bounded variation then $\mathbf{S_u}$ is also σ (i.e. countably) rectifiable.

From the Calderon & Zygmund [8] decomposition theorem we obtain the following expression of $D\mathbf{u}$, the distributional derivative of $\mathbf{u} \in \mathbf{SBV}(\Omega, R^n)$, seen as a measure:

$$D\mathbf{u} \ = \ \nabla \mathbf{u}(x) \ \mathrm{d}x \ + \ [\mathbf{u}] \otimes \mathbf{n} \ \mathrm{d}\mathcal{H}^{n-1}_{|_K} \quad .$$

We shall use further the notation $\mu \ll \lambda$ if the measure μ is absolutely continuous with respect to the measure λ.

Let us define the following Sobolev space associated to the crack set K (see [3]):

$$W^{1,2}_K \ = \ \left\{ \mathbf{u} \in \mathbf{SBV}(\Omega, R^n) : \ \int_\Omega \mid \nabla \mathbf{u} \mid^2 \ \mathrm{d}x + \int_K [\mathbf{u}]^2 \ \mathrm{d}\mathcal{H}^{n-1} < +\infty \ , \ \mid D^s\mathbf{u} \mid \ll \mathcal{H}^{n-1}_{|_K} \right\} \ .$$

It has been proved in [10] the following equality:

$$W^{1,2}(\Omega \setminus K, \mathbb{R}^n) \cap L^\infty(\Omega, R^n) \ = \ W^{1,2}_K(\Omega, R^n) \cap L^\infty(\Omega, R^n) \quad . \qquad (6.0.1)$$

A similar description can be made for the space of special functions with bounded deformation $\mathbf{SBD}(\Omega)$ can be found in [4]. For any function $\mathbf{u} \in L^1(\Omega, R^n)$ we denote by $E\mathbf{u}$ the symmetric part of the distributional derivative of $\mathbf{u}$, seen as a vector measure. We denote also by $\mathbf{J_u}$ the subset of Ω where $\mathbf{u}$ has different approximate limits with respect to a point-dependent direction. The difference between $\mathbf{S_u}$ and $\mathbf{J_u}$ is subtle. Let us quote only the fact that for a function $\mathbf{u} \in \mathbf{SBV}(\Omega, R^n)$ the difference of these sets is $\mathcal{H}^{n-1}$-negligible.

The definition of $\mathbf{SBD}(\Omega)$ is the following:

$$\mathbf{SBD}(\Omega, R^n) \ = \ \left\{ \mathbf{u} \in L^1(\Omega, R^n) : \ \mid E\mathbf{u} \mid (\Omega) < +\infty \ , \ \mid E^s\mathbf{u} \mid (\Omega \setminus \mathbf{J_u}) = 0 \right\} \ .$$

If $\mathbf{u}$ is a special function with bounded deformation then $\mathbf{J_u}$ is countably rectifiable. We have a decomposition theorem for $\mathbf{SBD}$ functions, similar to Calderon & Zygmund result applied for $\mathbf{SBV}$ functions. The decomposition theorem is due to Belletini, Coscia & Dal Maso [4] and asserts that

$$E\mathbf{u} \ = \ \epsilon(\mathbf{u})(x) \ \mathrm{d}x \ + \ [\mathbf{u}] \odot \mathbf{n} \ \mathrm{d}\mathcal{H}^{n-1}_{|_{\mathbf{J_u}}} \quad .$$

Here $\odot$ means the symmetric part of tensor product and $\epsilon(\mathbf{u})$ is the approximate symmetric gradient, hence the approximate limit of the symmetric part of the gradient of $\mathbf{u}$.

We sum up the main facts about functions with bounded variation or deformation, in the following three theorems.

Theorem 6.1 *Let* $\mathbf{u} \in L^1(\Omega, \mathbb{R}^m)$. *Then*

- *(De Giorgi) If* $\mathbf{u} \in \mathbf{BV}(\Omega, \mathbb{R}^m)$ *then* $\mathbf{S_u}$ *is countably rectifiable,* $\mathcal{H}^{n-1}(\mathbf{S_u} \setminus \mathbf{J_u}) = 0$ *and in* $\mathcal{H}^{n-1}$*-almost every point* $x \in \mathbf{S_u}$ *exists the approximate limits of* $\mathbf{u}$ *in the directions* $\nu(x)$ *and* $-\nu(x)$ *where* $\nu(x)$ *is the normal to* $\mathbf{S_u}$ *in* x.

- *(Kohn, Ambrosio, Coscia, Dal Maso) Let* $m = n$ *and* $\mathbf{u} \in \mathbf{BD}(\Omega)$. *Let* $\Theta_{\mathbf{u}}$ *be the Kohn set :*

$$\Theta_{\mathbf{u}} = \left\{ x \in \Omega \ : \ \limsup_{\rho \to 0^+} \frac{\mid E\mathbf{u} \mid (B_\rho(x))}{\rho^{n-1}} \ > \ 0 \right\}$$

Then $\Theta_{\mathbf{u}}$ *is countably rectifiable ,* $\mathbf{J_u} \subseteq \Theta_{\mathbf{u}}$ *and* $\mathcal{H}^{n-1}(\Theta_{\mathbf{u}} \setminus \mathbf{J_u}) = 0$.

Theorem 6.2 *Let* $\mathbf{u} \in L^1(\Omega, \mathbb{R}^m)$. *Then*

- *(Calderon, Zygmund) If* $\mathbf{u} \in \mathbf{BV}(\Omega, \mathbb{R}^m)$ *then* $\mathbf{u}$ *is approximately differentiable* $\mathcal{L}^n$*-a.e. in* Ω. *The approximate differential map* $x \mapsto \nabla \mathbf{u}(x)$ *is integrable.* $D\mathbf{u}$ *splits into three mutually singular measures on* Ω

$$D\mathbf{u} = \nabla \mathbf{u} \ dx \ + \ [\mathbf{u}] \otimes \nu \mathcal{H}^{n-1}_{|\mathbf{S_u}} \ + \ C\mathbf{u}$$

where $[\mathbf{u}]$ *is the jump of* $\mathbf{u}$ *in respect with the normal direction on* $\mathbf{S_u}$ ν. $C\mathbf{u}$ *is the Cantor part of* $D\mathbf{u}$ *defined by* $C\mathbf{u}(A) = D^s\mathbf{u}(A \setminus \mathbf{S_u})$ *where* $D^s\mathbf{u}$ *is the singular part of* $D\mathbf{u}$ *in respect to* $\mathcal{L}^n$.

- *(Belletini, Coscia, Dal Maso) Let* $m = n$ *and* $\mathbf{u} \in \mathbf{BD}(\Omega)$. *Then* $\mathbf{u}$ *has symmetric approximate differential* $\epsilon(\mathbf{u})$ $\mathcal{L}^n$*-a.e. in* Ω *and* $E\mathbf{u}$ *splits into three mutually singular measures on* Ω

$$E\mathbf{u} = \epsilon(\mathbf{u}) \ dx \ + \ [\mathbf{u}] \odot \nu \mathcal{H}^{n-1}_{|\mathbf{J_u}} \ + \ E^c\mathbf{u}$$

Moreover $\mathbf{u}$ *is approximately differentiable* $\mathcal{L}^n$*-a.e. in* Ω.

Theorem 6.3 *The following are true:*

- $W^{1,1}(\Omega, \mathbb{R}^m) \subset \mathbf{BV}(\Omega, \mathbb{R}^m)$. *The inclusion is continuous in respect with the Banach space topologies. If*

$$\mathbf{u} \in \mathbf{SBV}(\Omega, \mathbb{R}^m)$$

then

$$\mathbf{u} \in W^{1,1}(\Omega \setminus \mathbf{S_u}, \mathbb{R}^m)$$

Moreover if $\mathbf{u} \in W^{1,1}(\Omega \setminus K, \mathbb{R}^m) \cap L^\infty(\Omega, \mathbb{R}^m)$ *, where* K *is a closed , countably rectifiable set with* $\mathcal{H}^{n-1}(K) < +\infty$*, then* $\mathbf{u} \in \mathbf{SBV}(\Omega, \mathbb{R}^m)$ *and* $\mathcal{H}^{n-1}(K \setminus \mathbf{S_u}) = 0$.

- *Let* $LE^1(\Omega)$ *be the Banach space of* $L^1(\Omega, \mathbb{R}^n)$ *functions with* L^1 *symmetric differential. If* $\mathbf{u} \in \mathbf{SBD}(\Omega)$ *then* $\mathbf{u} \in LE^1(\Omega \setminus \mathbf{J_u})$. *Let* K *be a closed , countably rectifiable set with* $\mathcal{H}^{n-1}(K) < +\infty$. *If* $\mathbf{u} \in LE^1(\Omega \setminus K) \cap L^\infty(\Omega, \mathbb{R}^n)$ *then* $\mathbf{u} \in \mathbf{SBD}(\Omega)$ *and* $\mathcal{H}^{n-1}(K \setminus \mathbf{J_u}) = 0$.

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Existence and construction of bipotentials for graphs of multivalued laws

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Abstract

Based on an extension of Fenchel inequality, bipotentials are non smooth mechanics tools, used to model various non associative multivalued constitutive laws of dissipative materials (friction contact, soils, cyclic plasticity of metals, damage).

Let X, Y be dual locally convex spaces, with duality product $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$. Given the graph $M \subset X \times Y$ of a multivalued law $T : X \to 2^{Y}$, we state a simple necessary and sufficient condition for the existence of a bipotential b for which M is the set of (x, y) such that $b(x, y) = \langle x, y \rangle$.

If this condition is fulfilled, we use convex lagrangian covers in order to construct such a bipotential, generalizing a theorem due to Rockafellar, which states that a multivalued constitutive law admits a superpotential if and only if its graph is cyclically monotone.

1 Introduction

The basic tools of the mechanics of continua are the kinematical compatibility and equilibrium local equations but they are not sufficient to describe the deformation and motion of the continuous media. Additional information must be given through the constitutive laws traducing the material behaviour. In its simplest form, a constitutive law is given by a graph collecting couples of dual variables resulting from experimental testing.

For many physically relevant situations, the constitutive laws are multivalued, but also associated. The graph of the constitutive law is included in the graph of the subdifferential of a convex (and lower semi continuous) superpotential ϕ. The constitutive law takes the form of a differential inclusion, $y \in \partial\phi(x)$. Any superpotential ϕ has a polar function ϕ^{*} satisfying a fundamental relation, Fenchel's inequality,

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$\forall x, y, \phi(x) + \phi^*(y) \geq \langle x, y \rangle$. The constitutive law may be also written as $x \in \partial\phi^*(y)$. In the literature, this kind of materials are often called standard materials or generalized standard materials [12].

From the viewpoint of applications it is important to know whether there exists a superpotential for a given non smooth graph and how to construct it. The answer to this question is provided by a famous theorem due to Rockafellar [20] that ensures a graph admits a superpotential if and only if it is maximal cyclically monotone.

However, some of the constitutive laws are non-associated. They cannot be cast in the mould of the standard materials. To skirt this pitfall, a possible response, proposed first in [21], consists in constructing a function b of two variables, bi-convex and satisfying an inequality generalizing Fenchel's one, $\forall x, y, \quad b(x, y) \geq \langle x, y \rangle$. We call it a bipotential. Physically, it represents the dissipation. In the case of associated constitutive laws the bipotential has the expression $b(x, y) = \phi(x) + \phi^*(y)$.

As for the non associated constitutive laws which can be expressed with the help of bipotentials, they have the form of an implicit relation between dual variables, $y \in \partial b(\cdot, y)(x)$. In Mechanics they are called implicit, or weak, normality rules. The applications of bipotentials to Solid Mechanics are various: Coulomb's friction law [22] , non-associated Drücker-Prager [23] and Cam-Clay models [24] in Soil Mechanics, cyclic Plasticity ([22],[3]) and Viscoplasticity [16] of metals with non linear kinematical hardening rule, Lemaitre's damage law [2], the coaxial laws ([8],[30]). Such kind of materials are called implicit standard materials. A synthetic review of these laws expressed in terms of bipotentials can be found in [8] and [30].

The use of bipotentials in applications is particularly attractive in numerical simulations when using the finite element method, but the interest is not limited to this aspects. For instance, the bound theorems of the limit analysis ([26], [6]) and the plastic shakedown theory ([28], [8], [7], [4]) can be reformulated in a broader framework, precisely by means of weak normality rules. From an applied numerical viewpoint, the bipotential method suggests new algorithms, fast but robust, as well as variational error estimators assessing the accurateness of the finite element mesh ([14], [15], [25], [27], [5], [17], [18]). Applications to the contact Mechanics [9], the Dynamics of granular materials ([10], [11], [13][29]), the cyclic Plasticity of metals [25] and the Plasticity of soils ([1], [17]) illustrate the relevancy of this approach.

In all the papers already mentioned about the mechanical applications, bipotentials for certain multivalued constitutive laws were constructed. Nevertheless, in order to better understand the bipotential approach, one has to solve the following problems:

1) (existence) what are the conditions to be satisfied by a multivalued law such that it can be expressed with the help of a bipotential?

2) is there a procedure to construct a class of bipotentials for a multivalued law? We expect that generically the law does not uniquely determine the bipotential.

We give a first mathematical treatment of these problems and we prove results of existence (theorem 3.2) and construction (theorem 6.7) of bipotentials for a class of graphs of multivaluate laws.

One of the key ideas is constructing the bipotential as an inferior envelope. That could be considered as paradoxal because, in general, it is strongly unprobable that an inferior envelope, even of convex functions, would be convex. Nevertheless, we were convinced of the relevancy of this approach by examples inspired from mechanics and we wished to understand the reason. That led us to introduce the main tool of convex lagrangian covers (Definition 4.1) satisfying an implicit convexity condition.

The recipe that we give in this paper applies only to BB-graphs (Definition 3.1) admitting at least one convex lagrangian cover by maximal cyclically monotone graphs. This is an interesting class of graph of multivalued laws for the following two reasons:

(a) it contains the class of graphs of subdifferentials of convex lsc superpotentials,

(b) any of the graphs of non associated laws from the mentioned mechanical applications of bipotentials is a BB-graph and it admits a physically relevant convex lagrangian cover by cyclically monotone graphs.

Relating to point (b), it is important to know that the results from this paper don't apply to some BB-graphs of mechanical interest, such as the graph of the bipotential associated to contact with friction [21]. This is because we use in this paper only convex lagrangian covers with *maximal* cyclically monotone graphs, see also Remark 5.1.

This paper is only a first step into the subject of constructions of bipotentials. Our aim is to explain a general method of construction in a reasonably simple situation, interesting in itself, leaving aside for the moment certain difficulties appearing in the general method. Another article, in preparation, is dedicated to the extension of the method presented here to a more general class of BB-graphs, by relaxing the notion of convex lagrangian cover. In this way we shall be able to construct bipotentials even for some of the BB-graphs described in Remark 5.1.

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2 Notations and Definitions

X and Y are topological, locally convex, real vector spaces of dual variables $x \in X$ and $y \in Y$, with the duality product $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$. We shall suppose that X, Y have topologies compatible with the duality product, that is: any continuous linear functional on X (resp. Y) has the form $x \mapsto \langle x, y \rangle$, for some $y \in Y$ (resp. $y \mapsto \langle x, y \rangle$, for some $x \in X$).

For any convex and closed set $A \subset X$, its indicator function, χ_A, is defined by

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{otherwise} \end{cases}$$

The indicator function is convex and lower semi continuous. If the set A contains only one element $A = \{a\}$ then we shall use the notation χ_a for the indicator function of A.

We use the notation: $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.

Given a function $\phi : X \to \bar{\mathbb{R}}$, the polar $\phi^* : Y \to \bar{\mathbb{R}}$ is defined by:

$$\phi^*(y) = \sup \{\langle y, x\rangle - \phi(x) \mid x \in X\} \ .$$

The polar is always convex and lower semi continuous.

We denote by $\Gamma(X)$ the class of convex and lower semicontinuous functions $\phi : X \to \bar{\mathbb{R}}$. The class of convex and lower semicontinuous functions $\phi : X \to \mathbb{R}$ is denoted by $\Gamma_0(X)$.

The subgradient of a function $\phi : X \to \bar{\mathbb{R}}$ in a point $x \in X$ is the (possibly empty) set:

$$\partial\phi(x) = \{u \in Y \mid \forall z \in X \ \langle z - x, u\rangle \leq \phi(z) - \phi(x)\} \ .$$

In a similar way is defined the subgradient of a function $\psi : Y \to \bar{\mathbb{R}}$ in a point $y \in Y$, as the set:

$$\partial\psi(y) = \{v \in X \mid \forall w \in Y \ \langle v, w - y\rangle \leq \psi(w) - \psi(y)\} \ .$$

With these notations we have the Fenchel inequality: let $\phi : X \to \bar{\mathbb{R}}$ be a convex lower semicontinuous function. Then:

(i) for any $x \in X, y \in Y$ we have $\phi(x) + \phi^*(y) \geq \langle x, y\rangle$;

(ii) for any $(x, y) \in X \times Y$ we have the equivalences:

$$y \in \partial\phi(x) \iff x \in \partial\phi^*(y) \iff \phi(x) + \phi^*(y) = \langle x, y\rangle \ .$$

Definition 2.1 *We model* **the graph of a constitutive law** *by a set $M \subset X \times Y$. Equivalently, the law is given by the multivalued application*

$$X \ni x \mapsto m(x) \ = \ \{y \in Y \mid (x, y) \in M\} \ .$$

The **dual** *law is the multivalued application*

$$Y \ni y \mapsto m^*(y) \ = \ \{x \in X \mid (x, y) \in M\} \ .$$

The **domain** *of the law is the set $dom(M) = \{x \in X \mid m(x) \neq \emptyset\}$. The* **image** *of the law is the set $im(M) = \{y \in Y \mid m^*(y) \neq \emptyset\}$.*

For example, if $\phi : X \to \mathbb{R}$ is a convex lower semi continuous function, the associated law is the multivalued application $\partial\phi$, the subdifferential of ϕ, [19] Def. 10.1, that is the set of subgradients.The dual law is $\partial\phi^*$ (the subdifferential of the Legendre-Fenchel dual of ϕ) and the graph of the law is the set

$$M(\phi) \ = \ \{(x, y) \in X \times Y \ \mid \ \phi(x) + \phi^*(y) = \langle x, y\rangle\} \ . \qquad (2.0.1)$$

For any convex lower semi continuous function ϕ the graph $M(\phi)$ is *maximal cyclically monotone* ([20] Theorem 24.8. or [19] Proposition 12.2). Conversely, if M is closed and maximal cyclically monotone then there is a convex, lower semicontinuous ϕ such that $M = M(\phi)$.

Definition 2.2 *A* **bipotential** *is a function* $b : X \times Y \to \bar{\mathbb{R}}$*, with the properties:*

(a) b is convex and lower semicontinuos in each argument;

(b) for any $x \in X, y \in Y$ *we have* $b(x, y) \geq \langle x, y\rangle$*;*

(c) for any $(x, y) \in X \times Y$ *we have the equivalences:*

$$y \in \partial b(\cdot, y)(x) \iff x \in \partial b(x, \cdot)(y) \iff b(x, y) = \langle x, y\rangle \ . \tag{2.0.2}$$

The **graph** *of b is*

$$M(b) \ = \ \{(x, y) \in X \times Y \ \mid \ b(x, y) = \langle x, y\rangle\} \ . \tag{2.0.3}$$

Examples. (1.) (Separable bipotential) To any convex lower semicontinuous function ϕ we can associate the **separable bipotential**

$$b(x, y) = \phi(x) + \phi^*(y).$$

The bipotential b and the potential ϕ define the same law: $M(b) = M(\phi)$.

(2.) (Cauchy bipotential) Let $X = Y$ be a Hilbert space and let the duality product be equal to the scalar product. Then we define the **Cauchy bipotential** by the formula

$$b(x, y) = \|x\| \ \|y\|.$$

Let us check the Definition (2.2) The point (a) is obviously satisfied. The point (b) is true by the Cauchy-Schwarz-Bunyakovsky inequality. We have equality in the Cauchy-Schwarz-Bunyakovsky inequality $b(x, y) = \langle x, y\rangle$ if and only if there is $\lambda > 0$ such that $y = \lambda x$ or one of x and y vanishes. This is exactly the statement from the point (c), for the function b under study.

Remark 2.3 *The Cauchy bipotential is an ingredient in the construction of many bipotentials of mechanical interest, because the (graph of the) law associated to b is the set of pairs of collinear and with same orientation vectors. It can not be expressed by a separable potential because M(b) is not a cyclically monotone graph. We shall apply the results of this paper to the Cauchy bipotential, in order to show that we are able to recover the expression of this bipotential from the graph of its associated law.*

3 Existence of a bipotential

Given a non empty set $M \subset X \times Y$, Theorem 3.2 provides a necessary and sufficient condition on M for the existence of a bipotential b with $M = M(b)$. In order to shorten the notation we shall give a name to this condition:

Definition 3.1 *The non empty set* $M \subset X \times Y$ *is a* **BB-graph** *(bi-convex, bi-closed) if for all* $x \in$ *dom*(M) *and for all* $y \in$ *im*(M) *the sets* $m(x)$ *and* $m^*(y)$ *are convex and closed.*

The existence problem is easily settled by the following result.

Theorem 3.2 *Given a non empty set $M \subset X \times Y$, there is a bipotential b such that $M = M(b)$ if and only if M is a BB-graph.*

Proof. Let b be a bipotential such that $M(b)$ is not void. We first want to prove that for any $x \in X$ and $y \in Y$ the sets $m(x)$ and $m^*(y)$ are convex and closed.

Indeed, if $m(x)$ or $m(y)$ are empty or they contain only one element then there is nothing to prove. Let us suppose, for example, that $m(x)$ has more than one element. From the convexity and lower semi continuity hypothesis on b from Definition 2.2, it follows that $m(x)$ is closed and convex. Indeed, remark that $m(x)$ is a sub-level set for a convex and lower semi continuous mapping:

$$m(x) = \{y \in Y \, : \, b(x,y) - \langle x, y\rangle \leq 0\} \quad ,$$

thus a closed and convex set.

Let us consider now a non empty set $M \subset X \times Y$ such that for any $x \in X$ and $y \in Y$ the sets $m(x)$ and $m^*(y)$ are convex and closed. We define then the function $b_\infty : X \times Y \to \overline{\mathbb{R}}$ by:

$$b_\infty(x,y) = \begin{cases} \langle x, y\rangle & \text{if } (x,y) \in M \\ +\infty & \text{otherwise} \end{cases}$$

We have to prove that b_∞ is a bipotential and that $M = M(b_\infty)$. This last claim is trivial, so let us check the points from the Definition 2.2. For the point (a) notice that for any fixed $x \in X$ the function $b_\infty(x, \cdot)$ is the sum of a linear continuous function with the indicator function of $m(x)$. By hypothesis the set $m(x)$ is closed and convex, therefore its indicator function is convex and lower semicontinuous. It follows that the function $b_\infty(x, \cdot)$ is convex and lower semi continuous. In the same way we prove that for any fixed $y \in Y$ the function $b_\infty(\cdot, y)$ is convex and lower semi continuous. The points (b) and (c) are trivial by the Definition of the function b_∞. ■

Remark 3.3 *The uniqueness of b is not true. For example, in the case of the Cauchy bipotential we have two different bipotentials b and b_∞ with the same graph. Therefore the graph of the law alone is not sufficient to uniquely define the bipotential.*

4 Construction of a bipotential

Theorem 3.2 does not give a satisfying bipotential for a given multivalued constitutive law, because the bipotential b_∞ is definitely not interesting for applications.

The most important conclusion of preceding section is contained in the Remark 3.3: in the hypothesis of Theorem 3.2, the graph of the law is not sufficient to uniquely construct an associated bipotential. This is in contrast with the case of a maximal cyclically monotone graph M, when by Rockafellar theorem ([20] Theorem 24.8.) we have a method to reconstruct unambigously the associated separable bipotential (see point (a) below).

In our opinion this is the main reason why the bipotentials are not more often used in applications. Without a recipe for constructing the bipotential associated with (the experimental data contained in) the graph of a non associated mechanical law, there is little chance that one may guess a correct expression for this bipotential.

We are looking for a method of construction of a bipotential with the following properties:

(a) if the graph $M \subset X \times Y$ is maximal cyclically monotone then the constructed bipotential is separable (see Example (1.)),

(b) the method applied to the graph associated to the Cauchy bipotential allows to reconstruct the named bipotential (as mentioned in Remark 2.3, this bipotential appears in many applications),

(c) the method should use only hypothesis related to the graph $M \subset X \times Y$.

Relating to point (c), we noticed that in all applications we were able to reconstruct the bipotentials by knowing a little more than the graph $M \subset X \times Y$, namely a decomposition:

$$M = \bigcup_{\lambda \in \Lambda} M_\lambda \quad .$$

We have to mention that in all applications this decomposition stems out from physical considerations.

Thus we were led to the introduction of convex lagrangian covers.

Definition 4.1 *Let $M \subset X \times Y$ be a non empty set. A* **convex lagrangian cover** *of M is a function $\lambda \in \Lambda \mapsto \phi_\lambda$ from Λ with values in the set $\Gamma(X)$, with the properties:*

(a) The set Λ is a non empty compact topological space,

(b) Let $f : \Lambda \times X \times Y \to \bar{\mathbb{R}}$ be the function defined by

$$f(\lambda, x, y) = \phi_\lambda(x) + \phi^*_\lambda(y).$$

Then for any $x \in X$ and for any $y \in Y$ the functions $f(\cdot, x, \cdot) : \Lambda \times Y \to \bar{\mathbb{R}}$ and $f(\cdot, \cdot, y) : \Lambda \times X \to \bar{\mathbb{R}}$ are lower semi continuous on the product spaces $\Lambda \times Y$ and respectively $\Lambda \times X$ endowed with the standard topology,

(c) We have

$$M = \bigcup_{\lambda \in \Lambda} M(\phi_\lambda) \quad .$$

5 On the existence and uniqueness of convex lagrangian covers

Not any BB-graph admits a convex lagrangian cover. There are at least two sources of examples of such BB-graphs, described further. For more considerations along this line see the last section of the paper.

Remark 5.1 *Let M be a BB-graph with the property: for any ϕ, convex, lower semicontinuous function defined on X, we have $M(\phi) \setminus M \neq \emptyset$. Then M does not admit any convex lagrangian cover.*

As an example take any convex, lower semicontinuous $\phi : X \rightarrow \bar{\mathbb{R}}$ and consider $M \subset M(\phi)$, BB-graph, such that $M \neq M(\phi)$. Then M has the property described previously, therefore it does not admit any convex lagrangian cover.

Remark 5.2 *If M is a BB-graph and A is any linear, continuous transformation of $X \times Y$ into itself, such that $A(X \times \{0\}) \subset X \times \{0\}$ and $A(\{0\} \times Y) \subset \{0\} \times Y$, then $A(M)$ is also a BB-graph. However, it may happen that M admits convex lagrangian covers, but not $A(M)$.*

Indeed, we consider $X = Y = \mathbb{R}$ with natural duality and a $\mathcal{C}^2$ function $\phi : X \rightarrow \mathbb{R}$ with derivative ϕ' strictly increasing. Let us define $M = M(\phi)$ and $A(x, y) = (x, -y)$. The set $A(M)$ has a simple description as the graph of $-\phi'$. As ϕ' is stricly increasing, for any two different $x_1, x_2 \in \mathbb{R}$ and $y_i = -\phi'(x_i)$ $(i = 1, 2)$, we have

$$\langle x_1 - x_2, y_1 - y_2 \rangle = (x_1 - x_2)(y_1 - y_2) < 0 \quad .$$

This implies that $A(M)$ has the property described in Remark 5.1. For if there is a convex, lower semicontinuous $\psi : X \rightarrow \bar{\mathbb{R}}$ such that $M(\psi) \subset A(M)$ then for any two different $x_1, x_2 \in \mathbb{R}$ and $y_i \in \mathbb{R}$, $i = 1, 2$, such that $(x_i, y_i) \in M(\psi)$ we would have

$$\langle x_1 - x_2, y_1 - y_2 \rangle = (x_1 - x_2)(y_1 - y_2) \geq 0 \quad ,$$

which leads to contradiction.

The bipotential b_∞ from the proof of Theorem 3.2 does not come from a convex lagrangian cover. There exist BB-graphs admitting only one convex lagrangian cover (up to reparametrization), as well as BB-graphs which have infinitely many lagrangian covers.

In conclusion, we think it is a hard and challenging mathematical problem to describe all convex lagrangian covers of a BB-graph.

6 Implicit convexity and the main result

The main result of this paper is Theorem 6.7, which gives a recipe for constructing a bipotential not from the graph M of a multivalued law, but from a convex lagrangian cover. Therefore the results in this section apply only to BB-graphs admitting at least one convex lagrangian cover.

In the next section we shall apply this recipe for two convex lagrangian covers of $M(b)$, with b equal to the Cauchy bipotential.

Remark 6.1 *We give here a justification for the name "convex lagrangian cover". Suppose that for any $\lambda \in \Lambda$ the function ϕ_λ is smooth. Then it is well known that the graph (of the subdifferential of ϕ_λ) $M(\phi_\lambda)$ is a lagrangian manifold in the symplectic manifold $X \times Y$ with the canonical symplectic form*

$$\omega\left((x,y),(x',y')\right) = \langle x, y' \rangle - \langle y, x' \rangle$$

Therefore the set M is covered by the family of lagrangian manifolds $M(\phi_\lambda)$, $\lambda \in \Lambda$.

With the help of a convex lagrangian cover we shall define a function b. We intend to prove that (under a certain condition explained further) the function b is a bipotential and that $M = M(b)$.

Definition 6.2 *Let $\lambda \mapsto \phi_\lambda$ be a convex lagrangian cover of the BB-graph M. To the cover we associate the function $b : X \times Y \to \mathbb{R} \cup \{+\infty\}$ by the formula*

$$b(x,y) \; = \; \inf \left\{ \phi_\lambda(x) + \phi^*_\lambda(y) \, : \, \lambda \in \Lambda \right\} \; = \; \inf_{\lambda \in \Lambda} f(\lambda, x, y) \quad .$$

We have to check if the function b has the properties (a), (b), (c) from the Definition 2.2 of a bipotential.

Proposition 6.3 *Let $\lambda \mapsto \phi_\lambda$ be a convex lagrangian cover of the BB-graph M and b given by Definition 6.2. Then:*

(a) for all $(x,y) \in M$ we have $b(x,y) \; = \; \langle x, y \rangle$.

(b) for all $(x,y) \in X \times Y$ we have $b(x,y) \geq \langle x, y \rangle$.

Proof. For all $\lambda \in \Lambda$ and $(x,y) \in X \times Y$ we have the inequality:

$$\phi_\lambda(x) + \phi^*_\lambda(y) \geq \langle x, y \rangle \; .$$

As a consequence of this inequality and Definition 6.2 of the function b we obtain the point (b).

For proving the point (a) it is enough to show that if $(x,y) \in M$ then $b(x,y) \leq \langle x, y \rangle$. But this is true. Indeed, if $(x,y) \in M$ then there is a $\lambda \in \Lambda$ such that $(x,y) \in M(\phi_\lambda)$ and thus

$$\phi_\lambda(x) + \phi^*_\lambda(y) \; = \; \langle x, y \rangle \quad .$$

From the Definition 6.2 it follows that for any $\lambda \in \Lambda$ we have

$$b(x,y) \; \leq \; \phi_\lambda(x) + \phi^*_\lambda(y)$$

therefore $b(x,y) \leq \langle x, y \rangle$, which finishes the proof . ∎

Proposition 6.4 *Let $\lambda \mapsto \phi_\lambda$ be a convex lagrangian cover of the BB-graph M and b given by Definition 6.2.*

(a) Suppose that $x \in X$ is given and that $y \in Y$ has the minimum property

$$b(x,y) - \langle x,y\rangle \ \leq \ b(x,z) - \langle x,z\rangle$$

for any $z \in Y$. Then $b(x,y) \ = \ \langle x,y\rangle$.

(b) If $b(x,y) \ = \ \langle x,y\rangle$ then $(x,y) \in M$.

Proof. (a) We start from the Definition of b. We have

$$b(x,y) \ = \ \inf\left\{\phi_\lambda(x) + \phi^*_\lambda(z) \,:\, \lambda \in \Lambda\right\} \ .$$

We use the compactness of Λ (point (a) from Definition 4.1) to obtain a net $(\lambda_n)_n$ in Λ, which converges to $\bar\lambda \in \Lambda$, such that $b(x,y)$ is the limit of the net $\left(\phi_{\lambda_n}(x) + \phi^*_{\lambda_n}(y)\right)_n$.

From the lower semicontinuity of the cover (point (b) from Definition 4.1) we infer that

$$b(x,y) \ = \ \phi_{\overline{\lambda}}(x) \ + \ \phi^*_{\overline{\lambda}}(y) \ .$$

Remark that the value of the limit $\bar\lambda$ of the net $(\lambda_n)_n$ depends on (x,y).

The hypothesis from point (a) and the definition of the function b implies that for any $z \in Y$ and any $\lambda \in \Lambda$ we have

$$\phi_{\overline{\lambda}}(x) \ + \ \phi^*_{\overline{\lambda}}(y) - \langle x,y\rangle \ \leq \ \phi_\lambda(x) \ + \ \phi^*_\lambda(z) - \langle x,z\rangle.$$

In particular, for $\lambda = \overline{\lambda}$ we get that for all $z \in Y$

$$\phi^*_{\overline{\lambda}}(y) - \phi^*_{\overline{\lambda}}(z) \ \leq \ \langle x, y-z\rangle.$$

This means that $x \in \partial\phi^*_{\overline{\lambda}}(y)$, which implies that

$$b(x,y) \ = \ \phi_{\overline{\lambda}}(x) \ + \ \phi^*_{\overline{\lambda}}(y) \ = \ \langle x,y\rangle.$$

For the point (b), suppose that $b(x,y) = \langle x,y\rangle$. As we remarked before, there is $\overline{\lambda} \in \Lambda$ such that

$$b(x,y) \ = \ \phi_{\overline{\lambda}}(x) \ + \ \phi^*_{\overline{\lambda}}(y) \ .$$

Putting all together we see that

$$\phi_{\overline{\lambda}}(x) \ + \ \phi^*_{\overline{\lambda}}(y) \ = \ \langle x,y\rangle \ ,$$

therefore $(x,y) \in M(\phi_{\overline{\lambda}}) \subset M$. ■

We shall give now a sufficient hypothesis for the separate convexity of b. This is the last ingredient that we need in order to prove that b is a bipotential.

We shall use the following notion of implicit convexity.

Definition 6.5 *Let Λ be an arbitrary non empty set and V a real vector space. The function $f : \Lambda \times V \to \bar{\mathbb{R}}$ is* **implicitly convex** *if for any two elements $(\lambda_1, z_1), (\lambda_2, z_2) \in \Lambda \times V$ and for any two numbers $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ there exists $\lambda \in \Lambda$ such that*

$$f(\lambda, \alpha z_1 + \beta z_2) \ \leq \ \alpha f(\lambda_1, z_1) + \beta f(\lambda_2, z_2) \quad .$$

Let us state the last hypothesis from our construction, as a definition.

Definition 6.6 *Let $\lambda \mapsto \phi_\lambda$ be a convex lagrangian cover of the BB-graph M and $f : \Lambda \times X \times Y \to \mathbb{R}$ the associated function introduced in Definition 4.1, that is the function defined by*

$$f(\lambda, z, y) \ = \ \phi_\lambda(z) + \phi^*_\lambda(y) \quad .$$

The cover is bi-implicitly convex (or a **BIC-cover***) if for any $y \in Y$ and $x \in X$ the functions $f(\cdot, \cdot, y)$ and $f(\cdot, x, \cdot)$ are implicitly convex in the sense of Definition 6.5.*

In the case of $M = M(\phi)$, with ϕ convex and lower semi continuous (this corresponds to separable bipotentials), the set Λ has only one element $\Lambda = \{\lambda\}$ and we have only one potential ϕ. The associated bipotential from Definition 6.2 is obviously

$$b(x, y) \ = \ \phi(x) + \phi^*(y) \ .$$

This is a BIC-cover in a trivial way: the implicit convexity conditions are equivalent with the convexity of ϕ, ϕ^* respectively.

Therefore, in the case of separable bipotentials the BIC-cover condition is trivially true.

Our recipe concerning the construction of a bipotential is based on the following result.

Theorem 6.7 *Let $\lambda \mapsto \phi_\lambda$ be a BIC-cover of the BB-graph M and $b : X \times Y \to R$ defined by*

$$b(x, y) \ = \ \inf \{ \phi_\lambda(x) + \phi^*_\lambda(y) \ \mid \ \lambda \in \Lambda \} \ . \tag{6.0.1}$$

Then b is a bipotential and $M = M(b)$.

Proof. (Step 1.) We prove first that for any $x \in X$ and for any $y \in Y$, the functions $b(x, \cdot)$ and $b(\cdot, y)$ are convex.

For fixed $y \in Y$, for any $x_1, x_2 \in X$ and for any $\varepsilon > 0$, there are $\lambda_1, \lambda_2 \in \Lambda$ such that $(i = 1, 2)$

$$b(x_i, y) + \varepsilon \geq f(\lambda_i, x_i, y) \quad .$$

For the pairs $(\lambda_1, x_1), (\lambda_2, x_2)$ we use the implicit convexity of $f(\cdot, \cdot, y)$ to find that there is $\lambda \in \Lambda$ such that

$$f(\lambda, \alpha x_1 + \beta x_2, y) \leq \alpha f(\lambda_1, x_1, y) + \beta f(\lambda_2, x_2, y) \quad .$$

All in all we have:

$$b(\alpha x_1 + \beta x_2) \leq f(\lambda, \alpha x_1 + \beta x_2, y) \leq$$
$$\leq \alpha f(\lambda_1, x_1, y) + \beta f(\lambda_2, x_2, y) \leq \alpha b(x_1, y) + \beta b(x_2, y) + \varepsilon \quad .$$

As $\varepsilon > 0$ is an arbitrary chosen positive number, the convexity of the function $b(\cdot, y)$ is proven. The proof for the convexity of $b(x, \cdot)$ is similar.

(Step 2.) We shall prove now that for any $x \in X$ and for any $y \in Y$, the functions $b(\cdot, x)$ and $b(\cdot, y)$ are lower semicontinuous. Consider a net $(x_n)_n \in X$ which converges to x. We use the same reasoning as in the proof of Proposition 6.4 (a) to deduce that for each $n \in \mathbb{N}$ there exists a $\lambda_n \in \Lambda$ such that

$$b(x_n, y) \ = \ \phi_{\lambda_n}(x_n) + \phi_{\lambda_n}(y) = f(\lambda_n, x_n, y).$$

Λ is compact, therefore up to the choice of a subnet, there exists a $\lambda \in \Lambda$ such that $(\lambda_n)_n$ converges to λ. We use now the lower semicontinuity of $f(\cdot, \cdot, y)$ in order to get that

$$b(x, y) \leq f(\lambda, x, y) \leq \liminf_{n \rightarrow \infty} f(\lambda_n, x_n, y),$$

therefore the lower semicontinuity of $b(x, \cdot)$ is proven. For the function $b(\cdot, y)$ the proof is similar.

(Step 3.) $M = M(b)$. Indeed, this is true, by Propositions 6.3 (a) and 6.4 (b).

(Step 4.) By Proposition 6.3 (b) we have that for any $(x, y) \in X \times Y$ the inequality $b(x, y) \geq \langle x, y \rangle$ is true. Therefore the conditions (a), (b), from the Definition 2.2 of a bipotential, are verified.

(Step 5.) The only thing left to prove is the string of equivalences from Definition 2.2 (c). Using the knowledge that b is separately convex and lower semicontinuous, we remark that in fact we only have to prove two implications.

The first is: for any $x \in X$ suppose that $y \in Y$ has the minimum property

$$b(x, y) - \langle x, y \rangle \ \leq \ b(x, z) - \langle x, z \rangle$$

for any $z \in Y$. Then $b(x, y) \ = \ \langle x, y \rangle$.

The second implication is similar, only that we start with an arbitrary $y \in Y$ and with $x \in X$ satisfying the minimum property

$$b(x, y) - \langle x, y \rangle \ \leq \ b(z, y) - \langle z, y \rangle$$

for any $z \in X$. Then $b(x, y) \ = \ \langle x, y \rangle$.

The first implication is just Proposition 6.4 (a). The second implication has a similar proof. ■

The next proposition makes easier to check if a convex lagrangian cover satisfies the BIC condition.

Proposition 6.8 *Let $\lambda \mapsto \phi_\lambda$ be a BIC-cover of the BB-graph M. Consider any $\alpha, \beta \in [0, 1]$, $\alpha + \beta = 1$, any $y \in im(M)$, any $\lambda_1, \lambda_2 \in \Lambda$ and any $x_1 \in \partial \phi^*_{\lambda_1}(y)$, $x_2 \in \partial \phi^*_{\lambda_2}(y)$. According to the BIC condition there exists $\lambda \in \Lambda$ such that*

$$f(\lambda, \alpha x_1 + \beta x_2, y) \leq \alpha f(\lambda_1, x_1, y) + \beta f(\lambda_2, x_2, y) \quad . \qquad (6.0.2)$$

Then λ *has the property:*

$$\alpha x_1 + \beta x_2 \ \in \ \partial \phi^*_\lambda(y) \ .$$

Proof. Inequality (6.0.2) expresses as:

$$\phi_\lambda(\alpha x_1 + \beta x_2) + \phi^*_\lambda(y) \leq \alpha \phi_{\lambda_1}(x_1) + \beta \phi_{\lambda_2}(x_2) + \alpha \phi^*_{\lambda_1}(y) + \beta \phi^*_{\lambda_2}(y) \quad . \tag{6.0.3}$$

We have also ($i = 1, 2$)

$$\phi_{\lambda_i}(x_i) + \phi^*_{\lambda_i}(y) = \langle x_i, y \rangle \quad .$$

We use this in the inequality (6.0.3) to get

$$\phi_\lambda(\alpha x_1 + \beta x_2) + \phi^*_\lambda(y) \leq \langle \alpha x_1 + \beta x_2, y \rangle \quad ,$$

which shows that $\alpha x_1 + \beta x_2 \in \partial \phi^*_\lambda(y)$. Therefore the $\lambda \in \Lambda$ given by the implicit convexity inequality satisfies the conclusion of the proposition. ■

Remark 6.9 *Enforcing the satisfaction of the implicit convexity inequality for all values of* λ *which satisfy the conclusion of Proposition 6.8 would be too strong. This remark is supported by the second example in section 7, involving a family of non differentiable potentials for which there is no uniqueness for* λ.

7 Reconstruction of the Cauchy bipotential

In this section we shall reconstruct the Cauchy bipotential from two different convex lagrangian covers. As explained in Remark 2.3, it is important for applications that we are able to reconstruct the expression of the Cauchy bipotential from the graph of its associated law.

We shall take $X = Y = \mathbb{R}^n$ and the duality product is the usual scalar product in $\mathbb{R}$. The Cauchy bipotential is $\bar{b}(x, y) = \|x\|\|y\|$. By Cauchy-Schwarz-Bunyakovsky inequality the set $M = M(\bar{b})$ is

$$M = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \exists \lambda > 0 \, , x = \lambda y\} \cup (\{0\} \times \mathbb{R}^n) \times (\mathbb{R}^n \times \{0\}) \, .$$

Let us consider the topological compact set $\Lambda = [0, \infty]$ (with usual topology) and the function $\lambda \in \Lambda \mapsto \phi_\lambda$ defined as:

- if $\lambda \in [0, \infty)$ then $\phi_\lambda(x) = \dfrac{\lambda}{2}\|x\|^2$,
- if $\lambda = \infty$ then
$$\phi_\infty(x) = \chi_0(x) = \begin{cases} 0 & \text{if } x = 0 \\ +\infty & \text{otherwise} \end{cases}$$

A straightforward computation shows that the associated function f has the expression:

$$f(\lambda, x, y) = \begin{cases} \frac{\lambda}{2}\|x\|^2 + \frac{1}{2\lambda}\|y\|^2 & \text{if } \lambda \in (0, \infty) \\ \chi_0(y) & \text{if } \lambda = 0 \\ \chi_0(x) & \text{if } \lambda = \infty \end{cases} \quad (7.0.1)$$

It is easy to check that we have here a convex lagrangian cover of the set M. We shall prove now that we have a BIC-cover, according to Definition 6.6.

The cases $\lambda = 0$ and $\lambda = \infty$ will be treated separately.

Consider $y \in im(M) = \mathbb{R}^n$, $x_1, x_2 \in X$, $\alpha, \beta \in [0, 1]$, $\alpha + \beta = 1$, and $\lambda_1, \lambda_2 \in (0, \infty)$. We have to find $\lambda \in \Lambda$ such that

$$f(\lambda, \alpha x_1 + \beta x_2, y) \leq \alpha f(\lambda_1, x_1, y) + \beta f(\lambda_2, x_2, y) \quad . \quad (7.0.2)$$

We use Proposition 6.8, for $i = 1, 2$ and $x_i \in \partial\phi^*_{\lambda_i}$ in order to find the value of λ. Computation shows that there is only one such $\lambda \in \Lambda$, given by

$$\frac{1}{\lambda} \ = \ \frac{\alpha}{\lambda_1} + \frac{\beta}{\lambda_2} \quad . \quad (7.0.3)$$

As this value depends only on λ_1, λ_2, we shall try to see if this λ is good for any choice of x_1, x_2.

This is indeed the case: with λ given by (7.0.3) the relation (7.0.2) (multiplied by 2) becomes:

$$\lambda\|\alpha x_1 + \beta x_2\|^2 \leq \alpha\lambda_1\|x_1\|^2 + \beta\lambda_2\|x_2\|^2 \quad . \quad (7.0.4)$$

Remark that (7.0.3) can be written as:

$$\frac{\alpha\lambda}{\lambda_1} + \frac{\beta\lambda}{\lambda_2} \ = \ 1 \ .$$

Write then the fact that the square of the norm is convex, for the convex combination of $\lambda_1 x_1, \lambda_2 x_2$, with the coefficients $\dfrac{\alpha\lambda}{\lambda_1}, \dfrac{\beta\lambda}{\lambda_2}$. We get, after easy simplifications, the inequality (7.0.4).

If $\lambda_1 = 0$, $\lambda_2 \in (0, \infty)$ then y has to be equal to 0 and x_1 is arbitrary, $x_2 = 0$ and $\lambda = 0$. The inequality (7.0.2) is then trivial.

All other exceptional cases lead to trivial inequalities.

Remark that for any $\lambda \in \Lambda$ and any $x, y \in \mathbb{R}^n$ we have

$$f(\lambda, x, y) = f(\frac{1}{\lambda}, y, x)$$

with the conventions $1/0 = \infty$, $1/\infty = 0$. This symmetry and previous proof imply that we have a BIC-cover.

We compute now the function b from Definition 6.2. We know from Theorem 6.7 that b is a bipotential for the set M.

We have:

$$b(x, y) = \inf\left\{f(\lambda, x, y) \,:\, \lambda \in [0, \infty]\right\} .$$

From the relation (7.0.1) we see that actually

$$b(x,y) = \inf \left\{ \frac{\lambda}{2}\|x\|^2 + \frac{1}{2\lambda}\|y\|^2 \,:\, \lambda \in (0,\infty) \right\} .$$

By the arithmetic-geometric mean inequality we obtain that $b(x,y) = \|x\|\|y\|$, that is the Cauchy bipotential.

Here is a second example, supporting the Remark 6.9. We shall reconstruct the Cauchy bipotential starting from a family of non differentiable convex potentials.

Let $\lambda \geq 0$ be non negative and the closed ball of center 0 and radius λ be defined by

$$B(\lambda) \;=\; \{ y \in Y \,:\, \|y\| \leq \lambda \} \quad .$$

Defining $B(+\infty)$ as the whole space Y, one can suppose that λ belongs to the compact set $\Lambda = [0, +\infty]$.

For $\lambda \in [0, +\infty)$ we define the set:

$$M_\lambda \;=\; \{(0,y) \in X \times Y \,:\, \|y\| < \lambda\} \cup \{(x,y) \in X \times Y \,:\, \|y\| = \lambda \text{ and } \exists \eta \geq 0 \;\; x = \eta y\} \; .$$

One can recognize M_λ as the graph of the yielding law of a plastic material with a yielding threshold equal to λ. For $\lambda = +\infty$ we set $M_{+\infty} \;=\; \{0\} \times Y$.

It can be easily verified that the family $(M_\lambda)_{\lambda \in \Lambda}$ of maximal cyclically monotone graphs provides us a convex lagrangian cover of the set:

$$M \;=\; \{(x,y) \in X \times Y \,:\, \exists \alpha, \beta \geq 0 \;\; \alpha x = \beta y\} \; .$$

The corresponding convex lagrangian cover is given by:

- for $\lambda \in [0, +\infty)$, $\phi_\lambda(x) \;=\; \lambda \|x\|$, $\phi^*_\lambda(y) \;=\; \chi_{B(\lambda)}(y)$,
- $\phi_{+\infty}(x) \;=\; \chi_0(x)$, $\phi^*_{+\infty}(y) \;=0$.

The associated function f has the expression:

$$f(\lambda, x, y) = \left\{ \begin{array}{ll} \lambda \|x\| + \chi_{B(\lambda)}(y) & \text{if } \lambda \in (0, \infty) \\ \chi_0(y) & \text{if } \lambda = 0 \\ \chi_0(x) & \text{if } \lambda = +\infty \end{array} \right. \qquad (7.0.5)$$

All hypothesis excepting the BIC-cover condition are obviously satisfied. We check this condition further. Let $\lambda_1 < \lambda_2$, both in $[0, +\infty)$. We want first to determine the values of λ fulfilling the conclusion of Proposition 6.8. Let us recall that:

- if $\|y\| < \lambda$ then $\partial \phi^*_\lambda(y) \;=\; \{0\}$,
- if $\|y\| = \lambda$ then $x \in \partial \phi^*_\lambda(y)$ is equivalent to: $\exists \eta \geq 0$ such that $x = \eta y$,
- if $\|y\| > \lambda$ then $\partial \phi^*_\lambda(y) \;=\; \emptyset$.

Then the following events have to be considered:

(1) if $\|y\| < \lambda_1 < \lambda_2$ then $x_1 \in \partial \phi_{\lambda_1}(y)$ and $x_2 \in \partial \phi_{\lambda_2}(y)$ imply $x_1 = x_2 = 0$,

(2) if $\|y\| = \lambda_1 < \lambda_2$ then $x_1 \in \partial \phi_{\lambda_1}(y)$ and $x_2 \in \partial \phi_{\lambda_2}(y)$ imply: $\exists \eta \geq 0$ such that $x_1 = \eta y$ and $x_2 = 0$. Thus

$$\alpha x_1 + \beta x_2 \; = \; \alpha \eta x_1 \in \partial \phi^*_\lambda(y)$$

occurs for any $\lambda \geq \|y\|$ when $x_1 = 0$ and $\lambda = \|y\|$ otherwise.

(3) If $\lambda_1 < \|y\|$ then there is no x_1 such that $x_1 \in \partial \phi^*_{\lambda_1}(y)$. Likewise, if $\lambda_2 < \|y\|$ then there is no x_2 such that $x_2 \in \partial \phi^*_{\lambda_2}(y)$.

Consider $y \in im(M) = \mathbb{R}^n$, $x_1, x_2 \in X$, $\alpha, \beta \in [0,1]$, $\alpha + \beta = 1$, and $\lambda_1, \lambda_2 \in [0, \infty)$. For the verification of the implicit convexity inequality (7.0.2), we need only to consider the case $\|y\| \leq \min\{\lambda_1, \lambda_2\}$ and we shall choose $\lambda = \min\{\lambda_1, \lambda_2\} \geq \|y\|$. The relation (7.0.2) becomes

$$\min\{\lambda_1, \lambda_2\} \|\alpha x_1 + \beta x_2\| \leq \alpha \lambda_1 \|x_1\| + \beta \lambda_2 \|x_2\| \quad ,$$

which is true by the convexity of the norm. All the other cases turn out to be trivial.

The other half of the BIC-cover condition has a similar proof (remark though that the associated function f is not symmetric, as in the previous case).

By virtue of Theorem 6.7, the function given by Definition 6.2, namely

$$b(x,y) \; = \; \inf\{\phi_\lambda(x) + \phi^*_\lambda(y) \, : \, \lambda \in [0, +\infty]\} \; ,$$

is a bipotential. Computation shows that b is the Cauchy bipotential. Indeed:

$$b(x,y) \; = \; \inf\left\{\lambda\|x\| + \chi_{B(\lambda)}(y) \, : \, \lambda \in [0,\infty)\right\} \; =$$

$$= \; \inf\{\lambda\|x\| \, : \, \lambda \geq \|y\|\} \; = \; \|y\|\|x\| \; .$$

8 Conclusion and perspectives

Given (the graph of) a multivalued constitutive law M, there is a bipotential b such that $M = M(b)$ if and only if M is a BB-graph (Definition 3.1 and Theorem 3.2). If the BB-graph M admits a convex lagrangian cover (Definition 4.1) which is bi-implicitly convex (Definition 6.6) then we are able to construct an associated bipotential (Theorem 6.7).

Remarks 5.1 and 5.2 show that not any BB-graph admits a convex lagrangian cover. We would like to elaborate on the obstructions to the existence of such covers. We start with the example from the Remark 5.2, due to E. Ernst.

From a mechanical point of view, multivalued laws M with the property that for any two different pairs $(x_1, y_1), (x_2, y_2) \in M$ we have

$$\langle x_1 - x_2, y_1 - y_2 \rangle < 0$$

are not very interesting. Indeed, suppose that the evolution of a mechanical system is described by a sequence of states $(x_n, y_n) \in M$. Then, as the system passes from one state to another, the work done is always negative. Much more interesting seem to be multivalued laws with the property that for any $(x, y) \in M$ there is at least a different pair $(x', y') \in M$ such that

$$\langle x - x', y - y' \rangle \geq 0 \quad .$$

The BB-graphs admitting a convex lagrangian cover have this property.

There is another aspect, concerning the linear transformation A from the Remark 5.2. In the example given the transformation $A(x, y) = (x, -y)$ is not symplectic, but still it transforms lagrangian sets into lagrangian sets. In general, if the dimension of X is strictly greater than one then we can find linear endomorphisms of $X \times Y$ transforming lagrangian subsets of $X \times Y$ into sets which are not lagrangian, thus destroying lagrangian covers. Moreover, we can find linear symplectic transformations which transforms a convex lagrangian cover into a lagrangian cover which is no longer convex. For example, take $X = Y = \mathbb{R}$, $A(x, y) = (x, y - x)$ and the BB-graph $M = \mathbb{R} \times \{0\}$. Then $det(A) = 1$, therefore A is symplectic, and $A(M) = \{(x, -x) : x \in \mathbb{R}\}$. The set $A(M)$ is a BB-graph and a lagrangian set, but it does not admit a convex lagrangian cover. The reason for this phenomenon is that convexity is not a symplectic invariant. Nevertheless, there are famous theorems in Hamiltonian Dynamics which have a convexity assumption in the hypothesis, like the theorem of Rabinowitz stating that the Reeb vector field on the boundary of a convex domain which is bounded has at least a closed orbit (equivalently, a convex and coercive hamiltonian on $\mathbb{R}^{2n}$ admits a closed orbit on every level set). We can easily destroy the convexity assumption of this theorem but not the conclusion, by applying a nonlinear symplectomorphism.

For the notion of convex lagrangian cover we had the following source of inspiration. If M is a symplectic manifold and with convexity assumptions left aside, lagrangian covers as described in this paper resemble to (real) symplectic polarizations, which are a basic tool in some problems of symplectic geometry.

Much more interesting are cases relating with Remark 5.1. We may consider BB-graphs M not admitting convex lagrangian covers, but with the property that there is a family of convex, lower semicontinuous functions ϕ_λ, $\lambda \in \Lambda$ such that

$$M \subset \bigcup_{\lambda \in \Lambda} M(\phi_\lambda)$$

with strict inclusion. This is the case, for example, of the bipotential which appears in [21], related to contact with friction. In a future paper we shall extend this method of convex lagrangian cover to lagrangian covers by graphs which are cyclically monotone but not necessarily maximal cyclically monotone.

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Construction of bipotentials and a minimax theorem of Fan

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Abstract

The bipotential theory is based on an extension of Fenchel's inequality, with several powerful applications related to non associated constitutive laws in Mechanics: frictional contact [12], non-associated Drucker-Prager model [1], or Lemaitre plastic ductile damage law [2], to cite a few.

This is a second paper on the mathematics of the bipotentials, following [4]. We prove here another reconstruction theorem for a bipotential from a convex lagrangian cover, this time using a convexity notion related to a minimax theorem of Fan.

Key words: bipotentials, minimax theorems
MSC-class: 49J53; 49J52; 26B25

1 Introduction

In Mechanics, the theory of standard materials is a well-known application of Convex Analysis. However, the so-called non-associated constitutive laws cannot be cast in the mould of the standard materials.

From the mathematical viewpoint, a non associated constitutive law is a multivalued operator $T : X \to 2^Y$ *which is not supposed to be monotone.* Here X, Y are dual locally convex spaces, with duality product $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$.

A possible way to study non-associated constitutive laws by using Convex Analysis, proposed first in [12], consists in constructing a "bipotential" function b of two variables, which physically represents the dissipation.

A bipotential function b is bi-convex, satisfies an inequality generalizing Fenchel's one, $\forall x \in X, y \in Y, \; b(x, y) \geq \langle x, y \rangle$, and a relation involving partial subdifferentials of b with respect to variables x, y. In the case of associated constitutive laws the bipotential has the expression $b(x, y) = \phi(x) + \phi^*(y)$.

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The graph of a bipotential b is simply the set $M(b) \subset X \times Y$ of those pairs (x, y) such that $b(x, y) = \langle x, y \rangle$. A multivalued operator $T : X \to 2^Y$ is expressed with the help of the bipotential b if the graph of T (in the usual sense) equals $M(b)$.

The non associated constitutive laws which can be expressed with the help of bipotentials are called in Mechanics implicit, or weak, normality rules. They have the form of an implicit relation between dual variables, $y \in \partial b(\cdot, y)(x)$.

Among the applications of bipotentials to Solid Mechanics we cite: Coulomb's friction law [9] , non-associated Drücker-Prager [11] and Cam-Clay models [10] in Soil Mechanics, cyclic Plasticity ([9],[3]) and Viscoplasticity [6] of metals with non linear kinematical hardening rule, Lemaitre's damage law [2], the coaxial laws ([5],[13]). A review of these laws expressed in terms of bipotentials can be found in [5] and [13].

In order to better understand the bipotential approach, in the paper [4] we solved two key problems: (a) when the graph of a given multivalued operator can be expressed as the set of critical points of a bipotentials, and (b) a method of construction of a bipotential associated (in the sense of point (a)) to a multivalued, typically non monotone, operator.

Our main tool was the notion of convex lagrangian cover of the graph of the multivalued operator, and a related notion of implicit convexity of this cover.

In this paper we prove another reconstruction theorem for a bipotential from a convex lagrangian cover, this time using a convexity notion related to a minimax theorem of Fan.

2 Notations and definitions

X and Y are topological, locally convex, real vector spaces of dual variables $x \in X$ and $y \in Y$, with the duality product $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$. The topologies of the spaces X, Y are compatible with the duality product, that is: any continuous linear functional on X (resp. Y) has the form $x \mapsto \langle x, y \rangle$, for some $y \in Y$ (resp. $y \mapsto \langle x, y \rangle$, for some $x \in X$).

We use the notation: $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.

Given a function $\phi : X \to \bar{\mathbb{R}}$, the domain $dom\, \phi$ is the set of points with value other than $+\infty$. The polar of ϕ, or Fenchel conjugate, $\phi^* : Y \to \bar{\mathbb{R}}$ is defined by: $\phi^*(y) = \sup \{ \langle y, x \rangle - \phi(x) \mid x \in X \}$.

We denote by $\Gamma(X)$ the class of convex and lower semicontinuous functions $\phi : X \to \bar{\mathbb{R}}$. The class of convex and lower semicontinuous functions $\phi : X \to \mathbb{R}$ is denoted by $\Gamma_0(X)$.

The subdifferential of a function $\phi : X \to \bar{\mathbb{R}}$ in a point $x \in dom\, \phi$ is the (possibly empty) set:

$$\partial \phi(x) = \{ u \in Y \mid \forall z \in X \ \langle z - x, u \rangle \leq \phi(z) - \phi(x) \} \ .$$

In a similar way is defined the subdifferential of a function $\psi : Y \to \bar{\mathbb{R}}$ in a point $y \in dom\, \psi$, as the set:

$$\partial \psi(y) = \{ v \in X \mid \forall w \in Y \ \langle v, w - y \rangle \leq \psi(w) - \psi(y) \} \ .$$

With these notations we have the Fenchel inequality: let $\phi : X \to \bar{\mathbb{R}}$ be a convex lower semicontinuous function. Then:

(i) for any $x \in X, y \in Y$ we have $\phi(x) + \phi^*(y) \geq \langle x, y \rangle$;

(ii) for any $(x, y) \in X \times Y$ we have the equivalences:

$$y \in \partial \phi(x) \iff x \in \partial \phi^*(y) \iff \phi(x) + \phi^*(y) = \langle x, y \rangle \ .$$

Definition 2.1 *To a graph $M \subset X \times Y$ we associate the multivalued operators:*

$$X \ni x \mapsto m(x) \ = \ \{ y \in Y \mid (x,y) \in M \} \ ,$$
$$Y \ni y \mapsto m^*(y) \ = \ \{ x \in X \mid (x,y) \in M \} \ .$$

The domain of the graph M is by definition $dom(M) = \{ x \in X \mid m(x) \neq \emptyset \}$. The image of the graph M is the set $im(M) = \{ y \in Y \mid m^*(y) \neq \emptyset \}$.

3 Bipotentials

The notions and results in this section were introduced or proved in [4].

Definition 3.1 *A bipotential is a function $b : X \times Y \to \bar{\mathbb{R}}$ with the properties:*

(a) b is convex and lower semicontinuos in each argument;

(b) for any $x \in X, y \in Y$ we have $b(x, y) \geq \langle x, y \rangle$;

(c) for any $(x, y) \in X \times Y$ we have the equivalences:

$$y \in \partial b(\cdot, y)(x) \iff x \in \partial b(x, \cdot)(y) \iff b(x, y) = \langle x, y \rangle \ . \tag{3.0.1}$$

The graph of b is

$$M(b) \ = \ \{ (x,y) \in X \times Y \ \mid \ b(x,y) = \langle x, y \rangle \} \ . \tag{3.0.2}$$

Examples. (1.) (Separable bipotential) If $\phi : X \to \mathbb{R}$ is a convex, lower semicontinuous potential, consider the multivalued operator $\partial \phi$ (the subdifferential of ϕ). The graph of this operator is the set

$$M(\phi) \ = \ \{ (x,y) \in X \times Y \ \mid \ \phi(x) + \phi^*(y) = \langle x, y \rangle \} \ . \tag{3.0.3}$$

$M(\phi)$ is *maximally cyclically monotone* [8] Theorem 24.8. Conversely, if M is closed and maximally cyclically monotone then there is a convex, lower semicontinuous ϕ such that $M = M(\phi)$.

To the function ϕ we associate the *separable bipotential*

$$b(x, y) = \phi(x) + \phi^*(y).$$

Indeed, the Fenchel inequality can be reformulated by saying that the function b, previously defined, is a bipotential. More precisely, the point (b) (resp. (c)) in the definition of a bipotential corresponds to (i) (resp. (ii)) from Fenchel inequality.

The bipotential b and the function ϕ have the same graph: $M(b) = M(\phi)$.

(2.) (Cauchy bipotential) Let $X = Y$ be a Hilbert space and let the duality product be equal to the scalar product. Then we define the *Cauchy bipotential* by the formula

$$b(x, y) = \|x\| \, \|y\|.$$

Let us check the Definition (3.1) The point (a) is obviously satisfied. The point (b) is true by the Cauchy-Schwarz-Bunyakovsky inequality. We have equality in the Cauchy-Schwarz-Bunyakovsky inequality $b(x, y) = \langle x, y \rangle$ if and only if there is $\lambda > 0$ such that $y = \lambda x$ or one of x and y vanishes. This is exactly the statement from the point (c), for the function b under study.

The graph $M(b)$ is the set of pairs of collinear and with same orientation vectors. It can not be expressed by a separable bipotential because $M(b)$ is not a cyclically monotone graph.

Definition 3.2 *The non empty set $M \subset X \times Y$ is a BB-graph (bi-convex, bi-closed) if for all $x \in$ dom(M) and for all $y \in$ im(M) the sets $m(x)$ and $m^*(y)$ are convex and closed.*

The following theorem gives a necessary and sufficient condition for the existence of a bipotential associated to a constitutive law M.

Theorem 3.3 *Given a non empty set $M \subset X \times Y$, there is a bipotential b such that $M = M(b)$ if and only if M is a BB-graph.*

Given the BB-graph graph M, the uniqueness of bipotential b such that $M = M(b)$ *is not true*. For example, in the case of the Cauchy bipotential b, the proof of theorem 3.3 (theorem ... [4]) provides a bipotential, denoted by b_∞, such that $M(b) = M(b_\infty)$ but $b \neq b_\infty$. This is in contrast with the case of a maximal cyclically monotone graph M, when by Rockafellar theorem ([8] Theorem 24.8.) we have a method to reconstruct unambigously the associated separable bipotential.

We noticed that in mechanical applications, we were able to reconstruct the physically relevant bipotentials b starting from $M(b)$, by knowing a little more than the graph $M(b)$. This supplementary information is encoded in the following notion.

Definition 3.4 *Let $M \subset X \times Y$ be a non empty set. A convex lagrangian cover of M is a function $\lambda \in \Lambda \mapsto \phi_\lambda$ from Λ with values in the set $\Gamma(X)$, with the properties:*

(a) The set Λ is a non empty compact topological space,

(b) Let $f : \Lambda \times X \times Y \to \bar{\mathbb{R}}$ be the function defined by

$$f(\lambda, x, y) \;=\; \phi_\lambda(x) + \phi^*_\lambda(y).$$

Then for any $x \in X$ and for any $y \in Y$ the functions $f(\cdot, x, \cdot) : \Lambda \times Y \to \bar{\mathbb{R}}$ and $f(\cdot, \cdot, y) : \Lambda \times X \to \bar{\mathbb{R}}$ are lower semi continuous on the product spaces $\Lambda \times Y$ and respectively $\Lambda \times X$ endowed with the standard topology, *(c) We have*

$$M \ = \ \bigcup_{\lambda\in\Lambda} M(\phi_\lambda) \quad .$$

Not any BB-graph admits a convex lagrangian cover. There exist BB-graphs admitting only one convex lagrangian cover (up to reparametrization), as well as BB-graphs which have infinitely many lagrangian covers. The problem of describing the set of all convex lagrangian covers of a BB-graph seems to be difficult. We shall not discuss this problem here, but see the sections 5 and 8 in [4].

The results in this paper apply only to BB-graphs admitting at least one convex lagrangian cover.

To a a convex lagrangian cover we associate a function which will turn out to be a bipotential, under some supplementary hypothesis.

Definition 3.5 *Let $\lambda \mapsto \phi_\lambda$ be a convex lagrangian cover of the BB-graph M. To the cover we associate the function $b : X \times Y \to \mathbb{R} \cup \{+\infty\}$ by the formula*

$$b(x,y) \ = \ \inf\{\phi_\lambda(x) + \phi^*_\lambda(y) \, : \, \lambda \in \Lambda\} \ = \ \inf_{\lambda\in\Lambda} f(\lambda,x,y) \quad .$$

In [4] we imposed an implicit convexity inequality in order to get a function b which is a bipotential. We need two definitions.

Definition 3.6 *Let Λ be an arbitrary non empty set and V a real vector space. The function $f : \Lambda \times V \to \bar{\mathbb{R}}$ is implicitly convex if for any two elements $(\lambda_1, z_1), (\lambda_2, z_2) \in \Lambda \times V$ and for any two numbers $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ there exists $\lambda \in \Lambda$ such that*

$$f(\lambda, \alpha z_1 + \beta z_2) \ \leq \ \alpha f(\lambda_1, z_1) + \beta f(\lambda_2, z_2) \quad .$$

Definition 3.7 *Let $\lambda \mapsto \phi_\lambda$ be a convex lagrangian cover of the BB-graph M and $f : \Lambda \times X \times Y \to \mathbb{R}$ the associated function introduced in Definition 3.4, that is the function defined by*

$$f(\lambda, z, y) \ = \ \phi_\lambda(z) + \phi^*_\lambda(y) \quad .$$

The cover is bi-implicitly convex (or a BIC-cover) if for any $y \in Y$ and $x \in X$ the functions $f(\cdot,\cdot,y)$ and $f(\cdot,x,\cdot)$ are implicitly convex in the sense of Definition 3.6.

In the case of $M = M(\phi)$, with ϕ convex and lower semi continuous (this corresponds to separable bipotentials), the set Λ has only one element $\Lambda = \{\lambda\}$ and we have only one potential ϕ. The associated bipotential from Definition 3.5 is obviously

$$b(x,y) \ = \ \phi(x) + \phi^*(y) \ .$$

This is a BIC-cover in a trivial way: the implicit convexity conditions are equivalent with the convexity of ϕ, ϕ^* respectively.

With this convexity condition we obtained in [4] the following result.

Theorem 3.8 *Let $\lambda \mapsto \phi_\lambda$ be a BIC-cover of the BB-graph M and $b : X \times Y \rightarrow R$ defined by*

$$b(x,y) \ = \ \inf \left\{ \phi_\lambda(x) + \phi^*_\lambda(y) \ \mid \ \lambda \in \Lambda \right\} \ . \qquad (3.0.4)$$

Then b is a bipotential and $M = M(b)$.

4 Main result

For simplicity, in this section we shall work only with lower semi continuous convex functions ϕ with the property that ϕ and its Fenchel dual ϕ^* take values in $\mathbb{R}$.

We reproduce here the following definition of convexity (in a generalized sense), given by K. Fan [7] p. 42.

Definition 4.1 *Let X, Y be two arbitrary non empty sets. The function $f : X \times Y \rightarrow \mathbb{R}$ is convex on X in the sense of Fan if for any two elements $x_1, x_2 \in X$ and for any two numbers $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ there exists a $x \in X$ such that for all $y \in Y$:*

$$f(x,y) \ \leq \ \alpha f(x_1,y) + \beta f(x_2,y).$$

With the help of the previous definition we introduce a new convexity condition for a convex lagrangian cover.

Definition 4.2 *Let $\lambda \mapsto \phi_\lambda$ be a convex lagrangian cover of the BB-graph M. Consider the functions:*

$$g : X \times \Lambda \times X \rightarrow \mathbb{R} \quad , \quad h : Y \times \Lambda \times Y \quad ,$$

given by $g(x, \lambda, z) = \phi_\lambda(x) - \phi_\lambda(z)$, respectively $h(y, \lambda, u) = \phi^*_\lambda(y) - \phi^*_\lambda(u)$.

The cover is Fan bi-implicitly convex if for any $x \in X$, $y \in Y$, the functions $g(x, \cdot, \cdot)$, $h(y, \cdot, \cdot)$ are convex in the sense of Fan on $\Lambda \times X$, $\Lambda \times Y$ respectively.

Recall the following minimax theorem of Fan [7], Theorem 2. In the formulation of the theorem words "convex" and "concave" have the meaning given in definition 4.1 (more precisely f is concave if $-f$ is convex in the sense of the before mentioned definition).

Theorem 4.3 *(Fan) Let X be a compact Hausdorff space and Y an arbitrary set. Let f be a real valued function on $X \times Y$ such that, for every $y \in Y$, $f(\cdot, y)$ is lower semicontinuous on X. If f is convex on X and concave on Y, then the expressions $\min_{x \in X} \sup_{y \in Y} f(x,y)$ and $\sup_{y \in Y} \min_{x \in X} f(x,y)$ have meaning, and*

$$\min_{x \in X} \sup_{y \in Y} f(x,y) = \sup_{y \in Y} \min_{x \in X} f(x,y) \quad .$$

The difficulty of theorem 3.8 boils down to the fact the class of convex functions is not closed with respect to the inf operator. Nevertheless, by using Fan theorem 4.3 we get the following general result.

Theorem 4.4 *Let Λ be a compact Hausdorff space and $\lambda \mapsto \phi_\lambda \in \Gamma_0(X)$ be a convex lagrangian cover of the BB-graph M such that:*

(a) for any $x \in X$ and for any $y \in Y$ the functions $\Lambda \ni \lambda \mapsto \phi_\lambda(x) \in \mathbb{R}$ and $\Lambda \ni \lambda \mapsto \phi^*_\lambda(y) \in \mathbb{R}$ are continuous,

(b) the cover is Fan bi-implicitely convex in the sense of definition 4.2.

Then the function $b : X \times Y \to \mathbb{R}$ defined by

$$b(x, y) \;=\; \inf \{\phi_\lambda(x) + \phi^*_\lambda(y) \;\mid\; \lambda \in \Lambda\}$$

is a bipotential and $M = M(b)$.

Proof. For some of the details of the proof we refer to the proof of theorem 3.8 in [4] (in that paper theorem 4.12). There are five steps in that proof. In order to prove our theorem we have only to modify the first two steps: we want to show that for any $x \in dom(M)$ and any $y \in im(M)$ the functions $b(\cdot, y)$ and $b(x, \cdot)$ are convex and lower semi continuous.

For $(x, y) \in X \times Y$ let us define the function $\overline{xy} : \Lambda \times X \to \mathbb{R}$ by

$$\overline{xy}(\lambda, z) = \langle z, y\rangle + \phi_\lambda(x) - \phi_\lambda(z) \quad .$$

We check now that $\overline{xy}$ verifies the hypothesis of theorem 4.3. Indeed, the hypothesis (a) implies that for any $z \in X$ the function $\overline{xy}(\cdot, z)$ is continuous. Notice that

$$\overline{xy}(\lambda, z) = \langle z, y\rangle + g(x, \lambda, z) \quad .$$

It follows from hypothesis (b) that the function $\overline{xy}$ is convex on Λ in the sense of Fan.

In order to prove the concavity of $\overline{xy}$ on X, it suffices to show that for any $z_1, z_2 \in X$, for any $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$, we have the inequality

$$\overline{xy}(\lambda, \alpha z_1 + \beta z_2) \leq \alpha\overline{xy}(\lambda, z_1) + \beta\overline{xy}(\lambda, z_2)$$

for any $\lambda \in \Lambda$. This inequality is equivalent with

$$\langle \alpha z_1 + \beta z_2, y\rangle - \phi_\lambda(\alpha z_1 + \beta z_2) \leq \alpha\left(\langle z_1, y\rangle - \phi_\lambda(z_1) + \beta\left(\langle z_2, y\rangle - \phi_\lambda(z_2)\right.\right.$$

for any $\lambda \in \Lambda$. But this is implied by the convexity of ϕ_λ for any $\lambda \in \Lambda$.

In conclusion the function $\overline{xy}$ satisfies the hypothesis of theorem 4.3. We deduce that

$$\min_{\lambda\in\Lambda} \sup_{z\in X} \overline{xy}(\lambda, z) = \sup_{z\in X} \min_{\lambda\in\Lambda} \overline{xy}(\lambda, z) \quad .$$

Let us compute the two sides of this equality.

For the left hand side (LHS) we have:

$$LHS = min_{\lambda \in \Lambda} \sup_{z \in X} \{\langle z, y \rangle + \phi_\lambda(x) - \phi_\lambda(z)\} =$$

$$= \min_{\lambda \in \Lambda} \left\{ \phi_\lambda(x) + \sup_{z \in X} \{\langle z, y \rangle - \phi_\lambda(z)\} \right\} =$$

$$= \min_{\lambda \in \Lambda} \{\phi_\lambda(x) + \phi_\lambda^*(y)\} = b(x, y) \quad .$$

For the right hand side (RHS) we have:

$$RHS = \sup_{z \in X} \min_{\lambda \in \Lambda} \{\langle z, y \rangle + \phi_\lambda(x) - \phi_\lambda(z)\} =$$

$$= \sup_{z \in X} \left\{ \langle z, y \rangle - \max_{\lambda \in \Lambda} \{\phi_\lambda(z) - \phi_\lambda(x)\} \right\} \quad .$$

Let $\overline{x} : X \rightarrow \mathbb{R}$ be the function

$$\overline{x}(z) = \max_{\lambda \in \Lambda} \{\phi_\lambda(z) - \phi_\lambda(x)\} \quad .$$

Then the right hand side RHS is in fact:

$$RHS = \overline{x}^*(y) \quad .$$

Therefore we proved the equality:

$$b(x, y) = \overline{x}^*(y) \quad .$$

This shows that the function b is convex and lower semicontinuous in the second argument.

In order to prove that b is convex and lower semicontinuous in the first argument, replace ϕ_λ by ϕ_λ^* in the previous reasoning. ■

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LOWER SEMI-CONTINUITY OF INTEGRALS WITH G-QUASICONVEX POTENTIAL

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Abstract. This paper introduces the proper notion of variational quasiconvexity associated to a group of diffeomorphisms. We prove a lower semicontinuity theorem connected to this notion. In the second part of the paper we apply this result to a class of functions, introduced in [5]. Such functions are $GL(n,R)^+$ quasiconvex, hence they induce lower semicontinuous integrals.

MSC 2000: 49J45

1. Introduction

Lower semi-continuity of variational integrals

$$u \mapsto I(u) = \int_\Omega w(Du(x))\, \mathrm{d}x$$

defined over Sobolev spaces is connected to the convexity of the potential w. In the scalar case, that is for functions u with domain or range in R, the functional I is weakly $W^{1,p}$ lower semi-continuous (weakly * $W^{1,\infty}$) if and only if w is convex, provided it is continuous and satisfies some growth conditions. The notion which replaces convexity in the vector case is quasi-convexity (introduced by Morrey [14]).

We shall concentrate on the case $u : \Omega \subset R^n \to R^n$ which is interesting for continuum media mechanics. Standard notation will be used, like:

$gl(n,R)$	the linear space (Lie algebra) of $n \times n$ real matrices
$GL(n,R)$	the group of invertible $n \times n$ real matrices
$GL(n,R)^+$	the group of matrices with positive determinant
$sl(n,R)$	the algebra of traceless $n \times n$ real matrices
$SL(n,R)$	the group of real matrices with determinant one
$CO(n)$	the group of conformal matrices
id	the identity map
1	the identity matrix
$\circ$	function composition

In this frame Morrey's quasiconvexity has the following definition.

Definition 1.1. *Let $\Omega \subset R^n$ be an open bounded set such that $\mid \partial\Omega \mid = 0$ and $w : gl(n,R) \to R$ be a measurable function. The map w is quasiconvex if for any $H \in gl(n,R)$ and any Lipschitz $\eta : \Omega \to R^n$, such that $\eta(x) = 0$ on $\partial\Omega$, we have*

$$\int_\Omega w(H) \ \leq \ \int_\Omega w(H + D\eta(x)) \tag{1}$$

Translation and rescaling arguments show that the choice of Ω is irrelevant in the above definition.

Any quasiconvex function w is rank one convex. There are several ways to define rank one convexity but this is due to the regularity assumptions upon w. The most

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natural, physically meaningful and historically justified, is to suppose that w is C^2 and link rank one convexity with the ellipticity (cf. Hadamard [10]) of the Euler-Lagrange equation associated to w. There are well-known ways to show that one can get rid of any regularity assumption upon w, replacing it by some growth conditions. Rank one convexity becomes then just what the denomination means, that is convexity along any rank one direction.

Proposition 1.1. *Suppose that $w : gl(n,R) \rightarrow R$ is C^2 and quasiconvex. Then for any pair $a, b \in R^n$ the ellipticity inequality*

$$\frac{\partial^2 w}{\partial H_{ij} \partial H_{kl}}(H) a_i b_j a_k b_l \ \geq \ 0 \tag{2}$$

holds true.

Proof. Take any $\eta \in C^2(\Omega, R^n)$ such that $\eta(x) = 0$ on $\partial\Omega$ and $H \in gl(n,R)$. If w is quasiconvex then the function

$$t \mapsto \ f(t) \ = \ \int_\Omega w(H + tD\eta(x))$$

has a minimum in $t = 0$. Therefore $f'(0) = 0$ and $f''(0) \geq 0$. Straightforward computation shows that $f'(0) = 0$ anyway and $f''(0) \geq 0$ reads:

$$\frac{\partial^2 w}{\partial H_{ij} \partial H_{kl}}(H) \ \int_\Omega \eta_{i,j}(x) \eta_{k,l}(x) \ \geq \ 0$$

With the notation

$$\Delta(\eta) \ = \ \int_\Omega D\eta(x) \otimes D\eta(x)$$

remark that $\Delta(x) \in V = gl(n,R) \otimes gl(n,R)$, because V is a vectorspace and $D\eta(x) \otimes D\eta(x) \in V$ for any $x \in \Omega$. It follows that there is $P \in gl(n,R)$ such that:

$$\Delta(\eta)_{ijkl} = P_{ij} P_{kl}$$

Integration by parts shows that $\Delta(\eta)$ has more symmetry, namely:

$$\Delta(\eta)_{ijkl} = \Delta(\eta)_{ilkj}$$

which turns to be equivalent to $rank \ P \ \leq 1$. Therefore there are $a, b \in R^n$ such that $P = a \otimes b$.

All it has been left to prove is that for any $a, b \in R^n$ there is a $\lambda \neq 0$ and a vector field $\eta \in C^2(\Omega, R^n)$ such that $\eta(x) = 0$ on $\partial\Omega$ and $\Delta(\eta) = \lambda a \otimes b$. For this suppose that Ω is the unit ball in R^n, take $u : [0, \infty] \rightarrow R$ a C^∞ map, such that $u(1) = 0$ and define:

$$\eta(x) = u(\mid x \mid^2) \sin(b \cdot x) a$$

It is a matter of computation to see that η is well chosen to prove the thesis. □

In elasticity the elastic potential function w is not defined on the Lie algebra $gl(n,R)$ but on the Lie group $GL(n,R)$ or a subgroup of it. It would be therefore interesting to find the connections between lower semicontinuity of the functional and the (well chosen notion of) quasiconvexity in this non-linear context. This is a problem which floats in the air for a long time. Let us recall two different definitions of quasiconvexity which are relevant.

Definition 1.2. *Let $w : GL(n,R)^+ \rightarrow R$. Then:*

(a) *(Ball [2]) w is quasiconvex if for any $F \in GL(n,R)^+$ and any $\eta \in C_c^\infty(\Omega, R^n)$ such that $F + D\eta(x) \in GL(n,R)^+$ for almost any $x \in \Omega$ we have*

$$\int_\Omega w(F + D\eta(x)) \ \geq \mid \Omega \mid w(F)$$

(b) *(Giaquinta, Modica & Soucek [9], page 174, definition 3) w is Diff-quasiconvex if for any diffeomorphism $\phi : \Omega \rightarrow \phi(\Omega)$ such that $\phi(x) = Fx$ on $\partial\Omega$, for some $F \in GL(n,R)^+$ we have:*

$$\int_\Omega w(D\phi(x)) \ \geq \ \int_\Omega w(F)$$

These two definitions are equivalent.

It turns out that very little is known about the lower semicontinuity properties of integrals given by Diff-quasiconvex potentials. It is straightforward that Diff-quasiconvexity is a necessary condition for weakly * $W^{1,\infty}$ (or uniform convergence of Lipschitz mappings) (see [9] proposition 2, same page). All that is known reduces to the properties of polyconvex maps. A polyconvex map $w : GL(n,R)^+ \rightarrow R$ is described by a convex function $g : D \subset R^M \rightarrow R$ (the domain of definition D is convex as well) and M rank one affine functions $\nu_1, ..., \nu_M : GL(n,R)^+ \rightarrow R$ such that for any $F \in GL(n,R)^+$

$$w(F) = g(\nu_1(F), ..., \nu_M(F))$$

The rank one affine functions are known(cf. Edelen [7], Ericksen [8], Ball, Curie, Olver [4]): ν is rank one affine if and only if $\nu(F)$ can be expressed as a linear combination of subdeterminants of F (uniformly with respect to F). Any rank one convex function is also called a null Lagrangian, because it generates a trivial Euler-Lagrange equation.

Polyconvex function give lower semicontinuous functionals, as a consequence of Jensen's inequality and continuity of (integrals of) null lagrangians. This is a very interesting path to follow (cf. Ball [3]) and it leads to many applications. But it leaves unsolved the problem: are the integrals given by Diff-quasiconvex potentials lower semicontinuous?

In the case of incompressible elasticity one has to work with the group of matrices with determinant one, i.e. $SL(n,R)$. The "linear" way of thinking has been compensated by wonders of analytical ingenuity. One purpose of this paper is to show how a slight modification of thinking, from linear to nonlinear, may give interesting results in the case $w : G \rightarrow R$ where G is a Lie subgroup of $GL(n,R)$. Note that when n is even a group which deserves attention is $Sp(n,R)$, the group of symplectic matrices.

From now on linear transformations of R^n and their matrices are identified. G is a Lie subgroup of $GL(n,R)$.

Definition 1.3. *For any $\Omega \subset R^n$ open, bounded, with smooth boundary, we introduce the set $[G]^\infty(\Omega)$ of all bi-Lipschitz mappings u from Ω to R^n such that for almost any $x \in \Omega$ we have $Du(x) \in G$.*

The set $Q \subset R^n$ is the unit cube $(0,1)^n$.

The departure point of the paper is the following natural definition.

Definition 1.4. *The continuous function $w : G \rightarrow R$ is G-quasiconvex if for any $F \in G$ and $u \in [G]^\infty(Q)$ we have:*

$$\int_Q w(F) \ dx \ \leq \ \int_Q w(FDu(x)) \ dx \tag{3}$$

We describe now the structure of the paper. After the formulation of the lower semicontinuity theorem 2.1, in section 3 is shown that quasiconvexity in the sense of definition 1.2 is the same as $GL(r,n)^+$ quasiconvexity. Theorem 2.1 is proved in section 4; in the next section is described the rank one convexity (or ellipticity) notion associated to G quasiconvexity. The cases $GL(n,R)$ and $SL(n,R)$ are examined in detail. It turns out that classification of all universal conservation laws in incompressible elasticity is based on some unproved assumptions. In section 6 is described a class of $GL(n,R)^+$ quasiconvex functions introduced in Buliga [5]. Theorem 2.1 is used to prove that any such function induces a lower semicontinuous integral.

2. G-quasiconvexity and the lower semicontinuity result

We denote by $[G]_c^\infty$ the class of all Lipschitz mapping from R^n to R^n such that $u - \ id$ has compact support and for almost any $x \in R^n$ we have $Du(x) \in G$. The main result of the paper is:

Theorem 2.1. *Let G be a Lie subgroup of $GL(n,R)$, Ω an open, bounded set with $\mid \partial\Omega \mid = 0$ and $w : G \rightarrow R$ locally Lipschitz.*

a) *Suppose that for any sequence $u_h \in [G]_c^\infty$ weakly $* \, W^{1,\infty}$ convergent to id we have:*

$$\int_\Omega w(F) \ dx \ \leq \ \liminf_{h\rightarrow\infty} \int_\Omega w(FDu_h(x)) \ dx \tag{4}$$

Then for any bi-Lipschitz $u \in [G]_c^\infty$ and for any sequence u_h weakly $* \, W^{1,\infty}$ convergent to u we have:

$$\int_\Omega w(Du(x)) \ dx \ \leq \ \liminf_{h\rightarrow\infty} \int_\Omega w(Du_h(x)) \ dx \tag{5}$$

Moreover, if (5) holds for any bi-Lipschitz $u \in [G]_c^\infty$ and for any sequence u_h weakly $* \, W^{1,\infty}$ convergent to u then w is G-quasiconvex.

b) *Suppose that G contains the group $CO(R^n)$ of conformal matrices. Then (5) holds for any bi-Lipschitz $u \in [G]_c^\infty$ and for any sequence u_h weakly $* \, W^{1,\infty}$ convergent to u if and only if w is G-quasiconvex.*

The fact that weakly * lower semicontinuity implies G quasiconvexity (end of point (a)) is easy to prove by rescaling arguments (cf. proposition 2, Giaquinta, Modica and Soucek *op. cit.*).

The method of proving the point (a) of the theorem is well known (see Meyers [13]). Even if there is nothing new there from the pure analytical viewpoint, I think that the proof deserves attention.

3. G-quasiconvexity

This section contains preliminary properties of G-quasiconvex continuous functions.

Proposition 3.1. a) *In the definition of G-quasiconvexity the cube Q can be replaced by any open bounded set Ω such that $\mid \partial\Omega \mid = 0$.*

b) *The function w is G-quasiconvex if and only if for any $F \in G$ and $u \in [G]_c^\infty(Q)$ we have:*

$$\int_Q w(F) \ dx \ \leq \ \int_Q w(Du(x)F) \ dx \tag{6}$$

The converse is true.

c) *For any $U \in GL_n$ such that $UGU^{-1} \subset G$ and for any $W : G \rightarrow R$ G-quasiconvex, the mapping $W_U : G \rightarrow R$, $W_U(F) = W(UFU^{-1})$ is G-quasiconvex.*

Remark 3.1. The point b) shows that the non-commutativity of the multiplication operation does not affect the definition of G-quasiconvexity. The point c) is a simple consequence of the fact that G is a group.

Proof. The point a) has a straightforward proof by translation and rescaling arguments.

For b) let us consider $F \in G$ and an arbitrary open bounded $\Omega \subset R^n$ with smooth boundary. The application which maps $\phi \in [G]_c^\infty(\Omega)$ to $F^{-1}\phi F \in [G]_c^\infty(F^{-1}(\Omega))$ is well defined and bijective. By a), if the function w is G-quasiconvex then we have

$$\int_{F^{-1}(\Omega)} w(FD(F^{-1}\phi F)(x)) \, \mathrm{d}x \geq \mid F^{-1}(\Omega) \mid w(F)$$

The change of variables $x = F^{-1}y$ resumes the proof of b).

With U like in the hypothesis of c), the application which maps $\phi \in [G]_c^\infty(\Omega)$ to $U\phi U^{-1} \in [G]_c^\infty(U^{-1}(\Omega))$ is well defined and bijective. The proof resumes as for the point b). □

The following proposition shows that quasi-convexity in the sense of definition 1.2 is a particular case of G-quasiconvexity.

Proposition 3.2. *Let us consider $F \in GL(n, R)^+$. Then w is $GL(n, R)^+$-quasiconvex in F if and only if it is quasi-convex in F in the sense of Ball.*

Proof. Let $E \subset R^n$ be an open bounded set and $\phi \in [GL(n, R)^+]_c^\infty(E)$. The vector field $\eta = F(\phi - id)$ verifies the condition that almost everywhere $F + D\eta(x)$ is invertible. Therefore, if w is quasi-convex in F, we derive from the inequality:

$$\int_E w(FD\phi(y)) \, \mathrm{d}y \ \geq \mid E \mid W(F) \quad .$$

We implicitly used the chain of equalities

$$F + D\eta(y) \ = \ F + FD\phi(y) \ - \ F \ = \ FD\phi(y) \quad .$$

We have proved that quasi-convexity implies $GL(n, R)^+$-quasiconvexity.

In order to prove the inverse implication let us consider η such that almost everywhere $F + D\eta(x)$ is invertible. We have therefore $\phi = F^{-1}\psi \in [GL(n, R)^+](E)$ and $FD\phi = F + D\eta$. We use now the hypothesis that w is $GL(n, R)^+$-quasiconvex in F and we find that w is also quasi-convex. □

4. Proof of Theorem 2.1

The proof is divided into three steps. In the first step we shall prove the following:

(Step 1.) *Let $w : GL(n, R) \rightarrow R$ be locally Lipschitz. Suppose that for any Lipschitz bounded sequence $u_h \in [GL(n, R)]_c^\infty$ uniformly convergent to id on $\overline{\Omega}$ and for any $F \in GL(n, R)$ we have:*

$$\int_\Omega w(F) \ dx \ \leq \ \liminf_{h \to \infty} \int_\Omega w(FDu_h(x)) \ dx \tag{7}$$

Then for any bi-Lipschitz $u : R^n \rightarrow R^n$ and for any sequence $u_h \in [GL(n, R)]_c^\infty$ uniformly convergent to id on $\overline{\Omega}$ we have:

$$\int_\Omega w(Du(x)) \ dx \ \leq \ \liminf_{h \to \infty} \int_\Omega w(D(u_h \circ u)(x)) \ dx \tag{8}$$

Remark 4.1. This is just the point a) of the main theorem for the whole group of linear invertible transformations.

Proof. For $\varepsilon > 0$ sufficiently small consider the set:

$$U^{\varepsilon} \ = \ \left\{ B = \overline{B}(x,r) \subset \Omega \ : \ \exists \ A \in GL(n,R) \ , \ \int_B \mid Du(x) - A \mid < \varepsilon \mid B \mid \right\}$$

From the Vitali covering theorem and from the fact that u is bi-Lipschitz we deduce that there is a sequence $Bj = \overline{B}(x_j, r_j) \in U^{\varepsilon}$ such that:

- $\mid \Omega \setminus \bigcup_j B_j \mid = 0$
- for any j u is approximatively differentiable in x_j and $Du(x_j) \in GL(n,R)$
- we have

$$\int_{B_j} \mid Du(x) - Du(x_j) \mid < \ \varepsilon \mid B_j \mid$$

Choose N such that

$$\mid \Omega \setminus \bigcup_{j=1}^{N} B_j \mid < \ \varepsilon$$

We have therefore:

$$\int_{\Omega} w(D(u_h \circ u)(x)) \ \geq \ \sum_{j=1}^{N} \int_{B_j} w(D(u_h \circ u)(x)) \ - \ C\varepsilon$$

$$\sum_{j=1}^{N} \int_{B_j} w(D(u_h \circ u)(x)) = J_1 + J_2 + J_3$$

where the quantities J_i are given below, with their estimates.

$$J_1 = \sum_{j=1}^{N} \int_{B_j} \left[w(Du_h(u(x))Du(x)) - w(Du_h(u(x))Du(x_j)) \right]$$

$$\mid J_1 \mid \leq \sum_{j=1}^{N} \int_{B_j} \mid w(Du_h(u(x))Du(x)) - w(Du_h(u(x))Du(x_j)) \mid < \ C\varepsilon$$

$$J_2 = \sum_{j=1}^{N} \int_{B_j} \left[w(Du_h(u(x))Du(x_j)) - w(Du_h(\overline{u}_j(x))Du(x_j)) \right]$$

where $\bar{u}_j(x) = u(x_j) + Du(x_j)(x - x_j)$. We have the estimate:

$$\mid J_2 \mid \leq C\varepsilon$$

Indeed, by changes of variables we can write:

$$I'_j \ = \ \int_{B_j} w(Du_h(u(x))Du(x_j)) \ = \ \int_{u(B_j)} w(Du_h(y)Du(x_j) \ \mid \det Du^{-1}(y) \mid$$

$$I_j" \ = \ \int_{B_j} w(Du_h(\bar{u}_j(x))Du(x_j)) \ = \ \int_{\bar{u}_j(B_j)} w(Du_h(y)Du(x_j) \ \mid \det(Du(x_j))^{-1} \mid$$

The difference $\mid I'_j - I_j" \mid$ is majorised like this

$$\mid I'_j - I_j" \mid \leq \int_{u(B_j) \cap \bar{u}_j(B_j)} C \mid\mid \det Du^{-1}(y) \mid - \mid \det(Du(x_j))^{-1} \mid\mid \ + C \mid u(B_j) \Delta \bar{u}_j(B_j) \mid$$

The function $\mid \det \cdot \mid$ is rank one convex and satisfies the growth condition $\mid \det F \mid \leq c(1+ \mid F \mid^n)$ for any $F \in GL(n,R)$. Therefore this function satisfies also the inequality:

$$\mid\mid \det F \mid - \mid \det P \mid\mid \leq \ C \mid F - P \mid \left(1 + \mid F \mid^{n-1} + \mid P \mid^{n-1}\right)$$

Use now this inequality, the properties of the chosen Vitali covering and the uniform bound on Lipschitz norm of u, u_h, to get the claimed estimate.

$$J_3 = \sum_{j=1}^{N} \int_{B_j} w(Du_h(\overline{u}_j(x))Du(x_j))$$

By the change of variable $y = \overline{u}_j(x)$ and the hypothesis we have

$$\liminf_{h\to\infty} J_3 \geq \liminf_{h\to\infty} \sum_{j=1}^{N} \int_{B_j} w(Du(x_j))$$

Put all the estimates together and pass to the limit with $N \to \infty$ and then $\varepsilon \to 0$. □

(Step 2.) If we replace in **Step 1.** *the group $GL(n,R)$ by a Lie subgroup G the conclusion is still true.*

Proof. Indeed, remark that in the proof of the previous step it is used only the fact that $GL(n,R)$ is a group of invertible maps. □

Step 3. *The point b) of the Theorem 2.1 is true.*

Remark 4.2. In the classical setting of quasiconvexity, this step is proven by an argument involving Lipschitz extensions with controlled Lipschitz norm. In our case the corresponding Lipschitz extension assertion would be: *let $u \in [G]^\infty_c$ with Lipschitz norm $\| u - id \| = \varepsilon$. For $\delta > 0$ sufficiently big there exists $v \in [G](B(0,1+\delta))$ such that $v = u$ on $B(0,1)$ and $\| v - id \|$ controlled from above by ε.* This is not known to be true, even for $G = GL(n,R)$. That is why we shall use a different approach.

Proof. Because G is a group, it is sufficient to make the proof for $F = 1$.

Let $u_h \in [G]^\infty_c$ be a sequence weakly * convergent to id on Ω and $D \subset\subset \Omega$. For $\varepsilon > 0$ sufficiently small and $C > 1$ we have

$$D_{C\varepsilon} \ = \ \bigcup_{x\in D} B(x, C\varepsilon) \ \subset \ \Omega$$

It is not restrictive to suppose that

$$\lim_{h\to\infty} \int_{\Omega} w(Du_h) \, \mathrm{d}x$$

exists and it is finite. For any $\varepsilon > 0$ there is N_ε such that for any $h > N_\varepsilon$ $u_h(D) \subset D_\varepsilon$.

Take a minimal Lipschitz extension

$$\overline{u}_h : D_{C\varepsilon} \setminus C \to R^n \ , \ \overline{u}_h(x) = \begin{cases} u_h(x) & , \ x \in \partial D \\ x & , \ x \in \partial D_{C\varepsilon} \end{cases}$$

The Lipschitz norm of this extension, denoted by k_h, is smaller than some constant independent on h.

Now, for any h define:

$$\psi_h \ = \ \frac{1}{2k_h} \, \overline{u}_{h_{|D_{C\varepsilon}\setminus D}}$$

According to Dacorogna-Marcellini Theorem 7.28, Chapter 7.4. [6], there is a solution σ_h of the problem

$$\begin{cases} D\sigma_h \in O(n) & \text{a. e. in } D_{C\varepsilon} \setminus D \\ \sigma_h \ = \ \psi_h & \text{on } \partial(D_\varepsilon \setminus D \end{cases}$$

Let

$$v_h(x) \ = \ \begin{cases} u_h(x) & x \in D \\ k_h \sigma_h(x) & x \in \Omega \setminus D \end{cases}$$

Note that $Dv_h \in CO(n)$.

The following estimate is then true:

$$\mid \int_D w(Du_h) \, \mathrm{d}x \; - \; \int_\Omega w(Dv_h) \, \mathrm{d}x \mid = \mid \int_{D_{C\varepsilon} \setminus D} w(Dv_h) \, \mathrm{d}x \mid \leq$$

$$\leq \; \int_{D_{C\varepsilon} \setminus D} \mid w(Dv_h) \mid \; \mathrm{d}x \; \leq \; C \mid D_\varepsilon \setminus D \mid$$

w is G-quasiconvex, therefore:

$$\int_{D_\varepsilon} w(Dv_h) \, \mathrm{d}x \; \geq \mid D_\varepsilon \mid w(1)$$

We put all together and we get the inequality:

$$\lim_{h \to \infty} \int_D w(Du_h) \, \mathrm{d}x \; \geq \mid D_\varepsilon \mid w(1) - C \mid D_\varepsilon \setminus D \mid$$

The proof finishes after we pass ε to 0. □

5. Rank one convexity

The rank-one convexity notion associated to G quasi-convexity is described in the next proposition, for $w \in C^2(G, R)$. Before this, let us introduce a differential operator naturally connected to the group structure of G. Denote by $\mathcal{G}$ the Lie algebra of G. For any pair $(F, H) \in G \times \mathcal{G}$, the derivative of $w : G \to R$ in F with respect to H is

$$Dw(F)H \; = \; \frac{d}{dt}_{|t=0} w(F \exp(tH))$$

We shall also use the notation (for $F \in G$ and $H, P \in \mathcal{G}$):

$$D^2 w(F)(H, P) \; = \; D(Dw(\cdot)H)(F)P$$

Proposition 5.1. *A necessary condition for $w \in C^2(G, R)$ to be G quasi-convex is*

$$\int_\Omega D^2 w(F)(D\eta(x), D\eta(x)) \;\; = 0$$

for any $F \in G$ and $\eta \in C^2(\Omega, R^n)$, $D\eta(x) \in \mathcal{G}$ a.e. in Ω, supp $\eta \in \Omega$.

Proof. Given such an η, consider the solution of the o.d.e. problem:

$$\dot{\phi}_t \; = \eta \circ \phi_t \; , \; \phi_0 = \; id_{|\Omega}$$

This is an one-parameter group in the diffeomorphism class $[G]^\infty(\Omega)$. Define then:

$$f(t) \; = \; \int_\Omega w(F D\phi_t(x))$$

The G quasiconvexity of w implies that f has a minimum in $t = 0$. That means $f'(0) = 0$ and $f''(0) \geq 0$. The first condition is trivially satisfied and the second is, by straightforward computation, just the conclusion of the proposition. □

We shall call G rank one convex a function which satisfies the conclusion of the proposition 5.1.

Consider the vector space

$$V(\mathcal{G}) = \{(H, H) \in \mathcal{G} \times \mathcal{G} \; : \; H \in \mathcal{G}\}$$

and the set

$$RO(\mathcal{G}) = \{(a, b) \in R^n \times R^n \; : \; a \otimes b \in \mathcal{G}\}$$

Proposition 5.2. *Suppose that $w : G \rightarrow R$ is a C^2 function. If for any $a, b \in RO(\mathcal{G})$*

$$D^2 w(F)(a \otimes b, a \otimes b) \geq 0 \tag{9}$$

then w is G rank one convex.

Proof. We shall use the notations from the proof of the preceding proposition. We see that

$$\int_\Omega (D\eta(x), D\eta(x)) \ \in V(\mathcal{G})$$

Therefore there is an $X \in \mathcal{G}$ such that

$$(X, X) = \int_\Omega (D\eta(x), D\eta(x))$$

Using integration by parts we find that for any indices $i, j, k, l \in 1, ..., n$ we have:

$$X_{ij} X_{kl} = X_{il} X_{kj}$$

which implies that X has rank one. Hence there are $a, b \in R^n$ such that $X = a \otimes b$. Use the definition of G rank one convexity to prove that (9) implies the G rank one convexity. □

In the case $G = GL(n, R)$ we find that $GL(n, R)$ rank one convexity is equivalent to classical rank one convexity. To see this, take arbitrary $F \in GL(R^n)$, $a, b \in R^n$, $s > 0$ and $u \in C_c^\infty(\Omega, R)$. Define

$$\eta^s(x) = u(x) \sin\left[s(b \cdot x)\right] \ a$$

Because $GL(n, R)$ is an open set in the vectorspace of $n \times n$ real matrices, the $GL(n, R)$ rank one condition reads:

$$s^2 \ \frac{d^2 \ w}{dF_{ij} dF_{kl}}(F)(Fa)_i b_j (Fa)_k b_l \int_\Omega u^2 \ + B \ \geq 0$$

with B independent on s. We deduce that

$$\frac{d^2 \ w}{dF_{ij} dF_{kl}}(F)(Fa)_i b_j (Fa)_k b_l \geq 0$$

for any choice of F, a, b. This is the same as:

$$\frac{d^2 \ w}{dF_{ij} dF_{kl}}(F) a_i b_j (a_k b_l \geq 0$$

for any F, a, b.

For the group $SL(n, R)$ of matrices with determinant one we obtain a similar condition by imposing the constraint $div \ \eta^s = 0$. This can be done if $a \cdot b = 0$ and $Du(x) \cdot a = 0$. For simplicity suppose that w is defined in a neighbourhood of $SL(n, R)$. Then w is $SL(n, R)$ rank one convex implies

$$\frac{d^2 \ w}{dF_{ij} dF_{kl}}(F)(Fa)_i b_j (Fa)_k b_l \geq 0 \tag{10}$$

for any $F \in SL(n, R)$, $a, b \in R^n$, $a \cdot b = 0$.

5.1. **Rank one affine functions.** A map w is G rank one affine if w and $-w$ are G rank one convex. For the case $G = GL(n)$ we see that the rank one affines are known. This is very useful in several instances. The reason is that the Euler-Lagrange equation associated to the potential w does not change if one adds a rank affine function to w. At the action functional level

$$I_w(\phi) \ = \ \int_\Omega w(D\phi(x))$$

the addition of a $GL(n,R)$ rank one function means the addition of a closed form which cancels with the integral. This coincidence led to the development of formal calculus of variations in the frame of the jet bundle formalism, which permits to classify all universal conservation laws in elasticity. For this classification see Olver [15].

The case $G = SL(n,R)$ is equally important, because it is about incompressible elasticity. Or, in this case nothing is known, because it is not proven that the $SL(n,R)$ rank one affine functions correspond to closed forms. For this reason Olver's classification [15] of universal conservation laws is not proven to be complete.

We arrived to the following

Open problem: Describe all G rank one affine functions.

In particular situations the problem has been solved. For example if $G = GL(n,R)$ then any rank one affine function is a classical null lagrangian. In the case $SL(2,R)$ we have the following theorem:

Theorem 5.1. *Any $SL(2,R)$ rank one affine function is affine.*

Proof. We have to prove that if $w : SL(2,R) \to R$ is rank one affine then $w(F) = a_{ij}F_{ij} + b$. It is sufficient to prove the thesis for any F in an open dense set in $SL(2,R)$. We shall use the following maps:

$$(X,Y,Z) \in R^* \times R \times R \ \mapsto \ F \ = \ \left(\begin{array}{cc} X & Y \\ Z & \frac{1+YZ}{X} \end{array} \right)$$
$$(X',Y',Z') \in R^* \times R \times R \ \mapsto \ F \ = \ \left(\begin{array}{cc} \frac{1+Y'Z'}{X'} & Y' \\ Z' & X' \end{array} \right)$$

Take arbitrary $a = (a_1, a_2)$ and perpendicular $b = (-a_2, a_2)$. If w is SL_2 rank one affine then the mapping

$$t \mapsto f(t; a \otimes b, F) = w(F(1 + ta \otimes b))$$

is linear for any $F \in SL(2,R)$. We have used here the relation and the equality $\exp a \otimes b = 1 + a \otimes b$, for any orthogonal a, b. Rank one convexity of w means that the second derivative of $f(t; a \otimes b, F)$ with respect to t vanishes for any choice of F and a.

We express F in terms of the coordinates $F = F(X,Y,Z)$ and $F = F(X',Y',Z')$. After some elementary computation we obtain the following minimal system of equations for the function $w(X,Y,Z) = w(F(X,Y,Z))$:

$$(11) \qquad \left\{ \begin{array}{lcl} w_{XX}X^2 & = & 2w_{YZ}(1+YZ) \\ w_{ZZ}X & = & -w_{YZ}Y \\ w_{XY}X & = & -w_{YZ}Z \\ w_{YY} & = & 0 \\ w_{ZZ} & = & 0 \end{array} \right.$$

From equations (11.4) and (11.5) we find that w has the form

$$w(X,Y,Z) = A(X)YZ + B(X)Y + C(X)Z + D(X)$$

From (11.2) we obtain the equation

$$XC'(X) + XYA'(X) = -A(X)Y$$

From here we derive that $C(X) = c$ and $A(X) = k/X$. We update the form of w, use (11.3) to get $B(X) = b$ and (11.1) to get $D(X) = (k/X) + eX + f$. We collect all the information and we obtain that w has the expression:

$$w(X, Y, Z) = k\frac{1 + YZ}{X} + bY + cZ + eX + f$$

which proves the theorem. $\square$

Therefore, in the case $G = SL(2, R)$ we have proved that there are no rank one affine functions other than the classical ones. The proof is not adapted to generalizations. The case $G = SL(3, R)$ is open.

Other groups are equally significant, like the group $Sp(n, R)$ of symplectomorphisms. I don't know of any attempt to solve this problem.

5.2. **Rank one convexity and quasiconvexity.** The $GL(n, R)$ rank one convexity is not equivalent to $GL(n, R)$ quasiconvexity in any dimension.

Proposition 5.3. *The function $w : GL(n, R) \rightarrow R$ defined by*

$$w(F) \ = \ - \log \mid \det F \mid$$

is $GL(n, R)$ rank one convex but not $GL(n, R)$ quasiconvex.

Proof. The map is polyconvex hence it is rank one convex. It is not quasi-convex though. To see this fix $\varepsilon \in (0, 1)$, $A \in GL(n, R)$ and $\Omega = B(0, 1)$. There is a Lipschitz solution to the problem

$$\left\{ \begin{array}{ll} Dv(x) \in O(n) & \text{a.e. in } \Omega \\ v(x) = \varepsilon x & x \in \partial\Omega \end{array} \right.$$

We have then, for $u(x) = v(x)/\varepsilon \ \in [GL(n, R)]^{\infty}(\Omega)$:

$$\int_{\Omega} w(ADu(x)) \ = \ \int_{\Omega} - \log \mid \det A \mid \ + \ \int_{\Omega} n \log \varepsilon \ < \ \int_{\Omega} w(A)$$

$\square$

Next proposition justifies this result.

Proposition 5.4. *For any $w : G \rightarrow R$ define $\imath w : G \rightarrow R$ by:*

$$\imath w(F) = \mid \det F \mid \ w(F^{-1})$$

Then w is G rank one convex if and only if $\imath w$ is. Also, if w is G quasi-convex then for any $u \in [G]^{\infty}(\Omega)$ we have:

$$\int_{\Omega} w(FDu(x)) \ \geq \int_{\Omega} w(F)$$

Proof. Take u like in the hypothesis. Then for any (continuous) w we have

$$\int_{\Omega} w(Du^{-1}(x)) \ = \ \int_{\Omega} \imath w(Du(x))$$

by straightforward computation. Use now the proof of proposition 5.1 to deduce the first part of the conclusion. For the second part use the definition 1.4 and the proposition 3. $\square$

Let us apply this proposition to $w(F) = -\log \mid \det F \mid$. Remark that when $\det F$ goes to zero the function goes to $+\infty$. Now, $\imath w(F) \ = \mid \det F \mid \log \mid \det F \mid$ and this function can be continuously prolongated to matrices with determinant zero by setting $\imath w(F) = 0$ if $\det F = 0$. It is easy to see that the prolongation of $\imath w$ ceases to be rank one convex.

6. Application: a class of quasiconvex functions

The goal of this section is to give a class of quasi-convex isotropic functions which seem to be complementary to the polyconvex isotropic ones. We quote the following result of Thompson and Freede [16], Ball [2] (for a proof coherent with this paper see Le Dret [11]).

Theorem 6.1. *Let $g : [0,\infty)^n \rightarrow R$ be convex, symmetric and nondecreasing in each variable. Define the function w by*

$$w : gl(n,R) \rightarrow R \ , \quad w(F) = g(\sigma(F)).$$

Then w is convex.

We shall use the Theorem 6.2. Buliga [5]. We need a notation first. Let $x = (x_1, ..., x_n) \in R^n$ be a vector. Then the vector $x^{\downarrow} = (x_1^{\downarrow}, ..., x_n^{\downarrow}) \in R^n$ is obtained by rearanging in decreasing order the components of x. Remark that for any symmetric function $h : R^n \rightarrow R$ there exists and it is unique the function $p : R^n \rightarrow R$ defined by the relation:

$$p(\sum_{i=1}^{k} x_i^{\downarrow}) \ = \ h(x_k)$$

Theorem 6.2. *Let $g : (0,\infty)^n \rightarrow R$ be a continuous symmetric function and $h : R^n \rightarrow R$, $h(x_1, ..., x_n) = g(\exp x_1, ..., \exp x_n)$. Suppose that*

(a) *h is convex,*
(b) *The function p associated to h is nonincreasing in each argument.*

Let $\Omega \subset R^n$ be bounded, with piecewise smooth boundary and $\phi : \overline{\Omega} \rightarrow R$ be any Lipschitz function such that $D\phi(x) \in GL(n,R)^+$ a.e. and $\phi(x) = x$ on $\partial\Omega$. Define the function

$$w : GL(n,R)^+ \rightarrow R \ , \quad w(F) = g(\sigma(F))$$

Then for any $F \in GL(n,R)^+$ we have:

$$\int_\Omega w(FD\phi(x)) \ \geq \mid \Omega \mid w(F) \qquad (12)$$

A consequence of theorem 6.2 and Theorem 2.1 (a) is:

Proposition 6.1. *In the hypothesis of Theorem 6.2, let $\phi_h : \Omega \rightarrow R^n$ be a sequence of Lipschitz bounded functions such that*

(a) *for any h $D\phi_h(x) \in GL(n,R)^+$ a.e. in Ω.*
(b) *the sequence ϕ_h converges uniformly to $u : \Omega \rightarrow \Omega$, bi-Lipschitz function.*

Then

$$\liminf_{h\rightarrow\infty} \int_\Omega w(D\phi_h(x)) \ \geq \ \int_\Omega w(Du(x)) \qquad (13)$$

Proof. It is clear that theorem 6.2 implies the hypothesis of point (a), theorem 2.1. Indeed, the conclusion of theorem 6.2 can be written like this: for any $u \in [GL(n,R)^+](\Omega)$ such that

$$\bar{Du}(\Omega) = \frac{1}{\mid \Omega \mid} \int_\Omega Du(x) \ \mathrm{d}x \ \in GL(n,R)^+$$

we have the inequality

$$\int_\Omega w(Du(x)) \ \mathrm{d}x \geq \ \int_\Omega w(\bar{Du}(\Omega)) \ \mathrm{d}x$$

Take a sequence of mapping $(u_h) \subset [GL(n,R)^+](\Omega)$ uniformly convergent to $F \in GL(n,R)^+$. The previous inequality and the continuity of w imply:

$$\int_\Omega w(F) \, \mathrm{d}x \ \leq \ \int_\Omega w(Du_h(x)) \, \mathrm{d}x$$

Apply now theorem 2.1 (a) and obtain the thesis. □

The class of functions w described in theorem 6.2 and the class of polyconvex functions seem to be different. However, by picking h linear, we obtain a polyconvex function, like

$$w(F) = -\log \mid \det F \mid$$

We have seen in proposition 5.3 that this function is not $GL(n,R)$ quasiconvex but proposition 6.1 tells that w is $GL(n,R)^+$ quasiconvex.

We close with an example of another function which we can prove that it is $GL(n,R)^+$ quasiconvex. We use the notation $F = R_F U_F$ for the polar decomposition of $F \in GL(n,R)^+$, with U_F symmetric and positive definite. The example is the function:

$$w : GL(n,R)^+ \rightarrow R \ , \quad w(F) = \det F \ \log \left(trace \ U_F \right)$$

With the notation introduced in proposition 5.4, let's look to the the function $\hat{w} = \imath w$. It has the expression:

$$\hat{w} : GL(n,R)^+ \rightarrow R \ , \quad \hat{w}(F) \ = \ \log \left(\ trace \ U_F^{-1} \right)$$

It is a matter of straightforward computation to check that $\hat{w}$ verifies the hypothesis of theorem 6.2. It is therefore $GL(n,R)^+$ quasiconvex. By proposition 5.4 w is $GL(n,R)^+$ quasiconvex, too, hence lower semicontinuous in the sense of theorem 2.1 (a).

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Four applications of majorization to convexity in the calculus of variations

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Abstract

The resemblance between the Horn-Thompson theorem and a recent theorem by Dacorogna-Marcellini-Tanteri indicates that Schur convexity and the majorization relation are relevant for applications in the calculus of variations and its related notions of convexity, such as rank-one convexity or quasiconvexity.

We give in theorem 6.6 simple necessary and sufficient conditions for an isotropic objective function to be rank one convex on the set of matrices with positive determinant.

Majorization is used in order to give a very short proof of a theorem of Thompson and Freede [19], Ball [3], or Le Dret [13], concerning the convexity of a class of isotropic functions which appear in nonlinear elasticity.

Next we prove (theorem 7.3) a lower semicontinuity result for functionals with the form $\int_{\Omega} w(D\phi(x))\, \mathrm{d}x$, with $w(F) = h(\ln V_F)$. Here $F = R_F U_F = V_F R_F$ is the usual polar decomposition of $F \in gl(n, \mathbb{R})$, and $\ln V_F$ is Hencky's logarithmic strain.

We close this paper with a compact proof of Dacorogna-Marcellini-Tanteri theorem, based only on classical results about majorization. The mentioned resemblance of this theorem with the Horn-Thompson theorem is thus explained.

Keywords: convexity, majorization, Schur-convexity, quasiconvexity, Hencky's logarithmic strain

MSC classes: 74B20, 35Q72

1 Introduction

There is a strong resemblance between the following two theorems. The first theorem is (Horn, [11](1954), Thompson [18](1971), theorem 1.):

Theorem 1.1 *Let X, Y be any two positive definite $n \times n$ matrices and let $x_1 \geq x_2 \geq ... \geq x_n$ and $y_1 \geq y_2 \geq ... \geq y_n$ denote the respective sets of eigenvalues. Then there is an unitary matrix U such that XU and Y have the same spectrum if and only if:*

$$\prod_{i=1}^{k} x_i \geq \prod_{i=1}^{k} y_i \quad , \ k = 1, ..., n-1$$

$$\prod_{i=1}^{n} x_i = \prod_{i=1}^{n} y_i$$

The second theorem is Dacorogna-Marcellini-Tanteri [8](2000), Theorem 20, see also Dacorogna, Marcellini [7]. Rank one convexity and polyconvexity are fundamental notions in the calculus of variations, briefly explained in section 5.

Theorem 1.2 *Let $0 \leq \sigma_1(A) \leq ... \leq \sigma_n(A)$ denote the singular values of a matrix $A \in \mathbb{R}^{n \times n}$. For any string of given numbers $0 \leq a_1 \leq ... \leq a_n$ we define the set of $n \times n$ matrices:*

$$E(a) = \left\{ A \in \mathbb{R}^{n \times n} \ : \ \sigma_i(A) = a_i \ , \ i = 1, ..., n \ , \ \det A = \prod_{i=1}^{n} a_i \right\}$$

The following then holds

$$Pco\ E = \ Rco\ E(a) = \left\{ A \in \mathbb{R}^{n \times n} \ : \ \prod_{i=\nu}^{n} \sigma_i(A) \leq \prod_{i=\nu}^{n} a_i \ , \ \nu = 2, ..., n \ , \right.$$

$$\left. \det A = \prod_{i=1}^{n} a_i \right\}$$

where PCo, Rco stand for polyconvex, rank one convex envelope.

Both theorems can be understood as describing the set $\{y \ : \ y \prec\prec x\}$ where $\prec\prec$ is a preorder relation defined with the help of inequalities between products appearing in the formulations of the theorems.

It turns out that a common framework of these apparently unrelated results is the notion of majorization. This notion is familiar to mathematical fields like stochastic analysis, linear algebra, Lie groups theory. In this paper a first attempt is made to apply results connected to majorization to elasticity and the calculus of variations. We shall obtain simpler proofs of known results and new results as well.

It is significant to notice that most of the majorization results used in this paper are earlier or contemporary with the fundamental paper of Morrey (1952) [15] on quasiconvexity. However, it seems that there was not much interaction between these fields until now.

The content of the paper is described further. After the setting of notations in section 2, section 3 gives a brief passage trough basic properties of the majorization relation. Section 4 lists some properties of singular values and eigenvalues of matrices connected to majorization. In section 5 rank one convexity, quasiconvexity and polyconvexity are introduced as fundamental notions in the calculus of variations.

The paper continues with four applications of the classical results mentioned in sections 2–5.

The first application is in the field of nonlinear hyperelastic materials. Theorem 6.6 gives simple necessary and sufficient conditions for an isotropic objective function to be rank one convex on the set of matrices with positive determinant. The subject has a long history: the oldest citation used in this paper is Baker, Ericksen (1954) [1].

As a second application, we use majorization in order to give a very short proof of a theorem of Thompson and Freede [19], Ball [3], or Le Dret [13] (in this paper theorem 7.2).

We prove next (theorem 7.3) a lower semicontinuity result for functionals with the form $\int_\Omega w(D\phi(x))\ \mathrm{d}x$, with $w(F) = h(\ln V_F)$. Here $F = R_F U_F = V_F R_F$ is the usual polar decomposition of $F \in gl(n, \mathbb{R})$ and $\ln V_F$ is Hencky's logarithmic strain.

We close the paper with a proof of Dacorogna-Marcellini-Tanteri theorem, based only on classical results about majorization. This explains the resemblance between theorems 1.1 and 1.2. Related results can be found in [16] where Silhavy expresses Baker-Ericksen inequalities using multiplication instead of division, too.

Acknowledgements. I have learned about Horn-Thompson theorem with its amazing implications in Lie group convexity results from discussions with Tudor Ratiu. I want to thank Bernard Dacorogna for keeping me connected with parts of his research. Thanks are due to the anonymous referee for various suggestions leading to an improvement of the paper.

2 Notations

- A,B, ... real or complex matrices
- x,y,u,v, ... real or complex vectors
- $gl(n, K)$ space of all $n \times n$ real ($K = \mathbb{R}$) or complex ($K = \mathbb{C}$) matrices

- $GL(n, K)$ space of all $n \times n$ invertible real or complex matrices
- $GL(n, \mathbb{R})^+$ the group of all $n \times n$ invertible real matrices with strictly positive determinant
- $Sym(n, \mathbb{R})$ the space of all $n \times n$ invertible, real, symmetric matrices
- $SO(n)$ the group of all $n \times n$ real orthogonal matrices with positive determinant
- $\lambda(A)$ the vector of eigenvalues of A
- $\sigma(A)$ the vector of singular values of A
- A^* the conjugate transpose of A
- A^T the transpose of A
- $diag(A)$ the diagonal of A, seen as a vector
- $Diag(v)$ the diagonal matrix constructed from the vector v
- S_n the group of permutation of coordinates in $\mathbb{R}^n$
- $Conv(A)$ the convex hull of the set A
- $\circ$ function composition
- $f_{,i}$ partial derivative of the function f with respect to the coordinate x_i
- $f_{,ij}$ the second-order partial derivative of the function f with respect to the coordinates x_i, x_j

For any matrix $A \in gl(n, \mathbb{C})$, the matrix A^*A is Hermitian. The eigenvalues of the square root of A^*A are, by definition, the singular values of A. If the matrix A is Hermitian or real symmetric and positive definite then we denote by $\ln A$ the logarithm of A.

Matrices are identified with linear transformations.

For a vector $x \in \mathbb{R}^n$ we denote by $x^\downarrow$, $x^\uparrow$, the vectors obtained by rearranging the coordinates of x in decreasing, respectively increasing orders.

Let $f : A \subset \mathbb{R} \to \mathbb{R}$ be an arbitrary function and n a strictly positive integer. We shall use the notation $f : A^n \to \mathbb{R}^n$ for the function

$$x = (x_1, ..., x_n) \in A^n \ \mapsto \ f(x) \ = \ (f(x_1), ..., f(x_n)) \in \mathbb{R}^n$$

For example the logarithm function $f : (0, +\infty) \to \mathbb{R}$, $f(x) = \ln x$, has associated the function denoted by the same symbol $\ln : (0, +\infty)^n \to \mathbb{R}^n$ the function $\ln(x_1, ..., x_n) \ = \ (\ln x_1, ..., \ln x_n)$.

For any symmetric, positive definite, real matrix A let us denote by $\ln A$ the logarithm of A. Then , with the notations made before, we have $\lambda(\ln A) = \ln(\lambda(A))$.

Finally, $B(x,r)$ denotes the ball in $\mathbb{R}^n$, of radius $r > 0$ and center $x \in \mathbb{R}^n$. If Ω is an open, bounded set in $\mathbb{R}^n$ then $\mid \Omega \mid$ denotes its Lebesgue measure.

3 Basics about majorization

We have used Bhatia [4], Chapter 2, and Marshall and Olkin [14], Chapters 1-3. The results are given in the logical order.

Definition 3.1 *The following majorization notions are partial order relations in $\mathbb{R}^n$. Let $x, y \in \mathbb{R}^n$ be arbitrary vectors. Then:*

- $x \leq y$ *if* $x_i \leq y_i$ *for any* $i \in \{1, ..., n\}$.
- $x \prec_w y$ *if* $\sum_{j=1}^{k} {x_j}^{\downarrow} \leq \sum_{j=1}^{k} {y_j}^{\downarrow}$ *for any* $k \in \{1, ..., n\}$. *We say that x is weakly majorized by y.*
- $x \prec y$ *if* $x \prec_w y$ *and* $\sum_{j=1}^{n} {x_j}^{\downarrow} = \sum_{j=1}^{n} {y_j}^{\downarrow}$. *We say that x is majorised by y.*

The notion of majorization, the last in definition 3.1, is the most interesting. See Marshall and Olkin [14], Chapter 1, for the various places when one can encounter it.

Theorem 3.2 *(Hardy, Littlewood, Polya) The following statements are equivalent:*

(i) $x \prec y$

(ii) *x is in the convex hull of $S_n y$, where $S_n y$ is the set of all permutations of y,*

(iii) *for any convex function ϕ from $\mathbb{R}$ to $\mathbb{R}$ we have* $\sum_{i=1}^{n} \phi(x_i) \leq \sum_{i=1}^{n} \phi(y_i)$.

In the following definition we collect various notions useful in the following (notably monotonicity notions related to order relations).

Definition 3.3 *Consider a map Φ defined from an S_n invariant set in $\mathbb{R}^n$, with range in $\mathbb{R}^m$. We say that Φ is:*

- *symmetric if for any $P \in S_n$ there is $P' \in S_m$ such that $\Phi \circ P = P' \circ \Phi$,*

- *increasing if* $x \leq y \implies \Phi(x) \leq \Phi(y)$,
- *convex if for all* $t \in [0,1]$ $\Phi(tx + (1-t)y) \leq t\Phi(x) + (1-t)\Phi(y)$,
- *isotone if* $x \prec y \implies \Phi(x) \prec_w \Phi(y)$,
- *strongly isotone if* $x \prec_w y \implies \Phi(x) \prec_w \Phi(y)$,
- *strictly isotone if* $x \prec y \implies \Phi(x) \prec \Phi(y)$.

Any isotone Φ *with range in* $\mathbb{R}$ *is called Schur-convex. Note that convexity in the sense of this definition matches with the classical notion for functions* Φ *with range in* $\mathbb{R}$.

In particular a function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ symmetric if for any permutation matrix $P \in S_n$ we have $P(A) \subset A$ and $f \circ P = f$.

The next theorem shows that symmetric convex maps are isotone.

Theorem 3.4 *Let* $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ *be convex. If* Φ *is symmetric then it is isotone. If in addition* Φ *is monotone increasing then* Φ *is strictly isotone.*

In particular any L^p norm on $\mathbb{R}^n$ is Schur-convex. Not all isotone functions are convex, though. Important examples are the elementary symmetric polynomials, which are not convex but they are *Schur-concave.*

One can give three characterizations of isotone (or Schur convex) functions $f : \mathbb{R}^n \to \mathbb{R}$. Before that we need some notations.

Let us begin by noticing that the permutation group S_n acts on $GL(n, \mathbb{R})^+$ as follows: for any $P \in S_n$ and any $F \in GL(n, \mathbb{R})^+$ the matrix $P.F \in GL(n, \mathbb{R})^+$ has components $(P.F)_{ij} = F_{P(i)P(j)}$.

Let

$$\mathcal{D} = \{x \in \mathbb{R}^n \ : \ x_1 \geq x_2 \geq ... \geq x_n\} \quad ,$$

$$\mathcal{D}" = \{x \in \mathbb{R}^n \ : \ x_1 \geq x_2 - x_1 \geq ... \geq x_n - x_{n-1}\} \quad .$$

Let h: $\mathbb{R}^n \to \mathbb{R}$ be a symmetric function. Then there is a unique function $p : \mathcal{D}" \to \mathbb{R}$ such that for all $x \in \mathbb{R}^n$ we have

$$p\left((\sum_{i=1}^{k} y_i^{\downarrow})_{k=1,\ldots,n} \right) \ = \ h(y) \quad .$$

Indeed, for given h the function p is defined by

$$p(y_1, ..., y_n) \ = \ h(y_1, y_2 - y_1, ..., y_n - y_{n-1}) quad.$$

The Schur convexity of h is connected to the monotonicity of p. From the definitions we see that h is Schur convex if and only if p is increasing in the first $n-1$ arguments. This leads to the following theorem.

Theorem 3.5 *Let $A \subset \mathbb{R}^n$ be symmetric and let $f : A \to \mathbb{R}$. Then f is Schur convex if and only if f is symmetric and*

$$x_1 \mapsto f(x_1, s - x_1, x_3, ..., x_n)$$

is increasing in $x_1 \geq s/2$, for any fixed $s, x_2, ..., x_n$.

If in addition $A = I^n$ where I is an open interval of $\mathbb{R}$ and f is continuously differentiable on A, then f is Schur convex if and only if one of the following assertions is true:

(a) (Schur) f is symmetric and for any i and for all $x \in \mathcal{D} \cap I^n$ the function

$$t \mapsto f_{,i}(x_1, ..., x_i + t, ..., x_n)$$

is decreasing.

(b) (Schur) f is symmetric and for all $i \neq j$

$$(x_i - x_j)(f_{,i}(x) - f_{,j}(x)) \geq 0 \quad .$$

For weak majorization and strongly isotone functions we have the following theorem:

Theorem 3.6 *Let I be an open interval in $\mathbb{R}$ and let $f : I^n \to \mathbb{R}$.*

(a) (Ostrowski)Let f be continuously differentiable. Then f is strongly isotone if and only if f is symmetric and for all $x \in \mathcal{D} \cap I^n$ we have $Df(x) \in \mathcal{D} \cap \mathbb{R}^n_+$, that is:

$$f_{,1}(x) \geq f_{,2}(x) \geq ... \geq f_{,n}(x) \geq 0 \quad .$$

(b) Without differentiability assumptions, f is strongly isotone if and only if f is increasing and Schur convex.

4 Order relations for matrices

The results from this section have deep connections with Lie group theory. We shall give here only a minimal presentation, for matrix groups.

The main references are again Bhatia [4], Chapter 2, and Marshall and Olkin [14], Chapter 3; also Thompson [18]. The paper Kostant [12] gives an image of what is really happening from the Lie group point of view.

Definition 4.1 *We denote by $\mathcal{P}(n)$ the cone of Hermitian, positive definite matrices. In the class of Hermitian matrices we define the preorder relation $A \geq B$ by $A - B \in \mathcal{P}(n)$.*

The order relation $\leq$ between Hermitian matrices reflects into the order relation between the eigenvalues as it is shown in the next theorem, belonging to Weyl (theorem F1, chapter 16, Marshall and Olkin [14]).

Theorem 4.2 *(Weyl) If A, B are Hermitian matrices such that $A \leq B$ then*

$$\lambda^{\downarrow}(A) \leq \lambda^{\downarrow}(B) \quad .$$

In [18] Thompson introduces the following preorder relation on $GL(n, \mathbb{R})^{+}$:

$$X \prec Y \quad \text{if } \ln \sigma(X) \prec \ln \sigma(y) \quad .$$

With the use of this relation, the Horn-Thompson theorem 1.1, mentioned in the introduction of the paper, can be reformulated as:

Theorem 4.3 *(Horn,Thompson, theorem 1.1 reformulated) Let X, Y be any two positive definite $n \times n$ matrices and let $x_1 \geq x_2 \geq ... \geq x_n$ and $y_1 \geq y_2 \geq ... \geq y_n$ denote the respective sets of eigenvalues. Then there is an unitary matrix U such that XU and Y have the same spectrum if and only if $Y \prec X$.*

Another interesting majorization occurs between the absolute value of eigenvalues and singular values respectively.

Theorem 4.4 *(Weyl) For any matrix $F \in GL(n, \mathbb{C})$ we have the inequality:*

$$\ln \mid \lambda(F) \mid \prec \ \ln \sigma(F) \quad .$$

We end this section with two results of Fan (see [14] Theorem G.1, page 241 and G.1.d page 243).

Theorem 4.5 *(a) (Fan 1949) Let G, H be two Hermitian matrices. Then*

$$\lambda(G+H) \ \prec \ (\lambda_1^{\downarrow}(G) + \lambda_1^{\downarrow}(H), ..., \lambda_n^{\downarrow}(G) + \lambda_n^{\downarrow}(H)) \quad .$$

(b) (Fan 1951) if A and B are $n \times n$ matrices then

$$\sigma(A+B) \ \prec_w \ \sigma(A) + \sigma(B) \quad .$$

5 Notions of convexity in the calculus of variations

Morrey [15] introduced the notion of quasiconvexity in relation with the direct method in the calculus of variations for functionals in integral form defined over Sobolev spaces.

Let $m, n > 0$ be given natural numbers, $p \in [1, \infty]$ and $\Omega \subset \mathbb{R}^m$ an open, bounded set with piecewise smooth boundary. Let us consider the set $\bar{W}^{1,p}(\Omega, \mathbb{R}^m)$ of all functions ϕ defined almost everywhere (with respect to the Lebesgue measure) on Ω, with values in $\mathbb{R}^n$, which are L^1 integrable and with derivative in the sense of distribution being L^p integrable. The Sobolev space $W^{1,p}(\Omega, \mathbb{R}^m)$ is defined as the collection of equivalence classes of functions in $\bar{W}^{1,p}(\Omega, \mathbb{R}^m)$ with respect to equality almost everywhere in Ω.

By a well known theorem of Lebesgue, for any element $\phi \in W^{1,p}(\Omega, \mathbb{R}^m)$ the limit

$$\bar{\phi}(x) \ = \ \lim_{r \rightarrow 0^+} \frac{1}{\mid B(x,r) \mid} \int_{B(x,r)} \phi(y)$$

exists almost everywhere in Ω. Therefore any element $\phi \in W^{1,p}(\Omega, \mathbb{R}^m)$ has associated in a canonical way an element $\bar{\phi} \in \bar{W}^{1,p}(\Omega, \mathbb{R}^m)$. As it is customarily done, we identify ϕ with $\bar{\phi}$, which transforms $W^{1,p}(\Omega, \mathbb{R}^m)$ into a subspace of $\bar{W}^{1,p}(\Omega, \mathbb{R}^m)$. This identification has several nice properties, the most noticeable being that the space $W^{1,\infty}(\Omega, \mathbb{R}^m)$ identifies with the space of Lipschitz functions from Ω to $\mathbb{R}^m$ and the weak $*$ convergence in $W^{1,\infty}(\Omega, \mathbb{R}^m)$ becomes the uniform convergence. A function $\phi : \Omega \rightarrow \mathbb{R}^m$ is Lipschitz if there is a positive constant C such that for any $x, y \in \Omega$ we have

$$\| \phi(x) - \phi(y) \| \ \leq \ C \, \| x - y \| \quad .$$

Morrey's quasiconvexity is a necessary and sufficient condition for the lower semicontinuity of the functional

$$I : W^{1,\infty}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R} \quad , \quad I(\phi) \ = \ \int_{\Omega} w(D\phi(x)) \quad .$$

Definition 5.1 *Let $w : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$ be a measurable function and $\Omega = (0,1)^n$. The function w is quasiconvex if for any $F \in \mathbb{R}^{n \times m}$ and for any Lipschitz function $u : \Omega \rightarrow \mathbb{R}^m$, such that $u(x) = 0$ on $\partial \Omega$, we have the inequality:*

$$\int_{\Omega} w(F + Du(x)) \ \geq \ \int_{\Omega} w(F) \quad .$$

By a translation and rescaling argument, in this definition Ω can be replaced by any open bounded subset of $\mathbb{R}^n$. If the function w is continuous, then in the particular cases $n = 1$ or $m = 1$ quasiconvexity is equivalent with convexity of w (Tonelli [20]). In general quasiconvexity is a somewhat mysterious notion, very difficult to establish. That is why Morrey proposed the notion of polyconvexity, later used by Ball in several fundamental results in nonlinear elasticity. Polyconvex functions are quasiconvex. Further we explain what polyconvex functions are.

For any natural number $n > 0$ a multi-index α is a string $\alpha = (i_1, ..., i_k)$, $1 \leq i_1 < ... < i_k \leq n$. The length of α is $\mid \alpha \mid = k$.

For given natural numbers $m, n > 0$, $k \leq \min \{m, n\}$ and for any multi-indices $\alpha = (i_1, ..., i_k)$ and $\beta = (j_1, ..., j_k)$ of length k, we denote by $M_{\alpha\beta}$ the function which associates to any matrix $F \in \mathbb{R}^{n \times m}$ the minor

$$M_{\alpha\beta}(F) \ = \ \det \left(F_{i_p, j_q} \right)_{p,q=1,\ldots,k} \quad .$$

Moreover, we denote by $M(F)$ the ordered collection of all minors of the matrix F, in a given lexicographic order.

Definition 5.2 *A continuous function $w : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$ is polyconvex if it can be written as $w(F) = g(M(F))$, with g convex function.*

A necessary condition for quasiconvexity is rank one convexity. For matrices A, B, we denote by $[[A, B]]$ the line segment

$$[[A, B]] \ = \ \{(1 - t)A + tB \ : \ t \in [0, 1]\} \quad .$$

Definition 5.3 *The function $w : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$ is rank one convex if for any $A, B \in \mathbb{R}^{n \times m}$ such that rank $(A - B) = 1$ the function $t \in [0, 1] \mapsto w((1 - t)A + tB) \in \mathbb{R}$ is convex.*

If the function w is $\mathcal{C}^2$ then rank one convexity can be expressed as an ellipticity condition, see further (1).

Let us denote by *Rco*, *Qco*, *Pco*, *Conv* the classes of rank one convex, quasiconvex, polyconvex and convex functions respectively. We have then:

$$Conv \ \subset \ Pco \ \subset \ Qco \ \subset \ Rco \quad .$$

Definition 5.4 *To each notion of convexity corresponds a notion of convex hull:*

- *the rank one convex hull of a non empty set $A \subset \mathbb{R}^{n \times m}$ is*

$$Rco(A) \ = \ \left\{ H \in \mathbb{R}^{n \times m} : w(H) \leq \inf_{F \in A} w \quad , \forall w \in Rco \right\} \quad ,$$

- *the quasiconvex hull of a non empty set $A \subset \mathbb{R}^{n \times m}$ is*

$$Qco(A) \ = \ \left\{ H \in \mathbb{R}^{n \times m} : w(H) \leq \inf_{F \in A} w \quad , \forall w \in Qco \right\} \quad ,$$

- *the polyconvex hull of a non empty set* $A \subset \mathbb{R}^{n\times m}$ *is*

$$Pco(A) \ = \ \left\{ H \in \mathbb{R}^{n\times m} : w(H) \leq \inf_{F\in A} w \quad , \forall w \in Pco \right\} \quad ,$$

- *the convex hull of a non empty set* $A \subset \mathbb{R}^{n\times m}$ *is*

$$Conv(A) \ = \ \left\{ H \in \mathbb{R}^{n\times m} : w(H) \leq \inf_{F\in A} w \quad , \forall w \in Conv \right\} \quad .$$

We have the inclusions:

$$Rco(A) \ \subset \ Qco(A) \ \subset \ Pco(A) \ \subset \ Conv(A) \quad .$$

The particular case $m = n$ is important in applications to the elasticity theory. In this case functions $\phi \in W^{1,p}(\Omega, \mathbb{R}^n)$ represent displacements of the body with reference configuration $\Omega \subset \mathbb{R}^n$ and $w : gl(n, \mathbb{R}) \to \mathbb{R}$ is the potential of the elastic energy of the body.

From the point of view of mechanics we should consider only displacements ϕ which are inversible functions in some (weak or strong) sense. We shall not enter into details here, but we shall consider only the particular case of $W^{1,\infty}$ displacements, seen as Lipschitz functions, as explained previously. In applications we should only look at displacements ϕ which are bi-Lipschitz functions, that is functions $\phi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ such that there are constants $C, C' > 0$ with the property that for any $x, y \in \Omega$

$$C' \|x - y\| \ \leq \ \|\phi(x) - \phi(y)\| \ \leq \ C \|x - y\| \quad .$$

According to Rademacher theorem any Lipschitz function is derivable almost everywhere. We concentrate on bi-Lipschitz displacements $\phi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ such that almost everywhere $D\phi(x) \in GL(n, \mathbb{R})^+$. In this case the elastic potential w becomes a function $w : GL(n, \mathbb{R})^+ \to \mathbb{R}$. (Notice that such displacements ϕ are only locally inversible; global inversibility conditions lead to very difficult problems in the calculus of variations.)

In [5] we introduced the following notion of quasiconvexity.

Definition 5.5 *Let* $w : GL(n, \mathbb{R})^+ \to \mathbb{R}$ *be a function and* $\Omega = (0,1)^n$. w *is multiplicative quasiconvex if for any* $F \in GL(n, \mathbb{R})^+$ *and for any Lipschitz function* $u : \Omega \to \mathbb{R}$, *such that for almost any* $x \in \Omega$ $\det Du(x) > 0$ *and* $u(x) = x$ *on* $\partial\Omega$, *we have the inequality:*

$$\int_\Omega w(FDu(x)) \ \geq \ \int_\Omega w(F)$$

The notion of multiplicative quasiconvexity appears as Diff-quasiconvexity in Giaquinta, Modica, Soucek [10], page 174, definition 3. It can be found for the first time in Ball [3], in a disguised form. It is in fact the natural notion to be considered in connection with continuous media mechanics. Any polyconvex function is multiplicative quasiconvex.

6 Objective isotropic elastic potentials

In elasticity displacements are considered with respect to a reference frame. That is why the elastic potential $w : GL(n, \mathbb{R})^+ \to \mathbb{R}$ should be frame-indifferent (or objective), which is expressed as: for all $F \in GL(n, \mathbb{R})^+$ and all $Q \in SO(n)$ we have $w(QF) = w(F)$. The potential corresponds to an isotropic elastic material if for all $F \in GL(n, \mathbb{R})^+$ and all $Q \in SO(n)$ we have $w(FQ) = w(F)$.

If w is objective and isotropic then there is a symmetric function $g : \mathbb{R}^n_+ \to \mathbb{R}$ such that $w(F) = g(\sigma(F))$. If w is C^2 so is g, see Ball [2].

The definition 5.3 of rank one convexity for functions $w : GL(n, \mathbb{R})^+ \to \mathbb{R}$ has to be slightly modified.

Definition 6.1 *The function $w : GL(n, \mathbb{R})^+ \to \mathbb{R}$ is rank one convex if for any $A, B \in GL(n, \mathbb{R})^+$ such that rank $(A - B) = 1$ and such that $[[A, B]] \subset GL(n, \mathbb{R})^+$ the function $t \in [0, 1] \mapsto w((1 - t)A + tB) \in \mathbb{R}$ is convex.*

If w is C^2, then it is rank one convex if and only if it satisfies the ellipticity condition:

$$\sum_{i,j,k,l=1}^{n} \frac{\partial^2 w}{\partial F_{ij} \partial F_{kl}}(F) a_i b_j a_k b_l \ \geq \ 0 \tag{1}$$

for any $F \in GL(n, \mathbb{R})^+$, $a, b \in \mathbb{R}^n$.

There is a certain interest in giving necessary and sufficient conditions for an objective isotropic w to be rank one convex, especially in the cases $n = 2$ and $n = 3$. These conditions have been expressed in copositivity terms in Simpson and Spector [17] for $n = 3$, Silhavy [16] and Dacorogna [6] for arbitrary n (for an account on the history of results related to this problem see the [16] or [6]).

In this section we shall obtain simpler necessary and sufficient conditions for rank one convexity of isotropic functions. For this we need some preparations.

We shall introduce two auxiliary functions, h and l:

$$h : \mathbb{R}^n \to \mathbb{R} \quad , \quad h(x) = g(\exp x) \tag{2}$$

$$l : \mathbb{R}^n_+ \to \mathbb{R} \quad , \quad l(x) = g(\sqrt{x}) \tag{3}$$

The function h will be called "the diagonal of w".

Definition 6.2 *For any C^2, symmetric function $f : A \subset \mathbb{R}^n \to \mathbb{R}$, with A open, symmetric set, we define the function $Sch(f) : A \to Sym(n, \mathbb{R})$ by:*

(a) for (i, j) any pair of indices, with $i, j \in \{1, ..., n\}$, $i \neq j$ and any $x \in A$ such that $x_i \neq x_j$ we put

$$Sch_{ij}(f)(x) = \frac{f_{,i}(x) - f_{,j}(x)}{x_i - x_j}$$

(b) for (i,j) any pair of indices, with $i,j \in \{1,...,n\}$, $i \neq j$ and any $x \in A$ such that $x_i = x_j$ we put

$$Sch_{ij}(f)(x) = f_{,ij}(x) - f_{,jj}(x)$$

(c) if $i = j$ then we put $Sch_{ii}(f)(x) = 0$ for any $x \in A$.

By theorem 3.5 (b) the function f is Schur convex if and only if for any $i,j \in \{1,...,n\}$ and any $x \in A$ we have $Sch_{ij}(f)(x) \geq 0$. Remark also that the matrix function $Sch(f)$ is by definition continuous, namely the expression of $Sch(f)$ from (b) in previous definition is obtained from extension by continuity of the definition of $Sch(f)$ from the point (a).

In order to properly formulate the next result of Ball (here theorem 6.5) we need one more definition.

Definition 6.3 *Let $g : (0,+\infty)^n \to \mathbb{R}$ be any C^2, symmetric function. For any pair of indices (i,j) with $i,j \in \{1,...,n\}$, such that $i \neq j$, and for any $x \in (0,+\infty)^n$ with $x_i \neq x_j$ we define:*

$$G_{ij}(x) = \frac{x_i g_{,i}(x) - x_j g_{,j}(x)}{x_i^2 - x_j^2}$$

$$\overline{H}_{ij}(x) = \frac{x_j g_{,i}(x) - x_i g_{,j}(x)}{x_i^2 - x_j^2}$$

For $i = j$ and any $x \in (0,+\infty)^n$ we shall put $\overline{H}_{ii}(x) = G_{ii}(x) = 0$.

Lemma 6.4 *Let $g : (0,+\infty)^n \to \mathbb{R}$ be a C^2, symmetric function, and h, l the associated functions defined by relations (2), (3). Then for any pair of indices (i,j) with $i,j \in \{1,...,n\}$, such that $i \neq j$, and for any $x \in \mathbb{R}^n$ with $x_i \neq x_j$ we have:*

$$G_{ij}(\exp x)\frac{\exp(2x_i) - \exp(2x_j)}{x_i - x_j} = Sch_{ij}(h)(x)$$

For any $y \in (0,+\infty)^n$ with $y_i \neq y_j$ we have:

$$\overline{H}_{ij}(y) = 2y_i y_j \, Sch_{ij}(l)(x^2)$$

A direct consequence is that for any pair of indices (i,j) the functions G_{ij} and $\overline{H}_{ij}$ from definition 6.3 can be extended by continuity to all $x \in (0,+\infty)^n$.

Proof. By direct computation. □

The following is theorem 6.4 Ball [2], slightly reformulated.

Theorem 6.5 *For x with all components different, the ellipticity condition (1) for the objective isotropic function w can be expressed in terms of the associated function g as*

$$\sum_{i,j=1}^{n} g_{ij} a_i a_j b_i b_j \ + \ \sum_{i \neq j} G_{ij} a_i^2 b_j^2 \ + \ \sum_{i \neq j} \overline{H}_{ij} a_i a_j b_i b_j \ \geq 0$$

From lemma 6.4, by continuity arguments it follows that one can write the ellipticity condition for all $x \in \mathbb{R}^n_+$ as:

$$\sum_{i,j=1}^{n} H_{ij} a_i a_j b_i b_j \ + \ \sum_{i,j=1}^{n} G_{ij} a_i^2 b_j^2 \ \geq 0 \qquad (4)$$

where H is the matrix $H = \overline{H} + D^2 g$.

The main result of this section is written further.

Theorem 6.6 *Necessary and sufficient conditions for $w \in C^2$ to be rank one convex are:*

(a) h is Schur convex and

(b) for any $x \in \mathbb{R}^n$ we have

$$H_{ij} x_i x_j + G_{ij} \mid x_i \mid\mid x_j \mid \geq 0 \qquad (5)$$

Remark 6.7 *The condition (a) is equivalent with the Baker-Ericksen [1] set of inequalities*

$$\frac{x_i g_{,i}(x_i, x_j) - x_j g_{,j}(x_i, x_j)}{x_i^2 - x_j^2} \geq 0$$

for all $i \neq j$ and x such that $x_i \neq x_j$. Indeed, by theorem 3.5 (b), the function h is Schur convex if and only if

$$\left(h_{,i}(x_i, x_j) - h_{,j}(x_i, x_j)\right)(x_i - x_j) \geq 0$$

for all $i \neq j$ and $x_i \neq x_j$. By lemma 6.4 this is equivalent with $G_{ij} \geq 0$. In [16] Silhavy expresses Baker-Ericksen inequalities using multiplication instead of division, too.

Proof. We prove first the sufficiency. The hypothesis is that for all i, j $G_{ij} \geq 0$ and for all $x \in \mathbb{R}^n$ the relation (5) holds. We claim that for any $a, b \in \mathbb{R}^n$ the inequality

$$G_{ij} a_i a_j b_i b_j \leq G_{ij} a_i^2 b_j^2$$

is true. The ellipticity condition follows then from (5) by the choice $x_i = a_i b_i$, for each $i = 1, ..., n$. Indeed, we have the chain of inequalities

$$0 \leq H_{ij} a_i b_i a_j b_j + G_{ij} \mid a_i b_i \mid\mid a_j b_j \mid \leq \; H_{ij} a_i b_i a_j b_j + G_{ij} a_i^2 b_j^2$$

In order to prove the claim note that $G_{ij} \geq 0$ implies

$$-G_{ij} (a_j b_i - a_i b_j)^2 \; \leq 0$$

A straightforward computation which uses the relations $G_{ij} = G_{ji}$ gives

$$0 \geq \; -G_{ij} (a_j b_i - a_i b_j)^2 \; = \; 2 G_{ij} (a_j b_i - a_i b_j) a_i b_j$$

The sufficiency part is therefore proven.

For the necessity part choose first in the ellipticity condition $a_i = \delta_{iI}$, $b_i = \delta_{iJ}$. For $I \neq J$ we obtain $G_{IJ} \geq 0$, which implies the Schur convexity of h, as it is explained in remark (6.7). (For $I = J$ we obtain $g_{,II} \geq 0$, which is interesting but with no use in this proof.)

Next, suppose that $x, a \in (\mathbb{R}^*)^n$ and choose $b_i = x_i / a_i$ for each $i = 1, ..., n$. The ellipticity condition gives:

$$\sum_{i,j} H_{ij} x_i x_j \; + \; \sum_{i,j=1}^{n} G_{ij} \left(\frac{a_i}{a_j} \right)^2 x_j^2 \; \geq 0$$

Take $a_i^2 = \mid x_i \mid$ and get (5), but only for $x \in (\mathbb{R} \setminus \{0\})^n$. The expression from the left of (5) makes sense for any x. Evoking continuity with respect to x, the theorem is proved. $\square$

The conditions given in theorem 6.6 have some advantages compared with the ones available in the literature. The relation between rank one convexity and Schur convexity, which is rather obvious, can be used to obtain lower semicontinuity results. As for the condition (b), it concentrates in one inequality (containing absolute values) a family of 2^n inequalities expressing copositivity. Moreover, for $n = 2$ or $n = 3$, it can be used to obtain explicit conditions, as in theorem 5, Dacorogna [6].

At the end of this section we would like to discuss about nematic elastomers. For a mathematical treatment of these materials see for example DeSimone, Dolzmann [9]. Such a material is incompressible, isotropic and homogeneous. The elastic potential has the expression:

$$w(F) \; = \; \frac{\sigma_1{}^{\downarrow}(F)}{a_1} + \frac{\sigma_2{}^{\downarrow}(F)}{a_2} + \frac{\sigma_1{}^{\downarrow}(F)}{a_3}$$

with $a_1 > a_2 \geq a_3 > 0$. For the set of minimizers of the (quasiconvexification of the) associated energy functional the microstructure phenomenon appears.

A quick computation give directly the associated function p (see the notation introduced before theorem 3.5 in section 3). It has the expression:

$$p(y_1, y_2, y_3) = \frac{e^{y_1}}{a_1} + \frac{e^{y_2 - y_1}}{a_2} + \frac{e^{y_3 - y_2}}{a_3}$$

The function p is defined over $\mathcal{D}$" and it is not increasing in y_1, y_2, therefore h is not Schur convex. By theorem 6.6 the function w is not rank one convex, as expected. We think that the fact that h is not Schur convex explains the apparition of microstructure, which is seen in [9] as an "$SO(3)$ symmetry breaking". Such a symmetry breaking appears also in linear algebra: the Schur-Horn theorem states that the set

$$\left\{ (A_{11}, A_{22}, A_{33}) : A = Q Diag(a) Q^T, Q \in SO(3) \right\}$$

is a convex polygon. This theorem is the linear version of the Horn-Thompson theorem. It is conceivable then that a function h which is not Schur convex favorises deformations with singular values vector located on the edges of a (well chosen) convex polygon, leading thus to "symmetry breaking".

This makes me ask if there is any isotropic function w, with Schur convex associated function h, for which the microstructure phenomenon appears.

7 Majorization and Calculus of Variations

In this section we shall use majorization techniques in order to obtain simpler proofs of known results and to prove a new lower semicontinuity result which might have applications in Elasticity.

We start with a short proof of a classical theorem:

Theorem 7.1 *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a symmetric, convex function. Then $w : Sym(n, \mathbb{R}) \to \mathbb{R}$, $w(F) = h(\lambda(F))$ is convex.*

Proof. We use the inequality of Fan (1949): for any $A, B \in Sym(n, \mathbb{R})$ we have

$$\lambda(A + B) \prec (\lambda_1{}^{\downarrow}(A) + \lambda_1{}^{\downarrow}(B), ..., \lambda_n{}^{\downarrow}(A) + \lambda_n{}^{\downarrow}(B))$$

By hypothesis h is convex and symmetric, therefore it is Schur convex. From Fan majorization relation we get: for any $\alpha, \beta \in [0, 1]$, $\alpha + \beta = 1$

$$w(\alpha A + \beta B) = h(\lambda(\alpha A + \beta B)) \leq h(\lambda^{\downarrow}(\alpha A) + \lambda^{\downarrow}(\beta B))$$

The chain of inequalities continues by using first the convexity and then the symmetry of h:

$$h(\lambda^{\downarrow}(\alpha A) + \lambda^{\downarrow}(\beta B)) = h(\alpha \lambda^{\downarrow}(A) + \beta \lambda^{\downarrow}(B)) \leq \alpha h(\lambda(A)) + \beta h(\lambda(B)) \qquad \square$$

We quote next the following theorem of Thompson and Freede [19], Ball [3] (for a proof coherent with this paper see Le Dret [13]). We shall give a very easy proof of this theorem using weak majorization.

Theorem 7.2 *Let $g : [0, \infty)^n \to \mathbb{R}$ be convex, symmetric and nondecreasing in each variable. Define the function w by*

$$w : gl(n, \mathbb{R}) \to \mathbb{R} \ , \quad w(F) = g(\sigma(F))$$

Then w is convex.

Proof. This time we use the second inequality of Fan (1951): for any $A, B \in gl(n, \mathbb{R})$ we have

$$\sigma(A + B) \prec_w \sigma(A) + \sigma(B)$$

If g is symmetric, convex and nondecreasing in each variable then it is monotone with respect to weak majorization. The proof resumes exactly as before. □

The main result of this section is:

Theorem 7.3 *Let $g : (0, \infty)^n \to \mathbb{R}$ be a continuous symmetric function and $h : \mathbb{R}^n \to \mathbb{R}$, $h = g \circ \exp$. Suppose that h is convex. Define $w : gl(n, \mathbb{R}) \to \mathbb{R}$ by*

$$w(F) = \begin{cases} g(\sigma(F)) & \text{if } \det F > 0 \\ +\infty & \text{otherwise} \end{cases}$$

Let $F \in GL(n, \mathbb{R})^+$ and $\Omega \subset \mathbb{R}^n$ be bounded, with piecewise smooth boundary. Let $(\phi_h)_h \subset W^{1,1}(\Omega, \mathbb{R}^n)$ be any sequence of functions such that:

(a) for any h we have $\phi_h - \ id \in W_0^{1,1}(\Omega, \mathbb{R}^n)$ and

$$\int_\Omega w(FD\phi_h(x)) \ dx < +\infty$$

(b) Let $D\phi_h = \ R\phi_h U\phi_h = V\phi_h R\phi_h$ be the polar decomposition of $D\phi_h$. We shall suppose that $\ln V\phi_h$ converges weakly in $L^1(\Omega, M_{sym}^{n\times n})$ to 0.

Then we have:

$$\liminf_{h\to\infty} \int_\Omega w(FD\phi_h(x)) \ \geq \mid \Omega \mid w(F) \qquad (6)$$

For the lower semicontinuity properties of multiplicative quasiconvex functions see Buliga [5], theorem 2.1. It is proved there that if the potential w satisfies the inequality (6), but with respect to the usual $W^{1,\infty}$ weak $*$ convergence, then it induces a lower semicontinuous functional. In theorem 7.3 we use a different convergence. It would be interesting to see if a lower semicontinuity theorem similar with theorem 2.1 [5] holds for this convergence.

It is to be remarked that $\ln V\phi$ is the Hencky's logarithmic strain, a good measure of deformation which has been considered several times in the elasticity literature.

In order to prepare the proof of Theorem 7.3, two lemmata are given.

Lemma 7.4 *Let $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous, Schur convex and $g = h \circ \ln$. Define*

$$w : GL(n, \mathbb{R})^+ \rightarrow \mathbb{R} \ , \quad w(F) = g(\sigma(F))$$

$$\tilde{w} : GL(n, \mathbb{C}) \rightarrow \mathbb{R} \ , \quad \tilde{w}(F) = g(\mid \lambda(F) \mid)$$

Then for any F

$$w(F) \geq \tilde{w}(F)$$

Proof. This is a straightforward consequence of the Weyl inequality (theorem 4.4)

$$\ln \mid \lambda(F) \mid \prec \ \ln \sigma(F)$$

and of the Schur convexity of h. □

Lemma 7.5 *With the notations from the lemma 7.4, for any two symmetric matrices A, B, we have*

$$\tilde{w}(\exp A \ \exp B) \geq \tilde{w}(\exp(A + B))$$

Proof. We have to check the conditions from Thompson [18], lemma 6, which gives sufficient conditions on the function $\tilde{w}$ in order to satisfy the inequality we are trying to prove. These conditions are:

(1) for any X and any symmetric positive definite Y $\tilde{w}(XY) = \tilde{w}(YX)$. This is obvious from the definition of $\tilde{w}$.

(2) for any X and any $m = 1, 2, ...$

$$\tilde{w}\left([XX^*]^m\right) \geq \tilde{w}\left(X^{2m}\right)$$

This follows from the definition of $\tilde{w}$ and lemma 7.4. □

We give now the proof of the theorem 7.3.

Proof. It is not restrictive to suppose that $\mid \Omega \mid = 1$. To any $F \in GL(n,\mathbb{R})^+$ we associate its polar decomposition $F = R_F U_F = V_F R_F$. For any function ϕ such that $D\phi(x) \in GL(n,\mathbb{R})^+$ we shall use the (similar) notation

$$D\phi(x) = R\phi(x) U\phi(x) = V\phi(x) R\phi(x)$$

With the notations from the theorem, we have from the isotropy of w, hypothesis (a) and theorem 3.4 that h is Schur convex. From lemma 7.4 and lemma 7.5 we obtain the chain of inequalities, for any h:

$$\int_\Omega w(FD\phi_h(x)) \ = \ \int_\Omega w(U_F V\phi_h(x)) \ \geq \ \int_\Omega \tilde{w}(U_F V\phi_h(x)) \ \geq$$

$$\geq \ \int_\Omega \tilde{w}\left(\exp\left(\ln U_F + \ln V\phi_h(x)\right)\right) \qquad (7)$$

The proof continues using the definition of w and the Jensen inequality for convex h:

$$\int_\Omega \tilde{w}\left(\exp\left(\ln U_F + \ln V\phi(x)\right)\right) \ = \ \int_\Omega h(\ln \mid \lambda\left(\exp\left(\ln U_F + \ln V\phi(x)\right)\right) \mid)$$

$$= \ \int_\Omega h(\lambda\left(\ln U_F + \ln V\phi(x)\right)) \ \geq$$

$$\geq \ h\left(\int_\Omega \lambda\left(\ln U_F + \ln V\phi(x)\right) \ \mathrm{d}x\right) \qquad (8)$$

The proof ends by using the weak L^1 convergence hypothesis (b), when we pass to the limit $h \to \infty$. $\square$

The family of functions satisfying the hypothesis of theorem 7.3 is very big.

As an example of a function satisfying the hypothesis of theorem 7.3 take the polar decomposition $F = R_F U_F$ and define the function: $w(F) = \ \ln \ \mathit{trace} \ U_F$. Indeed, using the notations of theorem 7.3, by straightforward computation we find the associated function $g : (0,+\infty)^n \to \mathbb{R}$ as

$$g(y_1, ..., y_n) \ = \ \ln\left(\sum_{i=1}^n y_i\right)$$

hence the function $h(x) = g(\exp x)$ has the expression:

$$h(x_1, ..., x_n) \ = \ \ln\left(\sum_{i=1}^n \exp(x_i)\right)$$

It is easy to check that h is convex and nondecreasing in each argument.

Let us consider only the Schur convexity and componentwise convexity hypothesis related to w.

Proposition 7.6 *Let $h : \mathbb{R}^n \to \mathbb{R}$ be Schur convex and the function $x \in \mathbb{R} \mapsto h(\ln(x), ..., \ln(x))$ be convex, continuous. Let $\phi : \Omega \to \mathbb{R}$ be such that almost everywhere we have $D\phi(x) \in GL(n, \mathbb{R})^+$,*

$$\int_\Omega D\phi(x) \ = I_n$$

and the map $x \mapsto w(D\phi(x))$ is integrable. Then

$$\int_\Omega w(D\phi(x)) \ \geq \mid \Omega \mid w(I_n)$$

Proof. Because h is Schur convex and for almost any $x \in \Omega$

$$\frac{1}{n} \ln \det D\phi(x)(1, ...1) \prec \ \ln \sigma(D\phi(x))$$

we have the inequality

$$w(D\phi(x)) \geq w\left((\det D\phi(x))^{1/n} I_n\right)$$

Use the convexity hypothesis to obtain the desired inequality. □

8 Rank one convex hulls and majorization

In this section it is explained how majorization appears in the representation of some rank one convex hulls.

We give further a proof of theorem 1.2 using majorization. In this proof we use the fact that majorization relation

$$x \prec\prec y \ \text{ if } \ \ln x \prec \ln y$$

is defined using polyconvex maps. The isotropy of the set $E(a)$ from theorem 1.2 implies that the description of its rank one convex hull reduces to the description of the set of matrices $B \prec Diag(a)$, where $\prec$ is Thompson's order relation. These facts explain the resemblance between theorems 1.1 and 1.2.

Let $a \in (0, \infty)^n$. Denote by $E(a)$ the set of matrices F with positive determinant such that $\sigma(F) = Pa$ for some $P \in S_n$. We have to prove the equality of sets

$$Pco\ E(a) \ = \ Rco\ E(a) \ = \ K(a)$$

where

$$K(a) \ = \ \left\{ B \in GL(n, \mathbb{R})^+ \ : \ B \prec Diag\ (a) \right\}$$

The set $K(a)$ is polyconvex, being an intersection of preimages of $(-\infty, 0]$ by polyconvex functions. Therefore

$$Rco\ E(a) \ \subset \ Pco\ E(a) \ \subset \ K(a)$$

It is left to prove that $K(a) \subset Rco\ E(a)$. For this remark that $E(a)$ can be written as:

$$E(a) \ = \ \{R\,(P.Diag(a))\,Q \ : \ R, Q \in SO(n) \ , \ P \in S_n\}$$

Consider the convex cone of functions (Rco denotes the class of rank one convex functions)

$$Rco(a) \ = \ \{\phi \in Rco \ : \ \forall A \in E(a) \ \ \phi(A) = 0\}$$

This cone is closed with respect to *sup* operation. Moreover, it has the same symmetries as $E(a)$. Indeed for any $R, Q \in SO(n)$, any $P \in S_n$ and any $\phi : GL(n, \mathbb{R})^+$ define $(R, Q, P).\phi$ to be the function

$$F \in GL(n, \mathbb{R})^+ \mapsto (R, Q, P).\phi(F) = \phi(R(P.F)Q)$$

If $\phi \in \ Rco(a)$ then $(R, Q, P).\phi \in \ Rco(a)$.

For any $\phi \in \ Rco(a)$, let $\bar{\phi}$ be the objective isotropic function

$$\bar{\phi}(F) \ = \ \sup\{(R, Q, P).\phi(F) \ : \ R, Q \in SO(n) \ , \ P \in S_n\}$$

If $\phi \in Rco(a)$ then $\bar{\phi} \in Rco(a)$, by the previous remark about symmetries of $Rco(a)$.

Objective isotropic rank one convex functions have Schur convex diagonal, as a consequence of theorem 6.6 (a) (if the rank one convex w is not C^2 use a convolution argument). Therefore $F \in K(a)$ and $\phi \in Rco(a)$ imply

$$\phi(F) \leq \bar{\phi}(F) \leq \bar{\phi}(Diag(a)) = 0$$

This proves the inclusion $K(a) \subset Rco(a)$.

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Dilatation structures I. Fundamentals

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Abstract

A dilatation structure is a concept in between a group and a differential structure. In this article we study fundamental properties of dilatation structures on metric spaces. This is a part of a series of papers which show that such a structure allows to do non-commutative analysis, in the sense of differential calculus, on a large class of metric spaces, some of them fractals. We also describe a formal, universal calculus with binary decorated planar trees, which underlies any dilatation structure.

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Contents

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1 Introduction

The purpose of this paper is to introduce dilatation structures on metric spaces. A dilatation structure is a concept in between a group and a differential structure. Any metric space (X, d) endowed with a dilatation structure has an associated tangent bundle. The tangent space at a point is a conical group, that is the tangent space has a group structure together with a one-parameter group of automorphisms. Conical groups generalize Carnot groups, i.e nilpotent groups endowed with a graduation. Each dilatation structure leads to a non-commutative differential calculus on the metric space (X, d).

There are several important papers dedicated to the study of extra structures on a metric space which allows to do a reasonable analysis in such spaces, like Cheeger [6] or Margulis-Mostow [10, 11].

The constructions proposed in this paper first appeared in connection to problems in analysis on sub-riemannian manifolds. Parts of this article can be seen as a rigorous formulation of the considerations in the last section of Bellaïche [1].

A dilatation structure is simply a bundle of semigroups of (quasi-)contractions on the metric space (X, d), satisfying a number of axioms. The tangent bundle structure associated with a given dilatation structure on the metric space (X, d) is obtained by a passage to the limit procedure, starting from an algebraic structure which lives on the metric space.

With the help of the dilatation structure we construct a bundle (over the metric space) of (local) operations: to each $x \in X$ and parameter ε, for simplicity here $\varepsilon \in (0, +\infty)$, there is a natural non-associative operation

$$\Sigma_{\varepsilon}^{x} : U(x) \times U(x) \to U(x)$$

where $U(x)$ is a neighbourhood of x. The non-associativity of this operation is controlled by the parameter ε. As ε goes to 0 the operation Σ_{ε}^{x} converges to a group operation on the tangent space of (X, d) at x.

Denote by δ_{ε}^{x} the dilatation based at $x \in X$, of parameter ε. The bundle of operations satisfies a kind of weak associativity, even if for any fixed $y \in X$ the operation Σ_{ε}^{y} is non-associative. The weak associativity property, named also shifted associativity, is

$$\Sigma_{\varepsilon}^{x}(u, \Sigma_{\varepsilon}^{\delta_{\varepsilon}^{x}}(v, w)) = \Sigma_{\varepsilon}^{x}(\Sigma_{\varepsilon}^{x}(u, v), w)$$

for any $x \in X$ and any $u, v, w \in X$ sufficiently close to X. We shall describe also other objects (like a function satisfying a shifted inverse property) and algebraic identities related to the dilatation structure and the induced bundle of operations.

We briefly describe further the contents of the paper. In section 2 we give motivational examples of dilatation structures. Basic notions and results of metric geometry and groups endowed with dilatations are mentioned in section 3.

In section 4 we introduce a formalism based on decorated planar binary trees. This formalism will be used to prove the main results of the paper. We show that, from an algebraic point of view, dilatation structures (more precisely the formalism in section 4) induce a bundle of one parameter deformations of binary operations, which are not associative, but shifted associative. This is a structure which bears resemblance with the tangent bundle of a Lie group, but it is more general.

Section 5, 6 and 7 are devoted to dilatation structures. These sections contain the main results of the paper. After we introduce and explain the axioms of dilatation structures, we describe several key metric properties of such a structure, in section 5. In section 6 we prove that a dilatation structure induces a valid notion of tangent bundle. In section 7 we explain how a dilatation structure leads to a differential calculus.

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Section 8 is made of two parts. In the first part we show that dilatation structures induce differential structures, in a generalized sense. In the second part we turn to conical groups and we prove the curious result that, even if in a conical group left translations are smooth but right translations are generically non differentiable, the group operation is smooth if we well choose a dilatation structure.

2 Motivation

We start with a trivial example of a dilatation structure, then we briefly explain the occurence of such a structure in more unusual situations.

There is a lot of structure hiding in the dilatations of $\mathbb{R}^n$. For this space, the dilatation based at x, of coefficient $\varepsilon > 0$, is the function

$$\delta_\varepsilon^x : \mathbb{R}^n \to \mathbb{R}^n \quad \delta_\varepsilon^x y = x + \varepsilon(-x+y)$$

For fixed x the dilatations based at x form a one parameter group which contracts any bounded neighbourhood of x to a point, uniformly with respect to x.

Dilatations behave well with respect to the euclidean distance d, in the following sense: for any $x, u, v \in \mathbb{R}^n$ and any $\varepsilon > 0$ we have

$$\frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) = d(u,v)$$

This shows that from the metric point of view the space $(\mathbb{R}^n, d)$ is a metric cone, that is $(\mathbb{R}^n, d)$ looks the same at all scales.

Moreover, let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a function and $x \in \mathbb{R}^n$. The function f is differentiable in x if there is a linear transformation A (that is a group morphism which commutes with dilatations based at the neutral element 0) such that the limit

$$\lim_{\varepsilon \to 0} \delta_{\varepsilon^{-1}}^{f(x)} f \delta_\varepsilon^x (v) = f(x) + A(-x+v) \tag{2.1}$$

is uniform with respect to v in bounded neighbourhood of x. Really, let us calculate

$$\delta_{\varepsilon^{-1}}^{f(x)} f \delta_\varepsilon^x (v) = f(x) + \frac{1}{\varepsilon}(-f(x) + f(x + \varepsilon(-x+v)))$$

This shows that we get the usual definition of differentiability.

The relation (2.1) can be put in another form, using the euclidean distance:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d(\delta_\varepsilon^{f(x)} T(x)(v), f(\delta_\varepsilon^x v)) = 0$$

uniformly with respect to v in bounded neighbourhood of x. Here

$$T(x)(v) = x + A(-x+v)$$

In conclusion, dilatations are the fundamental object for doing differential calculus on $\mathbb{R}^n$.

Even the algebraic structure of $\mathbb{R}^n$ is encoded in dilatations. Really, we can recover the operation of addition from dilatations. It goes like this: for $x, u, v \in \mathbb{R}^n$ and $\varepsilon > 0$ define

$$\Delta_\varepsilon^x(u,v) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v, \qquad \Sigma_\varepsilon^x(u,v) = \delta_{\varepsilon^{-1}}^x \delta_\varepsilon^{\delta_\varepsilon^x u}(v), \qquad inv_\varepsilon^x(u) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} x$$

For fixed x, u, ε the functions $\Delta_\varepsilon^x(u,\cdot), \Sigma_\varepsilon^x(u,\cdot)$ are inverse one to another, but we don't insist on this for the moment (see Proposition 3).

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What is the meaning of these functions? Let us calculate

$$\begin{aligned}
\Delta_\varepsilon^x(u,v) &= \delta_\varepsilon^x u + \frac{1}{\varepsilon}\left(-\left(\delta_\varepsilon^x u\right) + \delta_\varepsilon^x v\right) \\
&= (x + \varepsilon(-x+u)) + \frac{1}{\varepsilon}\left(\varepsilon(-u+x) - x + x + \varepsilon(-x+v)\right) \\
&= x + \varepsilon(-x+u) + \frac{1}{\varepsilon}\varepsilon(-u+v) \\
&= x + \varepsilon(-x+u) + (-u+v) \\
\Sigma_\varepsilon^x(u,v) &= x + \frac{1}{\varepsilon}\left(-x + \delta_\varepsilon^x u + \varepsilon\left(-\left(\delta_\varepsilon^x u\right) + v\right)\right) \\
&= x + \frac{1}{\varepsilon}\left(\varepsilon(-x+u) + \varepsilon\left(\varepsilon(-u+x) - x + v\right)\right) \\
&= u + \varepsilon(-u+x) + (-x+v)
\end{aligned}$$

In the same way we get

$$inv_\varepsilon^x(u) = x + \varepsilon(-x+u) + (-u+x)$$

As $\varepsilon \to 0$ we have the following limits:

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \Delta_\varepsilon^x(u,v) = \Delta^x(u,v) = x + (-u+v) \\
&\lim_{\varepsilon\to 0} \Sigma_\varepsilon^x(u,v) = \Sigma^x(u,v) = u + (-x+v) \\
&\lim_{\varepsilon\to 0} inv_\varepsilon^x(u) = inv^x(u) = x - u + x
\end{aligned}$$

uniform with respect to x, u, v in bounded sets. The function $\Sigma^x(\cdot,\cdot)$ is a group operation, namely the addition operation translated such that the neutral element is x. Thus, for $x = 0$, we recover the group operation. The function $inv^x(\cdot)$ is the inverse function, and $\Delta^x(\cdot,\cdot)$ is the difference function.

Notice that for fixed x, ε the function $\Sigma_\varepsilon^x(\cdot,\cdot)$ is not a group operation, first of all because it is not associative. Nevertheless, this function satisfies a shifted associativity property, namely (see Proposition 5)

$$\Sigma_\varepsilon^x(\Sigma_\varepsilon^x(u,v),w) = \Sigma_\varepsilon^x(u, \Sigma_\varepsilon^{\delta_\varepsilon^x u}(v,w))$$

Also, the inverse function inv_ε^x is not involutive, but shifted involutive (Proposition 4),

$$inv_\varepsilon^{\delta_\varepsilon^x u}\left(inv_\varepsilon^x u\right) = u$$

These and other properties of dilatations allow to recover the structure of the tangent bundle of $\mathbb{R}^n$, which is trivial in this case.

Let us go to more elaborate examples. We may look to a riemannian manifold M, which is locally a deformation of $\mathbb{R}^n$. We can use charts for transporting (locally) the dilatation structure from $\mathbb{R}^n$ to the manifold. All the previously described metric and algebraic properties will hold in this situation, in a weaker form. For example the riemannian distance is no longer scalling invariant, but we still have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) = d^x(u,v)$$

uniform limit with respect tu x, u, v in (small) bounded sets. Here d^x is an euclidean distance which can be identified with the distance in the tangent space of M at x, induced by the metric

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tensor at x. In the same way we can construct the algebraic structure of the tangent space at x, using the functions $\Sigma_\varepsilon^x, \Delta_\varepsilon^x$. We will have a differentiability notion coming from the dilatations transported by the chart.

If we change charts or the riemannian metric then the dilatation structure will change too, but not very much, essentially because the change of charts is smooth, therefore we are still able to say what are tangent spaces and to describe their algebraic structure.

Let us go further with more complex examples. Consider the Heisenberg group $H(n)$. As a set $H(n) = \mathbb{R}^{2n} \times \mathbb{R}$. We shall use the following notation: an element of $H(n)$ will be denoted by $\tilde{x} = (x, \bar{x})$, with $x \in \mathbb{R}^{2n}, \bar{x} \in \mathbb{R}$. The group operation is

$$\tilde{x}\tilde{y} = (x + y, \bar{x} + \bar{y} + 2\omega(x, y))$$

where ω is the canonic symplectic 2-form on $\mathbb{R}^{2n}$.

The group $H(n)$ is nilpotent, in fact a 2 graded Carnot group. This means that $H(n)$ is nilpotent and that it admits a one-parameter group of isomorphisms

$$\delta_\varepsilon(x, \bar{x}) = (\varepsilon x, \varepsilon^2 \bar{x})$$

These are dilatations, more precisely we can construct dilatations based at $\tilde{x}$ by the formula

$$\delta_\varepsilon^{\tilde{x}} \tilde{u} = \tilde{x}\delta_\varepsilon \left(\tilde{x}^{-1}\tilde{u}\right)$$

We may also put a scalling invariant distance on $H(n)$, for example as follows:

$$d(\tilde{x}, \tilde{y}) = g(\tilde{x}^{-1}\tilde{y}), \qquad g(\tilde{u}) = \max\left\{\|u\|, \sqrt{|\ \bar{u}\ |}\right\}$$

We can repeat step by step the constructions explained before in this situation. There are some differences though.

First of all, from the metric point of view, $(H(n), d)$ is a fractal space, in the sense that the Hausdorff dimension of this space is equal to $2n+2$, therefore strictly greater than the topological dimension, which is $2n+1$. Second, the differential of a function defined by the dilatations is not the usual differential, but an essentially different one, called Pansu derivative (see [13]). This is part of a very active area of research in geometric analysis (among fundamental references one may cite [13, 6, 10, 11, 7]). A spectacular application of Pansu derivative was to prove a Rademacher theorem which in turn implies deep results about Mostow rigidity. The theory applies to general Carnot groups.

The Heisenberg group is not commutative. It is in fact the model for the tangent space of a contact metric manifold, as the euclidean $\mathbb{R}^n$ is the model of the tangent space of a riemannian manifold. We enter here in the realm of sub-riemannian geometry (see for example [1, 9]). In a future paper we shall deal with dilatation structures for sub-riemannian manifolds. An important problem in sub-riemannian geometry is to have good tangent bundle structures, which in turn allow us to prove basic theorems, like Poincaré inequality, Rademacher or Stepanov theorems.

We may even go further and find dilatation structures related with rectifiable sets, or with some self-similar sets. This is not the purpose of this paper though. In the sequel we shall define and study fundamental properties of dilatation structures.

3 Basic notions

We denote by $f \subset X \times Z$ a relation and we write $f(x) = y$ if $(x, y) \in f$. Therefore we may have $f(x) = y$ and $f(x) = y'$ with $y \neq y'$, if $(x, y) \in f$ and $(x, y') \in f$.

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The domain of f is the set of $x \in X$ such that there is $z \in Z$ with $f(x) = z$. We denote the domain by *dom* f. The image of f is the set of $z \in Z$ such that there is $x \in X$ with $f(x) = z$. We denote the image by *im* f.

By convention, when we state that a relation $R(f(x), f(y), ...)$ is true, it means that $R(x', y', ...)$ is true for any choice of $x', y', ...$, such that $(x, x'), (y, y'), ... \in f$.

In a metric space (X, d), the ball centered at $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$. If we need to emphasize the dependence on the distance d then we shall use the notation $B_d(x, r)$. In the same way, $\bar{B}(x, r)$ and $\bar{B}_d(x, r)$ denote the closed ball centered at x, with radius r.

We shall use the following convenient notation: by $\mathcal{O}(\varepsilon)$ we mean a positive function such that $\lim_{\varepsilon \to 0} \mathcal{O}(\varepsilon) = 0$.

3.1 Gromov-Hausdorff distance

There are several definitions of distances between metric spaces. For this subject see [5] (Section 7.4), [8] (Chapter 3) and [7].

We explain now a well-known alternative definition of the Gromov-Hausdorff distance, up to a multiplicative factor.

Definition 1. Let (X_i, d_i, x_i), $i = 1, 2$, be a pair of locally compact pointed metric spaces and $\mu > 0$. We shall say that μ is admissible if there is a relation $\rho \subset X_1 \times X_2$ such that

1. *dom* ρ is μ-dense in X_1,
2. *im* ρ is μ-dense in X_2,
3. $(x_1, x_2) \in \rho$,
4. for all $x, y \in$ *dom* ρ we have

$$\mid d_2(\rho(x), \rho(y)) - d_1(x, y) \mid \leq \mu \quad (3.1)$$

The Gromov-Hausdorff distance between (X_1, x_1, d_1) and (X_2, x_2, d_2) is the infimum of admissible numbers μ.

Denote by $[X, d_X, x]$ the isometry class of (X, d_X, x), that is the class of spaces (Y, d_Y, y) such that it exists an isometry $f : X \to Y$ with the property $f(x) = y$. Note that if (X, d_X, x) is isometric with (Y, d_Y, y) then they have the same diameter.

The Gromov-Hausdorff distance is in fact almost a distance between isometry classes of pointed metric spaces. Indeed, if two pointed metric spaces are isometric then the Gromov-Hausdorff distance equals 0. The converse is also true in the class of compact (pointed) metric spaces [8] (Proposition 3.6).

Moreover, if two of the isometry classes $[X, d_X, x]$, $[Y, d_Y, y]$, $[Z, d_Z, z]$ have (representants with) diameter at most equal to 3, then the triangle inequality is true. We shall use this distance and the induced convergence for isometry classes of the form $[X, d_X, x]$, with *diam* $X \leq 5/2$.

3.2 Metric profiles. Metric tangent space

We shall denote by CMS the set of isometry classes of pointed compact metric spaces. The distance on this set is the Gromov distance between (isometry classes of) pointed metric spaces and the topology is induced by this distance.

To any locally compact metric space we can associate a metric profile [3, 4].

Definition 2. The metric profile associated to the locally metric space (M, d) is the assignment (for small enough $\varepsilon > 0$)

$$(\varepsilon > 0, \ x \in M) \ \mapsto \ \mathbb{P}^m(\varepsilon, x) = \left[\bar{B}(x, 1), \frac{1}{\varepsilon} d, x \right] \in CMS$$

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We can define a notion of metric profile regardless to any distance.

Definition 3. A metric profile is a curve $\mathbb{P} : [0, a] \rightarrow CMS$ such that

(a) it is continuous at 0,
(b) for any $b \in [0, a]$ and $\varepsilon \in (0, 1]$ we have

$$d_{GH}(\mathbb{P}(\varepsilon b), \mathbb{P}^m_{d_b}(\varepsilon, x_b)) \; = \; O(\varepsilon)$$

The function $\mathcal{O}(\varepsilon)$ may change with b. We used the notations

$$\mathbb{P}(b) = [\bar{B}(x, 1), d_b, x_b] \quad \text{and} \quad \mathbb{P}^m_{d_b}(\varepsilon, x) = \left[\bar{B}(x, 1), \frac{1}{\varepsilon} d_b, x_b\right]$$

The metric profile is nice if

$$d_{GH}\left(\mathbb{P}(\varepsilon b), \mathbb{P}^m_{d_b}(\varepsilon, x)\right) = O(b\varepsilon)$$

Imagine that $1/b$ represents the magnification on the scale of a microscope. We use the microscope to study a specimen. For each $b > 0$ the information that we get is the table of distances of the pointed metric space $(\bar{B}(x, 1), d_b, x_b)$.

How can we know, just from the information given by the microscope, that the string of "images" that we have corresponds to a real specimen? The answer is that a reasonable check is the relation from point (b) of the definition of metric profiles 3.

Really, this point says that starting from any magnification $1/b$, if we further select the ball $\bar{B}(x, \varepsilon)$ in the snapshot $(\bar{B}(x, 1), d_b, x_b)$, then the metric space $(\bar{B}(x, 1), \frac{1}{\varepsilon} d_b, x_b)$ looks approximately the same as the snapshot $(\bar{B}(x, 1), d_{b\varepsilon}, x_b)$. That is: further magnification by ε of the snapshot (taken with magnification) b is roughly the same as the snapshot $b\varepsilon$. This is of course true in a neighbourhood of the base point x_b.

The point (a) from the Definition 3has no other justification than Proposition 1 in next subsection.

We rewrite definition 1 with more details, in order to clearly understand what is a metric profile. For any $b \in (0, a]$ and for any $\mu > 0$ there is $\varepsilon(\mu, b) \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon(\mu, b))$ there exists a relation $\rho = \rho_{\varepsilon,b} \subset \bar{B}_{d_b}(x_b, \varepsilon) \times \bar{B}_{d_{b\varepsilon}}(x_{b\varepsilon}, 1)$ such that

1. *dom* $\rho_{\varepsilon,b}$ is μ-dense in $\bar{B}_{d_b}(x_b, \varepsilon)$,
2. *im* $\rho_{\varepsilon,b}$ is μ-dense in $\bar{B}_{d_{b\varepsilon}}(x_{b\varepsilon}, 1)$,
3. $(x_b, x_{b\varepsilon}) \in \rho_{\varepsilon,b}$,
4. for all $x, y \in$ *dom* $\rho_{\varepsilon,b}$ we have

$$\left| \frac{1}{\varepsilon} d_b(x, y) - d_{b\varepsilon}\left(\rho_{\varepsilon,b}(x), \rho_{\varepsilon,b}(y)\right) \right| \leq \mu \tag{3.2}$$

In the microscope interpretation, if $(x, u) \in \rho_{\varepsilon,b}$ means that x and u represent the same "real" point in the specimen.

Therefore a metric profile gives two types of information:

- a distance estimate like (3.2) from point 4,
- an "approximate shape" estimate, like in the points 1–3, where we see that two sets, namely the balls $\bar{B}_{d_b}(x_b, \varepsilon)$ and $\bar{B}_{d_{b\varepsilon}}(x_{b\varepsilon}, 1)$, are approximately isometric.

The simplest metric profile is one with $(\bar{B}(x_b, 1), d_b, x_b) = (X, d_b, x)$. In this case we see that $\rho_{\varepsilon,b}$ is approximately an ε dilatation with base point x.

This observation leads us to a particular class of (pointed) metric spaces, namely the metric cones.

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Definition 4. A metric cone (X, d, x) is a locally compact metric space (X, d), with a marked point $x \in X$ such that for any $a, b \in (0, 1]$ we have

$$\mathbb{P}^m(a, x) = \mathbb{P}^m(b, x)$$

Metric cones have dilatations. By this we mean the following

Definition 5. Let (X, d, x) be a metric cone. For any $\varepsilon \in (0, 1]$ a dilatation is a function $\delta^x_\varepsilon : \bar{B}(x, 1) \to \bar{B}(x, \varepsilon)$ such that

- $\delta^x_\varepsilon(x) = x$,
- for any $u, v \in X$ we have

$$d\left(\delta^x_\varepsilon(u), \delta^x_\varepsilon(v)\right) = \varepsilon \, d(u, v)$$

The existence of dilatations for metric cones comes from the definition 4. Indeed, dilatations are just isometries from $(\bar{B}(x, 1), d, x)$ to $(\bar{B}, \frac{1}{a}d, x)$.

Metric cones are good candidates for being tangent spaces in the metric sense.

Definition 6. A (locally compact) metric space (M, d) admits a (metric) tangent space in $x \in M$ if the associated metric profile $\varepsilon \mapsto \mathbb{P}^m(\varepsilon, x)$ (as in definition 2) admits a prolongation by continuity in $\varepsilon = 0$, i.e if the following limit exists:

$$[T_x M, d^x, x] = \lim_{\varepsilon \to 0} \mathbb{P}^m(\varepsilon, x) \tag{3.3}$$

The connection between metric cones, tangent spaces and metric profiles in the abstract sense is made by the following proposition.

Proposition 1. *The associated metric profile $\varepsilon \mapsto \mathbb{P}^m(\varepsilon, x)$ of a metric space (M, d) for a fixed $x \in M$ is a metric profile in the sense of the definition 3 if and only if the space (M, d) admits a tangent space in x. In such a case the tangent space is a metric cone.*

Proof. A tangent space $[V, d_v, v]$ exists if and only if we have the limit from the relation (3.3). In this case there exists a prolongation by continuity to $\varepsilon = 0$ of the metric profile $\mathbb{P}^m(\cdot, x)$. The prolongation is a metric profile in the sense of definition 3. Indeed, we have still to check the property (b). But this is trivial, because for any $\varepsilon, b > 0$, sufficiently small, we have

$$\mathbb{P}^m(\varepsilon b, x) = \mathbb{P}^m_{d_b}(\varepsilon, x)$$

where $d_b = (1/b)d$ and $\mathbb{P}^m_{d_b}(\varepsilon, x) = [\bar{B}(x, 1), \frac{1}{\varepsilon}d_b, x]$.

Finally, let us prove that the tangent space is a metric cone. For any $a \in (0, 1]$ we have

$$\left[\bar{B}(x, 1), \frac{1}{a}d^x, x\right] = \lim_{\varepsilon \to 0} \mathbb{P}^m(a\varepsilon, x)$$

Therefore

$$\left[\bar{B}(x, 1), \frac{1}{a}d^x, x\right] = [T_x M, d^x, x]$$ □

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3.3 Groups with dilatations. Virtual tangent space

In section 6 we shall see that metric tangent spaces sometimes have a group structure which is compatible with dilatations. This structure, of a group with dilatations, is interesting by itself. The notion has been introduced in [2]; we describe it further.

We start with the following setting: G is a topological group endowed with an uniformity such that the operation is uniformly continuous. The description that follows is slightly non canonical, but is nevertheless motivated by the case of a Lie group endowed with a Carnot-Caratheodory distance induced by a left invariant distribution.

We introduce first the double of G, as the group $G^{(2)} = G \times G$ with operation

$$(x, u)(y, v) = (xy, y^{-1}uyv)$$

The operation on the group G, seen as the function

$$op : G^{(2)} \to G \ , \quad op(x, y) = xy$$

is a group morphism. Also the inclusions:

$$i' : G \to G^{(2)}, \quad i'(x) = (x, e)$$
$$i" : G \to G^{(2)}, \quad i"(x) = (x, x^{-1})$$

are group morphisms.

Definition 7. 1. G is a uniform group if we have two uniformity structures, on G and G^2, such that op, i', $i"$ are uniformly continuous.

2. A local action of a uniform group G on a uniform pointed space (X, x_0) is a function $\phi \in W \in \mathcal{V}(e) \mapsto \hat{\phi} : U_\phi \in \mathcal{V}(x_0) \to V_\phi \in \mathcal{V}(x_0)$ such that

   (a) the map $(\phi, x) \mapsto \hat{\phi}(x)$ is uniformly continuous from $G \times X$ (with product uniformity) to X,

   (b) for any $\phi, \psi \in G$ there is $D \in \mathcal{V}(x_0)$ such that for any $x \in D$ $\widehat{\phi\psi^{-1}}(x)$ and $\hat{\phi}(\hat{\psi}^{-1}(x))$ make sense and $\widehat{\phi\psi^{-1}}(x) = \hat{\phi}(\hat{\psi}^{-1}(x))$.

3. Finally, a local group is an uniform space G with an operation defined in a neighbourhood of $(e, e) \subset G \times G$ which satisfies the uniform group axioms locally.

Note that a local group acts locally at left (and also by conjugation) on itself.

This definition deserves an explanation. An uniform group, according to the Definition 7, is a group G such that left translations are uniformly continuous functions and the left action of G on itself is uniformly continuous too. In order to precisely formulate this we need two uniformities: one on G and another on $G \times G$.

These uniformities should be compatible, which is achieved by saying that i', $i"$ are uniformly continuous. The uniformity of the group operation is achieved by saying that the op morphism is uniformly continuous.

Definition 8. A group with dilatations (G, δ) is a local uniform group G with a local action of Γ (denoted by δ), on G such that

H0. the limit $\lim_{\varepsilon \to 0} \delta_\varepsilon x = e$ exists and is uniform with respect to x in a compact neighbourhood of the identity e.

H1. the limit

$$\beta(x, y) = \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)(\delta_\varepsilon y) \right)$$

is well defined in a compact neighbourhood of e and the limit is uniform.

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H2. the following relation holds:

$$\lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)^{-1} \right) = x^{-1}$$

where the limit from the left hand side exists in a neighbourhood of e and is uniform with respect to x.

These axioms are the prototype of a dilatation structure.

The "infinitesimal version" of an uniform group is a conical local uniform group.

Definition 9. A conical group N is a local group with a local action of $(0, +\infty)$ by morphisms δ_ε such that $\lim_{\varepsilon \to 0} \delta_\varepsilon x = e$ for any x in a neighbourhood of the neutral element e.

Here comes a proposition which explains why a conical group is the infinitesimal version of a group with dilatations.

Proposition 2. *Under the hypotheses H0, H1, H2 (G, β, δ) is a conical group, with operation β and dilatations δ.*

Proof. All the uniformity assumptions allow us to change at will the order of taking limits. We shall not insist on this further and we shall concentrate on the algebraic aspects.

We have to prove the associativity, existence of neutral element, existence of inverse and the property of being conical.

For the associativity $\beta(x, \beta(y, z)) = \beta(\beta(x, y), z)$ we calculate

$$\beta(x, \beta(y, z)) = \lim_{\varepsilon \to 0, \eta \to 0} \delta_\varepsilon^{-1} \left\{ (\delta_\varepsilon x) \delta_{\varepsilon/\eta} \left((\delta_\eta y)(\delta_\eta z) \right) \right\}$$

We take $\varepsilon = \eta$ and get

$$\beta(x, \beta(y, z)) = \lim_{\varepsilon \to 0} \left\{ (\delta_\varepsilon x)(\delta_\varepsilon y)(\delta_\varepsilon z) \right\}$$

In the same way

$$\beta(\beta(x, y), z) = \lim_{\varepsilon \to 0, \eta \to 0} \delta_\varepsilon^{-1} \left\{ (\delta_{\varepsilon/\eta} x) \left((\delta_\eta x)(\delta_\eta y) \right) (\delta_\varepsilon z) \right\}$$

and again taking $\varepsilon = \eta$ we obtain

$$\beta(\beta(x, y), z) = \lim_{\varepsilon \to 0} \left\{ (\delta_\varepsilon x)(\delta_\varepsilon y)(\delta_\varepsilon z) \right\} = \beta(x, \beta(y, z))$$

The neutral element is e, from H0 (first part) it follows that $\beta(x, e) = \beta(e, x) = x$. The inverse of x is x^{-1}, by a similar argument:

$$\beta(x, x^{-1}) = \lim_{\varepsilon \to 0, \eta \to 0} \delta_\varepsilon^{-1} \left\{ (\delta_\varepsilon x) \left(\delta_{\varepsilon/\eta} (\delta_\eta x)^{-1} \right) \right\}$$

and taking $\varepsilon = \eta$ we obtain

$$\beta(x, x^{-1}) = \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)(\delta_\varepsilon x)^{-1} \right) = \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1}(e) = e$$

Finally, β has the property

$$\beta(\delta_\eta x, \delta_\eta y) = \delta_\eta \beta(x, y)$$

which comes from the definition of β and commutativity of multiplication in $(0, +\infty)$. This proves that (G, β, δ) is conical. □

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In a sense (G, β, δ) is the tangent space of the group with dilatations (G, δ) at e. We can act with the conical group (G, β, δ) on (G, δ). Indeed, let us denote by $[f, g] = f \circ g \circ f^{-1} \circ g^{-1}$ the commutator of two transformations. For the group G we shall denote by $L^G_x y = xy$ the left translation and by $L^N_x y = \beta(x, y)$. The preceding proposition tells us that (G, β, δ) acts locally by left translations on G. We shall call the left translations with respect to the group operation β "infinitesimal". These infinitesimal translations admit an interesting commutator representation

$$\lim_{\lambda \to 0} \left[L^G_{(\delta_\lambda x)^{-1}}, \delta_\lambda^{-1} \right] = L^N_x \tag{3.4}$$

Definition 10. The group VT_eG formed by all transformations L^N_x is called the virtual tangent space at e to G.

As local groups, VT_eG and (G, β, δ) are isomorphic. We can easily define dilatations on VT_eG, by conjugation with dilatations δ_ε. Really, we see that

$$L^N_{\delta_\varepsilon x}(y) = \beta(\delta_\varepsilon x, y) = \delta_\varepsilon L^N_x \left(\delta_\varepsilon\right)^{-1}$$

The virtual tangent space VT_xG at $x \in G$ to G is obtained by translating the group operation and the dilatations from e to x. This means: define a new operation on G by

$$y \overset{x}{\cdot} z = yx^{-1}z$$

The group G with this operation is isomorphic to G with old operation and the left translation $L^G_x y = xy$ is the isomorphism. The neutral element is x. Introduce also the dilatations based at x by

$$\delta^x_\varepsilon y = x \delta_\varepsilon (x^{-1} y)$$

Then $G^x = (G, \overset{x}{\cdot})$ with the group of dilatations δ^x_ε satisfy the Axioms H0, H1, H2. Define then the virtual tangent space VT_xG to be: $VT_xG = VT_xG^x$.

4 Binary decorated trees and dilatations

We want to explore what happens when we make compositions of dilatations (which depends also on $\varepsilon > 0$). The ε variable apart, any dilatation $\delta^x_\varepsilon(y)$ is a function of two arguments: x and y, invertible with respect to the second argument. The functions we can obtain when composing dilatations are difficult to write, that is why we shall use a tree notation.

4.1 The formalism

Let X be a non empty set and $\mathcal{T}(X)$ be a class of binary planar trees with leaves in X and all nodes decorated with two colors $\{\circ, \bullet\}$. The empty tree, that is the tree with no nodes or leaves, belongs to $\mathcal{T}(X)$. For any $x \in X$ we accept that there is a tree in $\mathcal{T}(X)$ with no nodes and with x as the only leaf. That is $X \subset \mathcal{T}(X)$.

For any color $\mathbf{a} \in \{\circ, \bullet\}$, let $\bar{\mathbf{a}}$ be the opposite color. The colors $\circ$ and $\bullet$ are codes for the symbols ε and ε^{-1}.

The relation "$\approx$" is an equivalence relation on $\mathcal{T}(X)$, taken as a primitive notion for the axioms which will follow.

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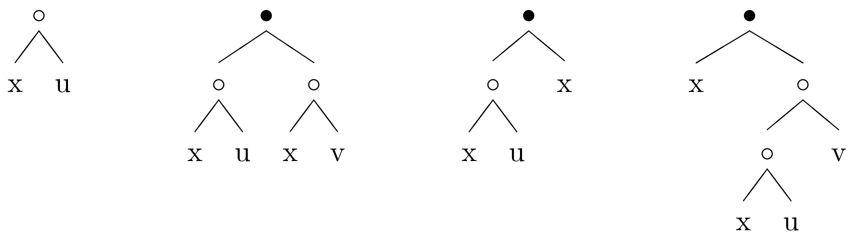
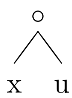
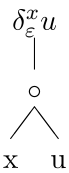
The equivalence class of of a tree $\mathcal{P} \in \mathcal{T}(X)$ is denoted by $\begin{array}{c} . \\ | \\ \mathcal{P} \end{array}$. In various diagrams that will follow we shall use the notation $\begin{array}{c} \Gamma \\ | \\ \mathcal{P} \end{array}$ for saying that Γ is the equivalence class of $\mathcal{P}$. For any $\mathcal{P}, \mathcal{R} \in \mathcal{T}(X)$, "$\mathcal{P} \approx \mathcal{R}$" or " $\begin{array}{c} \\ | \\ \mathcal{P} \end{array} = \begin{array}{c} \\ | \\ \mathcal{R} \end{array}$ " means the same thing.

Axiom T0. For any $x, u, v \in X$ the trees

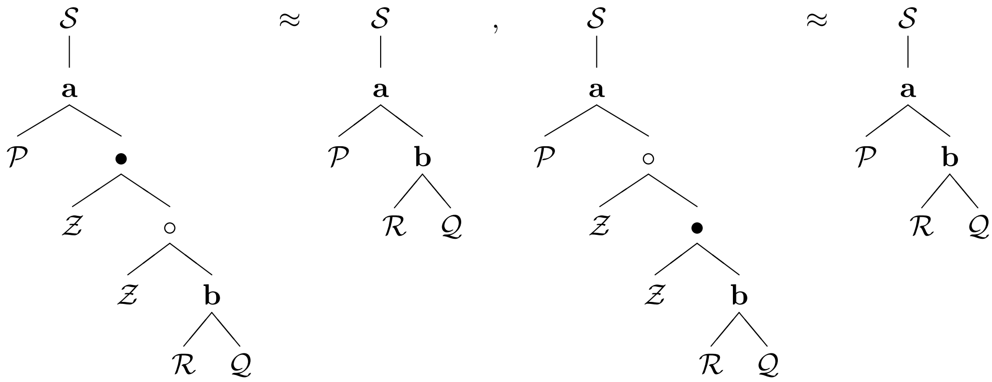


belong to $\mathcal{T}(X)$.

The equivalence class of  is denoted by $\delta^x_\varepsilon u$, that is we have



Axiom T1. Consider any trees $\mathcal{P}, \mathcal{R}, \mathcal{S}, \mathcal{Q}, \mathcal{Z} \in \mathcal{T}(X)$, any $x \in X$, and any colors $\mathbf{a}, \mathbf{b}$ such that the trees from the right hand sides of relations below belong to $\mathcal{T}(X)$. Then the trees from the left hand sides of relations below belong to $\mathcal{T}(X)$ and we have



(4.1)

Here, in all diagrams, the symbol $\begin{array}{c} \mathcal{S} \\ | \\ \mathbf{a} \end{array}$ means that the node colored with $\mathbf{a}$ is grafted at an arbitrary leaf of the tree $\mathcal{S}$.

The second axiom expresses the fact that the dilatation (of any coefficient ε) δ^x_ε has x as fixed point, that is $\delta^x_\varepsilon x = x$.

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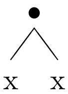
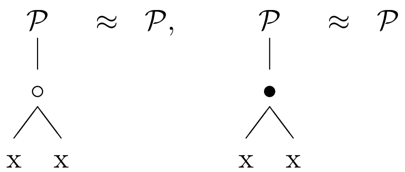
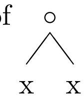
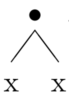
Axiom T2. For any $x \in X$ the tree  belongs to $\mathcal{T}(X)$. Moreover, consider any tree $\mathcal{P} \in \mathcal{T}(X)$ and any $x \in X$. Then the trees from the left hand sides of relations below belong to $\mathcal{T}(X)$ and we have

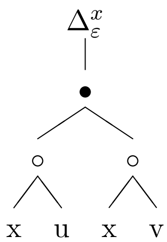


(4.2)

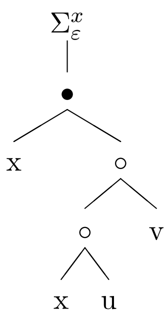
that is the equivalence class of x is the same as the equivalence class of  and the equivalence class of . As in Axiom T1, the symbol  means that the root of the tree $\mathcal{P}$ is grafted at an arbitrary leaf of the tree $\mathcal{S}$.

Definition 11. We define the difference, sum and inverse trees as follows:

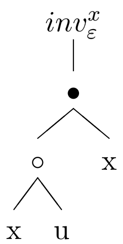
(a) the difference tree $\Delta_\varepsilon^x = \Delta_\varepsilon^x(u, v)$ is given by the relation



(b) the sum tree $\Sigma_\varepsilon^x = \Sigma_\varepsilon^x(u, v)$ is given by the relation



(c) the inverse tree $inv_\varepsilon^x = inv_\varepsilon^x(u)$ is given by the relation



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The next axiom states that T0, T1, T2 are sufficient for determining the class $\mathcal{T}(X)$ and the equivalence relation $\approx$.

Axiom T3. The class $\mathcal{T}(X)$ is the smallest class of trees obtained by grafting of trees listed in Axiom T0, and satisfying Axioms T1, T2. Moreover, two trees from $\mathcal{T}(X)$ are equivalent if and only if they can be proved equivalent after a finite string of applications of Axioms T1, T2.

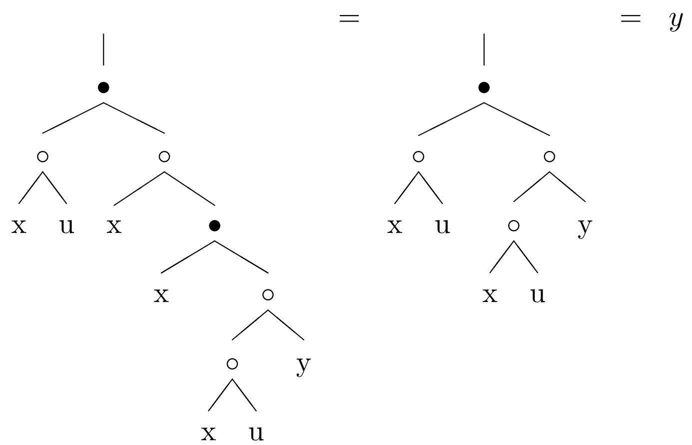
4.2 First consequences

We shall use the axioms in order to obtain results that we shall use later, for dilatation structures.

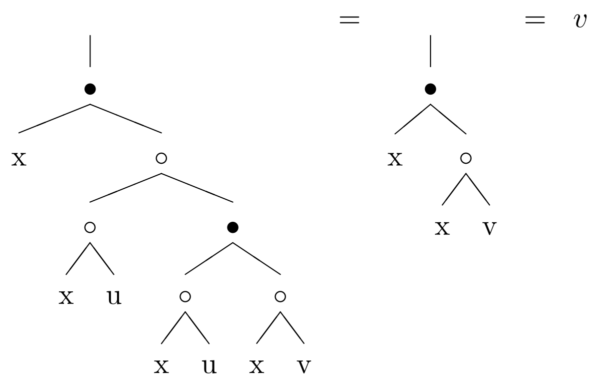
Proposition 3. *For any x, u, y and v we have*

(a) $\Delta^x_\varepsilon(u, \Sigma^x_\varepsilon(u, y)) = y$,
(b) $\Sigma^x_\varepsilon(u, \Delta^x_\varepsilon(u, v)) = v$.

Proof. We prove (a) by computations using the definition 11 of the sum and difference trees, and Axiom T1 several times.

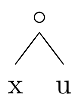


For (b) we proceed in the same way:



□

Proposition 4. *We have the relations*

$$\Delta^x_\varepsilon(u, v) = \Sigma_\varepsilon$$  $$(inv^x_\varepsilon(u), v) \quad (4.3)$$

$$inv^x_\varepsilon(u) = \Delta^x_\varepsilon(u, x) \quad (4.4)$$

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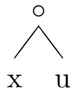
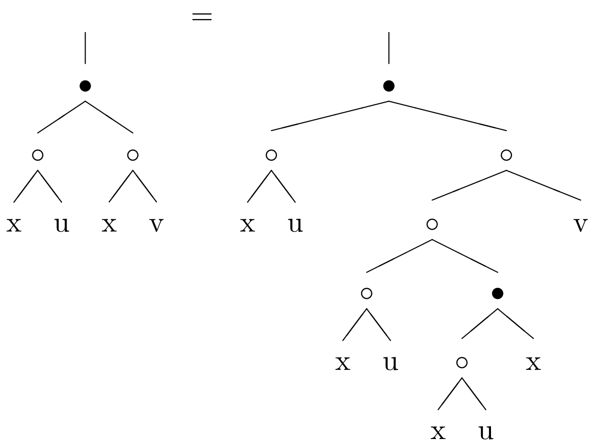
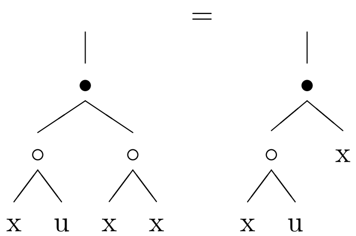
inv_ε  $((inv^x_\varepsilon(u)) = u$ (4.5)

Proof. Graphically, the relation (4.3) is



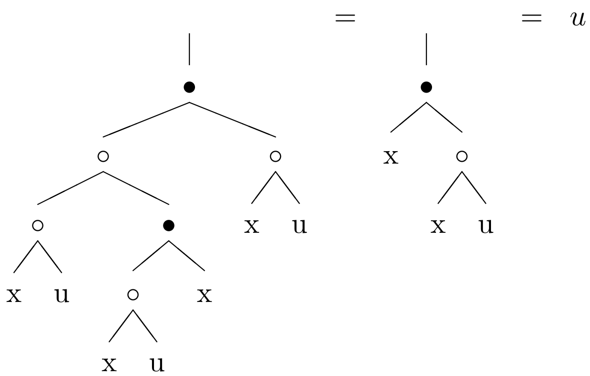
This is true by Axiom T1.

The relation (4.4) is



This is true by Axiom T2.

We prove the relation (4.5) by a string of equalities, starting from the left hand side to the right:



Here we have used the Axiom T1 several times. □

The relation (4.5) in last proposition shows that the "inverse function" inv^x_ε is not involutive, but shifted involutive.

The next proposition shows that the function $\Sigma^x_\varepsilon(\cdot, \cdot)$ satisfies a shifted associativity property.

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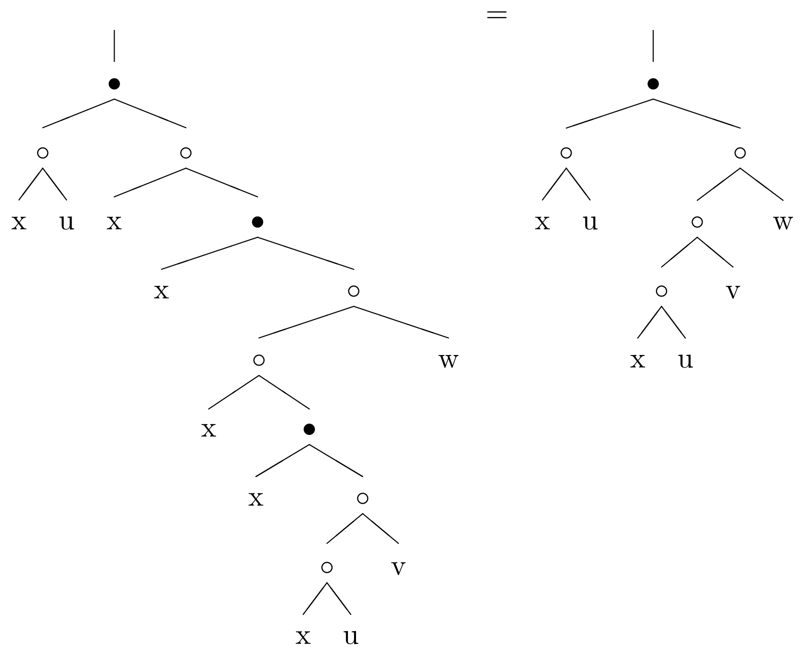
Proposition 5. *We have the following relations:*

$$\Delta_\varepsilon^x(u, \Sigma_\varepsilon^x(\Sigma_\varepsilon^x(u,v),w)) = \Sigma_\varepsilon^{\substack{\circ \\ x \quad u}}(v,w) \tag{4.6}$$

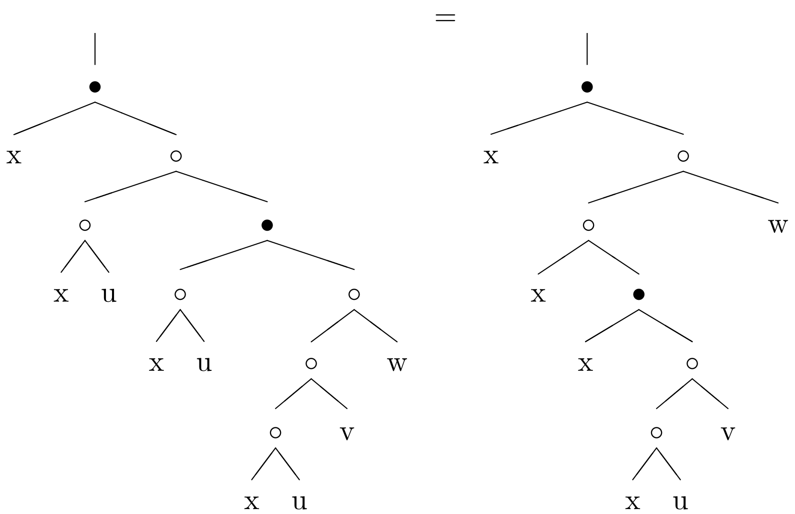
$$\Sigma_\varepsilon^x\left(u, \Sigma_\varepsilon^{\substack{\circ \\ x \quad u}}(v,w)\right) = \Sigma_\varepsilon^x(\Sigma_\varepsilon^x(u,v),w) \tag{4.7}$$

Proof. Graphically, the relation (4.6) is



This is true by Axiom T1.

The relation (4.7) is is equivalent to (4.6), by Proposition 3. We can also give a direct proof by graphically representing the relation



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This is true by the Axiom T1. □

5 Dilatation structures

The space (X, d) is a complete, locally compact metric space. This means that as a metric space (X, d) is complete and that small balls are compact.

5.1 Axioms of dilatation structures

The axioms of a dilatation structure (X, d, δ) are listed further. The first axiom is merely a preparation for the next axioms. That is why we counted it as Axiom 0.

A0. Depending on the parameter $\varepsilon \in (0, +\infty)$, dilatations are objects having the following description.

For any $\varepsilon \in (0, 1]$ the dilatations are functions

$$\delta^x_\varepsilon : U(x) \to V_\varepsilon(x)$$

All such dilatations are homeomorphisms (invertible, continuous, with continuous inverse).

We suppose that there is $1 < A$ such that for any $x \in X$ we have

$$\bar{B}_d(x, A) \subset U(x)$$

We suppose that for all $\varepsilon \in (0, 1)$, we have

$$B_d(x, \varepsilon) \subset \delta^x_\varepsilon B_d(x, A) \subset V_\varepsilon(x) \subset U(x)$$

For $\varepsilon \in (1, +\infty)$ the associated dilatation

$$\delta^x_\varepsilon : W_\varepsilon(x) \to B_d(x, B) \ ,$$

is an injective, continuous, with continuous inverse on the image. We shall suppose that $W_\varepsilon(x)$ is open,

$$V_{\varepsilon^{-1}}(x) \subset W_\varepsilon(x)$$

and that for all $\varepsilon \in [0, 1]$ and $u \in U(x)$ we have

$$\delta^x_{\varepsilon^{-1}} \ \delta^x_\varepsilon u = u$$

We remark that we have the following string of inclusions, for any $\varepsilon \in (0, 1]$ and any $x \in X$:

$$B_d(x, \varepsilon) \subset \delta^x_\varepsilon B_d(x, A) \subset V_\varepsilon(x) \subset W_{\varepsilon^{-1}}(x) \subset \delta^x_\varepsilon B_d(x, B)$$

A further technical condition on the sets $V_\varepsilon(x)$ and $W_\varepsilon(x)$ will be given just before the Axiom A4. (This condition will be counted as part of Axiom A0.)

A1. We have $\delta^x_\varepsilon x = x$ for any point x. We also have $\delta^x_1 = id$ for any $x \in X$.

Let us define the topological space

$$dom\, \delta = \{(\varepsilon, x, y) \in (0, \infty) \times X \times X \colon \text{ if } \varepsilon \in (0, 1] \text{ then } y \in U(x), \text{ else } y \in W_\varepsilon(x)\}$$

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with the topology inherited from the product topology on $\Gamma \times X \times X$. Consider also $Cl(dom\,\delta)$, the closure of $dom\,\delta$ in $[0,\infty) \times X \times X$ with product topology. The function

$$\delta : dom\,\delta \to X$$

defined by $\delta(\varepsilon, x, y) = \delta^x_\varepsilon y$ is continuous. Moreover, it can be continuously extended to $Cl(dom\,\delta)$ and we have

$$\lim_{\varepsilon \to 0} \delta^x_\varepsilon y \,=\, x$$

A2. For any $x, \in K$, $\varepsilon, \mu \in \Gamma_1$ and $u \in \bar{B}_d(x, A)$ we have

$$\delta^x_\varepsilon \delta^x_\mu u = \delta^x_{\varepsilon\mu} u$$

A3. For any x there is a function $(u, v) \mapsto d^x(u, v)$, defined for any u, v in the closed ball (in distance d) $\bar{B}(x, A)$, such that

$$\lim_{\varepsilon \to 0} \quad \sup \left\{ \left| \frac{1}{\varepsilon} d(\delta^x_\varepsilon u, \delta^x_\varepsilon v) - d^x(u, v) \right| : \, u, v \in \bar{B}_d(x, A) \right\} = 0$$

uniformly with respect to x in compact set.

Remark 12. The "distance" d^x can be degenerated. That means: there might be $v, w \in \bar{B}_d(x, A)$ such that $d^x(v, w) = 0$ but $v \neq w$. We shall use further the name "distance" for d^x, essentially by commodity, but keep in mind the possible degeneracy of d^x.

For the following axiom to make sense we impose a technical condition on the co-domains $V_\varepsilon(x)$: for any compact set $K \subset X$ there are $R = R(K) > 0$ and $\varepsilon_0 = \varepsilon(K) \in (0, 1)$ such that for all $u, v \in \bar{B}_d(x, R)$ and all $\varepsilon \in \Gamma$, $\nu(\varepsilon) \in (0, \varepsilon_0)$, we have

$$\delta^x_\varepsilon v \in W_{\varepsilon^{-1}}(\delta^x_\varepsilon u)$$

With this assumption the following notation makes sense:

$$\Delta^x_\varepsilon(u, v) = \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \delta^x_\varepsilon v \tag{5.1}$$

The next axiom can now be stated:

A4. We have the limit

$$\lim_{\varepsilon \to 0} \Delta^x_\varepsilon(u, v) = \Delta^x(u, v)$$

uniformly with respect to x, u, v in compact set.

Note that with the tree notation we may identify (5.1) with the difference tree from Definition 11 (a).

Definition 13. A triple (X, d, δ) which satisfies A0, A1, A2, A3, but d^x is degenerate for some $x \in X$, is called degenerate dilatation structure.

If the triple (X, d, δ) satisfies A0, A1, A2, A3 and d^x is non-degenerate for any $x \in X$, then we call it a weak dilatation structure.

If a weak dilatation structure satisfies A4 then we call it dilatation structure.

Note that it could be assumed, without great modification of the axioms, that

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(a) we may replace $(0, \infty)$ by Γ, a topological separated commutative group endowed with a continuous group morphism $\nu : \Gamma \rightarrow (0, +\infty)$ with $\inf \nu(\Gamma) = 0$. Here $(0, +\infty)$ is taken as a group with multiplication. The neutral element of Γ is denoted by 1. We use the multiplicative notation for the operation in Γ.

The morphism ν defines an invariant topological filter on Γ (equivalently, an end). Really, this is the filter generated by the open sets $\nu^{-1}(0, a)$, $a > 0$. From now on we shall name this topological filter (end) by "0" and we shall write $\varepsilon \in \Gamma \rightarrow 0$ for $\nu(\varepsilon) \in (0, +\infty) \rightarrow 0$.

The set $\Gamma_1 = \nu^{-1}(0, 1]$ is a semigroup. We note $\bar{\Gamma}_1 = \Gamma_1 \cup \{0\}$ On the set $\bar{\Gamma} = \Gamma \cup \{0\}$ we extend the operation on Γ by adding the rules $00 = 0$ and $\varepsilon 0 = 0$ for any $\varepsilon \in \Gamma$. This is in agreement with the invariance of the end 0 with respect to translations in Γ.

In the Axioms A0, A1 we therefore may replace $[0, 1]$ by $\bar{\Gamma}_1$, and so forth.

(b) we may leave some flexibility in Axiom A1 for the choice of base point of the dilatation, in the sense that

$$\lim_{\nu(\varepsilon) \rightarrow 0} \frac{1}{\nu(\varepsilon)} d(x, \delta_{\varepsilon}^{x} x) = 0$$

uniformly with respect to $x \in K$ compact set,

(c) we may relax the semigroup condition in the Axiom A2, in the sense: for any compact set $K \subset X$, for any $x, \in K$, ε, μ with $\nu(\varepsilon), \nu(\mu) \in (0, 1)$ and $u, v \in \bar{B}_d(x, A)$ we have

$$\frac{1}{\nu(\varepsilon\mu)} \mid d(\delta_{\varepsilon}^{x} \delta_{\mu}^{x} u, \delta_{\varepsilon}^{x} \delta_{\mu}^{x} v) - d(\delta_{\varepsilon\mu}^{x} u, \delta_{\varepsilon\mu}^{x} v) \mid \leq \mathcal{O}(\varepsilon\mu)$$

(d) in the Axioms A3 and A4 we may replace "$\varepsilon \rightarrow 0$" by "$\nu(\varepsilon) \rightarrow 0$" and "$1/\varepsilon$" by "$1/\nu(\varepsilon)$".

We shall write the proofs of further results such that these work even if we modify the axioms in the sense explained above. We shall nevertheless use ε and not $\nu(\varepsilon)$, in order to avoid a too heavy notation.

The axioms, as given in this section, are said to be in strong form. With the modifications explained at points (a), (b), (c), (d) above, the axioms are said to be in weak form.

Further, axioms are taken in weak form with the notational conventions explained above, unless it is explicitely stated that some axiom has to be taken in strong form.

5.2 Dilatation structures, tangent cones and metric profiles

We shall explain now what the axioms mean. The first Axiom A1 is stating that the distance between $\delta_{\varepsilon}^{x} x$ and x is negligible with respect to ε. If $\delta_{\varepsilon}^{x} x = x$ then this axiom is trivially satisfied.

The second Axiom A2. states that in an approximate sense the transformations δ_{ε}^{x} form an action of Γ on X. As previously, if we suppose that

$$\delta_{\varepsilon}^{x} \delta_{\mu}^{x} = \delta_{\varepsilon\mu}^{x}$$

then this axiom is trivially satisfied.

Remark now that the binary tree formalism described in section 4 underlies and simplifies the calculus with dilatation structures. More precisely, we shall use the results in section 4 in the proof of theorems in the next section.

The notation with binary trees for composition of dilatations is not directly adapted for taking limits as $\varepsilon \rightarrow 0$. An extension of the formalism can be made in this direction, but this would add length to this paper, which is devoted to first properties of dilatation structures. We reserve the full description of the formalism for a future paper.

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In Axiom A3 we take limits. In this subsection we shall look at dilatation structures from the metric point of view, by using Gromov-Hausdorff distance and metric profiles.

We state the interpretation of the Axiom A3 as a theorem. But before a definition: we denote by (δ, ε) the distance on

$$\bar{B}_{d^x}(x,1) = \{y \in X: \ d^x(x,y) \leq 1\}$$

given by

$$(\delta, \varepsilon)(u,v) = \frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v)$$

Theorem 6. *Let (X, d, δ) be a dilatation structure. The following are consequences of the Axioms A0 - A3 only:*

(a) for all $u, v \in X$ such that $d(x,u) \leq 1$ and $d(x,v) \leq 1$ and all $\mu \in (0, A)$ we have

$$d^x(u,v) = \frac{1}{\mu} d^x(\delta_\mu^x u, \delta_\mu^x v)$$

We shall say that d^x has the cone property with respect to dilatations.

(b) The curve $\varepsilon > 0 \mapsto \mathbb{P}^x(\varepsilon) = [\bar{B}_{d^x}(x,1), (\delta, \varepsilon), x]$ is a metric profile.

Proof. (a) For $\varepsilon, \mu \in (0,1)$ we have

$$\left| \frac{1}{\varepsilon\mu} d(\delta_\varepsilon^x \delta_\mu^x u, \delta_\varepsilon^x \delta_\mu^x v) - d^x(u,v) \right| \leq \left| \frac{1}{\varepsilon\mu} d(\delta_{\varepsilon\mu}^x u, \delta_\varepsilon^x \delta_\mu^x u) - \frac{1}{\varepsilon\mu} d(\delta_{\varepsilon\mu}^x v, \delta_\varepsilon^x \delta_\mu^x v) \right| + \left| \frac{1}{\varepsilon\mu} d(\delta_{\varepsilon\mu}^x u, \delta_{\varepsilon\mu}^x v) - d^x(u,v) \right|$$

Use now the Axioms A2 and A3 and pass to the limit with $\varepsilon \to 0$. This gives the desired equality.

(b)We have to prove that $\mathbb{P}^x$ is a metric profile. For this we have to compare two pointed metric spaces:

$$\left(\bar{B}_{d^x}(x,1), (\delta^x, \varepsilon\mu), x\right) \quad \text{and} \quad \left(\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,1), \frac{1}{\mu}(\delta^x, \varepsilon), x\right)$$

Let $u \in X$ such that

$$\frac{1}{\mu}(\delta^x, \varepsilon)(x,u) \leq 1$$

This means that

$$\frac{1}{\varepsilon} d(\delta_\varepsilon^x x, \delta_\varepsilon^x u) \ \leq \ \mu$$

Further use the Axioms A1, A2 and the cone property proved before:

$$\frac{1}{\varepsilon} d^x(\delta_\varepsilon^x x, \delta_\varepsilon^x u) \ \leq \ (\mathcal{O}(\varepsilon) + 1)\mu$$

therefore,

$$d^x(x,u) \ \leq \ (\mathcal{O}(\varepsilon) + 1)\mu$$

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It follows that for any $u \in \bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,1)$ we can choose $w(u) \in \bar{B}_{d^x}(x,1)$ such that

$$\frac{1}{\mu} d^x(u, \delta^x_\mu w(u)) = \mathcal{O}(\varepsilon)$$

We want to prove that

$$\left| \frac{1}{\mu}(\delta^x,\varepsilon)(u_1,u_2) - (\delta^x,\varepsilon\mu)(w(u_1),w(u_2)) \right| \leq \mathcal{O}(\varepsilon\mu) + \frac{1}{\mu}\mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon)$$

This goes as follows:

$$|\frac{1}{\mu}(\delta^x,\varepsilon)(u_1,u_2) - (\delta^x,\varepsilon\mu)(w(u_1),w(u_2))| = \left| \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon u_1, \delta^x_\varepsilon u_2) - \frac{1}{\varepsilon\mu} d(\delta^x_{\varepsilon\mu} w(u_1), \delta^x_{\varepsilon\mu} w(u_2)) \right|$$
$$\leq \mathcal{O}(\varepsilon\mu) + \left| \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon u_1, \delta^x_\varepsilon u_2) - \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon \delta^x_\mu w(u_1), \delta^x_\varepsilon \delta^x_\mu w(u_2)) \right|$$
$$\leq \mathcal{O}(\varepsilon\mu) + \frac{1}{\mu}\mathcal{O}(\varepsilon) + \frac{1}{\mu} \mid d^x(u_1,u_2) - d^x(\delta^x_\mu w(u_1), \delta^x_\mu w(u_2)) \mid$$

In order to obtain the last estimate we used twice the Axiom A3. We proceed as follows:

$$\mathcal{O}(\varepsilon\mu) + \frac{1}{\mu}\mathcal{O}(\varepsilon) + \frac{1}{\mu} \mid d^x(u_1,u_2) - d^x(\delta^x_\mu w(u_1), \delta^x_\mu w(u_2)) \mid \leq$$
$$\leq \mathcal{O}(\varepsilon\mu) + \frac{1}{\mu}\mathcal{O}(\varepsilon) + \frac{1}{\mu} d^x(u_1, \delta^x_\mu w(u_1)) + \frac{1}{\mu} d^x(u_1, \delta^x_\mu w(u_2))$$
$$\leq \mathcal{O}(\varepsilon\mu) + \frac{1}{\mu}\mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon)$$

This shows that the property (b) of a metric profile is satisfied. The property (a) is proved in the Theorem 7. □

The following theorem is related to Mitchell [12] Theorem 1, concerning sub-riemannian geometry.

Theorem 7. *In the hypothesis of theorem 6, we have the following limit:*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup \{ \mid d(u,v) - d^x(u,v) \mid : d(x,u) \leq \varepsilon, \ d(x,v) \leq \varepsilon \} = 0$$

Therefore if d^x is a true (i.e. nondegenerate) distance, then (X,d) admits a metric tangent space in x.

Moreover, the metric profile $[\bar{B}_{d^x}(x,1), (\delta,\varepsilon), x]$ is almost nice, in the following sense. Let $c \in (0,1)$. Then we have the inclusion

$$\delta^x_{\mu^{-1}} \left(\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c) \right) \subset \bar{B}_{d^x}(x,1)$$

Moreover, the following Gromov-Hausdorff distance is of order $\mathcal{O}(\varepsilon)$ for μ fixed (that is the modulus of convergence $\mathcal{O}(\varepsilon)$ does not depend on μ):

$$\mu \ d_{GH} \left([\bar{B}_{d^x}(x,1), (\delta^x,\varepsilon), x], \ [\delta^x_{\mu^{-1}} \left(\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c) \right), (\delta^x,\varepsilon\mu), x] \right) = \mathcal{O}(\varepsilon)$$

For another Gromov-Hausdorff distance we have the estimate

$$d_{GH} \left([\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c), \frac{1}{\mu}(\delta^x,\varepsilon), x] \ , \ [\delta^x_{\mu^{-1}} \left(\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c) \right), (\delta^x,\varepsilon\mu), x] \right) = \mathcal{O}(\varepsilon\mu)$$

when $\varepsilon \in (0, \varepsilon(c))$.

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Proof. We start from the Axioms A0, A3 and we use the cone property. By A0, for $\varepsilon \in (0,1)$ and $u, v \in \bar{B}_d(x, \varepsilon)$ there exist $U, V \in \bar{B}_d(x, A)$ such that

$$u = \delta_\varepsilon^x U, v = \delta_\varepsilon^x V.$$

By the cone property we have

$$\frac{1}{\varepsilon} \mid d(u,v) - d^x(u,v) \mid = \left| \frac{1}{\varepsilon} d(\delta_\varepsilon^x U, \delta_\varepsilon^x V) - d^x(U,V) \right|$$

By A2 we have

$$\left| \frac{1}{\varepsilon} d(\delta_\varepsilon^x U, \delta_\varepsilon^x V) - d^x(U,V) \right| \leq \mathcal{O}(\varepsilon)$$

This proves the first part of the theorem.

For the second part of the theorem take any $u \in \bar{B}_{\frac{1}{\mu}(\delta^x, \varepsilon)}(x, c)$. Then we have

$$d^x(x,u) \leq c\mu + \mathcal{O}(\varepsilon)$$

Then there exists $\varepsilon(c) > 0$ such that for any $\varepsilon \in (0, \varepsilon(c))$ and u in the mentioned ball we have

$$d^x(x,u) \leq \mu$$

In this case we can take directly $w(u) = \delta_{\mu^{-1}}^x u$ and simplify the string of inequalities from the proof of Theorem 6, point (b), to get eventually the three points from the second part of the theorem. □

6 Tangent bundle of a dilatation structure

In this section we shall use the calculus with binary decorated trees introduced in section 4, for a space endowed with a dilatation structure.

6.1 Main results

Theorem 8. *Let (X, d, δ) be a dilatation structure. Then the "infinitesimal translations"*

$$L_u^x(v) = \lim_{\varepsilon \to 0} \Delta_\varepsilon^x(u,v)$$

are d^x isometries.

Proof. The first part of the conclusion of Theorem 7 can be written as follows:

$$\sup \left\{ \frac{1}{\varepsilon} \mid d(u,v) - d^x(u,v) \mid : \ d(x,u) \leq \frac{3}{2}\varepsilon, \ d(x,v) \leq \frac{3}{2}\varepsilon \right\} \to 0 \tag{6.1}$$

as $\varepsilon \to 0$.

For $\varepsilon > 0$ sufficiently small the points $x, \delta_\varepsilon^x u, \delta_\varepsilon^x v, \delta_\varepsilon^x w$ are close one to another. Precisely, we have

$$d(\delta_x^\varepsilon u, \delta_x^\varepsilon v) = \varepsilon(d^x(u,v) + \mathcal{O}(\varepsilon))$$

Therefore, if we choose u, v, w such that $d^x(u,v) < 1$ and $d^x(u,w) < 1$, then there is $\eta > 0$ such that for all $\varepsilon \in (0, \eta)$ we have

$$d(\delta_x^\varepsilon u, \delta_x^\varepsilon v) \leq \frac{3}{2}\varepsilon, \qquad d(\delta_x^\varepsilon u, \delta_x^\varepsilon v) \leq \frac{3}{2}\varepsilon$$

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We apply the estimate (6.1) for the basepoint $\delta^x_\varepsilon u$ to get

$$\frac{1}{\varepsilon} \mid d(\delta^x_\varepsilon v, \delta^x_\varepsilon w) - d^{\delta^x_\varepsilon u}(\delta^x_\varepsilon v, \delta^x_\varepsilon w) \mid \to 0$$

when $\varepsilon \to 0$. This can be written, using the cone property of the distance $d^{\delta^x_\varepsilon u}$, like

$$\left| \frac{1}{\varepsilon} d(\delta^x_\varepsilon v, \delta^x_\varepsilon w) - d^{\delta^x_\varepsilon u}\left(\delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \delta^x_\varepsilon v, \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \delta^x_\varepsilon w\right) \right| \to 0 \qquad (6.2)$$

as $\varepsilon \to 0$. By the Axioms A1, A3, the function

$$(x, u, v) \mapsto d^x(u, v)$$

is an uniform limit of continuous functions, therefore uniformly continuous on compact sets. We can pass to the limit in the left hand side of the estimate (6.2), using this uniform continuity and Axioms A3, A4, to get the result. □

Let us define, in agreement with definition 11 (b)

$$\Sigma^x_\varepsilon(u, v) = \delta^x_{\varepsilon^{-1}} \delta^{\delta^x_\varepsilon u}_\varepsilon v$$

Corollary 9. *If for any x the distance d^x is non degenerate then there exists $C > 0$ such that for any x and u with $d(x, u) \leq C$ there exists a d^x isometry $\Sigma^x(u, \cdot)$ obtained as the limit*

$$\lim_{\varepsilon \to 0} \Sigma^x_\varepsilon(u, v) = \Sigma^x(u, v)$$

uniformly with respect to x, u, v in compact set.

Proof. From Theorem 8 we know that $\Delta^x(u, \cdot)$ is a d^x isometry. If d^x is non degenerate then $\Delta^x(u, \cdot)$ is invertible. Let $\Sigma^x(u, \cdot)$ be the inverse.

From Proposition 3 we know that $\Sigma^x_\varepsilon(u, \cdot)$ is the inverse of $\Delta^x_\varepsilon(u, \cdot)$. Therefore

$$\begin{aligned} d^x(\Sigma^x_\varepsilon(u, w), \Sigma^x(u, w)) &= d^x(\Delta^x(u, \Sigma^x_\varepsilon(u, w)), w) \\ &= d^x(\Delta^x(u, \Sigma^x_\varepsilon(u, w)), \Delta^x_\varepsilon(u, \Sigma^x_\varepsilon(u, w)) \end{aligned}$$

From the uniformity of convergence in Theorem 8 and the uniformity assumptions in axioms of dilatation structures, the conclusion follows. □

The next theorem is the generalization of Proposition 2. It is the main result of this paper.

Theorem 10. *Let (X, d, δ) be a dilatation structure which satisfies the strong form of the Axiom A2. Then for any $x \in X$ $(U(x), \Sigma^x, \delta^x)$ is a conical group. Moreover, left translations of this group are d^x isometries.*

Proof. We start by proving that $(U(x), \Sigma^x)$ is a local uniform group. The uniformities are induced by the distance d.

We shall use the general relations written in terms of binary decorated trees. According to relation (4.4) in Proposition 4, we can pass to the limit with $\varepsilon \to 0$ and define

$$inv^x(u) = \lim_{\varepsilon \to 0} \Delta^x_\varepsilon(u, x) = \Delta^x(u, x)$$

From relation (4.5) we get (after passing to the limit with $\varepsilon \to 0$)

$$inv^x(inv^x(u)) = u$$

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We shall see that $inv^x(u)$ is the inverse of u. Relation (4.3) gives

$$\Delta^x(u,v) = \Sigma^x(inv^x(u),v) \tag{6.3}$$

therefore relations (a), (b) from Proposition 3 give

$$\Sigma^x(inv^x(u),\Sigma^x(u,v)) = v \tag{6.4}$$
$$\Sigma^x(u,\Sigma^x(u,v)) = v \tag{6.5}$$

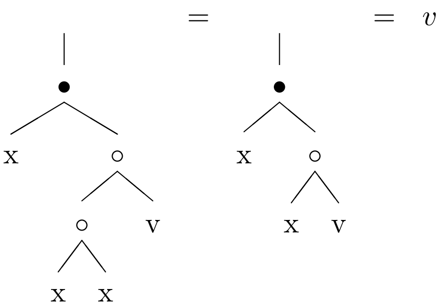
Relation (4.7) from Proposition 5 gives

$$\Sigma^x(u,\Sigma^x(v,w)) = \Sigma^x(\Sigma^x(u,v),w) \tag{6.6}$$

which shows that Σ^x is an associative operation. From (6.5), (6.4) we obtain that for any u, v

$$\Sigma^x(\Sigma^x(inv^x(u),u),v) = v \tag{6.7}$$
$$\Sigma^x(\Sigma^x(u,inv^x(u)),v) = v \tag{6.8}$$

Remark that for any x, v and $\varepsilon \in (0,1)$ we have $\Sigma^x(x,v) = v$. indeed, this means that



Therefore x is a neutral element at left for the operation Σ^x. From the definition of inv^x, relation (6.3) and the fact that inv^x is equal to its inverse, we get that x is an inverse at right too: for any x, v we have

$$\Sigma^x(v,x) = v$$

Replace now v by x in relations (6.7), (6.8) and prove that indeed $inv^x(u)$ is the inverse of u.

We still have to prove that $(U(x),\Sigma^x)$ admits δ^x as dilatations.In this reasoning we need the Axiom A2 in strong form.

Namely we have to prove that for any $\mu \in (0,1)$ we have

$$\delta^x_\mu \Sigma^x(u,v) = \Sigma^x(\delta^x_\mu u, \delta^x_\mu v)$$

For this is sufficient to notice that

$$\Delta^x_\varepsilon\left(\delta^x_\mu u, \delta^x_\mu v\right) = \delta_\mu^{\delta^x_{\varepsilon\mu} u} \Delta^x_{\varepsilon\mu}(u,v)$$

and pass to the limit as $\varepsilon \to 0$. Notice that here we used the fact that dilatations δ^x_ε and δ^x_μ exactly commute (Axiom A2 in strong form).

Finally, left translations L^x_u are d^x isometries. Really, this is a straightforward consequence of Theorem 8 and corollary 9. □

The conical group $(U(x),\Sigma^x,\delta^x)$ can be regarded as the tangent space of (X,δ,d) at x and denoted further by T_xX.

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6.2 Algebraic interpretation

In order to better understand the algebraic structure of the sum, difference, inverse operations induced by a dilatation structure, we collect previous results regarding the properties of these operations, into one place.

Theorem 11. *Let (X, d, δ) be a weak dilatation structure. Then, for any $x \in X$, $\varepsilon \in \Gamma$, $\nu(\varepsilon) < 1$, we have*

(a) For any $u \in U(x)$, $\Sigma_\varepsilon^x(x, u) = u$.

(b) For any $u \in U(x)$ the functions $\Sigma_\varepsilon^x(u, \cdot)$ and $\Delta_\varepsilon^x(u, \cdot)$ are inverse one to another.

(c) The inverse function is shifted involutive: for any $u \in U(x)$,

$$inv_\varepsilon^{\delta_\varepsilon^x u} \, inv_\varepsilon^x(u) = u$$

(d) The sum operation is shifted associative: for any u, v, w sufficiently close to x we have

$$\Sigma_\varepsilon^x \left(u, \Sigma_\varepsilon^{\delta_\varepsilon^x u}(v, w)\right) = \Sigma_\varepsilon^x(\Sigma^x(u, v), w)$$

(e) The difference, inverse and sum operations are related by

$$\Delta_\varepsilon^x(u, v) = \Sigma_\varepsilon^{\delta_\varepsilon^x u} \left(inv_\varepsilon^x(u), v\right)$$

for any u, v sufficiently close to x.

(f) For any u, v sufficiently close to x and $\mu \in \Gamma$, $\nu(\mu) < 1$, we have

$$\Delta_\varepsilon^x \left(\delta_\mu^x u, \delta_\mu^x v\right) = \delta_\mu^{\delta_{\epsilon\mu}^x u} \Delta_{\varepsilon\mu}^x(u, v)$$

7 Dilatation structures and differentiability

7.1 Equivalent dilatation structures

Definition 14. Two dilatation structures (X, δ, d) and $(X, \overline{\delta}, \overline{d})$ are equivalent if

(a) the identity map $id : (X, d) \to (X, \overline{d})$ is bilipschitz and

(b) for any $x \in X$ there are functions P^x, Q^x (defined for $u \in X$ sufficiently close to x) such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \overline{d} \left(\delta_\varepsilon^x u, \overline{\delta}_\varepsilon^x Q^x(u)\right) = 0 \tag{7.1}$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d \left(\overline{\delta}_\varepsilon^x u, \delta_\varepsilon^x P^x(u)\right) = 0 \tag{7.2}$$

uniformly with respect to x, u in compact sets.

Proposition 12. *Two dilatation structures (X, δ, d) and $(X, \overline{\delta}, \overline{d})$ are equivalent if and only if*

(a) the identity map $id : (X, d) \to (X, \overline{d})$ is bilipschitz and

(b) for any $x \in X$ there are functions P^x, Q^x (defined for $u \in X$ sufficiently close to x) such that

$$\lim_{\varepsilon \to 0} \left(\overline{\delta}_\varepsilon^x\right)^{-1} \delta_\varepsilon^x(u) = Q^x(u) \tag{7.3}$$

$$\lim_{\varepsilon \to 0} \left(\delta_\varepsilon^x\right)^{-1} \overline{\delta}_\varepsilon^x(u) = P^x(u) \tag{7.4}$$

uniformly with respect to x, u in compact sets.

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Proof. We make the notations

$$Q^x_\varepsilon(u) = \left(\overline{\delta}^x_\varepsilon\right)^{-1} \delta^x_\varepsilon(u), \qquad P^x_\varepsilon(u) = (\delta^x_\varepsilon)^{-1}\, \overline{\delta}^x_\varepsilon(u)$$

The relation (7.1) is equivalent to

$$\mathcal{O}\varepsilon) + \overline{d}^x \left(Q^x_\varepsilon(u), Q^x(u)\right) \to 0, \qquad \mathcal{O}\varepsilon) + d^x \left(P^x_\varepsilon(u), P^x(u)\right) \to 0$$

as $\varepsilon \to 0$, uniformly with respect to x, u in compact sets. The conclusion follows after passing $\varepsilon \to 0$. □

The next theorem shows a link between the tangent bundles of equivalent dilatation structures.

Theorem 13. *Let (X, δ, d) and $(X, \overline{\delta}, \overline{d})$ be equivalent dilatation structures. Suppose that for any $x \in X$ the distance d^x is non degenerate. Then for any $x \in X$ and any $u, v \in X$ sufficiently close to x we have:*

$$\overline{\Sigma}^x(u, v) = Q^x\left(\Sigma^x\left(P^x(u), P^x(v)\right)\right) \tag{7.5}$$

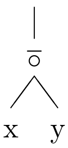
The two tangent bundles are therefore isomorphic in a natural sense.

Proof. We notice first that the hypothesis is symmetric: if d^x is non degenerate then $\overline{d}^x$ is non degenerate too. Really, this is straightforward from definition 14 (a) and Axiom A3 for the two dilatation structures.

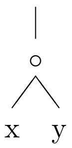
For the proof of relation (7.5) is enough to remark that for $\varepsilon > 0$ but sufficiently small we have

$$\overline{\Sigma}^x_\varepsilon(u, v) = Q^x_\varepsilon\left(\Sigma^x_\varepsilon\left(P^x_\varepsilon(v), P^{\overline{\delta}^x_\varepsilon u}_\varepsilon(v)\right)\right) \tag{7.6}$$

Really, with tree notation, let

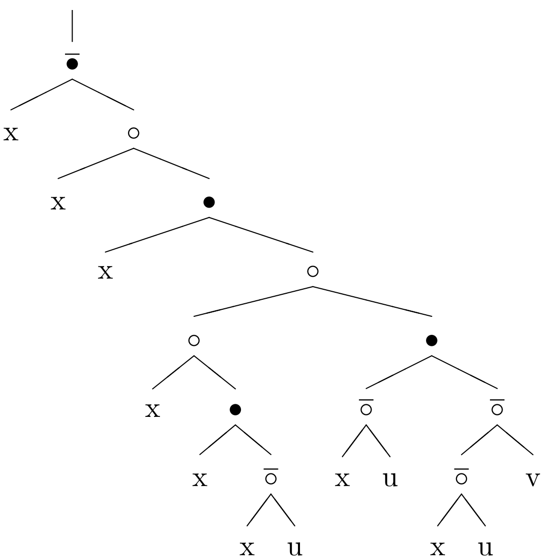


$= \overline{\delta}^x_\varepsilon y,$

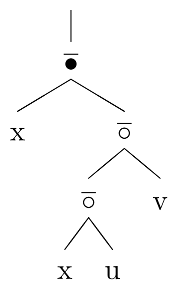


$= \delta^x_\varepsilon y$

The relation (7.6), written from right to left, is



$=$



But this is true by cancellations of dilatations and definitions of the operators P^x_ε and Q^x_ε. □

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7.2 Differentiable functions

Dilatation structures allow to define differentiable functions. The idea is to keep only one relation from definition 14, namely (7.1). We also renounce to uniform convergence with respect to x and u, and we replace this with uniform convergence in u and with a conical group morphism condition for the derivative.

First we need the natural definition below.

Definition 15. Let (N, δ) and $(M, \bar{\delta})$ be two conical groups. A function $f : N \to M$ is a conical group morphism if f is a group morphism and for any $\varepsilon > 0$ and $u \in N$ we have $f(\delta_\varepsilon u) = \bar{\delta}_\varepsilon f(u)$.

The definition of derivative with respect to dilatations structures follows.

Definition 16. Let (X, δ, d) and $(Y, \overline{\delta}, \overline{d})$ be two dilatation structures and $f : X \to Y$ be a continuous function. The function f is differentiable in x if there exists a conical group morphism $Q^x : T_x X \to T_{f(x)} Y$, defined on a neighbourhood of x with values in a neighbourhood of $f(x)$ such that

$$\lim_{\varepsilon \to 0} \sup \left\{ \frac{1}{\varepsilon} \overline{d} \left(f\left(\delta^x_\varepsilon u\right), \overline{\delta}^{f(x)}_\varepsilon Q^x(u) \right) : \, d(x,u) \leq \varepsilon \right\} = 0 \tag{7.7}$$

The morphism Q^x is called the derivative of f at x and will be sometimes denoted by $Df(x)$.

The function f is uniformly differentiable if it is differentiable everywhere and the limit in (7.7) is uniform in x in compact sets.

This definition deserves a short discussion. Let (X, δ, d) and $(Y, \overline{\delta}, \overline{d})$ be two dilatation structures and $f : X \to Y$ a function differentiable in x. The derivative of f in x is a conical group morphism $Df(x) : T_x X \to T_{f(x)} Y$, which means that $Df(x)$ is defined on a open set around x with values in a open set around $f(x)$, having the following properties:

(a) for any u, v sufficiently close to x

$$Df(x)\left(\Sigma^x(u,v)\right) = \Sigma^{f(x)}\left(Df(x)(u), Df(x)(v)\right)$$

(b) for any u sufficiently close to x and any $\varepsilon \in (0,1]$

$$Df(x)\left(\delta^x_\varepsilon u\right) = \bar{\delta}^{f(x)}_\varepsilon \left(Df(x)(u)\right)$$

(c) the function $Df(x)$ is continuous, as uniform limit of continuous functions. Indeed, the relation (7.7) is equivalent to the existence of the uniform limit (with respect to u in compact sets)

$$Df(x)(u) = \lim_{\varepsilon \to 0} \bar{\delta}^{f(x)}_{\varepsilon^{-1}} \left(f\left(\delta^x_\varepsilon u\right)\right)$$

From (7.7) alone and axioms of dilatation structures we can prove properties (b) and (c). We can reformulate therefore the definition of the derivative by asking that $Df(x)$ exists as an uniform limit (as in point (c) above) and that $Df(x)$ has the property (a) above. From these considerations the chain rule for derivatives is straightforward.

A trivial way to obtain a differentiable function (everywhere) is to modify the dilatation structure on the target space.

Definition 17. Let (X, δ, d) be a dilatation structure and $f : (X, d) \to (Y, \overline{d})$ be a bilipschitz and surjective function. We define then the transport of (X, δ, d) by f, named $(Y, f * \delta, \overline{d})$, by

$$(f * \delta)^{f(x)}_\varepsilon f(u) = f\left(\delta^x_\varepsilon u\right)$$

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The relation of differentiability with equivalent dilatation structures is given by the following simple

Proposition 14. *Let (X, δ, d) and $(X, \overline{\delta}, \overline{d})$ be two dilatation structures and $f : (X, d) \rightarrow (X, \overline{d})$ be a bilipschitz and surjective function. The dilatation structures $(X, \overline{\delta}, \overline{d})$ and $(X, f * \delta, \overline{d})$ are equivalent if and only if f and f^{-1} are uniformly differentiable.*

Proof. Straightforward from definitions 14 and 17. □

8 Differential structure, conical groups and dilatation structures

In this section we collect some facts which relate differential structures with dilatation structures. We resume then the paper with a justification of the unusual way of defining uniform groups (definition 7) by the fact that the *op* function (the group operation) is differentiable with respect to dilatation structures which are natural for a group with dilatations.

8.1 Differential structures and dilatation structures

A differential structure on a manifold is an equivalence class of compatible atlases. We show here that an atlas induces an equivalence class of dilatation structures and that two compatible atlases induce the same equivalence class of dilatation structures.

Let M be a $\mathcal{C}^1$ n-dimensional real manifold and $\mathcal{A}$ an atlas of this manifold. For each chart $\phi : W \subset M \rightarrow \mathbb{R}^n$ we shall define a dilatation structure on W.

Suppose that $\phi(W) \subset \mathbb{R}^n$ is convex (if not then take an open subset of W with this property). For $x, u \in W$ and $\varepsilon \in (0, 1]$ define the dilatation

$$\delta_\varepsilon^x u = \phi^{-1} \left(\phi(x) + \varepsilon(\phi(u) - \phi(x)) \right)$$

Otherwise said, the dilatations in W are transported from $\mathbb{R}^n$. Equally, we transport on W the euclidean distance of $\mathbb{R}^n$. We obviously get a dilatation structure on W.

If we have two charts $\phi_i : W_i \subset M \rightarrow \mathbb{R}^n$, $i = 1, 2$, belonging to the same atlas $\mathcal{A}$, then we have two equivalent dilatation structures on $W_1 \cap W_2$. Indeed, the atlas $\mathcal{A}$ is $\mathcal{C}^1$ therefore the distances (induced from the charts) are (locally) in bilipschitz equivalence. Denote by $\overline{\delta}$ the dilatation obtained from the chart ϕ_2. A short computation shows that (we use here the transition map $\phi_{21} = \phi_2 (\phi_1)^{-1}$)

$$Q_\varepsilon^x(u) = (\phi_2)^{-1} \left(\phi_2(x) + \frac{1}{\varepsilon} \left(\phi_{21} \left(\phi_1(x) + \varepsilon(f(u) - f(x)) \right) - \phi_2(x) \right) \right)$$

therefore, as $\varepsilon \rightarrow 0$, we have

$$\lim_{\varepsilon \rightarrow 0} Q_\varepsilon^x(u) = Q^x(u) = (\phi_2)^{-1} \left(\phi_2(x) + D\phi_{21}(f(x))(f(u) - f(x)) \right)$$

A similar computation shows that P^x also exists. The uniform convergence requirements come from the fact that we use a $\mathcal{C}^1$ atlas.

A similar reasoning shows that in fact two compatible atlases induce the same equivalence class of dilatation structures.

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8.2 Conical groups and dilatation structures

In a group with dilatations (G, δ) we define dilatations based in any point $x \in G$ by

$$\delta_\varepsilon^x u = x \delta_\varepsilon (x^{-1} u) \quad (8.1)$$

Definition 18. A normed group with dilatations $(G, \delta, \| \cdot \|)$ is a group with dilatations (G, δ) endowed with a continuous norm function $\| \cdot \| : G \to \mathbb{R}$ which satisfies (locally, in a neighbourhood of the neutral element e) the following properties:

(a) for any x we have $\|x\| \geq 0$; if $\|x\| = 0$ then $x = e$,
(b) for any x, y we have $\|xy\| \leq \|x\| + \|y\|$,
(c) for any x we have $\|x^{-1}\| = \|x\|$,
(d) the limit $\lim_{\varepsilon \to 0} \frac{1}{\nu(\varepsilon)} \| \delta_\varepsilon x \| = \|x\|^N$ exists, is uniform with respect to x in compact set,
(e) if $\|x\|^N = 0$, then $x = e$.

It is easy to see that if $(G, \delta, \| \cdot \|)$ is a normed group with dilatations then $(G, \beta, \delta, \| \cdot \|^N)$ is a normed conical group. The norm $\| \cdot \|^N$ satisfies the stronger form of property (d) of Definition 18: for any $\varepsilon > 0$, $\| \delta_\varepsilon x \|^N = \varepsilon \|x\|^N$.

Normed groups with dilatations can be encountered in sub-Riemannian geometry. Normed conical groups generalize the notion of Carnot groups.

In a normed group with dilatations we have a natural left invariant distance given by

$$d(x, y) = \|x^{-1} y\| \quad (8.2)$$

Theorem 15. *Let $(G, \delta, \| \cdot \|)$ be a locally compact normed group with dilatations. Then (G, δ, d) is a dilatation structure, where δ are the dilatations defined by (8.1) and the distance d is induced by the norm as in (8.2).*

Proof. The Axiom A0 is straightforward from definition 7, definition 8, Axiom H0, and because the dilatation structure is left invariant, in the sense that the transport by left translations in G, according to Definition 17, preserves the dilatations δ. We also trivially have Axioms A1 and A2 satisfied.

For the Axiom A3 remark that

$$d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) = d(x \delta_\varepsilon (x^{-1} u), x \delta_\varepsilon (x^{-1} u)) = d(\delta_\varepsilon (x^{-1} u), \delta_\varepsilon (x^{-1} v))$$

Denote $U = x^{-1} u$, $V = x^{-1} v$ and for $\varepsilon > 0$ let

$$\beta_\varepsilon (u, v) = \delta_\varepsilon^{-1} \left((\delta_\varepsilon u)(\delta_\varepsilon v) \right)$$

We have then:

$$\frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) = \frac{1}{\varepsilon} \| \delta_\varepsilon \beta_\varepsilon \left(\delta_\varepsilon^{-1} \left((\delta_\varepsilon V)^{-1} \right), U \right) \|$$

Define the function

$$d^x(u, v) = \| \beta(V^{-1}, U) \|^N$$

From Definition 8 Axioms H1, H2, and from definition 18 (d), we obtain that Axiom A3 is satisfied.

For the Axiom A4 we have to calculate

$$\begin{aligned} \Delta^x(u, v) &= \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v \\ &= (\delta_\varepsilon^x u) (\delta_\varepsilon)^{-1} \left((\delta_\varepsilon^x u)^{-1} (\delta_\varepsilon^x v) \right) \\ &= (x \delta_\varepsilon U) \beta_\varepsilon \left(\delta_\varepsilon^{-1} \left((\delta_\varepsilon V)^{-1} \right), U \right) \to x \beta \left(V^{-1}, U \right) \end{aligned}$$

as $\varepsilon \to 0$. Therefore the Axiom A4 is satisfied. □

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We remarked in the proof of the previous theorem that the transport by left translations in G, according to Definition 17, preserves the dilatation structure on G. This implies, according to Proposition 14, that left translations are differentiable. On the contrary, a short computation and examples from sub-Riemannian geometry indicate that right translations are not differentiable.

Nevertheless, the operation op is differentiable, if we endow the group $G^{(2)} = G \times G$ with a good dilatation structure. This will justify the non standard way to define local uniform groups in Definition 7.

Start from the fact that if G is a local uniform group then $G^{(2)}$ is a local uniform group too. If G is also normed, with dilatations, then we can easily define a similar structure on $G^{(2)}$. Really, the norm on $G^{(2)}$ can be taken as

$$\|(x,y)\|^{(2)} = \max\left\{\|x\|, \|y\|\right\}$$

and dilatations

$$\delta^{(2)}_{\varepsilon}(x,y) = (\delta_{\varepsilon} x, \delta_{\varepsilon} y)$$

We leave to the reader to check that $G^{(2)}$ endowed with this norm and these dilatations is indeed a normed group with dilatations.

Theorem 16. *Let $(G, \delta, \| \cdot \|)$ be a locally compact normed group with dilatations and let $(G^{(2)}, \delta^{(2)}, \| \cdot \|^{(2)})$ be the associated normed group with dilatation. Then the operation (op function) is differentiable.*

Proof. We start from the formula (easy to check in $G^{(2)}$)

$$(x,y)^{-1} = (x^{-1}, xy^{-1}x^{-1})$$

Then we have

$$\delta_{\varepsilon}^{(x,y)}(u,v) = \left(x\delta_{\varepsilon}(x^{-1}u), \left(\delta_{\varepsilon}(x^{-1}u)\right)^{-1} y\delta_{\varepsilon}(x^{-1}u)\delta_{\varepsilon}\left(u^{-1}xy^{-1}x^{-1}uv\right)\right)$$

Let us define

$$Q^{(x,y)}(u,v) = op(x,y)\beta((x,y)^{-1}(u,v))$$

Then we have

$$\frac{1}{\varepsilon}d\left(op\left(\delta^{(x,y)}(u,v)\right), \delta^{op(x,y)}Q^{(x,y)}(u,v)\right) = \frac{1}{\varepsilon}d\left(\delta_{\varepsilon}\beta_{\varepsilon}((x,y)^{-1}(u,v)), \delta_{\varepsilon}\beta((x,y)^{-1}(u,v))\right)$$

The right hand side of this equality converges then to 0 as $\varepsilon \to 0$. More precisely, we have

$$\sup\left\{\frac{1}{\varepsilon}d\left(op\left(\delta^{(x,y)}(u,v)\right), \delta^{op(x,y)}Q^{(x,y)}(u,v)\right) : \ ^{(2)}((x,y),(u,v)) \leq \varepsilon\right\} =$$

$$= \sup\left\{d^{e}\left(\beta_{\varepsilon}((x,y)^{-1}(u,v)), \beta((x,y)^{-1}(u,v))\right) : \ d^{(2)}((x,y),(u,v)) \leq \varepsilon\right\} + \mathcal{O}(\varepsilon) \qquad \square$$

In particular, we have $Q^{(e,e)}(u,v) = \beta(u,v)$, which shows that the operation β is the differential of the operation op calculated in the neutral element of $G^{(2)}$.

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Contractible groups and linear dilatation structures

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Abstract

A dilatation structure on a metric space, is a notion in between a group and a differential structure. The basic objects of a dilatation structure are dilatations (or contractions). The axioms of a dilatation structure set the rules of interaction between different dilatations.

There are two notions of linearity associated to dilatation structures: the linearity of a function between two dilatation structures and the linearity of the dilatation structure itself.

Our main result here is a characterization of contractible groups in terms of dilatation structures. To a normed conical group (normed contractible group) we can naturally associate a linear dilatation structure. Conversely, any linear and strong dilatation structure comes from the dilatation structure of a normed contractible group.

Keywords: contractible groups, Carnot groups, dilatation structures, metric tangent spaces

MSC classes: 22E20; 20E36; 20F65; 22A10; 51F99

Contents

1 Introduction

Dilatation structures on metric spaces, introduced in [3], describe the approximate self-similarity properties of a metric space. A dilatation structure is a notion related, but more general, to groups and differential structures.

The basic objects of a dilatation structure are dilatations (or contractions). The axioms of a dilatation structure set the rules of interaction between different dilatations.

A metric space (X, d) which admits a strong dilatation structure (definition 3.2) has a metric tangent space at any point $x \in X$ (theorem 4.2), and any such metric tangent space has an algebraic structure of a conical group (theorem 4.3). Conical groups are particular examples of contraction groups. The structure of contraction groups is known in some detail, due to Siebert [9], Wang [10], Glöckner and Willis [6], Glöckner [5] and references therein.

By a classical result of Siebert [9] proposition 5.4, we can characterize the algebraic structure of the metric tangent spaces associated to dilatation structures of a certain kind: they are Carnot groups, that is simply connected Lie groups whose Lie algebra admits a positive graduation (corollary 4.7).

Carnot groups appear in many situations, in particular in relation with sub-riemannian geometry cf. Bellaïche [1], groups with polynomial growth cf. Gromov [7], or Margulis type rigidity results cf. Pansu [8]. It is part of the author program of research to show that dilatation structures are natural objects in all these mathematical subjects. In this respect the corollary 4.7 represents a generalization of difficult results in sub-riemannian geometry concerning the structure of the metric tangent space at a point of a regular subriemannian manifold.

Linearity is also a property which can be explained with the help of a dilatation structure. In the second section of the paper we explain why linearity can be casted in terms of dilatations. There are in fact two kinds of linearity: the linearity of a function between two dilatation structures (definition 5.1) and the linearity of the dilatation structure itself (definition 5.7).

Our main result here is a characterization of contraction groups in terms of dilatation structures. To a normed conical group (normed contraction group) we can naturally associate a linear dilatation structure (proposition 5.8). Conversely, by theorem 5.11 any linear and strong dilatation structure comes from the dilatation structure of a normed contraction group.

2 Linear structure in terms of dilatations

Linearity is a basic property related to vector spaces. For example, if $\mathbb{V}$ is a real, finite dimensional vector space then a transformation $A : \mathbb{V} \rightarrow \mathbb{V}$ is linear if it is a morphism of groups $A : (\mathbb{V}, +) \rightarrow (\mathbb{V}, +)$ and homogeneous with respect to positive scalars. Furthermore, in a normed vector space we can speak about linear continuous transformations.

A transformation is affine if it is a composition of a translation with a linear transformation. In this paper we shall use the umbrella name "linear" for affine transformations too.

We try here to explain that linearity property can be entirely phrased in terms of dilatations of the vector space $\mathbb{V}$.

For the vector space $\mathbb{V}$, the dilatation based at $x \in \mathbb{V}$, of coefficient $\varepsilon > 0$, is the function

$$\delta^x_\varepsilon : \mathbb{V} \rightarrow \mathbb{V} \quad , \quad \delta^x_\varepsilon y = x + \varepsilon(-x + y) \quad .$$

For fixed x the dilatations based at x form a one parameter group which contracts any bounded neighbourhood of x to a point, uniformly with respect to x.

The algebraic structure of $\mathbb{V}$ is encoded in dilatations. Indeed, using dilatations we can recover the operation of addition and multiplication by scalars.

For $x, u, v \in \mathbb{V}$ and $\varepsilon > 0$ we define the following compositions of dilatations:

$$\Delta_\varepsilon^x(u,v) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v \quad , \quad \Sigma_\varepsilon^x(u,v) = \delta_{\varepsilon^{-1}}^x \delta_\varepsilon^{\delta_\varepsilon^x u}(v) \quad , \quad inv_\varepsilon^x(u) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} x \quad . \tag{2.0.1}$$

The meaning of this functions becomes clear if we compute:

$$\Delta_\varepsilon^x(u,v) = x + \varepsilon(-x+u) + (-u+v) \quad ,$$
$$\Sigma_\varepsilon^x(u,v) = u + \varepsilon(-u+x) + (-x+v) \quad ,$$
$$inv_\varepsilon^x(u) == x + \varepsilon(-x+u) + (-u+x) \quad .$$

As $\varepsilon \to 0$ we have the limits:

$$\lim_{\varepsilon\to 0} \Delta_\varepsilon^x(u,v) = \Delta^x(u,v) = x + (-u+v) \quad ,$$
$$\lim_{\varepsilon\to 0} \Sigma_\varepsilon^x(u,v) = \Sigma^x(u,v) = u + (-x+v) \quad ,$$
$$\lim_{\varepsilon\to 0} inv_\varepsilon^x(u) = inv^x(u) = x - u + x \quad ,$$

uniform with respect to x, u, v in bounded sets. The function $\Sigma^x(\cdot,\cdot)$ is a group operation, namely the addition operation translated such that the neutral element is x. Thus, for $x = 0$, we recover the usual addition operation. The function $inv^x(\cdot)$ is the inverse function with respect to addition, and $\Delta^x(\cdot,\cdot)$ is the difference function.

Notice that for fixed x, ε the function $\Sigma_\varepsilon^x(\cdot,\cdot)$ is not a group operation, first of all because it is not associative. Nevertheless, this function satisfies a shifted associativity property, namely (see theorem 4.1)

$$\Sigma_\varepsilon^x(\Sigma_\varepsilon^x(u,v),w) = \Sigma_\varepsilon^x(u, \Sigma_\varepsilon^{\delta_\varepsilon^x u}(v,w)) \quad .$$

Also, the inverse function inv_ε^x is not involutive, but shifted involutive (theorem 4.1):

$$inv_\varepsilon^{\delta_\varepsilon^x u}(inv_\varepsilon^x u) = u \quad .$$

Affine continuous transformations $A : \mathbb{V} \to \mathbb{V}$ admit the following description in terms of dilatations. (We could dispense of continuity hypothesis in this situation, but we want to illustrate a general point of view, described further in the paper).

Proposition 2.1 *A continuous transformation $A : \mathbb{V} \to \mathbb{V}$ is affine if and only if for any $\varepsilon \in (0,1)$, $x, y \in \mathbb{V}$ we have*

$$A\delta_\varepsilon^x y \; = \; \delta_\varepsilon^{Ax} Ay \quad . \tag{2.0.2}$$

The proof is a straightforward consequence of representation formulæ (2.0.1) for the addition, difference and inverse operations in terms of dilatations.

3 Dilatation structures

We present here a brief introduction into the subject of dilatation structures. For more details see Buliga [3]. The results with proofs are new.

3.1 Notations

Let Γ be a topological separated commutative group endowed with a continuous group morphism

$$\nu : \Gamma \to (0, +\infty)$$

with $\inf \nu(\Gamma) = 0$. Here $(0, +\infty)$ is taken as a group with multiplication. The neutral element of Γ is denoted by 1. We use the multiplicative notation for the operation in Γ.

The morphism ν defines an invariant topological filter on Γ (equivalently, an end). Indeed, this is the filter generated by the open sets $\nu^{-1}(0, a)$, $a > 0$. From now on we shall name this topological filter (end) by "0" and we shall write $\varepsilon \in \Gamma \to 0$ for $\nu(\varepsilon) \in (0, +\infty) \to 0$.

The set $\Gamma_1 = \nu^{-1}(0, 1]$ is a semigroup. We note $\bar{\Gamma}_1 = \Gamma_1 \cup \{0\}$ On the set $\bar{\Gamma} = \Gamma \cup \{0\}$ we extend the operation on Γ by adding the rules $00 = 0$ and $\varepsilon 0 = 0$ for any $\varepsilon \in \Gamma$. This is in agreement with the invariance of the end 0 with respect to translations in Γ.

The space (X, d) is a complete, locally compact metric space. For any $r > 0$ and any $x \in X$ we denote by $B(x, r)$ the open ball of center x and radius r in the metric space X.

On the metric space (X, d) we work with the topology (and uniformity) induced by the distance. For any $x \in X$ we denote by $\mathcal{V}(x)$ the topological filter of open neighbourhoods of x.

3.2 Axioms of dilatation structures

The axioms of a dilatation structure (X, d, δ) are listed further. The first axiom is merely a preparation for the next axioms. That is why we counted it as axiom 0.

A0. The dilatations

$$\delta^x_\varepsilon : U(x) \to V_\varepsilon(x)$$

are defined for any $\varepsilon \in \Gamma, \nu(\varepsilon) \leq 1$. The sets $U(x), V_\varepsilon(x)$ are open neighbourhoods of x. All dilatations are homeomorphisms (invertible, continuous, with continuous inverse).

We suppose that there is a number $1 < A$ such that for any $x \in X$ we have

$$\bar{B}_d(x, A) \subset U(x) \ .$$

We suppose that for all $\varepsilon \in \Gamma$, $\nu(\varepsilon) \in (0,1)$, we have

$$B_d(x, \nu(\varepsilon)) \subset \delta^x_\varepsilon B_d(x, A) \subset V_\varepsilon(x) \subset U(x) \ .$$

There is a number $B \in (1, A]$ such that for any $\varepsilon \in \Gamma$ with $\nu(\varepsilon) \in (1, +\infty)$ the associated dilatation

$$\delta^x_\varepsilon : W_\varepsilon(x) \to B_d(x, B) \ ,$$

is injective, invertible on the image. We shall suppose that $W_\varepsilon(x) \in \mathcal{V}(x)$, that $V_{\varepsilon^{-1}}(x) \subset W_\varepsilon(x)$ and that for all $\varepsilon \in \Gamma_1$ and $u \in U(x)$ we have

$$\delta^x_{\varepsilon^{-1}} \ \delta^x_\varepsilon u \ = \ u \ .$$

We have therefore the following string of inclusions, for any $\varepsilon \in \Gamma$, $\nu(\varepsilon) \leq 1$, and any $x \in X$:

$$B_d(x, \nu(\varepsilon)) \subset \delta^x_\varepsilon B_d(x, A) \subset V_\varepsilon(x) \subset W_{\varepsilon^{-1}}(x) \subset \delta^x_\varepsilon B_d(x, B) \quad .$$

A further technical condition on the sets $V_\varepsilon(x)$ and $W_\varepsilon(x)$ will be given just before the axiom A4. (This condition will be counted as part of axiom A0.)

A1. We have $\delta^x_\varepsilon x = x$ for any point x. We also have $\delta^x_1 = id$ for any $x \in X$.

Let us define the topological space

$$dom\, \delta = \{(\varepsilon, x, y) \in \Gamma \times X \times X : \quad \text{if } \nu(\varepsilon) \leq 1 \text{ then } y \in U(x) \ ,$$
$$\text{else } y \in W_\varepsilon(x)\}$$

with the topology inherited from the product topology on $\Gamma \times X \times X$. Consider also $Cl(dom\, \delta)$, the closure of $dom\, \delta$ in $\bar{\Gamma} \times X \times X$ with product topology. The function $\delta : dom\, \delta \to X$ defined by $\delta(\varepsilon, x, y) = \delta^x_\varepsilon y$ is continuous. Moreover, it can be continuously extended to $Cl(dom\, \delta)$ and we have

$$\lim_{\varepsilon \to 0} \delta^x_\varepsilon y \ = \ x \quad .$$

A2. For any $x, \in K$, $\varepsilon, \mu \in \Gamma_1$ and $u \in \bar{B}_d(x, A)$ we have:

$$\delta^x_\varepsilon \delta^x_\mu u = \delta^x_{\varepsilon\mu} u \ .$$

A3. For any x there is a function $(u,v) \mapsto d^x(u,v)$, defined for any u,v in the closed ball (in distance d) $\bar{B}(x,A)$, such that

$$\lim_{\varepsilon \to 0} \quad \sup \left\{ \mid \frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) \ - \ d^x(u,v) \mid \ : \ u,v \in \bar{B}_d(x,A) \right\} \ = \ 0$$

uniformly with respect to x in compact set.

Remark 3.1 *The "distance" d^x can be degenerated: there might exist $v,w \in U(x)$ such that $d^x(v,w) = 0$.*

For the following axiom to make sense we impose a technical condition on the co-domains $V_\varepsilon(x)$: for any compact set $K \subset X$ there are $R = R(K) > 0$ and $\varepsilon_0 = \varepsilon(K) \in (0,1)$ such that for all $u,v \in \bar{B}_d(x,R)$ and all $\varepsilon \in \Gamma$, $\nu(\varepsilon) \in (0,\varepsilon_0)$, we have

$$\delta_\varepsilon^x v \in W_{\varepsilon^{-1}}(\delta_\varepsilon^x u) \ .$$

With this assumption the following notation makes sense:

$$\Delta_\varepsilon^x(u,v) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v.$$

The next axiom can now be stated:

A4. We have the limit

$$\lim_{\varepsilon \to 0} \Delta_\varepsilon^x(u,v) = \Delta^x(u,v)$$

uniformly with respect to x,u,v in compact set.

Definition 3.2 *A triple (X,d,δ) which satisfies A0, A1, A2, A3, but d^x is degenerate for some $x \in X$, is called degenerate dilatation structure.*

If the triple (X,d,δ) satisfies A0, A1, A2, A3 and d^x is non-degenerate for any $x \in X$, then we call it a dilatation structure.

If a dilatation structure satisfies A4 then we call it strong dilatation structure.

3.3 Groups with dilatations. Conical groups

Metric tangent spaces sometimes have a group structure which is compatible with dilatations. This structure, of a group with dilatations, is interesting by itself. The notion has been introduced in [2]; we describe it further.

The following description of local uniform groups is slightly non canonical, but is motivated by the case of a Lie group endowed with a Carnot-Caratheodory distance induced by a left invariant distribution (see for example [2]).

We begin with some notations. Let G be a group. We introduce first the double of G, as the group $G^{(2)} = G \times G$ with operation

$$(x,u)(y,v) = (xy, y^{-1}uyv) \quad .$$

The operation on the group G, seen as the function $op : G^{(2)} \rightarrow G$, $op(x,y) = xy$ is a group morphism. Also the inclusions:

$$i' : G \rightarrow G^{(2)} \ , \quad i'(x) = (x,e)$$

$$i" : G \rightarrow G^{(2)} \ , \quad i"(x) = (x,x^{-1})$$

are group morphisms.

Definition 3.3 *1. G is an uniform group if we have two uniformity structures, on G and $G \times G$, such that op, i', $i"$ are uniformly continuous.*

2. A local action of a uniform group G on a uniform pointed space (X, x_0) is a function $\phi \in W \in \mathcal{V}(e) \mapsto \hat{\phi} : U_\phi \in \mathcal{V}(x_0) \rightarrow V_\phi \in \mathcal{V}(x_0)$ such that:

(a) the map $(\phi, x) \mapsto \hat{\phi}(x)$ is uniformly continuous from $G \times X$ (with product uniformity) to X,

(b) for any $\phi, \psi \in G$ there is $D \in \mathcal{V}(x_0)$ such that for any $x \in D$ $\phi\hat{\psi}^{-1}(x)$ and $\hat{\phi}(\hat{\psi}^{-1}(x))$ make sense and $\phi\hat{\psi}^{-1}(x) = \hat{\phi}(\hat{\psi}^{-1}(x))$.

3. Finally, a local group is an uniform space G with an operation defined in a neighbourhood of $(e,e) \subset G \times G$ which satisfies the uniform group axioms locally.

An uniform group, according to the definition (3.3), is a group G such that left translations are uniformly continuous functions and the left action of G on itself is uniformly continuous too.

Definition 3.4 *A group with dilatations (G, δ) is a local uniform group G with a local action of Γ (denoted by δ), on G such that*

H0. the limit $\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon x = e$ exists and is uniform with respect to x in a compact neighbourhood of the identity e.

H1. the limit

$$\beta(x,y) = \lim_{\varepsilon \rightarrow 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)(\delta_\varepsilon y) \right)$$

is well defined in a compact neighbourhood of e and the limit is uniform.

H2. *the following relation holds*

$$\lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)^{-1} \right) = x^{-1}$$

where the limit from the left hand side exists in a neighbourhood of e *and is uniform with respect to* x.

These axioms are in fact a particular version of the axioms for a dilatation structure.

Definition 3.5 *A (local) conical group* N *is a (local) group with a (local) action of* Γ *by morphisms* δ_ε *such that* $\lim_{\varepsilon \to 0} \delta_\varepsilon x = e$ *for any* x *in a neighbourhood of the neutral element* e.

A conical group is the infinitesimal version of a group with dilatations ([3] proposition 2).

Proposition 3.6 *Under the hypotheses H0, H1, H2* (G, β, δ) *is a conical group, with operation* β *and dilatations* δ.

Any group with dilatations has an associated dilatation structure on it. In a group with dilatations (G, δ) we define dilatations based in any point $x \in G$ by

$$\delta_\varepsilon^x u = x \delta_\varepsilon (x^{-1} u). \tag{3.3.1}$$

Definition 3.7 *A normed group with dilatations* $(G, \delta, \| \cdot \|)$ *is a group with dilatations* (G, δ) *endowed with a continuous norm function* $\| \cdot \| : G \to \mathbb{R}$ *which satisfies (locally, in a neighbourhood of the neutral element* e*) the properties:*

(a) *for any* x *we have* $\|x\| \geq 0$*; if* $\|x\| = 0$ *then* $x = e$,

(b) *for any* x, y *we have* $\|xy\| \leq \|x\| + \|y\|$,

(c) *for any* x *we have* $\|x^{-1}\| = \|x\|$,

(d) *the limit* $\lim_{\varepsilon \to 0} \frac{1}{\nu(\varepsilon)} \|\delta_\varepsilon x\| = \|x\|^N$ *exists, is uniform with respect to* x *in compact set,*

(e) *if* $\|x\|^N = 0$ *then* $x = e$.

It is easy to see that if $(G, \delta, \| \cdot \|)$ is a normed group with dilatations then $(G, \beta, \delta, \| \cdot \|^N)$ is a normed conical group. The norm $\| \cdot \|^N$ satisfies the stronger form of property (d) definition 3.7: for any $\varepsilon > 0$

$$\| \delta_\varepsilon x \|^N = \varepsilon \| x \|^N \quad .$$

In a normed group with dilatations we have a natural left invariant distance given by

$$d(x, y) = \| x^{-1} y \| \quad . \tag{3.3.2}$$

The following result is theorem 15 [3].

Theorem 3.8 *Let $(G, \delta, \| \cdot \|)$ be a locally compact normed group with dilatations. Then (G, δ, d) is a dilatation structure, where δ are the dilatations defined by (3.3.1) and the distance d is induced by the norm as in (3.3.2).*

3.4 Carnot groups

Normed conical groups generalize the notion of Carnot groups. A simply connected Lie group whose Lie algebra admits a positive graduation is also called a Carnot group. It is in particular nilpotent. Such objects appear in sub-riemannian geometry as models of tangent spaces, cf. [1], [7], [8].

Definition 3.9 *A Carnot (or stratified nilpotent) group is a pair (N, V_1) consisting of a real connected simply connected group N with a distinguished subspace V_1 of the Lie algebra $Lie(N)$, such that the following direct sum decomposition occurs:*

$$n \; = \; \sum_{i=1}^{m} V_i \; , \quad V_{i+1} \; = \; [V_1, V_i]$$

The number m is the step of the group. The number $Q \; = \; \sum_{i=1}^{m} i \; dim V_i$ is called the homogeneous dimension of the group.

Because the group is nilpotent and simply connected, the exponential mapping is a diffeomorphism. We shall identify the group with the algebra, if is not locally otherwise stated.

The structure that we obtain is a set N endowed with a Lie bracket and a group multiplication operation, related by the Baker-Campbell-Hausdorff formula. Remark that the group operation is polynomial.

Any Carnot group admits a one-parameter family of dilatations. For any $\varepsilon > 0$, the associated dilatation is:

$$x = \sum_{i=1}^{m} x_i \mapsto \delta_\varepsilon x = \sum_{i=1}^{m} \varepsilon^i x_i$$

Any such dilatation is a group morphism and a Lie algebra morphism.

In fact the class of Carnot groups is characterised by the existence of dilatations (see Folland-Stein [4], section 1).

Proposition 3.10 *Suppose that the Lie algebra $\mathfrak{g}$ admits an one parameter group $\varepsilon \in (0, +\infty) \mapsto \delta_\varepsilon$ of simultaneously diagonalisable Lie algebra isomorphisms. Then $\mathfrak{g}$ is the algebra of a Carnot group.*

We shall construct a norm on a Carnot group N. First pick an euclidean norm $\| \cdot \|$ on V_1. We shall endow the group N with a structure of a sub-Riemannian manifold now. For this take the distribution obtained from left translates of the space V_1. The metric on that distribution is obtained by left translation of the inner product restricted to V_1.

Because V_1 generates (the algebra) N then any element $x \in N$ can be written as a product of elements from V_1. An useful lemma is the following (slight reformulation of Lemma 1.40, Folland, Stein [4]).

Lemma 3.11 *Let N be a Carnot group and $X_1, ..., X_p$ an orthonormal basis for V_1. Then there is a a natural number M and a function $g : \{1, ..., M\} \rightarrow \{1, ..., p\}$ such that any element $x \in N$ can be written as:*

$$x = \prod_{i=1}^{M} \exp(t_i X_{g(i)}) \qquad (3.4.3)$$

Moreover, if x is sufficiently close (in Euclidean norm) to 0 then each t_i can be chosen such that $\mid t_i \mid \leq C \|x\|^{1/m}$

As a consequence we get:

Corollary 3.12 *The Carnot-Carathéodory distance*

$$d(x,y) = \inf \left\{ \int_0^1 \|c^{-1}\dot{c}\| \, dt \ : \ c(0) = x, \ c(1) = y, \right.$$

$$\left. c^{-1}(t)\dot{c}(t) \in V_1 \text{ for a.e. } t \in [0,1] \right\}$$

is finite for any two $x, y \in N$. The distance is obviously left invariant, thus it induces a norm on N.

3.5 Contractible groups

Conical groups are particular examples of (local) contraction groups.

Definition 3.13 *A contraction group is a pair (G, α), where G is a topological group with neutral element denoted by e, and $\alpha \in Aut(G)$ is an automorphism of G such that:*

- *α is continuous, with continuous inverse,*
- *for any $x \in G$ we have the limit $\lim_{n\to\infty} \alpha^n(x) = e$.*

We shall be interested in locally compact contraction groups (G, α), such that α is compactly contractive, that is: for each compact set $K \subset G$ and open set $U \subset G$, with $e \in U$, there is $N(K, U) \in \mathbb{N}$ such that for any $x \in K$ and $n \in \mathbb{N}$, $n \geq N(K, U)$, we have $\alpha^n(x) \in U$. If G is locally compact then a necessary and sufficient condition for (G, α) to be compactly contractive is: α is an uniform contraction, that is each identity neighbourhood of G contains an α-invariant neighbourhood.

A conical group is an example of a locally compact, compactly contractive, contraction group. Indeed, it suffices to associate to a conical group (G, δ) the contraction group (G, δ_ε), for a fixed $\varepsilon \in \Gamma$ with $\nu(\varepsilon) < 1$.

Conversely, to any contraction group (G, α), which is locally compact and compactly contractive, associate the conical group (G, δ), with $\Gamma = \left\{ \frac{1}{2^n} \,:\, n \in \mathbb{N} \right\}$ and for any $n \in \mathbb{N}$ and $x \in G$

$$\delta_{\frac{1}{2^n}} x \;=\; \alpha^n(x) \quad .$$

Finally, a local conical group has only locally the structure of a contraction group. The structure of contraction groups is known in some detail, due to Siebert [9], Wang [10], Glöckner and Willis [6], Glöckner [5] and references therein.

For this paper the following results are of interest. We begin with the definition of a contracting automorphism group [9], definition 5.1.

Definition 3.14 *Let G be a locally compact group. An automorphism group on G is a family $T = (\tau_t)_{t>0}$ in $Aut(G)$, such that $\tau_t\,\tau_s = \tau_{ts}$ for all $t, s > 0$.*

The contraction group of T is defined by

$$C(T) \;=\; \left\{ x \in G \,:\, \lim_{t\to 0} \tau_t(x) = e \right\} \quad .$$

The automorphism group T is contractive if $C(T) = G$.

It is obvious that a contractive automorphism group T induces on G a structure of conical group. Conversely, any conical group with $\Gamma = (0, +\infty)$ has an associated contractive automorphism group (the group of dilatations based at the neutral element).

Further is proposition 5.4 [9].

Proposition 3.15 *For a locally compact group G the following assertions are equivalent:*

(i) G admits a contractive automorphism group;

(ii) G is a simply connected Lie group whose Lie algebra admits a positive graduation.

4 Properties of dilatation structures

4.1 First properties

The sum, difference, inverse operations induced by a dilatation structure give to the space X almost the structure of an affine space. We collect some results from [3] section 4.2 , regarding the properties of these operations.

Theorem 4.1 *Let (X, d, δ) be a dilatation structure. Then, for any $x \in X$, $\varepsilon \in \Gamma$, $\nu(\varepsilon) < 1$, we have:*

(a) for any $u \in U(x)$, $\Sigma_{\varepsilon}^{x}(x, u) = u$.

(b) for any $u \in U(x)$ the functions $\Sigma_{\varepsilon}^{x}(u, \cdot)$ and $\Delta_{\varepsilon}^{x}(u, \cdot)$ are inverse one to another.

(c) the inverse function is shifted involutive: for any $u \in U(x)$,

$$inv_{\varepsilon}^{\delta_{\varepsilon}^{x} u} \, inv_{\varepsilon}^{x}(u) = u \quad .$$

(d) the sum operation is shifted associative: for any u, v, w sufficiently close to x we have

$$\Sigma_{\varepsilon}^{x} \left(u, \Sigma_{\varepsilon}^{\delta_{\varepsilon}^{x} u}(v, w) \right) = \Sigma_{\varepsilon}^{x}(\Sigma^{x}(u, v), w) \quad .$$

(e) the difference, inverse and sum operations are related by

$$\Delta_{\varepsilon}^{x}(u, v) = \Sigma_{\varepsilon}^{\delta_{\varepsilon}^{x} u} \left(inv_{\varepsilon}^{x}(u), v \right) \quad ,$$

for any u, v sufficiently close to x.

(f) for any u, v sufficiently close to x and $\mu \in \Gamma$, $\nu(\mu) < 1$, we have:

$$\Delta_{\varepsilon}^{x}\left(\delta_{\mu}^{x} u, \delta_{\mu}^{x} v\right) = \delta_{\mu}^{\delta_{\varepsilon\mu}^{x} u} \Delta_{\varepsilon\mu}^{x}(u, v) \quad .$$

4.2 Tangent bundle

A reformulation of parts of theorems 6,7 [3] is the following.

Theorem 4.2 *A dilatation structure (X, d, δ) has the following properties.*

(a) *For all $x \in X$, $u, v \in X$ such that $d(x, u) \leq 1$ and $d(x, v) \leq 1$ and all $\mu \in (0, A)$ we have:*

$$d^{x}(u, v) \;=\; \frac{1}{\mu} d^{x}(\delta_{\mu}^{x} u, \delta_{\mu}^{x} v) \; .$$

We shall say that d^x has the cone property with respect to dilatations.

(b) *The metric space (X, d) admits a metric tangent space at x, for any point $x \in X$. More precisely we have the following limit:*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup \left\{ \mid d(u, v) - d^{x}(u, v) \mid \; : d(x, u) \leq \varepsilon \; , \; d(x, v) \leq \varepsilon \right\} \;=\; 0 \; .$$

For the next theorem (composite of results in theorems 8, 10 [3]) we need the previously introduced notion of a normed conical (local) group.

Theorem 4.3 *Let (X, d, δ) be a strong dilatation structure. Then for any $x \in X$ the triple $(U(x), \Sigma^{x}, \delta^{x})$ is a normed local conical group, with the norm induced by the distance d^x.*

The conical group $(U(x), \Sigma^{x}, \delta^{x})$ can be regarded as the tangent space of (X, d, δ) at x. Further will be denoted by: $T_x X = (U(x), \Sigma^{x}, \delta^{x})$.

Definition 4.4 *Let (X, δ, d) be a dilatation structure and $x \in X$ a point. In a neighbourhood $U(x)$ of x, for any $\mu \in (0, 1)$ we defined the distances:*

$$(\delta^{x}, \mu)(u, v) = \frac{1}{\mu} d(\delta_{\mu}^{x} u, \delta_{\mu}^{x} v).$$

Proposition 4.5 *Let (X, δ, d) be a (strong) dilatation structure. For any $u, v \in U(x)$ let us define*

$$\hat{\delta}_{\varepsilon}^{u} v = \Sigma_{\mu}^{x}(u, \delta_{\varepsilon}^{\delta_{\mu}^{x} u} \Delta_{\mu}^{x}(u, v)) = \delta_{\mu^{-1}}^{x} \delta_{\varepsilon}^{\delta_{\mu}^{x} u} \delta_{\mu}^{x} v.$$

Then $(U(x), \hat{\delta}, (\delta^{x}, \mu))$ is a (strong) dilatation structure.

Proof. We have to check the axioms. The first part of axiom A0 is an easy consequence of theorem 4.2 for (X, δ, d). The second part of A0, A1 and A2 are true based on simple computations.

The first interesting fact is related to axiom A3. Let us compute, for $v, w \in U(x)$,

$$\frac{1}{\varepsilon}(\delta^x, \mu)(\hat{\delta}^u_\varepsilon v, \hat{\delta}^u_\varepsilon w) = \frac{1}{\varepsilon\mu} d(\delta^x_\mu \hat{\delta}^u_\varepsilon v, \delta^x_\mu \hat{\delta}^u_\varepsilon w) =$$

$$= \frac{1}{\varepsilon\mu} d(\delta^{\delta^x_\mu u}_\varepsilon \delta^x_\mu v, \delta^{\delta^x_\mu u}_\varepsilon \delta^x_\mu w) = \frac{1}{\varepsilon\mu} d(\delta^{\delta^x_\mu u}_{\varepsilon\mu} \Delta^x_\mu(u,v), \delta^{\delta^x_\mu u}_{\varepsilon\mu} \Delta^x_\mu(u,w)) =$$

$$= (\delta^{\delta^x_\mu u}, \varepsilon\mu)(\Delta^x_\mu(u,v), \Delta^x_\mu(u,w)).$$

The axiom A3 is then a consequence of axiom A3 for (X, δ, d) and we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\delta^x, \mu)(\hat{\delta}^u_\varepsilon v, \hat{\delta}^u_\varepsilon w) = d^{\delta^x_\mu u}(\Delta^x_\mu(u,v), \Delta^x_\mu(u,w)).$$

The axiom A4 is also a straightforward consequence of A4 for (X, δ, d) and is left to the reader. $\square$

The proof of the following proposition is an easy computation, of the same type as in the lines above, therefore we shall not write it here.

Proposition 4.6 *With the same notations as in proposition 4.5, the transformation $\Sigma^x_\mu(u, \cdot)$ is an isometry from $(\delta^{\delta^x_\mu u}, \mu)$ to (δ^x, μ). Moreover, we have $\Sigma^x_\mu(u, \delta^x_\mu u) = u$.*

These two propositions show that on a dilatation structure we almost have translations (the operators $\Sigma^x_\varepsilon(u, \cdot)$), which are almost isometries (that is, not with respect to the distance d, but with respect to distances of type (δ^x, μ)). It is almost as if we were working with a normed conical group, only that we have to use families of distances and to make small shifts in the tangent space (as in the last formula in the proof of proposition 4.5).

4.3 Topological considerations

In this subsection we compare various topologies and uniformities related to a dilatation structure.

The axiom A3 implies that for any $x \in X$ the function d^x is continuous, therefore open sets with respect to d^x are open with respect to d.

If (X, d) is separable and d^x is non degenerate then $(U(x), d^x)$ is also separable and the topologies of d and d^x are the same. Therefore $(U(x), d^x)$ is also locally compact (and a set is compact with respect to d^x if and only if it is compact with respect to d).

If (X, d) is separable and d^x is non degenerate then the uniformities induced by d and d^x are the same. Indeed, let $\{u_n : n \in \mathbb{N}\}$ be a dense set in $U(x)$, with $x_0 = x$. We can embed $(U(x), (\delta^x, \varepsilon))$ isometrically in the separable Banach space l^∞, for any $\varepsilon \in (0, 1)$, by the function

$$\phi_\varepsilon(u) = \left(\frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x x_n) - \frac{1}{\varepsilon} d(\delta_\varepsilon^x x, \delta_\varepsilon^x x_n) \right)_n .$$

A reformulation of point (a) in theorem 4.2 is that on compact sets ϕ_ε uniformly converges to the isometric embedding of $(U(x), d^x)$

$$\phi(u) = \left(d^x(u, x_n) - d^x(x, x_n) \right)_n .$$

Remark that the uniformity induced by (δ, ε) is the same as the uniformity induced by d, and that it is the same induced from the uniformity on l^∞ by the embedding ϕ_ε. We proved that the uniformities induced by d and d^x are the same.

From previous considerations we deduce the following characterisation of tangent spaces asociated to a dilatation structure.

Corollary 4.7 *Let (X, d, δ) be a strong dilatation structure with group $\Gamma = (0, +\infty)$. Then for any $x \in X$ the local group $(U(x), \Sigma^x)$ is locally a simply connected Lie group whose Lie algebra admits a positive graduation (a Carnot group).*

Proof. Use the facts: $(U(x), \Sigma^x)$ is a locally compact group (from previous topological considerations) which admits δ^x as a contractive automorphism group (from theorem 4.3). Apply then Siebert proposition 3.15. □

5 Linearity and dilatation structures

Definition 5.1 *Let (X, d, δ) be a dilatation structure. A transformation $A : X \to X$ is linear if it is Lipschitz and it commutes with dilatations in the following sense: for any $x \in X$, $u \in U(x)$ and $\varepsilon \in \Gamma$, $\nu(\varepsilon) < 1$, if $A(u) \in U(A(x))$ then*

$$A\delta_\varepsilon^x = \delta^{A(x)} A(u) \quad .$$

The group of linear transformations, denoted by $GL(X, d, \delta)$ is formed by all invertible and bi-lipschitz linear transformations of X.

$GL(X, d, \delta)$ is indeed a (local) group. In order to see this we start from the remark that if A is Lipschitz then there exists $C > 0$ such that for all $x \in X$

and $u \in B(x,C)$ we have $A(u) \in U(A(x))$. The inverse of $A \in GL(X,d,\delta)$ is then linear. Same considerations apply for the composition of two linear, bi-lipschitz and invertible transformations.

In the particular case of the first subsection of this paper, namely X finite dimensional real, normed vector space, d the distance given by the norm, $\Gamma = (0,+\infty)$ and dilatations $\delta_\varepsilon^x u = x + \varepsilon(u-x)$, a linear transformations in the sense of definition 5.1 is an affine transformation of the vector space X.

Proposition 5.2 *Let (X,d,δ) be a dilatation structure and $A : X \to X$ a linear transformation. Then:*

(a) for all $x \in X$, $u,v \in U(x)$ sufficiently close to x, we have:

$$A\,\Sigma_\varepsilon^x(u,v) = \Sigma_\varepsilon^{A(x)}(A(u),A(v)) \quad .$$

(b) for all $x \in X$, $u \in U(x)$ sufficiently close to x, we have:

$$A\,inv^x(u) = inv^{A(x)}A(u) \quad .$$

Proof. Straightforward, just use the commutation with dilatations. □

5.1 Differentiability of linear transformations

In this subsection we briefly recall the notion of differentiability associated to dilatation structures (section 7.2 [3]). Then we apply it for linear transformations.

First we need the natural definition below.

Definition 5.3 *Let (N,δ) and $(M,\bar{\delta})$ be two conical groups. A function $f : N \to M$ is a conical group morphism if f is a group morphism and for any $\varepsilon > 0$ and $u \in N$ we have $f(\delta_\varepsilon u) = \bar{\delta}_\varepsilon f(u)$.*

The definition of derivative with respect to dilatations structures follows.

Definition 5.4 *Let (X,δ,d) and $(Y,\overline{\delta},\overline{d})$ be two strong dilatation structures and $f : X \to Y$ be a continuous function. The function f is differentiable in x if there exists a conical group morphism $Q^x : T_xX \to T_{f(x)}Y$, defined on a neighbourhood of x with values in a neighbourhood of $f(x)$ such that*

$$\lim_{\varepsilon\to 0} \sup\left\{ \frac{1}{\varepsilon}\overline{d}\left(f\left(\delta_\varepsilon^x u\right), \overline{\delta}_\varepsilon^{f(x)} Q^x(u)\right) \,:\, d(x,u) \leq \varepsilon \right\} = 0, \qquad (5.1.1)$$

The morphism Q^x is called the derivative of f at x and will be sometimes denoted by $Df(x)$.

The function f is uniformly differentiable if it is differentiable everywhere and the limit in (5.1.1) is uniform in x in compact sets.

The following proposition has then a straightforward proof.

Proposition 5.5 *Let (X, d, δ) be a strong dilatation structure and $A : X \to X$ a linear transformation. Then A is uniformly differentiable and the derivative equals A.*

5.2 Linearity of strong dilatation structures

Remark that for general dilatation structures the "translations" $\Sigma_\varepsilon^x(u, \cdot)$ are not linear. Nevertheless, they commute with dilatation in a known way, according to point (f) theorem 4.1. This is important, because the transformations $\Sigma_\varepsilon^x(u, \cdot)$ really behave as translations, as explained in subsection 4.1.

The reason for which translations are not linear is that dilatations are generally not linear. Before giving the next definition we need to establish a simple estimate. Let $K \subset X$ be compact, non empty set. Then there is a constant $C(K) > 0$, depending on the set K such that for any $\varepsilon, \mu \in \Gamma$ with $\nu(\varepsilon), \nu(\mu) \in (0, 1]$ and any $x, y, z \in K$ with $d(x, y), d(x, z), d(y, z) \leq C(K)$ we have

$$\delta_\mu^y z \in V_\varepsilon(x) \quad , \quad \delta_\varepsilon^x z \in V_\mu(\delta_\varepsilon^x y) \quad .$$

Indeed, this is coming from the uniform (with respect to K) estimates:

$$d(\delta_\varepsilon^x y, \delta_\varepsilon^x z) \leq \varepsilon d^x(y, z) + \varepsilon \mathcal{O}(\varepsilon) \quad ,$$

$$d(x, \delta_\mu^y z) \leq d(x, y) + d(y, \delta_\mu^y z) \leq d(x, y) + \mu d^y(y, z) + \mu \mathcal{O}(\mu) \quad .$$

Definition 5.6 *A property $\mathcal{P}(x_1, x_2, x_3, ...)$ holds for $x_1, x_2, x_3, ...$ sufficiently closed if for any compact, non empty set $K \subset X$, there is a positive constant $C(K) > 0$ such that $\mathcal{P}(x_1, x_2, x_3, ...)$ is true for any $x_1, x_2, x_3, ... \in K$ with $d(x_i, x_j) \leq C(K)$.*

For example, we may say that the expressions

$$\delta_\varepsilon^x \delta_\mu^y z \quad , \quad \delta_\mu^{\delta_\varepsilon^x y} \delta_\varepsilon^x z$$

are well defined for any $x, y, z \in X$ sufficiently closed and for any $\varepsilon, \mu \in \Gamma$ with $\nu(\varepsilon), \nu(\mu) \in (0, 1]$.

Definition 5.7 *A dilatation structure (X, d, δ) is linear if for any $\varepsilon, \mu \in \Gamma$ such that $\nu(\varepsilon), \nu(\mu) \in (0, 1]$, and for any $x, y, z \in X$ sufficiently closed we have*

$$\delta_{\varepsilon}^{x} \, \delta_{\mu}^{y} z \; = \; \delta_{\mu}^{\delta_{\varepsilon}^{x} y} \delta_{\varepsilon}^{x} z \quad .$$

Linear dilatation structures are very particular dilatation structures. The next proposition gives a family of examples of linear dilatation structures.

Proposition 5.8 *The dilatation structure associated to a normed conical group is linear.*

Proof. Indeed, for the dilatation structure associated to a normed conical group we have, with the notations from definition 5.7:

$$\delta_{\mu}^{\delta_{\varepsilon}^{x} y} \delta_{\varepsilon}^{x} z \; = \; \left(x \delta_{\varepsilon}(x^{-1} y) \right) \, \delta_{\mu} \left(\delta_{\varepsilon}(y^{-1} x) \, x^{-1} \, x \, \delta_{\varepsilon}(x^{-1} z) \right) \; =$$

$$= \; \left(x \delta_{\varepsilon}(x^{-1} y) \right) \, \delta_{\mu} \left(\delta_{\varepsilon}(y^{-1} x) \, \delta_{\varepsilon}(x^{-1} z) \right) \; = \; \left(x \delta_{\varepsilon}(x^{-1} y) \right) \, \delta_{\mu} \left(\delta_{\varepsilon}(y^{-1} z) \right) \; =$$

$$= \; x \left(\delta_{\varepsilon}(x^{-1} y) \, \delta_{\varepsilon} \, \delta_{\mu}(y^{-1} z) \right) \; = \; x \, \delta_{\varepsilon} \left(x^{-1} y \, \delta_{\mu}(y^{-1} z) \right) \; = \; \delta_{\varepsilon}^{x} \, \delta_{\mu}^{y} z \quad .$$

Therefore the dilatation structure is linear. $\square$

In the proposition below we give a relation, true for linear dilatation structures, with an interesting interpretation. Let us think in affine terms: for closed points x, u, v, we think about let us denote $w = \Sigma_{\varepsilon}^{x}(u, v)$. We may think that the "vector" (x, w) is (approximatively, due to the parameter ε) the sum of the vectors (x, u) and (x, v), based at x. Denote also $w' = \Delta_{\varepsilon}^{u}(x, v)$; then the "vector" (u, w') is (approximatively) equal to the differrence between the vectors (u, v) and (u, x), based at u. In a classical affine space we would have $w = w'$. The same is true for a linear dilatation structure.

Proposition 5.9 *For a linear dilatation structure (X, δ, d), for any $x, u, v \in X$ sufficiently closed and for any $\varepsilon \in \Gamma$, $\nu(\varepsilon) \leq 1$, we have:*

$$\Sigma_{\varepsilon}^{x}(u, v) \; = \; \Delta_{\varepsilon}^{u}(x, v) \quad .$$

Proof. We have the following string of equalities, by using twice the linearity of the dilatation structure:

$$\Sigma_{\varepsilon}^{x}(u, v) \; = \; \delta_{\varepsilon^{-1}}^{x} \delta_{\varepsilon}^{\delta_{\varepsilon}^{x} u} v \; = \; \delta_{\varepsilon}^{u} \, \delta_{\varepsilon^{-1}}^{x} v \; =$$

$$= \; \delta_{\varepsilon^{-1}}^{\delta_{\varepsilon}^{u} x} \delta_{\varepsilon}^{u} v \; = \; \Delta_{\varepsilon}^{u}(x, v) \quad .$$

The proof is done. □

The following expression:

$$Lin(x,y,z;\varepsilon,\mu) \ = \ d\left(\delta^x_\varepsilon \, \delta^y_\mu z \, , \, \delta^{\delta^x_\varepsilon y}_\mu \delta^x_\varepsilon z\right) \qquad (5.2.2)$$

is a measure of lack of linearity, for a general dilatation structure. The next theorem shows that, infinitesimally, any dilatation structure is linear.

Theorem 5.10 *Let (X,d,δ) be a strong dilatation structure. Then for any $x,y,z \in X$ sufficiently close we have*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} Lin(x, \delta^x_\varepsilon y, \delta^x_\varepsilon z; \varepsilon, \varepsilon) \ = \ 0 \quad . \qquad (5.2.3)$$

Proof. From the hypothesis of the theorem we have:

$$\frac{1}{\varepsilon^2} Lin(x, \delta^x_\varepsilon y, \delta^x_\varepsilon z; \varepsilon, \varepsilon) \ = \ \frac{1}{\varepsilon^2} d\left(\delta^x_\varepsilon \, \delta^{\delta^x_\varepsilon y}_\varepsilon z \, , \, \delta^{\delta^x_{\varepsilon^2} y}_\varepsilon \delta^x_\varepsilon z\right) \ =$$

$$= \ \frac{1}{\varepsilon^2} d\left(\delta^x_{\varepsilon^2} \, \Sigma^x_\varepsilon(y,z) \, , \, \delta^x_{\varepsilon^2} \, \delta^x_{\varepsilon^{-2}} \, \delta^{\delta^x_{\varepsilon^2} y}_\varepsilon \delta^x_\varepsilon z\right) \ =$$

$$= \ \frac{1}{\varepsilon^2} d\left(\delta^x_{\varepsilon^2} \, \Sigma^x_\varepsilon(y,z) \, , \, \delta^x_{\varepsilon^2} \, \Sigma^x_{\varepsilon^2}(y \, , \, \Delta^x_\varepsilon(\delta^x_\varepsilon y, z))\right) \ =$$

$$= \ \mathcal{O}(\varepsilon^2) + d^x\left(\Sigma^x_\varepsilon(y,z) \, , \, \Sigma^x_{\varepsilon^2}(y \, , \, \Delta^x_\varepsilon(\delta^x_\varepsilon y, z))\right) \quad .$$

The dilatation structure satisfies A4, therefore as ε goes to 0 we obtain:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} Lin(x, \delta^x_\varepsilon y, \delta^x_\varepsilon z; \varepsilon, \varepsilon) \ = \ d^x\left(\Sigma^x(y,z) \, , \, \Sigma^x(y \, , \, \Delta^x(x,z))\right) \ =$$

$$= \ d^x\left(\Sigma^x(y,z) \, , \, \Sigma^x(y,z)\right) \ = \ 0 \quad . \qquad \square$$

The linearity of translations Σ^x_ε is related to the linearity of the dilatation structure, as described in the theorem below, point (a). As a consequence, we prove at point (b) that a linear and strong dilatation structure comes from a conical group.

Theorem 5.11 *Let (X,d,δ) be a dilatation structure.*

(a) If the dilatation structure is linear then all transformations $\Delta^x_\varepsilon(u,\cdot)$ are linear for any $u \in X$.

(b) If the dilatation structure is strong (satisfies A4) then it is linear if and only if the dilatations come from the dilatation structure of a conical group, precisely for any $x \in X$ there is an open neighbourhood $D \subset X$ of x such that $(\overline{D}, d^x, \delta)$ is the same dilatation structure as the dilatation structure of the tangent space of (X,d,δ) at x.

Proof. (a) If dilatations are linear, then let $\varepsilon, \mu \in \Gamma$, $\nu(\varepsilon), \nu(\mu) \leq 1$, and $x, y, u, v \in X$ such that the following computations make sense. We have:

$$\Delta_{\varepsilon}^{x}(u, \delta_{\mu}^{y} v) = \delta_{\varepsilon^{-1}}^{\delta_{\varepsilon}^{x} u} \delta_{\varepsilon}^{x} \delta_{\mu}^{y} v \quad .$$

Let $A_{\varepsilon} = \delta_{\varepsilon^{-1}}^{\delta_{\varepsilon}^{x} u}$. We compute:

$$\delta_{\mu}^{\Delta_{\varepsilon}^{x}(u,y)} \Delta_{\varepsilon}^{x}(u, v) = \delta_{\mu}^{A_{\varepsilon} \delta_{\varepsilon}^{x} y} A_{\varepsilon} \delta_{\varepsilon}^{x} v \quad .$$

We use twice the linearity of dilatations:

$$\delta_{\mu}^{\Delta_{\varepsilon}^{x}(u,y)} \Delta_{\varepsilon}^{x}(u, v) = A_{\varepsilon} \delta_{\mu}^{\delta_{\varepsilon}^{x} y} \delta_{\varepsilon}^{x} v = \delta_{\varepsilon^{-1}}^{\delta_{\varepsilon}^{x} u} \delta_{\varepsilon}^{x} \delta_{\mu}^{y} v \quad .$$

We proved that:

$$\Delta_{\varepsilon}^{x}(u, \delta_{\mu}^{y} v) = \delta_{\mu}^{\Delta_{\varepsilon}^{x}(u,y)} \Delta_{\varepsilon}^{x}(u, v) \quad ,$$

which is the conclusion of the part (a).

(b) Suppose that the dilatation structure is strong. If dilatations are linear, then by point (a) the transformations $\Delta_{\varepsilon}^{x}(u, \cdot)\delta$ are linear for any $u \in X$. Then, with notations made before, for $y = u$ we get

$$\Delta_{\varepsilon}^{x}(u, \delta_{\mu}^{u} v) = \delta_{\mu}^{\delta_{\varepsilon}^{x} u} \Delta_{\varepsilon}^{x}(u, v) \quad ,$$

which implies

$$\delta_{\mu}^{u} v = \Sigma_{\varepsilon}^{x}(u, \delta_{\mu}^{x} \Delta_{\varepsilon}^{x}(u, v)) \quad .$$

We pass to the limit with $\varepsilon \to 0$ and we obtain:

$$\delta_{\mu}^{u} v = \Sigma^{x}(u, \delta_{\mu}^{x} \Delta^{x}(u, v)) \quad .$$

We recognize at the right hand side the dilatations associated to the conical group $T_x X$.

By proposition 5.8 the opposite implication is straightforward, because the dilatation structure of any conical group is linear. □

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Linear dilatation structures and inverse semigroups

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Abstract

A dilatation structure encodes the approximate self-similarity of a metric space. A metric space (X, d) which admits a strong dilatation structure (definition 2.2) has a metric tangent space at any point $x \in X$ (theorem 4.1), and any such metric tangent space has an algebraic structure of a conical group (theorem 4.2). Particular examples of conical groups are Carnot groups: these are simply connected Lie groups whose Lie algebra admits a positive graduation.

The dilatation structures associated to conical (or Carnot) groups are linear, in the sense of definition 5.3. Thus conical groups are the right generalization of normed vector spaces, from the point of view of dilatation structures.

We prove that for dilatation structures linearity is equivalent to a statement about the inverse semigroup generated by the family of dilatations forming a dilatation structure on a metric space.

The result is new for Carnot groups and the proof seems to be new even for the particular case of normed vector spaces.

Keywords: inverse semigroups, Carnot groups, dilatation structures

MSC classes: 20M18; 22E20; 20F65

1 Inverse semigroups and Menelaos theorem

Definition 1.1 *A semigroup S is an inverse semigroup if for any $x \in S$ there is an unique element $x^{-1} \in S$ such that $x\, x^{-1} x = x$ and $x^{-1} x\, x^{-1} = x^{-1}$.*

An important example of an inverse semigroup is $I(X)$, the class of all bijective maps $\phi : dom\, \phi \to im\, \phi$, with $dom\, \phi, im\, \phi \subset X$. The semigroup operation is the composition of functions in the largest domain where this makes sense.

By the Vagner-Preston representation theorem [6] every inverse semigroup is isomorphic to a subsemigroup of $I(X)$, for some set X.

1.1 A toy example

Let $(\mathbb{V}, \| \cdot \|)$ be a finite dimensional, normed, real vector space. By definition the dilatation based at x, of coefficient $\varepsilon > 0$, is the function

$$\delta_{\varepsilon}^{x} : \mathbb{V} \to \mathbb{V} \quad , \quad \delta_{\varepsilon}^{x} y = x + \varepsilon(-x + y) \quad .$$

For fixed x the dilatations based at x form a one parameter group which contracts any bounded neighbourhood of x to a point, uniformly with respect to x.

With the distance d induced by the norm, the metric space $(\mathbb{V}, d)$ is complete and locally compact. For any $x \in \mathbb{V}$ and any $\varepsilon > 0$ the distance d behaves well with respect to the dilatation δ_{ε}^{x} in the sense: for any $u, v \in \mathbb{V}$ we have

$$\frac{1}{\varepsilon} d(\delta_{\varepsilon}^{x} u, \delta_{\varepsilon}^{x} v) \; = \; d(u, v) \quad . \tag{1.1.1}$$

Dilatations encode much more than the metric structure of the space $(\mathbb{V}, d)$. Indeed, we can reconstruct the algebraic structure of the vector space $\mathbb{V}$ from dilatations. For example let us define for any $x, u, v \in \mathbb{V}$ and $\varepsilon > 0$:

$$\Sigma_{\varepsilon}^{x}(u, v) = \delta_{\varepsilon^{-1}}^{x} \delta_{\varepsilon}^{\delta_{\varepsilon}^{x} u}(v) \quad .$$

A simple computation shows that $\Sigma_{\varepsilon}^{x}(u, v) = u + \varepsilon(-u + x) + (-x + v)$, therefore we can recover the addition operation in $\mathbb{V}$ by using the formula:

$$\lim_{\varepsilon \to 0} \Sigma_{\varepsilon}^{x}(u, v) = u + (-x + v) \quad . \tag{1.1.2}$$

This is the addition operation translated such that the neutral element is x. Thus, for $x = 0$, we recover the usual addition operation.

Affine continuous transformations $A : \mathbb{V} \to \mathbb{V}$ admit the following description in terms of dilatations. A continuous transformation $A : \mathbb{V} \to \mathbb{V}$ is affine if and only if for any $\varepsilon \in (0, 1)$, $x, y \in \mathbb{V}$ we have

$$A \, \delta_{\varepsilon}^{x} y \; = \; \delta_{\varepsilon}^{Ax} A y \quad . \tag{1.1.3}$$

Any dilatation is an affine transformation, hence for any $x, y \in \mathbb{V}$ and $\varepsilon, \mu > 0$ we have

$$\delta_{\mu}^{y} \, \delta_{\varepsilon}^{x} \; = \; \delta_{\varepsilon}^{\delta_{\mu}^{y} x} \delta_{\mu}^{y} \quad . \tag{1.1.4}$$

Moreover, some compositions of dilatations are dilatations. This is precisely stated in the next theorem, which is equivalent with the Menelaos theorem in euclidean geometry.

Theorem 1.2 *For any $x, y \in \mathbb{V}$ and $\varepsilon, \mu > 0$ such that $\varepsilon\mu \neq 1$ there exists an unique $w \in \mathbb{V}$ such that*

$$\delta_{\mu}^{y} \, \delta_{\varepsilon}^{x} \; = \; \delta_{\varepsilon\mu}^{w} \quad .$$

For the proof see Artin [1]. A straightforward consequence of this theorem is the following result.

Corollary 1.3 *The inverse subsemigroup of $I(\mathbb{V})$ generated by dilatations of the space $\mathbb{V}$ is made of all dilatations and all translations in $\mathbb{V}$.*

Proof. Indeed, by theorem 1.2 a composition of two dilatations with coefficients ε, μ with $\varepsilon\mu \neq 1$ is a dilatation. By direct computation, if $\varepsilon\mu = 1$ then we obtain translations. This is in fact compatible with (1.1.2), but is a stronger statement, due to the fact that dilatations are affine in the sense of relation (1.1.4).

Moreover any translation can be expressed as a composition of two dilatations with coefficients ε, μ such that $\varepsilon\mu = 1$. Finally, any composition between a translation and a dilatation is again a dilatation. □

1.2 Focus on dilatations

Suppose that we take the dilatations as basic data for the toy example above. Namely, instead of giving to the space $\mathbb{V}$ a structure of real, normed vector space, we give only the distance d and the dilatations δ^x_ε for all $x \in X$ and $\varepsilon > 0$. We should add some relations which prescribe:

- the behaviour of the distance with respect to dilatations, for example some form of relation (1.1.1),
- the interaction between dilatations, for example the existence of the limit from the left hand side of relation (1.1.2).

We denote such a collection of data by $(\mathbb{V}, d, \delta)$ and call it a dilatation structure (see further definition 2.2).

In this paper we ask if there is any relationship between dilatations and inverse semigroups, generalizing relation (1.1.4) and corollary 1.3.

Dilatation structures are far more general than our toy example. A dilatation structure on a metric space, introduced in [3], is a notion in between a group and a differential structure, expressing the approximate self-similarity of the metric space where it lives.

A metric space (X, d) which admits a strong dilatation structure (definition 2.2) has a metric tangent space at any point $x \in X$ (theorem 4.1), and any such metric tangent space has an algebraic structure of a conical group (theorem 4.2). Conical groups are particular examples of contractible groups. An important class of of conical groups is formed by Carnot groups: these are simply connected Lie groups whose Lie algebra admits a positive graduation. Carnot groups appear in many situations, in particular in relation with sub-riemannian geometry cf. Bellaïche [2], groups with polynomial growth cf. Gromov [5], or Margulis type rigidity results cf. Pansu [7].

The dilatation structures associated to conical (or Carnot) groups are linear, in the sense of relation (1.1.4), see also definition 5.3. We actually proved in [4] (here theorem 5.4) that a linear dilatation structure always comes from some associated conical group. Thus conical groups are the right generalization of normed vector spaces, from the point of view of dilatation structures.

2 Dilatation structures

We present here an introduction into the subject of dilatation structures, following Buliga [3].

2.1 Notations

Let Γ be a topological separated commutative group endowed with a continuous group morphism

$$\nu : \Gamma \rightarrow (0, +\infty)$$

with $\inf \nu(\Gamma) = 0$. Here $(0, +\infty)$ is taken as a group with multiplication. The neutral element of Γ is denoted by 1. We use the multiplicative notation for the operation in Γ.

The morphism ν defines an invariant topological filter on Γ (equivalently, an end). Indeed, this is the filter generated by the open sets $\nu^{-1}(0, a)$, $a > 0$. From now on we shall name this topological filter (end) by "0" and we shall write $\varepsilon \in \Gamma \rightarrow 0$ for $\nu(\varepsilon) \in (0, +\infty) \rightarrow 0$.

The set $\Gamma_1 = \nu^{-1}(0, 1]$ is a semigroup. We note $\bar{\Gamma}_1 = \Gamma_1 \cup \{0\}$ On the set $\bar{\Gamma} = \Gamma \cup \{0\}$ we extend the operation on Γ by adding the rules $00 = 0$ and $\varepsilon 0 = 0$ for any $\varepsilon \in \Gamma$. This is in agreement with the invariance of the end 0 with respect to translations in Γ.

The space (X, d) is a complete, locally compact metric space. For any $r > 0$ and any $x \in X$ we denote by $B(x, r)$ the open ball of center x and radius r in the metric space X.

On the metric space (X, d) we work with the topology (and uniformity) induced by the distance. For any $x \in X$ we denote by $\mathcal{V}(x)$ the topological filter of open neighbourhoods of x.

2.2 Axioms of dilatation structures

The first axiom is a preparation for the next axioms. That is why we counted it as axiom 0.

A0. The dilatations

$$\delta_\varepsilon^x : U(x) \rightarrow V_\varepsilon(x)$$

are defined for any $\varepsilon \in \Gamma, \nu(\varepsilon) \leq 1$. The sets $U(x), V_\varepsilon(x)$ are open neighbourhoods of x. All dilatations are homeomorphisms (invertible, continuous, with continuous inverse).

We suppose that there is a number $1 < A$ such that for any $x \in X$ we have

$$\bar{B}_d(x, A) \subset U(x) \ .$$

We suppose that for all $\varepsilon \in \Gamma$, $\nu(\varepsilon) \in (0,1)$, we have

$$B_d(x, \nu(\varepsilon)) \subset \delta^x_\varepsilon B_d(x, A) \subset V_\varepsilon(x) \subset U(x) \ .$$

There is a number $B \in (1, A]$ such that for any $\varepsilon \in \Gamma$ with $\nu(\varepsilon) \in (1, +\infty)$ the associated dilatation

$$\delta^x_\varepsilon : W_\varepsilon(x) \to B_d(x, B) \ ,$$

is injective, invertible on the image. We shall suppose that $W_\varepsilon(x) \in \mathcal{V}(x)$, that $V_{\varepsilon^{-1}}(x) \subset W_\varepsilon(x)$ and that for all $\varepsilon \in \Gamma_1$ and $u \in U(x)$ we have

$$\delta^x_{\varepsilon^{-1}} \ \delta^x_\varepsilon u \ = \ u \ .$$

We have therefore the following string of inclusions, for any $\varepsilon \in \Gamma$, $\nu(\varepsilon) \leq 1$, and any $x \in X$:

$$B_d(x, \nu(\varepsilon)) \subset \delta^x_\varepsilon B_d(x, A) \subset V_\varepsilon(x) \subset W_{\varepsilon^{-1}}(x) \subset \delta^x_\varepsilon B_d(x, B) \quad .$$

A further technical condition on the sets $V_\varepsilon(x)$ and $W_\varepsilon(x)$ will be given just before the axiom A4. (This condition will be counted as part of axiom A0.)

A1. We have $\delta^x_\varepsilon x = x$ for any point x. We also have $\delta^x_1 = id$ for any $x \in X$.

Let us define the topological space

$$dom\, \delta = \{(\varepsilon, x, y) \in \Gamma \times X \times X : \quad \text{if } \nu(\varepsilon) \leq 1 \text{ then } y \in U(x) \ , \ \text{else } y \in W_\varepsilon(x)\}$$

with the topology inherited from the product topology on $\Gamma \times X \times X$. Consider also $Cl(dom\, \delta)$, the closure of $dom\, \delta$ in $\bar{\Gamma} \times X \times X$ with product topology. The function $\delta : dom\, \delta \to X$ defined by $\delta(\varepsilon, x, y) = \delta^x_\varepsilon y$ is continuous. Moreover, it can be continuously extended to $Cl(dom\, \delta)$ and we have

$$\lim_{\varepsilon \to 0} \delta^x_\varepsilon y \ = \ x \quad .$$

A2. For any $x, \in K$, $\varepsilon, \mu \in \Gamma_1$ and $u \in \bar{B}_d(x, A)$ we have:

$$\delta^x_\varepsilon \delta^x_\mu u = \delta^x_{\varepsilon\mu} u \ .$$

A3. For any x there is a function $(u, v) \mapsto d^x(u, v)$, defined for any u, v in the closed ball (in distance d) $\bar{B}(x, A)$, such that

$$\lim_{\varepsilon \to 0} \quad \sup \left\{ \mid \frac{1}{\varepsilon} d(\delta^x_\varepsilon u, \delta^x_\varepsilon v) \ - \ d^x(u, v) \mid \ : \ u, v \in \bar{B}_d(x, A) \right\} \ = \ 0$$

uniformly with respect to x in compact set.

Remark 2.1 *The "distance" d^x can be degenerated: there might exist $v, w \in U(x)$ such that $d^x(v,w) = 0$.*

For the following axiom to make sense we impose a technical condition on the co-domains $V_\varepsilon(x)$: for any compact set $K \subset X$ there are $R = R(K) > 0$ and $\varepsilon_0 = \varepsilon(K) \in (0,1)$ such that for all $u, v \in \bar{B}_d(x,R)$ and all $\varepsilon \in \Gamma$, $\nu(\varepsilon) \in (0, \varepsilon_0)$, we have

$$\delta_\varepsilon^x v \in W_{\varepsilon^{-1}}(\delta_\varepsilon^x u) \ .$$

With this assumption the following notation makes sense:

$$\Delta_\varepsilon^x(u,v) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v.$$

The next axiom can now be stated:

A4. We have the limit

$$\lim_{\varepsilon \to 0} \Delta_\varepsilon^x(u,v) = \Delta^x(u,v)$$

uniformly with respect to x, u, v in compact set.

Definition 2.2 *A triple (X, d, δ) which satisfies A0, A1, A2, A3, but d^x is degenerate for some $x \in X$, is called degenerate dilatation structure.*

If the triple (X, d, δ) satisfies A0, A1, A2, A3 and d^x is non-degenerate for any $x \in X$, then we call it a dilatation structure.

If a dilatation structure satisfies A4 then we call it strong dilatation structure.

3 Normed conical groups

We shall need further the notion of normed conical group. Motivated by the case of a Lie group endowed with a Carnot-Carathéodory distance induced by a left invariant distribution, we shall use the following definition of a local uniform group.

Let G be a group. We introduce first the double of G, as the group $G^{(2)} = G \times G$ with operation

$$(x,u)(y,v) = (xy, y^{-1}uyv) \quad .$$

The operation on the group G, seen as the function $op : G^{(2)} \to G$, $op(x,y) = xy$ is a group morphism. Also the inclusions:

$$i' : G \to G^{(2)} \ , \quad i'(x) = (x, e)$$

$$i" : G \to G^{(2)} \ , \quad i"(x) = (x, x^{-1})$$

are group morphisms.

Definition 3.1 *1. G is an uniform group if we have two uniformity structures, on G and $G \times G$, such that op, i', $i"$ are uniformly continuous.*

2. *A local action of a uniform group G on a uniform pointed space (X, x_0) is a function $\phi \in W \in \mathcal{V}(e) \mapsto \hat{\phi} : U_\phi \in \mathcal{V}(x_0) \to V_\phi \in \mathcal{V}(x_0)$ such that:*

   (a) *the map $(\phi, x) \mapsto \hat{\phi}(x)$ is uniformly continuous from $G \times X$ (with product uniformity) to X,*

   (b) *for any $\phi, \psi \in G$ there is $D \in \mathcal{V}(x_0)$ such that for any $x \in D$ $\widehat{\phi\psi^{-1}}(x)$ and $\hat{\phi}(\hat{\psi}^{-1}(x))$ make sense and $\widehat{\phi\psi^{-1}}(x) = \hat{\phi}(\hat{\psi}^{-1}(x))$.*

3. *Finally, a local group is an uniform space G with an operation defined in a neighbourhood of $(e, e) \subset G \times G$ which satisfies the uniform group axioms locally.*

Definition 3.2 *A normed (local) conical group $(G, \delta, \| \cdot \|)$ is (local) group endowed with: (I) a (local) action of Γ by morphisms δ_ε such that $\lim_{\varepsilon \to 0} \delta_\varepsilon x = e$ for any x in a neighbourhood of the neutral element e; (II) a continuous norm function $\| \cdot \| : G \to \mathbb{R}$ which satisfies (locally, in a neighbourhood of the neutral element e) the properties:*

(a) *for any x we have $\|x\| \geq 0$; if $\|x\| = 0$ then $x = e$,*

(b) *for any x, y we have $\|xy\| \leq \|x\| + \|y\|$,*

(c) *for any x we have $\|x^{-1}\| = \|x\|$,*

(d) *for any $\varepsilon \in \Gamma$, $\nu(\varepsilon) \leq 1$ and any x we have $\|\delta_\varepsilon x\| = \nu(\varepsilon) \, \|x\|$.*

Particular cases of normed conical groups are:

- Carnot groups, that is simply connected real Lie groups whose Lie algebra admits a positive graduation,

- nilpotent p-adic groups admitting a contractive automorphism.

A very particular case of a normed conical group is described in the toy example: to any real, finite dimensional, normed vector space $\mathbb{V}$ we may associate the normed conical group $(\mathbb{V}, +, \delta, \| \cdot \|)$, with dilatations δ previously described.

In a normed conical group (G, δ) we define dilatations based in any point $x \in G$ by

$$\delta_\varepsilon^x u = x \delta_\varepsilon (x^{-1} u). \tag{3.0.1}$$

There is also a natural left invariant distance given by

$$d(x, y) = \|x^{-1} y\| \quad . \tag{3.0.2}$$

The following result is theorem 15 [3].

Theorem 3.3 *Let $(G, \delta, \| \cdot \|)$ be a locally compact normed group with dilatations. Then (G, δ, d) is a strong dilatation structure, where δ are the dilatations defined by (3.0.1) and the distance d is induced by the norm as in (3.0.2).*

4 Properties of dilatation structures

The following two theorems describe the most important metric and algebraic properties of a dilatation structure. As presented here these are condensed statements, available in full length as theorems 7, 8, 10 in [3].

Theorem 4.1 *Let (X, d, δ) be a dilatation structure. Then the metric space (X, d) admits a metric tangent space at x, for any point $x \in X$. More precisely we have the following limit:*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \sup \left\{ \mid d(u,v) - d^{x}(u,v) \mid \ : d(x,u) \leq \varepsilon \ , \ d(x,v) \leq \varepsilon \right\} \ = \ 0 \ .$$

Theorem 4.2 *Let (X, d, δ) be a strong dilatation structure. Then for any $x \in X$ the triple $(U(x), \Sigma^{x}, \delta^{x}, d^{x})$ is a normed local conical group. This means:*

(a) Σ^{x} is a local group operation on $U(x)$, with x as neutral element and inv^{x} as the inverse element function;

(b) the distance d^{x} is left invariant with respect to the group operation from point (a);

(c) For any $\varepsilon \in \Gamma$, $\nu(\varepsilon) \leq 1$, the dilatation δ^{x}_{ε} is an automorphism with respect to the group operation from point (a);

(d) the distance d^{x} has the cone property with respect to dilatations: foar any $u, v \in X$ such that $d(x,u) \leq 1$ and $d(x,v) \leq 1$ and all $\mu \in (0, A)$ we have:

$$d^{x}(u,v) \ = \ \frac{1}{\mu} d^{x}(\delta_{\mu}^{x} u, \delta_{\mu}^{x} v) \quad .$$

The conical group $(U(x), \Sigma^{x}, \delta^{x})$ can be regarded as the tangent space of (X, d, δ) at x.

By using proposition 5.4 [8] and from some topological considerations we deduce the following characterisation of tangent spaces asociated to some dilatation structures. The following is corollary 4.7 [4].

Corollary 4.3 *Let (X, d, δ) be a dilatation structure with group $\Gamma = (0, +\infty)$ and the morphism ν equal to identity. Then for any $x \in X$ the local group $(U(x), \Sigma^{x})$ is locally a simply connected Lie group whose Lie algebra admits a positive graduation (a Carnot group).*

5 Linearity and dilatation structures

In this section we describe the notion of linearity for dilatation structures, as in Buliga [4].

Definition 5.1 *Let (X, d, δ) be a dilatation structure. A transformation $A : X \rightarrow X$ is linear if it is Lipschitz and it commutes with dilatations in the following sense: for any $x \in X$, $u \in U(x)$ and $\varepsilon \in \Gamma$, $\nu(\varepsilon) < 1$, if $A(u) \in U(A(x))$ then*

$$A\delta_{\varepsilon}^{x} = \delta^{A(x)} A(u) \quad .$$

In the particular case of X finite dimensional real, normed vector space, d the distance given by the norm, $\Gamma = (0, +\infty)$ and dilatations $\delta_{\varepsilon}^{x} u = x + \varepsilon(u - x)$, a linear transformations in the sense of definition 5.1 is an affine transformation of the vector space X. More generally, linear transformations in the sense of definition 5.1 have the expected properties related to linearity, as explained in section 5 [4].

Convention 5.2 *Further we shall say that a property $\mathcal{P}(x_1,\ x_2,\ x_3, ...)$ holds for $x_1, x_2, x_3, ...$ sufficiently closed if for any compact, non empty set $K \subset X$, there is a positive constant $C(K) > 0$ such that $\mathcal{P}(x_1, x_2, x_3, ...)$ is true for any $x_1, x_2, x_3, ... \in K$ with $d(x_i, x_j) \leq C(K)$.*

For example, the expressions

$$\delta_{\varepsilon}^{x} \delta_{\mu}^{y} z \quad , \quad \delta_{\mu}^{\delta_{\varepsilon}^{x} y} \delta_{\varepsilon}^{x} z$$

are well defined for any $x, y, z \in X$ sufficiently closed and for any $\varepsilon, \mu \in \Gamma$ with $\nu(\varepsilon), \nu(\mu) \in (0, 1]$. Indeed, let $K \subset X$ be compact, non empty set. Then there is a constant $C(K) > 0$, depending on the set K such that for any $\varepsilon, \mu \in \Gamma$ with $\nu(\varepsilon), \nu(\mu) \in (0, 1]$ and any $x, y, z \in K$ with $d(x, y), d(x, z), d(y, z) \leq C(K)$ we have

$$\delta_{\mu}^{y} z \in V_{\varepsilon}(x) \quad , \quad \delta_{\varepsilon}^{x} z \in V_{\mu}(\delta_{\varepsilon}^{x} y) \quad .$$

Indeed, this is coming from the uniform (with respect to K) estimates:

$$d(\delta_{\varepsilon}^{x} y, \delta_{\varepsilon}^{x} z) \leq \varepsilon d^{x}(y, z) + \varepsilon \mathcal{O}(\varepsilon) \quad ,$$

$$d(x, \delta_{\mu}^{y} z) \leq d(x, y) + d(y, \delta_{\mu}^{y} z) \leq d(x, y) + \mu d^{y}(y, z) + \mu \mathcal{O}(\mu) \quad .$$

These estimates allow us to give the following definition.

Definition 5.3 *A dilatation structure (X, d, δ) is linear if for any $\varepsilon, \mu \in \Gamma$ such that $\nu(\varepsilon), \nu(\mu) \in (0, 1]$, and for any $x, y, z \in X$ sufficiently closed we have*

$$\delta_{\varepsilon}^{x} \delta_{\mu}^{y} z = \delta_{\mu}^{\delta_{\varepsilon}^{x} y} \delta_{\varepsilon}^{x} z \quad .$$

Linear dilatation structures are very particular dilatation structures. The next theorem is theorem 5.7 [4]. It is shown that a linear and strong dilatation structure comes from a normed conical group.

Theorem 5.4 *Let (X, d, δ) be a linear dilatation structure. Then the following two statements are equivalent:*

(a) For any $x \in X$ there is an open neighbourhood $D \subset X$ of x such that $(\overline{D}, d^x, \delta)$ is the same dilatation structure as the dilatation structure of the tangent space of (X, d, δ) at x;

(b) The dilatation structure is strong (that is satisfies A4).

6 Dilatation structures and inverse semigroups

Here we prove that for dilatation structures linearity is equivalent to a generalization of the statement from corollary 1.3. The result is new for Carnot groups and the proof seems to be new even for vector spaces.

Definition 6.1 *A dilatation structure (X, d, δ) has the Menelaos property if for any two sufficiently closed $x, y \in X$ and for any $\varepsilon, \mu \in \Gamma$ with $\nu(\varepsilon), \nu(\mu) \in (0, 1)$ we have*

$$\delta^x_\varepsilon \, \delta^y_\mu \; = \; \delta^w_{\varepsilon\mu} \quad ,$$

where $w \in X$ is the fixed point of the contraction $\delta^x_\varepsilon \delta^y_\mu$ (thus depending on x, y and ε, μ).

Theorem 6.2 *A linear dilatation structure has the Menelaos property.*

Proof. Let $x, y \in X$ be sufficiently closed and $\varepsilon, \mu \in \Gamma$ with $\nu(\varepsilon), \nu(\mu) \in (0, 1)$. We shall define two sequences $x_n, y_n \in X$, $n \in \mathbb{N}$.

We begin with $x_0 = x$, $y_0 = y$. Let us define by induction

$$x_{n+1} \; = \; \delta_\mu^{\delta_\varepsilon^{x_n} y_n} x_n \quad , \quad y_{n+1} \; = \; \delta^{x_n}_\varepsilon y_n \quad . \tag{6.0.1}$$

In order to check if the definition is correct we have to prove that for any $n \in \mathbb{N}$, if x_n, y_n are sufficiently closed then x_{n+1}, y_{n+1} are sufficiently closed too.

Indeed, due to the linearity of the dilatation structure, we can write the first part of (6.0.1) as:

$$x_{n+1} \; = \delta^{x_n}_\varepsilon \, \delta^{y_n}_\mu x_n \quad .$$

Then we can estimate the distance between x_{n+1}, y_{n+1} like this:

$$d(x_{n+1}, y_{n+1}) \; = \; d(\delta^{x_n}_\varepsilon \, \delta^{y_n}_\mu x_n, \delta^{x_n}_\varepsilon y_n) \; = \; \nu(\varepsilon) \, d(\delta^{y_n}_\mu x_n, y_n) \; = \; \nu(\varepsilon\mu) d(x_n, y_n) \quad .$$

From $\nu(\varepsilon\mu) < 1$ it follows that $d(x_{n+1}, y_{n+1}) < d(x_n, y_n)$, therefore x_{n+1}, y_{n+1} are sufficiently closed. We also find out that

$$\lim_{n\to\infty} d(x_n, y_n) \ = \ 0 \quad . \tag{6.0.2}$$

Further we use twice the linearity of the dilatation structure:

$$\delta_\varepsilon^{x_n} \, \delta_\mu^{y_n} \ = \ \delta_\mu^{\delta_\varepsilon^{x_n} y_n} \, \delta_\varepsilon^{x_n} \ = \ \delta_\varepsilon^{\delta_\mu^{\delta_\varepsilon^{x_n} y_n} x_n} \, \delta_\mu^{\delta_\varepsilon^{x_n} y_n} \quad .$$

By definition (6.0.1) we arrive at the conclusion that for any $n \in \mathbb{N}$

$$\delta_\varepsilon^{x_n} \, \delta_\mu^{y_n} \ = \ \delta_\varepsilon^{x} \, \delta_\mu^{y} \quad . \tag{6.0.3}$$

From relation (6.0.3) we deduce that the first part of (6.0.1) can be written as:

$$x_{n+1} \ = \delta_\varepsilon^{x_n} \, \delta_\mu^{y_n} x_n \ = \ \delta_\varepsilon^{x} \, \delta_\mu^{y} x_n \quad .$$

The transformation $\delta_\varepsilon^{x} \, \delta_\mu^{y}$ is a contraction of coefficient $\nu(\varepsilon\mu) < 1$, therefore we easily get:

$$\lim_{n\to\infty} x_n \ = \ w \quad , \tag{6.0.4}$$

where w is the unique fixed point of the contraction $\delta_\varepsilon^{x} \, \delta_\mu^{y}$.

We put toghether (6.0.2) and (6.0.4) and we get the limit:

$$\lim_{n\to\infty} y_n \ = \ w \quad , \tag{6.0.5}$$

Using relations (6.0.4), (6.0.5), we may pass to the limit with $n \to \infty$ in relation (6.0.3):

$$\delta_\varepsilon^{x} \, \delta_\mu^{y} \ = \ \lim_{n\to\infty} \delta_\varepsilon^{x_n} \, \delta_\mu^{y_n} \ = \ \delta_\varepsilon^{w} \, \delta_\mu^{w} \ = \ \delta_{\varepsilon\mu}^{w} \quad .$$

The proof is done. □

Corollary 6.3 *Let (X, d, δ) be a strong linear dilatation structure, with group $\Gamma = (0, +\infty)$ and the morphism ν equal to identity. Any element of the inverse subsemigroup of $I(X)$ generated by dilatations is locally a dilatation δ_ε^x or a left translation $\Sigma^x(y, \cdot)$.*

Proof. Let (X, d, δ) be a strong linear dilatation structure. From the linearity and theorem 6.2 we deduce that we have to care only about the results of compositions of two dilatations δ_ε^x, δ_μ^y, with $\varepsilon\mu = 1$.

The dilatation structure is strong, therefore by theorem 5.4 the dilatation structure is locally coming from a conical group. In a conical group we can make the following computation (here $\delta_\varepsilon = \delta_\varepsilon^e$ with e the neutral element of the conical group):

$$\delta_\varepsilon^x \delta_{\varepsilon^{-1}}^y z \ = \ x\delta_\varepsilon \left(x^{-1} y \delta_{\varepsilon^{-1}} \left(y^{-1} z\right)\right) \ = \ x\delta_\varepsilon \left(x^{-1} y\right) y^{-1} z \quad .$$

Therefore the composition of dilatations $\delta_\varepsilon^x \delta_\mu^y$, with $\varepsilon\mu = 1$, is a left translation.

Another easy computation shows that composition of left translations with dilatations are dilatations. The proof end by remarking that all the statements are local. □

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Dilatation structures with the Radon-Nikodym property

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Introduction

The notion of a dilatation structure stemmed out from my efforts to understand basic results in sub-Riemannian geometry, especially the last section of the paper by Bellaïche [2] and the intrinsic point of view of Gromov [5].

In these papers, as in other articles devoted to sub-Riemannian geometry, fundamental results admiting an intrinsic formulation were proved using differential geometry tools, which are in my opinion not intrinsic to sub-Riemannian geometry.

Therefore I tried to find a self-contained frame in which sub-Riemannian geometry would be a model, if we use the same manner of speaking as in the case of hyperbolic geometry (with its self-contained collection of axioms) and the Poincaré disk as a model of hyperbolic geometry.

An outcome of this effort are the notions of a dilatation structure and a pair of dilatation structures, one looking down to another. To the first notion are dedicated the papers [3], [4] (the second paper treating about a "linear" version of a generalized dilatation structure, corresponding to Carnot groups or more general contractible groups).

As it seems now, dilatation structures are a valuable notion by itself, with possible field of application strictly containing sub-Riemannian geometry, but also ultrametric spaces or contractible groups. A dilatation structure encodes the approximate self-similarity of a metric space and it induces non associative but approximately associative operations on the metric space, as well as a tangent bundle (in the metric sense) with group operations in each fiber (tangent space to a point).

In this paper I explain what is a pair of dilatation structures, one looking down to another, see definition 3.5. Such a pair of dilatation structures leads to the intrinsic definition of a distribution as a field of topological filters, definition 3.6.

To any pair of dilatation structures there is an associated notion of differentiability which generalizes the Pansu differentiability [8]. This allows the introduction of the Radon-Nikodym property for dilatation structures, which is the straightforward generalization of the Radon-Nikodym property for Banach spaces.

After an introducting section about length metric spaces and metric derivatives, is proved in theorem 3.4 that for a dilatation structure with the Radon-Nikodym property the length of absolutely continuous curves expresses as an integral of the norms of the tangents to the curve, as in Riemannian geometry.

Further it is shown that Radon-Nikodym property transfers from any "upper" dilatation structure looking down to a "lower" dilatation structure, theorem 3.7. Im my opinion this result explains intrinsically the fact that absolutely continuous curves in regular sub-Riemannian manifolds are derivable almost everywhere, as proved by Margulis, Mostow [7], Pansu [8] (for Carnot groups) or Vodopyanov [10].

The subject of application of these results for regular sub-Riemannian manifold will be left for a future paper, due to the unavoidable accumulation of technical estimates which are needed.

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1 Notations

Let Γ be a topological separated commutative group endowed with a continuous group morphism

$$\nu : \Gamma \rightarrow (0, +\infty)$$

with $\inf \nu(\Gamma) = 0$. Here $(0, +\infty)$ is taken as a group with multiplication. The neutral element of Γ is denoted by 1. We use the multiplicative notation for the operation in Γ.

The morphism ν defines an invariant topological filter on Γ (equivalently, an end). Indeed, this is the filter generated by the open sets $\nu^{-1}(0, a)$, $a > 0$. From now on we shall name this topological filter (end) by "0" and we shall write $\varepsilon \in \Gamma \rightarrow 0$ for $\nu(\varepsilon) \in (0, +\infty) \rightarrow 0$.

The set $\Gamma_1 = \nu^{-1}(0, 1]$ is a semigroup. We note $\bar{\Gamma}_1 = \Gamma_1 \cup \{0\}$ On the set $\bar{\Gamma} = \Gamma \cup \{0\}$ we extend the operation on Γ by adding the rules $00 = 0$ and $\varepsilon 0 = 0$ for any $\varepsilon \in \Gamma$. This is in agreement with the invariance of the end 0 with respect to translations in Γ.

The space (X, d) is a complete, locally compact metric space. For any $r > 0$ and any $x \in X$ we denote by $B(x, r)$ the open ball of center x and radius r in the metric space X.

By $\mathcal{O}(\varepsilon)$ we mean a positive function $f : \Gamma \rightarrow [0, +\infty)$ such that $\lim_{\varepsilon \rightarrow 0} f(\nu(\varepsilon)) = 0$.

2 Length and metric derivatives

For a detailed intrduction into the subject see for example [1], chapter 1.

Definition 2.1 *The (upper) dilatation of a map $f : X \rightarrow Y$ between metric spaces, in a point $u \in Y$ is*

$$Lip(f)(u) = \limsup_{\varepsilon \rightarrow 0} \ \sup \left\{ \frac{d_Y(f(v), f(w))}{d_X(v, w)} \ : \ v \neq w \ , \ v, w \in B(u, \varepsilon) \right\}$$

In the particular case of a derivable function $f : \mathbb{R} \rightarrow \mathbb{R}^n$ the upper dilatation is $Lip(f)(t) = \mid \dot{f}(t) \mid$. For any Lipschitz function $f : X \rightarrow Y$ and for any $x \in X$ we have the obvious relation:

$$Lip(f)(x) \ \leq \ Lip(f) \ .$$

A curve is a continuous function $f : [a, b] \rightarrow X$. The image of a curve is called path. Length measures paths. Therefore length does not depends on the reparametrisation of the path and it is additive with respect to concatenation of paths.

In a metric space (X, d) one can measure the length of curves in several ways.

Definition 2.2 *The length of a curve with L^1 dilatation $f : [a, b] \rightarrow X$ is*

$$L(f) = \int_a^b Lip(f)(t) \ dt$$

A different way to define a length of a curve is to consider its variation.

Definition 2.3 *The curve f has bounded variation if the quantity*

$$Var(f) \ = \ \sup \left\{ \sum_{i=0}^{n} d(f(t_i), f(t_{i+1})) \ \ : a = t_0 < t_1 < ... < t_n < t_{n+1} = b \right\}$$

(called variation of f) is finite.

There is a third, more basic way to introduce the length of a curve in a metric space.

Definition 2.4 *The length of the path $A = f([a,b])$ is the one-dimensional Hausdorff measure of the path. The definition is the following:*

$$l(A) \ = \ \lim_{\delta \rightarrow 0} \inf \left\{ \sum_{i \in I} diam \ E_i \ : diam \ E_i < \delta \ , \quad A \subset \bigcup_{i \in I} E_i \right\}$$

The definitions are not equivalent. The variation $Var(f)$ of a curve f and the length of a path L(f) do not agree in general. Consider for example: $f : [-1,1] \rightarrow R^2$, $f(t) \ = \ (t, \ sign(t))$. We have $Var(f) \ = \ 4$ and $L(f([-1,1]) = 2$. Another example: the Cantor staircase function is continuous, but not Lipschitz. It has variation equal to 1 and length of the graph equal to 2.

Nevertheless, for Lipschitz functions, the first two definitions agree. For injective Lipschitz functions (i.e. for simple Lipschitz curves) the last two definitions agree.

Theorem 2.5 *For each Lipschitz curve $f : [a,b] \rightarrow X$, we have $L(f) \ = \ Var(f)$.*

Theorem 2.6 *Suppose that $f \ : \ [a,b] \ \rightarrow \ X$ is a Lipschitz function and $A \ = \ f([a,b])$. Then $\mathcal{H}^1(A) \leq Var(f)$.*

If f is moreover injective then $\mathcal{H}^1(A) = Var(f)$.

An important tool used in the proof of the previous theorem is the geometrically obvious, but not straightforward to prove in this generality, Reparametrisation Theorem.

Theorem 2.7 *Any path $A \subset X$ with a Lipschitz parametrisation admits a reparametrisation $f : [a,b] \rightarrow A$ such that $Lip(f)(t) = 1$ for almost any $t \in [a,b]$.*

We shall denote by l_d the length functional, defined only on Lipschitz curves, induced by the distance d. The length induces a new distance d_l, say on any Lipschitz connected component of the space (X, d). The distance d_l is given by:

$$d_l(x,y) \ = \ \inf \ \{l_d(f([a,b])) \, : \, f : [a,b] \to X \ \text{ Lipschitz } ,$$

$$f(a) = x \ , \ f(b) = y\}$$

We have therefore two operators $d \mapsto l_d$ and $l \mapsto d_l$. This leads to the introduction of length metric spaces.

Definition 2.8 *A length metric space is a metric space* (X, d) *such that* $d = d_l$*.*

From theorem 2.5 we deduce that Lipschitz curves in complete length metric spaces are absolutely continuous. Indeed, here is the definition of an absolutely continuous curve (definition 1.1.1, chapter 1, [1]).

Definition 2.9 *Let* (X, d) *be a complete metric space. A curve* $c : (a, b) \to X$ *is absolutely continuous if there exists* $m \in L^1((a, b))$ *such that for any* $a < s \leq t < b$ *we have*

$$d(c(s), c(t)) \leq \int_s^t m(r) \ dr.$$

Such a function m *is called an upper gradient of the curve* c*.*

According to theorem 2.5, for a Lipschitz curve $c : [a, b] \to X$ in a complete lenght metric space such a function $m \in L^1((a, b))$ is the upper dilatation $Lip(c)$. More can be said about the expression of the upper dilatation. We need first to introduce the notion of metric derivative of a Lipschitz curve.

Definition 2.10 *A curve* $c : (a, b) \to X$ *is metrically derivable in* $t \in (a, b)$ *if the limit*

$$md(c)(t) = \lim_{s \to t} \frac{d(c(s), c(t))}{\mid s - t \mid}$$

exists and it is finite. In this case $md(c)(t)$ *is called the metric derivative of* c *in* t*.*

For the proof of the following theorem see [1], theorem 1.1.2, chapter 1.

Theorem 2.11 *Let* (X, d) *be a complete metric space and* $c : (a, b) \to X$ *be an absolutely continuous curve. Then* c *is metrically derivable for* $\mathcal{L}^1$*-a.e.* $t \in (a, b)$*. Moreover the function* $md(c)$ *belongs to* $L^1((a, b))$ *and it is minimal in the following sense:* $md(c)(t) \leq m(t)$ *for* $\mathcal{L}^1$*-a.e.* $t \in (a, b)$*, for each upper gradient* m *of the curve* c*.*

3 The Radon-Nikodym property

Definition 3.1 *A dilatation structure (X, d, δ) has the Radon-Nikodym property if any Lipschitz curve $c : [a, b] \rightarrow (X, d)$ is derivable almost everywhere.*

Example 3.1 For $(X, d) = (\mathbb{V}, d)$, a real, finite dimensional, normed vector space, with distance d induced by the norm, the (usual) dilatations δ_ε^x are given by:

$$\delta_\varepsilon^x y \ = \ x + \varepsilon(y - x)$$

Dilatations are defined everywhere. The group Γ is $(0, +\infty)$ and the function ν is the identity.

There are few things to check (see the appendix): axioms 0,1,2 are obviously true. For axiom A3, remark that for any $\varepsilon > 0$, $x, u, v \in X$ we have:

$$\frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) \ = \ d(u, v) \ ,$$

therefore for any $x \in X$ we have $d^x = d$.

Finally, let us check the axiom A4. For any $\varepsilon > 0$ and $x, u, v \in X$ we have

$$\delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v \ = \ x + \varepsilon(u - x) + \frac{1}{\varepsilon} \left(x + \varepsilon(v - x) - x - \varepsilon(u - x) \right) \ =$$

$$= \ x + \varepsilon(u - x) + v - u$$

therefore this quantity converges to

$$x + v - u \ = \ x + (v - x) - (u - x)$$

as $\varepsilon \rightarrow 0$. The axiom A4 is verified.

This dilatation structure has the Radon-Nikodym property. □

Example 3.2 Because dilatation structures are defined by local requirements, we can easily define dilatation structures on riemannian manifolds, using particular atlases of the manifold and the riemannian distance (infimum of length of curves joining two points). Note that any finite dimensional manifold can be endowed with a riemannian metric. This class of examples covers all dilatation structures used in differential geometry. The axiom A4 gives an operation of addition of vectors in the tangent space (compare with Bellaïche [2] last section). □

Example 3.3 Take $X = \mathbb{R}^2$ with the euclidean distance d. For any $z \in \mathbb{C}$ of the form $z = 1 + i\theta$ we define dilatations

$$\delta_\varepsilon x = \varepsilon^z x \ .$$

It is easy to check that $(\mathbb{R}^2, d, \delta)$ is a dilatation structure, with dilatations

$$\delta_\varepsilon^x y = x + \delta_\varepsilon(y - x) \quad .$$

Two such dilatation structures (constructed with the help of complex numbers $1+i\theta$ and $1+i\theta'$) are equivalent if and only if $\theta=\theta'$.

There are two other interesting properties of these dilatation structures. The first is that if $\theta \neq 0$ then there are no non trivial Lipschitz curves in X which are differentiable almost everywhere. It means that such dilatation structure does not have the Radon-Nikodym property.

The second property is that any holomorphic and Lipschitz function from X to X (holomorphic in the usual sense on $X = \mathbb{R}^2 = \mathbb{C}$) is differentiable almost everywhere, but there are Lipschitz functions from X to X which are not differentiable almost everywhere (suffices to take a $\mathcal{C}^\infty$ function from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is not holomorphic). $\square$

The Radon-Nikodym property can be stated in two equivalent ways.

Proposition 3.2 *Let (X, d, δ) be a dilatation structure. Then the following are equivalent:*

(a) (X, d, δ) has the Radon-Nikodym property;

(b) any Lipschitz curve $c' : [a', b'] \to (X, d)$ admits a reparametrization $c : [a, b] \to (X, d)$ such that for almost every $t \in [a, b]$ there is $\dot{c}(t) \in U(c(t))$ such that

$$\frac{1}{\varepsilon} d(c(t+\varepsilon), \delta_\varepsilon^{c(t)} \dot{c}(t)) \to 0$$

$$\frac{1}{\varepsilon} d(c(t-\varepsilon), \delta_\varepsilon^{c(t)} inv^{c(t)}(\dot{c}(t))) \to 0 \quad ;$$

(c) any Lipschitz curve $c' : [a', b'] \to (X, d)$ admits a reparametrization $c : [a, b] \to (X, d)$ such that for almost every $t \in [a, b]$ there is a conical group morphism

$$\dot{c}(t) : \mathbb{R} \to T_{c(t)} X$$

such that for any $a \in \mathbb{R}$ we have

$$\frac{1}{\varepsilon} d(c(t+\varepsilon a), \delta_\varepsilon^{c(t)} \dot{c}(t)(a)) \to 0.$$

Proof. It is straightforward that a conical group morphism $f : \mathbb{R} \to (N, \delta)$ is defined by its value $f(1) \in N$. Indeed, for any $a > 0$ we have $f(a) = \delta_a f(1)$ and for any $a < 0$ we have $f(a) = \delta_a f(1)^{-1}$. From the morphism property we also deduce that

$$\delta v = \left\{ \delta_a v \, : \, a > 0, v = f(1) \text{ or } v = f(1)^{-1} \right\}$$

is a one parameter group and that for all $\alpha, \beta > 0$ we have

$$\delta_{\alpha+\beta} u = \delta_\alpha u \delta_\beta u \quad \square$$

Definition 3.3 *In a conical group N we shall denote by $D(N)$ the set of all $u \in N$ with the property that $\varepsilon \in ((0,\infty),+) \mapsto \delta_\varepsilon u \in N$ is a morphism of semigroups .*

$D(N)$ is always non empty, because it contains the neutral element of N. $D(N)$ is also a cone, with dilatations δ_ε, and a closed set.

We shall always identify a conical group morphism $f : \mathbb{R} \rightarrow N$ with its value $f(1) \in D(N)$.

3.1 Length formula from Radon-Nikodym property

Theorem 3.4 *Let (X, d, δ) be a dilatation structure with the Radon-Nikodym property, over a complete length metric space (X, d). Then for any Lipschitz curve $c : [a, b] \rightarrow X$ the length of $\gamma = c([a, b])$ is*

$$L(\gamma) = \int_a^b d^{c(t)}(c(t), \dot{c}(t)) \ dt.$$

Proof. The upper dilatation of c in t is

$$Lip(c)(t) = \limsup_{\varepsilon \rightarrow 0} \ \sup \left\{ \frac{d(c(v), c(w))}{\mid v - w \mid} \ : \ v \neq w \ , \ \mid v - t \mid, \mid w - t \mid < \varepsilon \right\}.$$

From theorem 2.11 we deduce that for almost every $t \in (a, b)$ we have

$$Lip(c)(t) = \lim_{s \rightarrow t} \frac{d(c(s), c(t))}{\mid s - t \mid}.$$

If the dilatation structure has the Radon-Nikodym property then for almost every $t \in [a, b]$ there is $\dot{c}(t) \in D(T_{c(t)}X)$ such that

$$\frac{1}{\varepsilon} d(c(t + \varepsilon), \delta_\varepsilon^{c(t)} \dot{c}(t)) \rightarrow 0.$$

Therefore for almost every $t \in [a, b]$ we have

$$Lip(c)(t) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} d(c(t + \varepsilon), c(t)) = d^{c(t)}(c(t), \dot{c}(t)).$$

The formula for length follows from here. $\square$

3.2 A dilatation structure looking down to another

Consider two dilatation structures $\mathcal{A} = (X, d_A, \delta)$ and $\mathcal{B} = (X, d_B, \bar{\delta})$. We explain here in which sense $\mathcal{A}$ looks down at $\mathcal{B}$.

Definition 3.5 *Given dilatation structures $\mathcal{A} = (X, d_A, \delta)$ and $\mathcal{B} = (X, d_B, \bar{\delta})$, we write that $\mathcal{A} \geq \mathcal{B}$ if the following conditions are fulfilled:*

(a) the identity $id : (X, d_A) \rightarrow (X, d_B)$ is 1-Lipschitz,

(b) the identity $id : (X, d_A) \rightarrow (X, d_B)$ is derivable everywhere and for any point $x \in X$ the derivative $D\, id(x)$ is a projector,

(c) for any $x \in X$, any continuous curve $\varepsilon \in [0,1) \mapsto z(\varepsilon) \in X$, such that $d_A^x(z(0), x) \leq 3/2$, if

$$\lim_{\varepsilon \rightarrow 0} \left(d_A^x(x, z(\varepsilon)) - \frac{1}{\varepsilon} d_B^x(x, \delta_\varepsilon^x z(\varepsilon)) \right) \ = \ 0$$

then $\lim_{\varepsilon \rightarrow 0} d_A^x(Q_\varepsilon^x z(\varepsilon), z(\varepsilon)) \ = 0$, where $Q_\varepsilon^x = \ \bar{\delta}_{\varepsilon^{-1}}^x \delta_\varepsilon^x$.

We explain in more detail the meaning of this definition. Condition (a) says that for any $x, y \in X$ we have $d_B(x,y) \leq d_A(x,y)$. Condition (b) can be understood by using definition 4.10: for any $x \in X$ there exists a function $D\, id(x)$ defined on a neighbourhood of x with values in a neighbourhood of $f(x)$ such that

$$\lim_{\varepsilon \rightarrow 0} \sup \left\{ \frac{1}{\varepsilon} \overline{d} \left(\delta_\varepsilon^x u, \overline{\delta}_\varepsilon^x D\, id(x)(u) \right) \ : \ d(x,u) \leq \varepsilon \right\} = 0. \qquad (3.2.1)$$

From here we deduce that for any x and u such that $d_B(x,u)$ is sufficiently small

$$\lim_{\varepsilon \rightarrow 0} \overline{\delta}_{\varepsilon^{-1}}^x \delta_\varepsilon^x(u) = D\, id(x)(u)$$

and the limit is uniform with respect to u.

The second part of the condition (b) states that

$$D\, id(x) D\, id(x) = D\, id(x).$$

In order to understand the condition (c) we need to introduce the following topological version of a distribution.

Definition 3.6 *We denote by $TopD(x)$ the topological filter induced by the relatively open neighbourhoods of x in the closed ball $\{z \in X \ : \ d_A^x(x,z) \leq 2\}$, given by*

$$F(x, \varepsilon, \lambda) \ = \ \left\{ z \in X \ : \ d_A^x(x,z) \leq 2 \ , \ d_A^x(x,z) - \frac{1}{\varepsilon} d_B^x(x, \delta_\varepsilon^x z) \leq \lambda \right\} \quad .$$

This filter is called the topological distribution associated with the pair of dilatation structures $\mathcal{A} = (X, d_A, \delta)$ and $\mathcal{B} = (X, d_B, \bar{\delta})$, such that $\mathcal{A} \geq \mathcal{B}$.

With this notation we may rewrite the condition (c) definition 3.5 like this: let $z(\varepsilon)$ be a continuous curve such that (in the sense of topological filters)

$$\lim_{\varepsilon \rightarrow 0} z(\varepsilon) \ \in \ TopD(x) \quad .$$

Then $\lim_{\varepsilon \rightarrow 0} d_A^x(Q_\varepsilon^x z(\varepsilon), z(\varepsilon)) \ = 0$, where $Q_\varepsilon^x = \ \bar{\delta}_{\varepsilon^{-1}}^x \delta_\varepsilon^x$. This means that the "size of the vertical part" of $z(\varepsilon)$, which is $d_A^x(Q_\varepsilon^x z(\varepsilon), z(\varepsilon))$, becomes arbitrarily small as $\varepsilon \rightarrow 0$.

3.3 Transfer of Radon-Nikodym property

Suppose that (X, d_A) and (X, d_B) are complete, locally compact, length metric spaces and that we have two dilatation structures $\mathcal{A} = (X, d_A, \delta)$ and $\mathcal{B} = (X, d_B, \bar{\delta})$, such that $\mathcal{A} \geq \mathcal{B}$.

A sufficient condition to have (a) in definition 3.5 is the following (true in the case of sub-Riemannian manifolds):

(a') for any Lipschitz curve c, if $l_A(c) < +\infty$ then $l_B(c) = l_A(c)$. Here l_A and l_B denote the length functional associated to distance d_A, distance d_B respectively.

We prove here the following result concerning the transfer of Radon-Nikodym property.

Theorem 3.7 *Let (X, d_A) and (X, d_B) be complete, locally compact, length metric spaces. Suppose that we have two dilatation structures $\mathcal{A} = (X, d_A, \delta)$ and $\mathcal{B} = (X, d_B, \bar{\delta})$, such that $\mathcal{A} \geq \mathcal{B}$. Under the assumptions (a') and (b), (c), (d) from definition 3.5, if the dilatation structure $\mathcal{B} = (X, d_B, \bar{\delta})$ has the Radon-Nikodym property, then the dilatation structure $\mathcal{A} = (X, d_A, \delta)$ has the Radon-Nikodym property.*

Proof. Let $c : [0, 1] \rightarrow (X, d_A)$ be a Lipschitz curve. Because of hypothesis (a) it follows that $c : [0, 1] \rightarrow (X, d_B)$ is also Lipschitz. Moreover, we can reparametrize the curve c with the d_A lenght and so we can suppose that c is d_A 1-Lipschitz. Therefore we can suppose that c is d_B 1-Lipschitz.

The dilatation structure $\mathcal{B} = (X, d_B, \bar{\delta})$ has the Radon-Nikodym property. Then for almost any $t \in [0, 1]$ there is $\dot{c}(t)$ such that

$$\frac{1}{\varepsilon} d_B(c(t + \varepsilon), \bar{\delta}^x_\varepsilon \dot{c}(t)) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \tag{3.3.2}$$

$$\frac{1}{\varepsilon} d_B(c(t - \varepsilon), \bar{\delta}^x_\varepsilon \dot{c}(t)^{-1}) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \tag{3.3.3}$$

Further we shall give only the half of the proof, namely we shall use only relation (3.3.2). To get a complete proof, one has to repeat the reasoning starting from (3.3.3).

Because c is d_A 1- Lipschitz, it follows that

$$d_A^{c(t)}(\delta^{c(t)}_{\varepsilon^{-1}} c(t + \varepsilon), \dot{c}(t)) \leq 2$$

for any $\varepsilon < \varepsilon(t) \in (0, +\infty)$. From the local compactness with respect to $d_A^{c(t)}$ we find that for any $t \in [0, 1]$ there is a sequence $(\varepsilon_h)_h \subset (0, +\infty)$, converging to 0 as $h \rightarrow \infty$, and $u(t) \in X$ such that:

$$\lim_{h \rightarrow \infty} \delta^{c(t)}_{\varepsilon_h^{-1}} c(t + \varepsilon_h) \ = \ u(t)$$

Use equation (3.3.2) to get that

$$\lim_{h\to\infty} \bar{\delta}^{c(t)}_{\varepsilon_h^{-1}} c(t+\varepsilon_h) \ = \ \dot{c}(t)$$

Re-write this latter equation as:

$$\lim_{h\to\infty} \bar{\delta}^{c(t)}_{\varepsilon_h^{-1}} \delta^{c(t)}_{\varepsilon_h} \delta^{c(t)}_{\varepsilon_h^{-1}} c(t+\varepsilon_h) \ = \ \dot{c}(t)$$

and use the first part of hypothesis (b) to get

$$D\ id(c(t))u(t) \ = \ \dot{c}(t)$$

But according to the second part of the hypothesis (b) the operator $D\ id(c(t))$ is a projector, hence

$$D\ id(c(t))\dot{c}(t) \ = \ \dot{c}(t)$$

Because of the fact that the derivative commutes with dilatations we get the important fact that for any $\varepsilon > 0$

$$\delta^{c(t)}_{\varepsilon} \dot{c}(t) \ = \ \bar{\delta}^{c(t)}_{\varepsilon} \dot{c}(t) \qquad (3.3.4)$$

We wish to prove

$$\frac{1}{\varepsilon} d_A(\delta^{c(t)}_{\varepsilon} \bar{\delta}^{c(t)}_{\varepsilon^{-1}} c(t+\varepsilon), c(t+\varepsilon)) \to 0 \quad \text{as } \varepsilon \to 0 \qquad (3.3.5)$$

Suppose that (3.3.5) is true. Then we would have

$$d^{c(t)}_A(\bar{\delta}^{c(t)}_{\varepsilon^{-1}} c(t+\varepsilon), \delta^{c(t)}_{\varepsilon^{-1}} c(t+\varepsilon)) \to 0 \quad \text{as } \varepsilon \to 0$$

But relations (3.3.2) and (3.3.4) imply that

$$d^{c(t)}_A(\bar{\delta}^{c(t)}_{\varepsilon^{-1}} c(t+\varepsilon), \dot{c}(t)) \to 0 \quad \text{as } \varepsilon \to 0$$

therefore we would finally get

$$d^{c(t)}_A(\delta^{c(t)}_{\varepsilon^{-1}} c(t+\varepsilon), \dot{c}(t)) \to 0 \quad \text{as } \varepsilon \to 0$$

which is what we want to prove: that the curve c is derivable in t with respect to the dilatation structure $\mathcal{A}$.

Let us prove the relation (3.3.5). According to hypothesis (a') we have:

$$0 \leq \frac{1}{\varepsilon} d_A(c(t+\varepsilon), c(t)) - \frac{1}{\varepsilon} d_B(c(t+\varepsilon), c(t)) \ \leq \ \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mid \dot{c}(\tau) \mid_B \ \mathrm{d}\tau - \frac{1}{\varepsilon} d_B(c(t+\varepsilon), c(t))$$

where the quantity

$$\mid \dot{c}(s) \mid_B = \ \lim_{\varepsilon\to 0} \frac{d_B((c(s+\varepsilon), c(s))}{\varepsilon} \ = \ d^{c(s)}_B(c(s), \dot{c}(s))$$

exists for almost every $s \in [0,1]$, according to theorem 2.11.

We obtain therefore the relation:

$$\frac{1}{\varepsilon} d_A(c(t+\varepsilon), c(t)) - \frac{1}{\varepsilon} d_B(c(t+\varepsilon), c(t)) \ \to 0 \quad \text{as } \varepsilon \to 0 \tag{3.3.6}$$

Here is the moment to use the last hypothesis (d). Indeed, the relation (3.3.6) implies that

$$d_A^{c(t)}(c(t), \delta_{\varepsilon^{-1}}^{c(t)} c(t+\varepsilon)) - \frac{1}{\varepsilon} d_B^{c(t)}(c(t+\varepsilon), c(t)) \ \to 0 \quad \text{as } \varepsilon \to 0 \tag{3.3.7}$$

Denote by $z(t,\varepsilon) \ = \ \delta_{\varepsilon^{-1}}^{c(t)} c(t+\varepsilon)$. The relation (3.3.7) becomes:

$$d_A^{c(t)}(c(t), z(t,\varepsilon)) - \frac{1}{\varepsilon} d_B^{c(t)}(c(t), \delta_\varepsilon^x z(t,\varepsilon)) \ \to 0 \quad \text{as } \varepsilon \to 0 \tag{3.3.8}$$

We also have

$$d_A^{c(t)}(c(t), z(t,\varepsilon)) \ = \ \frac{1}{\varepsilon} d_A^{c(t)}(c(t), c(t+\varepsilon)) \leq 2$$

for ε sufficiently small, because we supposed that c was reparametrized with the length. Therefore, with the notations from definition 3.6 and the paragraph following it, we have

$$\lim_{\varepsilon \to 0} z(t,\varepsilon) \ \in \ TopD(c(t))$$

From the hypothesis (d) we deduce that

$$\lim_{\varepsilon \to 0} d_A^{c(t)}(z(t,\varepsilon), Q_\varepsilon^{c(t)} z(t,\varepsilon)) \ = \ 0$$

Let us see what this means:

$$\lim_{\varepsilon \to 0} d_A^{c(t)}(\bar{\delta}_\varepsilon^{c(t)} c(t+\varepsilon), \delta_\varepsilon^{c(t)} c(t+\varepsilon)) \ = \ 0$$

This relation is equivalent with (3.3.5), so the proof is done. ■

4 Appendix: Dilatation structures

For the sake of completeness we list in this appendix the definition and properties of a dilatation structure, according to [3], [4].

4.1 The axioms of a dilatation structure

The axioms of a dilatation structure (X, d, δ) are listed further. The first axiom is merely a preparation for the next axioms. That is why we counted it as axiom 0.

A0. The dilatations

$$\delta_\varepsilon^x : U(x) \to V_\varepsilon(x)$$

are defined for any $\varepsilon \in \Gamma, \nu(\varepsilon) \leq 1$. The sets $U(x), V_\varepsilon(x)$ are open neighbourhoods of x. All dilatations are homeomorphisms (invertible, continuous, with continuous inverse).

We suppose that there is a number $1 < A$ such that for any $x \in X$ we have

$$\bar{B}_d(x, A) \subset U(x) \ .$$

We suppose that for all $\varepsilon \in \Gamma$, $\nu(\varepsilon) \in (0, 1)$, we have

$$B_d(x, \nu(\varepsilon)) \subset \delta_\varepsilon^x B_d(x, A) \subset V_\varepsilon(x) \subset U(x) \ .$$

There is a number $B \in (1, A)$ such that for any $\nu(\varepsilon) \in (1, +\infty)$ the associated dilatation

$$\delta_\varepsilon^x : W_\varepsilon(x) \to B_d(x, B) \ ,$$

is injective, invertible on the image. We shall suppose that $W_\varepsilon(x)$ is a open neighbourhood of x,

$$V_{\varepsilon^{-1}}(x) \subset W_\varepsilon(x)$$

and that for all $\varepsilon \in \Gamma_1$ and $u \in U(x)$ we have

$$\delta_{\varepsilon^{-1}}^x \ \delta_\varepsilon^x u \ = \ u \ .$$

We have therefore the following string of inclusions, for any $\varepsilon \in \Gamma$, $\nu(\varepsilon) \leq 1$, and any $x \in X$:

$$B_d(x, \nu(\varepsilon)) \subset \delta_\varepsilon^x B_d(x, A) \subset V_\varepsilon(x) \subset W_{\varepsilon^{-1}}(x) \subset \delta_\varepsilon^x B_d(x, B) \quad .$$

A further technical condition on the sets $V_\varepsilon(x)$ and $W_\varepsilon(x)$ will be given just before the axiom A4. (This condition will be counted as part of axiom A0.)

A1. We have $\delta_\varepsilon^x x = x$ for any point x. We also have $\delta_1^x = id$ for any $x \in X$.

Let us define the topological space

$$dom\,\delta = \{(\varepsilon, x, y) \in \Gamma \times X \times X : \quad \text{if } \nu(\varepsilon) \leq 1 \text{ then } y \in U(x) \ ,$$

$$\text{else } y \in W_\varepsilon(x)\}$$

with the topology inherited from the product topology on $\Gamma \times X \times X$. Consider also $Cl(dom\,\delta)$, the closure of $dom\,\delta$ in $\bar{\Gamma} \times X \times X$ with product topology. The function $\delta : dom\,\delta \to X$ defined by $\delta(\varepsilon, x, y) = \delta_\varepsilon^x y$ is continuous. Moreover, it can be continuously extended to $Cl(dom\,\delta)$ and we have

$$\lim_{\varepsilon \to 0} \delta_\varepsilon^x y \ = \ x \quad .$$

A2. For any $x, \in K$, $\varepsilon, \mu \in \Gamma_1$ and $u \in \bar{B}_d(x, A)$ we have:

$$\delta_\varepsilon^x \delta_\mu^x u = \delta_{\varepsilon\mu}^x u \ .$$

A3. For any x there is a function $(u, v) \mapsto d^x(u, v)$, defined for any u, v in the closed ball (in distance d) $\bar{B}(x, A)$, such that

$$\lim_{\varepsilon \to 0} \quad \sup \left\{ \mid \frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) \ - \ d^x(u, v) \mid \ : \ u, v \in \bar{B}_d(x, A) \right\} \ = \ 0$$

uniformly with respect to x in compact set.

Remark 4.1 *The "distance" d^x can be degenerated: there might exist $v, w \in U(x)$ such that $d^x(v, w) = 0$.*

For the following axiom to make sense we impose a technical condition on the co-domains $V_\varepsilon(x)$: for any compact set $K \subset X$ there are $R = R(K) > 0$ and $\varepsilon_0 = \varepsilon(K) \in (0, 1)$ such that for all $u, v \in \bar{B}_d(x, R)$ and all $\varepsilon \in \Gamma$, $\nu(\varepsilon) \in (0, \varepsilon_0)$, we have

$$\delta_\varepsilon^x v \in W_{\varepsilon^{-1}}(\delta_\varepsilon^x u) \ .$$

With this assumption the following notation makes sense:

$$\Delta_\varepsilon^x(u, v) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v.$$

The next axiom can now be stated:

A4. We have the limit

$$\lim_{\varepsilon \to 0} \Delta_\varepsilon^x(u, v) = \Delta^x(u, v)$$

uniformly with respect to x, u, v in compact set.

Definition 4.2 *A triple (X, d, δ) which satisfies A0, A1, A2, A3, but d^x is degenerate for some $x \in X$, is called degenerate dilatation structure.*

If the triple (X, d, δ) satisfies A0, A1, A2, A3, A4 and d^x is non-degenerate for any $x \in X$, then we call it a dilatation structure.

4.2 Tangent bundle of a dilatation structure

The following two theorems describe the most important metric and algebraic properties of a dilatation structure. As presented here these are condensed statements, available in full length as theorems 7, 8, 10 in [3].

Theorem 4.3 *Let (X, d, δ) be a dilatation structure. Then the metric space (X, d) admits a metric tangent space at x, for any point $x \in X$. More precisely we have the following limit:*

$$\lim_{\varepsilon \to 0} \ \frac{1}{\varepsilon} \sup \left\{ \mid d(u, v) - d^x(u, v) \mid \ : d(x, u) \leq \varepsilon \ , \ d(x, v) \leq \varepsilon \right\} \ = \ 0 \ .$$

Theorem 4.4 *Let (X, d, δ) be a dilatation structure. Then for any $x \in X$ the triple $(U(x), \Sigma^x, \delta^x, d^x)$ is a normed local conical group. This means:*

(a) Σ^x is a local group operation on $U(x)$, with x as neutral element and inv^x as the inverse element function;

(b) the distance d^x is left invariant with respect to the group operation from point (a);

(c) For any $\varepsilon \in \Gamma$, $\nu(\varepsilon) \leq 1$, the dilatation δ^x_ε is an automorphism with respect to the group operation from point (a);

(d) the distance d^x has the cone property with respect to dilatations: foar any $u, v \in X$ such that $d(x,u) \leq 1$ and $d(x,v) \leq 1$ and all $\mu \in (0, A)$ we have:

$$d^x(u,v) \; = \; \frac{1}{\mu} d^x(\delta^x_\mu u, \delta^x_\mu v) \quad .$$

The conical group $(U(x), \Sigma^x, \delta^x)$ can be regarded as the tangent space of (X, d, δ) at x. Further will be denoted by: $T_x X = (U(x), \Sigma^x, \delta^x)$.

By using proposition 5.4 [9] and from some topological considerations we deduce the following characterisation of tangent spaces asociated to some dilatation structures. The following is corollary 4.7 [4].

Corollary 4.5 *Let (X, d, δ) be a dilatation structure with group $\Gamma = (0, +\infty)$ and the morphism ν equal to identity. Then for any $x \in X$ the local group $(U(x), \Sigma^x)$ is locally a simply connected Lie group whose Lie algebra admits a positive graduation (a Carnot group).*

4.3 Equivalent dilatation structures

Definition 4.6 *Two dilatation structures (X, δ, d) and $(X, \overline{\delta}, \overline{d})$ are equivalent if*

(a) the identity map $id : (X, d) \to (X, \overline{d})$ is bilipschitz and

(b) for any $x \in X$ there are functions P^x, Q^x (defined for $u \in X$ sufficiently close to x) such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \overline{d} \left(\delta^x_\varepsilon u, \overline{\delta}^x_\varepsilon Q^x(u) \right) = 0, \tag{4.3.1}$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d \left(\overline{\delta}^x_\varepsilon u, \delta^x_\varepsilon P^x(u) \right) = 0, \tag{4.3.2}$$

uniformly with respect to x, u in compact sets.

Proposition 4.7 *Two dilatation structures (X, δ, d) and $(X, \overline{\delta}, \overline{d})$ are equivalent if and only if* *(a) the identity map* $id : (X, d) \to (X, \overline{d})$ *is bilipschitz and*

(b) for any $x \in X$ *there are functions* P^x, Q^x *(defined for* $u \in X$ *sufficiently close to* x*) such that*

$$\lim_{\varepsilon \to 0} \left(\overline{\delta}_\varepsilon^x\right)^{-1} \delta_\varepsilon^x(u) = Q^x(u), \qquad (4.3.3)$$

$$\lim_{\varepsilon \to 0} \left(\delta_\varepsilon^x\right)^{-1} \overline{\delta}_\varepsilon^x(u) = P^x(u), \qquad (4.3.4)$$

uniformly with respect to x*,* u *in compact sets.*

The next theorem shows a link between the tangent bundles of equivalent dilatation structures.

Theorem 4.8 *Let* (X, δ, d) *and* $(X, \overline{\delta}, \overline{d})$ *be equivalent dilatation structures. Suppose that for any* $x \in X$ *the distance* d^x *is non degenerate. Then for any* $x \in X$ *and any* $u, v \in X$ *sufficiently close to* x *we have:*

$$\overline{\Sigma}^x(u, v) = Q^x\left(\Sigma^x\left(P^x(u), P^x(v)\right)\right). \qquad (4.3.5)$$

The two tangent bundles are therefore isomorphic in a natural sense.

4.4 Differentiable functions

Dilatation structures allow to define differentiable functions. The idea is to keep only one relation from definition 4.6, namely (4.3.1). We also renounce to uniform convergence with respect to x and u, and we replace this with uniform convergence in the "u" variable, with a conical group morphism condition for the derivative.

Definition 4.9 *Let* (N, δ) *and* $(M, \bar{\delta})$ *be two conical groups. A continuous function* $f : N \to M$ *is a conical group morphism if* f *is a group morphism and for any* $\varepsilon > 0$ *and* $u \in N$ *we have* $f(\delta_\varepsilon u) = \bar{\delta}_\varepsilon f(u)$*.*

Definition 4.10 *Let* (X, δ, d) *and* $(Y, \overline{\delta}, \overline{d})$ *be two dilatation structures and* $f : X \to Y$ *be a continuous function. The function* f *is differentiable in* x *if there exists a conical group morphism* $Q^x : T_x X \to T_{f(x)} Y$*, defined on a neighbourhood of* x *with values in a neighbourhood of* $f(x)$ *such that*

$$\lim_{\varepsilon \to 0} \sup \left\{ \frac{1}{\varepsilon} \overline{d}\left(f\left(\delta_\varepsilon^x u\right), \overline{\delta}_\varepsilon^{f(x)} Q^x(u) \right) \ : \ d(x, u) \leq \varepsilon \right\} = 0, \qquad (4.4.6)$$

The morphism Q^x *is called the derivative of* f *at* x *and will be sometimes denoted by* $Df(x)$*.*

The function f *is uniformly differentiable if it is differentiable everywhere and the limit in (4.4.6) is uniform in* x *in compact sets.*

A trivial way to obtain a differentiable function (everywhere) is to modify the dilatation structure on the target space.

Definition 4.11 *Let (X, δ, d) be a dilatation structure and $f : (X, d) \rightarrow (Y, \overline{d})$ be a bilipschitz and surjective function. We define then the transport of (X, δ, d) by f, named $(Y, f * \delta, \overline{d})$, by:*

$$(f * \delta)_{\varepsilon}^{f(x)} f(u) = f\left(\delta_{\varepsilon}^{x} u\right).$$

The relation of differentiability with equivalent dilatation structures is given by the following simple proposition.

Proposition 4.12 *Let (X, δ, d) and $(X, \overline{\delta}, \overline{d})$ be two dilatation structures and $f : (X, d) \rightarrow (X, \overline{d})$ be a bilipschitz and surjective function. The dilatation structures $(X, \overline{\delta}, \overline{d})$ and $(X, f * \delta, \overline{d})$ are equivalent if and only if f and f^{-1} are uniformly differentiable.*

We shall prove now the chain rule for derivatives, after we elaborate a bit over the definition 4.10.

Let (X, δ, d) and $(Y, \overline{\delta}, \overline{d})$ be two dilatation structures and $f : X \rightarrow Y$ a function differentiable in x. The derivative of f in x is a conical group morphism $Df(x) : T_x X \rightarrow T_{f(x)} Y$, which means that $Df(x)$ is defined on a open set around x with values in a open set around $f(x)$, having the properties:

(a) for any u, v sufficiently close to x

$$Df(x)\left(\Sigma^{x}(u, v)\right) = \Sigma^{f(x)}\left(Df(x)(u), Df(x)(v)\right),$$

(b) for any u sufficiently close to x and any $\varepsilon \in (0, 1]$

$$Df(x)\left(\delta_{\varepsilon}^{x} u\right) = \bar{\delta}_{\varepsilon}^{f(x)}\left(Df(x)(u)\right),$$

(c) the function $Df(x)$ is continuous, as uniform limit of continuous functions. Indeed, the relation (4.4.6) is equivalent to the existence of the uniform limit (with respect to u in compact sets)

$$Df(x)(u) = \lim_{\varepsilon \rightarrow 0} \bar{\delta}_{\varepsilon^{-1}}^{f(x)}\left(f\left(\delta_{\varepsilon}^{x} u\right)\right).$$

From (4.4.6) alone and axioms of dilatation structures we can prove properties (b) and (c). We can reformulate therefore the definition of the derivative by asking that $Df(x)$ exists as an uniform limit (as in point (c) above) and that $Df(x)$ has the property (a) above.

From these considerations the chain rule for derivatives is straightforward.

Proposition 4.13 *Let* (X, δ, d), $(Y, \overline{\delta}, \overline{d})$ *and* $(Z, \hat{\delta}, \hat{d})$ *be three dilatation structures and* $f : X \to Y$ *a continuous function differentiable in* x, $g : Y \to Z$ *a continuous function differentiable in* $f(x)$. *Then* $gf : X \to Z$ *is differentiable in* x *and*

$$Dgf(x) = Dg(f(x))Df(x).$$

Proof. Use property (b) for proving that $Dg(f(x))Df(x)$ satisfies (4.4.6) for the function gf and x. Both $Dg(f(x))$ and $Df(x)$ are conical group morphisms, therefore $Dg(f(x))Df(x)$ is a conical group morphism too. We deduce that $Dg(f(x))Df(x)$ is the derivative of gf in x. $\square$

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Dilatation structures in sub-riemannian geometry

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Abstract

Based on the notion of dilatation structure [2], we give an intrinsic treatment to sub-riemannian geometry, started in the paper [4]. Here we prove that regular sub-riemannian manifolds admit dilatation structures. From the existence of normal frames proved by Bellaïche we deduce the rest of the properties of regular sub-riemannian manifolds by using the formalism of dilatation structures.

Keywords: Carnot groups, dilatation structures, sub-riemannian manifolds

MSC classes: 53C17; 22E20; 20F65

1 Introduction

Sub-riemannian geometry is the modern incarnation of non-holonomic spaces, discovered in 1926 by the romanian mathematician Gheorghe Vrănceanu [21], [22]. The sub-riemannian geometry is the study of non-holonomic spaces endowed with a Carnot-Carathéodory distance. Such spaces appear in applications to thermodynamics, to the mechanics of non-holonomic systems, in the study of hypo-elliptic operators cf. Hörmander [14], in harmonic analysis on homogeneous cones cf. Folland, Stein [10], and as boundaries of CR-manifolds.

The interest in these spaces comes from several intriguing features which they have: from the metric point of view they are fractals (the Hausdorff dimension with respect to the Carnot-Carathéodory distance is strictly bigger than the topological dimension, cf. Mitchell [17]); the metric tangent space to a point of a regular sub-riemannian manifold is a Carnot group (a simply connected nilpotent Lie group with a positive graduation), also known classicaly as a homogeneous cone; the asymptotic space (in the sense of Gromov-Hausdorff distance) of a finitely generated group with polynomial growth is also a Carnot group, by a famous theorem of Gromov [11] wich leads to an inverse to the Tits alternative; finally, on such spaces we have enough structure to develop a differential calculus resembling to the one proposed by Cheeger [9] and to prove theorems like Pansu' version of Rademacher theorem [18], leading to an ingenious proof of a Margulis rigidity result.

There are several fundamental papers dedicated to the establisment of the sub-riemannian geometry, among them Mitchell [17], Bellaïche [1], a substantial paper of Gromov asking for an intrinsic point of view for sub-riemannian geometry [13], Margulis, Mostow [15], [16], dedicated to Rademacher theorem for sub-riemannian manifolds and to the construction of a tangent bundle of such manifolds, and Vodopyanov [19] (among other papers), concerning the same subject.

There is a reason for the existence of so many papers, written by important mathematicians, on the same subject: the fundamental geometric properties of sub-riemannian manifolds are very difficult to prove. Maybe the most difficult problem is to provide a rigorous construction of the tangent bundle of such a manifold, starting from the properties of the Carnot-Carathéodory distance, and somehow allowing to generalize Pansu' differential calculus.

In several articles devoted to sub-riemannian geometry, these fundamental results were proved using differential geometry tools, which are not intrinsic to sub-riemannian geometry, therefore leading to very long proofs, sometimes with unclear parts, corrected or clarified in other papers dedicated to the same subject.

The fertile ideas of Gromov, Bellaïche and other founders of the field of analysis in sub-riemannian spaces are now developed into a hot research area. For the study of sub-riemannian geometry under weaker than usual regularity hypothesis see for example the string of papers by Vodopyanov, among them [19], [20]. In these papers Vodopyanov constructs a tangent bundle structure for a sub-riemannian manifold, under weak regularity hypothesis, by using notions as horizontal convergence.

Based on the notion of dilatation structure [2], I tried to give a an intrinsic treatment to sub-riemannian geometry in the paper [4], after a series of articles [5], [6], [7] dedicated to the sub-riemannian geometry of Lie groups endowed with left invariant distributions.

In this article we show that normal frames are the central objects in the establishment of fundamental properties in sub-riemannian geometry, in the following precise sense. We prove that for regular sub-riemannian manifolds, the existence of normal frames (definition 3.7) implies that induced dilatation structures exist (theorems 6.3, 6.4). The existence of normal frames has been proved by Bellaïche [1], starting with theorem 4.15 and ending in the first half of section 7.3 (page 62). From these facts all classical results concerning the structure of the tangent space to a point of a regular sub-riemannian manifold can be deduced as straightforward consequences of the structure theorems 4.2, 4.3, 4.4, 4.5 from the formalism of dilatation structures.

In conclusion, our purpose is twofold: (a) we try to show that basic results in sub-riemannian geometry are particular cases of the abstract theory of dilatation structures, and (b) we try to minimize the contribution of classical differential calculus in the proof of these basic results, by showing that in fact the differential calculus on the sub-riemannian manifold is needed only for proving that normal frames exist and after this stage an intrinsic way of reasoning is possible.

If we take the point of view of Gromov, that the only intrinsic object on a sub-riemannian manifold is the Carnot-Carathéodory distance, the underlying differential structure of the manifold is clearly not intrinsic. Nevertheless in all proofs that I know this differential structure is heavily used. Here we try to prove that in fact it is sufficient to take as intrinsic objects of sub-riemannian geometry the Carnot-Carathéodory distance and dilatation structures compatible with it.

The closest results along these lines are maybe the ones of Vodopyanov. There is a clear correspondence between his way of defining the tangent bundle of a sub-riemannian manifold and the way of dilatation structures. In both cases the tangent space to a point is defined only locally, as a neighbourhood of the point, in the manifold, endowed with a local group operation. Vodopyanov proves the existence of the (locally defined) operation under very weak regularity assumptions on the sub-riemannian manifold. The main tool of his proofs is nevertheless the differential structure of the underlying manifold. In distinction, we prove in [2], in an abstract setting, that the very existence of a dilatation structure induces a locally defined operation. Here we show that the differential structure of the underlying manifold is important only in order to prove that dilatation structures can indeed be constructed from normal frames.

2 Metric profiles

Notations. The space CMS is the collection of isometry classes of pointed compact metric spaces. The notation used for elements of CMS is of the type $[X, d, x]$, representing the equivalence class of the pointed compact metric space (X, d, x) with respect to (pointed) isometry. The open ball of radius $r > 0$ and center $x \in (X, d)$ is denoted by $B(x, r)$ or $B_d(x, r)$ if we want to emphasize the dependence on the distance d. The notation for a closed ball is obtained by adding an overline to the notation for the open ball. The distance on CMS is the Gromov-Hausdorff distance d_{GH} between (isometry classes of) pointed metric spaces and the topology is induced by this distance. For the Gromov-Hausdorff distance see Gromov [12]. We denote by $\mathcal{O}(\varepsilon)$ a positive function such that $\lim_{\varepsilon \to 0} \mathcal{O}(\varepsilon) = 0$.

To any locally compact metric space there is an associated metric profile (Buliga [6], [7]).

Definition 2.1 *The metric profile associated to the locally compact metric space (M, d) is the assignment (for small enough $\varepsilon > 0$)*

$$(\varepsilon > 0, \ x \in M) \ \mapsto \ \mathbb{P}^m(\varepsilon, x) = [\bar{B}(x, 1), \frac{1}{\varepsilon} d, x] \in CMS$$

We may define a notion of metric profile which is more general than the previous one.

Definition 2.2 *A metric profile is a curve $\mathbb{P} : [0, a] \to CMS$ such that*

(a) it is continuous at 0,

(b) for any $\mu \in [0, a]$ and $\varepsilon \in (0, 1]$ we have

$$d_{GH}(\mathbb{P}(\varepsilon\mu), \mathbb{P}^m_{d_\mu}(\varepsilon, x_\mu)) \ = \ O(\mu)$$

The function $\mathcal{O}(\mu)$ may change with ε. We used the notations

$$\mathbb{P}(\mu) = [\bar{B}(x, 1), d_\mu, x_\mu] \quad \text{and} \quad \mathbb{P}^m_{d_\mu}(\varepsilon, x) = \left[\bar{B}(x, 1), \frac{1}{\varepsilon} d_\mu, x_\mu \right]$$

We shall unfold further the definition 2.2 in order to clearly understand what is a metric profile. For any $\mu \in (0, a]$ and for any $b > 0$ there is $\varepsilon(\mu, b) \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon(\mu, b))$ there exists a relation $\rho = \rho_{\varepsilon,\mu} \subset \bar{B}_{d_\mu}(x_\mu, \varepsilon) \times \bar{B}_{d_{\mu\varepsilon}}(x_{\mu\varepsilon}, 1)$ such that:

1. *dom* $\rho_{\varepsilon,\mu}$ is b-dense in $\bar{B}_{d_\mu}(x_\mu, \varepsilon)$,
2. *im* $\rho_{\varepsilon,\mu}$ is b-dense in $\bar{B}_{d_{\mu\varepsilon}}(x_{\mu\varepsilon}, 1)$,
3. $(x_\mu, x_{\mu\varepsilon}) \in \rho_{\varepsilon,\mu}$,
4. for all $x, y \in$ *dom* $\rho_{\varepsilon,\mu}$ we have $\mid \frac{1}{\varepsilon} d_\mu(x,y) - d_{\mu\varepsilon}(x',y') \mid \leq b$, for any x', y' such that $(x,x'), (y,y') \in \rho_{\varepsilon,\mu}$.

Therefore a metric profile gives two types of information:

- a distance estimate like the one from point 4 above,
- an "approximate shape" estimate, like in the points 1–3, where we see that two sets, namely the balls $\bar{B}_{d_\mu}(x_\mu, \varepsilon)$ and $\bar{B}_{d_{\mu\varepsilon}}(x_{\mu\varepsilon}, 1)$, are approximately isometric.

The metric profile $\varepsilon \mapsto \mathbb{P}^m(\varepsilon, x)$ of a metric space (M, d) for a fixed $x \in M$ is a metric profile in the sense of the definition 2.2 if and only if the space (M, d) admits a tangent space in x. Here is the general definition of a tangent space in the metric sense.

Definition 2.3 *A (locally compact) metric space (M, d) admits a (metric) tangent space in $x \in M$ if the associated metric profile $\varepsilon \mapsto \mathbb{P}^m(\varepsilon, x)$ (as in definition 2.1) admits a prolongation by continuity in $\varepsilon = 0$, i.e if the following limit exists:*

$$[T_x M, d^x, x] = \lim_{\varepsilon \to 0} \mathbb{P}^m(\varepsilon, x) \tag{2.0.1}$$

Metric tangent spaces are metric cones.

Definition 2.4 *A metric cone (X, d, x) is a locally compact metric space (X, d), with a marked point $x \in X$ such that for any $a, b \in (0, 1]$ we have*

$$\mathbb{P}^m(a, x) = \mathbb{P}^m(b, x)$$

Metric cones have the simplest metric profile, which is one with the property: $(\bar{B}(x_b, 1), d_b, x_b) = (X, d_b, x)$. In particular metric cones have dilatations.

Definition 2.5 *Let (X, d, x) be a metric cone. For any $\varepsilon \in (0, 1]$ a dilatation is a function $\delta^x_\varepsilon : \bar{B}(x, 1) \to \bar{B}(x, \varepsilon)$ such that*

(a) $\delta^x_\varepsilon(x) = x$,

(b) for any $u, v \in X$ we have

$$d\left(\delta^x_\varepsilon(u), \delta^x_\varepsilon(v)\right) = \varepsilon\, d(u, v)$$

The existence of dilatations for metric cones comes from the definition 2.4. Indeed, dilatations are just isometries from $(\bar{B}(x, 1), d, x)$ to $(\bar{B}, \frac{1}{a} d, x)$.

3 Sub-riemannian manifolds

Let M be a connected n dimensional real manifold. A distribution is a smooth subbundle D of M. To any point $x \in M$ there is associated the vector space $D_x \subset T_x M$. The dimension of the distribution D at point $x \in M$ is denoted by

$$m(x) = dim\, D_x$$

The distribution is smooth, therefore the function $x \in M \mapsto m(x)$ is locally constant. We suppose further that the dimension of the distribution is globally constant and we denote it by m (thus $m = m(x)$ for any $x \in M$). Clearly $m \leq n$; we are interested in the case $m < n$.

A horizontal curve $c : [a, b] \to M$ is a curve which is almost everywhere derivable and for almost any $t \in [a, b]$ we have $\dot{c}(t) \in D_{c(t)}$. The class of horizontal curves will be denoted by $Hor(M, D)$.

Further we shall use the following notion of non-integrability of the distribution D.

Definition 3.1 *The distribution D is completely non-integrable if M is locally connected by horizontal curves curves $c \in Hor(M, D)$.*

A sufficient condition for the distribution D to be completely non-integrable is given by Chow condition (C) [8].

Theorem 3.2 *(Chow) Let D be a distribution of dimension m in the manifold M. Suppose there is a positive integer number k (called the rank of the distribution D) such that for any $x \in X$ there is a topological open ball $U(x) \subset M$ with $x \in U(x)$ such that there are smooth vector fields $X_1, ..., X_m$ in $U(x)$ with the property:*

(C) the vector fields $X_1, ..., X_m$ span D_x and these vector fields together with their iterated brackets of order at most k span the tangent space $T_y M$ at every point $y \in U(x)$.

Then the distribution D is completely non-integrable in the sense of definition 3.1.

Definition 3.3 *A sub-riemannian (SR) manifold is a triple (M, D, g), where M is a connected manifold, D is a completely non-integrable distribution on M, and g is a metric (Euclidean inner-product) on the distribution (or horizontal bundle) D.*

3.1 The Carnot-Carathéodory distance

Given a distribution D which satisfies the hypothesis of Chow theorem 3.2, let us consider a point $x \in M$, its neighbourhood $U(x)$, and the vector fields $X_1, ..., X_m$ satisfying the condition (C).

One can define on $U(x)$ a filtration of bundles as follows. Define first the class of horizontal vector fields on U:

$$\mathcal{X}^1(U(x), D) \ = \ \{X \in \mathcal{X}^\infty(U) : \forall y \in U(x) \ , \ X(y) \in D_y\}$$

Next, define inductively for all positive integers j:

$$\mathcal{X}^{j+1}(U(x), D) \ = \ \mathcal{X}^j(U(x), D) \ + \ [\mathcal{X}^1(U(x), D), \mathcal{X}^j(U(x), D)]$$

Here $[\cdot, \cdot]$ denotes the bracket of vector fields. We obtain therefore a filtration $\mathcal{X}^j(U(x), D) \subset \mathcal{X}^{j+1}(U(x), D)$. Evaluate now this filtration at $y \in U(x)$:

$$V^j(y, U(x), D) \ = \ \left\{ X(y) \, : \, X \in \mathcal{X}^j(U(x), D) \right\}$$

According to Chow theorem there is a positive integer k such that for all $y \in U(x)$ we have

$$D_y = V^1(y, U(x), D) \subset V^2(y, U(x), D) \subset ... \subset V^k(y, U(x), D) = T_y M$$

Consequently, to the sub-riemannian manifold is associated the string of numbers:

$$\nu_1(y) = \dim V^1(y, U(x), D) < \nu_2(y) = \dim V^2(y, U(x), D) < ... < n = \dim M$$

Generally k, $\nu_j(y)$ may vary from a point to another.

The number k is called the step of the distribution at y.

Definition 3.4 *The distribution D is regular if $\nu_j(y)$ are constant on the manifold M. The sub-riemannian manifold $M, D, g)$ is regular if D is regular and for any $x \in M$ there is a topological ball $U(x) \subset M$ with $x \in U(M)$ and an orthonormal (with respect to the metric g) family of smooth vector fields $\{X_1, ..., X_m\}$ in $U(x)$ which satisfy the condition (C).*

The lenght of a horizontal curve is

$$l(c) \ = \ \int_a^b \left(g_{c(t)}(\dot{c}(t), \dot{c}(t)) \right)^{\frac{1}{2}} \ \mathrm{d}t$$

The length depends on the metric g.

Definition 3.5 *The Carnot-Carathéodory distance (or CC distance) associated to the sub-riemannian manifold is the distance induced by the length l of horizontal curves:*

$$d(x, y) \ = \ \inf \left\{ l(c) \, : \, c \in Hor(M, D) \ , \ c(a) = x \ , \ c(b) = y \right\}$$

The Chow theorem ensures the existence of a horizontal path linking any two sufficiently closed points, therefore the CC distance is locally finite. The distance depends only on the distribution D and metric g, and not on the choice of vector fields $X_1, ..., X_m$ satisfying the condition (C). The space (M, d) is locally compact and complete, and the topology induced by the distance d is the same as the topology of the manifold M. (These important details may be recovered from reading carefully the constructive proofs of Chow theorem given by Bellaïche [1] or Gromov [13].)

3.2 Normal frames

In the following we stay in a small open neighbourhood of an arbitrary, but fixed point $x_0 \in M$. All results are local in nature (that is they hold for some small open neighbourhood of an arbitrary, but fixed point of the manifold M). That is why we shall no longer mention the dependence of various objects on x_0, on the neighbourhood $U(x_0)$, or the distribution D.

We shall work further only with regular sub-riemannian manifolds, if not otherwise stated. The topological dimension of M is denoted by n, the step of the regular sub-riemannian manifold (M, D, g) is denoted by k, the dimension of the distribution is m, and there are numbers ν_j, $j = 1, ..., k$ such that for any $x \in M$ we have $dim\, V^j(x) = \nu_j$. The Carnot-Carathéodory distance is denoted by d.

Definition 3.6 *An adapted frame $\{X_1, ..., X_n\}$ is a collection of smooth vector fields which is obtained by the construction described below.*

We start with a collection $X_1, ..., X_m$ of vector fields which satisfy the condition (C). In particular for any point x the vectors $X_1(x), ..., X_m(x)$ form a basis for D_x. We further associate to any word $a_1....a_q$ with letters in the alphabet $1, ..., m$ the multi-bracket $[X_{a_1}, [..., X_{a_q}]...]$.

One can add, in the lexicographic order, $n - m$ elements to the set $\{X_1, ..., X_m\}$ until we get a collection $\{X_1, ..., X_n\}$ such that: for any $j = 1, ..., k$ and for any point x the set $\left\{X_1(x), ..., X_{\nu_j}(x)\right\}$ is a basis for $V^j(x)$.

Let $\{X_1, ..., X_n\}$ be an adapted frame. For any $j = 1, ..., n$ the degree $deg\, X_j$ of the vector field X_j is defined as the only positive integer p such that for any point x we have

$$X_j(x) \in V^p_x \setminus V^{p-1}(x)$$

Further we define normal frames. The name has been used by Vodopyanov [19], but for a slightly different object. The existence of normal frames in the sense of the following definition is the hardest technical problem in the classical establishment of sub-riemannian geometry.

Definition 3.7 *An adapted frame $\{X_1, ..., X_n\}$ is a normal frame if the following two conditions are satisfied:*

(a) we have the limit

$$\lim_{\varepsilon \to 0_+} \frac{1}{\varepsilon} d\left(\exp\left(\sum_1^n \varepsilon^{deg\, X_i} a_i X_i\right)(y), y\right) \in (0, +\infty)$$

uniformly with respect to y in compact sets and $a = (a_1, ..., a_n) \in W$, with $W \subset \mathbb{R}^n$ compact neighbourhood of $0 \in \mathbb{R}^n$,

(b) *for any compact set* $K \subset M$ *with diameter (with respect to the distance* d*) sufficiently small, and for any* $i = 1, ..., n$ *there are functions*

$$P_i(\cdot, \cdot, \cdot) : U_K \times U_K \times K \to \mathbb{R}$$

with $U_K \subset \mathbb{R}^n$ *a sufficiently small compact neighbourhood of* $0 \in \mathbb{R}^n$ *such that for any* $x \in K$ *and any* $a, b \in U_K$ *we have*

$$\exp\left(\sum_1^n a_i X_i\right)(x) \;=\; \exp\left(\sum_1^n P_i(a,b,y) X_i\right) \circ \exp\left(\sum_1^n b_i X_i\right)(x)$$

and such that the following limit exists

$$\lim_{\varepsilon \to 0_+} \varepsilon^{-deg\, X_i} P_i(\varepsilon^{deg\, X_j} a_j, \varepsilon^{deg\, X_k} b_k, x) \in \mathbb{R}$$

and it is uniform with respect to $x \in K$ *and* $a, b \in U_K$.

The existence of normal frames is proven in Bellaïche [1], starting with theorem 4.15 and ending in the first half of section 7.3 (page 62).

In order to understand normal frames let us look to the case of a Lie group G endowed with a left invariant distribution. The distribution is completely non-integrable if it is generated by the left translation of a vector subspace D of the algebra $\mathfrak{g} = T_e G$ which bracket generates the whole algebra $\mathfrak{g}$. Take $\{X_1, ..., X_m\}$ a collection of $m = dim\, D$ left invariant independent vector fields and define with their help an adapted frame, as explained in definition 3.6. Then the adapted frame $\{X_1, ..., X_n\}$ is in fact normal.

4 Dilatation structures

In this section we review the definition and main properties of a dilatation structure, according to [2], [3].

4.1 The axioms of a dilatation structure

Further are listed the axioms of a dilatation structure (X, d, δ), starting with axiom 0, which is a preparation for the axioms which follow.

We restrict the generality from [2] to the case which is related to sub-riemannian geometry, that is we shall consider only dilatations δ_ε^x with $\varepsilon \in (0, +\infty)$.

A0. The dilatations

$$\delta_\varepsilon^x : U(x) \to V_\varepsilon(x)$$

are defined for any $\varepsilon \in (0, 1]$. The sets $U(x), V_\varepsilon(x)$ are open neighbourhoods of x. All dilatations are homeomorphisms (invertible, continuous, with continuous inverse).

We suppose that there is a number $1 < A$ such that for any $x \in X$ we have

$$\bar{B}_d(x, A) \subset U(x) \ .$$

We suppose that for all $\varepsilon \in (0,1)$, we have

$$B_d(x,\varepsilon) \subset \delta^x_\varepsilon B_d(x,A) \subset V_\varepsilon(x) \subset U(x) \ .$$

There is a number $B \in (1, A)$ such that for any $\varepsilon \in (1, +\infty)$ the associated dilatation

$$\delta^x_\varepsilon : W_\varepsilon(x) \to B_d(x, B) \ ,$$

is injective, invertible on the image. We shall suppose that $W_\varepsilon(x)$ is a open neighbourhood of x,

$$V_{\varepsilon^{-1}}(x) \subset W_\varepsilon(x)$$

and that for all $\varepsilon \in (0,1)$ and $u \in U(x)$ we have

$$\delta^x_{\varepsilon^{-1}} \ \delta^x_\varepsilon u \ = \ u \ .$$

We have therefore the following string of inclusions, for any $\varepsilon \in (0,1)$, and any $x \in X$:

$$B_d(x,\varepsilon) \subset \delta^x_\varepsilon B_d(x,A) \subset V_\varepsilon(x) \subset W_{\varepsilon^{-1}}(x) \subset \delta^x_\varepsilon B_d(x,B) \quad .$$

A further technical condition on the sets $V_\varepsilon(x)$ and $W_\varepsilon(x)$ will be given just before the axiom A4. (This condition will be counted as part of axiom A0.)

A1. We have $\delta^x_\varepsilon x = x$ for any point x. We also have $\delta^x_1 = id$ for any $x \in X$.

Let us define the topological space

$$dom\, \delta = \{(\varepsilon, x, y) \in (0, +\infty) \times X \times X : \quad \text{if } \varepsilon \leq 1 \text{ then } y \in U(x) \ , \\ \text{else } y \in W_\varepsilon(x)\}$$

with the topology inherited from the product topology on $(0, +\infty) \times X \times X$. Consider also $Cl(dom\, \delta)$, the closure of $dom\, \delta$ in $[0, +\infty) \times X \times X$ with product topology. The function $\delta : dom\, \delta \to X$ defined by $\delta(\varepsilon, x, y) = \delta^x_\varepsilon y$ is continuous. Moreover, it can be continuously extended to $Cl(dom\, \delta)$ and we have

$$\lim_{\varepsilon \to 0} \delta^x_\varepsilon y \ = \ x \quad .$$

A2. For any $x, \in K$, $\varepsilon, \mu \in (0,1)$ and $u \in \bar{B}_d(x, A)$ we have:

$$\delta^x_\varepsilon \delta^x_\mu u = \delta^x_{\varepsilon\mu} u \ .$$

A3. For any x there is a function $(u,v) \mapsto d^x(u,v)$, defined for any u,v in the closed ball (in distance d) $\bar{B}(x,A)$, such that

$$\lim_{\varepsilon \rightarrow 0} \quad \sup \left\{ \mid \frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) \ - \ d^x(u,v) \mid \ : \ u,v \in \bar{B}_d(x,A) \right\} \ = \ 0$$

uniformly with respect to x in compact set.

Remark that d^x may be a degenerated distance: there might exist $v,w \in U(x)$ such that $d^x(v,w) = 0$.

For the following axiom to make sense we impose a technical condition on the co-domains $V_\varepsilon(x)$: for any compact set $K \subset X$ there are $R = R(K) > 0$ and $\varepsilon_0 = \varepsilon(K) \in (0,1)$ such that for all $u,v \in \bar{B}_d(x,R)$ and all $\varepsilon \in (0,\varepsilon_0)$, we have

$$\delta_\varepsilon^x v \in W_{\varepsilon^{-1}}(\delta_\varepsilon^x u) \ .$$

With this assumption the following notation makes sense:

$$\Delta_\varepsilon^x(u,v) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v.$$

The next axiom can now be stated:

A4. We have the limit

$$\lim_{\varepsilon \rightarrow 0} \Delta_\varepsilon^x(u,v) = \Delta^x(u,v)$$

uniformly with respect to x,u,v in compact set.

Definition 4.1 *A triple (X,d,δ) which satisfies A0, A1, A2, A3, but d^x is degenerate for some $x \in X$, is called degenerate dilatation structure.*

If the triple (X,d,δ) satisfies A0, A1, A2, A3, A4 and d^x is non-degenerate for any $x \in X$, then we call it a dilatation structure.

4.2 Metric profile of a dilatation structure

Here we describe the metric profile associated to a dilatation structure. This will be relevant further for understanding the geometry of the metric tangent spaces of regular sub-riemannian manifolds.

The following result is a reformulation of theorem 6 [2].

Theorem 4.2 *Let (X,d,δ) be a dilatation structure, $x \in X$ a point in X, $\mu > 0$ sufficiently small, and let (δ,μ,x) be the distance on $\bar{B}_{d^x}(x,1) = \{y \in X : \ d^x(x,y) \leq 1\}$ given by*

$$(\delta,\mu,x)(u,v) = \frac{1}{\mu} d(\delta_\mu^x u, \delta_\mu^x v)$$

Then the curve $\mu > 0 \mapsto \mathbb{P}^x(\mu) = [\bar{B}_{d^x}(x,1), (\delta, \mu, x), x]$ *admits an extension by continuity to a metric profile, by setting* $\mathbb{P}^x(0) = [\bar{B}_{d^x}(x,1), d^x, x]$. *More precisely we have the following estimate:*

$$d_{GH}\left([\bar{B}_{d^x}(x,1), (\delta, \varepsilon\mu, x), x], \left[\bar{B}_{\frac{1}{\varepsilon}(\delta^x, \mu, x)}(x,1), \frac{1}{\varepsilon}(\delta^x, \mu, x), x \right] \right) =$$
$$= \mathcal{O}(\varepsilon\mu) + \frac{1}{\varepsilon}\mathcal{O}(\mu) + \mathcal{O}(\mu)$$

uniformly with respect to x *in compact set.*

4.3 Tangent bundle of a dilatation structure

The following two theorems describe the most important metric and algebraic properties of a dilatation structure. As presented here these are condensed statements, available in full length as theorems 7, 8, 10 in [2].

Theorem 4.3 *Let* (X, d, δ) *be a dilatation structure. Then the metric space* (X, d) *admits a metric tangent space at* x*, for any point* $x \in X$*. More precisely we have the following limit:*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup \left\{ \mid d(u,v) - d^x(u,v) \mid \ : d(x,u) \leq \varepsilon \ , \ d(x,v) \leq \varepsilon \right\} \ = \ 0 \ .$$

Theorem 4.4 *Let* (X, d, δ) *be a dilatation structure. Then for any* $x \in X$ *the triple* $(U(x), \Sigma^x, \delta^x, d^x)$ *is a normed local conical group. This means:*

(a) Σ^x *is a local group operation on* $U(x)$*, with* x *as neutral element and* inv^x *as the inverse element function;*

(b) the distance d^x *is left invariant with respect to the group operation from point (a);*

(c) For any $\varepsilon \in \Gamma$, $\nu(\varepsilon) \leq 1$*, the dilatation* δ^x_ε *is an automorphism with respect to the group operation from point (a);*

(d) the distance d^x *has the cone property with respect to dilatations: foar any* $u, v \in X$ *such that* $d(x,u) \leq 1$ *and* $d(x,v) \leq 1$ *and all* $\mu \in (0, A)$ *we have:*

$$d^x(u,v) \ = \ \frac{1}{\mu} d^x(\delta^x_\mu u, \delta^x_\mu v) \quad .$$

The conical group $(U(x), \Sigma^x, \delta^x)$ can be regarded as the tangent space of (X, d, δ) at x. Further will be denoted by: $T_x X = (U(x), \Sigma^x, \delta^x)$.

The following is corollary 4.7 [3].

Theorem 4.5 *Let* (X, d, δ) *be a dilatation structure. Then for any* $x \in X$ *the local group* $(U(x), \Sigma^x)$ *is locally a simply connected Lie group whose Lie algebra admits a positive graduation (a Carnot group).*

5 Examples of dilatation structures

In this section we give some examples of dilatation structures, which share some common features. There are other examples, typically coming from iterated functions systems, which will be presented in another paper.

The first example is known to everybody: take $(X, d) = (\mathbb{R}^n, d_E)$, with usual (euclidean) dilatations δ_ε^x, with:

$$d_E(x,y) = \|x-y\| \quad , \quad \delta_\varepsilon^x y \ = \ x + \varepsilon(y-x) \ .$$

Dilatations are defined everywhere. There are few things to check: axioms 0,1,2 are obviously true. For axiom A3, remark that for any $\varepsilon > 0$, $x, u, v \in X$ we have:

$$\frac{1}{\varepsilon} d_E(\delta_\varepsilon^x u, \delta_\varepsilon^x v) \ = \ d_E(u,v) \ ,$$

therefore for any $x \in X$ we have $d^x = d_E$.

Finally, let us check the axiom A4. For any $\varepsilon > 0$ and $x, u, v \in X$ we have

$$\delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v \ = \ x + \varepsilon(u-x) + \frac{1}{\varepsilon}\left(x + \varepsilon(v-x) - x - \varepsilon(u-x)\right) \ =$$
$$= \ x + \varepsilon(u-x) + v - u$$

therefore this quantity converges to

$$x + v - u \ = \ x + (v-x) - (u-x)$$

as $\varepsilon \to 0$. The axiom A4 is verified.

We continue further with less obvious examples.

5.1 Riemannian manifolds

Take now $\phi : \mathbb{R}^n \to \mathbb{R}^n$ a bi-Lipschitz diffeomorphism. Then we can define the dilatation structure: $X \ = \ \mathbb{R}^n$,

$$d_\phi(x,y) \ = \ \|\phi(x) - \phi(y)\| \quad , \quad \delta_\varepsilon^x y \ = \ x + \varepsilon(y-x) \ ,$$

or the equivalent dilatation structure: $X \ = \ \mathbb{R}^n$,

$$d_\phi(x,y) \ = \ \|x-y\| \quad , \quad \delta_\varepsilon^x y \ = \ \phi^{-1}\left(\phi(x) + \varepsilon(\phi(y) - \phi(x))\right) .$$

In this example (look at its first version) the distance d_ϕ is not equal to d^x. Indeed, a direct calculation shows that

$$d^x(u,v) \ = \ \|D\phi(x)(v-u)\| \ .$$

The axiom A4 gives the same result as previously.

Because dilatation structures are defined by local requirements, we can easily define dilatation structures on riemannian manifolds, using particular atlases of the manifold and the riemannian distance (infimum of length of curves joining two points). This class of examples covers all dilatation structures used in differential geometry. The axiom A4 gives an operation of addition of vectors in the tangent space (compare with Bellaïche [1] last section).

5.2 Snowflakes

The next example is a snowflake variation of the euclidean case: $X = \mathbb{R}^n$ and for any $a \in (0,1]$ take

$$d_a(x,y) \ = \ \|x-y\|^{\alpha} \quad , \quad \delta_{\varepsilon}^{x} y \ = \ x + \varepsilon^{\frac{1}{a}}(y-x) \ .$$

We leave to the reader to verify the axioms.

More general, if (X,d,δ) is a dilatation structure then $(X,d_a,\delta(a))$ is also a dilatation structure, for any $a \in (0,1]$, where

$$d_a(x,y) \ = \ (d(x,y))^a \quad , \quad \delta(a)_{\varepsilon}^{x} \ = \ \delta_{\varepsilon^{\frac{1}{a}}}^{x} \ .$$

5.3 Nonstandard dilatations in the euclidean space

Take $X = \mathbb{R}^2$ with the euclidean distance. For any $z \in \mathbb{C}$ of the form $z = 1 + i\theta$ we define dilatations

$$\delta_{\varepsilon} x = \varepsilon^{z} x \ .$$

It is easy to check that $(X,\delta,+,d)$ is a conical group, equivalenty that the dilatations

$$\delta_{\varepsilon}^{x} y = x + \delta_{\varepsilon}(y-x) \ .$$

form a dilatation structure with the euclidean distance.

Two such dilatation structures (constructed with the help of complex numbers $1+i\theta$ and $1+i\theta'$) are equivalent if and only if $\theta = \theta'$.

There are two other surprising properties of these dilatation structures. The first is that if $\theta \neq 0$ then there are no non trivial Lipschitz curves in X which are differentiable almost everywhere. The second property is that any holomorphic and Lipschitz function from X to X (holomorphic in the usual sense on $X = \mathbb{R}^2 = \mathbb{C}$) is differentiable almost everywhere, but there are Lipschitz functions from X to X which are not differentiable almost everywhere (suffices to take a $\mathcal{C}^{\infty}$ function from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is not holomorphic).

6 Sub-riemannian dilatation structures

To any normal frame of a regular sub-riemannian manifold we associate a dilatation structure. (Technically this is a dilatation structure defined only locally, as in the case of riemannian manifolds.)

Definition 6.1 *To any normal frame $\{X_1, ..., X_n\}$ of a regular sub-riemannian manifold (M,D,g) we associate the dilatation structure (M,d,δ) defined by: d is the Carnot-Carathéodory distance, and for any point $x \in M$ and any $\varepsilon \in (0,+\infty)$ (sufficiently small if necessary), the dilatation δ_{ε}^{x} is given by:*

$$\delta_{\varepsilon}^{x}\left(\exp\left(\sum_{i=1}^{n} a_i X_i\right)(x)\right) \ = \ \exp\left(\sum_{i=1}^{n} a_i \varepsilon^{deg X_i} X_i\right)(x)$$

We shall prove that (M, d, δ) is indeed a dilatation structure. This allows us to get the main results concerning the infinitesimal geometry of a regular sub-riemannian manifold, as particular cases of theorems 4.2, 4.3, 4.4 and 4.5.

We only have to prove axioms A3 and A4 of dilatation structures. We do this in the next two theorems. Before this let us decribe what we mean by "sufficiently closed".

Convention 6.2 *Further we shall say that a property $\mathcal{P}(x_1, x_2, x_3, ...)$ holds for $x_1, x_2, x_3, ...$ sufficiently closed if for any compact, non empty set $K \subset X$, there is a positive constant $C(K) > 0$ such that $\mathcal{P}(x_1, x_2, x_3, ...)$ is true for any $x_1, x_2, x_3, ... \in K$ with $d(x_i, x_j) \leq C(K)$.*

In the following we prove a result similar to Gromov local approximation theorem [13], p. 135, or to Bellaïche theorem 7.32 [1]. Note however that here we take as a hypothesis the existence of a normal frame.

Theorem 6.3 *Consider $X_1, ..., X_n$ a normal frame and the associated dilatations provided by definition 6.1. Then axiom A3 of dilatation structures is satisfied, that is the limit*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d\left(\delta^x_\varepsilon u, \delta^x_\varepsilon v\right) \ = \ d^x(u, v)$$

exists and it uniform with respect to x,u,v sufficiently closed.

Proof. Let $x, u, v \in M$ be sufficiently closed. We write

$$u \ = \ \exp\left(\sum_1^n u_i X_i\right)(x) \qquad , \quad v \ = \ \exp\left(\sum_1^n v_i X_i\right)(x)$$

we compute, using definition 6.1:

$$\frac{1}{\varepsilon} d\left(\delta^x_\varepsilon u, \delta^x_\varepsilon v\right) \ = \ \frac{1}{\varepsilon} d\left(\delta^x_\varepsilon \exp\left(\sum_1^n u_i X_i\right)(x), \delta^x_\varepsilon \exp\left(\sum_1^n v_i X_i\right)(x)\right) \ =$$

$$= \ \frac{1}{\varepsilon} d\left(\exp\left(\sum_1^n \varepsilon^{deg\, X_i} u_i X_i\right)(x), \exp\left(\sum_1^n \varepsilon^{deg\, X_i} v_i X_i\right)(x)\right) \ = \ A_\varepsilon$$

Let us denote by $u_\varepsilon = \exp\left(\sum_1^n \varepsilon^{deg\, X_i} u_i X_i\right)(x)$. Use the first part of the property (b), definition 3.7 of a normal system, to write further:

$$A_\varepsilon \ = \ \frac{1}{\varepsilon} d\left(u_\varepsilon, \exp\left(\sum_1^n P_i(\varepsilon^{deg\, X_j} v_j, \varepsilon^{deg\, X_k} u_k, x) X_i\right)(u_\varepsilon)\right) \ =$$

$$= \ \frac{1}{\varepsilon} d\left(u_\varepsilon, \exp\left(\sum_1^n \varepsilon^{deg\, X_i}\left(\varepsilon^{-deg\, X_i} P_i(\varepsilon^{deg\, X_j} v_j, \varepsilon^{deg\, X_k} u_k, x)\right) X_i\right)(u_\varepsilon)\right)$$

We make a final notation: for any $i = 1, ..., n$

$$a_i^\varepsilon \;=\; \varepsilon^{-deg\, X_i}\, P_i(\varepsilon^{deg\, X_j} v_j, \varepsilon^{deg\, X_k} u_k, x)$$

thus we have:

$$\frac{1}{\varepsilon}\, d\,(\delta^x_\varepsilon u, \delta^x_\varepsilon v) \;=\; \frac{1}{\varepsilon}\, d\left(u_\varepsilon, \exp\left(\sum_1^n \varepsilon^{deg\, X_i} a_i^\varepsilon X_i\right)(u_\varepsilon)\right)$$

By the second part of property (b), definition 3.7, the vector $a^\varepsilon \in \mathbb{R}^n$ converges to a finite value $a^0 \in \mathbb{R}^n$, as $\varepsilon \to 0$, uniformly with respect to x, u, v in compact set. In the same time u_ε converges to x, as $\varepsilon \to 0$. The proof ends by using property (a), definition 3.7. □

Theorem 6.4 *Consider $X_1, ..., X_n$ a normal frame and the associated dilatations provided by definition 6.1. Then axiom A4 of dilatation structures is satisfied: as ε tends to 0 the quantity*

$$\Delta^x_\varepsilon(u, v) \;=\; \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \circ \delta^x_\varepsilon(v)$$

converges, uniformly with respect to x, u, v sufficiently closed.

Proof. We shall use the notations from definition 3.6, 3.7, 6.1.

Let $x, u, v \in M$ be sufficiently closed. We write

$$u \;=\; \exp\left(\sum_1^n u_i X_i\right)(x) \qquad , \quad v \;=\; \exp\left(\sum_1^n v_i X_i\right)(x)$$

We compute now $\Delta^x_\varepsilon(u, v)$:

$$\Delta^x_\varepsilon(u, v) \;=\; \delta_{\varepsilon^{-1}}^{\exp\left(\sum_1^n \varepsilon^{deg\, X_i} u_i X_i\right)(x)} \exp\left(\sum_1^n \varepsilon^{deg\, X_i} v_i X_i\right)(x)$$

Let us denote by $u_\varepsilon = \delta^x_\varepsilon u$. Thus we have

$$\Delta^x_\varepsilon(u, v) \;=\; \delta^{u_\varepsilon}_{\varepsilon^{-1}} \exp\left(\sum_1^n \varepsilon^{deg\, X_i} v_i X_i\right)(x)$$

We use the first part of the property (b), definition 3.7, in order to write

$$\exp\left(\sum_1^n \varepsilon^{deg\, X_i} v_i X_i\right)(x) \;=\; \exp\left(\sum_1^n P_i(\varepsilon^{deg\, X_j} v_j, \varepsilon^{deg\, X_k} u_k, x) X_i\right)(u_\varepsilon)$$

We finish the computation:

$$\Delta^x_\varepsilon(u, v) \;=\; \exp\left(\sum_1^n \varepsilon^{-\, deg\, X_i}\, P_i(\varepsilon^{deg\, X_j} v_j, \varepsilon^{deg\, X_k} u_k, x) X_i\right)(u_\varepsilon)$$

As ε goes to 0 the point u_ε converges to x uniformly with respect to x, u sufficiently closed (as a corollary of the previous theorem, for example). The proof therefore ends by invoking the second part of the property (b), definition 3.7. □

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