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Discrete Constraints in Control : Discontinuous feedback under Sampled-data and Switching implementations

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Preface

This report presents a selection of the results that I have developed since my recruitment as an Associate Researcher (Chargé de Rercherches - CR) with CNRS (Centre National de la Recherche Scientifique[1]), in October, 2008. The research activities are carried on in the group CO2 (Control and Scientific Computing), team SYNER (Systèmes hybrides, non-linéaires et à retard[2]) of CRIStAL (Centre de Recherche en Informatique, Signal et Automatique de Lille[3] - UMR CNRS 9189). I joined the team SYNER in October 2008 as a 2nd class Associate Researcher (CR2). This team is supervised by Prof. Lotfi Belkoura. Until December 2014, SYNER has been part of LAGIS (Laboratoire d'Automatique, Génie Informatique et Signal[4]) UMR CNRS 8219. On January 1st, 2015, LAGIS merged with LIFL (Laboratoire d'Informatique Fondamentale de Lille[5] - UMR CNRS 8022), creating CRIStAL. In the context of the creation of CRIStAL, SYNER is coordinating its research activities with the teams CFHP (Calcul Formel et Haute Performance[6]) and DEFROST (DEFormable RObotic SofTware) in the group CO2 - supervised by Prof. Jean-Pierre Richard.

The team SYNER addresses a large panel of problems related to the study of time-delay, hybrid dynamical systems and nonlinear systems. The activities of the team can be structured according to two main axes: on one side the members of SYNER develop estimation tools based on the use of differential algebra and operational calculation in the context of the INRIA project NON-A (Non-Asymptotic estimation for online systems). On the other side, the team proposes Lyapunov based methods for analysis and control design. My research activities are mainly concerned with this second axis of SYNER. At the national level, my activities contribute to the working groups on Hybrid Dynamical Systems and Time Delay System of GDR MACS (Groupe de Recherche du CNRS en Modélisation, Analyse et Conduite des Systèmes dynamiques[7]), and the regional research group GRAISYHM (Groupement de Recherche en Automatisation Intégrée

[1] National Center for Scientific Research, a public research organization under the responsibility of the French Ministry of Education and Research.
[2] Hybrid, nonlinear and time-delay systems.
[3] Center of Research on Computer Sciences, Signal Processing and Automatic Control.
[4] Laboratory of Automatic control, Computer Engineering and Signal processing.
[5] A theoretic Computer Science laboratory.
[6] Computer Algebra and High Performance Computing.
[7] A national research group on modelling analysis and control of dynamical systems under the responsibility of CNRS.

et Systèmes Homme-Machine[8]) of Région Hauts-de-France. At the international level, they have contributed to the HYCON Networks of Excellence (Highly-complex and networked control systems - FP6 HYCON and FP 7 HYCON2).

This document presents several contributions that have been obtained in collaboration with Emmanuel BERNUAU (Ass. Prof. Agro Paritech), Michael DEFOORT (Ass. Prof. UVHC, LAMIH), Mohamed DJEMAI (Prof. UVHC, LAMIH), Thierry FLOQUET (DR CNRS, CRIStAL), Emilia FRIDMAN (Prof. Univ. Tel-Aviv), Hisaya FUJIOKA (Ass. Prof. Univ. Kyoto), Alexandre KRUSZEWSKI (Ass. Prof. Centrale Lille, CRIStAL), Francoise LAMNABHI-LAGARRIGUE (DR CNRS, L2S), Silviu-Iulian NICULESCU (DR CNRS, L2S), Wilfrid PERRUQUETTI (Prof. Centrale Lille, CRIStAL), Mihaly PETRECZKY (CR CNRS, CRIStAL), Jean-Pierre RICHARD (Prof. Centrale Lille, CRIStAL), Alexandre SEURET (CR, CNRS, LAAS), and young researchers, PhDs and post-doctoral students, supervised at LAGIS and CRIStAL: Christophe FITER (PhD Centrale Lille, defended in November 2012, now Ass. Prof., Univ. Lille), Hassan OMRAN (PhD Centrale Lille, defended in March 2014, now Ass. Prof., TP Strasbourg), Srinath GOVINDASWAMY (post-doc Centrale Lille, 2012-2013), Romain DELPOUX (ATER Univ. Lille 1, 2013, now Ass. Prof., INSA Lyon). Other results, not mentioned in this document, have been obtained in collaboration with Denis EFIMOV (CR INRIA Non-A), Jamal DAAFOUZ (Prof. Univ. Lorraine, CRAN), Marieke CLOOSTERMAN (PhD, TU Eindhoven), Tijs DONKERS (Ass. Prof. TU Eindhoven), Maurice HEEMELS (Prof. TU Eindhoven), Marc JUNGERS (CR CNRS, CRAN), Ivan MALLOCI (PhD, CRAN), Sorin OLARU (Prof. Centrale SUPELEC Paris, L2S), Worody LOMBARDI (PhD, L2S), Andrey POLYAKOV (CR INRIA Non-A), Christophe PRIEUR (DR CNRS, GIPSA - lab), Patrick SZCZEPANSKI (Arcelor Mittal), Sophie TARBOURIECH (DR CNRS, LAAS), Nathan van de WOUW (Ass. Prof. TU Eindhoven). I would like to thank them all for their fruitful collaboration, dynamism and patience.

I am extremely grateful to Bernard BROGLIATO, Daniel LIBERZON and Luca ZACCARIAN for giving me the honour of reviewing this document, to the members of the committee, Olivier COLOT, Wim MICHIELS and Dimitri PEAUCELLE, for having accepted to participate in the evaluation of my research activity, and to Jean-Pierre RICHARD, for his guidance and support.

I would also like to thank all my colleagues from CRIStAL, INRIA and Centrale Lille who directly or indirectly influenced this work.

Finally, I wish to thank my family for their tremendous support.

[8] Regional group on automatic control and human-machine systems.

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Acronyms

- **IQC** – **I**ntegral **Q**uadratic **C**onstraint
- **LKF** – **L**yapunov-**K**rasovskii **F**unctional
- **LMI**– **L**inear **M**atrix **I**equalities
- **LPV** – **L**inear **P**arameter-**V**arying
- **LTI** – **L**inear **T**ime-**I**nvariant
- **MSI**– **M**aximum **S**ampling **I**nterval
- **NCS** – **N**etworked **C**ontrol **S**ystem
- **PDR** – **P**arameter **D**ependent **R**elay
- **PWM** – **P**ulse-**W**idth **M**odulation
- **SOS** – **S**um **O**f **S**quares
- **ZOH** – **Z**ero-**O**rder **H**old

Notations

- $\mathbb{R}_+$ denotes the set $\{\lambda \in \mathbb{R}, \lambda \geq 0\}$.
- $|c|$ denotes the absolute value of a scalar $c \in \mathbb{R}$.
- $\|x\|$ represents any norm of the vector x.
- $\|x\|_p$, $p \in \mathbb{N}$, denotes the p norm of a vector x.
- For a matrix M, M^T denotes the transpose of M and $M^\star$, its conjugate transpose.
- For square symmetric matrices M, N, $M \succeq N$ (resp. $M \succ N$) means that $M - N$ is a positive semi-definite (resp. definite positive) matrix. $M \preceq N$ (resp. $M \prec N$) means that $M - N$ is a negative semi-definite (resp. negative definite) matrix.
- For a matrix $M \in \mathbb{R}^{n \times n}$, we denote the Hermitian of M by $\mathcal{H}e\{M\} = M + M^T$.
- $*$ in a symmetric matrix represents elements that may be induced by symmetry.
- $\|M\|_p$, $p \in \mathbb{N}$ denotes the induced p-norm of a matrix M.
- $\bar{\sigma}(M)$ denotes the maximum singular value of M.
- $\mathcal{C}^0(X, Y)$, for two metric spaces X and Y, is the set of continuous functions from X to Y.
- $\mathcal{L}_p^n(a, b)$, $p \in \mathbb{N}$ denotes the space of functions $\phi : (a, b) \to \mathbb{R}^n$ with norm $\|\phi\|_{\mathcal{L}_p} = \left[\int_a^b \|\phi(s)\|^p \, ds\right]^{\frac{1}{p}}$.
- $\mathcal{L}_{2e}^n[0, \infty)$ is the space of functions $\phi : [0, \infty) \to \mathbb{R}^n$ which are square integrable on finite intervals.
- Given a set $\mathcal{S} \subset \mathbb{R}^n$, $\text{conv}\{\mathcal{S}\}$ denotes its closed convex hull and $\text{Int}\{\mathcal{S}\}$ its interior.
- For a convex polytope $\mathcal{S} \subset \mathbb{R}^n$ and a scalar $\alpha > 0$, we denote $\alpha\mathcal{S} := \{\alpha x, \ x \in \mathcal{S}\}$ and $\text{vert}\{\mathcal{S}\}$ the set of vertices of $\mathcal{S}$.
- The n dimensional open ball in $\mathbb{R}^n$ centred on $x \in \mathbb{R}^n$ with radius $c > 0$ is denoted $\mathcal{B}(x, c) := \{y \in \mathbb{R}^n : \|x - y\|_2 < c\}$.

General introduction

The practical implementation of control algorithms is always subject to various types of constraints: saturation, limited rate of actuators, digital implementation under quantization and finite sampling frequency, etc. This dissertation is concerned with a fundamental problem in modern control systems: the occurrence of *discrete constraints* in control loops. Two main aspects will be considered. On one side, we will discuss the occurrence of discrete-constraints in the time domain, related to *sampled-data control implementations* and fact that in practice the control action is computed sporadically, at *aperiodic* sampling instants. In this context, the main challenges are to determine the maximum sampling interval which preserves stability and to schedule the sampling instants so as to ensure desired performances. This topic is motivated by the uprising interest in networked and embedded control elements where real-time scheduling algorithms interact with control tasks and where communication and energetic constraints have to be taken into account. On the other side, we will present results concerning the design of feedback laws subject to discrete constraints in the sets of possible control values: the control signal is allowed to take only a finite number of values. Such constraints are typical in systems with *switches*, relays or binary (on-off) actuators. The main challenge here is to *design the switching surfaces* while guaranteeing desired safety constraints in terms of (local) stability. Both of these topics bring up open problems in the domain of hybrid dynamical systems. They involve the study of differential equations with discontinuous right-hand side and of systems with impulsive dynamics.

With respect to the research activity carried in the team SYNER, over the last eight years we have investigated the effect of *aperiodic sampling* on several classes of dynamical systems interacting with sampled-data implementations of both continuous and switching feedback laws. We have tried to address the main challenges in aperiodic sampled-data control using several different approaches. One of the main purposes of our work is to propose numerical tools for addressing the considered problems. We have dedicated some effort to express solutions to the analysis and control design problems in a form that is convenient to the derivation of computer-aided tools. A particular attention is given to the formulation of analysis and synthesis criteria as simple convex optimization problems which can be easily addressed numerically using powerful numerical algorithms.

First, the main contributions in the context of sampled-data systems are briefly presented as follows:

- *New conditions for the stability of linear time invariant (LTI) sampled-data systems with arbitrary time-varying sampling intervals* [Hetel 2011b]. The main idea is to use a discrete-time system model and quasi-quadratic Lyapunov functions previously encountered in the context of polytopic difference inclusions in order to provide stability conditions. The existence of a quasi-quadratic Lyapunov function decreasing at sampling instants is shown to be a necessary and sufficient condition for stability. Using approximations based on convex polytopes leads to sufficient stability criteria. This approach allows a very accurate numerical implementation of algorithms for evaluating the maximum allowable sampling interval which ensures stability.

- *A new framework for the analysis of sampled-data systems inspired by the Dissipativity Theory* [Omran 2014b, Omran 2014a, Omran 2016a]. The idea is to characterize the effect of sampling using "supply" functions. The method generalizes to the case of nonlinear affine systems several frequency domain criteria initially used for LTI systems. The advantage of this approach is its flexibility: the approach can be easily extended in order to take into account more complex performance and robustness specifications.

- *Optimization tools for sampled-data systems with controlled sampling sequences* [Fiter 2012a], [Fiter 2015]. In the literature, aperiodic sampled-data systems had been studied using either continuous-time or discrete-time models. We have proposed a continuous-time approach based on convex embeddings that is able to combine the advantages of the time-delay system modelling (inter-sampling behaviour, robustness to perturbations) with the ones of discrete-time models (accuracy of analysis). This approach has been used for the design of even-/self-triggered control algorithms. We have provided tools for optimizing the sampling maps so as to enlarge the minimum inter-event time between two sampling instants while ensuring desired performance and robustness properties.

In order to transfer our experience over this domain, we have gathered a collection of main results on aperiodic sampled-data systems in an overview of stability analysis approaches which has been presented as a tutorial paper at ECC [Fiter 2014a]. A detailed survey article [Hetel 2017] has been accepted for publication in Automatica.

Second, the document will present a more recent field of our activity: the design of switching surfaces under discrete constraints. While the study of systems with aperiodic sampling has now reached an advanced phase of development, the second main topic of research, the *design of switching surfaces* for systems subject to discrete constraints, represents an emerging research direction in the team SYNER. The design of switching controllers (relays, sliding mode controllers, variable structure systems, etc.) is an old problem in the control theory. However, very few numerical tools exist for optimizing the design of switching surfaces while optimizing the systems performances (domain of attraction, robustness to perturbations and delay, decay rate, etc.). We are currently investigating a recent research direction by addressing this topic from a hybrid system perspective. The main idea of our work is to use a simple convex optimization approach for the design of switching controllers based on Linear Matrix Inequalities (LMIs). We have addressed this problem for LTI, polytopic approximations of nonlinear systems, bilinear systems and switched affine systems. This new method has lead to several journal publications [Hetel 2015c, Hetel 2015a, Delpoux 2015, Hetel 2016]. For the case of linear systems it is shown that the robustness requirements of classical sliding mode controllers can be incorporated in the new design methodology while optimizing the domain of attraction and the robustness with respect to perturbations [Hetel 2015c]. For switched affine systems we provide a new point of view in the design of stabilizing state feedback laws: we show that the design of switching controllers can be re-stated as a classical design problem for nonlinear affine systems [Hetel 2015a]. The method allows to take into account some classes of switched affine system that can be stabilized only locally, on which the existing methods do not apply. Simple control design criteria are proposed for switched affine systems that do not satisfy the classic constraints related to the existence of Hurwitz convex combinations. The new methodology has potential in application to electro-magnetic systems (control of stepper motors [Delpoux 2015]) and energy management problems (DC/DC power converters [Hetel 2016]). The analysis of sampled-data implementations of switching controllers has equally been addressed [Hetel 2013b].

After this general introduction, the rest of this dissertation is organized into two major parts and a conclusion.

Part I deals with discrete constraints in the time domain. It is mainly concerned with the stability problem for sampled-data systems with aperiodic sampling. After presenting some generalities concerning systems with time-varying sampling in Chapter 1, the second chapter gives a overview of the literature on the field. Chapter 3 presents our main contributions to this topic. Our research effort has been dedicated to the analysis of various classes of systems (linear time invariant, polytopic, bilinear, polynomial, nonlinear affine, etc.) with both continuous

and switching controllers. We have tried to address the stability problem from different angles, through various competing methods. In this manuscript, a selection of the most significant results is given. For linear time invariant systems, we show in Chapter 3.1 how numerically efficient conditions can be derived using the exact system discretization and convex embeddings. Numerical tools for the optimization of (event/self-triggered) sampling maps are proposed, based on the used of Linear Matrix Inequalities (LMIs). In a more general context of bilinear (Chapter 3.2) and nonlinear affine systems (Chapter 3.3), we propose a new stability analysis framework inspired by Dissipativity Theory. Control design tools are presented for LTI systems with discontinuous controllers using a time-delay approach in Chapter 3.4.

Part II presents new results for systems with inputs constrained to a finite set of values. Chapter 4 deals with the design of switching controllers for linear systems and some approximations of nonlinear systems as linear polytopic systems. The case of switched affine systems is discussed in Chapter 5, while Chapter 6 presents results concerning bilinear systems. The potential of the approach is illustrated at the end of this part through experimental applications concerning the control of stepper motors and DC/DC power converters.

A conclusion summarizes the main results presented in this document. Finally, several ongoing research directions and open problems are presented.

Part I

Contributions to aperiodic sampled-data control

The last decade has witnessed an enormous interest in the study of networked and embedded control systems [Zhang 2001c, Hristu-Varsakelis 2005, Hespanha 2007, Chen 2011]. This interest is mainly due to the ubiquitous presence of embedded controllers in relevant application domains and the growing demand in industry on systematic methods to model, analyse and design systems where sensor and control data are transmitted over a digital communication channel. The study of systems with aperiodic sampling emerged as a modelling abstraction which allows to understand the behaviour of Networked Control Systems (NCS) with sampling jitters, packet drop-outs or fluctuations due to the inter-action between control algorithms and real-time scheduling protocols [Zhang 2001c, Antsaklis 2007, Astolfi 2008]. With the emergence of event-based and self-triggered control techniques [Heemels 2012], the study of aperiodic sampled-data systems constitutes nowadays a very popular research topic in control.

In this part, we focus on questions arising in the control of systems with time-varying sampling intervals. Important practical questions such as the choice of the minimal sampling bandwidth, the evaluation of necessary computational and energetic resources or the robust control synthesis are mainly related to stability issues. These issues often lead to the problem of estimating the Maximum Sampling Interval (MSI) for which the stability of a closed-loop sampled data system is ensured.

The study of aperiodic sampled-data systems has been addressed in several areas of research in Control Theory. Systems with aperiodic sampling can be seen as particular *time-delay systems*. Sampled-and-hold in control and sensor signals can be modelled using *hybrid systems* with impulsive dynamics. Aperiodic sampled-data systems have also been studied in the *discrete-time* domain. In particular, Linear Time Invariant (LTI) sampled-data systems with aperiodic sampling have been analysed using discrete-time *Linear Parameter Varying (LPV) models*, typically used in gain scheduling control. The effect of sampling can be modelled using *operators* and the stability problem can be addressed in the framework of Input/Output interconnections as typically done in modern *Robust Control*. While significant advances on this subject have been in the literature, problems related to both the fundamentals of such systems and the derivation of constructive methods for stability analysis remain open, even for the case of linear system.

The rest of this part is structured: Chapter 1 is dedicated to generalities concerning aperiodic sampled-data control. A state of the art on aperiodic sampled-data control will be given in Chapter 2 followed by our main contributions in Chapter 3.

Chapter 1

Generalities

1.1 System configuration

As follows we will study the properties of sampled-data systems consisting of a plant, a digital controller, and appropriate interface elements. A general configuration of such a sampled-data system is illustrated by the block diagram of Figure 1.1. In this configuration, $y(t)$ is a continuous-time signal representing the plant output (the plant variables that can be measured). This signal is represented as a function of time t, $y : \mathbb{R}_+ \to \mathbb{R}^p$.

The digital controller is usually implemented as an algorithm on an embedded computer. It operates with a sampled version of the plant output signal, $\{y_k\}_{k\in\mathbb{N}}$, obtained upon the request of a sampling trigger signal at discrete *sampling instants* t_k and using an analog-to-digital converter (the sampler block, S, in Figure 1.1). This trigger may represent a simple clock, as in the classical periodic sampling paradigm, or a more complex scheduling protocol which may take into account the sensor signal, a memory of its last sampled values, etc. The sampling instants are described by a monotone increasing sequence of positive real numbers $\sigma = \{t_k\}_{k\in\mathbb{N}}$ where

$$t_0 = 0,\ t_{k+1} - t_k > 0, \lim_{k\to\infty} t_k = \infty. \tag{1.1}$$

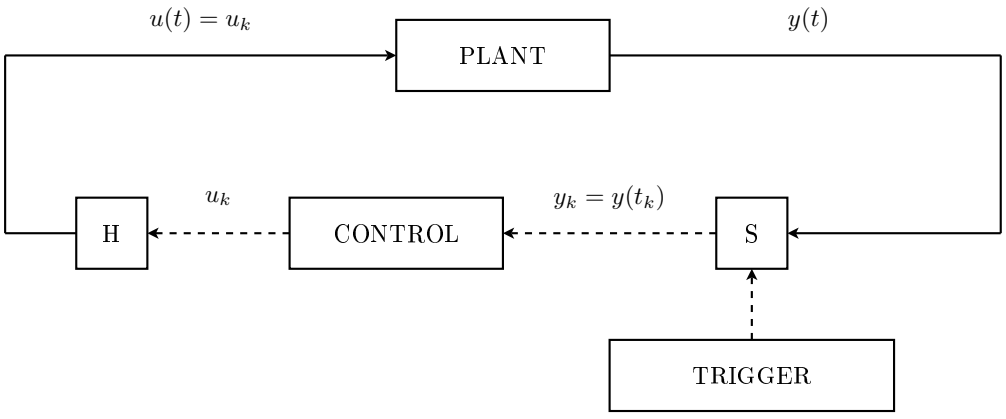


Figure 1.1: Classical sampled-data system configuration

The difference between two consecutive sampling times $h_k = t_{k+1} - t_k$ is called the k^{th} *sampling interval*. Assuming that the effect of quantizers may be neglected, the sampled version of the plant output is the sequence $\{y_k\}_{k\in\mathbb{N}}$ where $y_k = y(t_k)$.

In a sampled-data control loop, the digital controller produces a sequence of control values $\{u_k\}_{k\in\mathbb{N}}$ using the sampled version of the plant output signal $\{y_k\}_{k\in\mathbb{N}}$. This sequence is converted into a continuous-time signal $u(t)$, where $u : \mathbb{R}_+ \to \mathbb{R}^m$ (corresponding to the plant input) via a digital-to-analog interface. We consider that the digital-to-analog interface is a zero-order hold (the hold block, H, in Figure 1.1). Furthermore, we assume that there is no delay between the sampling instant t_k and the moment the control u_k (obtained based on the k^{th} plant output sample, y_k) is effectively implemented at the plant input. Then the input signal $u(t)$ is a piecewise constant signal $u(t) = u(t_k) = u_k, \forall t \in [t_k, t_{k+1})$.

Over the chapter, we will consider that the plant is modelled by a finite dimensional ordinary differential equation of the form

$$\left\{ \begin{array}{rcl} \dot{x} & = & F\left(t, x, u\right), \\ y & = & H\left(t, x, u\right), \end{array} \right. \tag{1.2}$$

where $x \in \mathbb{R}^n$ represents the plant state-variable. Here $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ with $F(t, 0, 0) = 0, \forall t \geq 0$, and $H : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$. It is assumed that for each constant control and each initial condition $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ the function F describing the plant model (1.2) is such that a unique solution exists for an interval $[t_0, t_0 + \epsilon)$ with ϵ large enough with respect to the maximum sampling interval. The discrete-time controller is considered to be described by an ordinary difference equation of the form

$$\left\{ \begin{array}{rcl} x^c_{k+1} & = & F^c_d\left(k, x^c_k, y_k\right), \\ u_k & = & H^c_d\left(k, x^c_k, y_k\right), \end{array} \right. \tag{1.3}$$

where $x^c_k \in \mathbb{R}^{n_c}$ is the controller state. Here, $F^c_d : \mathbb{N} \times \mathbb{R}^{n_c} \times \mathbb{R}^p \to \mathbb{R}^n_c$ and $H^c_d : \mathbb{N} \times \mathbb{R}^{n_c} \times \mathbb{R}^p \to \mathbb{R}^m$. We will use the denomination *sampled-data system* for the interconnection between the continuous-time plant (1.2) with the discrete-time controller (1.3) via the relations

$$y_k = y(t_k),\ u(t) = u_k, \forall t \in [t_k, t_{k+1}),\ \forall k \in \mathbb{N}, \tag{1.4}$$

under a sequence of sampling instants $\sigma = \{t_k\}_{k\in\mathbb{N}}$ satisfying (1.1).

The different concepts and results will be mostly illustrated on Linear Time Invariant (LTI) models

$$\dot{x} = Ax + Bu, \tag{1.5}$$

under a static linear state feedback,

$$u_k = Kx_k,\ k \in \mathbb{N}, \tag{1.6}$$

with $x_k = x(t_k)$. However, when possible, we will present the extensions to more general nonlinear systems.

1.2 Classical design methods

There are various approaches for the design of a sampled-data controller (1.3) (see the classical textbooks [Åström 1996, Chen 1993] and the tutorial papers [Monaco 2007, Monaco 2001, Nešić 2001, Laila 2006]).

Emulation. The simplest approach consists in designing first a continuous-time controller

using classical methods [Khalil 2002, Isidori 1995, Krstic 1995, Sastry 1999]. Next, a discrete-time controller of the form (1.3) is obtained by integrating the controller solutions over the interval $[t_k, t_{k+1})$.This approach is usually called *emulation*. Generally, it is difficult to compute in a formal manner the exact discrete-time model and approximations must be used [Monaco 2007, Laila 2006]. In the LTI case (1.5) with state feedback (1.6), the emulation simply means that the gain K is set such that the matrix $A + BK$ is Hurwitz and that the plant is driven by the control $u(t) = Kx(t_k), \forall t \in [t_k, t_{k+1}),\ k \in \mathbb{N}$. While the intuition seems to indicate that for sufficiently small sampling intervals the obtained sampled-data control gives an approximation of the continuous-time control problem, no guarantee can be given when the sampling interval increases, even for constant sampling intervals. In order to compensate the effect of controller discretisation, *re-design methods* may be used [Grüne 2008, Nešić 2005].

Discrete-time design. In this framework, a discrete-time model of the plant (1.2) is derived by integration. The obtained model represents the evolution of the plant state $x(t_k) = x_k$ at sampling times[9]. Then, a discrete-time controller (1.3) is designed using the obtained discrete-time model. In the simplest LTI case (1.5), (1.6), the evolution of the state between two consecutive sampling instants t_k and t_{k+1} is given by

$$x(t) \quad = \quad \Lambda(t - t_k)x(t_k),\ \forall t \in [t_k, t_{k+1}],\ k \in \mathbb{N}, \tag{1.7}$$

with a matrix function Λ defined on $\mathbb{R}$ as

$$\Lambda(\theta) = A_d(\theta) + B_d(\theta)K = \mathrm{e}^{A\theta} + \int_0^\theta \mathrm{e}^{As} ds BK. \tag{1.8}$$

Evaluating the closed-loop system's evolution at $t = t_{k+1}$ and using the notation $h_k = t_{k+1} - t_k$ leads to the linear difference equation

$$x_{k+1} = \Lambda(h_k)x_k,\ \forall k \in \mathbb{N} \tag{1.9}$$

representing the closed-loop system at sampling instants. When the sampling interval is constant, $h_k = T,\ \forall k \in \mathbb{N}$, a large variety of discrete-time control design methodologies is available in the literature (see [Åström 1996, Chen 1993] and the references within). It is well known for this case that system (1.9) is asymptotically stable if and only if the matrix $\Lambda(T)$ is Schur. In other words, to design a stabilizing control law (1.6), the matrix K must be set such as all the eigenvalues of $\Lambda(T)$ lay strictly in the unit circle.

For nonlinear systems with constant sampling intervals, an overview of control design methodologies and related issues can be found in [Monaco 2007, Monaco 2001, Nešić 2001, Laila 2006]. Note that the discrete-time models such as (1.9) do not take into consideration the inter-sampling behaviour of the system. Relations between the performances of the discrete-time model and the performances of the sampled-data loop, can be deduced using the methodology proposed in [Nešić 1999].

Sampled-data design. Infinite dimensional discrete-time models which take into account the inter-sampling system behaviour using *signal lifting* [Bamieh 1992, Bamieh 1991, Tadmor 1992, Toivonen 1992a, Yamamoto 1994] have been proposed in the literature for the case of linear systems. Specific design methodologies, that are able to take in consideration continuous-time

[9] Note that generally approximations of the system model must be used since the discretized plant model is difficult to compute formally [Monaco 1985, Veliov 1997]. Even for the case of LTI systems with constant sampling intervals, the numerical computation of the matrix exponential (or its integral) is subject to approximations [Moler 2003].

system performances, inter-sample ripples and robustness specifications, can be found in the textbook [Chen 1993] for the case of linear time invariant systems with periodic sampling.

1.3 Complex phenomena in aperiodic sampling

While in the last fifty years an intensive research has been dedicated to the analysis and design of sampled-data systems under periodic sampling, the study of systems with time-varying sampling intervals is quite underdeveloped compared to the periodic conterpart. The following examples illustrate the rich complexity of phenomena that may occur under aperiodic sampling.

Example 1.1 *[Zhang 2001a] Consider an LTI sampled-data system of the form* (1.5),(1.6) *where*

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}, \quad K = -\begin{bmatrix} 1 & 6 \end{bmatrix}. \tag{1.10}$$

For this example, system's (1.9) *transition matrix* $\Lambda(T)$ *is a Schur matrix for any constant sampling interval in* $T \in \mathcal{T} = \{T_1, T_2\}$*, with* $T_1 = 0.18$*, and* $T_2 = 0.54$*. Then, in the case of periodic sampling, the sampled-data system is stable for constant sampling intervals taking values in* $\mathcal{T}$*. An illustration of the system's evolution for constant sampling intervals* T_1*,* T_2*, is given in Figure 1.2. Clearly, when the sampling interval* h_k *is arbitrarily varying in* $\mathcal{T}$*, the Schur property of* $\Lambda(T)$*,* $\forall\ T \in \mathcal{T}$*, represents a necessary condition for stability of the sampled-data system* (1.1),(1.5),(1.6)*. However, it is not a sufficient one. For example, the sampled-data system with a sequence of periodically time-varying sampling intervals* $\{h_k\}_{k\in\mathbb{N}} = \{T_1, T_2, T_1, T_2, \ldots\}$ *is unstable, as it can be seen in Figure 1.3. This is due to the fact that the Schur property of matrices is not preserved under matrix product (i.e. the product of two Schur matrices is not necessarily Schur). Indeed, the discrete-time system representation over two sampling instants can be written as*

$$x_{k+2} = \Lambda(T_2)\Lambda(T_1)x_k,\ \forall k \in 2\mathbb{N},$$

and the transition matrix

$$\Lambda(T_2)\Lambda(T_1) = \begin{bmatrix} 0.8069 & -3.2721 \\ 0.6133 & -2.1125 \end{bmatrix}$$

over two sampling intervals T_1 *and* T_2*, is not Schur. This example shows the importance of taking into consideration the evolution of the sampling interval* h_k *when analysing the stability of sampled-data systems since arbitrary variations of the sampling interval* h_k *may induce instability.*

Example 1.2 *[Gu 2003a] Consider now an LTI system with*

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\ K = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{1.11}$$

Assume that the sampling interval h_k *is restricted to the set* $\mathcal{T} = \{T_1, T_2\}$ *with* $T_1 = 2.126$ *and* $T_2 = 3.950$*. The system is unstable for both constant sampling intervals* T_1 *and* T_2 *since for these values system's* (1.9) *transition matrix* $\Lambda(T)$*,* $T \in \mathcal{T}$ *is not a Schur matrix. However, the product of transition matrices* $\Lambda(T_1)\Lambda(T_2)$ *has the Schur property. Therefore, the sampled data system is stable under a periodic evolution of the sampling interval* $\{h_k\}_{k\in\mathbb{N}} = \{T_1, T_2, T_1, T_2, \ldots\}$*. An example of system evolution with this particular sampling sequence is provided in Figure 1.4. In this example the sampling* h_k *can act on the sampled-data system as a second control parameter*

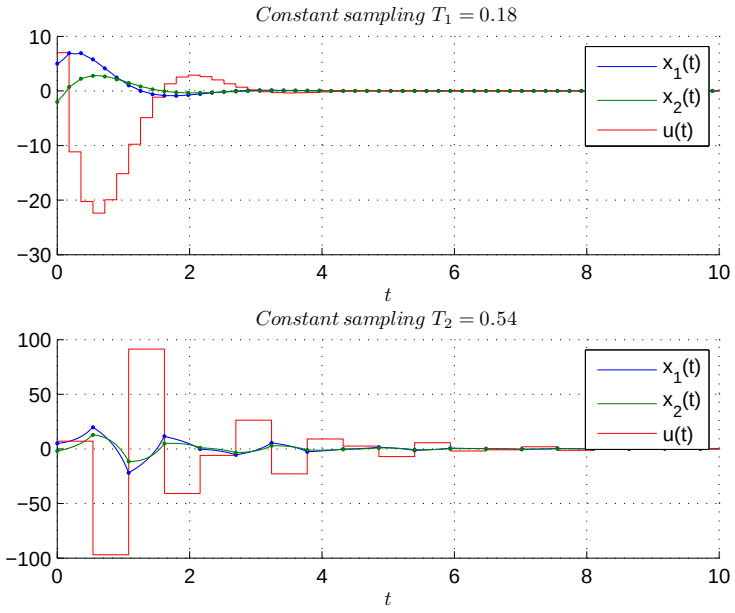


Figure 1.2: Stability of the system in Example 1.1 with periodic sampling intervals.

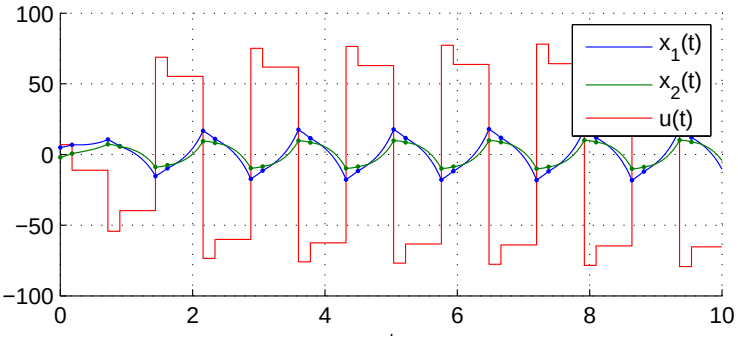


Figure 1.3: Instability of the system in Example 1.1 with a periodic sampling sequence $T_1 \to T_2 \to T_1 \cdots$.

which ensures the system's stability while the possible constant sampling configurations are not able to guarantee this property.

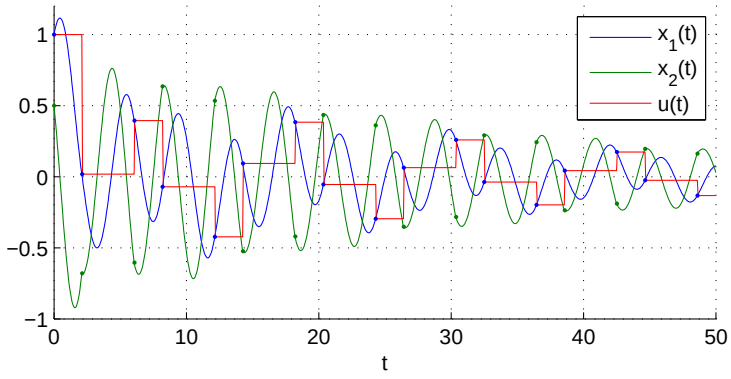


Figure 1.4: Periodic sampling sequence with a stable behaviour.

1.4 Problem set-ups

The core of Part I is dedicated to the *robust analysis* of sampled-data systems with sampling sequences of the form (1.1) where the sampling interval $h_k = t_{k+1} - t_k$ takes arbitrary values in some interval $\mathcal{T} = [\underline{h}, \overline{h}] \subset \mathbb{R}_+$. This first problem set-up may correspond, for example, to the sampling triggering mechanism from Figure 1.1 with a clock submitted to jitter [Wittenmark 1995], or with some scheduling protocol which is too complex to be modelled explicitly [Zhang 2001c, Hespanha 2007]. Basically, for the case of LTI models (1.5) with linear state feedback (1.6) under a sampling sequence (1.1) we will address the robust stability of the closed-loop system (1.12) given below

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + BKx(t_k),\ \forall t \in [t_k, t_{k+1}),\ \forall k \in \mathbb{N}, \\ t_{k+1} & = & t_k + h_k,\ \forall k \in \mathbb{N}, \\ t_0 & = & 0,\ x(t_0) = x_0 \in \mathbb{R}^n \end{array} \right. \tag{1.12}$$

as if h_k is a time-varying "perturbation" taking values in a bounded set $\mathcal{T}$.

We will also indicate some ideas concerning a recently emerging research topic where the sampling interval h_k plays the role of a control parameter that may be changed according to the plant state or output. This problem set-up corresponds to the design of a scheduling mechanism. For the case of system (1.12), h_k is considered as an additional input which, by an appropriate open/closed-loop choice, can ensure the system stability. In the following chapter, we will present an overview of approaches addressing these problems.

Chapter 2

State of the art on aperiodic sampled-data systems

This chapter presents basic concepts and recent research directions about the stability of sampled-data systems with aperiodic sampling[10]. We focus mainly on the stability problem for systems with arbitrary time-varying sampling intervals which has been addressed in several areas of research in Control Theory. Systems with aperiodic sampling can be seen as time-delay systems, hybrid systems, Input/Output interconnections, discrete-time systems with time-varying parameters, etc. The goal is to provide a structural overview of the progress made on the stability analysis of systems with aperiodic sampling. Without being exhaustive, which would be neither possible nor useful, we try to bring together results from diverse communities and present them in a unified manner. For each of the existing approaches the basic concepts, fundamental results and relations with the other approaches are discussed in detail. Results concerning extensions of Lyapunov and frequency domain methods for systems with aperiodic sampling are recalled, as they allow to derive constructive stability conditions. Furthermore, numerical criteria are presented while indicating the sources of conservatism, the problems that remain open and the possible directions of improvement. At last, some emerging research directions, such as the design of stabilizing sampling sequences, are briefly discussed.

2.1 Stability analysis under arbitrary time-varying sampling

In the following, we review some results which provide a qualitative estimation of the maximum sampling interval ensuring stability for sampled-data systems with sampling intervals that are arbitrary varying. More formally, over the section, we present results that address the following problem:

- **Problem A** (Arbitrary sampling problem): Consider the sampled-data system (1.1), (1.2), (1.3), (1.4) and a bounded subset $\mathcal{T} \subset \mathbb{R}_+$. Determine if the sampled-data system is stable (in some sense) for any arbitrary time-varying sampling interval $h_k = t_{k+1} - t_k$ with values in $\mathcal{T}$.

Often the set $\mathcal{T}$ is considered of the form $\mathcal{T} = (0, \overline{h}]$ where $\overline{h}$ is some positive scalar. The largest value of $\overline{h}$ for which the stability of the closed loop system is ensured is called Maximum Sampling Interval (MSI).

[10] The material presented in this chapter is part of a survey paper accepted for publication in Automatica [Hetel 2017].

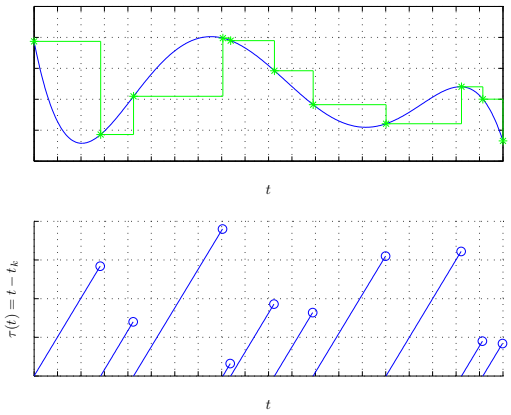


Figure 2.1: Sampling seen as a piecewise-continuous time-delay

Several perspectives for addressing Problem A exist. First, we present results that are based on a *time-delay modelling* of the sampled-data system (1.1),(1.2),(1.3),(1.4). Next, we show how the problem can be addressed from the point of view of *hybrid systems*. We continue with approaches that use the explicit system integration in-between successive sampling instants, such as the ones classically used in the *discrete-time* framework. Last, results addressing Problem A from the *robust control* theory point of view are presented.

2.1.1 Time-delay approach

To the best of our knowledge, this technique was initiated in [Mikheev 1988, Åström 1989], and further developed in [Fridman 1992, Teel 1998b, Louisell 2001] and in several other works. For the case of an LTI system with sampled-data state feedback (1.12), we may re-write

$$\begin{array}{l} u(t) = Kx(t_k) = Kx(t - \tau(t)), \\ \tau(t) = t - t_k, \quad \forall t \in [t_k, t_{k+1}), \end{array} \tag{2.1}$$

where the delay is piecewise-linear, satisfying $\dot{\tau}(t) = 1$ for $t \neq t_k$, and $\tau(t_k) = 0$. This delay indicates the time that has passed since the last sampling instant. An illustration of a typical delay evolution is given in Fig. 2.1. The LTI system with sampled-data (1.12) is then re-modeled as an LTI system with a time-varying delay

$$\dot{x}(t) = Ax(t) + BKx(t - \tau(t)), \ \forall t \geq 0. \tag{2.2}$$

This permits to adapt the tools for stability of systems with *fast varying delays* [Fridman 2003, Gu 2003b, Richard 2003, Niculescu 2004]. This model is equivalent to the original sampled-data system when considering that the sampling induced delay has a known derivative $\dot{\tau}(t) = 1$, for all $t \in [t_k, t_{k+1}), k \in \mathbb{N}$.

2.1.1.1 Basic results

For system (2.2) it is natural to consider, as a state variable, the functional $x_t(\theta) = x(t+\theta)$, $\forall \theta \in [-\bar{h}, 0]$, and, as state space, the set $\mathcal{C}^0\left(\left[-\overline{h}, 0\right], \mathbb{R}^n\right)$ of continuous functions mapping the interval

$\left[-\overline{h}, 0\right]$ into $\mathbb{R}^n$ [Fridman 2014, Niculescu 2001, Niculescu 1998]. In the general case of time-delay systems, it is difficult to apply the classical Lyapunov stability theory, because the The most popular generalization of the direct Lyapunov method for time-delay system has been proposed by Krasovskii [Krasovskiĭ 1963]. It uses the existence of *functionals* $V(t, x_t)$ depending on the state vector x_t. In the sampled-data case [Fridman 2004, Fridman 2010, Liu 2012a] functionals $V(t, x_t, \dot{x}_t)$ depending both on x_t and $\dot{x}_t$ (see [Kolmanovskii 1992], p.337) are useful.

Denote by $W[-\overline{h}, 0]$ the Banach space of absolutely continuous functions $\phi : [-\overline{h}, 0] \to \mathbb{R}^n$ with $\dot{\phi} \in \mathcal{L}_2^n(-\overline{h}, 0)$ (the space of square integrable functions) with the norm

$$\|\phi\|_W = \max_{s\in[-\overline{h},0]} \|\phi(s)\| + \left[\int_{-\overline{h}}^{0} \left\|\dot{\phi}(s)\right\|^2 ds\right]^{\frac{1}{2}}.$$

Theorem 2.1 (Lyapunov-Krasovskii Theorem) *[Kolmanovskii 1992] Consider $f : \mathbb{R}_+ \times \mathcal{C}^0[-\overline{h}, 0] \to \mathbb{R}^n$ continuous in both arguments and locally Lipschitz in the second argument. Assume that $f(t, 0) = 0$ for all $t \in \mathbb{R}_+$ and that f maps $\mathbb{R}\times$ (bounded sets in $\mathcal{C}^0[-\overline{h}, 0]$) into bounded sets of $\mathbb{R}^n$. Suppose that $\alpha, v, w : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous nondecreasing functions, $\alpha(s)$, $\beta(s)$ and $\gamma(s)$ are positive for $s > 0$, $\lim_{s\to\infty} \alpha(s) = \infty$ and $\alpha(0) = \beta(0) = 0$. The trivial solution of*

$$\dot{x}(t) = f\left(t, x_t\right)$$

is Globally Uniformly Asymptotically Stable if there exists a continuous functional $V : \mathbb{R} \times W[-\overline{h}, 0] \times \mathcal{L}_2^n(-\overline{h}, 0) \to \mathbb{R}_+$, which is positive-definite, i.e.

$$\alpha(\|\phi(0)\|) \leq V(t, \phi, \dot{\phi}) \leq \beta(\|\phi\|_W)$$

for all $\phi \in W[-\overline{h}, 0], t \in \mathbb{R}_+$, and such that its derivative along the system's solutions is non-positive

$$\dot{V}(t, x_t, \dot{x}_t) \leq -\gamma(\|x_t(0)\|). \tag{2.3}$$

The functional V satisfying the conditions of Theorem 2.1 is called a *Lyapunov-Krasovskii Functional* (LKF). In the general case of sampled-data nonlinear systems the underlying delay system $\dot{x} = f(t, x_t)$ used in Theorem 2.1 from [Kolmanovskii 1992] is described by a function f which is piecewise continuous with respect to t. However, the proof of the result in [Kolmanovskii 1992] can be adapted to cover this case.

2.1.1.2 Constructive stability conditions

Various generalisations of the Lyapunov-Krasovskii theorem have been proposed in the literature. For the case of sampled-data systems, in [Fridman 2004] the Lyapunov-Krasovskii Theorem was extended to linear systems with a discontinuous sawtooth delay by use of Barbalat lemma. Another extension to linear sampled-data systems has been provided in [Fridman 2010], where the LKF is allowed to have discontinuities at sampling times. It leads to an LKF of the form [Fridman 2010]:

$$V(t, x(t), \dot{x}_t) = x^T(t)Px(t) + (h_k - \tau(t)) \int_{t-\tau(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds \tag{2.4}$$

which improves the results from [Fridman 2004], as the information $\dot{\tau} = 1$ can be explicitly taken into account when evaluating its derivative.

Theorem 2.2 *[Fridman 2010] Let there exist $P \succ 0$, $R \succ 0$, P_2 and P_3 such that the LMI*

$$\begin{bmatrix} \Phi_s & P - P_2^T + (A+BK)^T P_3 \\ * & -P_3 - P_3^T + \overline{h} R \end{bmatrix} \prec 0 \tag{2.5}$$

$$\begin{bmatrix} \Phi_s & P - P_2^T + (A+BK)^T P_3 & -\overline{h} P_2^T A \\ * & -P_3 - P_3^T & -\overline{h} P_3^T A \\ * & * & -\overline{h} R \end{bmatrix} \prec 0 \tag{2.6}$$

with $\Phi_s = P_2^T(A+BK) + (A+BK)^T P_2$, are feasible. Then system (1.12) is Exponentially Stable for all sampling sequences $\sigma = \{t_k\}_{k\in\mathbb{N}}$ with $h_k = t_{k+1} - t_k \leq \bar{h}$.

The result takes into account information about the sawtooth shape of the delay, which is the specificity of the time-delay model (2.2) when representing exactly the sampled-data system (1.12). It can ensure the stability for time-varying delays $\tau(t)$ which are longer than any constant delay that preserves stability, provided that $\dot{\tau}(t) = 1$. See also [Seuret 2009] for an alternative LMI formulation.

2.1.1.3 An extension to nonlinear systems

Concerning nonlinear systems, [Mazenc 2013a] has extended the ideas in [Fridman 2004] for the case of control affine non-autonomous systems with sampled-data control. Consider the nonlinear system:

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t))u(t), \tag{2.7}$$

with the state $x(t) \in \mathbb{R}^n$ and the input $u(t) \in \mathbb{R}^m$, and with functions f, g that are locally Lipschitz with respect to x and piecewise continuous in t. Assume that the $\mathcal{C}^1$ controller $u(t) = K(t,x)$ is designed in order to make the system (2.7) Globally Uniformly Asymptotically Stable. Moreover, assume that there exist a $\mathcal{C}^1$ positive definite and radially unbounded function V, and a continuous positive definite function W such that:

$$-\Big[\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(f(t,x) + g(t,x)K(t,x))\Big] \geq W(x) \tag{2.8}$$

for all $t \geq t_0$ and $x \in \mathbb{R}^n$. Also, consider $K(t,0) = 0$ for all $t \in \mathbb{R}$. Hence, V is a strict Lyapunov function for

$$\dot{x} = f(t,x) + g(t,x)K(t,x)$$

and one can fix class $\mathcal{K}_\infty$ functions α_1 and α_2 such that $\alpha_1(\|x\|_2) \leq V(t,x) \leq \alpha_2(\|x\|_2)$, for all $t \geq t_0$ and $x \in \mathbb{R}^n$. Define the function

$$\rho(t,x) = \frac{\partial K}{\partial t}(t,x) + \frac{\partial K}{\partial x}\Big(f(t,x) + g(t,x)K(t,x)\Big). \tag{2.9}$$

Theorem 2.3 *(adapted from [Mazenc 2013a]) Suppose that there exist constants c_1, c_2, c_3 and c_4 such that:*

$$\left\|\frac{\partial K}{\partial x}(t,x)g(t,x)\right\|_2^2 \leq c_1, \left\|\frac{\partial V}{\partial x}(t,x)g(t,x)\right\|_2^2 \leq c_2,$$

$$\|\rho(t,x)\|_2^2 \leq c_3 W(x),$$

$$\left\|\frac{\partial V}{\partial x}(t,x)g(t,x)K(t,x)\right\|_2 \leq c_4(V(t,x)+1),$$

hold for all $t \geq t_0$ *and* $x \in \mathbb{R}^n$. *Consider the system* (2.7) *in closed-loop with:* $u(t) = K(t_k, x(t_k))$, $t \in [t_k, t_{k+1})$, $\sigma = \{t_k\}_{k\in\mathbb{N}}$ *as defined in* (1.1) *and* $h_k = t_{k+1} - t_k \in [\underline{h}, \overline{h}]$, $\forall k \in \mathbb{N}$. *Then, the closed-loop system is Globally Uniformly Asymptotically Stable if* $\overline{h} \leq (4c_1 + 8c_2c_3)^{-1/2}$.

The stability is proven by means of a Lyapunov functional of the form

$$U(t, x_t) = V\left(t, x(t)\right) + \frac{\epsilon}{\overline{h}} \int_{-\overline{h}}^{0} \int_{t+\theta}^{t} \|\Psi(s, x_s)\|_2^2 \, ds d\theta,$$

where

$$\Psi(t, x_t) = \frac{\partial K}{\partial t}\left(t, x_s(0)\right) + \frac{\partial K}{\partial x}\left(t, x_t(0)\right) \dot{x}_t(0).$$

This functional is reminiscent of the form (2.4) used in [Fridman 2004] to study LTI systems. However, differently from the LTI case, it is far more complex to determine how conservative the result is.

2.1.1.4 Further reading

The research on LKFs for sampled-data system is still a wide-open domain. Currently, an important effort is dedicated to finding better LKFs and better over-approximations of the derivatives. Note that the derivation of constructive stability conditions may be quite an elaborate analytical process and it is not always very intuitive. However, a notable advantage of this methodology is the fact that for linear systems the approach can be easily extended to control design [Fridman 2004, Suplin 2007, Liu 2012a] and to the case of systems with parameter uncertainties [Fridman 2010, Seuret 2012, Orihuela 2010, Gao 2010, Peng 2011], delays [Suplin 2009, Mazenc 2012, Gao 2008, Mazenc 2013b, Seuret 2011, de Wouw 2010] and schedulling protocols [Liu 2012b,Liu 2015b,Liu 2015a]. See also [Fridman 2012,Fridman 2013] for the control of semilinear 1-D heat equations.

Aside from the Lyapunov-Krasovskii method, the stability of sampled-data systems can also be analysed using the method proposed by Razumikhin [Razumikhin 1956]. Connections between Razumikhin's method and the *ISS* nonlinear small gain theorem [Sontag 1998] have been established in [Teel 1998a]. This relation has been used in [Teel 1998b] in order to show the preservation of ISS properties under sufficiently fast sampling for nonlinear systems with an emulated sampled-data controller. Razumikhin's method has been used in [Fiter 2012a] for the case of LTI sampled-data systems. In [Karafyllis 2009b], the Razumikhin method is explored for nonlinear sampled-data system on the basis of vector Lyapunov-Razumikhin Functions (LRF). For more general extensions to the control design problem, see [Karafyllis 2012a], concerning the case of nonlinear feed-forward systems and [Karafyllis 2012b], for nonlinear sampled-data system with input delays. At last, we would like to mention the *Input/Output* approach for the analysis of time-delay systems [Fu 1998,Gu 2003a,Kao 2004], which makes use of classical robust control tools [Zhou 1996,Megretski 1997]. The application of the Input/Output approach for the case of sampled-data systems has been discussed in [Mirkin 2007, Liu 2010, Michiels 2009]. The approach was further developed by [Fujioka 2009c,Omran 2012a,Omran 2014a,Omran 2014b,Omran 2013,Chen 2014] without passing through the time-delay system model. It will be presented in more detail in Section 2.1.4 and Chapter 3.

2.1.2 Hybrid system approach

Due to the existence of both continuous and discrete dynamics, it is quite natural to model sampled-data systems as hybrid dynamical systems [Goebel 2009, Goebel 2012, Haddad 2014, Brogliato 1996, Brogliato 2016]. The first mentions to sampled-data systems as *hybrid dynamical systems* date back to the middle of the '80s [Mousa 1986]. Later on, in the '90s, the use of hybrid models has been developed for linear sampled-data systems with uniform and multi-rate sampling as an interesting approach for the H_∞ and H_2 control problems [Kabamba 1993, Sun 1993, Toivonen 1992b]. The approach has also been developed for nonlinear sampled-data systems in [Hou 1997, Ye 1998]. For systems with aperiodic sampling, impulsive models had been used starting with [Toivonen 1992b, Dullerud 1999, Michel 1999]. Recently, more general hybrid models have been proposed in the context of Networked Controlled Systems by [Nešić 2004b, Nešić 2009]. A solid theoretic foundation has been established for hybrid systems in the framework proposed by [Goebel 2009, Goebel 2012] and it proves to be very useful in the analysis of sampled-data systems.

In this section we will present some basic hybrid models encountered in the analysis of sampled-data systems. The extensions of the Lyapunov stability theory for hybrid systems will be introduced together with constructive numerical and analytic stability analysis criteria.

2.1.2.1 Impulsive models for sampled-data systems

Consider the case of LTI sampled-data systems with linear state feedback, as in system (1.12). Let $\hat{x}$ denote a piecewise constant signal representing the most recent state measurement of the plant available at the controller, $\hat{x}(t) = x(t_k)$, for all $t \in [t_k, t_{k+1}), k \in \mathbb{N}$. Using the augmented system state $\chi(t) = [x^T(t),\ \hat{x}^T(t)]^T \in \mathbb{R}^{n_\chi}$ with $n_\chi = 2n$, the dynamics of the LTI sampled-data system (1.12) can be written under the form

$$\begin{cases} \dot{\chi}(t) &= F\chi(t), \quad t \neq t_k,\ k \in \mathbb{N}, \\ \chi(t_k) &= J\chi(t_k^-), \quad k \in \mathbb{N}, \end{cases} \tag{2.10}$$

with

$$\chi(t^-) = \lim_{\theta \uparrow t} \chi(\theta),\ F = \begin{bmatrix} A & BK \\ 0 & 0 \end{bmatrix},\ J = \begin{bmatrix} I & 0 \\ I & 0 \end{bmatrix}. \tag{2.11}$$

Similar models can be determined by considering an augmented state vector χ including the most recent control value implemented at the plant $\hat{u}(t) = u(t_k)$, the sampling error $e(t) = x(t) - \hat{x}(t)$, the actuation error $e_u(t) = u(t) - \hat{u}(t)$, etc. Models of the form (2.10),(2.11) fit into the framework of *impulsive dynamical systems* [Milman 1960, Haddad 2014, Lakshmikantham , Bainov 1993] (sometimes also called *discontinuous dynamical systems* or simply *jump systems*). More general nonlinear sampled-data systems lead to impulsive systems of the form [Naghshtabrizi 2008, Nešić 2004b]

$$\dot{\chi}(t) = F_k(t, \chi(t)), \quad t \neq t_k,\ k \in \mathbb{N}, \tag{2.12a}$$

$$\chi(t_k) = J_k(t_k, \chi(t_k^-)), \quad k \in \mathbb{N} \tag{2.12b}$$

where the augmented state may also include the controller state and some of its sampled components (state, output, etc.). Generally, for an impulsive system, (2.12a) is called the system's *flow dynamics* while (2.12b) is the *jump dynamics*.

2.1.2.2 Lyapunov methods for impulsive systems

The stability of equilibria for the impulsive systems of the form (2.12) can be ensured by the existence of candidate Lyapunov functions that depend both on the system state and on time, and evolve in a discontinuous manner at impulse instants [Bainov 1993, Haddad 2014, Naghshtabrizi 2008].

Theorem 2.4 *[Naghshtabrizi 2008] Consider system* (2.12) *and denote* $\tau(t) = t - t_k,\ \forall t \in [t_k, t_{k+1})$. *Assume that* F_k *and* J_k *are locally Lipschitz functions from* $\mathbb{R}_+ \times \mathbb{R}^{n_\chi}$ *to* $\mathbb{R}^{n_\chi}$ *such that* $F_k(t,0) = 0, J_k(t,0) = 0$, *for all* $t \geq 0$. *Let there exist positive scalars* c_1, c_2, c_3, b *and a Lyapunov function* $V : \mathbb{R}^{n_\chi} \times \mathbb{R} \to \mathbb{R}$, *such that*

$$c_1\|\chi\|^b \leq V(\chi,\tau) \leq c_2\|\chi\|^b, \tag{2.13}$$

for all $\chi \in \mathbb{R}^{n_\chi}, \tau \in [0, \overline{h}]$. *Suppose that for any impulse sequence* $\sigma = \{t_k\}_{k\in\mathbb{N}}$ *such that* $\underline{h} \leq t_{k+1} - t_k \leq \overline{h}$, $k \in \mathbb{N}$, *the corresponding solution* $\chi(\cdot)$ *to* (2.12) *satisfies:*

$$\frac{dV\left(\chi(t),\tau(t)\right)}{dt} \leq -c_3 V\left(\chi(t),\tau(t)\right), \quad \forall t \neq t_k,\ \forall k \in \mathbb{N},$$

and $V\left(\chi(t_k),0\right) \leq \lim_{t\to t_k^-} V\left(\chi(t),\tau(t)\right), \quad \forall k \in \mathbb{N}$. *Then, the equilibrium point* $\chi = 0$ *of system* (2.12) *is Globally Uniformly Exponentially Stable over the class of sampling impulse instants, i.e. there exist* $c, \lambda > 0$ *such that for any sequence* $\sigma = \{t_k\}_{k\in\mathbb{N}}$ *that satisfies* $\underline{h} \leq t_{k+1} - t_k \leq \overline{h}$, $k \in \mathbb{N}$,

$$\|\chi(t)\| \leq c\|\chi(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall t \geq t_0.$$

The previous stability theorem requires in (2.13) the candidate Lyapunov function to be positive at all times. For the case of system (2.12) with globally Lipschitz $F_k, k \in \mathbb{N}$, the condition can be relaxed by requiring the Lyapunov function to be positive only at impulse times [Naghshtabrizi 2008], i.e. $c_1\|\chi(t_k)\|^b \leq V\left(\chi(t_k),0\right) \leq c_2\|\chi(t_k)\|^b, \forall k \in \mathbb{N}$, instead of (2.13).

In the case of impulsive systems (2.10), with linear flow and jump dynamics, candidate Lyapunov functions of the form $V(\chi,\tau) = \chi^T P(\tau)\chi$, with $P : [0, \bar{h}] \to \mathbb{R}^{n_\chi \times n_\chi}$ a differentiable matrix function, have been used [Toivonen 1992a, Sun 1993, Briat 2013, Naghshtabrizi 2008]. Sufficient stability conditions can be obtained from Theorem 2.4 in terms of existence of a differentiable matrix function $P : [0, \overline{h}] \to \mathbb{R}^{n_\chi \times n_\chi}$, $c_1 I \prec P(\tau) \prec c_2 I$, satisfying the parametric set of LMIs

$$F^T P(\theta_1) + P(\theta_1)F + c_3 P(\theta_1) + \frac{\partial P}{\partial \tau}(\theta_1) \prec 0,$$
$$\forall\, \theta_1 \in [0, \overline{h}], \tag{2.14a}$$
$$J^T P(0) J - P(\theta_2) \prec 0,\ \forall\, \theta_2 \in [\underline{h}, \overline{h}], \tag{2.14b}$$

with positive scalars c_1, c_2, c_3. This formulation is reminiscent of the Riccati equation approach used for robust sampled-data control in [Toivonen 1992b, Sun 1993].

2.1.2.3 Numerically tractable criteria

In practice, the difficulty of checking the existence of candidate Lyapunov functions using LMI formulations such as (2.14) comes from the fact that the set of LMIs are parametrized by elements

in $[0, \bar{h}]$ or $[\underline{h}, \overline{h}]$, which leads to an infinite number of LMIs. As follows we will discuss the derivation of a finite number of LMIs from (2.14).

Concerning the parametric set of LMIs (2.14), a finite number of LMI conditions can be derived by considering particular forms for the matrix function $P(\tau)$. For example, consider a matrix $P(\tau)$ linear with respect to τ

$$P(\tau) = P_1 + (P_2 - P_1)\,\frac{\tau}{\overline{h}}, \tag{2.15}$$

for some positive definite matrices P_1, P_2, as in [Hu 2003, Allerhand 2011]. There, such a Lyapunov matrix has been used for sampled-data systems with multi-rate sampling and switched linear systems. For a candidate Lyapunov function $V(\chi, \tau) = \chi^T P(\tau)\chi$, with $P(\tau)$ as defined in (2.15), a finite set of LMIs that are sufficient for stability can be obtained from (2.14) using simple convexity arguments:

$$F^T P_1 + P_1 F + c_3 P_1 + \frac{P_2 - P_1}{\overline{h}} \prec 0, \tag{2.16a}$$

$$F^T P_2 + P_2 F + c_3 P_2 + \frac{P_2 - P_1}{\overline{h}} \prec 0, \tag{2.16b}$$

$$J^T P_1 J \prec P_2, \tag{2.16c}$$

$$J^T P_1 J \prec P_1 + (P_2 - P_1)\,\underline{h}/\overline{h}. \tag{2.16d}$$

For the particular case of LTI sampled-data systems represented by (2.10),(2.11), Lyapunov functions of the form $V(\chi, \tau) = \chi^T P(\tau)\chi$ are proposed in the literature by summing various terms such as:

$$V_1(\chi, \tau) = x^T P_0 x \tag{2.17}$$

$$V_2(\chi, \tau) = (x - \hat{x})^T\, Q\, (x - \hat{x})\, (\overline{h} - \tau) \tag{2.18}$$

$$V_3(\chi, \tau) = (x - \hat{x})^T\, R\, (x - \hat{x})\, e^{-\lambda\tau} \tag{2.19}$$

$$V_4(\chi, \tau) = \chi^T \Big(\int_{-\tau}^{0} (s + \overline{h})(Fe^{Fs})^T \tilde{U} (Fe^{Fs}) ds \Big) \chi, \tag{2.20}$$

where $\tilde{U} := \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}$, $\lambda > 0$ and P_0, R, U are symmetric positive definite matrices. Using such particular forms of Lyapunov functions, LMI stability conditions have been derived in the literature [Hu 2003, Naghshtabrizi 2008, Nešić 2009, Omran 2012b, Goebel 2012]. We point in particular to the term (2.20) used in [Naghshtabrizi 2008] which provided a significant improvement in what concerns the conservatism reduction. This term is inspired by Lyapunov-Krasovskii functionals from the input-delay approach, like the one in [Fridman 2004]. Note that the term (2.20) can also be written as $\int_{t-\tau}^{t} (s + \overline{h} - t)\dot{x}^T(s) U \dot{x}(s) ds$. It has been motivated by the term $\int_{-\overline{h}}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) U \dot{x}(s) ds d\theta$ used in the time-delay approach (see [Fridman 2004]). Vice versa, the hybrid system approach has also inspired the use of discontinuous Lyapunov functionals in the time-delay approach (see for example the functional (2.4) which is discontinuous at sampling times). Note that the term $(h_k - \tau) \int_{t-\tau}^{t} \dot{x}^T(s) R \dot{x}(s) ds$ in the functional (2.4) can be re-written as $(h_k - \tau)\chi^T \Big(\int_{-\tau}^{0} (Fe^{Fs})^T \tilde{R} (Fe^{Fs}) ds \Big) \chi$, with

$$\tilde{R} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}$$

and $R \succ 0$. Then, for the impulsive system (2.10), (2.11), the functional (2.4) can be interpreted as a Lyapunov function of the form $V(\chi, \tau, h_k) = \chi^T P(\tau, h_k)\chi$. Hybrid and input-delay approaches share the same advantages and drawbacks. Both of them are constructive, and LMI conditions are used to construct the Lyapunov functionals/functions. Similarly to the time delay approach, the LMI formulations can be adapted to cope with uncertainties in the system matrices. On the other hand, conservatism is added by the upper bounding introduced when studying the derivatives of Lyapunov functionals/functions.

2.1.2.4 More general hybrid models

A large variety of hybrid dynamical systems, including sampled-data and impulsive models, can be re-formulated in the unifying theoretical framework proposed by Goebel, Sanfelice and Teel [Goebel 2009, Goebel 2012]. Several fundamental properties have been investigated in this framework, providing a solid theory for hybrid dynamical systems. The main advantage of this generic hybrid formulation [Goebel 2009, Goebel 2012] is that the associated theoretic properties can be directly transferred to sampled-data systems with aperiodic sampling. The general formulation proposed in [Goebel 2009, Goebel 2012] considers models of the form

$$\dot{z} = F_z(z), \quad z \in C, \tag{2.21a}$$
$$z^+ = J_z(z), \quad z \in D, \tag{2.21b}$$

with state $z \in \mathbb{R}^{n_z}$. The system state evolves according to an ordinary differential equation (2.21a) when the state is in some subset C of $\mathbb{R}^{n_z}$ and according to a first order recurrence equation (2.21b) when the state is in the subset D of $\mathbb{R}^{n_z}$. z^+ denotes the next value of state given as a function of the current state z via the map $J_z(\cdot)$. C is called the *flow set* and D is called the *jump set*. Here, we assume that F_z and J_z are continuous functions from C to $\mathbb{R}^{n_z}$ and D to $\mathbb{R}^{n_z}$, respectively. C and D are assumed to be closed sets in $\mathbb{R}^{n_z}$.

Note that in the impulsive system formulation of sampled-data systems, the system jumps are time-triggered. However, the dynamic of the triggering mechanism is in some sense hidden. In the framework proposed by [Carnevale 2007, Dacic 2007, Nešić 2009, Goebel 2009, Goebel 2012], the mechanism triggering the system jumps is modelled explicitly by augmenting the system state with the clock variable $\tau(t) = t - t_k$, $\forall t \in [t_k, t_{k+1})$, $\forall k \in \mathbb{N}$. Consider the LTI sampled-data systems (1.12) with the notations $\hat{x}(t) = x(t_k)$, $\tau(t) = t - t_k$ for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. The system can be represented by the following hybrid model

$$\left\{\begin{array}{rcll} \left.\begin{array}{rcl} \dot{x} &=& Ax + BK\hat{x} \\ \dot{\hat{x}} &=& 0 \\ \dot{\tau} &=& 1 \end{array}\right\} & & \tau \in [0, \overline{h}], \\ \left.\begin{array}{rcl} x^+ &=& x \\ \hat{x}^+ &=& x \\ \tau^+ &=& 0 \end{array}\right\} & & \tau \in [\underline{h}, \overline{h}]. \end{array}\right. \tag{2.22}$$

Then, system (1.12) with $h_k \in [\underline{h}, \overline{h}]$ (or equivalently (2.10),(2.11)) can be re-modelled in the form (2.21) with $z^T = \begin{bmatrix} x^T & \hat{x}^T & \tau \end{bmatrix} = \begin{bmatrix} \chi^T & \tau \end{bmatrix}$,

$$C = \left\{ z \in \mathbb{R}^{n_z} : \tau \in [0, \overline{h}] \right\},$$

$$D = \left\{ z \in \mathbb{R}^{n_z} : \tau \in [\underline{h}, \overline{h}] \right\},$$

$$F_z(z) = \begin{bmatrix} Ax + BK\hat{x} \\ 0 \\ 1 \end{bmatrix}, \; J_z(z) = \begin{bmatrix} x \\ x \\ 0 \end{bmatrix}. \tag{2.23}$$

Solutions ϕ of the general hybrid system (2.21) are parametrized by both the continuous time t and the discrete time k: $\phi(t,k)$ represents the state of the hybrid system after t time units and k jumps. Such solutions are defined on a *hybrid time domain*, which for the case of sampled-data systems is given as the union of the intervals $[t_k, t_{k+1}]\times\{k\}$. A solution $\phi(\cdot,\cdot)$ is a function defined on a hybrid time domain such that $\phi(\cdot,k)$ is continuous on $[t_k, t_{k+1}]$, continuously differentiable on (t_k, t_{k+1}) for each k in the domain, and such that

$$\dot{\phi}(t,k) = F_z\left(\phi(t,k)\right),$$

if $\phi(t,k) \in C, \; t \in (t_k, t_{k+1}), k \in \mathbb{N}$, and

$$\phi(t_{k+1}, k+1) = J_z\left(\phi(t_{k+1}, k)\right),$$

if $\phi(t_{k+1}, k) \in D, \; k \in \mathbb{N}$. For sampled-data systems as (2.22) such solutions may be roughly interpreted as a generalization of the state lifting approach proposed in [Yamamoto 1994] for systems with periodic sampling.

A particularity of the model (2.22) in the context of stability analysis is the fact that although the matrix K is designed such that x (and consequently $\hat{x}$) converges to zero, the clock variable τ does not converge. For each sampling interval $[t_k, t_{k+1})$, the timer τ visits successively the points of the interval $[0, \overline{h}]$ up to $h_k = t_{k+1} - t_k$. The main consequence is that the hybrid system (2.22) does not have an asymptotically stable equilibrium point. For such systems the stability of the compact set $\mathcal{A} = \{0\}\times\{0\}\times[0,\overline{h}]$ is usually investigated instead. Studying this property allows to conclude on the convergence of x. One of the main results allowing to state the asymptotic stability of a set for hybrid systems is given below. This results is expressed in terms of the *pre-asymptotic stability* of a set $\mathcal{A}$ (see [Goebel 2009] for a detailed definition). The prefix "-pre" is used since the completeness of all system solutions[11] is not required. Only complete solutions need to converge to $\mathcal{A}$. The concept of pre-asymptotic stability used in the following theorem is equivalent to standard asymptotic stability of the set $\mathcal{A}$ when all system solutions are complete, which is the case for sampled-data systems.

Theorem 2.5 *[Goebel 2009] Consider the hybrid system* (2.21) *and the compact set* $\mathcal{A} \subset \mathbb{R}^{n_z}$ *such that* $J_z(\mathcal{A}\cap D) \subset \mathcal{A}$. *If there exists a candidate Lyapunov function*[12] V *such that*

$$\frac{\partial V}{\partial z} F_z(z) < 0 \text{ for all } z \in C \setminus \mathcal{A}, \tag{2.24a}$$

$$V\left(J_z(z)\right) - V(z) < 0 \text{ for all } z \in D \setminus \mathcal{A}, \tag{2.24b}$$

then the set $\mathcal{A}$ *is pre-asymptotically stable.*

Various relaxations of the above result are provided in Chapter 3 in [Goebel 2012]. A converse Lyapunov theorem is given below.

[11] A solution $\phi(t,k)$ is called *complete* if dom ϕ is unbounded.

[12] V is continuous and non-negative on $(C\cup D)\setminus\mathcal{A} \subset \text{dom}V$, it is continuously differentiable on an open set satisfying $C\setminus\mathcal{A} \subset \text{dom}V$, and $\lim_{z\to\mathcal{A}, z\in domV\cap(C\cup D)} V(z) = 0$. Furthermore, for global pre-asymptotic stability, the sublevel sets of $V(.)$ are required to be compact.

Theorem 2.6 *[Goebel 2009] For the hybrid system (2.21), if the compact set $\mathcal{A}$ is globally pre-asymptotically stable, then there exist a $\mathcal{C}^\infty$ function $V : \mathbb{R}^{n_z} \to \mathbb{R}_+$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(|z|_\mathcal{A}) \leq V(z) \leq \alpha_2(|z|_\mathcal{A})$, $\forall z \in \mathbb{R}^{n_z}$, where $|\cdot|_\mathcal{A}$ denotes the distance from the set $\mathcal{A}$, and*

$$\frac{\partial V}{\partial z} F_z(z) \leq -V(z), \ \forall z \in C, \tag{2.25a}$$

$$V(J_z(z)) \leq V(z)/2, \ \forall z \in D. \tag{2.25b}$$

Note that with respect to the case of sampled-data systems such as (2.22) (or equivalently (2.10), (2.11)) where solutions are complete, the previous theorem shows that asymptotic stability implies the existence of a C^∞ Lyapunov function of the form $V(z) = \tilde{V}(\chi, \tau)$, to be related with the sufficient conditions for stability in Theorem 2.4.

2.1.2.5 An estimation of the MSI for nonlinear systems

For nonlinear sampled-data systems the stability properties have been studied in the more general context of Networked Control Systems with scheduling protocols [Nešić 2004b, Carnevale 2007]. This approach has been particularized to the sampled-data case in [Nešić 2009]. Consider the plant:

$$\begin{cases} \dot{x} &= F(x,u), \\ y &= H(x,u), \end{cases} \tag{2.26}$$

where x is the plant state, u is the control input, y is the measured output. Suppose that asymptotic stability is guaranteed by the continuous-time output feedback:

$$\begin{cases} \dot{x}^c &= F^c(x^c,y), \\ u &= H^c(x^c,y), \end{cases} \tag{2.27}$$

where x^c is the controller state. Under an exact sampled-data implementation of the controller and a perfect knowledge of the sampling sequence $\sigma = \{t_k\}_{k\in\mathbb{N}}$, the sampled-data implementation of the closed-loop system can be written in the following impulsive system form:

$$\begin{cases} \dot{x} &= F(x,\hat{u}), & t \in [t_k, t_{k+1}), \\ y &= H(x), & t \in \mathbb{R}_+ \\ \dot{x}^c &= F^c(x^c,\hat{y}), & t \in [t_k, t_{k+1}), \\ u &= H^c(x^c), & t \in \mathbb{R}_+ \\ \dot{\hat{y}} &= 0, & t \in [t_k, t_{k+1}), \\ \dot{\hat{u}} &= 0, & t \in [t_k, t_{k+1}), \\ \hat{y}(t_k) &= y(t_k^-), \\ \hat{u}(t_k) &= u(t_k^-), \end{cases} \tag{2.28}$$

where $\hat{u}$ represents control being implemented at the plant and $\hat{y}$ the most recent plant output measurements that are available at the controller. In order to express the system in the general framework of [Goebel 2012], consider the augmented state vector $\eta(t) \in \mathbb{R}^{n_\eta}$ and the sampling-induced error $e(t) \in \mathbb{R}^{n_e}$:

$$\eta(t) := \begin{bmatrix} x(t) \\ x^c(t) \end{bmatrix}, \ e(t) = \begin{bmatrix} e^y(t) \\ e^u(t) \end{bmatrix} := \begin{bmatrix} \hat{y}(t) - y(t) \\ \hat{u}(t) - u(t) \end{bmatrix}$$

and a clock τ which evolves with respect to the sampling instants. The dynamics in (2.28) with $h_k \in [\underline{h}, \overline{h}]$ can be modelled by the following hybrid system:

$$\left\{\begin{array}{lcll} \left.\begin{array}{lcl} \dot{\eta} &=& f(\eta,e) \\ \dot{e} &=& g(\eta,e) \\ \dot{\tau} &=& 1 \end{array}\right\} & & \tau \in [0,\overline{h}], \\ \left.\begin{array}{lcl} \eta^+ &=& \eta \\ e^+ &=& 0 \\ \tau^+ &=& 0 \end{array}\right\} & & \tau \in [\underline{h},\infty), \end{array}\right. \tag{2.29}$$

with $\eta \in \mathbb{R}^{n_\eta}$, $e \in \mathbb{R}^{n_e}$, $\tau \in \mathbb{R}_+$. The functions f and g are obtained by direct calculations from the sampled-data system (2.28) (see [Nešić 2009]):

$$f(\eta,e) = \begin{bmatrix} F(x, H^c(x^c) + e^u) \\ F^c(x^c, H(x) + e^y) \end{bmatrix},$$

$$g(\eta,e) = \begin{bmatrix} -\frac{\partial H}{\partial x} F(x, H^c(x^c) + e^u) \\ -\frac{\partial H^c}{\partial x^c} F^c(x^c, H(x) + e^y) \end{bmatrix}.$$

It should be noted that $\dot{\eta} = f(\eta, 0)$ is the closed loop system without the sampled-data implementation. The following theorem provides a quantitative method to estimate the MSI, using model (2.29).

Theorem 2.7 *[Nešić 2009] Assume that f and g in (2.29) are continuous. Suppose there exist $\tilde{\Delta}_\eta$, $\tilde{\Delta}_e > 0$, a locally Lipschitz function $W : \mathbb{R}^{n_e} \to \mathbb{R}_+$, a locally Lipschitz, positive definite, radially unbounded function $V : \mathbb{R}^{n_\eta} \to \mathbb{R}_+$, a continuous function $\Theta : \mathbb{R}^{n_\eta} \to \mathbb{R}_+$, real numbers $L > 0$, $\gamma > 0$, functions $\underline{\alpha}_W$, $\overline{\alpha}_W \in \mathcal{K}_\infty$ and a continuous, positive definite function ϱ such that, for all $e \in \mathbb{R}^{n_e}$:*

$$\underline{\alpha}_W(\|e\|) \leq W(e) \leq \overline{\alpha}_W(\|e\|),$$

and for almost all $\|\eta\| \leq \tilde{\Delta}_\eta$ and $\|e\| \leq \tilde{\Delta}_e$:

$$\frac{\partial W}{\partial e} g(\eta,e) \leq L W(e) + \Theta(\eta),$$

$$\frac{\partial V}{\partial \eta} f(\eta,e) \leq -\varrho(\|\eta\|) - \varrho(W(e)) - \Theta^2(\eta) + \gamma^2 W^2(e).$$

Finally, consider that $0 < \underline{h} \leq \overline{h} < \mathcal{T}(\gamma, \mathcal{L})$, with

$$\mathcal{T}(\gamma, L) := \left\{\begin{array}{ll} \frac{1}{Lr} arctan(r) & \gamma > L, \\ \frac{1}{L} & \gamma = L, \\ \frac{1}{Lr} arctanh(r) & \gamma < L, \end{array}\right.$$

and $r = \sqrt{\left|\frac{\gamma^2}{L^2} - 1\right|}$. Then, for all sampling intervals less than $\overline{h}$ the set $\mathcal{A} = \{(\eta, e, \tau) : \eta = 0, e = 0, \tau \in [0, \overline{h}]\}$ is Uniformly Asymptotically Stable for system (2.29).

Theorem 2.7 provides an explicit formulation of the MSI for nonlinear sampled-data systems. It is applicable for both constant and variable sampling intervals. Moreover, it has the advantage of considering a general class of nonlinear systems. Nevertheless, for practical applications it is not obvious to construct the functions $V(\eta)$, $W(e)$ and $\Theta(\eta)$ which satisfy the hypotheses of the theorem.

2.1.2.6 Further reading

In the impulsive system framework control design conditions have been proposed in [Briat 2013]. For observer design conditions we point to the works in [Andrieu 2010,Dinh 2015,Ahmed-Ali 2009, Nadri 2013, Postoyan 2012, Ferrante 2014]. Some extensions of the hybrid systems approach for sampled-data systems with delay can be found in [Fridman 2000, Naghstabrizi 2010] and [Bauer 2012].

2.1.3 Discrete-time approach and convex-embeddings

In this sub-section we present several approaches which use the system integration over the sampling interval and convex embeddings of the transition matrix between sampling times in order to derive stability conditions.

2.1.3.1 Theoretical results for LTI systems using the discrete-time approach

Let us consider the LTI system with sampled linear static state feedback (1.12) where $h_k = t_{k+1} - t_k$ takes values in the set $\mathcal{T} = [\underline{h}, \overline{h}]$. Recall the notations $x_k = x(t_k)$,

$$\Lambda(\theta) = \mathrm{e}^{A\theta} + \int_0^\theta \mathrm{e}^{As} ds BK \tag{2.30}$$

for $\theta \in \mathbb{R}$. One can verify that the closed-loop system (1.12) satisfies

$$x_{k+1} = \Lambda(h_k) x_k \tag{2.31}$$

with $h_k \in \mathcal{T} = [\underline{h}, \overline{h}]$. Model (2.31) belongs to the class of discrete-time Linear Parameter Varying (LPV) systems [Rugh 2000,Kamen 1984,Molchanov 1989]. It captures the behaviour of system (1.12) at sampling times, without consideration of the intersample behavior. However, in [Fujioka 2009c], the following proposition has shown that for LTI sampled-data system, the asymptotic stability in continuous-time and in discrete-time are equivalent.

Proposition 2.8 *[Fujioka 2009c] Consider the sampled-data system* (1.12) *with* $h_k = t_{k+1} - t_k \in [\underline{h}, \overline{h}]$. *For a given* $x(t_0)$, *the following conditions are equivalent:*

1. $\lim_{t\to\infty} x(t) = 0$
2. $\lim_{k\to\infty} x(t_k) = 0$.

A simple stability criterion which is sufficient for stability can be obtained using classical quadratic Lyapunov functions, which are decreasing at each sample.

Theorem 2.9 *[Zhang 2001b] The origin of system* (2.31) *is Globally Uniformly Exponentially Stable for all sampling sequences* $\sigma = \{t_k\}_{k\in\mathbb{N}}$ *with* $h_k = t_{k+1} - t_k \in [\underline{h}, \overline{h}]$, $k \in \mathbb{N}$, *if there exists* $P \succ 0$ *such that*

$$\Lambda^T(\theta) P \Lambda(\theta) - P \prec 0, \ \forall \theta \in \mathcal{T} = [\underline{h}, \overline{h}]. \tag{2.32}$$

The LMI (2.32) ensures that the candidate Lyapunov function $V(x) = x^T P x$ satisfies the relation

$$\Delta V(k) = V(x_{k+1}) - V(x_k) < 0, \ \forall x_k \neq 0. \tag{2.33}$$

Note that, similarly to conditions (2.14) used for the hybrid system approach, the stability condition (2.32) represent a set of LMIs that are parametrized by $\theta \in \mathcal{T} = [\underline{h}, \overline{h}]$. This condition is not a computationally tractable problem by themselves. Approximate solutions, based on evaluation of the condition for a finite set of values of θ have been presented in [Zhang 2001b, Sala 2005, Skaf 2009]. A finite set of sufficient tractable numerical conditions can be obtained using normed-bounded and/or polytopic convex embeddings of the transition matrix $\Lambda(\theta)$.

2.1.3.2 Tractable criteria

In what follows, we try to give an idea about the manner to solve parametric LMIs involving matrix exponentials such as the one in (2.32). First, we present briefly the approach proposed by Fujioka in [Fujioka 2009a]. Consider a nominal sampling interval $T_0 \in [\underline{h}, \overline{h}]$. For a scalar δ, the transition matrix $\Lambda(\cdot)$ satisfies the relation

$$\Lambda(T_0 + \delta) = \Lambda(T_0) + \Delta(\delta)\Psi(T_0) \tag{2.34}$$

where $\Delta(\delta) := \int_0^\delta e^{As} ds$, $\Psi(T_0) = A\Lambda(T_0) + BK$. Using classical properties of the matrix exponential [Loan 1977], the induced Euclidean norm of $\Delta(\delta)$ can be over-bounded

$$\|\Delta(\delta)\|_2 \leq \int_0^\delta e^{\mu(A)s} ds \tag{2.35}$$

where $\mu(A)$ is the maximum eigenvalue of $\frac{A+A^T}{2}$. System (2.31) can be expressed as a nominal discrete-time LTI system with a norm-bounded uncertainty

$$x_{k+1} = \Lambda(T_0)x_k + \Delta(\delta_k)\Psi(T_0)x_k \tag{2.36}$$

where $\delta_k = h_k - T_0$, for which classical H_∞ criteria [Gahinet 1994] can be used. A simplified version of the main result in [Fujioka 2009a] is given as follows.

Theorem 2.10 *[Fujioka 2009a] Let $T_0 \in [\underline{h}, \overline{h}]$ be given. If there exists $X \succ 0$ and $\gamma > 0$ satisfying*

$$\mathcal{M}(T_0, X, \gamma) := \begin{bmatrix} \Lambda(T_0) & I \\ \Psi(T_0) & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Lambda(T_0) & I \\ \Psi(T_0) & 0 \end{bmatrix}^T - \begin{bmatrix} X & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec 0, \tag{2.37}$$

then (2.32) is satisfied with $P = X^{-1}$ for all $\theta \in \mathcal{T}(T_0, \gamma) := \big[\underline{h}(T_0, \gamma), \overline{h}(T_0, \gamma)\big]$ with

$$\underline{h}(T_0, \gamma) = \begin{cases} T_0 - \gamma^{-1}, \ \textit{if } \mu(-A) = 0, \\ -\infty, \ \textit{if } \mu(-A) \leq -\gamma, \\ T_0 - \frac{\log\left(1+\gamma^{-1}\mu(-A)\right)}{\mu(-A)}, \ \textit{otherwise,} \end{cases} \tag{2.38}$$

$$\overline{h}(T_0, \gamma) = \begin{cases} T_0 + \gamma^{-1}, \ \textit{if } \mu(A) = 0, \\ \infty, \ \textit{if } \mu(A) \leq -\gamma, \\ T_0 + \frac{\log\left(1+\gamma^{-1}\mu(A)\right)}{\mu(A)}, \ \textit{otherwise.} \end{cases} \tag{2.39}$$

Condition (2.37) is sufficient for the asymptotic stability of system (2.31) under time-varying sampling intervals $h_k \in [\underline{h}, \overline{h}]$ with $\underline{h}$ and $\overline{h}$ given in (2.38) and (2.39), respectively. Other norm-

bounded approximations of the transition matrix $\Lambda(\cdot)$ exist in the literature [Balluchi 2005, Suh 2008, Kao 2013, Fujioka 2011b, Zhang 2011]. For example, stability conditions have been provided using the Schur decomposition in [Suh 2008] while [Zhang 2011] uses the Jordan normal form. In [Fujioka 2011b] the transition matrix $\Lambda(T_0+\delta)$ is decomposed as

$$\Lambda(T_0+\delta) = \Lambda(T_0) + \delta L(T_0) + \Delta_2(\delta) A L(T_0)$$

with $L(T_0) = e^{AT_0}(A+BK)$, $\Delta_2(\delta) := \int_0^\delta \int_0^\rho e^{As} ds d\rho$, and stability conditions are provided by computing the induced Euclidean norm of $\Delta_2(\delta)$. See also [Kao 2013] where stability conditions have been derived using Integral Quadratic Constraints (IQC), by studying the positive realness of $\Delta(\delta)$. More general Lyapunov functions have been used in [Fujioka 2010b].

Alternatively to the use of norm bounded approximations, tractable numerical conditions can also be obtained using polytopic embeddings of the transition matrix $\Lambda(\cdot)$ in system (2.31). The set

$$\mathcal{W}_{[\underline{h},\overline{h}]} := \{\Lambda(\theta),\, \theta \in [\underline{h},\overline{h}]\}$$

is embedded in a larger convex polytope with a *finite* number of vertices Λ_i, $i \in \mathcal{I} := \{1, \cdots, N_v\}$,

$$\overline{\mathcal{W}} := \left\{ \sum_{i=1}^{N_v} \alpha_i \Lambda_i \,|\, \alpha_i \geq 0,\, i \in \mathcal{I},\, \sum_{i=1}^{N_v} \alpha_i = 1 \right\}, \tag{2.40}$$

in such a way that $\mathcal{W}_{[\underline{h},\overline{h}]} \subseteq \overline{\mathcal{W}}$. Using a polytopic embedding, system (2.31) can be expressed as a

$$x_{k+1} = \sum_{i=1}^{N_v} \alpha_i(h_k) \Lambda_i x_k, \tag{2.41}$$

where $\sum_{i=1}^{N_v} \alpha_i(h_k) = 1,\ \alpha_i(h_k) \geq 0, i \in \mathcal{I}$. This is a classical discrete-time system with polytopic uncertainty [Daafouz 2001]. Here

$$\alpha(h_k) = \begin{bmatrix} \alpha_1(h_k) & \alpha_2(h_k) & \dots & \alpha_{N_v}(h_k) \end{bmatrix}^T$$

represent the barycentric coordinates of $\Lambda(h_k)$ in the polytope $\overline{\mathcal{W}}$. The properties of the over-approximating polytopic set $\overline{\mathcal{W}}$ make it possible to derive a finite number of sufficient stability conditions from (2.32), by writing simple LMIs over the polytope vertices:

$$P \succ 0,\ \Lambda_i^T P \Lambda_i - P \prec 0,\ \forall i \in \mathcal{I}. \tag{2.42}$$

One of the advantages of the polytopic embedding is the fact that it allows the use of *parameter dependent Lyapunov functions* [Daafouz 2001, Hetel 2006, Cloosterman 2010] $\tilde{V}(x,\alpha) = x^T P(\alpha) x$, $P(\alpha) = \sum_{i=1}^{N_v} \alpha_i P_i$, which lead to refined stability conditions under a reasonable numerical complexity:

$$\exists\ P_i = P_i^T \succ 0, \quad \Lambda_i^T P_j \Lambda_i - P_i \prec 0, \forall\ (i,j) \in \mathcal{I} \times \mathcal{I}. \tag{2.43}$$

The main difficulty in constructing the polytope $\bar{\mathcal{W}}$ is the exponential dependence of the transition matrix $\Lambda(\theta) = e^{As} + \int_0^\theta e^{As} ds BK$ in the parameter θ over the the interval $[\underline{h},\overline{h}]$. Several approaches exist for the computation of a convex polytope embedding an uncertain matrix exponential. See for example [Olaru 2008, Oishi 2010, Cloosterman 2009, Cloosterman 2010, Lombardi 2012] for techniques based on the real Jordan form, [Gielen 2010] for a construction that uses the Cayley-Hamilton theorem and [Cloosterman 2006] for an approach studying interval

matrices. One may remark that the transition matrix $\Lambda(\cdot)$ can be re-expressed as

$$\Lambda(\theta) = I + \Delta(\theta)\left(A + BK\right) \tag{2.44}$$

which involves only one uncertain matrix term $\Delta(\theta) = \int_0^\theta e^{As}ds$. Then the stability problem can be addressed by constructing a polytopic approximation of $\Delta(\theta)$ for $\theta \in [\underline{h}, \overline{h}]$. To give an idea about the manner such a convex polytope can be constructed, let us consider a simple case where the matrix A has n real eigenvalues $\lambda_i \neq 0, i \in \{1, \ldots, n\}$ with multiplicity equal to one, i.e. where it takes the form

$$A = T^{-1}\begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \ldots & \ldots & 0 & \lambda_n \end{bmatrix} T \tag{2.45}$$

for some invertible matrix $T \in \mathbb{R}^{n\times n}$. Then the uncertain matrix $\Delta(\theta)$ takes the form:

$$\Delta(\theta) = T^{-1}\begin{bmatrix} \rho_1(\theta) & 0 & 0 & \ldots & 0 \\ 0 & \rho_2(\theta) & 0 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \ldots & \ldots & 0 & \rho_n(\theta) \end{bmatrix} T \tag{2.46}$$

where $\rho_i(\theta) = \frac{1}{\lambda_i}\left(e^{\lambda_i\theta} - 1\right), i = 1, \ldots, n$. By computing ρ_i^{min} and ρ_i^{max} the minimum and maximum values of $\rho_i(\theta)$ over $[\underline{h}, \overline{h}]$, the uncertain matrix $\Delta(\theta)$ is embedded in a convex polytope with $N_v = 2^n$ vertices

$$\begin{gathered}\Delta(\theta) \in \operatorname{conv}\left\{D_1, D_2, \ldots, D_{N_v}\right\} \\ := \operatorname{conv}\left\{T^{-1}diag(\rho_1, \ldots, \rho_n)T : \rho_i \in \{\rho_i^{min}, \rho_i^{max}\}, i = 1, \ldots, n\right\}.\end{gathered}$$

Using (2.44), the polytopic set (2.40) can be constructed with $\Lambda_i = I + D_i(A + BK), i \in \mathcal{I}$. A similar embedding procedure can be applied in the general case (when the eigenvalues of A have multiplicity different than one or when they are complex) - see [Cloosterman 2010].

As the numerical complexity of the obtained LMI conditions depends significantly on the number of vertices N_v of the polytopic approximation, one of the challenges is to provide accurate convex polytopes while reducing the number of vertices. For the Jordan decomposition procedure, the number of vertices N_v increases exponentially with the order of the system. A method for reducing the number of vertices has been provided in [Olaru 2008, Lombardi 2012, Lombardi 2009]. However, the method provides a larger polytopic embedding and may result in a conservative stability condition. The challenge is to find a convex embedding that provides a good trade-off between increased accuracy and reduced computational complexity. Methods that are independent of the order of the systems have been proposed by combining polytopic embeddings with norm bounded approximations [Hetel 2006, Hetel 2008, Donkers 2009, Donkers 2011a]. We present briefly an adaptation of the approach based on Taylor series approximation in [Hetel 2006, Hetel 2008], originally used for sampled-data systems with input delay. Note that the transition matrix $\Lambda(h_k)$ with $h_k \in [\underline{h}, \overline{h}]$ can be rewritten as

$$\Lambda(h_k) = \Lambda(\underline{h}) + \Delta(\rho_k)\Psi(\underline{h})$$

where $\rho_k = h_k - \underline{h} \in [0, \overline{h} - \underline{h}]$, $\Delta(\rho) = \int_0^\rho e^{As} ds$ and $\Psi(\underline{h}) = A\Lambda(\underline{h}) + BK$. Using a Taylor series approximation of the matrix exponential, $\Delta(\rho)$ can be expressed as

$$\Delta(\rho) = T_M(\rho) + R_M(\rho)$$

where $T_M(\rho) = \sum_{i=1}^{M} \frac{A^{i-1}\rho^i}{i!}$ is the M^{th} order Taylor series approximation and $R_M(\rho)$ is the reminder. The procedure proposed in [Hetel 2006, Hetel 2008] allows to embed $T_M(\rho)$ in a convex polytope with $N_v = M + 1$ vertices

$$T_M(\rho) \in \text{conv}\left\{U_i, i = 1, \ldots, M+1\right\}, \ \forall\ \rho \in [0, \overline{h} - \underline{h}],$$

where $U_0 = 0$, $U_{i+1} = \frac{(\overline{h}-\underline{h})^i A^{i-1}}{i!} + U_i$, $i = 1, \ldots, M$. Furthermore, an upper bound on the induced Euclidean norm of $R_M(\rho)$ can be computed using the method proposed in [Liou 1966]. To obtain an embedding with $\|R_M(\rho)\|_2 < \gamma_R$ for all $\rho \in [0, \overline{h} - \underline{h}]$ the approximation order M must be chosen such that

$$\frac{\|A\|_2 (\overline{h} - \underline{h})}{M+2} < 1$$

and

$$\frac{\left\|A^M\right\|_2 (\overline{h} - \underline{h})^{M+1}}{(M+1)!} \frac{M+2}{M+2-\|A\|_2 (\overline{h} - \underline{h})} \leq \gamma_R.$$

For this approach the number of vertices is linear in the order M of the Taylor approximation. Stability criteria are obtained in a direct manner by combining LMI methods for polytopic systems with the ones for systems with norm-bounded uncertainty.

Note that for both norm-bounded and polytopic embeddings approaches, the accuracy of the approximation may be significantly increased by dividing $[\underline{h}, \overline{h}]$ into several subintervals and applying the embedding procedure locally [Fujioka 2009a, Oishi 2010, Hetel 2013a, Donkers 2011a]. For example, in the case of the norm-bounded embedding used in Theorem 2.10, the idea is to consider a grid of r "nominal" sampling intervals $\{T_1 < T_2 < \cdots < T_r\}$ and to verify the existence of a symmetric positive definite matrix X and of r parameters $\gamma_i, i = 1, \ldots, r$, such that $\mathcal{M}(T_i, X, \gamma_i) \prec 0$ for all $i = 1, \ldots, r$. When this condition is satisfied, system (2.31) is stable for any time-varying sampling interval $h_k \in \cup_{i=1}^r \mathcal{T}(T_i, \gamma_i)$ where $\mathcal{T}(T_i, \gamma_i) = \left[\underline{h}(T_i, \gamma_i), \overline{h}(T_i, \gamma_i)\right]$ are defined using (2.38), (2.39). Furthermore, it has been shown in [Fujioka 2009a] that using this approach one can approximate the condition (2.32) as accurately as desired, in the sense that if the condition (2.32) holds for $\theta \in [\underline{h}, \overline{h}]$, then necessarily there exists a matrix $X = P^{-1}$, a sufficiently tight grid of parameters $T_i, i = 1, \ldots, r$ and positive scalars $\gamma_i, i = 1, \ldots, r$, such that $\mathcal{M}(T_i, X, \gamma_i) \prec 0$ for all $i = 1, \ldots, r$, and $[\underline{h}, \overline{h}] \subset \cup_{i=1}^r \mathcal{T}(T_i, \gamma_i)$. Such an *asymptotic exactness* property has also been discussed for other embedding approaches [Donkers 2011a, Oishi 2010, Skaf 2009]. The main issue is that using convex embeddings the conservatism with respect to the quadratic stability condition (2.32) can be reduced to any degree at the cost of increased computational complexity. However, the analysis of the asymptotic exactness property does not take into account all numerical implementation aspects. Most of the methods are based on the computation of the matrix exponential for nominal sampling intervals, on the use of the eigenvalue/eigenvectors of the state matrix A or of its characteristic polynomial, etc. Computing any of these elements introduces approximations [Moler 2003] which might influence the numerical implementation of the embedding. The effect of these approximations on the accuracy of the stability analysis needs to be further analysed.

2.1.3.3 A discrete-time approach for nonlinear systems

Results on discrete-time approaches for the control of nonlinear systems with time-varying sampling intervals are quite rare. We present as follows an adaptation of the result from [van de Wouw 2012] which extends earlier stability criteria from [Nešić 2004a, Nešić 1999, Nešić 1999]. Consider the nonlinear system

$$\dot{x}(t) = F\left(x(t), u(t)\right) \tag{2.47}$$

with $F(x,u)$ globally Lipschitz, i.e. there exists $\beta_f > 0$ such that

$$\|F(x_a,u_a) - F(x_b,u_b)\| \leq \beta_f \left(\|x_a - x_b\| + \|u_a - u_b\|\right)$$

for all $x_a, x_b \in \mathbb{R}^n$ and $u_a, u_b \in \mathbb{R}^m$. The control takes the form $u(t) = u_k$ for all $t \in [t_k, t_{k+1})$ and the sampling interval is bounded $h_k = t_{k+1} - t_k \in \mathcal{T} = [\underline{h}, \overline{h}]$, $\forall\ k \in \mathbb{N}$. The exact discrete-time model of the system over the sampling interval is given by

$$x_{k+1} = F^e_{h_k}(x_k, u_k) := x_k + \int_{t_k}^{t_k+h_k} F\left(x(s), u_k\right) ds \tag{2.48}$$

where $x_k = x(t_k)$. Note however that (2.48) is not known in general since it is rare to obtain an analytic solution to a nonlinear initial value problem. In practical problems, approximations are usually used [Stuart 1998, Nešić 2004a]. A simple example is given by the Euler model of (2.47):

$$x_{k+1} = x_k + h_k F\left(x_k, u_k\right).$$

Other approximations can be found in standard books [Stuart 1998] and tutorials [Monaco 2001, Monaco 2007]. The approach in [van de Wouw 2012] considers an approximate model

$$x_{k+1} = F^a_{h^*}\left(x_k, u_k\right), \tag{2.49}$$

obtained for some nominal sampling interval $h^* \in [\underline{h}, \overline{h}]$. Model (2.49) is assumed to be *one-step consistent* [Stuart 1998] with the exact discrete-time plant, i.e. there exists $\hat{\rho} \in \mathcal{K}_\infty$ such that $\|F^a_{h^*}(x,u) - F^e_{h^*}(x,u)\| \leq h^* \hat{\rho}(h^*)\left(\|x\| + \|u\|\right)$, for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ It is considered that the approximate model (2.49) has been used to design a controller

$$u_k = K_{h^*}\left(x_k\right) \tag{2.50}$$

parametrized by the nominal sampling interval h^* and that the closed-loop system (2.49),(2.50) is asymptotically stable. More formally, it is assumed that there exists a candidate Lyapunov function for the approximate closed-loop system (2.49),(2.50), i.e. a function $V_{h^*}(x)$ and $\alpha_i > 0, i = 1,2,3$ such that the involved conditions holds for some $r > 1 : \alpha_1 \|x\|^r \leq V_{h^*}(x) \leq \alpha_2 \|x\|^r$ and

$$\frac{V_{h^*}\left(F^a_{h^*}\left(x, K_{h^*}(x_k)\right)\right) - V_{h^*}(x)}{h} \leq -\alpha_3 \|x\|^r \tag{2.51}$$

for all $x \in \mathbb{R}^n$. Furthermore, the control law $K_{h^*}(\cdot)$ is considered to be linearly bounded, i.e. there exists $\beta_u > 0$ such that $\|K_{h^*}(x)\| \leq \beta_u \|x\|$ for all $x \in \mathbb{R}^n$. The following theorem provides generic results for the robust stability of the exact closed-loop system

$$x_{k+1} = F^e_{h_k}\left(x_k, K_{h^*}(x_k)\right), \tag{2.52}$$

using the fact that the control law $u_k = K_{h^*}(x_k)$ is a stabilizer for the approximate model (2.49).

Theorem 2.11 *[van de Wouw 2012] Consider system* (2.52) *with* $h_k \in [\underline{h}, \overline{h}]$ *for all* $k \in \mathbb{N}$. *Consider the following notation*

$$\beta_a = \Big(2 + \beta_u + (1 + \max(1, \beta_u))(e^{\beta_f \overline{h}} - 1)\Big) + h^* \hat{\rho}(h^*)(1 + \beta_u). \tag{2.53}$$

Assume that the Lyapunov candidate function $V_{h^*}(x)$ *is locally Lipschitz and there exists* $\beta_v > 0$ *such that*

$$\sup_{\zeta \in \partial V_{h^*}(x)} \|z\| \leq \beta_v \|x\|^r$$

for all $x \in \mathbb{R}^n$, *where* $\partial V_{h^*}(x)$ *denotes the generalized differential of Clarke. If there exists* $\beta \in (0, 1)$ *such that*

$$\frac{\beta_v \beta_a^{r-1}}{h^*} \Big(h^* \hat{\rho}(h^*)(1 + \beta_u) + \rho_h(h^*, M_h)\Big) \leq (1 - \beta)\alpha_3 \tag{2.54}$$

is satisfied where

$$\rho_h(h^*, M_h) = e^{\beta_f h^*} \Big((1 + \beta_u)\left(e^{\beta_f M_h} - 1\right)\Big)$$

with $M_h = \max_{h \in [\underline{h}, \overline{h}]} |h - h^*|$, *then there exist* $c, \lambda > 0$ *such that* $\|x_k\| \leq c\|x_0\| e^{-\lambda k \underline{h}}$. *In other words, system* (2.52) *is Globally Exponentially Stable, Uniformly for all* $h_k \in [\underline{h}, \overline{h}]$ *and all* $k \in \mathbb{N}$.

The above theorem is a natural extension of the result in [Nešić 1999, Nešić 2004a] for sampled-data systems with constant sampling intervals. The main condition (2.54) involves two terms. The first term $\beta_v \beta_a^{r-1} \hat{\rho}(h^*)(1 + \beta_u)$ reflects the effect of approximatively discretizing the nominal system using a nominal sampling interval h^*; the second one, $\frac{\beta_v \beta_a^{r-1}}{h^*} \rho_h(h^*, M_h)$ reflects the effect of uncertainty in the sampling interval.

2.1.3.4 Further reading

Control design methodologies based on convex embeddingsare given in [Hetel 2006], [Hetel 2008], [Cloosterman 2010], [Fujioka 2010a], [Mustafa 2013]. See also [Robert 2010] for an LPV design of controllers that are adapted in real time to the value of the sampling interval and [Hetel 2011a] for the case of systems with delay scheduled controllers. Extensions of the discrete-time approach for networked control systems with scheduling protocols can be found in [Donkers 2009, Li 2010, Donkers 2011a, Li 2014, Cela 2014]. For model predictive control of networked control systems see also [Olaru 2008, Gielen 2009, Lombardi 2012]. Lie algebraic criteria for the analysis of systems with time varying sampling have been proposed in [Felicioni 2008] using tools from [Liberzon 1999]. A mixed continuous-discrete approach has also been proposed in [Li 2011].

2.1.4 Input/Output stability approach

In this subsection we present several methods that study sampled-data systems from a robust control point of view. The main idea of the Input/Output stability approach is to consider the sampling error as a perturbation with respect to a nominal continuous-time control-loop. Classical robust control tools are used in order to assess the stability of the sampled-data systems [Zames 1966, Zhou 1996, Megretski 1997]. Some of the presented methods are reminiscent from the Input/Output stability approach used for the analysis of time delay systems [Huang 2000,

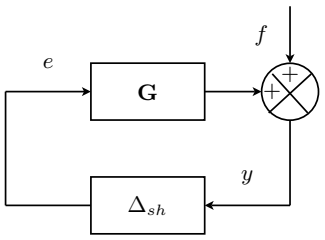


Figure 2.2: Equivalent representation of the sampled-data system, from a robust control theory point of view.

Jun 2001, Niculescu 2001, Gu 2003a, Kao 2004, Fridman 2006, Kao 2007], and have been further developed independently of the time delay approach.

2.1.4.1 Basic idea

Note that the LTI sampled-data system (1.12) can be re-expressed in the form [Mirkin 2007]

$$\dot{x}(t) = \big(\underbrace{A+BK}_{:=A_{cl}}\big)x(t) + \underbrace{BK}_{:=B_{cl}}\big(\underbrace{x(t_k)-x(t)}_{:=e(t)}\big). \tag{2.55}$$

where A_{cl} corresponds to the state matrix of the nominal continuous-time control loop while $e(t)$ represents the error induced by sampling. An essential fact in this approach is that the sampling induced error $e(t) = x(t_k) - x(t)$ can be equivalently re-expressed as

$$e(t) = -\int_{t_k}^{t} \dot{x}(\theta)d\theta, \ \forall t \in [t_k, t_{k+1}). \tag{2.56}$$

Considering $y(t) = \dot{x}(t)$ as an auxiliary output for system (2.55), the sampled-data system (1.12) can be represented equivalently by the feedback interconnection of the *operator* $\Delta_{sh} : \mathcal{L}^n_{2e}[0,\infty) \to \mathcal{L}^n_{2e}[0,\infty)$, $\Delta_{sh} : y \to e$, defined by:

$$e(t) = (\Delta_{sh}\, y)(t) = -\int_{t_k}^{t} y(\theta)d\theta, \quad \forall t \in [t_k, t_{k+1}), \tag{2.57}$$

with the system

$$\begin{cases} \dot{x}(t) = A_{cl}x(t) + B_{cl}e(t), \ x(0) = x_0 \in \mathbb{R}^n, \\ y(t) = C_{cl}x(t) + D_{cl}e(t) = \dot{x}(t), \end{cases} \tag{2.58}$$

where $C_{cl} = A_{cl} = A + BK$ and $D_{cl} = B_{cl} = BK$. Note that the nominal system (2.58) is LTI. It represents the dynamics of the continuous-time system with an additive input perturbation e. The operator Δ_{sh} captures both the effects of sampling and its variations. An alternative model can also be derived by considering the actuation error $e_u(t) = K(x(t_k) - x(t))$ (see [Fujioka 2009c]). The stability of the sampled-data system (1.12) can then be studied by analysing the interconnection (2.57),(2.58).

2.1.4.2 Small gain conditions

To provide constructive stability conditions, the *Small Gain Theorem* [Zames 1966, Zhou 1996, Huang 2000, Gu 2003a] constitutes a simple and powerful tool in the robust control framework. Let $\mathbf{G} : \mathcal{L}_2^n[0,\infty) \to \mathcal{L}_2^n[0,\infty)$ be the linear operator described by the transfer function

$$\hat{\mathbf{G}}(s) = s(sI - A_{cl})^{-1}B_{cl} \tag{2.59}$$

associated to system (2.58). The operator $\mathbf{G}$ captures the behaviour of (2.58) for null initial conditions. Considering the free response of system (2.58), $f(t) = A_{cl}e^{A_{cl}t}x_0,\ \forall\ t \geq 0$, the interconnection (2.57),(2.58) can be re-expressed as

$$\begin{cases} y = \mathbf{G}e + f \\ e = \Delta_{sh}y \end{cases} \tag{2.60}$$

(see Figure 2.2). A direct consequence of the Small Gain Theorem is the fact that if

$$\|\mathbf{G}\|_{2,2}\|\Delta_{sh}\|_{2,2} < 1, \tag{2.61}$$

then the interconnection (2.60) is $\mathcal{L}_2$ stable, i.e. there exist a positive scalar C such that

$$\int_0^t \left(\|y(\theta)\|^2 + \|e(\theta)\|^2\right) d\theta \leq C \int_0^t \|f(\theta)\|^2 \, d\theta \tag{2.62}$$

for any $t > 0$. Here $\|\mathbf{G}\|_{2,2}$, $\|\Delta_{sh}\|_{2,2}$ denote the induced $\mathcal{L}_2$ norms of $\mathbf{G}$ and Δ_{sh}, respectively [13]. The inequality (2.61) is known as the *small gain condition*. Due to the linearity of $\mathbf{G}$, its induced $\mathcal{L}_2$ norm can be readily computed [Zhou 1996] using the $\mathcal{H}_\infty$ norm of its transfer function:

$$\|\mathbf{G}\|_{2,2} = \|\mathbf{G}\|_\infty := \sup_{\omega\in\mathbb{R}} \bar{\sigma}\left(\hat{\mathbf{G}}(j\omega)\right).$$

Furthermore, for the case of LTI sampled-data systems, $\mathcal{L}_2$ stability of the interconnection (2.60) implies asymptotic stability [14] of the sampled-data control loop (1.12):

Theorem 2.12 *[Fujioka 2009c] Suppose that A_{cl} is Hurwitz. System* (1.12) *is Uniformly Asymptotically Stable if the feedback interconnection* (2.60) *is $\mathcal{L}_2$ stable.*

Therefore, providing tractable stability conditions for system (1.12) leads to providing an estimate for the induced $\mathcal{L}_2$ norm of the operator Δ_{sh}. An upper bound of this norm has been computed in [Kao 2004] using a more general uncertain delay operator:

$$\Delta_d : y(t) \to e(t) = (\Delta_d y)(t) := -\int_{t-\tau(t)}^{t} y(\theta)d\theta, \tag{2.63}$$

where $\tau(t) \in [0, \overline{h}]$. The operator Δ_{sh} is a particular case of Δ_d with $\tau(t) = t - t_k,\ \forall t \geq 0, k \in \mathbb{N}$.

Lemma 2.13 *[Kao 2004] The $\mathcal{L}_2$-induced norm of the operator Δ_d in* (2.63) *is bounded by $\overline{h}$.*

[13] Given an operator $\mathbf{G} : \mathcal{L}_2^n[0,\infty) \to \mathcal{L}_2^n[0,\infty)$, its induced $\mathcal{L}_2$ norm is defined as $\|\mathbf{G}\|_{2,2} := \sup_{u\neq 0} \frac{\|\mathbf{G}u\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}}$.

[14] For relations with exponential stability see also [Fridman 2014].

Using this property, and the fact that the operator Δ_d satisfies $M\Delta_d = \Delta_d M$ for all $M \in \mathbb{R}^{n\times n}$, Mirkin [Mirkin 2007] provided the following $\mathcal{L}_2$ stability conditions

$$\exists\, M \in \mathbb{R}^{n\times n},\ M \succ 0 \quad \text{such that } \|M\hat{\mathbf{G}}(s)M^{-1}\|_\infty < \frac{1}{\overline{h}}, \tag{2.64}$$

which is a consequence of the Scaled Small Gain Theorem [Skelton 1998]. Interestingly, it is also shown that (2.64) is related to the condition in [Fridman 2004]. The same LMI can be used to check both conditions. Mirkin then showed that the bound on the $\mathcal{L}_2$ induced norm can be enhanced by exploiting the properties of Δ_{sh}.

Lemma 2.14 *[Mirkin 2007]The $\mathcal{L}_2$-induced norm of the operator Δ_{sh} is bounded by $\delta_0 = \frac{2}{\pi}\overline{h}$, and thus*

$$\int_0^{+\infty} \|(\Delta_{sh}y)(\theta)\|^2 d\theta \leq \int_0^{+\infty} \delta_0^2 \|y(\theta)\|^2 d\theta, \tag{2.65}$$

for all $y \in \mathcal{L}_2^n[0,\infty)$.

This bound on the induced $\mathcal{L}_2$ norm of Δ_{sh} is actually exact and it is attained when there exists an index $k \in \mathbb{N}$ such that $t_{k+1} - t_k = \overline{h}$. This leads to the following sufficient $\mathcal{L}_2$ stability condition, improving (2.64):

$$\exists\, M \in \mathbb{R}^{n\times n},\ M \succ 0 \quad \text{such that } \|M\hat{\mathbf{G}}(s)M^{-1}\|_\infty < \frac{\pi}{2\overline{h}}. \tag{2.66}$$

Note that the upper bound on induced $\mathcal{L}_2$ norm of Δ_{sh} can also be related to the Wirtinger's inequalities [Liu 2010] used in the time delay approach. In practice, condition (2.66) is readily verifiable via standard LMI for the estimation of the $\mathcal{H}_\infty$ norm of LTI systems [Mirkin 2007, Skelton 1998, Gu 2003a]

$$\begin{bmatrix} XA_{cl} + A_{cl}^T X & \frac{2}{\pi}\overline{h}XBK & A_{cl}^T Y \\ * & -Y & \frac{2}{\pi}\overline{h}K^T B^T Y \\ * & * & -Y \end{bmatrix} \prec 0 \tag{2.67}$$

to be solved for $X, Y \succ 0$ (obtained with $Y = M^2$).

2.1.4.3 Integral Quadratic Constraints

For the case of LTI sampled-data systems (1.12), the properties of the operator Δ_{sh} in (2.57) can be further exploited in the framework of Integral Quadratic Constraints (IQC) [Megretski 1997, Ebihara 2015]. Less conservative stability conditions can be obtained. While very general definitions of IQCs are available in the literature [Megretski 1997], we restrict ourselves here to IQCs defined by symmetric matrices Π with real elements have been used for stability analysis. Roughly speaking, the bounded operator Δ_{sh} in (2.57), with input y and output e, is said to *satisfy the IQC defined by* the symmetric matrix Π if

$$\int_0^\infty \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix}^T \Pi \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix} d\theta \geq 0 \tag{2.68}$$

for all $y \in \mathcal{L}_2^n[0,\infty)$ and $e = \Delta_{sh}y$. We present as follows a simplified version of the classical IQC Theorem [Megretski 1997] that can be used in order to derive stability conditions for the interconnection (2.60).

Theorem 2.15 *[Megretski 1997] Consider the interconnection* (2.60) *describing the LTI sampled-data system* (1.12) *and the bounded operator* Δ_{sh} *in* (2.57). *Suppose that* $A_{cl} = A+BK$ *is Hurwitz and assume that there exists a matrix*

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^T & \Pi_{22} \end{bmatrix} \tag{2.69}$$

with $\Pi_{11}, \Pi_{12}, \Pi_{22} \in \mathbb{R}^{n\times n}$, $\Pi_{11} \succeq 0$, $\Pi_{22} \preceq 0$, *such that the operator* Δ_{sh} *satisfies the IQC defined by* Π; *there exists* $\epsilon > 0$ *such that*

$$\begin{bmatrix} \hat{\boldsymbol{G}}(j\omega) \\ I \end{bmatrix}^\star \Pi \begin{bmatrix} \hat{\boldsymbol{G}}(j\omega) \\ I \end{bmatrix} \preceq -\epsilon I, \ \forall\ \omega \in \mathbb{R}. \tag{2.70}$$

Then the interconnection (2.60) *is* $\mathcal{L}_2$ *stable.*

Using Theorem 2.12, the conditions of Theorem 2.15 also imply uniform asymptotic stability of the sampled-data system (1.12). Condition (2.70) can be converted into a frequency independent finite dimensional LMI using the Kalman-Yakubovich-Popov Lemma [Rantzer 1996]:

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} & P B_{cl} \\ B_{cl}^T P & 0 \end{bmatrix} + \begin{bmatrix} C_{cl} & D_{cl} \\ 0 & I \end{bmatrix}^T \Pi \begin{bmatrix} C_{cl} & D_{cl} \\ 0 & I \end{bmatrix} \prec 0 \tag{2.71}$$

to be solved for $P \succ 0$.

As an example, a simple IQC can be obtained directly from Lemma 2.14. Note that inequality (2.65) implies that Δ_{sh} satisfies the IQC defined by

$$\Pi = \begin{bmatrix} \left(\frac{2\overline{h}}{\pi}\right)^2 I & 0 \\ 0 & -I \end{bmatrix}. \tag{2.72}$$

For this IQC, condition (2.70) yields to the standard small gain criteria

$$\left(\frac{2\overline{h}}{\pi}\right)^2 \hat{\mathbf{G}}^\star(j\omega)\hat{\mathbf{G}}(j\omega) \prec I, \ \forall\ \omega \in \mathbb{R}, \tag{2.73}$$

which corresponds to a simple condition on the H_∞ norm of $\mathbf{G}$: $\|\hat{\mathbf{G}}(s)\|_\infty < \frac{\pi}{2\overline{h}}$.

Fujioka [Fujioka 2009c] showed that the operator Δ_{sh} also satisfies the following passivity-like property.

Lemma 2.16 *[Fujioka 2009c] The operator* Δ_{sh} *defined in* (2.57) *satisfies*

$$\int_0^{+\infty} y^T(\theta)(\Delta_{sh}y)(\theta)d\theta \leq 0, \tag{2.74}$$

for all $y \in \mathcal{L}_2^n[0,\infty)$.

It is important to note that if Δ_{sh} satisfies several IQC defined by matrices $\Pi_1, \Pi_2, \ldots, \Pi_r$, then a sufficient condition for stability that takes into account all the properties is given by the existence of positive scalars $\alpha_1, \alpha_2, \ldots, \alpha_r$ such that condition (2.70) holds with $\Pi = \alpha_1\Pi_2 + \alpha_2\Pi_2 + \ldots, \alpha_r\Pi_r$. The properties of Δ_{sh} in Lemma 2.14 and Lemma 2.16 can be generalized [Fujioka 2009c] using scaling matrices $0 \preceq Y \in \mathbb{R}^{n\times n}$, $0 \prec X \in \mathbb{R}^{n\times n}$ and grouped into the following IQC:

$$\int_0^\infty \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix}^T \begin{bmatrix} \delta_0^2 X & -Y \\ -Y & -X \end{bmatrix} \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix} d\theta \geq 0 \tag{2.75}$$

which holds for all $y \in \mathcal{L}_2^n[0,\infty)$ and $e = \Delta_{sh} y$ with $\delta_0 = \frac{2\overline{h}}{\pi}$. Using the integral property (2.75) and Theorem 2.12, Fujioka [Fujioka 2009c] has proposed the following stability condition.

Theorem 2.17 *[Fujioka 2009c] The system* (1.12) *is Globally Uniformly Asymptotically Stable for any sampling sequence with* $t_{k+1} - t_k \leq \overline{h}$ *if there exist* $0 \prec P \in \mathbb{R}^{n\times n}$, $0 \prec X \in \mathbb{R}^{n\times n}$, $0 \preceq Y \in \mathbb{R}^{n\times n}$ *satisfying*

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} & P B_{cl} \\ B_{cl}^T P & 0 \end{bmatrix} + \begin{bmatrix} C_{cl} & D_{cl} \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \delta_0^2 X & -Y \\ -Y & -X \end{bmatrix} \begin{bmatrix} C_{cl} & D_{cl} \\ 0 & I \end{bmatrix} \prec 0. \tag{2.76}$$

Taking into account more properties of the operator Δ_{sh} may lead to less conservative results. Nevertheless, since the analysis is of a frequency domain nature, the IQC approach is only applicable to LTI systems. However, one may note that input delays, several performance specifications and classical nonlinearities (sector bounded, saturations, etc.) can be characterized by elementary operators and IQCs [Megretski 1997]. A more complex system can be described by an interconnection of an LTI system and a single block diagonal operator representing the different perturbing elements. Once the IQCs for the different perturbing elements are available, stability of more complex systems is then a rather straightforward matter of defining a single aggregate IQC. This point enhances the applicability of the IQC approach.

2.1.4.4 Further reading

Some of the elements presented in Section 2.1.3 concerning the use of norm-bounded approximations of the matrix exponential [Fujioka 2009a] can also be interpreted in the Input/Output approach as the application of the Small Gain Theorem to a discrete-time model. Other IQCs can be found in [Fujioka 2009b, Fujioka 2011a]. An approach based on IQCs for the discrete-time model has been proposed recently in [Kao 2013]. For more general nonlinear networked systems, approaches considering sampling as a perturbations can be found in [Walsh 2001, Nešić 2004b, Chen 2014]. See also the work in [Liberzon 2006]. The boundedness properties of the sampling operator Δ_{sh} from Lemma 2.14 from [Mirkin 2007] can be related with the Wirtinger's inequalities used in the time delay approach [Liu 2010, Seuret 2013a, Seuret 2014]. Motivated by the approach presented in [Fridman 2010] in the input delay framework, the sampling effect has been recently described by a new operator in [Kao 2014].

2.2 Sampling as a control parameter

In this section we briefly present the main research directions and some problems concerning the case when the sampling interval h_k (or equivalently the sequence of sampling $\sigma = \{t_k\}_{k\in\mathbb{N}}$) is considered to be a control parameter that can be modified in order to ensure desired properties in terms of stability and resource utilization. From the real-time control point of view, this formulation corresponds to designing a scheduling mechanism that triggers the sampler [Velasco 2003]. The problem has attracted sporadically the attention of the control system's community since the early ages of sampled-data control [Jury 1959, Dorf 1962]. With the spring of *event-* and *self-triggered* control techniques [Årzén 1999, Åström 1999, Velasco 2003] it has become a very popular topic [Heemels 2012].

Let us consider the nonlinear system (1.2) and the controller (1.3) with a given sampling sequence $\sigma = \{t_k\}_{k\in\mathbb{N}}$. Clearly, the asymptotic stability of system holds when the sampling sequence σ satisfies $h_k = t_{k+1} - t_k \in (0, \bar{h}]$ for all $k \in \mathbb{N}$, where $\bar{h}$ represents the MSI for which the system is asymptotically stable under arbitrary sampling. A basic problem in designing a

sampling sequence $\sigma = \{t_k\}_{k\in\mathbb{N}}$ is to ensure the stability of the system while optimizing some Performance Index associated to the frequency of sampling. Most of the time, sampling sequences are compared in simulation based on the *mean sampling interval*. Given σ, one possible choice of Performance Index to be maximized could be

$$\mathcal{J}(\sigma) = \liminf_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} (t_{k+1} - t_k). \tag{2.77}$$

Generally, the goal is to find sequences that ensure stability and have the mean sampling interval larger then the maximum sampling interval admissible in the periodic and arbitrary varying case. Using the Performance Index (2.77), the following basic problem can be mathematically formalized:

- **Problem B** (Optimal sampling sequence): Consider the nonlinear system (1.2) and the controller (1.3). Design a sampling sequence σ maximizing the Performance Index $\mathcal{J}(\sigma)$ in (2.77) while ensuring the stability of the closed-loop system (1.1),(1.2),(1.3),(1.4).

Various alternative formalizations of Problem B can be imagined by considering other performance indexes or Cost Functions (e.g. $\mathcal{J}_c(\sigma) = \sum_{k=0}^{\infty} e^{-(t_{k+1}-t_k)}$) to be maximized or minimized (see for instance [Hsia 1974, Ma 1976] for a finite horizon formulation). A stochastic formulation of the problem can be found in [Cogill 2007, Molin 2013]. Additionally, it is possible to formulate a more complex problem in which one needs to find simultaneously the sampling sequence and system input, as in the *minimum attention control* formulation [Brockett 1997, Donkers 2011b, Marchand 2013].

While the research in the case of arbitrary sampling has reached an advanced phase of development, Problem B is largely open. Due to the complexity of Problem B, simplified versions are under study. For example, stability of sampled-data systems over periodic sequences of sampling has been investigated in [Jury 1959, Li 2010, Seuret 2012]. The optimization of sampling sequences over a finite horizon has been considered since the early works in [Hsia 1974, Ma 1976]. For both practical and theoretical reasons, the design of *state-dependent* (closed-loop) sampling sequences, in which the sampling is triggered according to the system state, represents a topic of interest. Basic ideas appeared in the '60s in the context of *adaptive sampling* [Dorf 1962, de la Sen 1996] and the topic is currently under study in the framework of *event-/self-triggered control* [Heemels 2012].

2.2.1 Event-Triggered (ET) Control

The basic idea of event-triggered control schemes [Årzén 1999], [Åström 1999], [Åström 2002], [Heemels 2012] is to continuously monitor the system state and to trigger the sampling only when necessary, according to the desired performance of the system. A sampling event is generated when the system's state crosses some frontier in the state-space. Let us re-consider the hybrid model of an LTI sampled-data system

$$\left\{\begin{array}{rcll} \left.\begin{array}{rcl} \dot{x} &=& Ax + BK\hat{x} \\ \dot{\hat{x}} &=& 0 \\ \dot{\tau} &=& 1 \end{array}\right\} & & & \left[x, \quad \hat{x}, \quad \tau\right] \in C, \\ \left.\begin{array}{rcl} x^+ &=& x \\ \hat{x}^+ &=& x \\ \tau^+ &=& 0 \end{array}\right\} & & & \left[x, \quad \hat{x}, \quad \tau\right] \in D \end{array}\right. \tag{2.78}$$

where $\hat{x}$ represents the sampled version of the state and τ the clock measuring the time since the last sampling instant. In the classical time-triggered sampling context (2.22), the sets C and D implicitly indicating the sampling moments are defined only according to the clock variable τ: when uniform sampling with period T is considered, C is defined by $\tau \in [0, T]$ and D by $\tau = T$. In event-triggered control the idea is to define the sampling triggering sets according to the state variable x and $\hat{x}$. For example, it may be of interest to trigger only when the error $x - \hat{x}$ becomes too large with respect to the system state, i.e. when $\|x(t) - x(t_k)\| \geq \gamma \|x(t)\|$ where $\gamma > 0$ is a design parameter (see [Tabuada 2007]). For this example the sets C and D are:

$$\begin{aligned} C &= \{(x, \hat{x}, \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} : \|x - \hat{x}\| \leq \gamma \|x\|\}, \\ D &= \{(x, \hat{x}, \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} : \|x - \hat{x}\| \geq \gamma \|x\|\}. \end{aligned}$$

Various other types of triggering conditions have been proposed in the literature: *send-on-delta* (Lesbegue sampling, absolute triggering) [Åström 2002, Otanez 2002, Cervin 2007], *send-on-energy* [Miśkowicz 2005], *send-on-area* [Miskowicz 2007], *Lyapunov sampling* [Velasco 2009, Seuret 2013b, Fiter 2015, Postoyan 2015], etc.

Note that in event-triggering control, the sampling sequence $\sigma = \{t_k\}_{k \in \mathbb{N}}$ is implicitly defined as:

$$t_{k+1} = \min \{t : t \geq t_k, (x, \hat{x}, \tau) \in D\}. \tag{2.79}$$

The value h^* for which $t_{k+1} - t_k \geq h^*$ for all $k \in \mathbb{N}$ and all initial conditions is called the *minimum inter-event time*. In the general case the implicit definition of the sampling sequence does not guarantee anything about the "well posedness" of the closed-loop system in terms of existence of solutions, or concerning the existence of a minimum interval between two consecutive events. In particular cases of event-triggered control Zeno phenomena may occur, i.e. the minimum inter-event time h^* is zero[15] [Marchand 2013, Donkers 2012, Borgers 2014]. This represents an important drawback since the system is converging to a continuous-time control implementation instead of a sampled-data one. To avoid it, various systematic design methodologies for event-triggered control with stability guarantees and no Zeno behavior have been proposed: see [Tabuada 2007, Wang 2008, Wang 2009, Lunze 2010] based on the *Input/Output stability approach*, [Donkers 2012, Seuret 2013b, Forni 2014, Postoyan 2015] using *hybrid models*, [Yue 2013, Peng 2013, Fiter 2015] based on the *time-delay approach*. See also [Michiels 2005] where the delay has a stabilizing affect on control. Note that Zeno phenomena can be easily avoided by including restrictions on the clock variable when defining the jump set D. For example, one may add next to the constraints on x and $\hat{x}$, a constraint that guarantees that sampling occurs only if τ is greater than some minimum desired inter-execution time [Forni 2014,Fiter 2015,Postoyan 2015]. Additionally, the triggering condition may be verified on a discrete sequence of time, as in the *Periodic Event-Trigger* (PET) control [Heemels 2013,Postoyan 2013], or in [Eqtami 2010], where the event-triggered control problem is formulated directly in discrete-time.

2.2.2 Self-Triggered (ST) Control

The term *self-triggered control* was initially proposed by [Velasco 2003] in the context of real time systems. The recent articles [Wang 2009, Anta 2010] have attracted the attention of the control system community. Note that basic ideas related to self-triggered control appeared in the '60s (see [Dorf 1962, Hsia 1974, de la Sen 1996] and the references therein). We point also to

[15] the system requires infinitely fast sampling

the pioneering work in [Hsu 1987] where elements concerning the use of Lyapunov arguments for the design of self-triggering control laws can be found.

In self-triggering, at each sampling time it is computed both the sampled-data control value (to be sent to the actuators) and the next sampling instant. The main idea is to use the value of the state at sampling times and knowledge about the system dynamics in order to predict the next time instant a control update is needed. A self-triggering control scheme is described by a sampling function $h : \mathbb{R}^n \to \mathbb{R}^+ \setminus \{0\}$ which, at each sampling time $t_k, k \in \mathbb{N}$, indicates the value of the current sampling interval according to the system state. The sampling sequence $\sigma = \{t_k\}_{k\in\mathbb{N}}$ is formulated explicitly as

$$t_{k+1} = t_k + h\left(x_k\right), \tag{2.80}$$

where $x_k = x(t_k)$. Very often, the synthesis of a self-triggered control scheme is based on a pre-existing event-triggered control mechanism. In this context, it is aimed at designing the sampling function by pre-computing, at each sampling instant, an estimation of the next time a sampling event has to be generated. For the example of the LTI system (1.12) with the event-triggered control condition $\|x(t) - x(t_k)\| \geq \gamma\|x(t)\|$, one may want to design the sampling function:

$$h(x_k) = \max\left\{\theta > 0 : \|(\Lambda(\theta) - I)x_k\| < \gamma\|\Lambda(\theta)x_k\|\right\} \tag{2.81}$$

where $\Lambda(\theta) = e^{A\theta} + \int_0^\theta e^{As}dsBK$. An important issue is the complexity of the algorithms used for the online implementation of the sampling function $h(x)$. Even for the simple case (2.81), the algorithms may be quite complex since they involve solving hyperbolic inequalities. In practice, simple approximations of such sampling function must be used.

Self-triggered control mechanisms with stability guarantees have been proposed in [Wang 2009, Wang 2010, Anta 2010, Forni 2010, Benedetto 2013] using the *Input/Output stability approach* and in [Tiberi 2013] using *discrete-time* Lyapunov functions. In the following chapter we will presents results from [Fiter 2012a, Fiter 2015, Fiter 2012b] using *convex embeddings* and the *time-delay system* approach.

However, the potential of the approaches used for the arbitrary sampling problem is far from being fully exploited. The tools presented in Section 2.1 may be useful for various aspects in Problem B: deriving new event-/self-triggering mechanisms, providing less conservative estimations of the minimum inter-event time h^*, etc.

2.3 Conclusion

This chapter has presented some of the basic concepts and recent research directions in sampled-data systems: time-delay, hybrid, discrete-time and input-output models; Lyapunov and frequency domain methods for the stability for systems with arbitrary sampling intervals. For the case of linear systems, it is shown that several pioneering approaches exist in the literature. These approaches share the advantage of using LMIs, thus they are numerically tractable. The maximum sampling interval that guarantees the stability can be estimated accurately using discrete-time methods. The robustness with respect to perturbation and the behaviour of the system between sampling times can be taken into account using time-delay, impulsive approaches or Input/Output approaches. However, the analyzis problem is still largely open and it is still a challenging problem to extend these methods to the nonlinear case where the main difficulty is to provide constructive methods for the quantitative estimation of the maximum sampling interval that preserves stability.

It is to be emphasized that this overview is far from being exhaustive. The research topic of systems with time-varying sampling is still wide open and continuously growing. In particular, the control of sampling is presently receiving a lot of attention, as it was shown in Section 2.2. It is worth noticing that the subject lies at the intersection of four important axes in Control Theory (time-delay, hybrid, LPV and input-output approaches) and it has a stimulating impact. As we will see in the following chapter, methods and tools can be transferred from one approach to another.

Chapter 3

Main contributions

In this chapter, we will present our main contributions to the study of aperiodic sampled-data systems. Over the last years, our research effort bas been dedicated to the analysis of various classes of systems (LTI, LPV, bilinear, polynomial, nonlinear affine) with both continuous and switching controllers (see Figure 3.1 for an illustration). We have tried to address the main chal-

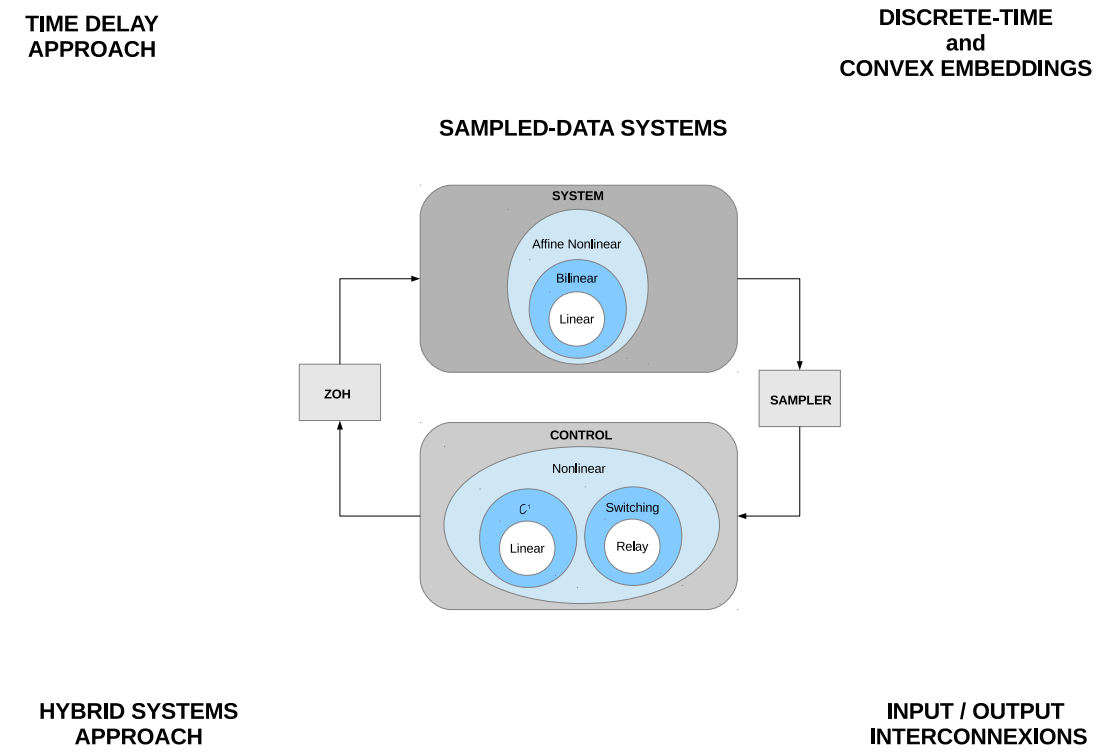


Figure 3.1: Illustration of main contributions: we have studied various classes of sampled-data systems (linear, bilinear, nonlinear affine) as time-delay systems, hybrid systems, input-output interconnexions or discrete-time (LPV) systems.

lenges of sampled-data systems from all possible angles. We have used the classical approaches (delay, hybrid, discrete-time, input/output interconnexions) for more complex classes of systems than the ones presented in the literature and we have proposed new approaches when needed. We present first the case of linear systems in order to show how the conservatism in the analysis can be reduced. Furthermore, we show a continuous-time approach based on convex embeddings that is able to combine the advantages of both time-delay methods (inter-sampling behaviour, robustness to perturbations) and discrete-time ones (numerical accuracy). The approach is applied to the self-triggering control problem allowing to optimize the design of sampling maps. Next, we present some contributions to the case of bilinear systems, which represents a simple class of nonlinear systems, and can be considered as an intermediate between linear and nonlinear systems. Two approaches are being considered for bilinear systems: the first one relies on the hybrid dynamical systems framework, while the second one is based on an extension of the Input/Output approach using tools inspired from the Dissipativity Theory. After that, we will consider a more general class of affine nonlinear systems, with aperiodic sampled-data control. The main contribution is to show how the frequency domain methods existing in the Input / Output stability approach can be extended in a constructive manner to more general nonlinear systems affine in the input. At last, we will discuss the sampled-data implementation of discontinuous controllers, as encountered in relay feedback control and switched systems.

3.1 Linear Time Invariant sampled-data system

As we have seen in Chapter 1, the control of sampled-data systems is a challenging problem, even for the case of Linear Time Invariant systems. As follows, some contributions to these systems are presented[16].

3.1.1 Discrete-time analysis based on quasi-quadratic Lyapunov functions

Let us recall the LTI sampled-data system

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & Ax(t)+BKx(t_k),\ \forall t\in[t_k,t_{k+1}),\ \forall k\in\mathbb{N},\\ t_{k+1} & = & t_k+h_k,\ \forall k\in\mathbb{N},\\ t_0 & = & 0,\ x(t_0)=x_0\in\mathbb{R}^n. \end{array}\right. \tag{3.1}$$

In what follows we consider that h_k takes values in a compact set $\mathcal{T}\subset\mathbb{R}_+$. Consider the discrete-time model associated to (3.1)

$$x_{k+1}=\Lambda(h_k)x_k \tag{3.2}$$

with

$$\Lambda(\theta)=\mathrm{e}^{A\theta}+\int_0^\theta \mathrm{e}^{As}dsBK \tag{3.3}$$

for $\theta\in\mathbb{R}$. For h_k arbitrarily varying in the compact set $\mathcal{T}$, system (3.2) is a discrete-time Linear Parameter Varying (LPV) system [Rugh 2000,Kamen 1984,Molchanov 1989], with the transition matrix $\Lambda(h_k)$ depending on the sampling interval h_k. Various methods are available for studying the stability of discrete-time LPV systems. For polytopic LPV systems, stability criteria have

[16] The results presented in this section have been developed in the context of the PhD Thesis of Christophe FITER as well as in collaboration with Prof. Jean-Pierre Richard, Prof. Wilfrid PERRUQUETTI and Ass. Prof. Alexandre KRUSZEWSKI.

been proposed by analysing the *joint spectral radius* [Blondel 2005] or by checking the existence of *quasi-quadratic* [Molchanov 1989, Hu 2010], *parameter dependent* [Daafouz 2001, Peaucelle 2000, Peaucelle 2001], *path-dependent* [Lee 2006], *non-monotonic* [Megretski 1996, Kruszewski 2008, Ahmadi 2008] and *composite quadratic* [Hu 2010] Lyapunov functions. Lie algebraic conditions can be found in [Gurvits 1995], [Liberzon 2003a]. However, system (3.2) is not a polytopic LPV system but an LPV system where the transition matrix $\Lambda(h_k)$ takes values in a compact set

$$\mathcal{W} := \{\Lambda(\theta),\, \theta \in \mathcal{T}\}. \tag{3.4}$$

The following theorem from [Hetel 2011b] addresses the case of (3.2) and provides necessary and sufficient stability conditions.

Theorem 3.1 *[Hetel 2011b] Consider the continuous-time system* (3.1) *and the discrete-time model* (3.2) *with* $\mathcal{T}$ *a compact subset of* $(0, \infty)$. *The following statements are equivalent:*

1) The equilibrium point $x = 0$ *of* (3.2) *is Globally Uniformly Exponentially Stable.*

2) There exist a $P \succ 0$ *and* $N > 0$ *such that*

$$\left(\prod_{i=1}^{N} \Lambda(\theta_i)\right)^T P \left(\prod_{i=1}^{N} \Lambda(\theta_i)\right) - P \prec 0, \tag{3.5}$$

for any N-length sequence $\{\theta_i\}_{i=1}^N$ *with values in* $\mathcal{T}$, *i.e. the function* $\bar{V}(x) = x^T P x$ *satisfies* $\bar{V}(x_{k+N}) < \bar{V}(x_k)$ *for all* $x_k \neq 0, k \in \mathbb{N}$.

3) There exists a positive definite function $V : \mathbb{R}^n \to \mathbb{R}^+$ *strictly convex, homogeneous (of the second order),* $V(x) = x^T \mathcal{P}_{[x]} x$, *with* $\mathcal{P}_{[\cdot]} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $\mathcal{P}_{[x]} = \mathcal{P}_{[x]}^T = \mathcal{P}_{[ax]}$, $\forall x \neq 0, a \in \mathbb{R}, a \neq 0$ *such that :*

$$V(x) - \max_{\theta\in\mathcal{T}} V\left(\Lambda(\theta)x\right) > 0,\ \forall x \neq 0. \tag{3.6}$$

Condition 2) in Theorem 3.1 corresponds to the existence of a *non-monotonic Lyapunov function* $\bar{V}(x) = x^T P x$, [Megretski 1996, Kruszewski 2008, Ahmadi 2008] which is decreasing every N samples. If the system is stable, then necessarily there exists a finite N and a matrix P such that (3.5) holds. However, checking the existence of a matrix P satisfying (3.5) for a given N represents a set of LMIs which are sufficient only for stability. Note that considering the case $N = 1$ reduces to the classical quadratic stability condition

$$\exists\ P \succ 0,\ \Lambda^T(\theta) P \Lambda(\theta) - P \prec 0,\ \forall \theta \in \mathcal{T} = [\underline{h}, \overline{h}]. \tag{3.7}$$

Condition 3) corresponds to the existence of a quasi-quadratic Lyapunov function [Hu 2010, Molchanov 1989] $V(x) = x^T \mathcal{P}_{[x]} x$. Theorem 3.1 shows the equivalence between quasi-quadratic Lyapunov functions and non-monotonic Lyapunov functions and provides necessary and sufficient conditions for the exponential stability of system (2.31). Note that the theorem goes beyond the results in [Megretski 1996, Kruszewski 2008, Molchanov 1989, Hu 2010] where only the case of polytopic LPV system is treated. In fact the result in Theorem 3.1 applies for any discrete-time LPV system with transition matrices defined on compact sets.

Taking a finite N and a larger convex polytope embedding with a *finite* number of vertices Λ_i, $i = 1, \cdots, N_v$,

$$\overline{\mathcal{W}} := \left\{ \sum_{i=1}^{N_v} \alpha_i \Lambda_i \,|\, \alpha_i \geq 0,\, i \in \mathcal{I},\, \sum_{i=1}^{N_v} \alpha_i = 1 \right\}, \tag{3.8}$$

in such a way that the set $\mathcal{W}$ in (3.4) is embedded in $\overline{\mathcal{W}}$, $\mathcal{W} \subseteq \overline{\mathcal{W}}$, Condition 2) in Theorem 3.1 leads to a numerically tractable LMI problem.

Given $k \in \mathbb{N}$, let $\mathcal{I}_k$, denote the set $\{1, 2, \ldots, k\} \subset \mathbb{N}$. For $k \in \mathbb{N}$, let us denote by

$$S_k = \left\{ \psi : \psi = \{\mu_i\}_{i=0}^{k-1},\ \mu_i \in \mathcal{I}_{N_v}, \forall i = 0, \ldots, k-1 \right\}$$

the set of all $k-$ length sequences with values in $\mathcal{I}_{N_v}$.

The following theorem provides constructive LMI conditions for stability analysis.

Theorem 3.2 *[Hetel 2011b] Consider system* (3.1)*, the discrete-time model* (3.2)*, the set of vertices* $\mathcal{Z} = \{\Lambda_i,\ i = 1, \ldots, N_v\}$ *of* $\overline{\mathcal{W}}$ *in* (3.8) *and the set*

$$\mathcal{Y}(\mathcal{Z}) = \left\{ Y : Y = \Pi_{i=0}^{N-1} Z_{\mu_i},\ Z_{\mu_i} \in \mathcal{Z},\ \mu_i \in \mathcal{I}_{N_v} \right\}. \tag{3.9}$$

If there exist a positive integer N *and a matrix* $P = P^T \succ 0$ *that satisfy*

$$P \succ Y^T P Y,\ \forall\ Y\ \in \mathcal{Y}(\mathcal{Z}),\ \textit{then} \tag{3.10}$$

1) the equilibrium point $x = 0$ *of* (3.2) *is Globally Uniformly Exponentially stable;*

2) there exists a quasi-quadratic Lyapunov function with the form

$$V(x) = \max_{i \in \mathcal{I}_M} x^T L_i x,\ M = {N_v}^{N-1}, \tag{3.11}$$

which is strictly decreasing along the solutions of (3.2)*. The matrices* $L_i,\ i \in \mathcal{I}_M$*, are obtained using an enumeration of the elements in the set*

$$\Omega(N) = \left\{ \sum_{j=1}^{N-1} \left(\Pi_{r=1}^{j} Z_{\mu_r} \right)^T P \left(\Pi_{r=1}^{j} Z_{\mu_r} \right) + P,\quad \psi = \{\mu_r\}_{r=1}^{N-1} \in S_{N-1} \right\}.$$

The test involves a finite number of LMI $(N_v^N + 1)$ that are sufficient for stability. The accuracy of the stability characterization from conditions (3.10) mainly depends on two factors: the length N of the horizon of analysis, and the accuracy of the polytopic embedding $\mathcal{W}$ described in (3.8) (for more details on such convex embedding see the survey in Chapter 1). The amount of conservatism introduced in the approach can be tuned according to these parameters.

Example 3.3 *Consider an LTI system* (3.1) *described by :*

$$A = \begin{bmatrix} -0.5 & 0 \\ 0 & 3.5 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \textit{and } K = \begin{bmatrix} 1.02 & -5.62 \end{bmatrix}.$$

$\Lambda(h)$ *is Schur for any sampling interval* $h \in (0, 0.46]$*. However, switching among different values of* h *in this interval may lead to an unstable behaviour: one can notice that although both* $\Lambda(0.25)$ *and* $\Lambda(0.45)$ *are Schur, the transition matrix*

$$\Phi = \Lambda(0.25)\Lambda(0.25)\Lambda(0.45)$$

has the eigenvalues outside the unit circle. This implies that when the sampling period varies in a periodic pattern $0.25 \to 0.25 \to 0.45 \to 0.25 \to 0.25 \to 0.45 \ldots$*, the closed-loop system is*

unstable. A similar unstable behaviour can be observed for $h \in \{0.1, 0.43\}$ *since the transition matrix*

$$\Phi = (\Lambda(0.1))^6 \, \Lambda(0.43)$$

is not Schur. Consider that the sampling interval arbitrary switches among two values $\{0.1,\ h_{max}\}$ *with* h_{max} *a parameter to estimate.*

Using the set of LMI (3.10)*, it is possible to find a quasi-quadratic Lyapunov function of the form* (3.11) *for* $N = 7$ *up to* $h_{max} = 0.41$ *(which is very close to the value* 0.43 *for which an unstable sampling path exists). For* $h_{max} = 0.41$*, using the existing LMI solvers, it is impossible to find a common quadratic Lyapunov function [Fujioka 2009a, Sala 2005] or a poly-quadratic one [Hetel 2007, Cloosterman 2010]. In fact, the maximum values of* h_{max} *that can be obtained from quadratic and poly-quadratic Lyapunov functions are* $h_{max} = 0.36$ *and* $h_{max} = 0.39$*, respectively.*

It is also interesting to compare the h_{max} *(computed using the LMIs* (3.10)*) with the maximum upper-bounds obtained in recent papers:* $h_{max} = 0.165$ *[Naghshtabrizi 2008],* 0.198 *[Seuret 2012],* 0.204 *[Fujioka 2009c], or* 0.259 *[Fridman 2010].*

Compared with continuous-time approaches such as the one based on time delay or impulsive models, discrete-time methods profit by involving the integration procedure that implicitly takes into account the continuous-time evolution of the sampling induced delay / sampling counter. Furthermore the accuracy can also be tuned according to the desired computational complexity. This is why, faced to numerical benchmarks, they seem to be less conservatives. However, discrete-time methods also present disadvantages with respect to the continuous-time analysis. The main drawback is the fact that they do not take into account the system behaviour in between sampling times. Besides they become numerically inaccurate when the sampling interval tends to zero.

Example 3.4 *Consider a continuous-time system* (3.1) *described by the following matrices:*

$$A = \begin{bmatrix} 1 & 15 \\ -15 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ K = [5.33\ \ -9.33]\,.$$

The unstable open-loop matrix A *has complex eigenvalues* $1 \pm 15i$*. The gain is obtained by pole assignment, in such a way that the ideal closed-loop system is stabilized and oscillations are reduced : the matrix* $A + BK$ *has the eigenvalues at* $-1 \pm i$*. When the sampling interval takes values in the set* $\mathcal{T} = \{0.91,\ 0.95\}$ *it is possible to find a common quadratic Lyapunov function that is strictly decreasing at the sampling times. Yet, this discrete-time Lyapunov function is increasing in between the sampling instants (see Figure 3.2). In this case a discrete-time analysis would be misleading from a performance point of view (i.e. in terms of the decay rate).*

The previous example shows that it is desirable to provide one method which is able to treat the analysis problem in continuous-time (for inter-sampling issues) and use the advantages of discrete-time methods (in terms of conservatism reduction). Such a method will be presented in the following section.

3.1.2 Continuous-time analysis based on convex embeddings

In the standard discrete-time analysis, the stability is guaranteed without consideration of the intersample behaviour. In practice it is important to provide an estimate of the system's performance in between sampling instants. Furthermore, one of the drawbacks of the discrete-time analysis is the fact that the transition matrix $\Lambda(\theta)$ is close to identity when θ is small. For small

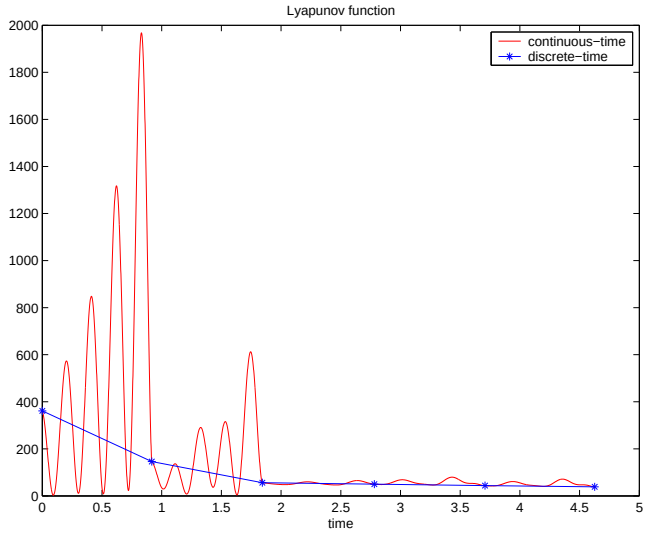


Figure 3.2: Evolution of the Lyapunov function for discrete-time representation in Example 3.4. The function is strictly decreasing at the sampling instants. However, it is increasing in between the sampling interval.

values of the lower bound of the sampling interval the existing stability conditions can be difficult to handle numerically. To avoid this numerical drawback, a continuous-time approach based on convexification arguments has been proposed in [Hetel 2011b, Fiter 2012a] for LTI systems. The approach takes into account the relation

$$x(t) = \Lambda(t - t_k)x(t_k),\ \forall\ t \in [t_k, t_{k+1}),\ k \in \mathbb{N}, \tag{3.12}$$

still referring to the definition of the transition matrix

$$\Lambda(t - t_k) = I + \int_0^{t-t_k} e^{As} ds (A + BK)$$

of system (3.1).

Lemma 3.5 *(adapted from [Fiter 2012a]) Consider system* (3.1) *with* $\mathcal{T} = (0, \overline{h}]$. *Given a positive scalar* λ, *if there exist a matrix* $P = P^T \succ 0$, *such that*

$$\begin{bmatrix} \Lambda(\theta) \\ I \end{bmatrix}^T \begin{bmatrix} A^T P + PA + 2\lambda P & PBK \\ K^T B^T P & 0 \end{bmatrix} \begin{bmatrix} \Lambda(\theta) \\ I \end{bmatrix} \prec 0, \forall \theta \in [0, \overline{h}], \tag{3.13}$$

then the origin of (3.1) *is Globally Exponentially Stable for any arbitrary sampling sequence with* $t_{k+1} - t_k \in (0, \overline{h}]$. *Furthermore, the function* $V(x) = x^T P x$ *satisfies the relation*

$$\dot{V}\left(x(t)\right) \leq -2\lambda V\left(x(t)\right)$$

along the system's solutions.

The lemma presents sufficient conditions for exponential decay of a quadratic Lyapunov function along the solutions of the continuous-time system (3.1) using the exact expression of the transition matrix $\Lambda(.)$. Similarly to the classical discrete-time approach, condition (3.27) is a parametric LMI which is not a computationally tractable by itself. However, it can be reduced to a finite number of LMI conditions using a polytopic embedding of the transition matrix $\Lambda(\theta)$ for $\theta \in [0, \overline{h}]$.

Theorem 3.6 *(adapted from [Hetel 2011b]) Consider system* (3.1) *with* $\mathcal{T} = (0, \overline{h}]$. *Assume that there exists a convex polytope*

$$\tilde{\mathcal{W}} := conv\{\tilde{\Lambda}_1, \tilde{\Lambda}_2, \cdots, \tilde{\Lambda}_{N_v}\}. \tag{3.14}$$

such that $\Lambda(\theta) \in \tilde{\mathcal{W}},\ \forall\ \theta \in [0, \overline{h}]$. *Given a positive scalar* λ, *if there exist matrices* $P = P^T \succ 0$, G_1, G_2 *solution to*

$$\begin{bmatrix} A^T P + PA + 2\lambda P + G_1 + G_1^T & PBK - G_1\tilde{\Lambda}_i + G_2^T \\ K^T B^T P - \tilde{\Lambda}_i^T G_1^T + G_2 & -G_2\tilde{\Lambda}_i - \tilde{\Lambda}_i^T G_2^T \end{bmatrix} \prec 0, \tag{3.15}$$

for all $i = 1, \ldots, N_v$ *then the origin of* (3.1) *is Globally Exponentially Stable for any arbitrary sampling sequence with* $t_{k+1} - t_k \in (0, \overline{h}]$. *Furthermore, the function* $V(x) = x^T P x$ *satisfies the relation*

$$\dot{V}(x(t)) \leq -2\lambda V(x(t))$$

along the system's solutions.

The previous theorem provides constructive conditions for checking the exponential stability of a sampled-data system with performance guarantees for the system's behaviour in between sampling times. However, conditions (3.15) are not feasible in the dead-beat control case, where for some $\theta \in [0, \overline{h}]$, $\Lambda(\theta)$ has eigenvalues at zero. A less conservative approach, combining convex embeddings with tools for time delay systems, has been proposed in [Hetel 2011b, Fiter 2012a], using the Lyapunov-Razumikhin method [Razumikhin 1956]. The originality of this approach is the fact that it is not necessary to require the exponential decay $\dot{V}(x(t)) \leq -\lambda V(x(t))$ everywhere along the system's solutions.

Proposition 3.7 *(adapted from [Fiter 2012a]). Consider system* (3.1) *with* $\mathcal{T} = (0, \overline{h}]$. *Given* $t \geq 0$ *and* $x_0 \in \mathbb{R}^n$, *let* $\varphi(t, x_0)$ *denote the solution of the open-loop system*

$$\dot{x} = Ax + BKx_0$$

at time t, *with the initial condition* $x(0) = x_0$, *i.e.* $\varphi(t, x_0) = \Lambda(t)x_0$. *Given scalars* $\alpha > 1$ *and* $0 < \lambda \leq \frac{\ln(\alpha)}{2\overline{h}}$, *if there exist a quadratic function* $V(x) = x^T P x$, $P = P^T \succ 0$, *such that for all* $x_0 \in \mathbb{R}^n$, *for all* $t \in [0, \overline{h}]$,

$$\frac{d}{dt}V(\varphi(t, x_0)) + 2\lambda V(\varphi(t, x_0)) \leq 0 \tag{3.16}$$

whenever

$$\alpha V(\varphi(t, x_0)) \geq V(x_0) \tag{3.17}$$

then the origin of (3.1) *is is Globally Exponentially Stable for any arbitrary sampling sequence with* $t_{k+1} - t_k \in (0, \overline{h}]$.

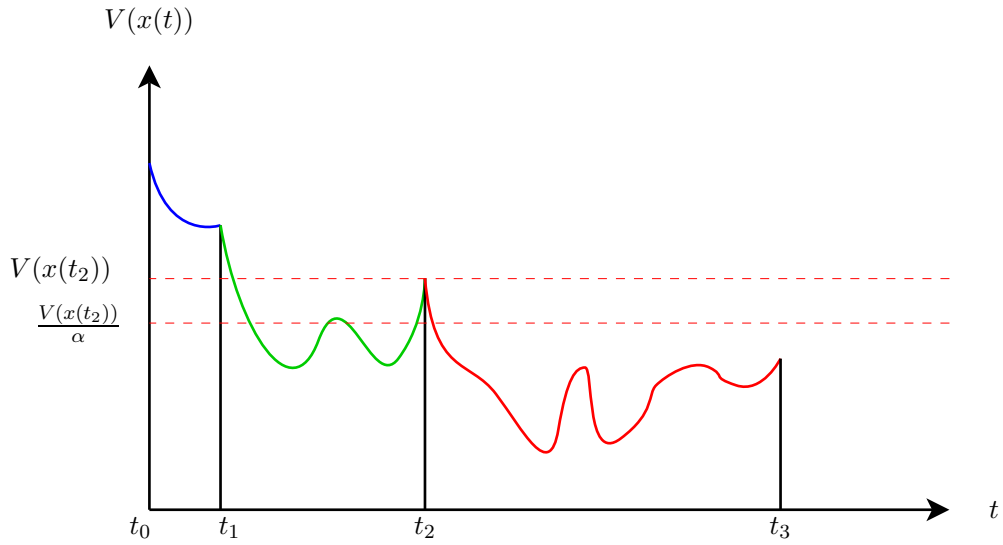


Figure 3.3: Illustration of the Lyapunov-Razumikhin approach: the derivative of $V(x(t))$ has to be negative only at time instants $t \in [t_k, t_{k+1})$ for which $V(x(t)) \geq \frac{1}{\alpha} V(x(t_k))$. The approach ensures that $V(x_{k+1}) < V(x_k)$. However, V is not required to be monotonously decreasing over the sampling interval.

Conditions (3.16),(3.17) in Proposition 3.7 ensure that

$$\dot{V}(x(t)) \leq -2\lambda V(x(t)) \tag{3.18}$$

is required only at times $t \in [t_k, t_{k+1})$ for which

$$V(x(t)) \geq \frac{1}{\alpha} V(x(t_k)).$$

This means that (3.18) has to be satisfied only when the system's solutions are outside a target level set defined according to the value of $V(.)$ at sampling times (a graphical illustration is given in Figure 3.3). α can be seen as a design parameter that can be chosen in order to enforce some performance. The smaller α is, the less restrictive the provided stability conditions will be. When α tends to infinity, the conditions of Proposition 3.7 reduce to the classical stability condition $\dot{V}(x(t)) \leq -2\lambda V(x(t))$ for all $t \in [t_k, t_{k+1})$. When α tends to 1 the provided condition ensure only stability but not attractivity. If λ is chosen to be null and condition (3.16) is enforced to be strict, asymptotic stability is granted. Using Proposition 3.7, the following stability parametric LMI condition is obtained.

Lemma 3.8 *(adapted from [Fiter 2012a]) Consider system* (3.1) *with* $\mathcal{T} = (0, \overline{h}]$. *Given positive scalars* $\alpha > 1$, $0 < \lambda \leq \frac{\ln(\alpha)}{2\overline{h}}$ *if there exist a matrix* $P = P^T \succ 0$ *and a scalar* $\epsilon \geq 0$, *such that*

$$\Phi(P,\theta) = \begin{bmatrix} \Lambda(\theta) \\ I \end{bmatrix}^T \begin{bmatrix} A^TP + PA + (2\lambda + \epsilon\alpha)P & PBK \\ K^TB^TP & -\epsilon P \end{bmatrix} \begin{bmatrix} \Lambda(\theta) \\ I \end{bmatrix} \prec 0, \forall \theta \in [0, \overline{h}], \tag{3.19}$$

then the origin of (3.1) *is Globally Exponentially Stable for any arbitrary sampling sequence with* $t_{k+1} - t_k \in (0, \overline{h}]$.

The conditions of Lemma 3.8 can be reduced to a finite number of LMI conditions using a polytopic embedding of the transition matrix $\Lambda(\theta)$ for $\theta \in [0, \overline{h}]$, similarly to the case treated in Theorem 3.6.

Theorem 3.9 *[Hetel 2011b] Consider system* (3.1) *with* $\mathcal{T} = (0, \overline{h}]$. *Assume that there exists a convex polytope*

$$\tilde{\mathcal{W}} := conv\{\tilde{\Lambda}_1, \tilde{\Lambda}_2, \cdots, \tilde{\Lambda}_{N_v}\}. \tag{3.20}$$

such that $\Lambda(\theta) \in \tilde{\mathcal{W}},\ \forall\ \theta \in [0, \overline{h}]$. *If there exist matrices* $P = P^T \succ 0\ G_1, G_2$ *and* $\epsilon \geq 0$ *solution to*

$$\begin{bmatrix} A^T P + PA + \epsilon P + G_1 + G_1^T & PBK - G_1\tilde{\Lambda}_i + G_2^T \\ K^T B^T P - \tilde{\Lambda}_i^T G_1^T + G_2 & -\epsilon P - G_2\tilde{\Lambda}_i - \tilde{\Lambda}_i^T G_2^T \end{bmatrix} \prec 0, \tag{3.21}$$

for all $i = 1, \ldots, N_v$ *then the origin of* (3.1) *is Globally Asymptotically Stable for any arbitrary sampling sequence with* $t_{k+1} - t_k \in (0, \overline{h}]$.

Note that conditions (3.21) can be expressed as a classical optimization problem that can be solved using a line search algorithm and LMI solvers. The theorem ensures that, within the sampling interval, the Lyapunov-Razumikhin function $V(x) = x^T P x$ is always less than its value at sampling times. However, it is not monotonously decreasing. It can be shown numerically that this approach is less conservative than several continuous-time approaches in the literature. In fact, this stability test is comparable to the one provided in discrete-time using a quadratic Lyapunov function. The advantage with respect to the discrete-time approach is the fact that intersampling behaviour is explicitly taken into account and that a sampling interval tending to zero can be considered as well. A less conservative approach has been proposed in [Fiter 2012a].

Example 3.10 *(Example 3.4 revisited) Consider a continuous-time system* (3.1) *described by the following matrices:*

$$A = \begin{bmatrix} 1 & 15 \\ -15 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ K = [5.33\ \ -9.33]\,.$$

In order to construct a polytopic set embedding $\Lambda(\theta)$, *we use the method proposed in [Hetel 2007] based on a Taylor series expansion. We use a uniform partition of the interval* $[0, \overline{h}]$ *into 10 subintervals and apply locally the embedding method (4th order development). Using Theorem 3.6, a quadratic Lyapunov function can be found up to* $\overline{h} = 0.09$ *(see Figure 3.4). For this example the matrix* $\Lambda(\theta)$ *is singular for* $\theta \approx 0.092$ *which shows that the obtained* $\overline{h}$ *is close to the theoretical bound for quadratic Lyapunov functions.*

The methods in [Mirkin 2007], [Naghshtabrizi 2008], [Fujioka 2009c] and [Fridman 2010] show that the system is stable for $\overline{h} = 0.014$, $\overline{h} = 0.033$, $\overline{h} = 0.07$ *and* $\overline{h} = 0.12$, *respectively. Theorem 3.9 proves the asymptotic stability for* $\theta \in [0, 0.14]$. *Note that using the discrete-time approach (Theorem 3.2), we are able to show the stability for any sequence with* $t_{k+1} - t_k \in [0.001, 0.15]$. *This means that Theorem 3.9 is almost as efficient as the discrete-time approach, with the additional advantage that it takes into account the intersample behavior and very small sampling intervals. Comparing now the number of LMI decision variables, [Mirkin 2007] and [Fujioka 2009c] have* $0.5(n^2+n)+m^2+m = 5$ *variables, [Naghshtabrizi 2008] has* $3.5n^2+1.5n = 17$ *while [Fridman 2010] has* $8n^2+n = 34$. *In Theorem 3.9 there are* $0.5(n^2+n)+2n^2 = 11$ *variables involved in* $N_v + 1 = 51$ *LMI constraints.*

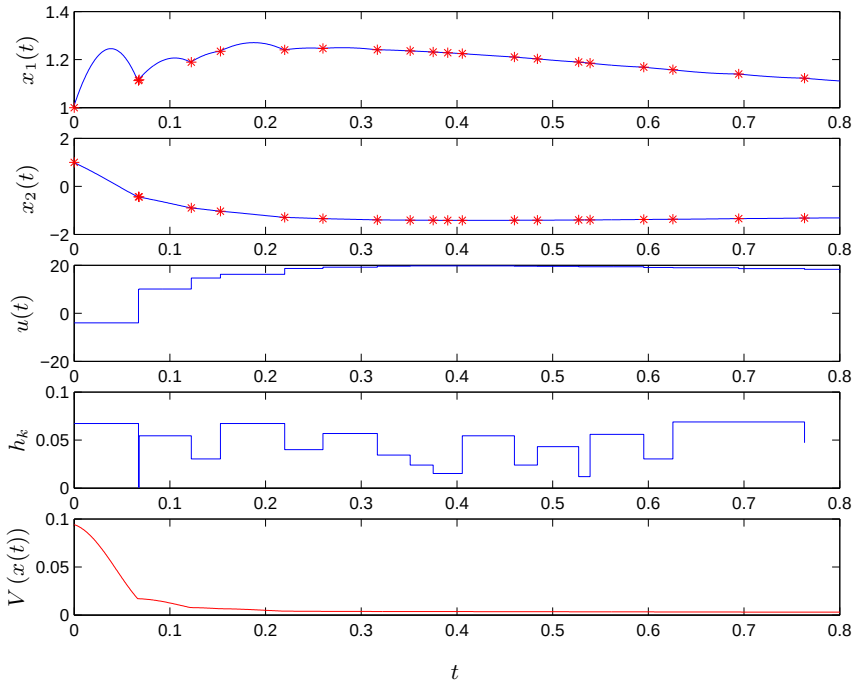


Figure 3.4: Simulation for an arbitrary sequence of sampling intervals with $\overline{h} = 0.09$ for the system in Example 3.10.

3.1.3 Extension to the sampling control problem

As follows we present an extension of the previously presented methodology for the problem of designing stabilizing sampling sequences (Problem B in Chapter 2). Consider the following LTI sampled-data system

$$\begin{cases} \dot{x}(t) &= Ax(t) + BKx_k, \ \forall t \in [t_k, t_{k+1}), \ \forall k \in \mathbb{N}, \\ t_{k+1} &= t_k + h_k, \ \forall k \in \mathbb{N}, \\ t_0 &= 0, \ x(t_0) = x_0 \in \mathbb{R}^n. \end{cases} \tag{3.22}$$

In what follows we consider that the sampling interval h_k is a control parameter that can be modified. We are interested in the design of feedback sampling mechanisms

$$h_k = \tau(x_k), \ \forall \ k \in \mathbb{N}, \tag{3.23}$$

where $\tau : \mathbb{R}^n \to \mathbb{R}_+$ is a sampling function such that $\delta_{min} \leq \tau(x) \leq \delta_{max}$, for strictly positive scalars $\delta_{min} < \delta_{max}$. This control configuration is usually called self-triggered control. The main contribution is to use tools for the robust stability analysis in order to optimize the design of sampling maps for the controlled sampling problem.

The following proposition provides sufficient stability conditions for the interconnection between the sampled-data system (3.22) and the sampling map (3.23).

Proposition 3.11 *(adapted from [Fiter 2012a]). Consider system* (3.22) *with the sampling map*

(3.23). *For $t \geq 0$ and $x_0 \in \mathbb{R}^n$, let $\varphi(t, x_0)$ denote the solution of the open-loop system*

$$\dot{x} = Ax + BKx_0$$

at time t, with the initial condition $x(0) = x_0$. Given positive scalars $\alpha > 1$, $0 < \lambda \leq \frac{\ln(\alpha)}{2\delta_{max}}$, if there exist a quadratic function $V(x) = x^T P x$, $P = P^T \succ 0$, such that for all $x_0 \in \mathbb{R}^n$,

$$\frac{d}{dt}V(\varphi(t, x_0)) + 2\lambda V(\varphi(t, x_0)) \leq 0, \ \forall \ t \in [0, \tau(x_0)] \tag{3.24}$$

whenever

$$\alpha V\left(\varphi(t, x_0)\right) \geq V(x_0) \tag{3.25}$$

then the origin of (3.22),(3.23) *is Globally Exponentially Stable.*

Furthermore, for any sampling function $\tilde{\tau} : \mathbb{R}^n \to \mathbb{R}_+$, with $\tilde{\tau}(x) \in [\delta_{min}, \tau(x)]$, the origin of (3.22) *with $h_k = \tilde{\tau}(x_k)$ is Globally Exponentially Stable.*

This result is an extension of the Lyapunov-Razumikhin approach from Proposition 3.7 to the case of controlled sampling. Using this approach, the function $V(x) = x^T P x$ satisfies

$$V(x_{k+1}) < V(x_k), \forall k \in \mathbb{N}.$$

The theorem ensures that $V\left(x(t)\right) \leq V(x_k), \forall t \in [t_k, t_{k+1})$. However, $V\left(x(t)\right)$ is not restricted to be monotonously decreasing over the sampling interval.

In order to provide tractable design conditions the following result is necessary.

Proposition 3.12 *(adapted from [Fiter 2012a]) Consider system* (3.22) *with the sampling map* (3.23). *Given positive scalars $\alpha > 1$, $0 < \lambda \leq \frac{\ln(\alpha)}{2\delta_{max}}$ if there exist a matrix $P = P^T \succ 0$, a scalar $\epsilon \geq 0$, assume that there exist a quadratic function $V(x) = x^T P x$, $P = P^T \succ 0$ and a positive scalar ϵ such that for all $x \in \mathbb{R}^n$, for all $\theta \in [0, \tau(x)]$,*

$$x^T \Phi(P, \theta) x \leq 0, \tag{3.26}$$

with

$$\Phi(P, \theta) = \begin{bmatrix} \Lambda(\theta) \\ I \end{bmatrix}^T \begin{bmatrix} A^T P + PA + (2\lambda + \epsilon\alpha)P & PBK \\ K^T B^T P & -\epsilon P \end{bmatrix} \begin{bmatrix} \Lambda(\theta) \\ I \end{bmatrix}, \tag{3.27}$$

and

$$\Lambda(\theta) = I + \int_0^\theta e^{sA} ds (A + BK). \tag{3.28}$$

Then the origin of (3.22),(3.23) *is is Globally Exponentially Stable.*

Furthermore, for any sampling function $\tilde{\tau} : \mathbb{R}^n \to \mathbb{R}_+$, with $\tilde{\tau}(x) \in [\delta_{min}, \tau(x)]$, the origin of (3.22) *with $h_k = \tilde{\tau}(x_k)$ is Globally Exponentially Stable.*

This result has several important consequences for the design of sampling maps $\tau(x)$. First, for any given matrix $P = P^T \succ 0$ and prescribed maximum sampling interval δ_{max}, the proposition motivates the design of sampling maps of the form

$$\tau(x) = \max\left\{\rho \in \mathbb{R} : \rho \leq \delta_{max}, \ x^T \Phi(P, \theta) x < 0, \forall \theta \in [0, \rho]\right\} \tag{3.29}$$

which, by definition, ensure that condition (3.26) holds. Second, if there exists P and ϵ solution

to the set of linear matrix inequalities

$$\Phi(P,\theta) \prec 0,\ \forall\theta \in [0, h^*]$$

for prescribed positive scalars h^*, α and $\lambda \leq \frac{\ln(\alpha)}{2\delta_{max}}$, the condition

$$x^T \Phi(P,\theta) x < 0,$$

is satisfied for any $x \neq 0$ and all $\theta \in [0, h^*]$. From the definition of τ in (3.29), this implies that the sampling function is lower bounded by h^*. The following result is obtained.

Corollary 3.13 *Consider system* (3.22). *Given positive scalars* h^*, $\delta_{max} \geq h^*$, $\alpha > 1$, $0 < \lambda \leq \frac{\ln(\alpha)}{2\delta_{max}}$ *let there exist a matrix* $P = P^T \succ 0$ *and a scalar* $\epsilon \geq 0$, *such that*

$$\Phi(P,\theta) = \begin{bmatrix}\Lambda(\theta)\\ I\end{bmatrix}^T \begin{bmatrix}A^TP + PA + (2\lambda + \epsilon\alpha)P & PBK\\ K^TB^TP & -\epsilon P\end{bmatrix} \begin{bmatrix}\Lambda(\theta)\\ I\end{bmatrix} \prec 0, \forall\theta \in [0, h^*]. \tag{3.30}$$

Then

- *the control-loop* (3.22), (3.23), (3.29) *has a minimum inter-event time of at least* h^*, *i.e. the sampling instants* $\{t_k\}_{k\in\mathbb{N}}$ *satisfy*

$$t_{k+1} - t_k \geq h^*,\ \forall\ k \in \mathbb{N};$$

- *system* (3.22), (3.23), (3.29) *is Globally Exponentially Stable;*
- *given* $\delta_{min} \in (0, h^*]$, *for any sampling function* $\tilde{\tau} : \mathbb{R}^n \to \mathbb{R}_+$, *with* $\tilde{\tau}(x) \in [\delta_{min}, \tau(x)]$, *the origin of* (3.22) *with* $h_k = \tilde{\tau}(x)$ *is Globally Exponentially Stable.*

One may remark that the conditions in Corollary 3.13 involve elements that are similar to the ones used for robust stability analysis. Note that the expression of Φ in (3.30) is the same as the one in (3.19) from Lemma 3.8. This means that the same optimization tools can be used for both the estimation of the maximum sampling interval preserving stability under an arbitrary sampling and for designing a sampling map while optimizing the minimum inter-event time. Then the design of a stabilizing sampling map $\tilde{\tau}$ satisfying the conditions in Corollary 3.13 can be addressed in two main steps:

- optimize the parameters P and ϵ which enlarge the minimum inter-event time h^* based on LMIs;
- for given parameters P and ϵ, provide a lower approximation $\tilde{\tau}$ of the sampling function τ in (3.29).

Several numerical tools based on convex embeddings have been proposed in [Fiter 2012a] for solving the set of LMIs (3.30) and for designing sampling functions $\tilde{\tau}$ approximating the map τ in (3.29). The approach has been further extended to deal with perturbations in [Fiter 2014b, Fiter 2015].

Example 3.14 *(Example 3.3 revisited) Consider the following system :*

$$\dot{x}(t) = \begin{bmatrix}-0.5 & 0\\ 0 & 3.5\end{bmatrix} x(t) + \begin{bmatrix}1\\ 1\end{bmatrix} Kx(t_k),$$

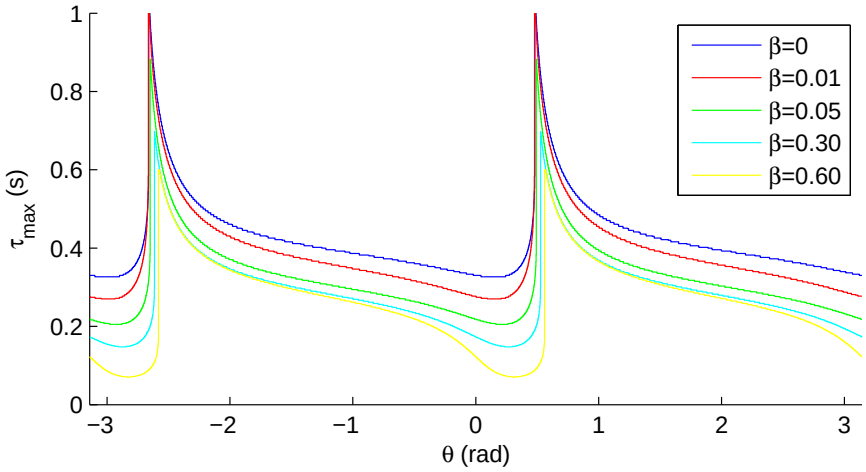


Figure 3.5: State-angle dependent sampling function τ for different decay rates λ (from top to bottom, $\lambda = 0,\ 0.01,\ 0.05,\ 0.30,\ 0.60$) as function of the angle θ in the polar coordinates $x = re^{i\theta}$.

$$K = \begin{bmatrix} -1.02 & 5.62 \end{bmatrix}.$$

Using the numerical methods in [Fiter 2012a], we can obtain a mapping of the state space that enlarges the minimum inter-event time for different values λ of the decay rate. For each decay rate λ, after fixing δ_{max}, we set the performance parameter $\alpha > 1$ (see Proposition 3.11) as small as possible and such that $\lambda \leq \frac{\ln(\alpha)}{2\delta_{max}}$. The state dependent sampling functions obtained offline and ensuring the stability of the system for different decay rates are presented in Figure 3.5.

For a constant sampling greater than $T_{max} = 0.469s$ the discrete-time dynamic matrix is not Schur, so the system becomes unstable. However, with the proposed technique, we can go beyond the limit T_{max} for some regions of the state space (up to $1s$ for $\lambda = 0$).

Figure 3.6 (resp. Figure 3.7) shows simulation results with $\lambda = 0$ (resp. $\lambda = 0.05$) and a random initial state. It first shows the sampling intervals (blue/piecewise constant curve), with the lower-bound of the offline computed state dependent sampling function (red/lower horizontal line), and the limit T_{max} of the periodic case (green/upper horizontal line), before showing the LRF evolution. The sampling times are represented by the red dots on each graph.

In Figure 3.6 ($\lambda = 0$), one can see that the number of actuations over the $20s$ time interval is 31 instead of 43 with T_{max}. For any (tested) initial condition in the simulation, the average sampling time converges to $T_{average} \simeq 0.726s \simeq 155\% T_{max}$.

For a given decay-rate $\lambda > 0$, the maximal constant sampling ensuring the exponential stability is given by

$$T^{\lambda}_{max} = argmax\{T > 0, -\frac{\ln(|eig_{max}|)}{T} \geq \lambda\} < T_{max}$$

, where eig_{max} is the eigenvalue of $\Lambda(T)$ with greatest modulus. In the simulation of Figure 3.7 ($\lambda = 0.05$), we can observe that

$$T^{\lambda=0.05}_{average\ over\ 20s} = 0.486s > T_{max} = 0.469s > T^{\lambda=0.05}_{max} = 0.457s$$

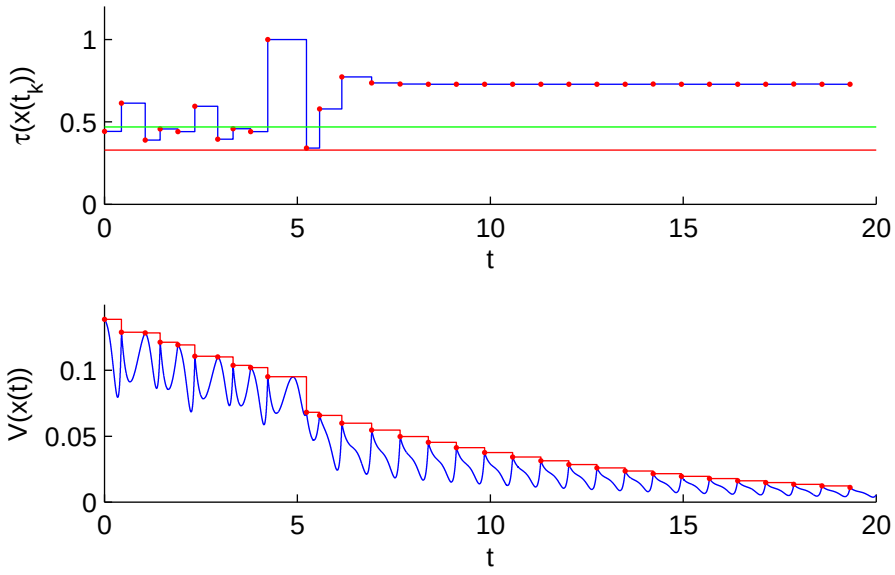


Figure 3.6: Inter-execution times $\tau(x(t_k))$ and LRF $V(x) = x^T P x$ for a decay rate $\lambda = 0$.

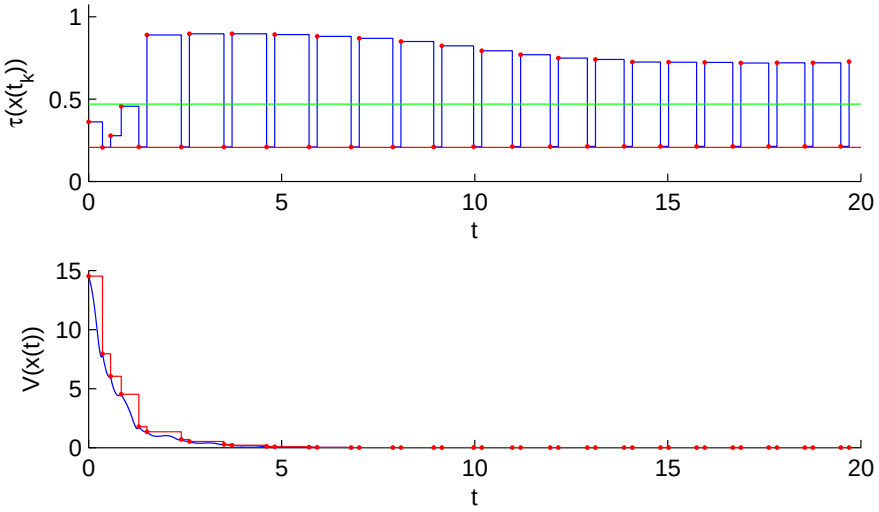


Figure 3.7: Inter-execution times $\tau(x(t_k))$ and LRF $V(x) = x^T P x$ for a decay rate $\lambda = 0.05$.

.

This means that it is possible to sample less in average than with the maximal periodic sampling T_{max} while still ensuring asymptotic or exponential stability. Although we can not guarantee that this will always be the case, the state-dependent sampling presents some advantages compared to

periodic sampling:
- It ensures some convergence performance (exponential stability for a given decay-rate λ, or asymptotic stability if $\lambda = 0$), whereas constant sampling with T_{max} only ensures marginal stability and doesn't give any hint about the inter-sampling state behaviour.
- It guarantees robustness regarding possible fluctuations of the sampling period, which is inherent to practical applications (due to scheduling issues for example). The state-dependent sampling approach ensures the system's stability for any time-varying sampling period satisfying $0 < \delta \leq \tilde{\tau}(t,x) \leq \tau(x)$, for all $t \in \mathbb{R}_+$ and for all $x \in \mathbb{R}^n$.

Example 3.15 *Consider the Batch Reactor system from [Mazo Jr. 2009]:*

$$\dot{x}(t) = \begin{bmatrix} 1.38 & -0.20 & 6.71 & -5.67 \\ -0.58 & -4.29 & 0 & 0.67 \\ 1.06 & 4.27 & -6.65 & 5.89 \\ 0.04 & 4.27 & 1.34 & -2.10 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 5.67 & 0 \\ 1.13 & -3.14 \\ 1.13 & 0 \end{bmatrix} u(t),$$

$$K = \begin{bmatrix} 0.1006 & -0.2469 & -0.0952 & -0.2447 \\ 1.4099 & -0.1966 & 0.0139 & 0.0823 \end{bmatrix}.$$

Using the numerical methods in [Fiter 2012a], a sampling map has been derived for a decay rate $\lambda = 0$ and $\delta_{max} = 1s$. This state space mapping (in dimension 4) provides a precise knowledge of the sampling function τ (which varies from $\tau^*_{sub} = 0.4409$ to 0.988. In comparison, the value of the maximal allowable constant sampling T_{max} is $0.5534s$. Using this mapping, we obtain the simulations shown in Figure 3.8. The number of actuations over the first $10s$ time interval (see Figure 3.8) is 17, which can be compared to the number of updates presented in [Mazo Jr. 2009] (32 in the best presented case), and the obtained average sampling time is $T_{average} = 0.5898 > T_{max}$. An illustration of the sampling map in polar coordinates is given in Figure 3.9.

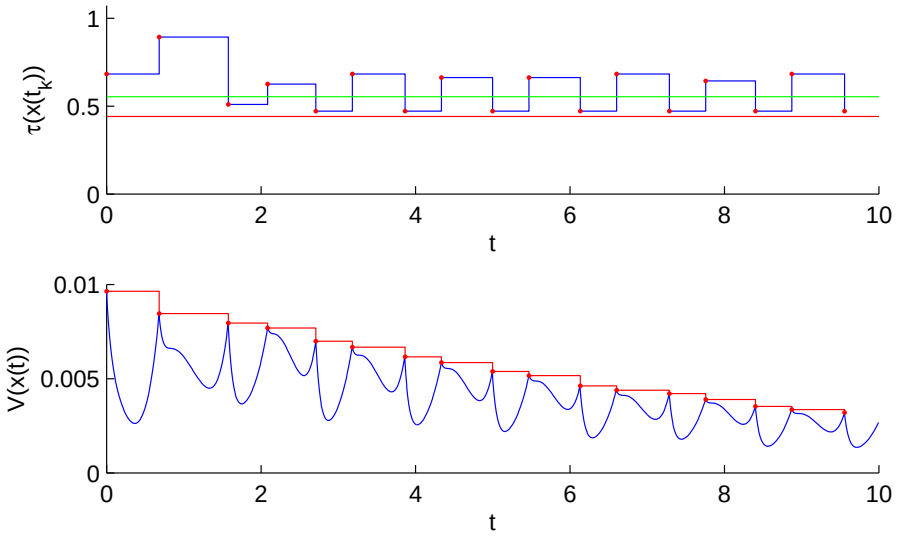


Figure 3.8: Example 3.15: Inter-execution times $\tau(x(t_k))$ and LRF $V(x) = x^T P x$ for a decay rate $\lambda = 0$.

3.2 Sampled-data control of bilinear systems

In this section, the stability problem is considered for sampled-data bilinear systems[17]. Bilinear systems [Mohler 1974, Elliott 2009] represent one of the most simple class of nonlinear affine systems. They are systems of ordinary differential equations of the form

$$\dot{x}(t) \quad = \quad A_0 x(t) + \sum_{i=1}^{m} [u(t)]_i N_i x(t) + B_0 u(t), \quad \forall t \geq t_0, \tag{3.31}$$

where $A_0 \in \mathbb{R}^{n\times n}, B_0 \in \mathbb{R}^{n\times m}$ and $N_i \in \mathbb{R}^{n\times n}, i = 1, \ldots, m$. The state vector is $x(t) \in \mathbb{R}^n$ and the control input is $u(t) \in \mathbb{R}^m$. Here we use the notation $[u(t)]_i$ to denote the i^{th} elements of the vector $u(t)$. More generally, over this section we will use the notation $[\chi]_i$ to denote the i^{th} elements of a vector χ.

Systems of the form (3.31) are linear with respect to the system state or to the control variable, but not in both of them jointly. The term $A_0 x$ is called the *drift*, $B_0 u$ is the *additive control* and $\sum_{i=1}^{m} [u]_i N_i x$ is the *multiplicative control*. Bilinear models appear naturally in a large variety of applications [Mohler 1974]. They can also be used as approximations to more complex nonlinear systems [Elliott 2009]. Various control methodologies have been proposed for bilinear systems. Constructive approaches for the design of linear [Mohler 1991, Andrieu 2013], quadratic [Gutman 1981], division [Mohler 1991] or sliding mode controllers [Al-Shamali 2007] can be found in the literature. LMI criteria have been proposed for the design of a locally stabilizing linear state feedback in [Andrieu 2013, Olalla 2011, Amato 2009, Valmorbida 2013]. Intuitively, the stability is preserved under a sampled-data implementation if the sampling frequency is sufficiently high.

[17] The results presented in this section have been developed in the context of the PhD Thesis of Hassan OMRAN, in collaboration with Prof. Jean-Pierre Richard and Francoise LAMNABHI-LAGARRIGUE (DR, CNRS).

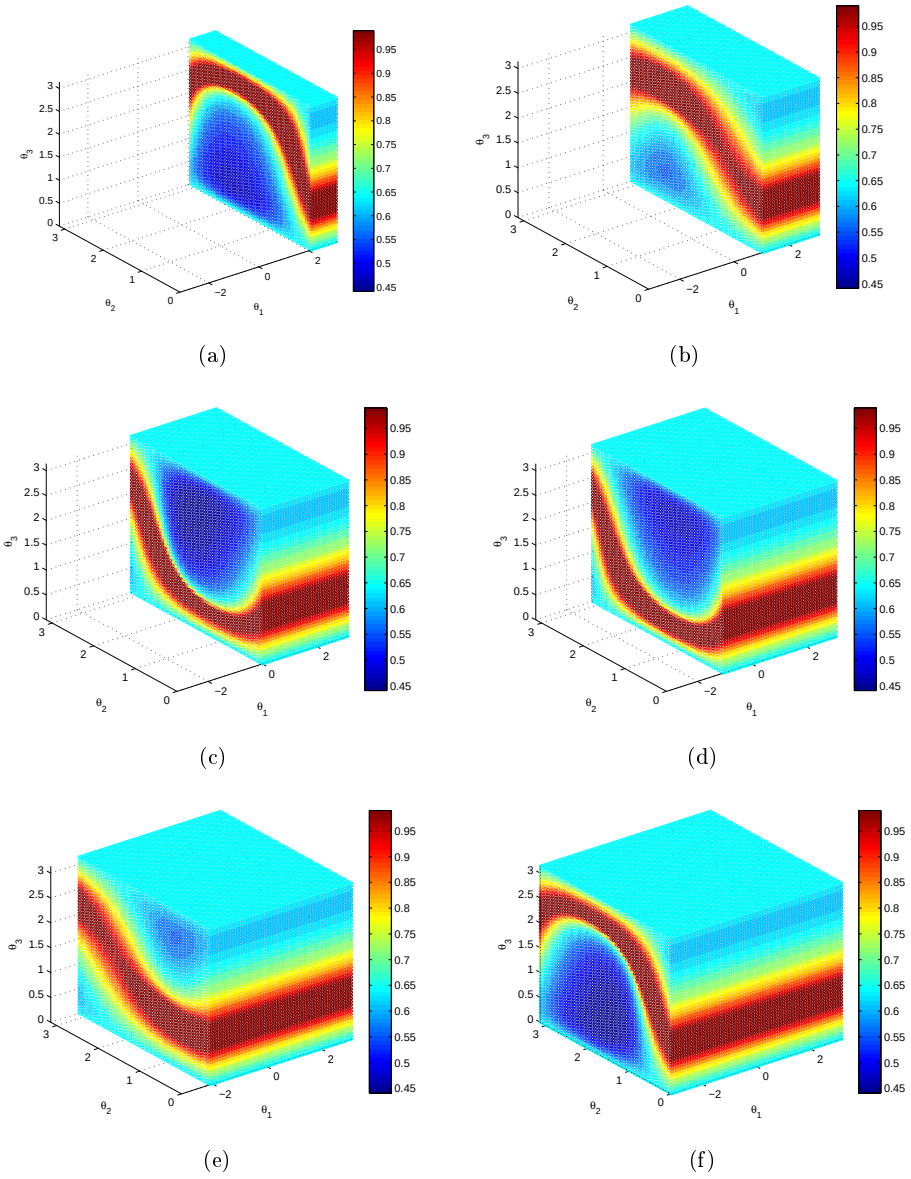


Figure 3.9: Illustration of the sampling map in Example 3.15 in polar coordinates.

However, there is a lack of formal tools for the analysis of bilinear sampled-data systems which provide a quantitative estimation of the Maximum Sampling Interval (MSI) preserving stability. As follows, several approaches providing an estimation of the MSI will be presented for the case of a linear state feedback controller with an aperiodic sampled-data implementation.

3.2.1 Hybrid system approach

Consider the bilinear system (3.31). We suppose that the following assumptions hold:

A.1 The control is a piecewise-constant control law

$$u(t) = Kx(t_k), \quad \forall t \in [t_k, t_{k+1}),$$

with a set of sampling instants $\{t_k\}_{k\in\mathbb{N}}$ satisfying:

$$0 < \epsilon \leq t_{k+1} - t_k \leq \overline{h}, \qquad \forall k \in \mathbb{N}, \tag{3.32}$$

where $\overline{h}$ is a given positive scalar.

A.2 The pair A_0, B_0 is stabilizable, and the linear feedback gain $K \in \mathbb{R}^{m\times n}$ is calculated so that the system (3.31) with the continuous state feedback $u(t) = Kx(t)$ has a locally asymptotically stable equilibrium point at $x = 0$. The actual domain of attraction (a connected neighbourhood of $x = 0$) is denoted $\mathcal{D}_0$.

A.3 The state variables are subject to constraints defined by a polytopic set $\mathcal{P} \subset \mathcal{D}_0$:

$$\begin{aligned} \mathcal{P} &= conv\{v_1, v_2, \ldots, v_p\} && (3.33)\\ &= \{x \in \mathbb{R}^n : a_j^T x \leq 1,\ j = 1, 2, \ldots, r\} && (3.34) \end{aligned}$$

corresponding to an admissible set in the state-space.

Under these assumptions, we obtain the closed-loop sampled-data system:

$$\dot{x}(t) = \big(A_0 + \sum_{i=1}^{m} [Kx(t_k)]_i N_i\big)x(t) + B_0 Kx(t_k), \quad \forall t \in [t_k, t_{k+1}), \quad \forall k \in \mathbb{N}. \tag{3.35}$$

System (3.35) may also be written as follows

$$\dot{x}(t) = \tilde{A}[x(t), e(t)]x(t) + Be(t), \quad \forall t \in [t_k, t_{k+1}) \tag{3.36}$$

with

$$e(t) = x(t_k) - x(t),$$

$$\tilde{A}[x, e] := A_0 + B_0 K + \sum_{i=1}^{m} [K(x + e)]_i N_i, \tag{3.37}$$

and

$$B = B_0 K. \tag{3.38}$$

The goal is to provide conditions that guarantee the asymptotic convergence of the system (3.35) solutions to the origin.

In the framework of [Goebel 2012], the hybrid model of the bilinear sampled-data system is determined by

$$\left.\begin{array}{rcl} \dot{x} &=& f(x,e) = \ \ \tilde{A}[x,e]x + Be \\ \dot{e} &=& g(x,e) = -\tilde{A}[x,e]x - Be \\ \dot{\tau} &=& 1 \end{array}\right\} \qquad \tau \in [0, \overline{h}]$$

$$\left.\begin{array}{rcl} x^+ &=& x \\ e^+ &=& 0 \\ \tau^+ &=& 0 \end{array}\right\} \qquad \tau \in [\epsilon, \overline{h}]. \tag{3.39}$$

Two methods are to be considered for analysing the stability of system (3.39). First, we introduce a method that is based on the application of results for general nonlinear sampled-data systems in [Nešić 2009] (Method 1). Next, to avoid the use of conservative bounds in the previous method, we look directly for a Lyapunov function by formalizing the conditions as LMIs (Method 2). In both of these methods, we will be dealing with local asymptotic stability.

3.2.1.1 Method 1: adaptation of a result on general nonlinear sampled-data systems

Considering the polytope $\mathcal{P}$ in (3.33), define the matrices

$$A_j = A_0 + B_0 K + \sum_{i=1}^{m} \left[K v_j \right]_i N_i, \ j = 1, \dots, p. \tag{3.40}$$

The following theorem proposes stability conditions using an adaptation of the results from Theorem 2.7 for the case of bilinear systems.

Theorem 3.16 *[Omran 2016b] Consider the bilinear sampled-data system* (3.39)*, the polytope* $\mathcal{P}$ *in* (3.33)*, and a function*

$$h^*(\gamma, L) := \begin{cases} \frac{1}{Lr} arctan(r), & \gamma > L \\ \frac{1}{L}, & \gamma = L \\ \frac{1}{Lr} arctanh(r), & \gamma < L \end{cases} \tag{3.41}$$

with

$$r = \sqrt{|\frac{\gamma^2}{L^2} - 1|} \tag{3.42}$$

where L *is given by*

$$L = \frac{1}{2} \max\{-\lambda_{min}(B^T + B), 0\} \tag{3.43}$$

and γ *is the solution to the following optimization problem:*

$$\gamma = \min \sqrt{\rho} \tag{3.44}$$

satisfying the constraints $\exists P \in \mathbb{R}^{n \times n}$ *a symmetric positive definite matrix ,* $\exists \rho > 0$ *and* $\exists \alpha > 0$*, such that*

$$M_{lj} = \begin{bmatrix} A_l^T P + P A_l + \frac{1}{2}(A_l^T A_j + A_j^T A_l) + \alpha I & PB \\ * & (\alpha - \rho) I \end{bmatrix} \prec 0, \\ \forall l, j \in \{1, 2, ..., p\}. \tag{3.45}$$

Assume that the sampling intervals are strictly bounded by $h^*(\gamma, L)$*, i.e.* $\overline{h} < h^*(\gamma, L)$*. Then, for the bilinear sampled-data system* (3.39)*, the set* $\{(x, e, \tau) : x = 0, e = 0, \tau \in [0, \bar{h}]\}$ *is Locally Uniformly Asymptotically Stable.*

In this method, the Maximum Sampling Interval is calculated by the expression (3.41), based on L and γ. L is calculated analytically, whereas γ is found by solving LMI conditions. The optimization problem is a minimization of γ' because for any constant L, $h^*(\cdot, L)$ is a strictly decreasing function. Note that since γ does not depend on L, and from the continuity of $h^*(\gamma, \cdot)$:

$$h^*(\gamma,0)=\lim_{L\to 0}h^*(\gamma,L)=\lim_{L\to 0}\frac{\arctan(\sqrt{\left|\frac{\gamma^2}{L^2}-1\right|})}{\sqrt{\left|\gamma^2-L^2\right|}}=\frac{\pi}{2\gamma}.$$

The stability conditions presented in this theorem are based on the generic inequalities from Theorem 2.7 in [Nešić 2009]. Our contribution is to provide a constructive manner to apply this result to the case of bilinear systems. We provide explicit forms of $H(x,e)$, $W(e)$, $V(x)$, and we find L, γ that gives the upper bound on Maximum Sampling Interval. We provide as well, an LMI formulation that allows us to obtain sufficient stability condition. Note that in order to obtain LMI based stability conditions the approach has been adapted to the bilinear case: the function $H(\cdot,\cdot)$ used here has been modified to depend both on the error $e(t)$ and the state $x(t)$, while in [Nešić 2009] it is only a function of x.

3.2.1.2 Method 2: direct Lyapunov function approach

In the previous method, the stability conditions are obtained using upper estimations of the derivative of a Lyapunov function. Such upper estimations may be found conservative. In order to avoid them, we provide as follows a second method which evaluates directly the derivative of a Lyapunov function.

Theorem 3.17 *[Omran 2016b] Consider the bilinear sampled-data system* (3.39)*. Suppose that Maximum Sampling Interval is bounded by a value h^*, i.e. $\overline{h}\leq h^*$. Assume that there exist symmetric positive definite matrices $P,Q,X,Y\in\mathbb{R}^{n\times n}$, such that the following LMIs are satisfied*

$$\begin{bmatrix}A_l^TP+PA_l+X & PB-A_l^TQ\\ * & -B^TQ-QB-\frac{1}{h^*}Q+Y\end{bmatrix}\prec 0,$$
$$\forall l\in\{1,2,...,p\}. \tag{3.46}$$

$$\begin{bmatrix}A_l^TP+PA_l+X & PB-A_l^TQ\exp(-1)\\ * & [-B^TQ-QB-\frac{1}{h^*}Q]\exp(-1)+Y\end{bmatrix}\prec 0,$$
$$\forall l\in\{1,2,...,p\}. \tag{3.47}$$

Then the set $\{(x,e,\tau):x=0,e=0\}$ of the bilinear sampled-data system (3.39) *is Locally Uniformly Asymptotically Stable.*

The theorem is based on the existence of a Lyapunov function

$$U(x,e,\tau)=V(x)+W(\tau,e) \tag{3.48}$$

with $V(x)=x^TPx$, and $W(\tau,e)=\exp(\frac{-\tau}{h^*})e^TQe$. In this method the Maximum Sampling Interval is found by solving a set of LMIs for the maximum value possible of h^*. The existence of a solution to the LMI conditions, guarantees the existence of a Lyapunov function that will yield the asymptotic stability. Note that the proposed conditions directly study the derivative of the Lyapunov function. Numerical examples will show the conservatism reduction in comparison with the approach in Method 1. Note that both the approach of Method 1 and Method 2

are robust not only to the sampled-data implementation but also to variations of the sampling intervals.

Example 3.18 *As follows we present a numerical comparison of the two proposed methods. Consider the following bilinear system, described by the matrices*

$$A_0 = \begin{bmatrix} -0.5 & 1.5 & 4 \\ 4.3 & 6.0 & 5.0 \\ 3.2 & 6.8 & 7.2 \end{bmatrix}, \quad B_0 = \begin{bmatrix} -0.7 & -1.3 \\ 0 & -4.3 \\ 0.8 & -1.5 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A continuous-time state feedback controller has been computed [Tarbouriech 2009] in order to locally stabilize the origin of the bilinear system

$$K = \begin{bmatrix} 0.0016 & 0.0035 & 0.0034 \\ 2.2404 & 3.2676 & 5.9199 \end{bmatrix}.$$

The controller was proven to establish the local stability for the bilinear system (in the continuous-time case), inside an ellipsoidal region $\mathcal{D}_0$. We consider a local polytopic region $\mathcal{P} \subset \mathcal{D}_0$

$$\mathcal{P} = [-1.35, +1.35] \times [-0.5, +0.5] \times [-0.5, +0.5].$$

Using Method 1, we found that the system is locally stable if $\overline{h} < h^* = 2.7 \times 10^{-3}$. This was calculated from (3.41) for $L = 29.79$, and $\gamma = 563.3$. The other variables in the optimization problem were $\alpha = 5.84$, and

$$P = \begin{bmatrix} 281.3 & 210.6 & 882.2 \\ 210.6 & 622 & 565.1 \\ 882.2 & 565.1 & 3688.3 \end{bmatrix}.$$

Using Method 2, we found that the sampled-data system is locally stable for a larger MSI, $\overline{h} \leq h^* = 12 \times 10^{-3}$. The LMIs in (3.46) and (3.47) have a solution for this value of MSI with

$$P = \begin{bmatrix} 1.2722 & 0.5769 & 3.8769 \\ 0.5769 & 2.4533 & 1.1283 \\ 3.8769 & 1.1283 & 16.9212 \end{bmatrix} \qquad Q = \begin{bmatrix} 5.6140 & 8.1180 & 14.7162 \\ 8.1180 & 12.0092 & 21.2460 \\ 14.7162 & 21.2460 & 39.7534 \end{bmatrix}.$$

The results illustrate the reduction of conservatism in Method 2 with respect to Method 1. Simulations show that the system is unstable for a larger sampling intervals. However, it is not clear how to improve the method in order to obtain a larger estimate of the MSI.

3.2.2 Input / Output approach

In the following, we present a different approach for the analysis of sampled-data bilinear system. The method is based on the extension of the frequency domain criteria from [Mirkin 2007, Fujioka 2009c] to the case of bilinear systems.

Let us remark that system (3.35) can be re-expressed as

$$\dot{x}(t) = \underbrace{\big(A_0 + B_0K + \sum_{i=1}^{m} [Kx(t_k)]_i N_i \big)}_{A(x(t_k))} x(t) + \underbrace{B_0K}_{B} \underbrace{(x(t_k) - x(t))}_{w(t)}$$

which can be further expressed by the feedback connection of the system

$$G := \begin{cases} \dot{x}(t) = A(x(t_k))x(t) + Bw(t) \\ y(t) = C(x(t_k))x(t) + Dw(t) \end{cases} \tag{3.49}$$

$$C(x(t_k)) = A(x(t_k)) = A_0 + B_0K + \sum_{i=1}^{m} [Kx(t_k)]_i N_i, \qquad D = B = B_0K \tag{3.50}$$

with the operator $\Delta_{sh} : y \to w$ defined by

$$w(t) = (\Delta_{sh}\, y)(t) = -\int_{t_k}^{t} y(\tau)d\tau, \quad \forall t \in [t_k, t_{k+1}). \tag{3.51}$$

We recall that the operator Δ_{sh} has been studied in the context of LTI sampled-data systems [Fujioka 2009c], [Mirkin 2007] and has two important properties. The first one concerns the gain, and the second is a passivity-type one (see Section 2.1.4). It was shown in [Mirkin 2007] that the operator is bounded on $\mathcal{L}_2^n[0,\infty)$. This property is based on the fact that for any $X = X^T \succ 0$, $v \in \mathcal{L}_2^n[0,\infty)$:

$$\int_{t_k}^{t} (\Delta_{sh}v)^T(\tau)X(\Delta_{sh}v)(\tau)d\tau \leq \delta_0^2 \int_{t_k}^{t} v^T(\tau)Xv(\tau)d\tau, \quad \forall t \in [t_k, t_{k+1}). \tag{3.52}$$

with $\delta_0 = \frac{2}{\pi}\overline{h}$. The passivity-type property given in [Fujioka 2009c] is based on the fact that for any $Y = Y^T \succ 0$, $v \in \mathcal{L}_2^n[0,\infty)$

$$\int_{t_k}^{t} v^T(\tau)Y(\Delta_{sh}v)(\tau)d\tau \leq 0, \quad \forall t \in [t_k, t_{k+1}). \tag{3.53}$$

In the LTI context, the two properties lead to LMI conditions for stability. However, these conditions are based on frequency domain criteria, and on the use of Kalman-Yakubovich-Popov lemma. The application of these techniques is restricted to the LTI case.

The main idea in [Omran 2013] is to re-interpret the properties of the operator Δ_{sh} in terms of "supply" functions $\mathcal{S}\big(y, w\big)$ such that

$$\int_{t_k}^{t} \mathcal{S}\big(y(\theta), w(\theta)\big)d\theta \leq 0, \forall t \in [t_k, t_{k+1}). \tag{3.54}$$

For the case of bilinear system, the properties (3.52), (3.53) can be used in order to show that for any $X^T = X \succ 0$ and $Y^T = Y \succ 0$, the sampled-data system satisfies the constraint

$$\int_{t_k}^{t} \mathcal{S}\left(\dot{x}(s), x(t_k) - x(s)\right) ds \leq 0, \ \forall t \in [t_k, t_{k+1}), \tag{3.55}$$

where

$$\mathcal{S}(\dot{x}(t), x(t_k) - x(t)) = \begin{bmatrix} \dot{x}(t) \\ x(t_k) - x(t) \end{bmatrix}^T \begin{bmatrix} -\delta_0^2 X & Y \\ Y & X \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ x(t_k) - x(t) \end{bmatrix}. \tag{3.56}$$

Inspired by the Dissipativity Theory [Brogliato 2007], the relation (3.55) can be useful to prove an invariance property when there exists quadratic function $V(x) = x^T P x$, with $P = P^T \succ 0$, such that

$$\frac{d}{dt} V(x(t)) \leq \mathcal{S}(\dot{x}(t), x(t_k) - x(t)), \forall\ t \in [t_k, t_{k+1}). \tag{3.57}$$

When this relation holds, $V(x(t)) \leq (x(t_k))$ for all $\forall\ t \in [t_k, t_{k+1})$. For the case of bilinear sampled-data systems (3.35) this leads to sufficient stability conditions that can be checked using LMIs.

Theorem 3.19 *[Omran 2014b] Consider system* (3.35)*. Assume that there exist symmetric positive definite matrices $X, Y, P \in \mathbb{R}^{n\times n}$, and matrices $P_2, P_3 \in \mathbb{R}^{n\times n}$ such that the following optimization problem is feasible*

$$\gamma^* = \min_{E_j \geq 0,\, M_q < 0} \gamma, \quad \forall j \in \{1, 2, ..., r\}, \quad \forall q \in \{1, 2, ..., p\}, \tag{3.58}$$

with

$$E_j = \begin{bmatrix} \gamma & a_j^T \\ a_j & P \end{bmatrix} \tag{3.59}$$

and

$$\begin{bmatrix} A_q^T P_2 + P_2 A_q & P - P_2^T + A_q^T P_3 & P_2^T B \\ P - P_2 + P_3^T A_q & -P_3 - P_3^T + \delta_0^2 X & P_3^T B - Y \\ B^T P_2 & B^T P_3 - Y & -X \end{bmatrix} \prec 0 \tag{3.60}$$

where the vertices $\{A_q\}_{q\in\{1,2,\cdots,p\}}$ are given in (3.40)*, and $\{a_j\}_{j\in\{1,2,...,r\}}$ are given in* (3.33)*. The equilibrium $x = 0$ of the system* (3.35) *is then Locally Asymptotically Stable for any arbitrary sampling sequence with $t_{k+1} - t_k \leq \overline{h}$.*

An estimate of a domain of attraction is given by the ellipsoid

$$\mathcal{E}(P, c^*) := \{x \in \mathbb{R}^n : x^T P x \leq c^*\} \subset \mathcal{P} \tag{3.61}$$

with $c^* = 1/\gamma^*$.

For given polytope $\mathcal{P}$ and MSI $\overline{h}$, the conditions in the previous theorem are LMIs. Note that the conditions only require the pair (A_0, B_0) to be stabilizable. Numerical examples illustrating the method are given below.

Example 3.20 *(Example 3.18 revisited) Consider the bilinear sampled-data system defined by*

$$A_0 = \begin{bmatrix} -0.5 & 1.5 & 4 \\ 4.3 & 6.0 & 5.0 \\ 3.2 & 6.8 & 7.2 \end{bmatrix}; \quad B_0 = \begin{bmatrix} -0.7 & -1.3 \\ 0 & -4.3 \\ 0.8 & -1.5 \end{bmatrix}; \quad N_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad N_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Using classical methods, the linear state feedback

$$K = \begin{bmatrix} 0.0016 & 0.0035 & 0.0034 \\ 2.2404 & 3.2676 & 5.9199 \end{bmatrix}$$

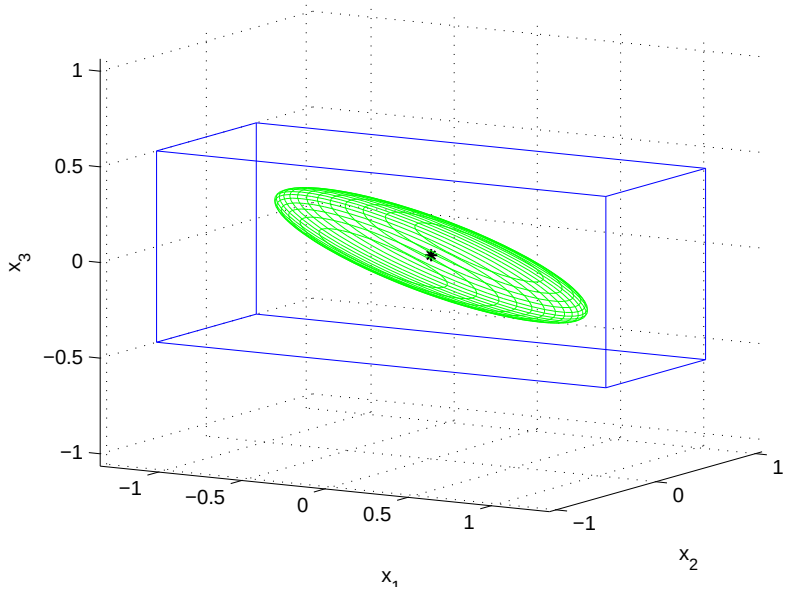


Figure 3.10: The polytope (blue boxes), and the corresponding region of stability $\mathcal{E}(P, c^*)$.

Table 3.1: The estimation of the MSI that guarantees the local stability of the system in Example 3.20.

	Theorem 3.16	Theorem 3.17	Theorem 3.19
MSI (ms)	5.6	13.8	51

was proven to establish the local stability for the continuous-time bilinear systems, inside an ellipsoidal region. Consider the box

$$\mathcal{P} = [-1.35, +1.35] \times [-0.5, +0.5] \times [-0.5, +0.5].$$

Our objective here is to find a MSI for which the local stability of the aperiodically bilinear sampled-data system is guaranteed. Using the method introduced in Theorem 3.19, we find that the LMI conditions in (3.60) *are feasible for* $\overline{h} = 51$ *ms, with*

$$P = 10^3 \begin{bmatrix} 34.27 & 10.82 & 92.73 \\ 10.82 & 50.43 & 28.41 \\ 92.73 & 28.41 & 394.23 \end{bmatrix}.$$

An estimate of the domain of attraction $\mathcal{E}(P, c^*)$ *is given by* (3.61) *for* $c^* = 0.1652$ *(see Figure 3.10). Considering the initial state* $x_0 = [-0.8 \ -0.2 \ +0.25]^T$*, two evolutions of the state are shown in Figure 3.11 and Figure 3.12. In Figure 3.11, a random sequence of sampling periods with* $\overline{h} = 51$ *ms was used for simulations. The stability is ensured as the initial state is located inside* $\mathcal{E}(P, c^*)$*. In Figure 3.12, a uniform sampling is considered, with a sampling period of* 89 *ms. We can notice that the sampled-data system becomes unstable.*

Considering the same box $\mathcal{P}$*, other methods are used to find the MSI that ensures the stability,*

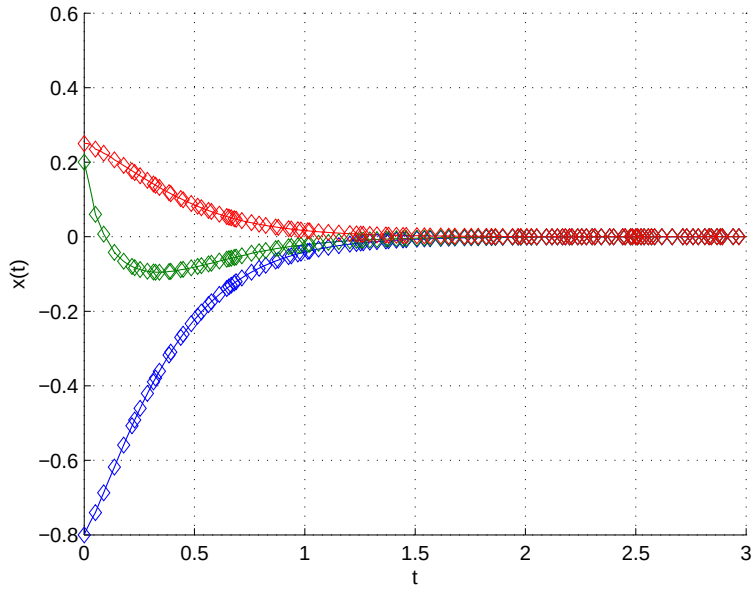


Figure 3.11: State evolution for the bilinear sampled-data system in Example 3.20, with a variable sampling which is bounded by $\overline{h} = 51\ ms$.

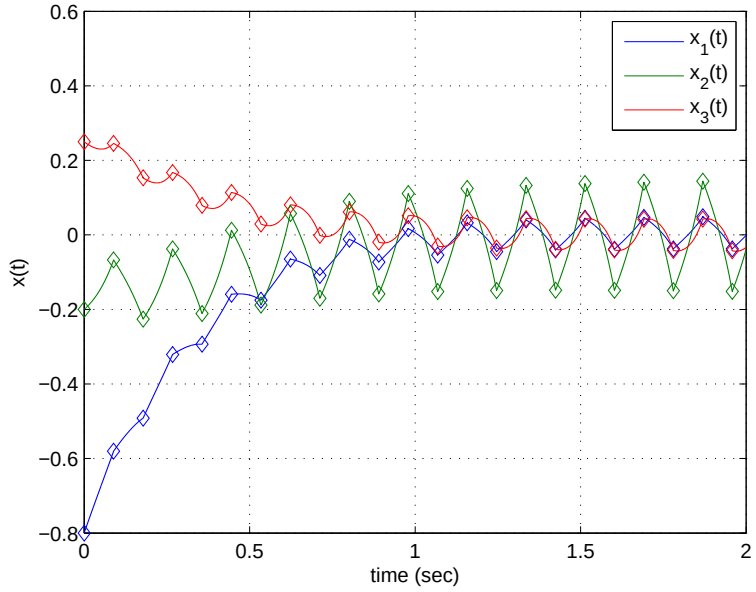


Figure 3.12: State evolution for the bilinear sampled-data system in Example 3.20, with uniform sampling $t_{k+1} - t_k = 89\ ms$.

and a comparison is given in Table 3.1. Note that methods based on the disspativity analysis, and the contractivity of invariant sets presented in [Omran 2012a] and Theorem 3.19, are less conservatives than the methods based on hybrid systems theory in [Omran 2012b]. The reduction of conservatism in Theorem 3.19 with respect to the Theorem 4 in [Omran 2012a], is due to using the descriptor method in formalizing the LMI conditions.

Example 3.21 *Consider the average values model of a buck-boost converter with pulse width modulator that adjust the duty cycle of the switching device.*

$$\dot{\bar{x}} = \big(DA_1 + (1-D)A_2\big)\bar{x} + \big(DB_1 + (1-D)B_2\big)v,$$

with the state $\bar{x} = [\bar{i_L}\ \bar{v_c}]^T$*, where* $\bar{i_L}$ *is the average inductor current, and* $\bar{v_c}$ *the average capacitor voltage. The average is taken over one switching period. The system matrices are*

$$A_1 = \begin{bmatrix} -\frac{R_{ON}+R_L}{L} & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix};\ A_2 = \begin{bmatrix} -\frac{R_L}{L} & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix};$$

$$B_1 = \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & 0 \end{bmatrix};\ B_2 = \begin{bmatrix} 0 & -\frac{1}{L} \\ 0 & 0 \end{bmatrix};\ v = \begin{bmatrix} V_{DC} \\ v_D \end{bmatrix}.$$

R_{ON} is the on-resistance of the switching device, v_D is the diode voltage, and V_{DC} is the source voltage. $D \in [D_1, D_2] \subset [0,1]$ is the duty cycle, representing the system input. Consider the following values: $V_{DC} = 6\,V$, $R = 50\,\Omega$, $L = 20\,mH$, $C = 220\,\mu F$, $R_{ON} = 0.08\Omega$, $R_L = 0.34\,\Omega$, and $v_D = 0.67\,V$. The system is subjected to saturation due to the hard limits on the duty cycle. For a certain working point $\bar{x}_0$, D_0 we have

$$0 = \big(D_0A_1 + (1-D_0)A_2\big)\bar{x}_0 + \big(D_0B_1 + (1-D_0)B_2\big)v.$$

Considering $\hat{x} = \bar{x} - \bar{x}_0$, and the input signal $u = D - D_0$, we can see that

$$\dot{\hat{x}} \ = \ A_0\hat{x} + B_0u + Nu\hat{x} \tag{3.62}$$

where

$$A_0 = (D_0A_1 + (1-D_0)A_2)$$

$$B_0 = ((A_1 - A_2)\bar{x}_0 + (B_1 - B_2)v)$$

and

$$N = (A_1 - A_2).$$

From the constraints over the duty cycle we see that u must be bounded by $-D_0 + D_1 \leq u \leq D_2 - D_0$. Usually, we consider $D_0 = (D_1 + D_2)/2$, and then $|u| \leq u_{max} = (D_2 - D_1)/2$. Using results [Olalla 2011] for the stabilization of the continuous-time system, we find the following controller

$$K = [-1.7329 \quad 0.0738].$$

We are interested in the state space region where a linear control $u = K\hat{x}$ is not saturated $\{\hat{x} \in \mathbb{R}^2 : |K\hat{x}| \leq u_{max}\}$. Furthermore, it is desired that the error with respect to the equilibrium point satisfies $|\bar{i_L}| < 0.5A$ and $|\bar{v_c}| < 3V$. This leads to considering the polytope

$$\mathcal{P} := \{[-0.42, -3], [-0.16, 3], [0.16, -3], [0.42, 3]\}.$$

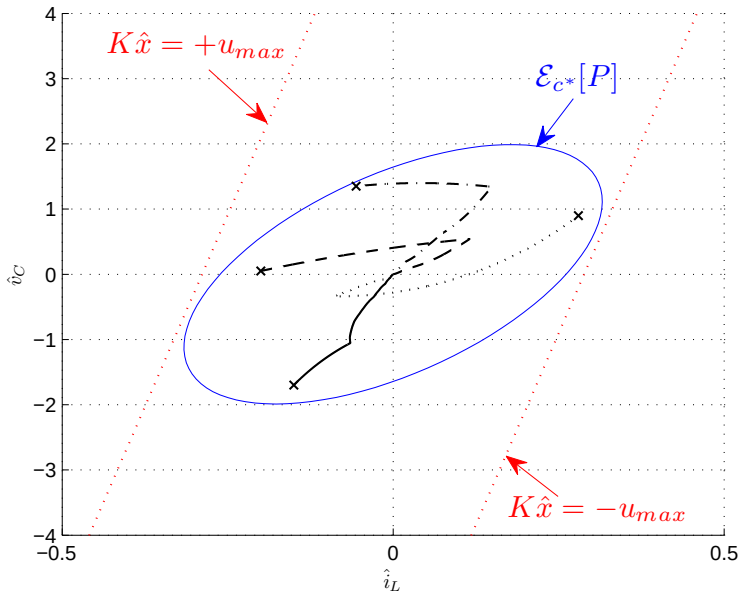


Figure 3.13: The domain of attraction for the system (3.62) when controlled with the static feedback controller, in the asynchronously sampled-data case $\mathcal{E}(P, c^*)$ with $\overline{h} = 1.5\ ms$. The curves in black are simulations of the sampled-data system, for different initial states.

In order to study the robustness with respect to asynchronous sampling, we apply Theorem 3.19. We find that the system is stable when implementing digitally the feedback controller K with variable sampling periods bounded by $\overline{h} = 1\ ms$. The guaranteed domain of attraction $\mathcal{E}(P, c^*)$ is given in (3.61)*, for $c^* = 37.81 \times 10^3$ and*

$$P = 10^3 \begin{bmatrix} 554.9 & -49.62 \\ -49.62 & 14.01 \end{bmatrix}.$$

The domain of attraction is shown in Figure 3.13, together with simulations of the evolutions of the state of the sampled-data system. Different initial conditions are considered, and random variable sampling periods, bounded by $\overline{h} = 1.5\,ms$ are used in the simulations. Note that by slightly increasing the sampling interval, the system becomes unstable. For example, with the initial condition $x_0 = [0.24\ \ 0.75]^T$, we obtain an unstable behaviour when choosing a constant sampling $t_{k+1} - t_k = 2.1\,ms$ as shown in the simulation in Figure 3.14. The same initial condition is considered in one of the simulations in Figure 3.13, and the the system is stable when respecting the bound $\overline{h} = 1.5\,ms$. The gap between the two values illustrates the conservatism of the proposed method.

3.3 Sampled-data control of input affine nonlinear systems

In the nonlinear case, an extension [Omran 2013, Omran 2014b, Omran 2016a, Omran 2014a] of the IQC approach is possible using methods inspired by the notion of Exponential Dissipativity

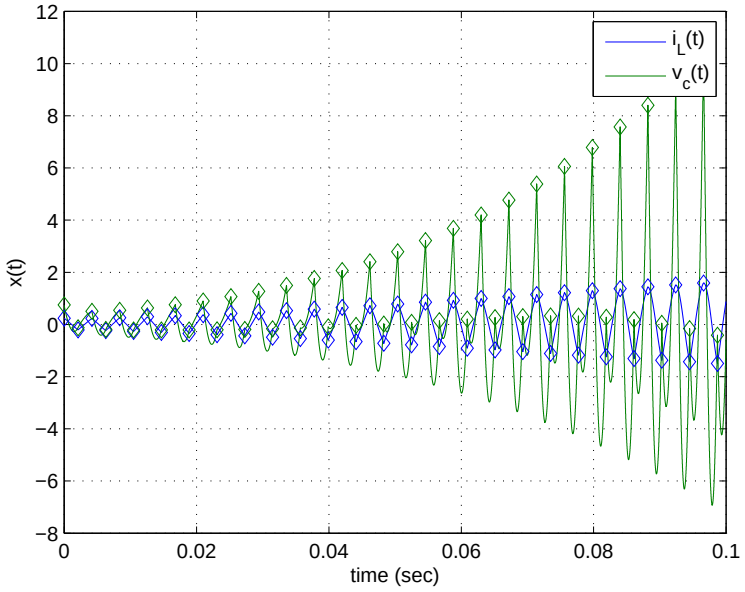


Figure 3.14: The instability of the sampled-data control of system (3.62) with uniform sampling of $t_{k+1} - t_k = 2.1\ ms$.

[VijaySekhar 2003]. The main ideas are presented as follows[18].

Consider the following nonlinear affine system:

$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)K\big(x(t_k)\big), \forall t \in [t_k, t_{k+1}),\ k \in \mathbb{N}. \tag{3.63}$$

The functions $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(0) = 0$, and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are considered to be sufficiently smooth and the controller $K : \mathbb{R}^n \to \mathbb{R}^m$ is a continuously differentiable function. Considering

$$f_{cl}(x) = f(x) + g(x)K(x),$$

$$w(t) = K\big(x(t_k)\big) - K\big(x(t)\big)$$

and an auxiliary output

$$y = \frac{\partial K}{\partial x}\dot{x},$$

system (3.63) can be represented by

$$\left\{\begin{array}{rcl} \dot{x} & = & f_{cl}\left(x\right) + g\left(x\right)w \\ y & = & \frac{\partial K}{\partial x}\left(f_{cl}\left(x\right) + g\left(x\right)w\right) \\ w & = & \Delta_{sh}y. \end{array}\right. \tag{3.64}$$

[18] The results presented in this section have been developed in the context of the PhD Thesis of Hassan OMRAN, in collaboration with Prof. Jean-Pierre Richard and Francoise LAMNABHI-LAGARRIGUE (DR CNRS).

Here we consider that $\Delta_{sh} : \mathcal{L}_{2e}^m[0,\infty) \to \mathcal{L}_{2e}^m[0,\infty)$, with Δ_{sh} defined similarly to (3.51):

$$\Delta_{sh} y(t) = -\int_{t_k}^{t} y(s)ds, \ \forall t \in [t_k, t_{k+1}). \tag{3.65}$$

Consider the following assumptions

- The nominal continuous-time system

  $$\dot{x} = f_{cl}(x), \ x(0) = x_0 \in \mathbb{R}^n$$

  has a well defined solution on $\mathbb{R}_+$ for any $x_0 \in \mathbb{R}^n$.

- The sequence of sampling instants $\{t_k\}_{k\in\mathbb{N}}$ satisfies $t_{k+1} - t_k \in (0, \overline{h}]$ for a given positive scalar $\overline{h}$.

- For any initial condition $x_0 \in \mathbb{R}^n$, the system

  $$\dot{x} = f(x) + g(x)K(x_0), x(0) = x_0,$$

  has a unique solution $x(t)$ defined on the interval $[0, \overline{h}]$.

The following result provides an extension of Theorem 2.17, Theorem 3.19 to the nonlinear affine case.

Theorem 3.22 *[Omran 2016a] Consider system* (3.63) *and the representation* (3.64),(3.65). *Assume that*

I. There exists a continuous function $\mathcal{S}(y,w)$ *which satisfies the integral property*

$$\int_{t_k}^{t} \mathcal{S}\left(y(s), w(s)\right) ds \leq 0, \ \forall \ t \in [t_k, t_{k+1}), \ k \in \mathbb{N}.$$

II. There exists a differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$, *class* $\mathcal{K}_\infty$ *functions* β_1, β_2 *and* $\alpha > 0$ *which satisfy*

$$\beta_1\left(\|x\|\right) \leq V(x) \leq \beta_2\left(\|x\|\right), \ \forall \ x \in \mathbb{R}^n,$$

$$\dot{V}\left(x(t)\right) + \alpha V\left(x(t)\right) \leq e^{-\alpha(t-t_k)} \mathcal{S}\left(y(t), w(t)\right), \ \forall \ t \in [t_k, t_{k+1})$$

along the solutions of (3.64).

Then the equilibrium point $x = 0$ *is Globally Asymptotically Stable for any sampling sequence* $\{t_k\}_{k\in\mathbb{N}}$ *with* $t_{k+1} - t_k \leq \overline{h}$.

The theorem provides generic conditions for stability based on the analysis of dissipation like inequalities to be satisfied along the system's (3.63) trajectories. These condition can lead to a constructive stability analysis method that only involves geometric properties of system (3.64).

Corollary 3.23 *[Omran 2016a] Consider system* (3.63) *and the representation* (3.64),(3.65). *Suppose that there exists symmetric positive definite matrices* X, Y, *a positive scalar* α, *class* $\mathbb{K}_\infty$ *functions* β_1, β_2 *and a differentiable function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *such that*

$$\beta_1\left(\|x\|\right) \leq V(x) \leq \beta_2\left(\|x\|\right), \ \forall \ x \in \mathbb{R}^n,$$

$$\frac{\partial V}{\partial x}\left(f_{cl}(x)+g(x)w\right) \leq \left(-\delta_0^2\left\|\frac{\partial K}{\partial x}\left(f_{cl}(x)+g(x)w\right)\right\|_X^2+\|w\|_X^2 \right.$$
$$\left.+2\left\langle\frac{\partial K}{\partial x}\left(f_{cl}(x)+g(x)w\right),w\right\rangle_Y\right)e^{-2\alpha l\overline{h}},\ l\in\{0,1\}, \tag{3.66}$$

for all $x\in\mathbb{R}^n$, $w\in\mathbb{R}^m$, *where* $\delta_0=\frac{2\overline{h}}{\pi}$, *and the notations* $\langle y_1,y_2\rangle_Y := y_1^TYy_2$, $\|y_1\|_X := \sqrt{y_1^TXy_1}$ *were used for vectors* $y_1,y_2\in\mathbb{R}^m$.

Then the equilibrium point $x=0$ *is Globally Asymptotically Stable for any sampling sequence* $\{t_k\}_{k\in\mathbb{N}}$ *with* $t_{k+1}-t_k\leq\overline{h}$.

The result is derived based on Theorem 3.22, using a "supply rate" function of a form similar to (3.56). Note that the proposed conditions do not require computing the system's solutions. Stability can be investigated by studying geometric properties of system (3.64). For the case when the functions f,g describing system (3.63) are polynomials, a numerically tractable sufficient condition can be obtained using Sum-Of-Squares decomposition.

In what follows, the notation $p(\chi)\in\mathbb{R}[\chi]$ with $\chi\in\mathbb{R}^n$, denotes that $p(\chi)$ belongs to the set of polynomials in the variables $\{\chi_1,\chi_2,\cdots,\chi_n\}$ with coefficients in $\mathbb{R}$.

Definition 3.24 *[Prajna 2004] A multivariate polynomial* $p(x)\in\mathbb{R}[x]$ *is said to be a sum of squares (SOS), if there exist some polynomials* $p_i(x)\in\mathbb{R}[x]$, $i\in\{1,\dots,M\}$, *such that* $p(x)=\sum_{i=1}^M p_i^2(x)$.

Corollary 3.25 *[Omran 2013] Consider the sampled-data system* (3.63) *in the case where* $f(x)$, $g(x)$ *and* $K(x)$ *are polynomial functions and the representation* (3.64),(3.65). *Denote*

$$F(x,w) := f_{cl}(x)+g(x)w$$

and

$$G(x,w) := \frac{\partial K}{\partial x}F(x,w).$$

Let $\mathcal{D}=\{x\in\mathbb{R}^n:\mu_l(x)\geq 0,\ l=1,2,\dots,s\}$ *be a neighbourhood of the origin* $x=0$ *where* $\mu_l(x)$, $l=1,2,\dots,s$, *are polynomial functions. Suppose that there exist a polynomial function* $V(x)\in\mathbb{R}[x]$ *of degree* $2d$, *sums of squares* $\sigma_l(\xi)$ *and* $\varsigma_l(\xi)$ *for* $l\in\{1,\dots,s\}$ *and* $\xi=(x,w)$, *such that the following polynomials are SOS*

$$\hat{V}(x)=V(x)-\varphi(x), \tag{3.67}$$
$$\rho_1(\xi)=-\sum_{l=1}^s\sigma_l(\xi)\mu_l(x)-\frac{\partial V}{\partial x}F(x,w)-\alpha V(x),$$
$$+\left[-\delta_0^2G^T(x,w)XG(x,w)+2G^T(x,w)Yw+w^TYw\right], \tag{3.68}$$
$$\rho_2(\xi)=-\sum_{l=1}^s\varsigma_l(\xi)\mu_l(x)-\frac{\partial V}{\partial x}F(x,w)-\alpha V(x),$$
$$+\left[-\delta_0^2G^T(x,w)XG(x,w)+2G^T(x,w)Yw+w^TYw\right]e^{-\alpha\overline{h}}, \tag{3.69}$$

with $\delta_0=\frac{2}{\pi}\overline{h}$, $0\succ X^T=X\in\mathbb{R}^{m\times m}$, $0\preccurlyeq Y^T=Y\in\mathbb{R}^{m\times m}$, *and* $\varphi(x)$ *a positive definite polynomial defined by*

$$\varphi(x)=\sum_{i=1}^n\sum_{j=1}^d\epsilon_{ij}x_i^{2j},\ \textit{such that}\quad\sum_{j=1}^d\epsilon_{ij}>\gamma,\quad\forall i=1,\dots,n. \tag{3.70}$$

Then, the equilibrium $x = 0$ of the system (3.63) is Locally Uniformly Asymptotically Stable. Moreover, the sub-level set $\mathcal{L}_{c^*}$ defined by $c^* = \max_{\mathcal{L}_c \subset \mathcal{D}} c$ with

$$\mathcal{L}_V(c) := \{x \in \mathbb{R}^n : V(x) \leq c\}, \tag{3.71}$$

is an estimate of the domain of attraction. Finally, if (3.68) and (3.69) are SOS while $\mu_l(x) = 0$, for all $l \in \{1, 2, \ldots, s\}$, then the equilibrium is Globally Uniformly Asymptotically Stable.

A numerical illustration of this result is presented below.

Example 3.26 *Consider the following system from [Nešić 2009]*

$$\dot{x} = dx^2 - x^3 + u,$$

with a bounded time-varying $|d| \leq 1$, and a stabilizing control $u = K(x) = -2x$. Emulating this controller results in a sampled-data system that can be represented by the operator Δ_{sh} in (3.65), and a system (3.64) described by

$$\begin{cases} \dot{x} = dx^2 - x^3 - 2x + w, \\ y = -2(dx^2 - x^3 - 2x + w). \end{cases}$$

We apply the Corollary 3.25 in order to find a storage function of the form $V(x) = ax^2 + bx^4$, such that (3.67), (3.68) and (3.69) are SOS. We choose $\varphi(x) = 10^{-3}x^2$, $\alpha = 0.1$ and $\overline{h} = 0.72$. We intend to test the global stability. In this case, the polynomials (3.68) and (3.69) are

$$\begin{aligned} \rho_1(\xi) = &-(2ax + 4bx^3)(dx^2 - x^3 - 2x + w) - \alpha(ax^2 + ax^4) \\ &+ \big[-4\delta_0^2 X(dx^2 - x^3 - 2x + w)^2 \\ &- 4Y(dx^2 - x^3 - 2x + w)w + Yw^2 \big], \end{aligned} \tag{3.72}$$

$$\begin{aligned} \rho_2(\xi) = &-(2ax + 4bx^3)(dx^2 - x^3 - 2x + w) - \alpha(ax^2 + ax^4) \\ &+ \big[-4\delta_0^2 X(dx^2 - x^3 - 2x + w)^2 \\ &- 4Y(dx^2 - x^3 - 2x + w)w + Yw^2 \big] e^{-\alpha\overline{h}}, \end{aligned} \tag{3.73}$$

where a, b, X, Y are decision variables. Note that the time-varying terms d and d^2 appear in the polynomial expressions. However, if both (3.72) and (3.73) are ensured to be SOS for all the values of $(d, d^2) \in \{(1, 0), (1, 1), (-1, 0), (-1, 1)\}$, then they will be SOS for any time-varying $|d| \leq 1$. This is found to be satisfied using the SOSTOOLS software [Prajna 2004], for the storage function $V(x) = 0.77402x^2 + 0.19911x^4$, and a supply function defined by $X = 0.47522$ and $Y = 0.62302\,10^{-3}$. By Corollary 3.25, we obtain the global uniform asymptotic stability of the equilibrium $x = 0$, of the sampled-data system. This result cannot be obtained when trying a quadratic storage function. Increasing α (the exponential decay rate of the storage function), results in the decrement of the maximum value of $\overline{h}$ for which the problem is feasible. This can be seen in Fig 3.15. Previous works considered this example in the literature for estimating the MSI. In [Nešić 2009], a bound of $\overline{h} = 0.368$ is found. In [Karafyllis 2009b], the proposed upper bound is $\overline{h} = 0.1428$. The conditions proposed in this paper are found feasible for $\overline{h} = 0.72$.

Example 3.27 *Consider the following system*

$$\dot{x} = x^2 + (x - 1)u,$$

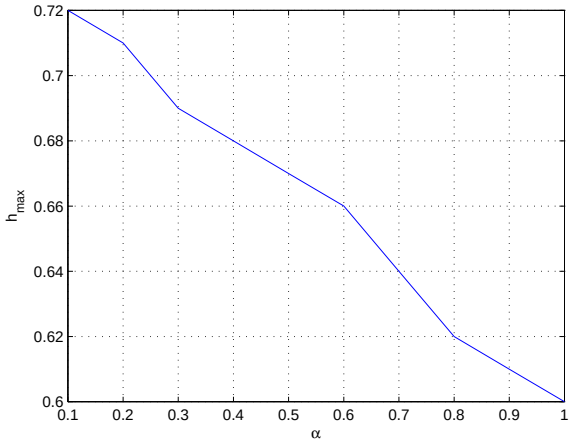


Figure 3.15: Trade-off between α (the exponential decay rate of the storage function), and the estimation of the MSI.

with the controller $u = K(x) = x + 2x^2$*, which stabilizes the equilibrium point* $x = 0$*. Note that, in the continuous-time case, this equilibrium is only locally stable. Our purpose is to find the maximum value of* $\overline{h}$ *that guarantees the local exponential stability of* $x = 0$*, when the controller is emulated. We consider the neighbourhood* $x \in [-0.4, +0.4]$*. The sampled-data system can be represented by the operator* Δ_{sh} *in* (3.65)*, and a system* (3.64) *described by*

$$\begin{cases} \dot{x} = -x + 2x^3 + (x-1)w, \\ y = (1+4x)(-x+2x^3+(x-1)w). \end{cases}$$

We consider applying Corollary 3.25 with a quadratic storage function $V(x) = ax^2$*. We choose* $\varphi(x) = 10^{-3}x^2$*,* $\alpha = 0.25$ *and* $\overline{h} = 0.6$*. The considered domain* $\mathcal{D}$ *is described by* $\{x \in \mathbb{R} : \mu_1(x) \geq 0\}$ *with* $\mu_1(x) = (x-0.4)(0.4-x)$*. The polynomials* (3.68) *and* (3.68) *are in this case*

$$\begin{aligned} \rho_1(\xi) = &-\sigma_1(\xi)\mu_1(x) - (2ax)(-x+2x^3+(x-1)w) - \alpha(ax^2) \\ &+ \big[-\delta_0^2 X(1+4x)^2(-x+2x^3+(x-1)w)^2 \\ &+ 2Y(1+4x)(-x+2x^3+(x-1)w)w + Yw^2\big], \qquad (3.74) \\ \rho_2(\xi) = &-\varsigma_1(\xi)\mu_1(x) - (2ax)(-x+2x^3+(x-1)w) - \alpha(ax^2), \\ &+ \big[-\delta_0^2 X(1+4x)^2(-x+2x^3+(x-1)w)^2 \\ &+ 2Y(1+4x)(-x+2x^3+(x-1)w)w + Yw^2\big]e^{-\alpha\overline{h}}, \qquad (3.75) \end{aligned}$$

where a, X, Y *are decision variables, and* $\sigma_1(\xi), \varsigma_1(\xi)$ *are decision SOS polynomials. Using the software SOSTOOLS we find that* (3.74) *and* (3.75) *are SOS with* $a = 0.12015$*,* $X = 0.25506$*,* $Y = 0.88456\,10^{-2}$*. The decision SOS polynomials are*

$$\begin{aligned}\sigma_1(\xi) &= 0.62335\,w^2 - 0.3616\,xw^2 + 1.6714\,x^2w^2 \\ &\quad - 0.67622\,x^3w + 2.0314\,x^4w + 3.228\,x^6, \\ \varsigma_1(\xi) &= 0.52025\,w^2 - 0.31686\,xw^2 + 1.4349\,x^2w^2 \\ &\quad - 0.54824\,x^3w + 1.60754\,x^4w + 2.8846\,x^6.\end{aligned}$$

Thus all the conditions of Corollary 3.25 are satisfied, and $x = 0$ is locally asymptotically stable. The domain of attraction $\mathcal{L}_V(c^*)$ can be easily seen to be equals to the studied domain $[-0.4, +0.4]$.

3.4 Switching controllers under sampled-data implementations

As follows, some contributions to the analysis and design of sampled-data control loops based on discontinuous feedback laws are presented[19]. First, we address the case of switched affine systems. Next, the sampled-data implementation of relay control laws is considered. The goal is to present for these classes of systems a continuous-time approach to sampled-data switching control design that ensures robustness with respect to sampling and to potential implementations imperfections (jitters, uncertainty etc.).

3.4.1 Switched affine systems

Consider matrices $A_1, A_2, \ldots, A_N \in \mathbb{R}^{n\times n}$ and vectors $b_1, b_2, \ldots, b_N \in \mathbb{R}^n$ where $N \in \mathbb{N}$. The matrices $A_i, i = 1, \ldots, N$, are not necessarily Hurwitz. We are interested in the class of switched affine systems described by

$$\dot{x}(t) = A_{\kappa(x_k)}x(t) + b_{\kappa(x_k)}, \forall t \in [t_k, t_{k+1}), \tag{3.76}$$

where $\kappa : \mathbb{R}^n \to \mathcal{I}_N := \{1, 2, \ldots, N\}$ represents a switching control. The goal is to design a control law κ which ensures stability (in some sense) of the system (3.76) under a sampled-data implementation. Note that in the switched affine system context, due to sampling, one can no longer drive the state exponentially towards the equilibrium point, but only towards a limit cycle or to some attractive compact set containing the equilibrium. Furthermore, classical switching control laws κ are often described by a discrete-event system with transitions ruled by a partition of the state space. Then the sampling usually induces a delay in the discrete-event system variable. This may imply a mismatch in the control: the system state may cross a frontier in the state space in between to sampling instants and one system mode may be active in other state zones than the one for which it has been designed. If not appropriately taken into account, the sampling may be a source of poor performance and even may lead to unbounded solutions.

The following theorem provides switching law design conditions that ensure the practical stability of the closed-loop switched affine system.

Theorem 3.28 *[Hetel 2013b] Consider the unit simplex*

$$\Delta_N = \left\{ \delta = [\delta_1, \delta_2, \ldots, \delta_N]^T \in \mathbb{R}^N,\ \delta_i \geq 0,\ \sum_{i=1}^{N} \delta_i = 1 \right\}, \tag{3.77}$$

[19] The results presented in this section have been developed in collaboration with Prof. Emilia FRIDMAN and Thierry FLOQUET (DR CNRS).

the notations $A(\delta)=\sum_{i=1}^{N}\delta_i A_i,\ b(\delta)=\sum_{i=1}^{N}\delta_i b_i,\ \delta\in\Delta_N$, *system* (3.76) *with* $t_{k+1}-t_k\in(0,\bar{h}]$ *and a given scalar tuning parameter* $\lambda>0$. *Assume that that there exists* $\delta\in\Delta_N$ *such that* $A(\delta)$ *is Hurwitz and* $b(\delta)=0$. *Let there exist matrices* $P,R\succ 0$ *in* $\mathbb{R}^{n\times n}$, *a scalar* $\beta>0$ *such that the LMIs*

$$\begin{bmatrix} A^T(\delta)P+PA(\delta)+2\lambda P+\bar{h}A_i^TRA_i & \bar{h}A_i^TRb_i \\ * & \bar{h}\left(b_i^TRb_i-\beta I\right)\end{bmatrix}\prec 0, \tag{3.78}$$

$$\begin{bmatrix} A^T(\delta)P+PA(\delta)+2\lambda P & 0 & -\bar{h}\Psi_i(\delta) \\ * & -\bar{h}\beta I & \bar{h}b_i^TP \\ * & * & -\bar{h}Re^{-2\lambda\bar{h}}+\bar{h}^2\Psi_i(\delta)\end{bmatrix}\prec 0, \tag{3.79}$$

$\forall i\in\mathcal{I}_N$, *with*

$$\Psi_i(\delta)=\left(A(\delta)-A_i\right)^TP+P\left(A(\delta)-A_i\right),\ i\in\mathcal{I}_N.$$

Then for

$$\kappa(x_k)\in\arg\min_{i\in\mathcal{I}_N}x_k^TP\left(A_ix_k+b_i\right) \tag{3.80}$$

the system solutions are exponentially attracted to the ellipsoid $\mathcal{E}\left(P,\bar{h}\dfrac{\beta}{2\lambda}\right)$, *i.e.*

$$\lim_{t\to\infty}x(t)\in\mathcal{E}\left(P,\bar{h}\frac{\beta}{2\lambda}\right),$$

where by $\mathcal{E}(P,c)$ *we denote the ellipsoid*

$$\mathcal{E}(P,c):=\left\{x\in\mathbb{R}^n:x^TPx<c\right\}. \tag{3.81}$$

The parameter λ from Theorem 3.28 corresponds to the system decay rate. For fixed λ, conditions (3.78), (3.79) represent LMIs. The optimization of the decay rate may be addressed by combining LMI-based methods with a line search on λ. The result is based on a Lyapunov-Krasovskii functional of the form (2.4). Note that the chattering set depends on value of the maximum sampling interval $\bar{h}$. Given $\bar{h}$, the feasibility of (3.78), (3.79) with some P,λ,β guarantees that for $t\to\infty$ the trajectories of the resulting system approach to the ball $\|x\|_2^2<C\bar{h}$, where

$$C=\beta\left(2eig_{min}(P)\right)\lambda)^{-1}$$

with $eig_{min}(P)$ the minimum eigenvalue of P.

It is important to highlight the fact that the conditions in Theorem 3.28 encompass the classical design conditions from [Bolzern 2004]. The set of conditions (3.78),(3.79) for $\bar{h}\to 0$ are reduced to

$$A^T(\delta)P+PA(\delta)+2\lambda P\prec 0, \tag{3.82}$$

which is the classical condition from [Bolzern 2004] ensuring the exponential stability of the continuous-time system. The approach can be easily extended to take into account uncertainties in the system matrices (see [Hetel 2013b] for details).

Example 3.29 *Consider a switched affine system consisting of four affine subsystems with and the following matrices [Bolzern 2004]:*

$$A_1=\begin{bmatrix}4.15 & -1.06 & -6.7\\ 5.74 & 4.78 & -4.68\\ 26.38 & -6.38 & -8.29\end{bmatrix},\ A_2=\begin{bmatrix}-3.2 & -7.6 & -2\\ 0.9 & 1.2 & -1\\ 1 & 6 & 5\end{bmatrix},$$

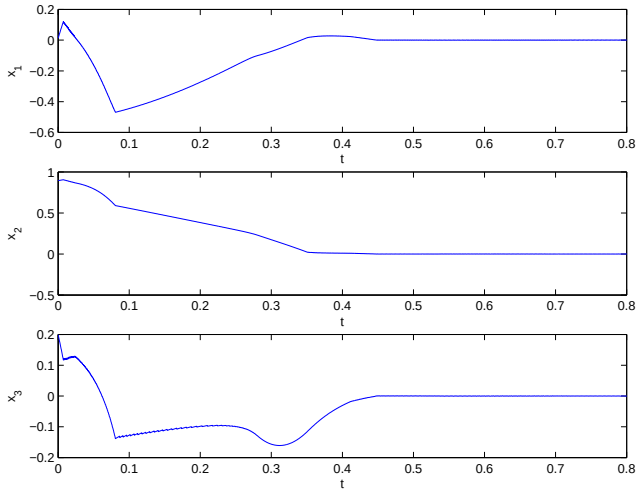


Figure 3.16: Evolution of system states under the control law based on Theorem 3.28 with a fixed sampling interval $h = 3.2 \cdot 10^{-4}$.

$$A_3 = \begin{bmatrix} 5.75 & -16.48 & 2.41 \\ 9.51 & -9.49 & 19.55 \\ 16.19 & 4.64 & 14.05 \end{bmatrix}, \; A_4 = \begin{bmatrix} -12.38 & 18.42 & 0.54 \\ -11.90 & 3.24 & -16.32 \\ -26.5 & -8.64 & -16.6 \end{bmatrix},$$

$$b_1 = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}, \; b_2 = \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix}, b_3 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}, \; b_4 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

Each individual subsystem is unstable. For $\delta_1 = 0.15, \delta_2 = 0.2, \delta_3 = 0.3$ *and* $\delta_4 = 0.35$, *the* $A(\delta)$ *is Hurwitz and* $b(\delta) = 0$. *Using Theorem 3.28 we find that the system is practically stabilizable under variable sampling with* $h_k \leq \overline{h} \leq 3.2 \cdot 10^{-4}$. *The LMI conditions are found to be feasible with*

$$P = \begin{bmatrix} 0.1 & -0.02 & 0 \\ -0.02 & 0.15 & 0.02 \\ 0 & 0.02 & 0.11 \end{bmatrix}, \; R = \begin{bmatrix} 0.13 & 0 & 0.02 \\ 0 & 0.17 & 0.03 \\ 0.02 & 0.03 & 0.16 \end{bmatrix}, \tag{3.83}$$

$\beta = 3.16$ *and* $\lambda = 0.022$. *An illustration of system evolution with an arbitrary initial condition is shown in Figure 3.16.*

Example 3.30 *We illustrate the applicability of this stabilization approach on an example from power electronics. Consider the DC-DC converter from [Hauroigne 2011], where the model has the form*

$$\dot{x}(t) = \mathcal{A}_\kappa x(t) + \mathcal{B}_\kappa$$

with

$$\mathcal{A}_1 = \begin{bmatrix} 0 & 1/L \\ -1/C & -1/(RC) \end{bmatrix}, \quad \mathcal{A}_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1/(RC) \end{bmatrix}, \tag{3.84}$$

$\mathcal{B}_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, $\mathcal{B}_2 = \begin{bmatrix} E/L & 0 \end{bmatrix}^T$ *with* $E = 6V$, $R = 50\Omega$, $L = 20mH$ *and* $C_0 = 220\mu F$. *For* $\delta_1 = \delta_2 = 0.5$, *the matrix* $A(\delta)$ *is Hurwitz and the system may be stabilized to the equilibrium point*

$$x_e = -A(\delta)^{-1} B(\delta) = [0.24 \quad -6]^T$$

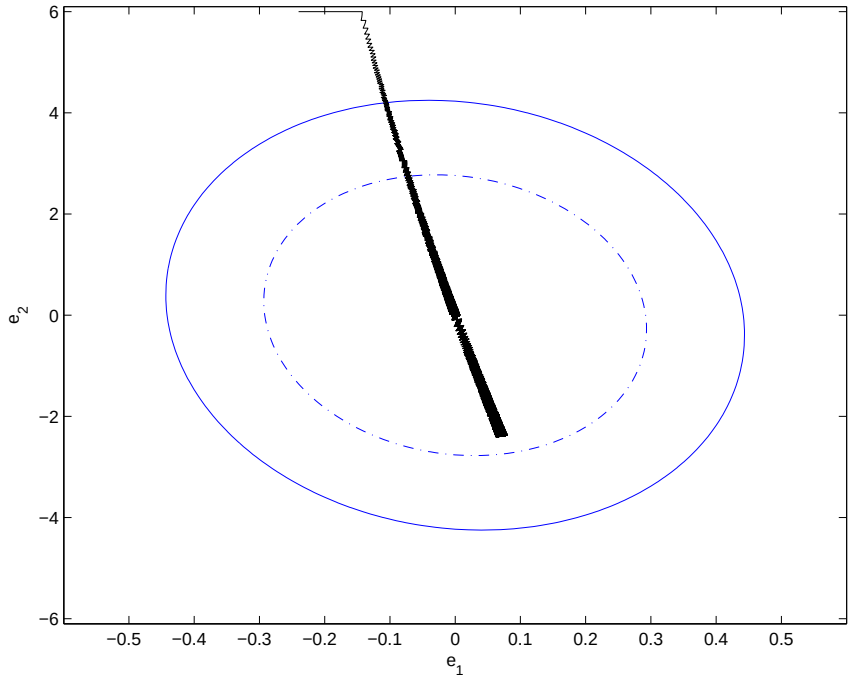


Figure 3.17: Trajectory in the error state space under variations in the resistor value from 35Ω to 65Ω with a fixed sampling interval $\mathbf{T}_{max} = 2.5 \cdot 10^{-5}s$ (solid black line), the attractive sets obtained for the continuous-time case (dashed line) and for $\mathbf{T}_{max} = 2.5 \cdot 10^{-5}s$ (solid line).

using a continuous-time switching law. Consider the error $e = x - x_e$ dynamics

$$\frac{de}{dt} = \mathcal{A}_\kappa e(t) + \mathcal{A}_\kappa x_e + \mathcal{B}_\kappa.$$

For the numerical tests, the time scale change $t = \epsilon \boldsymbol{t}$ with $\epsilon = 10^4$ is used to cope with large numerical values in the system matrices and to avoid ill conditioned matrix inequalities. The system of the form (3.76) is obtained with $A_i = \epsilon^{-1}\mathcal{A}_i$, $b_i = \epsilon^{-1}\left(\mathcal{A}_i x_e + \mathcal{B}_i\right)$, $\bar{h} = \epsilon \boldsymbol{T}_{max}$, $x = e$. Note that the trajectories are invariant with respect to time scaling. Furthermore, the switching laws are equivalent, since

$$\arg\min_{i\in\mathcal{I}_N}(x - x_e)^T P\left(\epsilon A_i x + \epsilon b_i\right) = \arg\min_{i\in\mathcal{I}_N}(x - x_e)^T P\left(A_i x + B_i\right).$$

Concerning the robust switching law design, the conditions of Theorem 3.28 are feasible for any (time-varying) sampling intervals with $\boldsymbol{T}_{max} \leq 1.5 \cdot 10^{-3}s$.

To illustrate the use of our method for uncertain systems, choose $\boldsymbol{T}_{max} = 2.5 \cdot 10^{-5}s$ and assume that the resistor is subject to unknown time-varying uncertainties $\delta R(t) \in [-15\Omega, +15\Omega]$. Then each of the matrices $\mathcal{A}_i$ is varying in a polytope corresponding to the two vertices $R \pm 15\Omega$. The robust stabilization conditions in Corollary 2 from [Hetel 2013b] are feasible with

$$P = \begin{bmatrix} 9.175 & 0.088 \\ 0.088 & 0.1 \end{bmatrix}, U_i = \begin{bmatrix} 7.75 & 0.161 \\ 0.161 & 0.048 \end{bmatrix}, i = 1, 2,$$

$\beta = 2.69 \cdot 10^{-2}, \gamma = 1.9 \cdot 10^{-3}$, which implies that $\|e(t)\| < 4.23$ as $t \to \infty$. The error system evolution with the initial condition $x(0) = 0$ is shown in Figure 3.17. The figure presents the attractive ellipsoids for both the sampled-data case and for the continuous-time switching implementation. Due to sampling and to parametric uncertainties the system state under a continuous-time control implementation (in black) does not converge to the equilibrium point (the center of the ellipsoid) but only to a bounded region. Numerical simulations under an uniform sampling $\boldsymbol{T}_{max}$ show that the same attractive ellipsoid is achieved for bigger $\boldsymbol{T}_{max} = 1.4 \cdot 10^{-3}$, to be compared with $\boldsymbol{T}_{max} = 2.5 \cdot 10^{-5}$ proved in theory under the variable sampling. The latter may illustrate the conservatism of the method.

In addition to the results presented here, the design of sampled-data switching controllers for switched affine systems has also been addressed using a hybrid system approach in [Hetel 2015b].

3.4.2 Relay control

As follows we present the method from [Hetel 2015c] which ensures local practical stabilization of LTI systems with relay control laws. Consider the system

$$\dot{x}(t) = Ax(t) - B\gamma \text{sign}(\Gamma x_k), \forall t \in [t_k, t_{k+1}), \tag{3.85}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $\gamma > 0$ and where $\Gamma \in \mathbb{R}^{1\times n}$ is a design parameter representing the switching hyperplane. We propose a simple design method based on the existence of a stabilizing linear state feedback and we show how it may be used in the sampled-data case in order to guarantee (locally) the practical stabilization to a bounded ellipsoid containing the origin. The main idea of the design procedure is to use the existence of an exponentially stabilizing state feedback as a reference control to be emulated (locally) by a relay feedback.

Proposition 3.31 *(adapted from [Hetel 2015c]) Consider system (3.85) with $t_{k+1} - t_k \leq \overline{h}$. Assume that the pair (A, B) is stabilizable and consider a gain matrix K such that $A_{cl} = A + BK$ is Hurwitz. Given tuning parameter λ, let there exist symmetric positive definite matrices P, R, and a positive scalar $\beta < 2\frac{\lambda}{\overline{h}}$ such that:*

$$\begin{bmatrix} I & \gamma^{-1}K \\ * & P \end{bmatrix} \succ 0, \tag{3.86}$$

$$\begin{bmatrix} A_{cl}^T P + PA_{cl} + 2\lambda P + \overline{h}A^T RA & \overline{h}A^T RBv \\ * & \overline{h}\left(B^T RB - \beta\right) \end{bmatrix} \prec 0, \tag{3.87}$$

$$\begin{bmatrix} A_{cl}^T P + PA_{cl} + 2\lambda P & 0 & -\left(PBK\right)^T \overline{h} \\ * & -\beta\overline{h} & \left(PBv\right)^T \overline{h} \\ * & * & -\overline{h}Re^{-2\lambda\overline{h}} \end{bmatrix} \prec 0, v \in \{-\gamma, \gamma\}. \tag{3.88}$$

Then for $\Gamma = B^T P$ any solution $x(t)$ of (3.85) with initial condition $x(0) \in \Omega_0 = \mathcal{E}(P, 1)$ converges exponentially to $\Omega_\infty = \mathcal{E}(P, c)$ as $t \to \infty$, with $c = (2\lambda)^{-1}\beta\overline{h}$.

Example 3.32 *Consider a linear time-invariant system with*

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3.89}$$

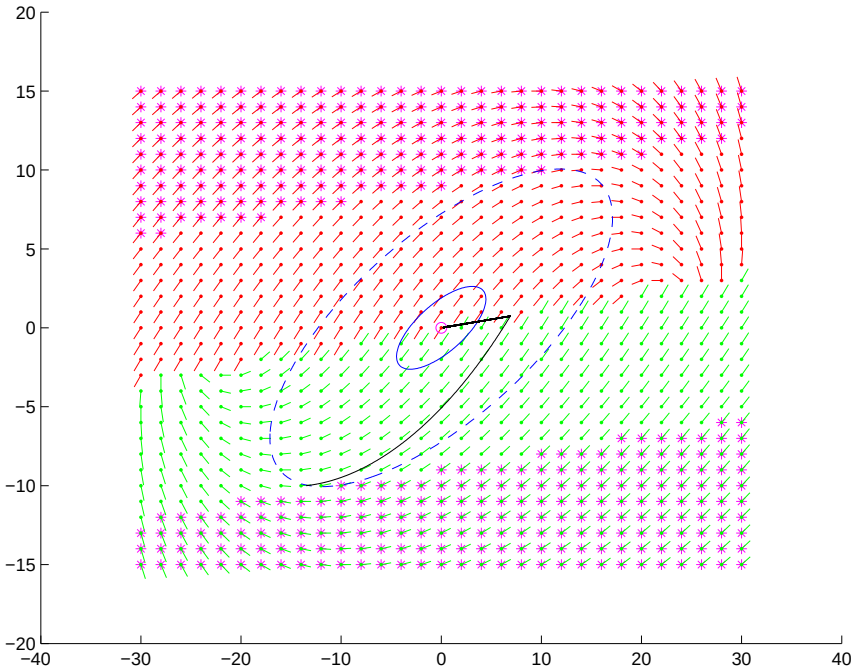


Figure 3.18: Illustration of evolution in the state space for a constant sampling interval $T = 10^{-3}$. Green: $u = \gamma$; red: $u = -\gamma$; ellipsoid in dashed line: estimation of the domain of attraction Ω_0; ellipsoid in solid line: attractive set for $t \to \infty$, Ω_∞; black line: trajectory from the initial condition $x_0 = [-13.5 \quad -10]^T$.

The A matrix has unstable eigenvalues $1 \pm i$. Consider that the control is constrained to the set $\mathcal{V} = \{-\gamma, \gamma\}$ with $\gamma = 25$. The pair (A, B) is fully controllable. The state feedback

$$K = [0.3125 \quad -2.8125]$$

ensures that $A_{cl} = A + BK$ is Hurwitz. Using $\lambda = 0.23$ and the gain K, it is possible to design a sampled-data relay control. For this set of parameters, with R and β as decision variables, the conditions of Proposition 3.31 are feasible for $\overline{h} \leq 1.9 \cdot 10^{-2}$. In particular, for $\overline{h} = 10^{-3}$, the LMIs are found feasible with $\beta = 15.63$ and

$$P = 10^{-2} \begin{bmatrix} 0.66 & -0.78 \\ -0.78 & 1.91 \end{bmatrix}, \tag{3.90}$$

which leads to $\Omega_\infty = \mathcal{E}(P, 0.068)$. A numerical illustration is shown in Figure 3.18.

Example 3.33 *As follows we illustrate the practical implementation of sampled-data controllers on a real cart-pendulum system platform from Ecole Centrale de Lille. We consider the following*

linearized model of the inverted pendulum on a cart:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{Ml} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{a}{M} \\ 0 \\ \frac{-a}{Ml} \end{bmatrix} u. \tag{3.91}$$

Here x represents the cart position, θ, the angle, $M = 3.9249Kg$ and $m = 0.2047Kg$, the cart and pendulum masses, respectively, $l = 0.2302m$ the distance from the pendulum center of mass to its pivot, $g = 9.81N/Kg$, the gravitational acceleration and $a = 25.3N/V$, the gain of the linear motor. We consider that the system input is restricted to $\mathcal{V} = \{-1, 1\}$. The control law is implemented on D-Space card with a sampling frequency $\overline{h} = 10^{-4}s$. The system can be stabilized by a continuous-time state feedback with $K = [6.4763\ 5.2313\ 15.4168\ 2.7498]$ for which

$$P = \begin{bmatrix} 117.66 & 77.84 & 171.63 & 27.93 \\ 77.84 & 57.947 & 133.258 & 21.425 \\ 171.63 & 133.258 & 347.978 & 54.504 \\ 27.93 & 21.425 & 54.504 & 9.18 \end{bmatrix} \tag{3.92}$$

is a Lyapunov matrix. Using Proposition 3.31 with $\lambda = 1.45$ it is possible to show that

$$\Gamma = PB = \begin{bmatrix} -280.33 & -226.412 & -667.228 & -118.95 \end{bmatrix}^T$$

ensures (in theory) local stabilization in $\mathcal{E}(P, 1)$. For the obtained sampled-data implementation, any system solution with initial conditions in $\Omega_0 = \mathcal{E}(P, 1)$ converges exponentially to $\Omega_\infty = \mathcal{E}(P, 0.07)$. The state evolution, illustrating practical stabilization of both pendulum angle and cart position, is shown in Figure 3.20 (to be compared with simulations in Figure 3.19). Differences between experimentations and simulations are due to the use of the linearized model of the inverted pendulum for control design, to imprecisions in the identification of system parameters and to perturbations due to friction.

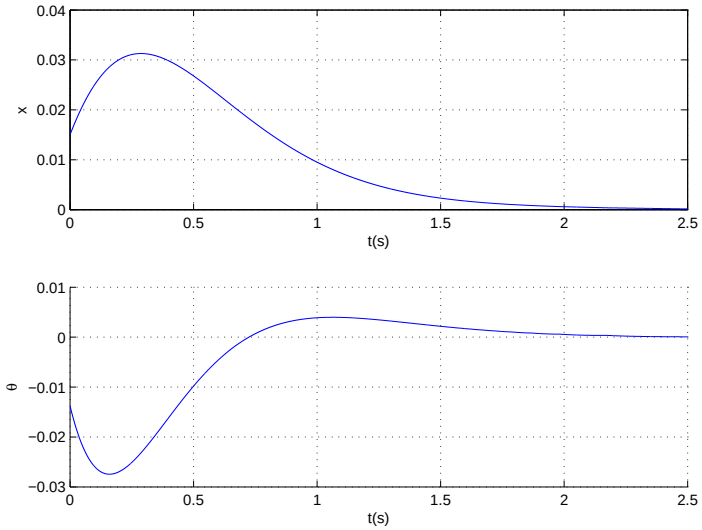


Figure 3.19: Numerical simulation based on the LTI model for the inverted pendulum on a cart: evolution of card position x (upper sub-plots in meters) and pendulum angle θ (lower sub-plots in radians) with a sampled-data relay control from the initial condition $x_0 = [0.015\ 0.115\ -0.014\ -0.142]$.

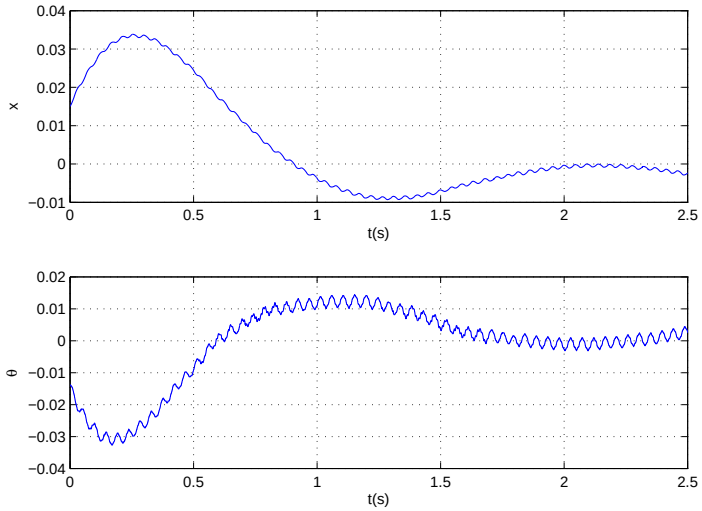


Figure 3.20: Experimental result for the inverted pendulum on a cart: evolution of card position x (upper sub-plots in meters) and pendulum angle θ (lower sub-plots in radians) with a sampled-data relay control from the initial condition $x_0 = [0.015\ 0.115\ -0.014\ -0.142]$.

Conclusion

In this part, several contributions to the study of aperiodic sampled-data systems have been presented. First, in the case of linear systems it is shown how the conservatism in the stability analysis can be reduced using discrete-time methods based on quasi-quadratic Lyapunov functions. A continuous-time approach combining the advantages of both time-delay methods and discrete-time ones has been presented and applied to the self-triggering control problem. The main issue is that the design of sampling maps can be optimized using robust control tools. Next, we have tackled the stability problem for nonlinear sampled-data systems, by studying the case of bilinear systems. Constructive stability analyzis conditions have been proposed using a hybrid system approach and a generalization of the Input/Output stability approach. The latter has been extended to a more general class of affine nonlinear systems, with aperiodic sampled-data control. Finally, the sampled-data implementation of some classed of discontinuous controllers has been studied.

It is to be emphasised that the interest of the presented results goes beyond the simple aperiodic sampling problem. In fact, this framework can be seen as an abstraction of more complex phenomena presented in Networked Control Systems. Many of the presented approaches can be extended to deal with delay, quantization or scheduling protocols [Donkers 2009], [Cloosterman 2010], [Hetel 2011a], [Donkers 2011a], [Lombardi 2012], [Liu 2012b], [Liu 2015a]. Furthermore, the presented approaches can be generalized to more complex hybrid dynamical systems [Hetel 2013a].

Part II

Design of switching controllers - an emerging research direction

The design of switching controllers represents an important problem in Control Theory. Simple ON/OFF, bang-bang and relay controllers are widely used in various technical domains. They represent the key components in variable structure systems [Emel'Yanov 1967] and sliding mode control [Edwards 1998] and have very interesting robustness properties faced to matched perturbations. Switching controllers are inherently *hybrid dynamical systems* which may describe complex behaviours [Goebel 2009], [Liberzon 2003a], [Bourdais 2007], [Acary 2014]. It is well known in the literature that even the simple relay feedback systems may tend to sliding modes [Utkin 1992, Wang 2015], Zeno solutions [van der Schaft 2000] or limit cycles [Johansson 1999].

As follows, we will present an emerging research direction concerning some classes of dynamical systems of the form

$$\dot{x} = f(x) + g(x)u \tag{1}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n,\ g : \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz continuous functions. Here $x \in \mathbb{R}^n$ is the systems state and $u \in \mathbb{R}^m$ is the input which is assumed to be constrained to take values in a finite set of vectors

$$\mathcal{V} = \{v_1, v_2, \ldots, v_N\}. \tag{2}$$

The goal is to design a control law

$$u = \kappa(x),\ \kappa : \mathbb{R}^n \to \mathcal{V}, \tag{3}$$

which ensures the stability of (1). Over this part, the system's solutions will be considered in the sense of Filippov (see [Filippov 1988]).

This problem formulation encompasses the classical relay feedback control design problem where the input u is constrained to take values in the set $\mathcal{V} = \{-v, v\}$, for some positive constant v [Flugge-Lotz 1953, Tsypkin 1984, Johansson 1999, Liberzon 2013]. It is important to highlight, that although relay feedback has been studied for a long time, there are still many unsolved issues. For the moment, very few numerical tools exist for designing switching surfaces while optimizing the system performances or the size of the domain of attraction.

For the case when the set $\mathcal{V}$ takes the form $\mathcal{V} = \{0, 1\}^m$, we encounter the case of ON/OFF actuators. Their study is motivated by the large number of applications in the domain of power electronics [Erickson 2001, Bacha 2014]. In this context, methods based on Pulse-Width Modulation (PWM) and averaging are often used for implementing classical continuous controllers while ignoring the ON/OFF nature of actuators.

The problem statement can also be related to the study of control loops with quantization [Brockett 2000, Liberzon 2003b, Liberzon 2005]. The set of control $\mathcal{V}$ can represent control value for systems that are subject to both saturation and quantization.

For the particular case when the set of vectors $\mathcal{V}$ form a simplex in $\mathbb{R}^m$ ($N = m + 1$, every subset of m vectors in $\mathcal{V}$ are linearly independent and there exists $m + 1$ positive scalars $\nu_i, i \in \mathcal{I}_{m+1}$ such that $\sum_{i=1}^{m+1} \nu_i v_i = 0$, $\sum_{i=1}^{m+1} \nu_i = 1$), the design of a control u with values constrained to the set $\mathcal{V}$ is a *simplex*-type variable structure control problem (see [Bartolini 2011, Bajda 1985] and the references with).

As follows, we will present a novel design strategy for the design of switching controllers defined on finite sets. The aim is to propose a convex optimization approach for the definition of switching surfaces. The methodology combines tools for systems with bounded controls and saturation [Tarbouriech 2011], [Blanchini 1999] with convex embedding arguments [Liberzon 2003a]. The main idea of the design procedure is to use the existence of a continuous stabilizer in order to re-design a switching control. It is based on simple convex optimization arguments and does

not need any computation of normal forms. For several classes of systems (LTI, LPV, switched affine, bilinear), the design of a switching controller can be formulated as a classical LMI problem, allowing to optimize the size of the domain of attraction and the robustness with respect to perturbations or parameter variations.

This part is structured as follows. Fist, some results are presented for the case of linear systems (time, invariant, polytopic uncertain and LPV) in Chapter 4. Next, in Chapter 5 the case of switched affine systems is considered. At last, some applications are presented in Chapter 6.

Chapter 4

Linear systems

In this chapter, we present some results concerning the design of switching surfaces for the case of linear system[20]. LMI stabilization conditions are given for linear (possibly uncertain) systems and LPV systems.

4.1 Simplified problem formulation

Consider $n, m \in \mathbb{N}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and the system

$$\dot{x} = Ax + B\left(u + d\right), \tag{4.1}$$

where $x \in \mathbb{R}^n$ represents the system state, $u \in \mathbb{R}^m$ the input and $d \in \mathbb{R}^m$ a matched perturbation. We adopt the following assumptions:

- (A.1) The pair (A, B) is stabilizable.
- (A.2) The input u is a static state feedback constrained to take values in a finite set of constant vectors $\mathcal{V} := \{v_1, v_2, \ldots, v_N\} \subset \mathbb{R}^m$, where N is a positive integer, i.e. $u = \kappa(x)$ with $\kappa : \mathbb{R}^n \to \mathcal{V}$.
- (A.3) The perturbation d is a measurable function taking values in the cube $\mathcal{P}\left(d_{max}\right)$ where $d_{max} \geq 0$ is a known scalar and $\mathcal{P}\left(c\right) := \left\{y \in \mathbb{R}^m : \|y\|_\infty \leq c\right\},\ \forall c \geq 0$.
- (A.4) $\mathrm{conv}\left\{\mathcal{V}\right\}$ is a nonempty closed subset in $\mathbb{R}^m$ containing the null vector in its interior: $0_m \in \mathrm{Int}\left\{\mathrm{conv}\left\{\mathcal{V}\right\}\right\}$.
- (A.5) There exists $\rho \in [0, 1)$ such that $\mathcal{P}\left(d_{max}\right) \subset \mathrm{conv}\left\{\rho\mathcal{V}\right\}$.

We are interested in the design of control laws $u = \kappa(x)$ of the form

$$u = \kappa(x) \in \arg\min_{v\in\mathcal{V}} x^T\Gamma v \tag{4.2}$$

where $\Gamma \in \mathbb{R}^{n\times m}$ is a matrix to be determined.

Note that for the case when the input u is a scalar constraint to the set $\mathcal{V} = \{-v, v\}$, with $v > 0$ a given constant, we obtain $u = \kappa(x) = v$ whenever $x^T\Gamma v \leq x^T\Gamma(-v)$, i.e. for $x^T\Gamma \leq 0$.

[20] The results presented in this chapter have been developed in collaboration with Prof. Emilia FRIDMAN, Thierry FLOQUET (DR, CNRS), Ass. Prof. Alexandre KRUSZEWSKI and Romain DELPOUX (ATER at LAGIS).

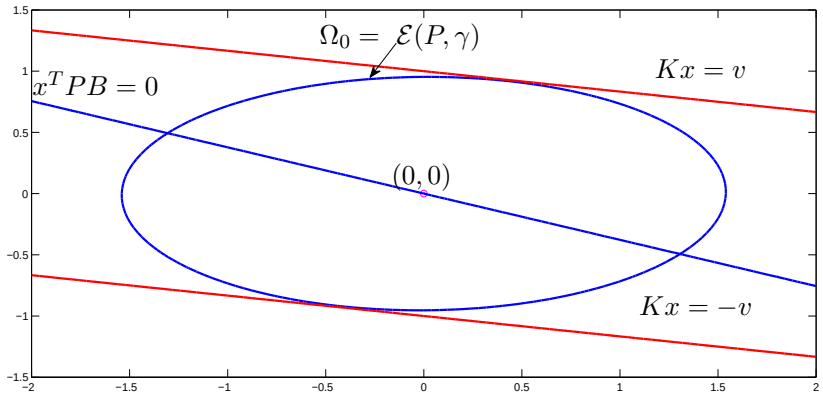


Figure 4.1: Basic idea for a system with $\mathcal{V} = \{v, -v\}$. Here $\mathcal{C}_{\mathcal{V}}(K)$ is the set delimited by the lines $Kx = v$ and $Kx = -v$. The ellipsoid $\Omega_0 := \mathcal{E}(P,\gamma)$ is chosen as the largest one contained in $\mathcal{C}_{\mathcal{V}}(K)$. Inside Ω_0 the continuous stabilizer $u = Kx$ can be replaced by a switching controller $u = -vsign(B^T Px)$.

Similarly, $u = \kappa(x) = -v$ whenever $x^T\Gamma \geq 0$. Then, for $\mathcal{V} = \{-v, v\}$, with $v > 0$, the control law (4.2) is reduced to the classical relay control $u = \kappa(x) \in -v$ sign $(\Gamma^T x)$.

Since the values of the input are restricted to a finite set, the closed loop system (4.1),(4.2) has a discontinuous right-hand side.

The goal is to provide criteria for the synthesis of a relay control law (4.2) that ensures *local stability* of Filippov solutions associated to the closed-loop system (4.1),(4.2). We provide optimization methods for control design while enlarging the domain of attraction. Finite time reachability properties to sliding manifolds and the robustness with respect to matched perturbations and time-varying uncertainties will be discussed.

4.2 Basic idea

Let us first consider the case when $d = 0$. Note that Assumption (A.1) is equivalent with

- (A.6) $\exists\ P \succ 0, K \in \mathbb{R}^{m\times n}, \delta > 0$, such that

$$(A + BK)^T P + P(A + BK) \prec -2\delta P. \tag{4.3}$$

Then $V(x) = x^T Px$ satisfies

$$\frac{\partial V}{\partial x}(A + BK)x < -2\delta V(x), \forall x \neq 0, \tag{4.4}$$

i.e. it is a Lyapunov function for system (4.1) with the state-feedback control law Kx.

For γ let

$$\mathcal{E}(P,\gamma) := \left\{x \in \mathbb{R}^n : x^T Px < \gamma\right\}$$

denote the γ level set of the function $V(x) = x^T P x$ and $\mathcal{C}_{\mathcal{V}}(K)$ the subset of the state space for which Kx belongs to the convex hull of $\beta\mathcal{V}$,

$$\mathcal{C}_{\mathcal{V}}(K) := \{x \in \mathbb{R}^n : Kx \in \text{conv}\,\{\mathcal{V}\}\}.$$

Since conv $\{\mathcal{V}\}$ is a non-empty closed subset in $\mathbb{R}^m$ containing the null vector in its interior, there exists a level set described by $\gamma > 0$ such that

$$\Omega_0 := \mathcal{E}(P, \gamma) \subset \mathcal{C}_{\mathcal{V}}(K). \tag{4.5}$$

The main idea is to use the existence of the linear state feedback gain K in order to design a locally stabilizing feedback of the form (4.2) (see also Figure 4.1 for a graphical illustration).

Remark that for any $x \in \Omega_0$ there exist N scalars $\alpha_j(x) \geq 0,\ \forall j \in \mathcal{I}_N$ with $\sum_{j=1}^N \alpha_j(x) = 1$ such that

$$Kx = \sum_{j=1}^{N} \alpha_j(x)\, v_j. \tag{4.6}$$

From (4.4), (4.5) and (4.6), we have

$$\sum_{j=1}^{N} \alpha_j(x) \frac{\partial V}{\partial x}(Ax + Bv_j) < -2\delta V(x), \tag{4.7}$$

for all $x \in \Omega_0 \setminus \{0\}$. Considering that $\alpha_j(x) \geq 0, j \in \mathcal{I}_N$, there must be at least one $j \in \mathcal{I}_N$ such that

$$\frac{\partial V}{\partial x}(Ax + Bv_j) < -2\delta V(x),\ \forall x \in \Omega_0 \setminus \{0\}. \tag{4.8}$$

Since Ω_0 represents a sub-level set of $V(x)$, local stabilization in Ω_0 with a control of the form (4.2) is ensured by choosing the control $\kappa(x)$ with the steepest descend of the Lyapunov function

$$\kappa(x) \in \arg\min_{v \in \mathcal{V}} x^T P B v \tag{4.9}$$

which leads to setting $\Gamma = PB$ in (4.2). Note that if there are several minimizers v in (4.9), they all ensure the decay of V. We arrive to the following:

Proposition 4.1 *[Hetel 2013c] Consider system* (4.1) *with* $d = 0$*, a control law* (4.2) *and Assumptions (A.2),(A.4),(A.6). Then there exit a function* $V(x) = x^T P x$ *, with* P *a symmetric positive definite matrix, and scalars* $\delta, \gamma > 0$ *such that for* $\Gamma = PB$

$$\frac{\partial V}{\partial x}(Ax + B\kappa(x)) < -2\delta V(x), \tag{4.10}$$

for all $\kappa(x) \in \arg\min_{v \in \mathcal{V}} x^T P B v, x \in \Omega_0 \setminus \{0\}$ *where* $\Omega_0 = \mathcal{E}(P, \gamma)$.

Using standard developments, it can be shown that (4.10) is a sufficient condition for the local asymptotic stability of Filippov solutions associated to system (4.1) with $d = 0$ and the control law (4.2). As follows, it will be shown that the provided control law also ensures robustness to perturbations and it presents a finite time reachable sliding dynamics.

4.3 Sliding dynamics and robustness to perturbations

The following theorem provides design conditions for the control law (4.2) in the case of non null perturbations.

Theorem 4.2 *[Hetel 2015c] Consider a set of co-vectors $h_i \in \mathbb{R}^{1\times m}, i \in \mathcal{I}_{n_h}$ describing the dual representation of the polytope $conv\{\mathcal{V}\}$:*

$$conv\{\mathcal{V}\} = \{y \in \mathbb{R}^m : \ h_i y \leq 1, i \in \mathcal{I}_{n_h}\}. \tag{4.11}$$

Consider Assumptions (A.2)-(A.6) and the closed-loop system (4.1),(4.2) with $\Gamma = PB$. Then for any

$$\gamma \leq \min_{i\in\mathcal{I}_{n_h}} (1-\rho)^2 \left(h_i K P^{-1} K^T h_i^T\right)^{-1} \tag{4.12}$$

a) the origin $x = 0$ of the closed-loop system is locally exponentially stable in $\Omega_0 := \mathcal{E}(P,\gamma)$;

b) if $rank(B) = m \leq n$ then, for $s = B^T P x$ the surface $s = 0$ is finite time reachable whenever $x(0) \in \mathcal{E}(P,\gamma)$, i.e. exists $t_f \in [0,\infty)$ such that $s(t) = 0$ for all $t \geq t_f$.

Furthermore, if for some P satisfying (4.3), $A^T P + PA$ is negative semi-definite then

c) the origin of the closed-loop system is globally asymptotically stable.

The theorem provides simple design conditions of a robust stabilizing controller under the simple assumptions (A.2)-(A.6). Note that the theorem guarantees that for the case $rank(B) = m \leq n$ the surface $s = B^T P x = 0$ is a sliding hyperplane that is reached in a finite time. The design procedure can be easily extended to deal with parametric uncertainties in the matrix A, that is when $A(\mu(t)) \in \mathcal{A} := \text{conv}\{A_1, A_2, \ldots, A_{n_v}\}$ where $\mu(t) = \left[\mu_1(t)\ \mu_2(t)\ \ldots \mu_{n_v}(t)\right]^T$ are the barycentric coordinates of A in $\mathcal{A}$.

Corollary 4.3 *[Hetel 2015c] For $c > 0$ and $x \in \mathbb{R}^n$, let*

$$\mathcal{B}(x,c) := \{y \in \mathbb{R}^n : \|x - y\|_2 < c\}.$$

Consider the system

$$\dot{x} = A(\mu)\, x + B(u + d), \tag{4.13}$$

where $\mu(\cdot)$ is measurable, Assumptions (A.2)-(A.5) and the dual representation of the polytope $conv\{\mathcal{V}\}$ in (4.11). Given $\delta > 0, \gamma > 0$, assume that there exists (Q, λ, ϵ) solution to the set of linear matrix inequalities

$$Q = Q^T \succ 0, \ \lambda > 0,$$

$$A_j Q + Q A_j^T - \lambda B B^T \prec -2\delta Q, \quad \forall j \in \mathcal{I}_{n_v}, \tag{4.14}$$

$$\begin{bmatrix} \epsilon I & I \\ * & Q\gamma \end{bmatrix} \succ 0, \tag{4.15}$$

$$\begin{bmatrix} 1 & \frac{\lambda}{2(1-\rho)} h_i B^T \gamma \\ * & Q\gamma \end{bmatrix} \succ 0, \qquad i \in \mathcal{I}_{n_h}. \tag{4.16}$$

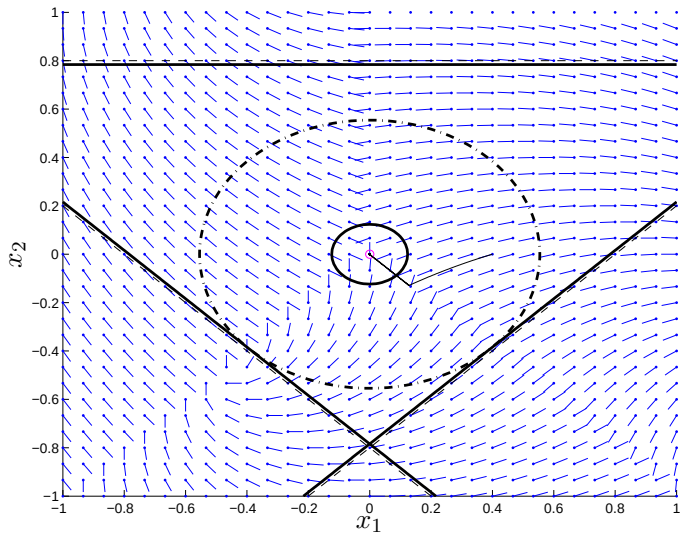


Figure 4.2: Phase space for the closed-loop system in Example 4.4 with $\|d\|_\infty \leq 0.01$. Ellipsoid in dotted-dashed line – Ω_0. Solid black lines – limiting hyperplanes of $\mathcal{C}_{(1-\rho)\mathcal{V}}(K)$.

Then the origin $x = 0$ of the closed-loop system (4.13),(4.2) with $\Gamma = Q^{-1}B$ is locally asymptotically stable in the ellipsoid $\mathcal{E}\left(Q^{-1}, \gamma\right)$ containing the ball $\mathcal{B}(0, c_B)$ with $c_B = 1/\sqrt{\epsilon}$. Furthermore, if $rank(B) = m \leq n$, the surface $s = B^T Q^{-1} x = 0$ is finite time reachable for any $x(0) \in \mathcal{E}\left(Q^{-1}, \gamma\right)$.

The existence of a solution (Q, λ, ϵ) to the LMI optimization problem inf ϵ under the constraints (4.14)-(4.16), guarantees that any Filippov solution of the closed-loop system (4.1), (4.2) (with $\Gamma = Q^{-1}B$), originating from $\mathcal{E}\left(Q^{-1}, \gamma\right)$ is exponentially converging to the origin. By minimizing ϵ, the size of the invariant ellipsoid is maximized. Note that without any loss of generality we may always consider $\gamma = 1$. If the LMIs (4.14)-(4.16) are satisfied for $(Q_0, \lambda_0, \epsilon_0)$, then they are also satisfied for $\gamma = 1$ with $(Q_0\gamma_0, \lambda_0\gamma_0, \epsilon_0)$. Given d_{max}, the minimum ρ s.t. $\mathcal{P}\left(d_{max}\right) \subset \text{conv}\left\{\rho\mathcal{V}\right\}$ can be computed from the standard optimization problem:

$$\inf \rho \text{ s.t. } h_i y \leq \rho, \forall\, y \in \text{vert}\left\{\mathcal{P}\left(d_{max}\right)\right\},\ i \in \mathcal{I}_{n_h}. \tag{4.17}$$

Example 4.4 *Consider a system (4.1) described by*

$$A = \begin{bmatrix} a & -1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{V} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$$

with $a = 1$, $\|d\|_\infty < d_{max} = 0.01$. The set conv $\{\mathcal{V}\}$ in (4.11) is characterized by $h_1 = [-1\ 1]$, $h_2 = [1\ \ 1]$, $h_3 = [0\ \ -1/2]$.

Addressing the optimization problems (4.17) and inf ϵ under the constraints (4.14)-(4.16) with $\gamma = 1, \delta = 0.25$, leads to a control law (4.2) with $\Gamma = PB$ and

$$P = \begin{bmatrix} 3.25 & 0 \\ 0 & 3.25 \end{bmatrix},$$

which ensures the local (robust) stabilization in $\Omega_0 = \mathcal{E}(P, 1)$*, containing the ball with the radius* $c_B = 0.55$*. For this example* $s = B^T P x$ *corresponds to the origin. Then the equilibrium point is finite-time reachable. Let us remark that the boundary of the domain of attraction is not far from the unstable equilibrium points of the closed-loop system:* $-A^{-1}Bv_2 = [1.5 \ \ -0.5]^T$*,* $-A^{-1}Bv_3 = [-0.5 \ 1.5]^T$*. Furthermore, for* $x(0) = [0.501 \ \ -0.501]^T$*, simulations with constant sampling interval* $t_{k+1} - t_k = 10^{-5}, \forall k \in \mathbb{N}$ *and* $d_{max} = 0$*, illustrate an unstable system behaviour. Note that* $\|x(0)\|_2 = 0.708$*, to be compared with* $c_B < 0.55$ *for which local stabilization is ensured. This gives an idea about the accuracy of the ellipsoidal estimation of the domain of attraction. An illustration is provided in Figure 4.4. A simulation from the initial condition* $x(0) = [0.4 \ 0]^T$ *is presented under arbitrary variations of the matched perturbation and with a sampling interval of* 10^{-3}*.*

Let us remark that for the system under study the matrix A is unstable. Therefore it is impossible to apply the classical global stabilization control design techniques based on the existence of a stable convex combination [Deaecto 2010, Bolzern 2004].

Assume now that the parameter a is time-varying in $[0.97, 1.03]$*. Let us consider a continuous-time control design based on Corollary 4.3 for* $\|d\|_\infty < d_{max} = 0.01$*. For* $\gamma = 1$*, solving the LMI problem* (4.14)-(4.16) *(for the two vertex of the A matrix) while minimizing* ϵ*, leads to a control law of the form* (4.2) *with* $\Gamma = PB$ *and*

$$P = \begin{bmatrix} 0.33 & 0 \\ 0 & 0.33 \end{bmatrix},$$

which ensures local stabilization of the continuous-time systems in $\Omega_0 = \mathcal{E}(P, 1)$ *for any* $\|d\|_\infty < d_{max} = 0.01$ *and any* $a(t) \in [0, 97, 1.03]$*.*

4.4 LPV case and Parameter Dependent Relay Control

The approach previously presented can be extended to case of Linear Parameter-Varying (LPV) systems with the state-space realization:

$$\dot{x} = A(\mu)x + B(\mu)u, \tag{4.18}$$

where $x \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}^m$ is the control vector, the matrices $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$ are polytopic matrices with the following form:

$$A(\mu(t)) = \sum_{i=1}^{n_v} \mu_i(t) A_i, B(\mu(t)) = \sum_{i=1}^{n_v} \mu_i(t) B_i, \tag{4.19}$$

with $A_1, \ldots, A_{n_v}, B_1, \ldots, B_{n_v}$ being known constant matrices. In what follows, the vector

$$\mu(t) = \begin{bmatrix} \mu_1(t) & \ldots & \mu_{n_v}(t) \end{bmatrix}^T$$

is a vector of real and known parameters which evolves piecewise continuously in the unit simplex Δ_{n_v}. Such models are interesting since they can be useful for absorbing locally the behaviour of more complex affine nonlinear systems [Rugh 2000].

Consider that for each $\mu \in \Delta_{n_v}$ the control u may only take values in a finite set which depends on the parameter μ. We define this set of finite values $\mathcal{V}_\mu$ by:

$$\mathcal{V}_\mu = \{v_i(\mu), i \in \mathcal{I}_N\}, v_i : \Delta_{n_v} \to \mathbb{R}^m, \forall i \in \mathcal{I}_N. \tag{4.20}$$

We consider that $\text{conv}\{\mathcal{V}_\mu\}$ is a non empty bounded set containing the origin in its interior for any $\mu \in \Delta_{n_v}$. The objective is to find a Parameter Dependent Relay (PDR) control $u = \kappa(x, \mu)$ which locally stabilizes the system (4.18):

$$\kappa(x, \mu) : \mathbb{R}^n \times \Delta_{n_v} \to \mathcal{V}_\mu. \tag{4.21}$$

Proposition 4.5 *(adapted from [Delpoux 2015]) Consider system* (4.18) *with the description* (4.19). *Consider $\mathcal{D} \subset \mathbb{R}^n$ a domain containing $x = 0$. Assume that there exists a control $u = K(x, \mu)$, with $K : \mathbb{R}^n \times \Delta_{n_v} \to \mathbb{R}^m$ such that $K(x, \mu) \in conv\{\mathcal{V}_\mu\}$, $\forall \mu \in \Delta_{n_v}, x \in \mathcal{D} \setminus \{0\}$. Let $V : \mathcal{D} \to \mathbb{R}$, be a continuously differentiable function such that*

$$0 < V(x), \ \forall \ x \in \mathcal{D} \setminus \{0\}, \tag{4.22}$$

$$\frac{\partial V}{\partial x}\left(A(\mu)x + B(\mu)K(x, \mu)\right) < -W(x), \forall \mu \in \Delta_{n_v}, x \in \mathcal{D} \setminus \{0\}, \tag{4.23}$$

where $W(x)$ is a continuous positive definite function on $\mathcal{D}$. Then system (4.18) *with the control:*

$$u = \kappa(x, \mu) \in \arg\min_{v \in \mathcal{V}_\mu} \frac{\partial V}{\partial x} B(\mu) v, \tag{4.24}$$

is locally asymptotically stable when solutions are understood in the sense of Filippov. Furthermore, for any level set $\mathcal{L}_V(c) = \{x \in \mathbb{R}^n : V(x) \leq c\}$ such that $\mathcal{L}_V(c) \subseteq \mathcal{D}$, the following relation is satisfied for any Filippov solution $x(t)$ originating from the initial condition x_0:

$$x_0 \in \mathcal{L}_V(c) \Rightarrow \lim_{t \to \infty} ||x(t)|| = 0, \tag{4.25}$$

i.e. $\mathcal{L}_V(c)$ is an inner estimation of the domain of attraction.

The previous result uses the existence of any stabilizer $K(x, \mu)$ (possibly continuous) in order to redesign a switching control $\kappa(x, \mu)$ which takes values only in the set $\mathcal{V}_\mu(x, \mu)$. Note that the switching control has at least the same guaranteed decay of the Lyapunov function as $K(x, \mu)$. The result provides a general theoretical framework for the design of switching controllers. In the following proposition we will show how this result can be used in a constructive manner.

Considering that for all $\mu \in \mathcal{V}_\mu$, $\text{conv}\{\mathcal{V}_\mu\}$ is non empty and contains the origin in its interior, remark that there exists a polytopic region:

$$\mathcal{Q} = \text{conv}\{q_1, q_2, \ldots, q_p\} = \{z \in \mathbb{R}^m : h_i z \leq 1, i \in \mathcal{I}_{N_h}\}, \tag{4.26}$$

such that

$$\mathcal{Q} \subset \text{conv}\{\mathcal{V}_\mu\}, \forall \mu \in \Delta_{n_v} \text{ and } 0 \in \text{Int}\{\mathcal{Q}\}. \tag{4.27}$$

Using the polytope $\mathcal{Q}$ one can adjust the design conditions to include an LMI based optimization of the domain of attraction.

Proposition 4.6 *[Delpoux 2015] Consider system* (4.18). *Assume that there exists $Q = Q^T \succ 0$, $Y_i \in \mathbb{R}^{m \times n}$, $i \in \mathcal{I}_N$ and a positive scalar δ such that:*

$$\mathcal{H}e\{(A_i + A_j)Q + B_i Y_j + B_j Y_i\} \prec -2\delta Q, \ i, j \in \mathcal{I}_{n_v}, \tag{4.28a}$$

$$\begin{bmatrix} 1 & h_i Y_j \\ * & Q \end{bmatrix} \succ 0, i \in \mathcal{I}_{N_h}, j \in \mathcal{I}_{n_v}, \tag{4.28b}$$

$$\begin{bmatrix} eI & I \\ * & Q \end{bmatrix} \succ 0. \tag{4.28c}$$

Let

$$u = \kappa(x,\mu) \in \arg\min_{v\in\mathcal{V}_\mu} x^T Q^{-1} B(\mu) v. \tag{4.29}$$

Then the equilibrium point $x = 0$ of the closed-loop system (4.18)-(4.29) is locally asymptotically stable. An estimation of the domain of attraction is provided by the ellipsoid $\mathcal{E}(Q^{-1},1)$ containing the ball

$$\mathcal{B}(0,\sqrt{\epsilon}) = \left\{y \in \mathbb{R}^n : \|y\|_2 < \sqrt{\epsilon}\right\}$$

with $\epsilon = \frac{1}{e}$, i.e.

$$\forall x(0) \in \mathcal{E}(Q^{-1},1), \lim_{t\to\infty} ||x(t)||_2 = 0.$$

Example 4.7 *In order to illustrate the results presented in this section, we propose to show simulations through a simple second order system so that the trajectories of the system can be plotted in a two dimensional phase portrait. We consider the system:*

$$\dot{x}(t) = A_0 x(t) + B_0(x_1(t))u(t), \tag{4.30}$$

with $x = [x_1 \quad x_2]^T$ in $\mathbb{R}^2$, $u \in \mathbb{R}^2$, $A_0 \in \mathbb{R}^{2\times 2}$ and $B_0 \in \mathbb{R}^{2\times 2}$ defined by

$$A_0 = \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix}, B_0(x_1(t)) = \begin{bmatrix} 1+0.5\sin(x_1(t)) & 0 \\ 0 & 1+0.5\sin(x_1(t)) \end{bmatrix}.$$

For each $x_1(t)$, the control $u(t)$ is constrained to switch among four different values in the set $\{R(x_1(t))\rho, \rho \in \Psi_2(v)\}$ where $\Psi_2(v) = \{u \in \mathbb{R}^2 : u_i \in \{-v, v\}, i = 1,2\}$, $v = 10$. The matrix $R(x_1(t))$ is the rotation matrix defined by

$$R(x_1(t)) = \begin{bmatrix} \cos(x_1(t)) & \sin(x_1(t)) \\ -\sin(x_1(t)) & \cos(x_1(t)) \end{bmatrix}. \tag{4.31}$$

Considering as bounded time-varying parameters $\sin(x_1)$, $\cos(x_1)$, the system (4.30) may be rewritten as an LPV system of the form (4.18) defined by:

$$\dot{x}(t) = Ax(t) + B(\mu(t))u(t) \tag{4.32}$$

with $A = A_0$ and $B(\mu(t)) = \sum_{i=1}^2 \mu_i(t) B_i = B_0(x_1(t))$, where

$$\mu_1(t) = \frac{1-\sin(x_1(t))}{2},\ \mu_2(t) = \frac{1+\sin(x_1(t))}{2}$$

and

$$B_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}.$$

The control u takes values in the finite set (4.20) defined by

$$\mathcal{V}_{\mu(t)} = \{v_i(\mu(t)), i \in 1,\dots,4\} = \{R(x_1(t))\rho, \rho \in \Psi_2(v)\}. \tag{4.33}$$

In order apply Proposition 4.6 we need to construct a polytopic region $\mathcal{Q}$ such that equation (4.26) is satisfied.

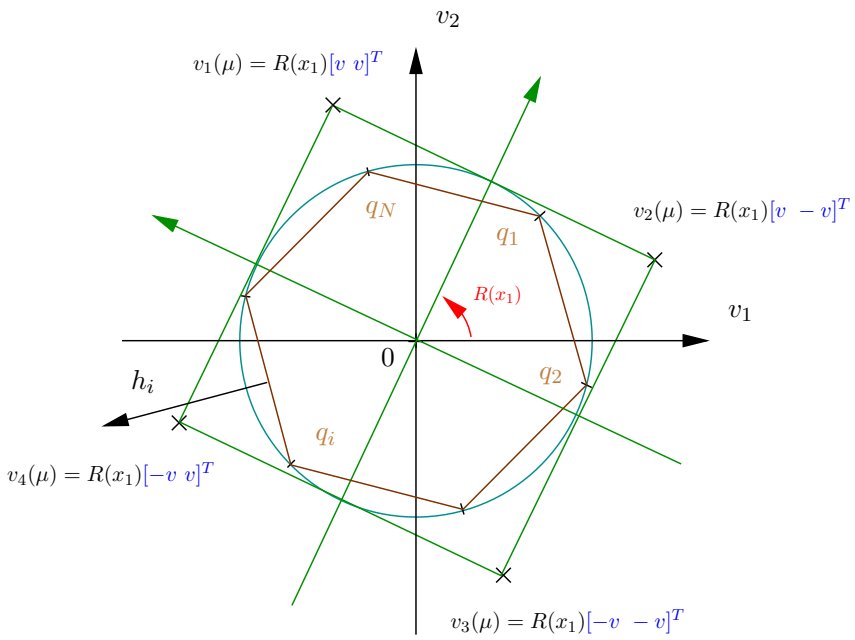


Figure 4.3: Representation of control sets $\mathcal{V}_\mu$ for the system in the simulation example.

Note that all squares defined by conv$\{\mathcal{V}_{\mu(t)}\}$ are centred at 0 and have the same size. Thus the disc of radius V centred at 0 belong to all conv$\{\mathcal{V}_{\mu(t)}\}$ (see Fig. 4.3). This disc can be approximated by the polytope $\mathcal{Q}$ represented in brown Fig. 4.3 for which the vertices q_i are given by

$$q_{i+1} = v \begin{bmatrix} \cos\left(\frac{2i\pi}{p}\right) \\ \\ \sin\left(\frac{2i\pi}{p}\right) \end{bmatrix}, i = 0, \ldots, p-1. \tag{4.34}$$

Each face of the polytope can be characterized by its normal:

$$h_{i+1} = \frac{q_i + q_{i+1}}{1 + \cos\left(\frac{2\pi}{p}\right)}, i = 0, \ldots, p-1. \tag{4.35}$$

For this example, to approximate the inscribed disc by a polytope $\mathcal{Q}$ we take $p = 15$. Choosing a decay rate $\delta = 4$ and applying Proposition 4.6, the LMI solver returns the matrices Q and $Y_i, i \in \mathcal{I}_2$ matrices:

$$Q = \begin{bmatrix} 43.17 & -18.86 \\ -18.86 & 9.77 \end{bmatrix}, Y_1 = \begin{bmatrix} -59.53 & 21.82 \\ 21.82 & -20.88 \end{bmatrix}, Y_2 = \begin{bmatrix} -21.70 & 7.66 \\ 7.66 & -8.17 \end{bmatrix}. \tag{4.36}$$

The Q matrix defines the parameter dependent relay control (4.29) and thus the switching regions. These regions are plotted Fig. 4.4 as function of the states x_1 and x_2. On this figure, r_1, r_2, r_3 and r_4 are the region for which the argument of the minimum is given for the control input

$$v_1\left(\mu(t)\right) = R\left(x_1(t)\right)\left[v \quad v\right]^T,$$

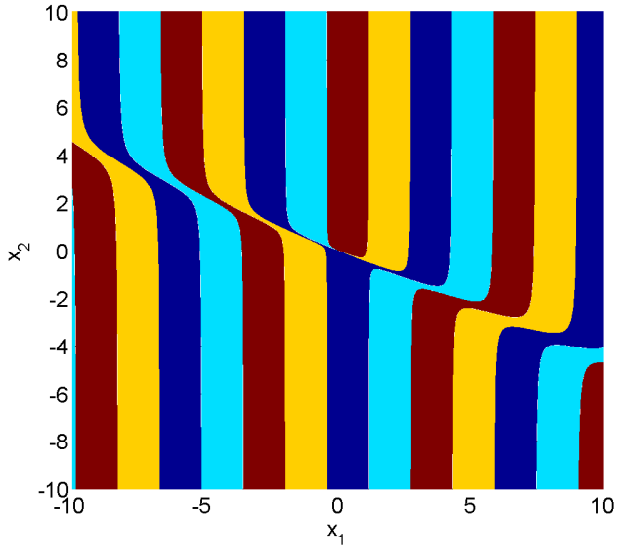


Figure 4.4: Representation of the switching regions, (the region r_1 in dark blue, r_2 in light blue, r_3 in yellow and r_4 in red).

$$v_2\left(\mu(t)\right) = R\left(x_1(t)\right)\begin{bmatrix} v & -v \end{bmatrix}^T,$$
$$v_3\left(\mu(t)\right) = R\left(x_1(t)\right)\begin{bmatrix} -v & v \end{bmatrix}^T$$

and

$$v_4\left(\mu(t)\right) = R\left(x_1(t)\right)\begin{bmatrix} -v & -v \end{bmatrix}^T,$$

respectively.

To illustrate the theoretical results, we compare the Continuous State Feedback (CSF) control law

$$K(x,\mu) = \sum_{i=1}^{n_v} \mu_i Y_i Q^{-1} x. \tag{4.37}$$

with the PDR control (4.29)*. In the continuous case, the control input applied to the system denoted by ρ in the description of the system is in $\mathbb{R}^2$ but it has elements saturated in the interval $[-v, v]$. The phase portrait of the states x_1 and x_2 for both cases are plotted Fig. 4.5. On theses figures, we have plotted in red the ellipsoid $\mathcal{E}(Q^{-1}, 1)$, characterizing the domain of attraction of the system. The brown lines represent the hyperplanes $h_i Y_j Q^{-1} = 1$.*

The first simulation is executed while taking initial conditions outside the attractive ellipsoid. On the figure, the initial condition is denoted by $x_{0,1}$. One observes that outside the attraction domain, the closed-loop system does not converge to the origin. The second simulation is realized with the initial condition $x_{0,2}$, near the domain of attraction, but outside. The figures show that in this case, the trajectories are converging to the origin. Finally, the initial condition $x_{0,3}$ is taken inside the domain of attraction. In this case the trajectories also converge to the origin. For this example the attractive ellipsoid contains the ball $\|x\|_2 < \epsilon$ with $\epsilon = 1.28$ and the initial condition $x_{0,3}$ with $\|x_{0,3}\|_2 = 4.14$. Note that $\|x_{0,1}\|_2 = 7.07$ and $\|x_{0,2}\|_2 = 4.24$, this gives an idea about the conservatism introduced in the estimation of the domain of attraction.

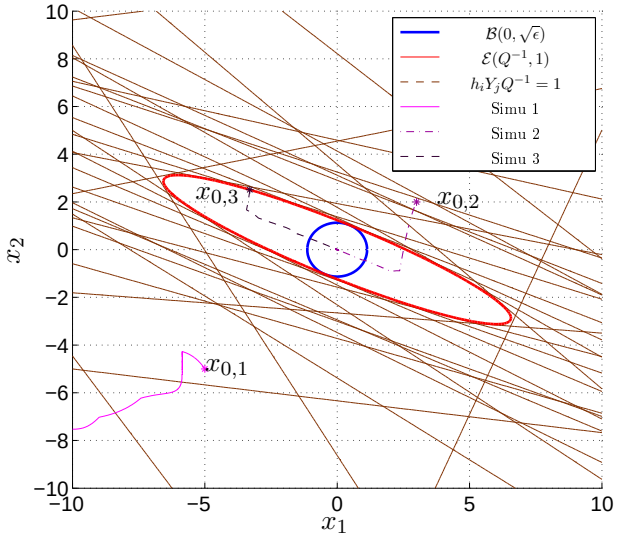


Figure 4.5: Characterization of the domain of attraction and simulation results.

The main advantage of the method is that it allows to optimize the design of nonlinear switching surfaces while providing a quantitative guarantees in terms of domain of attraction and performances. In Chapter 6, we will see how the proposed method can be applied to a practical example of a stepper motor.

Chapter 5

Switched affine systems

Switched systems represent a very popular area in hybrid dynamical systems. Generic results on this topic may be found in the book [Liberzon 2003a] and the survey papers [Shorten 2007, Lin 2009, Bourdais 2007]. Among the different problems encountered in the context of switched systems, in this chapter, we will focus on the problem of designing switching controllers. This problem is very challenging for the case of switched affine systems where, generally, the different subsystems do not share a common equilibrium point. Different stabilization solutions exist in the literature based on the existence of stable convex combinations [Bolzern 2004, Deaecto 2010], on optimal control methods [Seatzu 2006, Hauroigne 2011], or on the use of sliding modes [Sira-Ramirez 1994]. A characterization of the set of attainable equilibrium points using quadratic Lyapunov functions and conic switching laws has been provided in [Bolzern 2004, Deaecto 2010].

When dealing with the stabilization problem, the existing articles treat the global stabilization case. However, one may encounter switched affine systems that may be stabilized only locally. Consider system characterized by two vector fields,

$$f_1(x) = 3x + 1, f_2(x) = 2x - 1.$$

While global stabilization is not possible, local stabilization at the origin is possible for initial conditions in the ball $|x| < 1/3$, by choosing $f_1(x)$ for $x \leq 0$, and $f_2(x)$, whenever $x \geq 0$. Such systems cannot be considered using the existing methodology.

In what follows, we propose constructive methods for the derivation of state dependent switching laws that ensure local stabilization of switched affine systems at the origin[21]. The main idea is to reformulate the stabilization of switched affine systems as a classical stabilization problem for nonlinear systems affine in the input. The method derives state dependent switching laws by embedding, locally, the behaviour of a continuous controller. The classical restriction concerning the existence of a Hurwitz convex combination may be easily avoided. With respect to the existing results, the proposed methodology can be interpreted as an approach that uses convex combinations that depend on the system state.

[21] The results presented in this chapter have been developed in collaboration with Ass. Prof. Emmanuel BERNUAU.

5.1 System description

Let a set of couples $(A_i, b_i), i \in \mathcal{I}_N = \{1, 2, \ldots, N\}$ where $A_i \in \mathbb{R}^{n\times n}$, $b_i \in \mathbb{R}^n$, for positive integers n, N. Consider the system

$$\dot{x} = X(x) = A_{\kappa(x)}x + b_{\kappa(x)} \tag{5.1}$$

where $\kappa : \mathbb{R}^n \to \mathcal{I}_N$ represents the switching law. We assume that there exists

$$\delta^* \in \Delta_N = \left\{ \delta = \left[\delta_1, \ldots, \delta_N\right]^T \in \mathbb{R}^N : \sum_{i=1}^{N} \delta_i = 1,\ \delta_i \geq 0, i \in \mathcal{I}_N \right\}$$

such that $\sum_{i\in\mathcal{I}_N} \delta_i^* b_i = 0$. This is a necessary condition for the existence of an equilibrium at the origin when solutions are understood in the sense of Filippov (see [Filippov 1988]). The goal is to provide methods for the design of a local stabilizing switching law κ.

The main idea of the work is to re-formulate the switched affine system (5.1) in a classical nonlinear affine form

$$\dot{x} = f(x) + G(x)u$$

interconnected with a discontinuous control law $u = k(x)$ that is constrained to take values in a finite set of vectors

$$\mathcal{V} = \{v_1, v_2, \ldots, v_N\} \subset \mathbb{R}^{N-1}.$$

We propose such a system re-formulation in what follows. Furthermore, we show that the obtained nonlinear affine system has nice properties: if there exists a classical continuous feedback $k^c(x)$ such that the system

$$\dot{x} = f(x) + G(x)k^c(x)$$

is (locally or globally) stable, then there exists also a local discontinuous stabilizer, $k(x)$, taking values in $\mathcal{V}$, and in extenso, a switching law κ for the switched affine system (5.1)

5.2 Main results

In the following proposition, system (5.1) is rewritten in a classical nonlinear affine form interconnected with a discontinuous control law.

Proposition 5.1 *[Hetel 2015a] Consider system* (5.1)*,* $\delta^* \in \Delta_N$ *such that* $\sum_{j=1}^{N} \delta_j^* b_j = 0$ *and the notations* $m = N - 1$*,*

$$M = \left[I_{m\times m} \quad 0_{m\times 1}\right] \in \mathbb{R}^{m\times N}.$$

For $\psi_i, i \in \mathcal{I}_N$*, the vertex of* Δ_N*, define the set*

$$\mathcal{V} = \left\{v_i := M\left(\psi_i - \delta^*\right),\ i \in \mathcal{I}_N\right\}.$$

The switched affine system (5.1) *is equivalent to the interconnection between the nonlinear affine system*

$$\dot{x} = H(x, u) = f(x) + G(x)u,\ u \in \mathbb{R}^m, \tag{5.2}$$

and the control law

$$u = k(x),\ k : \mathbb{R}^n \to \mathcal{V} \tag{5.3}$$

with

$$f(x) = A(\delta^*)x = \sum_{j=1}^{N} \delta_j^* A_j x,$$

$$G(x) = \begin{bmatrix} g_1(x) & g_2(x) & \dots & g_m(x) \end{bmatrix},$$

$$g_j(x) = (A_j - A_N)x + (b_j - b_N), j \in \mathcal{I}_m$$

and

$$k(x) = v_{\kappa(x)}. \tag{5.4}$$

We may remark that designing a switching law σ leads to finding a discontinuous control law $k : \mathbb{R}^n \to \mathcal{V}$, such that system (5.2) with the control $u = k(x)$, is locally asymptotically stable. As follows we show how the existence of a continuous control $u = k^c(x)$ for system (5.2) can be used in order to derive a switching law $\kappa(x)$ for system (5.1) (or equivalently a discontinuous control (5.4) for the interconnection (5.2), (5.3)).

Theorem 5.2 *[Hetel 2015a] Consider the switched affine system* (5.1) *and the affine model* (5.2)*. Assume that:*

1. *there exists* $\delta^* = [\delta_1^* \ \delta_2^* \ \dots, \delta_N^*]^T \in \Delta_N$ *with* $\delta_i^* > 0, i \in \mathcal{I}_N$ *such that* $\sum_{i=1}^{N} \delta_i^* b_i = 0$*;*

2. *system* (5.2) *is continuously locally stabilizable at the origin by* $u = k^c(x)$*, with* $k^c(0) = 0$*.*

Then there exists a $\mathcal{C}^\infty$ *Lyapunov function* $V(x)$ *defined on some ball* $\mathcal{B}(0, \eta)$*,* $\eta > 0$*,* $V(0) = 0, V(x) > 0, \forall x \neq 0$*, and a measurable switching law*

$$\kappa(x) \in \arg\min_{i \in \mathcal{I}_N} \langle \nabla V(x), A_i x + b_i \rangle \tag{5.5}$$

such that system (5.1), (5.5) *(or equivalently* (5.2), (5.3) *with* $k(x)$ *as in* (5.4), (5.5)*) is locally asymptotically stable at the origin.*

The proof of Theorem 5.2 is constructive in the sense that if the affine nonlinear system (5.2) is stabilized by a controller k^c and admits a (local) Lyapunov function V, then the original switched system (5.1) can be (locally) stabilized by a switching law of the form (5.5) obtained based on the same Lyapunov function V. With respect to the classical convex combination approach [Bolzern 2004], [Deaecto 2010] the method that we propose can be interpreted as an extension where we look for a locally stable state dependent convex combination, with barycentric coordinates defined by

$$\delta_i(x) = \delta_i^* + k_i^c(x), i \in \mathcal{I}_{N-1},$$

$$\delta_N(x) = 1 - \sum_{i=1}^{N-1} \delta_i(x),$$

instead of a constant convex combination, with constant barycentric coordinates δ^* (as in [Bolzern 2004], [Deaecto 2010]).

Example 5.3 *(numerical illustration). Consider a system* (5.1) *described by the following ma-*

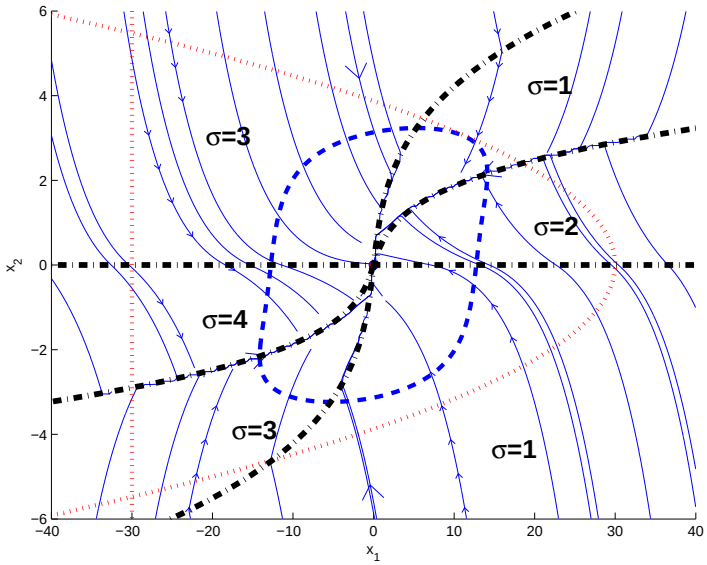


Figure 5.1: Illustration of the phase plane for the closed-loop switched affine system in Example 5.3: dotted line - border of the set where $k^c(x) \in \text{conv}\{\mathcal{V}\}$; dashed line - a level set of $V(x)$; dash-dotted lines - switching surfaces $\langle \nabla V(x), (A_i - A_j)x + b_i - b_j) \rangle = 0,\ i, j \in \mathcal{I}_N$. $\kappa(x) = \sigma$ - value of switching function in different regions of the state space.

trices:

$$A_1 = A_3 = \begin{bmatrix} 0 & 2 \\ 2 & -66 \end{bmatrix},\ b_1 = b_2 = \begin{bmatrix} -360 \\ 0 \end{bmatrix}, \tag{5.6}$$

$$A_2 = A_4 = \begin{bmatrix} 0 & 2 \\ 2 & 54 \end{bmatrix},\ b_3 = b_4 = \begin{bmatrix} 360 \\ 0 \end{bmatrix}, \tag{5.7}$$

For $\delta_i^* = 1/4,\ i \in \mathcal{I}_4$, *we have* $\sum_{i=1}^4 \delta_i^* b_i = 0$. *The obtained system* (5.2) *is described by*

$$A(\delta^*) = \begin{bmatrix} 0 & 2 \\ 2 & -6 \end{bmatrix}, \tag{5.8}$$

$$g_1(x) = g_2(x) + g_3(x),$$
$$g_2(x) = \begin{bmatrix} -720 \\ 0 \end{bmatrix},\ g_3(x) = \begin{bmatrix} 0 \\ -120x_2 \end{bmatrix}.$$

The matrix $A(\delta^*)$ *is not Hurwitz. Let*

$$k^c(x) = 1/120 \begin{bmatrix} 0 & x_1 & 2x_2^2 \end{bmatrix}^T.$$

The obtained closed-loop system, $H(x, k^c(x))$, *has the form*

$$\dot{x}_1 = -6x_1 + 2x_2 \tag{5.9}$$
$$\dot{x}_2 = 2x_1 - 6x_2 - 2x_2^3. \tag{5.10}$$

The stability of the closed-loop system can be shown using Krasovskii's method [Slotine 1991]. The method consists in using

$$V(x) = H^T(x, k^c(x))H(x, k^c(x)) = (-6x_1 + 2x_2)^2 + (2x_1 - 6x_2 - 2x_2^3)^2$$

as a candidates Lyapunov function and checking whether the Jacobian matrix

$$J(x) = \frac{\partial H(x, k^c(x))}{\partial x}$$

satisfies the relation

$$J^T(x) + J(x) \prec 0$$

in some neighbourhood of the origin. For system (5.9)

$$J^T(x) + J(x) = \begin{bmatrix} -12 & 4 \\ 4 & -12 - 12x^2 \end{bmatrix} \prec 0 \tag{5.11}$$

for all $x \in \mathbb{R}^n$. Then the closed-loop system $\dot{x} = H(x, k^c(x))$ is asymptotically stable. Since the conditions of Theorem 5.2 are satisfied, the function $V(x)$ can be used for constructing a switching law (5.5) *that ensures the local stabilization of the switched affine system. An illustration of the phase plane for the closed-loop switched affine system is provided in Figure 5.1.*

The main advantage of the proposed method is the fact that the difficult problem of existence of a stabilizing switching law for the switched affine system is reduced to the classical stabilization problem of a nonlinear affine system (5.2), on which a very large variety of control design methods are possible.

Example 5.4 *(stabilization based on the linearized model). As follows, simple stabilization conditions are given using the local linearized model of system* (5.2)*. Consider the notation*

$$B = \begin{bmatrix} b_1 - b_N & b_2 - b_N & \dots & b_{N-1} - b_N \end{bmatrix}. \tag{5.12}$$

System (5.2) *can be re-expressed as*

$$\dot{x} = A(\delta^*)x + Bu + w(x, u), \tag{5.13}$$

$$w(x, u) = D(u)x \tag{5.14}$$

where $w(x, u)$ is obtained from $w(x, u) = (G(x) - B)u$ and

$$D(u) = \sum_{i=1}^{N-1} (A_i - A_N)\, u_i. \tag{5.15}$$

Assume that the pair $(A(\delta^*), B)$ is stabilizable for some $\delta^* \in \Delta_N$ with $\delta_i^* > 0, i \in \mathcal{I}_N$. Then there exists a gain matrix K and functions $V(x) = x^T Px$, $W(x) = x^T Qx$, $P, Q \succ 0$, such that

$$\langle \nabla V(x), (A(\delta^*) + BK)\, x \rangle < -W(x). \tag{5.16}$$

The derivative of the function V along (5.13) *satisfies*

$$\langle \nabla V(x), (A(\delta^*) + BK)\, x + w(x, Kx) \rangle < -W(x) + 2x^T P w(x, Kx). \tag{5.17}$$

Let us remark that for any $\rho > 0$ there exists $r > 0$ such that $\|w(x, Kx)\|_2 < \rho\|x\|_2$ for any $\|x\|_2 < r$. Then

$$x^T P w(x, Kx) < \rho\|P\|_2\|x\|_2,$$

for all $\|x\|_2 < r$, which leads to

$$\begin{aligned}&\langle \nabla V(x), (A(\delta^*) + BK)\, x + w(x, Kx)\rangle \\ &\quad < -\left(eig_{min}(Q) - 2\rho\|P\|_2\right)\|x\|_2 \end{aligned} \tag{5.18}$$

for all $\|x\|_2 < r$, that is the state feedback $u = Kx$ ensures local stabilization of system (5.13) for ρ chosen such that $\rho < 1/2eig_{min}(Q)/\|P\|_2$, where $eig_{min}(Q)$ denotes the minimum eigenvalue of Q. Applying Theorem 5.2, one can conclude that the switched affine system can be locally stabilized. The obtained switching law has the form

$$\kappa(x) \in \arg\min_{i \in \mathcal{I}_N} x^T P(A_i x + b_i). \tag{5.19}$$

However, differently from [Bolzern 2004], [Deaecto 2010], $A(\delta^*)$ is not required to be a Hurwitz matrix. For local stabilization we only need the pair $(A(\delta^*), B)$ to be stabilizable.

The existence of a continuous stabilizing feedback k^c for system (5.2) is not very restrictive. In fact, for nonlinear affine systems such as (5.2), when the system can be stabilizable at the origin (in the sense of Filippov solutions) by means of a locally bounded, measurable feedback $u = k^b(x)$ such that $\lim_{\epsilon \to 0} \operatorname{ess\,sup}_{\|x\| < \epsilon} \|k^b(x)\| = 0$, there exists also a continuous stabilizer $u = k^c(x)$ for the same system (see [Bacciotti 2005], page 61). Furthermore, the non-existence of a stabilizing feedback for system (5.2) can be expressed as a certain topological obstruction. For the necessity of existence of continuous stabilizer k^c we point to the classical Brockett test. For the more general case of locally bounded, measurable stabilizers k, necessary condition may be found in [Ryan 1994]. Since for each subset $\mathcal{U} \subset \mathbb{R}^m$ and each $x \in \mathbb{R}^n$, system (5.2) satisfies

$$H\left(x, \operatorname{conv}\left(\mathcal{U}\right)\right) = \operatorname{conv}\left(H\left(x, \mathcal{U}\right)\right), \tag{5.20}$$

a necessary condition for the existence of a locally bounded, measurable feedback $u = k(x)$ which stabilizes the system (in the sense of Filippov) is that for each $\epsilon > 0$ there exists $\lambda > 0$ such that

$$\forall\, y \in \mathcal{B}\left(0, \lambda\right),\ \exists\, x \in \mathcal{B}\left(0, \epsilon\right), \exists\, u \in \mathbb{R}^m \text{ such that } y = H(x, u),$$

where $\mathcal{B}\left(x, c\right)$ denotes the n dimensional open ball in $\mathbb{R}^n$ centred on x with radius $c > 0$. This may be useful to determine the existence of stabilizing switching laws for the original switched affine system. This implies, for example, that switched affine systems for which $A(\delta^*) = 0$ whenever $\sum_{i=1}^N \delta_N^* b_i = 0$ and $rank\left(G(x)\right) = m < n$ cannot be stabilized by a static switching law $\kappa(x)$ if solutions are understood in the sense of Filippov.

Example 5.5 *stabilization obstruction for switched affine system. Consider a system affine*

system with

$$A_1 = -A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.5 & -0.5 & 0 \end{bmatrix},\ b_1 = -b_4 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}, \tag{5.21}$$

$$A_2 = -A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix},\ b_2 = -b_3 = \begin{bmatrix} -0.5 \\ 0.5 \\ 0 \end{bmatrix}. \tag{5.22}$$

For any $\delta^* \in \Delta_N$ such that $\sum_{i=1}^4 b_i \delta_i^* = 0$ we have $A(\delta^*) = 0$. The model (5.2) is characterized by $g_1(x) = g_2(x) + g_3(x)$, $g_2(x) = \begin{bmatrix} 0 & 1 & x_1 \end{bmatrix}^T, g_3(x) = \begin{bmatrix} 1 & 0 & -x_2 \end{bmatrix}^T$. This leads to

$$\dot{x}_1 = u_1 + u_3 \tag{5.23}$$

$$\dot{x}_2 = u_1 + u_2 \tag{5.24}$$

$$\dot{x}_3 = (u_1 + u_2)x_1 - (u_1 + u_3)x_2 \tag{5.25}$$

where the reader may recognize a classical non-holonomic integrator (see [Bacciotti 2005], p. 55) for which no point x of the form $x = (0\ 0\ \epsilon)^T$, $\epsilon \neq 0$, belongs to the image of H. We conclude that there is no switching law $\kappa(x)$ which makes the origin of the switched affine system locally asymptotically stable (in the sense of Filippov solutions).

5.3 Numerical issues

In practical applications it is of interest to provide numerical tools for the design of switching laws. For the system under study, we may be interested in optimizing the domain of attraction, the speed of convergence, etc. Here we present simple LMI based criteria for the design of a stabilizing switching law which optimizes an ellipsoidal estimation of the domain of attraction for given decay rate.

Consider the set of allowable control values $\mathcal{V}$. The set $\text{conv}\{\mathcal{V}\}$ is a convex polytope. It can be described by a finite number N_r of vectors $r_i \in \mathbb{R}^m, i \in \mathcal{I}_{N_r}$, such that

$$\text{conv}\{\mathcal{V}\} = \left\{u \in \mathbb{R}^N : r_i^T u \leq 1, i \in \mathcal{I}_{N_r}\right\}. \tag{5.26}$$

Proposition 5.6 *[Hetel 2015a] Consider the switched system (5.1), the equivalent representation (5.13) with controls u restricted to the set $\mathcal{V}$ and the polytope (5.26). Assume that $\delta_i^* > 0, i \in \mathcal{I}_N$. Given tuning parameters $\chi, c > 0$ assume that there exists $Q \succ 0, \theta > 0$ such that*

$$\left(A(\delta^*) + D(v_i)\right) Q + Q \left(A(\delta^*) + D(v_i)\right)^T - \theta B B^T \prec -2\chi Q, \tag{5.27}$$

$i \in \mathcal{I}_N$,

$$\begin{bmatrix} cI & I \\ I & Q \end{bmatrix} \succ 0, \tag{5.28}$$

and

$$\begin{bmatrix} 1 & \frac{\theta}{2} r_j^T B^T \\ \frac{\theta}{2} B r_j & Q \end{bmatrix} \succ 0,\ j \in \mathcal{I}_{N_r}. \tag{5.29}$$

Then the switched system (5.1) *with the switching law*

$$\kappa(x) \in \arg\min_{i\in\mathcal{I}_N} x^T Q^{-1} \left(A_i x + b_i\right) \tag{5.30}$$

is locally asymptotically stable at the origin. Furthermore, the domain of attraction includes the ball

$$\mathcal{B}(0, 1/\sqrt{c}) = \left\{y \in \mathbb{R}^n : \|y\|_2 < 1/\sqrt{c}\right\}$$

and there exists a positive scalar C *such that* $\|x(t)\|_2^2 \leq Ce^{-2\chi t}\|x(0)\|_2^2$ *for any* $x(0) \in \mathcal{B}(0, 1/\sqrt{c})$.

The feasibility of the LMIs (5.27),(5.28),(5.29) guarantees that any system solution originating in the ball $\mathcal{B}(0, 1/\sqrt{c})$ converges to the origin with a decay rate χ. The size of the domain of attraction can be optimized by considering the optimization problem

$$\inf c \text{ under the constraints } (5.27),(5.28),(5.29), \tag{5.31}$$

which is a standard optimization problem. The LMI criteria (5.27),(5.28),(5.29) represent sufficient condition for local stabilization in a domain that includes a prescribed ball $\mathcal{B}(0, 1/\sqrt{c})$. The set of LMIs implies that the local linearised model at $x = 0$ is stabilizable. The method is based on robust control arguments, in the sense that the term $w(x, u)$ in (5.13) is treated as a perturbation. This aspect may induce some conservatism in the design. Additional conservatism in the estimation of the domain of attraction may also stem from the choice of quadratic candidate Lyapunov functions. In terms of computational complexity, the approach requires solving $N + N_r + 3$ LMIs involving $0.5(n^2 + n) + 2$ variables.

Example 5.7 *LMI stabilization. Consider a switched affine system described by the matrices:*

$$A_1 = \begin{bmatrix} -3 & 0 \\ 0 & 12 \end{bmatrix}, b_1 = \begin{bmatrix} 0 \\ 7 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ -14 \end{bmatrix}.$$

For $\delta^* = \begin{bmatrix} 2/3 & 1/3 \end{bmatrix}^T$, $\sum_{i=1}^2 \delta_i^* b_i = 0$. *However,* $A(\delta^*)$ *is not Hurwitz therefore the example cannot be treated using the global stabilization approaches in [Bolzern 2004], [Deaecto 2010]. Using the formulation* (5.13) *and solving the optimization problem* (5.31) *for* $\chi = 0.25$ *leads to a switching law of the form* (5.30) *with*

$$Q = \begin{bmatrix} 2.87 & -3.62 \\ -3.62 & 17.45 \end{bmatrix} \times 10^{-2} \tag{5.32}$$

which guarantees local stabilization $\forall x(0) \in \mathcal{B}(0, 0.14)$.

Chapter 6

Applications

In this chapter we will present two experimental applications of the methodology proposed in Chapters 4 and 5. First, results concerning the control of a Permanent Magnet Synchronous Motor will be presented, based on the use of LPV models. Next, a methodology of control design a multi-level power converter will presented using similar nonlinear models as in Chapter 5.

6.1 Control of a Permanent Magnet Synchronous Motor

As follows we illustrate the proposed switching control methodology for the case of a Permanent Magnet Synchronous Motor (PMSM)[22]. Indeed, PMSM are usually controlled by relays and thus only a finite set of control values is available. However in most of classical control design methods the use of averaging and of PWM ignores the relay nature of the actuator [Bodson 1993], [Sira-Ramírez 2000]. Here we propose a direct relay control which may use the advantages of the switching actuator. The LPV framework encompasses the PMSM model. The obtained switching surfaces depend in a nonlinear manner on the motor speed.

The equations (6.1) give the standard PMSM model in the phase (or winding) variables [Marino 1995]:

$$\begin{cases} L\dfrac{di_\alpha}{dt} &= v_\alpha - Ri_\alpha + K\Omega\sin(n_p\theta), \\ L\dfrac{di_\beta}{dt} &= v_\beta - Ri_\beta - K\Omega\cos(n_p\theta), \\ J\dfrac{d\Omega}{dt} &= K\left(i_\beta\cos(n_p\theta) - i_\alpha\sin(n_p\theta)\right) - f_v\Omega - \tau, \end{cases} \tag{6.1}$$

where v_α and v_β are the voltages applied to the two phases of the PMSM, i_α and i_β are the two phase currents, L is the inductance of a phase winding, R is the resistance of a phase winding, K is the back-EMF constant (and also the torque constant), n_p is the number of pole pairs (or rotor teeth), J is the moment of inertia of the rotor (including the load), f_v is the coefficient of viscous friction and τ represents the load torque. The variable θ is the angular position of the rotor, $\Omega = d\theta/dt$ is the angular velocity of the rotor. While for particular applications the variable θ can be included in the state vector, here we consider only the speed control, justifying

[22] The results presented in this section have been developed in collaboration with Romain DELPOUX (ATER at LAGIS) and Ass. Prof. Alexandre KRUSZEWSKI.

the fact that θ is not in the state vector. We are interested in the stability of the velocity to a constant value. In this case the position θ is time varying. For this reason θ is not included in the state vector.

The non-linear state space representation of the system of equations (6.1) is given by:

$$\dot{x}_{\alpha\beta}(t) = f(x_{\alpha\beta}, t) + Bv_{\alpha\beta}(t) + D\varpi(t), \tag{6.2}$$

where $x_{\alpha\beta}^T = \begin{bmatrix} i_\alpha & i_\beta & \Omega \end{bmatrix}$, $v_{\alpha\beta}^T = \begin{bmatrix} v_\alpha & v_\beta \end{bmatrix}$ and $\varpi = \tau$. The function $f(x_{\alpha\beta}, t)$ is defined by:

$$f(x_{\alpha\beta}, t) = \begin{bmatrix} -\frac{R}{L} i_\alpha(t) + \frac{K}{L}\Omega(t)\sin(n_p\theta(t)) \\ -\frac{R}{L} i_\beta(t) - \frac{K}{L}\Omega(t)\cos(n_p\theta(t)) \\ \frac{K}{J}\left(i_\beta(t)\cos(n_p\theta(t)) - i_\alpha(t)\sin(n_p\theta(t))\right) - \frac{f_v}{J}\Omega(t) \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{L} \\ 0 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{J} \end{bmatrix}.$$

Considering that each motor phase is actuated via commutation, the control vector $v_{\alpha\beta}$ belongs to the set $\Psi_2(V)$, where $\Psi_2(V) = \{u \in \mathbb{R}^2 : u_i \in \{-V, V\}, i = 1, 2\}$ and where V represents the maximal voltage. In the phases frame the signals i_α and i_β vary at n_p times the frequency of rotation. This high frequency problem is alleviated by the use of the direct quadrature $(d-q)$ transformation, also known as the Park transformation [Park 1929]. This transformation changes the frame of reference from the fixed phase axes to axes moving with the rotor. Equation (6.3) gives the transformation performed to obtain the rotating frame:

$$R(\theta(t)) = \begin{bmatrix} \cos(n_p\theta(t)) & \sin(n_p\theta(t)) \\ -\sin(n_p\theta(t)) & \cos(n_p\theta(t)) \end{bmatrix},$$

$$\begin{bmatrix} i_d(t) \\ i_q(t) \end{bmatrix} = R(\theta(t)) \begin{bmatrix} i_\alpha(t) \\ i_\beta(t) \end{bmatrix} \text{ and } \begin{bmatrix} v_d(t) \\ v_q(t) \end{bmatrix} = R(\theta(t)) \begin{bmatrix} v_\alpha(t) \\ v_\beta(t) \end{bmatrix}. \tag{6.3}$$

The state space representation is then given by:

$$\dot{x}_{dq}(t) = A_{dq}(\Omega(t))x_{dq}(t) + Bv_{dq}(t) + D\varpi(t), \tag{6.4}$$

where $x_{dq}^T = \begin{bmatrix} i_d & i_q & \Omega \end{bmatrix}$, $v_{dq}^T = \begin{bmatrix} v_d & v_q \end{bmatrix}$, and,

$$A_{dq}(\Omega(t)) = \begin{bmatrix} -\frac{R}{L} & n_p\Omega(t) & 0 \\ -n_p\Omega(t) & -\frac{R}{L} & -\frac{K}{L} \\ 0 & \frac{K}{J} & -\frac{f_v}{J} \end{bmatrix}.$$

The matrices B and D remain unchanged. Consider that $\Omega(t)$ ranges between known extremal values $\Omega(t) \in [\underline{\Omega}, \overline{\Omega}]$. In this frame the PMSM can be described using an LPV state space representation. The state space representation of the system depends linearly on a vector of

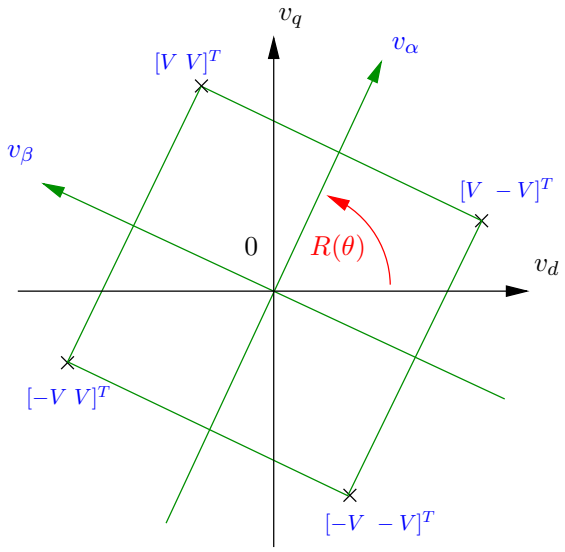


Figure 6.1: Finite set of control in the fixed frame and in the rotating frame.

time-varying parameters: $\Omega(t)$. The model may be represented as follows:

$$\begin{cases} \dot{x}_{dq}(t) & = & A(\alpha(t))x_{dq}(t) + Bv_{dq}(t) + D\varpi(t), \\ A(\alpha(t)) & = & \sum_{i=1}^{N_A} \alpha_i(t)A_i,\ \forall i,\ \alpha_i(t) \geq 0, \sum_i \alpha_i(t) = 1, \end{cases} \tag{6.5}$$

where $N_A = 2$, with $A_1 = A_{dq}(\underline{\Omega}), A_2 = A_{dq}(\overline{\Omega})$. The controls $v_{dq}(t)$ are defined for all $\theta \in [0, 2\pi]$ by:

$$v_{dq}(t) = \kappa(x_{dq}(t), \theta(t)),\ \kappa : \mathbb{R}^n \times [0, 2\pi] \to \mathbb{R}^m. \tag{6.6}$$

Note that the control $v_{dq}(t)$ is a PDR control which takes values in a finite set of vectors depending on θ: $\{u \in \mathbb{R}^2 : \exists v \in \Psi_2(V), u = R(\theta(t))v\}$. The input vector in the different frames is represented in Fig. 6.1. For a given V, the objective is to determine the switching surfaces in the state space, which ensure the closed loop stability of the system (6.5) with the control law (6.6).

The method proposed in Proposition 4.6 has been applied to a stepper motor benchmark at École Centrale de Lille (see Fig. 6.2). The parameters of the motor with coils in series have been identified using the offline procedure described in [Delpoux 2014], leading to $L = 9mH$, $R = 3.01\Omega$, $K = 0.27N.m.A^{-1}$ and $J = 3.18.10^{-4}kg.m^2$. The number of pole pairs is $n_p = 50$. The input voltages v_a and v_b of each coil are delivered by two D/A outputs of the dSpace card and amplified by two linear power amplifiers (this means that the controls are directly applied to the coils without a PWM implementation). The currents i_a and i_b are measured using Hall effect sensors with a precision of 1% of the nominal current $I_n = 3A$. The power supply provides a maximum voltage $v_{max} = 20V$ and $i_{max} = 3A$. The sampling period for this experiment is constant and equals to $10^{-4}s$ for the control.

We designed a control law where we consider that only four control inputs are available. The control design is considered with the assumption that there is no external torque (*i.e.* $\tau = 0$). An integral action is implemented with ζ the ouput of the integrator ($\zeta(0) = 0$)) to ensure tracking

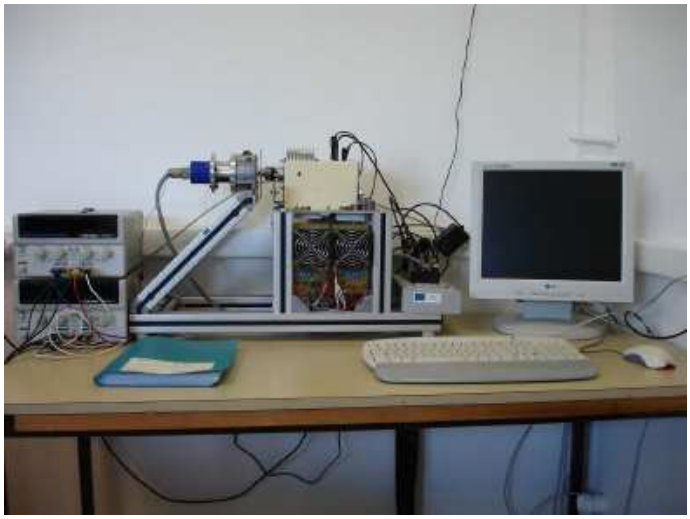


Figure 6.2: PMSM test-bench located at Centrale Lille.

performance with respect to a reference Ω_{ref}. The integral action given by:

$$\dot{\zeta} = \Omega - \Omega_{ref} = Cx - \Omega_{ref}, C = [0 \quad 0 \quad 1], \tag{6.7}$$

where ζ is the output of the integrator ($\zeta(0) = 0$). The combination of the state space representation (6.4) and the integral action without torque can be re-written as:

$$\underbrace{\begin{bmatrix} \dot{x}_{dq} \\ \dot{\zeta} \end{bmatrix}}_{\dot{z}} = \underbrace{\begin{bmatrix} A_{dq}(\Omega) & 0 \\ C & 0 \end{bmatrix}}_{\overline{A}(\Omega)} \underbrace{\begin{bmatrix} x_{dq} \\ \zeta \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\overline{B}} u - \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{\overline{I}} \Omega_{ref} \tag{6.8}$$

where u is constrained to switch among four different values in the set $\{R(\theta)\rho, \rho \in \Psi_2(V)\}$. The matrix $R(\theta)$ is defined by equation (6.3). Two different control strategies are proposed to show the experimental behavior of the PDR control applied to PMSM. Firstly, we are interested in the motor stabilization starting from non-zero initial conditions, next a velocity tracking strategy is proposed.

The stabilization is realized on the PMSM starting from different initial conditions to the origin. Here the PDR control is proposed based on Proposition 4.6 applied to model (6.8) with $\Omega_{ref} = 0$. It leads to a control law of the form

$$v_{\alpha\beta} = \arg \min_{\rho \in \Psi_2(V)} z^T Q^{-1} R(-\theta)\rho$$

where

$$Q = \begin{bmatrix} 29.6 & -4.8 & 9.4 & -0.012 \\ -4.8 & 26.6 & -15.9 & 0.038 \\ 9.4 & -16.0 & 208.4 & -2.8 \\ -0.012 & 0.038 & -2.8 & 0.069 \end{bmatrix}.$$

To compare the experimental behavior of the PDR control with the classical Continuous State

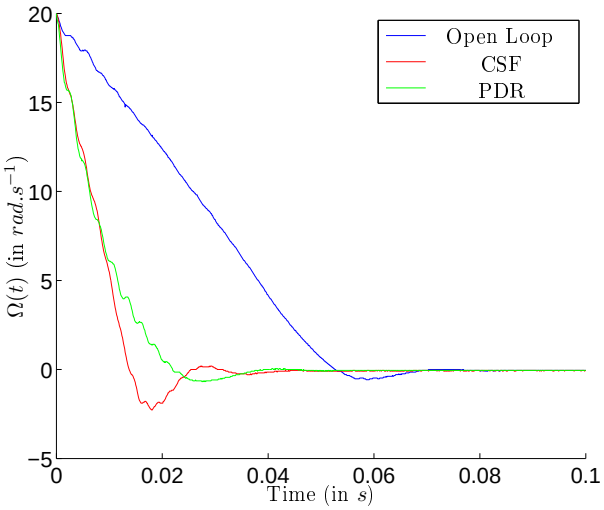


Figure 6.3: Comparison between the CSF and PDR control of the motor velocity stabilization.

Feedback (CSF) control,

$$v_{\alpha\beta} = R(-\theta)Y(\Omega)Q^{-1}z,$$

(obtained from (4.37)) starting from non null initial velocity, we have plotted in Fig. 6.3 the velocity evolution for three different cases (Open Loop, CSF and PDR). This control law is applied to the system by using linear amplifiers, without any PWM module. Knowing that the PMSM is a stable system, it is important to show that the stabilization performance are better than the open loop performance. For this reason, the blue curve represents the open-loop stabilization. The red line represents the CSF while the green one represents the PDR. The figure shows that the closed loop performances are better than the open loop performances (better settling-times and transient responses). The closed loop strategies show similar settling time given that the PDR uses only 4 inputs control values.

We compare the behaviour of the CSF and PDR for the tracking of a slowly varying velocity, although the proposed theoretical developments do not cover this case. The velocity profile is chosen according to industrial test trajectories [Hamida 2013]. The robustness of the proposed approach is also tested by applying an external torque to the motor produced by a Electromagnetic Particle Brake. Figure 6.4 exhibits the comparison between the CSF and the PDR when no external torque is applied to the motor.

Without additional torque the velocity tracking is accurate in both cases: it shows that at steady state the desired trajectory is tracked with a precision around $1 rad.s^{-1}$ for the PDR control. It must be noted that chattering phenomena appear in the PDR case leading to a sightly higher tracking error. However, in this case only four control inputs are used for the control.

Figure 6.5 shows the experimental result of the velocity tracking similarly to the previous figure. At time $t = 7s$ an unknown external torque is applied to the motor using an Electromagnetic Particle Brake. On this figure, the plot of the tracking errors shows that in the presence of external torque, the PDR is more robust to disturbances than the CSF. Indeed the perturbation is rejected only by the PDR control. This results is more clearly illustrated on Figure 6.6, where a focus on both trajectories tracking is represented. We can see that the PDR control (represented in red) provides a better velocity tracking performance. Moreover, for the CSF case, when the

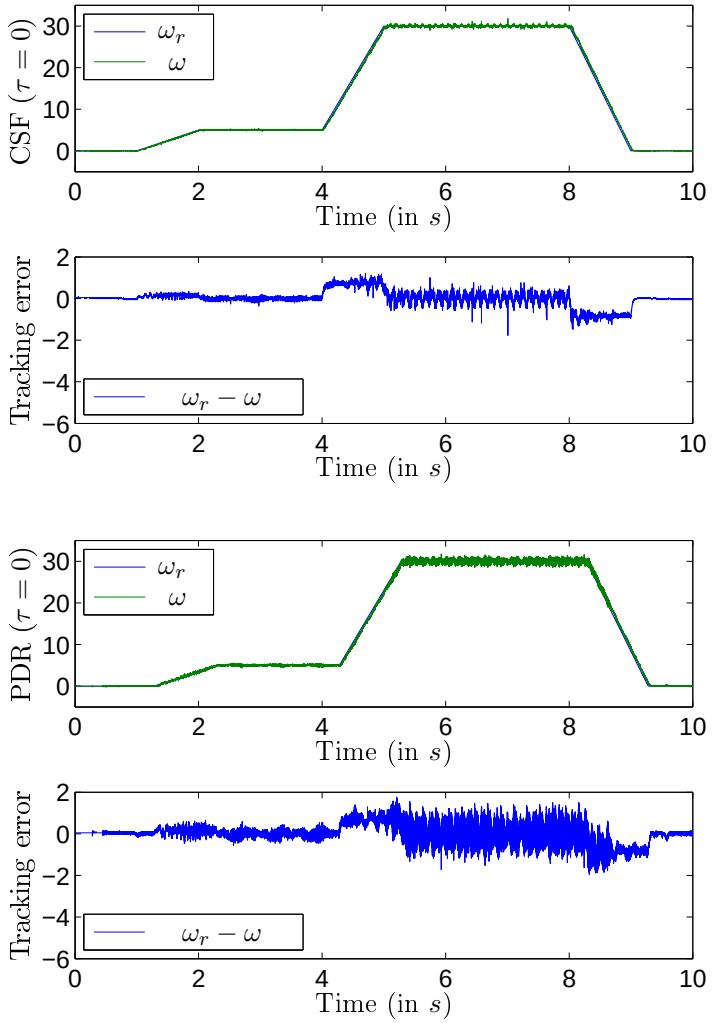


Figure 6.4: CSF and PDR experimental results without pertubations.

load torque is applied, the power amplifiers are in saturation.

6.2 Control of a multi-level power converter

Multi-level multicell power converters (also called *flying capacitors*), appeared at the beginning of the 1990s [Meynard 1991]. They are based on the association in series of the elementary cells of commutation with passive storage elements controlled by switches (transistors, diodes). During this last decade, these systems become more and more attractive to industrial applications, especially in high-power applications [Meynard 2002]. Indeed, the harmonic contents of the output signal are improved compared to the classical two levels converter technology using the same switching frequency [Rodriguez 2009]. Furthermore, this structure enables the reduction of the losses due to commutations of power semiconductors while allowing low cost

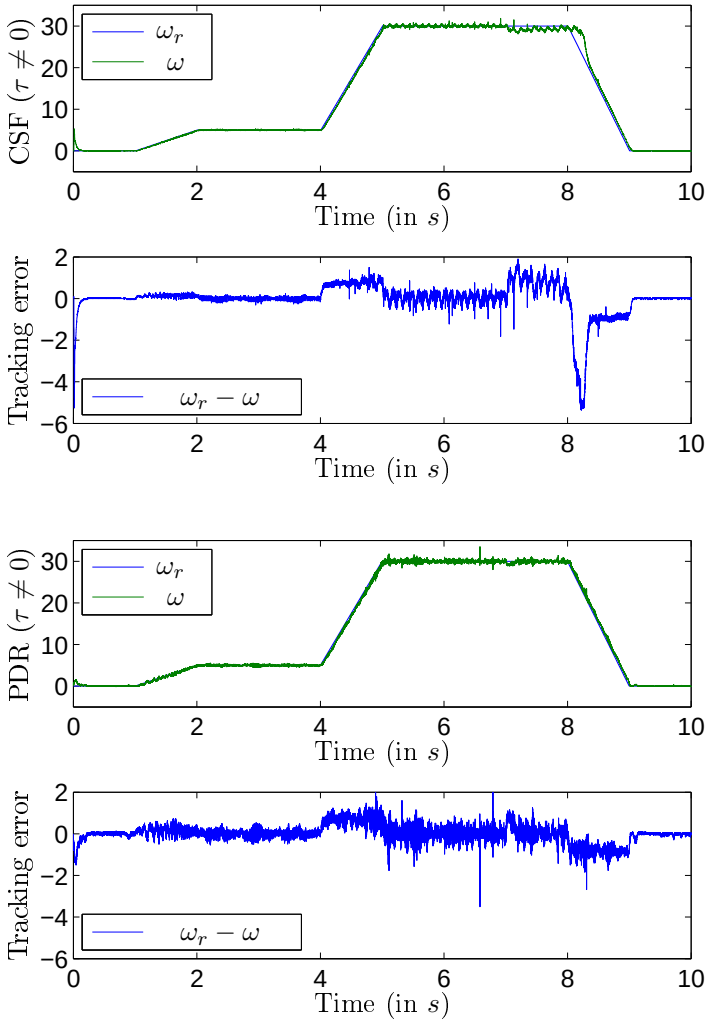


Figure 6.5: CSF and PDR experimental results with pertubations.

components [Bethoux 2002]. For multi-level power converters, the classical methodology lies on the use of *average models* [El Magri 2010, Bhagwat 1983], and continuous control design techniques [Gateau 2002, Sira-Ramirez 1994, Olalla 2011, Amato 2009] implemented via Pulse-Width-Modulation (PWM). *Direct control* techniques, addressing explicitly the design of binary signals, have been proposed in [Bethoux 2002], where a study of limit cycles was proposed, in [Hauroigne 2011], where optimal control techniques were given, and in [Gorp 2011] where sliding mode controllers are used.

As follows, we present an application of the proposed switching control methodology to the case of a multi-level power converter (also called flying capacitor)[23].

Figure 6.7 depicts the topology of a converter with p independent commutation cells associated to an inductive load. It consists of $(p-1)$ floating capacitors. The current flows from

[23] The results presented in this section have been developed in collaboration with Prof. Mohamed Djemai and Ass. Prof. Michael DEFOORT.

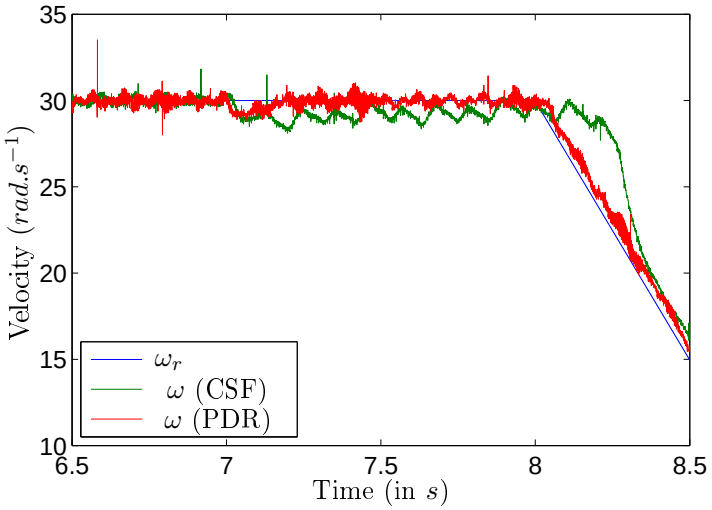


Figure 6.6: Velocity zoom during the perturbation.

the source E toward the output I through the different capacitors according to the switches configuration. The dynamics of the converter, with a load consisting in a resistance R and an inductance L, can be expressed by the following equations:

$$\frac{dV_{c_i}}{dt} = \frac{I}{C_i}(u_{i+1} - u_i), i \in \mathcal{I}_{p-1}, \tag{6.9}$$

$$\frac{dI}{dt} = -\frac{R}{L}I + \frac{E}{L}u_p + \frac{1}{L}\sum_{i=1}^{p-1} V_{c_i}(u_i - u_{i+1}), \tag{6.10}$$

where I is the load current, $C_i, i \in \mathcal{I}_{p-1}$ represent the value of capacitors, $V_{c_i}, i \in \mathcal{I}_{p-1}$ is the voltage of the $i-$th capacitor and E is the voltage of the source. Each commutation cell is controlled by the binary variable u_i which is constrained to take values in the set $\{0, 1\}$. Signal $u_i = 1$ means that the upper switch of the $i-$th cell is "on" and the lower switch is "off" whereas $u_i = 0$ means that the upper switch is "off" and the lower switch is "on". Model (6.9),(6.10) has

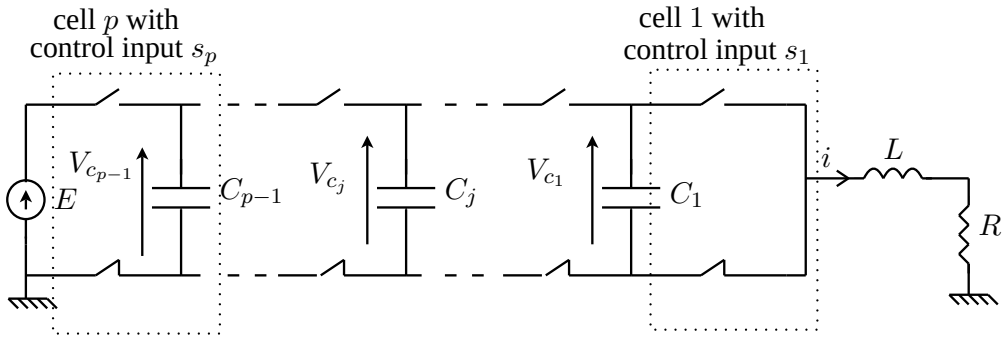


Figure 6.7: Flying capacitor converter associated to an inductive load.

as p state variables (the current I and the $p-1$ capacitor voltages $V_{c_i},\ i \in \mathcal{I}_{p-1}$) and p control variables $u_i, i \in \mathcal{I}_p$. Note that due to the presence of products between state variables I, V_{c_i} and inputs u_i, model (6.9),(6.10) is a nonlinear ordinary differential equation with a bilinear structure. Consider the generic bilinear model

$$\dot{x} = f(x,u) = A_0 x + \sum_{i=1}^{m} (A_i x + b_i)\, u_i \tag{6.11}$$

where $x \in \mathbb{R}^n$, $A_0, A_i \in \mathbb{R}^{n\times n}$, $b_i \in \mathbb{R}^n$, $i \in \mathcal{I}_m = \{1,2,\ldots,m\}$ and where the system input $u = [u_1, u_2, \ldots, u_m]^T \subset \mathbb{R}^m$ is constrained to take values in the discrete set $\mathcal{V} = \{0,1\}^m$. The multi-level power converter can be represented in the form (6.11) by considering a state vector $x = [V_{c_1}\ V_{c_2}\ \ldots V_{c_{p-1}}\ I]^T$. As an example for a converter with $p = 3$ cells, the corresponding matrices are as follows

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -R/L \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 & -1/C_1 \\ 0 & 0 & 0 \\ 1/L & 0 & 0 \end{bmatrix},\ b_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 1/C_1 \\ 0 & 0 & -1/C_2 \\ -1/L & 1/L & 0 \end{bmatrix},\ b_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/C_2 \\ 0 & -1/L & 0 \end{bmatrix},\ b_3 = \begin{bmatrix} 0 \\ 0 \\ E/L \end{bmatrix},$$

where $x = \begin{bmatrix} V_{c_1} & V_{c_2} & I \end{bmatrix}^T$. To define control objectives, it is important to highlight that, due to the industrial production standards, it is necessary to ensure a balanced distribution of the capacitor voltages [Gateau 2002]. Increasing power of static converters is generally obtained by increasing the voltage due to efficiency requirements. Multi-level converters enable to split the voltage constraints and to distribute them on several switches of smaller ratings in series. The equilibrium state of the p cells converter is reached when the voltage applied across the blocking switch of any cell (i.e. the differences of capacitor voltages $V_{c_i} - V_{c_{i-1}}$) takes the same value given by E/p. Under these conditions, the reference voltage of the $i-$th capacitor $i \in \mathcal{I}_{p-1}$ is given by $V_{c_i}^* = i\frac{E}{p}$. The control objective is to define the binary switching functions $u_i \in \{0,1\}$, $i \in \mathcal{I}_p$ such that the multicellular converter ensures:

1. the stabilization of I to a desired current of the form $I^*(\rho) = \frac{E}{R}\rho$, where $\rho \in (0,1)$,
2. the balanced distribution of the capacitor voltages across each cell, i.e. $\forall i \in \mathcal{I}_p$, V_{c_i} is stabilized toward $V_{c_i}^*$,
3. the robustness with respect to potential uncertainties in the load parameters R and L.

Thus the control of the multi-level power converter leads to the problem of designing a binary control law for a bilinear model.

For the generic model (6.11), this leads to designing a state feedback binary control

$$u = \kappa(x),\ \kappa : \mathbb{R}^n \to \mathcal{V}. \tag{6.12}$$

The problem of interest is the (local) stabilization of the Filippov solution of (6.11) via the

control law (6.12) to a point in the set of equilibria:

$$\mathcal{X} = \{x^* \in \mathbb{R}^n : \exists\, s^* \in \text{conv}\,\{\mathcal{V}\} \text{ s.t. } f(x^*, s^*) = 0\} \tag{6.13}$$

parametrized by inputs in $\text{conv}\,\{\mathcal{V}\}$.

6.2.1 LMI design for a generic bilinear model

As follows, we propose an LMI control design solution for a generic bilinear model with binary control law. For $s \in \mathbb{R}^m, x^* \in \mathcal{X}$, consider the following notations:

$$\tilde{A}(s) = A_0 + \sum_{i=1}^{m} A_i s_i, \tag{6.14}$$

$$\tilde{B}(x^*) = \left[\tilde{b}_1(x^*)\, \tilde{b}_2(x^*) \dots \tilde{b}_m(x^*)\right],\ \tilde{b}_i(x^*) = A_i x^* + b_i, i \in \mathcal{I}_m, \tag{6.15}$$

and the sets of vectors

$$\mathcal{H}_+ = \left\{h \in \mathbb{R}^m : h_i = 0, i \neq j,\ h_j = 1/s_j^*,\ j \in \mathcal{I}_m\right\} \tag{6.16}$$

$$\mathcal{H}_- = \left\{h \in \mathbb{R}^m : h_i = 0, i \neq j,\ h_j = 1/(1 - s_j^*),\ j \in \mathcal{I}_m\right\}. \tag{6.17}$$

Proposition 6.1 *[Hetel 2016] Consider system (6.11), $x^* \in \mathcal{X}$ and $s^* \in conv\,\{\mathcal{V}\}$ such that $f(x^*, s^*) = 0$. Given $\delta > 0$, let there exists (X, ψ, ϵ), $X \succ 0$, $\psi > 0$, $\epsilon > 0$, solution to the set of LMIs*

$$\begin{bmatrix} \epsilon I & I \\ * & X \end{bmatrix} \succ 0, \tag{6.18}$$

$$\tilde{A}(\sigma)X + X\tilde{A}^T(\sigma) - \psi\tilde{B}(x^*)\tilde{B}^T(x^*) \prec -\delta X, \sigma \in \mathcal{V}, \tag{6.19}$$

$$\begin{bmatrix} 1 & \frac{\psi}{2}h^T\tilde{B}^T(x^*) \\ * & X \end{bmatrix} \succ 0,\ h \in \mathcal{H}_+ \cup \mathcal{H}_-. \tag{6.20}$$

Consider the switching law $\kappa(x) = \left[\kappa_1(x),\ \kappa_2(x), \dots, \kappa_m(x)\right]$

$$\kappa_i(x) \in \begin{cases} \{1\}, & (x - x^*)^T\Gamma\,(A_i x^* + b_i) < 0, \\ \{0, 1\}, & (x - x^*)^T\Gamma\,(A_i x^* + b_i) = 0, \\ \{0\}, & (x - x^*)^T\Gamma\,(A_i x^* + b_i) > 0, \end{cases} \tag{6.21}$$

for $i = 1, \dots, m$, where $\Gamma = X^{-1}$. Then system (6.11) with the switching law (6.21) is locally exponentially stable at the equilibrium point $x = x^*$. Furthermore, an estimation of the domain of attraction is given by the ellipsoid

$$\mathcal{E}\left(x^*, X^{-1}, 1\right) = \left\{x \in \mathbb{R}^n : (x - x^*)^T X^{-1}(x - x^*) < 1\right\}$$

containing the ball $\mathcal{B}(x^*, C)$ with $C = 1/\sqrt{\epsilon}$.

The stabilization of the bilinear system (6.11) is expressed as an LMI optimization problem. The existence of solutions to the set of LMI conditions (6.18)-(6.20) can be verified using convex optimization software in Matlab. Minimizing ϵ such that a solution exists to the set of LMI conditions allows to design switching laws (6.21) while maximizing the size of the domain of attraction. The parameter δ, used in the set of LMIs, corresponds to the desired system decay

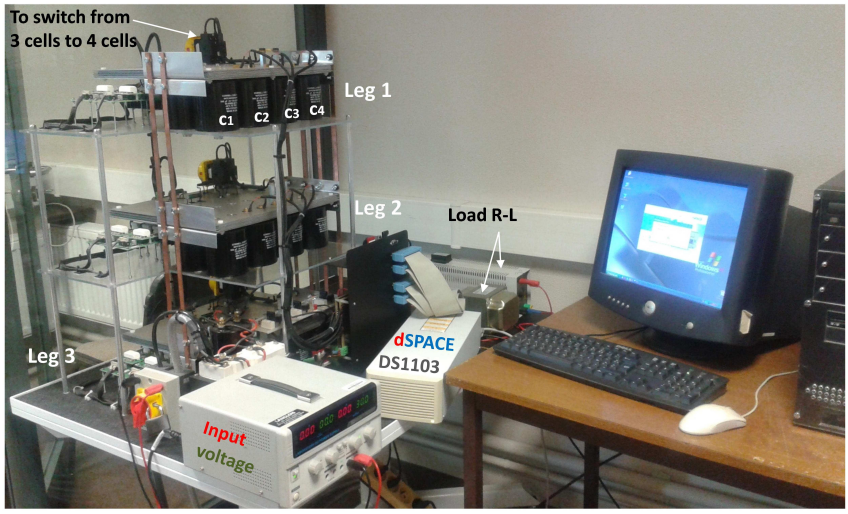


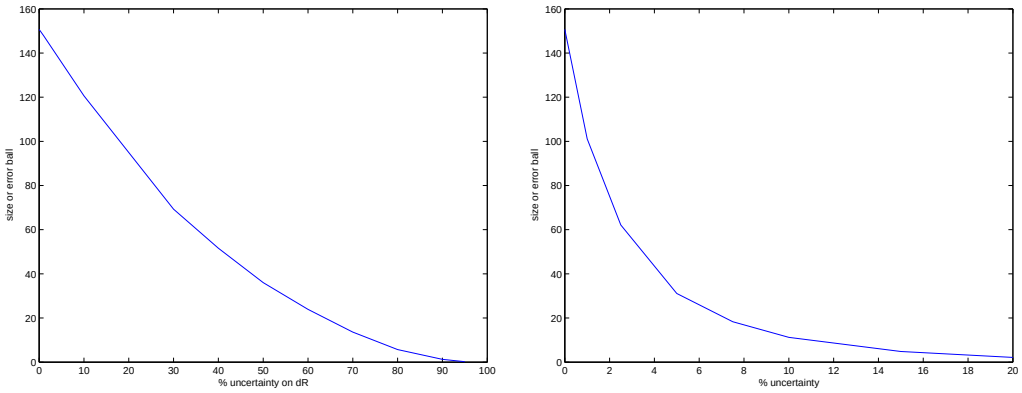
Figure 6.8: Experimental setup at LAMIH, Valenciennes: a 4-level 3-phase flying capacitor is adapted for illustrating the theoretical results. The experiments are performed using the first 3 cells of Leg 1.

rate inside the ellipsoidal estimation $\mathcal{E}(x^*, X^{-1}, 1)$ of the domain of attraction. The practical implementation of the obtained control law is quite simple since one only needs to compute the signs of $(x - x^*)^T \Gamma (A_i x^* + b_i), i = 1, \ldots, m$. The main intuition behind the control law (6.21) is that the control signals are chosen such that the gradient of the Lyapunov function $V(x - x^*) = (x - x^*)^T X^{-1} (x - x^*)$ is minimized.

6.2.2 Experimental results

Experiments have been carried out to illustrate the proposed binary controller, applied to a 3 cells multi-level power converter associated to an inductive load (an illustration is provided in Figure 6.8). The objective is to control each commutation cell such that the load current and the floating capacitor voltages are stabilized toward different equilibrium values. The LMI control design problems have been solved numerically using Sedumi as a numerical solver in Matlab. Hereafter, it will be shown that the proposed binary controller guarantees the stabilization of the closed-loop system even in the presence of parametric uncertainties.

To test the developed control strategy , a prototype of the topology in Figure 6.7 is built based on discrete insulated-gate bipolar transistors (IGBTs) SKM100GB12V. The relevant nominal bench parameters are $p = 3$, $C_1 = C_2 = 720.10^{-6} F$. The load is composed of an inductance and a resistance with nominal values $R_0 = 200\Omega$, $L_0 = 1H$. The control algorithm is implemented on a floating point DSP (TMS 320 F 240). An interface card allows to protect, by insulation, the DSP of the power electronics. The Dspace card DS1103 drives the peripheral devices (i.e. digital to analog devices, analog to digital devices, etc.). In order to obtain the best resolution, the minimum sampling period for the Dspace has been chosen, i.e. $T_{samp} = 7.10^{-5} s$. The measurement part is composed of voltage sensors to measure the voltage across the floating capacitors and a current transductor to measure the load current. A low pass filter with time constant $\tau = 5 \times 10^{-4} s$ and unitary gain has been added. The source voltage E is set to $30V$.



(a) (b)

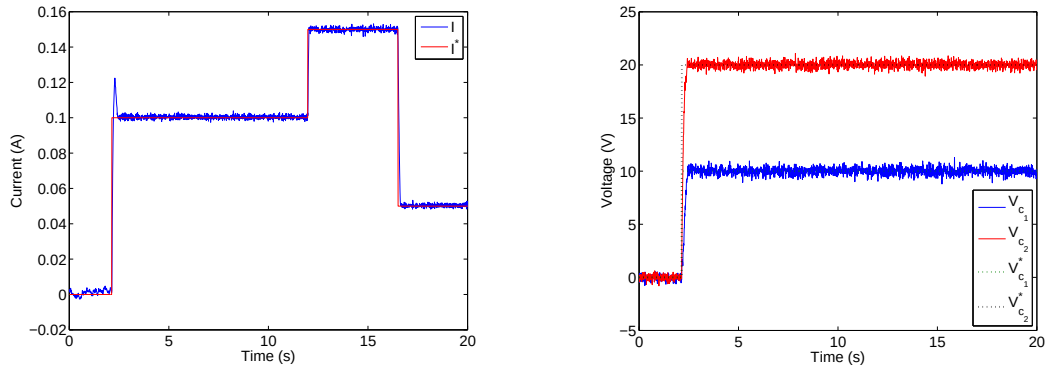
Figure 6.9: (a)-(b) Radius C of the stability domain as a function of the percentage of uncertainty for $I^* = 0.1A$ (a) when uncertainty affects the resistor R; (b) when both the resistor R and the inductance L are subject to uncertainty.

Applying the methodology based on convex optimization and LMIs, one can design the control laws that ensure local exponential stabilization of the system for several operating points of the form $[V_{c1}^* \; V_{c2}^* \; I^*]$. For example, let us consider the equilibrium point which consists of the desired load current $I^* = 0.1A$ and the desired floating capacitor voltages $V_{c_i}^* = \frac{E}{3} i, \; i \in \{1, 2\}$. It corresponds to the equilibrium point of (6.9)-(6.10) for an "averaged" input $s_i^* = \rho \approx 0.66$. We used an extension of Proposition 6.1 for systems with uncertain parameters (Proposition 2 from [Hetel 2016]). The set of LMI stabilization conditions are satisfied with a decay rate $\delta = 0.01$ and an uncertainty of 10% on the nominal values of the load ($R \in [180, 220], L \in [0.9, 1.1]$). A stabilizing switching law (6.21) is obtained with

$$\Gamma = X^{-1} = \begin{bmatrix} 0.5778 & 0.0156 & 0 \\ 0.0156 & 0.5778 & 0 \\ 0 & 0 & 822.0203 \end{bmatrix}. \tag{6.22}$$

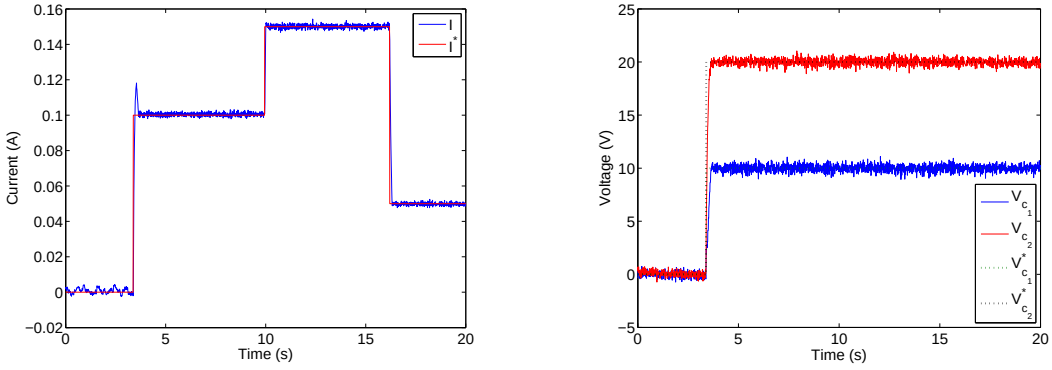
Note that by construction, switching laws of the form (6.21) satisfy the transition constraints classically encountered in multi-level power converters. On the intersection of switching hyperplanes, the usual adjacency can be ensured by using the automaton described in [Gorp 2011]. Applying Proposition 6.1 for the nominal values of the load, with X fixed as in (6.22) and a decay rate $\delta = 10^{-6}$, the obtained control law can be shown to ensure local stabilization in the ball $\mathcal{B}(x^*, C)$ with $C = 150.7$. Proposition 2 from [Hetel 2016] can also be used with the obtained X to compute the value of C for various values of uncertainties and illustrate the relation between the estimation of the domain of attraction and the robustness of the obtained control law to parametric uncertainties (see Figure 6.9). Furthermore, it is used to shown local stabilization for a set of equilibrium points corresponding to a uniform grid of 11 reference currents I^* in the set $[0.05, 0.15]$.

Hereafter, the experimental results obtained with a switching law (6.21) with Γ defined in (6.22) are presented. The control signals are chosen according to the theoretical developments in Section III, with switching surfaces (characterized by Γ) designed to ensure the steepest decent of the Lyapunov function $V(x - x^*) = (x - x^*)^T X^{-1} (x - x^*)$. Figures 6.10 (a),(b) show,



(a) (b)

Figure 6.10: Experimental results using the proposed binary control without perturbation. (a) - Load current I (b) - Internal voltages V_{c_1} and V_{c_2}.



(a) (b)

Figure 6.11: Experimental results using the proposed binary control with perturbations in the load. (a) - Load current I (b) - Internal voltages V_{c_1} and V_{c_2}.

respectively, the load current I, the internal voltages (V_{c_1}, V_{c_2}), and the control signals (s_1, s_2, s_3) using the obtained controller when no parameter uncertainty is considered. The control law ensures the stabilization for several equilibrium points corresponding to a reference current I^* in the interval $[0.05, 0.15]$. However, no overshoot constraints are included in the design procedure, which explains the peak at approximatively 2 seconds. The chattering phenomena in the steady state is mainly due to the sampled-data implementation of the binary control law.

In order to show the robustness of the proposed controller, the value of the load resistance has been tested for various configurations of the load. Figures 6.11 (a),(b) illustrate the system evolution with $R = 0.6 \cdot R_0 = 120\Omega$ (40% of uncertainty) and an inductance $L = 0.8 \cdot L_0 = 0.8H$ (20% of uncertainty). Note that the proposed binary controller manages to accomplish the control objective with good performance inspite of parametric variation with amplitude larger than the ones that have been shown in theory. In fact, for this configuration of perturbation on the load it can be shown theoretically the asymptotic stability for a tight grid of I^* in the interval $[0.07, 0.11]$. In practice, it is used for reference currents I^* in $[0.05, 0.15]$. This shows

that the proposed control design method has potential. However, it also shows that there is still place for improvement in what concerns the theoretic estimation of the rage of uncertainty for which the closed-loop system is stable.

Conclusion

In this part we have presented several contributions to the design of switching controllers for systems where the control signal is constrained to a finite number of values. Control design criteria have been presented for linear systems with relays, polytopic systems, bilinear systems with binary control and switched affine systems. The main idea of the proposed methodology is to use the existence of a continuous stabilizer in order to derive switching hyperplanes for the constrained control using convex optimization arguments. The approach is illustrated by experimental application to the control of multi-level power converters and stepper motors. This research line is still an emerging research direction and it leaves many open problem both from a theoretical and applicative point of view. Some of the perspectives will be mentioned after the concluding remarks of this manuscript.

General conclusions

This document has presented a selection of the research activities developed by Laurentiu HETEL and his collaborators since his recruitment as an Associate Researcher at CNRS. Two main topics have been addressed in relation with the occurrence of discrete constraints in the implementation of control laws. The first part of the manuscript has presented several contributions concerning the analysis and design of *sampled-data systems* with aperiodic sampling intervals. The core of this part was dedicated to the analysis of systems with arbitrarily varying sampling intervals. We have tried to be broad in outlook and address this problem from many different quarters (time-delay, discrete-time, hybrid, input/output approaches). Furthermore, we have investigated the effect of sampling for various classes of systems (linear, bilinear, nonlinear affine, switched, etc.). Some contribution to the design of state dependent sampling maps have equally been presented.

The second part of the manuscript was concerned with the design of *switching controllers* for systems where the control signal is only allowed to take a finite number of values in a discrete set. The main contributions are related to a new framework for the design of switching controllers based on the use of simple convex optimization arguments. This methodology provides new solutions for the design of sliding mode controllers and for the stabilization of switched affine systems. Furthermore, it has interesting applications for the control of some electronic and electro-mechanical devices. Several extensions are currently under study.

It is worth noticing that the subjects addressed here lie at the intersection of four important axes in Control Theory (time-delay, hybrid, LPV systems, input-output interconnections) and we hope this will have a stimulating impact in the control community. Methods and tools are being transferred from one research topic to another and the perspectives of cross-fertilisation and generalization are numerous. Several open problems that could be tackled in the future are discussed hereafter.

Perspectives

In what follows, I will indicate some of the research directions that we have discovered over the last few years. In order to provide a simple view, I would say that the core of my future research activities is centred on the use of the hybrid system framework for the study of sampled-data systems as abstractions of networked and embedded control systems. The main objectives of my future research activities can be roughly structured as follows:

Objective A. Fundamental study of Hybrid Systems. I intend to investigate the interaction between discrete algorithms and differential equations (as models of physical processes) through a fundamental study of dynamic and structural properties of hybrid dynamical systems. In particular, I will study the dynamics of switched systems and of impulsive differential equations.

In what concerns the study of switched systems, in my opinion, the main challenge now is to provide a solid theoretical framework for the design of switching surfaces in the case of *switched systems with non-common equilibria*. The study of such switched systems is relevant in many applicative domains. They are currently encountered in electronics, in energy management applications, for describing embedded power converters. Such systems are interesting since in practice they can be stabilized by fast switching to non-standard equilibrium points corresponding to convex combinations of the subsystems equilibria. However, the stabilization to such equilibrium points is not trivial. It requires a particular treatment, involving the study of specific solutions for discontinuous systems (such as Filippov solutions) and sliding dynamics, which is challenging from a theoretical point of view. For the moment, there is a serious lack of tractable theoretical tools for designing control algorithms in such cases. It is therefore interesting both from a theoretical point of view and for practical applications to generalize the existing theory on hybrid systems to cover this case. I will try to address this topic by combining tools from the study of nonlinear systems with (saturation) constraints with concepts previously used for the study of sliding dynamics in variable structure control.

For the case of *impulsive systems*, the research activities are strongly connected with the ones concerning the study of sampled-data systems. The two research lines mutually enrich each other. As we have seen in Part I, a large number of results have been provided on the analysis of sampled-data systems by re-formulating the systems dynamics in a time-delay or input-output interconnection framework. Our objective is to investigate the extension of these approaches to more general classes of hybrid systems with impulsive effects. The study of impulsive systems from the point of view of input-output interconnections would be an original perspective, with a particularly interesting potential in the development of numerical tools for analysis and design .

Objective B. Hybrid methodologies for Networked / Embedded Systems. Hybrid systems are not only used for the modelling of sampled-data systems. They can provide a natural theoretical framework for the analysis and design of networked and embedded systems. While an important effort is being made in the domain of Computer Sciences to enhance the design of embedded hardware, communication networks, real-time scheduling algorithms, etc., it is a challenging Control Theory problem to understand the interaction between the implementation of control algorithms (as codes in distributed microprocessors) and the physical processes. The challenge in Networked / Embedded Systems is to extend control theory so as to embrace the dynamics of software and networks. The aim is to provide methodological tools for the analysis and design of systems with embedded / distributed control implementations using the hybrid system formalism. The main idea is to provide reliability guarantees in terms of Lyapunov stability. This approach makes an interesting alternative to classical methods for which it becomes impossible to test the software under all possible conditions when the interaction with real physical processes is considered. Some generalizations of the approaches presented for systems with aperiodic sampling to more general networked systems are simple. Many others are not obvious,

at least not at this time.

Short term perspectives

As follows, I will describe some short term perspectives that are in line with the topics presented in the manuscript.

Continuous-discrete observers

While a large literature has addressed the stability analysis problem of systems with aperiodic sampling, less results are concerned with the design of observers. A promising research direction would be the extension of the existing methodologies for the design of continuous-discrete observers [Nadri 2003, Astorga 2002, Karafyllis 2009a, Nadri 2013]. In the case of LTI systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu_k, \ \forall t \in [t_k, t_{k+1}), \ h_k := t_{k+1} - t_k \in [\underline{h}, \bar{h}], \ k \in \mathbb{N} \quad (1) \\ y(t) &= Cx(t), \quad (2) \end{aligned}$$

one could investigate the design of observers of the form

$$\begin{aligned} \dot{\eta}(t) &= A\eta(t) + Bu(t_k), \ t \in [t_k, t_{k+1}) \quad (3) \\ \eta(t_k) &= \eta(t_k^-) + L\Big(y(t_k^-) - C\eta(t_k^-)\Big), t = t_k, \ k \in \mathbb{N}. \quad (4) \end{aligned}$$

The main objective is to derive constructive observer design criteria, non only in the linear case, but also in a more general nonlinear setting. The extensions of the impulsive system approach to this design problem is a challenging research direction.

New hybrid representations of sampled-data systems

Several approaches are available for the analysis and design of sampled-data systems and it is of interest to compare them and understand their significance. Some relations between the different approaches have been indicated in Chapter 2. For example, it has been shown that the stability criterion obtained using the time delay approach in [Fridman 2004] can be also deduced via the small gain theorem [Mirkin 2007]. However, it is more difficult to obtain such equivalence relations between the recent approaches in the literature. It would also be of interest to understand what is the significance of the existing work on time delay systems from the point of view of the hybrid formalism [Goebel 2012]. In particular, the approach presented in [Fridman 2010] seems to suggests a quite different hybrid representation of sampled-data systems with respect to the existing literature.

Consider the case of LTI sampled-data systems

$$\dot{x}(t) = Ax(t) + BKx(t_k), \ \forall t \in [t_k, t_{k+1}), \ h_k := t_{k+1} - t_k \in [\underline{h}, \bar{h}], \quad (5)$$

The approach in [Fridman 2010] uses Lyapunov-Krasovskii functionals of the form

$$V(t, x(t), \dot{x}_t) = x^T(t)Px(t) + \theta(t) \int_{t-\tau(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds. \quad (6)$$

with $\tau(t) = t - t_k$ and $\theta(t) = h_k - \tau(t)$. Note that the term

$$\theta \int_{t-\tau}^{t} \dot{x}^T(s) R \dot{x}(s) ds$$

in this functional can be re-written as

$$\theta \chi^T \Big(\int_{-\tau}^{0} (Fe^{Fs})^T \tilde{R} (Fe^{Fs}) ds \Big) \chi,$$

with

$$\tilde{R} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix},$$

$\chi(t) = [x^T(t),\, \hat{x}^T(t)]^T$ and

$$F = \begin{bmatrix} A & BK \\ 0 & 0 \end{bmatrix},$$

that is the functional (6) can be interpreted as a function of the form $V(\chi, \tau, \theta) = \chi^T P(\tau, \theta) \chi$ which depends non only on the increasing timer τ but also on a decreasing timer θ. In the hybrid framework proposed by [Goebel 2009], this function suggest a state representation of the form

$$\left\{ \begin{array}{rcl} \left.\begin{array}{rcl} \dot{x} &=& Ax + BK\hat{x} \\ \dot{\hat{x}} &=& 0 \\ \dot{\tau} &=& 1 \\ \dot{\theta} &=& -1 \end{array}\right\} & & \theta \in [0, \overline{h}], \\ \left.\begin{array}{rcl} x^+ &=& x \\ \hat{x}^+ &=& x \\ \tau^+ &=& 0 \\ \theta^+ &\in& (0, \overline{h}] \end{array}\right\} & & (s = 0) \bigwedge \left(\tau \in [\underline{h}, \overline{h}]\right). \end{array} \right. \tag{7}$$

While in (6) the use of the decreasing counter θ seems to be a technical artefact, it leads to a more profound reflection concerning the representation of sampled-data systems as hybrid systems. The use of such models for deriving stability analysis and control design conditions deserves to be further investigated. Furthermore, it is also necessary to provide more insight on the structural properties of sampled-data systems. Basic problems, related to the *realization* [Petreczky 2006] of an aperiodically sampled input-output map by a hybrid model and the characterization of *minimal representations* must also be addressed in the future.

Analyzis of hybrid models based on a 2D system representation

As we have seen in Chapter 2, in the general formulation proposed by Goebel, Teel and Sanfelice [Goebel 2009, Goebel 2012] the system solutions can be expressed in the form

$$\dot{z}_k(t) = F_z\left(z_k(t)\right)), \quad z_k(t) \in C, \forall t \in (t_k, t_{k+1}), k \in \mathbb{N} \tag{8a}$$

$$z_{k+1}(t_{k+1}) = J_z\left(z_k(t_{k+1})\right)), \quad z_k(t_{k+1}) \in D, k \in \mathbb{N}. \tag{8b}$$

Such systems are implicitly 2D systems [Owens 1999]. Solutions z of the general hybrid system (8) are parametrized by both the continuous time t and the discrete time k: $z_k(t)$ represents the state of the hybrid system after t time units and k jumps.

For 2D systems, it seems natural to adopt a stability analysis based on vector Laypunov

functions [Bellman 1962, Emelianova 2014]

$$V\left(z_k(t), z_{k+1}(t_{k+1})\right) = \begin{pmatrix} V_1\left(z_k(t)\right) \\ V_2\left(z_{k+1}(t_{k+1})\right) \end{pmatrix}$$

where V_1, V_2 are positive definite functions, and a divergence operator

$$divV = \frac{dV_1}{dt} + V_2\left(z_{k+1}(t_{k+1})\right) - V_2\left(z_k(t_{k+1})\right).$$

Such an approach might lead to new stability conditions for hybrid systems, and in particular for sampled-data systems. Some preliminary results in this direction have been obtained in [Ríos 2015].

Lur'e Lyapunov functions for relay control systems

This subject is related to the stabilization of switched systems with non-common equilibria. Before considering the general case, it is useful to tackle the application of new design tools by considering a simplified case of LTI systems with relays. The main idea of the methodology proposed in Part II is to use the existence of a continuous stabilizer in order to derive switching laws based on convex optimization arguments. However, the result only ensures local stabilization and the provided domain of attraction strongly depends on the choice of the continuous stabilizers and, implicitly, on the choice of the Lyapunov function. For the case of LTI systems

$$\dot{x} = Ax + Bu,\ u \in \mathcal{V} = \{-v, v\} \tag{9}$$

up to now, we have only used quadratic Lyapunov functions and, as continuous stabilisers, linear static state feedback controllers $u = Kx$. Numerical simulations show that there is still room for improvement. It is important to note that the literature on the design of continuous stabilizers for LTI systems with input constraints is quite rich [Blanchini 1999, Tarbouriech 2011, Hu 2006, Zaccarian 2011, Zaccarian 2002] and advanced numerical methods for enlarging the domain of attraction are available using more complex Lyapunov functions. To advance beyond the use of ellipsoidal estimations of the domain of attraction, one direction to be exploited is the use of Lur'e Lyapunov functions of the form

$$V(x) = x^T P x + 2\Omega \int_0^{Kx} \phi(s) ds \tag{10}$$

where $P \succ 0, \Omega < 0$ and

$$\phi(s) = \begin{cases} v - s, & s > v \\ 0, & s \in [-v, v] \\ -v - s, & s < -v. \end{cases} \tag{11}$$

Similar Lyapunov functions have been used for enlarging the domain of attraction of systems with saturation and they have a hight potential for the case of switching controllers. The approach would lead to nonlinear switching surfaces, i.e. a switching function of the form

$$u \in -vsign\left(x^T PB - \phi^2(Kx)\Omega KB\right) \tag{12}$$

which generalizes the switching law $u \in -vsign\left(x^T PB\right)$ presented in Part II. The challenge is to provide design LMI design conditions based on the existence of the Lur'e Lyapunov function.

Mid term directions

In what follows I will present some research direction that I intend to mathematically formalise in the future.

A new discrete-time approach for nonlinear sampled-data systems

In the control of classical sampled-data systems, the discrete-time framework is known to have several advantages with respect to the continuous-time approaches which are usually *indirect*: they are based on emulation of continuous controllers which have been design independently of the sampled-data implementation. The discrete framework allows a *direct design*, taking into account the value of the sampling interval for the control synthesis. Although the discrete-time approach has been shown to lead to efficient numerical conditions for the case of LTI systems with aperiodic sampling, very few results address the nonlinear case. In this context it would be of interest to generalise the classical geometric approaches proposed by [Monaco 2007, Monaco 2001] to the case of systems with aperiodic sampling. Furthermore, it would be useful to re-state the design conditions in the hybrid framework, in order to ensure also desired inter-sampling performances.

Dynamic sampled-data controller under uncertain sampled-data implementations

While the case of static controllers with aperiodic sampling has been extensively studied, few results exist concerning the case of dynamic controllers. To the best of our knowledge, up to now it is assumed that the discrete-time emulation of continuous dynamic controllers is perfect and that the sampling interval is known. In practice the controller discretization introduces approximations and the sampling interval is rarely available in real time. It is a very challenging theoretical problem to provide design conditions while taking into account these uncertainties in the control implementation.

A more general dissipativity framework

In the Input/Output stability approach, up to now the existing criteria are based on *static* IQCs. The use of *dynamic* IQCs might be a real source of improvement [Megretski 1997]. The generalization of such dynamic IQCs in the dissipativity framework is of interest, not only for the case of sampled-data systems, but also in a more general context, for the study of other robustness properties of nonlinear systems. In fact, before the emergence of powerful numerical tools for the analysis of dynamical system (LMIs, SOS optimization), a large variety of studies have been developed in the frequency domain. It would be useful to "translate" this literature in the time domain and enhance its applicability using optimization algorithms. The interpretation of this approach from the point of view of hybrid systems needs to be further investigated.

The best sampling pattern

The research in the case of systems with arbitrary sampling has reached a mature phase of development. However, the problem of designing stabilizing sequences of sampling is still largely open. In this context, the potential of the approaches used for the arbitrary sampling problem is far from being fully exploited. Nevertheless, a better mathematical formalization of what is required from "the best" sampling sequence is needed.

We would like to point out that that the interest of this study goes beyond the simple aperiodic sampling problem. The problematic has to be considered in a more general context of networked systems where we also have to deal to deal with (communication) delay, quantization or scheduling protocols. For large networks it is expensive, if not impossible, to control all systems individually, and mainstream centralized controllers are infeasible. Decentralized controllers must be considered, while taking into account the fact that overall dynamics are largely determined by the interactions of individual components. Instead of tuning controller gains, we could focus on optimizing the topology of the network, i.e. we determine which systems need to interact in order to optimize a global objective in an efficient way. The resulting challenge is to dynamically optimize the communication sequences of each link in the network so to ensure some desired fast/robust synchronizing control, while taking into account the costs in terms of computational load and/or energy consumption.

Constructive methods for switching law design

With respect to the stabilization problem of switched systems, it has been shown in Part II that the stabilization of switched affine systems can be related to the existence of a continuous stabilizer for a classical nonlinear affine (bilinear) model. This is a classical problem on which a large variety of results are available in the literature. While in this manuscript we have presented simple results based on the local linearisation of the underlying nonlinear model and the use of small gain arguments, the possibilities of extension are numerous. For example, in practical applications, we may consider "patchy" switching laws, using the existence of gain scheduled controllers associated to different equilibria.

Switched systems with spectral constraints

The experimental application to the control of DC/DC power converters has shown that it would be useful to design switching laws where the switching signal has to satisfies additional frequency domain constraints: for electro-magnetic compatibility reasons, the spectrum of the switching signal should be limited to a well defined spectrum range. Taking into account such spectral constrained when defining state-dependent switching controllers leads to a challenging theoretical problem.

Switching law design based on a geometric study

For a theoretical analysis of switched affine systems, the study of structural properties based on geometric tools [Isidori 1995] may be a promising research direction. For example, the use of the underlying bilinear model may lead to new necessary conditions for stabilization. The results presented for the case of switched affine systems should be extended to more general nonlinear switched systems where the system modes do not share the same equilibrium. It is also necessary to extend the convex embedding approach [Hetel 2015a] to cope with discontinuous stabilizers, for systems that fail the classical Brockett's condition or that cannot be stabilized by state feedback when solutions are considered in the sense of Filippov [Ryan 1994], [Clarke 1997]. This topic must be investigated in relation with the developments considering systems with aperiodic sampling and the joint synthesis of sampling patterns and stabilizing controllers.

Long term directions

There is no doubt that control theory is now a mature field of research. Furthermore, in practical applications embedded control devices are now widely spread. However, control theory is still an *invisible technology* and many of the academic advances in the last 20 years seem to be ignored outside a group of specialists. In fact, in many applications, the development of control laws (beyond the manual tuning of PIDs) requires at least a master level, if not a PhD. In my opinion, it is of high interest not only to provide new theoretical tools but also to take care about their applicability.

In the context of sampled-data systems, a large amount of results have been published, using various different approaches. However, the developments are sparse, in the sense that each of the existing approaches covers only some of the aspects in sampled-data control. Furthermore, the last textbooks concerning sampled-data systems have been published in the '90s and do not seem to cover important real-time implementation constraints that a control engineer needs to handle nowadays. Without a vulgarization effort from the researchers in the control community, the wide spread of use of networked and embedded control elements will develop on negligible theoretic foundations. My long term objective is to contribute to the establishment of a unifying theory of sampled-data control systems, which gathers the most significant results proposed in the different research communities and presents simple and theoretically solid tools for control engineers. I have the conviction that hybrid system will offer the appropriate fundamental framework for this unifying theory. However, making the fundamental results accessible is far from being obvious.

At a long term I would also like to dedicate more attention to the experimental research. Although up to now my research activities are mainly concerned with fundamental research, the scientific problems that I study are motivated by applications that are ubiquitous in industry and that respond to societal challenges. For example, the research on the stabilization of switched systems is relevant for the embedded control of power converters which are omnipresent in energy management applications. Furthermore, networked/embedded controllers are of interest in electric power networks where changes in the structure of the grid have to be taken into account in real time, in particular to support the introduction of renewable energy sources (wind farms and solar plants) while considering various constraints in terms of prices, demands and capacities. Since for the moment the research is situated at a fundamental level, it is not sufficiently mature for an industrial validation. However, I feel that it is time to return back to the starting point and to confront the developed concepts with their empirical sources. Some preliminary steps in this direction are planned via a joint PhD supervision with prof. Bogdan Marinescu (starting with November 2016) in the context of a RTE-Centrale Nantes Chair, and through the H2020 project UCOCOS - which involves industrial partners (EOS innovation, CITC, TNO) interested in networked systems. I strongly believe that such an experience with practical applications will be fruitful at long term. An original theoretical framework may emerge if we understand what are the limitations of current theories and we formalize mathematically what really works at the empirical level.

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