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Triangulated categories and applications

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Introduction

The context

The broad context of my research is the theory of triangulated categories.

Triangulated categories were introduced in the early 60's by Jean-Louis Verdier in his thesis in order to axiomatise derived categories, and also (minus the octahedral axiom) by Dieter Puppe in his axiomatisation of stable homotopy theory. Since then, triangulated categories have spread to numerous corners of mathematics, becoming a standard tool for topologists, geometers, representation theorists, and even some analysts, operator algebraists, and mathematical physicists. In a sense they can still be seen as offering a 'light' axiomatisation of stable homotopy theory, with the latter having nowadays gained a much broader meaning.

The success of triangulated categories has several roots. First of all, they offer an algebraic language which is simultaneously simple and sufficiently rich for formulating many important results and conjectures. But triangulated categories are not only a language: their abstract theory has turned out to be powerful enough for *proving* many of those results. Indeed, many ideas and techniques from different mathematical communities have been abstracted and subsumed into the theory of triangulated categories, making it possible to apply them elsewhere, often with striking results.

Triangulated categories are not perfect: because of certain serious defects (lack of limits and colimits, non-functoriality of cones, lack of constructions of new triangulated categories out of old ones, etc.) over the decades many mathematicians have called into question Verdier's axioms and proposed all sort of alternatives. Most notably, various kinds of 'enhancements' of triangulated categories are now competing on the market, such as $\mathcal{A}_\infty$-categories, dg categories, stable model categories, ∞-categories, and stable derivators. While these theories make a good job of repairing the technical deficiencies of triangulated categories, they pay the price of being harder to work with and less flexible with respect to the examples covered. In my opinion, triangulated categories are to be seen as *complementary to*, rather then competing with, such enhancements: while the added power of the latter is certainly useful and should be exploited, it would be foolish to forgo the simplicity and universality of triangulated categories, or to denigrate their undeniable successes.

The contents

In this memoir I present most of the research I have conducted since my PhD thesis, without giving any proofs but hopefully with reasonably contained statements. For all proofs and for more details I will refer to the published papers [2, 3, 5, 6, 7, 9, 11, 12, 13, 14] and the preprint [15] (see the list of my publications in Section 5). I have excluded from this overview the results of [4], [8] and [10] because they have nothing to do with triangulated categories, and those of [1] because they don't fit the main narrative.

The work surveyed here is divided, like Gaul, into three parts, each summarised by a slogan:

- *triangulated categories* are a convenient place for practicing homological algebra (Section 1).
- *tensor triangulated categories* are tools for formulating and proving classification results by geometric means (Section 2).
- *tensor exact functors* are tools for formulating, proving and comparing duality statements (Section 3).

We now briefly describe the contents of each part.

1. Triangulated categories and homological algebra

The first use of triangulated categories is to organise and facilitate homological computations. This is what they were designed for: the efficient handling of the long exact sequences arising in algebra, geometry and topology.

After recalling some basic definitions and techniques of the theory (§1.1), we explain how to use triangulated categories systematically for treating *universal coefficient theorems*, or UCT's (§1.2). In joint work with Greg Stevenson and Jan Šťovíček [15], we offer a new insight on this classical subject. UCT's are fundamental computational tools in the form of short exact sequences displaying the Hom spaces of a triangulated category as extensions of groups computed in some abelian category, via a suitable homological invariant. We build a theory for constructing such invariants systematically, mainly by making explicit a profound connection with Gorenstein algebra. We are able this way to unify and simplify the proofs of several UCT's used in the realm of Gennadi Kasparov's *KK-theory of C*-algebras* (§1.4). The same methods also yield a new proof and conceptual understanding of the Brown-Adams representability theorem in topology, in Amnon Neeman's general form [Nee97]: it is an example of a UCT, arising from the fact that any triangulated category with countably many morphisms behaves like a one-dimensional Gorenstein ring, in a precise sense (§1.3). Our proof also leads to a new variant of Brown-Adams representability which can be applied to triangulated categories having only countable coproducts and thus can be applied to the examples arising in KK-theory.

In §1.5 I present the results of [6], in which I make precise a new connection between representation theory, in the form of the theory of Mackey functors [Bou97], and the equivariant KK-theory of C*-algebras equipped with the action of a finite group. This connection is analogous to known results in equivariant stable homotopy, e.g. the fact that equivariant stable homotopy groups are naturally organised in Mackey functors. This observation allows me to construct some new spectral sequences for computing with equivariant KK-theory groups.

After joining forces with Heath Emerson and Ralf Meyer, I continue to apply triangular techniques to the study of equivariant KK-theory, now for general compact Lie groups (§1.6). Our goal is to obtain K-theoretic formulas for computing the traces of endomorphisms in the equivariant Kasparov category. When specialised to commutative C*-algebras, these formulas yield equivariant generalisations to G-manifolds of the classical Lefschetz-Hopf fixed-point formula for endo-maps on smooth compact manifolds. Our results are quite general and can also be applied to smooth correspondences, in the sense of Emerson and Meyer [EM10a] (building on ideas of Alain Connes and George Skandalis).

2. Tensor triangulated categories and classifications

One of the most basic invariants of a triangulated category $\mathcal{T}$ is its lattice of *thick subcategories*. On the one hand, it contains information about all the possible homological or exact images of $\mathcal{T}$, because thick subcategories are precisely the kernels of homological and exact functors. On the other hand, a classification of the thick subcategories yields a rough classification of the objects of $\mathcal{T}$: if two objects generate the same thick subcategory, this means that each can be obtained from the other by using a few natural operations. This is typically the best kind of classification that one can hope for in most examples, because the problem of classifying objects up to, say, isomorphism very quickly becomes intractable. If $\mathcal{T}$ admits infinite coproducts, one may also try to classify its localising subcategories, although this is usually a harder problem and is less well understood. (See §2.2.)

The first classification of thick subcategories appeared in stable homotopy theory, in the remarkable work of Devinatz, Hopkins and Smith on the Ravenel conjectures [DHS88] [HS98]. Their ideas were exported to commutative algebra by Hopkins and Neeman [Nee92], and globalised to schemes by Thomason [Tho97]. In representation theory, Benson, Carlson and Rickard [BCR97] proved a similar classification theorem for the stable module category of a finite group.

These first examples of classifications from the 90's have been subsumed into a general abstract theorem — at least for what concerns their statement, if not (yet) their proofs — by Paul Balmer [Bal05]. For each of these triangulated categories $\mathcal{T}$, the key to the classification is the presence of a compatible symmetric tensor product on it (§2.1) which can be used to define a space $\operatorname{Spc}\mathcal{T}$, the *spectrum* of $\mathcal{T}$. This abstract space contains precisely the same information as the lattice of thick tensor-ideal subcategories of $\mathcal{T}$; hence in order to obtain a classification theorem it only remains to find, for each given example, a relevant and manageable description of the spectrum (see §2.2). To this goal, the abstract theory at the level of tensor triangulated categories – Balmer's *tensor triangular geometry* [Bal10b] – can be of much help, by reducing the problem to easier ones or to already known computations. One such powerful tool is a natural continuous map $\rho\colon \operatorname{Spc}\mathcal{T} \to \operatorname{Spec} R$, which is always available and compares the triangular spectrum with the good old Zariski spectrum of the *central ring* $R = \operatorname{End}_{\mathcal{T}}(\mathbb{1})$ (see §2.3). This map is often surjective and one can study the fibers of ρ over each prime $\mathfrak{p}$ of R individually, after localising everything at $\mathfrak{p}$. Sometimes ρ is also injective, and thus yields a nice computation of $\operatorname{Spc}\mathcal{T}$. If this fails, one can also consider graded versions of ρ.

This second part contains my work in this subject, taken from the series of articles [2, 3, 7, 9, 14]. Using (generalised versions of) Balmer's comparison map ρ, Greg Stevenson and I proved classifications theorems for thick and localising subcategories in derived categories of graded commutative rings, generalising the above-mentioned work of Hopkins, Neeman and Thomason (§2.6.1). By applying ideas of Benson, Iyengar and Krause [BIK08] [BIK11], Don Stanley and I also obtained similar theorems for derived categories of regular noetherian commutative dg rings and ring spectra, thus generalising some results of Benson-Iyengar-Krause and Shamir (see §2.7). I also proved the first classification in the domain of Kasparov's KK-theory (§2.4.2) and, together with Gonçalo Tabuada, produced the first (very partial) results in the theory of noncommutative motives (§2.8). Along the way we also make contributions to abstract tensor triangular geometry, for instance by reducing the computation of the spectrum to the existence of a well-behaved support function on an ambient compactly generated category (§2.4), or by constructing a more flexible version of Balmer's graded comparison maps ρ (§2.5).

3. Tensor exact functors and duality

Triangulated categories have always been connected with duality. Indeed, derived and triangulated categories were first thought up by Grothendieck and Verdier mainly in order to formulate a relative version of Serre duality in algebraic geometry, what is now called Grothendieck duality (see [Har66] for the classical presentation and [Nee10] for a modern survey). Verdier went on to prove similar results in the topological setting, obtaining the generalisation of Poincaré duality which is now known as Verdier duality.

More and more aspects of the theory have been subsequently abstracted to the pure triangular realm, most notably by Neeman [Nee96] who added to the mix the powerful topological techniques of Brown representability and Bousfield localisation. More recently, Fausk, Hu and May [FHM03] noticed the formal similarities between Grothendieck-Verdier duality and the Wirthmüller isomorphism in equivariant stable homotopy.

In this third part we present our contribution to this subject, taken from [13] and [12], both written jointly with Paul Balmer and Beren Sanders. Our own work can be seen as a continuation of Neeman's and Fausk-Hu-May's, in that we continue the process of triangular formalisation and simplification of proofs. Moreover, we further clarify the relation between Grothendieck duality and the Wirthmüller isomorphism, and explain how to unify in this framework various sorts of other duality phenomena in algebra and topology.

Concretely, we consider a coproduct-preserving tensor exact functor $f^*\colon \mathcal{D} \to \mathcal{C}$ between 'big' tensor triangulated categories (see §3.1), as often arise in practice by pulling back representations, sheaves, spectra, etc., along some 'underlying' map f of groups, spaces, schemes, etc. (As we exclusively work at the level of tensor triangulated categories, though, no role is played by such an f and in fact it may well not exist.) As in algebraic geometry, we often think of $\mathcal{D}$ as the base category over which $\mathcal{C}$ is defined, but in some examples it may rather be the other way round.

In this generality, we study the conditions for the existence of right or left adjoints of f^*, and adjoints of these adjoints, etc.:

$$\cdots \quad \uparrow \quad \downarrow \quad \overset{\mathcal{C}}{\underset{\mathcal{D}}{\uparrow}} f^* \quad \downarrow f_* \quad \uparrow \quad \downarrow \quad \cdots \tag{0.1}$$

As it turns out, the more adjoints exist, the stronger these functors must be related by various canonical formulas. Surprisingly, we also discover that there are only three possible distinct stages of adjunction:

Theorem (Trichotomy of adjoints)**.** *If f^* is a coproduct-preserving tensor exact functor between rigidly-compactly generated tensor triangulated categories (see Hypothesis 3.1.1 below for details), then* exactly one *of the following three possibilities must hold:*

(1) *There are two adjunctions as follows and no more: $f^* \dashv f_* \dashv f^{(1)}$.*

(2) *There are four adjunctions as follows and no more:*

$$f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)} \dashv f_{(-1)} \,.$$

(3) *There is an infinite tower of adjunctions in both directions:*

$$\cdots f^{(-1)} \dashv f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)} \dashv f_{(-1)} \cdots f^{(n)} \dashv f_{(-n)} \dashv f^{(n+1)} \cdots$$

All three stages often occur in examples. This theorem is an immediate consequence of the results of §3.1-§3.3, which also characterise in much detail what happens at each stage.

A central rôle in our theory will be played by a certain *relative dualizing object* ω_f (Definition 3.1.3), which also serves to unify the dualizing complexes in Grothendieck duality and the canonical twist in the Wirthmüller isomorphism.

We then proceed in §3.4 to develop a general theory of dualizing objects and show how to pull back or push forward subcategories with duality along the 'push forward' right adjoint f_*. This extends work of Calmès and Hornbostel [CH09]. In §3.5 we show how a generalisation of *Serre functors* appears naturally in this context. We include in §3.6 some concrete examples of dualities from various domains of mathematics that fall under the mantle of our theory; more can be found in [13].

We then conclude with the results of [12], where – under more relaxed hypotheses allowing us to also include examples from KK-theory – we look at the "ambidextrous" situation, namely the situation when f^* has isomorphic left and right adjoints (corresponding to the case $\omega_f \cong \mathbb{1}$). We show that, in many examples in which the functor f^* arises as a restriction functor to a finite-index subgroup, a strong monoidal version of monadicity holds, which lets us see these functors as analogs of finite étale coverings in algebraic geometry.

Examples

Because of the innate transdisciplinary nature of triangulated categories, examples from disparate realms of mathematics are to be found throughout this memoir. The following summary should facilitate a topic-by-topic consultation.

Algebra: derived categories of (graded) commutative rings **§2.6**, §2.6.1, §3.6; derived categories of dg rings §2.7.6; stable module categories §3.6.1; local algebra and Matlis duality §3.6.8.

Geometry: derived categories of schemes **§2.6**, §2.6.4; Grothendieck duality **§3.6.4**; Serre duality §3.6.7; equivariant sheaves §3.7.5.

Topology: the homotopy category of spectra §1.3.1; derived categories of ring spectra §2.7, **§3.6.2**; homotopy categories of G-spectra **§3.6.3**, §3.7.3; generalised Matlis duality §3.6.8.

KK-theory: Kasparov categories of C*-algebras **§1.3.3**; universal coefficient theorems §1.4; Mackey functors in equivariant Kasparov categories §1.5; trace computations in equivariant Kasparov categories §1.6; classifications of subcategories §2.4.2.

The ordinary Kasparov category §1.4.1, §2.4.2; filtrated KK-theory §1.4.3; equivariant KK-theory §1.4.5, §1.5, §1.6, §3.7.4.

Motives: noncommutative motives §2.8.

1 Triangulated categories and homological algebra

We present here the results of [15] (joint with Greg Stevenson and Jan Stovicek), [6], and [11] (joint with Heath Emerson and Ralf Meyer).

1.1 Preliminaries on triangulated categories

We collect in this section the basic facts about triangulated categories that will be used throughout. We refer to [Nee01] for most of the unproved statements to be found here.

A *triangulated category* (in the sense of Verdier [Ver96]) is an additive category $\mathcal{T}$ equipped with an auto-equivalence $\Sigma\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$, called *suspension* (translation, shift, ...), and a distinguished collection of diagrams of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

called the *triangulation* of $\mathcal{T}$. The triangulation must satisfy a short list of reasonable existence and closure axioms, which we do not repeat here, which guarantee that the elements of the triangulation (called *exact* or *distinguished triangles*, or just *triangles*) can be used to perform some basic homological algebra reasoning. Note that the usual categorical tools of homological algebra are quite useless in a triangulated category; for instance, the only monomorphisms and epimorphisms in a triangulated category are the split ones! In fact, the only limits and colimits one can expect to exist in a triangulated category are products and coproducts.

It follows easily from the axioms that, in a distinguished triangle as above, we have $gf = 0$, and f behaves as a weak kernel for g and g as a weak cokernel for f; here "weak" means that only the existence part, not the uniqueness part, of the universal property of a (co)kernel holds. If $f = 0$ then g is a split mono, and if $g = 0$ then f is a split epi.

The object Z in a triangle containing f, as above, is uniquely determined up to a non-unique isomorphism and is called the *cone of* f, written $\mathrm{Cone}(f)$. A basic result says that f is an isomorphism if and only if $Z = 0$. This often allows to translate properties and constructions involving morphisms into ones only involving objects, which can be a great simplification. An important example of this is localisation; see §1.1.2.

1.1.1. Homological and exact functors. A crucial and motivating consequence of the axioms is that if we take any distinguished triangle and use the suspension functor Σ to unroll it into a doubly infinite sequence

$$\ldots \xrightarrow{-\Sigma^{-1}g} \Sigma^{-1}Z \xrightarrow{-\Sigma^{-1}h} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma g} \ldots$$

and apply to this sequence the Hom functors $\mathcal{T}(U, -)$ or $\mathcal{T}(-, U)$ for any object $U \in \mathcal{T}$, then we obtain long exact sequences of abelian groups. Most (doubly infinite) exact sequences appearing in Nature arise this way, and triangulated categories make it easy to handle these sequences when they are still uncoiled, so to speak, tightly packed within triangles.

More generally, an additive functor $\mathcal{T} \to \mathcal{A}$ (or $\mathcal{T}^{\mathrm{op}} \to \mathcal{A}$) into some abelian category is said to be *homological* (or *cohomological*) if it sends every (unrolled) distinguished triangle to a long exact sequence in $\mathcal{A}$.

Thus the (co)representable functors $\mathcal{T}(U,-)$ and $\mathcal{T}(-,U)$ are homological functors that send products, respectively coproducts, to products of abelian groups. One of the most useful features of triangulated categories is that, provided they admit sufficiently many coproducts and are nicely generated in some sense, the converse is true: *every* cohomological functor (and, a little less frequently, every homological functor) sending products (coproducts) to products is representable by an object of the category; see §1.1.3.

Note that the (additive) Yoneda functor $\mathcal{T} \to \operatorname{Mod}\mathcal{T}$ is homological, because exactness in the functor category $\operatorname{Mod}\mathcal{T}$ is detected objectwise. More generally, if $\mathcal{C}$ is an (essentially small) full subcategory of $\mathcal{T}$, we will consider the homological functor

$$h_{\mathcal{C}} \colon \mathcal{T} \to \operatorname{Mod}\mathcal{C}$$

obtained by sending $X \in \mathcal{T}$ to the restriction of $\mathcal{T}(-,X)$ to $\mathcal{C}$. Although typically not fully faithful anymore, such *restricted Yoneda functors* $h_{\mathcal{C}}$ are very useful tools for approximating triangulated categories by abelian categories; see §1.2 and §1.5.1.

A functor $F \colon \mathcal{T} \to \mathcal{S}$ between two triangulated categories is *exact* (or *triangulated*) if it preserves the suspension up to a natural isomorphism $\Sigma F \cong F\Sigma$ (which may or may not be considered part of the data together with F, according to one's needs) and if it sends distinguished triangles to distinguished triangles. Clearly, if we compose an exact functor with a homological one we again get a homological functor. Quite conveniently, one can show that a functor which is left or right adjoint to an exact one is automatically exact.

1.1.2. Thick and localising subcategories and localisation. If F is an exact or homological functor on a triangulated category $\mathcal{T}$, the closure properties of its (full) kernel on objects

$$\operatorname{Ker}(F) := \{X \in \mathcal{T} \mid FX \cong 0\}$$

are those of a *thick subcategory*: it is a full, replete (i.e. closed under isomorphic objects) additive subcategory of $\mathcal{T}$ stable under Σ and taking cones of maps (i.e. it is a *triangulated subcategory*), and moreover it stable under taking direct summands: if $X \oplus Y \in \operatorname{Ker}(F)$ then $X, Y \in \operatorname{Ker}(F)$.

Conversely, every thick subcategory $\mathcal{S} \subseteq \mathcal{T}$ is $\operatorname{Ker} F$ for some exact functor F (at least, if we ignore set-theoretical issues – which can usually be done safely e.g. by passing to a higher universe). Indeed, the class of maps $\mathcal{W}_{\mathcal{S}} := \{f \in \operatorname{Mod}\mathcal{T} \mid \operatorname{Cone}(f) \in \mathcal{S}\}$ turns out to satisfy a calculus of left and right fractions and to be compatible with the triangulation of $\mathcal{T}$; this compatibility ensures that the localised category $\mathcal{T}[\mathcal{W}_{\mathcal{S}}^{-1}]$, obtained from $\mathcal{T}$ by formally inverting the maps of $\mathcal{W}_\mathcal{S}$, can be endowed with a unique suspension and a unique triangulation making the canonical functor

$$q_\mathcal{S}\colon \mathcal{T} \longrightarrow \mathcal{T}[\mathcal{W}_\mathcal{S}^{-1}]$$

exact; then $\mathcal{S} = \mathrm{Ker}(q_\mathcal{S})$.

Note that $\mathcal{W}_\mathcal{S}^{-1}$ consists precisely of the maps which are inverted by $q_\mathcal{S}$, and that $\mathcal{S}$ and $\mathcal{W}_\mathcal{S}$ determine each other, as $\mathcal{S} = \{X \mid \exists f \in \mathcal{W}_\mathcal{S} \text{ s.t. } X \cong \mathrm{Cone}(f)\}$. This triangulation-compatible localisation is called a *Verdier quotient*, and the category $\mathcal{T}[\mathcal{W}_\mathcal{S}^{-1}]$ is usually denoted by $\mathcal{T}/\mathcal{S}$.

If the triangulated categories $\mathcal{T}$ and $\mathcal{U}$ admit coproducts and $F\colon \mathcal{T} \to \mathcal{U}$ is the exact functor preserves them, then $\mathrm{Ker}(F)$ is closed under the formation of coproducts. Conversely, if a thick subcategory is closed under coproducts — in which case it is said to be a *localising* subcategory of $\mathcal{T}$ — then it gives rise to a coproduct-preserving Verdier quotient $q_\mathcal{S}\colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$.

1.1.3. Generators and Brown representability. A triangulated category $\mathcal{T}$ is (*classically*) *generated* by a subset $\mathcal{G}$ of objects if the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{G}$ is the whole of $\mathcal{T}$.

If $\mathcal{T}$ admits arbitrary small coproducts, one says instead that $\mathcal{T}$ is (*weakly*) *generated* by $\mathcal{G}$ if $\mathcal{T}$ is the smallest localising subcategory containing $\mathcal{G}$. We write $\mathrm{Loc}(\mathcal{S})$ for the localising subcategory generated by $\mathcal{S} \subseteq \mathcal{T}$. An object $C \in \mathcal{T}$ is *compact* if $\mathcal{T}(C, -)$ preserves all coproducts, and $\mathcal{T}$ is *compactly generated* if it admits a small set $\mathcal{G}$ of compact generators. Equivalently, $\mathcal{T}$ is compactly generated if it admits small coproducts and a small set $\mathcal{G}$ of compact objects such that, for any $X \in \mathcal{T}$, $\mathcal{T}(C, X) = 0$ for all $C \in \mathcal{G}$ implies $X = 0$.

By Neeman's basic *Brown representability theorem* [Nee96], if $\mathcal{T}$ is compactly generated, then every cohomological functor $H\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Mod}\,\mathbb{Z}$ which sends the coproducts of $\mathcal{T}$ to products of abelian groups is representable, i.e., $H \cong \mathcal{T}(-, X)$ for some $X \in \mathcal{T}$. Similarly, socalled *Brown representability for the dual* also holds (see [Nee01]), saying that every product-preserving homological functor $H\colon \mathcal{T} \to \mathrm{Mod}\,\mathbb{Z}$ is of the form $H \cong \mathcal{T}(X, -)$ for some X.

These results can be easily translated into the existence of adjoint functors. Say $F\colon \mathcal{T} \to \mathcal{S}$ is an exact functor between triangulated categories with $\mathcal{T}$ compactly generated. Then F admits a right adjoint (resp. a left adjoint) if and only if F preserves coproducts (resp. products). Note that the latter makes sense, because arbitrary products automatically exist in a compactly generated category.

Let $\aleph$ be a regular cardinal, and say a set is $\aleph$-*small* if its cardinality is strictly less then $\aleph$. It sometimes happens that a triangulated category $\mathcal{T}$ only has arbitrary $\aleph$-small coproducts; see e.g. §1.3.3 for many examples with $\aleph = \aleph_1$. In this case we say that $\mathcal{T}$ is *compactly*$_\aleph$ *generated* if it admits a set of generators of cardinality less than $\aleph$ and consisting of *compact*$_\aleph$ *objects*, i.e., objects C such that $\mathcal{T}(C, -)$ preserves coproducts and moreover $\mathcal{T}(C, X)$ is $\aleph$-small for all $X \in \mathcal{T}$. In a compactly$_\aleph$ generated category, Brown representability and the existence of right adjoints still holds, provided the cohomological functors in question take $\aleph$-small values (see [MN06]). On the other hand, Brown representability for the dual, the existence of left adjoints, or even the existence of infinite products, may fail in a compactly$_\aleph$ generated category (see [2]).

1.2 Universal coefficient theorems

Very roughly speaking, a universal coefficient theorem is a statement expressing a triangulated category $\mathcal{T}$ as the extension of an abelian category $\mathcal{A}$ by a nilpotent ideal of "phantom" maps. Concretely, this amounts to a natural short exact sequence computing the Hom groups of the triangulated category as the extension of one Hom group and one Ext group computed in $\mathcal{A}$ via a suitable homological functor $\mathcal{T} \to \mathcal{A}$. Virtually all "universal coefficient theorems" encountered in Nature arise this way, although this may not be obvious at first sight.

Let us give some more details.

1.2.1. The UCT in the triangular setting. Let $\mathcal{T}$ be a triangulated category and let $\mathcal{C} \subset \mathcal{T}$ be a small full subcategory. We consider the abelian category $\mathrm{Mod}\,\mathcal{C}$ of right $\mathcal{C}$-modules (i.e. additive functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}\,\mathbb{Z}$) and the "restricted Yoneda" homological functor

$$h_\mathcal{C} \colon \mathcal{T} \longrightarrow \mathrm{Mod}\,\mathcal{C} \qquad X \mapsto \mathcal{T}(-,X)|_\mathcal{C}$$

(see §1.1.1). For any two objects X, Y of $\mathcal{T}$, we can consider the subgroup $\ker(h_C)(X,Y) := \{f \in X \to Y \mid h_\mathcal{C}(f) = 0\}$ of $\mathcal{T}(X,Y)$ of *$\mathcal{C}$-phantom maps* between them. To every such $\mathcal{C}$-phantom f we can associate an element $\xi(f)$ of $\mathrm{Ext}^1_\mathcal{C}(h_\mathcal{C}\Sigma X, h_\mathcal{C} Y)$ by completing f to a distinguished triangle $X \to Y \to Z \to \Sigma X$ and applying $h_\mathcal{C}$ to obtain an exact sequence $0 \to h_\mathcal{C} Y \to h_\mathcal{C} Z \to h_\mathcal{C}\Sigma X \to 0$ in $\mathrm{Mod}\,\mathcal{C}$; one sees easily that its Ext class $\xi(f)$ does not depend on the choice of the distinguished triangle. We thus obtain a canonical diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker(h_\mathcal{C})(X,Y) & \longrightarrow & \mathcal{T}(X,Y) & \xrightarrow{h_\mathcal{C}} & \mathrm{Hom}_\mathcal{C}(h_\mathcal{C}X, h_\mathcal{C}Y) \\ & & \downarrow{\scriptstyle \xi} & & & & \\ & & \mathrm{Ext}^1_\mathcal{C}(h_\mathcal{C}\Sigma X, h_\mathcal{C}Y) & & & & \end{array}$$

where the first row is is exact.

We say that *the universal coefficient theorem (=: UCT) for X and Y with respect to $\mathcal{C}$ holds* if the map $h_\mathcal{C}$ is surjective and the map ξ is invertible. The resulting short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_\mathcal{A}(h_\mathcal{C}\Sigma X, h_\mathcal{C}Y) \longrightarrow \mathcal{T}(X,Y) \longrightarrow \mathrm{Hom}_\mathcal{C}(h_\mathcal{C}X, h_\mathcal{C}Y) \longrightarrow 0$$

is *the associated UCT exact sequence.*

Typically then, a "universal coefficient theorem" is a result stating that, under some hypotheses, some particular instance of the UCT holds, in the above sense. In order to obtain a very general form of the UCT, one essentially only needs to assume that X belongs to the localising subcategory generated by $\mathcal{C}$ and that the $\mathcal{C}$-module $h_{\mathcal{C}}X$ has projective dimension at most one. This is quite well-known (see e.g. [Chr98] [MN10]), although for some of our applications we will need the following refinement to a cardinality-relative statement.

1.2.2 Terminology. Fix an infinite regular cardinal $\aleph$. A set is $\aleph$-*small* if it has cardinality strictly less then $\aleph$. An $\aleph$-*small* coproduct is one indexed by an $\aleph$-small set. In the abelian category $\mathrm{Mod}\,\mathcal{C}$, we say an object X is $\aleph$-*generated*, respectively $\aleph$-*presentable*, if there is an exact sequence $C \to X \to 0$, resp. $D \to C \to X \to 0$, where C, D are $\aleph$-small coproducts of representable $\mathcal{C}$-modules. We write $\mathrm{Loc}_{\aleph}(\mathcal{S})$ for the $\aleph$-*localising* subcategory generated by $\mathcal{S} \subset \mathcal{T}$, i.e., the smallest triangulated subcategory containing $\mathcal{S}$ and closed under the formation of (all available) $\aleph$-small coproducts in $\mathcal{T}$. In the following, we may choose $\aleph$ to be 'the cardinal of a proper class', in which case the above terminology reduces to the usual 'absolute' one.

1.2.3 Theorem ([15, Thm. 9.4])**.** *Let $\mathcal{T}$ be an idempotent complete triangulated category admitting arbitrary $\aleph$-small coproducts, and let $\mathcal{C}$ be a full subcategory of $\mathcal{T}$ which we assume consists of compact objects (so that $h_{\mathcal{C}}$ commutes with coproducts) and is suspension closed (i.e. $\Sigma\mathcal{C} = \mathcal{C}$). Suppose moreover that $h_{\mathcal{C}}(X)$ is $\aleph$-generated for every object X of $\mathrm{Loc}_{\aleph}(\mathcal{C}) \subseteq \mathcal{T}$. Then the UCT holds with respect to $\mathcal{C}$ for all $X \in \mathrm{Loc}_{\aleph}(\mathcal{C})$ such that $\mathrm{pdim}_{\mathcal{C}}\, h_{\mathcal{C}}(X) \leqslant 1$ and for arbitrary $Y \in \mathcal{T}$.*

The above UCT is 'local', in that it considers one object X at a time. We can easily derive from it a 'global' version:

1.2.4 Corollary ([15, Thm. 9.5])**.** *Let $\mathcal{T}$ and $\mathcal{C}$ be as in Theorem 1.2.3 with the further assumption that $h_{\mathcal{C}}X$ is $\aleph$-generated and $\mathrm{pdim}_{\mathcal{C}}\, h_{\mathcal{C}}X \leqslant 1$ for all $X \in \mathcal{T}$. Then the essential image of $h_{\mathcal{C}}$ is the following hereditary exact category:*

$$\mathcal{E} := \{M \in \mathrm{Mod}\,\mathcal{C} \mid \mathrm{pdim}_{\mathcal{C}} M \leqslant 1 \text{ and } M \text{ is } \aleph\text{-presentable}\}\,.$$

Moreover, the functor $h_{\mathcal{C}}\colon \mathrm{Loc}_{\aleph}(\mathcal{C}) \to \mathcal{E}$ is full, essentially surjective, and reflects isomorphisms. In particular, it induces a bijection on isomorphism classes of objects.

In order to apply the abstract UCT of Theorem 1.2.3 in a given triangulated subcategory $\mathcal{T}$, one must be able to effectively identify those subcategories $\mathcal{C} \subset \mathcal{T}$ for which there are enough interesting $X \in \mathrm{Loc}(\mathcal{C})$ with $\mathrm{pdim}_{\mathcal{C}}\, h_{\mathcal{C}}X \leqslant 1$.

Classically, one considers a compact object $G \in \mathcal{T}$ whose graded endomorphism ring $R = \mathrm{End}^*_{\mathcal{T}}(G)$ is hereditary; taking $\mathcal{C} := \{\Sigma^n G \mid n \in \mathbb{Z}\}$ to be the suspension closure of G, we see that $\mathrm{Mod}\,\mathcal{C}$ is just the category of graded right R-modules, which by hypothesis has global dimension one. The UCT follows.

In some more 'exotic' situations, however, one encounters small categories $\mathcal{C}$ such that $\mathrm{Mod}\,\mathcal{C}$ is *not* hereditary (indeed, it may have infinite global dimension), but the UCT still holds. As it turns out, in all these examples this is because of the combination of two reasons:

(A) every object M of $\operatorname{Mod}\mathcal{C}$ has either projective dimension 1 or ∞, and

(B) the homological functor $h_{\mathcal{C}}$ can only take values in objects of finite projective dimension.

Thus we have to understand which subcategories $\mathcal{C}$ are nice enough for (A) and (B) to occur. To understand (A), it suffices to consult the theory of *Gorenstein categories* (see §1.2.5), while part (B) is more subtle and leads us to the notion of a *Gorenstein closed subcategory* (see §1.2.8). By combining them, we will obtain a new powerful form of the UCT, namely Theorem 1.2.11.

1.2.5. Gorenstein categories. Classically, a Gorenstein ring is a commutative noetherian ring having finite injective dimension as a module over itself (at least locally). This condition imposes some strong symmetries on the homological algebra over that ring, the first of them being that the maximal finite projective dimension and the maximal finite injective dimension of its modules coincide, and are equal to the injective dimension of the ring. Hence regular (noetherian and commutative) rings are Gorenstein as they have finite global dimension, but the singular case is even more interesting, because then one has a dichotomy: the projective or injective dimension of modules can be infinite, but whenever it is finite it must be uniformly bounded by the same number – the Gorenstein dimension of the ring.

For our purposes, we need a simultaneous non-commutative, non-noetherian and several-objects generalisation of this notion. Luckily, such a theory already exists for arbitrary Grothendieck categories; it was introduced and studied by Enochs, Estrada and García-Rozas [EEGR08]. We apply it to the Grothendieck category $\operatorname{Mod}\mathcal{C}$ of right $\mathcal{C}$-modules over a small category $\mathcal{C}$.

According to this definition, we say the small category $\mathcal{C}$ (or the abelian category $\operatorname{Mod}\mathcal{C}$) is *Gorenstein* if:

- for all $M \in \operatorname{Mod}\mathcal{C}$, the projective dimension of M is finite if and only if its injective dimension is finite: $\operatorname{pdim}_{\mathcal{C}} M < \infty \Leftrightarrow \operatorname{idim}_{\mathcal{C}} M < \infty$; and
- the finitary injective and finitary projective dimensions of $\operatorname{Mod}\mathcal{C}$ are finite:

$$\sup\{\operatorname{pdim}_{\mathcal{C}} M \mid M \in \operatorname{Mod}\mathcal{C}, \operatorname{pdim}_{\mathcal{C}} M < \infty\} < \infty\,,$$
$$\sup\{\operatorname{pdim}_{\mathcal{C}} M \mid M \in \operatorname{Mod}\mathcal{C}, \operatorname{idim}_{\mathcal{C}} M < \infty\} < \infty\,.$$

If this holds the infinitary injective and projective dimensions automatically coincide. This number $n \in \mathbb{N}$ is called the *Gorenstein dimension of* $\mathcal{C}$, and we say that $\mathcal{C}$ is n*-Gorenstein*.

For instance, if R is a ring which happens to be both left and right noetherian, then $R = \mathcal{C}$ is Gorenstein in our sense iff it has finite injective dimension over itself both as a left and a right module (see [15, Ex. 2.2]), i.e., iff R is an *Iwanaga-Gorenstein ring* in the sense of [EJ00]; this is a well-studied standard notion of Gorensteinness for noncommutative rings.

In [15] we prove a few new criteria for recognising Gorenstein categories, thereby unifying some previous results. Here we only present two of them.

Although, in view of point (A) in §1.2.1, we are really only interested in 1-Gorenstein categories, our criteria work equally well in all dimensions.

1.2.6 Theorem ([15, Thm. 4.6]). *Let $\mathcal{C}$ be a small $\Bbbk$-category for some commutative ring $\Bbbk$. Assume the following:*

1. *$\mathcal{C}$ is bounded, that is, for any fixed object $C \in \mathcal{C}$ there are only finitely many objects of $\mathcal{C}$ mapping nontrivially into or out of C.*
2. *The Hom $\Bbbk$-modules $\mathcal{C}(C,D)$ are all finitely generated projective.*
3. *Each unit map $\Bbbk \to \mathrm{End}_{\mathcal{C}}(C)$, $C \in \mathcal{C}$, admits a $\Bbbk$-linear retraction.*
4. *$\mathcal{C}$ admits a Serre functor relative to $\Bbbk$, that is, a self-equivalence $S\colon \mathcal{C} \overset{\sim}{\to} \mathcal{C}$ equipped with a natural isomorphism*

$$\mathcal{C}(C,D) \cong \mathcal{C}(D,SC)^{\star}$$

where $(-)^{\star} = \mathrm{Hom}_{\Bbbk}(-,\Bbbk)$ denotes the $\Bbbk$-linear dual.

If R is any n-Gorenstein $\Bbbk$-algebra (for instance $n = 1$ and $R = \Bbbk = \mathbb{Z}$), then the R-category $R \otimes_{\Bbbk} \mathcal{C}$ (extend scalars Hom-wise) is n-Gorenstein as well.

The following known examples, which look quite dissimilar at first sight, can all be obtained as easy applications of the above theorem (see [15, §4]):

- If G is a finite group and R an is n-Gorenstein ring, the group algebra RG is n-Gorenstein.
- If R is an n-Gorenstein ring, then the abelian category $\mathrm{Ch}(R)$ of chain complexes of right R-modules is n-Gorenstein; and similarly for left modules, and for categories of π-periodic complexes of any period $\pi \in \mathbb{N}$.
- Let $\Bbbk$ be a commutative n-Gorenstein ring, and let $\mathcal{T}$ be a $\Bbbk$-linear triangulated category with a Serre functor. Then any small bounded full subcategory $\mathcal{C} \subset \mathcal{T}$ closed under the Serre functor and such that $\mathrm{End}_{\mathcal{C}}(C) \cong \Bbbk$ for all $C \in \mathcal{C}$, is n-Gorenstein.

The other criterion we mention here is specific to triangulated categories:

1.2.7 Theorem ([15, Thm. 6.1]). *Let $\mathcal{C}$ be a triangulated category which admits a skeleton with at most $\aleph_n$ morphisms. Then $\mathcal{C}$ is an m-Gorenstein category for some $m \leq n+1$.*

The most famous example of such a triangulated category $\mathcal{C}$, with $n = 0$, is the Spanier-Whitehead category, i.e. the stable homotopy category of finite pointed CW-complexes — in other words, the homotopy category of finite spectra — which is well-known to admit a countable skeleton. See e.g. [Mar83].

1.2.8. Gorenstein closed subcategories of triangulated categories. We now consider point (B) of §1.2.1: if $\mathcal{C}$ is a small 1-Gorenstein category occurring as a full subcategory of a triangulated category $\mathcal{T}$, what conditions ensure that the restricted Yoneda functor $h_{\mathcal{C}}\colon \mathcal{T} \to \operatorname{Mod}\mathcal{C}$ takes values in modules of finite (hence necessarily $\leqslant 1$) projective dimension?

The next theorem provides several necessary and sufficient conditions for this to happen which, as it turns out, work uniformly for categories of any Gorenstein dimension n. In order to formulate our result we need to recall one more definition: an object $M \in \operatorname{Mod}\mathcal{C}$ is *Gorenstein projective* if it admits a complete projective resolution, that is, if there exists an exact complex $P^\bullet = (P^i, d^i)_{i\in\mathbb{Z}}$ of projectives with $M = \operatorname{Ker}(d^0)$ and such that the complex $\mathcal{C}(P^\bullet, Q)$ is exact for every projective module Q.

We denote by $\operatorname{add}\mathcal{C}$ the additive closure of $\mathcal{C}$ in $\mathcal{T}$, i.e., the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{C}$ and closed under the formation in $\mathcal{T}$ of direct sums and retracts.

1.2.9 Theorem ([15, Thm. 8.6 and Prop. 8.2])**.** *Consider a small suspension-closed full subcategory $\mathcal{C}$ of a triangulated category $\mathcal{T}$, and assume that* $\operatorname{Mod}\mathcal{C}$ *is Gorenstein (§1.2.5) and locally coherent (e.g. locally noetherian). Then the following four conditions are equivalent:*

1. *The $\mathcal{C}$-module $h_{\mathcal{C}}X$ has finite projective dimension for every object $X \in \mathcal{T}$.*
2. *For every finitely presented Gorenstein projective $\mathcal{C}$-module M, there exists a distinguished triangle $X \to Y \to Z \to \Sigma X$ in $\operatorname{add}\mathcal{C}$ such that $M \cong \operatorname{Im}(h_{\mathcal{C}}X \to h_{\mathcal{C}}Y)$.*
3. *There exists a set $\mathcal{S}$ of finitely presented Gorenstein projective $\mathcal{C}$-modules such that:*
    - *Every $M \in \mathcal{S}$ occurs as a syzygy of a triangle in $\operatorname{add}\mathcal{C}$, as in (2).*
    - *The modules of $\mathcal{S}$, together with the finitely presented projectives, generate all finitely presented Gorenstein projectives by taking extensions and retracts.*
4. *If $X \to Y$ is a morphism in $\operatorname{add}\mathcal{C}$ such that the image of $h_{\mathcal{C}}X \to h_{\mathcal{C}}Y$ is Gorenstein projective, then the cone of $X \to Y$ is also in $\operatorname{add}\mathcal{C}$.*

Definition. If the equivalent conditions of the theorem are satisfied, we say that $\mathcal{C}$ is *Gorenstein closed in $\mathcal{T}$*.

Thus $\mathcal{C}$ is Gorenstein closed in $\mathcal{T}$ if its additive closure contains 'sufficiently many' distinguished triangles.

Here is a suggestive way to understand our theorem (see [15, §2] for details). As $\mathcal{C}$ is Gorenstein, the subcategory $\operatorname{GProj}\mathcal{C} \subset \operatorname{Mod}\mathcal{C}$ of Gorenstein projective modules is a Frobenius exact category and therefore its stable category $\underline{\operatorname{GProj}\mathcal{C}}$ (the quotient of $\operatorname{GProj}\mathcal{C}$ by the ideal of all maps factoring through a projective) is a triangulated category with arbitrary small coproducts. If $\mathcal{C}$ is also locally coherent, the image in $\underline{\operatorname{GProj}\mathcal{C}}$ of the finitely presented Gorenstein projectives are precisely its compact objects, $(\underline{\mathrm{GProj}\,\mathcal{C}})^c$. In many situations, the triangulated category $(\underline{\mathrm{GProj}\,\mathcal{C}})^c$ is called the *singularity category of* $\mathcal{C}$ and is thought of as measuring how far $\mathcal{C}$ is from being regular. What our theorem says is that $h_\mathcal{C}$ takes values in modules of finite projective dimension precisely when, among the (necessarily finitely presented and Gorenstein projective) syzygies of the distinguished triangles contained in $\mathrm{add}\,\mathcal{C}$, we find a set of generating objects for the singularity category of $\mathcal{C}$.

1.2.10. The Gorenstein UCT. It is now immediate to combine the abstract UCT of Theorem 1.2.3 with our answers to the points (A) and (B) as given in the previous two sections. The result is a new criterion for recognising more concrete universal coefficient theorems, which we may call "the Gorenstein UCT":

1.2.11 Theorem ([15, Thm. 9.16]). *Let $\mathcal{T}$ be an idempotent complete triangulated category with $\aleph$-small coproducts, for some infinite regular cardinal $\aleph$. Let $\mathcal{C}$ be a Gorenstein closed (§1.2.8) and suspension closed full subcategory of compact objects of $\mathcal{T}$ such that $\mathrm{Mod}\,\mathcal{C}$ is locally coherent and 1-Gorenstein (§1.2.5). Then the UCT with respect to $\mathcal{C}$ holds for all pairs of objects $X, Y \in \mathcal{T}$ provided that $X \in \mathrm{Loc}_\aleph(\mathcal{C})$.*

Moreover, the following dichotomy holds for any $\aleph$-presented $\mathcal{C}$-module M:

- *either $\mathrm{pdim}_\mathcal{C}\, M \leqslant 1$ and $M \cong h_\mathcal{C} X$ for some $X \in \mathrm{Loc}_\aleph(\mathcal{C}) \subseteq \mathcal{T}$,*
- *or $\mathrm{pdim}_\mathcal{C}\, M = \infty$ and M is not of the form $h_\mathcal{C} X$ for any object $X \in \mathcal{T}$.*

Note that, in the absolute case (with $\aleph$ the 'cardinality of a proper class'), the condition on $\mathcal{C}$-modules to be $\aleph$-presented is void.

In the next two sections we present some applications of this result.

1.3 Applications: Brown-Adams representability

In §1.2.5 we have given some evidence that it often possible to recognise 1-Gorenstein categories. In order to apply Theorem 1.2.11 though, we must also be able to recognise *Gorenstein closed* subcategories of a given triangulated category $\mathcal{T}$, i.e., we must somehow recognise them in concrete examples; it is not clear whether the abstract characterisation of Theorem 1.2.9 can be used for this purpose. Indeed, this seems to be a much harder problem, but there are at least two cases where it actually becomes trivial.

The first case is the regular one: if $G \in \mathcal{T}$ is a compact object with a *hereditary* graded endomorphism ring — i.e. such that $\mathrm{Mod}\,\mathcal{C}$ has global dimension $\leqslant 1$ for $\mathcal{C} = \{\Sigma^i G \mid i \in \mathbb{Z}\}$ — then the only Gorenstein projective $\mathcal{C}$-modules are the projective ones, hence $\mathcal{C}$ is obviously Gorenstein closed, and we obtain a UCT, the most classical instance of all being obtained with $\mathcal{T} = \mathrm{D}(R)$ the derived category of a hereditary ring (or dg-algebra) and $G = R$. Of course, we didn't need all this Gorenstein business to prove it, so this first case is not so interesting — although it does show that that the classical UCT's used in algebra, topology and KK-theory are also covered by Theorem 1.2.11 (see [15, Examples 9.11 and 9.12] for details).

As we explain next, the second obvious case occurs at the opposite end of the spectrum, where we take $\mathcal{C}$ to consist of *all* compact objects.

1.3.1. A new proof of Neeman's Brown-Adams representability. Let $\mathcal{C} = \mathcal{T}^c$ be the subcategory of all compact objects in $\mathcal{T}$. As it is a triangulated category in its own right, Theorem 1.2.7 tells us that $\mathcal{C}$ is Gorenstein provided it is not too big; in particular, it is 1-Gorenstein if it admits a skeleton with countably many maps. Moreover, it being triangulated immediately implies that it is Gorenstein closed (see Theorem 1.2.9), and also locally coherent (because triangulated categories have weak kernels). Therefore we can now specialise Theorem 1.2.11, or rather its global version as in Corollary 1.2.4, to the following representability result which was first proved by Neeman [Nee97].

1.3.2 Theorem ([15, Thm. 9.17]). *Let $\mathcal{T}$ be a triangulated category admitting arbitrary small coproducts and such that its category of compact objects, $\mathcal{T}^c$, admits a skeleton with only countably many morphisms. Then all cohomological functors on $\mathcal{T}^c$ are represented by objects in $\mathcal{T}$, and all natural transformations between them can be lifted to morphisms in $\mathcal{T}$. That is, every natural transformation $H \to H'$ between cohomological functors $H, H' \colon (\mathcal{T}^c)^{\mathrm{op}} \to \mathrm{Mod}\,\mathbb{Z}$ is isomorphic to one of the form $\mathcal{T}(-,X)|_{\mathcal{T}^c} \to \mathcal{T}(-,X')|_{\mathcal{T}^c}$ induced by some morphism $f \colon X \to X'$ of $\mathcal{T}$.*

The example where $\mathcal{T}$ is the homotopy category of spectra, whose compact objects — finite spectra — are well-known to admit a countable skeleton, recovers the original representability theorem of Brown and Adams [Ada71].

An advantage of our approach, which starts out by identifying $\mathcal{T}^c$ as a small 1-Gorenstein category, is that it gives a neat conceptual explanation for some phenomena surrounding this theorem, such as the following dichotomy for a functor $F \colon (\mathcal{T}^c)^{\mathrm{op}} \to \mathrm{Mod}\,\mathbb{Z}$: either F is homological and representable, in which case it has projective and injective dimension at most one over $\mathcal{T}^c$, or it is neither representable nor homological and it has infinite projective and injective dimension. (Cf. [CS98, Prop. 1.4] for this result in the example of spectra, or more generally, for a "monogenic Brown category".)

1.3.3. The countable case and Kasparov's KK-theory. Another advantage of our approach is that it is more flexible. For instance, our cardinal-reckoning lets us easily find a version of the Brown-Adams representability theorem which holds in categories where only *countable* coproducts are available. Recall the definitions of $\aleph_1$-generated and $\mathrm{Loc}_{\aleph_1}(-)$ from Terminology 1.2.2.

1.3.4 Theorem ([15, Thm. 9.18]). *Let $\mathcal{T}$ be a triangulated category admitting arbitrary countable coproducts and such that $\mathcal{T}^c$ is essentially small and $\mathcal{T} = \mathrm{Loc}_{\aleph_1}(\mathcal{T}^c)$. Then all $\aleph_1$-generated cohomological functors on $\mathcal{T}^c$ and all natural transformations between them are representable in $\mathcal{T}$.*

This can be applied, for instance, to the triangulated categories that arise in connection with Kasparov's KK-theory of C*-algebras. We refer to them collectively as *Kasparov categories* and point the reader to [MN06] and especially [Mey08] for introductions.

These triangulated categories have the following features in common. The objects are (non-necessarily unital) separable complex C*-algebras, possibly equipped with some extra structure, and their Hom groups are given by some variant of Kasparov's bivariant topological K-groups $KK_0(-,-)$. The suspension functor is given by the function algebra $\Sigma A = C_0(\mathbb{R}, A)$ and satisfies *Bott periodicity*: $\Sigma^2 \cong \mathrm{id}$. Because of this, all homological functors on Kasparov categories naturally take values in $\mathbb{Z}/2$-graded rather than $\mathbb{Z}$-graded objects. The distinguished triangles all arise (up to isomorphism) from a noncommutative generalisation of the Puppe sequences of topological spaces or, equivalently, from a suitably nicely behaved class of C*-algebra extensions.

Because of the separability hypothesis on the C*-algebras, Kasparov categories only admit *countable* ($= \aleph_1$-small) coproducts, rather than arbitrary small ones. Moreover, they are typically not compactly$_{\aleph_1}$ generated, nor nicely generated in any manner. This can be easily remedied by choosing a suitable countable (or finite) generating subset $\mathcal{G} \subset \mathcal{T}$ of compact$_{\aleph_1}$ objects and by considering instead the triangulated subcategory $\mathcal{B} := \mathrm{Loc}_{\aleph_1}(\mathcal{G})$. Such compactly$_{\aleph_1}$ generated triangulated categories $\mathcal{B}$ are usually called *bootstrap categories* by operator algebraists.

Let us give some examples:

- The first example is the original Kasparov category KK of separable C*-algebras [Kas80]. By choosing $\mathcal{C} = \{\mathbb{C}\}$ we obtain the original bootstrap category $\mathcal{B} \subseteq \mathrm{KK}$ studied by Rosenberg and Schochet as the domain of application of their eponymous UCT [RS87]. Note that $\mathcal{B}$ contains all C*-algebras which are KK-equivalent to a commutative one.
- Given a suitable topological space T, there is a Kasparov category $\mathrm{KK}(T)$ of bundles of C*-algebras over T. With the goal of extending the classification program to non-simple C*-algebras, Meyer and Nest [MN09] [MN12] have studied the case when T is a finite poset with the associated Alexandrov topology, and have introduced a bootstrap category $\mathcal{B} \subseteq \mathrm{KK}(T)$ whose algebras are those bundles whose fibres belong to the Rosenberg-Schochet bootstrap class.
- If G is a second countable locally compact group [Kas88], there is a Kasparov category KK^G of *G-C*-algebras*, which are C*-algebras equipped with a continuous G-action. (Le Gall has also defined a Kasparov category for groupoids, which generalises simultaneously that for G-C*-algebras and for bundles; see [LG99] [MN06].) For G a finite group, the bootstrap category $\mathcal{B} \subseteq \mathrm{KK}^G$ defined by $\mathcal{G} = \{C(G/H) \mid H \leqslant G\}$ has been studied by Köhler [Köh11] and myself [6]. A larger equivariant Bootstrap category $\mathcal{B} \subset \mathrm{KK}^G$ has been considered by myself, Emerson and Meyer [11].

As these $\mathcal{B}$ lack arbitrary coproducts, the usual version of Brown-Adams representability, Theorem 1.3.2, cannot be used here — indeed, its conclusion would be wrong. The correct result is obtained by applying Theorem 1.3.4:

1.3.5 Theorem ([15, Thm. 10.25]). *Let $\mathcal{B} = \mathrm{Loc}_{\aleph_1}(\mathcal{G})$ be any of the bootstrap triangulated categories of C*-algebras as above, where the full suspension-closed subcategory of generators $\Sigma^{0,1}\mathcal{G}$ has at most countably many objects and maps. Then every natural transformation $\alpha\colon H \to H'$ between two cohomological functors $H, H'\colon (\mathcal{B}^c)^{\mathrm{op}} \to \mathrm{Mod}\,\mathbb{Z}$ is represented by a map $X \to X'$ of $\mathcal{B}$, provided $H(\Sigma^i C)$ and $H'(\Sigma^i C)$ are countable for all $C \in \mathcal{G}$ and $i \in \mathbb{Z}/2\mathbb{Z}$.*

1.4 UCT's for the KK-theory of C*-algebras

Our Gorenstein universal coefficient theorem (§1.2.10) generalises the classical UCT's of algebra, topology, and KK-theory so as to include the Brown-Adams representability theorem (§1.3.1). Our original motivation, however, was to unify and explain the 'exotic' UCT's discovered for certain variants of KK-theory of C*-algebras by Ralf Meyer and collaborators. We now briefly present these examples and show how they fit in our theory, referring to §1.3.3 for basic facts about the Kasparov categories and bootstrap categories involved.

1.4.1. The universal multi-coefficient theorem. Let $\mathcal{T} = \mathrm{KK}$ be the Kasparov category of separable C*-algebras. The Rosenberg-Schochet UCT for C*-algebras [RS87] computes the KK-theory of any separable C*-algebras A, B with A in the bootstrap category $\mathcal{B} = \mathrm{Loc}(\mathcal{C})$ by a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_{*-1}A, K_*B) \longrightarrow \mathrm{KK}(A,B) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_*A, K_*B) \longrightarrow 0\,.$$

It can be obtained immediately from Theorem 1.2.3 by setting $\mathcal{C} = \{\mathbb{C}, \Sigma\mathbb{C}\}$, so that $\mathrm{Mod}\,\mathcal{C}$ is the category of $\mathbb{Z}/2$-graded abelian groups and $h_{\mathcal{C}} = K_*$ is topological K-theory.

In order to study finer structures on the KK-theory groups, Dadarlat and Loring [DL96] have proved a "multi-coefficient" UCT short exact sequence (again for $A \in \mathcal{B}$ and B arbitrary)

$$0 \longrightarrow \mathrm{Ext}^1_{\Lambda}(\underline{K}_{*-1}A, \underline{K}_*B) \longrightarrow \mathrm{KK}(A,B) \longrightarrow \mathrm{Hom}_{\Lambda}(\underline{K}_*A, \underline{K}_*B) \longrightarrow 0\,.$$

where the invariant $\underline{K}_*(-) = \bigoplus_{n=0}^{\infty} K_*(-;\mathbb{Z}/n)$ collects all K-theory groups with cyclic or integer coefficients, and is considered as a (left) module over the category of "generalised Bockstein operations". The group Ext^1_Λ is also known as *pure ext*, Pext. One sees by direct inspection that Λ^{op} is isomorphic to the full subcategory $\mathcal{C} := \{\mathrm{Cone}(n \cdot \mathrm{id}_{\mathbb{C}}), \Sigma\,\mathrm{Cone}(n \cdot \mathrm{id}_{\mathbb{C}}) \mid n \geq 0\}$ of KK and that $\underline{K}_*$ identifies with $h_{\mathcal{C}}$.

1.4.2 Theorem ([15, Thm .10.1]). *The above category $\mathcal{C}$ is 1-Gorenstein and is Gorenstein closed in the ambient triangulated category* KK.

Therefore we can obtain a new proof of the Dadarlat-Loring universal multi-coefficient theorem by combining the latter theorem with our Gorenstein UCT, Theorem 1.2.11 (with $\aleph = \aleph_1$).

1.4.3. Filtrated KK-theory. Consider the Kasparov category $\mathrm{KK}(T_n)$ of bundles of C*-algebras over a certain finite topological space T_n, designed so that such bundles amount to C*-algebras equipped with a chosen filtration by $n-1$ ideals (thus e.g. $\mathrm{KK}(T_1) = \mathrm{KK}$ and the objects of $\mathrm{KK}(T_2)$ are the C*-algebra extensions $0 \to J \to A \to A/J \to 0$).

Meyer and Nest [MN12] have proved for such filtered C*-algebras $A, B \in \mathrm{KK}(T_n)$, with A in a suitable bootstrap class $\mathcal{B}$, a UCT

$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{NT}_*}(FK_{*-1}A, FK_*B) \longrightarrow \mathrm{KK}(T_n)(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{NT}_*}(FK_*A, FK_*B) \longrightarrow 0$$

where the invariant $A \mapsto FK_*(A)$, *filtrated K-theory*, assembles the ordinary K-theory groups of all subquotients of the given filtration of A, and is considered as a (left $\mathbb{Z}/2$-graded) module over its graded category $\mathcal{NT}_*$ of natural transformations. As it turns out, FK_* is the restricted Yoneda functor $h_{\mathcal{C}}$ for a certain full subcategory $\mathcal{C} := \{\Sigma^i R_{[a,b]} \mid i \in \{0,1\}, 1 \leqslant a \leqslant b \leqslant n\} \subset \mathrm{KK}(T_n)$, also introduced by Meyer and Nest.

1.4.4 Theorem ([15, Thm. 10.9]). *Let $\mathcal{C} \subset \mathrm{KK}(T_n)$ be the full subcategory representing filtrated K-theory, as above. Then $\mathcal{C}$ is 1-Gorenstein and is Gorenstein closed in $\mathrm{KK}(T_n)$.*

By combining this with the Gorenstein UCT, as before, we obtain a new proof of Meyer and Nest's UCT for filtered C*-algebras.

1.4.5. Equivariant KK-theory. Let C_p be a cyclic group of prime order p. Köhler [Köh11] proved a remarkable UCT for KK^{C_p}, the C_p-equivariant Kasparov category, using as invariant (what amounts to) the restricted Yoneda functor $h_{\mathcal{C}}$ for the six-objets full subcategory $\mathcal{C} = \Sigma^{0,1}\{\mathbb{C}, C(C_p), \mathrm{Cone}(u)\}$, where $u \colon \mathbb{C} \to C(C_p)$ is the unit map of the function algebra $C(C_p)$.

Although we were not able to give a new and simpler proof of Köhler UCT (as we did for the previous examples), the next theorem implies that it too falls nicely within our framework:

1.4.6 Theorem ([15, Thm. 10.16]). *Köhler's category $\mathcal{C}$ is 1-Gorenstein.*

Indeed, $\mathcal{C}$ is 1-Gorenstein by the theorem and, by Köhler's UCT and the characterisation of Theorem 1.2.9, it must also be Gorenstein closed in KK^G.

1.5 Equivariant KK-theory and Mackey functors

In this subsection we present the results of [6], where we apply the theory of Mackey functors, whose origins are in representation theory, to the study of equivariant topological KK-theory of finite groups. The general framework for this application is relative homological algebra in triangulated categories, which we explain first.

1.5.1. General relative homological algebra. Universal coefficient theorems as in §1.2 are actually quite rare, and most of the time one has to make do with spectral sequences. The theory that neatly handles the general situation is *relative homological algebra in triangulated categories*, an adaptation to triangulated categories of the classical (relative) homological algebra in abelian categories, and which is mostly due to Christensen [Chr98], Beligiannis [Bel00] and Meyer-Nest [MN10] [Mey08], although its roots go back to work of Adams in stable homotopy [Ada74] (see also Brinkmann [Bri68] for an early axiomatisation).

Rather then present the general theory, we briefly describe those definitions and results that will be used in the later sections. Assume we are given a triangulated category $\mathcal{T}$ equipped with, at least, all countable coproducts; it follows in particular that $\mathcal{T}$ is idempotent complete. Fix also a full subcategory $\mathcal{C} \subset \mathcal{T}^c$ of compact objects and consider, as in §1.2, the $\mathcal{C}$-restricted Yoneda homological functor

$$h_{\mathcal{C}} \colon \mathcal{T} \longrightarrow \operatorname{Mod} \mathcal{C} \qquad X \mapsto \mathcal{T}(-,X)|_{\mathcal{C}} =: h_{\mathcal{C}} X$$

as well as its kernel on morphisms

$$\mathcal{I} := \ker(h_{\mathcal{C}}) = \{ f \in \operatorname{Mor} \mathcal{T} \mid h_{\mathcal{C}}(f) = 0 \} .$$

By construction, the latter is a *stable homological ideal*, that is, a categorical ideal (consisting of subgroups $\mathcal{I}(X,Y) \subseteq \mathcal{T}(X,Y)$ and closed under arbitrary compositions on both sides) which moreover is closed under suspensions, desuspension, coproducts and some other operations involving triangles, translating the fact that it is the kernel of a homological functor. Such an ideal $\mathcal{I}$ of "phantom maps" is precisely the data necessary to define a notion of relative homological algebra in $\mathcal{T}$, where the collection of all distinguished triangles

$$X \rightarrowtail Y \twoheadrightarrow Z \xrightarrow{f} \Sigma X$$

with $f \in \mathcal{I}$ takes on a rôle similar to that of the admissible short exact sequences in an exact category. We thus have associated notions of $\mathcal{I}$-*monomorphisms* ($\rightarrowtail$), $\mathcal{I}$-*epimorphisms* ($\twoheadrightarrow$), $\mathcal{I}$-*projective objects* — those $X \in \mathcal{T}$ such that $\mathcal{T}(X, f) = 0$ for all $f \in \mathcal{I}$ — and so on.

Because $\mathcal{I}$ is not any abstract stable homological ideal but is actually defined by $\mathcal{C}$, we are guaranteed the existence of enough $\mathcal{I}$-projective objects: they are simply given by $\operatorname{Add} \mathcal{C}$, the closure of $\mathcal{C}$ in $\mathcal{T}$ under retracts and coproducts. We can then use $\mathcal{I}$-projective resolutions to define the $\mathcal{I}$-*relative left derived functors* $\mathrm{L}^{\mathcal{I}}_n F$ of any homological functor $F \colon \mathcal{T} \to \mathcal{A}$, as well as the relative *right* derived functors $\mathrm{R}^n_{\mathcal{I}} G$ of any cohomological functor $G \colon \mathcal{T}^{\mathrm{op}} \to \mathcal{A}$. As [Mey08] explains in detail, provided that F preserves coproducts and G sends coproducts to products, these derived functors assemble into exact triples and yield two spectral sequences of the form

$$E^2_{p,q} = \mathrm{L}^{\mathcal{I}}_p F(\Sigma^q X) \overset{n=p+q}{\Longrightarrow} F(\Sigma^n X) \qquad E_2^{p,q} = \mathrm{R}^p_{\mathcal{I}} G(\Sigma^{-q} X) \overset{n=p+q}{\Longrightarrow} G(\Sigma^n X)$$

for every $X \in \mathrm{Loc}(\mathcal{C})$.

By choosing $G := \mathcal{T}(-, Y)$ for any $Y \in \mathcal{T}$, we obtain from the second spectral sequence a very general *universal coefficient theorem spectral sequence* computing $\mathcal{T}(X, Y)$, where the input $E_2^{p,q}$ identifies with certain $\mathcal{I}$-relative Ext functors (see [6, Thm. 5.15] for details). In case $\mathrm{pdim}_{\mathcal{C}}\, h_{\mathcal{C}} X \leqslant 1$, this spectral sequence collapses and specialises to the UCT short exact sequence already encountered in §1.2.1.

If $\mathcal{T}$ happens to be equipped with a biexact tensor product $\otimes\colon \mathcal{T} \times \mathcal{T} \to \mathcal{T}$, i.e. is a tensor triangulated category in the sense of Section 2, then we can choose $F := h_{\mathcal{C}}(- \otimes Y)$ for any $Y \in \mathcal{T}$ and obtain this way a *Künneth spectral sequence* computing the value of restricted Yoneda at a tensor product, $h_{\mathcal{C}}(X, Y)$. In this case the input $E^2_{p,q}$ is given by certain Tor functors.

1.5.2. Mackey and Green functors. (See [Bou97] [Lew81].) Fix a finite group G. Denote by $\mathcal{B}$ the *Burnside category*; this is the additive category whose objects are all finite G-sets and where a morphism $X \to Y$ is a formal linear combination of isomorphism classes of spans $X \leftarrow S \to Y$ of G-equivariant maps. The composition of $\mathcal{B}$ is induced by the pullback of G-sets. A *Mackey functor* (for G and with coefficients in $\mathbb{Z}$) is an additive functor $\mathcal{B}^{\mathrm{op}} \to \mathrm{Mod}\, \mathbb{Z}$. We denote by $\mathrm{Mack}_{\mathbb{Z}}(G) := \mathrm{Mod}\, \mathcal{B}$, or simply by Mack, the abelian category of Mackey functors. Alternatively, one may choose any reasonable abelian category of values instead of $\mathrm{Mod}\, \mathbb{Z}$. A typical choice is $\mathrm{Mod}\, \Bbbk$ for any commutative ring $\Bbbk$; in the next subsection we will use $\mathbb{Z}/2$-graded abelian groups.

The cartesian product of G-sets induces a bilinear symmetric monoidal structure on $\mathcal{B}$, which by taking suitable Kan extensions ("Day convolution") can be extended to an additive closed symmetric monoidal structure on Mack. This is usually called the *box product* of Mackey functors. The tensor unit $\mathbb{1} = \mathcal{B}(-, G/G)$ is known as the *Burnside ring* functor; its value on $X = G/H$ is the Grothendieck group $K_0(H\text{-}\mathrm{set})$ of finite H-sets with sum given by disjoint union. We denote it by *Bur* (a more canonical, but for us rather inconvenient, notation is A). Concretely, a Mackey functor M is entirely determined by its values on the orbits G/H and by the *restriction*, *transfer* and *conjugation* maps

$$r^H_L := M(G/L \leftarrow G/L \to G/H) \qquad t^H_L := M(G/H \leftarrow G/L \to G/L)$$

$$c_{g,H} := M(G/{}^gH \leftarrow G/H \to G/H)$$

for all $L \subseteq H \subseteq G$ and $g \in G$.

A (commutative) *Green functor* is by definition a (commutative) monoid $R = (R, m, u)$ in the tensor category Mack, i.e., it is an object R equipped with maps $m\colon R \otimes R \to R$ and $u\colon Bur \to R$ making the evident associativity, unit and commutativity diagrams commute in Mack. Concretely, a Green functor is the same as a Mackey functor R taking values in rings and such that its restriction and conjugation maps are ring morphisms and its transfers satisfy the Frobenius formulas $t^H_L(x) \cdot y = t^H_L(x \cdot r^H_L(y))$.

Similarly, a (left) *Mackey module*, or simply *module*, over a Green functor R is a Mackey functor M equipped with an action map $R \otimes M \to M$ which makes an associativity and unit diagrams commute; as before, a more concrete description can also be given. Mackey modules over R and action-preserving maps assemble into an abelian category R-Mack.

By a result of Bouc (see [Bou97, §3.2]), the category R-Mack is itself the abelian module category $\mathrm{Mod}\,\mathcal{B}_R$ over a certain essentially small additive category $\mathcal{B}_R$, which we call the *Burnside-Bouc category* of R. As with $\mathcal{B}$, the objects of $\mathcal{B}_R$ are finite G-sets; its Hom groups are $\mathcal{B}_R(X,Y) = R(X \times Y)$ and composition can be described by an explicit formula in terms of R.

1.5.3 Examples. We need only consider two examples of commutative Green functors, which are also probably the most classical and well-known of all.

- If $R = Bur$ is the Burnside ring, as above, then clearly Bur-Mack = Mack, because being the tensor unit Bur acts canonically and uniquely on all Mackey functors. Indeed, we have the equality $\mathcal{B}_{Bur} = \mathcal{B}$.
- The *representation Green functor*, Rep, is given by its value on G/H, namely the integral complex representation ring $R_{\mathbb{C}}(H)$ of H (or equivalently, the ring of complex characters on H), together with its usual restriction, transfer (induction) and conjugation maps.

1.5.4. Mackey modules from topological K-theory. We now explain how the representation Green functor Rep and its module category Rep-Mack are present at the very heart of equivariant KK-theory, similarly to the way the Burnside Green functor and its modules (i.e., Mackey functors) are central aspects of equivariant stable homotopy. Here below, Mackey functors will take values in $\mathbb{Z}/2$-graded abelian groups.

Fix a finite group G as before.

As in §1.3.3, consider the G-equivariant Kasparov category KK^G. Recall that G-equivariant topological K-theory is a functor K_*^G defined on G-C*-algebras and corepresented in the equivariant Kasparov category KK^G by the tensor unit $\mathbb{C}$: $\mathrm{KK}^G(\Sigma^n\mathbb{C}, -) = K_n^G$ (see [Phi87]). More generally we have $K_n^H(\mathrm{Res}^G_H A) = \mathrm{KK}^G(\Sigma^n C(G/H), A)$, i.e., the equivariant K-theory of actions restricted to a subgroup H is corepresented by the finite dimensional G-algebras $C(G/H)$ of complex functions on the orbit G/H.

Definition. For any G-C*-algebra A, we can define a Mackey functor $k_*^G(A)$ by setting $k_*^G(A)(G/H) := K_*^H(\mathrm{Res}^G_H A)$, by choosing the evident restriction and conjugation maps, and by taking the transfer maps to be those introduced by Phillips in [Phi87, §5.1].

1.5.5 Theorem ([6, Prop. 4.5, Lem. 4.7, Thm. 4.9])**.** *The above definition extends naturally to a functor*

$$k_*^G \colon \mathrm{KK}^G \to Rep\text{-Mack} \qquad A \mapsto k_*^G(A) = k_0^G(A) \oplus k_1^G(A)$$

from the Kasparov category of separable G-C-algebras to the category of Mackey modules over Rep, the complex representation Green functor for G with values in $\mathbb{Z}/2$-graded abelian groups (§1.5.2). Moreover, the restriction of $k_*^G \colon \mathrm{KK}^G \to$ Rep-Mack to the full subcategory $\{C(G/H) \colon H \leqslant G\}$ of KK^G is fully faithful.*

Consider now the following full subcategories of KK^G

$$\mathrm{Perm}^G := \{C(X) \mid X \in G\text{-}\mathrm{Set}\} \qquad \mathrm{perm}^G := \{C(X) \mid X \in G\text{-}\mathrm{set}\}$$

consisting of the 'permutation algebras' associated with all, respectively all finite, G-sets (here $C(X)$ denotes the C*-algebra of complex functions on X with the left G-action induced by the one on X).

1.5.6 Theorem ([6, Thm. 4.11]). *For every finite group G, the functor of Theorem 1.5.5 restricts to an equivalence (in fact, an isomorphism)*

$$k_*^G = k_0^G \colon \mathrm{perm}^G \xrightarrow{\sim} \mathcal{B}_{Rep}$$

of linear tensor categories between perm^G and the subcategory of representable Rep-modules, i.e., with the Burnside-Bouc category $\mathcal{B}_{Rep}$ associated with G and the representation Green functor Rep (see §1.5.2).

In particular, it now follows immediately from the properties of $\mathcal{B}_{Rep}$:

1.5.7 Corollary ([6, Cor. 4.12]). *There is a canonical equivalence between the category of additive functors $(\mathrm{perm}^G)^{\mathrm{op}} \to \mathrm{Mod}\,\mathbb{Z}$ (or equivalently, of coproduct-preserving additive functors $(\mathrm{Perm}^G)^{\mathrm{op}} \to \mathrm{Mod}\,\mathbb{Z}$), and the category of (ungraded) Mackey modules over the representation Green functor Rep. If we equip the functor category with the Day convolution tensor product, we have a symmetric monoidal equivalence.*

All our results of this subsection are close KK-theoretic analogues of well-known results in stable homotopy theory. In particular, finite G-sets can be mapped as suspension spectra into the G-equivariant stable homotopy category $\mathrm{SH}(G)$, where they form a full subcategory $\{\Sigma^\infty_+ X \mid X \in G\text{-}\mathrm{set}\}$ isomorphic to the Burnside category $\mathcal{B}$ for G. It follows that the equivariant homotopy groups naturally take values in Mackey functors (i.e., *Bur*-modules), just like topological K-theory takes values in *Rep*-modules as we explained above.

1.5.8. UCT and Künneth spectral sequences for KK^G. We now want to approximate the triangulated equivariant Kasparov category KK^G by the abelian category *Rep*-Mack of $\mathbb{Z}/2$-graded modules over the representation Green functor. We accomplish this by applying the relative homological algebra of §1.5.1 with the following choice of generating compact objects:

$$\mathcal{C} := \{C(G/H), \Sigma C(G/H) \mid H \leqslant G\}$$

By construction, the best results will only hold for G-C*-algebras in the bootstrap category $\mathrm{Loc}(\mathcal{C}) = \mathrm{Loc}_{\aleph_1}(\mathcal{C}) = \mathrm{Loc}_{\aleph_1}(\mathsf{Perm}^G)$. We denote this subcategory by Cell^G and call its objects *G-cell algebras.* Note that Cell^G, being generated by the 'canonical orbits' $C(G/H)$, is a better analogue of the equivariant stable homotopy category $\mathrm{SH}(G)$ than KK^G — which, as far as is known, has no good generation properties. Moreover, it can be shown that Cell^G contains many G-C*-algebras of interest, e.g. all commutative ones.

By combining Theorem 1.5.5 with the classical Brauer and Artin induction theorems for the representation theory of finite groups, we can easily reduce the number of generators. Recall that a finite group is *elementary* if it is a product of a p-group for some prime number p and a cyclic group of order prime to p.

1.5.9 Theorem ([6, Prop. 2.11]). *For every finite group, we have*

$$\mathrm{Cell}^G = \mathrm{Loc}_{\aleph_1}(\{C(G/E) \mid E \textit{ is an elementary subgroup of } G\})$$
$$\mathrm{Cell}^G_{\mathbb{Q}} = \mathrm{Loc}_{\aleph_1}(\{C(G/C) \mid C \textit{ is a cyclic subgroup of } G\})$$

where $\mathrm{Cell}^G_{\mathbb{Q}}$ *denotes the coproduct-compatible rationalisation ([2, Thm. 2.33]).*

As explained in §1.5.1, we obtain from this setup spectral sequences associated to all reasonable (co)homological functors on KK^G. Below we only mention a UCT and a Künneth spectral sequence, the latter associated to the minimal tensor product $A \otimes B$ of G-C*-algebras (G acts diagonally). Thanks to the results of §1.5.4, it is not difficult to identify the relative derived functors on the second pages with Ext^n_{Rep} and Tor_n^{Rep} respectively, that is, the ($\mathbb{Z}/2$-graded) Ext and Tor functors computed in the tensor-abelian category *Rep*-Mack.

1.5.10 Theorem ([6, Thm. 5.16]). *Let G be a finite group. For all $A, B \in \mathrm{KK}^G$ such that A is a G-cell algebra, and depending functorially on them, there exists a conditionally convergent, cohomologically indexed right half-plane spectral sequence of the form*

$$E_2^{p,q} = \mathrm{Ext}^p_{Rep}(k^G_* A, k^G_* B)_{-q} \quad \overset{n=p+q}{\Longrightarrow} \quad \mathrm{KK}^G_n(A,B)\,.$$

This spectral sequence converges strongly if A is such that $\mathrm{KK}^G(A,f) = 0$ for every morphism f which can be written, for each $n \geqslant 1$, as a composition of n maps each of which vanishes under k^G_*. If A is such that the Rep-module $k^G_* A$ has a projective resolution of finite length $m \geqslant 1$, then the spectral sequence is confined in the region $0 \leqslant p \leqslant m+1$ and thus collapses at the page $E^{*,*}_{m+1} = E^{*,*}_\infty$.

1.5.11 Theorem ([6, Thm. 5.17]). *Let G be a finite group. For all $A, B \in \mathrm{KK}^G$ with A a G-cell algebra, and depending functorially in them, there is a strongly convergent, homologically indexed right half-plane spectral sequence of the form*

$$E^2_{p,q} = \mathrm{Tor}_p^{Rep}(k^G_* A, k^G_* B)_q \quad \overset{n=p+q}{\Longrightarrow} \quad K^G_n(A \otimes B)\,.$$

If moreover A is such that $k^G_* A$ has a projective resolution of finite length $m \geqslant 1$, then the spectral sequence is confined in the region $0 \leqslant p \leqslant m$ and thus collapses at the page $E^{m+1}_{*,*} = E^\infty_{*,*}$.

An easy corollary of Theorem 1.5.9 and the latter Künneth spectral sequence is the following vanishing result:

1.5.12 Theorem ([6]). *Let A and B be two G-C*-algebras for a finite group G, and assume that either A or B is a G-cell algebra. If $K^E_*(\mathrm{Res}^G_E A) = 0$ for all elementary subgroups E of G, then $K^G_*(A \otimes B) = 0$. In a similar fashion, if $K^C_*(\mathrm{Res}^G_C A) \otimes \mathbb{Q} = 0$ for all cyclic subgroups C of G, then $K^G_*(A \otimes B) \otimes \mathbb{Q} = 0$.*

One could proceed to derive more elaborate applications of our spectral sequences, but some serious efforts would first have to be made to obtain nontrivial computations of Ext and Tor groups for specific *Rep*-modules.

1.6 Trace computations in equivariant Kasparov categories

We now present the results of [11].

Let X be a compact orientable smooth manifold and let $f\colon X \to X$ be a smooth self-map with simple isolated fixed points. The classical Lefschetz-Hopf formula equates the sum of the indices of all fixed points of f with its cohomological trace:

$$\sum_x \mathrm{ind}_f(x) = \sum_n (-1)^n \operatorname{tr} H^n(f;\mathbb{Q})\,.$$

The formula identifies a local and geometric computation with a global and homological one.

Let G be a compact group. Our main result is a G-equivariant generalisation of the fixed-point formula for endo-maps $f\colon X \to X$ on a compact smooth G-manifold X, and even to correspondences.

Note that, by the Chern isomorphism, the right-hand side can also be expressed as the super-trace $\operatorname{tr} K^*(f)\otimes\mathbb{Q} := \operatorname{tr} K^0(f)\otimes\mathbb{Q} - \operatorname{tr} K^1(f)\otimes\mathbb{Q}$ of the map induced by f on rationalised K-theory $K^*(X)\otimes\mathbb{Q}$. The homological side of our generalised formula will involve equivariant topological K-theory. According to the nature of the group G, we will provide different computations; see, §1.6.7, §1.6.12 and §1.6.15. The geometric side will be computed in a certain category of smooth correspondences; see §1.6.2.

The proofs make use of the G-equivariant stable Kasparov category KK^G, already encountered in §1.3.3. The point is that KK^G is a *tensor triangulated category*, meaning that it comes equipped with a symmetric monoidal structure

$$-\otimes -\colon \mathrm{KK}^G \times \mathrm{KK}^G \longrightarrow \mathrm{KK}^G$$

which preserves exact triangles in each variable (see §2.1.1 for details). As already mentioned when discussing the Künneth Theorem 1.5.11, $A\otimes B$ is obtained by endowing the minimal tensor product of C*-algebras with the diagonal G-action. The tensor unit is $\mathbb{C}$ equipped with the trivial G-action.

The map f induces in the category KK^G an endomorphism of the G-C*-algebra $C(X)$. The geometric hypotheses on X ensure that $C(X)$ is a rigid object for the tensor structure of KK^G so that f has a well-defined trace $\operatorname{tr}(f)$, which is an element of $\mathrm{End}_{\mathrm{KK}^G}(\mathbb{C}) \cong R_{\mathbb{C}}(G)$, the complex representation ring of G. The equivariant Lefschetz-Hopf formula is then an equation in $R_{\mathbb{C}}(G)$ between different computations of this element $\operatorname{tr}(f)$.

1.6.1. Dualisable objects and the monoidal trace. We begin by recalling the definition of a monoidal trace, introduced by Dold and Puppe [DP80] in order to unify several notions of trace in algebra and topology.

Let $(\mathcal{M}, \otimes, \mathbb{1})$ be a symmetric monoidal category with symmetry isomorphisms $\gamma = \gamma_{X,Y} \colon X \otimes Y \overset{\sim}{\to} Y \otimes X$ (see [ML98, §VII.7]). An object X of $\mathcal{M}$ is *dualisable* (or *strongly dualisable*, *rigid*) if it admits a *tensor dual*, i.e., if we can find in $\mathcal{M}$ an object $X^\vee$ and maps $\eta \colon \mathbb{1} \to X^\vee \otimes X$ and $\varepsilon \colon X \otimes X^\vee \to \mathbb{1}$ such that (omitting the coherent associativity and unit isomorphisms) we have

$$(\varepsilon \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes \eta) = \mathrm{id}_X \quad \text{ and } \quad (\mathrm{id}_{X^\vee} \otimes \varepsilon)(\eta \otimes \mathrm{id}_{X^\vee}) = \mathrm{id}_{X^\vee} \ .$$

This is equivalent to the functor $X \otimes - \colon \mathcal{M} \to \mathcal{M}$ having a right adjoint given by $X^\vee \otimes -$, with η and ε providing the unit and counit of this adjunction. In particular, if such a triple $(X^\vee, \eta, \varepsilon)$ exists for X then it is uniquely determined up to a canonical isomorphism.

If $f \colon X \to X$ is any endomorphism of a dualisable object X with dual $X^\vee$, the (*monoidal*) *trace of f* is the endomorphism $\mathrm{tr}(f)$ of $\mathbb{1}$ defined as the composite

$$\mathbb{1} \xrightarrow{\ \eta\ } X^\vee \otimes X \xrightarrow{\ \mathrm{id} \otimes f\ } X^\vee \otimes X \xrightarrow[\sim]{\ \gamma\ } X \otimes X^\vee \xrightarrow{\ \varepsilon\ } \mathbb{1} \, , \qquad X^\vee \otimes X \xrightarrow[\sim]{\ \gamma\ } X \otimes X^\vee \xrightarrow{\ f \otimes \mathrm{id}\ } X \otimes X^\vee .$$

If $\mathcal{M}$ is additive, $\mathrm{End}_{\mathcal{M}}(X)$ is a ring for every X, and the above definition yields a family of trace maps $\mathrm{tr} \colon \mathrm{End}(X) \to \mathrm{End}(\mathbb{1})$ with nice properties, e.g. cyclicity $\mathrm{tr}(fg) = \mathrm{tr}(gf)$, and which specialises for vector spaces to the usual trace of matrices. See [PS14] for a nice modern exposition of the resulting theory and its relation to fixed points.

An important advantage of this definition is that it is invariant under tensor functors. Indeed, if $F \colon \mathcal{M} \to \mathcal{N}$ is a *symmetric monoidal functor*, i.e. if it comes equipped with natural isomorphisms $FX \otimes FY \overset{\sim}{\to} F(X \otimes Y)$ and $\mathbb{1} \overset{\sim}{\to} F\mathbb{1}$ compatible with the associativity, unit and symmetry isomorphisms of the tensor categories $\mathcal{M}$ and $\mathcal{N}$, then the image FX of a dualisable object X is itself dualisable with dual $FX^\vee$, and the induced ring morphism $F \colon \mathrm{End}_{\mathcal{M}}(\mathbb{1}) \to \mathrm{End}_{\mathcal{N}}(\mathbb{1})$ preserves traces: $F\,\mathrm{tr}(f \colon X \to X) = \mathrm{tr}(Ff \colon FX \to FX)$.

This means that we can try to simplify the computation of traces by exchanging the ambient tensor category for a simpler one. And as long as we ensure that the morphism $\mathrm{End}_{\mathcal{M}}(\mathbb{1}) \to \mathrm{End}_{\mathcal{N}}(\mathbb{1})$ is injective, we have not lost any information.

We follow this basic strategy below, by localising the tensor triangulated category KK^G in a suitable fashion.

1.6.2. The geometric computation. Extending work of Connes and Skandalis [CS84], my collaborators Emerson and Meyer [EM10a] [EM10b] have developed a theory of smooth correspondences which computes equivariant KK-theory of commutative C*-algebras in a very topological, rather then analytic, way. For every compact group G, they construct a 'geometric' Kasparov category $\widehat{\mathrm{KK}}^G$ whose objects are non-necessarily compact (Hausdorff) G-manifolds, possibly with boundary.

A morphism $\varphi\colon X \to Y$ in $\widehat{\mathrm{KK}}^G$ is the data of a *smooth correspondence* $X \overset{b}{\leftarrow} (M,\xi) \overset{f}{\rightarrow} Y$, which consists of: a G-space M; a G-map $b\colon M \to X$ (inducing the usual contravariant functoriality of $X \mapsto C_0(X)$); a G-equivariant K-theory class $\xi \in RK^*_{G,X}(M)$ with X-compact support via b; and a K_G-oriented normally nonsingular G-map $f\colon M \to Y$ (inducing the "wrong-way" functoriality). Correspondences are up to a certain equivalence relation, and their composition is defined by an intersection product after deformation to a transverse position. We refer to [11, §2.1-2] [EM10b, Def. 2.3] for more details.

The category $\widehat{\mathrm{KK}}^G$ is additive (but not triangulated) and is also a symmetric monoidal category with respect to the cartesian product $X \times Y$. The usual complex function algebra $X \mapsto C(X)$ extends to a functor $\widehat{\mathrm{KK}}^G \to \mathrm{KK}^G$ which is fully faithful when restricted to compact G-manifolds. Moreover, it is symmetric monoidal: $C_0(X \times Y) \cong C_0(X) \otimes C_0(Y)$.

Correspondences can be used to explicitly compute tensor duals, in a similar way to Atiyah duality in equivariant stable homotopy:

1.6.3 Theorem ([EM10b] [11, Thm. 2.7])**.** *Let X be a compact smooth G-manifold, possibly with boundary. Then X is dualisable in $\widehat{\mathrm{KK}}^G$ with dual $N\mathring{X}$, the normal bundle for an embedding $\mathring{X} \to E$ of its interior $\mathring{X} = X \smallsetminus \partial X$ into a linear G-representation. The unit and counit can be given by explicit correspondences. In particular, the G-C^*-algebra $C(X) = C_0(X)$ is dualisable in KK^G.*

The following theorem gives the geometric side of our equivariant Lefschetz-Hopf formulas. For simplicity, we assume that the manifold X is closed (i.e. without boundary). Again, we refer to the cited works for details and for more general statements.

1.6.4 Theorem ([11, Thm. 2.18])**.** *Let X be a smooth compact G-manifold. Let $\varphi \in \mathrm{KK}^G(C(X), C(X))$ be represented by a smooth correspondence as above: $\varphi = [X \overset{b}{\leftarrow} (M,\xi) \overset{f}{\rightarrow} Y]$. Assume that b and f intersect smoothly with K_G-oriented* excess intersection bundle $\eta := \mathrm{Coker}(Df - Db) \in K^*_G(Q)$ *(the latter measures the failure of b and f to intersect transversally; if they do, $\eta = 0$).*

Then the monoidal trace of φ is represented by the correspondence

$$[\mathrm{pt} \leftarrow (Q, \xi|_Q \otimes e(\eta)) \to \mathrm{pt}] \quad \in \widehat{\mathrm{KK}}(\mathrm{pt},\mathrm{pt})^G$$

where the (by hypothesis, smooth) manifold Q is the intersection space $Q := \{m \in M \mid b(m) = f(m)\}$, *equipped with a canonical K_G-orientation, and $e(\eta)$ is the Euler class of η.*

As an element of $\mathrm{KK}^G(\mathbb{C},\mathbb{C}) = R_{\mathbb{C}}(G)$, the monoidal trace $\mathrm{tr}(\varphi)$ is given by the index of the Dirac operator on Q with coefficients in $\xi|_Q \otimes e(\eta)$.

For $G = 1$ trivial and for the correspondence $X \overset{f}{\leftarrow} X \overset{\mathrm{id}}{\rightarrow} X$ given by a self-map with isolated non-singular fixed points on an oriented X, then this reduces to the number $\sum_{x\in Q} \mathrm{ind}_f(x) \in \mathbb{Z}$ as expected.

Note however that the above computation as smooth correspondence is also of interest in more general situations, e.g. for maps without fixed points.

1.6.5. Localisation and general computation. Note that in any additive (not necessarily symmetric) monoidal category $\mathcal{M}$ the ring $R := \mathrm{End}_{\mathcal{M}}(\mathbb{1})$ is automatically commutative. Moreover, it acts on the whole category via the tensor product: for all $r \in R$ and $f \in \mathcal{M}(X,Y)$, simply define $r \cdot f \in \mathcal{M}(X,Y)$ to be the composite

$$X \cong \mathbb{1} \otimes X \xrightarrow{r \otimes f} \mathbb{1} \otimes Y \cong Y\,.$$

This turns each $\mathcal{M}(X,Y)$ into an R-module and makes the composition of $\mathcal{M}$ R-bilinear, so that $\mathcal{M}$ is canonically an *R-linear category* (i.e. a category enriched over R-modules [Kel05]). In particular, if $S \subseteq R$ is a multiplicative system we can localise each Hom R-module of $\mathcal{M}$ to obtain a localised $S^{-1}R$-linear category, $S^{-1}\mathcal{M}$. Note that the canonical functor $\mathcal{M} \to S^{-1}\mathcal{M}$ restricts on $\mathbb{1}$ to the localisation map of rings $R = \mathrm{End}_{\mathcal{M}}(\mathbb{1}) \to \mathrm{End}_{S^{-1}\mathcal{M}}(\mathbb{1}) = S^{-1}R$.

If $\mathcal{M}$ is a tensor triangulated category, then $S^{-1}\mathcal{M}$ is also the Verdier quotient by the thick tensor ideal of $\mathcal{M}$ generated by $\{\mathrm{Cone}(s) \mid s \in S\}$, so it is again a tensor triangulated category and the localisation functor is tensor exact; see §2.1.4.

We can apply this to $\mathcal{M} = \mathrm{KK}^G$ and $R = R_{\mathbb{C}}(G)$ as follows. By choosing the multiplicative system S of all non-zero divisors of $R_{\mathbb{C}}(G)$ we obtain the *total ring of fractions*, which splits as a finite product of fields

$$S^{-1}R_{\mathbb{C}}(G) \cong \prod_{\mathfrak{p}_i} F_i$$

where the product runs over the minimal prime ideals $\mathfrak{p}_i$ of the Zariski spectrum $\mathrm{Spec}\, R_{\mathbb{C}}(G)$ and $F_i := \mathrm{Frac}(R_{\mathbb{C}}(G)/\mathfrak{p}_i)$ denotes the residue field at $\mathfrak{p}_i$. Thus the localised category $S^{-1}\mathrm{KK}^G$ correspondingly splits into a product. In particular, by localising equivariant K-theory $K_n^G = \mathrm{KK}^G(\Sigma^n\mathbb{C}, -)$, we obtain for all $A \in \mathrm{KK}^G$ a natural splitting

$$S^{-1}K_*^G(A) \cong \bigoplus_{\mathfrak{p}_i} K_{*,i}^G(A)$$

where each $K_{*,i}^G(A)$ is a $\mathbb{Z}/2$-graded F_i-vector space. We thus obtain:

1.6.6 Theorem ([11, Thm. 3.4])**.** *Let $A \in \mathrm{KK}^G$ belong to the thick subcategory generated by $\mathbb{C}$, and let $\varphi \in \mathrm{KK}^G(A,A)$ be any endomorphism. Then A is a dualisable object, and the trace $\mathrm{tr}(\varphi)$ is uniquely determined by $\mathrm{tr}\, S^{-1}K_*^G(f) \in S^{-1}R_{\mathbb{C}}(G)$, whose components are given by the finitely many matrix super-traces $\mathrm{tr}\, K_{0,i}^G(\varphi) - \mathrm{tr}\, K_{1,i}^G(\varphi)$ over the vector spaces F_i.*

The above hypothesis on A ensures that it is a dualisable object and that traces are well-defined, but we still need to connect this with compact manifolds. We begin with the simplest case.

1.6.7. The computation for Hodgkin Lie groups. Assume now that G is a compact connected Lie group with torsionfree fundamental group, i.e., a

Hodgkin Lie group. For instance, G could be a torus, or any of a vast choice of classical Lie groups. For such a group, the representation ring $R_{\mathbb{C}}(G)$ has a particularly pleasant structure, in particular it is a domain and thus the localisation $S^{-1}R_{\mathbb{C}}(G)$ in Theorem 1.6.6 is just its field of fractions.

We have:

1.6.8 Theorem ([11, Thm. 3.5])**.** *Let G be a Hodgkin Lie group. Then an object $A \in \mathrm{KK}^G$ belongs to the thick subcategory generated by $\mathbb{C}$ if and only if A is dualisable and $\mathrm{Res}^G_1 A \in \mathrm{KK}$ belongs to the Rosenberg-Schochet bootstrap class.*

Every commutative C*-algebra belongs to the non-equivariant bootstrap class, and we know from Theorem 1.6.3 that compact smooth G-manifolds yield dualisable algebras $C(X)$. Hence we may apply Theorem 1.6.6 to $A = C(X)$, and by combining this homological computation of the monoidal trace with the geometric one of Theorem 1.6.4, we immediately obtain a nice equivariant Lefschetz-Hopf formula for Hodgkin Lie groups:

1.6.9 Theorem ([11, Thm. 1.1])**.** *Let G be a Hodgkin Lie group and denote by $F := \mathrm{Frac}(R_{\mathbb{C}}(G))$ the field of fractions of its complex representation ring. Let X be a closed G-manifold, and $X \overset{b}{\leftarrow} (M,\xi) \overset{f}{\rightarrow} X$ be a smooth correspondence on X with $\xi \in K_G^{\dim M - \dim X}$; it represents a Kasparov class $\varphi \in \mathrm{KK}^G(C(X), C(X))$. Assume that b and f intersect smoothly with K_G-oriented excess intersection bundle η, and equip the incidence space $Q = \{m \in M \mid b(m) = f(m)\}$ with its induced K_G-orientation. Then the $R_{\mathbb{C}}(G)$-valued index of the Dirac operator on Q twisted by the bundle $\xi|_Q \otimes e(\eta)$ is equal to the super-trace of the F-linear map on $K^*_G(X) \otimes F$ induced by φ.*

If G is any connected Lie group, there exists a finite covering $\hat{G} \to G$ where $\hat{G}$ is a Hodgkin Lie group. Hence, by exploiting the resulting restriction functor $\mathrm{KK}^G \to \mathrm{KK}^{\hat{G}}$ and the associated injective morphism $R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(\hat{G})$, we can proceed to derive a more general trace formula for any connected Lie group.

1.6.10. An equivariant bootstrap class. We should note that Theorem 1.6.8 fails badly for group which are not connected. If we want to obtain Lefschetz-Hopf formulas for more general compact group G (e.g. finite groups) and interesting objects $A \in \mathrm{KK}^G$ (e.g. $A = C(X)$ for compact smooth G-manifolds X) we are led to consider the following equivariant version of the classical bootstrap class.

Definition. For any compact group G, define the *G-equivariant bootstrap category* $\mathcal{B}^G$ to be the localizing$_{\aleph_1}$ subcategory of KK^G (see §1.1.3) generated by all *elementary* G-C*-algebras, i.e., those of the form $A = \mathrm{Ind}^G_H(B) = C(G,B)^H$ where B is a matrix algebra $M_n(\mathbb{C})$ $(n \geqslant 1)$ equipped with some action of some closed subgroup $H \leqslant G$.

Thus $\mathcal{B}^G$ contains all algebras that can be obtained by starting with those G-C*-algebras which are induced from actions of closed subgroups on matrix algebras and by taking mapping cones (extensions) and direct sums. There is a more intrinsic characterisation: $\mathcal{B}^G$ contains precisely those G-C*-algebras which are KK^G-equivalent to some G-action on a Type I C*-algebra ([11, Thm. 3.10]).

Theorem 1.6.8 holds because, if G is Hodgkin, it can be shown that $\mathcal{B}^G = \mathrm{Loc}_{\aleph_1}(\mathbb{C})$. In general, since $\mathcal{B}^G$ is by construction a compactly$_{\aleph_1}$ generated category, we have that an object $A \in \mathcal{B}^G$ is dualisable if and only if it is compact$_{\aleph_1}$, if and only if it belongs to the thick subcategory generated by the elementary G-C*-algebras ([11, Prop. 3.13]).

Let us denote the latter category by $\mathcal{B}^G_c = \mathcal{B}^G \cap (\mathrm{KK}^G)^c$. In particular, since commutative algebras are Type I, we deduce from the above discussion:

1.6.11 Corollary. *If X is a compact smooth G-manifold, the G-C*-algebra $C(X)$ belongs to $\mathcal{B}^G_c$.*

Note that, contrary to the non-equivariant case, in general not all algebras in $\mathcal{B}^G$ are equivalent to a commutative one (this fails already for the circle group $G = U(1)$). Ultimately, this is the reason why $\widehat{\mathrm{KK}}^G$ is not triangulated.

1.6.12. The computation for general compact Lie groups. Thanks to Corollary 1.6.11, in order to deduce a Lefschetz-Hopf formula for compact smooth G-manifold with G a non-connected group, it now suffices to provide a homological computation of the monoidal trace which holds for the G-C*-algebras belonging to $\mathcal{B}^G_c$.

The crucial result in the context of non-connected group is the following reduction of generators for *topologically cyclic groups*, i.e. groups admitting an element g which generates a dense subgroup. A compact Lie group is topologically cyclic iff it is isomorphic to a product of a finite cyclic group and a torus.

1.6.13 Theorem ([11, Thm. 3.17])**.** *Let G be a topologically cyclic compact Lie group, as above. Then:*

- *The equivariant bootstrap class $\mathcal{B}^G$ is already generated by the finitely many G-C*-algebras $C(G/H)$ for all open subgroups $H \leqslant G$.*
- *An object $A \in \mathcal{B}^G$ is dualisable iff it is compact$_{\aleph_1}$ iff it belongs to the thick subcategory generated by $C(G/H)$ for open subgroups $H \leqslant G$.*

The relevance of such groups is due to the classical results of Segal [Seg68], according to which, for any compact Lie group G, the minimal primes $\mathfrak{p}_i$ of $R_{\mathbb{C}}(G)$ are in a canonical bijection with the (conjugacy classes of) the *Cartan subgoups* of G, i.e. the topologically cyclic subgroups which have finite index in their normalizer. Segal's theory allows us to deduce from Theorem 1.6.13 the following general, improved version of the homological computation of Theorem 1.6.6. It is crucial here that we use the equivariant K-theory for *restricted* actions, otherwise the result would be wrong (already with $G = \mathbb{Z}/2$).

1.6.14 Theorem ([11, Thm. 3.23])**.** *Let G be any compact Lie group, let $A \in \mathcal{B}^G_c$ be a dualisable object in the equivariant bootstrap class (e.g. $A = C(X)$ for a* *compact smooth G-manifold X), and let $\varphi \in \mathrm{KK}^G(A, A)$ be any endomorphism of A (e.g. one induced by a self-map $f\colon X \to X$, or a smooth correspondence). For any Cartan subgroup $H \leqslant G$, let $\mathfrak{p}_H$ denote the corresponding minimal prime ideal of $\operatorname{Spec} R_{\mathbb{C}}(G)$ and let $F_H := \operatorname{Frac}(R_{\mathbb{C}}(G)/\mathfrak{p}_H)$ be its residue field.*

Then the monoidal trace $\operatorname{tr}(\varphi)$ is uniquely determined by the super-traces

$$\operatorname{tr}\left(K_0^H(\varphi) \otimes_{R_{\mathbb{C}}(G)} F_H\right) - \operatorname{tr}\left(K_1^H(\varphi) \otimes_{R_{\mathbb{C}}(G)} F_H\right)$$

of the endomorphisms induced by φ on the finite dimensional $\mathbb{Z}/2$-graded F_H-vector spaces $K_*^H(A) \otimes_{R_{\mathbb{C}}(G)} \otimes F_H$, with H running through the finitely many conjugacy classes of Cartan subgroups of G.

By combining this theorem with the geometric computation of the trace, we obtain a generalisation of Theorem 1.6.9 for arbitrary compact Lie groups, including finite groups.

1.6.15. The computation as a Hattori-Stallings trace. We also provide another way to compute the trace, by means of the Hattori-Stallings trace for modules of finite projective dimension; see e.g. [Bas76].

In general, we can use this approach by working in any tensor triangulated category $\mathcal{T} = (\mathcal{T}, \otimes, \mathbb{1})$ for which the following assumption holds:

Hypothesis (Additivity of traces)**.** Let $A \to B \to C \to \Sigma A$ a distinguished triangle in $\mathcal{T}$, assume that A and B are dualisable objects, and assume also that the left square in the following diagram is commutative:

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Sigma f} \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \end{array}$$

Then C is dualisable and there exists a map $h\colon C \to C$ making the diagram commute and such that $\operatorname{tr}(f) - \operatorname{tr}(g) + \operatorname{tr}(h) = 0$.

Now let $R := \bigoplus_{n\in\mathbb{Z}} \operatorname{End}_{\mathcal{T}}(\mathbb{1})$ be the *graded* endomorphism ring of $\mathbb{1}$ as before. Assuming only the most basic compatibility between the suspension Σ and the tensor $\otimes$, it is a graded commutative ring (see §2.1.1). For every object A, the graded Hom group $M(A) := \mathcal{T}_*(\mathbb{1}, A)$ is canonically a graded R-module, and every map $f \in \mathcal{T}(\Sigma^n A, A)$ induces a degree-n endomorphism $M(f) := \mathcal{T}_*(\mathbb{1}, f)$ of $M(A)$. Our next theorem will say that, under some hypotheses, the monoidal trace of f equals the Hattori-Stallings trace of $M(f)$, so in particular it only depends on the module map $M(f)$.

In order to state the theorem, we first need to define a graded version of the usual Hattori-Stallings trace for modules over a ring. Let $\alpha\colon M \to M$ be an endomorphism of a finitely generated R-module M. Assume first that M is a free R-module, i.e., it is a finite sum $M = \bigoplus_{i=0}^k R(n_i)$ of copies of the shifted modules $R(n)$ defined by $R(n)_m = R(n+m)$, and that α is homogeneous of degree d. Then the *Hattori-Stallings* (*super-*) *trace* of α is defined to be

$$\operatorname{tr}_{HS}(\alpha) := \sum_{i=0}^{k} (-1)^{n_i} \operatorname{tr}(\alpha_{ii}) \quad \in R_d$$

where the α_{ij} denote the components $R(n_j) \to R(n_i)$ of α and $\operatorname{tr}(\alpha_{ii})$ is the degree-d element of R such that right multiplication by it gives the map $\alpha_{ii}\colon R(n_i) \to R(n_i)$. If M is a finitely generated projective module, then $M \oplus N$ is (isomorphic to) a free module as above for some module N, and we can define $\operatorname{tr}_{HS}(\alpha) := \operatorname{tr}_{HS}(\alpha \oplus 0)$ via this isomorphism. More generally, if M admits a finite resolution

$$0 \longrightarrow P_\ell \xrightarrow{d_\ell} P_{\ell-1} \xrightarrow{d_{\ell-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

by finitely generated projectives, in which we have arranged the differentials d_i to have odd degree, choose liftings $\alpha_i\colon P_i \to P_i$ of α to an endomorphism of the resolution and set $\operatorname{tr}_{HS}(\alpha) := \sum_{i=0}^{\ell} \operatorname{tr}_{HS}(\alpha_i)$. One can check that this definition is independent of all the choices we have made.

The proof of the next theorem uses the *phantom tower* for A, i.e. a relative homological algebra resolution of A in $\mathcal{T}$ (a construction which is also used to derive the spectral sequences of §1.5.1), combined with the the hypothesis on the additivity of traces.

To understand the extent of the theorem, note that if the graded commutative ring R is coherent (e.g. noetherian) and has finite global dimension, then an object $A \in \operatorname{Loc}(\mathbb{1})$ is dualisable iff $M(A)$ is finitely generated.

1.6.16 Theorem ([11, Thm. 4.2])**.** *Let $f \in \mathcal{T}(A, A)$ be any endomorphism of an object $A \in \mathcal{T}$, and assume that $A \in \operatorname{Loc}(\mathbb{1})$. If the graded R-module $M(A)$ admits a finite resolutions by finitely generated projectives, then A is dualisable in $\mathcal{T}$ and*

$$\operatorname{tr} f = \operatorname{tr}_{HS} M(f)$$

i.e. the monoidal trace of f is equal to the Hattori-Stallings trace of the map it induces on the graded module $M(A) = \mathcal{T}_*(\mathbb{1}, A)$.

This result can be applied to equivariant Kasparov theory as follows. First notice that our bootstrap category $\mathcal{B}^G$ satisfies the additivity of traces. This is a consequence of Theorem 1.6.14; it can also be proved, more directly, by embedding $\mathcal{B}^G$ in the derived category of a highly structured ring spectrum, as was done in [DEKM11]. As usual, since the graded endomorphism ring is $R_{\mathbb{C}}(G)[\beta, \beta^{-1}]$ with $\beta\colon \mathbb{C} \overset{\sim}{\to} \Sigma^2\mathbb{C}$, we can equally work with $\mathbb{Z}/2$-graded modules over the ungraded ring $R_{\mathbb{C}}(G)$ (this doesn't change the above sign choices).

Now if G is a Hodgkin Lie group, its representation ring is a regular noetherian ring and $\mathcal{B}^G_c = \operatorname{Thick}(\mathbb{C})$, hence we may use the Hattori-Stallings trace to compute traces of endomorphisms on any dualisable algebra in the G-equivariant bootstrap class.

2 Tensor triangulated categories and classifications

We present here results taken from [2], [3], [7, 9] (joint with Greg Stevenson) and [14] (joint with Donald Stanley). We prove classifications theorems for thick and localising subcategories in derived categories of graded commutative rings, generalising the work of Hopkins, Neeman and Thomason (§2.6.1), and similar theorems for derived categories of regular noetherian commutative dg rings or ring spectra, generalising work of Benson-Iyengar-Krause and Shamir (see §2.7). We also prove the first classification in the domain of Kasparov's KK-theory (§2.4.2), and produce the first (very partial) results in the realm of noncommutative motives (§2.8). Along the way we also make some contributions to abstract tensor triangular geometry, presented in §2.4 and §2.5.

2.1 Preliminaries on tensor triangulated categories

A *tensor triangulated category* is commonly understood to be a triangulated category $\mathcal{T}$ which is equipped with a compatible symmetric[1] monoidal structure. The precise meaning of "compatible" is open to some debate.

Most of the naturally occurring examples of tensor triangulated categories — morally, all of them — arise from some kind of model category or enhancement which is itself equipped with a compatible symmetric monoidal structure. What "compatible" means at the level of models is usually clear, and it entails many known and useful compatibility properties at the homotopy level. Ideally one would like to capture these properties axiomatically, once and for all, in order to obtain model-independent statements. But this is a subtle enterprise, suffering from some of the same defects that plague the axioms of a triangulated category themselves. Notably, poor functoriality, and different choices for the possible strength levels for the axioms. The most courageous such axiomatisation has been attempted by May [May01] in order to prove the additivity of traces (see §1.6.15), but his axioms are rather unwieldy. His work has been streamlined by Keller and Neeman [KN02], but the most elegant approach so far is probably that of monoidal derivators. A monoidal derivator is simply a pseudo-monoid in the Cartesian 2-category of derivators, just as a monoidal category is a pseudo-monoid in the Cartesian 2-category of categories. From such a small seed, one can obtain May's axioms and the additivity of traces [GPS14]. Another fashionable solution, although perhaps closer in spirit to the use of a monoidal model category, is to follow Lurie [Lur16] and work at the level of an underlying symmetric monoidal stable ∞-category.

Instead of fixing the 'perfect' setting, we take the minimalist road and add along the way those axioms that are needed. This allows us to choose the correct generality for each result, and also, more importantly, to cover the less well-behaved tensor triangulated categories arising from KK-theory, for which

[1] Non-symmetrical tensor structures on triangulated categories are also of great interest, but will not concern us here.

models or enhancements have mostly not been worked out yet (although the work of Mahanta [Mah15b] [Mah15a] in this direction looks quite promising).

2.1.1. The axioms of a tensor triangulated category. The most basic compatibility between the triangulation and the tensor structure is to require the tensor functor $-\otimes -\colon \mathcal{T}\times\mathcal{T}\to\mathcal{T}$ to be exact in each variable; this is all we will always assume with the words *tensor triangulated category*, because it suffices for defining the spectrum of $\mathcal{T}$ as a topological space and for inferring most of its properties (see §2.2).

If we want to consider some 'geometry' (e.g. sheaves of rings and modules) on the spectrum, it is a good idea to ask for some compatibility between $\otimes$ and the suspension Σ ensuring that the graded endomorphism ring of the unit

$$\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}) = \bigoplus_{n\in\mathbb{Z}} \mathcal{T}(\mathbb{1},\Sigma^n\mathbb{1})$$

is graded commutative: $fg = (-1)^{|f||g|}gf$. To this goal, the simple axioms of [SA04] suffice, requiring the isomorphisms $r\colon X\otimes\Sigma Y \overset{\sim}{\to} \Sigma(X\otimes Y)$ and $\ell\colon \Sigma X\otimes Y \overset{\sim}{\to} \Sigma(X\otimes Y)$ (which are part of the biexactness of $-\otimes-$) to make the two triangles below commute

$$\begin{array}{ccc} \mathbb{1}\otimes\Sigma X & \xrightarrow{\lambda} & \Sigma X \\ {\scriptstyle r}\downarrow & \nearrow{\scriptstyle \Sigma\lambda} & \\ \Sigma(\mathbb{1}\otimes X) & & \end{array} \qquad \begin{array}{ccc} \Sigma X & \xleftarrow{\rho} & \Sigma X\otimes\mathbb{1} \\ & \nwarrow{\scriptstyle \Sigma\rho} & \downarrow{\scriptstyle \ell} \\ & & \Sigma(X\otimes\mathbb{1}) \end{array} \qquad \begin{array}{ccc} \Sigma X\otimes\Sigma Y & \xrightarrow{\ell} & \Sigma(X\otimes\Sigma Y) \\ {\scriptstyle r}\downarrow & -1 & \downarrow{\scriptstyle \Sigma r} \\ \Sigma(\Sigma X\otimes Y) & \xrightarrow{\Sigma\ell} & \Sigma^2(X\otimes Y) \end{array}$$

and the square anti-commute (so far, this also makes sense if the monoidal structure is not symmetric). Omitting parentheses, the anti-commutativity of the square can be expressed as the commutativity of

$$\begin{array}{ccc} \Sigma^n X\otimes\Sigma^m Y & \xrightarrow{\sim} & \Sigma^{n+m}(X\otimes Y) \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle (-1)^{n+m}} \\ \Sigma^m Y\otimes\Sigma^n Y & \xrightarrow{\sim} & \Sigma^{n+m}(X\otimes Y) \end{array} \tag{2.1}$$

for all $n,m\in\mathbb{Z}$, where γ is the symmetry and the horizontal arrows are iterations of r and ℓ. If the tensor structure is closed, i.e. if it admits an internal Hom functor

$$\mathsf{hom}(-,-)\colon \mathcal{T}^{\mathrm{op}}\times\mathcal{T}\longrightarrow\mathcal{T}$$

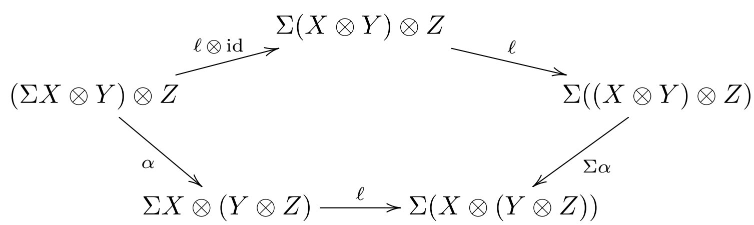
defined by the existence of natural isomorphisms

$$\mathcal{T}(X\otimes Y,Z)\cong\mathcal{T}(X,\mathsf{hom}(Y,Z))$$

then we may also want it to be exact and to interact nicely with the suspension. On the whole, the following properties are commonly required to hold.

2.1.2 Definition ([HPS97, App. A.2])**.** A triangulated structure and a closed tensor structure on a category $\mathcal{T}$ are *compatible* if the following are satisfied:

- The tensor product preserves suspensions: there are natural isomorphisms $r\colon X \otimes \Sigma Y \overset{\sim}{\to} \Sigma(X \otimes Y)$ and $\ell\colon \Sigma X \otimes Y \overset{\sim}{\to} \Sigma(X \otimes Y)$, determining each other via $r = (\Sigma\gamma)\ell\gamma^{-1}$, and making the above triangles commute as well as the following pentagon (α being the coherent associator):



  (the latter may be used e.g. to justify omitting parentheses in (2.1)).
- The tensor product $- \otimes -$ is exact in each variable (via ℓ and r).
- The internal Hom, $\mathsf{hom}(-,-)$, is exact in the second variable and is anti-exact in the first one (i.e. each functor $\mathsf{hom}(-,X)$ sends an exact triangle to a triangle which is exact after changing the sign of one arrow).
- The graded commutativity rule (2.1) for $\otimes$ and Σ holds.

A consequence of the above conditions is that the rigid (= dualisable §1.6.1) objects of $\mathcal{T}$ form a tensor triangulated thick subcategory $\mathcal{T}^d$, which moreover is *rigid*: the internal Hom is given by $\mathsf{hom}(X,Y) = X^\vee \otimes Y$ for a tensor exact equivalence $(-)^\vee\colon \mathcal{T}^{\mathrm{op}} \overset{\sim}{\to} \mathcal{T}$.

If $\mathcal{T}$ is a big category admitting infinite coproducts we certainly want to consider its compact objects, which always form a thick subcategory $\mathcal{T}^c$. However it does not follow in general from the above axioms that compact and rigid objects coincide; for one thing, $\mathbb{1}$ may fail to be compact. If $\mathbb{1} \in \mathcal{T}^c$ then $\mathcal{T}^d \subseteq \mathcal{T}^c$ follows easily, but one still has to assume the other inclusion in order to have equality. The equality $\mathcal{T}^d = \mathcal{T}^c$ is a common feature in the examples and is very desirable for the abstract theory, which leads to the following notion. It will be crucial for most results of Section 3.

2.1.3 Definition. A *rigidly-compactly generated tensor triangulated category* (or sometimes, simply, *compactly generated tensor category*) is a compactly generated (§1.1.3) triangulated category $\mathcal{T}$ equipped with a compatible closed tensor structure as in Definition 2.1.2 and such that its subcategories of rigid and compact objects coincide; hence, in particular, $\mathcal{T}^c$ is a rigid tensor triangulated category, as above.

2.1.4. Central graded rings and localisation. The most compact way to understand the axioms in Definition 2.1.2 is that the suspension Σ should be given by $\Sigma(\mathbb{1}) \otimes -$, that the symmetry γ on $\Sigma(\mathbb{1}) \otimes \Sigma(\mathbb{1})$ is equal to -1, and that the opposite category $\mathcal{T}^{\mathrm{op}}$ should be equipped with the opposite triangulation (which involves a sign change); the rest then follows by the coherence theorem of symmetric tensor categories and by requiring all functors to be exact (see [Del08, §2.1]).

More generally, instead of grading the endomorphism ring of $\mathbb{1}$, or other Hom sets, by $\Sigma \cong \Sigma(\mathbb{1}) \otimes -$, we can instead use any tensor-invertible object G:

$$\mathcal{T}^{G,*}(X,Y) := \bigoplus_{n\in\mathbb{Z}} \mathcal{T}(X, G^{\otimes n} \otimes Y)\,.$$

The composition of $\mathcal{T}$ extends in a 'G-graded' (or rather $\mathbb{Z}$-graded with respect to G) way, simply by setting

$$gf = g \circ f\colon\; X \xrightarrow{\;f\;} G^{\otimes n} \otimes Y \xrightarrow{\mathrm{id}\otimes g} G^m \otimes (G^{\otimes n} \otimes Z) \cong G^{m+n} \otimes Z$$

for all $f\colon X \to G^{\otimes n} \otimes Y$ and $g\colon Y \to G^{\otimes m} \otimes Z$. In particular, we obtain a 'G-graded' ring

$$R^{G,*}_{\mathcal{T}} := \mathrm{End}^{G,*}_{\mathcal{T}}(\mathbb{1})$$

which is called by Balmer [Bal10a] the *graded central ring* (with respect to G) of the tensor triangulated category $\mathcal{T}$. One can show that this ring is graded commutative in the slightly generalised sense that $rs = \epsilon^{n+m} sr$ where n and m are the G-degrees of r and s and $\epsilon := \gamma_{G,G}\colon G \otimes G \overset{\sim}{\to} G \otimes G$ is the symmetry at G. The graded central ring acts canonically on all graded Hom groups by

$$r \cdot f\colon\; X \cong \mathbb{1} \otimes X \xrightarrow{r\otimes f} G^{\otimes n} \otimes \mathbb{1} \otimes G^{\otimes m} \otimes Y \cong G^{\otimes n+m} Y$$

($r\colon \mathbb{1} \to G^{\otimes n} \otimes \mathbb{1}$, $f\colon X \to G^{\otimes m} \otimes Y$). Graded composition is bilinear for this action, up to the above ϵ-sign rule, and its action on itself coincides with the multiplication, i.e. with composition. For $G = \mathbb{1}$ we recover the usual action of the (plain) endomorphism ring $\mathrm{End}(\mathbb{1})$ on the Hom groups.

The latter is *the* (plain) central ring of $\mathcal{T}$, and the above observations generalise the well-known fact that the endomorphism ring of any (additive) monoidal category is commutative and the category is automatically enriched in modules over it.

An important application of the above remarks is that one can alway localise $\mathcal{T}$ at a (homogeneous) multiplicative subset $S \subseteq R^{G,*}_{\mathcal{T}}$ of the central ring, by localising each Hom-module, and the resulting category $S^{-1}\mathcal{T}$ is again tensor triangulated; indeed, it is the Verdier quotient $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ by the tensor-ideal subcategory generated by the mapping cones of all $s \in S$.

We will show in §2.5 how to generalise this to multi-gradings, *not* by trying to fit multiple grading objects G_i into a ring (which in general seems a pretty hopeless task), but rather by categorifying the notion of a graded ring and by generalising basic commutative algebra to such objects.

2.2 Classification problems and the spectrum

Let $\mathcal{T}$ be a tensor triangulated category. A *thick tensor ideal* of $\mathcal{T}$ is a thick subcategory $\mathcal{S}$ which is an ideal with respect to the tensor product: if $X \in \mathcal{J}$ and $Y \in \mathcal{T}$ is any object then $X \otimes Y \in \mathcal{J}$. If $F\colon \mathcal{T} \to \mathcal{S}$ is an exact tensor functor, then its full kernel is a thick tensor ideal. Conversely, the Verdier quotient $\mathcal{T}/\mathcal{J}$ by any thick tensor ideal inherits a structure of tensor triangulated category which makes the quotient functor tensor exact. Just like the collection of all thick subcategories, the thick tensor ideals of $\mathcal{T}$ are partially ordered by inclusion and form a lattice where the meet is given by intersection and the join by the smallest thick tensor ideal containing the union.

An easy but fundamental observation is that if $\mathbb{1}$ generates $\mathcal{T}$ as a thick subcategory, then every thick subcategory of $\mathcal{T}$ is automatically a tensor ideal, hence the two lattices coincide.

As we will see in a moment, the lattice of thick tensor ideals has the remarkable property of arising from a topological space, i.e. it is a coherent frame. Hence it is amenable to be studied by geometric means, and because of this it can be computed explicitly in many important examples. On the other hand, the lattice of all thick subcategories usually isn't so nice, and there is no known general strategy for studying it. Indeed, only a few examples have been completely computed where the thick subcategories do not coincide with all tensor ideals for some tensor product (e.g. $\mathrm{D^b}(\mathrm{coh}\,\mathbb{P}^1_k)$ or $\mathrm{D^b}(R)$ for some hereditary Artin algebras; see [Kra12]).

2.2.1. The spectrum of a tensor triangulated category. Let $\mathcal{T}$ be a tensor triangulated category, which we assume to be essentially small. The crucial tool for getting a hold on the thick tensor ideals of $\mathcal{T}$ is a topological space defined by Balmer, the *spectrum of* $\mathcal{T}$, denoted $\mathrm{Spc}\,\mathcal{T}$. We now recall its definition and its first properties ([Bal05]).

As a set, the spectrum consists of all thick tensor ideals which are *prime* with respect to the tensor product, in the usual sense:

$$\mathrm{Spc}\,\mathcal{T} := \{\mathcal{P} \subsetneq \mathcal{T} \mid \mathcal{P} \text{ thick tensor ideal s.t. } X \otimes Y \in \mathcal{P} \Rightarrow X \in \mathcal{P} \text{ or } Y \in \mathcal{P}\}.$$

For every object $X \in \mathcal{T}$ there is a distinguished subset of $\mathrm{Spc}\,\mathcal{T}$, called the *support* of X:

$$\mathrm{Supp}(X) := \{\mathcal{P} \in \mathrm{Spc}\,\mathcal{T} \mid X \notin \mathcal{P}\} \quad (X \in \mathrm{Obj}\,\mathcal{T}).$$

The complements $\mathrm{U}(X) := \mathrm{Spc}\,\mathcal{T} \smallsetminus \mathrm{Supp}\,X$ ($X \in \mathcal{T}$) form the basis for a topology on $\mathrm{Spc}\,\mathcal{T}$, the *Zariski topology*. This topological space is *spectral* in the sense of Hochster [Hoc69]. This means the it has the following properties:

- it is quasi-compact,
- it admits a basis of quasi-compact opens (e.g. the above $\mathrm{U}(X)$), and
- every irreducible closed subset is the closure of a unique point (its *generic point*).

Spectral spaces are precisely those spaces that arise as the Zariski spectrum of some commutative ring, and have very nice and distinctive properties.

The assignment $X \mapsto \mathrm{Supp}(X)$ is compatible with all the operations of the tensor triangulated category $\mathcal{T}$, in the following way:

(SD1) $\mathrm{Supp}\, 0 = \emptyset$ and $\mathrm{Supp}\, \mathbb{1} = \mathrm{Spc}\, \mathcal{T}$.

(SD2) $\mathrm{Supp}(X \oplus Y) = \mathrm{Supp}\, X \cup \mathrm{Supp}\, Y$.

(SD3) $\mathrm{Supp}\, \Sigma X = \mathrm{Supp}\, X$.

(SD4) $\mathrm{Supp}\, Y \subseteq \mathrm{Supp}\, X \cup \mathrm{Supp}\, Z$ for every exact triangle $X \to Y \to Z \to \Sigma X$.

(SD5) $\mathrm{Supp}(X \otimes Y) = \mathrm{Supp}\, X \cap \mathrm{Supp}\, Y$.

More generally, a *support data* on $\mathcal{T}$ is a pair (T, σ) consisting of a (spectral) topological space T and a function σ assigning to each object of $\mathcal{T}$ a closed subset $\sigma(X) \subseteq T$ satisfying the analogues of (SD1)-(SD5). Thus the pair $(\mathrm{Spc}\, \mathcal{T}, \mathrm{supp})$ is a support data. As it turns out, given any other support data (T, σ) on $\mathcal{T}$, there exists a unique continuous map

$$f \colon T \longrightarrow \mathrm{Spc}\, \mathcal{T}$$

such that $\sigma(X) = f^{-1}\, \mathrm{Supp}(X)$ for all objects $X \in \mathcal{T}$. This universal property shows that $(\mathrm{Spc}\, \mathcal{T}, \mathrm{Supp})$ is the 'finest' of all possible support data on $\mathcal{T}$, and characterises it uniquely up to a unique support-compatible homeomorphism.

For every tensor exact functor $F \colon \mathcal{T} \to \mathcal{S}$ we obtain an induced map

$$\mathrm{Spc}\, F \colon\ \mathrm{Spc}\, \mathcal{S} \to \mathrm{Spc}\, \mathcal{T} \qquad \mathcal{P} \mapsto F^{-1}\mathcal{P}$$

which is continuous and *spectral*, which means that the preimage under $\mathrm{Spc}\, F$ of a quasi-compact open subset is again quasi-compact open. This defines a contravariant functor Spc from tensor triangulated categories and tensor exact functors to spectral spaces and spectral maps.

2.2.2. The abstract classification of thick tensor ideals. The main interest of the universal support data $(\mathrm{Spc}\, \mathcal{T}, \mathrm{Supp})$ is that it classifies the thick tensor ideals of $\mathcal{T}$. More precisely, a thick tensor ideal $\mathcal{J} \subseteq \mathcal{T}$ is said to be *radical* if $X^{\otimes n} \in \mathcal{J}$ for some n already implies $X \in \mathcal{J}$. A subset $S \subset \mathrm{Spc}\, \mathcal{T}$ is said to be *Thomason* if S is a union of closed subsets, all of which have quasi-compact open complements.

Balmer [Bal05] proved that, for *any* essentially small tensor triangulated category $\mathcal{T}$, we have the following canonical inclusion-preserving bijection between Thomason subsets and radical ideals:

$$\left\{ \begin{array}{c} \text{Thomason subsets} \\ S \subseteq \mathrm{Spc}\, \mathcal{T} \end{array} \right\} \underset{\sim}{\overset{\longrightarrow}{\longleftarrow}} \left\{ \begin{array}{c} \text{radical thick tensor} \\ \text{ideals } \mathcal{J} \subseteq \mathrm{Spc}\, \mathcal{T} \end{array} \right\}$$

$$S \mapsto \{X \in \mathcal{T} \mid \mathrm{Supp}\, X \subseteq S\}$$

$$\bigcup_{X \in \mathcal{J}} \mathrm{Supp}\, X \leftarrow\!\shortmid \mathcal{J}$$

In particular, every radical thick tensor ideal is uniquely determined by the support of its objects.

The converse statement of the above classification result is also true ([Bal05, BKS07]). Say a support data (T, σ) on $\mathcal{T}$ is *classifying* if T is a spectral space and if the analogues for σ of the above two maps also yield mutually inverse bijections. Then one can show that there must be a support-preserving homeomorphism $T \cong \operatorname{Spc}\mathcal{T}$. Thus the universal support data is also characterised as being the unique classifying support data. This can be used to compute the spectrum from already available classification theorems.

The following simplifications often occur in examples:

- If $\mathcal{T}$ is a *rigid* tensor triangulated category (i.e. every object admits a tensor dual), then each thick tensor ideal is automatic radical. For instance $\mathcal{T}$ is rigid if it is generated by its tensor unit $\mathbb{1}$ as a thick subcategory.
- As we have already mentioned, if $\mathcal{T}$ is generated by $\mathbb{1}$ then every thick subcategory is automatically a (radical) thick tensor ideal, so in this particular case the spectrum classifies *all* thick subcategories.
- If the spectrum is a noetherian space (every open is quasi-compact), then a Thomason subset S is the same as a union of closed subsets, i.e., a *specialisation closed* subset: $\mathcal{P} \in S \Rightarrow \overline{\{\mathcal{P}\}} \subseteq S$.

We should also note that, like every spectral space, $\operatorname{Spc}\mathcal{T}$ has a *Hochster dual*, which is the space with the same underlying set as $\operatorname{Spc}\mathcal{T}$ and with the topology where the opens are the Thomason subsets. Hence some authors prefer to express the above classification in terms of the dual. The choice is mainly a matter of taste, although ours is compatible with many examples where supports occur naturally as closed subsets rather than open ones. For instance, if $\mathrm{D}^{\mathrm{perf}}(X)$ is the tensor triangulated category of perfect complexes over a quasi-compact and quasi-separated scheme X (see §2.6), then Thomason's classification of thick tensor ideals yields a homeomorphism $\operatorname{Spc}(\mathrm{D}^{\mathrm{perf}}(X)) \cong X$.

2.3 Comparison maps from triangular to Zariski spectra

Let $\mathcal{T}$ be an essentially small tensor triangulated category. In order to describe the spectrum $\operatorname{Spc}\mathcal{T}$, one can try to compare it with other topological spaces. By the universal property, any available support data (T, σ) on $\mathcal{T}$ yields a continuous comparison map $T \to \operatorname{Spc}\mathcal{T}$. As shown by Balmer [Bal10a], it is also possible in full generality to construct continuous maps *out* of $\operatorname{Spc}\mathcal{T}$ whose target spaces are ordinary Zariski spectra of (graded) commutative rings.

More precisely, let $G \in \mathcal{T}$ be an invertible object of $\mathcal{T}$, and recall the graded central ring $R^{G,*}_{\mathcal{T}} = \bigoplus_{n\in\mathbb{Z}} \mathcal{T}(\mathbb{1}, G^{\otimes n})$ of §2.1.4. Write $\operatorname{Spec}^{\mathrm{h}}(R^{G,*}_{\mathcal{T}})$ for its Zariski spectrum of all homogeneous prime ideals. Then

$$\rho := \rho^{G,*}_{\mathcal{T}} \colon \operatorname{Spc}\mathcal{T} \longrightarrow \operatorname{Spec}^{\mathrm{h}} R^{G,*}_{\mathcal{T}} \qquad \mathcal{P} \mapsto \{r \in R^{G,*}_{\mathcal{T}} \mid \operatorname{Cone}(r) \notin \mathcal{P}\}$$

is a well-defined continuous and spectral map. Moreover, it is natural with respect to tensor exact functors $F \colon \mathcal{T} \to \mathcal{S}$, in the evident way, and behaves well with respect to central localisation. The latter means that if S is any homogeneous multiplicative system in $R_{\mathcal{T}}^{G,*}$ and $S^{-1}\mathcal{T}$ is the tensor triangulated category localised at S, as in §2.1.4, then the resulting commutative square

$$\begin{array}{ccc} \operatorname{Spc} S^{-1}\mathcal{T} & \longrightarrow & \operatorname{Spc}\mathcal{T} \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} \\ \operatorname{Spec^h} S^{-1}R_{\mathcal{T}}^{G,*} & \longrightarrow & \operatorname{Spec^h} R_{\mathcal{T}}^{G,*} \end{array}$$

has horizontal inclusions and is a pullback of spaces. By localising at primes ideals of the ring, this allows us to simplify arguments by reducing them to the situation of categories with local (graded) endomorphism ring. This strategy will be used to prove the results of §2.7.

By choosing $G = \Sigma(\mathbb{1})$, the above specialises to a comparison map

$$\rho\colon \operatorname{Spc}\mathcal{T} \longrightarrow \operatorname{Spec^h}\operatorname{End}^*_{\mathcal{T}}(\mathbb{1})$$

for the ordinary graded endomorphism ring of $\mathbb{1}$, and all of the above can be done also for the ordinary (ungraded) endomorphism ring, yielding a map

$$\rho\colon \operatorname{Spc}\mathcal{T} \longrightarrow \operatorname{Spec}\operatorname{End}_{\mathcal{T}}(\mathbb{1})\,.$$

All these variants have their uses in some examples.

We note that, while the maps $T \to \operatorname{Spc}\mathcal{T}$ obtained from support data tend to be injective, the comparison maps ρ are more often surjective for quite general reasons, e.g. whenever the graded ring is noetherian or even just coherent. Their injectivity holds less frequently and is harder to prove.

We will see in §2.5.8 how the definition of Balmer's comparison maps can be extended to allow more general target spaces.

2.4 Supports for big categories

Balmer's axioms for a support data are very well-adapted for 'small' triangulated categories, but it is less clear what a support theory should be for a 'big' one. The currently available formal approaches require at least a compactly generated category $\mathcal{T}$ (§1.1.3). Benson, Iyengar and Krause [BIK08] [BIK11] [BIK12], by assuming only that a graded noetherian ring R acts on $\mathcal{T}$ compatibly with the suspension, have developed a very successful theory where supports are subsets of $\operatorname{Spec^h} R$. For most of their applications, however, they require R to be (a subring of) the graded endomorphism ring of a rigidly-compactly generated tensor triangulated category (Def. 2.1.3), acting canonically as in §2.1.4. In this setup some basic examples cannot be covered, such as the derived category of a non-affine noetherian scheme. Also, the noetherian hypothesis on the ring is essential to their theory.

Balmer and Favi [BF11] begin instead with a rigidly-compactly generated tensor triangulated category $\mathcal{T}$, so that the compact objects form a rigid tensor subcategory $\mathcal{T}^c$, and construct a support for all objects of $\mathcal{T}$ with values in the spectrum $\operatorname{Spc}(\mathcal{T}^c)$. Their theory has been successfully furthered by Stevenson, who also generalised it to include actions $\mathcal{T} \times \mathcal{S} \to \mathcal{S}$ of a tensor triangulated category $\mathcal{T}$ on another triangulated category $\mathcal{S}$; see [Ste13]. Assuming that $\operatorname{Spc}\mathcal{T}^c$ is a noetherian space (i.e., every open set is quasi-compact), Balmer and Favi show that their support, as a map $\sigma\colon \operatorname{Obj}(\mathcal{T}) \to 2^{\operatorname{Spc}\mathcal{T}^c}$, satisfies the following properties (cf. the axioms (SD1)-(SD5) of a support data §2.2.1):

(S1) $\sigma(0) = \emptyset$ and $\sigma(\mathbb{1}) = T$.

(S2) $\sigma(\coprod_i X_i) = \bigcup_i \sigma(X_i)$ for every family $\{X_i\}_i$ of objects.

(S3) $\sigma(\Sigma X) = \sigma(X)$.

(S4) $\sigma(Y) \subseteq \sigma(X) \cup \sigma(Z)$ if there is an exact triangle $X \to Y \to Z \to \Sigma X$.

(S5) $\sigma(X \otimes Y) \subseteq \sigma(X) \cap \sigma(Y)$ for all X, Y, with equality if $X \in \mathcal{T}^c$.

Moreover, on compact objects σ agrees with the universal support Supp.

Our next theorem provides a rough converse of this: provided it also detects objects, any mapping σ satisfying the above conditions must coincide with the universal support data on compact objects. In fact we don't need any noetherian hypothesis, as long as the space is spectral. Moreover, the category $\mathcal{T}$ can be allowed to merely be compactly$_\aleph$ generated for some regular infinite cardinal $\aleph$ (see §1.1.3). Like Balmer and Favi, we assume that $\mathcal{T}^c$ is a rigid tensor triangulated subcategory, which amounts to requiring that $\mathbb{1} \in \mathcal{T}^c$ and that every compact object has a tensor dual.

2.4.1 Theorem ([2, Thm. 3.1])**.** *Let $\mathcal{T}$ be a compactly$_\aleph$ generated tensor triangulated category, in the above sense. Let $\sigma : \operatorname{Obj}(\mathcal{T}) \to 2^T$ be a function assigning to every object of $\mathcal{T}$ a subset of T, where T is some spectral topological space in the sense of Hochster (§2.2.1). Assume that the pair (T, σ) satisfies the properties (S1)-(S5) above and also:*

(S6) *It detects objects: $\sigma(X) = \emptyset$ implies $X \cong 0$.*

(S7) *A subset $U \subseteq T$ is quasi-compact and open if and only if it has the form $U = T \smallsetminus \sigma(C)$ for some compact object $C \in \mathcal{T}^c$.*

Then the restriction of (T, σ) to $\mathcal{T}^c$ is a classifying support datum; in particular, the canonical map $T \to \operatorname{Spc}(\mathcal{T}^c)$ is a homeomorphism (see §2.2.2).

This theorem can sometimes be used to compute the spectrum $\operatorname{Spc}(\mathcal{T}^c)$, for instance for the stable module category of a finite group (see [BCR97] — this was the original example that inspired our abstract result) or the bootstrap category in KK-theory (see §2.4.2 below). More recently, it has been applied to triangulated categories of representations of certain classical Lie superalgebras (see [BKN14]). Typically, the hardest properties to prove are (a subset of) the "half tensor product formula" (S5), the detection of objects (S6) or the realisation by compact objects (S7), whereas the other ones are immediate.

2.4.2. Un easy example: the Bootstrap category of C*-algebras. Let $\mathcal{B} \subseteq \mathrm{KK}$ denote the Rosenberg-Schochet bootstrap category of separable C*-algebra, as in §1.3.3. Its subcategory of compact objects $\mathcal{B}^c$ consists of all C*-algebras in the bootstrap class having finitely generated K-theory groups.

The classification problems for $\mathcal{B}$ and $\mathcal{B}^c$ have very easy solutions, especially when compared with the extremely complex situation of the topological analogue, i.e. the stable homotopy category.

Let $\mathbb{F}_p$ be the prime field of characteristic p, including $\mathbb{F}_0 := \mathbb{Q}$. For all $A \in \mathcal{B}$ and $p \in \operatorname{Spec} \mathbb{Z}$, set $K_*(A; \mathbb{F}_p) := K_*(A \otimes \kappa(p))$, where $\kappa(p) \in \mathcal{B}$ is any object with $K_*(\kappa(p)) \cong \mathbb{F}_p$. These are the classical topological K-theories with mod p and rational coefficients for C*-algebras.

The next result takes care of the classification of thick subcategories of compact objects, and can be obtained from Theorem 2.4.1:

2.4.3 Theorem ([2, Thm. 1.2])**.** *There is a homeomorphism between* $\operatorname{Spc}(\mathrm{Boot}_c)$ *and* $\operatorname{Spec}(\mathbb{Z})$ *identifying the universal support datum of a compact object* $A \in \mathcal{B}$ *with the set* $\{p \in \operatorname{Spec} \mathbb{Z} \mid K_*(A; \mathbb{F}_p) \neq 0\}$.

The localising subcategories are also easy to describe in terms of $\operatorname{Spec} \mathbb{Z}$, as follows. The proof is an adaptation of ideas of Neeman [Nee92].

2.4.4 Theorem ([3, Thm. 1.1])**.** *There is an inclusion-preserving bijection between localizing subcategories of the Bootstrap category* Boot *and arbitrary subsets of* $\operatorname{Spec} \mathbb{Z}$. *It sends a localising subcategory* $\mathcal{L} \subseteq B$ *to the subset*

$$\{p \in \operatorname{Spec} \mathbb{Z} \mid \exists\, A \in \mathcal{L} \ s.\,t.\ K_*(A; \mathbb{F}_p) \neq 0\}$$

and a subset $S \subseteq \operatorname{Spec} \mathbb{Z}$ *to the subcategory*

$$\{A \in \mathcal{B} \mid \forall p \notin S, K_*(A; \mathbb{F}_p) = 0\}\,.$$

Very recently, this classification theorem has been extended by Nadareishvili [Nad16] to the Meyer-Nest bootstrap category $\mathcal{B} \subseteq \mathrm{KK}(T_n)$ of filtered C*-algebras (see §1.3.3), by combining our results above with more combinatorial methods; notice also that $\mathrm{KK}(T_n)$ does not have a tensor.

2.5 Graded 2-rings and generalised comparison maps

As we have seen in §2.3, Balmer's comparison map $\rho\colon \operatorname{Spc} \mathcal{T} \to \operatorname{Spec} \operatorname{End}_{\mathcal{T}}(\mathbb{1})$ and its graded versions are often surjective for simple-minded reasons. In the cases where they are injective, this is usually much harder to prove. Moreover, the graded versions have more chances of being injective then the ungraded one. In [9] we take the latter observation seriously and construct more general comparison maps, by allowing more sophisticated gradings, in the hope of obtaining embeddings of the triangular spectrum $\operatorname{Spc} \mathcal{T}$ into some more ameanable space. We will give here the abstract results, followed in §2.6 by two applications.

In order to handle the extra gradings, we use the following categorification (and generalisation) of the usual notion of graded commutative rings.

2.5.1. Commutative 2-groups and 2-rings. Following [BL04], we define a (symmetric) *2-group* to be an (essentially) small symmetric monoidal category $\mathcal{G} = (\mathcal{G}, \otimes, \mathbb{1})$ in which every morphism is invertible and every object is tensor invertible. We write $g^\vee$ for a tensor inverse of $g \in \operatorname{Obj} \mathcal{G}$, which is also its tensor dual as in §1.6.1.

Definition ([7])**.** A *graded commutative 2-ring* is a $\mathbb{Z}$-linear symmetric monoidal category $\mathcal{R}$ in which every object is tensor invertible. In particular, its maximal subgroupoid is a 2-group $\mathcal{G} = \mathcal{G}(\mathcal{R})$ having the same objects as $\mathcal{R}$.

We consider $\mathcal{R}$ as being a fancy version of a graded commutative ring, graded by the objects of $\mathcal{G}$. (One could allow the grading 2-group to vary, but we stick here with the canonical choice for each $\mathcal{R}$.)

Here are our basic examples:

- A commutative ring R can be seen as a $\mathbb{Z}$-linear category with a single object $*$ and $\operatorname{End}(*) = R$. It is a tensor category by $* \otimes * = * = \mathbb{1}$ and $r \otimes s = rs$, and a commutative 2-ring with trivial grading (2-)group.
- More generally, let R be graded commutative ring; we can allow it to be graded by any abelian group G, and its commutativity $sr = (-1)^{\epsilon(|s|,|r|)} rs$ to be ruled by any symmetric bilinear map $\epsilon\colon G \times G \to \mathbb{Z}/2$ (more general formulations are also possible). Then the *companion category* $\mathcal{C}_R$ of R is a graded commutative 2-ring, as follows: its objects are the elements $g \in G$, its Hom groups are $\mathcal{C}_R(g,h) = R_{h-g}$, and its composition is the product of homogeneous elements of R. The tensor product is strictly symmetric, given by $g \otimes h = g + h$ on objects and by $r \otimes s = (-1)^{\epsilon(g, g'-h')} rs$ on maps $r\colon g \to h$ and $s\colon g' \to h'$. The companion category $\mathcal{C}_R$ has the same representation theory as R (see [7, §2]); it will reappear in §2.6.1.
- Let $\mathcal{A}$ be any additive symmetric monoidal category and $\mathcal{L}$ any family of invertible objects $\ell \in \mathcal{A}$. Denote by $\underline{\mathcal{L}}$ the maximal subgroupoid of $\mathcal{A}$ whose objects are the tensor-multiplicative closure of $\mathcal{L}$, and let $\mathcal{R}(\underline{\mathcal{L}})$ be the full subcategory of $\mathcal{A}$ on the same objects. Then $\mathcal{R}(\underline{\mathcal{L}})$ is a graded commutative 2-ring, graded by the 2-group $\underline{\mathcal{L}}$.
- For a concrete example of the previous construction, we can take $\mathcal{A} = \mathrm{D}(X)$ to be the derived category of a scheme, and $\mathcal{L}$ a family of line bundles on X. This example will be used in §2.6.4.

If $r, r' \in \mathcal{R}$ are two maps in a graded commutative 2-ring $\mathcal{R}$ with 2-group $\mathcal{G}$, we say r is a *translate of* r' if r can be obtained from r' by twisting (i.e. tensoring) with objects (of $\mathcal{G}$) and composing or precomposing with invertible maps (i.e. maps of $\mathcal{G}$), repeatedly and in any order. This defines an equivalence relation, *translation*, on the maps of $\mathcal{R}$. The important observation is that *any two maps r, s commute up to translation*: there always exist translates r' of r and s' of s such that $rs = s'r'$. This pseudo-commutativity allows us to generalise many results and constructions from graded rings to graded commutative 2-rings.

2.5.2. The spectrum of a graded commutative 2-ring. Let $\mathcal{R}$ be any commutative graded commutative 2-ring, as in Definition 2.5.1. A *homogeneous ideal* in $\mathcal{R}$ is a categorical ideal $\mathcal{I} \subseteq \mathcal{R}$ of maps (closed under sums of parallel maps and under composition on both sides with arbitrary maps) which moreover is closed under all twists, i.e., under tensoring with objects $g \otimes -$ (or equivalently, with any maps). We say $\mathcal{I}$ is *prime* if it is proper and if $r \circ s \in \mathcal{I}$ implies $r \in \mathcal{I}$ or $s \in \mathcal{I}$.

The *spectrum of* $\mathcal{R}$, written $\operatorname{Spec} \mathcal{R}$, is the set of all prime homogeneous ideals of $\mathcal{R}$ equipped with the evident Zariski topology.

2.5.3 Theorem ([7])**.** *The spectrum defines a contravariant functor* $\mathcal{R} \mapsto \operatorname{Spec} \mathcal{R}$ *from the category of graded commutative 2-rings and additive tensor functors to the category of spectral spaces and spectral maps, in the sense of Hochster (see §2.2.1). Moreover, it interacts in the expected way with quotients* $\mathcal{R}/\mathcal{I}$ *by homogeneous ideals and localisations* $\mathcal{R}_{\mathfrak{p}}$ *at homogeneous primes.*

The quotient mentioned in the theorem is the additive categorical quotient $\mathcal{R}/\mathcal{I}$ equipped with the induced tensor product. Similarly, localisation $\mathcal{R}_{\mathfrak{p}}$ at a prime ideal is, by definition, simply the categorical localisation inverting all the maps in $\mathcal{R} \smallsetminus \mathfrak{p}$, equipped with the induced tensor product. More generally, one can localise $\mathcal{R}$ at any *homogeneous multiplicative system* S, i.e. a set of maps containing all isomorphisms and stable under translation (see §2.5.1). The resulting localisation $\mathcal{R} \to S^{-1}\mathcal{R}$ satisfies a calculus of left and right fractions.

2.5.4. Central 2-rings in triangulated categories. We now apply the above theory to triangulated categories.

Let $\mathcal{R}$ be a graded commutative 2-ring. By an *algebra over* $\mathcal{R}$ (or $\mathcal{R}$-*algebra*) we mean an additive symmetric monoidal category $\mathcal{A}$ equipped with an additive symmetric monoidal functor $F \colon \mathcal{R} \to \mathcal{A}$. (Note that the objects of $\mathcal{A}$ are not required to be invertible.)

If S be a homogeneous multiplicative system in $\mathcal{R}$ and $F \colon \mathcal{R} \to \mathcal{A}$ is an $\mathcal{R}$-algebra, write $S_{\mathcal{A}}$ for the smallest class of maps in $\mathcal{A}$ containing FS and all isomorphisms of $\mathcal{A}$ and which is closed under composition and twisting with objects of $\mathcal{A}$.

2.5.5 Theorem ([9, Thm. 2.47])**.** *The set* $S_{\mathcal{A}}$ *is a homogeneous multiplicative set satisfying in* $\mathcal{A}$ *both a left and a right calculus of fractions.*

Definition. Let $\mathcal{T}$ be tensor triangulated category. A *central 2-ring* of $\mathcal{T}$ is any graded commutative 2-ring $\mathcal{R}$ occurring as a full tensor subcategory of $\mathcal{T}$. The inclusion functor $\mathcal{R} \to \mathcal{T}$ turns $\mathcal{T}$ into an $\mathcal{R}$-algebra in the above sense.

Together with Theorem 2.5.5, our next result shows that Balmer's technique of central localisation of tensor triangulated categories (§2.1.4) generalises smoothly to any central 2-ring.

2.5.6 Theorem ([9, Thm. 3.6])**.** *Let* $\mathcal{R}$ *be any central 2-ring of a tensor triangulated category* $\mathcal{T}$*, as above, and let* S *be a homogeneous multiplicative system* *in $\mathcal{R}$. Then localisation induces a canonical isomorphism*

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{q} & \mathcal{T}/\mathcal{J}_S \\ {\scriptstyle \mathrm{loc}}\downarrow & \nearrow_{\cong} & \\ S_{\mathcal{T}}^{-1}\mathcal{T} & & \end{array}$$

between $\mathcal{T}$ localized at S as an $\mathcal{R}$-algebra (see Theorem 2.5.5) and the Verdier quotient of $\mathcal{T}$ by the thick tensor ideal

$$\mathcal{J}_S := \langle \mathrm{cone}(s) \mid s \in S \rangle_{\otimes} = \{ x \in \mathcal{T} \mid \exists s \in S \text{ such that } s \otimes \mathrm{id}_x = 0 \}$$

generated by the cones of maps in S.

Moreover, the central 2-ring of these categories on the objects of $\mathcal{R}$ is canonically isomorphic to the localized graded commutative 2-ring $S^{-1}\mathcal{R}$.

It follows in particular that the localisation $S_{\mathcal{T}}^{-1}\mathcal{T}$ inherits from $\mathcal{T}$ a canonical tensor triangulated structure. Or we can view this the other way round: the tensor triangular quotient $\mathcal{T}/\mathcal{J}_S$ has a computationally easier model $S_{\mathcal{T}}^{-1}\mathcal{T}$.

In order to compare the geometry of the tensor triangulated category with that of its central rings, we need to understand the local case. This is the content of the next theorem. A tensor triangulated category (just like a commutative ring, or a graded commutative 2-ring . . .) is said to be *local* if its spectrum has a unique closed point (see [Bal10a]).

2.5.7 Theorem ([9, Thm. 3.3])**.** *If $\mathcal{T}$ is a local tensor triangulated category, then every central 2-ring $\mathcal{R}$ of $\mathcal{T}$ is local as a graded commutative 2-ring, i.e. it has a unique maximal homogeneous ideal. Moreover, this maximal ideal consists precisely of the non-invertible arrows of $\mathcal{R}$.*

2.5.8. Generalised comparison maps. As was the case for central localisation, we can also generalise Balmer's comparison map $\rho\colon \mathrm{Spc}\,\mathcal{T} \to \mathrm{Spec}\,R$ to arbitrary graded commutative 2-rings. This provides us with a vast new choice of maps with which one can try to compute the triangular spectrum.

2.5.9 Theorem ([9, Thm. 3.10])**.** *For every central 2-ring $\mathcal{R}$ of $\mathcal{T}$ there is a continuous spectral map $\rho_{\mathcal{T}}^{\mathcal{R}}\colon \mathrm{Spc}\,\mathcal{T} \to \mathrm{Spec}\,\mathcal{R}$ which sends the prime thick $\otimes$-ideal $\mathcal{P} \subset \mathcal{T}$ to the prime ideal*

$$\rho_{\mathcal{T}}^{\mathcal{R}}(\mathcal{P}) := \{ r \in \mathrm{Mor}\,\mathcal{R} \mid \mathrm{cone}(r) \notin \mathcal{P} \}\,.$$

Moreover, the map $\rho_{\mathcal{T}}^{\mathcal{R}}$ is natural in the following sense. If $F\colon \mathcal{T} \to \mathcal{T}'$ is a tensor-exact functor and $\mathcal{R}'$ is a central 2-ring of $\mathcal{T}'$ such that $F\mathcal{R} \subseteq \mathcal{R}'$, then the square of spectral continuous maps

$$\begin{array}{ccc} \mathrm{Spc}\,\mathcal{T}' & \xrightarrow{\mathrm{Spc}\,F} & \mathrm{Spc}\,\mathcal{T} \\ {\scriptstyle \rho_{\mathcal{T}'}^{\mathcal{R}'}}\downarrow & & \downarrow{\scriptstyle \rho_{\mathcal{T}}^{\mathcal{R}}} \\ \mathrm{Spec}\,\mathcal{R}' & \xrightarrow{\mathrm{Spec}\,F} & \mathrm{Spec}\,\mathcal{R} \end{array}$$

is commutative.

With our comparison maps, we also have at our disposal a general abstract criterion for injectivity (first observed by Stevenson for the original map ρ):

2.5.10 Proposition ([9, Prop. 3.11])**.** *Suppose the collection of subsets*

$$\mathcal{B} = \{\mathrm{Supp}(\mathrm{Cone}(r)) \mid r \in \mathrm{Mor}\,\mathcal{R}\}$$

gives a basis of closed subsets for the topology of $\mathrm{Spc}\,\mathcal{T}$*. Then the map* $\rho^{\mathcal{R}}_{\mathcal{T}}$ *is a homeomorphism onto its image; in particular, it is injective.*

Although it is quite abstract, this criterion can actually be used in some situations, for instance in the two examples discussed in §2.6.

2.6 Applications to derived categories

We now apply the previous abstract nonsense to derived categories of (graded) rings and schemes.

Recall that the derived category of an abelian category is by definition the localisation of the category $\mathrm{Ch}(\mathcal{A})$ of complexes $X = (X^n, d^n_X \colon X^n \to X^{n+1})_{n\in\mathbb{Z}}$ in $\mathcal{A}$ obtained by formally inverting the quasi-isomorphisms, i.e., chain maps $X \to Y$ which induce an isomorphism $H^nX \overset{\sim}{\to} H^nY$ on all cohomology objects. The derived category $\mathrm{D}(\mathcal{A})$ is a triangulated category where the suspension functor ΣX is given by degree shift and a sign change on differentials: $(\Sigma X)^n = X^{n+1}$ and $d^n_{\Sigma X} = -d^{n+1}_X$. The distinguished triangles all arise (up to quasi-isomorphism) from short exact sequences of complexes. One often considers variations where some conditions are imposed on the complexes, e.g. boundedness, nice cohomology objects, etc.

If X is a quasi-compact and quasi-separated scheme, we consider its derived category $\mathrm{D}(X) := \mathrm{D}_{\mathrm{qc}}(X)$ of (unbounded) complexes with quasi-coherent cohomology sheaves. It is compactly generated, and its compact objects are the *perfect complexes*, $\mathrm{D}(X)^c = \mathrm{D}^{\mathrm{perf}}(X)$, i.e. those complexes which are locally quasi-isomorphic to a bounded complex of vector bundles. The left derived tensor product $-\otimes^{\mathrm{L}}_{\mathcal{O}_X} -$ turns $\mathrm{D}(X)$ and $\mathrm{D}(X)^c$ into tensor triangulated categories with unit $\mathcal{O}_X$.

In the affine case $X = \mathrm{Spec}(R)$, this amounts to the derived category $\mathrm{D}(R)$ of all complexes of R-modules over a commutative ring R, which is compactly generated by the tensor unit R. A complex of modules is perfect iff it is quasi-isomorphic to a bounded complex of finitely generated projectives.

We consider also the case of a graded commutative ring R. Here we can allow R to be graded by a general abelian group G and its commutativity rule $rs = (-1)^{\epsilon(|r|,|s|)}sr$ to use any signing bilinear form $\epsilon \colon G \times G \to \mathbb{Z}/2$. Then $\mathrm{D}(R)$ is the derived category of complexes of (left, say) graded R-modules and degree-preserving R-linear maps. Again, $\mathrm{D}(R)$ is a compactly generated tensor triangulated category, but now it is generated by the family $R(g)$ ($g \in G$) of all inner shifts $R(g)$ of the tensor unit R (see [7]). In particular, $\mathrm{D}^{\mathrm{perf}}(R)$ is not anymore generated by its tensor unit.

As usual, we will consider the Balmer spectrum only for the tensor triangulated subcategories of compact objects.

2.6.1. Graded commutative rings. The following result amounts to a graded generalisation of the Hopkins-Neeman-Thomason classification of the thick subcategories of perfect complexes over a commutative ring [Tho97]. A global version, for quite general 'super-schemes', can also be derived, although we haven't written up the details.

2.6.2 Theorem ([9, Thm. 4.7]). *Let R be any graded commutative ring, in the above flexible sense. Then there is a (unique) homeomorphism*

$$\operatorname{Spc} \mathrm{D}^{\mathrm{perf}}(R) \cong \operatorname{Spec}^{\mathrm{h}} R$$

which identifies the support in the sense of Balmer with the usual homological support given by $\operatorname{Supp}_R(X) = \{\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}} R \mid H^*(X)_{\mathfrak{p}} \neq 0\}$.

Our proof is similar in spirit to that of Thomason in that it reduces the problem to the case of a *noetherian* graded commutative ring, which was treated separately in [7]. The main difference is that we deduce the general case by working almost exclusively at the tensor triangular level, rather then by more ring- and module-theoretic arguments. The tool that allows us to perform this more formal (hence, hopefully, also more reusable) reduction step is our generalised comparison map $\rho^{\mathcal{R}}_{\mathcal{T}}$ of §2.5.8 for graded commutative 2-rings, thanks to its flexibility and naturality. Another ingredient is the companion category $\mathcal{C}_R$ of R, as in §2.5.1, and the fact that it can be easily identified, as a graded 2-ring, with the full subcategory $\mathcal{R} := \{R(g) \mid g \in G\}$ of $\mathrm{D}(R)$. Thus the identification in the theorem is actually a chain of homeomorphisms

$$\operatorname{Spc} \mathrm{D}^{\mathrm{perf}}(R) \overset{\sim}{\to} \operatorname{Spec} \mathcal{R} \cong \operatorname{Spec} \mathcal{C}_R \cong \operatorname{Spec}^{\mathrm{h}} R$$

where the leftmost map is our comparison map $\rho^{\mathcal{R}}_{\mathrm{D}^{\mathrm{perf}}(R)}$.

In [7], the noetherian case is proved by extending to the graded setting some well-known arguments involving the classification of indecomposable injective modules over noetherian commutative rings, which is used to define the *small support* ssupp for the objects of $\mathrm{D}(R)$, and by applying to it our recognition criterion of Theorem 2.4.1. The result is a (slightly more abstract) variant of Neeman's origin proof [Nee92] for the ungraded case. As in *loc. cit.*, the argument simultaneously classifies the localising subcategories of the ambient compactly generated category $\mathrm{D}(R)$. We present our graded statement below.

The small support of any $X \in \mathrm{D}(R)$ is given by

$$\operatorname{ssupp}(X) = \{\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}} R \mid k(\mathfrak{p}) \otimes^{\mathrm{L}}_R X \neq 0\}$$

where $k(\mathfrak{p})$ is the graded residue field of R at $\mathfrak{p}$.

Note that a localising subcategory in $\mathrm{D}(R)$ which is a tensor ideal is the same as one which is stable under internal degree shift by any $g \in G$.

2.6.3 Theorem ([7, Thm. 5.7]). *There are mutually inverse and inclusion preserving bijections*

$$\Big\{ \textit{ subsets of } \operatorname{Spec}^{\mathrm{h}} R \Big\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \Big\{ \textit{ localizing } \otimes\textit{-ideals of } \mathrm{D}(R) \Big\},$$

and

$$\left\{ \begin{array}{c} \textit{specialization closed} \\ \textit{subsets of } \operatorname{Spec}^{\mathrm{h}} R \end{array} \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \begin{array}{c} \textit{localizing } \otimes\textit{-ideals of } \mathrm{D}(R) \\ \textit{generated by objects of } \mathrm{D}(R)^c \end{array} \right\}$$

where for a subset W of $\operatorname{Spec}^{\mathrm{h}} R$ and a localizing $\otimes$-ideal $\mathcal{L}$ we set

$$\tau(W) = \{X \in \mathrm{D}(R) \mid \operatorname{ssupp} X \subseteq W\}$$

and

$$\sigma(\mathcal{L}) = \{\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}} R \mid k(\mathfrak{p}) \otimes^{\mathrm{L}}_R \mathcal{L} \neq 0\}.$$

The second bijection is equivalent to a classification for perfect complexes which, in its turn, amounts to the identification of $\operatorname{Spc} \mathrm{D}(R)^c$ and $\operatorname{Spec}^{\mathrm{h}} R$ because of the general abstract classification of §2.2.2.

The classification results of the latter theorem can be applied to derived categories of "weighted projective schemes" (certain algebraic stacks), which notably involve gradings by more general abelian groups than $\mathbb{Z}$ (see [7, §6]).

2.6.4. Schemes with an ample family of line bundles. For another, rather straightforward application of our generalised comparison map, we may consider as in §2.6 above a scheme X and its derived category $\mathrm{D}(X)$. If $\mathcal{L} = \{\mathcal{L}_\lambda\}_{\lambda\in\Lambda}$ is a collection of line bundles on X, we can consider as in §2.5.1 the full tensor subcategory $\mathcal{R}(\underline{\mathcal{L}})$ of $\mathrm{D}(X)$ generated by $\mathcal{L}$. It is a graded commutative 2-ring and therefore has a nice spectrum $\operatorname{Spec} \mathcal{R}(\underline{\mathcal{L}})$.

2.6.5 Theorem ([9, Thm. 4.11]). *Let X be a quasi-compact and quasi-separated scheme equipped with an ample family of line bundles $\mathcal{L} = \{\mathcal{L}_\lambda\}_{\lambda\in\Lambda}$. Then the comparison map*

$$\rho\colon \operatorname{Spc} \mathrm{D}^{\mathrm{perf}}(X) \to \operatorname{Spec} \mathcal{R}(\underline{\mathcal{L}})$$

is a homeomorphism onto its image. In particular, there is an injective morphism $\rho^{\underline{\mathcal{L}}}_X \colon X \to \operatorname{Spec} \mathcal{R}(\underline{\mathcal{L}})$ and X has the subspace topology relative to this injection.

The injectivity of the comparison map follows from the criterion in Proposition 2.5.10 and the definition of an ample family. The second statement follows by Balmer's reconstruction's theorem $X \cong \operatorname{Spc} \mathrm{D}^{\mathrm{perf}}(X)$.

As it turns out, the above theorem recovers, with a new proof, the embeddings due to Brenner and Schröer [BS03] of divisorial schemes into generalised projective spaces. An advantage of our definition is that it is clearly functorial in the pair $(X, \mathcal{L})$ and does away with some finiteness hypothesis.

2.7 Classifications in the affine regular case

Concerning Balmer's original graded comparison map §2.3, we have the following general result stating that it is a homeomorphism, provided that the tensor unit generates the category and has a *regular* and *noetherian* graded endomorphism ring. We also have a version classifying the localising subcategories in a compactly generated category.

2.7.1 Theorem ([14, Thm. 1.1])**.** *Let $\mathcal{K}$ be an essentially small tensor triangulated category and denote by R its graded endomorphism ring $R := \mathrm{End}^*_{\mathcal{K}}(\mathbb{1})$. Assume that $\mathcal{K}$ and R satisfy the next two conditions:*

(1) *$\mathcal{K}$ is classically generated by $\mathbb{1}$, i.e., as a thick subcategory:* $\mathrm{Thick}(\mathbb{1}) = \mathcal{K}$.

(2) *R is a graded noetherian ring concentrated in even degrees and, for every homogeneous prime ideal $\mathfrak{p}$ of R, the maximal ideal of the local ring $R_{\mathfrak{p}}$ is generated by a finite regular sequence of homogeneous non-zero-divisors.*

Then the graded comparison map $\rho\colon \mathrm{Spc}\,\mathcal{K} \overset{\sim}{\to} \mathrm{Spec^h}\,R$ is a homeomorphism.

The noetherianity of R already guarantees that ρ is surjective. The key to proving injectivity is to first use general tensor triangular geometry to reduce the statement to the localised category $\mathcal{K}_{\mathfrak{p}}$. Then one shows that the fiber of ρ over any $\mathfrak{p} \in \mathrm{Spec}\,R$ consists of a single point, by using the *residue field object*

$$K(\mathfrak{p}) := \mathrm{Cone}(f_1) \otimes \cdots \otimes \mathrm{Cone}(f_n)$$

defined by the regular sequence $f_1, \ldots, f_n$ provided by hypothesis (2). It has the property that its cohomology (homotopy) R-module $\mathcal{K}^*_{\mathfrak{p}}(\mathbb{1}, K(\mathfrak{p}))$ is just $k(\mathfrak{p})$, the graded residue field. We deduce that the thick subcategory generated by $K(\mathfrak{p})$ is minimal, and the result follows.

2.7.2 Remark. We can also slightly change hypothesis (2) of the theorem, by replacing the evenness hypothesis with a more abstract one, relating to the properties of *algebraic triangulated categories* [Sch10]. The variant (2)' says: R is noetherian, and for every $\mathfrak{p} \in \mathrm{Spec^h}\,R$ the ideal $\mathfrak{p}R_{\mathfrak{p}} \subseteq R_{\mathfrak{p}}$ is generated by a finite regular sequence $f_1, \ldots, f_n$ such that each f_i acts as zero on $\mathrm{Cone}(f_i)$.

As usual this yields a classification of the thick tensor ideals as follows, in terms of the usual big Zariski support Supp_R. Since $\mathcal{K}$ is generated by its tensor unit, every thick subcategory is automatically a tensor ideal. Therefore:

2.7.3 Corollary ([14, Cor. 1.2])**.** *If $\mathcal{K}$ and R are as in the theorem, then there exists a canonical inclusion-preserving bijection*

$$\{\textit{thick subcategories } \mathcal{C} \textit{ of } \mathcal{K}\} \overset{\sim}{\rightleftarrows} \{\textit{specialization closed subsets } V \textit{ of } \mathrm{Spec}\,R\}$$

mapping a thick subcategory $\mathcal{C}$ to $V = \bigcup_{X\in\mathcal{C}} \mathrm{Supp}_R\, H^*X$ and a specialization closed subset V to $\mathcal{C} = \{X \in \mathcal{K} \mid \mathrm{Supp}_R\, H^*X \subseteq V\}$.

Assume now that, as in many natural examples, $\mathcal{K}$ is the subcategory $\mathcal{T}^c$ of compact objects in a compactly generated tensor triangulated category $\mathcal{T}$. Here we simply mean that $\mathcal{T}$ is compactly generated, and equipped with an exact and coproduct preserving tensor product, such that $\mathcal{T}^c$ is a tensor subcategory. The next classification uses the support supp_R of objects $X \in \mathcal{T}$ defined by

$$\operatorname{supp}_R X := \{\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}} R \mid K(\mathfrak{p}) \otimes X \neq 0\}\,.$$

With our hypotheses, this coincides with the support defined by Benson, Iyengar and Krause [BIK08], and we employ several of their results in proving the following theorem:

2.7.4 Theorem ([14, Thm. 1.3])**.** *Let $\mathcal{T}$ be a compactly generated tensor triangulated category such that its subcategory of compact objects $\mathcal{K} := \mathcal{T}^c$ and its graded central ring R satisfy the conditions* (1) *and* (2) *in Theorem 2.7.1. Then we have the following canonical inclusion-preserving bijection:*

$$\big\{\textit{localizing subcategories } \mathcal{L} \subseteq \mathcal{T}\big\} \xrightarrow[\longleftarrow]{\sim} \big\{\textit{subsets } S \subseteq \operatorname{Spec}^{\mathrm{h}} R\big\}\,.$$

The correspondence sends a localizing subcategory $\mathcal{L}$ to $S = \bigcup_{X \in \mathcal{L}} \operatorname{supp}_R X$, and an arbitrary subset S to $\mathcal{L} = \{X \in \mathcal{T} \mid \operatorname{supp}_R X \subseteq S\}$. Moreover, the bijection restricts, on the left, to localizing subcategories $\mathcal{L} = \operatorname{Loc}(\mathcal{L} \cap \mathcal{K})$ which are generated by compact objects and, on the right, to specialization closed subsets $S = \bigcup_{\mathfrak{p} \in S} \overline{\{\mathfrak{p}\}}$.

Recall that a localising subcategory if *smashing* if its inclusion functor has a coproduct preserving right adjoint. The following statement is ofter referred to as the (generalised) *telescope conjecture*:

2.7.5 Corollary ([14, Cor. 1.4])**.** *In the same situation as in Theorem 2.7.4, every smashing subcategory of $\mathcal{T}$ is generated by a set of compact objects of $\mathcal{T}$.*

For $\mathcal{T} := \mathrm{D}(A)$ the derived category of a (highly structured) commutative ring spectrum A (see §3.6.2), the tensor unit A generates, hence we can apply all the above results as soon as the homotopy algebra $R = \pi_* A$ satisfies the regularity hypothesis (2). This example was first treated by Shamir [Sha12], using some more model-theoretic arguments.

2.7.6. Application to commutative dg-algebras. Let A be a commutative differential graded (= dg) algebra and $\mathrm{D}(A)$ its derived category of dg modules. Then $\mathrm{D}(A)$ is a compactly generated tensor triangulated category with respect to the standard tensor product $\otimes = \otimes^{\mathrm{L}}_A$. It is generated by its tensor unit A so it satisfies hypothesis (1) in Theorem 2.7.1. The graded central ring $R = H^*A$ is the cohomology algebra of A. Hence if the latter also satisfies hypothesis (2) all previous results apply to $\mathrm{D}(A)$. Since every $f \in H^*A$ acts as zero on its cone, we can apply our results in the variant of Remark 2.7.2. We obtain:

2.7.7 Theorem. *Let A be a commutative dg algebra such that its graded cohomology ring $R = H^*A$ is noetherian and such that for every homogeneous prime $\mathfrak{p}$ the maximal ideal $\mathfrak{p}R_\mathfrak{p} \subset R_\mathfrak{p}$ is generated by a finite regular sequence. Then all the conclusions of Theorems 2.7.1 and 2.7.4 and of Corollaries 2.7.3 and 2.7.5 hold for $\mathcal{T} = D(A)$ and $\mathcal{K} = D(A)^c$: namely, the spectrum of H^*A classifies the thick tensor ideals of perfect dg modules as well as the localising subcategories of* $\mathrm{D}(A)$*, and the telescope conjecture holds in* $\mathrm{D}(A)$*.*

The advantage of not having the evenness hypothesis is that this theorem can also be applied, for instance, to a graded polynomial algebra with any choice of grading for the variables, seen a strictly commutative formal dg algebra.

2.8 Another example: noncommutative motives

We now briefly present the results of [5]. This only represents a modest first attempt towards a description of the triangular spectrum of non-commutative motives, but it serves to illustrate the order of difficulty of such an enterprise.

Fix a base commutative ring k, and let Mot^a_k and Mot^ℓ_k denote the triangulated categories of *noncommutative motives*, in their additive and localising versions (see [Tab08]). There are functors

$$U^a_k \colon \mathrm{dgcat}_k \to \mathrm{Mot}^a_k \qquad U^\ell_k \colon \mathrm{dgcat}_k \to \mathrm{Mot}^\ell_k$$

from the category of small dg categories over k which are, in a precise sense, the universal additive and localising invariant, respectively. In particular, Hochschild homology HH, topological cyclic homology THH, and nonconnective algebraic K-theory $I\!\!K$ (as functors into spectra) are all localising invariants, hence factor uniquely through Mot^ℓ_k and (since localising invariants are additive) also through Mot^a_k. Quillen K-theory K is only additive and factors through Mot^a_k.

Both Mot^a_k and Mot^ℓ_k are tensor triangulated categories with arbitrary coproducts [CT12]. Mot^a_k is known to be compactly generated, with a set of compact generators given by the motives $U^a_k(\mathcal{A})$ of homotopically finite dg categories $\mathcal{A}$, while this is not known for Mot^ℓ_k. In the following we will consider the *monogenic cores* $\mathrm{Core}^a_k \subset \mathrm{Mot}^a_k$ and $\mathrm{Core}^\ell_k \subset \mathrm{Mot}^\ell_k$, defined to be the thick subcategories generated by the tensor unit $\mathbb{1} = U^a_k(k)$ and $\mathbb{1} = U^\ell_k(k)$, respectively. Their rationalisations are denoted by $\mathrm{Core}^a_{k;\mathbb{Q}}$ and $\mathrm{Core}^\ell_{k;\mathbb{Q}}$.

After these drastic simplifications, we obtain:

2.8.1 Theorem ([5, Thm. 1.1])**.** *Assume that the base ring k is finite or that it is the algebraic closure of a finite field. Then, if we denote by n the number of prime ideals in k, we have an equivalence* $\mathrm{Core}^a_{k;\mathbb{Q}} \cong \mathrm{D}^{\mathrm{perf}}(\mathbb{Q}^n)$ *of tensor triangulated categories. Hence* $\mathrm{Spc}(\mathrm{Core}^a_{k;\mathbb{Q}})$ *has precisely n distinct points. The same result holds for* $\mathrm{Core}^\ell_{k;\mathbb{Q}}$ *if we further assume that the base ring k is regular.*

Without rationalising the monogenic cores, we can exploit some classical (non-trivial!) computations of the above-mentioned invariants to obtain a little extra information. Balmer's ungraded comparison map ρ is also used here.

We denote by HP periodic cyclic homology, which, although not a localising invariant, can be factors through Mot_k^ℓ if we consider it as taking values in the derived category of 2-periodic complexes.

2.8.2 Theorem ([5, Thm. 1.2])**.** *Assume that k is a finite field $\mathbb{F}_q$ or an algebraic closure $\overline{\mathbb{F}_q}$ (where $q = p^r$ with p a prime number and r a positive integer). Then:*

(i) We have a continuous surjective map $\rho\colon \mathrm{Spc}(\mathrm{Core}_k^a) \twoheadrightarrow \mathrm{Spec}(\mathbb{Z})$.

(ii) The fiber $\rho^{-1}(\{0\})$ at the prime ideal $\{0\} \subset \mathbb{Z}$ has a single point given by

$$\{M \in \mathrm{Core}_k^a \,|\, K_*(M)_{\mathbb{Q}} = 0\}\,,$$

where $K_*(M)$ stands for the homotopy groups of the spectrum $K(M)$.

(iii) The fiber $\rho^{-1}(p\mathbb{Z})$ at the prime ideal $p\mathbb{Z} \subset \mathbb{Z}$ has a single point. Moreover, this point admits the following three different descriptions:

$$\begin{array}{ll} (a) & \{M \in \mathrm{Core}_k^a \,|\, HH(M) = 0\} \\ (b) & \{M \in \mathrm{Core}_k^a \,|\, HP(M) = 0\} \\ (c) & \{M \in \mathrm{Core}_k^a \mid K_*(M)_{(p)} = 0\}\,, \end{array}$$

where $K_*(M)_{(p)}$ is the localization of $K_*(M)$ at the prime $p\mathbb{Z}$.

Moreover, the same results hold for Core_k^ℓ, with K replaced by $I\!K$.

Now contrast the above results with the following one, which says that things get more complicated as soon as we add polynomial algebras to the monogenic core. Let $\mathrm{ECore}_k^a \subset \mathrm{Mot}_k^a$ ('extended core') be the tensor triangulated subcategory generated by the tensor unit together with the motive $U_k^a(k[t])$.

2.8.3 Theorem ([5, Thm. 1.4, Prop.])**.** *Let k be a field and $\mathrm{char}(k) = 0$. Then:*

(i) We have a continuous surjective map $\rho : \mathrm{Spc}(\mathrm{ECore}_k^a) \twoheadrightarrow \mathrm{Spec}(\mathbb{Z})$.

(ii) The two subcategories

$$\{M \in \mathrm{ECore}_k^a \,|\, HH(M) = 0\} \quad \text{and} \quad \{M \in \mathrm{ECore}_k^a \,|\, HP(M) = 0\}$$

define two distinct points in $\mathrm{Spc}(\mathrm{ECore}_k^a)$, both in the same fiber $\rho^{-1}(\{0\})$.

It seems to us that any significant progress in the tensor triangular geometry of noncommutative motives beyond these first observations will need the input of some radically new idea.

3 Tensor exact functors and duality

In this last section we describe the results of [13, 12], joint work with Paul Balmer and Beren Sanders, where we systematically study the properties of nice tensor exact functors $f^*\colon \mathcal{D} \to \mathcal{C}$ and connect them with various duality theories throughout mathematics.

3.1 Preliminaries on tensor exact functors

Here are the precise hypotheses we will need for most of our results:

3.1.1 Hypothesis. We assume that the tensor triangulated categories $\mathcal{C}$ and $\mathcal{D}$ are *rigidly-compactly generated* as in Definition 2.1.3: they are compactly generated, they admit a compatible closed monoidal structure, and the subcategories of compact and rigid objects coincide. Of the given functor $f^*\colon \mathcal{D} \to \mathcal{C}$ we assume that it preserves distinguished triangles, tensor products (it is a symmetric monoidal functor) and also arbitrary coproducts.

Remark. Unfortunately, we cannot weaken the above hypothesis to include the $\aleph_1$-relative version of Brown representability, hence we cannot include the examples of Kasparov categories of C*-algebras that we had encountered so far, which only admit countable coproducts. This is because our proofs use Brown representability for the dual, which cannot hold in such categories.

In particular, our hypothesis implies that the subcategory of compact objects $\mathcal{C}^c$ (and similarly for $\mathcal{D}^c$) is a rigid tensor triangulated category, which comes equipped with the following canonical tensor exact duality functor

$$\Delta := \mathsf{hom}(-, \mathbb{1})\colon (\mathcal{C}^c)^{\mathrm{op}} \xrightarrow{\sim} \mathcal{C}^c .$$

This is the starting point of all our dualities. One of the main goals of our duality theory presented below in §3.4 is to identify bigger subcategories $\mathcal{C}_0 \subset \mathcal{C}$ admitting such a duality functor $\Delta_\kappa := \mathsf{hom}(-, \kappa)$, typically twisted by some more general dualizing object κ.

As we explained in the Introduction, we are going to study the existence of adjoint functors to f^*, adjoints to those adjoints, etc., and will show that only three possible distinct configurations of adjoints are possible. To begin with, since we assume f^* to preserve coproducts Neeman's Brown representability (§1.1.3) already implies the existence of a right adjoint f_* for it. Since f^* is a tensor functor, it preserves rigid and hence compact objects, and therefore, by another well-known trick, the right adjoint f_* preserves coproducts; hence by Brown representability it admits itself a right adjoint, $f^{(1)}$. This already establishes the basic first stage of adjunction:

3.1.2 Theorem ([13, Cor. 2.14, Prop. 2.15])**.** *Under Hypothesis 3.1.1, the functor $f^* : \mathcal{D} \to \mathcal{C}$ admits a right adjoint $f_* : \mathcal{C} \to \mathcal{D}$, which itself admits a right adjoint $f^{(1)} : \mathcal{D} \to \mathcal{C}$: Moreover, the three functors*

$$f^* \dashv f_* \dashv f^{(1)}$$

are related by the following canonical isomorphisms:

$$x \otimes f_*(y) \cong f_*(f^*(x) \otimes y) \tag{3.1}$$
$$\mathsf{hom}_{\mathcal{D}}(x, f_* y) \cong f_* \,\mathsf{hom}_{\mathcal{C}}(f^* x, y) \tag{3.2}$$
$$\mathsf{hom}_{\mathcal{D}}(f_* x, y) \cong f_* \,\mathsf{hom}_{\mathcal{C}}(x, f^{(1)} y) \tag{3.3}$$
$$f^{(1)} \mathsf{hom}_{\mathcal{D}}(x, y) \cong \mathsf{hom}_{\mathcal{C}}(f^* x, f^{(1)} y)\,.$$

Here and in the following, $\mathsf{hom}_{\mathcal{C}}$ and $\mathsf{hom}_{\mathcal{D}}$ denote the internal hom functors on $\mathcal{C}$ and $\mathcal{D}$ respectively. The isomorphism (3.1) is usually called the (*right*) *projection formula*. The isomorphisms (3.2) and (3.3) are simply internal versions of the two adjunctions, from which the adjunctions can be recovered by applying $\mathsf{hom}_{\mathcal{D}}(\mathbb{1}_{\mathcal{D}}, -)$.

The following terminology refers to the dualizing complexes of algebraic geometry; see [Lip09] and [Nee96, Nee10].

3.1.3 Definition ([13])**.** The object $f^{(1)}(\mathbb{1}_{\mathcal{D}})$ of $\mathcal{C}$ will be called the *relative dualizing object* (for $f^* \colon \mathcal{D} \to \mathcal{C}$) and will be denoted by ω_f.

Note that by definition ω_f is uniquely determined by the existence of a natural bijection

$$\mathrm{Hom}_{\mathcal{D}}(f_*(-), \mathbb{1}) \cong \mathrm{Hom}_{\mathcal{C}}(-, \omega_f)$$

or equivalently, using the internal adjunction, by the existence of a natural isomorphism in $\mathcal{D}$

$$\mathsf{hom}_{\mathcal{D}}(f_*(-), \mathbb{1}) \cong f_* \,\mathsf{hom}_{\mathcal{C}}(-, \omega_f)\,.$$

Writing $\Delta = \mathsf{hom}(-, \mathbb{1})$ and $\Delta_{\omega_f} = \mathsf{hom}(-, \omega_f)$ for the usual duality functor on $\mathcal{D}$ and for the ω_f-twisted duality on $\mathcal{C}$, this becomes the equation

$$\Delta \circ f_* \cong f_* \circ \Delta_{\omega_f}$$

telling us that ω_f is precisely the object needed to make the direct image f_* a duality preserving functor.

3.2 Grothendieck-Neeman duality

We now come to the second possible stage of adjunction. This is the content of the following mammuth theorem. When the three equivalent conditions (1)-(3) of the theorem hold true, we say that f^* *satisfies Grothendieck-Neeman duality.*

3.2.1 Theorem ([13, Thm. 3.3])**.** *Let $f^* : \mathcal{D} \to \mathcal{C}$ be as in Hypothesis 3.1.1 and consider the automatic adjoints $f^* \dashv f_* \dashv f^{(1)}$ of Thm. 3.1.2. Then the following conditions are equivalent:*

(1) Grothendieck duality: *There is a natural isomorphism*

$$\omega_f \otimes f^*(-) \cong f^{(1)}(-)$$

identifying the mysterious new functor $f^{(1)}$ with the twist of the given pull-back functor f^* by the relative dualising object ω_f.

(2) Neeman's criterion: *The functor f_* preserves compact objects, or equivalently its right adjoint $f^{(1)}$ preserves coproducts, or equivalently by Brown Representability $f^{(1)}$ admits a right adjoint $f_{(-1)}$.*

(3) *The original functor $f^* : \mathcal{D} \to \mathcal{C}$ preserves products, or equivalently by Brown Representability for the dual, f^* admits a left adjoint $f_{(1)}$.*

Moreover, when these conditions hold, the five functors

$$\begin{array}{c} \mathcal{C} \\ f_{(1)}\big\downarrow \quad f^*\big\uparrow \quad f_*\big\downarrow \quad f^{(1)}\big\uparrow \quad \big\downarrow f_{(-1)} \\ \mathcal{D} \end{array}$$

are related by the following additional canonical natural isomorphisms:

$$f_{(-1)} \cong f_* \,\mathsf{hom}_{\mathcal{C}}(\omega_f, -)$$

$$\begin{aligned} f^{(1)}(x \otimes y) &\cong f^{(1)}(x) \otimes f^*(y) \\ \mathsf{hom}_{\mathcal{D}}(x, f_{(-1)}y) &\cong f_* \,\mathsf{hom}_{\mathcal{C}}(f^{(1)}x, y) \\ \mathsf{hom}_{\mathcal{D}}(x, f_{(-1)}y) &\cong f_{(-1)} \,\mathsf{hom}_{\mathcal{C}}(f^*x, y) \end{aligned}$$

$$f^*(-) \cong \mathsf{hom}_{\mathcal{C}}(\omega_f, f^{(1)}(-)) \tag{3.4}$$

$$\boxed{f_{(1)}(-) \cong f_*(\omega_f \otimes -)} \qquad \textit{(ur-Wirthmüller)}$$

$$\boxed{x \otimes f_{(1)}(y) \cong f_{(1)}(f^*(x) \otimes y)} \qquad \textit{(left projection formula)}$$

$$\begin{aligned} f^*\mathsf{hom}_{\mathcal{D}}(x, y) &\cong \mathsf{hom}_{\mathcal{C}}(f^*x, f^*y) \\ \mathsf{hom}_{\mathcal{D}}(f_{(1)}x, y) &\cong f_* \,\mathsf{hom}_{\mathcal{C}}(x, f^*y). \end{aligned}$$

There is a logic to all these formulas, and they are organised in 'conjugate' families as indicated in our display (see [13, Rem. 2.21]).

We have highlighted the *left projection formula* for the left adjoint $f_{(1)}$ of f^*, analogous to the already encountered projection formula for the right adjoint f^*, as well as the *ur-Wirthmüller formula*, which gives rise to the Wirthmüller isomorphism relating the left and the right adjoints via a canonical twist. It is remarkable that the same canonical object, namely ω_f, giving rise to the Wirthmüller isomorphims, is the same object appearing in the Grothendieck-Neeman duality formula. This had not been observed before.

A Wirthmüller isomorphism is usually understood to be a twist by a tensor invertible object (hence our "ur-" qualifier), so it is only natural to ask when ω_f becomes invertible. The answer is, precisely when the third and last stage of adjunction occurs!

3.3 The Wirthmüller isomorphism

There is an isomorphism

$$\mathbb{1}_{\mathcal{C}} \cong \mathsf{hom}_{\mathcal{C}}(\omega_f, \omega_f)$$

for the relative dualising object ω_f, which can be derived for instance by inserting $y = \mathbb{1}$ in relation (3.4) of the Grothendieck-Neeman Duality Theorem 3.2.1. This easily implies that ω_f is tensor invertible precisely when it is a rigid, i.e. compact object. In fact, there is a surprising number of equivalent characterisations of the invertibility of ω_f, and they are the content of the next theorem. One of them is the existence of a further adjoint at the right end, or of a further adjoint at the left end, but each of these already implies the existence of the complete doubly infinite tower of adjoints. Hence the third phase of adjunction is also the last possible one.

If the six equivalent conditions in the theorem are true, we say that *the Wirthmüller isomorphism holds for f^*.*

3.3.1 Theorem ([13, Thm. 1.6])**.** *Suppose that we have the five adjoint functors $f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)} \dashv f_{(-1)}$ of Grothendieck-Neeman duality (see Theorem 3.2.1). Then the following six conditions are equivalent:*

(1) *The left-most functor $f_{(1)}$ admits itself a left adjoint, or equivalently by Brown Representability it preserves arbitrary products.*

(2) *The right-most functor $f_{(-1)}$ admits itself a right adjoint, or equivalently by Brown Representability it preserves arbitrary coproducts, or equivalently its left adjoint $f^{(1)}$ preserves compact objects.*

(3) *The relative dualizing object ω_f is a compact object of $\mathcal{C}$.*

(4) *The relative dualizing object ω_f is $\otimes$-invertible in $\mathcal{C}$.*

(5) *There exists a* (strong) Wirthmüller isomorphism *between f_* and $f_{(1)}$; that is, there exists a $\otimes$-invertible object $\omega \in \mathcal{C}$ such that $f_{(1)} \cong f_*(\omega \otimes -)$, or equivalently such that $f_* \cong f_{(1)}(\omega^{-1} \otimes -)$.*

(6) *There exists an infinite tower of adjoints on both sides:*

$$\cdots \; \Big\uparrow f^{(-n)} \; \Big\downarrow f_{(n)} \; \cdots \; \Big\uparrow f^{(-1)} \; \Big\downarrow f_{(1)} \; \overset{\mathcal{C}}{\underset{\mathcal{D}}{\Big\uparrow}} f^* \; \Big\downarrow f_* \; \Big\uparrow f^{(1)} \; \Big\downarrow f_{(-1)} \; \Big\uparrow f^{(2)} \; \cdots \; \Big\uparrow f^{(n)} \; \Big\downarrow f_{(-n)} \; \cdots$$

which necessarily preserve all coproducts, products and compact objects.

Moreover, when these conditions hold, the tower of adjoints appearing in (6) is necessarily given for all $n \in \mathbb{Z}$ by the formulas

$$f^{(n)} = \omega_f^{\otimes n} \otimes f^* \qquad \textit{and} \qquad f_{(n)} = f_*(\omega_f^{\otimes n} \otimes -)\,. \tag{3.5}$$

Finally, (1)-(6) hold true as soon as the functor $f_* : \mathcal{C} \to \mathcal{D}$ satisfies, in addition to Grothendieck-Neeman duality, any one of the following three properties:

(a) *The functor f_* is faithful (i.e. f^* is surjective up to direct summands).*

(b) *The functor f_* detects compact objects: any $x \in \mathcal{C}$ is compact if $f_*(x)$ is.*

(c) *Any $x \in \mathcal{C}$ is compact if $f_*(x \otimes y)$ is compact for every compact $y \in \mathcal{C}$.*

These conditions are ordered in increasing generality, because (a)⇒(b)⇒(c).

Note that the two equations (3.5) describing the infinitely many adjoints necessarily follow from the Grothendieck-Neeman duality formula $f^{(1)} \cong \omega_f \otimes f^*$, the ur-Wirthmüller formula $f_{(1)} \cong f_*(\omega_f \otimes -)$, and the uniqueness of adjoints. They also justify our notations.

The "Trichotomy of Adjoints" theorem stated in the Introduction should now be obvious.

As we will see in the examples, in algebraic geometry the sufficient condition (c) is related to the regularity of schemes.

3.4 Grothendieck duality on subcategories

As we have seen, the canonical dualizing object ω_f is characterised by the fact that it lifts the canonical duality $\mathsf{hom}_{\mathcal{D}}(-, \mathbb{1})$ to $\mathcal{C}$ along the push-forward functor $f_* \colon \mathcal{C} \to \mathcal{D}$. We now study more general dualizing objects κ, the subcategories $\mathcal{C}_0 \subset \mathcal{C}$ that they dualize, and their functorial behaviour.

3.4.1. Dualizing objects. We will always assume that the subcategory $\mathcal{C}_0 \subset \mathcal{C}$ is a $\mathcal{C}^c$*-submodule*, *i.e.* a thick triangulated subcategory of the big category $\mathcal{C}$ such that $c \otimes x \in \mathcal{C}_0$ for all $x \in \mathcal{C}_0$ and all compact objects $c \in \mathcal{C}^c$. (This is justified by the examples.)

Definition. An object $\kappa \in \mathcal{C}_0$ is called a *dualizing object for* $\mathcal{C}_0$ if the κ*-twisted duality* $\Delta_\kappa := \mathsf{hom}_{\mathcal{C}}(-, \kappa)$ defines an anti-equivalence on $\mathcal{C}_0$:

$$\Delta_\kappa := \mathsf{hom}_{\mathcal{C}}(-, \kappa) : (\mathcal{C}_0)^{\mathrm{op}} \xrightarrow{\sim} \mathcal{C}_0.$$

If the latter holds for an object $\kappa \notin \mathcal{C}$ not belonging to $\mathcal{C}_0$, we say that it is an *external* dualizing object for $\mathcal{C}_0$. (If $\mathbb{1} \in \mathcal{C}_0$ then necessarily $\kappa \cong \Delta_\kappa(\mathbb{1}) \in \mathcal{C}_0$.)

This definition is inspired by dualizing complexes in algebraic geometry.

A primordial example is, of course, the dualizing object $\kappa = \mathbb{1}$ for the subcategory of rigid-compact objects $\mathcal{C}^c$. In general, an object $\kappa \in \mathcal{C}^c$ is dualizing for $\mathcal{C}_0 := \mathcal{C}^c$ if and only if it is tensor invertible.

Under a choice of mild hypotheses, dualizing objects can be shown to be unique up to tensoring with some invertible object.

For any object $\kappa \in \mathcal{C}$, we deduce from the tensor-Hom adjunction a canonical double dual comparison map

$$\varpi_\kappa : x \longrightarrow \Delta_\kappa \Delta_\kappa(x)$$

and it is not hard to see that $\kappa \in \mathcal{C}_0$ is an (internal) dualizing object for $\mathcal{C}_0$ if and only if $\Delta_\kappa(x) \in \mathcal{C}_0$ and $\varpi_\kappa \colon x \xrightarrow{\sim} \Delta_\kappa^2(x)$ for all $x \in \mathcal{C}_0$.

3.4.2. Pulling back dualizing objects. We now generalise our previous observations about ω_f and duality, and make precise what it means for the functor f_* to preserve duality.

3.4.3 Theorem. *Assume the functor $f^* : \mathcal{D} \to \mathcal{C}$ satisfies our basic Hypothesis 3.1.1 and let $\kappa \in \mathcal{D}$. Recall the two adjunctions $f^* \dashv f_* \dashv f^{(1)}$ of Theorem 3.1.2, as well as their internal realizations* (3.2) *and* (3.3)*. The latter yields a canonical natural isomorphism*

$$\zeta : \Delta_\kappa \circ f_* \xrightarrow{\sim} f_* \circ \Delta_{f^{(1)}(\kappa)}. \tag{3.6}$$

This isomorphism is compatible with the canonical maps ϖ of Δ_κ and $\Delta_{\kappa'}$ for $\kappa' = f^{(1)}(\kappa)$. This means that the following diagram commutes, for all $x \in \mathcal{C}$:

$$\begin{array}{ccc} f_*(x) & \xrightarrow{f_*(\varpi_x)} & f_*\Delta_{\kappa'}\Delta_{\kappa'}(x) \\ {\scriptstyle \varpi_{f_*(x)}}\downarrow & & \downarrow{\scriptstyle \cong\ \zeta_{\Delta_{\kappa'}(x)}} \\ \Delta_\kappa\Delta_\kappa f_*(x) & \xrightarrow[\Delta_\kappa(\zeta)]{\cong} & \Delta_\kappa f_* \Delta_{\kappa'}(x). \end{array}$$

In other words, $f_* : \mathcal{C} \to \mathcal{D}$ is a duality-preserving functor in the sense of [CH09].

The latter compatibility was systematically studied by Calmès and Hornbostel [CH09] and has to be taken into account, for instance, when applying derived categories to the theory of quadratic forms over schemes[2]; see [Bal04].

3.4.4. The compact pullback of a subcategory. If $\kappa \in \mathcal{D}$ is a dualizing object for some subcategory $\mathcal{D}_0 \subset \mathcal{D}$ of the base category, in view of Theorem 3.4.3 we may wish to somehow pull the subcategory $\mathcal{D}_0$ back along f_* in order to obtain a subcategory of $\mathcal{C}$ which is dualized by $f^{(1)}(\kappa)$.

We now explain the correct way to do it.

Definition. If $\mathcal{D}_0$ is a $\mathcal{D}^c$-submodule of $\mathcal{D}$, define its *compact pullback along f_** as the following full subcategory of $\mathcal{C}$:

$$f^\#(\mathcal{D}_0) := \{x \in \mathcal{C} \mid f_*(c \otimes x) \in \mathcal{D}_0 \text{ for all } c \in \mathcal{C}^c\}\,.$$

One sees immediately that $f^\#(\mathcal{D}_0)$ is a $\mathcal{C}^c$-submodule of $\mathcal{C}$.

We note that the compact pullback *of compact objects* is a good measure for the three stages of adjunction: the functor f^* satisfies Grothendieck-Neeman duality (stage two) if and only if $\mathcal{C}^c \subseteq f^\#(\mathcal{D}^c)$; and the weakest sufficient condition (c) in Theorem 3.3.1 guaranteeing that f^* satisfies the Wirthmüller isomorphism (stage three) is equivalent to having equality: $\mathcal{C}^c = f^\#(\mathcal{D}^c)$.

Our next result is the following general *Grothendieck duality theorem*. Indeed, as we will see, it specialises the Grothendieck duality in its form of an anti-equivalence $\mathrm{D^b}(\mathrm{coh}\,X)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{D^b}(\mathrm{coh}\,X)$ on the category of bounded complexes of coherent modules over a (nice) scheme.

[2] Incidentally, my first mathematical work [1] – not included here – was done in this context.

3.4.5 Theorem ([13, Thm. 5.25])**.** *Let $f^*\colon \mathcal{D} \to \mathcal{C}$ be as in our basic Hypothesis 3.1.1 and let $\kappa \in \mathcal{D}$. Recall the functors $f^* \dashv f_* \dashv f^{(1)}$, and suppose that f^* satisfies Grothendieck-Neeman duality (the second stage of adjunction, Theorem 3.2.1) and that $\mathcal{D}_0 \subset \mathcal{D}$ is a $\mathcal{D}^c$-submodule which admits $\kappa \in \mathcal{D}_0$ as a dualizing object. Then*

$$\kappa' := f^{(1)}(\kappa) \cong \omega_f \otimes f^*(\kappa)$$

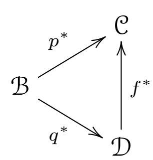
is a dualizing object for the above-defined compact pullback $f^\#(\mathcal{D}_0) \subset \mathcal{C}$. In particular,

$$f_* : f^\#(\mathcal{D}_0) \longrightarrow \mathcal{D}_0$$

is a duality-preserving exact functor between categories with duality, where $f^\#(\mathcal{D}_0)$ is equipped with the duality $\Delta_{\kappa'}$ and $\mathcal{D}_0$ with Δ_κ.

3.4.6. Categories over a base. We also have a more general version of our Grothendieck duality theorem, in which we work relative to a base category $\mathcal{B}$. Instead of assuming that $f^*\colon \mathcal{D} \to \mathcal{C}$ satisfies Grothendieck-Neeman duality, we now require that $\mathcal{C}$ and $\mathcal{D}$ satisfy it relative to the base $\mathcal{B}$.

More precisely, assume that we have a commutative triangle



of functors, all satifying the basic Hypothesis 3.1.1 so that we have adjunctions $f^* \dashv f_* \dashv f^{(1)}$, as well as $p^* \dashv p_* \dashv p^{(1)}$ and $q^* \dashv q_* \dashv q^{(1)}$. We say f^* is a *morphism of $\mathcal{B}$-categories.*

3.4.7 Theorem. *Let $f^*\colon \mathcal{D} \to \mathcal{C}$ be a morphism of $\mathcal{B}$-categories as above. Assume that the structure functors p^* and q^* both satisfy Grothendieck-Neeman duality (stage two of adjunction). Let $\mathcal{B}_0 \subset \mathcal{B}$ be a $\mathcal{B}^c$-subcategory with dualizing object $\kappa \in \mathcal{B}_0$. Let $\mathcal{C}_0 := p^\#\mathcal{B}_0$ and $\mathcal{D}_0 := q^\#\mathcal{B}_0$ be its compact pullbacks in $\mathcal{C}$ and $\mathcal{D}$ respectively (see §3.4.4), which admit the dualizing objects*

$$\gamma := \omega_p \otimes p^*(\kappa) \in \mathcal{C}_0 \qquad \textit{and} \qquad \delta := \omega_q \otimes q^*(\kappa) \in \mathcal{D}_0$$

respectively, by Theorem 3.4.5. Then we have the equality $f^\#(\mathcal{D}_0) = \mathcal{C}_0$, and f_* restricts to a well-defined exact functor $f_* : \mathcal{C}_0 \to \mathcal{D}_0$ which is duality-preserving with respect to Δ_γ and Δ_δ.

3.4.8. Pushing forward dualizing objects. Instead of starting with a subcategory with duality $\mathcal{D}_0 \subseteq \mathcal{D}$ and pulling it back to $\mathcal{C}$ along the functor $f_*\colon \mathcal{C} \to \mathcal{D}$, as we have done so far, we may instead already have a subcategory with duality $\mathcal{C}_0 \subseteq \mathcal{C}$, in which case the question is how to push it forward towards $\mathcal{D}$.

An answer is given by the next theorem:

3.4.9 Theorem ([13, Thm. 7.1]). *Let $f^*\colon \mathcal{D} \to \mathcal{C}$ be a functor satisfying our basic Hypothesis 3.1.1 and hence the first stage of adjunction $f^* \dashv f_* \dashv f^{(1)}$. Let $\mathcal{C}_0$ be a subcategory of $\mathcal{C}$ admitting a dualizing object $\kappa' \in \mathcal{C}$, which could be internal, $\kappa' \in \mathcal{C}$, or also external, $\kappa' \notin \mathcal{C}_0$. Assume moreover that κ' admits a* Matlis lift *κ, that is, an object $\kappa \in \mathcal{D}$ such that $f^{(1)}(\kappa) \cong \kappa'$. Then the Matlis lift κ is a (possibly external) dualizing object for the subcategory $\mathcal{D}_0 := \mathrm{Thick}(f_*\mathcal{C}_0)$, the thick subcategory generated by the image of $\mathcal{C}_0$ under push-forward.*

The name 'Matlis lift' is taken from the related work of Dwyer, Greenlees and Iyengar [DGI06], cf. the examples in §3.6.8.

3.5 Relative Serre duality

Grothendieck duality generalises Serre duality to the relative situation, i.e. to morphisms of schemes. However, there is another useful way to generalise Serre duality, by using the notion of a Serre functor on a triangulated category. We now explain how our framework offers a natural generalisation of Serre functors which can be applied to morphisms. This only requires that we don't insist on working over a base field, as is traditionally done with Serre functors.

In general, if we are given any adjunction $f^* : \mathcal{D} \rightleftarrows \mathcal{C} : f_*$ between closed tensor categories with f^* a (strong) tensor functor, the category $\mathcal{C}$ inherits an enrichment over $\mathcal{D}$ ([Kel05]): the Hom-objects of this enrichment are given by

$$\underline{\mathcal{C}}(x,y) := f_* \mathsf{hom}_{\mathcal{C}}(x,y) \quad \in \mathcal{D}$$

and the unit and composition morphisms $\mathbb{1}_{\mathcal{D}} \to \underline{\mathcal{C}}(x,y)$ and $\underline{\mathcal{C}}(y,z) \otimes_{\mathcal{D}} \underline{\mathcal{C}}(x,y) \to \underline{\mathcal{C}}(x,z)$ in $\mathcal{D}$ are obtained by adjunction, in the evident way.

We have the following *relative Serre duality theorem*:

3.5.1 Theorem ([13, Thm. 6.12]). *Let $f^*\colon \mathcal{D} \to \mathcal{C}$ be a functor as in Hypothesis 3.1.1 and let $\underline{\mathcal{C}}$ denote the resulting $\mathcal{D}$-enriched category as above. Then there is a canonical natural isomorphism in $\mathcal{D}$*

$$\sigma_{x,y} : \Delta\underline{\mathcal{C}}(x,y) \overset{\sim}{\to} \underline{\mathcal{C}}(y, x \otimes \omega_f) \tag{3.7}$$

for all $x \in \mathcal{C}^c$ and $y \in \mathcal{C}$, where $\Delta := \mathsf{hom}_{\mathcal{D}}(-,\mathbb{1})$ is the standard duality. In particular, if f^* satisfies the Wirthmüller isomorphism (Theorem 3.3.1), the pair $(\mathbb{S} := (-)\otimes\omega_f, \sigma)$ defines a Serre functor on $\mathcal{C}^c$ relative to $\mathcal{D}^c$, *by which we mean that $\mathbb{S}$ is an equivalence $\mathbb{S}\colon \mathcal{C}^c \overset{\sim}{\to} \mathcal{C}^c$ and that σ is a natural isomorphism $\Delta\underline{\mathcal{C}}(x,y) \cong \underline{\mathcal{C}}(y, \mathbb{S}x)$ in the tensor-category $\mathcal{D}^c$ for all $x, y \in \mathcal{C}^c$.*

Serre duality, in a more traditional sense, is then the special case where the base is (the derived category of) a field:

3.5.2 Corollary ([13, Cor. 6.12]). *Let $f^*\colon \mathcal{D} \to \mathcal{C}$ satisfy the Wirthmüller isomorphism (Theorem 3.3.1), and assume moreover that $\mathcal{D} = \mathrm{D}(\Bbbk)$ is the derived category of a field $\Bbbk$. Then $\mathcal{C}^c$ is $\Bbbk$-linear and endowed with a Serre functor*

$$\mathbb{S} = (-) \otimes \omega_f \colon \mathcal{C}^c \overset{\sim}{\longrightarrow} \mathcal{C}^c \qquad \sigma\colon \mathcal{C}(x,y)^\star \overset{\sim}{\longrightarrow} \mathcal{C}(y, \mathbb{S}x)$$

in the original sense of Bondal and Kapranov [BK89] (see also[BO01]), where $(-)^\star = \mathrm{Hom}_{\Bbbk}(-, \Bbbk)$ *denotes the* $\Bbbk$*-linear dual.*

We note that if a Serre functor $(\mathbb{S}, \sigma)$ exists, even in the general sense of Theorem 3.5.1, then it is uniquely determined up to a canonical isomorphism, by an application of the Yoneda lemma.

3.6 Examples and applications

We illustrate the preceding theory with some examples taken from representation theory, algebraic geometry, and commutative algebra. We should note that, in some cases, our point of view allows us to clarify things and uncover some new features. Even when the results are known, as is mostly the case, our abstract proofs are sometimes simpler then those found in the literature.

Other examples mentioned in [13] include motivic homotopy theory, cohomology rings of classifying spaces, highly structured cochain algebras, and Brown-Comenetz duality for the stable homotopy category. A point worth mentioning about the latter is that it can be shown, using K-theoretic obstructions, that the functor f^* in question is *not*, in that case, induced by any underlying functor at the level of models. Hence it provides an example of an f^* which exists purely at the triangular level, without any 'underlying map f'.

3.6.1. Representation theory. Let G be a finite group and let $\Bbbk$ be a field. Then $\mathcal{D} := \mathrm{Stab}(\Bbbk G)$, the stable category of $\Bbbk G$-modules modulo projectives, is rigidly-compactly generated. More generally, G could be a finite group scheme over $\Bbbk$ (see e.g. [HPS97, Theorem 9.6.3]). (Note however that the derived category $\mathrm{D}(\Bbbk G)$, though compactly generated, is not *rigidly*-compactly generated because its unit object $\mathbb{1} = \Bbbk$ is not compact).

Let H be a finite subgroup (a closed subgroup scheme) of G and consider its stable category $\mathrm{Stab}(\Bbbk H)$. As explained in [Jan87, Chapter 8], the restriction functor

$$f^* \colon \mathcal{D} = \mathrm{Stab}(\Bbbk G) \longrightarrow \mathrm{Stab}(\Bbbk H) = \mathcal{C}$$

provides an example of Theorem 3.3.1, i.e. satisfies the Wirthmüller isomorphism. Indeed, if δ_G denotes the unimodular character of the finite group scheme G then the relative dualizing object ω_f is $\delta_G|_H \cdot \delta_H^{-1}$. A finite group scheme is said to be *unimodular* if its unimodular character is trivial, $\omega_f \cong \mathbb{1}$, which is equivalent to the group algebra being a symmetric algebra; this is the case for instance for (discrete) finite groups.

3.6.2. ("Brave New") commutative algebra. Let A be a commutative ring or, more generally, a "Brave New" commutative ring, that is, a highly structured commutative ring spectrum. To be more precise, we can for instance understand A to be a commutative $\mathbb{S}$-algebra in the sense of [EKMM97]. Then its derived category $\mathrm{D}(A)$, i.e. the homotopy category of A-modules (coinciding with the derived category of chain complexes of A-modules in the case of a plain ring), is a rigidly-compactly generated category generated by its tensor unit A (see e.g. [HPS97, Example 1.2.3(f)] and [SS03, Example 2.3(ii)]). For example, every commutative dg ring has an associated commutative $\mathbb{S}$-algebra (its Eilenberg-MacLane spectrum) whose derived category is equivalent, as a tensor triangulated category, to the derived category of dg modules (see [Shi07] and [SS03, Theorem 5.1.6]).

Consider now a morphism $\phi\colon B \to A$ of commutative $\mathbb{S}$-algebras, or commutative dg rings. Then ϕ induces a functor

$$f^* := A \otimes_B - \colon \mathcal{D} = \mathrm{D}(B) \longrightarrow \mathrm{D}(A) = \mathcal{C}$$

satisfying our basic Hypotheses. (We do not write ϕ^* to preserve the correct variance; think "$f = \operatorname{Spec}\phi$", which is literally true in the case of a plain commutative ring.) The right adjoint f_* is obtained simply by considering A-modules as B-modules through ϕ and the next right adjoint $f^{(1)}$ is given by the formula $f^{(1)} = \operatorname{Hom}_B(A, -)$. (All functors considered here are derived, of course.) Since $\mathrm{D}(A)^c$ is the thick subcategory generated by A, we see by Neeman's criterion (point (2) in Theorem 3.2.1) that f^* satisfies Grothendieck-Neeman duality if and only if $f_*(A)$ is compact.

For usual rings, this simply means that A admits a finite resolution by finitely generated B-modules. Assume this is the case, and assume further that $B = \Bbbk$ is a field, so that A is a finite-dimensional commutative $\Bbbk$-algebra. Then ω_f is the A-module $\operatorname{Hom}_B(A, B) = \operatorname{Hom}_{\Bbbk}(A, \Bbbk) =: A^\star \in \mathrm{D}(A)$, the $\Bbbk$-linear dual of A, and ω_f is invertible iff it is a perfect complex, iff $A^\star \cong A$ as A-modules. For a standard example where this is not true, we can take the $\Bbbk$-algebra $A = \Bbbk[t, s]/(s^2, t^2, st)$. Hence the latter is the example of an f^* satisfying Grothendick-Neeman duality but not the Wirthmüller isomorphism.

3.6.3. Equivariant stable homotopy. Let G be a compact Lie group. Then $\mathcal{D} := \mathrm{SH}(G)$, the homotopy category of "genuine" G-spectra indexed on a complete G-universe (see [HPS97, §9.4]), is rigidly-compactly generated. The suspension G-spectra $\Sigma^\infty_+ G/H$, with H running through all closed subgroups of G, is a set of rigid-compact generators which includes the tensor unit $\mathbb{1} = \Sigma^\infty_+ G/G$.

Let H be a closed subgroup of G and let

$$f^* \colon \mathcal{D} = \mathrm{SH}(G) \longrightarrow \mathrm{SH}(H) = \mathcal{C}$$

denote the associated restriction functor. Then f^* satisfies Wirthmüller duality, as in Theorem 3.3.1. This example gives the phenomenon its name. The relative dualizing object ω_f can be computed to be the H-sphere S^L, where L denotes the tangent H-representation of the smooth G-manifold G/H at the identity coset eH (see [LMSM86, Chapter III]). The (ur-)Wirthmüller formula takes the form $G_+ \wedge_H X \cong F_H(G_+, X \wedge S^L)$ and provides the original Wirthmüller isomorphism between induction and coinduction, up to a twist by S^L. If H has finite index in G (e.g. if G is a finite group) then $L = 0$ and $\omega_f \cong \mathbb{1}$.

Fausk, Hu and May [FHM03] already provide an axiomatisation of the Wirthmüller isomorphism, which however requires a God-given object C and isomorphism $f_*(\mathbb{1}) \cong f_{(1)}(C)$, from which they derive a formula $f_* \cong f_{(1)}(C \otimes -)$. Contrary to our setting, this leaves open the question of the canonicity of such an isomorphism. We prove that, in the context of their axiomatisation, one necessarily must have $C \cong \omega_f^{-1}$ as soon as C is compact, which seems to generally be the case (see [13, Rem. 4.3, Prop. 4.4] for details).

3.6.4. Algebraic geometry. Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{C} := \mathrm{D}(X) = \mathrm{D}_{\mathrm{Qcoh}}(X)$ be the derived category of complexes of $\mathcal{O}_X$-modules having quasi-coherent homology (see [Lip09]). It is rigidly-compactly generated, and its compact objects are precisely the perfect complexes: $\mathrm{D}(X)^c = \mathrm{D}^{\mathrm{perf}}(X)$ (see [BvdB03]). The tensor functor is given by the derived tensor product $\otimes = \otimes^{\mathrm{L}}_{\mathcal{O}_X}$. If moreover X is separated, there is an equivalence $\mathrm{D}_{\mathrm{Qcoh}}(X) \cong \mathrm{D}(\mathrm{Qcoh}\, X)$ with the derived category of complexes of quasi-coherent $\mathcal{O}_X$-modules (see [BN93]). If $X = \mathrm{Spec}(A)$ is affine, then $\mathrm{D}(\mathrm{Qcoh}\, X) \cong \mathrm{D}(A\text{-}\mathrm{Mod})$ with compact objects $\mathrm{D}(A\text{-}\mathrm{Mod})^c \cong \mathrm{K}^{\mathrm{b}}(A\text{-}\mathrm{proj})$, the homotopy category of bounded complexes of finitely generated projectives, and we recover the (non-Brave New case of) §3.6.2.

Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes as above, and consider the (derived) inverse image functor

$$f^* : \mathcal{D} = \mathrm{D}(Y) \longrightarrow \mathrm{D}(X) = \mathcal{C}\,.$$

It is easy to see that f^* satisfies our basic Hypothesis 3.1.1; its right adjoint is the derived push-forward $f_* = \mathrm{R}f_*$, whose right adjoint $f^{(1)}$ is the twisted inverse image functor, usually written $f^\times$ or $f^!$ (see [Lip09]).

Then the functor f^* satisfies Grothendieck-Neeman duality precisely when the f_* preserves compact objects, i.e. perfect complexes. By definition, this means that the morphism f is *quasi-perfect* [LN07, Def. 1.1]. Thus in this context, Theorem 3.2.1 recovers the original results of Neeman that have inspired us. Even when specialized to algebraic geometry, our theorem is somewhat stronger, because it includes the extra information about the left adjoint $f_{(1)}$ of f^*, whose existence is equivalent to the quasi-perfection of f and which is necessarily given by the ur-Wirthmüller formula $f_{(1)} \cong \omega_f \otimes f_*$. (This left adjoint was only observed in a few case; we now know it exists in full generality.) Among other things, it is shown in [LN07] that f is quasi-perfect iff it is proper and of finite tor-dimension. Hence, if $f\colon X \to Y$ is finite (e.g. in the affine case), then it is quasi-perfect iff $f_*(\mathbb{1}_{\mathcal{C}}) = \mathrm{R}f_*(\mathcal{O}_X)$ is a perfect complex.

Under some reasonable hypotheses, the stage of the Wirthmüller isomorphism is closely related to the scheme being Gorenstein. To see this, let us first see how the abstract notion of compact pullback (§3.4.4) specialises in algebraic geometry in the following nice way.

3.6.5 Theorem ([13, Thm. 5.21])**.** *Let $f\colon X \to Y$ be a morphism of noetherian schemes and consider the functor $f^*\colon \mathcal{D} = \mathrm{D}(Y) \longrightarrow \mathrm{D}(X) = \mathcal{C}$ as above.*

(1) *Suppose that $f\colon X \to Y$ is proper. Then $f_*\colon \mathcal{C} \to \mathcal{D}$ maps $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$ into $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Y)$. Moreover, for every object $x \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$ and every perfect $c \in \mathrm{D}(X)^c$ we have $f_*(c \otimes x) \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Y)$.*

(2) *Suppose that $f\colon X \to Y$ is projective. Then the following converse to (1) holds: If $x \in \mathrm{D}(X)$ is such that $f_*(c \otimes x) \in \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y)$ for every perfect $c \in \mathrm{D}(X)^c$ then $x \in \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$.*

Thus in our notation we have for a projective morphism $f\colon X \to Y$ that

$$f^{\#}\big(\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y)\big) = \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X).$$

By all rights, this equality should hold for all proper morphisms, but we don't have a general proof yet.

In any case, for projective varieties X at least we obtain in particular the following purely tensor-triangular description of bounded complexes of coherent sheaves. Since X is regular iff the inclusion $\mathrm{D}^{\mathrm{perf}}(X) \subseteq \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ is an equality, this also yields a tensor-triangular characterisation of regularity.

3.6.6 Corollary. *Consider a projective morphism $f\colon X \to S$ of noetherian schemes, with regular base S (for instance $S = \operatorname{Spec}(\Bbbk)$ for a field $\Bbbk$). Then*

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) = \big\{\, x \in \mathrm{D}(X) \,\big|\, f_*(c \otimes x) \in \mathrm{D}^{\mathrm{perf}}(S) \text{ for all } c \in \mathrm{D}^{\mathrm{perf}}(X) \,\big\}\,.$$

3.6.7. Grothendieck and Serre duality for algebraic varieties. Accordingly, and for simplicity, we now restrict attention to a projective variety X over a field $\Bbbk$ with structure map $p\colon X \to \Bbbk$ and induced pullback functor $p^*\colon \mathcal{B} = \mathrm{D}(\Bbbk) \to \mathrm{D}(X) = \mathcal{C}$. By the abstract theory, the inclusion $\mathcal{C}^c = \mathrm{D}^{\mathrm{perf}}(X) \subseteq \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) = p^{\#}(\mathcal{B}^c)$ tells us that p^* must satisfy Grothendieck-Neeman duality. By another abstract characterisation ([13, Thm. 5.23]), we know that the subcategory $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ consists of ω_p-reflexive objects in $\mathrm{D}(X)$, that is, those $x \in \mathrm{D}(X)$ whose canonical map $\varpi_{\omega_p}\colon x \to \Delta^2_{\omega_p}(x)$ is invertible. Hence by the Grothendieck duality Theorem 3.4.5, the object ω_p is dualizing for the subcategory $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$, i.e. it is a *dualizing complex for X* (see [Nee10]). This is one of the central results of classical Grothendieck duality.

If X is *Gorenstein* (e.g. regular, or a complete intersection), then by [Har66, p. 299] the structure sheaf $\mathcal{O}_X$ is also a dualizing complex for X. But then, by the uniqueness of dualizing complexes up to twist, there exists a tensor invertible $\ell \in \mathrm{D}(X)$ and an isomorphism $\omega_p \cong \mathcal{O}_X \otimes \ell = \ell$, so in the Gorenstein case ω_p is invertible and therefore p^* satisfies the Wirthmüller isomorphism, Thm. 3.3.1. Indeed, it can be shown in general that Gorenstein varieties are characterized by having an invertible dualizing complex (see [AIL10, §8.3]).

If we assume further that X is *regular*, we can even determine ω_p up to isomorphism. Indeed, it is a basic classical result that $\mathcal{C}^c = \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ admits a Serre functor $- \otimes \Sigma^n \omega_X$, where $\omega_X = \Lambda^n \Omega_{X/\Bbbk}$ is the canonical sheaf on X (see e.g. [Rou10, Lemma 4.18]; here we assume X is of pure dimension n, for simplicity). By Corollary 3.5.2, the functor $- \otimes \omega_p$ is also a Serre functor on $\mathcal{C}^c$. Therefore, by the uniqueness of Serre functors, we must have $\omega_p \cong \Sigma^n \omega_X$.

Finally, suppose now that $f\colon X \to Y$ is a $\Bbbk$-morphism of projective varieties. By Grothendieck duality over the base $\mathcal{B} = \mathrm{D}(\Bbbk)$ (Theorem 3.4.7), we have a well-defined functor $f_* : \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \to \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y)$ compatible with the dualities

$\Delta_{\omega_p} = \mathsf{hom}(-,\omega_p)$ and $\Delta_{\omega_q} = \mathsf{hom}(-,\omega_q)$ as determined above (where we denote by $q : Y \to \mathrm{Spec}(\Bbbk)$ the structure map of Y). This conclusion is another major aspect of classical Grothendieck duality.

3.6.8. Pontryagin and Matlis duality in local algebra. Consider again the example of commutative algebra §3.6.2, only from the distinct *local* point of view: let $R \to k$ be the quotient morphism of a commutative noetherian local ring R to its residue field k, and let

$$f^* \colon \mathcal{D} = \mathrm{D}(R) \longrightarrow \mathrm{D}(k) = \mathcal{C}$$

be the induced tensor triangular functor satisfying our Hypothesis 3.1.1. Then $E(k)$, the injective hull of the R-module k, is a Matlis lift of k: $f^{(1)}(E(k)) = \mathrm{RHom}_R(k, E(k)) \cong k$ in $\mathrm{D}(k)$ in the sense of the push-forward Theorem 3.4.9. Hence, by the same theorem, the functor $\Delta_{E(k)} = \mathrm{RHom}_R(-, E(k))$ induces a duality on the thick subcategory of $\mathrm{D}(R)$ generated by $f_*(k)$. This contains the complexes whose homology is bounded and consists of finite length modules. As $E(k)$ is injective, we may restrict this duality to the category of finite length modules. This is the classical *Matlis duality* in local commutative algebra.

Note that the dualizing object $E(k)$ above is typically external, it often lies outside the subcategory it dualizes: $E(k) \notin \mathrm{Thick}(f_*k)$. This already happens in the archetypical example of (discrete p-local) *Pontryagin duality*, where $R \to k$ is the quotient map $\mathbb{Z}_{(p)} \to \mathbb{Z}/p$ and $E(k)$ is the Prüfer group $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}_{(p)}$, which has infinite length.

Nothing forbids us now, in this argument, to replace the morphism $R \to k$ of ordinary rings with a more general morphism of ring spectra (§3.6.2):

3.6.9 Corollary ([13, Cor. 7.6])**.** *Let $R \to k$ be any morphism of commutative $\mathbb{S}$-algebras, and let I be any Matlis lift of k, i.e., an object $I \in \mathrm{D}(R)$ admitting an isomorphism*

$$\mathrm{D}(k)(x,k) \cong \mathrm{D}(R)(f_*x, I)$$

natural in $x \in \mathrm{D}(k)$. Then I is a (possibly external) dualizing object for the thick subcategory of $\mathrm{D}(R)$ generated by $f_*(k)$. □

Using models, Dwyer, Greenlees and Iyengar [DGI11] can find many examples of such Matlis lifts along morphisms of $\mathbb{S}$-algebras, and develop for them a rather sophisticated theory.

Finally, let us use local rings to illustrate another phenomenon.

In the trichotomy of adjoints for a coproduct-preserving tensor-triangulated functor $f^* \colon \mathcal{D} \to \mathcal{C}$ (see the Introduction), the third stage (what we call the Wirthmüller isomorphism) is reached precisely when the relative dualising object ω_f is invertible (cf. Theorem 3.3.1). However, this is conditional on having first reached stage two (Grothendieck-Neeman duality). It can actually happen that ω_f is already invertible at the first stage, although Grothendieck-Neeman duality does not hold. For a commonly occurring example, consider the derived extension-of-scalars functor $f^* \colon \mathrm{D}(R) \to \mathrm{D}(k)$, where R is a local commutative

Gorenstein ring with residue field k. In this case $\omega_f \cong \Sigma^d k$ is invertible (d being the Krull dimension of R), but f^* satisfies Grothendieck-Neeman duality only if R is a regular ring (see [13, Ex. 3.25]).

3.7 Restriction to finite-index subgroups

In many domains of equivariant mathematics, one can assign to every (sufficiently nice) group G a tensor triangulated category $\mathcal{C}(G)$, and to every subgroup $H \leqslant G$ a tensor exact restriction functor $\mathrm{Res}^G_H \colon \mathcal{C}(G) \to \mathcal{C}(H)$.

It was first observed by Balmer, in the context of linear representation theory, that restriction functors to a *finite index* subgroup are formally equivalent to extension of scalars functors for finite étale extensions. This allowed him to successfully import descent techniques into representation theory [Bal15].

As it turns out, this phenomenon is rather formal and has little to do with linear representation theory, and in fact holds in various examples from topology, analysis and geometry. This is the observation made in [13] that we present in this final section.

What we mean by *étale extension of scalars* is the following: for each of our examples, there exists a monoid (a.k.a. algebra) object A^G_H in $\mathcal{C}(G)$ such that the category of A^G_H-modules *within the triangulated category* $\mathcal{C}$ (so in particular we don't use any underlying models here) is canonically equivalent to $\mathcal{C}(H)$:

$$\mathcal{C}(H) \cong A^G_H \text{-}\mathrm{Mod}_{\mathcal{C}}$$

Moreover, the equivalence identifies the restriction functor Res^G_H with the extension of scalars functor $F_A = A^G_H \otimes - \colon \mathcal{C} \to A^G_H\text{-}\mathrm{Mod}$, and the monoid is compact and separable (its multiplication admits a bilinear section), hence we say "finite étale".

This result is closely related to the restriction-coinduction adjunction being monadic (see §3.7.1), a question which is rather orthogonal to those explored in the preceding sections. Our assumptions will also be much weaker that those we had in Hypothesis 3.1.1, and as a consequence we will be able to include examples from KK-theory.

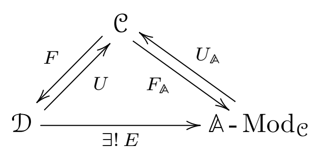
Let us note that, although we will not make it explicit, on can show that from the data of the tensor triangulated category $\mathcal{C}(G)$ and the monoid A^G_H it is possible to recover not only the category $\mathcal{C}(H)$ but also its tensor triangulated structure. It is therefore all the more impressive that, in all examples, the separable monoid A^G_H is extremely simple, basically just a domain-specific variant of the finite dimensional function algebra $k^{G/H}$ equipped with the pointwise multiplication and its evident bilinear section.

3.7.1. Monadicity and monoids. Recall that a monad $\mathbb{A} = (\mathbb{A}, \mu, \eta)$ on a category $\mathcal{C}$ consists of an endofunctor $\mathbb{A} \colon \mathcal{C} \to \mathcal{C}$ together with two natural transformations $\eta \colon \mathrm{id}_{\mathcal{C}} \to \mathbb{A}$ and $\mu \colon \mathbb{A} \circ \mathbb{A} \to \mathbb{A}$ satisfying the associativity and unit axioms with respect to composition:

$$\mu \circ (\mathbb{A}\mu) = (\mathbb{A}\mu) \circ \mu \qquad \mu \circ (\mathbb{A}\eta) = \mathrm{id}_{\mathcal{C}} = \mu \circ (\eta\mathbb{A})\,.$$

A (left) $\mathbb{A}$-*module* is a pair (x, ρ) consisting of an object $x \in \mathcal{C}$ and a map $\rho\colon \mathbb{A}x \to x$ satisfying the evident analogue associativity and unit axiom. Together with the action-preserving maps of $\mathcal{C}$ as morphisms, $\mathbb{A}$-modules form a category $\mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}}$, which comes equipped with a faithful functor $U_{\mathbb{A}}\colon \mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}} \to \mathcal{C}$ forgetting the action and its left adjoint $F_{\mathbb{A}}\colon \mathcal{C} \to \mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}}$ sending an object x to the *free module* $(\mathbb{A}x, \mu_x\colon \mathbb{A}^2x \to \mathbb{A}x)$. This is called the *Eilenberg-Moore adjunction* for $\mathbb{A}$.

If $F : \mathcal{C} \leftrightarrows \mathcal{D} : U$ is any pair of adjoint functors between two categories, with unit $\eta\colon \mathrm{id}_{\mathcal{C}} \to UF$ and counit $\varepsilon\colon FU \to \mathrm{id}_{\mathcal{D}}$, then $\mathbb{A} := (UF, \mu := U\varepsilon F, \eta)$ defines a monad on $\mathcal{C}$, and one says that this monad is *realised by the adjunction* $F \dashv U$. Every monad is realised by some adjunction, for instance by the associated Eilenberg-Moore adjunction. The latter is characterised as being *final* among all adjunctions realising the given monad, in the sense that for any $F \dashv U$ realising $\mathbb{A}$ as above there exists a unique comparison functor $E\colon \mathcal{D} \to \mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}}$ such that $EF = F_{\mathbb{A}}$ and $U_{\mathbb{A}}E = U$:



Concretely, $E(x) = (Ux, U\varepsilon_x)$ on objects and $U(f)$ on morphisms. One can start with the adjunction $F \dashv U$ and construct the above diagram for the associated monad $\mathbb{A} = UF$; if it so happens that the comparison functor E is an equivalence, one says that the adjunction (or the functor U) is *monadic*.

If $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1})$ is a tensor category and $A = (A, m, u)$ is any ring object (monoid) in it, then tensoring with A defines a monad on $\mathcal{C}$ in the evident way. In this case we write $F_A : \mathcal{C} \leftrightarrows \mathcal{A}\text{-}\mathrm{Mod}_{\mathcal{C}} : U_A$ for the Eilenberg-Moore adjunction, where an object is given by a pair $(x, \rho\colon A \otimes x \to x)$ where the action map ρ satisfies the usual unit and associativity axioms with respect to multiplication m and unit u.

Consider now an adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : U$ of additive tensor categories $\mathcal{C}$ and $\mathcal{D}$, and assume that F is a (strong) tensor functor. By adjunction, U inherits a lax monoidal structure $\lambda\colon Ux \otimes Uy \to U(x \otimes y)$, $\iota\colon \mathbb{1}_{\mathcal{D}} \to U\mathbb{1}_{\mathcal{C}}$. In particular we get a natural morphism

$$\pi\colon (Ux) \otimes y \xrightarrow{\mathrm{id} \otimes \eta} (Ux) \otimes (UFy) \xrightarrow{\lambda} U(x \otimes Fy)$$

for every $y \in \mathcal{D}$. Similarly, the lax monoidal functor U sends monoids to monoids, hence in particular it sends $\mathbb{1} \in \mathcal{D}$ to a monoid $A := U\mathbb{1}$ in $\mathcal{D}$.

We thus have two *a priori* different monads on $\mathcal{C}$, namely UF and $A \otimes -$, as well as natural map $A \otimes (-) \to UF$ comparing them (use π with $x = \mathbb{1}$).

The next easy result is a 'separable monoidal' alternative to the Beck monadicity theorem for recognising monadic adjunctions:

3.7.2 Theorem ([12, Thm. 2.9]). *Let $F : \mathcal{C} \leftrightarrows \mathcal{D} : U$ be an adjunction of idempotent-complete additive tensor categories, where F is a tensor functor. Assume moreover that:*

(a) The monad is separable*, i.e. the counit $\varepsilon : FU \to \mathrm{Id}_{\mathcal{D}}$ of the adjunction admits a natural section.*

(b) The projection formula holds*, i.e. the natural map $U(x) \otimes y \overset{\sim}{\to} U(x \otimes F(y))$ defined above is invertible.*

Then the adjunction is monadic and the associated monad is isomorphic to the one induced by the commutative ring object $A = U\mathbb{1}$ in $\mathcal{C}$. Thus there is a (unique) equivalence $E : \mathcal{D} \overset{\sim}{\to} A\text{-}\mathrm{Mod}_{\mathcal{C}}$ identifying the given adjunction $F \dashv U$ with the free-forgetful adjunction $F_A \dashv U_A$. Explicitly, the quasi-inverse $E^{-1} \colon A\text{-}\mathrm{Mod}_{\mathcal{C}} \overset{\sim}{\to} \mathcal{D}$ of E sends a module $(x, \rho \colon A \otimes x \to x)$ to the image of the idempotent map $e^2 = e := (F\rho)(\sigma F x)$, where σ is a section of ε as in (a).

We are now going to apply this theorem to three examples of restriction functors $F = \mathrm{Res}^G_H \colon \mathcal{C} = \mathcal{C}(G) \to \mathcal{C}(H) = \mathcal{D}$ admitting a "co-induction" right adjoint $U = \mathrm{CoInd}^G_H$. In all of them, the natural splitting of $\varepsilon \colon FU \to \mathrm{Id}_{\mathcal{D}}$ passes to the object $U(\mathbb{1})$ (which *a priori* is not automatic) and turns it into a separable ring object. One way to unify this fact is to observe that in all cases, the right adjoint U is also *left* adjoint to restriction, and the section in hypothesis (a) is actually given by the unit $\mathrm{id}_{\mathcal{D}} \to FU$ for this extra adjunction; i.e., $U \dashv F \dashv U$ is a Frobenius adjunction as in [Str04]. Such "ambidextrous" adjunctions, as well as the projection formula, are closely related to the Wirthmüller isomorphism and the projection formulas appearing in the previous sections. But our understanding of tensor-triangulated functors arising from restriction to subgroups is still not mature enough to say how exactly this all fits in a more general theory.

3.7.3. Equivariant stable homotopy. Recall the equivariant stable homotopy categories $\mathrm{SH}(G)$ of "genuine" G-equivariant spectra, already encountered in §3.6.3.

Theorem ([12, Thm 1.1]). *Let G be a compact Lie group and let $H \leqslant G$ be a closed subgroup of finite index. Then the suspension G-spectrum $A^G_H := \Sigma^\infty G/H_+$ is a commutative separable ring object in the equivariant stable homotopy category $\mathrm{SH}(G)$. Moreover, there is an equivalence of categories $\mathrm{SH}(H) \cong A^G_H\text{-}\mathrm{Mod}_{\mathrm{SH}(G)}$ between $\mathrm{SH}(H)$ and the category of left A^G_H-modules in $\mathrm{SH}(G)$ under which the restriction functor $\mathrm{SH}(G) \to \mathrm{SH}(H)$ becomes isomorphic to the extension-of-scalars functor $\mathrm{SH}(G) \to A^G_H\text{-}\mathrm{Mod}_{\mathrm{SH}(G)}$.*

3.7.4. Equivariant KK-theory. Recall the equivariant Kasparov categories of KK^G of separable G-C*-algebras, already encountered in §1.5 and §1.6.

Theorem ([12, Thm. 1.2]). *Let G be a second countable locally compact Hausdorff group and let $H \leqslant G$ be a closed subgroup of finite index. Then the finite-dimensional algebra $A^G_H := \mathbb{C}(G/H)$ is a commutative separable ring object in* *the equivariant Kasparov category* $\mathrm{KK}(G)$ *of* G*-*C^**-algebras. Moreover, there is an equivalence of categories* $\mathrm{KK}(H) \cong A^G_H\text{-}\mathrm{Mod}_{\mathrm{KK}(G)}$ *between* $\mathrm{KK}(H)$ *and the category of left* A^G_H*-modules in* $\mathrm{KK}(G)$ *under which the restriction functor* $\mathrm{KK}(G) \to \mathrm{KK}(H)$ *becomes isomorphic to the extension-of-scalars functor* $\mathrm{KK}(G) \to A^G_H\text{-}\mathrm{Mod}_{\mathrm{KK}(G)}$*.*

3.7.5. Equivariant derived categories. Consider a ringed space $S = (S, \mathcal{O}_S)$ equipped with the action of a discrete group G. The G-equivariant derived category of S, here denoted $\mathrm{D}(G;S)$, is the derived category of the abelian category of equivariant sheaves, an equivariant sheaf on S consisting of a sheaf M of $\mathcal{O}_S$-modules equipped with a family $\varphi_g \colon M \overset{\sim}{\to} g^*M$ $(g \in G)$ of isomorphisms satisfying certain coherence condition. The derived tensor product $\otimes^{\mathrm{L}}_{\mathcal{O}_S}$ turns $\mathrm{D}(G;S)$ into a tensor triangulated category, and for each subgroup $H \leqslant G$ one can easily define a tensor exact restriction functor Res^G_H and a right adjoint CoInd^G_H. If S is a (noetherian) scheme, one can also restrict attention to quasi-coherent $\mathcal{O}_S$-modules, as usual.

3.7.6 Theorem ([12, Thm. 1.3])**.** *Let* G *be a discrete group acting on a ringed space* S *(for instance, a scheme) and let* $H \leqslant G$ *be a subgroup of finite index. Then the free* $\mathcal{O}_S$*-module* $A^G_H := \mathcal{O}_S(G/H)$ *on* G/H *is a commutative separable ring object in the derived category* $\mathrm{D}(G;S)$ *of* G*-equivariant sheaves of* $\mathcal{O}_S$*-modules. Moreover, there is an equivalence of categories* $\mathrm{D}(H;S) \cong A^G_H\text{-}\mathrm{Mod}_{\mathrm{D}(G;S)}$ *between* $\mathrm{D}(H;S)$ *and the category of left* A^G_H*-modules in* $\mathrm{D}(G;S)$ *under which the restriction functor* $\mathrm{D}(G;S) \to \mathrm{D}(H;S)$ *becomes isomorphic to the extension-of-scalars functor* $\mathrm{D}(G;S) \to A^G_H\text{-}\mathrm{Mod}_{\mathrm{D}(G;S)}$*.*

We should note that monadicity certainly does not come for free in these families of examples. For instance, one can show that if G is a compact connected Lie group and $H \leqslant G$ is a non-trivial discrete subgroup, then the right adjoint of the restriction functor Res^G_H is not faithful, hence *a fortiori* not monadic (see [12, Thm. 1.5]).

In general, we morally expect of any such "equivariant context" that the restriction-coinduction adjunction $\mathcal{C}(G) \leftrightarrows \mathcal{C}(H)$ is monadic if G/H is discrete, and that this is implemented by a separable ring object if G/H is moreover compact. But we don't have yet a general complete understanding of this pattern, not least because of the various technical hypotheses currently needed to even be able to construct the various examples of equivariant triangulated categories and the functors between them.

4 Prospectives

I will very briefly mention some of the work in progress, and suggest possible future directions, that would directly continue the works presented here.

Universal coefficient theorems. The techniques of §1.2 should be further exploited in order to produce more examples of UCT's, beyond those described in §1.3-1.4. Ralf Meyer has suggested extending Manuel Köhler's UCT in equivariant KK-theory to the case of a product of two cyclic groups of prime order. Another, perhaps more straightforward, application would be to extend Ralf Meyer and Ryszard Nest's UCT for filtrated KK-theory to cover all cases where the base space is finite with Hasse diagram an ADE Dynkin quiver.

Mackey functors in KK-theory. It would be worthwhile to generalise to compact (Lie) groups the connection of §1.5 between Mackey functors and equivariant KK-theory, similarly to what is known to hold in equivariant stable homotopy. An extension to *locally compact*, or even just infinite discrete, groups would be more subtle, e.g. it is not clear a priori what such Mackey functors should be, and stable homotopy wouldn't be a guide here. But it would also be much more interesting, as it would probably connect with the Baum-Connes conjecture (e.g. it could be used for comparing variants for different families of compact subgroups).

Classifications in derived categories. The theorems of §2.7, together with similar results of Akhil Mathew for E_∞-spectra defined over the rationals, suggest that the classification of thick and localising subcategories of the derived category $\mathrm{D}(A)$ via the Zariski spectrum of the (graded) central ring $\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}) = H^*A$ could perhaps generalise from (graded) noetherian commutative rings to 'algebraic enough' commutative dg rings (or ring spectra) with noetherian cohomology. In any case, and also in view of the known examples where $\mathrm{Spec}^{\mathrm{h}} H^*A$ does *not* account for all thick or localising subcategories, it would be interesting to find the precise point where the 'formal' approach to either classification problem breaks down, and to better understand the reasons for it.

Classifications in Kasparov categories. In the realm of KK-theory, by combining the results of §2.7 with ideas recently used by Paul Balmer and Beren Sanders in equivariant stable homotopy, I believe I can extend the classifications in §2.4.2 for the Rosenberg-Schochet bootstrap category to the category $\mathrm{Cell}^G \subseteq \mathrm{KK}^G$ of G-cell algebras (see §1.5.8) for small cyclic groups G. This should be written down. I also have a fairly good idea on how to reduce the case of a general finite group G to that of cyclic group, so the complete picture for the equivariant KK-theory of finite groups (at least for cell algebras — which is probably all one can hope for anyway) seems to be within reach.

Equivariant triangulated categories. At several occasions throughout this memoir we have encountered triangulated categories arising in families $\{\mathcal{T}(G)\}_G$ indexed by groups G: stable module categories and derived categories of group algebras, equivariant stable homotopy categories, equivariant Kasparov categories, ... The categories in each family are connected by induction, restriction, conjugation and possibly inflation, fixed point and cross product functors, and it seems to me that this kind of rich and extremely useful structure cries out for a conceptual explanation and for an effective organisation, of the kind provided by derivators for categories of diagrams. The work of §1.5 and especially §3.7 should be viewed as fitting within this wider and more ambitious program.

A partial answer to this problem, aiming to provide an effective handling of categorified versions of Mackey and Green functors for finite groups, is current work in progress with Paul Balmer. Another part of the problem, namely to establish the proper universal rôle of equivariant stable homotopy theory among such equivariant families of categories, is currently being investigated jointly with Paul Balmer and Beren Sanders.

5 Author's publications

Article published during PhD thesis:

[1] The Witt groups of the spheres away from two (with Jean Fasel). Journal of Pure and Applied Algebra **212** (2008) 1039–1045.

Article containing main results of PhD thesis:

[2] Tensor triangular geometry and KK-theory. Journal of Homotopy and Related Structures **5** (2010) 319–358.

Articles published after PhD thesis:

[3] Localizing subcategories in the Bootstrap category of separable C*-algebras. Journal of K-Theory **8** (2011) 493-505.

[4] The unitary symmetric monoidal model category of small C*-categories. Homology, Homotopy and Applications **14** (2012) 101–127.

[5] Tensor triangular geometry of noncommutative motives (with Gonçalo Tabuada). Advances in Mathematics **229** (2012) 1329–1357.

[6] Equivariant Kasparov theory of finite groups via Mackey functors. Journal of Noncommutative Geometry **8** (2014) 837–871.

[7] On the derived category of a graded commutative noetherian ring (with Greg Stevenson). Journal of Algebra **373** (2013) 356–376.

[8] Morita homotopy theory of C*-categories (with Gonçalo Tabuada). Journal of Algebra **398** (2014) 162–199.

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