Université de Lille 1 Sciences et Technologies

Smooth curves in toric surfaces

(Les courbes lisses dans les surfaces toriques)

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English summary

Let k be an algebraically closed field, let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a bivariate Laurent polynomial that is supported on Δ and that is sufficiently generic. Let C be the algebraic curve over k that is defined by f. In this manuscript we study connections between the birational geometry of C and the combinatorics of Δ. The starting point of this research topic is a theorem from 1893 due to Baker (improved in 1977 by Khovanskii) which states that the geometric genus of C equals the number of lattice points in the interior of Δ. Some other entries to the geometry-combinatorics dictionary were added by Koelman in 1991 and by Kawaguchi in 2012. The presented thesis gathers a number of recent research papers that are devoted to extending this dictionary further, by providing combinatorial interpretations for the gonality, the Clifford index, the Clifford dimension, and the scrollar invariants associated to a gonality pencil. Under certain restrictions on Δ we also give combinatorial interpretations for the canonical graded Betti table and for the first scrollar Betti numbers. The last part of the manuscript deals with a notion called "intrinsicness": given the many combinatorial features of Δ that can be told from the abstract geometry of C, we study to which extent it is possible to recover all of Δ.

Résumé en français

Soit k un corps algébriquement clos, soit Δ un polygone entier deux-dimensionnel et soit $f \in k[x^{\pm 1}, y^{\pm 1}]$ un polynôme de Laurent bivarié qui est supporté sur Δ et qui est suffisamment générique. Soit C la courbe algébrique sur k définie par f. Dans ce manuscrit, nous étudions les connexions entre la géométrie birationnelle de C et la combinatoire de Δ. Le point de départ de ce sujet de recherche est un théorème de 1893 dû à Baker (amélioré en 1977 par Khovanskii) qui dit que le genre géométrique de C est égal au nombre de points entiers à l'intérieur de Δ. Quelques autres entrées au dictionnaire géométrie-combinatoire ont été ajoutées par Koelman en 1991 et par Kawaguchi en 2012. La thèse présentée rassemble un nombre de travaux de recherche récents qui sont dédiés à étendre davantage ce dictionnaire, en fournissant des interprétations combinatoires pour la gonalité, l'indice de Clifford, la dimension de Clifford, et les invariants scrollaires associés à un pinceau qui réalise la gonalité. Sous certaines restrictions sur Δ, nous fournissons également des interprétations pour le tableau de Betti canonique et pour les premiers nombres de Betti scrollaires. La dernière partie du manuscrit traite d'une notion que l'on appelle "intrinsèqualité" : étant donné les nombreuses caractéristiques combinatoires de Δ qui peuvent être prédites en considérant la géométrie abstraite de C, nous étudions dans quelle mesure il est possible de récupérer Δ complètement.

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EXTENDED PREFACE TO:

Smooth curves in toric surfaces

(Les courbes lisses dans les surfaces toriques)

by Wouter CASTRYCK

Let k be an algebraically closed field and let

$$f = \sum_{(i,j)\in\mathbf{Z}^2} c_{ij}x^iy^j \in k[x^{\pm1}, y^{\pm1}]$$

be an irreducible bivariate Laurent polynomial, defining a curve U_f inside the two-dimensional torus $\mathbf{T}^2 := (k^*)^2 = \mathbf{A}^2 \setminus$ coordinate axes. This manuscript is devoted to connections between the birational geometry of U_f and the combinatorics of the Newton polygon

$$\Delta(f) = \mathrm{conv}\{\,(i,j)\in\mathbf{Z}^2 \,|\, c_{ij}\neq 0\,\} \subseteq \mathbf{R}^2$$

(assumed to be two-dimensional) of f. The earliest such connection is surprisingly old, dating back to 1893, when Baker observed [Bak93] that the geometric genus of U_f is bounded by the number of lattice points (= $\mathbf{Z}^2$-valued points) in the interior of $\Delta(f)$. In the 1970s, after toric geometry had made its appearance, a more satisfactory proof was given by Khovanskii [Kho77], who moreover showed that Baker's bound is generically met. Recently developed tools such as tropical geometry and Berkovich theory conceptualized this remarkable result further, although these topics will not be addressed here.

A well-known generically satisfied condition which is sufficient for meeting Baker's bound [CDV06, Prop. 1] is that f is **nondegenerate with respect to its Newton polygon**, meaning that for all faces $\tau \subseteq \Delta(f)$ of any dimension (i.e. vertices, edges and $\Delta(f)$ itself), the system of equations

$$f_\tau = x\frac{\partial f_\tau}{\partial x} = y\frac{\partial f_\tau}{\partial y} = 0 \qquad \text{with} \qquad f_\tau = \sum_{(i,j)\in\tau\cap\mathbf{Z}^2} c_{ij}x^iy^j$$

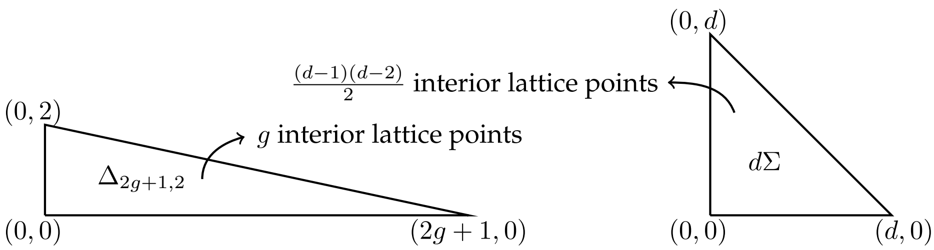
has no solutions in $\mathbf{T}^2$. For Δ a lattice polygon (= the convex hull in $\mathbf{R}^2$ of finitely many points in $\mathbf{Z}^2$) we say that f is **Δ-nondegenerate** if it is nondegenerate with respect to its Newton polygon and $\Delta(f) = \Delta$. In general, the condition of nondegeneracy is not strictly needed for meeting Baker's bound, which leads to the following slight and seemingly bland relaxation:

Definition 1. *Let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm1}, y^{\pm1}]$ be irreducible. We say that f is* **weakly Δ-nondegenerate** *if*

- *$\Delta(f) \subseteq \Delta$ and for each edge $\tau \subseteq \Delta$ one has $\Delta(f) \not\subseteq \Delta \setminus \tau$,*
- *the genus of U_f equals the number of lattice points in the interior of Δ.*

Weak nondegeneracy is the assumption underlying most of the results presented in this manuscript. Besides being (slightly) weaker than nondegeneracy and thereby leading to stronger statements, the notion allows for more combinatorial freedom, in the sense that a weakly Δ-nondegenerate Laurent polynomial might also be weakly Δ'-nondegenerate for some other (potentially easier) lattice polygon Δ', which has important proof-technical advantages. This freedom does not apply to $\Delta^{(1)}$, the convex hull of the lattice points in the interior of Δ, which is fixed and in fact turns out to play a more important role than Δ itself.

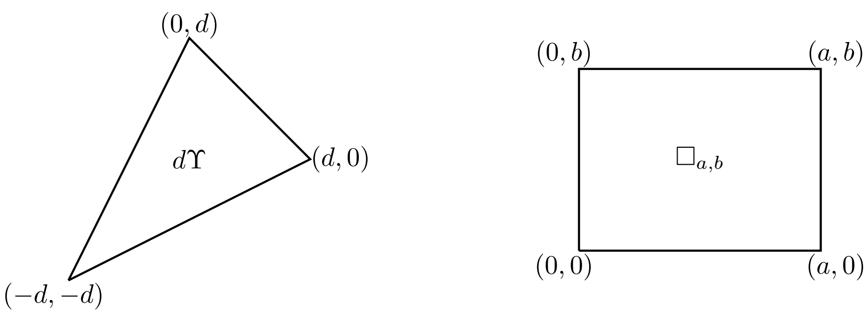
Well-known examples. Familiar examples include the Weierstrass polynomials $f = y^2 - h(x)$, where $\operatorname{char} k \neq 2$ and $h(x) \in k[x]$ is squarefree of degree $2g+1$ for some integer $g \geq 1$: these are weakly $\Delta_{2g+1,2}$-nondegenerate. Other examples are the dehomogeniza-



tions $f \in k[x,y]$ with respect to z of the homogeneous degree $d \geq 1$ forms $F \in k[x,y,z]$ that define a smooth curve in $\mathbf{P}^2$: such polynomials are weakly $d\Sigma$-nondegenerate. In both cases the reader sees that Baker's bound confirms the well-known formula for the genus.

Remark. More generally for $a, b \in \mathbf{Z}_{\geq 1}$ we use $\Delta_{a,b}$ to denote $\operatorname{conv}\{(0,0),(a,0),(0,b)\}$. If $\gcd(a,b) = 1$ then the corresponding curves are said to be $C_{a,b}$; this notion was introduced by Miura in the context of coding theory [Miu93].

More examples. Other recurring examples are weakly $d\Upsilon$-nondegenerate Laurent polynomials and weakly $\square_{a,b}$-nondegenerate Laurent polynomials, where $d, a, b \geq 1$, which



define curves of genus $\frac{3}{2}d^2 - \frac{3}{2}d + 1$ and $(a-1)(b-1)$, respectively.

For an irreducible (not necessarily smooth or complete) algebraic curve C/k and a two-dimensional lattice polygon Δ, we say that C is **weakly Δ-nondegenerate** if it is birationally equivalent to U_f for some weakly Δ-nondegenerate Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$ — similarly we say that C is **Δ-nondegenerate** if f can moreover be taken Δ-nondegenerate.

The presented work groups together a number of research papers that are devoted to connections between the birational geometry of such a weakly Δ-nondegenerate curve

C and the combinatorics of Δ. Their joint goal is to extend the geometry-combinatorics dictionary that started with Baker's formula for the genus, although we stress that several entries remain to be added and/or enhanced by future researchers. For reasons of coauthorship and efficiency I have left the papers in their original shape, even though when put together the treatment is not entirely uniform: some statements assume nondegeneracy rather than weak nondegeneracy, while others are presented subject to the condition that the base field k is of characteristic 0. One source for this non-uniformity is that the material has matured over time, with some insights postdating the publication of the earliest papers. Another cause is that several important references assume that $\operatorname{char} k = 0$ or even $k = \mathbf{C}$, and unfortunately I was not able to sift each of these to the bottom to verify the need for this (possibly often unneeded) assumption.

In view of these considerations, the goal of this preface is not only to give an overview of the results obtained, but also to update the exposition: in the text below, all main results are stated under the weak nondegeneracy assumption, which is always sufficient, and certain characteristic zero statements have been reformulated in arbitrary characteristic, along with some lines of explanation why this is allowed.

Remark on terminology. Unfortunately the non-uniformity also affects the terminology of being *weakly Δ-nondegenerate*, for which a.o. in Chapters 5 and 11 the phrasing *Δ-toric* is used.

Contents. Concretely, the following papers are included in this HDR thesis:

- **Chapter 1**: *On nondegeneracy of curves*, Algebra & Number Theory **3**(3), pp. 255-281 (2009), written jointly with John Voight
- **Chapter 2**: *Moving out the edges of a lattice polygon*, Discrete and Computational Geometry **47**(3), pp. 496-518 (2012)
- **Chapter 3**: *The lattice size of a lattice polygon*, Journal of Combinatorial Theory, Series A **136**, pp. 64-95 (2015), written jointly with Filip Cools
- **Chapter 4**: *A minimal set of generators for the canonical ideal of a non-degenerate curve*, Journal of the Australian Mathematical Society **98**(3), pp. 311-323 (2015), written jointly with Filip Cools
- **Chapter 5**: *Linear pencils encoded in the Newton polygon*, to appear in International Mathematics Research Notices (2017), written jointly with Filip Cools
- **Chapter 6**: *Computing graded Betti tables of toric surfaces*, preprint, written jointly with Filip Cools, Jeroen Demeyer and Alexander Lemmens
- **Chapter 7**: *A lower bound for the gonality conjecture*, preprint
- **Chapter 8**: *On graded Betti tables of curves in toric surfaces*, preprint, written jointly with Filip Cools, Jeroen Demeyer and Alexander Lemmens
- **Chapter 9**: *A combinatorial interpretation for Schreyer's tetragonal invariants*, Documenta Mathematica **20**, pp. 903-918 (2015), written jointly with Filip Cools
- **Chapter 10**: *Intrinsicness of the Newton polygon for smooth curves on $\mathbf{P}^1 \times \mathbf{P}^1$*, to appear in Revista Matemática Complutense, written jointly with Filip Cools
- **Chapter 11**: *Curves in characteristic 2 with non-trivial 2-torsion*, Advances in Mathematics of Communications **8**(4), pp. 479-495 (2014), written jointly with Marco Streng and Damiano Testa

I have also made these chapters, as well as the current preface, available in electronic form on http://math.univ-lille1.fr/~castryck/HDR/.

Acknowledgements. My Ph.D. thesis was on the development of a Kedlaya-style algorithm for computing Hasse-Weil zeta functions of nondegenerate curves over finite fields of small characteristic [CDV06], which is how I got acquainted with the world of smooth curves in toric surfaces. I wish to thank my former supervisor Jan Denef for his enthusiastic introduction to this beautiful topic, and him and my collaborator Frederik Vercauteren for their guidance over the first hurdles. The direct provocation for the currently presented work was to gain a better understanding of to which curves exactly our algorithm applies, a problem which I attacked together with John Voight. Later the research diverged in the direction of linear systems on smooth curves in toric surfaces, sparked by connections with tropical geometry [Bak08; CC12] and by recent work of Kawaguchi [Kaw16]; here, most of the results were obtained in collaboration with Filip Cools. I would like to thank John and Filip, and also my other coauthors Jeroen Demeyer, Alexander Lemmens, Marco Streng and Damiano Testa for the fruitful collaboration. My hope is that our work turns out useful for future algebraic geometers in verifying hypotheses and proving existence results, and as such contributes to Fulton's qualification of toric geometry as a remarkably fertile testing ground for general theories [Ful93, Pref.]. Finally I would like to express my gratitude to my garant Raf Cluckers, for his stimulating and genuinely positive attitude, to Pierre Dèbes, Anne Moreau, Sam Payne, Josef Schicho and Frank-Olaf Schreyer for willing to be part of the jury, and to my parents, sister, brother in law, niece, and other family and friends, for their continuous support and for the moments of much-welcomed relaxation.

1. Weakly nondegenerate curves as smooth curves in toric surfaces (Chapters 4 and 5)

To every two-dimensional lattice polygon Δ one can associate a projectively embedded toric surface X_Δ over k, obtained by taking the Zariski closure of the image of

$$\varphi_\Delta : \mathbf{T}^2 \hookrightarrow \mathbf{P}^{\#(\Delta\cap\mathbf{Z}^2)-1} : (x,y) \mapsto (x^i y^j)_{(i,j)\in\Delta\cap\mathbf{Z}^2}.$$

If $f \in k[x^{\pm1}, y^{\pm1}]$ is weakly Δ-nondegenerate then $\varphi_\Delta(U_f)$ closes along with $\varphi_\Delta(\mathbf{T}^2)$ to the smooth hyperplane section

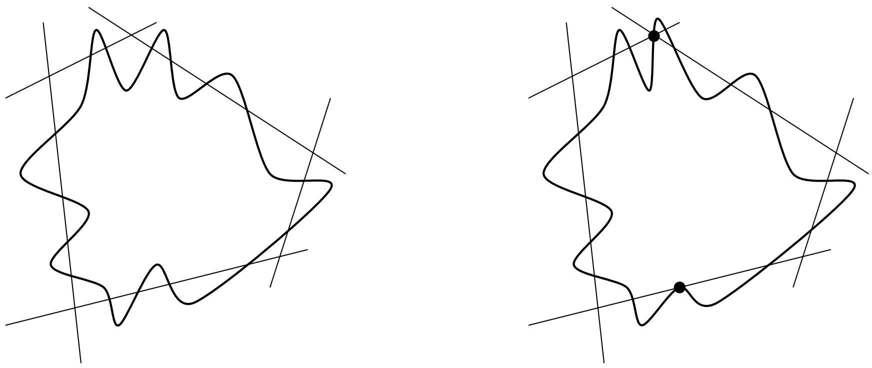
$$\sum_{(i,j)\in\Delta\cap\mathbf{Z}^2} c_{ij} X_{i,j} = 0$$

of X_Δ, where $X_{i,j}$ denotes the projective coordinate corresponding to the lattice point (i,j). Thus, weakly Δ-nondegenerate Laurent polynomials f allow for an explicit smooth complete model of U_f, which we denote by C_f.

Remark. Informally one can think of a weakly Δ-nondegenerate Laurent polynomial f as defining a smooth curve in $\mathbf{T}^2$, the singularities of whose planar completion are 'no worse' than what Δ prescribes and that therefore can be resolved using toric geometry.

When viewed as a divisor on X_Δ the curve C_f is Cartier and very ample. From the theory of toric varieties [CLS11; Ful93] it follows that C_f is linearly equivalent to a torus-invariant divisor D, to which one can naturally associate a polygon $P_D \subseteq \mathbf{R}^2$. It turns out that this polygon is precisely Δ, modulo translation over an element of $\mathbf{Z}^2$ that depends on the specific choice of D; this issue will be ignored from now on. Conversely consider a smooth complete Cartier curve C on a toric surface $X \supseteq \mathbf{T}^2$, where to avoid certain pathologies we assume that C is non-rational. Consider a torus-invariant divisor $D \sim C$ and let $P_D \subseteq \mathbf{R}^2$ be the associated polygon. Then this is automatically a two-dimensional lattice polygon and $C \cap \mathbf{T}^2$ is defined by a weakly P_D-nondegenerate Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$.

In this sense weak nondegeneracy is a geometrically more pleasing notion than nondegeneracy, which on top of smoothness requires that the curve intersects toric infinity $X \setminus \mathbf{T}^2$ transversally. For instance while $d\Sigma$-nondegenerate Laurent polynomials merely



nondegenerate

weakly nondegenerate
but not nondegenerate

correspond to smooth degree d curves in $\mathbf{P}^2$, $d\Sigma$-nondegeneracy moreover forces the curve not to pass through the coordinate points and to be non-tangent to the coordinate axes. On the other hand every weakly $d\Sigma$-nondegenerate *curve* (i.e. when considered modulo birational equivalence) is also $d\Sigma$-nondegenerate because using an automorphism of $\mathbf{P}^2$ one can enforce appropriate intersection behaviour with the coordinate axes. This trick does not always work: there exist two-dimensional lattice polygons Δ along with weakly Δ-nondegenerate curves that are genuinely *non*-Δ-nondegenerate. An example is given in Chapter 5.

Remark on the non-Cartier case. Let C be a smooth complete non-rational curve on a toric surface $X \supseteq \mathbf{T}^2$ which is not necessarily Cartier. Let D be a linearly equivalent torus-invariant divisor. Then P_D need not be a lattice polygon, which complicates matters slightly. Nevertheless C is weakly nondegenerate, as one can show that $C \cap \mathbf{T}^2$ is defined by a weakly $\mathrm{conv}(P_D \cap \mathbf{Z}^2)$-nondegenerate Laurent polynomial.

Khovanskii's proof of Baker's formula for the genus $g(U_f)$ essentially amounts to an application of the adjunction formula to the inclusion $C_f \subseteq X_\Delta$, in combination with a well-known combinatorial interpretation for the Riemann-Roch space associated to a torus-invariant divisor D (the statement involves the polygon P_D). In fact this yields much finer information than merely $g(U_f) = g(C_f) = \#(\Delta^{(1)} \cap \mathbf{Z}^2)$: it entails an explicit canonical divisor K_Δ on C_f that satisfies

$$H^0(C_f, K_\Delta) = \langle x^i y^j \rangle_{(i,j) \in \Delta^{(1)} \cap \mathbf{Z}^2},$$

where x, y are viewed as functions on C_f through φ_Δ. This leads to the following classification based on $\dim \Delta^{(1)}$:

(i) C_f is rational if and only if $\Delta^{(1)} = \emptyset$.

(ii) C_f is elliptic if and only if $\dim \Delta^{(1)} = 0$.

(iii) C_f is hyperelliptic if and only if $\dim \Delta^{(1)} = 1$.

(iv) If $\dim \Delta^{(1)} = 2$ then the canonical embedding 'factors' through $\varphi_{\Delta^{(1)}}$ and therefore the canonical image C_f^{can} is contained in the toric surface

$$X_{\Delta^{(1)}} \subseteq \mathbf{P}^{\#(\Delta^{(1)} \cap \mathbf{Z}^2)-1}.$$

Even though these four claims are easy consequences of Khovanskii's proof method, as far as we know, prior to our articles the only explicit mention of the last two statements can be found in Koelman's (otherwise unpublished) Ph.D. thesis see [Koe91, Lem. 3.1.3 and Lem. 3.2.9] and are therefore not well-known. We hope that our work helps to publicize these interesting facts.

Chapters 4 and 5 contain a number of new accompanying facts, one of which is the following geometric interpretation in case (iv) of $\Delta^{\max}$, the **maximal polygon** with respect to inclusion whose interior polygon equals $\Delta^{(1)}$ — from a combinatorial perspective the existence of such a maximum was observed by Koelman [Koe91, §2.2] and rediscovered by Haase and Schicho [HS09] (see also the next section).

Lemma 2. *If in case* (iv) *one considers a torus-invariant divisor on $X_{\Delta^{(1)}}$ that is linearly equivalent to C_f^{can}, then its associated polygon equals $\Delta^{\max}$.*

Another contribution is an explicit minimal set of generators for the ideal $\mathcal{I}(C_f^{\mathrm{can}})$ of C_f^{can}, again in case (iv). These are obtained by starting from a minimal set of generators for the ideal of $X_{\Delta^{(1)}}$, consisting of

$$\binom{g-1}{2} - 2\,\mathrm{vol}(\Delta^{(1)}) \text{ quadrics} \quad \text{and} \quad \begin{cases} 0 & \text{if } \Delta^{(1)} \not\cong \Upsilon \\ 1 & \text{if } \Delta^{(1)} \cong \Upsilon \end{cases} \text{ cubics}$$

(here $\cong$ denotes unimodular equivalence). Extending this to a minimal set of generators for the canonical ideal of C_f can be done following a so-called rolling factors recipe. This amounts to adding

$$\begin{cases} 1 \text{ quartic} & \text{if } \Delta^{(1)} \cong \Sigma, \\ g-3 \text{ cubics} & \text{if } \Delta^{(2)} = \emptyset \text{ but } \Delta^{(1)} \not\cong \Sigma, \\ \#(\Delta^{(2)} \cap \mathbf{Z}^2) \text{ quadrics} & \text{if } \Delta^{(2)} \neq \emptyset \end{cases}$$

(where $\Delta^{(2)}$ abbreviates $\Delta^{(1)(1)}$). For instance in the last case the quadrics are

$$Q_w = \sum_{(i,j) \in \Delta \cap \mathbf{Z}^2} c_{ij} X_{u_{ij}} X_{v_{ij}} \in k[\, X_{i,j} \,|\, (i,j) \in \Delta^{(1)} \cap \mathbf{Z}^2 \,]$$

where w runs through $\Delta^{(2)} \cap \mathbf{Z}^2$ and u_{ij}, v_{ij} are chosen such that $(i,j) - w = (u_{ij} - w) + (v_{ij} - w)$. For more details we refer to Chapter 4. We have implemented the resulting algorithm in Magma [BCP97], allowing for a quick computation of a minimal set of generators for the canonical ideal of any concretely given weakly nondegenerate curve of genus up to about 100. For general curves within this range this is currently an infeasible task.

2. The combinatorics of lattice polygons (Chapters 2, 3 and 5)

Even though there is always some toric geometric motivation in the background, several parts of the presented chapters are purely combinatorial. Mostly these parts are concerned with the question of how the operation $\Delta \mapsto \Delta^{(1)}$ affects certain *combinatorial invariants*, i.e. quantities that do not change when applying a unimodular transformation.

One example of such a combinatorial invariant is the number of lattice points on the boundary, in which case the question amounts to relating $\#(\partial\Delta \cap \mathbf{Z}^2)$ to $\#(\partial\Delta^{(1)} \cap \mathbf{Z}^2)$. An answer was obtained through a beautiful application of Poonen and Rodrigues-Villegas' 12 theorem, by Haase and Schicho [HS09], from whom we have copied the superscript notation $^{(1)}$. We omit a detailed statement. Indirectly their work also treats the number of lattice points in the interior $\#(\Delta^\circ \cap \mathbf{Z}^2) = \#(\Delta^{(1)} \cap \mathbf{Z}^2)$, which in view of Baker's theorem is called the **genus**. An important property of the genus is given by the following result:

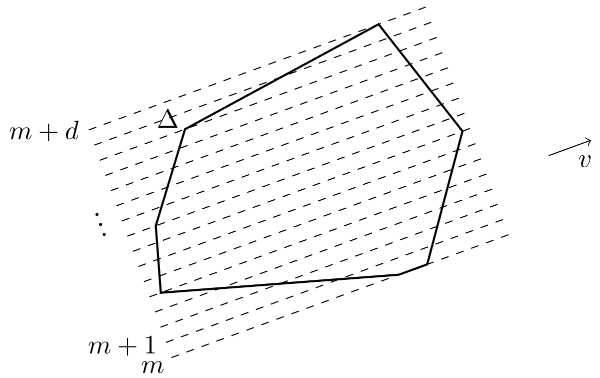
Lemma 3. *Up to unimodular equivalence the number of lattice polygons having a given genus* $g \geq 1$ *is finite.*

See e.g. [LZ91]. An alternative proof can be found in Chapter 2.

For our needs, besides the genus the most important combinatorial invariant is the **lattice width**, which is defined as follows. For each primitive vector $v = (a, b) \in \mathbf{Z}^2$ define the width $w(\Delta, v)$ to be the smallest integer d for which there exists an $m \in \mathbf{Z}$ such that

$$m \leq aj - bi \leq m + d \qquad \text{for all } (i, j) \in \Delta, \tag{1}$$

as illustrated below. This definition assumes that Δ is a non-empty lattice polygon; one



lets $w(\emptyset, v) = -1$. The lattice width $\mathrm{lw}(\Delta)$ is then defined as $\min_v w(\Delta, v)$. Alternatively, if $\Delta \neq \emptyset$ then $\mathrm{lw}(\Delta)$ is the minimal height $d \in \mathbf{Z}_{\geq 0}$ of a horizontal strip $\mathbf{R} \times [0, d]$ in which Δ can be mapped using a unimodular transformation. The question of relating $\mathrm{lw}(\Delta)$ to $\mathrm{lw}(\Delta^{(1)})$ has the following surprisingly simple answer: if Δ is two-dimensional then

$$\mathrm{lw}(\Delta) = \begin{cases} \mathrm{lw}(\Delta^{(1)}) + 3 & \text{if } \Delta \cong d\Sigma \text{ for some } d \geq 2, \\ \mathrm{lw}(\Delta^{(1)}) + 2 & \text{if not.} \end{cases} \tag{2}$$

Note that this allows one to compute $\mathrm{lw}(\Delta)$ recursively; we have implemented this in Magma.

Remark. This implies that $\mathrm{lw}(\Delta) = \mathrm{lw}(\Delta^{\max})$ whenever $\Delta^{(1)}$ is two-dimensional, except possibly if $\Delta^{\max} \cong d\Sigma$ for some $d \geq 4$.

A proof of the recursive formula (2) can be found in the paper [CC12], which is not considered part of this thesis because independently a more complete result, discussing the concrete primitive vectors v for which $\mathrm{lw}(\Delta) = w(\Delta, v)$, was obtained by Lubbes and Schicho [LS11, Thm. 13]. Note that such vectors always arise in pairs $\pm v$. One can prove [DMN12] that the number of pairs realizing the lattice width is at most 4 as soon as Δ is two-dimensional. These and some accompanying properties are reported upon in more detail in Chapter 5, which also includes a number of new facts and introduces the following refined quantity.

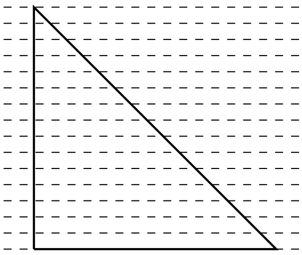
Definition 4. *The multi-set of* **width invariants** *of a lattice polygon Δ with non-empty interior associated to a primitive $v \in \mathbf{Z}^2$ is defined as*

$$E(\Delta, v) = \left\{ -1 + \#\{ (i,j) \in \Delta^{(1)} \cap \mathbf{Z}^2 \,|\, aj - bi = m + \ell \} \,\middle|\, \ell = 1, \ldots, d-1 \right\},$$

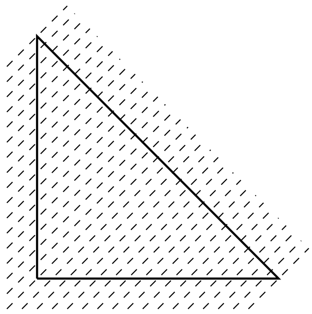
where $m, d \in \mathbf{Z}$ are the values from (1). (Its cardinality is $w(\Delta, v) - 1$, counting multiplicities.)

Using a unimodular transformation if needed one can always assume that $v = (1,0)$ and $\Delta \subseteq \mathbf{R} \times [0, w(\Delta, v)]$. In this setting the width invariants are given by the multi-set $\{E_j\}_{j=1,\ldots,w(\Delta,v)-1}$, with E_j the number of lattice points minus one that are contained in $\Delta^{(1)}$ at height j.

Example. Consider $d\Sigma$ for some $d \geq 3$. Then $\mathrm{lw}(d\Sigma) = d$ and there are three pairs of primitive vectors realizing the lattice width, namely $\pm(1,0), \pm(0,1), \pm(1,-1)$. For each of these vectors v the multi-set $E(d\Sigma, v)$ of width invariants equals $\{-1, 0, 1, 2, \ldots, d-4, d-3\}$. On the other hand one verifies that $w(d\Sigma, (1,1)) = 2d$ and $E(d\Sigma, (1,1)) =$



$w(d\Sigma, (1,0)) = d$



$w(d\Sigma, (1,1)) = 2d$

$\{-1^4, 0^4, 1^4, 2^4, \ldots, \lfloor (d-4)/2 \rfloor^4, \lfloor (d-2)/2 \rfloor^\epsilon\}$, where the superscripts denote the multiplicities and $\epsilon = 1$ or 3, depending on whether d is odd or even, respectively.

Note that the width invariants are elements of $\mathbf{Z}_{\geq -1}$. In Chapter 5 it is shown that if v realizes the lattice width and $\Delta \not\cong d\Sigma$ for any $d \geq 3$, then the width invariants associated to v are all non-negative.

Chapter 3 introduces the following generalization of the lattice width:

Definition 5. *The* **lattice size** $\mathrm{ls}_X(\Delta)$ *of a non-empty lattice polygon Δ with respect to a given set $X \subseteq \mathbf{R}^2$ having positive Jordan measure is defined as the minimal $d \in \mathbf{Z}_{\geq 0}$ such that Δ can be mapped inside dX by means of a unimodular transformation.*

For $X = \mathbf{R} \times [0,1]$ one recovers the lattice width. The chapter focuses entirely on $X = \Sigma$ and $X = \square := \square_{1,1}$. In order to state our main results relating $\mathrm{ls}_X(\Delta)$ to $\mathrm{ls}_X(\Delta^{(1)})$ it is convenient to define $\mathrm{ls}_\Sigma(\emptyset) = -2$ and $\mathrm{ls}_\square(\emptyset) = -1$.

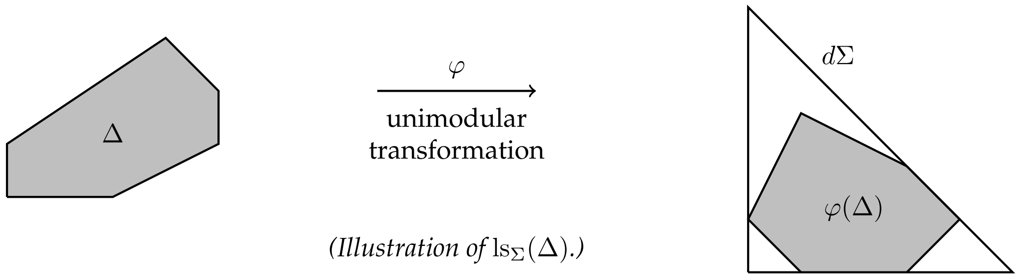
Theorem 6. *Let Δ be a two-dimensional lattice polygon. Then $\mathrm{ls}_\Sigma(\Delta) = \mathrm{ls}_\Sigma(\Delta^{(1)}) + 3$, except in the following situations:*

- $\Delta \cong \mathrm{conv}\{(0,0), (a,0), (b,1), (0,1)\}$ *where $a = b = 1$ or $2 \leq a \geq b \geq 0$, in which case* $\mathrm{ls}_\square(\Delta^{(1)}) = -2$ *while*

$$\mathrm{ls}_\Sigma(\Delta) = \begin{cases} a+1 & \text{if } a = b \\ a & \text{if } a > b. \end{cases}$$

- $\Delta \cong 2\Sigma$ *in which case* $\mathrm{ls}_\Sigma(\Delta^{(1)}) = -2$ *while* $\mathrm{ls}_\Sigma(\Delta) = 2$.
- $\Delta \cong \Delta_{4,2}$ *in which case* $\mathrm{ls}_\Sigma(\Delta^{(1)}) = 0$ *while* $\mathrm{ls}_\Sigma(\Delta) = 4$.
- $\Delta \cong \square_{a,b}$ *for* $a, b \geq 2$, *in which case* $\mathrm{ls}_\Sigma(\Delta^{(1)}) = a + b - 4$ *while* $\mathrm{ls}_\Sigma(\Delta) = a + b$.
- *There exist parallel edges* $\tau \subseteq \Delta$ *and* $\tau' \subseteq \Delta^{(1)}$ *whose supporting lines are at integral distance* 1 *of each other, such that*
$$\#(\tau \cap \mathbf{Z}^2) - \#(\tau' \cap \mathbf{Z}^2) \geq 4,$$
in which case $\mathrm{ls}_\Sigma(\Delta^{(1)}) = \#(\tau' \cap \mathbf{Z}^2)$ *and* $\mathrm{ls}_\Sigma(\Delta) = \#(\tau \cap \mathbf{Z}^2)$.

As in the case of the lattice width, this result can be converted into a recursive algorithm for computing $\mathrm{ls}_\Sigma(\Delta)$ in practice, which we have again implemented in Magma. From the



(Illustration of $\mathrm{ls}_\Sigma(\Delta)$*.)*

proof of the foregoing theorem one sees that Schicho's algorithm for simplifying rational surface parametrizations [Sch03a] works optimally.

Theorem 7. *Let* Δ *be a two-dimensional lattice polygon. Then* $\mathrm{ls}_\square(\Delta) = \mathrm{ls}_\square(\Delta^{(1)}) + 2$, *except in the following situations:*

- $\Delta \cong \mathrm{conv}\{(0,0),(a,0),(b,1),(0,1)\}$ *where* $2 \leq a \geq b \geq 0$, *in which case* $\mathrm{ls}_\square(\Delta^{(1)}) = -1$ *while* $\mathrm{ls}_\square(\Delta) = a$.
- $\Delta \cong 2\Sigma$ *in which case* $\mathrm{ls}_\square(\Delta^{(1)}) = -1$ *while* $\mathrm{ls}_\square(\Delta) = 2$.
- $\Delta \cong 3\Sigma, \Delta_{3,2}, \mathrm{conv}\{(0,0),(3,0),(2,1)(0,2)\}$ *or* $\mathrm{conv}\{(0,0),(3,0),(1,2),(0,2)\}$ *in which case* $\mathrm{ls}_\square(\Delta^{(1)}) = 0$ *while* $\mathrm{ls}_\square(\Delta) = 3$.
- $\Delta \cong \Delta_{4,2}$ *in which case* $\mathrm{ls}_\square(\Delta^{(1)}) = 0$ *while* $\mathrm{ls}_\square(\Delta) = 4$.
- *There exist parallel edges* $\tau \subseteq \Delta$ *and* $\tau' \subseteq \Delta^{(1)}$ *whose supporting lines are at integral distance* 1 *of each other, such that*
$$\#(\tau \cap \mathbf{Z}^2) - \#(\tau' \cap \mathbf{Z}^2) \geq 3,$$
in which case $\mathrm{ls}_\square(\Delta^{(1)}) = \#(\tau' \cap \mathbf{Z}^2)$ *and* $\mathrm{ls}_\square(\Delta) = \#(\tau \cap \mathbf{Z}^2)$.

Again the resulting recursive method has been implemented in Magma. The proof of the foregoing theorem has a remarkable byproduct: it turns out that the unimodular transformation mapping Δ inside $\mathrm{ls}_\square(\Delta) \cdot \square$ can be chosen such that it also maps inside $\mathbf{R} \times [0, \mathrm{lw}(\Delta)]$. As a consequence:

Corollary 8. *For each non-empty lattice polygon* Δ *the set*
$$\left\{ (a,b) \in \mathbf{Z}^2_{\geq 0} \;\middle|\; a \leq b \text{ and } \exists \Delta' \cong \Delta \text{ with } \Delta' \subseteq [0,a] \times [0,b] \right\}$$
has a minimum with respect to the product order on $\mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$, *namely* $(\mathrm{lw}(\Delta), \mathrm{ls}_\square(\Delta))$.

We conclude by stressing that the map $\Delta \mapsto \Delta^{(1)}$ is not surjective. In fact for a two-dimensional lattice polygon Γ to be of the form $\Delta^{(1)}$ for some larger lattice polygon Δ is a rather restrictive property. The following criterion was proved by Haase and Schicho [HS09]. Each edge $\tau \subseteq \Gamma$ lies on the boundary of a unique half-plane $a_\tau X + b_\tau Y \leq c_\tau$ containing Γ, where $a_\tau, b_\tau, c_\tau \in \mathbf{Z}$ are chosen to satisfy $\gcd(a_\tau, b_\tau) = 1$. Consider the polygon

$$\Gamma^{(-1)} = \bigcap_\tau (\text{half-plane } a_\tau X + b_\tau Y \leq c_\tau + 1)\,,$$

said to be obtained from Γ by moving out the edges. Then $\Gamma = \Delta^{(1)}$ for some lattice polygon Δ if and only if $\Gamma^{(-1)}$ is a lattice polygon. If this is the case then one can simply let $\Delta = \Gamma^{(-1)}$, and this is the maximal possible choice with respect to inclusion. In other words if Δ is a lattice polygon having a two-dimensional interior then $\Delta^{\max} = \Delta^{(1)(-1)}$.

For any lattice polygon Δ, a repeated application of $\Delta \mapsto \Delta^{(1)}$ eventually leads to a lattice polygon whose interior is at most one-dimensional. Such polygons have been classified explicitly by Koelman [Koe91, §4]. Conversely, starting from these basic cases one can algorithmically produce all lattice polygons up to a given genus by repeatedly applying $\Delta \mapsto \Delta^{(-1)}$, verifying Haase and Schicho's criterion and making local tweaks (clipping off vertices). The details can be found in Chapter 2, which comes along with a Magma implementation by means of which we have produced a list containing exactly one representative within each unimodular equivalence class of lattice polygons of genus $1 \leq g \leq 30$. This list is useful for testing hypotheses and detecting patterns; we have mainly applied this to the study of syzygies of toric surfaces (and of smooth curves therein) in Chapters 6 and 8. But there are also some purely combinatorial consequences which seem interesting in their own right. For instance prior to our work the concrete number of equivalence classes of lattice polygons of genus $g \geq 1$ was unknown for g as small as 3, even though asymptotically for $g \to \infty$ it was shown to be $O(\exp(g^{1/3}))$ by Bárány [BT04]. Another consequence (albeit slightly indirect; see Chapter 2 for the details) is:

Lemma 9. *The minimal genus of a lattice* 15*-gon is* 45.

This fills in the smallest open entry of a list whose study began with Arkinstall [Ark80]. Recently our data set was used to give tight bounds on the generalized Helly numbers of $\mathbf{Z}^2$; see [Ave+15].

3. The number of moduli (Chapter 1)

The generic Laurent polynomial that is supported on a given two-dimensional lattice polygon $\Delta \subseteq \mathbf{R}^2$ is Δ-nondegenerate, and weakly Δ-nondegenerate in particular. As a consequence, if the man in the street would be asked to scribble down a random curve, the outcome is likely to be weakly nondegenerate, and most curves that can be found *in the wild* are indeed of this kind, including all hyperelliptic curves and smooth curves in $\mathbf{P}^2$ as we have seen above, but also all trigonal curves, $C_{a,b}$ curves, and several more well-studied families. For a moment this might tempt one to conclude that the generic curve, in the proper moduli-theoretic sense, is weakly non-degenerate. But a second thought quickly reveals that this is far from true. Some obstructions are:

- The moduli space $\mathcal{M}_g$ of curves of genus g is not unirational for $g \geq 22$ [Far09].
- The gonality of a weakly Δ-nondegenerate genus g curve is bounded by $\text{lw}(\Delta)$, which is $O(\sqrt{g})$ by [FTM74], while the general curve has gonality $\lfloor (g+3)/2 \rfloor$ by Brill-Noether theory. This was recently elaborated in detail by Smith [Smi15] who

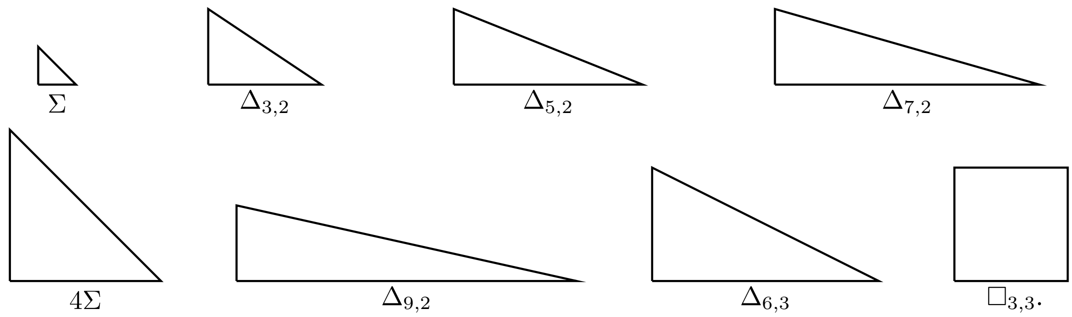
proved that weakly nondegenerate curves cannot be Brill-Noether general from $g \geq 7$ onwards.

- The canonical ideal of a weakly nondegenerate curve contains (many) quadratic binomials, i.e. quadrics of rank 3 or 4.

The latter obstruction seems best-suited for proving that a certain concretely given curve C/k is *not* weakly nondegenerate.

In Chapter 1 we try to obtain a more precise understanding of which curves are weakly nondegenerate. For curves of genus at most four, we prove:

Theorem 10. *Every curve C/k of genus $g \leq 4$ is weakly Δ-nondegenerate for exactly one choice of Δ among the lattice polygons listed below.*



(The polygons referred to in the statement of Theorem 10.)

The theorem remains true upon replacement of 'weakly Δ-nondegenerate' by 'Δ-nondegenerate'. Also, slightly modified versions hold over fields that are not necessarily algebraically closed. For instance over finite fields the theorem is true except when C is of genus 4 and canonically embeds into an elliptic quadric in $\mathbf{P}^3$; see also [CV10].

The main result of Chapter 1 is a determination of the number of moduli of the family of weakly nondegenerate curves, through a parameter count that builds on Haase and Schicho's aforementioned work [HS09] and a combinatorial description of the automorphism group of X_Δ due to Bruns and Gubeladze [BG09]. For a two-dimensional lattice polygon Δ, we denote by $\mathcal{M}_\Delta$ the Zariski closure of the locus inside $\mathcal{M}_g$ of all weakly Δ-nondegenerate curves; these spaces had already been introduced and studied by Koelman [Koe91, §2]. For each $g \geq 1$ let

$$\mathcal{M}_g^{\text{wnd}} = \bigcup_{\substack{\Delta \text{ for which} \\ \sharp(\Delta^{(1)} \cap \mathbf{Z}^2) = g}} \mathcal{M}_\Delta,$$

which in view of Lemma 3 is a finite union because unimodularly equivalent lattice polygons give rise to the same curves. We show:

Theorem 11. *One has*

$$\begin{cases} \dim \mathcal{M}_1^{\text{wnd}} = 1, \\ \dim \mathcal{M}_2^{\text{wnd}} = 3, \\ \dim \mathcal{M}_3^{\text{wnd}} = 6, \\ \dim \mathcal{M}_7^{\text{wnd}} = 16, \\ \dim \mathcal{M}_g^{\text{wnd}} = 2g+1 \quad \textit{if } g \geq 4 \textit{ and } g \neq 7. \end{cases}$$

In particular from genus five on the generic curve is not weakly nondegenerate (let alone nondegenerate). For $g \geq 4$ a top-dimensional subvariety of $\mathcal{M}_g^{\text{wnd}}$ is given by the trigonal locus $\mathcal{M}_g^{\text{tri}}$, except when $g = 7$, where the trigonal curves are beaten by the trinodal plane sextics. Recently Brodsky, Joswig, Morrison and Sturmfels used a similar approach to obtain the same moduli count for tropical plane curves [Bro+15].

4. Gonality, Clifford index, and related invariants (Chapters 3 and 5)

This section adds a number of entries to the geometry-combinatorics dictionary for weakly nondegenerate curves, related to linear systems. The main reference for the results presented below is Chapter 5. The most important new entry is the **gonality**, which is defined as the minimal possible degree of a non-constant rational map to $\mathbf{P}^1$:

Theorem 12. *Let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. Then the gonality of U_f equals $\operatorname{lw}(\Delta^{(1)})+2$, unless $\Delta^{(1)} \cong \Upsilon$ in which case it is $\operatorname{lw}(\Delta^{(1)}) + 1$.*

This theorem arises as a consequence of a stronger result. Let f be a weakly Δ-nondegenerate Laurent polynomial and let $v = (a, b) \in \mathbf{Z}^2$ be a primitive vector. We define the **combinatorial pencil** g_v on C_f associated to v as the trace of the linear system on X_Δ swept out by $\mathbf{T}^2 \to \mathbf{T}^1 : (x, y) \mapsto x^a y^b$. Notice that $g_v = g_{-v}$ is of degree $w(\Delta, v)$, in other words it concerns a $g^1_{w(\Delta,v)}$.

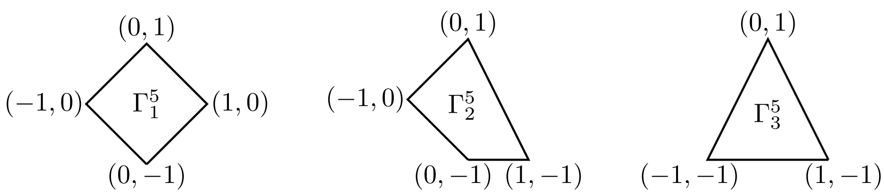
Remark. In almost all cases g_v equals the basepoint free pencil associated to the map $U_f \to \mathbf{T}^1 : (x, y) \mapsto x^a y^b$. However when $\Delta(f) \subsetneq \Delta$, in certain cases this needs to be extended by basepoints.

Our strengthening of Theorem 12 reads as follows:

Theorem 13. *Let Δ be a two-dimensional lattice polygon such that $\Delta^{(1)}$ is not unimodularly equivalent to any of the following:*

$$\emptyset, \quad (d-3)\Sigma \text{ (for some } d \geq 3), \quad \Upsilon, \quad 2\Upsilon, \quad \Gamma^5_1, \quad \Gamma^5_2, \quad \Gamma^5_3.$$

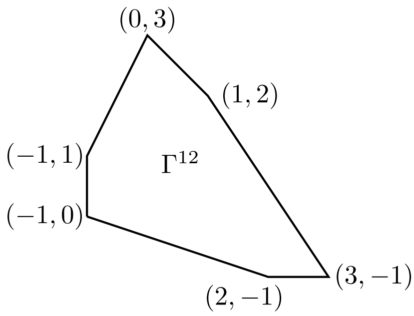
If $\operatorname{char} k > 0$ then we also exclude Γ^{12}. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. Then every linear pencil on C_f which realizes the gonality is combinatorial.



(Polygons excluded in the statement of Theorem 13, corresponding to curves of genus 5.)

One byproduct of the above theorem is that besides the gonality itself, one also knows the **number of gonality pencils** by merely looking at the Newton polygon, except possibly when $\Delta^{(1)} \cong \Upsilon$ in which case U_f is always tetragonal but the number of gonality pencils depends on the concrete choice of f (it is either 1 or 2), and except possibly when $\operatorname{char} k > 0$ and $\Delta^{(1)} \cong \Gamma^{12}$ where the situation is not fully understood. If $\Delta^{(1)} = \emptyset$ then there is a unique gonality pencil. In the other exceptional cases $\Delta^{(1)} \cong (d-3)\Sigma, 2\Upsilon, \Gamma^5_i$ the number of gonality pencils can be shown to be infinite.

Our main proof ingredient is a result due to Serrano [Ser87] which given a curve C inside some surface X, provides sufficient conditions under which a morphism $C \to \mathbf{P}^1$



(Polygon excluded by Theorem 13 in positive characteristic, corresponding to curves of genus 12.)

can be extended to a morphism $X \to \mathbf{P}^1$. We stress that this approach, and as a matter of fact the entire statement of Theorem 13, is due to Kawaguchi [Kaw16], modulo two relaxations:

- Kawaguchi made the technical assumption that U_f is not birationally equivalent to a smooth plane curve of degree $d \geq 5$. We got rid of this condition, essentially by invoking the formula (2) at the proof step where this assumption was used.

- Both Kawaguchi and we proved these statements subject to $\operatorname{char} k = 0$. However one can obtain the same results in positive characteristic by using [Ser87, Rmk. 3.12] and the fact that every morphism $C_f \to \mathbf{P}^1$ decomposes into a purely inseparable part and a separable part
$$C_f \to C_f^{\text{Frob}} \to \mathbf{P}^1,$$
along with the observation that Frobenius preserves weak nondegeneracy. This approach does not work when $\Delta^{(1)} \cong \Gamma^{12}$, in which case the extra condition mentioned in [Ser87, Rmk. 3.12] is violated. Therefore Γ^{12} pops up as a new exception, although this may well be just a proof artefact (unlike the other exclusions, which are really needed). More details will be included in the forthcoming version of [CT].

In addition, our database of polygons having small genus allowed us to skip a large and combinatorially tedious part of Kawaguchi's proof.

Using the same techniques we can deduce an analogous result for **near-gonal pencils**, by which we mean base-point free linear pencils of degree $\gamma + 1$, where γ is the gonality of U_f:

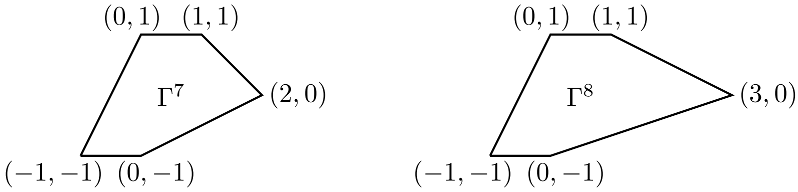
Theorem 14. *Let Δ be a two-dimensional lattice polygon such that*

$$\operatorname{ls}_\Sigma(\Delta^{(1)}) \geq \operatorname{lw}(\Delta^{(1)}) + 2$$

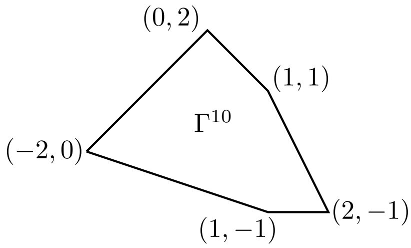
and such that $\Delta^{(1)} \not\cong 2\Upsilon, 3\Upsilon, \Gamma^7, \Gamma^8$. If $\operatorname{char} k > 0$ then we also exclude Γ^{10}. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. Then every near-gonal pencil on C_f is combinatorial.

Again the exclusion of Γ^{10} might be a proof artefact; as explained in Chapter 5 the other exclusions are necessary. Also, one can verify that the list of excluded polygons is a strict extension of its counterpart from Theorem 13.

In principle it should be possible to obtain similar statements for basepoint free $g^1_{\gamma+n}$'s with $n = 2, 3, \ldots$, but we expect the proof to become increasingly case-distinctive and the number of excluded polygons to grow. Nevertheless for small n it might be worth the try, in order to gain some feeling on how $\dim W^1_{\gamma+n}$ can grow with n, a question which has



(Polygons excluded in the statement of Theorem 14, corresponding to curves of genus 7 and 8.)



(Polygon excluded by Theorem 14 in positive characteristic, corresponding to curves of genus 10.)

sparked much interest in view of connections with Green's canonical syzygy conjecture, through Aprodu's linear growth condition [Apr05].

Another entry to the dictionary is given by the **scrollar invariants** associated to a combinatorial pencil (e.g. any gonality pencil in the case of a polygon that is non-exceptional for Theorem 13). The scrollar invariants associated to a linear pencil g^1_d on a non-hyperelliptic genus g curve C/k are defined as follows. View the g^1_d as a 1-dimensional family of effective divisors D on the canonical model $C_f^{\text{can}} \subseteq \mathbf{P}^{g-1}$ and let $S \subseteq \mathbf{P}^{g-1}$ be the ruled variety obtained by taking the union of all linear spans $\langle D \rangle$. A theorem by Eisenbud and Harris [EH87, Thm. 2] states that S is a rational normal scroll. The scrollar invariants associated to g^1_d are defined as the multi-set of invariants (= the degrees of the spanning rational normal curves) of this scroll. If our g^1_d is complete and basepoint free then the $\langle D \rangle$'s are planes of dimension $d-2$, and S is of dimension $d-1$. In this case the scrollar invariants $0 \leq e_0 \leq e_1 \leq \ldots \leq e_{d-1}$ satisfy $e_{d-1} \leq (2g-2)/d$.

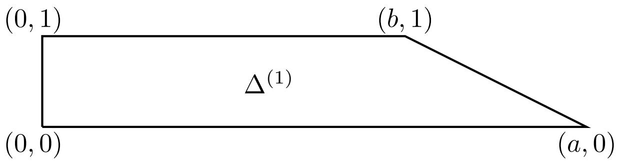
Theorem 15. *Let Δ be a lattice polygon such that $\Delta^{(1)}$ is two-dimensional. Let $v \in \mathbf{Z}^2$ be a primitive vector and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. Then the multi-set of scrollar invariants of C_f with respect to g_v equals the multi-set of non-negative width invariants of Δ with respect to v.*

The proof can be found in Chapter 5 and has the following corollary:

Corollary 16. *The rank of the complete linear system spanned by g_v equals the number of negative width invariants (counting multiplicities) plus one. In particular g_v is complete if and only if all width invariants are non-negative.*

Example. Consider $d\Sigma$ for some $d \geq 4$ along with the primitive vector $v = (1,0)$. Recall from Section 2 that $E(d\Sigma, v) = \{-1, 0, 1, 2, \ldots, d-3\}$. By Theorem 15 the scrollar invariants associated to $g_{(1,0)}$ are $\{0, 1, 2, \ldots, d-3\}$. By Corollary 16 our $g_{(1,0)}$ is a subsystem of a g^2_d, hence it is not complete. But this just confirms a well-known fact, because C_f is a smooth projective degree d curve in $\mathbf{P}^2$ and $g_{(1,0)}$ is cut out by the pencil of lines through a fixed point outside the curve. By varying the point one obtains the g^2_d.

Example (Maroni invariants). Consider a lattice polygon Δ with $\text{lw}(\Delta) = 3$ and $\Delta \not\cong 3\Sigma$. Then up to unimodular equivalence $\Delta^{(1)}$ is of the form below, for certain integers $1 \leq$



$a \geq b \geq 0$. Notice that weakly Δ-nondegenerate Laurent polynomials $f \in k[x^{\pm 1}, y^{\pm 1}]$ give rise to trigonal curves C_f of genus $g = \#(\Delta^{(1)} \cap \mathbf{Z}^2) = a + b + 2$. Then $g_{(1,0)}$ is a gonality pencil and the corresponding scrollar invariants are seen to be $\{a, b\}$; if $a = 1$ then there may exist other combinatorial gonality pencils but the associated scrollar invariants are the same. The numbers a and b are classical invariants called the **Maroni invariants**[1] of C_f. Every trigonal curve arises as a weakly Δ-nondegenerate curve with Δ a lattice polygon of the above form. A fun fact is that the well-known bound $a \leq (2g-2)/3$ which is usually proven through the Riemann-Roch theorem, can also be obtained in a purely combinatorial way, using Haase and Schicho's criterion for $\Delta^{(1)}$ to be an interior polygon.

Another observation is that Theorem 13 can be combined with results of Coppens and Martens [CM91] to obtain combinatorial interpretations for the **Clifford index** and the **Clifford dimension**, which are defined for curves of genus $g \geq 4$ only: the Clifford index is

$$\min\{\, d - 2r \mid C_f \text{ carries a divisor } D \text{ with } |D| = g^r_d \text{ and } h^0(C_f, D), h^0(C_f, K_\Delta - D) \geq 2 \,\}$$

which is a non-negative integer due to Clifford's theorem. The Clifford dimension is the smallest r for which the minimum is realized; this concept was introduced in [Eis+89]. Coppens and Martens assume $\operatorname{char} k = 0$; we inherit this condition since it is not clear to us how to circumvent it.

Theorem 17. *Let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. Assume that $\#(\Delta^{(1)} \cap \mathbf{Z}^2) \geq 4$ and $\operatorname{char} k = 0$. Then:*

- *The Clifford index of U_f equals $\operatorname{lw}(\Delta^{(1)})$, unless $\Delta^{(1)} \cong (d-3)\Sigma$ for some $d \geq 5$, $\Delta^{(1)} \cong \Upsilon$, or $\Delta^{(1)} \cong 2\Upsilon$, in which cases it is $\operatorname{lw}(\Delta^{(1)}) - 1$.*
- *The Clifford dimension of U_f equals 2 if $\Delta^{(1)} \cong (d-3)\Sigma$ for some $d \geq 5$, it equals 3 if $\Delta^{(1)} \cong 2\Upsilon$, and it is 1 in all other cases.*

The possibility of combining Theorem 13 with [CM91] was already mentioned by Kawaguchi [Kaw16]. However we recall Kawaguchi's assumption that U_f is not birationally equivalent to a smooth plane curve of degree $d \geq 5$, or in other words that the Clifford dimension is different from 2. So by getting rid of this condition we end up with a more complete and pleasing statement. The main ingredient taken from [CM91] is that there always exist infinitely many gonality pencils as soon as the Clifford dimension is at least 2. Because the number of combinatorial pencils is necessarily finite, through Theorem 13 this reduces one's task to analyzing the exceptions $\Delta^{(1)} \cong \emptyset, (d-3)\Sigma, \Upsilon, 2\Upsilon, \Gamma^5_1, \Gamma^5_2, \Gamma^5_3$. For smooth plane curves Coppens and Martens' result is classical and holds in any characteristic [Har86], allowing one to obtain the following corollary:

Corollary 18. *Let Δ be a lattice polygon with non-empty interior. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be weakly Δ-nondegenerate and assume that U_f is birationally equivalent to a smooth projective curve in $\mathbf{P}^2$, say of degree $d \geq 3$. Then $\Delta^{(1)} \cong (d-3)\Sigma$.*

[1] The existing literature is ambiguous at this point: sometimes one talks about a single Maroni invariant, in which case one means either b or $a - b$.

Therefore, in the case of smooth plane curves, one can view $\Delta^{(1)}$ in some sense as a geometric invariant. We refer to this property as **intrinsicness** of the interior polygon and will state a few more results of this kind in Section 8.

Remark. Stated more geometrically, this means that if a toric surface X contains a smooth projective curve C that is abstractly isomorphic to a smooth plane projective curve, then there exists a *toric* blow-down $\pi : X \to \mathbf{P}^2$ such that $\pi|_C : C \to \mathbf{P}^2$ is an embedding.

A consequence of Theorems 12 and 17 is that the gonality and (if $\mathrm{char}\, k = 0$) the Clifford index and dimension of U_f depend on Δ only, rather than on the specific choice of our weakly Δ-nondegenerate Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$. This is an a priori non-trivial fact that can be rephrased as constancy among the smooth curves in linear systems of curves on toric surfaces. The existing literature contains other results of this type, which are usually stated in characteristic zero only. For instance recent work of Lelli-Chiesa proves constancy of the gonality and the Clifford index for curves in certain linear systems on other types of rational surfaces [LC13]. An important theorem by Green and Lazarsfeld states that constancy of the Clifford index holds in linear systems on K3 surfaces [GL87], although here constancy of the gonality is not necessarily true. In the next section we will state a constancy result for the entire canonical graded Betti table (which subject to Green's canonical syzygy conjecture is a vast generalization of the Clifford index).

We end this section with a brief discussion (more details to be found in Chapter 3) of two other invariants that we have put to a combinatorial analysis, albeit with less conclusive results:

- The minimal degree $s_2(U_f)$ of a possibly singular curve in $\mathbf{P}^2$ that is birationally equivalent to U_f; equivalently this asks for the minimal degree of a simple linear system of rank 2.
- The minimum $s_{1,1}(U_f)$ of

  $$\left\{ (a,b) \in \mathbf{Z}^2_{\geq 0} \,\middle|\, a \leq b \text{ and } \exists\, C \subseteq \mathbf{P}^1 \times \mathbf{P}^1 \text{ of bidegree } (a,b) \text{ with } C \simeq U_f \right\} \qquad (3)$$

  where $\simeq$ denotes birational equivalence. The minimum is taken with respect to the lexicographic order on $\mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$ (but see the 'open question' at the end of this section).

Unfortunately we can only provide upper bounds, and leave it as an unsolved problem whether these statements are sharp. In particular we do not know whether the quantities $s_2(U_f)$ and $s_{1,1}(U_f)$ are independent of the concrete choice of $f \in k[x^{\pm 1}, y^{\pm 1}]$.

Theorem 19. *Let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. Then $s_2(U_f) \leq \mathrm{ls}_\Sigma(\Delta^{(1)}) + 3$. If $\Delta^{(1)} \cong d\Upsilon$ for some $d \geq 1$ then the sharper bound $s_2(U_f) \leq \mathrm{ls}_\Sigma(\Delta^{(1)}) + 2$ applies.*

Theorem 20. *Let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. Then $s_{1,1}(U_f) \leq (\mathrm{lw}(\Delta^{(1)}) + 2, \mathrm{ls}_\Box(\Delta^{(1)}) + 2)$. If $\Delta^{(1)} \cong \Upsilon$ then the sharper bound $s_{1,1}(U_f) \leq (3,4)$ applies.*

Remark that by Theorem 12 the first components of the upper bounds stated in Theorem 20 are equal to the gonality. Therefore this part of the statement is optimal and one sees that the bounds necessarily hold with respect to the product order on $\mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$. In particular, if it is indeed true that Theorem 20 is always sharp, then the set (3) always admits a minimum with respect to the product order. Please compare this to Corollary 8, which can be viewed as a combinatorial analogue of this statement.

Open question. As an even wilder shot in the dark, we wonder whether it is true for *every* algebraic curve C/k (i.e. not necessarily weakly nondegenerate) that the set of bidegrees (a, b) with $a \leq b$ of curves in $\mathbf{P}^1 \times \mathbf{P}^1$ that are birationally equivalent to C admits a minimum with respect to the product order on $\mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$.

5. Canonical graded Betti numbers (Chapters 6, 7 and 8)

Let Δ be a lattice polygon with two-dimensional interior and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. Recall from Section 1 that $C_f \subseteq X_\Delta$ is non-hyperelliptic and that its canonical model satisfies

$$C_f^{\text{can}} \subseteq X_{\Delta^{(1)}} \subseteq \mathbf{P}^{g-1} = \text{Proj}\, S_{\Delta^{(1)}} \tag{4}$$

where $S_{\Delta^{(1)}} = k[\, X_{i,j} \,|\, (i,j) \in \Delta^{(1)} \cap \mathbf{Z}^2 \,]$ and $g = \#(\Delta^{(1)} \cap \mathbf{Z}^2)$ denotes the genus of C_f. In this section we report on a combinatorial analysis of the Betti numbers β_{ij} appearing in a minimal free resolution of the homogeneous coordinate ring of C_f^{can} as a graded $S_{\Delta^{(1)}}$-module:

$$\cdots \to \bigoplus_{q \geq 2} S_{\Delta^{(1)}}(-q)^{\beta_{2,q}} \to \bigoplus_{q \geq 1} S_{\Delta^{(1)}}(-q)^{\beta_{1,q}} \to \bigoplus_{q \geq 0} S_{\Delta^{(1)}}(-q)^{\beta_{0,q}} \to S_{\Delta^{(1)}} / \mathcal{I}(C_f^{\text{can}}) \to 0.$$

These numbers are usually gathered in what is called the **canonical graded Betti table** of C_f, by writing $\beta_{p,p+q}$ in the pth column and the qth row. Alternatively, and often more conveniently, this entry equals the dimension of the Koszul cohomology space $K_{p,q}(C_f, K_\Delta)$. The canonical graded Betti table is known to be of the form

	0	1	2	3	...	$g-4$	$g-3$	$g-2$
0	1	0	0	0	...	0	0	0
1	0	a_1	a_2	a_3	...	a_{g-4}	a_{g-3}	0
2	0	a_{g-3}	a_{g-4}	a_{g-5}	...	a_2	a_1	0
3	0	0	0	0	...	0	0	1,

(5)

where omitted entries are understood to be zero. If one assumes Green's canonical syzygy conjecture [Gre84], the settlement of which is arguably the most important unsolved problem concerning linear series on algebraic curves, then the canonical graded Betti table is a vast generalization of the Clifford index. Indeed, Green's conjecture predicts that the latter is equal to $\min\{\, \ell \,|\, a_{g-\ell} \neq 0 \,\} - 2$.

Remark. Assume that $\text{char}\, k = 0$. If $X_{\Delta^{(1)}}$ carries an anticanonical pencil, or equivalently if the polygon P_{-K} associated to an anticanonical torus-invariant divisor $-K$ on $X_{\Delta^{(1)}}$ contains at least two lattice points, then one can invoke a result of Lelli-Chiesa [LC13] to settle Green's conjecture for all weakly Δ-nondegenerate curves. This includes the cases where $X_{\Delta^{(1)}}$ is Gorenstein and weak Fano, which are discussed further down. The details of these claims are explained in Chapter 8.

Remark. It is known that Green's conjecture may fail over fields of very small positive characteristic [Sch03b], but we do not know of any weakly nondegenerate counterexamples.

Having a combinatorial description of the Clifford index at hand (at least if $\operatorname{char} k = 0$, see Theorem 17), it is a natural step to look for a similar description of the entire canonical Betti table. At this moment this seems to be an infeasible task, both from a combinatorial and a geometric perspective. In view of (4) we hope for an explicit relationship with the graded Betti table of $X_{\Delta^{(1)}}$, which is of the form

$$\begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & \dots & g-4 & g-3 \\ \hline 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & b_1 & b_2 & b_3 & \dots & b_{g-4} & c_{g-3} \\ 2 & 0 & c_{g-3} & c_{g-4} & c_{g-5} & \dots & c_2 & c_1. \end{array} \tag{6}$$

Concretely, we distill the following three research questions, each of which we leave unanswered in their general form, although we can offer several partial results:

(i) **What would such an explicit relationship look like?**

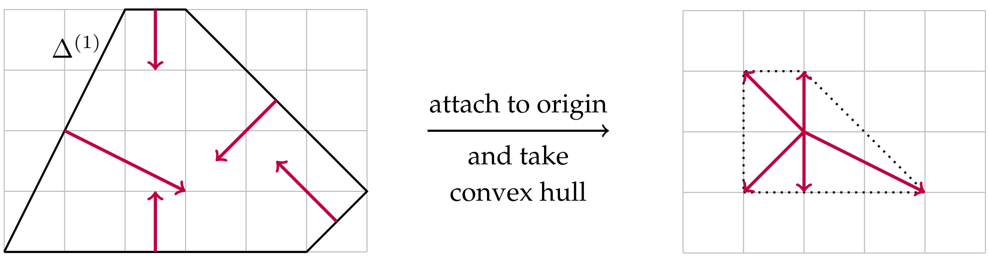
The inclusion (4) gives rise to an exact sequence

$$0 \longrightarrow b_\ell \longrightarrow a_\ell \longrightarrow c_\ell \xrightarrow{\mu_{\ell,f}} c_{g-1-\ell} \longrightarrow a_{g-1-\ell} \longrightarrow b_{g-1-\ell} \longrightarrow 0 \tag{7}$$

for each value of $\ell = 1, 2, \dots, g-2$. Here we abusingly write the dimensions of the Koszul cohomology spaces, rather than the spaces themselves, and it is understood that $a_{g-2} = b_{g-2} = c_{g-2} = 0$. The map $\mu_{\ell,f}$ is a morphism between two cohomology spaces associated to $X_{\Delta^{(1)}}$ that is induced by multiplication by f; we refer to Chapter 8 for the precise construction. This shows that $a_\ell = b_\ell + c_\ell - \dim \operatorname{im} \mu_{\ell,f}$, and the question reduces to a determination of the last term. Our main theorem is that $\mu_{\ell,f} = 0$ in the cases where $X_{\Delta^{(1)}}$ is Gorenstein and weak Fano.

Theorem 21. *Let Δ be a lattice polygon with two-dimensional interior. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial and let $g = \#(\Delta^{(1)} \cap \mathbf{Z}^2)$. Denote by $a_1, a_2, \dots, a_{g-3}$ the canonical graded Betti numbers of C_f as in (5), and similarly let $b_1, c_1, b_2, c_2, \dots, b_{g-3}, c_{g-3}$ be the graded Betti numbers of $X_{\Delta^{(1)}}$ as in (6). If $X_{\Delta^{(1)}}$ is Gorenstein and weak Fano then for all $\ell = 1, 2, \dots, g-3$ we have $a_\ell = b_\ell + c_\ell$.*

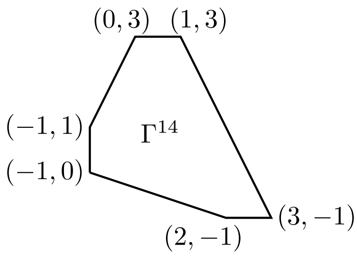
Being Gorenstein and weak Fano has an easy combinatorial interpretation: it means that the convex hull of the primitive inward pointing normal vectors to the edges is a reflexive polygon (= a lattice polygon of genus one). An example is depicted below. This is a rather strong condition, but we note that Theorem 21 applies to



(Illustration of the Gorenstein weak Fano property from the combinatorial viewpoint.)

most of our introductory examples, including the cases where $\Delta \cong d\Sigma$ for some $d \geq 4$, where $\Delta \cong d\Upsilon$ for some $d \geq 2$, where $\Delta \cong \square_{a,b}$ for some $a, b \geq 3$, and so on. Moreover, experimentally we observe that $\mu_{\ell,f} = 0$ much more frequently than under the Gorenstein weak Fano assumption. Of course, an obvious reason could be that $c_\ell = 0$ or $c_{g-1-\ell} = 0$: by Theorem 24 below we perfectly understand when this happens. But often $\mu_{\ell,f} = 0$ for reasons we do not know.

Example. In this example we let k be (the algebraic closure of) the finite field $\mathbf{F}_{10007}$; this is mainly for computational efficiency, we expect the same analysis to apply over $\mathbf{C}$. The toric surface $X_{\Delta^{(1)}}$ over k corresponding to the interior polygon $\Delta^{(1)}$ shown below is Gorenstein but not weak Fano. One computationally verifies that



the corresponding graded Betti table is

	0	1	2	3	4	5	6	7	8	9	10	11
0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	59	363	1100	2013	2310	1525	343	24	0	0	0
2	0	0	0	0	0	7	112	574	561	265	66	7,

while the canonical graded Betti table of C_f for an aimlessly chosen[2] Laurent polynomial $f \in \mathbf{F}_{10007}[x^{\pm 1}, y^{\pm 1}]$ that is weakly $\Delta^{\max}$-nondegenerate when considered over k was found to be

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	66	429	1365	2574	2884	1637	350	24	0	0	0	0
2	0	0	0	0	24	350	1637	2884	2574	1365	429	66	0
3	0	0	0	0	0	0	0	0	0	0	0	0	1.

(The computation started from our explicit minimal set of generators for the canonical ideal; see Section 2.) Thus for $\ell = 6$ the exact sequence (7) reads

$$0 \longrightarrow 1525 \longrightarrow 1637 \longrightarrow 112 \xrightarrow{\mu_{6,f}} 7 \longrightarrow 350 \longrightarrow 343 \longrightarrow 0,$$

implying that $\mu_{6,f} = 0$, and similarly one sees that $\mu_{7,f} = 0$. We do not understand why, as this is not explained by Theorem 21. In all other cases $\mu_{\ell,f} = 0$ because either $c_\ell = 0$ or $c_{g-1-\ell} = 0$.

[2] We equipped the lattice points on the boundary of $\Delta^{\max}$ with the prime coefficients $2, 3, 5, 7, 11, 13, 17, 19$, starting at the left-most vertex and proceeding counterclockwise. The coefficients corresponding to the interior lattice points were chosen 0 in view of Lemma 23 below.

Example. The same computer experiment, when applied to the polygon Γ^{12} from Section 4, respectively resulted in the graded Betti tables

	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	0	45	231	550	693	399	69	0	0	0	0
2	0	0	0	0	69	399	693	550	231	45	0
3	0	0	0	0	0	0	0	0	0	0	1

and

	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	0	39	186	414	504	295	69	0	0	0
2	0	0	0	0	1	105	189	136	45	6.

Here one sees that the exact sequence (7) for $\ell = 5$ reads:

$$0 \longrightarrow 295 \longrightarrow 399 \longrightarrow 105 \xrightarrow{\mu_{5,f}} 1 \longrightarrow 69 \longrightarrow 69 \longrightarrow 0.$$

So $\mu_{5,f}$ is *not* trivial in this case, but rather surjective onto its one-dimensional codomain. In fact, $\Delta^{(1)}$ is the only interior polygon for which we have observed deviating behavior with respect to the formula $a_\ell = b_\ell + c_\ell$, although we expect more exceptions to pop up beyond the range of polygons that we have computed (if not then Green's conjecture would be violated, as explained at the end of this section).

(ii) **Is it true at all that the canonical graded Betti table of C_f only depends on the graded Betti table of $X_{\Delta^{(1)}}$, rather than on the specific choice of f?**

In other words, do we have constancy in the sense discussed in Section 4? It is clear from Theorem 21 that the answer is yes if $X_{\Delta^{(1)}}$ is Gorenstein and weak Fano:

Corollary 22. *Let Δ be a lattice polygon with two-dimensional interior and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. If $X_{\Delta^{(1)}}$ is Gorenstein and weak Fano then the canonical graded Betti numbers of C_f do not depend on the specific choice of f.*

For example, this implies that the canonical graded Betti table of a smooth plane projective degree $d \geq 4$ curve depends on d only. For general lattice polygons Δ we can show that only the coefficients that are supported on the boundary potentially matter:

Lemma 23. *Let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. Then the canonical graded Betti table of C_f at most depends on the coefficients of f that are supported on $\partial\Delta \cap \mathbf{Z}^2$.*

See again Chapter 8 for a proof. As a modest new application of this, we obtain constancy of the canonical graded Betti table for triangles whose only lattice points on the boundary are its vertices. Indeed, using the action of $\mathbf{T}^2$ the three corresponding coefficients can always be set to 1.

Remark. If the answer to (ii) is no, then question (i) still makes sense by restricting to *sufficiently generic* weakly Δ-nondegenerate Laurent polynomials $f \in k[x^{\pm 1}, y^{\pm 1}]$.

(iii) **What does the graded Betti table of $X_{\Delta^{(1)}}$ look like?**

In order to have a combinatorial description of the canonical graded Betti table of C_f it does not suffice to merely relate it to the graded Betti numbers of $X_{\Delta^{(1)}}$: we also need to describe these numbers in a combinatorial way. This is a difficult question in its own right, with several partial results available in the existing literature, most notably in the Ph.D. thesis [Her06] of Hering (who in fact studied syzygies of toric varieties of arbitrary dimension). Much of our recent research time was devoted to complementing the existing statements, but the overall picture remains far from complete. Because of the independent interest we studied graded Betti numbers of *arbitrary* projectively embedded toric surfaces, i.e. not necessarily of the form $X_{\Delta^{(1)}}$. An overview of our findings can be found in the chart on the next page. For an accompanying discussion and proofs we refer to Chapter 6, but let us highlight two statements that can be viewed as analogues of Green's canonical syzygy conjecture. At the lower-left end of the graded Betti table we have:

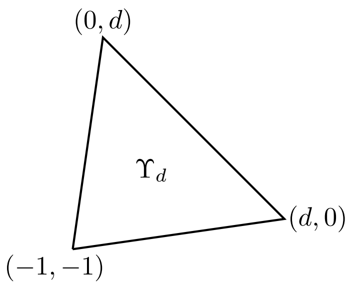
Theorem 24 (Hering, Schenck, Lemmens)**.** *Let Δ be a lattice polygon such that $\Delta^{(1)} \neq \emptyset$. The number of leading zeroes on row $q = 2$ of the graded Betti table of the toric surface X_Δ over k, counting from column index $p = 1$, is given by $\#(\partial\Delta \cap \mathbf{Z}^2) - 3$.*

(If $\Delta^{(1)} = \emptyset$ then the entire row $q = 2$ is trivial.) In characteristic zero the above theorem was proven by Hering [Her06], following an observation of Schenck [Sch04] and building on work of Gallego–Purnaprajna [GP01]. Recently Lemmens [Lem] gave a proof that works in arbitrary characteristic; he also provided an explicit formula for the first non-zero entry. At the upper-right end of the graded Betti table we conjecture:

Conjecture 25. *Let Δ be a two-dimensional lattice polygon such that $\Delta \not\cong \Sigma, \Upsilon$. The number of concluding zeroes on row $q = 1$ of the graded Betti table of the toric surface X_Δ over k, counting backwards from column index $p = \#(\Delta \cap \mathbf{Z}^2) - 3$, is given by $\mathrm{lw}(\Delta) - 1$, except if*

$$\Delta \cong d\Sigma \text{ for some } d \geq 2 \quad \text{or} \quad \Delta \cong \Upsilon_d \text{ for some } d \geq 2 \quad \text{or} \quad \Delta \cong 2\Upsilon \tag{8}$$

in which case it is given by $\mathrm{lw}(\Delta) - 2$.



(If $\Delta \cong \Sigma, \Upsilon$ then the entire row $q = 1$ is trivial.) We have the following evidence in favour of this conjecture:

Theorem 26. *Let Δ be a two-dimensional lattice polygon such that $\Delta \not\cong \Sigma, \Upsilon$. The number of concluding zeroes on row $q = 1$ of the graded Betti table of the toric surface X_Δ over k is at most the quantity predicted by Conjecture 25. Moreover:*

- *If $\mathrm{char}\, k = 0$ and $\#(\Delta \cap \mathbf{Z}^2) \leq 32$ then equality holds.*

Graded Betti table of the toric surface X_Δ associated to a two-dimensional lattice polygon $\Delta \not\cong \Sigma, \Upsilon$: known and conjectural facts.
Notation: $N_\Delta = \#(\Delta \cap \mathbf{Z}^2)$.

$b_1 = \binom{N_\Delta - 1}{2} - 2\mathrm{vol}(\Delta)$

$b_2 = 2\binom{N_\Delta - 1}{3} - 2(N_\Delta - 3)\mathrm{vol}(\Delta) + c_{N_\Delta - 3}$

Conjecture: number of concluding zeroes on first row is $\mathrm{lw}(\Delta) - 1$, unless $\Delta \cong 2\Upsilon$, $\Delta \cong d\Sigma$ for some $d \geq 2$, or $\Delta \cong \Upsilon_d$ for some $d \geq 2$, in which case it is $\mathrm{lw}(\Delta) - 2$

Conjecture: if $\Delta \cong d\Sigma$ for some $d \geq 2$ then last non-zero b-value equals $d^3(d^2-1)/8$
(no general conjecture)

$b_{N_\Delta - 4} = (N_\Delta - 4)B_\Delta$ where B_Δ is given by

$$\begin{cases} 0 & \text{if } \dim \Delta^{(1)} = 2 \text{ and } \Delta \not\cong \Upsilon_2 \\ 1 & \text{if } \dim \Delta^{(1)} = 1 \text{ or } \Delta \cong \Upsilon_2 \\ (N_\Delta - 1)/2 & \text{if } \dim \Delta^{(1)} = 0 \\ N_\Delta - 2 & \text{if } \Delta^{(1)} = \emptyset \end{cases}$$

$b_{N_\Delta - 3} = \begin{cases} 0 \text{ if } \Delta^{(1)} \neq \emptyset \\ N_\Delta - 3 \text{ else} \end{cases}$

	0	1	2	3	4	5	...	$N_\Delta - 6$	$N_\Delta - 5$	$N_\Delta - 4$	$N_\Delta - 3$
0	1	0	0	0	0	0	...	0	0	0	0
1	0	b_1	b_2	b_3	b_4	b_5	...	$b_{N_\Delta-6}$	$b_{N_\Delta-5}$	$b_{N_\Delta-4}$	$b_{N_\Delta-3}$
2	0	$c_{N_\Delta-3}$	$c_{N_\Delta-4}$	$c_{N_\Delta-5}$	$c_{N_\Delta-6}$	$c_{N_\Delta-7}$	...	c_4	c_3	c_2	c_1

c-row is zero iff $\Delta^{(1)} = \emptyset$

$c_1 = N_{\Delta^{(1)}}$

$c_2 = \max\left(0, (N_\Delta - 3)(N_{\Delta^{(1)}} - 1)\right)$

$c_3 = (N_\Delta - 4)\left((N_\Delta - 3)\mathrm{vol}(\Delta) - \binom{N_\Delta - 1}{2} + B_\Delta\right)$
with B_Δ as defined above

$c_{N_\Delta - 3} =$
$$\begin{cases} 0 & \text{if } \#(\partial\Delta \cap \mathbf{Z}^2) > 3 \\ 1 & \text{if } \#(\partial\Delta \cap \mathbf{Z}^2) = 3 \text{ and } \dim \Delta^{(1)} = 2 \\ N_\Delta - 3 & \text{if } \#(\partial\Delta \cap \mathbf{Z}^2) = 3 \text{ and } \dim \Delta^{(1)} \leq 1 \end{cases}$$

number of leading zeroes on second row is
$\#(\Delta \cap \mathbf{Z}^2) - 3$
(Hering – Schenck in char 0, Lemmens in gen'l)

if $\Delta^{(1)} \neq \emptyset$ then first non-zero c-value equals
$$\binom{N_{\Delta^{(1)}} + t - 2}{N_{\Delta^{(1)}} - 1}$$
with t the number of integral translates of $\Delta^{(1)}$ that fit inside Δ (Lemmens)

antidiagonal differences $b_i - c_{N_\Delta - 1 - i} = i\binom{N_\Delta - 1}{i+1} - 2\binom{N_\Delta - 3}{i-1}\mathrm{vol}(\Delta)$

- *If* $\operatorname{char} k = 0$, $\Delta = \Gamma^{(1)}$ *for a larger lattice polygon* Γ, *and Green's canonical syzygy conjecture holds for some weakly* Γ*-nondegenerate curve (e.g. if* X_Δ *carries an anti-canonical pencil [LC13]), then equality holds.*

  In particular, if $\operatorname{char} k = 0$ *and* X_Δ *is Gorenstein and weak Fano then equality holds.*
- *If equality holds for a certain instance of* Δ *not among (8), then it also holds for every lattice polygon containing* Δ *and having the same lattice width.*

  Using [CL] and (a) it follows that if $\operatorname{char} k = 0$ *and* $\operatorname{lw}(\Delta) \leq 6$ *then equality holds.*

For proofs we refer to Chapter 6, although let us note that the first claim was obtained using an explicit determination of the relevant entries in the graded Betti table of X_Δ, for all two-dimesional lattice polygons Δ containing at most 32 lattice points, using our database from Chapter 2. For reasons of efficiency the computation was carried out in finite characteristic, leading to the stated result through a semi-continuity argument.

Remark. The underlying algorithm can be used to gather all sorts of related data. It explicitly computes the Koszul cohomology of X_Δ, using duality, the action of $\mathbf{T}^2$ and some of the features stated in the chart on the previous page to reduce the time and memory requirements. It was implemented in SageMath and for instance allowed us to explicitly determine the graded Betti numbers of the Veronese surface $X_{6\Sigma} = \nu_6(\mathbf{P}^2)$ in characteristic $40\,009$; we expect this to match with characteristic zero. Up to $\nu_5(\mathbf{P}^2)$ these data were recently gathered by Greco and Martino [GM16].

We end this section with two applications. Recall that Green's conjecture helped us in settling special instances of Conjecture 25. But there is also an implication in the opposite direction: the instances of Conjecture 25 that were established through explicit computation in turn imply new cases of Green's conjecture.

Theorem 27. *Let* $\operatorname{char} k = 0$, *let* X/k *be a toric surface, and let* $C \subseteq X$ *be a non-hyperelliptic smooth projective curve of genus* $4 \leq g \leq 32$. *Then Green's canonical syzygy conjecture is true for* C.

Omitting exceptional cases, the proof uses that

$$a_{g-\operatorname{lw}(\Delta^{(1)})-1} = b_{g-\operatorname{lw}(\Delta^{(1)})-1}$$

as soon as

$$c_{g-\operatorname{lw}(\Delta^{(1)})-1} = 0, \tag{9}$$

which is immediate from the exact sequence (4). From Theorem 24 we know that (9) holds if and only if $\#(\partial\Delta^{(1)} \cap \mathbf{Z}^2) \geq \operatorname{lw}(\Delta^{(1)}) + 2$. Using our database of lattice polygons we computationally verified that this last inequality is true whenever $\#(\Delta^{(1)} \cap \mathbf{Z}^2) \leq 32$, except if $\Delta^{(1)} \cong \Upsilon$. The result then follows from Theorem 26, which says that $b_{g-\operatorname{lw}(\Delta^{(1)})-1} = 0$, and our combinatorial interpretation for the Clifford index stated in Theorem 17, which says that Green's conjecture amounts to $a_{g-\operatorname{lw}(\Delta^{(1)})-1} = 0$. We refer to Chapter 8 for additional details.

Remark. In higher genus there exist more counterexamples to $\#(\partial\Delta^{(1)} \cap \mathbf{Z}^2) \geq \operatorname{lw}(\Delta^{(1)}) + 2$, with $\Delta = \operatorname{conv}\{(4,0),(10,4),(0,10)\}$ being the smallest instance that we have found, corresponding to $g = 36$. Here $\#(\partial\Delta^{(1)} \cap \mathbf{Z}^2) = 9$ and $\operatorname{lw}(\Delta^{(1)}) = 8$, and as a consequence $c_{27} \neq 0$. In this case Green's canonical syzygy conjecture amounts to $a_{27} = 0$, but unlike the foregoing cases it is insufficient to verify that $b_{27} = 0$. In fact, if the sum formula $a_{27} = b_{27} + c_{27}$ from Theorem 21 would be true here (which we do not think it is), then this would show that $a_{27} \neq 0$ and hence that weakly Δ-nondegenerate curves are counterexamples to Green's conjecture!

A second application is concerned with the gonality conjecture due to Green and Lazarsfeld [GL86], which was recently proven by Ein and Lazarsfeld [EL15].

Theorem 28 (Gonality conjecture, proven by Ein–Lazarsfeld)**.** *Let* $\operatorname{char} k = 0$ *and let* C/k *be a smooth projective curve of gonality* $\gamma \geq 2$*. Let* L *be a globally generated divisor on* C *of sufficiently large degree, and assume that* $C \subseteq \mathbf{P}^{\operatorname{rk}|L|}$ *is embedded using the linear system* $|L|$*. Then the number of non-zero entries on row* $q = 1$ *of the graded Betti table of the homogeneous coordinate ring of* C *equals* $h^0(C, L) - \gamma - 1$*.*

Concretely Ein and Lazarsfeld showed that $\deg L \geq g^3$ is sufficient, a bound which was recently improved to $\deg L \geq 4g - 3$ by Rathmann [Rat]. It is expected that this is not yet optimal, although Green and Lazarsfeld already noted that one needs at least $\deg L \geq 2g + \gamma - 1$. In a first draft of [FK] Farkas and Kemeny speculated that the latter bound might always be sufficient. However we show:

Theorem 29. *For each* $\gamma \geq 3$ *there exists a curve* C/k *of genus* $g = \gamma(\gamma - 1)/2$ *along with a very ample divisor* L *of degree* $2g + \gamma - 1$ *such that the number of non-zero entries on row* $q = 1$ *of the graded Betti table of the homogeneous coordinate ring of the correspondingly embedded curve is at least* $h^0(C, L) - \gamma$*.*

The curve C we construct is weakly $\Upsilon_{\gamma-1}$-nondegenerate, and its exceptional behaviour is tightly connected with the fact that $\Upsilon_{\gamma-1}$ is exceptional for Conjecture 25. We refer to Chapter 7 for the details.

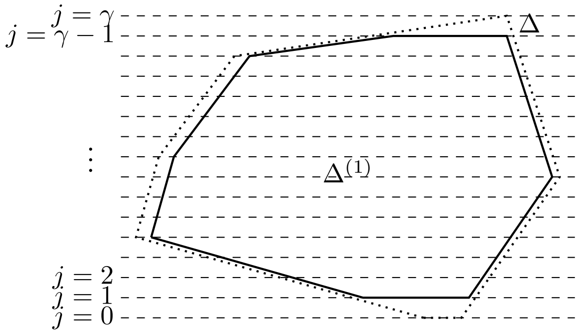
6. First scrollar Betti numbers (Chapters 9 and 10)

As in the previous section we consider a lattice polygon Δ with two-dimensional interior along with a weakly Δ-nondegenerate Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$. Write $g = \#(\Delta^{(1)} \cap \mathbf{Z}^2)$ and let $\gamma \geq 3$ denote the lattice width $\operatorname{lw}(\Delta)$. Assume that the latter equals the gonality of C_f, or in other words that there exists at least one combinatorial gonality pencil g_v.

Remark. In view of the material from Chapter 5 this means that we exclude $\Delta \cong 2\Upsilon$ and $\Delta \cong d\Sigma$ for any $d \geq 4$. However, in the last case one can circumvent this by noting that a weakly $d\Sigma$-nondegenerate curve is also weakly $\operatorname{conv}\{(1,0),(d,0),(0,d),(0,1)\}$-nondegenerate: simply replace f by $f(x + x_0, y + y_0)$ for some $(x_0, y_0) \in U_f$. This does not affect the interior polygon, in terms of which all results below are stated.

For convenience we assume that $v = (1, 0)$ and that $\Delta \subseteq \mathbf{R} \times [0, \gamma]$; in particular $g_v = |\text{fiber of } (x, y) \mapsto x|$. This can always be achieved by means of a unimodular transformation. It is not hard to see that the rational normal scroll $S \subseteq \mathbf{P}^{g-1}$ swept out by g_v equals the $(\gamma - 1)$-dimensional toric variety associated to the polytope that one obtains from $\Delta^{(1)}$ by 'forgetting' that horizontal lines are coplanar, which amounts to omitting certain defining equations; we refer to Chapter 5 for more details. In fact this observation is the central ingredient in the proof of Theorem 15, which in our case states that the scrollar invariants are given by the width invariants

$$E_j := i^{(+)}(j) - i^{(-)}(j)$$



for $j = 1, \ldots, \gamma - 1$, where

$$i^{(-)}(j) = \min\{i \in \mathbf{Z} \,|\, (i,j) \in \Delta^{(1)}\} \quad \text{and} \quad i^{(+)}(j) = \max\{i \in \mathbf{Z} \,|\, (i,j) \in \Delta^{(1)}\}.$$

For our current purposes an important conclusion is that the series of inclusions (4) extends to $C_f^{\text{can}} \subseteq X_{\Delta^{(1)}} \subseteq S \subseteq \mathbf{P}^{g-1}$.

Recall that in Chapter 4 we provided a recipe for obtaining a minimal set of generators of the canonical ideal of C_f. In this section we report on a similar method for determining a minimal set of defining equations for our canonical curve *relative to the scroll* S. This concept was introduced and made precise by Schreyer [Sch86]. Informally spoken, the goal is to realize C_f^{can} as the scheme-theoretic intersection of as few divisors on S as possible. Modulo linear equivalence these divisors can be expressed as linear combinations of the hyperplane section class H and the ruling class R, which generate the Picard group. The coefficient of H matches with our intuitive notion of degree and therefore has little added value. But the coefficient of R can give interesting new discrete information.

Example. In the trigonal case C_f^{can} is a divisor itself. It concerns a 'cubic', as it is contained in the class $3H - (g-4)R$.

Subtlety. If one of the rational normal curves spanning S is of degree 0 (i.e. is a point) then the Picard group may not be *freely* generated by H and R [Fer01]. To give a rather degenerated example, consider $\Delta^{(1)} = \Sigma$: here the scroll is just $\mathbf{P}^2$ where $R = H$, and we recover that $C_f^{\text{can}} \in 3H + R = 4R$ is a plane quartic. To avoid non-unique expansions and various other theoretical issues one should actually work with the strict transforms of C_f^{can} and $X_{\Delta^{(1)}}$ under the natural birational morphism $j : S' \to S$ induced by increasing the degrees of the spanning rational normal curves by some fixed positive amount, but we will ignore this technicality here.

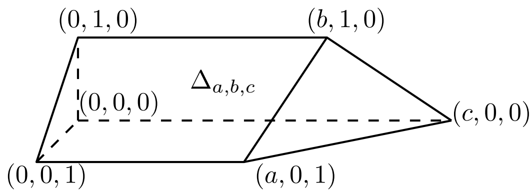
From $\gamma \geq 4$ on it turns out that our curve is minimally cut out by $(\gamma^2 - 3\gamma)/2$ 'quadrics', i.e. divisors whose classes are of the form

$$2H - b_1 R, \quad 2H - b_2 R, \quad 2H - b_3 R, \quad \ldots, \quad 2H - b_{(\gamma^2-3\gamma)/2} R$$

for integers b_i that sum up to $(\gamma - 3)(g - \gamma - 1)$ and which turn out to be independent of the chosen divisors, up to order. See [BH15; Sch86] for more details. We have called these numbers the **first scrollar Betti numbers** with respect to g_v because they appear in the first step of a minimal free resolution of $\mathcal{O}_{C_f}$ as an $\mathcal{O}_S$-module.[3]

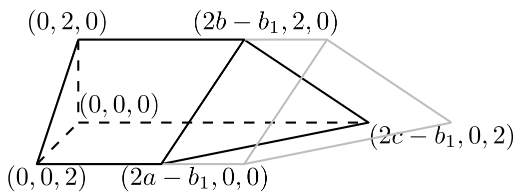
[3]However these numbers do not appear as dimensions of cohomology spaces: therefore this terminology is up for improvement.

We first concentrate on the case $\gamma = 4$, which is particularly interesting. Assume that g_v has scrollar invariants $0 \leq a \leq b \leq c$, so that $S \subseteq \mathbf{P}^{g-1}$ is the toric threefold associated to the polytope $\Delta_{a,b,c}$ depicted below. Note that we can view it as a completion of affine



space $\mathbf{A}^3$ rather than just the torus $\mathbf{T}^3$. Inside S our curve C_f^{can} arises as a complete intersection of two divisors Y, Z whose respective classes are $2H - b_1 R$ and $2H - b_2 R$ with $b_1 + b_2 = g - 5$, where we can assume that $b_1 \geq b_2$. It can be shown that $b_2 \geq -1$, where equality occurs if and only if C_f is isomorphic to a smooth plane quintic [Sch86, §6], i.e. if and only if $\Delta^{(1)} \cong 2\Sigma$.

In terms of defining equations this means that $Y \cap \mathbf{A}^3$ is defined by a polynomial $f_Y \in k[x, y, z]$ which is supported on the horizontally shrunk version of $2\Delta_{a,b,c}$ shown below (and that this is no longer true if we shrink it further). The analogous claim applies



to the polynomial $f_Z \in k[x, y, z]$ associated to Z. Since Z corresponds to the bigger polytope it moves in a family: one is free to replace f_Z by $f_Z + g f_Y$ for some $g \in k[x]$ of degree at most $b_1 - b_2$. On the other hand if $b_1 > b_2$ then Y is immovable. In fact Schreyer proved that the invariants b_1, b_2 are independent of the chosen g_4^1 and that the same is true for the surface Y as soon as $b_1 > b_2$. The main result of Chapter 9 is:

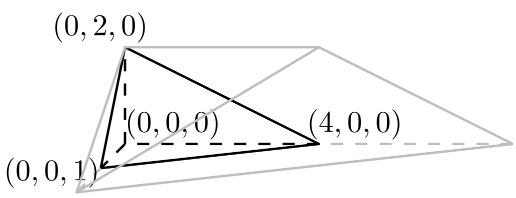
Theorem 30. *Let Δ be a two-dimensional lattice polygon, let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be weakly Δ-nondegenerate, and assume that C_f is tetragonal. Then the first scrollar Betti numbers $\{b_1, b_2\}$ of C_f are given by*

$$\left\{ \#(\partial \Delta^{(1)} \cap \mathbf{Z}^2) - 4 \, , \, \#(\Delta^{(2)} \cap \mathbf{Z}^2) - 1 \right\}.$$

Moreover if $\#(\partial \Delta^{(1)} \cap \mathbf{Z}^2) - 4 > \#(\Delta^{(2)} \cap \mathbf{Z}^2) - 1$ then the surface Y is given by $X_{\Delta^{(1)}}$.

The last equality is almost always satisfied, with all counterexamples to be found in Chapter 9.

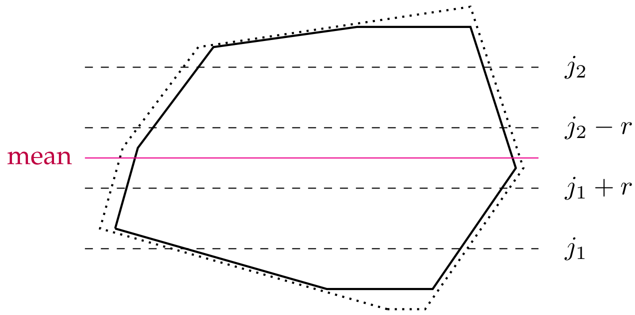
Example. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly $\Delta_{4,7}$-nondegenerate Laurent polynomial. This is a $C_{4,7}$ curve, which by Baker's bound is of genus 9, and by Theorem 13 carries a unique g_4^1. According to Theorem 15 the corresponding scrollar invariants are $0, 2, 4$ and the



above theorem says that $b_1 = 4$ and $b_2 = 0$. The surface Y, whose corresponding polytope is depicted above, is the toric surface $X_{\Delta^{(1)}_{4,7}}$, which on $\mathbf{A}^3$ is cut out by $f_Y = y^2 - z$.

Next in Chapter 10 we work towards a combinatorial interpretation for the first scrollar Betti numbers of a weakly nondegenerate curve of gonality $\gamma \geq 5$. For this we assume that Δ and v satisfy two technical conditions $\mathcal{P}_1(v)$ and $\mathcal{P}_2(v)$, which are explained in more detail below. Essentially these conditions impose on $\Delta^{(1)}$ a certain combinatorial compatibility between its 'left-hand side' and its 'right-hand side', i.e. between the numbers $i^{(-)}(j)$ and the numbers $i^{(+)}(j)$.

In a first phase we look for divisors that cut out our toric surface $X_{\Delta^{(1)}}$. Recall that S was obtained from $X_{\Delta^{(1)}}$ by forgetting that horizontal lines are coplanar, so the idea is to rivet these lines gradually back together. Concretely for each pair $j_1, j_2 \in \{1, 2, \ldots, \gamma - 1\}$ such that $j_2 - j_1 \geq 2$ we define a toric $(\gamma - 2)$-fold $D_{j_1,j_2} \subseteq S$ which reminds the scroll of the fact that the pair of lines at heights j_1, j_2 and the pair of lines at heights $j_1 + r, j_2 - r$ have the same 'mean':



Here $r \in \{1, 2, \ldots, (j_2 - j_1)/2\}$ should be chosen carefully, which is where the condition $\mathcal{P}_1(v)$ shows up. Concretely, for each r we define

$$\begin{aligned}\epsilon^{(-)}_{j_1,j_2,r} &= \begin{cases} 0 & \text{if} \quad i^{(-)}(j_1 + r) + i^{(-)}(j_2 - r) \leq i^{(-)}(j_1) + i^{(-)}(j_2) \\ 1 & \text{if} \quad i^{(-)}(j_1 + r) + i^{(-)}(j_2 - r) > i^{(-)}(j_1) + i^{(-)}(j_2) \end{cases} \\ &= \max\{0, (i^{(-)}(j_1 + r) + i^{(-)}(j_2 - r)) - (i^{(-)}(j_1) + i^{(-)}(j_2))\}\end{aligned}$$

and

$$\begin{aligned}\epsilon^{(+)}_{j_1,j_2,r} &= \begin{cases} 0 & \text{if} \quad i^{(+)}(j_1 + r) + i^{(+)}(j_2 - r) \geq i^{(+)}(j_1) + i^{(+)}(j_2) \\ 1 & \text{if} \quad i^{(+)}(j_1 + r) + i^{(+)}(j_2 - r) < i^{(+)}(j_1) + i^{(+)}(j_2) \end{cases} \\ &= \max\{0, (i^{(+)}(j_1) + i^{(+)}(j_2)) - (i^{(+)}(j_1 + r) + i^{(+)}(j_2 - r))\}.\end{aligned}$$

Condition $\mathcal{P}_1(v)$ imposes that

- either we can find an r for which $\epsilon^{(-)}_{j_1,j_2,r} = \epsilon^{(+)}_{j_1,j_2,r} = 0$, in which case we let r be the smallest such choice and we define $\epsilon_{j_1,j_2} = 0$,
- or there is no r for which $\epsilon^{(-)}_{j_1,j_2,r} = 0$ but there is an r for which $\epsilon^{(+)}_{j_1,j_2,r} = 0$, in which case we let r be the smallest such choice and we define $\epsilon_{j_1,j_2} = 1$,
- or, similarly, there is no r for which $\epsilon^{(+)}_{j_1,j_2,r} = 0$ but there is an r for which $\epsilon^{(-)}_{j_1,j_2,r} = 0$, in which case we let r be the smallest such choice and we define $\epsilon_{j_1,j_2} = 1$.

This should be true all $(\gamma - 2)(\gamma - 3)/2$ pairs j_1, j_2. We refer to Chapter 10 for further details on the construction of D_{j_1,j_2}, for some clarifying examples, and eventually for a proof of the following statement:

Theorem 31. *Inheriting the notation from above and assuming that condition $\mathcal{P}_1(v)$ is satisfied, one has that the $(\gamma-2)(\gamma-3)/2$ divisors $D_{j_1,j_2} \subseteq S$ together cut out $X_{\Delta^{(1)}}$. Moreover*

$$D_{j_1,j_2} \in 2H - B_{j_1,j_2}R \text{ for all } j_1, j_2, \text{ where } B_{j_1,j_2} = E_{j_1} + E_{j_2} - \epsilon_{j_1,j_2},$$

and the B_{j_1,j_2}'s sum up to $(\gamma-4)g - (\gamma^2 - 3\gamma) + \#(\partial\Delta^{(1)} \cap \mathbf{Z}^2)$.

The next step is to add divisors that slice this further down to C_f^{can}. Recall from Section 2 that the canonical ideal of $C_f \subseteq \mathbf{P}^{g-1}$ is spanned by the ideal of $X_{\Delta^{(1)}}$ and the quadrics

$$Q_w = \sum_{(i,j)\in\Delta\cap\mathbf{Z}^2} c_{ij} X_{u_{ij}} X_{v_{ij}}, \qquad w \in \Delta^{(2)} \cap \mathbf{Z}^2,$$

where u_{ij}, v_{ij} should be chosen such that $(i,j) - w = (u_{ij} - w) + (v_{ij} - w)$. Typically the choice of u_{ij} and v_{ij} is not unique. Condition $\mathcal{P}_2(v)$ amounts to the existence for each horizontal line L of horizontal lines M_1 and M_2 such that it is possible to choose $u_{ij} \in M_1$ and $v_{ij} \in M_2$ for all $w \in L$. If this is indeed possible then we obtain our requested divisors by grouping together all Q_w's that correspond to lattice points on the same horizontal line L. More precisely we define for each $j = 2, 3, \ldots, \gamma - 2$ a divisor

$$D_j := S \cap \{\, Q_w \,|\, w \in \Delta^{(2)} \cap \mathbf{Z}^2 \text{ lies at height } j \,\}.$$

We refer to Chapter 10 for explanation why this indeed results in a subscheme of codimension one (in contrast with what happens if one does not choose the u_{ij}'s and v_{ij}'s in a consistent way), and for a proof of the following statement:

Theorem 32. *Inheriting the notation from above and assuming that condition $\mathcal{P}_2(v)$ is satisfied, one has that the $\gamma - 3$ divisors $D_j \subseteq S$ together with $X_{\Delta^{(1)}} \subseteq S$ cut out C_f^{can}. Moreover*

$$D_j \in 2H - B_j R \text{ for all } j, \text{ where } B_j = -1 + \#\{\, i \in \mathbf{Z} \,|\, (i,j) \in \Delta^{(2)} \cap \mathbf{Z}^2 \,\},$$

and the B_j's sum up to $\#(\Delta^{(2)} \cap \mathbf{Z}^2) - (\gamma - 3)$.

Notice that the multi-set of B_j's equals the multi-set of width invariants $E(\Delta^{(1)}, v)$. Finally, by combining both results and noticing that

$$(\gamma-2)(\gamma-3)/2 + (\gamma-3) = (\gamma^2 - 3\gamma)/2,$$

we arrive at our desired interpretation of the first scrollar Betti numbers:

Corollary 33. *Inheriting the notation from above and assuming that conditions $\mathcal{P}_1(v)$ and $\mathcal{P}_2(v)$ are satisfied, one has that the first scrollar Betti numbers of C_f with respect to g_v are given by*

$$\{B_j\}_{j\in\{2,\ldots,\gamma-2\}} \cup \{B_{j_1,j_2}\}_{\substack{j_1,j_2\in\{1,\ldots,\gamma-1\} \\ j_2-j_1\geq 2}}.$$

These scrollar Betti numbers indeed add up to $(\gamma-3)(g-\gamma-1)$, as announced.

Remark. The conditions $\mathcal{P}_1(v)$ and $\mathcal{P}_2(v)$ are milder than one might fear at a first glance. In fact we believe that they are void for $\gamma = 5$ and $\gamma = 6$, although we could not prove this. The smallest pair Δ, v violating $\mathcal{P}_2(v)$ that we managed to find corresponds to curves of genus 46 and gonality 10; see Chapter 10.

7. Arithmetic features (Chapter 11)

Consider a field k that is not necessarily algebraically closed. Let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a Laurent polynomial that is weakly Δ-nondegenerate when considered over $k^{\text{alg.cl.}}$. Then $C_f \subseteq X_\Delta$ is a smooth projective curve that is defined over k. In this section we wish to illustrate that besides geometric information, our polygon Δ potentially also contains some arithmetic data, even though we do not expect the existence of a large one-to-one arithmetic-combinatorics dictionary as in the geometric case.

A first arithmetic feature is that if an edge $\tau \subseteq \Delta$ has integral length one then the corresponding torus-invariant divisor $D_\tau \subseteq X_\Delta$ contains a k-rational point of C_f. The reason is that the intersection locus of C_f with D_τ is locally given by a linear equation with coefficients in k. Thus if there are many such edges then this yields meaningful lower bounds on $\#C_f(k)$. This was used by Kresch, Wetherell and Zieve to prove the following fact:

Theorem 34 (Kresch, Wetherell, Zieve [KWZ02]). *For every integer $g \geq 0$ and every prime power q define $N_q(g) := \max_C \#C(\mathbf{F}_q)$, where C ranges over all smooth projective curves of genus g over $\mathbf{F}_q$. Then $\lim_{g\to\infty} N_q(g) = \infty$. More precisely $\liminf_{g\to\infty} N_q(g)/g^{1/3} > 0$.*

This statement is no longer the best available: in 2004 the same authors, in cooperation with Elkies, Howe and Poonen [Elk+04], managed to replace the denominator $g^{1/3}$ by g, which is optimal in view of the Hasse-Weil bound. Unfortunately this relies on other techniques, but nevertheless Theorem 34 remains a beautiful application of smooth curves in toric surfaces.

Remark. More generally the presence of an edge $\tau \subseteq \Delta$ of integeral length r ensures the existence of a k-rational divisor of degree r. Using a classification due to Fisher [Fis08], we used this in Chapter 1 to prove that a genus one curve C/k is k-birationally equivalent to a weakly nondegenerate curve if and only if it has a k-rational divisor of degree at most 3.

A second arithmetic feature is that the k-rational gonality of C_f, by which we mean the minimal degree of a k-rational map $C_f \to \mathbf{P}^1$, equals its geometric gonality, except possibly if $\Delta \cong 2\Upsilon$ or if $\Delta \cong d\Sigma$ for some $d \geq 2$. This is a trivial consequence of Theorem 13, because combinatorial pencils are clearly k-rational. In particular C_f is hyperelliptic if and only if it is geometrically hyperelliptic.

Remark. By letting $k = \mathbf{C}(\!(t)\!)$, through specialization of divisors this gives (very) prudent support in the case of planar graphs in favour of a conjecture by Baker [Bak08, Conj. 3.14], saying that the gonality of a graph equals the gonality of its metrization.

If $\Delta \cong 2\Upsilon$ then the k-rational gonality equals the geometric gonality (namely 3) except if C_f canonically embeds into an elliptic quadric in $\mathbf{P}^3$ in which case it equals 4; the occurrence of this event may depend on the specific choice of f. If $\Delta \cong d\Sigma$ for some $d \geq 2$ then the k-rational gonality equals the geometric gonality (namely $d-1$) except if $\#C_f(k) = \emptyset$ in which case it equals d; again this may depend on the specific choice of f.

Chapter 11 is devoted to yet another arithmetic phenomenon, which has a geometric intake. One can verify that the canonical divisor K_Δ on C_f that one obtains from adjunction theory (see Section 1) equals

$$\sum_\tau (-\langle \nu_\tau, p_\tau \rangle - 1)(D_\tau \cap C_f), \tag{10}$$

where:

- the sum runs through all edges $\tau \subseteq \Delta$,
- D_τ denotes the torus-invariant divisor on X_Δ associated to τ,
- $\nu_\tau \in \mathbf{Z}^2$ is the primitive inward pointing normal vector to τ,
- p_τ is any point on $\tau \cap \mathbf{Z}^2$.

Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbf{R}^2$. It is the divisor of the differential

$$\frac{dx}{xy\frac{\partial f}{\partial y}},$$

unless $\partial f / \partial y = 0$, which happens if x is not a separating variable, in which case[4] one should exchange the role of x and y. Similarly one verifies that the set

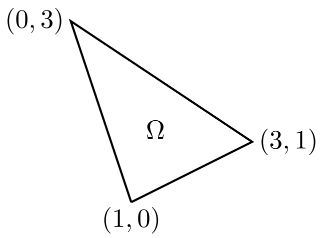
$$\left\{ x^i y^j \frac{dx}{xy\frac{\partial f}{\partial g}} \right\}_{(i,j) \in \Delta^{(1)} \cap \mathbf{Z}^2} \tag{11}$$

is a basis of holomorphic differentials. We refer to [CDV06] for more details.

One observes that if all $\langle \nu_\tau, p_\tau \rangle$'s are odd then all coefficients in (10) are even, so that we obtain a theta characteristic Θ_{geom} by halving them. Note that Θ_{geom} is k-rational. If not all inner products are odd then it might be possible to achieve this by translating Δ over some $(i, j) \in \mathbf{Z}^2$, amounting to multiplying f by the monomial $x^i y^j$. If this is indeed possible then Δ is called **canonically even**. If it is moreover possible to do this in such a way that $(0, 0)$ becomes contained in Δ, then Δ is called **effectively canonically even** because the resulting theta characteristic is effective. (Note that then $(0, 0)$ is automatically contained in $\Delta^{(1)}$, otherwise one of the inner products would be zero, hence even.)

Remark. If translation over a vector $v \in \mathbf{Z}^2$ makes all $\langle \nu_\tau, p_\tau \rangle$'s odd, then so does translating over $v + w$ for any $w \in (2\mathbf{Z})^2$.

Examples. All triangles $\Delta_{a,b}$ where $\gcd(a, b, 2) = 1$ are canonically even. If moreover $a, b \geq 2$ then they are effectively canonically even. This covers all smooth plane curves of odd degree and all $C_{a,b}$ curves. Also all triangles $\Delta_{2,2g+2}$ where $g \geq 3$ is odd are effectively canonically even; this corresponds to hyperelliptic curves of odd genus. The polygon Ω depicted below is an example of a lattice polygon which is canonically even but not effectively canonically even.



Now assume $\operatorname{char} k = 2$. Then C_f automatically carries another k-rational theta characteristic Θ_{arith}, which was introduced by Mumford [Mum71]. It is simply defined as

[4] This event is extremely unlikely but it can happen. Example: $f = y^2 + x^2 + x + 1$ in characteristic 2.

$\Theta_{\text{arith}} := (\operatorname{div} dx)/2$, where we note that dx indeed has even orders of vanishing because only even terms remain when differentiating a Laurent series in characteristic two.[5] A theorem by Stöhr and Voloch [SV87] says that $h^0(C_f, \Theta_{\text{arith}}) = g - r$, where g is the genus of C_f and r is the rank of the Cartier operator acting on the space of holomorphic differentials. It is well-known that $r = g$ if and only if C_f is an ordinary curve. This implies:

Lemma 35. *Let Δ be a two-dimensional effectively canonically even lattice polygon and let k be a field of characteristic 2. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. If C_f is ordinary then $\operatorname{Jac}(C_f)$ carries a non-trivial k-rational 2-torsion point.*

Indeed, if C_f is ordinary then $h^0(C_f, \Theta_{\text{arith}}) = 0$ and therefore Θ_{arith} is not linearly equivalent to an effective divisor, in particular not to Θ_{geom}. Then the divisor $\Theta_{\text{arith}} - \Theta_{\text{geom}}$ maps to a non-trivial k-rational 2-torsion point on the Jacobian.

Alternatively, we obtain Lemma 35 as a corollary to the following stronger result, which is proven in Chapter 11 by explicit computation, using the basis (11).

Theorem 36. *Let Δ be a two-dimensional canonically even lattice polygon and let k be a field of characteristic 2. Let*

$$f = \sum_{(i,j) \in \Delta \cap \mathbf{Z}^2} c_{ij} x^i y^j \in k[x^{\pm 1}, y^{\pm 1}]$$

be a weakly Δ-nondegenerate Laurent polynomial. Let P be the set of vectors $(i, j) \in \Delta \cap \mathbf{Z}^2$ such that translating over $(-i, -j)$ makes all $\langle \nu_\tau, p_\tau \rangle$'s odd. Let $\rho = \#P$.

- *If $c_{i,j} \neq 0$ for at least one $(i, j) \in P$ then Θ_{arith} and Θ_{geom} are not linearly equivalent, and therefore $\operatorname{Jac}(C_f)$ carries a non-trivial k-rational 2-torsion point.*
- *If $c_{i,j} = 0$ for all $(i, j) \in P$ then the rank of the Cartier operator is at most $g - \rho$.*

In particular if Δ is effectively canonically even, then $\rho > 0$ and

$$C_f \text{ ordinary} \quad \Rightarrow \quad c_{i,j} \neq 0 \text{ for some } (i,j) \in P \quad \Rightarrow \quad \operatorname{Jac}(C_f)(k)[2] \neq 0.$$

As a consequence, in characteristic two, a sufficiently generic Laurent polynomial that is supported on an effectively canonically even lattice polygon defines a curve with a non-trivial k-rational two-torsion point on its Jacobian. This observation was first made by Cais, Ellenberg and Zureick-Brown in the case of smooth plane curves of odd degree [CEZB13]. In the case of $\Delta_{a,b}$ with $\gcd(a, b) = 1$ this explains why Denef and Vercauteren [DV06] had to tolerate a factor 2 in $\#\operatorname{Jac}(C)(k)$ when trying to generate cryptographically secure $C_{a,b}$ curves C over finite fields of characteristic two.

We actually conjecture that under the assumptions of the theorem the rank of the Cartier operator is *at least* $g - \rho$, where equality holds if and only if $c_{i,j} = 0$ for all $(i, j) \in P$. Chapter 11 contains proofs of this conjecture for $\Delta \cong \Delta_{2g+2,2}$ with g odd (hyperelliptic curves of odd genus), for $\Delta \cong d\Sigma$ with d odd (smooth plane curves of odd degree), and also for $\Delta \cong \Omega$. In the latter case $\rho = 0$, so this converts into the following fact:

Lemma 37. *Let k be a field of characteristic 2 and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be weakly Ω-nondegenerate. Then C_f is ordinary.*

8. Intrinsicness (Chapters 5, 9 and 10)

In this section we reinstall the assumption that k is an algebraically closed field. Let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate

[5] If x is not a separating variable then $dx = 0$, in which case we again exchange the role of x and y.

Laurent polynomial. Given the long list of geometric invariants that can be told from the combinatorics of Δ, one can wonder to what extent it is possible to recover the polygon *itself* from the abstract birational geometry of U_f (or of C_f). The best one can hope for is to find back Δ up to unimodular equivalence, because unimodular transformations correspond to automorphisms of $\mathbf{T}^2$. Another relaxation is that (usually) one can only expect to recover $\Delta^{(1)}$, rather than all of Δ. For example, recall from the first remark in Section 6 that every weakly $d\Sigma$-nondegenerate Laurent polynomial is also weakly Δ-nondegenerate, where Δ is obtained from $d\Sigma$ by clipping off the point $(0,0)$. More generally, pruning a vertex off a two-dimensional lattice polygon Δ without affecting its interior boils down to forcing the curve through a certain non-singular point of X_Δ, which is usually not an intrinsic property. One is naturally led to the following definition.

Definition 38. *Let Δ be a two-dimensional lattice polygon and let C/k be a weakly Δ-nondegenerate curve. We say that $\Delta^{(1)}$ is **intrinsic to C** if for all two-dimensional lattice polygons Δ' for which C is weakly Δ'-nondegenerate it holds that $\Delta^{(1)} \cong \Delta'^{(1)}$. We say that $\Delta^{(1)}$ is **intrinsic** if it is intrinsic to every weakly Δ-nondegenerate curve.*

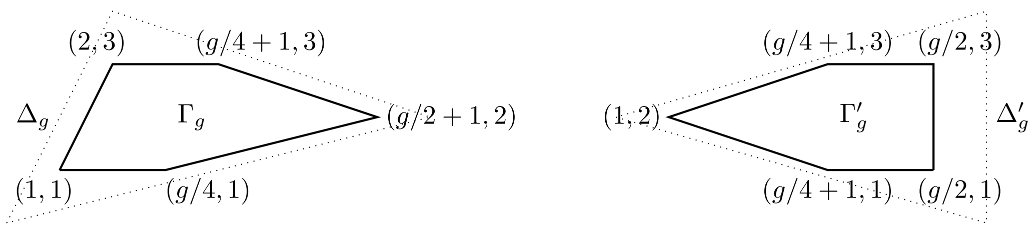
A few first cases in which $\Delta^{(1)}$ is intrinsic are:

- $\Delta^{(1)} = \emptyset$, which occurs if and only if C_f is rational,
- $\dim \Delta^{(1)} = 0$, which holds if and only if C_f is elliptic,
- $\dim \Delta^{(1)} = 1$, which holds if and only if C_f is hyperelliptic of genus $\#(\Delta^{(1)} \cap \mathbf{Z}^2)$,
- $\Delta^{(1)} \cong (d-3)\Sigma$ for some $d \geq 3$, which by Corollary 18 occurs if and only if C_f is birationally equivalent to a smooth projective plane curve of degree d.

From Theorem 17 we see that if $\operatorname{char} k = 0$ then also $\Delta^{(1)} \cong 2\Upsilon$ is intrinsic, because this occurs if and only if C_f is of Clifford index 3. Most likely this result is also true in positive characteristic.

As with many statements in this manuscript, the case where $\Delta^{(1)} \cong \Upsilon$ turns out to be an exception. Indeed, recall from Theorem 10 that every genus 4 curve is either weakly $\square_{3,3}$-nondegenerate or weakly $\Delta_{6,3}$-nondegenerate. The respective interiors of these polygons are $\square_{1,1}$ and $\Delta_{2,1}$, while Υ is equivalent to neither of both. Since both cases occur, this turns all three interior genus 4 polygons $\square_{1,1}, \Delta_{2,1}, \Upsilon$ into exceptions.

More counterexamples. Our polygon Υ belongs to a larger family of counterexamples. Let $g \geq 4$ satisfy $g \equiv 0 \bmod 4$, and consider the lattice polygons Γ_g and Γ'_g depicted below, which are non-equivalent. We note that $\Gamma_4 \cong \Upsilon$. If $\operatorname{char} k > g/2+1$ then the polynomi-



als $f = 1 - x^2y^4 - x^{\frac{g}{2}+2}y^2$ and $f' = (y^4-1)x^{\frac{g}{2}+1} + 4y^2$ are weakly Δ_g-nondegenerate and weakly Δ'_g-nondegenerate, respectively. Here Δ_g and Δ'_g are as depicted above and satisfy $\Delta_g^{(1)} = \Gamma_g$ and $\Delta_g'^{(1)} = \Gamma'_g$. Since the rational maps

$$U_f \to U_{f'} : (x,y) \mapsto \left(x, \frac{1-xy^2}{x^{\frac{g}{4}+1}y}\right)$$

$$U_{f'} \to U_f : (x,y) \mapsto \left(x, \frac{2y}{x^{\frac{g}{4}+1}(1+y^2)} \right)$$

are inverses of each other, we conclude that C_f and $C_{f'}$ are isomorphic. Therefore Γ_g is not intrinsic to C_f, and neither is Γ'_g.

In spite of these exceptions we believe that 'most' interior lattice polygons are intrinsic, but making this statement precise (let along proving this) seems to be a hard task. Using Theorem 15 and Theorem 30 we can settle some additional cases, though:

- $\#(\Delta^{(1)} \cap \mathbf{Z}^2) \geq 5$ and $\Delta^{(2)} = \emptyset$, which holds if and only if C_f is trigonal of genus $g \geq 5$, or isomorphic to a smooth plane quintic,
- $\mathrm{lw}(\Delta^{(1)}) = 2$ and $\#(\partial\Delta^{(1)} \cap \mathbf{Z}^2) \geq \#(\Delta^{(2)} \cap \mathbf{Z}^2) + 5$, which holds if and only if C_f is tetragonal and $b_1 \geq b_2 + 2$.

In both cases, the bare line of accompanying text does not suffice to conclude intrinsicness: more details and refined statements can be found in Chapter 9. Let us remark that in both situations $X_{\Delta^{(1)}}$ can be easily recovered from the canonical model $C_f^{\mathrm{can}} \subseteq \mathbf{P}^{g-1}$. Indeed, in the former case it arises as the intersection of all quadrics containing C_f^{can}. In the latter case it is the unique surface containing C_f^{can} that is linearly equivalent to $2H - b_1R$, when viewed as a divisor inside the scroll spanned by a g^1_4. Our most subtle intrinsicness result, which strongly relies on Corollary 33, is:

Theorem 39. *Let $a, b \geq 1$ be integers that are not both equal to 1. Then the interior polygon $\square_{a,b}$ is intrinsic. More precisely let Δ be a two-dimensional lattice polygon, let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial, and assume that U_f is birationally equivalent to a smooth projective curve in $\mathbf{P}^1 \times \mathbf{P}^1$ of bidegree $(a+2, b+2)$. Then $\Delta^{(1)} \cong \square_{a,b}$.*

A proof can be found in Chapter 10.

Remark. One can also target weaker intrinsicness questions, by only distinguishing between polygons that belong to some given family:

- A weakly $\Delta_{a,b}$-nondegenerate curve cannot be weakly $\Delta_{a',b'}$-nondegenerate for distinct pairs of coprime positive integers $\{a, b\}$ and $\{a', b'\}$. This is immediate from our combinatorial interpretations for the genus

  $$\#(\Delta_{a,b}^{(1)} \cap \mathbf{Z}^2) = (a-1)(b-1)/2$$

  and the gonality

  $$\mathrm{lw}(\Delta_{a,b}^{(1)}) + 2 = \mathrm{lw}(\Delta_{a,b}) = \min\{a, b\}.$$

  In other words a $C_{a,b}$ curve cannot be $C_{a',b'}$.
- A similar reasoning involving the scrollar invariants shows that if a smooth non-hyperelliptic curve C/k of genus $g \geq 2$ can be embedded in the n^{th} Hirzebruch surface $\mathcal{H}_n$ for some $n \geq 0$, then this value of n is unique and can therefore be considered an invariant of C. We refer to Chapter 5 for an elaboration of the details.

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ON NONDEGENERACY OF CURVES

WOUTER CASTRYCK AND JOHN VOIGHT

Abstract. We study the conditions under which an algebraic curve can be modelled by a Laurent polynomial that is nondegenerate with respect to its Newton polytope. We prove that every curve of genus $g \leq 4$ over an algebraically closed field is nondegenerate in the above sense. More generally, let $\mathcal{M}_g^{\mathrm{nd}}$ be the locus of nondegenerate curves inside the moduli space of curves of genus $g \geq 2$. Then we show that $\dim \mathcal{M}_g^{\mathrm{nd}} = \min(2g+1, 3g-3)$, except for $g = 7$ where $\dim \mathcal{M}_7^{\mathrm{nd}} = 16$; thus, a generic curve of genus g is nondegenerate if and only if $g \leq 4$.
Subject classification: 14M25, 14H10

Let k be a perfect field with algebraic closure $\overline{k}$. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be an irreducible Laurent polynomial, and write $f = \sum_{(i,j) \in \mathbb{Z}^2} c_{ij} x^i y^j$. We denote by $\mathrm{supp}(f) = \{(i,j) \in \mathbb{Z}^2 : c_{ij} \neq 0\}$ the support of f, and we associate to f its Newton polytope $\Delta = \Delta(f)$, the convex hull of $\mathrm{supp}(f)$ in $\mathbb{R}^2$. We assume throughout that Δ is 2-dimensional. For a face $\tau \subset \Delta$, let $f|_\tau = \sum_{(i,j) \in \tau} c_{ij} x^i y^j$. We say that f is *nondegenerate* if, for every face $\tau \subset \Delta$ (of any dimension), the system of equations

$$f|_\tau = x \frac{\partial f|_\tau}{\partial x} = y \frac{\partial f|_\tau}{\partial y} = 0 \tag{1}$$

has no solutions in $\overline{k}^{*2}$.

From the perspective of toric varieties, the condition of nondegeneracy can be rephrased as follows. The Laurent polynomial f defines a curve $U(f)$ in the torus $\mathbb{T}_k^2 = \mathrm{Spec}\, k[x^{\pm 1}, y^{\pm 1}]$, and $\mathbb{T}_k^2$ embeds canonically in the projective toric surface $X(\Delta)_k$ associated to Δ over k. Let $V(f)$ be the Zariski closure of the curve $U(f)$ inside $X(\Delta)_k$. Then f is nondegenerate if and only if for every face $\tau \subset \Delta$, we have that $V(f) \cap \mathbb{T}_\tau$ is smooth of codimension 1 in $\mathbb{T}_\tau$, where $\mathbb{T}_\tau$ is the toric component of $X(\Delta)_k$ associated to τ. (See Proposition 1.2 for alternative characterizations.)

Nondegenerate polynomials have become popular objects in explicit algebraic geometry, owing to their connection with toric geometry [4]: a wealth of geometric information about $V(f)$ is contained in the combinatorics of the Newton polytope $\Delta(f)$. The notion was initially employed by Kouchnirenko [24], who studied nondegenerate polynomials in the context of singularity theory. Nondenegerate polynomials emerge naturally in the theory of sparse resultants [15] and admit a linear effective Nullstellensatz [8, Section 2.3]. They make an appearance in the study of real algebraic curves in maximal position [28] and in the problem of enumerating curves through a set of prescribed points [29]. In the case where k is a finite field, they arise in the construction of curves with many points [6, 25], in the p-adic cohomology theory of Adolphson and Sperber [2], and in explicit methods for computing zeta functions of varieties over k [8]. Despite their utility and seeming ubiquity, the *intrinsic* property of nondegeneracy has not seen detailed study, with the exception of the Ph.D. thesis of Koelman [22] from 1991, otherwise unpublished (see Section 12 below).

We are therefore led to the central problem of this article: *Which curves are nondegenerate?* To the extent that toric varieties are generalizations of projective space, this question asks us to generalize the characterization of nonsingular plane curves amongst all curves. An immediate provocation for this question was to understand the locus of curves to which the point counting algorithm of Castryck–Denef–Vercauteren [8] actually applies. Our results are collected in two parts.

In the first part, comprising Sections 3–7, we investigate the nondegeneracy of some interesting classes of curves (hyperelliptic, C_{ab}, and low genus curves). Our conclusions can be summarized as follows.

Theorem. *Let V be a curve of genus g over a perfect field k. Suppose that one of the following conditions holds:*

(i) $g = 0$;
(ii) $g = 1$ *and* $V(k) \neq \emptyset$;
(iii) $g = 2, 3$, *and either* $17 \leq \#k < \infty$, *or* $\#k = \infty$ *and* $V(k) \neq \emptyset$;
(iv) $g = 4$ *and* $k = \overline{k}$.

Then V is nondegenerate.

Remark. The condition $\#k \geq 17$ in (iii) ensures that k is large enough to allow nontangency to the toric boundary of $X(\Delta)_k$, but is most likely superfluous; see Remark 7.2.

In the second part, consisting of Sections 8–12, we restrict to algebraically closed fields $k = \overline{k}$ and consider the locus $\mathcal{M}_g^{\mathrm{nd}}$ of nondegenerate curves inside the coarse moduli space of all curves of genus $g \geq 2$. We prove the following theorem.

Theorem. *We have* $\dim \mathcal{M}_g^{\mathrm{nd}} = \min(2g+1, 3g-3)$, *except for* $g = 7$ *where* $\dim \mathcal{M}_7^{\mathrm{nd}} = 16$. *In particular, a generic curve of genus g is nondegenerate if and only if $g \leq 4$.*

Our methods combine ideas of Bruns–Gubeladze [7] and Haase–Schicho [17] and are purely combinatorial—only the universal property of the coarse moduli space is used.

Conventions and notations. Throughout, $\Delta \subset \mathbb{R}^2$ will denote a polytope with $\dim \Delta = 2$. The coordinate functions on the ambient space $\mathbb{R}^2$ will be denoted by X and Y. A *facet* or *edge* of a polytope is a face of dimension 1. A *lattice polytope* is a polytope with vertices in $\mathbb{Z}^2$. Two lattice polytopes Δ and Δ' are *equivalent* if there is an affine map

$$\begin{aligned} \varphi : \mathbb{R}^2 &\to \mathbb{R}^2 \\ v &\mapsto Av + b \end{aligned}$$

such that $\varphi(\Delta) = \Delta'$ with $A \in GL_2(\mathbb{Z})$ and $b \in \mathbb{Z}^2$. Two Laurent polynomials f and f' are *equivalent* if f' can be obtained from f by applying such a map to the exponent vectors. Note that equivalence preserves nondegeneracy. For a polytope $\Delta \subset \mathbb{R}^2$, we let $\mathrm{int}(\Delta)$ denote the interior of Δ. We denote the standard 2-simplex in $\mathbb{R}^2$ by $\Sigma = \mathrm{conv}(\{(0,0),(1,0),(0,1)\})$.

ON NONDEGENERACY OF CURVES

1. Nondegenerate Laurent polynomials

In this section, we review the geometry of nondegenerate Laurent polynomials. We retain the notation used in the introduction: in particular, k is a perfect field, $f = \sum c_{ij} x^i y^j \in k[x^{\pm 1}, y^{\pm 1}]$ is an irreducible Laurent polynomial, and Δ is its Newton polytope. Our main implicit reference on toric varieties is Fulton [14].

Let $k[\Delta]$ denote the graded semigroup algebra over k generated in degree d by the monomials that are supported in $d\Delta$, i.e.

$$k[\Delta] = \bigoplus_{d=0}^{\infty} \langle x^i y^j t^d \mid (i,j) \in (d\Delta \cap \mathbb{Z}^2) \rangle_k.$$

Then $X = X(\Delta)_k = \operatorname{Proj} k[\Delta]$ is the projective toric surface associated to Δ over k. This surface naturally decomposes into toric components as

$$X = \bigsqcup_{\tau \subset \Delta} \mathbb{T}_\tau,$$

where τ ranges over the faces of Δ and $\mathbb{T}_\tau \cong \mathbb{T}_k^{\dim \tau}$. The surface X is nonsingular except possibly at the zero-dimensional toric components associated to the vertices of Δ. The Laurent polynomial f defines a curve in $\mathbb{T}_k^2 \cong \mathbb{T}_\Delta \subset X$, and we denote by $V = V(f)$ its closure in X. Alternatively, if we denote $A = \Delta \cap \mathbb{Z}^2$, then X can be canonically embedded in $\mathbb{P}_k^{\#A-1} = \operatorname{Proj} k[t_{ij}]_{(i,j)\in A}$, and V is the hyperplane section $\sum c_{ij} t_{ij} = 0$ of X.

We abbreviate $\partial_x = x\dfrac{\partial}{\partial x}$ and $\partial_y = y\dfrac{\partial}{\partial y}$.

Definition 1.1. *The Laurent polynomial f is* nondegenerate *if for each face $\tau \subset \Delta$, the system*

$$f|_\tau = \partial_x f|_\tau = \partial_y f|_\tau = 0$$

has no solution in $\overline{k}^{*2}$.

We will sometimes write that f is Δ-nondegenerate to emphasize that $\Delta(f) = \Delta$.

Proposition 1.2. *The following statements are equivalent.*

(i) *f is nondegenerate.*

(ii) *For each face $\tau \subset \Delta$, the ideal of $k[x^{\pm 1}, y^{\pm 1}]$ generated by*

$$f|_\tau, \partial_x f|_\tau, \partial_y f|_\tau$$

is the unit ideal.

(iii) *For each face $\tau \subset \Delta$, the intersection $V \cap \mathbb{T}_\tau$ is smooth of codimension 1 in the torus orbit $\mathbb{T}_\tau$ associated to τ.*

(iv) *The sequence of elements $f, \partial_x f, \partial_y f$ (in degree one) forms a regular sequence in $k[\Delta]$.*

(v) *The quotient of $k[\Delta]$ by the ideal generated by $f, \partial_x f, \partial_y f$ is finite of k-dimension equal to $2\operatorname{vol}(\Delta)$.*

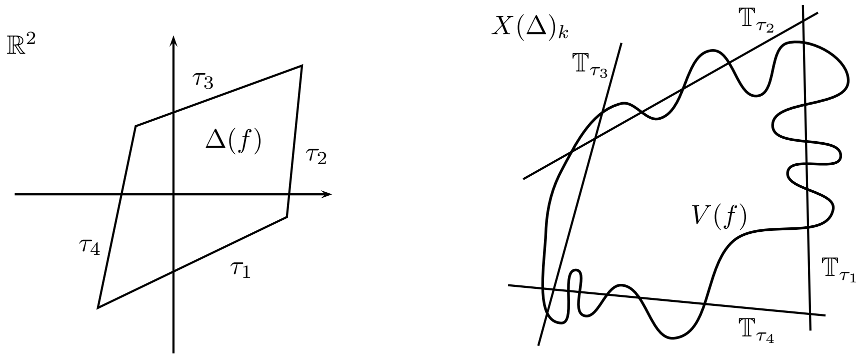
Remark 1.3. Condition (iii) can also be read as: V is smooth and intersects $X \setminus \mathbb{T}_k^2$ transversally and outside the zero-dimensional toric components associated to the vertices of Δ.

Proof. See Batyrev [3, Section 4] for a proof of these equivalences and further discussion. □

Remark 1.4. Some authors refer to nondegenerate as Δ-*regular*, though we will not employ this term. The use of *nondegenerate* to indicate a projective variety which is not contained in a smaller projective space is unrelated to our present usage.

Example 1.5. Let $f(x,y) \in k[x,y]$ be a bivariate polynomial of degree $d \in \mathbb{Z}_{\geq 1}$ with Newton polytope $\Delta = d\Sigma = \operatorname{conv}(\{(0,0),(d,0),(0,d)\})$. The toric variety $X(\Delta)_k$ is the d-uple Veronese embedding of $\mathbb{P}^2_k$ in $\mathbb{P}^{d(d+3)/2}_k$, and $V(f)$ is the projective curve in $\mathbb{P}^2_k$ defined by the homogenization $F(x,y,z)$ of f. We see that $f(x,y)$ is Δ-nondegenerate if and only if $V(f)$ is nonsingular, does not contain the coordinate points $(0,0,1)$, $(0,1,0)$ and $(1,0,0)$, and is not tangent to any coordinate axis.

Example 1.6. The following picture illustrates nondegeneracy in case of a quadrilateral Newton polytope.



Proposition 1.7. *If $f \in k[x^{\pm 1}, y^{\pm 1}]$ is nondegenerate, then there exists a k-rational canonical divisor K_Δ on $V = V(f)$ such that $\{x^i y^j : (i,j) \in \operatorname{int}(\Delta) \cap \mathbb{Z}^2\}$ is a k-basis for the Riemann-Roch space $\mathcal{L}(K_\Delta) \subset k(V)$. In particular, the genus of V is equal to $\#(\operatorname{int}(\Delta) \cap \mathbb{Z}^2)$.*

Proof. See Khovanskiĭ [21] or Castryck–Denef–Vercauteren [8, Section 2.2]. □

Remark 1.8. In general, if f is irreducible (but not necessarily nondegenerate), one has that the geometric genus of $V(f)$ is bounded by $\#(\operatorname{int}(\Delta) \cap \mathbb{Z}^2)$: this is also known as Baker's inequality [6, Theorem 4.2].

We conclude this section with the following intrinsic definition of nondegeneracy.

Definition 1.9. *A curve V over k is* Δ-nondegenerate *if V is birational over k to a curve $U \subset \mathbb{T}^2_k$ defined by a nondegenerate Laurent polynomial f with Newton polytope Δ. The curve V is* nondegenerate *if it is Δ-nondegenerate for some Δ. The curve V is* geometrically nondegenerate *if $V \times_k \overline{k}$ is nondegenerate over $\overline{k}$.*

2. Moduli of nondegenerate curves

We now construct the moduli space of nondegenerate curves of given genus $g \geq 2$. Since in this article we will be concerned with dimension estimates only, we restrict to the case $k = \overline{k}$.

We denote by $\mathcal{M}_g$ the coarse moduli space of curves of genus $g \geq 2$ over k, with the property that for any flat family $\mathcal{V} \to M$ of curves of genus g, there is a (unique) morphism $M \to \mathcal{M}_g$ which maps each closed point $f \in M$ to the isomorphism class of the fiber $\mathcal{V}_f$. (See e.g. Mumford [31, Theorem 5.11].)

Let $\Delta \subset \mathbb{R}^2$ be a lattice polytope with g interior lattice points. We will construct a flat family $\mathcal{V}(\Delta) \to M_\Delta$ which parametrizes all Δ-nondegenerate curves over k. The key ingredient is provided by the following result of Gel'fand–Kapranov–Zelevinsky. Let $A = \Delta \cap \mathbb{Z}^2$ and define the polynomial ring $R_\Delta = k[c_{ij}]_{(i,j)\in A}$.

Proposition 2.1 (Gel'fand–Kapranov–Zelevinsky [15]). *There exists a polynomial $E_A \in R_\Delta$ with the property that for any Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$ with $\operatorname{supp}(f) \subset \Delta$, we have that f is Δ-nondegenerate if and only if $E_A(f) \neq 0$.*

Proof. The proof of Gel'fand–Kapranov–Zelevinsky [15, Chapter 10] is over $\mathbb{C}$; however, the construction yields a polynomial over $\mathbb{Z}$ which is easily seen to characterize nondegeneracy for an arbitrary field. □

The polynomial E_A is known as the *principal A-determinant* and is given by the *A-resultant* $\operatorname{res}_A(F, \partial_1 F, \partial_2 F)$. It is homogeneous in the variables c_{ij} of degree $6\operatorname{vol}(\Delta)$, and its irreducible factors are the *face discriminants* D_τ for faces $\tau \subset \Delta$.

Example 2.2. Consider the universal plane conic

$$F = c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2,$$

associated to the Newton polytope 2Σ as in Example 1.5.

Then

$$E_A = c_{00}c_{02}c_{20}(c_{11}^2 - 4c_{02}c_{20})(c_{10}^2 - 4c_{00}c_{20})(c_{01}^2 - 4c_{00}c_{02})D_\Delta$$

where

$$D_\Delta = 4c_{00}c_{20}c_{02} - c_{00}c_{11}^2 - c_{10}^2c_{02} - c_{01}^2c_{20} + c_{10}c_{01}c_{11}.$$

The nonvanishing of the factor $c_{00}c_{02}c_{20}$ (corresponding to the discriminants of the zero-dimensional faces) ensures that the curve does not contain a coordinate point, and in particular does not have Newton polytope smaller than 2Σ; the nonvanishing of the quadratic factors (corresponding to the one-dimensional faces) ensures that the curve intersects the coordinate lines in two distinct points; and the nonvanishing of D_Δ ensures that the curve is smooth.

Let M_Δ be the complement in $\mathbb{P}_k^{\#A-1} = \operatorname{Proj} R_\Delta$ of the algebraic set defined by E_A. By the above, M_Δ parameterizes nondegenerate polynomials having Δ as Newton polytope. One can show that

$$\dim M_\Delta = \#A - 1, \tag{2}$$

which is a non-trivial statement if k is of finite characteristic (and false in general for an arbitrary number of variables), see [8, Section 2]. Let $\mathcal{V}(\Delta)$ be the closed subvariety of

$$X(\Delta)_k \times M_\Delta \subset \operatorname{Proj} k[t_{ij}] \times \operatorname{Proj} k[c_{ij}]$$

defined by the universal hyperplane section

$$\sum_{(i,j)\in A} c_{ij}t_{ij} = 0.$$

Then the universal family of Δ-nondegenerate curves is realized by the projection map $\varphi : \mathcal{V}(\Delta) \to M_\Delta$. The fiber $\mathcal{V}(\Delta)_f$ above a nondegenerate Laurent polynomial $f \in M_\Delta$ is precisely the corresponding curve $V(f)$, realized as the corresponding hyperplane section of $X(\Delta)_k \subset \operatorname{Proj} k[t_{ij}]$. Note that φ is indeed flat [19, Theorem III.9.9], since the Hilbert polynomial of $\mathcal{V}(\Delta)_f$ is independent of f: its degree is equal to $\deg X(\Delta)_k$ and its genus is g by Proposition 1.7.

Thus by the universal property of $\mathcal{M}_g$, there is a morphism $h_\Delta : M_\Delta \to \mathcal{M}_g$, the image of which consists precisely of all isomorphism classes containing a Δ-nondegenerate curve. Let $\mathcal{M}_\Delta$ denote the Zariski closure of the image of h_Δ. Finally, let

$$\mathcal{M}_g^{\mathrm{nd}} = \bigcup_{g(\Delta)=g} \mathcal{M}_\Delta,$$

where the union is taken over all polytopes Δ with g interior lattice points, of which there are finitely many up to equivalence (see Hensley [20]).

The aim of Sections 8–12 is to estimate $\dim \mathcal{M}_g^{\mathrm{nd}}$. This is done by first refining the obvious upper bounds $\dim \mathcal{M}_\Delta \leq \dim M_\Delta = \#(\Delta \cap \mathbb{Z}^2) - 1$, taking into account the action of the automorphism group $\mathrm{Aut}(X(\Delta)_k)$, and then estimating the outcome in terms of g.

Remark 2.3. It follows from the fact that $\mathcal{M}_g$ is of general type for $g \geq 23$ (see e.g. [18]) that $\dim \mathcal{M}_g^{\mathrm{nd}} < \dim \mathcal{M}_g = 3g - 3$ for $g \geq 23$, since each component of $\mathcal{M}_g^{\mathrm{nd}}$ is unirational. Below, we obtain much sharper results which do not rely on this deep statement.

3. Triangular nondegeneracy

In Sections 4–6, we study the nondegeneracy of certain well-known classes, such as elliptic, hyperelliptic and C_{ab} curves. In many cases, classical constructions provide models for these curves that are supported on a triangular Newton polytope; the elementary observations in this section will allow us to prove that these models are nondegenerate when $\#k$ is not too small.

Lemma 3.1. *Let $f(x, y) \in k[x, y]$ define a smooth affine curve of genus g and suppose that $\#k > 2(g + \max(\deg_x f, \deg_y f) - 1) + \min(\deg_x f, \deg_y f)$. Then there exist $x_0, y_0 \in k$ such that the translated curve $f(x - x_0, y - y_0)$ does not contain $(0, 0)$ and is also nontangent to both the x- and the y-axis.*

Proof. Suppose $\deg_y f \leq \deg_x f$. Applying the Riemann-Hurwitz theorem to the projection map $(x, y) \mapsto x$, one verifies that there are at most $2(g + \deg_y f - 1)$ points with a vertical tangent. Therefore, we can find an $x_0 \in k$ such that $f(x - x_0, y)$ is nontangent to the y-axis. Subsequently, there are at most $2(g + \deg_x f - 1) + \deg_y f$ values of $y_0 \in k$ for which $f(x - x_0, y - y_0)$ is tangent to the x-axis and/or contains $(0, 0)$. □

Lemma 3.2. *Let $a \leq b \in \mathbb{Z}_{\geq 2}$ be such that $\gcd(a, b) \in \{1, a\}$, and let Δ be the triangular lattice polytope $\mathrm{conv}(\{(0, 0), (b, 0), (0, a)\})$. Let $f(x, y) \in k[x, y]$ be an irreducible polynomial such that:*

- *f is supported on Δ, and*
- *the genus of $V(f)$ equals $g = \#(\mathrm{int}(\Delta) \cap \mathbb{Z}^2)$.*

Then if $\#k > 2(g + b - 1) + a$, we have that $V(f)$ is Δ-nondegenerate.

Proof. First suppose that $\gcd(a, b) = 1$. The coefficients of x^b and y^a must be nonzero, because else $\#(\mathrm{int}(\Delta(f)) \cap \mathbb{Z}^2) < g$, which contradicts Baker's inequality. For the same reason, f must define a smooth affine curve: if (x_0, y_0) is a singular point (over $\overline{k}$), then $\#(\mathrm{int}(\Delta(f(x - x_0, y - y_0)) \cap \mathbb{Z}^2)) < g$. The result now follows from Lemma 3.1. Note that the nonvanishing of the face discriminant D_τ, where τ

is the edge connecting $(b, 0)$ and $(0, a)$, follows automatically from the fact that τ has no interior lattice points.

Next, suppose that $\gcd(a, b) = a$. Then we may assume that the coefficients of x^b and y^a are nonzero. Indeed, if $a < b$ then the coefficient of y^a must be nonzero. Let $g(t) \in k[t]$ be the coefficient of x^b in $f(x, y + tx^{b/a})$. It is of degree a and therefore has a non-root $t_0 \in k$. Then substituting $y \leftarrow y + t_0x^{b/a}$ ensures that the coefficient of x^b is nonzero as well. If $a = b$ then the coefficient of y^a might be zero, but f must contain at least one non-zero term of total degree a, and a similar argument proves the claim.

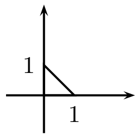
Then as above, we have that f defines a smooth affine curve. So by applying Lemma 3.1, we may assume that the face discriminants decomposing $E_{\Delta \cap \mathbb{Z}^2}$ are nonvanishing at f, with the possible exception of D_τ, where τ is the edge connecting $(b, 0)$ and $(0, a)$. However, under the equivalence

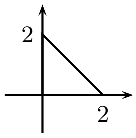
$$\mathbb{R}^2 \to \mathbb{R}^2 : (X, Y) \mapsto (b - X - \frac{b}{a}Y, Y),$$

τ is interchanged with the edge connecting $(0, 0)$ and $(0, a)$. By applying Lemma 3.1 again, we obtain full nondegeneracy. □

4. Nondegeneracy of curves of genus at most one

Curves of genus 0. Let V be a curve of genus 0 over k. The anticanonical divisor embeds $V \hookrightarrow \mathbb{P}^2_k$ as a smooth conic. If $\#k = \infty$, then by Lemma 3.2 and Proposition 1.2, we see that V is nondegenerate. If $\#k < \infty$ then $V(k) \neq \emptyset$ by Wedderburn, hence $V \cong \mathbb{P}^1_k$ can be embedded as a nondegenerate line in $\mathbb{P}^2$. Therefore, any curve V of genus 0 is Δ-nondegenerate, where Δ is one of the following:

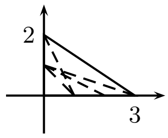




Curves of genus 1. Let V be a curve of genus 1 over k. First suppose that $V(k) \neq \emptyset$. Then V is an elliptic curve and hence can be defined by a nonsingular Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{3}$$

with $a_i \in k$. The corresponding Newton polytope Δ is



where one of the dashed lines appears as a facet if $a_6 = 0$. By Lemma 3.2, we have that V is nondegenerate if $\#k \geq 9$. With some extra work we can get rid of this condition.

For $A = \Delta \cap \mathbb{Z}^2$, the principal A-determinant has 7 or 9 face discriminants D_τ as irreducible factors. The nonvanishing of D_Δ corresponds to the fact that our curve is smooth in $\mathbb{T}^2_k$. In case τ is a vertex or a facet containing no interior lattice points, the nonvanishing of D_τ is automatic. Thus it suffices to consider the discriminants D_τ for τ a facet supported on the X-axis (denoted τ_X) or the

Y-axis (denoted τ_Y). First, suppose that $\operatorname{char} k \neq 2$. After completing the square, we have $a_1 = a_3 = 0$ and the nonvanishing of D_{τ_X} follows from the fact that the polynomial $p(x) = x^3 + a_2x^2 + a_4x + a_6$ is squarefree. The nonvanishing of D_{τ_Y} (if τ_Y exists) is clear. Now suppose $\operatorname{char} k = 2$. Let δ be the number of distinct roots (over $\overline{k}$) of $p(x) = x^3 + a_2x^2 + a_4x + a_6$. If $\delta = 3$ then D_{τ_X} is non-vanishing. For the nonvanishing of D_{τ_Y}, it then suffices to substitute $x \leftarrow x + 1$ if necessary, so that a_3 is nonzero (note that not both a_1 and a_3 can be zero). If $\delta < 3$ then $p(x)$ has a root x_0 of multiplicity at least 2. Since k is perfect, this root is k-rational and after substituting $x \mapsto x + x_0$ we have $p(x) = x^3 + a_2x^2$. In particular, D_{τ_X} (if τ_X exists) and D_{τ_Y} do not vanish.

In conclusion, we have shown that every genus 1 curve V over a field k is nondegenerate, given that $V(k) \neq \emptyset$. This condition is automatically satisfied if k is a finite field (by Hasse–Weil) or if k is algebraically closed. In particular, every genus 1 curve is geometrically nondegenerate. More generally, we define the *index* of a curve V over a field k to be the least degree of an effective non-zero k-rational divisor on V (equivalently, the least extension degree of a field $L \supset k$ for which $V(L) \neq \emptyset$). We then have the following criterion.

Lemma 4.1. *A curve V of genus 1 is nondegenerate if and only if V has index at most 3.*

Proof. First, assume that V is nondegenerate. There are exactly 16 equivalence classes of polytopes with only 1 interior lattice point; see [34, Figure 2] or the appendix at the end of this article. So we may assume that V is Δ-nondegenerate with Δ in this list. Now for every facet $\tau \subset \Delta$, the toric component $\mathbb{T}_\tau$ of $X(\Delta)_k$ cuts out an effective k-rational divisor of degree $\ell(\tau)$ on V, where $\ell(\tau) + 1$ is the number of lattice points on τ. The result then follows, since one easily verifies that every polytope in the list contains a facet τ with $\ell(\tau) \leq 3$.

Conversely, suppose that V has index $\imath \leq 3$. If $\imath = 1$, we have shown above that V is nondegenerate. If $\imath = 2$ (resp. $\imath = 3$), using Riemann-Roch one can construct a plane model $f \in k[x, y]$ with $\Delta(f) \subset \operatorname{conv}(\{(0,0),(4,0),(0,2)\})$ (resp. $\Delta(f) \subset 3\Sigma$); see e.g. Fisher [13, Section 3] for details. Then since $V(k) = \emptyset$ and hence $\#k = \infty$, an application of Lemma 3.2 concludes the proof. □

Remark 4.2. There exist genus 1 curves of arbitrarily large index over every number field; see Clark [9]. Hence there exist infinitely many genus 1 curves which are not nondegenerate.

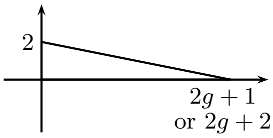
5. Nondegeneracy of hyperelliptic curves and C_{ab} curves

Hyperelliptic curves. A curve V over k of genus $g \geq 2$ is *hyperelliptic* if there exists a nonconstant morphism $V \to \mathbb{P}^1_k$ of degree 2. The morphism is automatically separable [19, Proposition IV.2.5] and the curve can be defined by a Weierstrass equation

$$y^2 + q(x)y = p(x). \tag{4}$$

Here $p(x), q(x) \in k[x]$ satisfy $2 \deg q(x) \leq \deg p(x)$ and $\deg p(x) \in \{2g+1, 2g+2\}$, as long as $k \neq \mathbb{F}_2$: see Enge [12, Theorem 7]. (This condition will fail for any hyperelliptic curve C over $k = \mathbb{F}_2$ for which the degree 2 morphism $\pi : C \to \mathbb{P}^1$ splits completely over k, meaning that above each point $0, 1, \infty \in \mathbb{P}^1(k)$ there are two distinct k-rational points of C.) For the rest of this subsection, we suppose $k \neq \mathbb{F}_2$, and we leave the small modifications in this case to the reader.

The universal such curve has Newton polytope as follows:



By Lemma 3.2, if $\#k \geq 6g+5$ then V is nondegenerate. In particular, if $\#k \geq 17$ then every curve of genus 2 is nondegenerate.

If $\operatorname{char} k \neq 2$, we can drop the condition on $\#k$ by completing the square, as in the elliptic curve case. This observation immediately weakens the condition to $\#k \geq 2^{\lfloor \log_2(6g+5) \rfloor} + 1$. As a consequence, $\#k \geq 17$ is also sufficient for every hyperelliptic curve of genus 3 or 4 to be nondegenerate.

Conversely, any curve defined by a nondegenerate polynomial as in (4) is hyperelliptic. We conclude that $\dim \mathcal{M}_\Delta = \dim \mathcal{H}_g = 2g-1$ [19, Example IV.5.5.5].

One can decide if a nondegenerate polynomial f defines a hyperelliptic curve according to the following criterion, which also appears in Koelman [22, Lemma 3.2.9] with a more complicated proof.

Lemma 5.1. *Let $f \in k[x^{\pm}, y^{\pm}]$ be nondegenerate and suppose $\# \operatorname{int}(\Delta(f) \cap \mathbb{Z}^2) \geq 2$. Then $V(f)$ is hyperelliptic if and only if the interior lattice points of $\Delta(f)$ are collinear.*

Proof. We may assume that $\Delta = \Delta(f)$ has $g \geq 3$ interior lattice points, since all curves of genus 2 are hyperelliptic and any two points are collinear.

Let $L \subset k(V)$ be the subfield generated by all quotients of functions in $\mathcal{L}(K)$, where K is a canonical divisor on V. Then L does not depend on the choice of K, and L is isomorphic to the rational function field $k(\mathbb{P}^1_k)$ if and only if V is hyperelliptic.

We now show that $L \cong k(\mathbb{P}^1_k)$ if and only if the interior lattice points of Δ are collinear. We may assume that $(0,0)$ is in the interior of Δ. Then from Proposition 1.7, we see that L contains all monomials of the form $x^i y^j$ for $(i,j) \in \operatorname{int}(\Delta) \cap \mathbb{Z}^2$. In particular, if the interior lattice points of Δ are not collinear then after a transformation we may assume further that $(0,1),(1,0) \in \operatorname{int}(\Delta)$, whence $L \supset k(x,y) = k(V)$; and if they are collinear, then clearly $L \cong k(\mathbb{P}^1_k)$. The result then follows. $\square$

For this reason, we call a lattice polytope *hyperelliptic* if its interior lattice points are collinear.

A curve V over k of genus $g \geq 2$ is called *geometrically hyperelliptic* if $V_{\overline{k}} = V \times_k \overline{k}$ is hyperelliptic. Every hyperelliptic curve is geometrically hyperelliptic, but not conversely: if $V \to C \subset \mathbb{P}^{g-1}_k$ is the canonical morphism, then V is hyperelliptic if and only if $C \cong \mathbb{P}^1_k$. This latter condition is satisfied if and only if $C(k) \neq \emptyset$, which is guaranteed when k is finite, when $V(k) \neq \emptyset$, and when g is even.

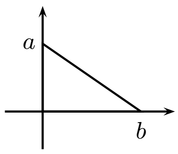
Lemma 5.2. *Let V be a geometrically hyperelliptic curve which is nonhyperelliptic. Then V is not nondegenerate.*

Proof. Suppose that V is geometrically hyperelliptic and Δ-nondegenerate for some lattice polytope Δ. Then applying Lemma 5.1 to $V_{\overline{k}}$, we see that the interior lattice points of Δ are collinear. But then again by Lemma 5.1 (now applied to V itself), V must be hyperelliptic. □

$\mathbf{C_{ab}}$ curves. Let $a, b \in \mathbb{Z}_{\geq 2}$ be coprime. A C_{ab} *curve* is a curve having a place with Weierstrass semigroup $a\mathbb{Z}_{\geq 0} + b\mathbb{Z}_{\geq 0}$ (see Miura [30]). Any C_{ab} curve is defined by a Weierstrass equation

$$f(x,y) = \sum_{\substack{i,j \in \mathbb{N} \\ ai+bj \leq ab}} c_{ij} x^i y^j = 0. \tag{5}$$

with $c_{0a}, c_{b0} \neq 0$. By Lemma 3.2, if $\#k \geq 2(g + a + b - 2)$ then we may assume that this polynomial is nondegenerate with respect to its Newton polytope Δ_{ab}:



Conversely, every curve given by a Δ_{ab}-nondegenerate polynomial is C_{ab}, and the unique place dominating the point at projective infinity has Weierstrass semigroup $a\mathbb{Z}_{\geq 2} + b\mathbb{Z}_{\geq 2}$ (see Matsumoto [27]). Note that if k is algebraically closed, the class of hyperelliptic curves of genus g coincides with the class of $C_{2,2g+1}$ curves.

The moduli space of all C_{ab} curves (for varying a and b) of fixed genus g is then a finite union of moduli spaces $\mathcal{M}_{\Delta_{ab}}$. One can show that its dimension equals $2g-1$ by an analysis of the Weierstrass semigroup, which has been done in Rim–Vitulli [35, Corollary 6.3]. This dimension equals $\dim \mathcal{H}_g = \dim \mathcal{M}_{\Delta_{2,2g+1}}$ and in fact this is the dominating part: in Example 8.7 we will show that $\dim \mathcal{M}_{\Delta_{ab}} < 2g - 1$ if $a, b \geq 3$ and $g \geq 6$.

6. Nondegeneracy of curves of genus three and four

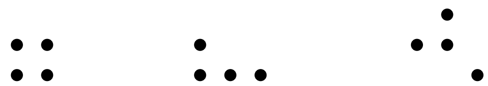
Curves of genus 3. A genus 3 curve V over k is either geometrically hyperelliptic or it canonically embeds in $\mathbb{P}^2_k$ as a plane quartic.

If V is geometrically hyperelliptic, then V may not be hyperelliptic and hence (by Lemma 5.2) not nondegenerate. For example, over $\mathbb{Q}$ there exist degree 2 covers of the imaginary circle having genus 3. However, if k is finite or $V(k) \neq 0$ then every geometrically hyperelliptic curve is hyperelliptic. If moreover $\#k \geq 17$ we can conclude that V is nondegenerate. See Section 5 for more details.

If V is embedded as a plane quartic, then assuming $\#k \geq 17$, we can apply Lemma 3.2 and see that V is defined by a 4Σ-nondegenerate Laurent polynomial.

Curves of genus 4. Let V be a curve of genus 4 over k. If V is a geometrically hyperelliptic curve then it is hyperelliptic, since the genus is even; thus if $\#k \geq 17$ then V is nondegenerate (see Section 5). Assume therefore that V is nonhyperelliptic. Then it canonically embeds as a curve of degree 6 in $\mathbb{P}^3_k$ which is the complete intersection of a unique quadric surface Q and a (non-unique) cubic surface C [19, Example IV.5.2.2].

First, we note that if V is Δ-nondegenerate for some nonhyperelliptic lattice polytope $\Delta \subset \mathbb{R}^2$, then Q or C must have combinatorial origins as follows. Let $\Delta^{(1)} = \mathrm{conv}(\mathrm{int}(\Delta) \cap \mathbb{Z}^2)$. Up to equivalence, there are three possible arrangements for these interior lattice points:



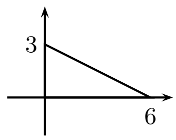
(a) (b) (c)

By Proposition 1.7, one verifies that V canonically maps to $X^{(1)} = X(\Delta^{(1)})_k \subset \mathbb{P}^3_k$. In (a), $X^{(1)}$ is nothing else but the Segre product $\mathbb{P}^1_k \times \mathbb{P}^1_k$ defined by the equation $xz = yw$ in $\mathbb{P}^3_k$, and by uniqueness it must equal Q. For (b), $X^{(1)}$ is the singular quadric cone $yz = w^2$, which again must equal Q. For (c), $X^{(1)}$ is the singular cubic $xyz = w^3$, which must be an instance of C. Note that a curve V can be Δ-nondegenerate with $\Delta^{(1)}$ as in (a) or (b), but not both: whether Q is smooth or not is intrinsic, since Q is unique. The third type (c) is special, and we leave it as an exercise to show that the locus of curves of genus 4 which canonically lie on such a singular cubic surface is a codimension ≥ 2 subspace of $\mathcal{M}_4$ (use the dimension bounds from Section 8).

With these observations in mind, we work towards conditions under which our given nonhyperelliptic genus 4 curve V is nondegenerate. Suppose first that the quadric Q has a (necessarily k-rational) singular point T; then V is called *conical*. This corresponds to the case where $V_{\overline{k}} = V \times_k \overline{k}$ has a unique g^1_3, and represents a codimension 1 subscheme of $\mathcal{M}_4$ [19, Exercise IV.5.3]. If $Q(k) = \{T\}$ then V cannot be nondegenerate with respect to any polytope with $\Delta^{(1)}$ as in (a) or (b), since then Q is not isomorphic to either of the corresponding canonical quadric surfaces $X^{(1)}$. If $Q(k) \supsetneq \{T\}$, which is guaranteed if k is finite or if $V(k) \neq \emptyset$, then after a choice of coordinates we can identify Q with the weighted projective space $\mathbb{P}(1, 2, 1)$. Our degree 6 curve V then has an equation of the form

$$f(x, y, z) = y^3 + a_2(x, z)y^2 + a_4(x, z)y + a_6(x, z)$$

with $\deg a_i = i$; the equation is monic in y because $T \notin V$. By Lemma 3.2, if $\#k \geq 23$ then we may assume that $f(x, y, 1)$ is nondegenerate with respect to its Newton polytope Δ as follows:

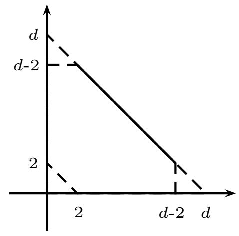


Remark 6.1. This argument shows that every conical genus 4 curve is *potentially nondegenerate*, i.e., becomes nondegenerate after a finite extension of k. In fact, we need only take a degree 2 extension which splits the quadric Q: after a k-rational linear change of variable, Q is the cone over a conic C over k, so we may take any field over which C acquires a rational point. This argument works even when $\operatorname{char} k = 2$.

Next, suppose that Q is smooth; then V is called *hyperboloidal*. This corresponds to the case where $V_{\overline{k}}$ has two g^1_3's, and represents a dense subscheme of $\mathcal{M}_4$ [19, Exercise IV.5.3]. If $Q \ncong \mathbb{P}^1_k \times \mathbb{P}^1_k$ (e.g. this will be the case whenever the discriminant of Q is nonsquare), then again V cannot be nondegenerate with respect to Δ with $\Delta^{(1)}$ as in (a) or (b). Therefore suppose that k is algebraically closed. Then $Q \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$ and V can be projected to a plane quintic with 2 nodes [19, Exercise IV.5.4].

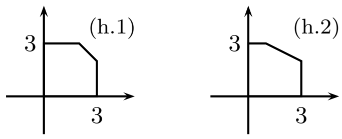
Consider the line connecting these nodes. Generically, it will intersect the nodes with multiplicity 2, i.e. it will intersect all branches transversally. By Bezout, the line will then intersect the curve transversally in one other point. This observation fits within the following general phenomenon. Let $d \in \mathbb{Z}_{\geq 4}$, and consider the polytope $\Delta = d\Sigma$ with up to three of its angles pruned as follows:

(6)



Let $f \in k[x, y]$ be a nondegenerate polynomial with Newton polytope Δ. If we prune no angle of $d\Sigma$, then $X(\Delta)_k \cong \mathbb{P}^2_k$ (it is the image of the d-uple embedding) and $V(f)$ is a smooth plane curve of degree d. Pruning an angle has the effect of blowing up $X(\Delta)_k$ at a coordinate point; the image of $V(f)$ under the natural projection $X(\Delta)_k \to \mathbb{P}^2_k$ has a node at that point. If we prune $m = 2$ (resp. $m = 3$) angles, then we likewise obtain the blow-up of $\mathbb{P}^2_k$ at m points and the image of $V(f)$ in $\mathbb{P}^2_k$ has m nodes. Since f is nondegenerate, the line connecting any two of these nodes intersects the curve transversally elsewhere, and due to the shape of Δ the intersection multiplicity at the nodes will be 2. Conversely, every projective plane curve having at most 3 nodes such that the line connecting any two nodes intersects the curve transversally (also at the nodes themselves), is nondegenerate. Indeed, after an appropriate projective transformation, it will have a Newton polytope as in (6).

Exceptionally, the line connecting the two nodes of our quintic may be tangent to one of the branches at a node. Using a similar reasoning, we conclude that V is Δ-nondegenerate, with Δ equal to polytope (h.2) from Section 7 below.



genus 4 hyperboloidal

Remark 6.2. Again, this argument can be used to show that any hyperboloidal curve of genus 4 is potentially nondegenerate. Standard results in the theory of quadratic forms over fields k with char$k \neq 2$ imply that Q splits, so that $Q \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$, if and only if $Q(k) \neq \emptyset$ and the discriminant of Q is a square in k: if $Q(k) \neq \emptyset$ then Q splits a hyperbolic plane; by scaling, the orthogonal complement is of the form $x^2 - dy^2$, so if $d \in k^{\times 2}$ then Q splits, and conversely. It follows that any quadric over k splits over an at most quadratic extension. To proceed, we then project V to a plane quintic, which requires $\#k$ to be sufficiently large: one could make this explicit, one could use the Bertini theorem over finite fields due to Poonen [33] and analyze explicitly the finitely many exceptions. Assuming that V has been so projected (extending k further, if necessary), the rest of the argument holds.

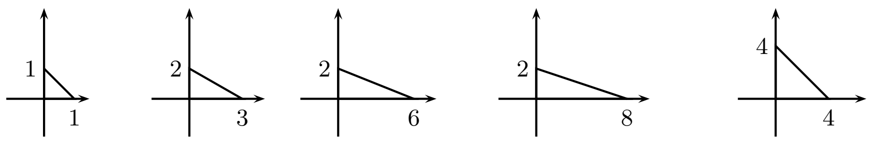
Now, however, we can make a further change of variables to bring all genus 4 hyperboloidal curves under a single polytope. Indeed, if $f(x, y)$ has a Newton polytope of type (h.1) or (h.2), then applying a change of variables to $x^3y^3f(x^{-1}, y^{-1})$

of the form $(x, y) \mapsto (x+a, y+b)$ for $a, b \in k$ yields a square 3×3 Newton polytope. So replacing the two polytopes of class (h) by the single polytope in the summary below.

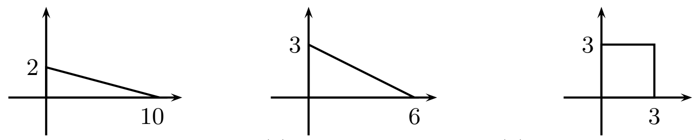
Remark 6.3. As in Remark 4.2, an argument based on the index shows that there exist genus 4 curves which are not nondegenerate. A result by Clark [10] states that for every $g \geq 2$, there exists a number field k and a genus g curve V over k, such that the index of V is equal to $2g - 2$, the degree of the canonical divisor. In particular, there exists a genus 4 curve V of index 6. Such a curve cannot be nondegenerate. Indeed, for each of the above arrangements (a)–(c), $X^{(1)}$ contains the line $z = w = 0$, which cuts out an effective divisor on V of degree 3 in cases (a) and (b) and degree 2 in case (c).

7. Nondegeneracy of low genus curves: summary

We now summarize the results of the preceding sections. If k is an algebraically closed field, then every curve V of genus at most 4 over k can be modeled by a nondegenerate polynomial having one of the following as Newton polytope:



(a) genus 0 (b) genus 1 (c) genus 2 (d) genus 3 hyperelliptic (e) genus 3 planar



(f) genus 4 hyperelliptic (g) genus 4 conical (h) genus 4 hyperboloidal

Moreover, each of these classes are disjoint. For the polytopes (c)–(h), we have $\dim \mathcal{M}_\Delta = 3, 5, 6, 7, 8, 9$, respectively. All hyperelliptic curves and C_{ab} curves are nondegenerate.

For an arbitrary perfect field k, if V is not hyperboloidal and has genus at most 4, then V is nondegenerate whenever k is a sufficiently large finite field, or when k is infinite and $V(k) \neq \emptyset$; for the former, the condition $\#k \geq 23$ is sufficient but most likely superfluous (see Remark 7.2).

Remark 7.1. We can situate the nonhyperelliptic C_{ab} curves that lie in this classification. In genus 3, we have $C_{3,4}$ curves, which have a smooth model in $\mathbb{P}^2_k$, since $\Delta_{3,4}$ is nonhyperelliptic. In genus 4, we have $C_{3,5}$ curves, which are conical: this can be seen by analyzing the interior lattice points of $\Delta_{3,5}$, as in Section 6.

Remark 7.2. In case $\#k < \infty$, we proved (without further condition on $\#k$) that if V is not hyperboloidal then it can be modeled by a polynomial $f \in k[x, y]$ with Newton polytope contained in one of the polytopes (a)–(g). The condition on $\#k$ then came along with an application of Lemma 3.2 to deduce nondegeneracy. In the $g = 1$ case, we got rid of this condition by using non-linear transformations (completing the square) and allowing smaller polytopes. Similar techniques can be used to improve (and probably even remove) the bounds on $\#k$ in genera $2 \leq g \leq 4$.

For example, using naive brute force computation we have verified that in genus 2, all curves are nondegenerate whenever $\#k = 2, 4, 8$.

8. An upper bound for $\dim \mathcal{M}_g^{\mathrm{nd}}$

From now on, we assume $k = \overline{k}$. In this section, we prepare for a proof of Theorem 11.1, which gives an upper bound for $\dim \mathcal{M}_g^{\mathrm{nd}}$ in terms of g.

For a lattice polytope $\Delta \subset \mathbb{Z}^2$ with $g \geq 2$ interior lattice points, we sharpen the obvious upper bound $\dim \mathcal{M}_\Delta \leq \dim M_\Delta = \#(\Delta \cap \mathbb{Z}^2) - 1$ (see (2)) by incorporating the action of the automorphism group of $X(\Delta)_k$, which has been explicitly described by Bruns and Gubeladze [7, Section 5]. In Sections 9–11 we then work towards a bound in terms of g, following ideas of Haase and Schicho [17].

The automorphisms of $X(\Delta)_k = \operatorname{Proj} k[\Delta] \hookrightarrow \mathbb{P}^{\#(\Delta\cap\mathbb{Z}^2)-1}$ correspond to the graded k-algebra automorphisms of $k[\Delta]$, and admit a combinatorial description as follows.

Definition 8.1. *A nonzero vector $v \in \mathbb{Z}^2$ is a* column vector *of Δ if there exists a facet $\tau \subset \Delta$ (the* base facet*) such that*

$$v + ((\Delta \setminus \tau) \cap \mathbb{Z}^2) \subset \Delta.$$

We denote by $c(\Delta)$ the number of column vectors of Δ.

Example 8.2. Any multiple of the standard 2-simplex Σ has 6 column vectors. The octagonal polytope below shows that a polytope may have no column vectors.

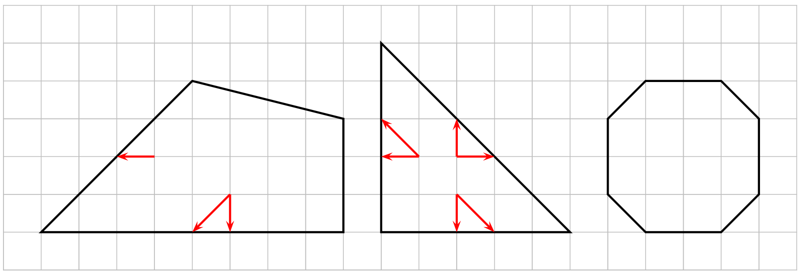


Figure 8.2: Column vectors of some lattice polytopes

The dimension of the automorphism group $\operatorname{Aut}(X(\Delta)_k)$ is then determined as follows.

Proposition 8.3 (Bruns–Gubeladze [7, Theorem 5.3.2])**.** *We have*

$$\dim \operatorname{Aut}(X(\Delta)_k) = c(\Delta) + 2.$$

Proof sketch. One begins with the 2-dimensional subgroup of $\operatorname{Aut}(X(\Delta)_k)$ induced by the inclusion $\operatorname{Aut}(\mathbb{T}^2) \hookrightarrow \operatorname{Aut}(X(\Delta)_k)$. On the $k[\Delta]$-side, this corresponds to the graded automorphisms induced by $(x, y) \mapsto (\lambda x, \mu y)$ for $\lambda, \mu \in k^{*2}$.

Next, column vectors of Δ correspond to automorphisms of $X(\Delta)_k$ in the following way. If v is a column vector, modulo equivalence we may assume that $v = (0, -1)$, that the base facet is supported on the X-axis, and that Δ is contained in the positive quadrant $\mathbb{R}^2_{\geq 0}$. Let $f(x, y) \in k[x, y]$ be supported on Δ. Since the vector $v = (0, -1)$ is a column vector, the polynomial $f(x, y + \lambda)$ will again be supported on Δ, for any $\lambda \in k$. Hence v induces a family of graded automorphisms $k[\Delta] \to k[\Delta]$, corresponding to a one-dimensional subgroup of $\mathrm{Aut}(X(\Delta)_k)$.

It then remains to show that these subgroups are algebraically independent from each other and from $\mathrm{Aut}(\mathbb{T}^2)$, and that together they generate $\mathrm{Aut}(X(\Delta)_k)$ (after including the finitely many automorphisms coming from $\mathbb{Z}$-affine transformations mapping Δ to itself). □

Using the fact that a curve of genus $g \geq 2$ has finitely many automorphisms we obtain the following corollary. We leave the details as an exercise.

Corollary 8.4. *We have* $\dim \mathcal{M}_\Delta \leq m(\Delta) := \#(\Delta \cap \mathbb{Z}^2) - c(\Delta) - 3$.

Remark 8.5. In order to have equality, it is sufficient that Δ is a so-called *maximal polytope* (see Section 10 for the definition). This is the main result of Koelman's thesis [22, Theorem 2.5.12].

Example 8.6. Let $\Delta = \mathrm{conv}(\{(0,0),(2g+2,0),(0,2)\})$ as in section 5, so that $\dim \mathcal{M}_\Delta = 2g-1$. One verifies that $c(\Delta) = g+4$, so the upper bound in Corollary 8.4 reads $m(\Delta) = (3g+6)-(g+4)-3 = 2g-1$; so in this case, the bound is sharp. It is easy to verify that the bound is also sharp if $\Delta = d\Sigma$, $d \in \mathbb{Z}_{\geq 4}$; then $\dim \mathcal{M}_\Delta$ reads $(d+1)(d+2)/2 - 9 = g + 3d - 9 \leq 2g$. The latter are examples of maximal polytopes. Opposed to this, let $(d\Sigma)_0$ be obtained from $d\Sigma$ by pruning off $(0,0)$. This reduces the number of lattice points by 1 and the number of column vectors by 2. Hence our bound increases, although $d\Sigma$ and $(d\Sigma)_0$ give rise to the same moduli space. Indeed, pruning off $(0,0)$ only forces our curves in $X(d\Sigma)_k \cong \mathbb{P}^2$ to pass through $(0,0,1)$.

Example 8.7. We now use Corollary 8.4 to show that the dimension of the moduli space of nonhyperelliptic C_{ab} curves of genus g (where a and b may vary) has dimension strictly smaller than $2g-1 = \dim \mathcal{H}_g$ whenever $g \geq 6$. Consider $\Delta_{ab} = \mathrm{Conv}\{(0,a),(b,0),(0,0)\}$ with $a, b \in \mathbb{Z}_{\geq 3}$ coprime. Then we have

$$g = (a-1)(b-1)/2, \quad \#(\Delta \cap \mathbb{Z}^2) = g + a + b + 1,$$

and the set of column vectors is given by

$$\{(n,-1) : n = 0, \ldots, \lfloor b/a \rfloor\} \cup \{(-1,m) : m = 0, \ldots, \lfloor a/b \rfloor\}.$$

Suppose without loss of generality that $a < b$. Then a is bounded by $\sqrt{2g}+1$. Corollary 8.4 yields

$$\dim \mathcal{M}_\Delta \leq m(\Delta) = g + a + b + 1 - \left(\left\lfloor \frac{b}{a} \right\rfloor + 2 \right) - 3 < a + \frac{2g-1}{a} + g - 2.$$

As a (real) function of a, this upper bound has a unique minimum at $a = \sqrt{2g-1}$. Therefore, to deduce that it is strictly smaller than $2g-1$ for all $a \in [3, \sqrt{2g}+1]$, it suffices to verify so for the boundary values $a = 3$ and $a = \sqrt{2g}+1$, which is indeed the case if $g \geq 6$.

9. A bound in terms of the genus

Throughout the rest of this article, we will employ the following notation. Let $\Delta^{(1)}$ be the convex hull of the interior lattice points of Δ. Let r (resp. $r^{(1)}$) denote the number of lattice points on the boundary of Δ (resp. $\Delta^{(1)}$), and let $g^{(1)}$ denote the number of interior lattice points in $\Delta^{(1)}$, so that $g = g^{(1)} + r^{(1)}$.

We now prove the following preliminary bound.

Proposition 9.1. *If Δ has at least $g \geq 2$ interior lattice points, then* $\dim \mathcal{M}_\Delta \leq 2g+3$.

Proof. We may assume that Δ is nonhyperelliptic, because otherwise $\dim \mathcal{M}_\Delta \leq 2g-1$ by Lemma 5.1. We may also assume that $\Delta^{(1)}$ is not a multiple of Σ, since otherwise Δ-nondegenerate curves are canonically embedded in $X(\Delta^{(1)})_k \cong \mathbb{P}^2_k$ using Proposition 1.7; then from Example 8.6 it follows that $\dim \mathcal{M}_\Delta \leq 2g$.

An upper bound for $\dim \mathcal{M}_\Delta$ in terms of g then follows from a lemma by Haase and Schicho [17, Lemma 12], who proved that $r \leq r^{(1)} + 9$, in which equality holds if and only if $\Delta = d\Sigma$ for some $d \in \mathbb{Z}_{\geq 4}$ (a case which we have excluded). Hence

$$\#(\Delta \cap \mathbb{Z}^2) = g + r \leq g + r^{(1)} + 8 = 2g + 8 - g^{(1)}, \tag{7}$$

and thus

$$\dim \mathcal{M}_\Delta \leq m(\Delta) = \#(\Delta \cap \mathbb{Z}^2) - c(\Delta) - 3 \leq 2g + 5 - c(\Delta) - g^{(1)} \leq 2g + 5. \tag{8}$$

This bound improves to $2g+3$ if $g^{(1)} \geq 2$, so we remain with two cases: $g^{(1)} = 0$ and $g^{(1)} = 1$.

Suppose first that $g^{(1)} = 0$. Then by Lemma 9.2 below, any Δ-nondegenerate curve is either a smooth plane quintic (excluded), or a trigonal curve. Since the moduli space of trigonal curves has dimension $2g+1$ (a classical result, see also Section 12 below), the bound holds.

Next, suppose that $g^{(1)} = 1$. Then, up to equivalence, there are only 16 possibilities for $\Delta^{(1)}$, which are listed in [34, Figure 2] or in the appendix below. Hence, there are only finitely many possibilities for Δ, and for each of these polytopes we find that $\#(\Delta \cap \mathbb{Z}^2) - c(\Delta) - 3 \leq 2g + 2$. $\square$

In fact, for all but the 5 polytopes in Figure 9 (up to equivalence), we find that the stronger bound $\#(\Delta \cap \mathbb{Z}^2) - c(\Delta) - 3 \leq 2g + 1$ holds.

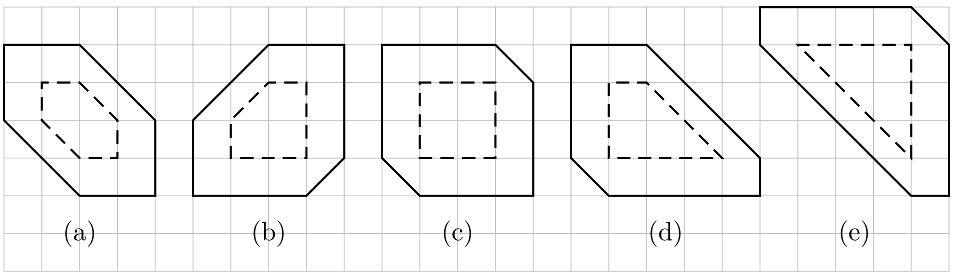


Figure 9: Polytopes with $g^{(1)} = 1$ and $\#(\Delta \cap \mathbb{Z}^2) - c(\Delta) - 3 = 2g + 2$

Lemma 9.2. *Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be nondegenerate and suppose that the interior lattice points of $\Delta(f)$ are not collinear. Let $\Delta^{(1)}$ be the convex hull of these interior lattice points.*

(a) *If $\Delta^{(1)}$ has no interior lattice points, then $V(f)$ is either trigonal or isomorphic to a smooth plane quintic.*
(b) *If $V(f)$ is trigonal or isomorphic to a smooth plane quintic, and $\Delta^{(1)}$ has at least 4 lattice points on the boundary, then $\Delta^{(1)}$ has no interior lattice points.*

Proof. Koelman gives a proof of this in his Ph.D. thesis [22, Lemma 3.2.13], based on Petri's theorem. A more combinatorial argument uses the fact that lattice polytopes of genus 0 are equivalent with either 2Σ, or with a polytope that is caught between two horizontal lines of distance 1. This was proved independently by Arkinstall, Khovanskii, Koelman, and Schicho (see the generalized statement by Batyrev-Nill [5, Theorem 2.5]).

In the first case, Δ-nondegenerate curves are canonically embedded in $X(2\Sigma)_k \cong \mathbb{P}^2_k$, hence they are isomorphic to smooth plane quintics.

In the second case, it follows that Δ is caught between two horizontal lines of distance 3. This may actually fail if $\Delta^{(1)} = \Sigma$, which corresponds to smooth plane quartics. But in both situations, Δ-nondegenerate curves are trigonal.

For (b), using the canonical divisor K_Δ from Proposition 1.7, one sees that the canonical embedding of $V(f)$ in $\mathbb{P}^{g-1}_k$ is contained in $X(\Delta^{(1)})_k$. According to a theorem of Koelman [23], the condition of having at least 4 lattice points on the boundary ensures that $X(\Delta^{(1)})$ is generated by quadrics. Now since $V(f)$ is trigonal or isomorphic to a smooth plane quintic, by Petri's theorem the intersection of *all* quadrics containing $V(f)$ is a surface of sectional genus 0. Hence this surface must be $X(\Delta^{(1)})_k$ and $\Delta^{(1)}$ must have genus 0. □

Remark 9.3. The condition that $\Delta^{(1)}$ should have at least 4 lattice points on the boundary is necessary in Lemma 9.2. For example, let k be algebraically closed and let $\Delta = \mathrm{conv}\{(2,0),(0,2),(-2,-2)\}$. Then Δ is a lattice polytope of genus 4, hence all Δ-nondegenerate curves are trigonal. However, $\Delta^{(1)}$ contains $(0,0)$ in its interior. Note that $X(\Delta^{(1)})_k \subset \mathbb{P}^3_k$ is the cubic $xyz = w^3$.

10. Refining the upper bound: Maximal polytopes

We further refine the bound in Proposition 9.1 by adapting the proof of the Haase–Schicho bound $r \leq r^{(1)} + 9$ in order to obtain an estimate for $r - c(\Delta)$ instead of just r. We first do this for *maximal* polytopes, and treat nonmaximal polytopes in the next section.

Definition 10.1. *A lattice polytope $\Delta \subset \mathbb{Z}^n$ is* maximal *if Δ is not properly contained in another lattice polytope with the same interior lattice points, i.e., for all lattice polytopes $\Delta' \supsetneq \Delta$, we have*

$$\mathrm{int}(\Delta') \cap \mathbb{Z}^n \neq \mathrm{int}(\Delta) \cap \mathbb{Z}^n.$$

We define the *relaxed polytope* $\Delta^{(-1)}$ of a lattice polytope $\Delta \subset \mathbb{Z}^2$ as follows. Assume that $0 \in \Delta$. To each facet $\tau \subset \Delta$ given by an inequality of the form $a_1X + a_2Y \leq b$ with $a_i \in \mathbb{Z}$ coprime, we define the *relaxed inequality* $a_1X + a_2Y \leq b+1$ and let $\Delta^{(-1)}$ be the intersection of these relaxed inequalities. If p is a vertex of Δ given by the intersection of two such facets, we define the *relaxed vertex* $p^{(-1)}$ to be the intersection of the boundaries of the corresponding relaxed inequalities.

Lemma 10.2 (Haase–Schicho [17, Lemmas 9–10], Koelman [22, Section 2.2]). *Let $\Delta \subset \mathbb{Z}^2$ be a 2-dimensional lattice polytope. Then $\Delta^{(-1)}$ is a lattice polytope if and only if $\Delta = \Delta'^{(1)}$ for some lattice polytope Δ'. Furthermore, if Δ is nonhyperelliptic, then Δ is maximal if and only if $\Delta = (\Delta^{(1)})^{(-1)}$.*

The proof of the Haase–Schicho bound $r \leq r^{(1)} + 9$ utilizes a theorem of Poonen and Rodriguez-Villegas [34], which we now introduce.

A *legal move* is a pair (v, w) with $v, w \in \mathbb{Z}^2$ such that $\mathrm{conv}(\{0, v, w\})$ is a 2-dimensional triangle whose only nonzero lattice points lie on $e(v, w)$, the edge between v and w. The *length* of a legal move (v, w) is

$$\ell(v, w) = \det \begin{pmatrix} v \\ w \end{pmatrix},$$

which is of absolute value $r - 1$, where $r = \#(e(v, w) \cap \mathbb{Z}^2)$ is the number of lattice points on the edge between v and w. Note that the length can be negative.

A *legal loop* $\mathcal{P}$ is a sequence of vectors $v_1, v_2, \ldots, v_n \in \mathbb{Z}^2$ such that for all $i = 1, \ldots, n$ and indices taken modulo n, we have:

- (v_i, v_{i+1}) is a legal move, and
- v_{i-1}, v_i, v_{i+1} are not contained in a line.

The *length* $\ell(\mathcal{P})$ of a legal loop $\mathcal{P}$ is the sum of the lengths of its legal moves.

The *winding number* of a legal loop is its winding number around 0 in the sense of algebraic topology. The *dual loop* $\mathcal{P}^\vee$ is given by $w_1, \ldots, w_n$, where $w_i = \ell(v_i, v_{i+1})^{-1} \cdot (v_{i+1} - v_i)$ for $i = 1, \ldots, n$. One can check that this is again a legal loop with the same winding number as $\mathcal{P}$ and that $\mathcal{P}^{\vee\vee} = \mathcal{P}$ after a 180° rotation.

Theorem 10.3 (Poonen–Rodriguez-Villegas [34, Section 9.1])**.** *Let $\mathcal{P}$ be a legal loop with winding number w. Then $\ell(\mathcal{P}) + \ell(\mathcal{P}^\vee) = 12w$.*

Now let $\Delta \subset \mathbb{Z}^2$ be a maximal polytope with 2-dimensional interior $\Delta^{(1)}$. We associate to Δ a legal loop $\mathcal{P}(\Delta)$ as follows. By Lemma 10.2, Δ is obtained from $\Delta^{(1)}$ by relaxing the edges. Let $p_1, \ldots, p_n$ be the vertices of $\Delta^{(1)}$, enumerated counterclockwise; then $\mathcal{P}(\Delta)$ is given by the sequence $q_i = p_i^{(-1)} - p_i$ where $p_i^{(-1)}$ is the relaxed vertex of p_i.

Example 10.4. The following picture, inspired by Haase–Schicho [17, Figure 20], is illustrative: it shows a polytope Δ with 2-dimensional interior $\Delta^{(1)}$, the associated legal loop $\mathcal{P}(\Delta)$, and its dual $\mathcal{P}(\Delta)^\vee$. In this example, $\ell(\mathcal{P}(\Delta)) = \ell(\mathcal{P}(\Delta)^\vee) = 6$.

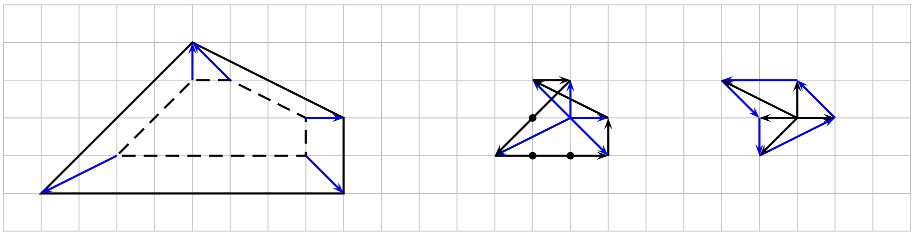


Figure 10.4: The legal loop $\mathcal{P}(\Delta)$ associated to a lattice polytope Δ

A crucial observation is that the bold-marked lattice points of $\mathcal{P}(\Delta)$ are column vectors of Δ. This holds in general and lies at the core of our following refinement of the Haase–Schicho bound.

Lemma 10.5. *If Δ is maximal and nonhyperelliptic, then:*

(a) $r - r^{(1)} = \ell(\mathcal{P}(\Delta)) \leq 9$.
(b) $r - r^{(1)} - c(\Delta) \leq \min\left(\ell(\mathcal{P}(\Delta)), \ell(\mathcal{P}(\Delta)^\vee)\right) \leq 6$.

Proof. We abbreviate $\mathcal{P} = \mathcal{P}(\Delta)$.

Inequality (a) is by Haase–Schicho [17, Lemma 11] and works as follows. The length of the legal move (q_i, q_{i+1}) measures the difference between the number of lattice points on the facet of Δ connecting $p_i^{(-1)}$ and $p_{i+1}^{(-1)}$, and the number of lattice points on the edge of $\Delta^{(1)}$ connecting p_i and p_{i+1}. Therefore $r - r^{(1)} = \ell(\mathcal{P})$. The dual loop $\mathcal{P}^\vee$ walks (in a consistent and counterclockwise-oriented way) through the direction vectors of the edges of $\Delta^{(1)}$, therefore each move has positive length and we have $\ell(\mathcal{P}(\Delta)^\vee) \geq 3$. Since $\mathcal{P}^\vee$ has winding number 1, the statement follows from Theorem 10.3. (One can further show that equality holds if and only if Δ is a multiple of the standard 2-simplex Σ.)

To prove inequality (b), we first claim: there is a bijection between lattice points v which lie properly on a counterclockwise-oriented (positive length) legal move $q_i q_{i+1}$ of $\mathcal{P}$, and column vectors of Δ with base facet $p_i^{(-1)} p_{i+1}^{(-1)}$. Indeed, after an appropriate transformation, we may assume as in Proposition 8.3 that $v = (0, -1)$, that $p_i^{(-1)}$ and $p_{i+1}^{(-1)}$ lie on the X-axis, and that Δ is contained in the positive quadrant $\mathbb{R}^2_{\geq 0}$; after these normalizations, the claim is straightforward.

Now, since the dual loop $\mathcal{P}^\vee$ consists of counterclockwise-oriented legal moves only, it has at most $\ell(\mathcal{P}^\vee)$ vertices. Since $\mathcal{P} = \mathcal{P}^{\vee\vee}$ (after 180° rotation), $\mathcal{P}$ has at most $\ell(\mathcal{P}^\vee)$ vertices. By the claim, we have $\ell(\mathcal{P}) \leq \ell(\mathcal{P}^\vee) + c$, and the result follows by combining this with part (a) and Theorem 10.3. ☐

Corollary 10.6. *If Δ is maximal, then $\dim \mathcal{M}_\Delta \leq 2g + 3 - g^{(1)}$. In particular, if $g^{(1)} \geq 2$ then $\dim \mathcal{M}_\Delta \leq 2g + 1$.*

Proof. By Lemma 10.5, we have $m(\Delta) = g + r - 3 - c(\Delta) \leq g + r^{(1)} + 3 \leq 2g + 3 - g^{(1)}$. ☐

Remark 10.7. Note that Lemma 10.5(a) immediately extends to nonmaximal polytopes ($r - r^{(1)}$ can only decrease), so the Haase–Schicho bound holds for arbitrary nonhyperelliptic polytopes. This we cannot conclude for part (b): if r decreases, $c(\Delta)$ may decrease more quickly so that the bound no longer holds. An example of such behaviour can be found in Figure 9(c).

11. Refining the upper bound: general polytopes

We are now ready to prove the main result of Sections 8–11.

Theorem 11.1. *If $g \geq 2$, then $\dim \mathcal{M}_g^{\mathrm{nd}} \leq 2g + 1$ except for $g = 7$ where we have $\dim \mathcal{M}_7^{\mathrm{nd}} \leq 16$.*

Proof. It suffices to show that the claimed bounds hold for all polytopes Δ with g interior lattice points. By the proof of Proposition 9.1, we may assume that $\Delta^{(1)}$ is two-dimensional, that it is not a multiple of Σ, and that it has $g^{(1)} \geq 1$ interior lattice points.

Let us first assume that $g^{(1)} \geq 2$. We will show that $\dim \mathcal{M}_\Delta \leq 2g + 1$. From Corollary 10.6, we know that this is true if Δ is maximal. Therefore, suppose that Δ is nonmaximal; then it is obtained from a maximal polytope $\widetilde{\Delta}$ by taking away points on the boundary (keeping the interior lattice points intact). If two or more boundary points are taken away, then as in (8) we have

$$m(\Delta) \leq \#(\Delta \cap \mathbb{Z}^2) - 3 \leq \#(\widetilde{\Delta} \cap \mathbb{Z}^2) - 2 - 3 \leq 2g + 5 - g^{(1)} - 2 \leq 2g + 1.$$

So we may assume that $\Delta = \mathrm{conv}(\widetilde{\Delta} \cap \mathbb{Z}^2 \setminus \{p\})$ for a vertex $p \in \widetilde{\Delta}$. Similarly, we may assume that $c(\Delta) < c(\widetilde{\Delta})$, for else

$$m(\Delta) = \#(\Delta \cap \mathbb{Z}^2) - c(\Delta) - 3 \leq \#(\widetilde{\Delta} \cap \mathbb{Z}^2) - c(\widetilde{\Delta}) - 3 = m(\widetilde{\Delta}) \leq 2g + 1.$$

Let v be a column vector of $\widetilde{\Delta}$ that is no longer a column vector of $\Delta = \mathrm{conv}(\widetilde{\Delta} \cap \mathbb{Z}^2 \setminus \{p\})$. Then p must lie on the base facet τ of v. After an appropriate transformation, we may assume that $p = (0,0)$, that $v = (0,-1)$, that τ lies along the X-axis, and that $\widetilde{\Delta}$ lies in the positive quadrant, as follows.

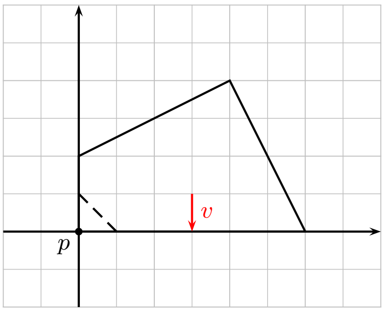


Figure 11.2: An almost maximal polytope.

Note that $(1,1) \in \mathrm{int}(\widetilde{\Delta})$ since otherwise v would still be a column vector of Δ. But then the other facet of $\widetilde{\Delta}$ which contains p must be supported on the Y-axis, for else $(1,1)$ would no longer be in $\mathrm{int}(\Delta)$. One can now verify that if $f(x,y)$ is Δ-nondegenerate, then for all but finitely many $\lambda \in k$, the polynomial $f(x, y+\lambda)$ will have Newton polytope $\widetilde{\Delta}$ and all but finitely of those will be $\widetilde{\Delta}$-nondegenerate. Therefore, we have $\mathcal{M}_\Delta \subset \mathcal{M}_{\widetilde{\Delta}}$, and the dimension estimate follows.

Now suppose that $g^{(1)} = 1$. From the finite computation in the proof of Proposition 9.1, we know that the bound $\dim \mathcal{M}_\Delta \leq 2g+1$ holds if Δ is not among the polytopes listed in Figure 9. Now in this list, the polytopes (b)–(e) are not maximal, and for these polytopes the same trick as in the $g^{(1)} \geq 2$ case applies. However, polytope (a) is maximal and contains 7 interior lattice points: therefore, we can only prove $\dim \mathcal{M}_7^{\mathrm{nd}} \leq 16$. □

Let Δ be a nonmaximal nonhyperelliptic lattice polytope, and let $\widetilde{\Delta} = (\Delta^{(1)})^{(-1)}$ be the smallest maximal polytope containing Δ. Let $f \in k[x^\pm, y^\pm]$ be a Δ-nondegenerate Laurent polynomial. Since $\Delta \subset \widetilde{\Delta}$, we can consider the (degree 1) locus $\widetilde{V}$ of $f = 0$ in $X(\widetilde{\Delta})_k = \mathrm{Proj}\, k[\widetilde{\Delta}]$. Then one can wonder whether the observation we made in the proof of Theorem 11.1 holds in general: is there always a $\sigma \in \mathrm{Aut}(X(\widetilde{\Delta})_k)$ such that $\sigma(\widetilde{V}) \cap \mathbb{T}_k^2$ is defined by a $\widetilde{\Delta}$-nondegenerate polynomial? The answer is no, because it is easy to construct examples where the only automorphisms of $X(\widetilde{\Delta})_k$ are those induced by $\mathrm{Aut}(\mathbb{T}_k^2)$. Then $\sigma(\widetilde{V}) \cap \mathbb{T}_k^2$ is always defined by $f(\lambda x, \mu y)$ (for some $\lambda, \mu \in k^*$), which does not have $\widetilde{\Delta}$ as its Newton polytope and hence cannot be $\widetilde{\Delta}$-nondegenerate.

However, f is very close to being $\widetilde{\Delta}$-nondegenerate, and this line of thinking leads to the following observation. Let p be a vertex of $\widetilde{\Delta}$ that is not a vertex of Δ, and let q_1, q_2 be the closest lattice points to p on the respective facets of $\widetilde{\Delta}$ containing p. The triangle spanned by p, q_1, q_2 cannot contain any other lattice points, because otherwise removing p would affect the interior of $\widetilde{\Delta}$. Thus the volume of this triangle is equal to 1/2 by Pick's theorem, and the affine chart of $X(\widetilde{\Delta})_k$ attached to the cone at p is isomorphic to $\mathbb{A}_k^2$. In particular, $X(\widetilde{\Delta})_k$ is nonsingular in the zero-dimensional torus $\mathbb{T}_p$ corresponding to p. Then f fails to be $\widetilde{\Delta}$-nondegenerate only because $\widetilde{V}$ passes through $\mathbb{T}_p$ (i.e. passes through $(0,0) \in \mathbb{A}^2_k$), elsewhere it fulfils the conditions of nondegeneracy: $\widetilde{V}$ is smooth, intersects the 1-dimensional tori associated to the facets of $\widetilde{\Delta}$ transversally, and does not contain the singular points of $X(\widetilde{\Delta})_k$. Now following the methods of Section 2, one could construct the bigger moduli space of curves satisfying this weaker nondegeneracy condition. Its dimension would still be bounded by $\#(\widetilde{\Delta} \cap \mathbb{Z}^2) - c(\widetilde{\Delta}) - 3$, which by Lemma 10.5 is at most $2g + 3 - g^{(1)}$ because $\widetilde{\Delta}$ is maximal. Therefore $\dim \mathcal{M}_\Delta \leq 2g + 3 - g^{(1)}$ for nonmaximal Δ, and this yields an alternative proof of Theorem 11.1. Related observations have been made by Koelman [22, Section 2.6].

12. Trigonal curves, trinodal sextics, and sharpness of our bounds

For $g \geq 2$, we implicitly proved in Section 5 that $\dim \mathcal{M}^{\text{nd}}_g \geq 2g - 1$. But already in genera 3 and 4, by the results in Section 6 we have $\dim \mathcal{M}^{\text{nd}}_3 = 6$ and $\dim \mathcal{M}^{\text{nd}}_4 = 9$, so this lower bound is an underestimation. For higher genera, we prove in this last section that the bounds given in Theorem 11.1 are sharp, mainly by investigating spaces of trigonal curves. Our main result is the following.

Theorem 12.1. *If $g \geq 4$, then $\dim \mathcal{M}^{\text{nd}}_g = 2g + 1$ except for $g \neq 7$ where $\dim \mathcal{M}^{\text{nd}}_7 = 16$.*

Proof. It suffices to find for every genus $g \geq 5$ a lattice polytope Δ with g interior lattice points, for which $\dim \mathcal{M}_\Delta = 2g + 1$ if $g \neq 7$, and $\dim \mathcal{M}_\Delta = 16$ if $g = 7$. If $g = 2h$ is even, let Δ be the rectangle

$$\text{conv}\left(\{(0,0),(0,3),(h+1,3),(h+1,0)\}\right). \tag{9}$$

Note that then $\#(\Delta \cap \mathbb{Z}^2) = 2g + 8$ and $c(\Delta) = 4$. If $g = 2h + 1$ is odd but different from 7, let Δ be the trapezium

$$\text{conv}\left(\{(0,0),(0,3),(h,3),(h+3,0)\}\right). \tag{10}$$

Again, $\#(\Delta \cap \mathbb{Z}^2) = 2g + 8$ and $c(\Delta) = 4$. Finally, if $g = 7$ then let Δ be

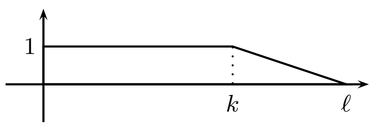
$$\text{conv}\{(2,0),(0,2),(-2,2),(-2,0),(0,-2),(2,-2)\} \tag{11}$$

(i.e. the polytope given in Figure 9(a)). Here, $\#(\Delta \cap \mathbb{Z}^2) = 19$ and $c(\Delta) = 0$.

We first prove that

$$\dim \mathcal{M}_\Delta = \#(\Delta \cap \mathbb{Z}^2) - 1 - \dim \text{Aut}(X(\Delta)_k), \tag{12}$$

holds for the families of polytopes (9) and (10), for which the result then follows from Proposition 8.3. This can be achieved using the well-known theory of trigonal curves [11, 26]. More generally, let $k, \ell \in \mathbb{Z}_{\geq 2}$ satisfy $k \leq \ell$, let $\Delta^{(1)}$ be the trapezium



and let $\Delta = \Delta^{(1)(-1)}$. In general, $\Delta^{(1)(-1)}$ need not be a lattice polygon: it may take some of its vertices outside $\mathbb{Z}^2$; but when $k = \ell$ and $k = \ell - 1$, corresponding to (9) and (10), respectively, the polygon $\Delta^{(1)(-1)}$ takes its vertices in $\mathbb{Z}^2$.

Remark 12.2. In fact, using the combinatorial criterion from Lemma 10.2, one can verify that $\Delta^{(1)(-1)}$ is a lattice polygon if and only if $\ell \leq (2g-2)/3$, where $g = k + \ell + 2$. This confirms a well-known inequality on the Maroni invariants of a trigonal curve (where the inequality is proven using the Riemann-Roch theorem).

Then in these cases, if a curve V is Δ-nondegenerate, it is trigonal of genus $g = k+\ell+2$. By Proposition 1.7, it can be canonically embedded in $X(\Delta^{(1)})_k$, which is the rational surface scroll $S_{k,\ell} \subset \mathbb{P}_k^{g-1}$. By Petri's theorem [1], this scroll is the intersection of all quadrics containing the canonical embedding. As a consequence, two different such canonical embeddings must differ by an automorphism of $\mathbb{P}_k^{g-1}$ that maps $X(\Delta^{(1)})_k$ to itself; in other words, any two canonical embeddings of V must differ by an automorphism of $X(\Delta^{(1)})_k$.

Now let $f_1, f_2 \in k[x^\pm, y^\pm]$ be Δ-nondegenerate polynomials such that $V(f_1)$ and $V(f_2)$ are isomorphic as abstract curves. Since the fans associated to Δ and $\Delta^{(1)}$ are the same, we have $X(\Delta)_k = X(\Delta^{(1)})_k$. Under this identification, $V(f_1)$ and $V(f_2)$ become canonical curves that must differ by an automorphism of $X(\Delta)_k$. Thus we can conclude (12). (We note that although any trigonal curve is canonically embedded in some rational normal scroll $S_{k,\ell}$ and hence in some $X(\Delta)_k$, it might fail to be nondegenerate because it can be impossible to avoid tangency to $X(\Delta)_k \setminus \mathbb{T}_k^2$.)

To conclude, suppose that Δ is as in (11). We refer to the pruned simplex (6) and the accompanying discussion; here we have $d = 6$. It follows that if f is a Δ-nondegenerate polynomial, then f gives rise to a plane sextic V with three nodes (at the coordinate points) and no other singularities. Conversely, any trinodal sextic where any line connecting two nodes intersects the curve transversally elsewhere, is Δ-nondegenerate. Since the latter is an open condition, $\mathcal{M}_\Delta$ is the Zariski closure of the moduli space $\mathcal{V}_{3,6}$ of trinodal plane sextics. The variety $\mathcal{V}_{3,6}$ is in its turn the image of a Severi variety [37], and it is classical that $\dim \mathcal{V}_{3,6} = 16$—for a modern treatment, see Sernesi [36]. □

Remark 12.3. In his Ph.D. thesis, Koelman [22, Theorem 2.5.12] proves that Equation (12) holds for any polytope $\Delta \subset \mathbb{R}^2$ which is maximal and nonhyperelliptic. In fact, Koelman assumes $k = \mathbb{C}$, but his methods extend to an arbitrary algebraically closed field $k = \overline{k}$. This provides another proof of Theorem 12.1, but we are content to prove our results in the above more elementary (and classical) way.

Acknowledgements

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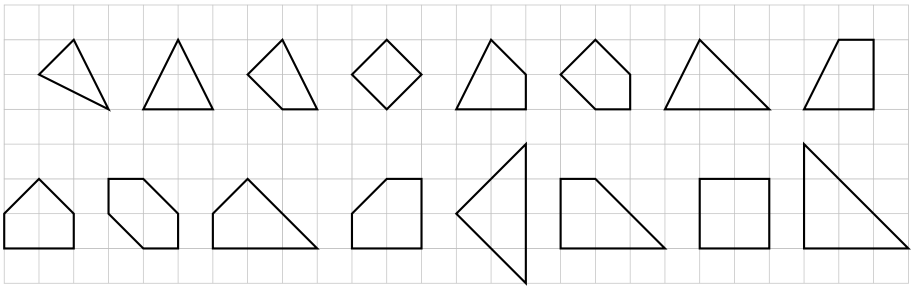
Appendix: lattice polytopes of genus one

There are 16 equivalence classes of lattice polytopes having one interior lattice point. Polytopes representing these are drawn below. This is a copy of [34, Figure 2], we include the list here for sake of self-containedness. It is an essential ingredient in the proofs of Lemma 4.1 and Proposition 9.1.



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Moving out the edges of a lattice polygon

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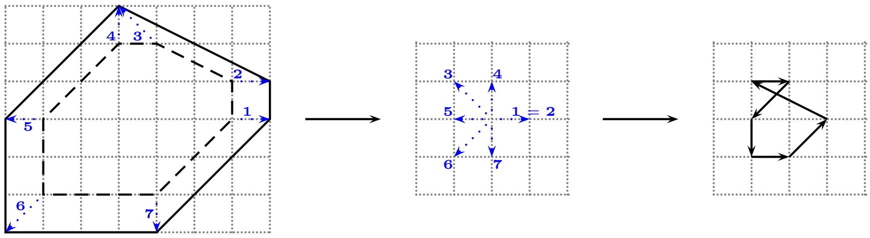
Abstract

We review previous work of (mainly) Koelman, Haase and Schicho, and Poonen and Rodriguez-Villegas on the dual operations of *(i)* taking the interior hull and *(ii)* moving out the edges of a two-dimensional lattice polygon. We show how the latter operation naturally gives rise to an algorithm for enumerating lattice polygons by their genus. We then report on an implementation of this algorithm, by means of which we produce the list of all lattice polygons (up to equivalence) whose genus is contained in $\{1, \ldots, 30\}$. In particular, we obtain the number of inequivalent lattice polygons for each of these genera. As a byproduct, we prove that the minimal possible genus for a lattice 15-gon is 45.

1 Introduction

(1.1) A *lattice polygon* is a (nonempty) convex polygon in $\mathbb{R}^2$ with vertices in $\mathbb{Z}^2$. Points of $\mathbb{Z}^2$ are called *lattice points*. The *dimension* of a polygon Δ is the dimension of the smallest affine subspace of $\mathbb{R}^2$ containing Δ. The *genus* of a two-dimensional lattice polygon is the number of lattice points in its topological interior (when equipped with the subspace topology of $\mathbb{R}^2$). The genus of a lower-dimensional lattice polygon is considered 0. A $\mathbb{Z}$-*affine transformation* of $\mathbb{R}^2$ is a map of the form $\varphi : \mathbb{R}^2 \to \mathbb{R}^2 : p \mapsto pA + b$ with $A \in \mathrm{GL}_2(\mathbb{Z})$ and $b \in \mathbb{Z}^2$. Two lattice polygons Δ and Δ' are called *equivalent* if and only if there exists a $\mathbb{Z}$-affine map φ such that $\varphi(\Delta) = \Delta'$.

(1.2) We review a useful tool in the study of the combinatorics of lattice polygons. The rough idea is to gradually peel off the lattice polygon by consecutively considering the convex hull of the interior lattice points. Although this 'onion skin' principle dates (at least) back to the work of Rabinowitz [24] and Koelman [15], Haase and Schicho [12] noticed that to each step of the peeling, one can associate a so-called legal loop catching the corresponding global information. This allows one to apply a remarkable theorem due to Poonen and Rodriguez-Villegas [23], in which the number 12 makes an intriguing appearance.



The interior hull of a lattice polygon and the associated legal loop.

Whereas Haase and Schicho worked towards a refinement of a theorem by Scott [25], our endpoint is a new and conceptual proof of a more precise conjecture due to Coleman [8].

Theorem 1 (Coleman's conjecture, 1978) *Let Δ be a lattice n-gon of genus $g \geq 1$. Let R be the number of lattice points on the boundary of Δ. Then $R \leq 2g + 10 - n$.*

The first complete proof was provided in 2006 by Kołodziejczyk and Olszewska [17]. However, already in his 1991 Ph.D. thesis [15, Lemma 4.5.2(2)], Koelman must have been unaware of the existence of this conjecture and was only one sentence left from a proof. His argument heavily relies on another '12 theorem' due to Oda [21, Remark on page 45], and can in fact be extended to cover a non-trivial part of Poonen and Rodriguez-Villegas' result, see **(2.6)**. In their turn, Haase and Schicho must have been unaware of this entire story: our proof will merely add a couple of lines to their arguments. We therefore do not claim many credits, but hope that this proof gives an indication of the unacknowledged potential of the machinery. At the same time, we smoothen the theoretical and historical framework.

(1.3) We then reverse the process of gradually peeling off a lattice polygon by instead consecutively moving out its edges, following ideas that were discovered by Koelman [15, Section 2.2], Haase and Schicho [12], and Kołodziejczyk and Olszewska [16]. This gives a natural and efficient way of enumerating lattice polygons by their genus, up to equivalence. We will report on an implementation of this procedure using the Magma computer algebra system [5], by means of which we produced the list of all equivalence classes of lattice polygons of genus $1 \leq g \leq 30$. This comprises approximately 368 MB of data that we made available for download at `http://wis.kuleuven.be/algebra/castryck/`. As a consequence, we can now answer virtually every reasonable question on lattice polygons of genus $1 \leq g \leq 30$. In particular, we obtain the number of inequivalent lattice polygons for each of these genera. Up to our knowledge, these numbers did not appear in the literature thus far, even for g as small as 3. Among the other consequences, we find:

Theorem 2 *The minimal genus of a lattice 15-gon is 45.*

This fills in the smallest open entry of a list whose study was initiated by Arkinstall [1] and that since invoked fair interest. E.g. only recently, it was proven that the minimal genus of a lattice 11-gon is 17 (see [22]). Finally, we introduce the *lifespan* of a lattice polygon, which measures how often its edges can be moved out without tumbling off the lattice. We prove the following fact:

Theorem 3 *A lattice n-gon has finite lifespan as soon as $n \geq 10$.*

For each $3 \leq n \leq 9$, there exists an n-gon having infinite lifespan. Explicit examples are provided in **(3.6)** below.

2 Legal loops associated to a lattice polygon

(2.1) In 1976, Scott proved the following theorem [25].

Theorem 4 (Scott, 1976) *Let $\Delta \subset \mathbb{R}^2$ be a lattice polygon having $g \geq 1$ lattice points in its interior. Let R be the number of lattice points on the boundary of Δ. Then $R \leq 2g + 7$.*

Moreover, Scott proved that equality holds if and only if Δ is equivalent to $\text{Conv}\{(0,0),(3,0),(0,3)\}$. Two years later, Coleman conjectured that the refinement mentioned in Theorem 1 of **(1.2)** should hold. The usage of the word 'theorem' is now justified, due to a recent proof by Kołodziejczyk and Olszewska [17]. However, as already mentioned, it should be attributed in part to Koelman [15, Lemma 4.5.2(2)]: see **(2.6)**.
In 2009, Haase and Schicho revisited Scott's bound and provided an alternative proof. Implicitly though, they gave a new proof of Coleman's conjecture, along with a substantial refinement of the statement. An explicit version of this proof will be given in **(2.5)**. Along the way, we give an overview of the machinery used, adapt certain definitions, and prove a number of facts that were merely sketched in the literature before.

(2.2) The following two basic operations on lattice polygons will be crucial throughout.

Definition (moving out the edges) Let $\Delta \subset \mathbb{R}^2$ be a two-dimensional lattice polygon. Then each of its edges $\tau \subset \Delta$ corresponds to a unique half-plane

$$\mathcal{H}_\tau = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, a_\tau x + b_\tau y \leq c_\tau \right\}$$

jointly satisfying $\Delta = \cap_\tau \mathcal{H}_\tau$. In this, $a_\tau, b_\tau, c_\tau \in \mathbb{Z}$ are uniquely determined by the condition $\gcd(a_\tau, b_\tau) = 1$. Then we define

$$\Delta^{(-1)} := \cap_\tau \mathcal{H}_\tau^{(-1)}$$

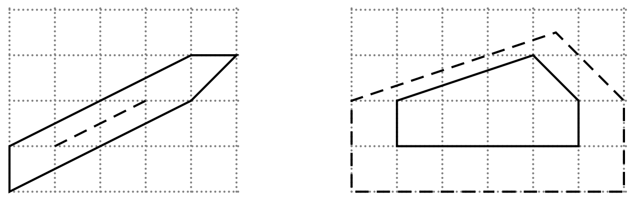
where

$$\mathcal{H}_\tau^{(-1)} = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, a_\tau x + b_\tau y \leq c_\tau + 1 \right\}.$$

We say that $\Delta^{(-1)}$ is obtained from Δ by *moving out the edges*.

Definition (interior hull) Let $\Delta \subset \mathbb{R}^2$ be a two-dimensional lattice polygon of genus at least 1. Then we define $\Delta^{(1)}$ as the convex hull of the lattice points in the interior of Δ. We say that $\Delta^{(1)}$ is the *interior hull* of Δ.

We will abuse notation and write $\Delta^{(k)}$ for $\Delta^{(1)(1)\cdots(1)}$ (the interior hull taken k times consecutively), given that this is well-defined: note that $\Delta^{(1)}$ need not have interior lattice points. Likewise, we will write $\Delta^{(-k)}$ for $\Delta^{(-1)(-1)\cdots(-1)}$ (moving out the edges k times consecutively). Again, this may not be well-defined since $\Delta^{(-1)}$ need not be a lattice polygon: it may take vertices outside $\mathbb{Z}^2$. It is sometimes convenient to write $\Delta^{(0)}$ for Δ.



Examples where $\Delta^{(2)}$ resp. $\Delta^{(-2)}$ would not be well-defined.

A crucial property of lattice polygons is the following.

Theorem 5 (Koelman, 1991) *Let $\Delta \subset \mathbb{R}^2$ be a two-dimensional lattice polygon, such that $\Delta^{(1)}$ is again two-dimensional. Then $\Delta^{(1)(-1)}$ is a lattice polygon containing Δ.*

Proof. See Koelman [15, Lemma 2.2.13] or Haase–Schicho [12, Lemma 11]. ■

Note that this theorem gives a criterion for a two-dimensional lattice polygon Γ to satisfy that $\Gamma^{(-1)}$ is a lattice polygon: this will be the case *if and only if* there exists a lattice polygon Δ such that $\Delta^{(1)} = \Gamma$. Although the notions of moving out the facets and taking the interior hull straightforwardly generalize to higher dimensions, Theorem 5 does not. This is the main reason why we restrict to dimension two in this article.

Definition (maximal polygon) A lattice polygon Δ for which $\Delta^{(1)}$ is two-dimensional is called *maximal* if $\Delta = \Delta^{(1)(-1)}$.

(2.3) We now review the theory of legal loops, in the sense of [23]. Throughout, we will write o for the origin $(0,0) \in \mathbb{R}^2$.

Definition (legal move) A *legal move* is a couple of points (p_1, p_2) with $p_1, p_2 \in \mathbb{Z}^2$ such that $\mathrm{Conv}\{o, p_1, p_2\} \cap \mathbb{Z}^2 = \{o\} \sqcup \big(\mathrm{Conv}\{p_1, p_2\} \cap \mathbb{Z}^2\big)$.

Note that $p_1 = p_2$ is a priori allowed. If $p_1 \neq p_2$, then the condition reads that the line connecting p_1 and p_2 lies at integral distance 1 from o, i.e. it has an equation of the form $aX + bY = 1$ for (necessarily coprime) $a, b \in \mathbb{Z}$.

Definition (legal loop) A *legal loop* is a finite sequence $\mathcal{P} = (p_0, \ldots, p_{n-1})$, where $n \geq 1$, such that (p_i, p_{i+1}) is a legal move for all $i = 0, \ldots, n-1$. For any primitive vector $p_0 \in \mathbb{Z}^2$, the legal loop (p_0) will be called *trivial*.

In the above, indices should be considered modulo n, i.e. $p_n = p_0$. Such abuse of notation will be repeated throughout.

Definition (length) The *length* of a legal move $s = (p_1, p_2)$ is defined to be

$$\det \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

and will be denoted by $\ell(s)$. The *length* of a legal loop $\mathcal{P}$ is the sum of the lengths of its legal moves and will be denoted by $\ell(\mathcal{P})$.

A legal loop $\mathcal{P} = (p_0, \ldots, p_{n-1})$ gives rise to a closed curve $\gamma(\mathcal{P})$ in $\mathbb{R}^2 \setminus \{o\}$ by 'connecting the dots'. One way of making this precise is

$$[0,1] \to \mathbb{R}^2 \setminus \{o\} : t \mapsto (nt - \lfloor nt \rfloor) p_{\lfloor nt \rfloor + 1} + (1 - nt + \lfloor nt \rfloor) p_{\lfloor nt \rfloor},$$

although we will only be interested in $\gamma(\mathcal{P})$ up to homotopy.

Definition (winding number) The *winding number* of a legal loop $\mathcal{P}$ is the winding number of $\gamma(\mathcal{P})$ around o in the sense of algebraic topology, i.e. the image of its homotopy class under the unique isomorphism $\pi_1(\mathbb{R}^2 \setminus \{o\}) \to \mathbb{Z}$ mapping the class of a counterclockwise loop around o to 1.

Definition (inverse loop) Let $\mathcal{P} = (p_0, p_1, \ldots, p_{n-1})$ be a legal loop. Then we define the *inverse loop* $\mathcal{P}^{-1}$ to be $(p_{n-1}, p_{n-2}, \ldots, p_0)$.

Taking the inverse of a legal loop alters the sign of both the length and the winding number.

Definition (equivalence) We equip the set of legal loops with the smallest equivalence relation satisfying

1. *(shifting)* a legal loop $(p_0, p_1, \ldots, p_{n-1})$ is equivalent to $(p_1, \ldots, p_{n-1}, p_0)$;
2. *(merging and splitting moves)* a legal loop $(p_0, p_1, \ldots, p_{n-1})$ is equivalent to the legal loop $(p_0, q, p_1, \ldots, p_{n-1})$, where q is any lattice point on a line through p_0 and p_1 at integral distance 1 from o;
3. *(orientation-preserving lattice equivalence)* a legal loop $(p_0, p_1, \ldots, p_{n-1})$ is equivalent to $(p_0 A, p_1 A, \ldots, p_{n-1} A)$ for any matrix $A \in \mathrm{SL}_2(\mathbb{Z})$.

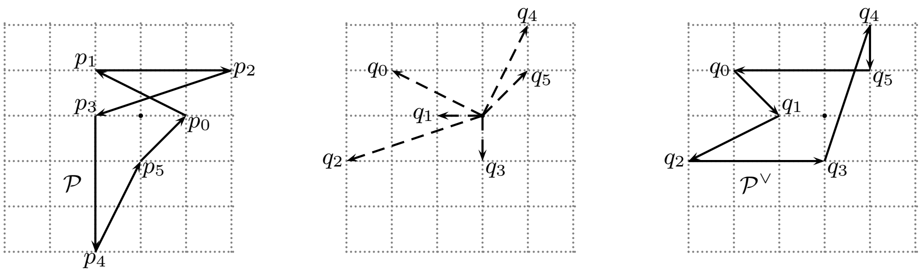
One easily verifies that equivalence preserves the length and the homotopy class of the corresponding curve. For that reason, we can unambiguously talk about the length and the winding number of an equivalence class of legal loops $\overline{\mathcal{P}}$. The former will be denoted by $\ell(\overline{\mathcal{P}})$.
A pathological remark is that all trivial legal loops are equivalent. Indeed, if $p_0, p_1 \in \mathbb{Z}^2$ are primitive vectors, one can always find a distinct point $p_2 \in \mathbb{Z}^2$ such that both the line through p_0 and p_2 and the line through p_1 and p_2 lie at integral distance 1 from o. By merging and splitting, $(p_0) \sim (p_0, p_2) \sim (p_2) \sim (p_2, p_1) \sim (p_1)$. The corresponding equivalence class will be called the *trivial class*.

Definition (dual class) Let $\overline{\mathcal{P}}$ be an equivalence class of legal loops. Take a representant $\mathcal{P} = (p_0, \ldots, p_{n-1})$ for which $p_i \neq p_{i+1}$ for all $i = 0, \ldots, n-1$. Define

$$q_i = \frac{p_{i+1} - p_i}{\det \begin{pmatrix} p_{i+1} \\ p_i \end{pmatrix}}.$$

Then the dual class $\overline{\mathcal{P}}^\vee$ is defined to be the class of $(q_0, \ldots, q_{n-1})$.

The reader can check that this is well-defined. Note that the trivial class is self-dual. In the non-trivial case, one can take a representant having no two consecutive moves along the same line, by means of which one easily verifies that $\overline{\mathcal{P}}^{\vee\vee} = \overline{\mathcal{P}}$.



The above series of figures illustrates the construction of the dual class. The notation $\mathcal{P}^\vee$ should in principle be read as 'a representant of $\overline{\mathcal{P}}^\vee$', but since $\mathcal{P}$ does not contain any moves of length 0 we can unambiguously write $\mathcal{P}^\vee$. Note that the length of $\mathcal{P}$ is $1 - 3 + 1 + 3 + 1 + 1 = 4$, whereas the length of $\mathcal{P}^\vee$ is $1 + 1 + 3 + 1 - 1 + 3 = 8$.

Theorem 6 (Poonen and Rodriguez-Villegas, 2000) *Let $\mathcal{P}$ be a legal loop of winding number ω. Then $\ell(\overline{\mathcal{P}}) + \ell(\overline{\mathcal{P}}^\vee) = 12\omega$.*

<u>Proof.</u> This is hinted in the paper by Poonen and Rodriguez-Villegas [23], which contains the details of the case where $\mathcal{P}$ is the boundary of a lattice polygon of genus 1, ran through counterclockwise. Since the necessary adaptations for the general case are not entirely trivial, we include the details here.

The only external fact we need concerns the set $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$, an element of which is a pair $(M, [\gamma])$. Here $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and γ is a homotopy class of paths in $\mathbb{R}^2 \setminus \{o\}$ from $(0,1)$ to (c,d). If $(c,d) \neq (0,-1)$, one can consider a straight-line path γ from $(0,1)$ to (c,d). In that case, we simply write $\widetilde{M}$ instead of $(M,[\gamma])$. The set $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ is turned into a group by the rule

$$(M_1, [\gamma_1]) \cdot (M_2, [\gamma_2]) = (M_1 M_2, [\gamma_2 * \gamma_1^{M_2}]),$$

where $*$ is the concatenation of paths sharing an endpoint, and where $\gamma_1^{M_2}$ is the path obtained by composing γ_1 with $(a,b) \mapsto (a,b)M_2$. A prominent role is played by the element $(\mathbb{I}, \mathrm{loop})$, where $\mathbb{I}$ is the identity matrix and 'loop' is the homotopy class of a counterclockwise loop around the origin. Namely, the property of $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ that we need is the existence of a group homomorphism $\Phi : \widetilde{\mathrm{SL}_2(\mathbb{Z})} \to \mathbb{Z}$ under which $(\mathbb{I}, \mathrm{loop})$ is mapped to 12. This can be achieved in various ways; see [23, Section 8.4] for a fancy proof in which the '12' appears as the weight of the modular discriminant.
By merging, splitting and switching to $\mathcal{P}^{-1}$ if necessary, we may assume that $\mathcal{P} = (p_0, \ldots, p_{n-1})$ consists of moves of length 1 or -1 only, and that the move (p_0, p_1) has length 1. Then an orientation-preserving lattice transformation brings us to the case where $p_0 = (1,0)$ and $p_1 = (0,1)$. For $i = 0, \ldots, n-1$, let s_i denote the legal move from p_i to p_{i+1}, and let $M_i \in \mathrm{SL}_2(\mathbb{Z})$ be defined inductively by

$$M_i \cdot M_{i-1} \cdots M_0 = \begin{pmatrix} \ell(s_{i+1}) \cdot p_{i+1} \\ p_{i+2} \end{pmatrix}.$$

In particular $M_{n-1} \cdots M_0 = \mathbb{I}$, but note that one even has

$$\widetilde{M}_{n-1} \cdots \widetilde{M}_0 = (\mathbb{I}, [\gamma(\mathcal{P})]).$$

It follows that

$$\sum_{i=0}^{n-1} \Phi(\widetilde{M}_i) = 12\omega.$$

Now there are two types of M_i:

$$\text{either } M_i = \begin{pmatrix} 0 & 1 \\ -1 & d_i \end{pmatrix} \text{ or } M_i = \begin{pmatrix} 0 & -1 \\ 1 & d_i \end{pmatrix},$$

depending on whether $\ell(s_{i+1}) = 1$ or $\ell(s_{i+1}) = -1$. Using that $\Phi((\mathbb{I}, \mathrm{loop})) = 12$, one accordingly finds that

$$\Phi(\widetilde{M}_i) = 3 - d_i \quad \text{resp.} \quad \Phi(\widetilde{M}_i) = d_i - 3.$$

Thus we conclude

$$\sum_{i=0}^{n-1} \ell(s_{i+1}) \cdot (3 - d_i) = 12\omega. \tag{1}$$

On the other hand, if we define q_i as in the above definition, and if we let $s_i^\vee$ be the legal move from q_i to q_{i+1}, then d_i contains information about $\ell(s_i) + \ell(s_i^\vee)$. Namely,

1. if $\ell(s_i) = \ell(s_{i+1}) = 1$, then $\ell(s_i) + \ell(s_i^\vee) = 3 - d_i$;
2. if $\ell(s_i) = \ell(s_{i+1}) = -1$, then $\ell(s_i) + \ell(s_i^\vee) = d_i - 3$;
3. if $\ell(s_i) = 1$ and $\ell(s_{i+1}) = -1$, then $\ell(s_i) + \ell(s_i^\vee) = d_i - 1$;
4. if $\ell(s_i) = -1$ and $\ell(s_{i+1}) = 1$, then $\ell(s_i) + \ell(s_i^\vee) = 1 - d_i$.

Since a closed loop must switch as many times from being positively oriented to being negatively oriented as conversely, we find that

$$\sum_{i=0}^{n-1} \ell(s_i) + \ell(s_i^\vee) = \sum_{i=0}^{n-1} \ell(s_{i+1})(3 - d_i).$$

Together with (1), this concludes the proof. ■

Remark. In case $\mathcal{P}$ is the boundary of a genus 1 polygon, Theorem 6 can be easily proven by exhaustively verifying it for the 16 representants of Theorem 10*(b)*. In a quest for explaining the '12', Poonen and Rodriguez-Villegas gave three alternative proofs. One of these is, essentially, the proof produced above. Of the four approaches, it seems best-suited for addressing the general case. A fifth proof manages to deal with the intermediate case where $\mathcal{P}$ has winding number 1 and consists of positively oriented moves only, and is implicitly contained in the work of Koelman [15]; this is briefly elaborated in **(2.6)** below.

Remark. The notion of a legal loop can be generalized by dropping the condition that the endpoints should coincide. Such curves have been studied by Karpenkov in the context of lattice trigonometry, where they are called *o-broken lines* [14, Definition 3.1]. In the same trigonometric philosophy, Poonen and Rodriguez-Villegas suggested a connection between Theorem 6 and the Gauss-Bonnet theorem. See [23, Section 10].

(2.4) Let Δ be a lattice polygon such that $\Delta^{(1)}$ is two-dimensional. To $\Delta^{(1)}$, we can associate in a natural way two legal loops, up to shifting.

Definition (edge-moving loop) Let $p_0, p_1, \ldots, p_{n-1}$ be the vertices of $\Delta^{(1)}$, enumerated counterclockwise. Let τ_i and τ_i' be the edges adjacent to p_i, so that p_i is the top of the cone $\mathcal{H}_{\tau_i} \cap \mathcal{H}_{\tau_i'}$. Let $p_i^{(-1)}$ be the top of the cone $\mathcal{H}_{\tau_i}^{(-1)} \cap \mathcal{H}_{\tau_i'}^{(-1)}$. The *edge-moving loop* $\mathcal{P}(\Delta^{(1)})$ is defined to be $(p_0^{(-1)} - p_0, \ldots, p_{n-1}^{(-1)} - p_{n-1})$. This is well-defined up to shifting.

Definition (normal fan loop) Let $t_0, \ldots, t_{n-1}$ be the primitive generators of the rays of the normal fan of $\Delta^{(1)}$, enumerated counterclockwise. Along with

o, two consecutive primitive generators always span a triangle having no interior lattice points. Therefore, $\mathcal{N}(\Delta^{(1)}) := (t_0, \ldots, t_{n-1})$ is a legal loop, well-defined up to shifting. It will be called the *normal fan loop* of $\Delta^{(1)}$.

Remark. If, conversely, the primitive generators of the normal fan of a two-dimensional lattice polygon Γ span a legal loop, then this does not guarantee that $\Gamma = \Delta^{(1)}$ for some lattice polygon Δ. However, it does guarantee that $k\Gamma = \Delta^{(1)}$ for some lattice polygon Δ and some Minkowski multiple $k\Gamma$. In other words, there are two distinct reasons why a lattice polygon Δ can fail to be interior to a larger lattice polygon: either its normal fan is not a legal loop (i.e. the fan is not *Gorenstein* using the terminology of **(4.2)**), or the polygon is just too small.

The following lemma gives relationships between $\mathcal{P}(\Delta^{(1)})$ and $\mathcal{N}(\Delta^{(1)})$. We call a legal loop $\mathcal{P} = (p_0, p_1, \ldots, p_{n-1})$ *convex* if every move has positive length and each p_i lies on an edge of $\text{Conv}\{p_0, p_1, \ldots, p_{n-1}\}$. We call $\mathcal{P}$ *strictly convex* if moreover each p_i appears as a vertex. Note that convexity nor strict convexity are properties of the equivalence class: they are not invariant under merging and splitting. Observation *(d)* below is crucial and is essentially due to Haase and Schicho.

Lemma 1 *(a)* $\mathcal{N}(\Delta^{(1)})$ *has moves of strictly positive length only.*

(b) The following are equivalent.

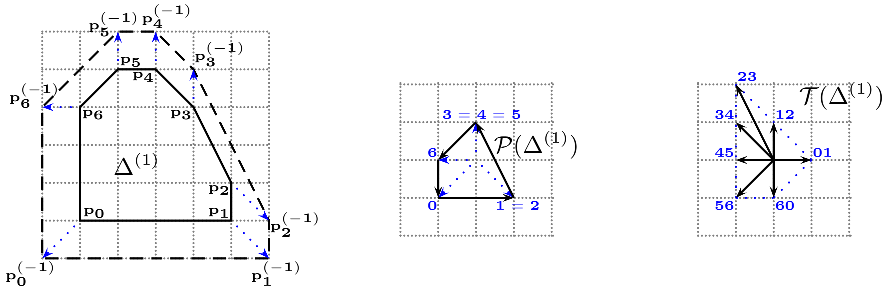
- $\mathcal{P}(\Delta^{(1)})$ *has moves of positive length only,*
- $\mathcal{P}(\Delta^{(1)})$ *is convex,*
- $\mathcal{P}(\Delta^{(1)})$ *is strictly convex,*
- $\mathcal{N}(\Delta^{(1)})$ *is convex.*

(c) The following are equivalent.

- $\mathcal{P}(\Delta^{(1)})$ *has moves of strictly positive length only,*
- $\mathcal{N}(\Delta^{(1)})$ *is strictly convex.*

(d) $\overline{\mathcal{P}(\Delta^{(1)})}^{\vee} = \overline{\mathcal{N}(\Delta^{(1)})}$

Proof. We only include the details for *(d)*. Instead of $\mathcal{N}(\Delta^{(1)})$, we will consider the legal loop $\mathcal{T}(\Delta^{(1)})$ obtained by considering the consecutive counterclockwise direction vectors of the edges of $\Delta^{(1)}$. Since $\mathcal{T}(\Delta^{(1)})$ is obtained from $\mathcal{N}(\Delta^{(1)})$ by applying a 90° counterclockwise rotation, both legal loops are clearly equivalent. If $\mathcal{P}(\Delta^{(1)})$ contains no legal moves of length 0, it follows by construction that $\overline{\mathcal{P}(\Delta^{(1)})}^{\vee} = \overline{\mathcal{T}(\Delta^{(1)})}$. In general, the situation is more subtle, and it is convenient to start from $\mathcal{T}(\Delta^{(1)})$ instead. The latter has no moves of length 0, so it makes sense to talk of the dual $\mathcal{T}(\Delta^{(1)})^{\vee}$ of $\mathcal{T}(\Delta^{(1)})$, rather than of its class.



An example where $\mathcal{P}(\Delta^{(1)})$ has several moves of length 0.

After an orientation-preserving lattice transformation and a translation over an integral vector if necessary, we may assume that

$$p_0 = (a, 0), \quad p_1 = (0, 0), \quad p_2 = (p, q),$$

for integers $a > 0$ and coprime $p, q < 0$ such that $q \leq p < 0$. Because moving out the edges should result in a lattice polygon again, we must have $p = -1$. Then $p_1^{(-1)} - p_1 = (0, 1)$. On the other hand, the corresponding move of $\mathcal{T}(\Delta^{(1)})$ is from $(-1, 0)$ to $(-1, q)$. This gives a vertex $(0, -1)$ on the dual $\mathcal{T}(\Delta^{(1)})^\vee$. One concludes that $\mathcal{T}(\Delta^{(1)})$ equals $\mathcal{P}(\Delta^{(1)})$ modulo a 180° rotation (and modulo shifting). ■

Lemma 2 *Let Δ be a lattice polygon such that $\Delta^{(1)}$ is two-dimensional. Suppose that Δ is maximal. Let R be the number of lattice points on the boundary of Δ and let $R^{(1)}$ be the number of lattice points on the boundary of $\Delta^{(1)}$. Then $\ell(\mathcal{P}(\Delta^{(1)})) = R - R^{(1)}$.*

Proof. Using a normalization as above, one easily verifies that the length of $(p_i^{(-1)} - p_i, p_{i+1}^{(-1)} - p_{i+1})$ equals the difference between the number of lattice points on the face (edge or vertex) of $\Delta = \Delta^{(1)(-1)}$ connecting $p_i^{(-1)}$ and $p_{i+1}^{(-1)}$ and the number of lattice points on the edge of $\Delta^{(1)}$ connecting p_i and p_{i+1}. ■

Remark. As pointed out in [27], there is a natural way of associating a legal loop to *any* lattice polygon Δ for which $\Delta^{(1)}$ is two-dimensional, in such a way that its length still measures $R - R^{(1)}$: for each vertex $p_i^{(-1)}$ of $\Delta^{(1)(-1)}$, let a_i and b_i be the nearest-by lattice points on the adjacent edges of $\Delta^{(1)(-1)}$ that are contained in Δ (considered counterclockwise). Then in the definition of $\mathcal{P}(\Delta^{(1)})$, one should replace $p_i^{(-1)} - p_i$ by $a_i - p_i, b_i - p_i$.

(2.5) We are now ready to prove Coleman's conjecture.

Proof of Theorem 1. Using the classification given in Theorem 10 below, the statement is easily verified in case $\Delta^{(1)}$ is not two-dimensional. So suppose to the contrary that $\Delta^{(1)}$ is two-dimensional. Then a second observation is that it suffices to give a proof for the case where Δ is maximal, i.e. $\Delta = \Delta^{(1)(-1)}$.

Indeed, if not, Δ is obtained from $\Delta^{(1)(-1)}$ by repeatedly clipping off a vertex. At each step, the number of lattice points on the boundary is reduced by one, whereas the number of vertices increases by at most one. Hence the validity of Coleman's conjecture for Δ follows from its validity for $\Delta^{(1)(-1)}$.

Now let $n^{(1)}$ be the number of vertices of $\Delta^{(1)}$ and let $R^{(1)}$ be the number of lattice points on its boundary. From the definition of moving out the edges, we see that $n \leq n^{(1)}$. From Lemmata 1 and 2, it follows that

$$R - R^{(1)} = \ell(\mathcal{P}(\Delta^{(1)})) = 12 - \ell(\mathcal{N}(\Delta^{(1)})) \leq 12 - n^{(1)} \leq 12 - n.$$

The statement then follows from $R^{(1)} \leq g$ and $g \geq 2$. ■

Note that the proof yields the much stronger statement that

$$R \leq R^{(1)} + 12 - n \tag{2}$$

as soon as $\Delta^{(1)}$ is two-dimensional (regardless of whether Δ is maximal or not).

(2.6) Building on work of Oda [21, Remark on page 45], Koelman proved a statement which immediately implies Coleman's conjecture. Let Δ be a lattice polygon with two-dimensional interior $\Delta^{(1)}$. Let η be number of rays of the smooth completion of the normal fan of Δ. Let R resp. $R^{(1)}$ be the number of lattice points on the boundary of Δ resp. $\Delta^{(1)}$. Then [15, Lemma 4.5.2(2)] states

$$R^{(1)} = R + \eta - 12. \tag{3}$$

Since $\eta \geq n$, with n the number of vertices of Δ, Coleman's conjecture follows. Equality (3) even implies the '12 theorem' for legal loops of winding number 1, all of whose segments are positively oriented (and for their duals, of course). Indeed, let $\mathcal{P} = (p_0, \ldots, p_{n-1})$ be such a legal loop, then the p_i can be thought of as the generators of the rays of a fan. This fan can always be realized as the normal fan of a certain two-dimensional lattice polygon. By the legal-loop-properties of the fan, a sufficiently large Minkowski multiple Δ of this lattice polygon will be such that $\Delta^{(-1)}$ takes vertices in $\mathbb{Z}^2$. Applying (3) to $\Delta^{(-1)}$ then implies the theorem, by noting that $\eta = \ell(\mathcal{T}(\Delta))$. Digging into Oda's work, one sees that this proof is somehow related to Poonen and Rodriguez-Villegas' second proof [23, Section 6] and the exercises in Fulton's book [11, Section 2.5] to which they refer. Using work of Hille and Skarke [13], it should be possible to generalize the above to arbitrary winding numbers.

(2.7) We end this section by briefly commenting on Haase and Schicho's 'onion skin theorem' [12, Theorem 8]. Using $n \geq 3$, inequality (3) yields $R \leq R^{(1)} + 9$, which is the above-mentioned refinement of Scott's bound that Haase and Schicho obtained. In this case, one additionally checks that equality holds if and only if Δ is equivalent to $d\Sigma$ for some integer $d \geq 4$. Here $\Sigma = \text{Conv}\{(0,0),(1,0),(0,1)\}$ is the standard 2-simplex. By recursively applying

$R \leq R^{(1)} + 9$, whilst gradually 'peeling off' the lattice polygon, one obtains an inequality relating R to the genus g of Δ and to the 'number of onion skins'. This led Haase and Schicho to introducing the notion of *level*. Let $n \geq 0$ be the maximal integer for which $\Delta^{(n)}$ is defined. The level of Δ is *(i)* equal to n if $\Delta^{(n)}$ is a point or a line segment, *(ii)* equal to $n + 1/3$ if $\Delta^{(n)}$ is equivalent to Σ, *(iii)* equal to $n+2/3$ if $\Delta^{(n)}$ is equivalent to 2Σ, and *(iv)* is equal to $n+1/2$ if $\Delta^{(n)}$ is any other two-dimensional lattice polygon of genus 0. Then the onion skin theorem reads:

Theorem 7 (Haase and Schicho, 2009) *Let Δ be a convex lattice polygon of level $\ell \geq 1$ and genus g, containing R lattice points on the boundary. Then $(2\ell - 1)R \leq 2g + 9\ell^2 - 2$.*

However, although the proof is beautiful, the resulting statement is not as deep as one might hope. The reason is that applying $R \leq R^{(1)} + 9$ at each step is too rough; it would be more powerful to include the number of vertices in the argument, although we did not find an elegant way of doing so.
An alternative, more classically flavored measure for the number of onion skins is the *lattice width* of Δ, which is the minimal integer $s \geq 0$ for which there is a $\mathbb{Z}$-affine transformation mapping Δ into the strip $\{ (x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq s \}$; it is denoted $\mathrm{lw}(\Delta)$. Indeed, one can prove (see [6, Theorem 4] or [19, Theorem 13]) that for every lattice polygon Δ of genus at least 1 one has $\mathrm{lw}(\Delta) = \mathrm{lw}(\Delta^{(1)})+2$, unless Δ is equivalent to $d\Sigma$ for some integer $d \geq 3$, in which case $\mathrm{lw}(\Delta) = \mathrm{lw}(\Delta^{(1)}) + 3 = d$. Then by redoing the onion skin argument, using the lattice width rather than the level, one obtains a closely related statement:

Theorem 8 *Let Δ be a two-dimensional lattice polygon of genus g, containing R lattice points on its boundary. Then* $(\mathrm{lw}(\Delta)-1)\cdot R \leq 2g + 2\cdot(\mathrm{lw}(\Delta)^2-1)$.

<u>Proof.</u> If Δ is a lattice polygon which has genus 0, or for which $\Delta^{(1)}$ is not two-dimensional, then the inequality can be verified by hand (using, e.g. Theorem 10 below). Therefore, suppose that the lemma holds for all lattice widths up to $k-1$, $k \geq 3$. Let Δ be a lattice polygon with $\mathrm{lw}(\Delta) = k$. If Δ is equivalent to $k\Sigma$, then the inequality holds by explicit verification (use $\mathrm{lw}(\Delta) = k$, $g = (k-1)(k-2)/2$, $R = 3k$). If not, then the result easily follows by induction, using $R \leq R^{(1)} + 8$. ■

As said, this is not a deep statement. Indeed, imagine Δ being caught in a horizontal strip $\{ (x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq \mathrm{lw}(\Delta) \}$. Assume for ease of exposition that both $\Delta \cap \{y = 0\}$ and $\Delta \cap \{y = \mathrm{lw}(\Delta)\}$ are line segments of length $a \geq 3$ and $b \geq 3$ respectively. By focusing on the interior lattice points of these, one sees that $R \leq (a-1) + (b-1) + 2 \cdot (\mathrm{lw}(\Delta) + 1)$. Applying Pick's theorem to the trapezoid spanned by these interior lattice points then yields

$$R \leq \frac{2}{\mathrm{lw}(\Delta) - 1} \cdot g + 2 \cdot (\mathrm{lw}(\Delta) + 1)$$

which is indeed a rephrasing of Theorem 8.

3 Consecutively moving out the edges

(3.1) Although it is not entirely clear whom it should be attributed to, the following fact is well-known.

Theorem 9 *Let g be a positive integer. If $g \geq 1$ then there exists only a finite number of equivalence classes of lattice polygons of genus g.*

The classical argument works by noting that a lattice polygon of genus $g \geq 1$ satisfies $\mathrm{Vol}(\Delta) \leq 2g + \frac{5}{2}$ (using Scott's bound from Theorem 4 along with Pick's theorem) and that a lattice polygon with volume V can always be caught in a lattice square of side length $4V$ – see Lagarias and Ziegler for an account that deals with arbitrary dimension [18]. This yields an algorithm for enumerating all equivalence classes of lattice polygons of genus g: consider all lattice polygons that are contained in

$$[0, 8g + 10] \times [0, 8g + 10]$$

and filter out unique representants of each conjugacy class. However, this is too slow to be of any practical use.

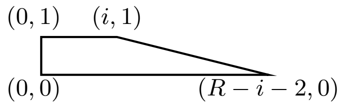
(3.2) We present an alternative proof of Theorem 9 that leads to a more efficient algorithm. The idea is to proceed by induction on g, based on Theorem 5. We call a lattice polygon *elliptic* if it contains a unique lattice point in its interior. A lattice polygon of genus $g \geq 2$ is called *hyperelliptic* if its interior lattice points are contained in a line.

Proof of Theorem 9. Suppose that the theorem holds for $0, 1, \ldots, g - 1$, where $g \geq 1$. We partition the set of (equivalence classes of) lattice polygons of genus g as

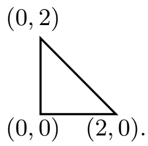
$$\begin{array}{lll} \text{(i)} & & \{\text{elliptic or hyperelliptic lattice polygons}\} \\ \text{(ii)} & \sqcup & \left\{\text{lattice polygons } \Delta \text{ for which } \Delta^{(1)} \text{ is two-dimensional of genus } 0\right\} \\ \text{(iii)} & \sqcup & \left\{\text{lattice polygons } \Delta \text{ for which } \Delta^{(1)} \text{ has genus} \geq 1\right\} \end{array}$$

and prove the finiteness of each subset. For set (i), this follows from Theorem 10*(b-c)* below. For set (ii), Theorem 10*(a)* shows that there is only a finite number of possibilities for $\Delta^{(1)}$; for each $\Delta^{(1)}$, Theorem 5 states that $\Delta \subset \Delta^{(1)(-1)}$, hence there is only a finite number of possibilities for Δ. Finally, for set (iii), the induction hypothesis shows that there is only a finite number of possibilities for $\Delta^{(1)}$, and again Theorem 5 shows that there is only a finite number of possibilities for Δ. ■

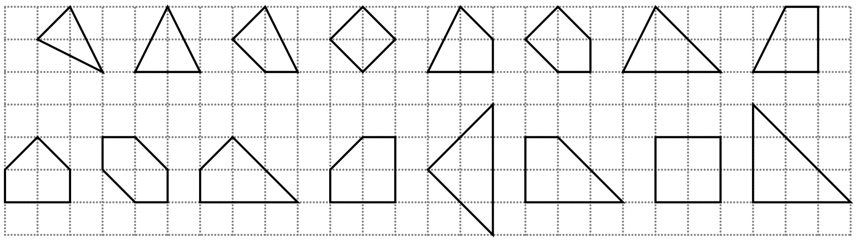
Theorem 10 (Koelman, 1991) *(a) Every two-dimensional lattice polygon of genus 0 having $R \geq 3$ lattice points on the boundary is equivalent to exactly one of the following $\lfloor R/2 \rfloor$ polygons:*



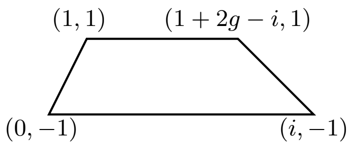
for $i \in \{0, \dots, \lfloor R/2 \rfloor - 1\}$*, except if* $R = 6$*, where in addition one has the possibility*



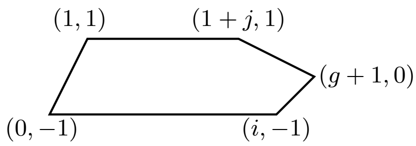
(b) *Every lattice polygon of genus* 1 *is equivalent to exactly one of the following* 16 *polygons:*



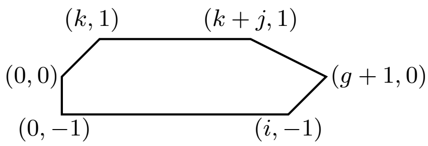
(c) *Every hyperelliptic lattice polygon of genus* $g \geq 2$ *is equivalent to exactly one of the following* $\frac{1}{6}(g+3)(2g^2+15g+16)$ *polygons:*



$g \leq i \leq 2g$



$0 \leq i \leq g \;\&\; 0 \leq j \leq i$
$g < i \leq 2g+1 \;\&\; 0 \leq j \leq 2g - i + 1$



$0 \leq k \leq g+1 \;\&\; 0 \leq i \leq g+1-k \;\&\; 0 \leq j \leq i$
$0 \leq k \leq g+1 \;\&\; g+1-k < i \leq 2g+2-2k \;\&\; 0 \leq j \leq 2g-i-2k+2.$

Proof. A complete proof can be found in Chapter 4 of Koelman's Ph.D. thesis [15], but since there are no surprising ingredients, the proof could also be left as a patience-involving exercise. Note that *(a)* and *(b)* have been (re)discovered multiple times before and since. ■

For our alternative proof of Theorem 9, a strongly simplified version of Theorem 10 only involving finiteness statements would have been sufficient.

(3.3) Our proof of Theorem 9 results in the following algorithm for enumerating all lattice polygons of genus at most g up to equivalence.

INPUT: an integer $g \geq 1$.
OUTPUT: a list $[L_1, L_2, \dots, L_g]$ where each L_i is a list containing a unique representant of each equivalence class of lattice polygons of genus i.

In fact, the algorithm produces a list $\mathfrak{L} = [\mathfrak{L}_1, \mathfrak{L}_2, \dots, \mathfrak{L}_g]$, where each $\mathfrak{L}_i$ is a list

$$[\ell_{i,3}, \ell_{i,4}, \dots, \ell_{i,2i+7}],$$

and each $\ell_{i,R}$ is a list containing a unique representant of each equivalence class of lattice polygons of genus i having R lattice points on the boundary. The lists L_i are then obtained by concatenating the $\ell_{i,R}$'s for $R = 3, \dots, 2i+7$. Remark that the number of lattice points on the boundary of a genus $i \geq 1$ lattice polygon is indeed at most $2i+7$ by Theorem 4.

1. $\mathfrak{L} := [\,[\,],[\,],\dots,[\,]\,]$ (g entries, indexed by $1, \dots, g$);
2. for $i = 1, \dots, g$ do;
3. $\mathfrak{L}[i] := [\,[\,],[\,],\dots,[\,]\,]$ ($2i+5$ entries, indexed by $3, \dots, 2i+7$)
4.   for $R = 3, \dots, 2i+7$ do
5. $\mathfrak{L}[i][R] :=$ list of (hyper)elliptic polygons of genus i
6.   with R boundary points (using Theorem 10*(b-c)*);
7.   end for;
8.   for all two-dimensional Δ of genus 0 having i boundary points
9.   (using Theorem 10*(a)*) do
10. $L := [$ unique representants of all polygons with interior $\Delta\,]$;
11.   for $\Gamma \in L$ do
12. $\mathfrak{L}[i][\#\partial\Gamma \cap \mathbb{Z}^2]$ cat:= $[\Gamma]$;
13.   end for;
14.   end for;
15.   for all (j, R) such that $j + R = i$ and $3 \leq R \leq 2j + 7$ do
16.   for all $\Delta \in \mathfrak{L}[j][R]$ do
17. $L := [$ unique representants of all polygons with interior $\Delta\,]$;
18.   for $\Gamma \in L$ do
19. $\mathfrak{L}[i][\#\partial\Gamma \cap \mathbb{Z}^2]$ cat:= $[\Gamma]$;
20.   end for;
21.   end for
22.   end for;
23. end for;

The three disjoint for-loops correspond respectively to the cases (i), (ii), (iii) of our proof of Theorem 9. The operation 'cat:=' abbreviates 'concatenate with'. Listing all polygons with interior Δ (see steps **10.** and **17.**) is done using Theorem 5: one checks whether $\Delta^{(-1)}$ is a lattice polygon. If not, then the resulting set L is empty. If yes, then L consists of all lattice polygons Γ that can be

obtained from $\Delta^{(-1)}$ by taking away boundary points without affecting the interior. Each time a lattice polygon is to be added, one checks whether or not it is equivalent to a polygon that is already contained in the list. In our implementation below, checking the equivalence of two lattice polygons Γ and Γ' is done very naively: we simply try for all triples of consecutive vertices $v_1, v_2, v_3 \in \Gamma$ (ordered counterclockwise) and $v_1', v_2', v_3' \in \Gamma$ (ordered either clockwise or counterclockwise) whether there is a $\mathbb{Z}$-affine transformation φ taking v_i to v_i' (for $i = 1, 2, 3$); if yes, it is necessarily unique. We then check whether $\varphi(\Gamma) = \Gamma'$. The algorithm can be sped up by instead keeping track of the automorphisms of Δ (i.e., $\mathbb{Z}$-affine transformations φ for which $\varphi(\Delta) = \Delta$): indeed, if there is a $\mathbb{Z}$-affine transformation taking Γ to Γ', it must be an automorphism of $\Delta = \Gamma^{(1)}$. For almost all polygons, this automorphism group will consist of the identity map only, resulting in a substantial speed-up.
While much more efficient than the naive method suggested in **(3.1)**, note that the problem is in itself exponential and that one cannot expect being able to push the computation very far: if $N(g)$ denotes the number of equivalence classes of lattice polygons of genus g, then one can show that $\log N(g)$ grows like $\sqrt[3]{g}$, although it is unknown whether $\lim_{g\to\infty} \log N(g)/\sqrt[3]{g}$ exists. See Bárány's survey paper [2] for some discussions on this matter.
We finally remark that Koelman already briefly described and implemented a similar algorithm for enumerating lattice polygons by their total number of lattice points, rather than their genus – see [15, Section 4.4].

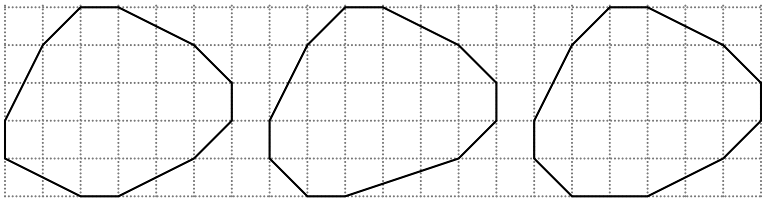
(3.4) We have implemented the above algorithm in Magma [5], along with several basic functions for dealing with lattice polygons, such as functions for computing the genus, the number of boundary points, the interior hull, the polygon obtained by moving out the edges, ... We have also implemented an algorithm due to Feschet [10, Section 3] for computing the lattice width of a lattice polygon Δ. The code can be found at `http://wis.kuleuven.be/algebra/castryck/`. The intention is to make the code cleaner and more efficient in the future. We have executed our current implementation on the input $g = 30$. This took roughly one month of computation, although it is likely that keeping track of the automorphism groups, as explained at the end of **(3.3)**, would have shortened this span considerably. The resulting output is stored in a file of approximately 368 MB, which has also been made available for download. We include some summarizing data here – see Table 1, but of course our output can be used to answer virtually every reasonable question on lattice polygons of genus ≤ 30. Many of these questions have been asked explicitly in the literature before, see e.g. [16, 24]. It is somewhat remarkable that the merit of exhaustive computation in tackling these questions has not been fully acknowledged thus far, with the exception of the Ph.D. thesis of Koelman and some preliminary attempts by Rabinowitz.

(3.5) We now focus on one particular problem: for every positive integer n, what is the minimal genus $g(n)$ of a lattice n-gon? Arkinstall [1], Rabinowitz [24], Simpson [26], and Olszewska [22] elaborated this for various small values

g	N	$N^{(1)}$	$n_{\max}$	$n_{\max}^{(1)}$	$n_{\min}^{(1)}$	$\mathrm{lw}_{\max}$	$\mathrm{lw}_{\max}^{(1)}$	$R_{\min}^{(1)}$
1	16	16	6	6	3	3	3	3
2	45	22	6	6	4	2	2	5
3	120	63	6	6	3	4	4	5
4	211	78	8	8	3	4	4	6
5	403	122	7	7	3	4	4	7
6	714	192	8	8	3	5	5	6
7	1023	239	9	9	3	4	4	7
8	1830	316	8	8	4	4	4	8
9	2700	508	8	8	3	5	5	8
10	3659	509	10	10	3	6	6	8
11	6125	700	9	9	4	5	5	8
12	8101	1044	9	9	4	6	6	8
13	11027	1113	10	10	3	6	6	9
14	17280	1429	10	10	4	6	6	9
15	21499	2052	10	10	3	7	7	9
16	28689	1962	10	10	3	6	6	9
17	43012	2651	11	11	4	6	6	9
18	52736	3543	10	10	4	7	7	10
19	68557	3638	12	12	3	8	8	9
20	97733	4594	12	11	4	7	7	9
21	117776	5996	12	12	3	8	8	10
22	152344	6364	11	11	4	8	8	10
23	209409	7922	11	11	4	8	7	10
24	248983	9693	12	12	4	8	8	10
25	319957	10208	12	12	3	8	8	10
26	420714	12727	12	12	4	8	8	11
27	497676	15431	13	12	4	8	8	9
28	641229	15918	12	12	3	9	9	10
29	813814	20354	12	12	4	8	8	11
30	957001	23874	13	12	4	9	9	11

Table 1: A superscript $^{(1)}$ denotes that the corresponding invariant was obtained by restricting the count to those polygons that are *interior* to another polygon (which can be easily checked using Theorem 5). Then for each integer $1 \leq g \leq 30$, the table shows the number of equivalence classes (N) of polygons of genus g, the maximal resp. minimal number of vertices ($n_{\max}$ resp. $n_{\min}$), the maximal lattice width ($\mathrm{lw}_{\max}$) and the minimal number of lattice points on the boundary ($R_{\min}$) that are possible for that genus. Note that $n_{\min}$ and $R_{\min}$ without superscript are always equal to 3, hence not included in the table. The maximal number of points on the boundary (in both settings) is $2g + 6$ except if $g = 1$ (where it is 9) by Theorem 4, so again we did not include this.

of n, leaving $n = 15$ as the smallest open entry. Up to $n = 13$, their results are immediately confirmed by our computation. E.g. using Table 1, one can check that $g(11) = 17$, a case which has provoked particular interest in the past and was settled in 2006 only [22]. In fact, our output shows that there are three inequivalent 11-gons realizing the bound:



Note that the second polygon is obtained from the third by clipping off the right-most of the lower-most vertices. This is a general phenomenon: every lattice n-gon having minimal genus can be transformed to a lattice n-gon having minimal genus and all of whose boundary lattice points are vertices. This implies that the minimal genus $g(n)$ and the minimal volume $V(n)$ of a lattice n-gon are related through Pick's theorem by $V(n) = g(n) + n/2 - 1$. These are the contents of [26, Theorem 1]. In our case, it yields that every lattice 11-gon has an area of at least $43/2$. This bound is achieved by (and only by) the first two of the above polygons.
Using a refined search, we managed to settle the case $n = 15$.

<u>Proof of Theorem 2B.</u> We first make some general observations. Let Δ be a lattice n-gon such that $\Delta^{(1)}$ is two-dimensional. Write $\Delta^{\max} = \Delta^{(1)(-1)}$, let $n^{(1)}$ resp. $n^{\max}$ be the number of vertices of $\Delta^{(1)}$ and $\Delta^{\max}$, and let R, $R^{(1)}$ resp. $R^{\max}$ be the number of lattice points on the boundaries of Δ, $\Delta^{(1)}$ and $\Delta^{\max}$. One obviously has

$$n^{\max} \leq n^{(1)}.$$

We also have

$$R^{\max} \geq n^{\max} + 2(n - n^{\max}),$$

which follows because Δ is obtained from $\Delta^{\max}$ by taking away a number of boundary points (indeed, $\Delta \subset \Delta^{\max}$ by Theorem 5), and that each introduction of a new vertex requires the existence of two lattice points on the boundary of $\Delta^{\max}$ that are not vertices. From the proof of Theorem 1 in **(2.5)** we see that

$$R^{\max} \leq R^{(1)} + 12 - n^{(1)}.$$

Then combining the three inequalities yields

$$R^{(1)} \geq 2n - 12,$$

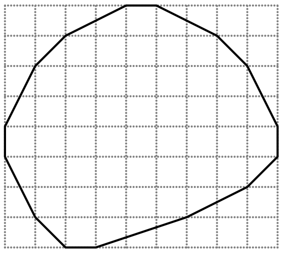
while it is also clear that

$$n^{(1)} \geq \lceil n/2 \rceil.$$

Now let Δ be a lattice 15-gon of genus $g = g(15)$. Note that by Theorem 10*(c)*, Δ is non-hyperelliptic, thus the above applies. In particular, we have

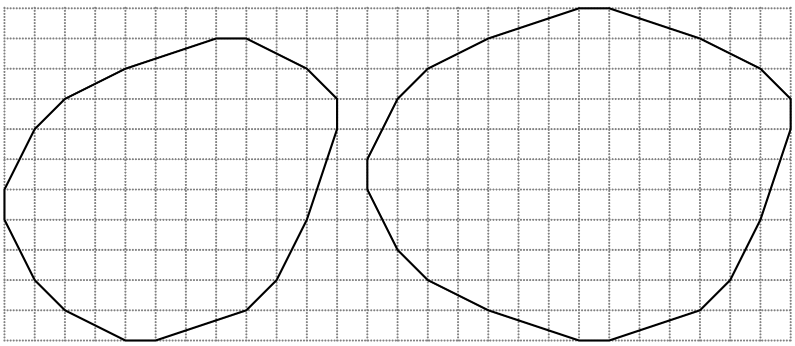
$$n^{(1)} \geq 8 \quad \text{and} \quad R^{(1)} \geq 18. \tag{4}$$

Now since



is a lattice 15-gon of genus 45, it follows that $g \leq 45$. We also know that $g \geq 43$ by [26, Corollary 11(c)]. So suppose that $g \in \{43, 44\}$. Then with $g^{(1)}$ the genus of $\Delta^{(1)}$ we have that $g^{(1)} \geq 1$ because of Theorem 10*(a)* (note that $n^{(1)} \geq 8$), and $g^{(1)} \leq 26$ because of (4). In particular, $\Delta^{(1)}$ must be contained in our list produced in **(3.4)**.
The remainder of the proof is computational. Out of our list, we have selected those lattice polygons Γ that are of the form $\Delta^{(1)}$, i.e. for which $\Gamma^{(-1)}$ takes vertices in the lattice (following Theorem 5), that have at least 8 vertices, that contain at least 18 lattice points on the boundary, that have genus at most 26, and for which the sum of the latter two invariants is contained in $\{43, 44\}$. This resulted in 1929 polygons. For each such polygon Γ, we enumerated all polygons Δ for which $\Delta^{(1)} = \Gamma$, in a similar way as described in **(3.3)**, and checked whether any of these has 15 vertices. In each case, the answer was no. ■

Finally, the picture below proves that $g(17) \leq 72$ (Simpson's previous upper bound was 79) and that $g(19) \leq 105$ (versus 112). Our guess is that these bounds are not yet optimal.



A summarizing update of the currently known values of $g(n)$ can be found in Table 2. We conclude by remarking that the asymptotic behavior of $g(n)$ is

n	3	4	5	6	7	8	9	10	11	12
$g(n)$	0	0	1	1	4	4	7	10	17	19
n	13	14	15	16	17	18	19	20	21	22
$g(n)$	27	34	45^*	52	$[66, 72^*]$	79	$[96, 105^*]$	112	$[133, 154]$	154

Table 2: Known values for $g(n)$; values marked with an asterisk are new contributions.

well-understood. It is known that for all $n \geq 3$

$$\frac{1}{8\pi^2} < \frac{V(n)}{n^3} \leq \frac{1}{54}(1 + o(1))$$

and that

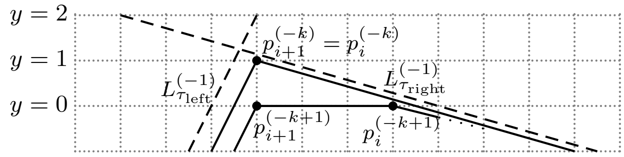
$$\lim_{n\to\infty} \frac{g(n)}{n^3} = \lim_{n\to\infty} \frac{V(n)}{n^3}$$

exists and lies close to (but is not equal to) 1/54. See [3] and the references therein.

(3.6) A notion dual to the level, as introduced by Haase and Schicho and reviewed in **(2.7)**, is the *lifespan* of a two-dimensional lattice polygon $\Delta \subset \mathbb{R}^2$, which is defined to be the maximal $k \in \mathbb{Z}_{\geq 0}$ for which $\Delta^{(-k)}$ is well-defined and takes vertices in $\mathbb{Z}^2$, provided such a k exists. If no such k exists, the lifespan is said to be *infinite*. Theorem 3 claims that an infinite lifespan can only occur if the number of vertices n is at most 9.

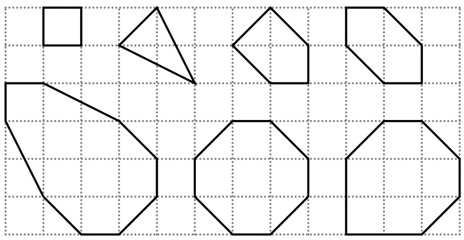
Proof of Theorem 3. Suppose that Δ has infinite lifespan. It suffices to prove that all moves of the edge-moving loop $\mathcal{P}(\Delta)$ have positive length. Indeed, by Lemma 1 this implies that the normal fan loop $\mathcal{N}(\Delta)$ is convex. Then the convex hull of its primitive generators (which are in 1-to-1 correspondence with the edges of Δ) must be contained in the list of Theorem 10*(b)*. In particular, the maximal number of primitive generators is 9, hence so is the maximal number of edges (equalling the number of vertices).
So suppose by contradiction that there are two consecutive vertices p_i and p_{i+1} of Δ such that the move $(p_i^{(-1)} - p_i, p_{i+1}^{(-1)} - p_{i+1})$ has negative length. As explained in Lemma 2, this means that the edge of $\Delta^{(-1)}$ connecting to $p_i^{(-1)}$ and $p_{i+1}^{(-1)}$ must have become shorter, unless it has even disappeared, i.e. $p_{i+1}^{(-1)} = p_i^{(-1)}$. If no edge disappears, then it is easy to see that $\mathcal{P}(\Delta) = \mathcal{P}(\Delta^{(-1)})$. By repeating the argument, one eventually must have $p_{i+1}^{(-k)} = p_i^{(-k)}$ for some $k \in \mathbb{Z}_{\geq 1}$ (where ${}^{(-k)}$ abbreviates ${}^{(-1)(-1)\cdots(-1)}$). We claim that $\Delta^{(-k-1)}$ takes at least one vertex outside $\mathbb{Z}^2$.

To see this, choose i such that $p_{i+1}^{(-k)} = p_i^{(-k)}$, but $p_{i+2}^{(-k)} \neq p_{i+1}^{(-k)}$. Modulo a $\mathbb{Z}$-affine transformation we may assume that $p_{i+1}^{(-k+1)} = (0,0)$, that $p_i^{(-k+1)} = (a, 0)$ for some integer $a > 0$, and that $p_{i+1}^{(-k)} = p_i^{(-k)} = (0, 1)$.



Let τ_{left} be the edge of $\Delta^{(-k)}$ that is left adjacent to $p_{i+1}^{(-k)} = p_i^{(-k)}$, and let $L_{\tau_{\text{left}}}$ be its supporting line. Move it out (in the sense of **(2.2)**) to obtain a line $L_{\tau_{\text{left}}}^{(-1)}$. Because of our choice of i, $L_{\tau_{\text{left}}}^{(-1)}$ contains the point $(0, 2)$. Now similarly define $L_{\tau_{\text{right}}}$ and $L_{\tau_{\text{right}}}^{(-1)}$. The slope σ of $L_{\tau_{\text{right}}}$ (hence of $L_{\tau_{\text{right}}}^{(-1)}$) satisfies $0 > \sigma > -1$. Since $(1, 1)$ is a lattice point that is not contained in $\Delta^{(-k)}$, the lines $L_{\tau_{\text{left}}}^{(-1)}$ and $L_{\tau_{\text{right}}}^{(-1)}$ must intersect in a point that lies strictly between $y = 1$ and $y = 2$. ■

Note that the above proof actually gives a criterion for a two-dimensional lattice polygon Δ to have infinite lifespan. This will be the case *if and only if* $\Delta^{(-1)}$ is a lattice polygon and $\mathcal{N}(\Delta)$ is convex. Examples of n-gons (for $n = 3, \ldots, 9$) having infinite lifespan are given in the picture below.



4 Concluding comments

(4.1) Coleman's conjecture (or at least parts of the proof given in **(2.5)**) can be extended to certain *non-convex lattice polygons*. By a non-convex lattice polygon we mean a closed region in $\mathbb{R}^2$ that can be bounded by a closed non-self-intersecting curve that is piece-wise linear, with the endpoints of the linear parts contained in $\mathbb{Z}^2$. For such a non-convex lattice polygon Δ, it makes sense to define $\Delta^{(-1)}$ by extending the corresponding notion of **(2.2)**. Suppose it takes vertices in $\mathbb{Z}^2$ and let R be the number of lattice points on its boundary. Then with $R^{(1)}$ the number of lattice points on the boundary of Δ, we will again have $R = R^{(1)} + 12 - \ell(\mathcal{T}(\Delta))$, with $\mathcal{T}(\Delta)$ the legal loop spanned by the direction vectors of the piece-wise linear boundary components of Δ.

(4.2) Much (if not all) of the foregoing can be related to toric geometry. It lies beyond the scope of this article to go into much detail here, but we briefly mention a few facts. We fully rely on the according references for the background.

For a two-dimensional lattice polygon Δ, we denote the according toric surface over $\mathbb{C}$ by $X(\Delta)$, which we assume to be naturally embedded in $\mathbb{P}_{\mathbb{C}}^{\#(\Delta\cap\mathbb{Z}^2)-1}$.
(i) If the primitive generators of the normal fan of Δ span a legal loop (see the corresponding remark in **(2.4)**), then by definition this is a *Gorenstein fan*, which is equivalent to saying that $X(\Delta)$ has only Gorenstein singularities [4, Proposition 2.7]. In particular, if Δ is the interior hull of another lattice polygon, then $X(\Delta)$ has only Gorenstein singularities. The converse is not true.
(ii) If this legal loop is moreover convex, then $X(\Delta)$ is *weak Fano*, meaning that the anticanonical bundle $-K_{X(\Delta)}$ is nef and big. If it is strictly convex, then $X(\Delta)$ is *Fano*, meaning that $-K_{X(\Delta)}$ is ample. See [20, Section 2.3]. In particular, if a lattice polygon Δ has infinite lifespan, then $X(\Delta)$ is Gorenstein and weak Fano. In this case, the converse holds as well.
(iii) Since convex legal loops (of winding number 1) have length at most 9, the above implies that in the weak Fano case, Δ can have no more than 9 edges and vertices. This also follows from a well-known degree bound for weak Fano surfaces X (namely, $(-K_X)^2 \leq 9$).
(iv) If $\Delta^{(1)}$ is well-defined and two-dimensional, then $X(\Delta^{(1)})$ is the so-called *adjoint* of $X(\Delta)$. That is, $X(\Delta^{(1)})$ is obtained from $X(\Delta)$ by taking its image under the map corresponding to $\mathcal{O}_{X(\Delta)}(1)+K_{X(\Delta)}$. See [9, 12] for more details.
(v) Conversely, if Δ is two-dimensional and $\Delta^{(-1)}$ has the same number of edges as Δ, then $X(\Delta^{(-1)}) \cong X(\Delta)$. The former is then embedded by the ample line bundle $\mathcal{O}_{X(\Delta)}(1) - K_{X(\Delta)}$. Similarly, for $k \geq 0$, if $\Delta^{(-k)}$ has the same number of edges as Δ, then $X(\Delta^{(-k)})$ corresponds to the ample line bundle $\mathcal{O}_{X(\Delta)}(1) - kK_{X(\Delta)}$. If this works for arbitrary k, one must have that $-K_{X(\Delta)}$ is nef (and big, which is automatic), i.e. $X(\Delta)$ is weak Fano. Along with *(iii)*, this gives some geometric insight in Theorem 3.
(vi) If $\Delta^{(1)}$ is well-defined and two-dimensional, then the dimension of the automorphism group $\mathrm{Aut}(X(\Delta))$ is determined by the number of lattice points that lie in the interior of a positively oriented move of $\mathcal{P}(\Delta^{(1)})$. See [7, Lemma 10.5] and [15, phrase following (2.99)]. This was used in [7] to determine the dimension of the moduli space of generic hyperplane sections of $X(\Delta)$.
(vii) The genus of a two-dimensional lattice polygon Δ is equal to the genus of a generic hyperplane section of $X(\Delta)$. Such a generic hyperplane section will be (hyper)elliptic if and only if Δ is (hyper)elliptic. See [15, Section 3.2] or [7, Lemma 5.1].

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The lattice size of a lattice polygon

Wouter Castryck and Filip Cools

Abstract

We give upper bounds on the minimal degree of a model in $\mathbb{P}^2$ and the minimal bidegree of a model in $\mathbb{P}^1 \times \mathbb{P}^1$ of the curve defined by a given Laurent polynomial, in terms of the combinatorics of the Newton polygon of the latter. We prove in various cases that this bound is sharp as soon as the polynomial is sufficiently generic with respect to its Newton polygon.

MSC2010: Primary 14H45, Secondary 14H51, 14M25

1 Introduction

Let k be an algebraically closed field and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be an irreducible Laurent polynomial whose Newton polygon, denoted by $\Delta(f)$, we assume to be two-dimensional. Let $\mathbb{T}^2 = k^* \times k^*$ be the two-dimensional torus over k, and denote by $U_f \subset \mathbb{T}^2$ the curve defined by f. (Throughout this paper, all curves are understood to be irreducible, but not necessarily non-singular and/or projective.) For a curve C/k we define $s_2(C)$ as the minimum of

$$S_2(C) = \left\{ \, d \in \mathbb{N} \mid C \simeq \text{a curve of degree } d \text{ in } \mathbb{P}^2 \, \right\}$$

and $s_{1,1}(C)$ as the lexicographic minimum of

$$S_{1,1}(C) = \left\{ \, (a,b) \in \mathbb{N}^2 \, \middle| \, a \leq b \text{ and } C \simeq \text{a curve of bidegree } (a,b) \text{ in } \mathbb{P}^1 \times \mathbb{P}^1 \, \right\},$$

where $\simeq$ denotes birational equivalence. The aim of this article is to give upper bounds on the invariants $s_2(U_f)$ and $s_{1,1}(U_f)$ purely in terms of the combinatorics of $\Delta(f)$.

The invariant $s_2(C)$ has seen study in the past [11, 17, 19] but is not well-understood. On the other hand we are unaware of existing literature explicitly devoted to $s_{1,1}(C)$, even though for hyperelliptic curves the notion has made an appearance [14] in the context of cryptography. Note that at first sight, the definition of $s_{1,1}(C)$ has a non-canonical flavor: instead of lexicographic, one could also consider the minimum with respect to other types of monomial orders on $\mathbb{N}^2$. But in fact we conjecture:

Conjecture 1.1. *For each curve C/k the set $S_{1,1}(C)$ admits a minimum with respect to the product order $\leq \times \leq$ on $\mathbb{N}^2$.*

Because the product order is coarser than every monomial order, this would mean that the term 'lexicographic' can be removed without ambiguity. In Section 2 we will state a number of basic facts on $s_2(C)$ and $s_{1,1}(C)$, along with some motivation in favor of Conjecture 1.1.

Our central combinatorial notion is the *lattice size* $\mathrm{ls}_X(\Delta)$ of a lattice polygon Δ with respect to a set $X \subset \mathbb{R}^2$ with positive Jordan measure. In case $\Delta \neq \emptyset$ we define it as the smallest integer $d \geq 0$ for which there exists a unimodular transformation $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that

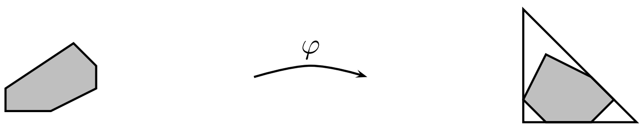
$$\varphi(\Delta) \subset dX.$$

A unimodular transformation that attains this minimum is said to compute the lattice size. We will restrict ourselves to three instances of X, namely

$$[0,1] \times \mathbb{R}, \quad \Sigma = \mathrm{conv}\{(0,0),(1,0),(0,1)\}, \quad \square = \mathrm{conv}\{(0,0),(1,0),(0,1),(1,1)\},$$

where it is convenient to define $\mathrm{ls}_X(\emptyset) = -1, -2, -1$, respectively.

In the case of $X = \Sigma$ the lattice size measures the smallest standard triangle containing a unimodular copy of Δ.



This was studied by Schicho [25], who designed an algorithm for finding a unimodular transformation that maps a given polygon Δ inside a small standard triangle. He did this in the context of simplifying parametrizations of rational surfaces. Our results below show that Schicho's algorithm works optimally, that is, its output computes the lattice size $\mathrm{ls}_\Sigma(\Delta)$. In the case of $X = [0,1] \times \mathbb{R}$ the lattice size is nothing else than the commonly studied lattice width, which we denote by $\mathrm{lw}(\Delta)$ rather than $\mathrm{ls}_{[0,1]\times\mathbb{R}}(\Delta)$. See [7, Lem. 5.2] for some of its properties, such as Fejes Tóth and Makai Jr.'s result [12] that $\mathrm{lw}(\Delta)^2 \leq 8\,\mathrm{Vol}(\Delta)/3$. In the case of $X = \square$ the notion implicitly appears in the work of Arnold [1] and Lagarias–Ziegler [22, Thm. 2] in the context of counting lattice polygons (up to unimodular equivalence) with a given volume; they found that $\mathrm{ls}_\square(\Delta) \leq 4\,\mathrm{Vol}(\Delta)$ as soon as Δ is two-dimensional. Note that this implies the bound $\mathrm{ls}_\Sigma(\Delta) \leq 8\,\mathrm{Vol}(\Delta)$, which is most likely not sharp.

Recently, Lubbes and Schicho [23, Thm. 13] and the current authors [5, Thm. 4] independently provided an explicit formula for $\mathrm{lw}(\Delta)$ in terms of $\mathrm{lw}(\Delta^{(1)})$, where $\Delta^{(1)}$ denotes the convex hull of the lattice points in the interior of Δ; see Lemma 5.1 for a precise statement. This yields a recursive method for computing the lattice width in practical situations, by gradually 'peeling off' the polygon.[1] The biggest part of this article (Sections 3 and 4) is devoted to proving similar recursive formulas for $\mathrm{ls}_\Sigma(\Delta)$ and $\mathrm{ls}_\square(\Delta)$, which can again be used for computing the lattice size in practice. In the former case one recovers Schicho's algorithm. In the latter case the proof entails that the unimodular transformations computing $\mathrm{ls}_\square(\Delta)$ essentially also compute $\mathrm{lw}(\Delta)$. This is made precise in Section 5, where as a corollary we obtain:

Theorem 1.2. *For each non-empty lattice polygon Δ the set*

$$S_{1,1}(\Delta) = \left\{ (a,b) \in \mathbb{N}^2 \,\middle|\, a \leq b \text{ and } \exists\, \Delta' : \Delta \simeq \Delta' \text{ with } \Delta' \subset [0,a] \times [0,b] \right\}$$

admits a minimum with respect to the product order on $\mathbb{N}^2$, namely $s_{1,1}(\Delta) := (\mathrm{lw}(\Delta), \mathrm{ls}_\square(\Delta))$.

Here $\simeq$ denotes unimodular equivalence. We will sometimes (but not always) write $\square_{a,b}$ instead of $[0,a] \times [0,b]$. The reader can view Theorem 1.2 as a combinatorial version of Conjecture 1.1.

Now if we write

$$f = \sum_{(i,j) \in \mathbb{Z}^2} c_{i,j} x^i y^j \quad \in k[x^{\pm 1}, y^{\pm 1}]$$

then for every unimodular transformation $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ the Laurent polynomial

$$f^\varphi = \sum_{(i,j) \in \mathbb{Z}^2} c_{i,j} x^{\varphi_1(i,j)} y^{\varphi_2(i,j)}$$

(with φ_1 and φ_2 the component functions of φ) satisfies $\Delta(f^\varphi) = \varphi(\Delta(f))$. Since U_f and U_{f^φ} are isomorphic it follows that

$$s_2(U_f) \leq \mathrm{ls}_\Sigma(\Delta(f)) \quad \text{and} \quad s_{1,1}(U_f) \leq s_{1,1}(\Delta(f)), \tag{1}$$

where the second inequality should be read lexicographically. While the first bound is straightforward, we note that the second bound relies on Theorem 1.2. Our main result, which shows up as a consequence to our recursive formulas for the lattice size, refines these bounds:

Theorem 1.3. *One has*

$$s_2(U_f) \leq \mathrm{ls}_\Sigma(\Delta(f)^{(1)}) + 3 \quad \text{and} \quad s_{1,1}(U_f) \leq s_{1,1}(\Delta(f)^{(1)}) + (2,2). \tag{2}$$

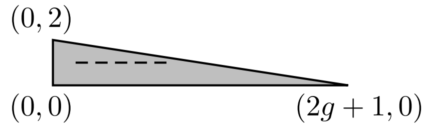
If $\Delta(f) \simeq d\Upsilon$ for some $d \geq 2$ then the first bound sharpens to $s_2(U_f) \leq 3d-1$; if $d = 2$ then also the second bound sharpens to $s_{1,1}(U_f) \leq (3,4)$.

[1] We remark that for very large polygons there exist more effective methods; see e.g. [13].

Here $\Upsilon = \text{conv}\{(-1,-1),(1,0),(0,1)\}$. The proof of Theorem 1.3 is given in Section 6. We will see below that $\text{ls}_\Sigma(\Delta^{(1)}) + 3 \leq \text{ls}(\Delta)$ and $s_{1,1}(\Delta^{(1)}) + (2,2) \leq s_{1,1}(\Delta)$ as soon as Δ is two-dimensional, and that the difference can be arbitrarily large. Thus Theorem 1.3 can be seen as a considerable improvement over the bounds (1). As a teasing example, consider a hyperelliptic curve C of genus $g \geq 2$ defined by a Weierstrass equation

$$f := y^2 + h_1(x)y + h_2(x) = 0,$$

with $h_2 \in k[x]$ of degree $2g+1$ and $h_1 \in k[x]$ of degree at most g. Assume for simplicity that $h_2(0) \neq 0$, so that the Newton polygon $\Delta(f)$ equals



The interior polygon $\Delta(f)^{(1)}$ equals $\text{conv}\{(1,1),(g,1)\}$; it is indicated by the dashed line. In this case the bounds (1) read $s_2(C) \leq 2g+1$ and $s_{1,1}(C) \leq (2, 2g+1)$, while Theorem 1.3 yields $s_2(C) \leq g+2$ and $s_{1,1}(C) \leq (2, g+1)$. The latter bounds are actually sharp; see Section 2. More generally, we conjecture:

Conjecture 1.4. *If f is sufficiently generic with respect to its Newton polygon $\Delta(f) \not\cong 2\Upsilon$, then the (smallest applicable) bounds of Theorem 1.3 are met.*

In Section 7, where we will be more precise on what is meant by 'sufficiently generic', we will prove this conjecture in a number of special cases.

Acknowledgments

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2 Basic facts on the minimal (bi)degree

Let C be a curve of (geometric) genus g over an algebraically closed field k. In this section we discuss a number of basic properties of the invariants $s_2(C)$ and $s_{1,1}(C)$. Throughout we make the assumption that $\text{char}\, k = 0$, because several of our references rely on it.

In the case of $s_2(C)$ it is known that

$$\frac{3+\sqrt{8g+1}}{2} \leq s_2(C) \leq g+2.$$

The lower bound is met if and only if C is birationally equivalent to a non-singular projective plane curve. As for the upper bound one has $s_2(C) = g + 2$ if and only if C is elliptic or hyperelliptic. If $g \geq 6$ then $s_2(C) = g + 1$ if and only if C is bi-elliptic. See [17] and the references therein for proofs.

In the case of $s_{1,1}(C)$ we prove an analogous statement:

Lemma 2.1. *One has $s_{1,1}(C) = (c, d)$, where c is the gonality of C and d satisfies*

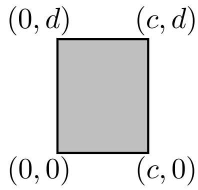
$$\frac{g}{c-1} + 1 \leq d \leq g + 1$$

unless $c = 1$, in which case $d = 1$. The lower bound is met if and only if C is birationally equivalent to a non-singular curve in $\mathbb{P}^1 \times \mathbb{P}^1$. If C is elliptic or hyperelliptic then the upper bound is met.

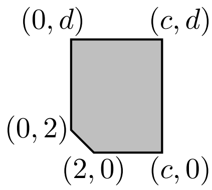
Proof. If $c = 1$ then the statement is trivial, so we can assume that $c \geq 2$.

As for the upper bound, fix a g^1_c on C and pick a point $P \in C$. Let $D \in g^1_c$ be such that P is in the support. Now construct a divisor D' by gradually adding points that are not in the support of D to the point P, until $\dim H^0(C, D') = 2$. By the Riemann-Roch theorem this happens after at most g steps, i.e. $d := \deg D' \leq g + 1$. By construction, the corresponding base-point free g^1_d does not have a factor in common with our given g^1_c, so we can use $g^1_c \times g^1_d$ to map C to a birationally equivalent curve of bidegree (c, d).

As for the other inequality, consider Baker's bound [3], which says that the genus of the curve defined by an irreducible Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$ is bounded by $\sharp(\Delta(f)^{(1)} \cap \mathbb{Z}^2)$. Now the Newton polygon of a polynomial of bidegree (c, d) is contained in the rectangle:



Hence $g \leq (c-1)(d-1)$, from which the lower bound follows. If there is a singularity in $\mathbb{P}^1 \times \mathbb{P}^1$, then without loss of generality we may assume that it concerns an affine point (x_0, y_0). But then the Newton polygon of $f(x + x_0, y + y_0)$ is contained in:



Therefore $g \leq (c-1)(d-1)-1$, which shows that the lower bound cannot be attained in this case.

Finally, if C is elliptic or hyperelliptic then $c = 2$ so that the lower bound meets the upper bound. ■

We think that for $c \geq 2$ the upper bound can be improved to $g+3-c$. Namely, by Brill-Noether theory the space of g^1_{g+3-c}'s on C has dimension $g-2c+4$, while the subspace of pencils of the form g^1_c + base points has dimension $g-2c+3$. This gives plenty of base-point free g^1_d's with $d \leq g+3-c$ that do not obviously have a factor in common with the given g^1_c. But we did not succeed in proving that there indeed always exists such a truly independent g^1_d. The bi-elliptic case illustrates the subtlety of the argument: here one has a full-dimensional component of dependent g^1_{g+3-c}'s. Nevertheless the bound $g+3-c = g-1$ is valid here (and met); see [11, Ex. 1.13].

As a special cases of Conjecture 1.1, we note:

Lemma 2.2. *If the gonality of C is a prime number then $S_{1,1}(C)$ admits a minimum with respect to the product order $\leq \times \leq$ on $\mathbb{N}^2$.*

Proof. Fix a gonality pencil g^1_c. It suffices to show that if $(a,b) \in S_{1,1}(C)$ then $(c,a) \in S_{1,1}(C)$ or $(c,b) \in S_{1,1}(C)$. In other words, it is sufficient to prove that at least one of the given g^1_a and g^1_b is independent of our g^1_c. But if g^1_a and g^1_c have a common factor, then by primality this factor must be g^1_c itself, and similarly for g^1_b. Because g^1_a and g^1_b are mutually independent, the claim follows. ■

We do not have much additional evidence in favor of Conjecture 1.1, except that all our attempts to construct a counterexample failed in a suspicious way: each time unexpected linear pencils popped up that made the statement true. As a typical example, we considered the fiber product

$$C : \begin{cases} y_1^3 = f_1(x) \\ y_2^3 = f_2(x) \end{cases}$$

of two cyclic degree 3 covers of the projective line, with $f_1(x), f_2(x)$ degree 6 polynomials that are squarefree and mutually coprime. This is a 9-gonal curve of genus 28 by Riemann-Hurwitz, so in view of Lemma 2.1 we have $(9,d) \in S_{1,1}(C)$ with $d \leq 29$. On the other hand both covers naturally admit a g^1_4, which when composed with the g^1_3 of the other curve gives rise to two independent g^1_{12}'s on C, each of which has a component in common with our g^1_9. So we also find that $(12,12) \in S_{1,1}(C)$, while it is not obvious that $(9,e) \in S_{1,1}(C)$ with $e \leq 12$, especially because the genus is so high. However, in all concrete versions that we tried the substitution $y_1 \leftarrow y_1 + y_2$, when followed by a projection on the (x, y_1)-plane, resulted in a plane degree 15 curve having several triple points, each of which corresponds to a g^1_{12} by projection. In this way we always found that $(9, 12) \in S_{1,1}(C)$.

3 A recursive formula for $\mathrm{ls}_\Sigma(\Delta)$

We begin by discussing some first properties. For $d \in \mathbb{Z}_{\geq 0}$ one has

$$\mathrm{ls}_\Sigma\left(\mathrm{conv}\{(0,0),(d,0)\}\right) = d.$$

Indeed, it is immediate that $\mathrm{conv}\{(0,0),(d,0)\} \subset d\Sigma$ and that the integral distance $\gcd(a_2 - a_1, b_2 - b_1)$ between two points (a_1, b_1), $(a_2, b_2) \in (d-1)\Sigma$ cannot exceed $d-1$. More generally, every lattice polygon that contains a line segment of integral length d must have lattice size at least d with respect to Σ. In particular $\mathrm{ls}_\Sigma(d\Sigma) = d$.

Lemma 3.1. *Let Δ be a non-empty lattice polygon. Then $\mathrm{lw}(\Delta) \leq \mathrm{ls}_\Sigma(\Delta)$, and equality holds if and only if $\Delta \simeq d\Sigma$ for some integer $d \geq 0$.*

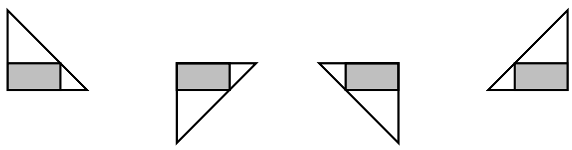
Proof. This follows because $\mathrm{lw}(d\Sigma) = d$, while every strict subpolygon $\Gamma \subset d\Sigma$ satisfies $\mathrm{lw}(\Gamma) < d$. ■

A less straightforward lattice size calculation is:

Lemma 3.2. *Let $a, b \in \mathbb{Z}_{\geq 0}$ and consider $\square_{a,b} = [0,a] \times [0,b]$. Then $\mathrm{ls}_\Sigma(\square_{a,b}) = a+b$.*

Proof. The case where $a = 0$ or $b = 0$ follows from the above considerations, so we can assume that $a, b \geq 1$. Instead of looking for the minimal d such that $\square_{a,b}$ can be mapped inside $d\Sigma$ through a unimodular transformation, we will look for the minimal d such that $\square_{a,b}$ is contained in a unimodular transform of $d\Sigma$. More precisely, we will prove the following assertion by induction on $a + b$:

> *We have $\mathrm{ls}_\Sigma(\square_{a,b}) = a + b$. Moreover, there are exactly four ways of fitting $\square_{a,b}$ inside a unimodular transform of $(a+b)\Sigma$:*



The basis of our induction is the case $a = b = 1$. Here, the first part of the assertion holds because $\square_{1,1} \subset 2\Sigma$ and $\mathrm{Vol}(\square_{1,1}) > \mathrm{Vol}(\Sigma)$. The second part follows because 2Σ contains only 3 lattice points that are non-vertices. Therefore, when fitting $\square_{1,1}$ inside a transform of 2Σ, at least one of its vertices must coincide with a vertex of 2Σ, and the two adjacent vertices of $\square_{1,1}$ must coincide with the interior lattice points of the respective adjacent edges of 2Σ. From this the claim follows easily.

Now assume that $a, b \geq 1$ and (without loss of generality) that $a \geq 2$. Clearly $\square_{a,b} \subset (a+b)\Sigma$. Suppose that $\square_{a,b}$ sits inside a unimodular transform of $(a+b-1)\Sigma$. By applying the induction hypothesis to $\square_{a-1,b} \subset \square_{a,b}$ we find that $(a+b-1)\Sigma$ must enclose this subpolygon in one of the four manners above. But for each of these four configurations, it is clear that $\square_{a,b}$ itself could not have been contained in $(a+b-1)\Sigma$: contradiction. As for the second assertion, let Σ' be a unimodular transform of $(a+b)\Sigma$ containing $\square_{a,b}$. Then

- each edge of Σ' must contain at least one vertex of $\square_{a,b}$: otherwise we could crop Σ' to a unimodular transform of $(a+b-1)\Sigma$ that still contains $\square_{a,b}$;
- at least one vertex v of Σ' does not appear as a vertex of $\square_{a,b}$: otherwise the latter would be a triangle;
- the edges of Σ' that are adjacent to v cannot contain two vertices of $\square_{a,b}$ each: otherwise $\square_{a,b}$ would contain two non-adjacent non-parallel edges.

So there must be an edge $\tau \subset \Sigma'$ that contains exactly one vertex v of $\square_{a,b}$. Then the transform of $(a+b-1)\Sigma$ obtained from Σ' by shifting τ inwards contains $\square_{a,b} \setminus \{v\}$. In particular it contains (a translate of) $\square_{a-1,b}$. By applying the induction hypothesis we find that Σ' must be positioned in one of the four standard ways above. ■

We now investigate the relation between $\mathrm{ls}_\Sigma(\Delta)$ and $\mathrm{ls}_\Sigma(\Delta^{(1)})$. Since for $d \geq 3$ one has $(d\Sigma)^{(1)} \simeq (d-3)\Sigma$, we have that

$$\mathrm{ls}_\Sigma(\Delta^{(1)}) \leq \mathrm{ls}_\Sigma(\Delta) - 3 \tag{3}$$

as soon as Δ is two-dimensional (this includes the case where $\Delta^{(1)} = \emptyset$, which can be verified separately). Typically, one expects equality to hold, but there are many exceptions, which are classified by Theorem 3.5 below.

In what follows, we will make use of the following terminology and facts; see [15, §4] or [21, §2.2] for proofs. An edge τ of a two-dimensional lattice polygon Γ is always supported on a line $a_\tau X + b_\tau Y = c_\tau$ with $a_\tau, b_\tau, c_\tau \in \mathbb{Z}$ and a_τ, b_τ coprime. When signs are chosen appropriately, we can moreover assume that Γ is contained in the half-plane $a_\tau X + b_\tau Y \leq c_\tau$. The line $a_\tau X + b_\tau Y = c_\tau + 1$ is called the *outward shift* of τ. It is denoted by $\tau^{(-1)}$, and the polygon (which may take vertices outside $\mathbb{Z}^2$) that arises as the intersection of the half-planes $a_\tau X + b_\tau Y \leq c_\tau + 1$ is denoted by $\Gamma^{(-1)}$. If $\Gamma = \Delta^{(1)}$ for some lattice polygon Δ, then the outward shifts of two adjacent edges of Γ always intersect in a lattice point, and in fact $\Gamma^{(-1)} = \Delta^{(1)(-1)}$ is a lattice polygon. Moreover, $\Delta \subset \Delta^{(1)(-1)}$, i.e. $\Delta^{(1)(-1)}$ is the maximal lattice polygon (with respect to inclusion) for which the convex hull of the interior lattice points equals $\Delta^{(1)}$.

Before stating Theorem 3.5, let us prove two auxiliary lemmas:

Lemma 3.3. *Assume that there exist parallel edges $\tau \subset \Delta$ and $\tau' \subset \Delta^{(1)}$ whose supporting lines are at integral distance 1 of each other, of respective lengths r and s. If $r \geq s+3$ then $\mathrm{ls}_\Sigma(\Delta^{(1)}) = s$ and $\mathrm{ls}_\Sigma(\Delta) = r$.*

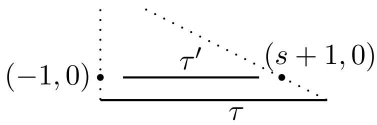
Remark. As usual, by an edge we mean a one-dimensional face. In particular, if $\Delta^{(1)}$ is one-dimensional then it is an edge of itself. Example: consider the hyperelliptic Weierstrass polygon

$$\mathrm{conv}\{(0,0),(2g+1,0),(0,2)\}$$

from the introduction. Then $\mathrm{ls}_\Sigma(\Delta^{(1)}) = g$ and $\mathrm{ls}_\Sigma(\Delta) = 2g+1$. This shows that the difference between $\mathrm{ls}_\Sigma(\Delta)$ and $\mathrm{ls}_\Sigma(\Delta^{(1)})$ can be arbitrarily large.

Proof of Lemma 3.3. By using a unimodular transformation if needed, we can assume that $\tau = \mathrm{conv}\{(-1,-1),(r-1,-1)\}$ and $\tau' = \mathrm{conv}\{(0,0),(s,0)\}$. Since $r \geq s+3$ and $\Delta^{(1)}$ cannot contain any lattice points on the line $Y=0$ apart from those contained in τ',

- the edge of Δ that is left-adjacent to τ must pass through or to the right of $(-1,0)$, and
- the edge of Δ that is right-adjacent to τ must pass through or to the left of $(s+1,0)$.



From the convexity of Δ one immediately sees that $\Delta \subset (-1,-1)+r\Sigma$, and similarly that $\Delta^{(1)} \subset s\Sigma$. Therefore $\mathrm{ls}_\Sigma(\Delta^{(1)}) \leq s$ and $\mathrm{ls}_\Sigma(\Delta) \leq r$, and equality follows from the considerations preceding Lemma 3.2. ■

Lemma 3.4. *Assume that $\Delta^{(1)}$ is two-dimensional. Let $s \geq 1$ be an integer such that $\Delta^{(1)} \subset s\Sigma$, and assume that $\Delta^{(1)}$ has an edge τ' in common with $s\Sigma$. Let $\tau'^{(-1)}$ be its outward shift, and consider the face $\tau = \Delta \cap \tau'^{(-1)}$ of Δ, whose integral length we denote by r. Then*

$$\mathrm{ls}_\Sigma(\Delta^{(1)}) = s \qquad \text{and} \qquad \mathrm{ls}_\Sigma(\Delta) = \max\{r, s+3\}.$$

Remark. The face τ is either a vertex or an edge. In the former case, its integral length is understood to be 0.

PROOF. The fact that $\text{ls}_\Sigma(\Delta^{(1)}) = s$ follows immediately from the considerations preceding Lemma 3.2. As for $\text{ls}_\Sigma(\Delta)$, in case $r \geq s+3$ the statement follows from Lemma 3.3. So assume that $r \leq s+3$ (we reinclude the case $r = s+3$ for the sake of the symmetry of the argument below). Without loss of generality we may suppose that $\tau' = \text{conv}\{(0,0),(s,0)\}$. We claim that we can moreover assume that $\tau \subset \text{conv}\{(-1,-1),(s+2,-1)\}$, while still keeping $\Delta^{(1)} \subset s\Sigma$.

Assuming the claim, we can make the following reasoning.

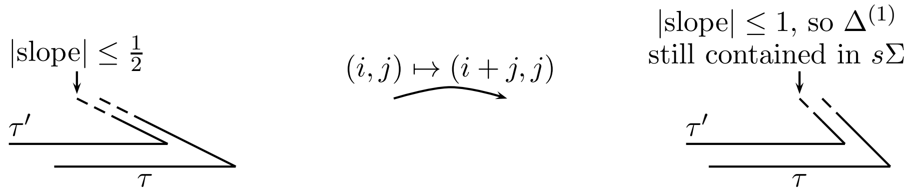
- Clearly Δ is contained in the half-plane $Y \geq -1$.
- Suppose that Δ contains a lattice point (a,b) for which $a < -1$. Because $b = -1$ contradicts our claim, while $b = 0$ contradicts that $\Delta^{(1)} \subset s\Sigma$ (indeed, it implies that $(-1,0) \in \Delta^{(1)}$), we must have $b \geq 1$. Along with the fact that $\Delta^{(1)}$ is two-dimensional (so that it must contain a lattice point on or above the line $Y = 1$) this implies that $(0,1) \in \Delta^{(1)}$. But then, apart from the point (a,b) itself, all lattice points which are contained in the triangle spanned by (a,b), $(0,0)$ and $(0,1)$ must be elements of $\Delta^{(1)}$. The volume of this triangle being at least 1, Pick's theorem implies that it must contain a lattice point different from (a,b), $(0,0)$ and $(0,1)$. This contradicts $\Delta^{(1)} \subset s\Sigma$.

  We conclude that Δ is contained in the half-plane $X \geq -1$.
- By applying the unimodular transformation $(i,j) \mapsto (s-i-j,j)$, one sees that the foregoing reasoning also allows to conclude that Δ is contained in the half-plane $X + Y \leq s+1$.

So the claim implies that $\Delta \subset (-1,-1) + (s+3)\Sigma$, and hence that $\text{ls}_\Sigma(\Delta) \leq s+3$, which together with (3) proves the lemma.

To prove the claim, note that because $r \leq s+3$, again using the transformation $(i,j) \mapsto (s-i-j,j)$ if needed, we can assume that τ is contained in the half-plane $X \geq -1$. Let $(a,-1)$ be the right-most vertex of τ. As long as $a > s+2$, we can apply a unimodular transformation of the form $(i,j) \mapsto (i+j,j)$ to Δ, while

- keeping τ in the half-plane $X \geq -1$ (here we again used that $r \leq s+3$);
- keeping $\Delta^{(1)}$ inside $s\Sigma$: indeed, because $a > s+2$ and $(s+1,0) \notin \Delta^{(1)}$, the edge of Δ that is right-adjacent to τ must have a slope that is smaller than $1/2$ (in absolute value), and hence the same must be true for the edge of $\Delta^{(1)}$ that is right-adjacent to τ'.



This decreases the value of a by 1. So the claim follows by repeating this step until $a \leq s+2$. ■

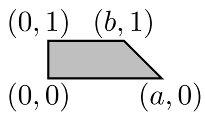
We are now ready to state and prove our recursive expression.

Theorem 3.5. *Let Δ be a two-dimensional lattice polygon. Then*

$$\mathrm{ls}_\Sigma(\Delta) = \mathrm{ls}_\Sigma(\Delta^{(1)}) + 3,$$

except in the following situations:

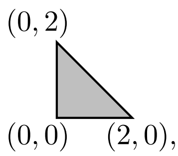
- Δ *is equivalent to a Lawrence prism*



where $a = b = 1$ or $2 \leq a \geq b \geq 0$, in which case $\mathrm{ls}_\Sigma(\Delta^{(1)}) = -2$ and

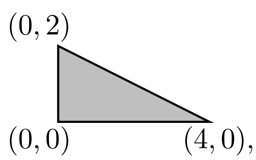
$$\begin{cases} \mathrm{ls}_\Sigma(\Delta) = a+1 & \text{if } a = b, \\ \mathrm{ls}_\Sigma(\Delta) = a & \text{if } a > b; \end{cases}$$

- Δ *is equivalent to*



in which case $\mathrm{ls}_\Sigma(\Delta^{(1)}) = -2$ and $\mathrm{ls}_\Sigma(\Delta) = 2$;

- Δ *is equivalent to*



in which case $\mathrm{ls}_\Sigma(\Delta^{(1)}) = 0$ and $\mathrm{ls}_\Sigma(\Delta) = 4$;

- $\Delta \simeq \square_{a,b}$ *for certain* $a, b \geq 2$*, in which case*

$$\mathrm{ls}_\Sigma(\Delta^{(1)}) = a + b - 4 \qquad \textit{and} \qquad \mathrm{ls}_\Sigma(\Delta) = a + b;$$

- *there exist parallel edges* $\tau \subset \Delta$ *and* $\tau' \subset \Delta^{(1)}$ *whose supporting lines are at integral distance* 1 *of each other, such that*

$$\sharp(\tau \cap \mathbb{Z}^2) - \sharp(\tau' \cap \mathbb{Z}^2) \geq 4;$$

in this case $\mathrm{ls}_\Sigma(\Delta^{(1)}) = \sharp(\tau' \cap \mathbb{Z}^2)$ *and* $\mathrm{ls}_\Sigma(\Delta) = \sharp(\tau \cap \mathbb{Z}^2)$.

Remark. The third case $\mathrm{conv}\{(0,0),(4,0),(0,2)\}$ can in some sense be viewed as a special case of the last item, with τ' having length 0.

Proof. For the Lawrence prisms and the two explicit polygons the statement is immediate, while the polygons $\square_{a,b}$ are covered by Lemma 3.2 and the observation that $(\square_{a,b})^{(1)} \simeq \square_{a-2,b-2}$. The last statement follows from Lemma 3.3.

By (3) it remains to show that in all other situations $\mathrm{ls}_\Sigma(\Delta^{(1)}) \geq \mathrm{ls}_\Sigma(\Delta) - 3$. The cases where $\Delta^{(1)}$ is not two-dimensional can be analyzed explicitly using Koelman's classification: see [4, Thm. 10] or [21, Ch. 4]. We can therefore assume that $\Delta^{(1)}$ is two-dimensional. Let $s = \mathrm{ls}_\Sigma(\Delta^{(1)})$, so that we can suppose that $\Delta^{(1)} \subset s\Sigma$. If

$$\Delta^{(1)(-1)} \subset (s\Sigma)^{(-1)} \tag{4}$$

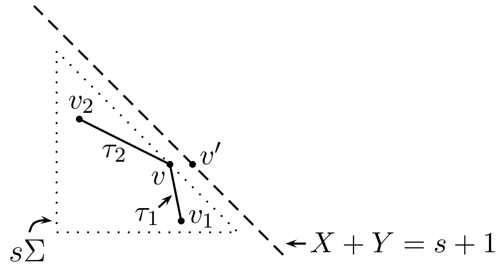
then the theorem follows because $\Delta \subset \Delta^{(1)(-1)}$ and $(s\Sigma)^{(-1)} \simeq (s+3)\Sigma$. So let us assume that (4) is not satisfied. Without loss of generality we may then suppose that $\Delta^{(1)(-1)}$ is not contained in the half-plane

$$X + Y \leq s + 1.$$

This means that the edge of $s\Sigma$ connecting $(s, 0)$ and $(0, s)$ cannot contain two vertices of $\Delta^{(1)}$. But it must contain at least one vertex v of $\Delta^{(1)}$: if not, $\Delta^{(1)}$ would be contained in $(s-1)\Sigma$, contradicting $s = \mathrm{ls}_\Sigma(\Delta^{(1)})$.

Write $v = (a, s-a)$ for some $a \in \{0, \ldots, s\}$. We distinguish between two cases.

- Assume that v lies in the interior of the edge of $s\Sigma$ that connects $(s, 0)$ and $(0, s)$, i.e. $a \notin \{0, s\}$. Let $v_1 = (a_1, b_1)$ and $v_2 = (a_2, b_2)$ be the vertices of $\Delta^{(1)}$ that are adjacent to v, ordered counterclockwise, and for $i = 1, 2$ let τ_i be the edge connecting v_i and v. Note that $b_1 < s - a$: otherwise $\Delta^{(1)}$ would be contained in $\mathrm{conv}\{(0, s-a), (a, s-a), (0, s)\} \simeq a\Sigma$, which would contradict $s = \mathrm{ls}_\Sigma(\Delta^{(1)})$. This means that the outward shift $\tau_1^{(-1)}$ must intersect the line segment spanned by $v = (a, s-a)$ and $v' = (a+1, s-a)$.

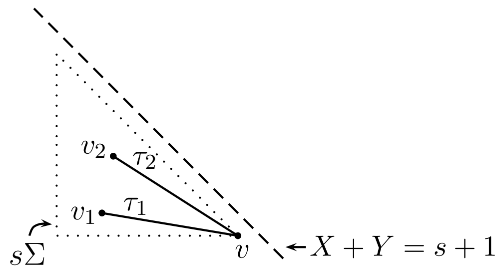


But then $b_2 \leq s - a$, otherwise $\tau_2^{(-1)}$ would also pass in between v and v', implying that $\tau_1^{(-1)}$ and $\tau_2^{(-1)}$ intersect in the half-plane $X + Y \leq s + 1$: a contradiction. We conclude that $\Delta^{(1)}$ must be contained below the line $Y = s - a$. By symmetry of arguments, it must also lie to the left of $X = a$. Thus $\Delta^{(1)}$ is contained in the rectangle

$$\operatorname{conv}\{(0,0),(a,0),(a,s-a),(0,s-a)\}.$$

Now if any of these four vertices would not appear as an actual vertex of $\Delta^{(1)}$ then we would again contradict $s = \operatorname{ls}_\Sigma(\Delta^{(1)})$. Thus $\Delta^{(1)}$ must be exactly this rectangle, and $\Delta^{(1)(-1)} \simeq \square_{a+2,s-a+2}$. The case $\Delta = \Delta^{(1)(-1)}$ being among our exceptions, we can assume that at least one of the four vertices of $\Delta^{(1)(-1)}$ does not appear as an actual vertex of Δ. But then $\operatorname{ls}_\Sigma(\Delta) \leq s+3$, as desired.

- Assume that v is an endpoint of the edge of $s\Sigma$ connecting $(s,0)$ and $(0,s)$, i.e. $a \in \{0,s\}$. Without loss of generality we may assume that $a = s$. Again let $v_1 = (a_1,b_1)$ and $v_2 = (a_2,b_2)$ be the vertices of $\Delta^{(1)}$ that are adjacent to v, ordered counterclockwise, and for $i = 1,2$ let τ_i be the edge connecting v_i and v.



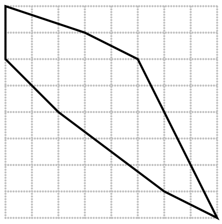
We claim that $v_1 = (0,0)$, i.e. $a_1 = b_1 = 0$. Indeed:

  – Assume that $b_1 = 0$. Then $\tau_1^{(-1)}$ is the line $Y = -1$. Since $\tau_2^{(-1)}$ must intersect this line in a lattice point outside the half-plane $X + Y \leq s + 1$ we find (as in the proof of Lemma 3.4) that τ_2 has slope at most $1/2$ (in absolute value), i.e. $a_2 \leq s - 2b_2$. From this it follows that $a_1 = 0$: if not, the unimodular transformation $(i,j) \mapsto (i+j-1,j)$ maps $\Delta^{(1)}$ inside $(s-1)\Sigma$, contradicting $s = \operatorname{ls}_\Sigma(\Delta^{(1)})$.

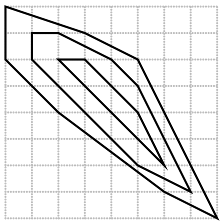
– Assume that $b_1 \neq 0$. If $a_2 \leq s - 2b_2$ then we would again find a contradiction with $s = \mathrm{ls}_\Sigma(\Delta^{(1)})$. Therefore $a_2 > s - 2b_2$, and by symmetry of arguments also $a_1 < s - 2b_1$. But then $\tau_1^{(-1)}$ passes through or above the point $(s+2, -1)$, while $\tau_2^{(-1)}$ passes through or to the left of $(s+2, -1)$. Taking into account their respective slopes, one sees that these lines must intersect in the half-plane $X + Y \leq s + 1$: a contradiction. So this case cannot occur.

Thus $\tau_1 = \mathrm{conv}\{(0,0),(s,0)\}$. Now consider the face $\tau = \tau_1^{(-1)} \cap \Delta$ of Δ. The case $\sharp(\tau \cap \mathbb{Z}^2) \geq s + 4$ being among our exceptions, we can assume that $\sharp(\tau \cap \mathbb{Z}^2) \leq s + 3$. The theorem then follows from Lemma 3.4. ■

Theorem 3.5 gives a recursive method for computing the lattice size with respect to Σ in practice. For example, let Δ be the lattice polygon below.



By taking consecutive interiors, we find the following 'onion skins'.



The inner polygon is (equivalent to) a Lawrence prism with $a = 4$ and $b = 2$, while the subsequent steps are not exceptional. We find $\mathrm{ls}_\Sigma(\Delta) = \mathrm{ls}_\Sigma(\emptyset) + 6 + 3 + 3 = 10$. We remark that this is in fact a rephrasing of Schicho's algorithm for simplifying rational parametrizations of toric surfaces [25, §4]. Whereas Schicho proved that the output of the algorithm is at worst twice the lattice size [25, Thm. 10], our result shows that the result is actually optimal.

A Magma implementation of this method can be found in the file `basic_commands.m` that accompanies [7]. For instance, the above example can be treated as follows:

```
> load "basic_commands.m";
```



```
Loading "basic_commands.m"
> P := LatticePolytope([<8,0>,<6,1>,<2,4>,<0,6>,<0,8>,<3,7>,<5,6>]);
> LatticeSizeRecursiveSigma(P);
10
```

4 A recursive formula for $\mathrm{ls}_\square(\Delta)$

Some basic properties of the lattice size with respect to $\square$ are that

$$\mathrm{ls}_\square(\mathrm{conv}\{(0,0),(d,0)\}) = d$$

for any $d \in \mathbb{Z}_{\geq 0}$ (in particular every lattice polygon that contains a line segment of integral length d must have lattice size at least d with respect to $\square$), and that for each non-empty lattice polygon Δ we have

$$\mathrm{lw}(\Delta) \leq \mathrm{ls}_\square(\Delta) \leq \mathrm{ls}_\Sigma(\Delta) \leq 2\,\mathrm{ls}_\square(\Delta). \tag{5}$$

By Lemma 3.1 the first two inequalities become equalities for (and only for) $\Delta \simeq d\Sigma$ with $d \in \mathbb{Z}_{\geq 0}$.

The aim is again to relate $\mathrm{ls}_\square(\Delta)$ to $\mathrm{ls}_\square(\Delta^{(1)})$. Our treatment is very similar to that of the previous section. Because $(d\square)^{(1)} \simeq (d-2)\square$ for $d \geq 2$, we have that

$$\mathrm{ls}_\square(\Delta^{(1)}) \leq \mathrm{ls}_\square(\Delta) - 2 \tag{6}$$

as soon as Δ is two-dimensional (this includes the case where $\Delta^{(1)} = \emptyset$, which can be verified explicitly). Typically one expects equality to hold, so our task amounts to classifying the exceptions. We again rely on two auxiliary lemmas. The first is a literal rephrasing of Lemma 3.3:

Lemma 4.1. *Assume that there exist parallel edges $\tau \subset \Delta$ and $\tau' \subset \Delta^{(1)}$ whose supporting lines are at integral distance 1 of each other, of respective lengths r and s. If $r \geq s+3$ then $\mathrm{ls}_\square(\Delta^{(1)}) = s$ and $\mathrm{ls}_\square(\Delta) = r$.*

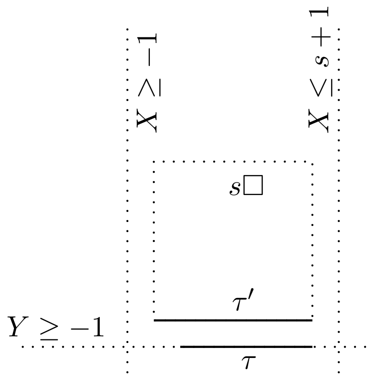
Proof. By Lemma 3.3 we know that $\mathrm{ls}_\Sigma(\Delta^{(1)}) = s$ and $\mathrm{ls}_\Sigma(\Delta) = r$, so by (5) we find $\mathrm{ls}_\square(\Delta^{(1)}) \leq s$ and $\mathrm{ls}_\square(\Delta) \leq r$. Equality follows from the considerations at the beginning of this section. ■

(Instead of invoking Lemma 3.3 one can also just copy its proof, basically.) Our second lemma is analogous to Lemma 3.4, but the statement is slightly more subtle:

Lemma 4.2. *Assume that $\Delta^{(1)}$ is two-dimensional. Let $s \geq 1$ be an integer such that $\Delta^{(1)} \subset s\square$, and assume that $\Delta^{(1)}$ has at least one edge in common with $s\square$. Choose such an edge τ' for which the integral length r of the face $\tau = \Delta \cap \tau'^{(-1)}$ of Δ is maximal. Then*

$$\mathrm{ls}_\square(\Delta^{(1)}) = s \qquad \textit{and} \qquad \mathrm{ls}_\square(\Delta) = \max\{r, s+2\}.$$

PROOF. The fact that $\mathrm{ls}_\square(\Delta^{(1)}) = s$ follows immediately from the considerations at the beginning of this section. As for $\mathrm{ls}_\square(\Delta)$, in case $r \geq s+3$ the statement follows from Lemma 4.1. So assume that $r \leq s+2$. Without loss of generality we may suppose that $\tau' = \mathrm{conv}\{(0,0),(s,0)\}$. In complete analogy with the proof of Lemma 3.4 we can moreover assume that $\tau \subset \mathrm{conv}\{(-1,-1),(s+1,-1)\}$, while still keeping $\Delta^{(1)} \subset s\square$. Still copying the reasoning from that proof, we conclude that Δ must be in the half-planes $Y \geq -1$, $X \geq -1$ and $X \leq s+1$.



Now suppose that Δ contains a lattice point (a,b) for which $b > s+1$. If $0 \leq a \leq s$ then the point $(a, s+1)$ is contained in the triangle spanned by (a,b), $(0,0)$ and $(s,0)$, therefore it must be contained in $\Delta^{(1)}$, contradicting that $\Delta^{(1)} \subset s\square$. We can therefore make the following case distinction:

- Δ is contained in the half-plane $Y \leq s+1$. But this means that $\Delta \subset (-1,-1)+(s+2)\square$ and hence that $\mathrm{ls}_\square(\Delta) \leq s+2$, which together with (6) allows us to conclude.

- Δ contains a point $(-1,b)$ with $b > s+1$. By considering the convex hull of $(-1,b)$, $(0,0)$ and $(s,0)$ this implies that $s\Sigma \subset \Delta^{(1)}$. Now if

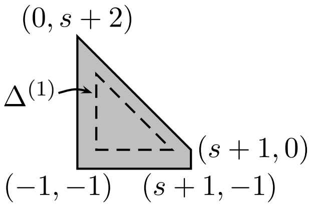
  - the latter inclusion would be strict, or
  - if $b > s+2$,

  then one would obtain that $(0, s+1) \in \Delta^{(1)} \subset s\square$, a contradiction. Therefore $b = s+2$ and $\Delta^{(1)} = s\Sigma$. In particular $\Delta^{(1)}$ also has the vertical edge $\mathrm{conv}\{(0,0),(0,s)\}$ in common with $s\square$. This means that $(-1,-1) \notin \Delta$, for otherwise the corresponding face of Δ would contain $\mathrm{conv}\{(-1,-1),(-1,s+2)\}$ which has integral length $s+3$, contradicting the maximality of r. But then the unimodular transformation $(i,j) \mapsto (i, i+j)$ maps Δ inside $(-1,-1)+(s+2)\square$. Hence $\mathrm{ls}_\square(\Delta) \leq s+2$, which together with (6) allows us to conclude.

- Δ contains a point $(s+1, b)$ with $b > s+1$. This case follows from the previous one, by symmetry.

This proves the lemma. ■

In the statement of Lemma 4.2, the condition of maximality is necessary. For instance let Δ be the polygon



so that $\Delta^{(1)} = s\Sigma \subset s\Box$. Both $\text{conv}\{(0,0),(0,s)\}$ and $\text{conv}\{(0,0),(s,0)\}$ are common edges, but the corresponding faces τ of Δ have different integral lengths, namely $s+3$ resp. $s+2$. So in this case the lattice size of Δ with respect to $\Box$ is $s+3$.

Let us include the following corollary to (the proof of) Lemma 4.2, for use in Section 5. Define a *horizontal* resp. *vertical skewing* as a unimodular transformation of the form

$$\begin{pmatrix} i \\ j \end{pmatrix} \mapsto \begin{pmatrix} \pm 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} + \begin{pmatrix} b \\ 0 \end{pmatrix} \qquad \text{resp.} \qquad \begin{pmatrix} i \\ j \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ a & \pm 1 \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix}$$

for some $a, b \in \mathbb{Z}$ (i.e. leaving the second resp. first coordinate invariant).

Corollary 4.3. *Assume that $\Delta^{(1)}$ is two-dimensional and contained in $\text{ls}_\Box(\Delta^{(1)}) \cdot \Box$. Suppose that these polygons have a unique edge in common. If this edge is horizontal (resp. vertical) then there exists a horizontal (resp. vertical) skewing φ for which $\varphi(\Delta) \subset (-1,-1) + \text{ls}_\Box(\Delta) \cdot \Box$.*

Proof. Let τ' be the common edge with $s\Box$ where $s = \text{ls}_\Box(\Delta^{(1)})$, and let $\tau = \tau'^{(-1)} \cap \Delta$. Denote the integral length of the latter by r. Without loss of generality we can assume that τ' is a horizontal edge of $s\Box$.

We actually claim the stronger statement that there exists a horizontal skewing φ such that

$$\varphi(\Delta) \subset [-1, \text{ls}_\Box(\Delta) - 1] \times [-1, s+1].$$

To prove this it suffices to assume that τ' is the bottom edge of $s\Box$, so that we are in the set-up from the proof of Lemma 4.2. We make a case distinction:

- either $r \leq s+2$, in which case the proof of Lemma 4.2 yields that $\text{ls}_\Box(\Delta) = s+2$ and, through the proof of Lemma 3.4, that there exists a horizontal skewing φ such that

$$\varphi(\Delta^{(1)}) \subset s\Box \qquad \text{and} \qquad \varphi(\Delta) \subset (-1,-1) + (s+2)\Box :$$

indeed, the unicity of τ' excludes that $s\Sigma \subset \varphi(\Delta^{(1)})$, so the last part of the proof of Lemma 4.2 can be omitted;

- or $r \geq s+3$, in which case Lemma 4.1 yields that $\mathrm{ls}_\square(\Delta) = r$ and, through the the proof of Lemma 3.3, that there exists a horizontal skewing φ such that $\varphi(\Delta) \subset (-1,-1) + r\square$.

In both cases the claim follows. ∎

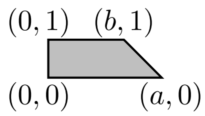
We now state and prove our recursive expression.

Theorem 4.4. *Let Δ be a two-dimensional lattice polygon. Then*

$$\mathrm{ls}_\square(\Delta) = \mathrm{ls}_\square(\Delta^{(1)}) + 2,$$

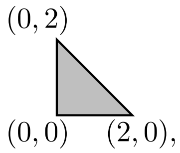
except in the following situations:

- Δ *is equivalent to a Lawrence prism*



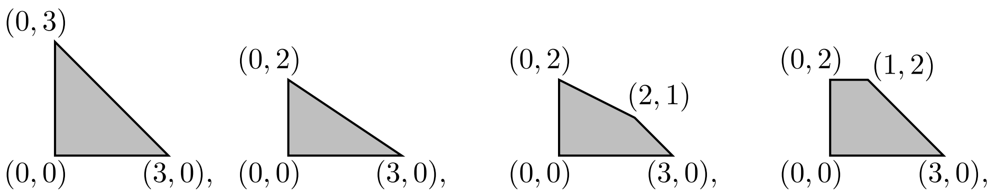
where $2 \leq a \geq b \geq 0$, in which case $\mathrm{ls}_\square(\Delta^{(1)}) = -1$ and $\mathrm{ls}_\square(\Delta) = a$;

- Δ *is equivalent to*



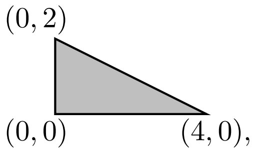
in which case $\mathrm{ls}_\square(\Delta^{(1)}) = -1$ and $\mathrm{ls}_\square(\Delta) = 2$;

- Δ *is equivalent to one of*



in which case $\mathrm{ls}_\square(\Delta^{(1)}) = 0$ and $\mathrm{ls}_\square(\Delta) = 3$;

- Δ *is equivalent to*



in which case $\mathrm{ls}_\square(\Delta^{(1)}) = 0$ *and* $\mathrm{ls}_\square(\Delta) = 4$;

- *there exist parallel edges* $\tau \subset \Delta$ *and* $\tau' \subset \Delta^{(1)}$ *whose supporting lines are at integral distance* 1 *of each other, such that*

$$\sharp(\tau \cap \mathbb{Z}^2) - \sharp(\tau' \cap \mathbb{Z}^2) \geq 3;$$

in this case $\mathrm{ls}_\square(\Delta^{(1)}) = \sharp(\tau' \cap \mathbb{Z}^2)$ *and* $\mathrm{ls}_\square(\Delta) = \sharp(\tau \cap \mathbb{Z}^2)$.

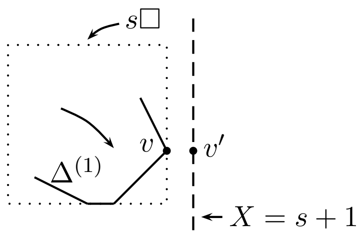
Remark. Except for 2Σ, the explicitly given polygons can in some sense be viewed as special cases of the last item, with τ' having length 0.

Proof. For the Lawrence prisms and the six explicitly given polygons, the theorem is immediate, while the last statement follows from Lemma 4.1.

By (6) it remains to show that in all other situations $\mathrm{ls}_\square(\Delta^{(1)}) \geq \mathrm{ls}_\square(\Delta) - 2$. The cases where $\Delta^{(1)}$ is not two-dimensional can again be analyzed explicitly using Koelman's classification: see [4, Thm. 10] or [21, Ch. 4]. We can therefore assume that $\Delta^{(1)}$ is two-dimensional. Let $s = \mathrm{ls}_\square(\Delta^{(1)})$ and suppose that $\Delta^{(1)} \subset s\square$. If

$$\Delta^{(1)(-1)} \subset (s\square)^{(-1)} \tag{7}$$

then the theorem follows because $\Delta \subset \Delta^{(1)(-1)}$ and $(s\square)^{(-1)} \simeq (s+2)\square$. So let us assume that (7) is not satisfied. Without loss of generality we may then suppose that $\Delta^{(1)(-1)}$ is not contained in the half-plane $X \leq s+1$. By using a translation if needed, we can assume that the both the lower edge and the right edge of $s\square$ contain at least one vertex of $\Delta^{(1)}$.



By our assumption the right edge then contains *exactly* one such vertex, that we denote by $v = (s, a)$, for some $a \in \{0, \ldots, s\}$.

We first reduce to the case where $a = 0$. Suppose that $a > 0$ and let $v_1 = (a_1, b_1)$ and $v_2 = (a_2, b_2)$ be the vertices of $\Delta^{(1)}$ that are adjacent to v, ordered counterclockwise. For $i = 1, 2$ let τ_i be the edge connecting v_i and v. By our assumption that the lower edge of $s\square$ contains at least one vertex of $\Delta^{(1)}$ we have that $b_1 < a$. This means that the outward shift $\tau_1^{(-1)}$ must intersect the line segment spanned by $v = (s, a)$ and $v' = (s+1, a)$. But then $b_2 \leq a$, otherwise $\tau_2^{(-1)}$ would also pass in between v and v', implying that $\tau_1^{(-1)}$ and $\tau_2^{(-1)}$ intersect in the half-plane $X \leq s + 1$: a contradiction. We conclude that $\Delta^{(1)}$ lies in the half-plane $Y \leq a$. But then a vertical flip followed by a vertical translation positions v at $(s, 0)$, while leaving our other assumptions unaffected.

So we can assume that $v = (s, 0)$. We claim that this implies that at least one of $\text{conv}\{(0, 0), (s, 0)\}$ or $\text{conv}\{(0, s), (s, 0)\}$ appears as an edge of $\Delta^{(1)}$. Assuming the claim we can conclude quickly. Indeed, in the former case we see that $\Delta^{(1)}$ has an edge in common with $s\square$, so that the theorem follows from Lemma 4.2 (using that we excluded the cases where $r \geq s + 3$). In the latter case either $(i, j) \mapsto (i + j, j)$ or $(i, j) \mapsto (i, i + j - s)$ positions $\Delta^{(1)}$ inside $s\square$ in such a way that there is an edge in common:

So the theorem again follows from Lemma 4.2.

To prove the claim, as before let $v_1 = (a_1, b_1)$ and $v_2 = (a_2, b_2)$ be the vertices of $\Delta^{(1)}$ that are adjacent to v, ordered counterclockwise, and denote by τ_1, τ_2 the corresponding edges. We make a case distinction.

- Assume that $b_1 = 0$. Then $\tau_1^{(-1)}$ is the line $Y = -1$. Since $\tau_2^{(-1)}$ must intersect this line in a lattice point outside the half-plane $X \leq s+1$ we find that τ_2 has slope at most 1 (in absolute value), i.e. $a_2 \leq s - b_2$. It follows that $\Delta^{(1)} \subset s\Sigma$. Now:
  - if $(0, 0) \in \Delta^{(1)}$ or $(0, s) \in \Delta^{(1)}$ then the claim follows;
  - if not then the transformation $(i, j) \mapsto (i + j - 1, j)$ maps $\Delta^{(1)}$ inside $(s - 1)\square$, contradicting that $s = \text{ls}_\square(\Delta^{(1)})$.
- Assume that $b_1 \neq 0$. Then τ_1 and τ_2 cannot lie at opposite sides of the line connecting $(s, 0)$ and $(0, s)$, i.e. one cannot simultaneously have $a_1 < s - b_1$ and $a_2 > s - b_2$, because otherwise $\tau_1^{(-1)}$ and $\tau_2^{(-1)}$ would intersect in the half-plane $X \leq s + 1$. But then either $a_2 \leq s - b_2$, in which case $\Delta^{(1)} \subset s\Sigma$ and we can proceed as before, or $a_1 \geq s - b_1$, in which case the situation is entirely analogous.

This completes the proof. ■

Theorem 4.4 gives a recursive method for computing the lattice size with respect to $\square$ in practice. Using the example Δ from the end of the previous section, we see that $\mathrm{ls}_\square(\Delta) = \mathrm{ls}_\square(\emptyset) + 5 + 2 + 2 = 8$. A Magma implementation can be found in the file `basic_commands.m` that accompanies [7]. For instance, the foregoing example can be treated as follows:

```
> load "basic_commands.m";
Loading "basic_commands.m"
> P := LatticePolytope([<8,0>,<6,1>,<2,4>,<0,6>,<0,8>,<3,7>,<5,6>]);
> LatticeSizeRecursiveSquare(P);
8
```

We include an immediate corollary to the above proof, for use in the next section. Let $s \in \mathbb{Z}_{\geq 1}$. Then by a *slice* of $s\square$ we mean a line segment of the form $\mathrm{conv}\{(a,0),(a,s)\}$ or $\mathrm{conv}\{(0,a),(s,a)\}$ for some $a \in \{0,\ldots,s\}$. By a *diagonal* we mean $\mathrm{conv}\{(0,s),(s,0)\}$ or $\mathrm{conv}\{(0,0),(s,s)\}$. Then we have:

Corollary 4.5. *Suppose that $\Delta^{(1)}$ is two-dimensional and contained in $\mathrm{ls}_\square(\Delta^{(1)})\cdot\square$. Assume that there is no edge of $\Delta^{(1)}$ that is a slice or a diagonal of the latter. Then $\mathrm{ls}_\square(\Delta) = \mathrm{ls}_\square(\Delta^{(1)}) + 2$ and $\Delta \subset (-1,-1) + \mathrm{ls}_\square(\Delta)\cdot\Delta$.*

5 A minimum with respect to the product order

This section is devoted to our combinatorial version of Conjecture 1.1, namely that for each non-empty lattice polygon Δ the set $S_{1,1}(\Delta)$ admits a minimum with respect to the product order $\leq \times \leq$ on $\mathbb{N}^2$. It suffices to show that Δ admits a unimodular copy inside the rectangle

$$\square_{\mathrm{lw}(\Delta),\mathrm{ls}_\square(\Delta)} = [0,\mathrm{lw}(\Delta)] \;\times\; [0,\mathrm{ls}_\square(\Delta)]. \tag{8}$$

Indeed, then $(\mathrm{lw}(\Delta),\mathrm{ls}_\square(\Delta)) \in S_{1,1}(\Delta)$, and from the respective definitions of $\mathrm{lw}(\Delta)$ and $\mathrm{ls}_\square(\Delta)$ it is clear that this concerns a minimum with respect to the product order.

We need the following properties of the lattice width:

Lemma 5.1. *If $\Delta^{(1)} \neq \emptyset$ then $\mathrm{lw}(\Delta) = \mathrm{lw}(\Delta^{(1)}) + 2$, except if $\Delta \simeq d\Sigma$ for some $d \geq 3$, in which case $d = \mathrm{lw}(\Delta) = \mathrm{lw}(\Delta^{(1)}) + 3$. If moreover $\Delta^{(1)} \not\simeq d\Sigma$ for any $d \geq 0$ and*

$$\Delta^{(1)} \;\subset\; [0,\mathrm{lw}(\Delta^{(1)})] \;\times\; \mathbb{R}$$

then

$$\Delta \;\subset\; [-1,\mathrm{lw}(\Delta^{(1)})+1] \;\times\; \mathbb{R} \;=\; [-1,\mathrm{lw}(\Delta)-1] \;\times\; \mathbb{R}.$$

Proof. See [23, Thm. 13], where the second statement is phrased as follows: an optimal viewangle for $\Delta^{(1)}$ is also an optimal viewangle for Δ. ■

Due to the special role of standard triangles, we treat the following case separately:

Lemma 5.2. *Let Δ be a two-dimensional lattice polygon such that $\Delta^{(1)} \simeq d\Sigma$ for some $d \geq 1$. Then there exists a unimodular transformation mapping Δ inside* (8).

Proof. If $\Delta^{(1)} = d\Sigma$ then $\Delta \subset (d\Sigma)^{(-1)} \simeq (d+3)\Sigma$. If equality holds then $\mathrm{lw}(\Delta) = \mathrm{ls}_\square(\Delta) = d+3$ and Δ is indeed contained in a box of size $(d+3) \times (d+3)$. If not then at least one of the vertices of $(d+3)\Sigma$ is not contained in Δ. By applying a unimodular transformation if needed we can assume that it concerns the right-most vertex. We now make a case distinction:

- If the left-most edge of $(d+3)\Sigma$ is contained in Δ, then $\mathrm{ls}_\square(\Delta) = d+3$ by Lemma 4.1. On the other hand $\mathrm{lw}(\Delta) = d+2$ by Lemma 5.1. We see that Δ is contained in a box of size $(d+2) \times (d+3)$, as wanted.

  (Remark: the example following the proof of Lemma 4.2 is of this kind.)

- If the left-most edge does not appear, then without loss of generality we can assume that the top vertex is missing. Then $\mathrm{ls}_\square(\Delta) = d+2$ by Lemma 4.2, while still $\mathrm{lw}(\Delta) = d+2$. We see that Δ is contained in a box of size $(d+2) \times (d+2)$, as wanted.

The lemma follows. ■

We can now treat the general case.

Proof of Theorem 1.2. We will proceed by induction on $\mathrm{lw}(\Delta^{(1)})$. The base case is where $\mathrm{lw}(\Delta^{(1)}) \leq 0$, for which the theorem can be verified explicitly using Koelman's classification: see [4, Thm. 10] or [21, Ch. 4].

So assume that $\Delta^{(1)}$ is two-dimensional. Because $\mathrm{lw}(\Delta^{(1)(1)}) < \mathrm{lw}(\Delta^{(1)})$ we can apply the induction hypothesis to find that $\Delta^{(1)}$ can be positioned inside the box

$$[0, \mathrm{lw}(\Delta^{(1)})] \;\times\; [0, \mathrm{ls}_\square(\Delta^{(1)})]. \tag{9}$$

The foregoing lemma allows us to assume that $\Delta^{(1)}$ is not a standard triangle. But then Δ must be contained in the strip

$$[-1, \mathrm{lw}(\Delta) - 1] \;\times\; \mathbb{R} \tag{10}$$

by Lemma 5.1. We make a case distinction:

- Suppose that the box (9) is a square, i.e. $\mathrm{lw}(\Delta^{(1)}) = \mathrm{ls}_\square(\Delta^{(1)})$. Then by symmetry Δ must also be contained in the strip

$$\mathbb{R} \times [-1, \mathrm{lw}(\Delta) - 1].$$

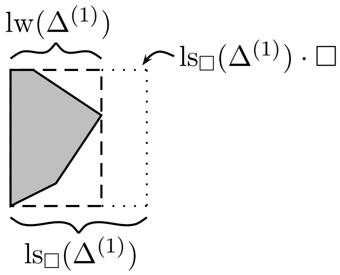
  So it is contained in the intersection

$$[-1, \mathrm{lw}(\Delta) - 1] \times [-1, \mathrm{lw}(\Delta) - 1] \simeq [0, \mathrm{lw}(\Delta)] \times [0, \mathrm{lw}(\Delta)].$$

  Therefore $\mathrm{ls}_\square(\Delta) = \mathrm{lw}(\Delta)$, and the statement follows.

- Suppose that the box (9) is not a square, i.e. $\mathrm{lw}(\Delta^{(1)}) < \mathrm{ls}_\square(\Delta^{(1)})$. We make a further distinction:

  – Suppose that an edge of $\Delta^{(1)}$ arises as a slice of $\mathrm{ls}_\square(\Delta^{(1)}) \cdot \square$. Because $\mathrm{lw}(\Delta^{(1)}) < \mathrm{ls}_\square(\Delta^{(1)})$ it necessarily concerns one of the two vertical edges of our box (9). By flipping horizontally if needed we can assume that it concerns the left edge, which is then a common edge of $\Delta^{(1)}$ with $\mathrm{ls}_\square(\Delta^{(1)}) \cdot \square$.



  It is the unique such edge, so we can apply Corollary 4.3 to find a vertical skewing φ such that $\varphi(\Delta) \subset (-1,-1) + \mathrm{ls}_\square(\Delta) \cdot \square$. But the strip (10) is invariant under vertical skewings. By taking the intersection, we find that

$$\varphi(\Delta) \subset [-1, \mathrm{lw}(\Delta) - 1] \times [-1, \mathrm{ls}_\square(\Delta) - 1] \simeq [0, \mathrm{lw}(\Delta)] \times [0, \mathrm{ls}_\square(\Delta)],$$

  as wanted.

  – Suppose that $\Delta^{(1)}$ does not have an edge arising as a slice of $\mathrm{ls}_\square(\Delta^{(1)}) \cdot \square$. Because our box (9) is not a square, it cannot have a diagonal edge either. So from Corollary 4.5 we see that

$$\Delta \subset (-1,-1) + (\mathrm{ls}_\square(\Delta^{(1)}) + 2) \cdot \square = (-1,-1) + \mathrm{ls}_\square(\Delta) \cdot \square,$$

  and we conclude as above.

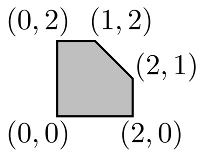
The theorem follows. ■

We conclude this section by remarking that the above material can be used to design an algorithm for simplifying rational parametrizations of toric surfaces, following Schicho [25], where the focus now lies on the bidegree rather than the total degree.

6 Proof of the main theorem

After this large chunk of combinatorics, let us return to algebraic geometry. As in the introduction, let k be an algebraically closed field (of arbitrary characteristic), let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be an irreducible Laurent polynomial, and assume that $\Delta(f)$ is two-dimensional. Let $U_f \subset \mathbb{T}^2$ be the curve defined by f. Our aim is to prove Theorem 1.3.

Proof of Theorem 1.3. First remark that the inequalities (2) are trivial as soon as U_f is a rational curve, because the right-hand sides are at least 1 resp. $(1,1)$. In particular, by Baker's bound [3] we can assume that $\Delta(f)^{(1)}$ is not empty. But then the right-hand sides are at least 3 and $(2,2)$. For curves of genus one these bounds can be met simultaneously. Indeed, pick a cubic (e.g. Weierstrass) model and apply a projective transformation ensuring that the curve passes through the two coordinate points at infinity. Then its affine part is defined by a polynomial whose Newton polygon is contained in



and therefore both in 3Σ and $2\square$. Thus we can assume that U_f is of genus $g \geq 2$. By Baker's bound this implies that $\sharp(\Delta(f)^{(1)} \cap \mathbb{Z}^2) \geq 2$.

Let us begin with proving the first inequality $s_2(U_f) \leq \mathrm{ls}_\Sigma(\Delta(f)^{(1)}) + 3$. By the trivial bound (1) it suffices to analyze the exceptional polygons listed in Theorem 3.5. Since $\sharp(\Delta(f)^{(1)} \cap \mathbb{Z}^2) \geq 2$ this leaves us with two cases:

- Assume that $\Delta(f) = \square_{a,b}$ for certain $a, b \geq 2$. Pick a point $(x_0, y_0) \in U_f$. Then the Newton polygon of $f(x + x_0, y + y_0)$ is contained in

$$\mathrm{conv}\{(1,0), (a,0), (a,b), (0,b), (0,1)\}.$$

But then $x^a y^b f(x^{-1} + x_0, y^{-1} + y_0)$ is a polynomial of degree at most $a + b - 1$. So $s_2(U_f) \leq \mathrm{ls}_\Sigma(\Delta(f)) - 1 = \mathrm{ls}_\Sigma(\Delta(f)^{(1)}) + 3$.

- Assume that there exist parallel edges $\tau \subset \Delta(f)$ and $\tau' \subset \Delta(f)^{(1)}$ whose supporting lines are at integral distance 1 of each other, of respective lengths r and s, such that $r \geq s+4$. From Lemma 3.3 and its proof we see that $s = \mathrm{ls}_\Sigma(\Delta(f)^{(1)})$ and that we can assume that

$$\tau = \mathrm{conv}\{(0,0),(r,0)\} \qquad \text{and} \qquad \tau' = \mathrm{conv}\{(1,1),(s+1,1)\}.$$

This configuration implies that $\Delta(f)$ is contained in the half-planes $X \geq 0$, $Y \geq 0$ and $X + (r-s-2)Y \leq r$. In other words,

$$f = \sum_{j=0}^{\lfloor r/(r-s-2) \rfloor} g_j(x) y^j$$

for polynomials $g_j \in k[x]$ satisfying $\deg g_j \leq r - (r-s-2)j$ and $\deg g_0 = r$. Now factor $g_0(x) = g_0'(x)h_0'(x)$ with $\deg g_0' = s+3$ and $\deg h_0' = r-s-3$, substitute $y \leftarrow y h_0'(x)$, and kill a factor $h_0'(x)$ to obtain

$$g_0'(x) + \sum_{j=1}^{\lfloor r/(r-s-2) \rfloor} g_j(x) h_0'(x)^{j-1} y^j.$$

One verifies that each term has degree at most $s+3$, which proves that $s_2(U_f) \leq s+3 = \mathrm{ls}_\Sigma(\Delta(f)^{(1)}) + 3$.

As for the case where $\Delta(f) \simeq d\Upsilon$ for some $d \geq 2$, note that by Theorem 3.5 we have $\mathrm{ls}_\Sigma(d\Upsilon) = 3d$, so the bound we need to prove is sharper. Consider the embedding

$$\psi : \mathbb{T}^2 \hookrightarrow \mathbb{P}^3 = \mathrm{Proj}\, k[X_0, X_1, X_2, X_3] : (x,y) \mapsto (x^{-1}y^{-1} : x : y : 1).$$

It embeds U_f in a projective curve C_f which arises as the intersection of the cubic $X_0X_1X_2 - X_3^3$ and an irreducible hypersurface of degree d, whose concrete equation depends on f. In particular it is a curve of degree $3d$. By [16, IV.Prop. 3.8 and IV.Thm. 3.9] we can find a point on C_f, the general secant line through which is not a multisecant. Projecting from such a point yields a birational equivalence between C_f and a plane curve of degree $3d-1$, as wanted.

Next we address the inequality

$$s_{1,1}(U_f) \leq s_{1,1}(\Delta(f)^{(1)}) + (2,2) = \left(\mathrm{lw}(\Delta(f)^{(1)}) + 2, \mathrm{ls}_\Box(\Delta(f)^{(1)}) + 2\right).$$

We make a case distinction.

- Assume that $\Delta(f)^{(1)} \simeq (d-3)\Sigma$ for some $d \geq 4$, so that

$$\mathrm{ls}_\Box(\Delta(f)^{(1)}) = \mathrm{ls}_\Sigma(\Delta(f)^{(1)}) = d-3.$$

By the foregoing U_f has a plane model of degree d. Using a projective transformation we can ensure that this model passes through the coordinate points at infinity. As in the genus one case we end up with a model of bidegree $(d-1, d-1)$, as wanted.

- Suppose that $\Delta(f)^{(1)}$ is not a standard triangle. By Lemma 5.1 we have $\mathrm{lw}(\Delta(f)) = \mathrm{lw}(\Delta(f)^{(1)}) + 2$. If we are not among the exceptions listed in Theorem 4.4 then also $\mathrm{ls}_\square(\Delta(f)) = \mathrm{ls}_\square(\Delta(f)^{(1)}) + 2$, and the statement follows from the bound (1).

  Because $\sharp(\Delta(f)^{(1)} \cap \mathbb{Z}^2) \geq 2$ only the last exception is a concern. Assume that there exist parallel edges $\tau \subset \Delta(f)$ and $\tau' \subset \Delta(f)^{(1)}$ whose supporting lines are at integral distance 1 of each other, of respective lengths r and s, such that $r \geq s+3$. By Lemma 4.1 we know that $s = \mathrm{ls}_\square(\Delta(f)^{(1)})$. Thus our aim is to apply a birational change of variables to f so that the result has bidegree $(\mathrm{lw}(\Delta(f)), s+2)$.

  Again, as in the proof of Lemma 3.3 we can assume that

  $$\tau = \mathrm{conv}\{(0,0),(r,0)\} \qquad \text{and} \qquad \tau' = \mathrm{conv}\{(1,1),(s+1,1)\},$$

  so that $\Delta(f)$ is contained in the half-planes $X \geq 0$, $Y \geq 0$ and $X + (r-s-2)Y \leq r$. This implies that $\Delta(f)^{(1)}$ is contained in $(1,1) + s\Sigma$, and because we excluded standard triangles the top vertex of the latter cannot occur, from which one sees that $\mathrm{lw}(\Delta(f)^{(1)}) < s$.

  If we now use Theorem 1.2 to position $\Delta(f)^{(1)}$ inside a box

  $$[1, s+1] \times [1, \mathrm{lw}(\Delta(f)^{(1)}) + 1],$$

  then τ' necessarily arises as a horizontal line segment; we can assume it to be the bottom segment $\mathrm{conv}\{(1,1),(s+1,1)\}$. By Lemma 5.1 our Newton polygon $\Delta(f)$ is then contained in the strip

  $$\mathbb{R} \times [0, \mathrm{lw}(\Delta(f))]. \tag{11}$$

  Now once again as in the proof of Lemma 3.3 we can apply a horizontal skewing to position τ at $\mathrm{conv}\{(0,0),(r,0)\}$. We again obtain that $\Delta(f)$ is contained in the half-planes $X \geq 0$, $Y \geq 0$ and $X + (r-s-2)Y \leq r$, while it is also kept in the strip (11). In other words,

  $$f = \sum_{j=0}^{\mathrm{lw}(\Delta(f))} g_j(x) y^j$$

for polynomials $g_j \in k[x]$ satisfying $\deg g_j \leq r - (r-s-2)j$, $\deg g_0 = r$ and $g_{\mathrm{lw}(\Delta(f))} \neq 0$. Now factor $g_0(x) = g_0'(x)h_0'(x)$ with $\deg g_0' = s+2$ and $\deg h_0' = r-s-2$, substitute $y \leftarrow yh_0'(x)$, and kill a factor $h_0'(x)$ to obtain a polynomial

$$g_0'(x) + \sum_{j=1}^{\mathrm{lw}(\Delta(f))} g_j(x)h_0'(x)^{j-1}y^j$$

of degree $s+2$ in x and degree $\mathrm{lw}(\Delta(f)))$ in y, as wanted.

It remains to show that $s_{1,1}(U_f) \leq (3,4)$ when $\Delta(f) = 2\Upsilon$. By Baker's bound U_f is a curve of genus at most 4. If U_f is hyperelliptic then the bound follows trivially (because of the lexicographic order). If U_f is non-hyperelliptic of genus 3 then U_f is birationally equivalent to a non-singular quartic in $\mathbb{P}^2$, and one can construct a model of bidegree $(3,3)$ by forcing it through the coordinate points at infinity. Finally if U_f is non-hyperelliptic of genus 4 then it is birationally equivalent to a singular quintic in $\mathbb{P}^2$ by [16, IV.Ex. 5.4]. Using a projective transformation we can assume that the curve passes through the coordinate points at infinity, one of these being a singularity. Dehomogenizing yields an affine model of bidegree $(3,4)$ as wanted. ■

7 Cases where the bounds are sharp

In this section we again restrict to $\mathrm{char}\, k = 0$, because of some references on which we will rely. One of these references is a subsequent, more elaborate paper [7] of ours, in which we study linear pencils that are encoded in the Newton polygon. At some point in that paper, the lattice size with respect to Σ pops up as a convenient notion [7, Thm. 7.2]. This is how we came up with the first inequality from Theorem 1.3, which meant the start of this project.

We will make extensive reference to [7], even though it concerns a successive paper. But we stress that no circular reasoning is being made: no statements in [7] make use of any of the results of this section. Moreover, some of the results of [7] that we need appear (in more disguised terms) in an earlier article by Kawaguchi [18]. Finally, we emphasize that the primary aim of this section is to convince the reader that the bounds from Theorem 1.3 often give the correct values of $s_2(U_f)$ and $s_{1,1}(U_f)$, and to give some evidence in favor of Conjecture 1.4; we will not push the limits of our exposition.

Let us specify what we mean by f being sufficiently generic with respect to its Newton polygon $\Delta(f)$. To each two-dimensional lattice polygon Δ there is a standard way of associating a toric surface $X(\Delta)$ over k (along with an embedding in $\mathbb{P}^{\#(\Delta \cap \mathbb{Z}^2)-1}$). This is a completion of the torus $\mathbb{T}^2$, so it is natural to consider the closure of U_f inside it. It turns out that for almost all Laurent polynomials f the closure C_f of U_f inside $X(\Delta(f))$ is non-singular. More precisely, if one fixes a two-dimensional lattice polygon Δ, then the locus of the Laurent polynomials f for which $\Delta(f) = \Delta$ and C_f is non-singular is dense in the according $\#(\Delta \cap \mathbb{Z}^2)$-dimensional coefficient space. We refer to [9, §2] and [7, §4] for more background.

We now rephrase Conjecture 1.4 as follows.

Conjecture 7.1. *If C_f is a non-singular curve and $\Delta(f) \not\simeq 2\Upsilon$ then*

$$s_2(U_f) = \mathrm{ls}_\Sigma(\Delta(f)^{(1)}) + 3 \quad \textit{and} \quad s_{1,1}(U_f) = s_{1,1}(\Delta(f)^{(1)}) + (2,2),$$

unless $\Delta(f) \simeq d\Upsilon$ for some $d \geq 3$, in which case $s_2(U_f) = 3d - 1$.

This would extend the list of geometric invariants that are known to be encoded in the Newton polygon. We mention some of its current entries: if C_f is a non-singular curve then

(i) its (geometric) genus g equals $\sharp(\Delta(f)^{(1)} \cap \mathbb{Z}^2)$; this is due to Khovanskii [20];

(ii) its gonality c equals $\mathrm{lw}(\Delta(f)^{(1)}) + 2$, unless $\Delta(f) \simeq 2\Upsilon$ in which case the gonality equals 3; this is [7, Cor. 6.2], whose proof strongly builds on previous work of Kawaguchi [18];

(iii) it is isomorphic to a non-singular plane curve if and only if $\Delta(f)^{(1)} = \emptyset$ or $\Delta(f)^{(1)} \simeq (d-3)\Sigma$ for some $d \geq 3$; this is [7, Cor. 8.2].

For an extension of this list we refer to [6, 7, 8]. Note the similarity between statement (ii) and Conjecture 7.1.

An intuitive reason for the fact that many invariants are encoded in the Newton polygon is that C_f canonically embeds inside $X(\Delta^{(1)}) \subset \mathbb{P}^{g-1}$, and that the defining equations of the latter are so special (quadrics of very low rank) that they can often be recovered from the canonical ideal of C_f itself. We refer to the introduction of [6] for an extended discussion. Up to equivalence, the polygon Υ is the unique two-dimensional polygon of the form $\Delta^{(1)}$ for which the ideal of $X(\Delta^{(1)})$ is *not* generated by quadrics. This explains the special role of 2Υ, which is the only polygon having Υ as its interior. If $\Delta(f) \simeq 2\Upsilon$ then from (i) we find that C_f is a genus four curve, for which

- $s_2(U_f) = 5$, so the formula $s_2(U_f) = 3d - 1$ is actually correct here: the existence of a degree 5 model follows from Theorem 1.3, while degree 4 or less would contradict that the genus is 5;

- $s_{1,1}(U_f) = (3,4)$ or $s_{1,1}(U_f) = (3,3)$, depending on whether the unique quadric in which C_f canonically embeds is singular or not: in this case by [9, §6] there exists an $f' \in k[x^{\pm 1}, y^{\pm 1}]$ with

$$\Delta(f') = \mathrm{conv}\{(0,0),(6,0),(0,3)\} \quad \text{resp.} \quad \Delta(f') = [0,3] \times [0,3],$$

  such that $C_{f'}$ is non-singular and birationally equivalent to U_f; the formulas then follow from Theorem 7.3 below.

Alternatively, these formulas can be proved along the lines of [16, IV.Ex. 5.4].

Note that by Lemma 3.1 and (5) we have

$$-1 \le \mathrm{lw}(\Delta(f)^{(1)}) \le \mathrm{ls}_\Sigma(\Delta(f)^{(1)}) \quad \text{and} \quad -1 \le \mathrm{lw}(\Delta(f)^{(1)}) \le \mathrm{ls}_\square(\Delta(f)^{(1)}).$$

We can prove Conjecture 7.1 near both ends of these ranges.

Theorem 7.2. *If C_f is non-singular and*

$$\mathrm{lw}(\Delta(f)^{(1)}) \le 1 \quad \textit{or} \quad \mathrm{lw}(\Delta(f)^{(1)}) \ge \mathrm{ls}_\Sigma(\Delta(f)^{(1)}) - 2,$$

then $s_2(U_f) = \mathrm{ls}_\Sigma(\Delta(f)^{(1)}) + 3$, unless $\Delta(f) \simeq 2\Upsilon, 3\Upsilon$, in which case $s_2(U_f) = 5$ and $s_2(U_f) = 8$, respectively.

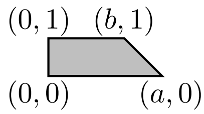
Proof. By the above discussion we can assume that $\Delta(f) \not\simeq 2\Upsilon$.

At the lower end, we can argue as follows:

- If $\mathrm{lw}(\Delta(f)^{(1)}) = -1$, or in other words if $\Delta(f)^{(1)} = \emptyset$, then U_f is rational because of (i), and there is nothing to prove.

- If $\mathrm{lw}(\Delta(f)^{(1)}) = 0$, then $\Delta(f)^{(1)}$ is a line segment, say of integral length $g-1$. By (i) and (ii) we find that U_f is hyperelliptic of genus g. So $s_2(U_f) = g+2$, which indeed equals $\mathrm{ls}_\Sigma(\Delta(f)^{(1)}) + 3$.

- If $\mathrm{lw}(\Delta(f)^{(1)}) = 1$ then $\Delta(f)^{(1)}$ is equivalent to a Lawrence prism



  with $1 \le a \ge b \ge 0$. In this case C_f is a trigonal curve with scrollar invariants a, b by [7, Thm. 9.1]. From [19, Lem. 2.1], which is expressed in terms of the Maroni invariant $b = \min(a,b)$, we conclude that $s_2(U_f) = g+1-b = a+3$ if $a > b$, and $s_2(U_f) = g+2-b = a+4$ if $a = b$. By Theorem 3.5, in both cases this exactly matches with $\mathrm{ls}_\Sigma(\Delta(f)^{(1)}) + 3$.

At the other end, we make the following reasonings.

- Assume that $\mathrm{lw}(\Delta(f)^{(1)}) = \mathrm{ls}_\Sigma(\Delta(f)^{(1)})$. Then by Lemma 3.1 we have that $\Delta(f)^{(1)} \cong (d-3)\Sigma$ for some integer $d \geq 3$. But then by (iii) our curve C_f is isomorphic to a non-singular plane curve of degree d, and therefore $s_2(U_f) = d = \mathrm{ls}_\Sigma(\Delta(f)^{(1)}) + 3$.

- If $\mathrm{lw}(\Delta(f)^{(1)}) = \mathrm{ls}_\Sigma(\Delta(f)^{(1)}) - 1$ then by (ii) the gonality of U_f equals

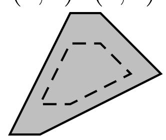
$$c = \mathrm{ls}_\Sigma(\Delta(f)^{(1)}) + 1,$$

unless $\Delta(f) \simeq 2\Upsilon$, but this case was excluded. On the other hand, again by (iii) every plane model is necessarily singular. This means that $s_2(U_f) \geq c+2$, because otherwise projection from a singular point on the plane model would give a map to $\mathbb{P}^1$ of degree strictly less than c. We conclude that $s_2(U_f) \geq \mathrm{ls}_\Sigma(\Delta(f)^{(1)}) + 3$, and by Theorem 1.3 equality holds.

- If $\mathrm{lw}(\Delta(f)^{(1)}) = \mathrm{ls}(\Delta(f)^{(1)}) - 2$ then by (ii) the gonality of U_f equals

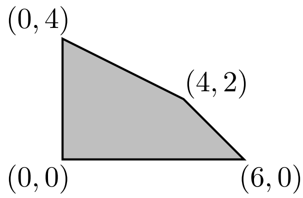
$$c = \mathrm{ls}_\Sigma(\Delta(f)^{(1)}).$$

By (iii) every plane model is singular and we find as above that $s_2(U_f) \geq c+2$. In case $\Delta(f) \simeq 3\Upsilon$ this matches with the upper bound from Theorem 1.3, and we are done. We would like to show that $s_2(U_f) \geq c+3$ in the other cases. So suppose that $\Delta(f) \not\simeq 3\Upsilon$ and assume by contradiction that $s_2(U_f) = c+2$. In this case we see that the curve carries infinitely many base-point free g^1_{c+1}'s, obtained by projection from the non-singular points of a plane degree $c+2$ model. By [7, Thm. 7.2] this is possible only if $\Delta(f)^{(1)} \cong \mathrm{conv}\{(0,0),(1,0),(3,1),(2,2),(1,2)\}$, so that $\Delta(f)$ is of the form



(the dashed polygon indicates $\Delta(f)^{(1)}$). By (i) and (ii) our curve C_f has gonality $c = 4$ and geometric genus $g = 7$. By [7, Cor. 6.3] and [7, Thm. 9.1] the gonality pencil is unique and its scrollar invariants are 1, 1, and 2. Now take a curve in $\mathbb{P}^2$ of degree $d = c+2 = 6$ that is birationally equivalent to C_f. Because the gonality is 4 and the gonality pencil is unique, the curve must have a unique singular point P, of multiplicity 2. The point P cannot be an ordinary node or a cusp, otherwise the genus would be $(d-1)(d-2)/2-1 = 9$.

Thus there is a unique tangent line at P which intersects the curve at P with multiplicity at least 4. Using a transformation of $\mathbb{P}^2$ we can assume that $P = (0:1:0)$ and that this line is at infinity. Dehomogenizing the defining equation then results in a polynomial $g(x, y)$ that is supported on the following polygon:



The coefficient at y^4 is non-zero because the gonality is 4. In particular our g^1_4 is given by the projection $(x, y) \mapsto x$. Also note that at least one of the coefficients at x^4y^2, x^5y, x^6 is non-zero, because the degree is 6. Now let $D_x \in g^1_4$ be the zero divisor of x^{-1}, and similarly let D_y be the zero divisor of y^{-1}. The steepness of the above polygon ensures that $D_y \leq 2D_x$. In particular $H^0(2D_x) \supset \{1, x^{-1}, x^{-2}, y^{-1}\}$ is at least 4-dimensional. This shows that 0 must be among the scrollar invariants of our g^1_4: a contradiction. We conclude that $s_2(U_f) \geq c + 3$: a contradiction. ■

Theorem 7.3. *If C_f is non-singular, $\Delta(f) \not\simeq 2\Upsilon$ and*

$$\mathrm{lw}(\Delta(f)^{(1)}) \leq 1 \quad or \quad \mathrm{lw}(\Delta(f)^{(1)}) \geq \mathrm{ls}_\square(\Delta(f)^{(1)}) - 1,$$

then $s_{1,1}(U_f) = s_{1,1}(\Delta(f)^{(1)}) + (2, 2)$.

Proof. The case of rational and (hyper)elliptic curves follows from Lemma 2.1. In the trigonal case:

- If $g = 3$ then $\Delta(f)^{(1)} \simeq \Sigma$ and the upper bound from Theorem 1.3 reads $(3, 3)$, which is clearly sharp in the case of a trigonal curve.
- If $g = 4$ then either $\Delta(f)^{(1)} \simeq \mathrm{conv}\{(0, 0), (2, 0), (0, 1)\}$ or $\Delta(f)^{(1)} \simeq \square_{1,1}$; indeed $\Delta(f)^{(1)} \simeq \Upsilon$ was excluded in the statement of the theorem. In the latter case the upper bound from Theorem 1.3 reads $(3, 3)$, which is clearly optimal in the case of a trigonal curve. In the former case the upper bound reads $(3, 4)$, which is also optimal because by [7, Cor. 6.3] the gonality pencil is unique, while the existence of a model of bidegree $(3, 3)$ would contradict that.

- If $g \geq 5$ then the g^1_3 is always unique. From [24, Prop. 1] (see also [10, Ex. 1.2.7]) one sees that there exists a base-point free g^1_d that is independent from this g^1_3 if and only if $g-d$ does not exceed the Maroni invariant. Using the same notation as in the foregoing proof, this condition reads $g - d \leq a$, which is equivalent with $d \geq b + 2$. Thus $s_{1,1}(U_f) = (3, b+2) = s_{1,1}(\Delta(f)) + (2, 2)$, as wanted.

At the other end, we make the following reasonings.

- If $\mathrm{lw}(\Delta(f)^{(1)}) = \mathrm{ls}_\square(\Delta(f)^{(1)})$ then by (ii) the gonality of U_f equals

  $$c = \mathrm{lw}(\Delta(f)^{(1)}) + 2 = \mathrm{ls}_\square(\Delta(f)^{(1)}) + 2$$

  unless $\Delta(f) \cong 2\Upsilon$, but this case was excluded. So $s_{1,1}(\Delta(f)^{(1)}) + (2, 2) = (c, c)$ is clearly a lower bound for $s_{1,1}(U_f)$, and by Theorem 1.3 equality holds.

- If $\mathrm{lw}(\Delta(f)^{(1)}) = \mathrm{ls}_\square(\Delta(f)^{(1)}) - 1$ then

  $$c = \mathrm{lw}(\Delta(f)^{(1)}) + 2 = \mathrm{ls}_\square(\Delta(f)^{(1)}) + 1,$$

  and we similarly find that $s_{1,1}(\Delta(f)^{(1)}) + (2, 2) = (c, c + 1)$. So it is sufficient to show that $(c, c) \notin S_{1,1}(U_f)$, i.e. our curve does not carry two independent gonality pencils. But by [7, Thm. 6.1] every gonality pencil is combinatorial, i.e. it corresponds to projecting along some lattice width direction of $\Delta(f)$. In particular $\mathrm{lw}(\Delta(f)) = c$, and if C_f would admit two gonality pencils then $\Delta(f)$ would admit two $\mathbb{R}$-linearly independent lattice width directions. By [7, Lem. 5.2(v)] this would mean that $\mathrm{ls}_\square(\Delta(f)) = c$. But then

  $$c - 1 = \mathrm{ls}_\square(\Delta(f)^{(1)}) \leq c - 2,$$

  a contradiction. ■

There is room for improvement in Theorems 7.2 and 7.3, in order to cover larger ranges of $\mathrm{lw}(\Delta(f)^{(1)})$. At the lower end this seems difficult however. Whereas it is well-understood which base-point free pencils occur in the hyperelliptic and trigonal cases [24], for curves of higher gonality not much seems known, although Coppens and Martens proved some potentially useful facts in the tetragonal case [11]. At the upper end more seems possible: one can try to extend the results of [7, §7] in order to describe the g^1_{c+n}'s on smooth curves in toric surfaces, for $n = 2, 3, \ldots$ It is expected that these are always combinatorially determined, except for a finite (but increasing) number of polygons. This would help in pushing the above arguments. The finite number of exceptions can then hopefully be treated using an ad hoc idea, such as the one used in the proof of Theorem 7.2. We expect this to become increasingly difficult and case-distinctive, however. Alternatively, it might be possible to obtain some results by using specialization of linear systems from curves to graphs [2, 5] to reduce Conjecture 1.4 to a purely combinatorial statement.

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A minimal set of generators for the canonical ideal of a non-degenerate curve

Wouter Castryck* and Filip Cools

Abstract

We give an explicit way of writing down a minimal set of generators for the canonical ideal of a non-degenerate curve, or of a more general smooth projective curve in a toric surface, in terms of its defining Laurent polynomial.

MSC2010: Primary 14H45, Secondary 14M25

Accompanying Magma file: `canonical.m`

1 Introduction

Let k be an algebraically closed field and consider the affine torus $\mathbb{T}^2 = (k \setminus \{0\})^2$. Let $\Delta \subset \mathbb{R}^2$ be a two-dimensional lattice polygon and define $N = \sharp(\Delta \cap \mathbb{Z}^2)$. In this article we are concerned with algebraic curves $U_f \subset \mathbb{T}^2$ that are cut out by a sufficiently generic Laurent polynomial

$$f = \sum_{(i,j) \in \Delta \cap \mathbb{Z}^2} c_{i,j} x^i y^j \in k[x^{\pm 1}, y^{\pm 1}].$$

Here 'sufficiently generic' means that f is contained in a certain Zariski dense subset of the corresponding N-dimensional coefficient space. More precisely, to each $(i, j) \in \Delta \cap \mathbb{Z}^2$ we associate a formal variable $X_{i,j}$, and we let

$$\mathbb{P}^{N-1} = \operatorname{Proj} k[X_{i,j}]_{(i,j) \in \Delta \cap \mathbb{Z}^2}.$$

We have a natural embedding

$$\varphi_\Delta : \mathbb{T}^2 \hookrightarrow \mathbb{P}^{N-1} : (x, y) \mapsto \left(x^i y^j\right)_{(i,j) \in \Delta \cap \mathbb{Z}^2},$$

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the Zariski closure of the image of which is a toric surface that we denote by $\mathrm{Tor}(\Delta)$. Note that $\varphi_\Delta(U_f)$ is contained in the hyperplane section

$$H : \sum_{(i,j) \in \Delta \cap \mathbb{Z}^2} c_{i,j} X_{i,j} = 0$$

of $\mathrm{Tor}(\Delta) \subset \mathbb{P}^{N-1}$. Then by 'sufficiently generic' we mean that the Zariski closure C_f of $\varphi_\Delta(U_f)$ is a smooth projective curve that equals this hyperplane section. Bertini's theorem implies that this is indeed a Zariski dense condition. Alternatively and more explicitly, for C_f to arise as a smooth hyperplane section of $\mathrm{Tor}(\Delta)$, it suffices that f is *non-degenerate* with respect to Δ, in the sense that for each face $\tau \subset \Delta$ (vertex, edge, or Δ itself) the system

$$f_\tau = \frac{\partial f_\tau}{\partial x} = \frac{\partial f_\tau}{\partial y} = 0$$

has no solutions in $\mathbb{T}^2$. Here f_τ is obtained from f by restricting to those terms that are supported on τ. Non-degeneracy is known to be generically satisfied; see [3, Prop. 1].

Remark. Every (nef and big) smooth projective curve C on a toric surface X arises as such a toric hyperplane section. Indeed, let D_C be a torus-invariant divisor on X that is linearly equivalent to C, and let Δ be the two-dimensional lattice polygon associated to D_C (here we use that C is nef and big). Then the $\mathbb{T}^2$-part of C is cut out by a Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$ that is supported on Δ. The above construction then yields a hyperplane section C_f of $\mathrm{Tor}(\Delta)$ that is isomorphic to C.

We refer to [4, §3-4] and the references therein for more background, both on curves in toric surfaces and on non-degenerate Laurent polynomials. Various of these references assume the base field k to be of characteristic 0, but we emphasize that the material presented below is valid in any characteristic.

The main result of this paper is an explicit recipe for writing down a minimal set of generators for the canonical ideal of curves of the form C_f, where $f \in k[x^{\pm 1}, y^{\pm 1}]$ satisfies the above generic condition (e.g. non-degeneracy) with respect to a given two-dimensional lattice polygon Δ.

A quick implementation of the resulting algorithm already heavily outperforms Magma's built-in function for computing canonical ideals [1]. The latter relies on general lattice basis reduction algorithms that were developed by Hess [9]. Our code can be found in the file `canonical.m` that accompanies the article. It allows one to compute the canonical ideal of a non-degenerate curve of genus $g \approx 100$ in a matter of minutes, whereas everything beyond $g = 20$ looks hopeless using the Magma intrinsic, both in terms of time and memory. Of course, this comes at the cost of working in less generality, but note that the condition of non-degeneracy is generically satisfied (for a fixed instance of Δ), and easy to verify. It therefore seems useful to begin the computation of the canonical ideal with a test for whether the input polynomial is non-degenerate or not, and if yes, to proceed with the method presented here.

Our starting point is a theorem by Khovanskii [10], stating that there exists a canonical divisor K_Δ on C_f such that a basis for $H^0(C_f, K_\Delta)$ is given by

$$\left\{x^i y^j\right\}_{(i,j)\in\Delta^{(1)}\cap\mathbb{Z}^2},$$

where $\Delta^{(1)}$ denotes the convex hull of the interior lattice points of Δ. Here x, y are viewed as functions on C_f through φ_Δ. See [5, Prop. 10.5.8] for a modern proof. Two notable corollaries are:

- The genus of C_f equals $g = \sharp(\Delta^{(1)} \cap \mathbb{Z}^2)$.
- If $g \geq 2$ then the linear system $|K_\Delta|$ maps U_f inside the image of $\varphi_{\Delta^{(1)}}$. In particular:
  - if $\Delta^{(1)}$ is one-dimensional then the canonical image of C_f is a rational normal curve of degree $g-1$, hence C_f is hyperelliptic;
  - if $\Delta^{(1)}$ is two-dimensional, then C_f is non-hyperelliptic and the canonical image of C_f is contained in the toric surface $\mathrm{Tor}(\Delta^{(1)}) \subset \mathbb{P}^{g-1}$.

See [4, §4] and its references for more details.

In what follows we assume that C_f is non-hyperelliptic or, equivalently, that $\Delta^{(1)}$ is two-dimensional. Then the generators for the canonical ideal of C_f are gathered in two steps.

- In Section 2, which can be seen as an addendum to previous work by Koelman [12, 13], we will describe a method for finding a minimal set of generators for the ideal of $\mathrm{Tor}(\Delta^{(1)})$. We also provide explicit formulas for the number of generators in each degree. Because of the independent interest, we will do this for toric surfaces $\mathrm{Tor}(\Gamma)$ where Γ is an *arbitrary* two-dimensional lattice polygon (not necessarily of the form $\Delta^{(1)}$).
- Then in Section 3, we will explicitly describe which generators have to be added in order to obtain a minimal set of generators for the canonical ideal of C_f. These can be seen as analogues of Reid's rolling factors [15], where the 'rolling' now happens in two directions, rather than one.

Notation and terminology. We use a special notation for two recurring polygons

$$\Sigma = \text{conv}\{(0,0),(1,0),(0,1)\}, \qquad \Upsilon = \text{conv}\{(-1,-1),(1,0),(0,1)\},$$

and write $\cong$ to indicate unimodular equivalence. For instance, $\Delta \cong \Sigma$ if and only if Δ is a unimodular simplex. We recall that the convex hull of the interior lattice points of a two-dimensional lattice polygon Δ is denoted by $\Delta^{(1)}$. If the latter is again two-dimensional, we abbreviate $\Delta^{(1)(1)}$ by $\Delta^{(2)}$. We use Δ° to denote the topological interior of Δ, and write $\partial\Delta$ for its boundary. A two-dimensional lattice polygon Δ is said to be *hyperelliptic* if $\Delta^{(1)}$ is one-dimensional. If X is a projectively embedded variety over k, we write $\mathcal{I}(X)$ for its defining ideal. For each non-negative integer d we use $\mathcal{I}_d(X)$ to denote the k-vector space of homogeneous degree d polynomials that are contained in $\mathcal{I}(X)$.

2 The ideal of a toric surface

Let $\Gamma \subset \mathbb{R}^2$ be a two-dimensional lattice polygon and let $N = \#(\Gamma \cap \mathbb{Z}^2)$. Define $\text{Tor}(\Gamma)$ as the Zariski closure inside $\mathbb{P}^{N-1}$ of the image of φ_Γ. A result due to Koelman [12, 13] states that the ideal $\mathcal{I}(\text{Tor}(\Gamma))$ is generated by all binomials

$$\prod_{\ell=1}^{n} X_{i_\ell,j_\ell} - \prod_{\ell=1}^{n} X_{i'_\ell,j'_\ell} \qquad \text{for which} \qquad \sum_{\ell=1}^{n}(i_\ell,j_\ell) = \sum_{\ell=1}^{n}(i'_\ell,j'_\ell)$$

where $n \in \{2,3\}$. Moreover one can restrict to $n=2$ if and only if $\sharp(\partial\Gamma \cap \mathbb{Z}^2) \geq 4$ or Γ is a unimodular simplex. This result was generalized to property N_p for arbitrary p by Hering and Schenck; see [8, Thm. 4.20].

The current section can be seen as an addendum to Koelman's work: we give explicit formulas for the *number* of quadrics and cubics in a minimal set of homogeneous generators for $\mathcal{I}(\text{Tor}(\Gamma))$.

Lemma 1. *For all integers $d \geq 0$ one has:*

$$\dim \mathcal{I}_d\left(\text{Tor}(\Gamma)\right) = \binom{\sharp(\Gamma \cap \mathbb{Z}^2) + d - 1}{d} - \sharp(d\Gamma \cap \mathbb{Z}^2).$$

Proof. The k-vector space morphism

$$\chi_d : \mathcal{I}_d(\mathbb{P}^{N-1}) \to k[x^{\pm 1}, y^{\pm 1}] : X_{i_1,j_1} \cdots X_{i_d,j_d} \mapsto x^{i_1+\cdots+i_d} y^{j_1+\cdots+j_d}$$

has kernel $\mathcal{I}_d(\text{Tor}(\Gamma))$ and surjects onto $\langle x^i y^j \rangle_{(i,j)\in d\Gamma\cap\mathbb{Z}^2}$ (here we use that two-dimensional lattice polygons are always normal [2, Prop. 1.2.2-4], i.e. every lattice point in $d\Gamma$ is the sum of d lattice points in Γ). □

The main result of this section is:

Theorem 2. *A minimal set of generators for* $\mathcal{I}(\mathrm{Tor}(\Gamma))$ *consists of*

$$\binom{\sharp(\Gamma \cap \mathbb{Z}^2)+1}{2} - \sharp(2\Gamma \cap \mathbb{Z}^2) \text{ quadrics and } c_\Gamma \text{ cubics,}$$

where

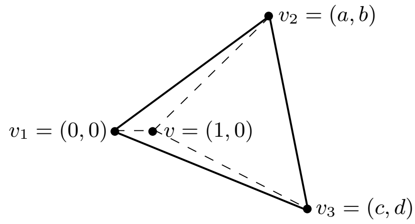
$$c_\Gamma = \begin{cases} 0 & \text{if } \sharp(\partial\Gamma \cap \mathbb{Z}^2) \geq 4 \text{ or } \Gamma \cong \Sigma, \\ 1 & \text{if } \sharp(\partial\Gamma \cap \mathbb{Z}^2) = 3,\ \Gamma \not\cong \Sigma, \text{ and } \Gamma \text{ is non-hyperelliptic,} \\ \sharp(\Gamma \cap \mathbb{Z}^2) - 3 & \text{if } \sharp(\partial\Gamma \cap \mathbb{Z}^2) = 3,\ \Gamma \not\cong \Sigma, \text{ and } \Gamma \text{ is hyperelliptic.} \end{cases}$$

Proof. The formula for the number of quadrics follows from Lemma 1 along with the fact that $\mathrm{Tor}(\Gamma)$ is not contained in any hyperplane of $\mathbb{P}^{N-1}$. By Koelman's result, it remains to prove the formula for the number of cubics c_Γ when $\sharp(\partial\Gamma \cap \mathbb{Z}^2) = 3$ and $\Gamma \not\cong \Sigma$. We moreover know that $c_\Gamma \geq 1$ in these cases. Also recall that $\mathcal{I}(\mathrm{Tor}(\Gamma))$ is generated by binomials.

First assume that Γ is non-hyperelliptic and $\Gamma \not\cong \Upsilon$. Along with $\sharp(\partial\Gamma \cap \mathbb{Z}^2) = 3$ and $\Gamma \not\cong \Sigma$ this implies that $\Gamma^{(1)}$ is two-dimensional; see e.g. Koelman's classification [11, Ch. 4], although this could also serve as an easy exercise. Let $\{v_1, v_2, v_3\}$ be the three vertices of Γ and consider

$$\Gamma' = \mathrm{conv}\left((\Delta \setminus \{v_1\}) \cap \mathbb{Z}^2\right).$$

Then $\Gamma' \supset \Gamma^{(1)}$ is again a two-dimensional lattice polygon. We claim that there are at least 4 lattice points on its boundary. Indeed, if there would only be 3 such lattice points, then Γ' would be a triangle whose vertices are $\{v, v_2, v_3\}$, where v is contained in the interior of Γ, and the triangles v_1-v-v_2 and v_1-v-v_3 are unimodular simplices (i.e. they do not contain any lattice points besides the vertices).



We may assume that $v_1 = (0,0)$, $v = (1,0)$, $v_2 = (a,b)$ and $v_3 = (c,d)$, where $b > 0 > d$. By Pick's theorem the unimodularity of v_1-v-v_2 and v_1-v-v_3 implies that $b = 1$ and $d = -1$, and hence that Γ is contained in a horizontal strip of width 2: a contradiction with the fact that $\Gamma^{(1)}$ is two-dimensional. So the claim follows. Now consider a binomial

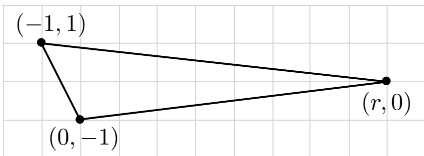
$$C = X_{i_1,j_1}X_{i_2,j_2}X_{i_3,j_3} - X_{i'_1,j'_1}X_{i'_2,j'_2}X_{i'_3,j'_3} \in \mathcal{I}_3(\mathrm{Tor}(\Gamma)) \tag{1}$$

and define $\Gamma_C = \text{conv}\{(i_1, j_1), (i_2, j_2), (i_3, j_3), (i'_1, j'_1), (i'_2, j'_2), (i'_3, j'_3)\}$.

- If $\Gamma_C \subsetneq \Gamma$, then by the above $\Gamma_C \subset \Gamma'$ for a subpolygon Γ' that contains at least 4 lattice points on the boundary. So by Koelman's result applied to Γ' our cubic C can be written as a linear combination of a number of elements of $\mathcal{I}_2(\text{Tor}(\Gamma))$.
- If $\Gamma_C = \Gamma$ then it is not hard to see that either $(i_1, j_1), (i_2, j_2), (i_3, j_3)$ or $(i'_1, j'_1), (i'_2, j'_2), (i'_3, j'_3)$ are the three vertices of Γ; see [13, Lem. 2.6].

It follows that the sum or difference of two binomials $C_1, C_2 \in \mathcal{I}_3(\text{Tor}(\Gamma))$ that are independent of $\mathcal{I}_2(\text{Tor}(\Gamma))$ is again a cubic binomial C. But the latter satisfies $\Gamma_C \subsetneq \Gamma$, so by the first observation C is expressible as a linear combination of elements of $\mathcal{I}_2(\text{Tor}(\Gamma))$. This proves that one cubic is sufficient, i.e. $c_\Gamma = 1$.

Next assume that Γ is hyperelliptic or $\Gamma \cong \Upsilon$. Using that $\sharp(\partial\Gamma \cap \mathbb{Z}^2) = 3$ we find that it is unimodularly equivalent to



where $r = \#(\Gamma \cap \mathbb{Z}^2) - 3$. One verifies that the irreducible binomials in $\mathcal{I}_3(\text{Tor}(\Gamma))$ involving $X_{-1,1}$ or $X_{0,-1}$ must involve both variables in the same monomial. This monomial is necessarily among

$$X_{-1,1}X_{0,-1}X_{i,0} \qquad i = 1, \ldots, r$$

and conversely, for each of these monomials there is a corresponding binomial $C_i \in \mathcal{I}_3(\text{Tor}(\Gamma))$. As before we find that the difference or sum of two cubic binomials involving the same monomial $X_{-1,1}X_{0,-1}X_{i,0}$ is a linear combination of elements of $\mathcal{I}_2(\text{Tor}(\Gamma))$. So we conclude that $\mathcal{I}(\text{Tor}(\Gamma))$ is generated by $\mathcal{I}_2(\text{Tor}(\Gamma)) \cup \{C_1, \ldots, C_r\}$. Because the quadratic binomials in $\mathcal{I}(\text{Tor}(\Gamma))$ do neither involve $X_{-1,1}$ nor $X_{0,-1}$, the latter r cubics are independent of $\mathcal{I}_2(\text{Tor}(\Gamma))$. □

We have included Magma code for computing such a minimal set of (binomial) generators; see our accompanying file `canonical.m`. As for the quadratic generators, this is done by naively gathering all relations of the form

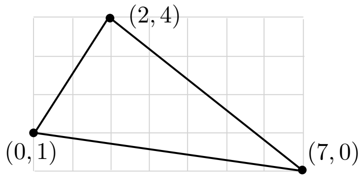
$$(i_1, j_1) + (i_2, j_2) = (i'_1, j'_1) + (i'_2, j'_2)$$

for $(i_1, j_1), (i_2, j_2), (i'_1, j'_1), (i'_2, j'_2) \in \Gamma \cap \mathbb{Z}^2$, and then finding a k-linearly independent subset of the set of corresponding binomials

$$X_{i_1,j_1}X_{i_2,j_2} - X_{i'_1,j'_1}X_{i'_2,j'_2}.$$

In the case where $\sharp\,(\partial\Gamma \cap \mathbb{Z}^2) = 3$, $\Gamma \ncong \Sigma$ and Γ is non-hyperelliptic, a single binomial of the form (1) with $(i_1, j_1), (i_2, j_2), (i_3, j_3)$ the vertices of Γ is added by exhaustive search. In the hyperelliptic case the explicit construction from the above proof is followed.

Example. The code below carries this out for the following lattice polygon (over $k = \overline{\mathbb{Q}}$):




```
> load "canonical.m"
Loading "canonical.m"
Loading "basic_commands.m"
> P := LatticePolytope([<0,1>,<7,0>,<2,4>]);
> time I := TorIdeal(P, Rationals());
Time: 0.110
```

This can be used as input to more advanced functions, such as the Magma intrinsic for computing the Betti diagram:

```
> BettiTable(GradedModule(Ideal(I)));
[
    [ 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
    [ 0, 55, 320, 891, 1424, 1470, 972, 315, 16, 0, 0, 0 ],
    [ 0, 1, 11, 71, 480, 1302, 1932, 1886, 1221, 485, 110, 11 ]
]
```

Remark. From the point of view of efficiency the above method leaves room for improvement. Especially the gathering of the quadratic generators can be done more systematically, for instance using Gröbner bases computations. These are implicitly invoked by the code below (a continuation of the above example):

```
> AA<x,y> := AffinePlane(Rationals());
> latticepoints := ConvexHull(P); N := #latticepoints;
> PP := ProjectiveSpace(Rationals(), N-1);
> phi_P := map< AA->PP | [x^p[1]*y^p[2] : p in latticepoints] >;
> time I := Ideal(Image(phi_P));
Time: 0.080
```

This produces a reduced Gröbner basis for $\mathrm{Tor}(\Gamma)$. In general this is *not* a minimal set of generators, but its quadratic elements do form a basis of $\mathcal{I}_2(\mathrm{Tor}(\Gamma))$, so that one can obtain a minimal set of generators by proceeding as above.

Remark. Up to unimodular equivalence, the only two-dimensional instances of $\Delta^{(1)}$ for which $\sharp(\partial\Delta^{(1)} \cap \mathbb{Z}^2) = 3$ are Σ and Υ. This can be shown using [7, Lem. 9-11]. Therefore, for the purposes of describing the canonical ideal of curves in toric surfaces, the above general treatment is more elaborate than needed. We have included it because we believe it to be of independent interest.

3 An explicit description of the canonical ideal

Let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a Laurent polynomial satisfying the sufficiently generic condition from the introduction (e.g. non-degeneracy). Assume that the corresponding curve C_f is non-hyperelliptic of genus $g \geq 3$, i.e. $\Delta^{(1)}$ is two-dimensional and $\sharp(\partial\Delta^{(1)} \cap \mathbb{Z}^2) \geq 3$. Let C_f^{can} be the canonical model of C_f obtained using $|K_\Delta|$.

We already know that $\mathcal{I}(C_f^{\text{can}})$ contains $\mathcal{I}(\text{Tor}(\Delta^{(1)}))$, and from the previous section we know how to find a minimal set of generators for the latter. In this section we describe which generators have to be added in order to obtain a minimal set of generators for $\mathcal{I}(C_f^{\text{can}})$. A priori, it is not entirely trivial that it suffices to merely *add* some generators, but note from the previous remark that $\text{Tor}(\Delta^{(1)})$ is almost always generated by quadrics, in which case this is clear. The only exception is when $\Delta^{(1)} \cong \Upsilon$, which corresponds to curves of genus 4, and is therefore well-understood.

Our main auxiliary tool is:

Theorem 3. *The equality*

$$\dim \mathcal{I}_d(C_f^{\text{can}}) - \dim \mathcal{I}_d(\text{Tor}(\Delta^{(1)})) = \sharp\left(\left((d-1)\Delta^{(1)}\right)^{(1)} \cap \mathbb{Z}^2\right)$$

holds for all integers $d \geq 2$.

Proof. From Lemma 1 it follows that

$$\dim \mathcal{I}_d(\text{Tor}(\Delta^{(1)})) = \binom{g+d-1}{d} - \sharp(d\Delta^{(1)} \cap \mathbb{Z}^2).$$

On the other hand, let $H(d)$ be the Hilbert function of the homogeneous coordinate ring of $C_f^{\text{can}} \subset \mathbb{P}^{g-1}$. Then $H(d) = (2g-2)d + (1-g) = (2d-1)(g-1)$ if $d \geq 2$ (see [6, Cor. 9.4]), hence

$$\dim \mathcal{I}_d(C) = \binom{g+d-1}{d} - (2d-1)(g-1).$$

So we are left with proving that

$$\sharp(d\Delta^{(1)} \cap \mathbb{Z}^2) - \sharp\left(\left((d-1)\Delta^{(1)}\right)^{(1)} \cap \mathbb{Z}^2\right) = (2d-1)(g-1).$$

For this, write $R^{(1)} = \sharp(\partial\Delta^{(1)} \cap \mathbb{Z}^2)$ and consider the Ehrhart polynomial

$$\mathrm{Ehr}_{\Delta^{(1)}}(k) = \mathrm{Vol}(\Delta^{(1)}) \cdot k^2 + \frac{R^{(1)}}{2} \cdot k + 1$$

of $\Delta^{(1)}$. Since $\sharp(k\Delta^{(1)} \cap \mathbb{Z}^2) = \mathrm{Ehr}_{\Delta^{(1)}}(k)$ and $\sharp\left(\partial\left(k\Delta^{(1)}\right) \cap \mathbb{Z}^2\right) = kR^{(1)}$ for all $k \in \mathbb{Z}_{\geq 1}$, we have that

$$\begin{aligned} &\sharp(d\Delta^{(1)} \cap \mathbb{Z}^2) - \sharp\left(\left((d-1)\Delta^{(1)}\right)^{(1)} \cap \mathbb{Z}^2\right) \\ &= \mathrm{Ehr}_{\Delta^{(1)}}(d) - \mathrm{Ehr}_{\Delta^{(1)}}(d-1) + \sharp\left(\partial\left((d-1)\Delta^{(1)}\right) \cap \mathbb{Z}^2\right) \\ &= (2d-1)\left(\mathrm{Vol}(\Delta^{(1)}) + \frac{R^{(1)}}{2}\right) \\ &= (2d-1)(g-1). \end{aligned}$$

This concludes the proof. □

Remark. Some readers may prefer the following cohomological proof of Theorem 3 (brief). Assume for ease of exposition that $\mathrm{Tor}(\Delta)$ is smooth; if not the argument below has to be preceded by a toric blow-up. Let D_{C_f} be a torus-invariant divisor on $\mathrm{Tor}(\Delta)$ that is linearly equivalent to C_f, let K be a torus-invariant canonical divisor on $\mathrm{Tor}(\Delta)$, and define $L = D_{C_f} + K$. When tensoring the exact sequence

$$0 \to \mathcal{O}_{\mathrm{Tor}(\Delta)}(-D_{C_f}) \to \mathcal{O}_{\mathrm{Tor}(\Delta)} \to \mathcal{O}_{C_f} \to 0$$

with $\mathcal{O}_{\mathrm{Tor}(\Delta)}(dL)$, taking cohomology and using the standard toric vanishing theorems for H^1 we get

$$0 \to H^0(\mathrm{Tor}(\Delta), (d-1)L + K) \to H^0(\mathrm{Tor}(\Delta), dL) \to H^0(C_f, dL|_{C_f}) \to 0.$$

The respective dimensions of these spaces are seen to be

$$\sharp\left(\left((d-1)\Delta^{(1)}\right)^{(1)} \cap \mathbb{Z}^2\right), \ \dim \frac{\mathcal{I}_d(\mathbb{P}^{g-1})}{\mathcal{I}_d(\mathrm{Tor}(\Delta^{(1)}))}, \ \text{and} \ \dim \frac{\mathcal{I}_d(\mathbb{P}^{g-1})}{\mathcal{I}_d(C_f^{\mathrm{can}})},$$

(indeed, by adjunction theory $L|_{C_f}$ is a canonical divisor on C_f), so that the theorem follows.

Now write

$$f = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} c_{i,j}x^iy^j \in k[x^{\pm 1}, y^{\pm 1}]$$

and define $\mathcal{W}_d = \left(\Delta^{(1)}\right)^\circ \cap \left(\frac{1}{d-1}\mathbb{Z}\right)^2$. Note that

$$\sharp\,\mathcal{W}_d = \sharp\left(\left((d-1)\Delta^{(1)}\right)^{(1)} \cap \mathbb{Z}^2\right).$$

To every $w \in \mathcal{W}_d$ we can associate a homogeneous degree d polynomial, as follows. For each $(i,j) \in \Delta \cap \mathbb{Z}^2$ there exist

$$v_{1,(i,j)}, \ldots, v_{d,(i,j)} \in \Delta^{(1)} \cap \mathbb{Z}^2$$

such that

$$(i,j) - w = (v_{1,(i,j)} - w) + \ldots + (v_{d,(i,j)} - w). \tag{2}$$

This follows from the inclusion $\left((d-1)\Delta^{(1)}\right)^{(1)} + \Delta \subset d\Delta^{(1)}$ and the normality of the polygon $\Delta^{(1)}$. The d-form

$$\mathcal{F}_{d,w} = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} c_{i,j}X_{v_{1,(i,j)}}\cdots X_{v_{d,(i,j)}}$$

is well-defined modulo the ideal of $\mathrm{Tor}(\Delta^{(1)})$. It clearly vanishes on $\varphi_{\Delta^{(1)}}(U_f)$, hence it is contained in the ideal of C_f^{can}.

The forms $\mathcal{F}_{d,w}$ with $w \in \mathcal{W}_d$ are k-linearly independent of each other and of the forms in $\mathcal{I}_d(\mathrm{Tor}(\Delta^{(1)}))$. Indeed, this holds because

$$\chi_d(\mathcal{F}_{d,w}) = (x,y)^{(d-1)w} \cdot f,$$

where χ_d is the vector space morphism from the proof of Lemma 1. Hence any linear combination in which the $\mathcal{F}_{d,w}$'s appear non-trivially is mapped to a non-zero multiple of f, and must therefore be non-zero itself. By Theorem 3, we can conclude that a basis for $\mathcal{I}_d(C_f^{\mathrm{can}})$ is obtained by adjoining $\{\mathcal{F}_{d,w}\}_{w\in\mathcal{W}_d}$ to a basis for $\mathcal{I}_d(\mathrm{Tor}(\Delta^{(1)}))$. In other words:

$$\mathcal{I}_d(C_f^{\mathrm{can}}) = \mathcal{I}_d(\mathrm{Tor}(\Delta^{(1)})) \oplus \langle \mathcal{F}_{d,w}\rangle_{w\in\mathcal{W}_d}. \tag{3}$$

We are now ready to prove our main theorem.

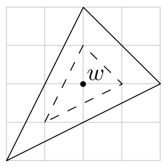
Theorem 4. *Let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a Laurent polynomial satisfying the sufficiently generic condition from the introduction (e.g. non-degeneracy). Assume that $\Delta^{(1)}$ is two-dimensional and let $g = \sharp(\Delta^{(1)}\cap\mathbb{Z}^2)$.*

- *If $\Delta^{(2)} \neq \emptyset$ and $\Delta^{(1)} \not\cong \Upsilon$, then a minimal set of generators for $\mathcal{I}(C_f^{\mathrm{can}})$ is given by a basis for $\mathcal{I}_2(\mathrm{Tor}(\Delta^{(1)}))$ and the quadrics $\{\mathcal{F}_{2,w}\}_{w \in \Delta^{(2)} \cap \mathbb{Z}^2}$.*
- *If $\Delta^{(1)} \cong \Upsilon$ then a minimal set of generators for $\mathcal{I}(C_f^{\mathrm{can}})$ is given by the cubic defining $\mathrm{Tor}(\Delta^{(1)}) \subset \mathbb{P}^3$ and the quadric $\mathcal{F}_{2,w}$ with $\Delta^{(2)} = \{w\}$.*
- *If $\Delta^{(1)} \cong \Sigma$ then a minimal set of generators for $\mathcal{I}(C_f^{\mathrm{can}})$ is given by the single quartic $\mathcal{F}_{4,w}$ with $\left(\Delta^{(1)}\right)^{\circ} \cap \left(\frac{1}{3}\mathbb{Z}\right)^2 = \{w\}$.*
- *If $\Delta^{(1)} \cong 2\Sigma$ then a minimal set of generators for $\mathcal{I}(C_f^{\mathrm{can}})$ is given by a basis for $\mathcal{I}_2(\mathrm{Tor}(\Delta^{(1)}))$ and the three cubics $\mathcal{F}_{3,w}, \mathcal{F}_{3,w'}, \mathcal{F}_{3,w''}$ with $\left(\Delta^{(1)}\right)^{\circ} \cap \left(\frac{1}{2}\mathbb{Z}\right)^2 = \{w, w', w''\}$.*
- *In the other cases a minimal set of generators for the ideal $\mathcal{I}(C_f^{\mathrm{can}})$ is given by a basis for $\mathcal{I}_2(\mathrm{Tor}(\Delta^{(1)}))$ and the $g-3$ cubics $\mathcal{F}_{3,w}$ with $w \in \left(\Delta^{(1)}\right)^{\circ} \cap \left(\frac{1}{2}\mathbb{Z}\right)^2$.*

Proof. From [4, Thm. 8.1], the assumptions $\Delta^{(2)} \neq \emptyset$ and $\Delta^{(1)} \not\cong \Upsilon$ imply that the Clifford index of C_f is at least 2. In this case Petri's theorem [14] guarantees that $\mathcal{I}(C_f^{\mathrm{can}})$ is generated by quadrics and the statement follows from (3).

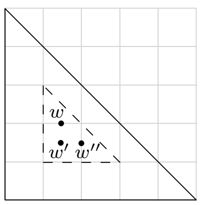
As for the other cases:

- If $\Delta^{(1)} \cong \Upsilon$, the claim follows by noting that $\mathrm{Tor}(\Upsilon)$ is cut out by the cubic $X_{-1,-1}X_{1,0}X_{0,1} - X_{0,0}^3$ and that a canonical curve of genus 4 is of degree 6, so that a single (necessarily unique) quadric suffices.



$\Delta^{(1)} = \Upsilon$

- The case $\Delta^{(1)} \cong \Sigma$ corresponds to smooth plane quartics and is obvious.
- If $\Delta^{(1)} \cong 2\Sigma$ then C_f is a smooth plane quintic. By Petri's theorem we know that $\mathcal{I}(C_f^{\mathrm{can}})$ is generated by quadrics and cubics. Since $\mathcal{I}(\mathrm{Tor}(\Delta^{(1)})$ is generated by quadrics, the statement follows from (3). (Note that $\mathrm{Tor}(\Delta^{(1)})$ is just the Veronese surface.)

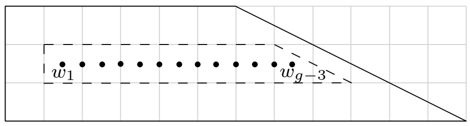


$\Delta^{(1)} = 2\Sigma$

- In the other cases C_f is a trigonal curve and $\sharp\left(\partial\Delta^{(1)} \cap \mathbb{Z}^2\right) \geq 4$, so that $\mathrm{Tor}(\Delta^{(1)})$ is generated by quadrics. By Petri's theorem we know that $\mathcal{I}(C_f^{\mathrm{can}})$ is generated by quadrics and cubics, so that the statement again follows from (3). (Note that $\mathrm{Tor}(\Delta^{(1)})$ is a rational normal surface scroll.) Remark that

$$\sharp\left(\left(\Delta^{(1)}\right)^\circ \cap \left(\frac{1}{2}\mathbb{Z}\right)^2\right) = \sharp\left(\left(2\Delta^{(1)}\right)^{(1)} \cap \mathbb{Z}^2\right) = g-3$$

by Pick's theorem.



This concludes the proof. □

We remark that in the last case of trigonal curves, the generators $\mathcal{F}_{3,w}$ are just the 'rolling factors' that were introduced by Reid; see [15]. For more general polygons, our forms $\mathcal{F}_{d,w}$ can be viewed as analogues of these, where the 'rolling' is done in two directions instead of one.

Theorem 4 immediately gives rise to an efficient algorithm for computing a minimal set of generators for the canonical ideal of C_f, for a given Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$ that is non-degenerate with respect to its Newton polygon $\Delta(f)$. As before we assume that

- $\sharp(\Delta(f)^{(1)} \cap \mathbb{Z}^2) \geq 3$, so that C_f is of genus $g \geq 3$, and
- $\Delta(f)^{(1)}$ is two-dimensional, so that C_f is non-hyperelliptic, or equivalently that its Clifford index is at least 1 (otherwise the canonical image is just a rational normal curve).

In case $\Delta(f)^{(1)} \cong \Sigma$ the output consists of a single quartic. If not, it consists of independent quadratic and cubic generators of the canonical ideal, i.e. $\binom{g-2}{2}$ quadrics and $g-3$ cubics in the case of Clifford index 1, and just $\binom{g-2}{2}$ quadrics in the case of Clifford index at least 2. Indeed, all one needs to do is adding the appropriate $\mathcal{F}_{d,w}$'s to a minimal set of generators for $\mathrm{Tor}(\Delta^{(1)})$. Finding these $\mathcal{F}_{d,w}$'s boils down to finding relations of the form (2), which can be done by exhaustive search. An implementation can be found in the Magma file `canonicalideal.m` that accompanies this paper. The function of interest is called `NondegIdeal()`.

Example. The following sample code computes the canonical ideal of a genus 14 curve in a fraction of a second:

```
> load "canonical.m"
Loading "canonical.m"
Loading "basic_commands.m"
> R<x,y> := PolynomialRing(Rationals(),2);
> f :=  13*x^6*y^5 - 6*x^6*y^4 + 2*x^3*y^5 + 4*x^3*y^4 + x^3 + 3*y^4;
> AA := AffineSpace(Rationals(),2);
> C := Curve(AA,f);
> Genus(C);
14
> time I := Ideal(NondegIdeal(f));
Time: 0.130
```

In sharp contrast, it takes the Magma intrinsic way over an hour.

```
> time I := Ideal(Image(CanonicalMap(C)));
Time: 5405.360
```

Note moreover that in the latter case, in general, the output does not consist of a minimal set of generators.

Remark. Here again, the method can be slightly improved by taking into account the corresponding remark from Section 2, i.e. by computing a set of generators for $\mathcal{I}(\mathrm{Tor}(\Delta^{(1)}))$ using Gröbner bases. It is also possible to do this at once for the entire ideal $\mathcal{I}(C_f^{\mathrm{can}})$, as below (continuation of the above example):

```
> latticepoints := ConvexHull(InnerPoints(NewtonPolytope(f)));
> g := #latticepoints;
> PP := ProjectiveSpace(Rationals(), g-1);
> phi_can := map< C->PP | [x^p[1]*y^p[2] : p in latticepoints] >;
> time I := Ideal(Image(phi_can));
Time: 0.370
```

This is already much faster than the Magma intrinsic, but slower than the previous method (the difference in timing increases as the genus grows). Note again that the output does not necessarily consist of a minimal set of generators.

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LINEAR PENCILS ENCODED IN THE NEWTON POLYGON

WOUTER CASTRYCK AND FILIP COOLS

Abstract. Let C be an algebraic curve defined by a sufficiently generic bivariate Laurent polynomial with given Newton polygon Δ. It is classical that the geometric genus of C equals the number of lattice points in the interior of Δ. In this paper we give similar combinatorial interpretations for the gonality, the Clifford index and the Clifford dimension, by removing a technical assumption from a recent result of Kawaguchi. More generally, the method shows that apart from certain well-understood exceptions, every base-point free pencil whose degree equals or slightly exceeds the gonality is combinatorial, in the sense that it corresponds to projecting C along a lattice direction. Along the way we prove various features of combinatorial pencils. For instance, we give an interpretation for the scrollar invariants associated to a combinatorial pencil, and show how one can tell whether the pencil is complete or not.

Among the applications, we find that every smooth projective curve admits at most one Weierstrass semi-group of embedding dimension 2, and that if a non-hyperelliptic smooth projective curve C of genus $g \geq 2$ can be embedded in the n^{th} Hirzebruch surface $\mathcal{H}_n$, then n is actually an invariant of C.

MSC2010: Primary 14H45, Secondary 14H51, 14M25

Contents

[1]Available at `http://users.ugent.be/~wcastryc/`

1. Introduction

Let k be an algebraically closed field of characteristic zero, let $\mathbb{T}^2 = (k^*)^2$ be the 2-dimensional torus over k, and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be an irreducible Laurent polynomial. Denote by $U(f)$ the curve in $\mathbb{T}^2$ defined by f. Let $\Delta(f) \subset \mathbb{R}^2$ be the Newton polygon of f, which we always assume to be two-dimensional. We say that f is *non-degenerate with respect to its Newton polygon* if for every face $\tau \subset \Delta(f)$ (including $\Delta(f)$ itself) the system

$$f_\tau = \frac{\partial f_\tau}{\partial x} = \frac{\partial f_\tau}{\partial y} = 0$$

has no solutions in $\mathbb{T}^2$. (Here f_τ is obtained from f by only considering those terms that are supported on τ.) For a two-dimensional lattice polygon $\Delta \subset \mathbb{R}^2$, we say that f is Δ-*non-degenerate* if it is non-degenerate with respect to its Newton polygon and $\Delta(f) = \Delta$. For Laurent polynomials that are supported on Δ, the condition of Δ-non-degeneracy is generically satisfied, in the sense that it is characterized by the non-vanishing of

$$\mathrm{Res}_\Delta \left(f, x\frac{\partial f}{\partial x}, y\frac{\partial f}{\partial y} \right)$$

(where Res_Δ is the sparse resultant; see [11, Prop. 1.2] and [23, Thm. 10.1.2] for an according discussion). An algebraic curve C/k is called Δ-*non-degenerate* if it is birationally equivalent to $U(f)$ for some Δ-non-degenerate Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$.

Remark. Sometimes in the existing literature a projectively embedded variety is called *non-degenerate* if it is not contained in a hyperplane. Our notion of non-degeneracy is unrelated to this.

It is well-known that if C is Δ-non-degenerate, then several of its geometric properties are encoded in the combinatorics of Δ. The most prominent example is that the geometric genus equals the number of lattice points in the interior of Δ [30, §4 Ass. 2]. The proof of this fact is briefly recalled at the beginning of Section 4, because it entails an explicit description of the canonical map that will play a role in Section 9. Other known examples are that one can tell from Δ whether C is hyperelliptic or not [31, Lem. 3.2.9], and whether it is trigonal or not [8, Lem. 3]. Recently, this was extended to arbitrary gonalities by Kawaguchi [29, Thm. 1.3] under the technical assumption that C is not birationally equivalent to a smooth plane projective curve.

In Section 6 we revisit Kawaguchi's proof, while making a more explicit connection with the language of Newton polygons and getting rid of the above technical assumption. Kawaguchi's method yields that apart from some well-understood exceptional instances of Δ, every gonality pencil on C is *combinatorial*, in the sense that it corresponds to a projection of the form $(x, y) \mapsto x^a y^b$ for coprime $a, b \in \mathbb{Z}$. In this case, the gonality is easily seen to equal the *lattice width* of Δ (this notion will be recalled in Section 5). This settles a conjecture by the current authors [8, Conj. 1], although most cases, including all lattice polygons whose number of interior lattice points is not of the form $(d-1)(d-2)/2$, were already covered by Kawaguchi's work.

In Section 7 we apply the same method to near-gonal pencils, i.e. base-point free linear systems of the form $g^1_{\gamma+1}$, where γ is the gonality. It again turns out that, apart from some reasonably well understood exceptions, every such pencil is combinatorial.

Then in Section 8, we prove that also the Clifford index and the Clifford dimension of C are fully determined by the combinatorics of Δ. This is again inspired by [29], but thanks to our coverage of the case of smooth projective plane curves (i.e., curves of Clifford dimension 2) we are able to fill in the missing spots. In particular, we obtain a purely combinatorial criterion for determining whether C is birationally equivalent to a smooth projective curve in $\mathbb{P}^2$ or not.

Note that, as an immediate corollary to all this, we obtain that the gonality, the Clifford index and the Clifford dimension do not depend on the specific choice of our Δ-non-degenerate curve C. This is an extension to arbitrary toric surfaces of a recent theorem by Lelli-Chiesa [34, Thm. 1.2] on families of curves on rational (e.g. toric) surfaces that carry an anticanonical pencil.

Next, in Section 9, we show that the scrollar invariants associated to a combinatorial pencil (which specialize to the classical Maroni invariants in the case of a g^1_3) have a natural combinatorial interpretation. The same interpretation allows one to decide whether a given combinatorial pencil is complete or not.

Finally, Section 10 discusses a number of applications. One potential use of our results is as a tool for constructing examples of curves having certain prescribed invariants (and for finding lower bounds on the dimension of the corresponding moduli space). Among the other byproducts we find that

- any curve (not necessarily non-degenerate) admits at most one Weierstrass semigroup of embedding dimension two,
- if C is a non-hyperelliptic smooth projective curve of genus $g \geq 2$ in the n^{th} Hirzebruch surface $\mathcal{H}_n$, then n is actually an invariant of C.

2. Notation, terminology and conventions

For lattice polygons $\Delta, \Delta' \subset \mathbb{R}^2$, we say that Δ is *equivalent* to Δ' (notation: $\Delta \cong \Delta'$) if Δ' is obtained from Δ through a unimodular transformation, i.e. through a transformation of the form

$$\upsilon : \mathbb{R}^2 \to \mathbb{R}^2 : \begin{pmatrix} i \\ j \end{pmatrix} \mapsto A \begin{pmatrix} i \\ j \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad A \in \mathrm{GL}_2(\mathbb{Z}),\ a_1, a_2 \in \mathbb{Z}.$$

If A can be taken the unit matrix, we sometimes write $\Delta \cong_t \Delta'$ to emphasize that Δ is obtained from Δ' through a translation. Note that if a Laurent polynomial

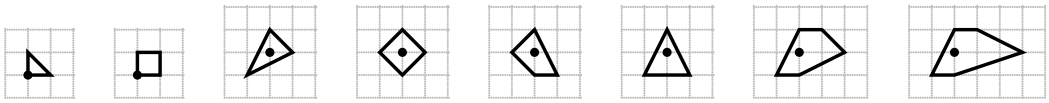
$$f = \sum_{(i,j) \in \Delta \cap \mathbb{Z}^2} c_{i,j} (x, y)^{(i,j)}$$

is Δ-non-degenerate (where $(x, y)^{(i,j)}$ means $x^i y^j$) and υ is a unimodular transformation, then

$$f^{\upsilon} = \sum_{(i,j) \in \Delta \cap \mathbb{Z}^2} c_{i,j} (x, y)^{\upsilon(i,j)}$$

is $\upsilon(\Delta)$-non-degenerate, and $U(f) \cong U(f^\upsilon)$. (Every unimodular transformation induces an automorphism of $\mathbb{T}^2$.)

It is convenient to introduce a special notation for certain recurring polygons:



Σ $\quad\square\quad$ Υ $\quad\Gamma_1^5\quad$ Γ_2^5 $\quad\Gamma_3^5\quad$ Γ^7 $\quad\Gamma^8.$

Here the bold-marked lattice point indicates the point $(0,0) \in \mathbb{R}^2$, although we are usually interested in lattice polygons up to equivalence only. Thus Σ is the standard simplex, and $d\Sigma$ (Minkowski multiple) is the Newton polygon of a generic degree d polynomial. If Δ is a two-dimensional lattice polygon, we denote by $\Delta^{(1)}$ the convex hull of its interior lattice points. The boundary of Δ is denoted by $\partial\Delta$.

Example. For $d \geq 3$ one has $(d\Sigma)^{(1)} \cong (d-3)\Sigma$. For $d \geq 1$ one has $(d\Upsilon)^{(1)} \cong (d-1)\Upsilon$.

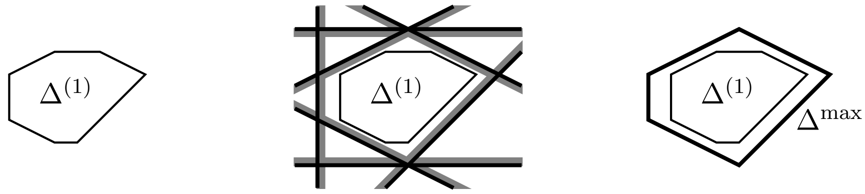
Remark. Occasionally, we will also apply the notation $\Delta^{(1)}$ to convex polygons Δ that are lower-dimensional and/or take vertices outside the lattice $\mathbb{Z}^2$. Here again we mean the convex hull of the lattice points in the interior of Δ, where the interior is understood to be empty in the lower-dimensional case.

If $\Delta^{(1)}$ is two-dimensional, then the set of lattice polygons Γ for which $\Gamma^{(1)} = \Delta^{(1)}$ admits a maximum with respect to inclusion [24, Lem. 9]. We denote this maximum by $\Delta^{\max}$. It can be characterized as follows. Write $\Delta^{(1)}$ as an intersection of half-spaces

$$\bigcap_{\ell=1}^{r} H_\ell, \qquad \text{with } H_\ell = \left\{ \, (i,j) \in \mathbb{R}^2 \, \middle| \, <(i,j), v_\ell> \, \geq -a_\ell \, \right\},$$

where $< \cdot , \cdot >$ denotes the standard inner product on $\mathbb{R}^2$ and $v_1, \ldots, v_r$ are primitive inward pointing normal vectors of the edges of $\Delta^{(1)}$. Then

$$\Delta^{\max} = \bigcap_{\ell=1}^{r} H_\ell^{(-1)}, \qquad \text{where } H_\ell^{(-1)} = \left\{ \, (i,j) \in \mathbb{R}^2 \, \middle| \, <(i,j), v_\ell> \, \geq -a_\ell - 1 \, \right\}.$$



When applying this construction to an *arbitrary* two-dimensional lattice polygon Γ, one ends up with a polygon $\Gamma^{(-1)}$ that is a lattice polygon if and only if $\Gamma = \Delta^{(1)}$ for some lattice polygon Δ; see [24, Lem. 10] for a proof of this convenient criterion. (If we call

$$\left\{ \, (i,j) \in \mathbb{R}^2 \, \middle| \, <(i,j), v_\ell> \, = -a_\ell - 1 \, \right\}$$

the *outward shift* of the edge corresponding to index ℓ, then a necessary, but generally insufficient condition for $\Gamma^{(-1)}$ to be a lattice polygon is that the outward shifts of any pair of adjacent edges intersect in a lattice point [24, Lem. 9].)

Remark. The criterion yields a method for algorithmically enumerating lattice polygons, as elaborated in [6] and [31, §4.4]. We will use this in the proofs of Theorem 6.1 and Theorem 7.2.

In Lemma 4.1 we will give a geometric interpretation of $\Delta^{\max}$.

We use the notation $\mathcal{Z}(\cdot)$ to denote the algebraic set associated to an ideal, and $\mathcal{I}(\cdot)$ to denote the ideal of an algebraic set.

A *curve* is always assumed irreducible, but we don't a priori require it to be complete and/or smooth. By the *genus* of a curve C, which we denote by $g(C)$, we mean its geometric genus unless otherwise stated. The gonality of C will be denoted by $\gamma(C)$. A *canonical curve* is a curve that arises as the canonical image of a non-hyperelliptic smooth projective curve of genus $g \geq 3$. A *canonical model* of a curve C is a canonical curve that is birationally equivalent to C.

3. Divisors on toric surfaces

This section gathers some facts on divisors on toric surfaces. Our primary objective is to fix notation and terminology, but we also group some statements that are somewhat sprawled across our main references [15, 22].

To a two-dimensional lattice polygon Δ we can associate a projective toric surface $\mathrm{Tor}(\Delta)$ over k, in two ways:

- One can consider the (inner) normal fan Σ_Δ, and let $\mathrm{Tor}(\Delta) = \mathrm{Tor}(\Sigma_\Delta)$ be the toric surface associated to it.
- One can define $\mathrm{Tor}(\Delta)$ as the Zariski closure of the image of

$$\varphi_\Delta : \mathbb{T}^2 \hookrightarrow \mathbb{P}^N : (x, y) \mapsto \left(x^i y^j\right)_{(i,j)\in\Delta\cap\mathbb{Z}^2} \tag{1}$$

(where $N = \sharp(\Delta \cap \mathbb{Z}^2) - 1$). Explicit equations for $\mathrm{Tor}(\Delta)$ can be read from the combinatorics of Δ, as follows. To each $(i, j) \in \Delta \cap \mathbb{Z}^2$ one associates a variable $X_{i,j}$. Then the ideal of $\mathrm{Tor}(\Delta)$ is generated by the binomials

$$\prod_{\ell=1}^{n} X_{i_\ell,j_\ell} - \prod_{\ell=1}^{n} X_{i'_\ell,j'_\ell} \qquad \text{for which} \qquad \sum_{\ell=1}^{n} (i_\ell, j_\ell) = \sum_{\ell=1}^{n} (i'_\ell, j'_\ell)$$

(apply [15, Prop. 2.1.4.(b,d)] to $\Delta \times \{1\} \subset \mathbb{R}^3$). A result of Koelman states that one can restrict to $n \in \{2, 3\}$, and to $n = 2$ as soon as $\partial\Delta \cap \mathbb{Z}^2 \geq 4$, see [32, 41].

Examples.
- $\mathrm{Tor}(\Upsilon) = \mathcal{Z}(X_{0,0}^3 - X_{-1,-1}X_{1,0}X_{0,1}) \subset \mathbb{P}^3$,
- $\mathrm{Tor}(\square) = \mathcal{Z}(X_{0,0}X_{1,1} - X_{1,0}X_{0,1}) \subset \mathbb{P}^3$,
- $\mathrm{Tor}(\Gamma_1^5) = \mathcal{Z}(X_{0,0}^2 - X_{-1,0}X_{1,0}, X_{0,0}^2 - X_{0,-1}X_{0,1}) \subset \mathbb{P}^4$.

Both constructions give rise to the same geometric object by [15, Cor. 2.2.19.(b)] and the series of equivalences in the proof of [15, Prop. 6.1.10]. But the second construction comes along with an embedding $\psi : \mathrm{Tor}(\Delta) \hookrightarrow \mathbb{P}^N$, i.e. a very ample invertible sheaf $\psi^* \mathcal{O}_{\mathbb{P}^N}(1)$ on $\mathrm{Tor}(\Delta)$. Note that every complete fan in $\mathbb{R}^2$ arises as some Σ_Δ.

The self-action of $\mathbb{T}^2$ yields an action of $\mathbb{T}^2$ on $\varphi_\Delta(\mathbb{T}^2)$ that naturally extends to an action on all of $\mathrm{Tor}(\Delta)$. The orbits of the latter are in a dimension-preserving one-to-one correspondence with the faces of Δ. Denote the Zariski closures of the one-dimensional orbits (corresponding to the edges of Δ and to the rays of Σ_Δ) by $D_1, \ldots, D_r$. A Weil divisor that arises as a $\mathbb{Z}$-linear combination of the D_ℓ's is called *torus-invariant*. An important example is $K = -\sum_\ell D_\ell$, which is a canonical divisor; see [15, Thm. 8.2.3] or [22, §4.4]. To a torus-invariant Weil divisor $D = \sum_\ell a_\ell D_\ell$ one can associate the polygon

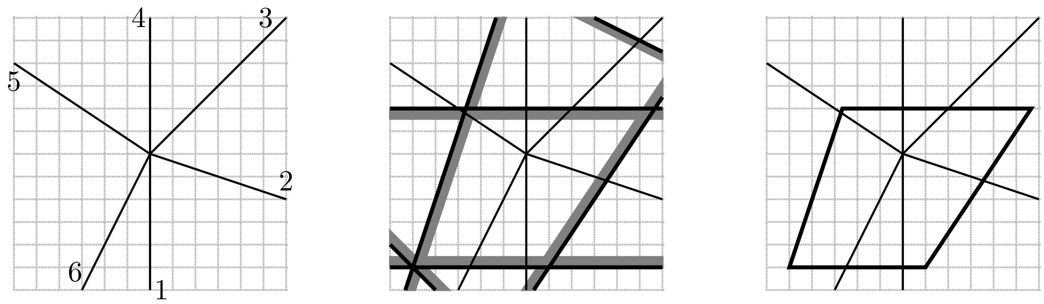
$$\Delta_D = \bigcap_{\ell=1}^{r} H_\ell, \qquad \text{with } H_\ell = \left\{ \, (i,j) \in \mathbb{R}^2 \, \middle| \, <(i,j), v_\ell> \, \geq -a_\ell \, \right\}, \tag{2}$$

where v_ℓ is the primitive generator of the corresponding ray in Σ_Δ. It can be proven [15, Prop. 4.3.3] that

$$H^0(\mathrm{Tor}(\Delta), D) = \{ \, f \in k(x,y)^* \mid \mathrm{div}(f) + D \geq 0 \, \} \cup \{0\} = \left\langle x^i y^j \right\rangle_{(i,j) \in \Delta_D \cap \mathbb{Z}^2}$$

(here $\langle \cdot \rangle$ denotes the k-linear span; we view x and y as functions on $\mathrm{Tor}(\Delta)$ through φ_Δ).

Example. Let Σ be the fan given on the left, where the rays are enumerated as indicated.



Let $D = 2D_1 + D_2 + 5D_3 + 5D_4 + D_5 + 3D_6$. Then the corresponding half-planes are drawn in the middle, and Δ_D is depicted on the right. Remark that Δ_D is not a lattice polygon.

One can also show that D is Cartier if and only if the apex of $H_\ell \cap H_m$ is an element of $\mathbb{Z}^2$ for each pair ℓ, m corresponding to adjacent edges of Δ [15, Thm. 4.2.8.(a,c)]. If moreover every such apex is a vertex of Δ_D then D is called *convex* (in particular, if D is a convex torus-invariant Cartier divisor then Δ_D is a lattice polygon). If this gives a bijective apex-vertex correspondence then D is called *strictly convex*.

A torus-invariant Cartier divisor D is convex iff it is nef iff it is base-point free (i.e. $\mathcal{O}_{\mathrm{Tor}(\Delta)}(D)$ is generated by global sections) by [15, Thm. 6.1.7 and Thm. 6.3.12]. It is strictly convex iff it is ample iff it is very ample [15, Thm. 6.1.14]. If D is convex then all higher cohomology spaces are trivial [15, Thm. 9.2.3]. If D_1 and D_2 are convex torus-invariant Cartier divisors, then their intersection number can be interpreted in terms of a mixed volume:

$$D_1 \cdot D_2 = \mathrm{MV}(\Delta_{D_1}, \Delta_{D_2}) = \mathrm{Vol}(\Delta_{D_1} + \Delta_{D_2}) - \mathrm{Vol}(\Delta_{D_1}) - \mathrm{Vol}(\Delta_{D_2}),$$

where $\mathrm{Vol}(\cdot)$ denotes the Euclidean area, and the addition of polygons is in Minkowski's sense (see [22, §5.3, first Cor.] and the reasoning preceding [22, §5.5, (2)]). This is an instance of the Bernstein–Khovanskii–Koushnirenko (BKK) theorem.

Every Weil divisor on $\mathrm{Tor}(\Delta)$ is linearly equivalent to a torus-invariant Weil divisor and two equivalent torus-invariant Weil divisors D_1 and D_2 differ by some $\mathrm{div}(x^i y^j)$ [15, Thm. 4.1.3], so that the corresponding polygons Δ_{D_1} and Δ_{D_2} are translates of each other. Therefore, if one is willing to work modulo $\cong_t$, one can associate a polygon Δ_D to *any* Weil divisor D (and a polygon $\Delta_{\mathcal{L}}$ to any invertible sheaf $\mathcal{L}$). All definitions and statements above carry through.

Example. We have $\Delta_{\psi^*\mathcal{O}_{\mathbb{P}^N}(1)} \cong_t \Delta$. Indeed, using (2) it is straightforward to construct a convex torus-invariant Cartier divisor D_Δ such that $\Delta_{D_\Delta} = \Delta$. But then the global sections of $\mathcal{O}_{\mathrm{Tor}(\Delta)}(D_\Delta)$ and $\psi^*\mathcal{O}_{\mathbb{P}^N}(1)$ are naturally identified. Since both sheaves are globally generated, we find that $\mathcal{O}_{\mathrm{Tor}(\Delta)}(D_\Delta) \cong \psi^*\mathcal{O}_{\mathbb{P}^N}(1)$, from which the claim follows.

Example. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be an irreducible Laurent polynomial and let $U(f)$ be the curve in $\mathbb{T}^2$ that it cuts out. Let Δ be any two-dimensional lattice polygon and let C be the Zariski closure of $\varphi_\Delta(U(f))$ in $\mathrm{Tor}(\Delta)$. Let $P(f, \Delta)$ be the smallest convex polygon such that

- $\Delta(f) \subset P(f, \Delta)$, and
- all edges of $P(f, \Delta)$ are parallel to an edge of Δ.

We claim that $\Delta_C \cong_t P(f, \Delta)$. Indeed, consider the torus-invariant Weil divisor $D_C = C - \mathrm{div}(f)$, so that we can assume that $\Delta_C = \Delta_{D_C}$. Then $f \in H^0(\mathrm{Tor}(\Delta), D_C)$ and therefore Δ_C must contain the support of f. Moreover, as we are working on $\mathrm{Tor}(\Delta)$, every edge of Δ_C must be parallel to an edge of Δ. Each such edge must meet at least one point of the support of f, because otherwise the pole order of f at the corresponding torus-invariant prime divisor would be too large [15, Prop. 4.1.1]. So Δ_C must be the tightest fit, which is precisely $P(f, \Delta)$.

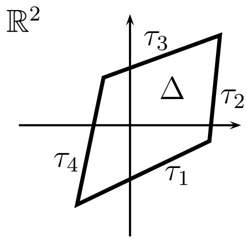
4. Non-degenerate curves as smooth curves on toric surfaces

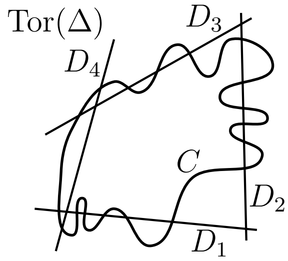
We show how non-degenerate Laurent polynomials naturally give rise to smooth curves in toric surfaces, and discuss how the non-degeneracy condition can be relaxed slightly. Much of the material below can be found (possibly in disguised terms) in [3, 10, 15]. On the other hand, Lemmata 4.1–4.4 seem genuinely new.

Non-degenerate curves. Let Δ be a two-dimensional lattice polygon and consider a Δ-non-degenerate Laurent polynomial

$$f = \sum_{(i,j) \in \Delta \cap \mathbb{Z}^2} c_{i,j} x^i y^j \in k[x^{\pm 1}, y^{\pm 1}].$$

Let $C \subset \mathrm{Tor}(\Delta)$ be the Zariski closure of $\varphi_\Delta(U(f))$. From the non-degeneracy of f one sees that C cuts out a smooth codimension 1 subscheme in every $\mathbb{T}^2$-orbit of $\mathrm{Tor}(\Delta)$. Because $\mathrm{Tor}(\Delta)$ is normal [15, Thm. 3.1.5], this is equivalent to saying that C is a smooth curve not containing any of the zero-dimensional toric orbits and intersecting the one-dimensional orbits transversally.





Note that C is just the hyperplane section

$$\sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} c_{i,j}X_{i,j} \;=\; 0.$$

Therefore $\mathcal{O}_{\mathrm{Tor}(\Delta)}(C) \cong \psi^*\mathcal{O}_{\mathbb{P}^N}(1)$. In particular, C is a strictly convex Cartier divisor and $\Delta_C \cong_t \Delta$.

Toric surfaces are Cohen-Macaulay [15, Thm. 9.2.9] and therefore enjoy a nice adjunction theory, which we will use in the following form. Let D_C be a torus-invariant divisor that is linearly equivalent to C; for instance one may take $D_C = C - \mathrm{div}(f)$. Then there is a canonical divisor K_C on C along with an exact sequence

$$0 \to \mathcal{O}_{\mathrm{Tor}(\Delta)}(K) \to \mathcal{O}_{\mathrm{Tor}(\Delta)}(D_C + K) \to \mathcal{O}_C(K_C) \to 0 \tag{3}$$

of morphisms of sheaves of $\mathcal{O}_{\mathrm{Tor}(\Delta)}$-modules; locally the maps are given by $\cdot f$ and restriction to C, respectively.

The existence of such an exact sequence is (in far greater generality) well-known to specialists in birational geometry; for example, this is essentially covered by [33, Prop. 5.73]. However we could not find a ready-to-use statement in the literature, so let us include the following flexible argument, which was explained to us by Karl Schwede. Consider the short exact sequence

$$0 \to \mathcal{O}_{\mathrm{Tor}(\Delta)}(-D_C) \xrightarrow{\cdot f} \mathcal{O}_{\mathrm{Tor}(\Delta)} \to \mathcal{O}_C \to 0$$

and note that $\mathcal{O}_{\mathrm{Tor}(\Delta)}(K)$ is a so-called dualizing sheaf for $\mathrm{Tor}(\Delta)$; see [22, §4.4]. We apply the sheafy $\mathcal{H}om(\cdot, \mathcal{O}_{\mathrm{Tor}(\Delta)}(K))$-functor to form a long exact sequence

$$0 \to \mathcal{H}om(\mathcal{O}_C, \mathcal{O}_{\mathrm{Tor}(\Delta)}(K)) \to \mathcal{H}om(\mathcal{O}_{\mathrm{Tor}(\Delta)}, \mathcal{O}_{\mathrm{Tor}(\Delta)}(K)) \to \mathcal{H}om(\mathcal{O}_{\mathrm{Tor}(\Delta)}(-D_C), \mathcal{O}_{\mathrm{Tor}(\Delta)}(K))$$
$$\to \mathcal{E}xt^1(\mathcal{O}_C, \mathcal{O}_{\mathrm{Tor}(\Delta)}(K)) \to \mathcal{E}xt^1(\mathcal{O}_{\mathrm{Tor}(\Delta)}, \mathcal{O}_{\mathrm{Tor}(\Delta)}(K)).$$

The first term vanishes because $\mathcal{O}_C$ is torsion while $\mathcal{O}_{\mathrm{Tor}(\Delta)}(K)$ is not. The last term vanishes by [27, III.Prop. 6.3(b)]. Finally because $\mathrm{Tor}(\Delta)$ is Cohen-Macaulay, by [39, Thm. 2.12(1)] the fourth term is a dualizing sheaf for C. This is just $\mathcal{O}_C(K_C)$ and (3) follows.

Now note that $\Delta_K = \emptyset$, so that $H^0(\mathrm{Tor}(\Delta), K) = 0$. Also $H^1(\mathrm{Tor}(\Delta), K) = 0$, because by toric Serre duality [15, Thm. 9.2.10] the left-hand side is isomorphic to $H^1(\mathrm{Tor}(\Delta), 0)$, which vanishes by Demazure's theorem [15, Thm. 9.2.3]. Thus by taking the cohomology of (3) one finds that the restriction map

$$H^0(\mathrm{Tor}(\Delta), D_C + K) \to H^0(C, K_C) \tag{4}$$

is an isomorphism. Since the polygon associated to $D_C + K$ equals $\Delta^{(1)}$, we recover the well-known fact that $g(C) = \sharp(\Delta^{(1)} \cap \mathbb{Z}^2)$. In fact, the isomorphism (4) also shows that

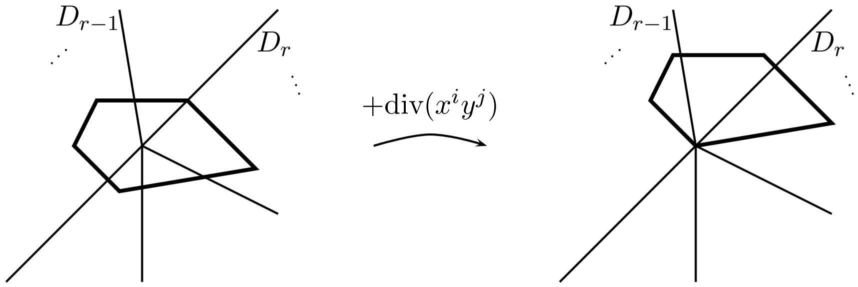
$$\varphi_{\Delta^{(1)}}|_{U(f)} = \kappa \circ \varphi_\Delta|_{U(f)}, \tag{5}$$

where $\kappa : C \to \mathbb{P}^{g(C)-1}$ is a canonical morphism. This seems less readily known, and will play an important role in Section 9. Using that the canonical image is rational iff C is hyperelliptic, this observation implies the aforementioned fact that C is hyperelliptic iff the interior lattice points of Δ are collinear; see [31, Lem. 3.2.9] or [8, Lem. 2] for more details. If C is non-hyperelliptic (i.e. $\Delta^{(1)}$ is two-dimensional) it follows that the canonical image $\kappa(C)$ lies in $\mathrm{Tor}(\Delta^{(1)}) \subset \mathbb{P}^{g(C)-1}$.

Remark. If C is an arbitrary (possibly singular, possibly non-Cartier) complete curve on $\mathrm{Tor}(\Delta)$ then the above adjunction process remains valid: one can still pick a torus-invariant divisor D_C that is equivalent to C, say with polygon Δ_C (not necessarily a lattice polygon!), and one will still find that the restriction map (4) is an isomorphism. When interpreting the outcome, some prudence is needed:

- In the non-Cartier case, note that in general $\Delta_C^{(1)}$ is *not* the polygon associated to $D_C + K$, which is the polygon obtained from Δ_C by shifting the edges inwards: this could result in a polygon having vertices outside the lattice. But the lattice points of both polygons are the same, so in the smooth case it remains justified to say that $g(C) = \sharp(\Delta_C^{(1)} \cap \mathbb{Z}^2)$.
- In the singular case we find that $\sharp(\Delta_C^{(1)} \cap \mathbb{Z}^2)$ is the *arithmetic* genus of C, rather than its geometric genus.

We note that classical adjunction theory, as elaborated in most textbooks, requires the ambient surface to be smooth. Even though $\mathrm{Tor}(\Delta)$ need not be smooth, it is possible to prove the genus formula $g(C) = \sharp(\Delta^{(1)} \cap \mathbb{Z}^2)$ in this way, by first resolving the singularities using a toric blow-up. This is the approach that is followed in [15, §10.5], for instance. We will briefly sketch this blow-up process and show that it does not affect the combinatorics of Δ, because for the application of Serrano's Theorem 6.4 later on, we *will* need that the ambient toric surface is smooth. (Serrano's theorem plays the key role in the proofs of Theorems 6.1 and 7.2.) So pick a subdivision Σ' of Σ_Δ such that the induced birational morphism $\mu : \mathrm{Tor}(\Sigma') \to \mathrm{Tor}(\Sigma_\Delta)$ is a resolution of singularities [15, Thm. 10.1.10]. Let C' be the strict transform of C under μ. By non-degeneracy C' does not meet the exceptional locus of μ, so $C' = \mu^* C \cong C$. Note that C' is again Cartier and convex, although not strictly convex (unless the subdivision is trivial). It moreover remains true that $\Delta_{C'} \cong_t \Delta$. To prove this, we can suppose that Σ' is obtained from Σ_Δ by inserting a single ray σ' (the general case then follows by repeating the argument). Let $D_1, \ldots, D_{r-2}$ be the torus-invariant prime divisors on $\mathrm{Tor}(\Sigma_\Delta)$ corresponding to the rays of Σ_Δ that are non-adjacent to σ', and let $D_1', D_2', \ldots, D_{r-2}'$ be the according torus-invariant prime divisors on $\mathrm{Tor}(\Sigma')$. Then $D_i' = \mu^* D_i$ for all $i = 1, \ldots, r-2$ (since D_i' does not meet the exceptional locus of μ). Now by adding a divisor of the form $\mathrm{div}(x^i y^j)$ if needed, we see that C is linearly equivalent to a torus-invariant Weil divisor of the form $\sum_{\ell=1}^{r-2} a_\ell D_\ell$.



But then $C' = \mu^*C \sim \sum_{\ell=1}^{r-2} a_\ell \mu^* D_\ell = \sum_{\ell=1}^{r-2} a_\ell D'_\ell$, from which it follows that $\Delta_{C'} \cong_t \Delta_C \cong_t \Delta$.

Δ-toric curves. We now present a (slight) relaxation of the non-degeneracy condition. Let Δ be a two-dimensional lattice polygon. We say that an irreducible Laurent polynomial $f \in k[x^{\pm1}, y^{\pm1}]$ is Δ-*toric* if

(i) $\Delta(f) \subset \Delta$,
(ii) $\Delta(f)$ contains at least one point of every edge of Δ, i.e. $P(f, \Delta) = \Delta$, and
(iii) the Zariski closure C of $\varphi_\Delta(U(f))$ is a smooth curve in $\mathrm{Tor}(\Delta)$.

The condition that $P(f, \Delta) = \Delta$ ensures that C again arises as a hyperplane section of $\mathrm{Tor}(\Delta)$. We therefore still find that $\Delta_C \cong_t \Delta$. All other conclusions of the preceding section remain valid, except for the part on resolutions of singularities, where we add the assumption that Σ' does not subdivide any of the smooth cones of Σ_Δ. Indeed, if it would, then this could affect Δ_C. But since in practice there is no need for subdividing smooth cones, this is not an issue. We also still obtain that $g(C) = \sharp(\Delta^{(1)} \cap \mathbb{Z}^2)$ and that there is a canonical map $\kappa : C \to \mathbb{P}^{g(C)-1}$ satisfying (5). Remark that H. Baker's bound [4] implies $g(C) \leq \sharp\left(\Delta(f)^{(1)} \cap \mathbb{Z}^2\right)$, which together with $\Delta(f) \subset \Delta$ yields $\Delta(f)^{(1)} = \Delta^{(1)}$, a fact which can also be proved directly by making a local analysis at the zero-dimensional $\mathbb{T}^2$-orbits of $\mathrm{Tor}(\Delta)$.

Geometrically, the only difference with Δ-non-degeneracy is that we allow C to contain some of the non-singular zero-dimensional orbits, or to be tangent to some of the one-dimensional orbits. It cannot pass through any of the singular zero-dimensional orbits however: otherwise C would be singular as well.

A curve C/k is called Δ-*toric* if it is birationally equivalent to $U(f)$ for a Δ-toric Laurent polynomial f. This notion captures all smooth projective curves on toric surfaces, as we will prove in Lemma 4.2 below (while this is not true for non-degenerate curves: see Lemma 4.4).

Remark. In the definition of being Δ-toric, condition (iii) can be replaced by requiring that

(iii') $g(C) = \sharp(\Delta^{(1)} \cap \mathbb{Z}^2)$.

Indeed, in this case C is automatically smooth, because by adjunction theory $\sharp(\Delta^{(1)} \cap \mathbb{Z}^2)$ equals the arithmetic genus, which in the case of singular curves is always strictly less than the geometric genus [27, IV.Ex. 1.8]. Recall that (iii') also implies $\Delta(f)^{(1)} = \Delta^{(1)}$

by Baker's bound, which in turn implies (ii) as soon as $\Delta^{(1)} \neq \emptyset$.

Here is a geometric interpretation for the polygon $\Delta^{\max}$ from Section 2.

Lemma 4.1. *Let Δ be a lattice polygon and assume that $\Delta^{(1)}$ is two-dimensional. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be Δ-toric and let C be the Zariski closure of $\varphi_\Delta(U(f))$ in $\mathrm{Tor}(\Delta)$. Let κ be as in (5), so that $\kappa(C)$ can be viewed as a curve in the toric surface $\mathrm{Tor}(\Delta^{(1)})$. Then $\Delta_{\kappa(C)} \cong_t \Delta^{\max}$.*

Proof. We see from (5) that $\kappa(C)$ is the Zariski closure of $\varphi_{\Delta^{(1)}}(U(f))$ in $\mathrm{Tor}(\Delta^{(1)})$. From the remark concluding Section 3 it follows that $\Delta_{\kappa(C)}$ is equivalent to $P(f, \Delta^{(1)})$, the tightest polygon containing $\Delta(f)$ all of whose edges are parallel to an edge of $\Delta^{(1)}$. But this polygon is clearly $\Delta^{\max} = \Delta^{(1)(-1)}$. ■

We now show that all smooth curves on toric surfaces are Δ-toric, for an appropriate instance of Δ.

Lemma 4.2. *Let C be a non-rational smooth projective curve on a toric surface, and let*

$$\tilde{\Delta}_C = \mathrm{conv}(\Delta_C \cap \mathbb{Z}^2).$$

Then C is $\tilde{\Delta}_C$-toric.

Note that if Δ_C is a lattice polygon (i.e. if C is Cartier) then $\tilde{\Delta}_C = \Delta_C$. The non-rationality condition is not really a restriction: all smooth rational curves are isomorphic to $\mathbb{P}^1$, hence Σ-toric.

Proof. Let X be our toric surface, containing the torus $\mathbb{T}^2$ as an open subset. Since C is non-rational, it is non-torus-invariant. So $C \cap \mathbb{T}^2$ is defined by an irreducible Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$. The torus-invariant divisor $D_C = C - \mathrm{div}(f)$ is equivalent to C, so that we can assume that Δ_C is the polygon associated to D_C. Because $f \in H^0(X, D_C)$ we see that f is supported on Δ_C, and because $\Delta(f)$ is a lattice polygon we even have that

$$\Delta(f) \subset \tilde{\Delta}_C \subset \Delta_C$$

and in particular that

$$\sharp(\Delta(f)^{(1)} \cap \mathbb{Z}^2) \leq \sharp(\tilde{\Delta}_C^{(1)} \cap \mathbb{Z}^2) \leq \sharp(\Delta_C^{(1)} \cap \mathbb{Z}^2). \tag{6}$$

By adjunction theory the genus of C equals $\sharp(\Delta_C^{(1)} \cap \mathbb{Z}^2)$. On the other hand by Baker's bound it is at most $\sharp(\Delta(f)^{(1)} \cap \mathbb{Z}^2)$. Thus the inequalities in (6) are equalities, and in particular the genus of C also equals $\sharp(\tilde{\Delta}_C^{(1)} \cap \mathbb{Z}^2)$. In other words, with respect to the lattice polygon $\tilde{\Delta}_C$, our polynomial f satisfies condition (iii') mentioned above, and therefore it is $\tilde{\Delta}_C$-toric. ■

From the proof we see that C is in fact also $\Delta(f)$-toric, but we chose to provide a polygon that depends on the divisor class of C only (up to translation). As an immediate corollary to the previous lemmata and their proofs, we obtain:

Lemma 4.3. *Let Δ be a lattice polygon and assume that $\Delta^{(1)}$ is two-dimensional. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be Δ-toric. Then f is also $\Delta^{\max}$-toric.*

This lemma will play an important role in the proofs of Theorems 6.1 and 7.2. It is in the same vein as Kawaguchi's notion of *relative minimality* [29, Def. 3.9], and can be proven more directly, by noting that Δ is obtained from $\Delta^{\max}$ by clipping off a number of vertices, without affecting the interior. From Pick's theorem it follows that such a vertex is necessarily smooth, i.e. that the primitive normal vectors of its adjacent edges form a basis of $\mathbb{Z}^2$. Then locally around the corresponding zero-dimensional orbit, $\text{Tor}(\Delta^{\max})$ looks like $\mathbb{A}^2$ with C passing smoothly through the origin. The smoothness of C outside these zero-dimensional orbits then just follows from the fact that f is Δ-toric.

Clearly every Δ-non-degenerate curve is Δ-toric. The converse implication may fail:

Lemma 4.4. *There exist instances of two-dimensional lattice polygons Δ, along with Δ-toric curves that are not Δ-non-degenerate. More precisely, let*

$$f = 1 + x^5 + y^2 + x^2y^3 \in k[x^{\pm 1}, y^{\pm 1}] \quad \textit{and} \quad \Delta = \text{conv}\{(0,0),(5,0),(2,3),(0,3)\}.$$

Then f is Δ-toric, but $U(f)$ is not Δ-non-degenerate, that is, it is not birationally equivalent to $U(f')$ for some Δ-non-degenerate Laurent polynomial $f' \in k[x^{\pm 1}, y^{\pm 1}]$

Proof. Our proof uses the theory of trigonal curves. We need the following facts. If C/k is a trigonal curve of genus $g \geq 5$, then the intersection of all quadrics containing its canonical model $C_{\text{can}} \subset \mathbb{P}^{g-1}$ is a rational normal surface scroll S spanned by two rational normal curves R_1 and R_2 of respective degrees e_1 and e_2, where $e_1 \leq e_2$. These numbers are uniquely determined and are called the Maroni invariants of C_{can}. See [40, (4.11)] for a proof, and [26, Ex. 8.17] and Section 9 for more background on this terminology. For our needs it is important that if $e_1 < e_2$ then R_1 is uniquely determined by S [26, Prop. 8.20(b)]. It follows that in the case where $e_1 < e_2$, the number of points at which C_{can} is tangent to R_1 is an invariant of C, which we denote by t_C.

Now the reader can verify that f is indeed Δ-toric, i.e. the Zariski closure C of $\varphi_\Delta(U(f))$ is a smooth curve in $\text{Tor}(\Delta)$. Note that C is a trigonal curve of genus 5, since it is non-hyperelliptic by [31, Lem. 3.2.9] and the map $U(f) \to \mathbb{T}^1 : (x, y) \mapsto x$ is of degree 3. Let C_{can} be the canonical model obtained by taking the Zariski closure of $\varphi_{\Delta^{(1)}}(U(f))$ inside $\text{Tor}(\Delta^{(1)}) \subset \mathbb{P}^4$. Since the latter surface is generated by quadrics, it must be our rational normal scroll S. The scrollar structure can easily be made explicit in this case. In particular, one verifies that $e_1 = 1$ and $e_2 = 2$, and that the line R_1 is the torus-invariant prime divisor of $\text{Tor}(\Delta^{(1)})$ corresponding to the top edge of $\Delta^{(1)}$. Now remark that $\Sigma_\Delta = \Sigma_{\Delta^{(1)}}$, so we have a natural isomorphism $\mu : \text{Tor}(\Delta^{(1)}) \to \text{Tor}(\Delta)$, which is compatible with the respective embeddings of $\mathbb{T}^2$ in $\text{Tor}(\Delta^{(1)})$ and $\text{Tor}(\Delta)$, i.e. $\varphi_\Delta = \mu \circ \varphi_{\Delta^{(1)}}$. In particular $\mu(C_{\text{can}}) = C$, and because μ behaves well with respect to the toric orbits we find that t_C can be interpreted as the number of points at which C is tangent to the torus-invariant prime divisor of $\text{Tor}(\Delta)$ corresponding to the top edge of Δ. Using this, one easily checks that $t_C = 1$. On the other hand, the same reasoning shows that if $U(f)$ were Δ-non-degenerate, then t_C would be 0. ■

Remarks.

- It is not possible to construct similar counterexamples from arbitrary two-dimensional lattice polygons. For instance, let $\Delta = d\Sigma$ for some integer $d \geq 1$, so that $\mathrm{Tor}(\Delta) \cong \mathbb{P}^2$. Then every Δ-toric curve is Δ-non-degenerate. Indeed, using an automorphism of $\mathbb{P}^2$, every smooth projective plane curve can be positioned in such a way that it does not contain any of the coordinate points, and such that it intersects the coordinate axes transversally.
- In all theorems and lemmata appearing in Sections 6 to 9 of this paper (which contain our main results), the notions of being Δ-non-degenerate and Δ-toric are interchangeable, i.e. only the property of being Δ-toric is used in the proofs. For instance:

  Corollary 6.2. *Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$. Then the gonality of $U(f)$ equals* $\mathrm{lw}(\Delta^{(1)}) + 2$, *unless $\Delta^{(1)} \cong \Upsilon$ (i.e. $\Delta \cong 2\Upsilon$), in which case it equals* 3.

  from Section 6 can be replaced by the slightly stronger statement that the gonality of a Δ-toric curve equals $\mathrm{lw}(\Delta^{(1)}) + 2$, unless $\Delta^{(1)} \cong \Upsilon$, in which case it equals 3. We have chosen to state our main results in a toric-geometry-free language, however.

5. Lattice directions and combinatorial pencils

Lattice directions. A *lattice direction* is just a primitive element of $\mathbb{Z}^2$. For a non-empty lattice polygon Δ and a lattice direction $v = (a, b)$, the *width of Δ with respect to v* is the minimal d for which there exists an $m \in \mathbb{Z}$ such that Δ is contained in the strip

$$m \leq aY - bX \leq m + d.$$

Note that $w(\Delta, v) = w(\Delta, -v)$. If $w(\Delta, v) = d$, we will sometimes say that v *computes* d. It is convenient to define $w(\emptyset, v) = -1$. (This notion appeared in [35, Def. 5] where it is called the *viewangle width.*)

Example. The width of $d\Sigma$ with respect to $(1, 1)$ is $2d$, while its width with respect to $(1, -1)$ is d.

Lemma 5.1. *If Δ is two-dimensional, then for a given $d \in \mathbb{Z}_{\geq 0}$, the number of lattice directions computing d is finite.*

Proof. It suffices to prove that for each d, the number of lattice directions v for which $w(\Delta, v) \leq d$ is finite. Since Δ is two-dimensional we may assume that it contains the standard simplex Σ (see e.g. [5, Prop. 1.2.4.(b)], although this easily follows from Pick's theorem), so that $w(\Sigma, v) \leq w(\Delta, v)$ for every v. Thus it suffices to prove that for each d, the number of lattice directions v for which $w(\Sigma, v) \leq d$ is finite. But this is straightforward. ∎

Assume that $w(\Delta, v) = d \geq 2$. Write $v = (a, b)$ and assume that Δ is contained in the strip $m \leq aY - bX \leq m + d$. Then we define the *width invariants of* Δ *with respect to* v as the tuple

$$E(\Delta, v) \;=\; (E_\ell)_{\ell=1,\ldots,d-1}$$

where

$$E_\ell = \sharp \left\{ \, (i,j) \in \Delta^{(1)} \cap \mathbb{Z}^2 \, \middle| \, aj - bi = m + \ell \, \right\} - 1.$$

(The reason for the -1 term will become clear in Section 9.)

Example. Let $v = (1, 0)$ and $d \in \mathbb{Z}_{\geq 2}$. Then $w(d\Sigma, v) = d$ and $E(d\Sigma, v) = (d-3, d-4, d-5, \ldots, 1, 0, -1)$.

The *lattice width of* Δ is

$$\mathrm{lw}(\Delta) = \min_v w(\Delta, v).$$

Equivalently, $\mathrm{lw}(\Delta)$ is the minimal d such that Δ is unimodularly equivalent to a lattice polygon that is contained in a horizontal strip of height d; for the latter, two lattice directions computing the lattice width are $(\pm 1, 0)$. If a lattice direction computes the lattice width, we call it a *lattice width direction for* Δ.

Example. Let $d \in \mathbb{Z}_{\geq 0}$. Then $\mathrm{lw}(d\Sigma) = d$. Indeed, clearly $\mathrm{lw}(d\Sigma) \leq d$, while $\mathrm{lw}(\Delta) \geq d$ follows from the fact that every edge of $d\Sigma$ contains $d + 1$ lattice points.

A convenient tool for computing $\mathrm{lw}(\Delta)$ is given by (i) from Lemma 5.2 below, which gathers some useful facts about the lattice width:

Lemma 5.2. *Let* Δ *be a two-dimensional lattice polygon.*

(i) *One has* $\mathrm{lw}(\Delta^{(1)}) = \mathrm{lw}(\Delta) - 2$*, unless* $\Delta \cong d\Sigma$ *for some* $d \geq 2$ *in which case* $\mathrm{lw}(\Delta^{(1)}) = \mathrm{lw}(\Delta) - 3 = d - 3$.
(ii) *A lattice width direction for* Δ *is also a lattice width direction for* $\Delta^{(1)}$*; if moreover* $\Delta^{(1)} \neq \emptyset$ *and* $\Delta^{(1)} \not\cong (d-3)\Sigma$ *for any* $d \geq 3$*, then the converse holds as well.*
(iii) *Assume* $\mathrm{lw}(\Delta) \geq 2$ *and* $\Delta \not\cong d\Sigma$ *for any* $d \geq 2$*. Then the width invariants of* Δ *with respect to a lattice width direction are all non-negative.*
(iv) *There are at most* 4 *pairs* $\pm v$ *of lattice width directions for* Δ*; the bound is met if and only if* $\Delta \cong d\Gamma_1^5$ *for some* $d \in \mathbb{Z}_{\geq 1}$.
(v) *If* v_1, v_2 *are lattice width directions for* Δ*, then* $|\det(v_1, v_2)| \leq 2$*; if equality holds then* $\Delta \cong d\Gamma_1^5$ *for some* $d \in \mathbb{Z}_{\geq 1}$.
(vi) *One has* $\mathrm{lw}(\Delta)^2 \leq \frac{8}{3}\mathrm{Vol}(\Delta)$*, and equality holds if and only if* $\Delta \cong d\Upsilon$ *for some* $d \geq 1$.

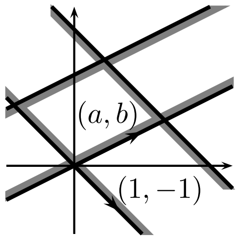
Proof. For (i) and (ii) see [8, Thm. 4] or [35, Thm. 13].

Claim (iii) can be proved by induction, as follows. Let $v = (a, b)$ be a lattice width direction for Δ and let m be such that Δ is contained in the strip $m \leq aY - bX \leq m + \mathrm{lw}(\Delta)$. We have to show that for each $\ell = 1, \ldots, \mathrm{lw}(\Delta) - 1$ there exists an $(i, j) \in \Delta^{(1)} \cap \mathbb{Z}^2$ such that $aj - bi = m + \ell$. Now (i) implies that this must be the case for $\ell = 1$ and $\ell = \mathrm{lw}(\Delta) - 1$, while from (ii) it follows that v is also a lattice width direction for $\Delta^{(1)}$.

So the claim follows by recursively applying it to $\Delta^{(1)}$; if at some point $\Delta^{(1)}$ happens to be of the form $d\Sigma$ for some $d \geq 2$ then the claim can be verified explicitly.

For (iv) see [17].

To prove (v), let v_1, v_2 be lattice width directions for Δ for which $|\det(v_1, v_2)| > 1$. Using a unimodular transformation if needed, we can assume that $v_1 = (1, -1)$ and $v_2 = (a, b)$ with $a, b > 0$, and that Δ is contained in the strips $0 \leq Y + X \leq \mathrm{lw}(\Delta)$ and $0 \leq aY - bX \leq \mathrm{lw}(\Delta)$.



Thus Δ is contained in the parallelogram

$$\mathrm{conv}\left\{(0,0), \left(\frac{a\mathrm{lw}(\Delta)}{a+b}, \frac{b\mathrm{lw}(\Delta)}{a+b}\right), \left(\frac{(a-1)\mathrm{lw}(\Delta)}{a+b}, \frac{(b+1)\mathrm{lw}(\Delta)}{a+b}\right), \left(-\frac{\mathrm{lw}(\Delta)}{a+b}, \frac{\mathrm{lw}(\Delta)}{a+b}\right)\right\}.$$

The horizontal width of this parallelogram equals $(a+1)\mathrm{lw}(\Delta)/(a+b)$, while its vertical width equals $(b+1)\mathrm{lw}(\Delta)/(a+b)$. By the definition of $\mathrm{lw}(\Delta)$ it follows that $a = b = 1$, so that $|\det(v_1, v_2)| = 2$. Moreover, these four vertices must be actual vertices of Δ. In particular, they must be contained in $\mathbb{Z}^2$, from which one sees that $\mathrm{lw}(\Delta)$ is even, and $\Delta \cong \frac{\mathrm{lw}(\Delta)}{2}\Gamma_1^5$.

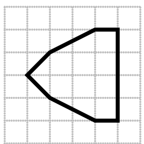
For (vi) see [21]. ■

Note that Lemma 5.2.(v) implies that if Δ has two linearly independent lattice width directions v_1, v_2, then there is a unimodular transformation mapping Δ inside $\mathrm{lw}(\Delta)\square$. (Indeed, if $|\det(v_1, v_2)| = 1$ then one can take a $\mathbb{Z}$-linear transformation mapping v_1 to $(1, 0)$ and v_2 to $(0, 1)$, and compose it with an appropriate translation; if $|\det(v_1, v_2)| \neq 1$ then Δ is of the form $d\Gamma_1^5$, and the statement can be verified explicitly.) In particular, it follows that

$$\sharp(\Delta^{(1)} \cap \mathbb{Z}^2) \leq (\mathrm{lw}(\Delta) - 1)^2 \tag{7}$$

in this case.

Example. Let Δ be the lattice polygon



for which $\mathrm{lw}(\Delta) = 4$ (as can be seen by applying Lemma 5.2.(i)). Clearly $\pm(1, 0)$ and $\pm(0, 1)$ are lattice directions computing $\mathrm{lw}(\Delta)$. It is also immediate that $\Delta \not\cong d\Gamma_1^5$ for any $d \in \mathbb{Z}_{\geq 1}$, so that by Lemma 5.2.(iv) the number of pairs $\pm v$ of lattice width directions is either two or three. But three is impossible, because by Lemma 5.2.(v) the third pair would need to be among $\pm(1, 1), \pm(1, -1)$, both of which correspond to widths that strictly exceed 4.

Remark. Lemma 5.2.(iii) can also be proven using the well-known geometric fact that gonality pencils are always complete, by combining Theorem 6.1 and Corollary 9.4 below.

Combinatorial pencils. Returning to the geometric side, let Δ be a two-dimensional lattice polygon, let $v = (a, b)$ be a lattice direction, let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a Δ-non-degenerate or Δ-toric Laurent polynomial, and let $C \subset \mathrm{Tor}(\Delta)$ be the corresponding smooth projective curve (i.e. the Zariski closure of $\varphi_\Delta(U(f))$, where φ_Δ is as in Section 3). We associate to v a linear pencil g_v on C as follows. For each $c \in \mathbb{P}^1 = \mathbb{T}^1 \cup \{0, \infty\}$ we have a function $x^a y^b - c$ on $\mathrm{Tor}(\Delta)$ (where $x^a y^b - \infty$ should be read as $x^{-a} y^{-b}$) whose zero divisor $\mathcal{F}_c$ cuts out a divisor D_c on C. Then

$$g_v = \{D_c\}_{\substack{c \in \mathbb{P}^1 \\ \mathcal{F}_c \neq C}}.$$

In other words this is the trace on C of the linear system $\{\mathcal{F}_c\}_c$, in the sense of [27, p. 158].

Remark. The subscript $\mathcal{F}_c \neq C$ is usually superfluous, but it could indeed happen that $\mathcal{F}_c = C$ for some c. Example: $f = x + 1$, $\Delta = \Sigma$, $v = (1, 0)$ and $c = -1$. In this example g_v is just the linear system consisting of one base point, namely the point $(0 : 1 : 0) \in \mathbb{P}^2 = \mathrm{Tor}(\Sigma)$ (note the abuse of language here). By genus considerations $\mathcal{F}_c = C$ can occur only if $\Delta^{(1)} = \emptyset$. Since from Section 6 on, all our theorems and lemmata that involve combinatorial pencils exclude the case $\Delta^{(1)} = \emptyset$ (for other reasons), the reader can in fact ignore the possibility of this event.

There are several ways of seeing that g_v has degree $w(\Delta, v)$. One approach, the details of which we leave to the reader, uses the BKK theorem along with the fact that $w(\Delta, v) = \mathrm{MV}(\Delta, \mathrm{conv}\{0, v\})$. We will give a more elementary argument that gives finer information.

Lemma 5.3. *The pencil g_v is of degree $w(\Delta, v)$. More precisely, it splits into a base-point free part of degree $w(\Delta(f), v)$ and a fixed part of degree $w(\Delta, v) - w(\Delta(f), v)$ that is supported on at most two points. In particular, if*

- *f is Δ-non-degenerate, or*
- *v is a lattice width direction, $\mathrm{lw}(\Delta) \geq 2$, and $\Delta \not\cong d\Sigma$ for all $d \geq 2$,*

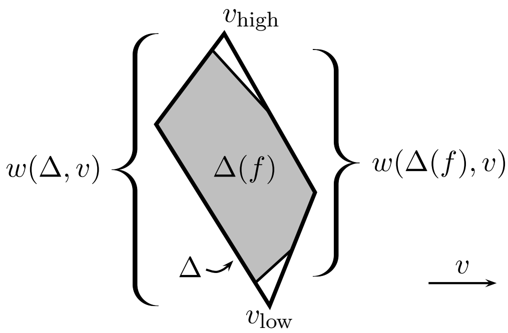
then g_v is base-point free.

Proof. We will assume that $\mathcal{F}_c \neq C$ for all $c \in \mathbb{P}^1$, and leave the details of the other case to the reader. Then the rational map $U(f) \to \mathbb{T}^1 : (x, y) \mapsto x^a y^b$ extends to a degree $w(\Delta(f), v)$ morphism $C \to \mathbb{P}^1$. Its fibers determine a base-point free pencil that necessarily matches with the base-point free part of g_v.

As for the fixed part, suppose that Δ and $\Delta(f)$ are contained in the strips

$$m \leq aY - bX \leq m + w(\Delta, v) \qquad \text{and} \qquad m_f \leq aY - bX \leq m_f + w(\Delta(f), v),$$

respectively. If Δ has a unique vertex v_{low} lying on the line $m = aY - bX$, the corresponding zero-dimensional orbit $O(v_{\mathrm{low}})$ is contained in every divisor $\mathcal{F}_c$. Similarly, if there is a unique vertex v_{top} on the line $m + w(\Delta, v) = aY - bX$ then $O(v_{\mathrm{top}})$ is contained in every $\mathcal{F}_c$. All other points of $\operatorname{Tor}(\Delta)$ lie on a unique $\mathcal{F}_c$. This means that the fixed part of g_v is supported on at most these two points.



Now if there is indeed a unique lower-most vertex v_{low} of Δ, then a local analysis shows that a generic $\mathcal{F}_c$ intersects C in $O(v_{\text{low}})$ with multiplicity $m_f - m$, or in other words, the order of the fixed part of g_v at $O(v_{\text{low}})$ equals $m_f - m$. If there is no unique lower-most vertex, then necessarily $m = m_f$, otherwise there would be an edge of Δ not supporting any term of f, contradicting that f is Δ-toric. A similar analysis at the top then yields that the fixed part of g_v has degree $w(\Delta, v) - w(\Delta(f), v)$.

For the last claim it suffices to note that if f is Δ-non-degenerate then $\Delta(f) = \Delta$, and therefore $w(\Delta(f), v) = w(\Delta, f)$, while if v is a lattice width direction, $\operatorname{lw}(\Delta) \geq 2$, and $\Delta \not\cong d\Sigma$ for all $d \geq 2$, then

$$w(\Delta(f), v) = w(\Delta(f)^{(1)}, v) + 2 = w(\Delta^{(1)}, v) + 2 = w(\Delta, v),$$

where the outer equalities follow from Lemma 5.2.(iii). ■

A pencil on C that arises as g_v for some lattice direction v is called *combinatorial.* The number of combinatorial pencils is countable; in fact, by Lemma 5.1 there is only a finite number of combinatorial pencils of each given degree. Note that the minimal degree of a combinatorial pencil is $\operatorname{lw}(\Delta)$, from which we immediately find that the gonality $\gamma(C)$ of C is bounded from above by $\operatorname{lw}(\Delta)$. As we will see in Section 6, equality typically holds.

The correspondence between pairs $\pm v$ of lattice directions and combinatorial pencils is usually 1-to-1, but there are counterexamples. For instance, let Δ be a primitive lattice parallelogram, i.e. a polygon of the form $\operatorname{conv}\{(0,0), v_1, v_2, v_1 + v_2\}$ for linearly independent primitive vectors $v_1, v_2 \in \mathbb{Z}^2$. Then

$$w(\Delta, v_1) = w(\Delta, v_2) = |\det(v_1, v_2)|.$$

Assume that f is supported on the vertices of Δ only, i.e.

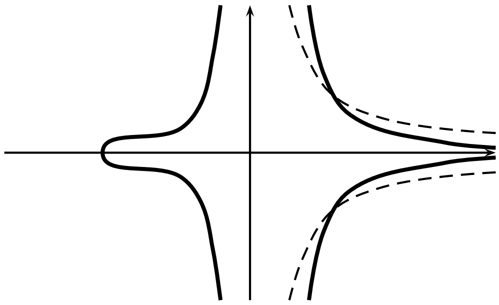
$$f = c_{0,0} + c_{1,0}(x,y)^{v_1} + c_{0,1}(x,y)^{v_2} + c_{1,1}(x,y)^{v_1+v_2},$$

and that the coefficients $c_{i,j}$ are sufficiently generic. Then the fiber of $U(f) \to \mathbb{T}^1 : (x,y) \mapsto (x,y)^{v_1}$ above a point $c \in \mathbb{T}^1 \setminus \{-c_{0,1}c_{1,1}^{-1}\}$ matches with the fiber of $U(f) \to \mathbb{T}^1 : (x,y) \mapsto (x,y)^{v_2}$ above

$$-\frac{c_{0,0} + cc_{1,0}}{c_{0,1} + cc_{1,1}}.$$

From this it follows that $g_{v_1} = g_{v_2}$. The same construction works for the primitive lattice triangle $\Delta = \text{conv}\{(0,0), v_1, v_2\}$.

Example with $v_1 = (3,2)$ and $v_2 = (1,0)$. The graph below shows the (real affine) zero locus of $f = 3 + x + x^3y^2 - x^4y^2 \in \mathbb{C}[x,y]$.



The dashed line cuts out a typical fiber of x^3y^2, which is also a fiber of x.

Clearly, by degree considerations, $w(\Delta, v_1) \neq w(\Delta, v_2)$ is a sufficient condition for v_1, v_2 to give rise to different combinatorial pencils. Another sufficient condition is as follows.

Lemma 5.4. *Let Δ be a two-dimensional lattice polygon and let f be a Δ-toric Laurent polynomial. Let $v_1 \neq \pm v_2$ be lattice directions and let g_{v_1} and g_{v_2} be the corresponding combinatorial pencils. If*

$$w(\Delta^{(1)}, v_1) \; > \; |\det(v_1, v_2)| \, - \, 2 \tag{8}$$

then $g_{v_1} \neq g_{v_2}$.

Proof. Fibers of

$$\mathbb{T}^2 \to \mathbb{T}^1 : (x,y) \mapsto (x,y)^{v_1} \qquad \text{and} \qquad \mathbb{T}^2 \to \mathbb{T}^1 : (x,y) \mapsto (x,y)^{v_2}$$

intersect each other in at most $|\det(v_1, v_2)|$ points. Now because $\Delta(f)^{(1)} = \Delta^{(1)}$, condition (8) implies that $w(\Delta(f), v_1) > |\det(v_1, v_2)|$. We conclude that a general fiber of

$$U(f) \to \mathbb{T}^2 : (x,y) \mapsto (x,y)^{v_1}$$

cannot be contained in a fiber of $U(f) \to \mathbb{T}^2 : (x,y) \mapsto (x,y)^{v_2}$. The lemma follows. ■

In the case of lattice width directions we obtain:

Corollary 5.5. *Let Δ be a two-dimensional lattice polygon and assume that $\sharp(\Delta^{(1)} \cap \mathbb{Z}^2) > 1$. Let f be a Δ-toric Laurent polynomial, let $v_1 \neq \pm v_2$ be lattice width directions, and let g_{v_1} and g_{v_2} be the according combinatorial pencils. Then $g_{v_1} \neq g_{v_2}$.*

Proof. If $|\det(v_1, v_2)| = 1$ then condition (8) amounts to $\Delta^{(1)} \neq \emptyset$, which is clearly the case. So by Lemma 5.2.(iv) it remains to analyze the case where $|\det(v_1, v_2)| = 2$ and $\Delta \cong d\Gamma_1^5$ for some integer $d \geq 2$ (indeed, $d = 1$ is excluded in the statement of the corollary). But here

$$w(\Delta^{(1)}, v_1) = w((d-1)\Gamma_1^5, v_1) \geq \text{lw}((d-1)\Gamma_1^5) = 2(d-1),$$

so again condition (8) is satisfied. ■

6. Gonality

We can now state our refinement of Kawaguchi's theorem [29, Thm. 1.3].

Theorem 6.1. *Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$. Suppose that $\Delta^{(1)}$ is not equivalent to any of the following:*

$$\emptyset, \quad (d-3)\Sigma \text{ (for some integer } d \geq 3), \quad \Upsilon, \quad 2\Upsilon, \quad \Gamma_1^5, \quad \Gamma_2^5, \quad \Gamma_3^5. \tag{9}$$

Then every gonality pencil on (the smooth projective model of) $U(f)$ is combinatorial.

Remark. In case $\Delta^{(1)}$ is among $\Upsilon, 2\Upsilon, \Gamma_1^5, \Gamma_2^5, \Gamma_3^5$, there is only a single corresponding Δ, namely, $2\Upsilon, 3\Upsilon, 2\Gamma_1^5, 2\Gamma_2^5$ and $2\Gamma_3^5$, respectively.

Before we proceed to the proof of Theorem 6.1, let us discuss some corollaries. From the énoncé it follows that if $\Delta^{(1)}$ is non-equivalent to any of the polygons listed in (9), then the gonality of $U(f)$ equals the lattice width of Δ. Thus by Lemma 5.2.(i), if $\Delta^{(1)}$ is not among the polygons listed in (9) then the gonality of $U(f)$ equals $\mathrm{lw}(\Delta^{(1)}) + 2$. The other instances of $\Delta^{(1)}$ can be analyzed case by case:

- If $\Delta^{(1)} = \emptyset$ then $U(f)$ is rational, hence of gonality 1.
- If $\Delta^{(1)} \cong (d-3)\Sigma$ then $U(f)$ is birationally equivalent to a smooth projective plane curve of degree d, hence of gonality $d-1$ by a result of Namba [38] (a proof can also be found in [43, Prop. 3.13]).
- If $\Delta^{(1)} \cong \Upsilon$ then $U(f)$ is a non-hyperelliptic curve of genus 4, hence of gonality 3.
- If $\Delta^{(1)} \cong 2\Upsilon$ then $U(f)$ is birationally equivalent to a smooth intersection of two cubics in $\mathbb{P}^3$, hence of gonality 6 by a result of Martens (see [8, Thm. 9] for more details).
- If $\Delta^{(1)} \cong \Gamma_i^5$ $(i = 1, 2, 3)$ then $U(f)$ is a non-hyperelliptic, non-trigonal curve of genus 5 by [8, Lem. 3], hence of gonality 4.

We conclude:

Corollary 6.2. *Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$. Then the gonality of $U(f)$ equals $\mathrm{lw}(\Delta^{(1)})+2$, unless $\Delta^{(1)} \cong \Upsilon$ (i.e. $\Delta \cong 2\Upsilon$), in which case it equals 3.*

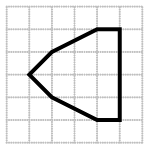
Unless $\Delta^{(1)} \cong \Upsilon$ we can even read off the number of gonality pencils:

Corollary 6.3. *Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$.*

- *If $\Delta^{(1)} = \emptyset$ then there is a unique gonality pencil.*
- *If $\Delta^{(1)} \cong \Upsilon$ then the number of gonality pencils is at most 2.*
- *If $\Delta^{(1)} \cong (d-3)\Sigma$ for some $d \geq 3$, or if $\Delta^{(1)} \cong 2\Upsilon, \Gamma_1^5, \Gamma_2^5, \Gamma_3^5$, then there are infinitely many gonality pencils.*
- *In all other cases the number of gonality pencils equals the number of lattice width directions. In particular, the number of gonality pencils is at most 4, and the bound is met iff $\Delta^{(1)} \cong d\Gamma_1^5$ for some $d \geq 2$.*

PROOF. The first three claims follow from the considerations above: rational curves have a unique gonality pencil, non-hyperelliptic genus 4 curves carry one or two g^1_3's [27, Ex. IV.5.5.2], smooth plane degree d curves admit infinitely many g^1_{d-1}'s [43, Prop. 3.13], smooth intersections of cubics in $\mathbb{P}^3$ carry infinitely many g^1_6's [20, pp. 174-175], and non-hyperelliptic, non-trigonal curves of genus 5 have infinitely many g^1_4's [1, Ex. IV.F]. The last claim follows from Theorem 6.1, combined with Lemma 5.2.(iv) and Corollary 5.5. ■

Example (revisited, see Section 5). Let Δ be the lattice polygon



and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a Δ-non-degenerate (or Δ-toric) Laurent polynomial. Then $U(f)$ is a 4-gonal genus 7 curve carrying exactly two g^1_4's.

Remarks.

- Corollary 6.2 implies a conjecture by the current authors [8, Conj. 1]. It does not imply the corresponding conjecture on metric graphs [8, Conj. 3 + Err.].
- Corollary 6.2 also implies that if $\Delta^{(1)} \cong \Upsilon$ (i.e. if $\Delta \cong 2\Upsilon$), then a combinatorial gonality pencil cannot exist. The same conclusion holds for $\Delta \cong d\Sigma$ for $d \geq 2$. In all other cases, there exists at least one combinatorial gonality pencil.
- In case $\Delta^{(1)} \cong \Upsilon$ then both one and two g^1_3's can occur, depending on whether the quadric on which the curve canonically embeds is singular or not [27, Ex. IV.5.5.2]; see [7, Thm. 4] for an explicit description of this quadric.
- Let k' be an arbitrary field of characteristic 0 with algebraic closure k. Suppose that $f \in k'[x^{\pm 1}, y^{\pm 1}]$ is non-degenerate with respect to its Newton polygon when considered as a Laurent polynomial over k. If $\Delta(f) \not\cong 2\Upsilon, d\Sigma$ then the above remark implies that $\gamma(U(f)) = \gamma_{k'}(U(f))$, where $\gamma_{k'}(U(f))$ is the minimal degree of a k'-*rational* map to $\mathbb{P}^1$. If $\Delta(f) \cong 2\Upsilon$ or $\Delta(f) \cong d\Sigma$ for some $d \geq 2$ then this may not be true. (Example: $x^2 + y^2 + 1 \in \mathbb{R}[x, y]$.)
- By letting $k' = \mathbb{C}(\!(t)\!)$, the preceding remark lends prudent support in favor of a conjecture by M. Baker (stating that the gonality of a graph equals the gonality of the associated metric graph [2, Conj. 3.14]) in the case of graphs associated to regular subdivisions of lattice polygons [8, Err. §1].
- If $\Delta^{(1)}$ is neither among the polygons excluded in Theorem 6.1, nor of the form $d\Gamma^5_1$ for some $d \geq 2$, then Lemma 5.2.(v) implies that two different gonality pencils on $U(f)$ are always *independent*, in the sense that they span a base-point free linear system of rank 2, defining a morphism $U(f) \to \mathbb{P}^2$ that induces a birational equivalence between $U(f)$ and its image. (For general lattice directions $v_1 \neq \pm v_2$ the morphism $C \to \mathbb{P}^2$ defined by g_{v_1} and g_{v_2} induces a degree $|\det(v_1, v_2)|$ cover.) See [12, (1.2)] for more background on this terminology.

We now give a proof of Theorem 6.1. We recall that the main ideas are taken from Kawaguchi [29], but that our proof covers the case where $U(f)$ is birationally equivalent to a smooth projective plane curve (the key ingredient here being the block of text surrounding (13) below).

PROOF OF THEOREM 6.1. Let $g = \sharp(\Delta^{(1)} \cap \mathbb{Z}^2)$ be the geometric genus of $U(f)$. Note that our assumptions imply $g \geq 2$. Recall that $\mathrm{lw}(\Delta^{(1)}) = 0$ if and only if $U(f)$ is hyperelliptic. By Lemma 5.2.(i) this holds if and only if $\mathrm{lw}(\Delta) = 2$, hence a g^1_2 can be computed by projection along some lattice direction. Since the g^1_2 of a hyperelliptic curve is unique, Theorem 6.1 follows in this case. Thus we may assume that $\Delta^{(1)}$ is two-dimensional and that $U(f)$ is of gonality $\gamma \geq 3$.

From Lemma 4.3 we know that f is $\Delta^{\max}$-toric, i.e. $\varphi_{\Delta^{\max}}(U(f))$ completes to a smooth projective curve $C \subset \mathrm{Tor}(\Delta^{\max})$. Let Σ' be a minimal smooth subdivision of $\Sigma_{\Delta^{\max}}$ and let $\mu : \mathrm{Tor}(\Sigma') \to \mathrm{Tor}(\Sigma_{\Delta^{\max}})$ be the corresponding birational morphism. Let C' be the strict transform of C under μ. Because the smooth subdivision was chosen minimal, C' does not meet the exceptional locus of μ. In particular, $\mu|_{C'}$ is an isomorphism of curves and $\Delta_{C'} \cong_t \Delta^{\max}$. Since $\mathrm{Tor}(\Sigma')$ is smooth, every Weil divisor is Cartier.

By the BKK theorem (recall that C' is a convex divisor),

$$C'^2 = \mathrm{MV}(\Delta^{\max}, \Delta^{\max}) = 2\mathrm{Vol}(\Delta^{\max}) \geq \frac{3}{4}\mathrm{lw}(\Delta^{\max})^2 = \frac{3}{4}\mathrm{lw}(\Delta)^2,$$

where the third and fourth (in)equalities follow from Lemma 5.2.(i,vi). For small lattice widths this bound can be improved: using the data from [6] one can computationally verify that

$$C'^2 = 2\mathrm{Vol}(\Delta^{\max}) \geq \begin{cases} 18 & \text{if } \mathrm{lw}(\Delta^{\max}) = 3, \\ 20 & \text{if } \mathrm{lw}(\Delta^{\max}) = 4, \\ 25 & \text{if } \mathrm{lw}(\Delta^{\max}) = 5, \\ 28 & \text{if } \mathrm{lw}(\Delta^{\max}) = 6 \end{cases} \tag{10}$$

(remark that by Pick's theorem it suffices to verify these inequalities for small genus only). Magma code assisting the reader in this can be found in the accompanying file `gonal.m`. The patient reader can also do an elaborate analysis by hand, following Kawaguchi [29, Props. 3.10–3.12,4.3]. We stress that for these bounds it is essential that $\Delta^{\max}$ is maximal and that $\Delta^{(1)}$ is not among the polygons listed in (9).

We now come to the heart of the proof. Fix a gonality pencil g^1_γ and let $p : C' \to \mathbb{P}^1$ be a corresponding morphism of degree γ. A theorem by Serrano [43, Thm. 3.1] states that if $C'^2 > (\gamma+1)^2$ then p can be extended to a morphism $\mathrm{Tor}(\Sigma') \to \mathbb{P}^1$. From this it will follow that p is combinatorial (as explained in the last paragraph of the proof). Unfortunately, in general we only have that[2]

$$C'^2 \geq \frac{3}{4}\mathrm{lw}(\Delta)^2 \geq \frac{3}{4}\gamma^2. \tag{11}$$

To bridge this, we follow an approach of Harui [28], who dug into Serrano's proof to extract Theorem 6.4 below.

We proceed by contradiction: assume that p cannot be extended to all of $\mathrm{Tor}(\Sigma')$. Then by Theorem 6.4 (note that $C'^2 > 4\gamma$) there exists an effective divisor V on $\mathrm{Tor}(\Sigma')$

[2]But note that for 'most' lattice polygons, the stronger bound $C'^2 > (\gamma+1)^2$ *does* hold, in which case the proof simplifies a lot.

satisfying

$$1 \leq s < C' \cdot V - s \leq \gamma \qquad \text{and} \qquad C'^2 \leq \frac{(\gamma + s)^2}{s}, \tag{12}$$

where $s = V^2$. We may assume that V is torus-invariant, i.e. $V = \sum a_\ell D_\ell$ for certain integers a_ℓ (where the D_ℓ's are the torus-invariant prime divisors of $\mathrm{Tor}(\Sigma')$). From our bounds (11) and (12) we see that

$$\frac{3}{4}\gamma^2 \leq \frac{(\gamma + s)^2}{s} < \frac{(\gamma + \gamma)^2}{s} = \frac{4}{s}\gamma^2$$

which implies that $s \leq 5$. Rewrite the first inequality as $(3s - 4)\gamma^2 - 8s\gamma - 4s^2 \leq 0$: if $s \geq 2$ then the largest real root of the left-hand side, when viewed as a polynomial in γ, is given by $(4s + 2s\sqrt{3s})/(3s - 4)$ which for $s \leq 5$ is strictly less than 9. We conclude that if $\gamma \geq 9$ then $s = 1$. A finer analysis using the better bounds (10) shows that $\gamma \geq 4$, and that $s = 1$ except possibly if $\gamma \in \{6, 7, 8\}$ in which case $s \in \{1, 2\}$.

We claim that this implies $h^0(\mathrm{Tor}(\Sigma'), V) \leq s + 1$. Suppose not, then Δ_V contains at least $s + 2$ lattice points. Let Γ be the convex hull of the lattice points in Δ_V and let $D_\Gamma = \sum_\ell a'_\ell D_\ell$ be the torus-invariant divisor obtained by taking the a'_ℓ's minimal such that

$$\Gamma \subset \left\{ \, (i,j) \in \mathbb{R}^2 \, \middle| \, <(i,j), v_\ell> \, \geq -a'_\ell \, \right\}.$$

One verifies that D_Γ is convex, that $\Delta_{D_\Gamma} = \Gamma$, and that $a_\ell \geq a'_\ell$ for all ℓ, i.e. $V - D_\Gamma$ is effective.

- Suppose that, up to a unimodular transformation, Γ contains a horizontal line segment I of length ≥ 2. Then $C' \cdot V = C' \cdot (D_\Gamma + (V - D_\Gamma))$ is bounded from below by

  $$C' \cdot D_\Gamma = \mathrm{MV}(\Delta^{\max}, \Gamma) \geq \mathrm{MV}(\Delta^{\max}, I) \geq 2\,\mathrm{lw}(\Delta^{\max}) = 2\,\mathrm{lw}(\Delta) \geq 2\gamma,$$

  where the first inequality follows because MV is an increasing function.This contradicts $C' \cdot V \leq \gamma + s$.
- So we can assume that Γ does not contain such a line segment.
  - If $s = 1$ then Γ contains at least 3 non-collinear lattice points. But then, by applying a unimodular transformation if needed, we may assume that $\Sigma \subset \Gamma$. One finds

    $$C' \cdot V \geq \mathrm{MV}(\Delta^{\max}, \Gamma) \geq \mathrm{MV}(\Delta^{\max}, \Sigma) = d \tag{13}$$

    where d is the smallest integer such that $\Delta^{\max}$ is contained in a translate of $d\Sigma$ (indeed, this follows from Bézout's theorem). Then $\Delta^{(1)} \subset (d-3)\Sigma$, and by our assumptions this inclusion is strict. It follows that $\mathrm{lw}(\Delta^{(1)}) \leq d - 4$, hence by Lemma 5.2.(i) that $\mathrm{lw}(\Delta) = \mathrm{lw}(\Delta^{\max}) \leq d - 2$. From (13) we conclude that $C' \cdot V \geq \mathrm{lw}(\Delta) + 2$. This contradicts $C' \cdot V \leq \gamma + 1$.
  - If $s = 2$ then Γ contains at least 4 lattice points. By our assumption that it contains no line segment of integral length 2, we can assume $\square \subset \Gamma$ or $\Upsilon \subset \Gamma$, again by applying a unimodular transformation if needed. In the former case we have

    $$C' \cdot V \geq \mathrm{MV}(\Delta^{\max}, \Gamma) \geq \mathrm{MV}(\Delta^{\max}, \square) = a + b,$$

where (a, b) is the 'bidegree' of f, i.e. the minimal couple of values for which $\Delta^{\max}$ is contained in a translate of $[0, a] \times [0, b]$. This follows from the BKK theorem applied to $\mathrm{Tor}(\square) = \mathbb{P}^1 \times \mathbb{P}^1$, and implies that $C' \cdot V \geq 2\,\mathrm{lw}(\Delta) \geq 2\gamma$. In the latter case, by the BKK theorem applied to $\mathrm{Tor}(\Upsilon)$, one similarly finds

$$C' \cdot V \geq \mathrm{MV}(\Delta^{\max}, \Gamma) \geq \mathrm{MV}(\Delta^{\max}, \Upsilon) = 3d,$$

where d is the smallest integer such that $\Delta^{\max}$ is contained in a translate of $d\Upsilon$. Because

$$2d = \mathrm{lw}(d\Upsilon) \geq \mathrm{lw}(\Delta^{\max}) \geq \gamma \tag{14}$$

we find that $C' \cdot V \geq \frac{3}{2}\gamma$. In both cases this contradicts $C' \cdot V \leq \gamma + 2$; recall that $\gamma \geq 6$ in the $s = 2$ case.

Remark. The bound $C' \cdot V \geq \frac{3}{2}\gamma$ can be proven more easily by noting that Υ contains a line segment of integral length $\frac{3}{2}$; however, the argument using (14) will reappear in the proof of Theorem 7.2, so we have included it for the sake of consistency.

Our claim that $h^0(\mathrm{Tor}(\Sigma'), V) \leq s + 1$ follows.

Because a lattice polygon having at most 3 lattice points cannot have any interior lattice points, we deduce that $h^0(\mathrm{Tor}(\Sigma'), V + K) = 0$, with $K = -\sum_\ell D_\ell$ the canonical divisor from Section 3. The Riemann-Roch theorem yields that

$$\frac{1}{2}(V + K) \cdot V =$$

$$h^0(\mathrm{Tor}(\Sigma'), V + K) - h^1(\mathrm{Tor}(\Sigma'), V + K) + h^0(\mathrm{Tor}(\Sigma'), -V) - \chi(\mathcal{O}_{\mathrm{Tor}}(\Sigma'))$$

is bounded by $-\chi(\mathcal{O}_{\mathrm{Tor}}(\Sigma')) = -1$, i.e. $K \cdot V \leq -s - 2$. But then Riemann-Roch also tells us that

$$h^0(\mathrm{Tor}(\Sigma'), V) = h^1(\mathrm{Tor}(\Sigma'), V) - h^0(\mathrm{Tor}(\Sigma'), K - V) + \frac{1}{2}V \cdot (V - K) + 1$$

is at least $s + 2$.

Thus we run into the desired contradiction, and we conclude that p can be extended to all of $\mathrm{Tor}(\Sigma')$. Let $\tilde{p} : \mathrm{Tor}(\Sigma') \to \mathbb{P}^1$ be such that $\tilde{p}|_{C'} = p$. Let F be a fiber of $\tilde{p}$, so that $F \cdot C' = \gamma$. Then $C' \cdot (F - C') \leq \gamma - \frac{3}{4}\gamma^2 < 0$. Since C' is nef it follows that $h^0(\mathrm{Tor}(\Sigma'), F - C') = 0$. Now by tensoring the short exact sequence

$$0 \to \mathcal{O}_{\mathrm{Tor}(\Sigma')}(-C') \to \mathcal{O}_{\mathrm{Tor}(\Sigma')} \to \mathcal{O}_{C'} \to 0$$

with $\mathcal{O}_{\mathrm{Tor}(\Sigma')}(F)$ and taking cohomology, we find the exact sequence

$$0 \to H^0(\mathrm{Tor}(\Sigma'), F - C') \to H^0(\mathrm{Tor}(\Sigma'), F) \to H^0(C', F|_{C'}) \to \dots,$$

which proves that $h^0(\mathrm{Tor}(\Sigma'), F) \leq 2$; here we used that $h^0(C', F|_{C'}) = 2$ because g^1_γ is complete. Thus $|F|$ is a linear system of rank 1, i.e. every element of $|F|$ is a fiber of $\tilde{p}$. Let D be a torus-invariant divisor that is equivalent to F. By translating if necessary we may assume that $(0, 0) \in \Delta_D$, so that D is effective. But then $D \in |F|$ and

$$H^0(\mathrm{Tor}(\Sigma'), D) = \left\langle 1, x^a y^b \right\rangle$$

for some primitive $(a, b) \in \mathbb{Z}^2$. We find that $\tilde{p}|_{\mathbb{T}^2} : (x, y) \mapsto x^a y^b$ (up to an automorphism of $\mathbb{P}^1$), i.e. $g^1_\gamma = g_{(a,b)}$. ■

Theorem 6.4 (Serrano, 1987)**.** *Let C be a smooth projective curve on a smooth projective surface S, and let $p : C \to \mathbb{P}^1$ be a surjective morphism of degree d. Suppose that $C^2 > 4d$ and that p cannot be extended to a morphism $S \to \mathbb{P}^1$. Then there exists an effective divisor V on S for which*

$$0 < V^2 < V \cdot (C - V) \leq d \qquad \text{and} \qquad C^2 \leq \frac{(d + V^2)^2}{V^2}.$$

Proof. By contradiction. Suppose that such an effective divisor V does not exist, then one can replace *Claim 6* in Serrano's proof [43, p. 401] by the following reasoning (the text below does not make sense without Serrano's paper at hand):

Claim 6: $a = 0$. Suppose that $a > 0$. Then V_1 is an effective divisor such that $0 < V_1^2 < V_1 \cdot V_2 \leq d$ because $a < e$. On the other hand,

$$C^2 = a + 2e + b \leq a + 2e + \frac{e^2}{a} \leq a + 2d + \frac{d^2}{a} = \frac{(a + V_1^2)^2}{V_1^2}.$$

Since $V_2 = C - V_1$ this contradicts our hypothesis. Hence, $a = 0$.

The rest of the proof can be copied word by word. ■

7. Near-gonal pencils

By a *near-gonal pencil* on a smooth projective curve C/k we mean a base-point free $g^1_{\gamma(C)+1}$ (note that such pencils need not exist). The method of the previous section can be adapted to show that, apart from some reasonably well-understood exceptional instances of Δ, every near-gonal pencil on a Δ-non-degenerate curve is combinatorial.

It is convenient to state our main result in terms of the *lattice size*, a notion to which we have devoted a separate paper [9]. If $\Delta \neq \emptyset$, then its lattice size is defined as the minimal integer $d \geq 0$ such that Δ is equivalent to a lattice polygon that is contained in $d\Sigma$. We denote this integer by $\mathrm{ls}(\Delta)$, and let $\mathrm{ls}(\emptyset) = -2$. If Δ is two-dimensional then, as in the case of the lattice width (cf. Lemma 5.2.(i)), there exists an expression for $\mathrm{ls}(\Delta)$ in terms of $\mathrm{ls}(\Delta^{(1)})$, allowing one to compute $\mathrm{ls}(\Delta)$ by gradually peeling off the polygon [9, Thm. 3.5]. For our needs, one of the main results of [9] can be reformulated as follows:

Theorem 7.1. *Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$. Then the minimal degree of a (possibly singular) projective plane curve that is birationally equivalent to $U(f)$ is bounded by $\mathrm{ls}(\Delta^{(1)}) + 3$. If $\Delta^{(1)} \cong (d-1)\Upsilon$ for a certain integer $d \geq 2$ (i.e. $\Delta \cong d\Upsilon$), then it is moreover bounded by $3d - 1$.*

Proof. See [9, Thm. 1.3]. ■

Remarks.

- If $\Delta^{(1)} \cong (d-1)\Upsilon$ then $\mathrm{ls}(\Delta^{(1)}) + 3 = 3d$ (as can be verified using [9, Thm. 3.5]). So the second bound is sharper in this case.

- We expect that the (smallest applicable) bound of Theorem 7.1 is in fact sharp; see [9, §7] for a discussion.

Our main result is as follows:

Theorem 7.2. *Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$, and let γ be the gonality of $U(f)$. Suppose that*

$$\mathrm{ls}(\Delta^{(1)}) \geq \mathrm{lw}(\Delta^{(1)}) + 2 \tag{15}$$

and that $\Delta^{(1)} \ncong 2\Upsilon, 3\Upsilon, \Gamma^7, \Gamma^8$. Then every base-point free $g^1_{\gamma+1}$ on the smooth projective model of $U(f)$ is combinatorial.

Before we proceed to the proof of Theorem 7.2, let us analyze the list of polygons that are excluded in the statement (this is a strict extension of the list of polygons that were exluded in the statement of Theorem 6.1). First note that Theorem 7.2 implies that if Δ is not among the excluded polygons, the number of base-point free $g^1_{\gamma+1}$'s is finite. Opposed to that, we have:

Lemma 7.3. *If Δ violates condition (15) or $\Delta^{(1)} \cong 2\Upsilon, \Gamma^7$, then the number of base-point free $g^1_{\gamma+1}$'s is infinite.*

Proof. A violation of condition (15) implies that $U(f)$ is birationally equivalent to a (possibly singular) plane curve of degree at most $\gamma + 2$. Indeed:

- If $\Delta^{(1)} \cong \Upsilon$ then $U(f)$ is a non-hyperelliptic genus 4 curve, hence of gonality 3. It is known that such curves admit a plane model of degree 5; see e.g. [27, Ex. IV.5.4].
- If $\Delta^{(1)} \ncong \Upsilon$ but $\mathrm{ls}(\Delta^{(1)}) < \mathrm{lw}(\Delta^{(1)}) + 2$, then by Corollary 6.2 the assumption can be rephrased as $\mathrm{ls}(\Delta^{(1)}) < \gamma$. Along with Theorem 7.1 this implies that $U(f)$ has a projective plane model of degree at most $\gamma + 2$.

It follows that $U(f)$ must have a plane model of degree exactly $\gamma + 1$ or $\gamma + 2$, because a model of degree at most γ would contradict that γ equals the gonality (by projecting from a point on this plane model). But then there exist infinitely many base-point free $g^1_{\gamma+1}$'s, obtained either by projection from a point outside the plane model, or by projection from a non-singular point on the plane model.

If $\Delta^{(1)} \cong 2\Upsilon$, so that $\Delta \cong 3\Upsilon$, then $U(f)$ is a 6-gonal curve that is birationally equivalent to a smooth intersection of two cubics in $\mathbb{P}^3 = \mathrm{Proj}\, k[X_{0,0}, X_{-1,-1}, X_{1,0}, X_{0,1}]$, where one of the cubics is just $\mathrm{Tor}(\Upsilon)$, i.e. it is given by $X_{0,0}^3 - X_{-1,-1}X_{1,0}X_{0,1}$ (see the according remark following Theorem 6.1). By the trisecant lemma [27, IV.Prop. 3.8 and IV.Thm. 3.9] we can find a point on this curve, the general secant through which is not a multisecant. Projecting from this point gives a birational equivalence with a plane curve of degree 8, and hence we again obtain infinitely many g^1_7's.

Finally, if $\Delta^{(1)} \cong \Gamma^7$ then $\gamma = 4$. Now there exists at least one base-point free g^1_5 (namely $g_{(0,1)}$). By Brill-Noether theory it then follows that the number of base-point free g^1_5's is infinite. ∎

The exclusion of 3Υ (in which case $\gamma = 8$) is also necessary:

Lemma 7.4. *If $\Delta^{(1)} \cong 3\Upsilon$ then there exists a base-point free g^1_9, while there are no combinatorial g^1_9's.*

Proof. If $\Delta^{(1)} \cong 3\Upsilon$, then $\Delta \cong 4\Upsilon$ and $U(f)$ is a curve of genus 19 that is birationally equivalent to a smooth intersection of $\mathrm{Tor}(\Upsilon)$ and a quartic in $\mathbb{P}^3$. By Theorem 6.1 our curve is 8-gonal, and there are exactly three g^1_8's. Geometrically, the three g^1_8's can be visualized as pencils of planes through the three lines of $\mathrm{Tor}(\Upsilon)$. By the trisecant lemma we can find a point on the curve that is

(1) not contained in any of these three lines, and
(2) the general secant line through which is not a multisecant.

Projecting from such a point gives a birational map to a plane curve of degree 11, the map being birational because of condition (2). Genus considerations yield that the curve must be singular. Moreover, the singular points all have multiplicity 2. Indeed, if there were a singularity of multiplicity 3, the pencil of lines through this point would cut out one of our g^1_8's, which is impossible by condition (1). On the other hand, a singularity of higher multiplicity would contradict that the gonality is 8. Then projecting from such a singular point of multiplicity 2 yields a base-point free g^1_9. We leave it to the reader to verify that there are indeed no combinatorial g^1_9's. ■

Finally, if $\Delta^{(1)} \cong \Gamma^8$, so that $\Delta \cong \mathrm{conv}\{(0,0),(6,2),(2,4)\}$, then $\gamma = 4$ and it can be checked that there are no combinatorial g^1_5's. On the other hand, the Laurent polynomial $f = 1 - x^6y^2 - x^2y^4$ is non-degenerate with respect to its Newton polygon, while $U(f)$ admits a rational map

$$U(f) \to \mathbb{A}^1 : (x,y) \mapsto \frac{1-xy^2}{x^3y}$$

of degree 5, and therefore carries a base-point free g^1_5. Moduli-theoretic considerations then allow one to draw the same conclusion for a non-empty open subset of the space of Laurent polynomials $f \in k[x^{\pm 1}, y^{\pm 1}]$ that are supported on Δ. Unfortunately, this does not prove the corresponding statement for all Δ-non-degenerate (or Δ-toric) Laurent polynomials, even though we believe that it should be true. But in any case this shows that the exclusion of Γ^8 is also necessary.

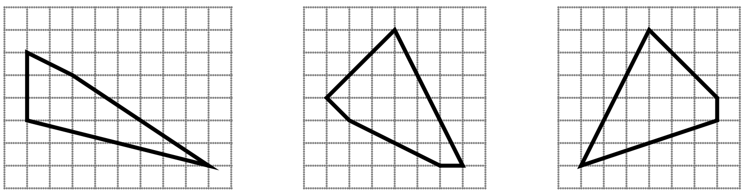
We now prove Theorem 7.2:

Proof of Theorem 7.2. This is very similar to the proof of Theorem 6.1. Let $g = \sharp(\Delta^{(1)} \cap \mathbb{Z}^2)$ be the geometric genus of $U(f)$. The assumptions imply that $g \geq 3$. Because hyperelliptic curves of genus at least 3 never carry a base-point free g^1_3, we can assume that $\Delta^{(1)}$ is two-dimensional and that $U(f)$ is of gonality $\gamma \geq 3$.

As before, let C be the Zariski closure of $\varphi_{\Delta^{\max}}(U(f))$ inside $\mathrm{Tor}(\Delta^{\max})$, let Σ' be a minimal smooth subdivision of $\Sigma_{\Delta^{\max}}$, let $\mu : \mathrm{Tor}(\Sigma') \to \mathrm{Tor}(\Sigma_{\Delta^{\max}})$ be the corresponding birational morphism, and let C' be the strict transform of C under μ. Recall that $C'^2 \geq \frac{3}{4}\gamma^2$. Using the data from [6], our list of sharpened lower bounds (10) can be adapted and extended to

$$C'^2 = 2\mathrm{Vol}(\Delta^{\max}) \geq \begin{cases} 24 & \text{if } \mathrm{lw}(\Delta^{\max}) = 3, \\ 24 & \text{if } \mathrm{lw}(\Delta^{\max}) = 4, \\ 30 & \text{if } \mathrm{lw}(\Delta^{\max}) = 5, \\ 34 & \text{if } \mathrm{lw}(\Delta^{\max}) = 6, \\ 46 & \text{if } \mathrm{lw}(\Delta^{\max}) = 7, \\ 55 & \text{if } \mathrm{lw}(\Delta^{\max}) = 8, \end{cases} \tag{16}$$

unless $\Delta^{\max}$ is equivalent to one of the following three polygons,



Δ_1 Δ_2 Δ_3

whose respective lattice widths and doubled volumes are $5, 6, 6$ and $25, 32, 33$. See the accompanying Magma file `neargonal.m` for assistance in verifying these bounds. It is again essential that $\Delta^{\max}$ is maximal and that $\Delta^{(1)}$ is not among the polygons excluded in the énoncé (recall that this is a strict extension of the list of polygons that were excluded in Theorem 6.1).

For now, assume that $\Delta^{\max} \not\cong \Delta_1, \Delta_2, \Delta_3$: we will deal with these polygons later. Consider a base-point free $g^1_{\gamma+1}$ on C' and let $p : C' \to \mathbb{P}^1$ be a corresponding morphism of degree $\gamma + 1$ (which exists precisely because our $g^1_{\gamma+1}$ is base-point free). Assume that p cannot be extended to all of $\mathrm{Tor}(\Sigma')$. Because $C'^2 > 4(\gamma + 1)$ we can apply Serrano's Theorem 6.4 to obtain the existence of an effective divisor V on $\mathrm{Tor}(\Sigma')$ for which

$$0 < s < C' \cdot V - s \leq \gamma + 1 \qquad \text{and} \qquad C'^2 \leq \frac{(\gamma + 1 + s)^2}{s}, \tag{17}$$

where $s = V^2$. The bounds on C'^2 imply that $s = 1$, except possibly if $\gamma \in \{4, \ldots, 13\}$ in which case $s \in \{1, 2\}$.

We claim that this implies $h^0(\mathrm{Tor}(\Sigma'), V) \leq s + 1$. Suppose not, and let Γ be as in the proof of Theorem 6.1, i.e., it is a lattice polygon containing at least $s + 2$ lattice points, with the property that $C' \cdot V \geq \mathrm{MV}(\Delta^{\max}, \Gamma)$.

- If Γ contains a line segment of integral length 2, then as before it follows that $C' \cdot V \geq 2\gamma$, which contradicts $C' \cdot V \leq \gamma + 1 + s$ (note that $s = 1$ in case $\gamma = 3$).
- So we can assume that Γ does not contain such a line segment.
  - If $s = 1$ it therefore suffices to consider the case where Γ contains Σ (after performing a unimodular transformation if needed). We again find $C' \cdot V \geq d$ where $d \geq 0$ is the smallest integer such that $\Delta^{\max}$ is contained in a translate of $d\Sigma$. By definition of the lattice size, it follows that

    $$C' \cdot V \geq \mathrm{ls}(\Delta^{\max}) \geq \mathrm{ls}(\Delta^{(1)}) + 3 \geq \mathrm{lw}(\Delta^{(1)}) + 5 \geq \mathrm{lw}(\Delta) + 3 \geq \gamma + 3.$$

    Here the second inequality follows from [9, Eq. (2)], the third inequality follows from (15), and the fourth inequality follows from Lemma 5.2.(i). This contradicts that $C' \cdot V \leq \gamma + 1 + s = \gamma + 2$.

– If $s = 2$ then we can assume that $\gamma \geq 4$ and that Γ contains a unimodular copy of either $\square$ or Υ. As before we respectively find that $C' \cdot V \geq 2\gamma$ and $C' \cdot V \geq \frac{3}{2}\gamma$. In the former case this contradicts $C' \cdot V \leq \gamma + 1 + s$. In the latter case, the contradiction follows for $\gamma \geq 7$ only. To deal with the case where $\gamma \leq 6$, note that (14) can be rewritten as

$$2d = \mathrm{lw}(d\Upsilon) > \mathrm{lw}(\Delta^{\max}) = \gamma,$$

where the last equality follows from Corollary 6.2, and the strict inequality in the middle holds because the lattice width of a strict subpolygon of $d\Upsilon$ is strictly less than $2d$ (we excluded the possibility that $\Delta^{\max} \cong 2\Upsilon, 3\Upsilon$ in the énoncé). It follows that for $\gamma \leq 6$, the bound $C' \cdot V \geq \frac{3}{2}\gamma$ can be refined to $C' \cdot V \geq \frac{3}{2}(\gamma + 1)$, which is now sufficient to contradict $C' \cdot V \leq \gamma + 1 + s$.

So we conclude that indeed $h^0(\mathrm{Tor}(\Sigma'), V) \leq s+1$. As in the proof of Theorem 6.1, along with $s \leq 2$ this again implies that $h^0(\mathrm{Tor}(\Sigma'), V + K) = 0$. The remainder of the proof is an exact copy of the corresponding part of the proof of Theorem 6.1 (except in the last paragraph, where now $F \cdot C' = \gamma + 1$, but this doesn't affect the argument). Remark that for this part we need $g^1_{\gamma+1}$ to be complete, which is true because the contrary would lead to infinitely many g^1_γ's, contradicting Corollary 6.3.

It remains to deal with the case where $\Delta^{\max}$ is among $\Delta_1, \Delta_2, \Delta_3$. Here (17) only allows us to conclude $s \in \{1, 2, 3\}$. If $s \in \{1, 2\}$ then the above proof applies, so we can assume $s = 3$. We claim that in this case $h^0(\mathrm{Tor}(\Sigma'), V) \leq 3$. Suppose not, then there exists a lattice polygon Γ containing at least 4 lattice points, with the property that $C' \cdot V \geq \mathrm{MV}(\Delta^{\max}, \Gamma)$.

- If Γ contains a line segment of integral length 2 then we again run into a contradiction (note that we only consider $\gamma = 5$ and $\gamma = 6$).
- If not then we can again assume that $\square \subset \Gamma$ or $\Upsilon \subset \Gamma$. In the former case the bound $C' \cdot V \geq 2\gamma$ suffices to run into contradiction (again using that $\gamma = 5, 6$). In the case $\Upsilon \subset \Gamma$, the above sharpened bound $C' \cdot V \geq \frac{3}{2}(\gamma + 1)$ results in a contradiction for Δ_2 and Δ_3, but remains insufficient in the case of Δ_1. Now it is not hard to see that there is no unimodular transformation mapping Δ_1 inside 3Υ. Indeed, because the lattice width of a subpolygon of 3Υ that misses two vertices of 3Υ is at most 4, we find that a unimodular copy of Δ_1 inside 3Υ should have an edge in common with 3Υ. But Δ_1 contains only one edge having 4 lattice points, and the width of Δ_1 with respect to the direction of this edge is 8. So Δ_1 can indeed impossibly fit inside 3Υ. It follows that the smallest multiple of Υ containing a unimodular copy of Δ_1 is 4Υ, from which

$$C' \cdot V \geq \mathrm{MV}(\Delta^{\max}, \Gamma) \geq \mathrm{MV}(\Delta^{\max}, \Upsilon) \geq 3 \cdot 4 = 12.$$

  This gives the desired contradiction.

So we conclude that indeed $h^0(\mathrm{Tor}(\Sigma'), V) \leq 3$. This implies that $h^0(\mathrm{Tor}(\Sigma'), V + K) = 0$, and the rest of the argument can again be copied word by word, essentially. ■

Remark. Kawaguchi's proof technique should in principle allow one to obtain similar theorems on base-point free $\gamma^1_{\gamma+n}$'s for $n = 2, 3, \ldots$ Here condition (15) will have to be replaced by

$$\mathrm{ls}(\Delta^{(1)}) \geq \mathrm{lw}(\Delta^{(1)}) + n + 1.$$

However, an increasing number of exceptional polygons are expected to come into play, both for geometric reasons (definitely, more and more multiples of Υ will show up) and for proof-technical reasons (as in the case of $\Delta_1, \Delta_2, \Delta_3$ in the above proof). This might be feasible for $n = 2$, although we did not try this in detail. For higher values of n we expect a complete classification to become very complicated.

8. Clifford index and Clifford dimension

To a smooth projective curve C/k of genus $g \geq 4$ one can associate its Clifford index

$$\mathrm{ci}(C) = \min\{\, d - 2r \,|\, C \text{ carries a divisor } D \text{ with } |D| = g^r_d \text{ and } h^0(C,D), h^0(C,K-D) \geq 2 \,\}$$

(where K is a canonical divisor on C) and its Clifford dimension

$$\mathrm{cd}(C) = \min\{\, r \,|\, \text{there exists a } g^r_d \text{ realizing } \mathrm{ci}(C)\};$$

see [20]. In the case of a singular and/or non-complete curve C/k, we define $\mathrm{ci}(C)$ and $\mathrm{cd}(C)$ to be the corresponding quantities associated to its smooth complete model. In this section we give a combinatorial interpretation for the Clifford index and the Clifford dimension. Again the key trick is due to Kawaguchi [29, Proof of Thm. 1.3.(iii)], but thanks to our more careful analysis of the planar curve case we obtain a complete statement.

Theorem 8.1. *Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$ and suppose that $\sharp(\Delta^{(1)} \cap \mathbb{Z}^2) \geq 4$. Then*

- *if $\Delta^{(1)} \cong (d-3)\Sigma$ for $d \geq 5$ then $\mathrm{ci}(U(f)) = d - 4$ and $\mathrm{cd}(U(f)) = 2$,*
- *if $\Delta^{(1)} \cong \Upsilon$ then $\mathrm{ci}(U(f)) = 1$ and $\mathrm{cd}(U(f)) = 1$,*
- *if $\Delta^{(1)} \cong 2\Upsilon$ then $\mathrm{ci}(U(f)) = 3$ and $\mathrm{cd}(U(f)) = 3$,*
- *in all other cases $\mathrm{ci}(U(f)) = \mathrm{lw}(\Delta^{(1)})$ and $\mathrm{cd}(U(f)) = 1$.*

Proof. The first three cases correspond to smooth projective plane curves of degree $d \geq 5$, non-hyperelliptic curves of genus 4, resp. smooth intersections of pairs of cubics in $\mathbb{P}^3$, while the cases $\Delta^{(1)} \cong \Gamma^5_1, \Gamma^5_2, \Gamma^5_3$ correspond to non-hyperelliptic, non-trigonal curves of genus 5. In these situations the Clifford index and the Clifford dimension are well-known; see [20, pp. 174-175] and [18, p. 225]. In all other cases Corollary 6.3 yields that the number of gonality pencils is finite, while from Corollary 6.2 we know that $\gamma(U(f)) = \mathrm{lw}(\Delta^{(1)}) + 2$. A result by Coppens and Martens [14] (see the discussion preceding [14, Thm. B]) then implies that $\mathrm{ci}(U(f)) = \mathrm{lw}(\Delta^{(1)})$. By definition of the Clifford dimension, this implies $\mathrm{cd}(U(f)) = 1$. ∎

Remark. For curves C/k of genus $1 \leq g \leq 3$ one sometimes defines

- $\mathrm{ci}(C) = 1$ if C is a non-hyperelliptic genus 3 curve, and $\mathrm{ci}(C) = 0$ if not,
- $\mathrm{cd}(C) = 1$.

With these conventions, Theorem 8.1 remains valid when one replaces the condition $\sharp(\Delta^{(1)} \cap \mathbb{Z}^2) \geq 4$ with $\sharp(\Delta^{(1)} \cap \mathbb{Z}^2) \geq 1$.

Corollary 8.2. *Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its (two-dimensional) Newton polygon $\Delta = \Delta(f)$. Then $U(f)$ is birationally equivalent to a smooth projective plane curve if and only if $\Delta^{(1)} = \emptyset$ or $\Delta^{(1)} \cong (d-3)\Sigma$ for some integer $d \geq 3$.*

Proof. The 'if' part is easily verified. As for the 'only if' part, let g be the geometric genus of $U(f)$, which is necessarily of the form $(d-1)(d-2)/2$ for some $d \geq 2$. If $d \geq 5$ then $\mathrm{cd}(U(f)) = 2$ and the corollary follows from Theorem 8.1. If $d = 2$ or $d = 3$ then the statement is trivial. If $d = 4$ then the claim follows because $U(f)$ is non-hyperelliptic, and because Σ is the only two-dimensional lattice polygon containing $g = 3$ lattice points (up to unimodular equivalence). ■

9. Scrollar invariants

We begin by recalling some facts on rational normal scrolls and on scrollar invariants. Our main references are [19], [26, §8.26-29] and [42, §1-4].

Let $n \in \mathbb{Z}_{\geq 1}$ and let $\mathcal{E} = \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_n)$ be a locally free sheaf of rank n on $\mathbb{P}^1$. Denote by $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ the corresponding $\mathbb{P}^{n-1}$-bundle. We assume that $0 \leq e_1 \leq e_2 \leq \ldots \leq e_n$ and that $e_1 + e_2 + \cdots + e_n \geq 2$. Set $N = e_1 + e_2 + \ldots + e_n + n - 1$. A *rational normal scroll of type* $(e_1, \ldots, e_n)$ in $\mathbb{P}^N$ is the image of the induced morphism

$$\mu : \mathbb{P}(\mathcal{E}) \to \mathbb{P}H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)),$$

composed with an isomorphism $\mathbb{P}H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \to \mathbb{P}^N$.

The dimension of a rational normal scroll of type $(e_1, \ldots, e_n)$ equals n, while its degree equals $e_1 + \ldots + e_n = N - n + 1$. This means that the classical lower bound $\deg(X) \geq \mathrm{codim}_{\mathbb{P}^N}(X) + 1$ for projective varieties $X \subset \mathbb{P}^N$ that are not contained in any hyperplane is attained. Varieties for which this holds are said to *have minimal degree.* They have been classified by Del Pezzo (the surface case, 1886) and Bertini (1907): any projective variety of minimal degree is a cone over a smooth such variety, and the smooth such varieties are exactly the rational normal scrolls with $e_1 > 0$, the quadratic hypersurfaces, and the Veronese surface in $\mathbb{P}^5$. See [19] for a modern proof.

There is an easy geometric way of describing rational normal scrolls. Consider linear subspaces $\mathbb{P}^{e_1}, \ldots, \mathbb{P}^{e_n} \subset \mathbb{P}^N$ that span $\mathbb{P}^N$. In each $\mathbb{P}^{e_\ell}$, take a rational normal curve[3] of degree e_ℓ, e.g. parameterized by

$$\nu_\ell : \mathbb{P}^1 \to \mathbb{P}^{e_\ell} : (X : Z) \mapsto \left(Z^{e_\ell} : XZ^{e_\ell - 1} : \cdots : X^{e_\ell}\right). \tag{18}$$

Then

$$S = \bigcup_{P \in \mathbb{P}^1} \langle \nu_1(P), \ldots, \nu_n(P) \rangle \subset \mathbb{P}^N$$

is a rational normal scroll of type $(e_1, \ldots, e_n)$, and conversely every rational normal scroll arises in this way. The scroll is smooth if and only if $e_1 > 0$. In this case $\mu : \mathbb{P}(\mathcal{E}) \to S$ is an isomorphism. If $0 = e_1 = \ldots = e_\ell < e_{\ell+1}$ with $1 \leq \ell < n$, then the scroll is a cone with an $(\ell - 1)$-dimensional vertex. In this case $\mu : \mathbb{P}(\mathcal{E}) \to S$ is a resolution of singularities. Outside the exceptional locus, our $\mathbb{P}^{n-1}$-bundle $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ corresponds to

$$S \setminus S^{\mathrm{sing}} \to \mathbb{P}^1 : Q \in \langle \nu_1(P), \ldots, \nu_n(P) \rangle \mapsto P.$$

[3] If $e_\ell = 0$ then this 'curve' is just a point, in fact. We will keep making this abuse of language.

Abusing notation, we denote this map also by π. Abusing terminology, when talking about the fiber of π above a point P, we mean the whole space $\langle \nu_1(P), \ldots, \nu_n(P)\rangle$.

Now let $C \subset \mathbb{P}^{g-1}$ be a canonical curve of genus $g \geq 3$ and fix any pencil g^1_d on C. Let $K \sim \mathcal{O}_C(1)$ be a canonical divisor on C. For an effective divisor $D \in g^1_d$, denote by $\langle D \rangle$ its linear span (if D is the sum of γ distinct points, the linear span of D is just the linear span of these points; in general one defines it as the intersection of all hyperplanes whose intersection divisor with C is at least D, see [42, §2.3]). The Riemann-Roch theorem implies that $h^0(C, K - D) = g - d - 1 + h^0(C, D)$, from which it follows that the dimension of $\langle D \rangle$ equals

$$d - h^0(C, D). \tag{19}$$

This does not depend the specific choice of D. In particular, if our g^1_d is complete, then the dimension of $\langle D \rangle$ is $d - 2$.

Consider

$$S = \bigcup_{D \in g^1_d} \langle D \rangle \subset \mathbb{P}^{g-1}. \tag{20}$$

Then S is a rational normal scroll by [19, Thm. 2] or [42, (2.5)], and it contains the curve C. In most interesting cases $\dim S = d - h^0(C, D) + 1$, but it may happen that $\dim S = d - h^0(C, D)$, which holds iff $h^0(C, K - D) = 0$, i.e. iff $\langle D \rangle = \mathbb{P}^{g-1}$. If g^1_d is base-point free then C does not meet the singular locus of S (in which case the restriction of π to C is a dominant rational map of degree d).

Let $(e_1, \ldots, e_n)$ be the type of S. Then the numbers $e_1, \ldots, e_n$ are called the *scrollar invariants of C with respect to g^1_d*. When we talk about the *scrollar invariants of C*, without making reference to a specific pencil, we always mean the scrollar invariants with respect to a gonality pencil, but note that this may depend on the choice of the latter, in which case the terminology is avoided. In the trigonal case the notion is well-behaved, and here the scrollar invariants are better known under the name *Maroni invariants*.[4] The scrollar invariants of an arbitrary non-hyperelliptic curve C/k of genus $g \geq 3$ with respect to a pencil g^1_d are then defined to be the corresponding invariants of a canonical model.

If $g^1_d = |D|$ is complete and base-point free then $n = d - 1$, and the scrollar invariants can alternatively be described as follows:

$$h^0(C, mD) = \begin{cases} h^0(C, (m-1)D) + 1 = m + 1 & \text{if } 0 \leq m \leq e_1 + 1, \\ h^0(C, (m-1)D) + 2 & \text{if } e_1 + 1 < m \leq e_2 + 1, \\ \qquad \vdots & \qquad \vdots \\ h^0(C, (m-1)D) + d - 1 & \text{if } e_{d-2} + 1 < m \leq e_{d-1} + 1, \\ h^0(C, (m-1)D) + d = md - g + 1 & \text{if } m > e_{d-1} + 1. \end{cases}$$

See [42, (2.4)] for more details, as well as a treatment of the general case (where our g^1_d is not necessarily complete and/or base-point free).

[4] Unfortunately, the existing literature is ambiguous at this point: sometimes one talks about *the* Maroni invariant of a trigonal curve, in which case one could mean either e_1 or $e_2 - e_1$.

Remark. From this description it follows that if our g^1_d is complete and base-point free then $e_{d-1} \leq \frac{2g-2}{d}$. Indeed, if $m > \frac{2g-2}{d}$ then $h^0(C, mD) = md - g + 1$ and by the above characterization, the smallest m for which $h^0(C, mD) = md - g + 1$ is $m = e_{d-1} + 1$.

The main result of this section is as follows.

Theorem 9.1. *Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$, and assume that $\Delta^{(1)}$ is two-dimensional. Let v be a lattice direction. Then the multiset of scrollar invariants of $U(f)$ with respect to g_v equals the multiset of non-negative width invariants of Δ with respect to v.*

Remark. As mentioned at the end of Section 4, our main results stay true if one weakens the assumption of being Δ-non-degenerate to being Δ-toric. This also applies to Theorem 9.1, but the argument becomes more technical due to the potential presence of base points. For the sake of clarity, the proof below only handles the case of Δ-non-degenerate Laurent polynomials. The extra ingredients in the Δ-toric case are then sketched in a following remark.

Proof. Write $d = w(\Delta, v)$, so that g_v is a base-point free g^1_d. Using a unimodular transformation if needed, we may assume that $v = (a, b) = (1, 0)$ and that Δ is contained in the horizontal strip $\mathbb{R} \times [0, d] \subset \mathbb{R}^2$. Then the width invariants of Δ with respect to v are the numbers

$$E_\ell = \sharp\{(i, j) \in \Delta^{(1)} \cap \mathbb{Z}^2 \,|\, j = \ell\} \,-\, 1,$$

where $\ell = 1, \ldots, d-1$. We have to show that the scrollar invariants with respect to the pencil cut out by $p : U(f) \mapsto \mathbb{T}^1 : (x, y) \mapsto x$ are given by the multiset $\{E_\ell\}_{\ell=1,\ldots,d-1} \cap \mathbb{Z}_{\geq 0}$. Denote the cardinality of this multiset by n.

Let C be the canonical model of $U(f)$ obtained by taking the Zariski closure of its image under the morphism $\varphi_{\Delta^{(1)}}$, as described in (5). For all $\ell \in \{1, \ldots, d-1\}$ for which $E_\ell \geq 0$, let $\mathbb{P}^{E_\ell} \subset \mathbb{P}^{g-1}$ be the linear subspace defined by $X_{i,j} = 0$ for all $(i, j) \in \Delta^{(1)} \cap \mathbb{Z}^2$ for which $j \neq \ell$. That is, $\mathbb{P}^{E_\ell}$ is the subspace corresponding to the projective coordinates $(X_{i,\ell})_{(i,\ell) \in \Delta^{(1)} \cap \mathbb{Z}^2}$. Also consider the rational normal curves parameterized by $\nu_\ell : \mathbb{P}^1 \to \mathbb{P}^{E_\ell}$ as in (18), i.e.

$$\forall x \in k^* : \nu_\ell(x) = (1 : x : \ldots : x^{E_\ell}).$$

Then $\varphi_{\Delta^{(1)}}$ maps every $(x, y) \in \mathbb{T}^2$ inside the $(n-1)$-dimensional linear subspace of $\mathbb{P}^{g-1}$ spanned by the points $\nu_\ell(x)$. Indeed, abusing notation, one sees that when the $\nu_\ell(x)$'s are scaled by an appropriate power of x, the point $\varphi_{\Delta^{(1)}}(x, y)$ arises as the linear combination

$$\sum_{\substack{\ell=1 \\ E_\ell \geq 0}}^{d-1} y^\ell \nu_\ell(x).$$

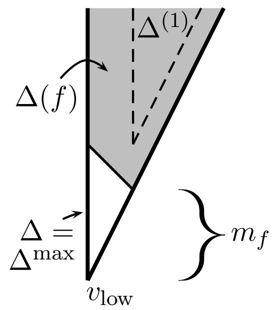
Now for all but finitely many $c \in k^*$, the inverse image divisor $p^{-1}(c)$ consists of d distinct points $(c, y_1), \ldots, (c, y_d)$ of $U(f)$. For these c, the linear span $\langle D_c \rangle$ of $D_c = \varphi_{\Delta^{(1)}}(p^{-1}(c))$ is contained in $\langle \nu_\ell(c) \rangle_\ell$, and since the matrix

$$\left(y_i^\ell \right)_{\substack{i=1,\ldots,d \\ \ell=1,\ldots,d-1 \\ E_\ell \geq 0}}$$

has rank n (indeed, its columns are linearly independent because by adding a number of columns one obtains a $(d \times d)$-Vandermonde matrix), we find that actually $\langle D_c \rangle = \langle \nu_\ell(c) \rangle_\ell$. We conclude that the scroll $S \subset \mathbb{P}^{g-1}$ swept out by our g^1_d is exactly the rational normal scroll parameterized by the ν_ℓ's. Hence we obtain that the multiset of scrollar invariants with respect to g^1_d equals the multiset consisting of the non-negative E_ℓ's, which is exactly what we wanted. ■

Remark (continued). If f is only Δ-toric, rather than Δ-non-degenerate, it may happen that $d' < d$, where $d' = w(\Delta(f), v)$ and $d = w(\Delta, v)$. In this case g_v decomposes into a base-point free $g^1_{d'}$ and a fixed part F which is supported on at most two zero-dimensional toric orbits, as explained in the proof of Lemma 5.3. The base-point free part corresponds to the morphism $p : U(f) \to \mathbb{T}^1 : (x, y) \mapsto x$, and the above reasoning shows that for all but finitely many $c \in k^*$, the linear span $\langle D_c \rangle$ of $D_c = \varphi_{\Delta^{(1)}}(p^{-1}(c))$ equals $\langle \nu_\ell(c) \rangle_\ell$. For each of these D_c one clearly has $\langle D_c \rangle \subset \langle D_c + F \rangle$. We claim that actually equality holds. This implies that the scroll swept out by g_v coincides with the scroll swept out by its base-point free part, so that Theorem 9.1 also follows in the Δ-toric case. Note that it suffices to prove the claim under the assumption that $\Delta = \Delta^{\max}$. Indeed, from Lemma 4.3 (and the consequent remark) we see that if f is Δ-toric, then it is also $\Delta^{\max}$-toric. Of course switching from Δ to $\Delta^{\max}$ may have an influence on g_v, but it can only affect the fixed part F, and if it does then F becomes replaced by F' with $F' > F$. So if we can prove that $\langle D_c \rangle = \langle D_c + F' \rangle$ then necessarily $\langle D_c \rangle = \langle D_c + F \rangle$.

Let $\Delta(f)$ be contained in the strip $m_f \leq Y \leq m_f + d'$ and suppose that $0 < m_f$. Recall that Δ has a unique lower-most vertex v_{low}. Our assumption $\Delta = \Delta^{\max} = \Delta^{(1)(-1)}$ ensures that also $\Delta^{(1)}$ has a unique lower-most vertex and that the adjacent cones are similar. Denote the corresponding zero-dimensional orbit by P.

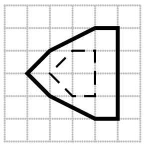


Then locally around $O(v_{\text{low}})$ we have a natural isomorphism $\text{Tor}(\Delta) \to \text{Tor}(\Delta^{(1)})$ under which $O(v_{\text{low}})$ corresponds to P. From the proof of Lemma 5.3 we conclude that $C \subset \text{Tor}(\Delta^{(1)})$ intersects the zero divisor $\mathcal{F}_c$ of $x^a y^b - c$, with $c \in k^*$ sufficiently generic, with multiplicity m_f in P. Our task is to prove that every hyperplane H containing the support of D_c intersects C in P with multiplicity at least m_f. But this follows from

$$I_{P,\mathbb{P}^{g-1}}(H, C) = I_{P,\text{Tor}(\Delta^{(1)})}(H \cap \text{Tor}(\Delta^{(1)}), C) \geq I_{P,\text{Tor}(\Delta^{(1)})}(\mathcal{F}_c, C) = m_f,$$

where $I_{P,X}(\cdot, \cdot)$ denotes the intersection multiplicity of the arguments in P when viewed as schemes inside X, and the inequality holds because $H \supset \langle D_c \rangle = \langle \nu_\ell(c) \rangle_\ell \supset \mathcal{F}_c$. A similar reasoning at the top (if needed) then proves that $\langle D_c \rangle = \langle D_c + F \rangle$.

Example (revisited, see Sections 5 and 6). Let Δ be the lattice polygon



where $\Delta^{(1)}$ is marked in dashed lines. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a Δ-non-degenerate (or Δ-toric) Laurent polynomial. Then $U(f)$ is a 4-gonal genus 7 curve carrying exactly two g^1_4's, namely $g_{(1,0)}$ and $g_{(0,1)}$. In the former case the scrollar invariants are $\{1, 1, 2\}$ while in the latter case they read $\{0, 2, 2\}$.

As a corollary to the proof of Theorem 9.1 we find:

Corollary 9.2. *Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$, where we assume that $\Delta^{(1)}$ is two-dimensional. Let v be a lattice direction. Then the rank of the complete linear system spanned by g_v equals the number of negative width invariants of Δ with respect to v (counting multiplicities) plus* 1*.*

Proof. Let $d = w(\Delta, v)$ and let $D \in g_v$, and assume that we work on the canonical model C of $U(f)$ from the proof of Theorem 9.1. By (19) we know that $\langle D \rangle$ is $(d - h^0(C, D))$-dimensional, while the proof of Theorem 9.1 tells us that the dimension equals the number of non-negative lattice width invariants minus 1. From this the statement follows. ■

In particular we find the following combinatorial characterization of completeness:

Corollary 9.3. *Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$, where we assume that $\Delta^{(1)}$ is two-dimensional. Let v be a lattice direction. Then g_v is complete if and only if the width invariants of Δ with respect to v are all non-negative.*

Example. Let $\Delta = d\Sigma$ for some $d \geq 2$, so that $U(f)$ is birationally equivalent to a smooth plane curve of degree d. The width invariants of Δ with respect to $(1, 0)$ are $(d-3, d-4, \ldots, 1, 0, -1)$, so that $g_{(1,0)}$ is not complete. (Indeed: it is a subsystem of the g^2_d cut out by all line sections of $\mathbb{P}^2$.)

Corollary 9.4. *Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$, where we assume that $\Delta^{(1)}$ is two-dimensional. Then the dimension of the scroll spanned by g_v equals the number of non-negative lattice width invariants of Δ with respect to v, unless this number is g (i.e. there are no strictly positive lattice width invariants) in which case the dimension equals $g - 1$.*

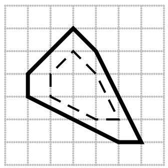
Proof. This follows from the considerations below formula (20), along with the combinatorial interpretation for $d - h^0(C, D)$ stated in the proof of Corollary 9.2. ■

Remarks.

- Inheriting the notation of the proof of Theorem 9.1, we have $C \subset \mathrm{Tor}(\Delta^{(1)}) \subset S \subset \mathbb{P}^{g-1}$. One can verify that $\mathrm{Tor}(\Delta^{(1)})$ intersects the fiber of π above a point $x \in k^*$ in a rational normal curve of degree $\gamma - 2$. Above $(1:0),(0:1) \in \mathbb{P}^1$ this fiber may degenerate.
- Through Corollary 6.2 and Theorem 9.1, the upper bound $\frac{2g-2}{\gamma}$ on the scrollar invariants with respect to a gonality pencil g^1_γ implies the purely combinatorial inequality
$$\mathrm{lw}(\Delta) \cdot E_\ell \leq 2\,\sharp(\Delta^{(1)} \cap \mathbb{Z}^2) - 2,$$
where Δ is understood to be contained in
$$\{(i,j) \in \mathbb{R}^2 \,|\, 0 \leq j \leq \mathrm{lw}(\Delta)\}$$
and the E_ℓ's are the width invariants of Δ with respect to any lattice width direction. This inequality holds as soon as $\sharp(\Delta^{(1)} \cap \mathbb{Z}^2) \geq 1$ (including the cases $\Delta = 2\Upsilon$ and $\Delta = d\Sigma$, which can be verified separately). The bound can be attained. For example, consider the lattice polygon $\Delta_{a,b} = \mathrm{conv}\{(b,0),(0,a),(0,0)\}$, where $a \geq 2$ and b is of the form $ak-1$ with $k \in \mathbb{Z}_{\geq 2}$. In this case, $\gamma = \mathrm{lw}(\Delta_{a,b}) = a$ is computed by $(1,0)$, and $E_1 = ak - k - 2 = \frac{2g-2}{\gamma}$.

10. Applications

Curves with prescribed invariants. The results of this article might serve as a tool in proving certain existence results in Brill-Noether theory. The number of inequivalent lattice polygons grows very quickly with the genus (for instance, in genus 30 this number is 957 001; see [6, Tab. 1]), resulting in a wide variety of Brill-Noether types, that (at least in principle) can be scanned by exhaustive search. To highlight one example, let Δ be the following polygon.



Every Δ-non-degenerate (or Δ-toric) curve is a 5-gonal curve of genus 9 admitting exactly three g^1_5's (corresponding to the lattice directions $(1,0)$, $(0,1)$ and $(1,-1)$), that are independent of each other, and with respect to each of which the scrollar invariants are $\{0,1,2,2\}$. Moreover, by [31, Thm. 2.5.12] the locus of such curves inside the moduli space $\mathcal{M}_9$ of curves of genus 9 has dimension 15. See [13] for a related discussion; note that each of our g^1_5's is of 'type II' (i.e. 0 is among the scrollar invariants), as opposed to the 'type I' pencils that are the main object of study in [13].

We want to stress that many Brill-Noether types are not represented in the toric world. For instance, Lemma 5.2.(vi) shows that the gonality of a smooth curve in a toric surface is $O(\sqrt{g})$, while general curves of genus g have gonality $\lceil g/2 \rceil + 1$. So the class of curves that we are considering in this article is rather special. In terms of moduli, the locus of curves of genus $g \geq 4$ that admit a smooth embedding in a toric surface has dimension $2g+1$, with the exception of $g = 7$, where the dimension reads 16; see [11]. Recall that $\dim \mathcal{M}_g = 3g - 3$.

Weierstrass semi-groups of embedding dimension 2. The Weierstrass semi-group of a point P on a smooth projective curve C is the set of possible pole orders at P of functions that are regular on $C \setminus \{P\}$. This is a numerical semi-group, i.e. a sub-semi-group of $\mathbb{N}$ with finite complement. A numerical semi-group is said to have *embedding dimension* 2 if it is of the form $a\mathbb{N} + b\mathbb{N}$ for coprime integers $a, b \geq 2$. Using Corollary 6.2 we can prove the following:

Theorem 10.1. *If a smooth projective curve C/k carries a point P having a Weierstrass semi-group of embedding dimension 2, then this semi-group does not depend on the choice of P.*

Remark. This is well-known in the case of hyperelliptic curves of genus $g \geq 2$, all of whose Weierstrass points have semi-group $2\mathbb{N} + (2g+1)\mathbb{N}$.

Proof. If C has a Weierstrass point with semi-group $a\mathbb{N} + b\mathbb{N}$ for coprime integers $a, b \geq 2$, then it is of genus $(a-1)(b-1)/2$ (by Riemann-Roch – this is the number of gaps in the semi-group). We claim that C has gonality $\min\{a, b\}$. Together, this implies that a and b are indeed uniquely determined (up to order). To prove the claim, we use a result of Miura stating that C is birationally equivalent to a smooth affine curve of the form

$$c_{b,0}x^b + c_{0,a}y^a + \sum_{ia+jb<ab} c_{i,j}x^i y^j, \qquad c_{b,0}c_{0,a} \neq 0.$$

See [37, Thm. 5.17, Lem. 5.30] or [36]. From this it is clear that C is $\Delta_{a,b}$-toric, where

$$\Delta_{a,b} = \mathrm{conv}\{(b,0), (0,a), (0,0)\}$$

(in fact C is even $\Delta_{a,b}$-non-degenerate, since an affine translation ensures appropriate behavior with respect to the toric boundary). By Corollary 6.2, we have that the gonality of C equals $\mathrm{lw}(\Delta_{a,b}) = \min\{a, b\}$. ■

Remark. Miura studied curves having a Weierstrass semi-group of the form $a\mathbb{N} + b\mathbb{N}$ in the context of coding theory; he called them $C_{a,b}$ *curves*. (In a recent past, $C_{a,b}$ curves have enjoyed fair interest from researchers in explicit algebraic geometry [16, 25, 37]). Then another way to state Theorem 10.1 is that a curve cannot be simultaneously $C_{a,b}$ and $C_{a',b'}$ for distinct pairs $\{a, b\}$ and $\{a', b'\}$.

Curves in Hirzebruch surfaces. We can use Theorem 9.1 to compute the scrollar invariants of smooth curves on Hirzebruch surfaces. An immediate corollary to this computation is that if a non-hyperelliptic smooth projective curve C of genus $g \geq 2$ can be embedded in the n^{th} Hirzebruch surface $\mathcal{H}_n$, then n is actually an invariant of C (that is, it cannot be embedded in $\mathcal{H}_{n'}$ for $n' \neq n$).

Theorem 10.2.
- *The scrollar invariants (with respect to any gonality pencil) of a smooth projective plane curve C/k of degree $d \geq 4$ are $\{0, 1, \ldots, d-3\}$.*

- *The scrollar invariants (with respect to any gonality pencil) of a smooth projective curve C/k of genus $g \geq 2$ and gonality γ in the n^{th} Hirzebruch surface $\mathcal{H}_n$ are*

$$\left\{ \frac{g}{\gamma - 1} + \left(\ell - \frac{\gamma}{2} \right) n - 1 \right\}_{1 \leq \ell \leq \gamma - 1}.$$

In particular, if $\gamma > 2$ then

$$n = \frac{2g - 2(\gamma - 1)(e_1 + 1)}{(\gamma - 1)(\gamma - 2)}$$

is an invariant of the curve.

PROOF. Because $\mathcal{H}_1$ is a blow-up of $\mathbb{P}^2$, the first statement is actually a corollary to the second. Nevertheless, we will treat it separately.

Let $C \subset \mathbb{P}^2$ be a smooth projective curve of degree d and fix a gonality pencil g^1_{d-1} on C. By [43, Prop. 3.13(ii)], the latter is computed by projecting from a point of the curve. Using a projective transformation we may assume that this point is $(0 : 1 : 0)$. Let $F(X, Y, Z)$ be a corresponding defining homogeneous degree d polynomial. Then $F(x, y, 1)$ is Δ-toric, with

$$\Delta = \mathrm{conv}\{(0,0), (d,0), (1, d-1), (0, d-1)\},$$

and our g^1_{d-1} corresponds to $(x, y) \mapsto x$, i.e. it equals $g_{(1,0)}$. The statement now follows from Theorem 9.1.

Next, let C be a smooth projective curve in $\mathcal{H}_n$. Due to the toric description of Hirzebruch surfaces [15, Ex. 3.1.16] we may assume that our curve C is Δ-toric, with

$$\Delta = \mathrm{conv}\{(0,0), (a + dn, 0), (a, d), (0, d)\}$$

for integers $a \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{\geq 2}$. Now

- If $a = 0$ and $n = 1$ then C is isomorphic to a smooth projective plane curve (of degree d) and the statement follows from the first part.
- If $a > 0$ or $n > 1$ then by Theorem 6.1 there exists only one gonality pencil, corresponding to vertical projection (i.e. $\gamma = d$). One finds that

  $$g = \sharp(\Delta^{(1)} \cap \mathbb{Z}^2) = \frac{\gamma(\gamma - 1)}{2} n + (\gamma - 1)(a - 1)$$

  and, by Theorem 9.1,

  $$e_\ell = a - 2 + \ell n \text{ (for } 1 \leq \ell \leq \gamma - 1).$$

  From these two equalities the statement follows.
- If $n = 0$ then $\Delta = [0, a] \times [0, d]$ is a standard rectangle. If $a \neq d$ then by Theorem 6.1 there exists only one gonality pencil, corresponding to horizontal or vertical projection (i.e. $\gamma = d$ or $\gamma = a$). If $a = d$ then there are two gonality pencils. In both cases the statement follows similarly from Theorem 9.1. ∎

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Computing graded Betti tables of toric surfaces

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Abstract

We present various facts on the graded Betti table of a projectively embedded toric surface, expressed in terms of the combinatorics of its defining lattice polygon. These facts include explicit formulas for a number of entries, as well as a lower bound on the length of the linear strand that we conjecture to be sharp (and prove to be so in several special cases). We also present an algorithm for determining the graded Betti table of a given toric surface by explicitly computing its Koszul cohomology, and report on an implementation in SageMath. It works well for ambient projective spaces of dimension up to roughly 25, depending on the concrete combinatorics, although the current implementation runs in finite characteristic only. As a main application we obtain the graded Betti table of the Veronese surface $\nu_6(\mathbb{P}^2) \subseteq \mathbb{P}^{27}$ in characteristic 40 009. This allows us to formulate precise conjectures predicting what certain entries look like in the case of an arbitrary Veronese surface $\nu_d(\mathbb{P}^2)$.

Contents

1 Introduction

Let k be a field of characteristic 0 and let $\Delta \subseteq \mathbb{R}^2$ be a lattice polygon, by which we mean the convex hull of a finite number of points of the standard lattice $\mathbb{Z}^2$. We write $\Delta^{(1)}$ for the convex hull of the lattice points in the interior of Δ. Assume that Δ is two-dimensional, write $N_\Delta = |\Delta \cap \mathbb{Z}^2|$, and let $S_\Delta = k[X_{i,j} \,|\, (i,j) \in \Delta \cap \mathbb{Z}^2]$, so that $\mathbb{P}^{N_\Delta - 1} = \operatorname{Proj} S_\Delta$. The toric surface over k associated to Δ is the Zariski closure of the image of

$$\varphi_\Delta : (k^*)^2 \hookrightarrow \mathbb{P}^{N_\Delta - 1} : (a,b) \mapsto (a^i b^j)_{(i,j)\in\Delta\cap\mathbb{Z}^2}.$$

We denote it by X_Δ and its ideal by I_Δ. It has been proved by Koelman [24] that I_Δ is generated by binomials of degree 2 and 3, where degree 2 suffices if and only if $|\partial\Delta \cap \mathbb{Z}^2| > 3$.

Our object of interest is the graded Betti table of X_Δ, which gathers the exponents appearing in a minimal free resolution

$$\cdots \to \bigoplus_{q\geq 2} S_\Delta(-q)^{\beta_{2,q}} \to \bigoplus_{q\geq 1} S_\Delta(-q)^{\beta_{1,q}} \to \bigoplus_{q\geq 0} S_\Delta(-q)^{\beta_{0,q}} \to {S_\Delta}/{I_\Delta} \to 0$$

of the homogeneous coordinate ring of X_Δ as a graded S_Δ-module, obtained by taking syzygies. Traditionally one writes $\beta_{p,p+q}$ in the pth column and the qth row. Alternatively and often more conveniently, the Betti numbers $\beta_{p,p+q}$ are the dimensions of the Koszul cohomology spaces $K_{p,q}(X_\Delta, \mathcal{O}(1))$, which will be described in detail in Section 2.

Remark 1.1. If Δ and Δ' are lattice polygons, we say that they are *unimodularly equivalent* (denoted by $\Delta \cong \Delta'$) if they are obtained from one another using a transformation from the affine group $\mathrm{AGL}_2(\mathbb{Z})$, that is a map of the form

$$\mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \mapsto (x,y)A + (a,b) \text{ with } A \in \mathrm{GL}_2(\mathbb{Z}) \text{ and } a,b \in \mathbb{Z}.$$

Unimodularly equivalent polygons yield projectively equivalent toric surfaces, which have the same graded Betti table. So we are interested in lattice polygons up to unimodular equivalence only.

In Section 3 we prove/gather some first facts on the graded Betti table. To begin with, we show that it has the following shape:

Lemma 1.2. *The graded Betti table of X_Δ has the form*

	0	1	2	3	…	$N_\Delta - 4$	$N_\Delta - 3$
0	1	0	0	0	…	0	0
1	0	b_1	b_2	b_3	…	$b_{N_\Delta-4}$	$b_{N_\Delta-3}$
2	0	$c_{N_\Delta-3}$	$c_{N_\Delta-4}$	$c_{N_\Delta-5}$	…	c_2	c_1

, (1)

where omitted entries are understood to be 0*. Moreover* $(\forall \ell : c_\ell = 0) \Leftrightarrow \Delta^{(1)} = \emptyset$.

We also provide a closed formula for the antidiagonal differences:

Lemma 1.3. *For* $\ell = 1, \ldots, N_\Delta - 2$ *one has*

$$b_\ell - c_{N_\Delta-1-\ell} = \ell\binom{N_\Delta - 1}{\ell + 1} - 2\binom{N_\Delta - 3}{\ell - 1}\operatorname{vol}(\Delta)$$

where it is understood that $b_{N_\Delta-2} = c_{N_\Delta-2} = 0$.

This reduces the determination of the graded Betti numbers to that of the b_i's (or of the c_i's). Finally we give explicit formulas for the entries b_1, b_2, and $b_{N_\Delta-4}$, $b_{N_\Delta-3}$, which then also yield explicit descriptions of c_1, c_2, c_3, and $c_{N_\Delta-3}$. The precise statements are a bit lengthy and can be found in Section 3.3.

We mentioned Koelman's result on the generators of I_Δ: this was vastly generalized in the Ph.D. thesis of Hering [22, Thm. IV.20], building on an observation due to Schenck [34] and invoking a theorem of Gallego–Purnaprajna [18, Thm. 1.3]. She provided a combinatorial interpretation for the number of leading zeroes in the quadratic strand (the row $q = 2$).

Theorem 1.4 (Hering, Schenck)**.** *If* $\Delta^{(1)} \neq \emptyset$ *then* $\min\{\, \ell \,|\, c_{N_\Delta-\ell} \neq 0 \,\} = |\partial\Delta \cap \mathbb{Z}^2|$, *where* $\partial\Delta$ *denotes the boundary of* Δ*.*

In Green's language of property N_p, this reads that S_Δ/I_Δ satisfies N_p if and only if $|\partial\Delta \cap \mathbb{Z}^2| \geq p+3$. Hering's thesis contains several other statements of property N_p type for toric varieties of any dimension.

In Section 4 we work towards a similar combinatorial expression for the number of zeroes at the end of the linear strand (the row $q = 1$). We are unable to provide a definitive answer, but we formulate a concrete conjecture that we can prove in many special cases. The central combinatorial notion is the following:

Definition 1.5. *Let* Δ *be a lattice polygon. If* $\Delta \neq \emptyset$*, then the* lattice width *of* Δ*, denoted* $\mathrm{lw}(\Delta)$*, is the minimal height* d *of a horizontal strip* $\mathbb{R} \times [0, d]$ *in which* Δ *can be mapped using a unimodular transformation. If* $\Delta = \emptyset$*, we define* $\mathrm{lw}(\Delta) = -1$*.*

Remark that $\mathrm{lw}(\Delta) = 0$ if and only if Δ is zero- or one-dimensional. The lattice width can be computed recursively; see [8, Thm. 4] or [28, Thm. 13]: if Δ is two-dimensional then

$$\mathrm{lw}(\Delta) = \begin{cases} \mathrm{lw}(\Delta^{(1)}) + 3 & \text{if } \Delta \cong d\Sigma \text{ for some } d \geq 2, \\ \mathrm{lw}(\Delta^{(1)}) + 2 & \text{if not,} \end{cases}$$

where $\Sigma := \mathrm{conv}\{(0,0),(1,0),(0,1)\}$.

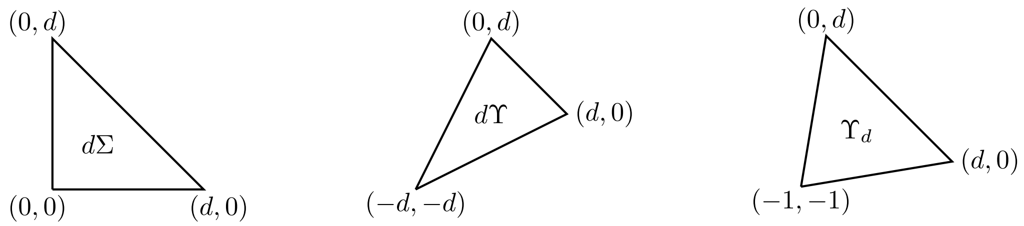
The multiples of Σ, whose associated toric surfaces are the Veronese surfaces (more precisely $X_{d\Sigma}$ is the image of $\mathbb{P}^2$ under the d-uple embedding ν_d), will keep playing a special role throughout the rest of this paper. Another important role is attributed to multiples of $\Upsilon = \mathrm{conv}\{(-1,-1),(1,0),(0,1)\}$. Finally we also introduce the polygons $\Upsilon_d = \mathrm{conv}\{(-1,-1),(d,0),(0,d)\}$, where we note that $\Upsilon_1 = \Upsilon$. For the sake of overview, these polygons are depicted in Figure 1, along with some elementary combinatorial properties.

Our conjecture is as follows:

Conjecture 1.6. *If* $\Delta \not\cong \Sigma, \Upsilon$ *then one has* $\min\{\, \ell \,|\, b_{N_\Delta-\ell} \neq 0 \,\} = \mathrm{lw}(\Delta) + 2$*, unless*

$$\Delta \cong d\Sigma \text{ for some } d \geq 2 \qquad \text{or} \qquad \Delta \cong \Upsilon_d \text{ for some } d \geq 2 \qquad \text{or} \qquad \Delta \cong 2\Upsilon$$

in which case it is $\mathrm{lw}(\Delta) + 1$*.*



$\mathrm{lw}(d\Sigma) = d$ $\qquad$ $\mathrm{lw}(d\Upsilon) = 2d$ $\qquad$ $\mathrm{lw}(\Upsilon_d) = d+1$

$(d\Sigma)^{(1)} \cong (d-3)\Sigma$ $\qquad$ $(d\Upsilon)^{(1)} \cong (d-1)\Upsilon$ $\qquad$ $\Upsilon_d^{(1)} \cong (d-1)\Sigma$

Figure 1: Three recurring families of polygons

In other words we conjecture that the number of zeroes at the end of the linear strand equals $\mathrm{lw}(\Delta) - 1$, unless Δ is of the form $d\Sigma$, Υ_d or 2Υ, in which case it equals $\mathrm{lw}(\Delta) - 2$.

Remark 1.7. The excluded cases $\Delta \cong \Sigma, \Upsilon$ are pathological: the Betti tables are

	0
0	1
1	0
2	0

resp.

	0	1
0	1	0
1	0	0
2	0	1

,

i.e. the entire linear strands are zero.

As explained in Section 4 the upper bound $\min\{\,\ell \,|\, b_{N_\Delta - \ell} \neq 0\,\} \leq \mathrm{lw}(\Delta) + 2$ follows from the fact that our toric surface X_Δ is naturally contained in a rational normal scroll of dimension $\mathrm{lw}(\Delta) + 1$, which is known to have non-zero linear syzygies up to column $p = N_\Delta - \mathrm{lw}(\Delta) - 2$. Then also X_Δ must have non-zero linear syzygies up to that point, yielding the desired bound. Thus another way of reading Conjecture 1.6 is that the natural bound coming from this ambient rational normal scroll is usually sharp. This is in the philosophy of Green's $K_{p,1}$ theorem [1, Thm. 3.31] that towards the end of the resolution, 'most' linear syzygies must come from the smallest ambient variety of minimal degree. In the exceptional cases $d\Sigma$, Υ_d and 2Υ we can prove the sharper bound $\min\{\,\ell \,|\, b_{N_\Delta - \ell} \neq 0\,\} \leq \mathrm{lw}(\Delta) + 1$ by following a slightly different argument, using explicit computations in Koszul cohomology.

We can prove sharpness of these bounds in a considerable number of special situations, overall leading to the following partial result towards Conjecture 1.6.

Theorem 1.8. *If* $\Delta \ncong \Sigma, \Upsilon$ *then one has* $\min\{\,\ell \,|\, b_{N_\Delta - \ell} \neq 0\,\} \leq \mathrm{lw}(\Delta) + 2$. *If*

$$\Delta \cong d\Sigma \text{ for some } d \geq 2 \qquad \text{or} \qquad \Delta \cong \Upsilon_d \text{ for some } d \geq 2 \qquad \text{or} \qquad \Delta \cong 2\Upsilon$$

then moreover one has the sharper bound $\mathrm{lw}(\Delta) + 1$. *In other words the sharpest applicable upper bound predicted by Conjecture 1.6 holds. Moreover:*

- *If* $N_\Delta \leq 32$ *then the bound is met.*

- *If a certain non-exceptional lattice polygon Δ (i.e. not of the form $d\Sigma, \Upsilon_d, 2\Upsilon$) meets the bound then so does every lattice polygon containing Δ and having the same lattice width. In particular if* $\operatorname{lw}(\Delta) \leq 6$ *then the bound is met.*

† *If $\Delta = \Gamma^{(1)}$ for some larger lattice polygon Γ and if Green's canonical syzygy conjecture holds for smooth curves on X_Δ (known to be true if $H^0(X_\Delta, -K_{X_\Delta}) \geq 2$) then the bound is met.*

Sharpness in the cases where $N_\Delta \leq 32$ is obtained by explicit verification, based on the data from [7] and using the algorithm described in Section 8; this covers more than half a million (unimodular equivalence classes of) small lattice polygons. The statement involving $\operatorname{lw}(\Delta) \leq 6$ relies on this exhaustive verification, along with the classification of inclusion-minimal lattice polygons having a given lattice width, which is elaborated in [12].

Remark 1.9. The statement marked with † will not be proven in the current paper, even though it is actually the reason why we came up with Conjecture 1.6 in the first place. To date, Green's canonical syzygy conjecture for curves in toric surfaces remains open in general, but the cases where $H^0(X_\Delta, -K_{X_\Delta}) \geq 2$ are covered by recent work of Lelli-Chiesa [25], which allows one to deduce Conjecture 1.6 for all multiples of Υ, for all multiples of Σ, for all polygons $[0, a] \times [0, b]$ with $a, b \geq 1$, and so on. The details of this are discussed in a subsequent paper [11], which is devoted to syzygies of curves in toric surfaces. For the sake of conciseness we have chosen to keep the present document curve-free.

Next we describe our algorithm for determining the graded Betti table of $X_\Delta \subseteq \mathbb{P}^{N_\Delta - 1}$ upon input of a lattice polygon Δ, by explicitly computing its Koszul cohomology. The details can be found in Section 8, but in a nutshell the ingredients are as follows. The most dramatic speed-up comes from incorporating the torus action, which decomposes the cohomology spaces into eigenspaces, one for each bidegree $(a, b) \in \mathbb{Z}^2$, all but finitely many of which are trivial. Another important speed-up comes from toric Serre duality, enabling a meet-in-the-middle approach where one fills the graded Betti table starting from the left and from the right simultaneously. A third speed-up comes from the explicit formula for the antidiagonal differences given in Lemma 1.3, thanks to which it suffices to determine half of the graded Betti table only. Moreover if $|\partial\Delta \cap \mathbb{Z}^2|$ is large (which is particularly the case for the Veronese polygons $d\Sigma$) then many of these entries come for free using Hering and Schenck's Theorem 1.4. A fourth theoretical ingredient is a combinatorial description of certain exact subcomplexes of the Koszul complex that can be quotiented out, resulting in smaller vector spaces, thereby making the linear algebra more manageable. Because this seems interesting in its own right, we have devoted the separate Section 7 to it. Final ingredients include sparse linear algebra, using symmetries, and working in finite characteristic. More precisely, most of the data gathered in this article, some of which can be found in Appendix A, are obtained by computing modulo 40 009, the smallest prime number larger than 40 000.

Remark 1.10. By semi-continuity the entries of the graded Betti table cannot decrease upon reduction of X_Δ modulo some prime number. Therefore working in finite characteristic is fine for proving that certain entries are zero, as is done in our partial verification of Conjecture 1.6. But entries that are found to be non-zero might a priori be too large, even though we do not expect them to be. Therefore the non-zero entries of some of the graded Betti tables given in Appendix A are conjectural. For technical reasons our current implementation does not straightforwardly adapt to characteristic zero, but we are working on fixing this issue. Although it would come at the cost of some efficiency, this should enable us to confirm all of the data from Appendix A in characteristic zero.

In view of the wide interest in syzygies of Veronese modules [5, 14, 19, 27, 30, 31, 32], the most interesting new graded Betti table that we obtain is that of $X_{6\Sigma} \subseteq \mathbb{P}^{27}$, i.e. the image of $\mathbb{P}^2$ under the 6-uple embedding ν_6, in characteristic 40 009. Up to 5Σ this data was recently gathered (in characteristic zero) by Greco and Martino [19]. An extrapolating glance at these Betti tables naturally leads to the following conjecture:

Conjecture 1.11. *Consider the graded Betti table of the d-fold Veronese surface $X_{d\Sigma}$. If $d \geq 2$ then the last non-zero entry on the linear strand is*

$$b_{d(d+1)/2} = \frac{d^3(d^2-1)}{8},$$

while if $d \geq 3$ then the first non-zero entry on the quadratic strand is

$$c_g = \binom{N_{(d\Sigma)^{(1)}} + 8}{9}$$

where $N_{(d\Sigma)^{(1)}} = |(d\Sigma)^{(1)} \cap \mathbb{Z}^2| = (d-1)(d-2)/2$.

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2 Koszul cohomology of toric surfaces

As is well-known, instead of using syzygies, the entries of the graded Betti table can also be defined as dimensions of Koszul cohomology spaces, which we now explicitly describe in the specific case of toric surfaces. We refer to the book by Aprodu and Nagel [1] for an introduction to Koszul cohomology, and to the books by Fulton [17] and Cox, Little and Schenck [13] for more background on toric geometry.

For a lattice polygon Δ we write V_Δ for the space of Laurent polynomials

$$\sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} c_{i,j}x^iy^j \in k[x^{\pm 1}, y^{\pm 1}],$$

which we view as functions on X_Δ through φ_Δ. This equals the space $H^0(X_\Delta, L_\Delta)$ of global sections of $\mathcal{O}(L_\Delta)$, where L_Δ is some concrete very ample torus-invariant divisor on X_Δ satisfying $\mathcal{O}(L_\Delta) \cong \mathcal{O}(1)$. More generally $V_{q\Delta} = H^0(X_\Delta, qL_\Delta)$ for each $q \geq 0$.

Then the entry in the pth column and the qth row of the graded Betti table of X_Δ is the dimension of the Koszul cohomology space $K_{p,q}(X_\Delta, L_\Delta)$, defined as the cohomology in the middle of

$$\bigwedge^{p+1} H^0(X_\Delta, L_\Delta) \otimes H^0(X_\Delta, (q-1)L_\Delta) \xrightarrow{\delta} \bigwedge^{p} H^0(X_\Delta, L_\Delta) \otimes H^0(X_\Delta, qL_\Delta)$$
$$\xrightarrow{\delta'} \bigwedge^{p-1} H^0(X_\Delta, L_\Delta) \otimes H^0(X_\Delta, (q+1)L_\Delta)$$

which can be rewritten as

$$\bigwedge^{p+1} V_\Delta \otimes V_{(q-1)\Delta} \xrightarrow{\delta} \bigwedge^{p} V_\Delta \otimes V_{q\Delta} \xrightarrow{\delta'} \bigwedge^{p-1} V_\Delta \otimes V_{(q+1)\Delta}. \qquad (2)$$

Here the coboundary maps δ and δ' are defined by

$$v_1 \wedge v_2 \wedge v_3 \wedge v_4 \wedge \cdots \otimes w \mapsto \sum (-1)^s v_1 \wedge v_2 \wedge v_3 \wedge v_4 \wedge \cdots \wedge \widehat{v_s} \wedge \cdots \otimes v_s w \qquad (3)$$

where s ranges from 1 to $p+1$ resp. 1 to p, and $\widehat{v_s}$ means that v_s is being omitted. In particular one sees that b_ℓ is the dimension of the cohomology in the middle of

$$\bigwedge^{\ell+1} V_\Delta \xrightarrow{\delta} \bigwedge^{\ell} V_\Delta \otimes V_\Delta \xrightarrow{\delta'} \bigwedge^{\ell-1} V_\Delta \otimes V_{2\Delta}, \qquad (4)$$

where we note that the left map is always injective. On the other hand c_ℓ is the dimension of the cohomology in the middle of

$$\bigwedge^{N_\Delta-1-\ell} V_\Delta \otimes V_\Delta \xrightarrow{\delta} \bigwedge^{N_\Delta-2-\ell} V_\Delta \otimes V_{2\Delta} \xrightarrow{\delta'} \bigwedge^{N_\Delta-3-\ell} V_\Delta \otimes V_{3\Delta}, \qquad (5)$$

for all $\ell = 1, \ldots, N_\Delta - 3$.

2.1 Duality

A more concise description of the c_ℓ's is obtained using Serre duality. Because the version that we will invoke requires us to work with smooth surfaces, we consider a toric resolution of singularities $X \to X_\Delta$ and let L be the pullback of L_Δ. Then L may no longer be very ample, but it remains globally generated by the same global sections V_Δ. Let K be the canonical divisor on X obtained by taking minus the sum of all torus-invariant prime divisors. By Demazure vanishing one has $H^1(X, qL) = 0$ for all $q \geq 0$, so that we can apply the duality formula from [1, Thm. 2.25], which in our case reads

$$K_{p,q}(X,L)^\vee \cong K_{N_\Delta-3-p,3-q}(X;K,L),$$

to conclude that

$$\begin{aligned} b_\ell &= \dim K_{\ell,1}(X_\Delta, L_\Delta) = \dim K_{\ell,1}(X,L) = \dim K_{N_\Delta-3-\ell,2}(X;K,L), \\ c_\ell &= \dim K_{N_\Delta-2-\ell,2}(X_\Delta, L_\Delta) = \dim K_{N_\Delta-2-\ell,2}(X,L) = \dim K_{\ell-1,1}(X;K,L), \end{aligned}$$

again for all $\ell = 1, \ldots, N_\Delta - 3$. Here the attribute '$;K$' denotes Koszul cohomology twisted by K, which is defined as before, except that each appearance of $\cdot \otimes H^0(X, qL)$ is replaced by $\cdot \otimes H^0(X, qL + K)$. Using that $H^0(X, qL + K) = V_{(q\Delta)^{(1)}}$ for $q \geq 1$ and that $H^0(X, K) = 0$ we find that b_ℓ is the cohomology in the middle of

$$\bigwedge^{N_\Delta-2-\ell} V_\Delta \otimes V_{\Delta^{(1)}} \xrightarrow{\delta} \bigwedge^{N_\Delta-3-\ell} V_\Delta \otimes V_{(2\Delta)^{(1)}} \xrightarrow{\delta'} \bigwedge^{N_\Delta-4-\ell} V_\Delta \otimes V_{(3\Delta)^{(1)}} \qquad (6)$$

and, more interestingly, that c_ℓ is the dimension of the kernel of

$$\bigwedge^{\ell-1} V_\Delta \otimes V_{\Delta^{(1)}} \xrightarrow{\delta'} \bigwedge^{\ell-2} V_\Delta \otimes V_{(2\Delta)^{(1)}}. \qquad (7)$$

For example this gives a quick way of seeing that $c_1 = \dim \ker(V_{\Delta^{(1)}} \to 0) = N_{\Delta^{(1)}}$.

2.2 Bigrading

For $(a,b) \in \mathbb{Z}^2$ we call an element of

$$\bigwedge^p V_\Delta \otimes V_{q\Delta}$$

homogeneous of bidegree (a,b) if it is a k-linear combination of elementary tensors of the form

$$x^{i_1}y^{j_1} \wedge \cdots \wedge x^{i_p}y^{j_p} \otimes x^{i'}y^{j'}$$

satisfying $(i_1,j_1) + \cdots + (i_p,j_p) + (i',j') = (a,b)$. The coboundary morphisms δ and δ' send homogeneous elements to homogeneous elements of the same bidegree, i.e. the Koszul complex is naturally bigraded. Thus the Koszul cohomology spaces decompose as

$$K_{p,q}(X,L) = \bigoplus_{(a,b)\in\mathbb{Z}^2} K_{p,q}^{(a,b)}(X,L)$$

where in fact it suffices to let (a,b) range over $(p+q)\Delta \cap \mathbb{Z}^2$. Similarly, we have a decomposition of the twisted cohomology spaces

$$K_{p,q}(X;K,L) = \bigoplus_{(a,b)\in\mathbb{Z}^2} K_{p,q}^{(a,b)}(X;K,L)$$

where now (a,b) in fact runs over $\left(p\Delta + (q\Delta)^{(1)}\right) \cap \mathbb{Z}^2$. In particular also the b_ℓ's and the c_ℓ's, and as a matter of fact the entire graded Betti table, decompose as sums of smaller instances. We will write

$$b_{\ell,(a,b)} = \dim K_{\ell,1}^{(a,b)}(X,L), \qquad b_{\ell,(a,b)}^\vee = \dim K_{N_\Delta-3-\ell,2}^{(a,b)}(X;K,L),$$
$$c_{\ell,(a,b)} = \dim K_{N_\Delta-2-\ell,2}^{(a,b)}(X,L), \qquad c_{\ell,(a,b)}^\vee = \dim K_{\ell-1,1}^{(a,b)}(X;K,L),$$

so that

$$b_\ell = \sum_{(a,b)\in\mathbb{Z}^2} b_{\ell,(a,b)} = \sum_{(a,b)\in\mathbb{Z}^2} b_{\ell,(a,b)}^\vee \quad \text{and} \quad c_\ell = \sum_{(a,b)\in\mathbb{Z}^2} c_{\ell,(a,b)} = \sum_{(a,b)\in\mathbb{Z}^2} c_{\ell,(a,b)}^\vee.$$

Example 2.1. For $\Delta = 4\Sigma$ one can compute that $c_3 = \dim K_{2,1}(X;K,L) = 55$, which decomposes as the sum of the following numbers.

$$\begin{array}{cccccccccc}
0 \\
0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 2 & 2 & 2 & 0 \\
0 & 2 & 3 & 3 & 2 & 0 \\
0 & 2 & 3 & 4 & 3 & 2 & 0 \\
0 & 1 & 2 & 3 & 3 & 2 & 1 & 0 \\
0 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Here the entry in the ath column (counting from the left) and the bth row (counting from the bottom) is the dimension $c_{3,(a,b)}^\vee$ of the degree (a,b) part. In other words we think of the above triangle as being in natural correspondence with the lattice points (a,b) inside $2\Delta + \Delta^{(1)} = (1,1) + 9\Sigma$.

2.3 Duality versus bigrading

An interesting observation that came out of a joint discussion with Milena Hering is that duality respects the bigrading along the rule

$$K_{p,q}^{(a,b)}(X,L)^\vee \cong K_{N_\Delta-3-p,3-q}^{\sigma_\Delta-(a,b)}(X;K,L),$$

where σ_Δ denotes the sum of all lattice points in Δ. We postpone a proof to [2], but note that taking dimensions yields the formulas

$$b_{\ell,(a,b)} = b_{\ell,\sigma_\Delta-(a,b)}^\vee \quad \text{and} \quad c_{\ell,(a,b)} = c_{\ell,\sigma_\Delta-(a,b)}^\vee. \tag{8}$$

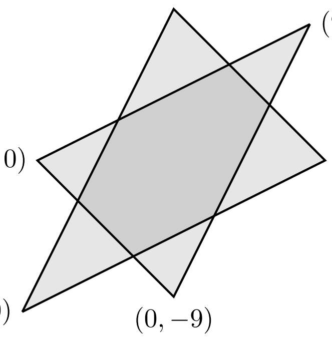
These imply that $K_{p,q}(X, L)$ is actually supported on the degrees (a, b) that are contained in

$$(p+q)\Delta \ \cap \ \left(\sigma_\Delta - (N_\Delta - 3 - p)\Delta - ((3-q)\Delta)^{(1)}\right),$$

and similarly that $K_{p,q}(X; K, L)$ vanishes outside

$$\left(p\Delta + (q\Delta)^{(1)}\right) \cap (\sigma_\Delta - (N_\Delta - p - q)\Delta).$$

The image below illustrates this for $\Delta = 2\Upsilon$, $p = 4$, $q = 1$, where $K_{p,q}(X; K, L)$ is supported on $9\Upsilon \cap (-10\Upsilon)$:



In principle this could be used to speed up our computation of the graded Betti table, because it says that certain bidegrees can be omitted. Unfortunately the vanishing happens in a range of bidegrees that is dealt with relatively easily anyway. Therefore, the computational advantage is negligible and we will not use this in our algorithm.

3 First facts on the graded Betti table

3.1 Overall shape of the graded Betti table

We prove the shape of the graded Betti table of X_Δ announced in Lemma 1.2, by invoking some well-known theorems from the existing literature. It is also possible to give a more elementary, handcrafted proof using Koszul cohomology.

Proof of Lemma 1.2. Hochster has proven that S_Δ/I_Δ is a Cohen–Macaulay module [13, Ex. 9.2.8]. Its Krull dimension equals 3, and therefore the Auslander–Buchsbaum formula [15, Thm. A.2.15] implies that the graded Betti table has non-zero entries up to column $p = N_\Delta - 3$. Now it is well-known that the Hilbert polynomial $P_{X_\Delta}(d)$ of X_Δ is given by the Ehrhart polynomial

$$|d\Delta \cap \mathbb{Z}^2| = \mathrm{vol}(\Delta)d^2 + \frac{|\partial\Delta \cap \mathbb{Z}^2|}{2}d + 1, \tag{9}$$

and that this matches with the Hilbert function $H_{X_\Delta}(d)$ for all integers $d \geq 0$. In fact, the smallest integer s such that $P_{X_\Delta}(d) = H_{X_\Delta}(d)$ for all $d \geq s$ is

$$\begin{cases} 0 & \text{if } \Delta^{(1)} \neq \emptyset, \\ -1 & \text{if } \Delta^{(1)} = \emptyset. \end{cases}$$

From [15, Cor. 4.8] we conclude that the Castelnuovo–Mumford regularity of X_Δ equals 2, unless $\Delta^{(1)} = \emptyset$ in which case it equals 1. □

The polygons for which $\Delta^{(1)} = \emptyset$ have the following geometric characterization:

Lemma 3.1. *The surface $X_\Delta \subseteq \mathbb{P}^{N_\Delta - 1}$ is a variety of minimal degree if and only if $\Delta^{(1)} = \emptyset$.*

Proof. By definition X_Δ has minimal degree if and only if $\deg X_\Delta = 1 + \operatorname{codim} X_\Delta$. By the above formula (9) for the Hilbert polynomial this can be rewritten as

$$2 \operatorname{vol}(\Delta) = N_\Delta - 2$$

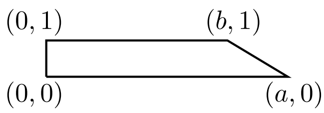
which by Pick's theorem holds if and only if $\Delta^{(1)} = \emptyset$. □

It follows that if $\Delta^{(1)} = \emptyset$ then the graded Betti table of X_Δ is of the form

	0	1	2	3	...	$N_\Delta - 4$	$N_\Delta - 3$
0	1	0	0	0	...	0	0
1	0	$\binom{N_\Delta-2}{2}$	$2\binom{N_\Delta-2}{3}$	$3\binom{N_\Delta-2}{4}$	...	$(N_\Delta-4)\binom{N_\Delta-2}{N_\Delta-3}$	$(N_\Delta-3)\binom{N_\Delta-2}{N_\Delta-2}$,

(10)

because the Eagon–Northcott complex is exact in this case; see for instance [15, App. A2H]. It also follows that if $\Delta^{(1)} \neq \emptyset$ then $b_{N_\Delta - 3} = 0$; see [1, Thm. 3.31(i)]. From a combinatorial viewpoint the two-dimensional lattice polygons Δ for which $\Delta^{(1)} = \emptyset$ were classified in [23, Ch. 4]: up to unimodular equivalence they are 2Σ and the Lawrence prisms



for integers $a \geq b \geq 0$ with $a > 0$.

The respective corresponding X_Δ's are the Veronese surface in $\mathbb{P}^5$ and the rational normal surface scrolls of type (a, b). One thus sees that Conjecture 1.6 is true if $\Delta^{(1)} = \emptyset$.

3.2 Antidiagonal differences

From the explicit shape (9) of the Hilbert polynomial, the closed formula

$$b_\ell - c_{N_\Delta - 1 - \ell} = \ell \binom{N_\Delta - 1}{\ell + 1} - 2 \binom{N_\Delta - 3}{\ell - 1} \operatorname{vol}(\Delta)$$

for the antidiagonal differences, which was announced in Lemma 1.3, can be proved by induction. We will give a slightly more convenient argument using Koszul cohomology.

Proof of Lemma 1.3. The proof relies on three elementary facts:

(i) Pick's theorem,

(ii) for any bounded complex of finite-dimensional vector spaces V_j one has

$$\sum_j (-1)^j \dim V_j = \sum_j (-1)^j \dim H^j,$$

where H^j is the cohomology of the complex at place j,

(iii) for all $n, k, N \geq 0$ we have $\sum_{j=0}^n (-1)^j \binom{N}{n-j}\binom{j}{k} = (-1)^k \binom{N-k-1}{n-k}$.

We compute

$$\begin{aligned}
b_\ell - c_{N_\Delta - 1 - \ell} &= \sum_{j=0}^{\ell+1} (-1)^{j+1} \dim K_{\ell-j+1,j}(X_\Delta, L_\Delta) \\
&\overset{\text{(ii)}}{=} \sum_{j=0}^{\ell+1} (-1)^{j+1} \dim \left(\bigwedge^{\ell+1-j} V_\Delta \otimes V_{j\Delta} \right) \\
&= \sum_{j=0}^{\ell+1} (-1)^{j+1} \binom{N_\Delta}{\ell+1-j} N_{j\Delta} \\
&\overset{\text{(i)}}{=} -\sum_{j=0}^{\ell+1} (-1)^j \binom{N_\Delta}{\ell+1-j} \left(j^2 \operatorname{vol}(\Delta) + \frac{j}{2} \left|\partial\Delta \cap \mathbb{Z}^2\right| + 1\right) \\
&\overset{\text{(i)}}{=} -\sum_{j=0}^{\ell+1} (-1)^j \binom{N_\Delta}{\ell+1-j} \left(j^2 \operatorname{vol}(\Delta) + j(N_\Delta - \operatorname{vol}(\Delta) - 1) + 1\right) \\
&= -\sum_{j=0}^{\ell+1} (-1)^j \binom{N_\Delta}{\ell+1-j} \left(2\operatorname{vol}(\Delta)\binom{j}{2} + (N_\Delta - 1)\binom{j}{1} + \binom{j}{0}\right) \\
&\overset{\text{(iii)}}{=} -2\operatorname{vol}(\Delta)\binom{N_\Delta - 3}{\ell - 1} + (N_\Delta - 1)\binom{N_\Delta - 2}{\ell} - \binom{N_\Delta - 1}{\ell + 1} \\
&= -2\operatorname{vol}(\Delta)\binom{N_\Delta - 3}{\ell - 1} + \ell\binom{N_\Delta - 1}{\ell + 1},
\end{aligned}$$

which equals the desired expression. □

We note the following corollary to Lemma 1.3:

Corollary 3.2. *For all ℓ one has that $b_\ell \geq c_{N_\Delta - 1 - \ell}$ if and only if*

$$\ell \leq \frac{(N_\Delta - 1)(N_\Delta - 2)}{2\operatorname{vol}(\Delta)} - 1.$$

Remark 3.3. Note that $2\operatorname{vol}(\Delta) = 2N_\Delta - |\partial\Delta \cap \mathbb{Z}^2| - 2$ by Pick's theorem. This is typically $\approx 2N_\Delta$, so the point where the c_ℓ's take over from the b_ℓ's is about halfway the Betti table. If $|\partial\Delta \cap \mathbb{Z}^2|$ is relatively large then $2\operatorname{vol}(\Delta)$ becomes smaller when compared to N_Δ, and the takeover point is shifted to the right.

3.3 Explicit formulas for some entries

We can give a complete combinatorial characterization of eight entries. Six of these are rather straightforward:

Corollary 3.4. *On the quadratic strand one has*

$$c_1 = N_{\Delta^{(1)}}, \qquad c_2 = \begin{cases} (N_\Delta - 3)(N_{\Delta^{(1)}} - 1) & \text{if } \Delta^{(1)} \neq \emptyset, \\ 0 & \text{if } \Delta^{(1)} = \emptyset, \end{cases}$$

$$c_{N_\Delta - 3} = \begin{cases} 0 & \text{if } |\partial\Delta \cap \mathbb{Z}^2| > 3, \\ 1 & \text{if } |\partial\Delta \cap \mathbb{Z}^2| = 3 \text{ and } \dim \Delta^{(1)} = 2, \\ N_\Delta - 3 & \text{if } |\partial\Delta \cap \mathbb{Z}^2| = 3 \text{ and } \dim \Delta^{(1)} \leq 1. \end{cases}$$

On the linear strand one has

$$b_1 = \binom{N_\Delta - 1}{2} - 2\operatorname{vol}(\Delta), \qquad b_{N_\Delta - 3} = \begin{cases} 0 & \text{if } \Delta^{(1)} \neq \emptyset, \\ N_\Delta - 3 & \text{if } \Delta^{(1)} = \emptyset, \end{cases}$$

$$b_2 = 2\binom{N_\Delta - 1}{3} - 2(N_\Delta - 3)\operatorname{vol}(\Delta) + c_{N_\Delta - 3}.$$

Proof. The formulas for b_1 and c_1 follow immediately from Lemma 1.3, where in the latter case we use that $N_\Delta - 2 - 2\operatorname{vol}(\Delta) = N_{\Delta^{(1)}}$ by Pick's theorem. The entry $c_{N_\Delta - 3}$ equals the number of cubics in a minimal set of generators of I_Δ, which was determined in [9, §2]. Together with Lemma 1.3 this then gives the formula for b_2. The formula for $b_{N_\Delta - 3}$ was discussed above, and the formula for c_2 then again follows using Lemma 1.3 in combination with Pick's theorem. □

In Section 6 we will extend this list as follows. This will take considerably more work, and depends on our proof of Conjecture 1.6 for polygons of small lattice width, given in Section 5.

Theorem 3.5. *Assume that $N_\Delta \geq 4$, or equivalently that $\Delta \not\cong \Sigma$. Then we have*

$$b_{N_\Delta - 4} = (N_\Delta - 4) \cdot B_\Delta \quad \text{where} \quad B_\Delta = \begin{cases} 0 & \text{if } \dim \Delta^{(1)} = 2,\ \Delta \not\cong \Upsilon_2, \\ 1 & \text{if } \dim \Delta^{(1)} = 1 \text{ or } \Delta \cong \Upsilon_2, \\ (N_\Delta - 1)/2 & \text{if } \dim \Delta^{(1)} = 0, \\ N_\Delta - 2 & \text{if } \Delta^{(1)} = \emptyset \end{cases}$$

and

$$c_3 = (N_\Delta - 4)\left((N_\Delta - 3)\operatorname{vol}(\Delta) - \frac{(N_\Delta - 1)(N_\Delta - 2)}{2} + B_\Delta\right).$$

4 Bound on the length of the linear strand

4.1 Bound through rational normal scrolls

Let $\Delta \subseteq \mathbb{R}^2$ be a two-dimensional lattice polygon and apply a unimodular transformation in order to have $\Delta \subseteq \mathbb{R} \times [0, d]$ with $d = \mathrm{lw}(\Delta)$. For each $j = 0, \ldots, d$ consider

$$m_j = \min\{a \,|\, (a, j) \in \Delta \cap \mathbb{Z}^2\} \quad \text{and} \quad M_j = \max\{a \,|\, (a, j) \in \Delta \cap \mathbb{Z}^2\}.$$

These are well-defined, i.e. on each height j there is at least one lattice point in Δ, see for instance [10, Lem. 5.2]. Recall that X_Δ is the Zariski closure of the image of

$$\begin{aligned}\varphi_\Delta : (k^*)^2 \hookrightarrow \mathbb{P}^{N_\Delta - 1} : (\alpha, \beta) \mapsto (&\alpha^{m_0}\beta^0, \alpha^{m_0+1}\beta^0, \ldots, \alpha^{M_0}\beta^0,\\ &\alpha^{m_1}\beta^1, \alpha^{m_1+1}\beta^1, \ldots, \alpha^{M_1}\beta^1,\\ &\vdots\\ &\alpha^{m_d}\beta^d, \alpha^{m_d+1}\beta^d, \ldots, \alpha^{M_d}\beta^d).\end{aligned}$$

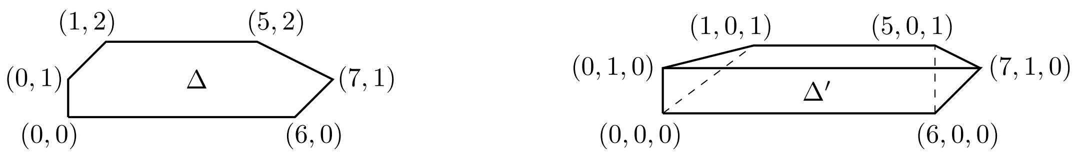
It is clear that this is contained in the Zariski closure of the image of

$$\begin{aligned}(k^*)^{1+d} \hookrightarrow \mathbb{P}^{N_\Delta - 1} : (\alpha, \beta_1, \ldots, \beta_d) \mapsto (&\alpha^{m_0}\beta_0, \alpha^{m_0+1}\beta_0, \ldots, \alpha^{M_0}\beta_0,\\ &\alpha^{m_1}\beta_1, \alpha^{m_1+1}\beta_1, \ldots, \alpha^{M_1}\beta_1,\\ &\vdots\\ &\alpha^{m_d}\beta_d, \alpha^{m_d+1}\beta_d, \ldots, \alpha^{M_d}\beta_d)\end{aligned}$$

where $\beta_0 = 1$. This is a $(d + 1)$-dimensional rational normal scroll, spanned by rational normal curves of degrees $M_0 - m_0$, $M_1 - m_1$, $\ldots$, $M_d - m_d$ (some of these degrees may be zero, in which case the 'curve' is actually a point). Its ideal is obtained from I_Δ by restricting to those binomial generators that remain valid if one forgets about the vertical structure of Δ. More precisely, we associate to Δ a lattice polytope $\Delta' \subseteq \mathbb{R}^{d+1}$ by considering for each $(a, b) \in \Delta \cap \mathbb{Z}^2$ the lattice point

$$(a, 0, 0, \ldots, 1, \ldots, 0), \qquad \text{where the 1 is in the } (b + 1)\text{st place (omitted if } b = 0),$$

and taking the convex hull. For example:



Then our scroll is just the toric variety $X_{\Delta'}$ associated to Δ'; this is unambiguously defined because Δ' is normal, as is easily seen using [6, Prop. 1.2.2]. We denote its defining ideal viewed inside $I_\Delta \subseteq S_\Delta$ by $I_{\Delta'}$.

As a generalization of (10), it is known that a minimal free resolution of the coordinate ring $S_\Delta/I_{\Delta'}$ of a rational normal scroll is given by the Eagon–Northcott complex, from which it follows that the graded Betti table of $X_{\Delta'}$ has the following shape:

$$\begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & \dots & f-2 & f-1 \\ \hline 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \binom{f}{2} & 2\binom{f}{3} & 3\binom{f}{4} & \dots & (f-2)\binom{f}{f-1} & (f-1)\binom{f}{f} \end{array} \tag{11}$$

where $f = \deg X_{\Delta'} = N_{\Delta'} - d - 1 = N_\Delta - d - 1$. Because all syzygies are linear, this must be a summand of the graded Betti table of X_Δ, from which it follows that:

Lemma 4.1. $\min\{\,\ell \,|\, b_{N_\Delta - \ell} \neq 0\,\} \leq \mathrm{lw}(\Delta) + 2.$

4.2 Explicit construction of non-exact cycles

We can give an alternative proof of Lemma 4.1 by explicitly constructing non-zero elements in Koszul cohomology. From a geometric point of view this approach is less enlightening, but it allows us to prove the sharper bound $\min\{\,\ell \,|\, b_{N_\Delta - \ell} \neq 0\,\} \leq \mathrm{lw}(\Delta) + 1$ in the cases $\Delta \cong d\Sigma, \Upsilon_d$ ($d \geq 2$) and $\Delta \cong 2\Upsilon$. As we will see, the sharper bound for $d\Sigma$ immediately implies the sharper bound for Υ_d.

For $\ell = 1, \dots, N_\Delta - 3$ recall that b_ℓ is the cohomology in the middle of

$$\bigwedge^{\ell+1} V_\Delta \xrightarrow{\delta} \bigwedge^{\ell} V_\Delta \otimes V_\Delta \xrightarrow{\delta'} \bigwedge^{\ell-1} V_\Delta \otimes V_{2\Delta}.$$

It is convenient to view this as a subcomplex of

$$\bigwedge^{\ell+1} V_\Delta \otimes V_{\mathbb{Z}^2} \xrightarrow{\delta_{\mathbb{Z}^2}} \bigwedge^{\ell} V_\Delta \otimes V_{\mathbb{Z}^2} \xrightarrow{\delta'_{\mathbb{Z}^2}} \bigwedge^{\ell-1} V_\Delta \otimes V_{\mathbb{Z}^2},$$

where $V_{\mathbb{Z}^2} = k[x^{\pm 1}, y^{\pm 1}]$. In what follows we will abuse notation and describe the basis elements of V_Δ and $V_{\mathbb{Z}^2}$ using the points $(i,j) \in \mathbb{Z}^2$ rather than the monomials $x^i y^j$.

Our technique to construct an element of $\ker \delta' \setminus \mathrm{im}\,\delta$ will be to apply $\delta_{\mathbb{Z}^2}$ to an element of $\bigwedge^{\ell+1} V_\Delta \otimes V_{\mathbb{Z}^2}$ such that the result is in $\bigwedge^{\ell} V_\Delta \otimes V_\Delta$. This result will then automatically be contained in $\ker \delta'$, but it might land outside $\mathrm{im}\,\delta$. We first state an easy lemma that will be helpful in proving that certain elements are indeed not contained in $\mathrm{im}\,\delta$. Fix a strict total order $<$ on $\Delta \cap \mathbb{Z}^2$ and consider the bases

$$B = \{P_1 \wedge \dots \wedge P_{\ell+1} \,|\, P_1 < \dots < P_{\ell+1},\ P_1, \dots, P_{\ell+1} \in \Delta \cap \mathbb{Z}^2\},$$

$$B' = \{P_1 \wedge \dots \wedge P_\ell \otimes P \,|\, P_1 < \dots < P_\ell,\ P, P_1, \dots, P_\ell \in \Delta \cap \mathbb{Z}^2\}$$

of $\bigwedge^{\ell+1} V_\Delta$ and $\bigwedge^{\ell} V_\Delta \otimes V_\Delta$, respectively.

Lemma 4.2. *If $x \in \bigwedge^{\ell+1} V_\Delta$ has n non-zero coordinates with respect to B, then $\delta(x)$ has $(\ell+1)n$ non-zero coordinates with respect to B'.*

Proof. Write $x = \sum_{i=1}^{n} a_i P_{i,1} \wedge \ldots \wedge P_{i,\ell+1}$, $a_i \in k \setminus \{0\}$, where the $P_{i,1} \wedge \ldots \wedge P_{i,\ell+1}$'s are distinct elements of B. Then

$$\delta(x) = \sum_{i=1}^{n} \sum_{j=1}^{\ell+1} (-1)^j a_i P_{i,1} \wedge \ldots \wedge \widehat{P_{i,j}} \wedge \ldots \wedge P_{i,\ell+1} \otimes P_{i,j}$$

Each term in this sum is $\pm a_i$ times an element of B', and the number of terms is $(\ell+1)n$, so we just have to verify that these elements of B' are mutually distinct, but that is easily done. □

Our alternative proof of the upper bound $\min\{\, \ell \,|\, b_{N_\Delta - \ell} \neq 0 \,\} \leq \mathrm{lw}(\Delta) + 2$ now goes as follows.

Alternative proof of Lemma 4.1. As before, we can assume that $\Delta \subseteq \mathbb{R} \times [0, d]$ with $d = \mathrm{lw}(\Delta)$. Let $\ell = N_\Delta - d - 2$ and let $P_1, \ldots, P_{\ell+1}$ be the points $(i, j) \in \Delta$ for which $i > m_j$, indexed so that $P_1 < \ldots < P_{\ell+1}$. Now consider

$$\begin{aligned} y &= \delta_{\mathbb{Z}^2}(P_1 \wedge \ldots \wedge P_{\ell+1} \otimes (-1, 0)) \\ &= \sum_{s=1}^{\ell+1} (-1)^s P_1 \wedge \ldots \wedge \widehat{P_s} \wedge \ldots \wedge P_{\ell+1} \otimes (P_s + (-1, 0)). \end{aligned}$$

Clearly $y \in \bigwedge^\ell V_\Delta \otimes V_\Delta$ and therefore $y \in \ker \delta'$. So it remains to show that $y \notin \operatorname{im} \delta$. Suppose $y = \delta(x)$ for some $x \in \bigwedge^{\ell+1} V_\Delta$. Since y has $\ell + 1$ nonzero coordinates with respect to the basis B', by the previous lemma x has just one non-zero coordinate with respect to the basis B. Therefore we can write

$$x = a P'_1 \wedge \ldots \wedge P'_{\ell+1}, \quad a \in k \setminus \{0\}, \quad P'_1 < \ldots < P'_{\ell+1},$$

so that

$$y = \delta(x) = \sum_{s=1}^{\ell+1} a(-1)^s P'_1 \wedge \ldots \wedge \widehat{P'_s} \wedge \ldots \wedge P'_{\ell+1} \otimes P'_s.$$

Comparing both expressions for y, we deduce that $\{P_1, \ldots, P_{\ell+1}\} = \{P'_1, \ldots, P'_{\ell+1}\}$. This gives us a contradiction since the two expressions for y have a different bidegree. Summing up, we have shown that $b_{N_\Delta - d - 2} \neq 0$, from which Lemma 4.1 follows. □

The same proof technique enables us to deduce a sharper bound in the exceptional cases $d\Sigma$ ($d \geq 2$) and 2Υ.

Lemma 4.3. *If* $\Delta \cong d\Sigma$ *for some* $d \geq 2$ *then* $\min\{\, \ell \,|\, b_{N_\Delta - \ell} \neq 0 \,\} \leq \mathrm{lw}(\Delta) + 1$.

Proof. We can of course assume that $\Delta = d\Sigma$. Recall that $N_\Delta = (d+1)(d+2)/2$ and that $\mathrm{lw}(\Delta) = \mathrm{lw}(d\Sigma) = d$. Let $\ell = N_\Delta - d - 1 = d(d+1)/2$. Let $P_1, \ldots, P_\ell$ be the elements of $(d-1)\Sigma \cap \mathbb{Z}^2$ and define

$$y = \delta_{\mathbb{Z}^2}\Big((d-1, 1) \wedge P_1 \wedge \ldots \wedge P_\ell) \otimes (1, 0) - (d, 0) \wedge P_1 \wedge \ldots \wedge P_\ell \otimes (0, 1)\Big)$$

$$= \sum_{s=1}^{\ell} (-1)^s (d,0) \wedge P_1 \wedge \ldots \wedge \widehat{P_s} \wedge \ldots \wedge P_\ell \otimes (P_s + (0,1))$$
$$- \sum_{s=1}^{\ell} (-1)^s (d-1,1) \wedge P_1 \wedge \ldots \wedge \widehat{P_s} \wedge \ldots \wedge P_\ell \otimes (P_s + (1,0)).$$

As in the previous proof, since $y \in \bigwedge^\ell V_\Delta \otimes V_\Delta$ we have $y \in \ker \delta'$. The fact that $y \notin \operatorname{im} \delta$ follows from the fact that the number of nonzero coordinates with respect to B' is 2ℓ. If y were in the image, then by our lemma 2ℓ should be divisible by $\ell + 1$, hence $\ell \leq 2$. But $\ell = d(d+1)/2 \geq 3$ because $d \geq 2$: contradiction, and the lemma follows. □

Lemma 4.4. *If* $\Delta \cong 2\Upsilon$ *then* $\min\{\, \ell \,|\, b_{N_\Delta - \ell} \neq 0 \,\} \leq \operatorname{lw}(\Delta) + 1$.

Proof. Here we can assume $\Delta = 2\Upsilon$ and note that $N_\Delta = 10$ and $\operatorname{lw}(\Delta) = \operatorname{lw}(2\Upsilon) = 4$. With $\ell = N_\Delta - d - 1 = 5$, in exactly the same way as before we see that

$$\delta_{\mathbb{Z}^2}\Big((1,0) \wedge (0,1) \wedge (0,0) \wedge (-1,-1) \wedge (-1,0) \wedge (0,-1) \otimes (-1,-1)$$
$$+ (1,0) \wedge (0,1) \wedge (0,0) \wedge (-1,-1) \wedge (0,-1) \wedge (-2,-2) \otimes (0,1)$$
$$- (1,0) \wedge (0,1) \wedge (0,0) \wedge (-1,-1) \wedge (-1,0) \wedge (-2,-2) \otimes (1,0) \Big)$$

is a non-zero cycle: it has $12 = 2(\ell + 1)$ terms, so if it were in $\operatorname{im} \delta$, then any preimage should have two terms, and we leave it to the reader to verify that this again leads to a contradiction. Alternatively, the reader can just look up the graded Betti table of $X_{2\Upsilon}$ in Appendix A. □

Lemma 4.5. *If* $\Delta \cong \Upsilon_d$ *for some* $d \geq 2$ *then* $\min\{\, \ell \,|\, b_{N_\Delta - \ell} \neq 0 \,\} \leq \operatorname{lw}(\Delta) + 1$.

Proof. From the combinatorics of Υ_d it is clear that if one restricts to those equations of X_{Υ_d} not involving $X_{-1,-1}$, one obtains a set of defining equations for $X_{d\Sigma}$. Thus the linear strand of the graded Betti table of $X_{d\Sigma}$ is a summand of the linear strand of the graded Betti table of X_Δ. From Lemma 4.3 we conclude that

$$\min\{\, \ell \,|\, b_{N_\Delta - \ell} \neq 0 \,\} \leq \min\{\, \ell \,|\, b_{N_{d\Sigma} - \ell} \neq 0 \,\} + 1 \leq \operatorname{lw}(d\Sigma) + 2 = d + 2.$$

The lemma follows from the observation that $\operatorname{lw}(\Delta) = d + 1$. □

4.3 Conclusion

Summarizing the results in this section, we state:

Theorem 4.6. *If* $\Delta \not\cong \Sigma, \Upsilon$ *then one has* $\min\{\, \ell \,|\, b_{N_\Delta - \ell} \neq 0 \,\} \leq \operatorname{lw}(\Delta) + 2$. *If*

$$\Delta \cong d\Sigma \text{ for some } d \geq 2 \qquad \text{or} \qquad \Delta \cong \Upsilon_d \text{ for some } d \geq 2 \qquad \text{or} \qquad \Delta \cong 2\Upsilon$$

then moreover one has the sharper bound $\operatorname{lw}(\Delta) + 1$. *In other words the sharpest applicable upper bound predicted by Conjecture 1.6 holds.*

5 Pruning off vertices without changing the lattice width

Theorem 5.1. *Let Δ be a two-dimensional lattice polygon and let $p \geq 1$. Let P be a vertex of Δ and define $\Delta' = \mathrm{conv}(\Delta \cap \mathbb{Z}^2 \setminus \{P\})$, where we assume that Δ' is two-dimensional. If $K_{p,1}(X_{\Delta'}, L_{\Delta'}) = 0$ then also $K_{p+1,1}(X_\Delta, L_\Delta) = 0$.*

Proof. Consider

$$\bigwedge^{p+1} V_{\Delta'} \xrightarrow{\delta_1} \bigwedge^{p} V_{\Delta'} \otimes V_{\Delta'} \xrightarrow{\delta_2} \bigwedge^{p-1} V_{\Delta'} \otimes V_{2\Delta'}$$

and

$$\bigwedge^{p+2} V_{\Delta} \xrightarrow{\delta_3} \bigwedge^{p+1} V_{\Delta} \otimes V_{\Delta} \xrightarrow{\delta_4} \bigwedge^{p} V_{\Delta} \otimes V_{2\Delta}$$

where the δ_i's are the usual coboundary maps. Assuming that $\ker \delta_2 = \mathrm{im}\, \delta_1$ we will show that $\ker \delta_4 = \mathrm{im}\, \delta_3$. Suppose the contrary: we will find a contradiction. Let $L : \mathbb{R}^n \to \mathbb{R}$ be a linear form that maps different lattice points in Δ to different numbers, such that P attains the maximum of L on Δ. This exists because P is a vertex. For any $x \in \bigwedge^{p+1} V_\Delta \otimes V_\Delta$ define its support as the convex hull of the set of $P_{j,i}$'s occurring when expanding x in the form

$$x = \sum_i \lambda_i P_{1,i} \wedge \ldots \wedge P_{p+1,i} \otimes Q_i.$$

Here as in Section 4 we take the notational freedom to write points rather than monomials, and of course we do not write any redundant terms. Choose an $x \in \ker \delta_4 \setminus \mathrm{im}\, \delta_3$ such that the maximum that L attains on the support of x is minimal, and let $P' \in \Delta \cap \mathbb{Z}^2$ be the unique point attaining this maximum. Rearrange the above expansion as follows:

$$x = \sum_i \lambda_i P' \wedge P_{1,i} \wedge \ldots \wedge P_{p,i} \otimes Q_i + \text{ terms not containing } P' \text{ in the } \wedge \text{ part} \tag{12}$$

where all $P_{j,i}$'s are in Δ' and $Q_i \in \Delta$. We claim that in fact $Q_i \in \Delta'$, i.e. none of the Q_i's equals P. Indeed, otherwise when applying δ_4 the term $-\lambda_i P_{1,i} \wedge \ldots \wedge P_{p,i} \otimes (P' + Q_i)$ of $\delta_4(x)$ has nothing to cancel against, contradicting that $\delta_4(x) = 0$. Let

$$y = \sum_i \lambda_i P_{1,i} \wedge \ldots \wedge P_{p,i} \otimes Q_i \in \bigwedge^{p} V_{\Delta'} \otimes V_{\Delta'}. \tag{13}$$

We have

$$0 = \delta_4(x) = -P' \wedge \delta_2(y) + \text{ terms not containing } P' \text{ in the } \wedge \text{ part.}$$

Because terms of $P' \wedge \delta_2(y)$ cannot cancel against terms without P' in the $\wedge$ part, $\delta_2(y)$ must be zero, and therefore $y \in \mathrm{im}\, \delta_1$ by the exactness assumption. So write $y = \delta_1(z)$ with

$$z = \sum_i \mu_i P'_{1,i} \wedge \ldots \wedge P'_{p+1,i} \in \bigwedge^{p+1} V_{\Delta'}.$$

Let P'' be the point occurring in this expression such that $L(P'')$ is maximal. Since there is no cancellation when applying δ_1 one sees that P'' is in the support of y, hence in the support of x and therefore $L(P'') < L(P')$. This means that L achieves a smaller maximum on the support of z than on the support of x. Finally, let

$$x' = x + \delta_3(P' \wedge z) = x - P' \wedge y - z \otimes P'.$$

Since $x \in \ker \delta_4 \setminus \operatorname{im} \delta_3$ we have $x' \in \ker \delta_4 \setminus \operatorname{im} \delta_3$ and by (12) and (13) one concludes that L will achieve a smaller maximum on the support of x' than on the support of x, namely $L(P'')$. This contradicts the choice of x. □

This immediately implies the following corollary, which is included in the statement of Theorem 1.8 in the introduction.

Corollary 5.2. *Let Δ and Δ' be as in the statement of the above theorem. Assume that $\operatorname{lw}(\Delta) = \operatorname{lw}(\Delta')$, that $\Delta' \ncong d\Sigma, \Upsilon_d$ for any $d \geq 1$ and that $\Delta' \ncong 2\Upsilon$. If Conjecture 1.6 holds for Δ' then it also holds for Δ.*

In order to deduce Conjecture 1.6 for polygons having a small lattice width, we note the following.

Lemma 5.3. *Let Δ be a two-dimensional lattice polygon, let $d = \operatorname{lw}(\Delta)$, and assume that removing an extremal lattice point makes the lattice width decrease, i.e. for every vertex $P \in \Delta$ it holds that*

$$\operatorname{lw}(\operatorname{conv}(\Delta \cap \mathbb{Z}^2 \setminus \{P\})) < d.$$

Then there exists a unimodular transformation mapping Δ into $[0, d] \times [0, d]$.

Proof. The cases where $\Delta^{(1)} \cong \emptyset$ or where $\Delta^{(1)} \cong d\Sigma$ for some $d \geq 0$ are easy to verify. In the other cases $\operatorname{lw}(\Delta^{(1)}) = \operatorname{lw}(\Delta) - 2 = d - 2$ and the lattice width directions for Δ and $\Delta^{(1)}$ are the same [28, Thm. 13]. Assume that $\Delta \subseteq \mathbb{R} \times [0, d]$, fix a vertex on height 0 and a vertex on height d, and let P be any other vertex. Then $\operatorname{lw}(\operatorname{conv}(\Delta \cap \mathbb{Z}^2 \setminus \{P\})) \leq d - 1$, where we note that a corresponding lattice width direction is necessarily non-horizontal, and that along such a direction the width of $\Delta^{(1)}$ is at most $d - 2$. But then equality must hold, and in particular it must also concern a lattice width direction for $\Delta^{(1)}$, hence it must concern a lattice width direction for Δ. We conclude that Δ has two independent lattice width directions, and the lemma follows from the remark following [10, Lem. 5.2]. □

Let us call a lattice polygon Δ as in the statement of the foregoing lemma 'minimal', and note that this attribute applies to each of the exceptional polygons $d\Sigma, \Upsilon_d, 2\Upsilon$ mentioned in the statement of Conjecture 1.6. In order to prove Conjecture 1.6 for a certain non-exceptional polygon Δ, by Corollary 5.2 it suffices to do this for any lattice polygon obtained by repeatedly pruning off vertices without changing the lattice width. Thus the proof reduces to verifying the case of a minimal lattice polygon, unless it concerns one of the exceptional cases $d\Sigma, \Upsilon_d, 2\Upsilon$, in which case one needs to stop pruning one step earlier (otherwise this strategy has no chance of being successful).

In other words the above lemma implies that if Conjecture 1.6 is true for all lattice polygons Δ for which $N_\Delta \leq (d+1)^2+1$, then it is true for all lattice polygons Δ with $\mathrm{lw}(\Delta) \leq d$. This observation, along with our exhaustive verification in the cases where $N_\Delta \leq 32$, reported upon in Section 8, allows us to conclude that Conjecture 1.6 is true as soon as $\mathrm{lw}(\Delta) \leq 4$. This fact will be used in the proof of our explicit formula for $b_{N_\Delta - 4}$.

But one can do better: in a spin-off paper [12] devoted to minimal polygons, the second and the fourth author show that if Δ is a minimal lattice polygon with $\mathrm{lw}(\Delta) \leq d$ then

$$N_\Delta \leq \max\left\{(d-1)^2+4, (d+1)(d+2)/2\right\}.$$

From this, using a similar reasoning, the conjecture follows for $\mathrm{lw}(\Delta) \leq 6$, as announced in the statement of Theorem 1.8.

6 Explicit formula for $b_{N_\Delta-4}$

In this section we will prove Theorem 3.5, whose statement distinguishes between the following four cases:

$$\begin{cases} \Delta^{(1)} = \emptyset, \\ \dim \Delta^{(1)} = 0, \\ \dim \Delta^{(1)} = 1 \text{ or } \Delta \cong \Upsilon_2, \\ \dim \Delta^{(1)} = 2 \text{ and } \Delta \ncong \Upsilon_2. \end{cases}$$

We will treat these cases in the above order, which as we will see corresponds to increasing order of difficulty. The first case where $\Delta^{(1)} = \emptyset$ follows trivially from (10), so we can skip it. Now recall from (6) that $b_{N_\Delta-4}$ is the dimension of the cohomology in the middle of

$$\bigwedge^2 V_\Delta \otimes V_{\Delta^{(1)}} \xrightarrow{\delta} V_\Delta \otimes V_{(2\Delta)^{(1)}} \xrightarrow{\delta'} V_{(3\Delta)^{(1)}}.$$

Because $K_{0,3}(X;K,L) \cong K_{N_\Delta-3,0}(X,L) = 0$, where we use that $\Delta \ncong \Sigma$, we have that the map δ' is surjective. In particular we obtain the formula

$$b_{N_\Delta-4} = \dim \operatorname{coker} \delta - |(3\Delta)^{(1)} \cap \mathbb{Z}^2|.$$

Case $\dim \Delta^{(1)} = 0$

If $\dim \Delta^{(1)} = 0$ then δ is injective, so

$$b_{N_\Delta-4} = \dim(V_\Delta \otimes V_{(2\Delta)^{(1)}}) - \dim(\bigwedge^2 V_\Delta) - |(3\Delta)^{(1)} \cap \mathbb{Z}^2| = (N_\Delta - 4)(N_\Delta - 1)/2,$$

as can be calculated using Pick's theorem, thereby yielding Theorem 3.5 in this case (alternatively, one can give an exhaustive proof by explicitly computing the graded Betti tables of the toric surfaces associated to the 16 reflexive lattice polygons).

Case $\dim \Delta^{(1)} = 1$ **or** $\Delta \cong \Upsilon_2$

The graded Betti table of X_{Υ_2} can be found in Appendix A, where one verifies that $b_{N_{\Upsilon_2}-4} = b_3 = 3$, as indeed predicted by the statement of Theorem 3.5. Therefore we can assume that $\dim \Delta^{(1)} = 1$. The polygons Δ having a one-dimensional interior were explicitly classified by Koelman [24, §4.3], but in any case it is easy to see that, using a unimodular transformation if needed, we can assume that

$$\Delta = \operatorname{conv}\{(m_1, 1), (M_1, 1), (m_0, 0), (M_0, 0), (m_{-1}, -1), (M_{-1}, -1)\}$$

for some $m_i \leq M_i \in \mathbb{Z}$. Here $m_0 < M_0$ can be taken such that

$$\Delta \cap (\mathbb{Z} \times \{0\}) = \{m_0, m_0 + 1, \ldots, M_0\} \times \{0\}.$$

Write $\Delta^{(1)} = [u, v] \times \{0\}$, then

$$(2\Delta)^{(1)} = \Delta + \Delta^{(1)} = \operatorname{conv}\{(m_i + u, i), (M_i + v, i) \,|\, i = 1, 0, -1\}.$$

Now consider $V_{\mathbb{Z}} = k[x^{\pm 1}]$ and define a morphism

$$f : V_\Delta \otimes V_{(2\Delta)^{(1)}} \to k[x_{-1}, x_0, x_1] \otimes V_{\mathbb{Z}}$$

by letting $(a, b) \otimes (c, d) \mapsto x_b x_d \otimes (a + c)$, where again we abusingly describe the basis elements of V_Δ, $V_{(2\Delta)^{(1)}}$ and $V_{\mathbb{Z}}$ using lattice points rather than monomials. Note that

$$f(\delta((a, b) \wedge (c, d) \otimes (e, 0))) = f((a, b) \otimes (c + e, d) - (c, d) \otimes (a + e, b)) = 0,$$

so $\operatorname{im} \delta \subseteq \ker f$.

We claim that actually equality holds. First note that every element $\alpha \in \ker f$ decomposes into elements

$$\sum_j \lambda_j (a_j, b_j) \otimes (c_j, d_j)$$

for which $(\{b_j, d_j\}, a_j + c_j)$ is the same for all j: indeed, terms for which these are different cannot cancel out when applying f. Note that $\sum_j \lambda_j = 0$, so one can rewrite the above as a linear combination of expressions either of the form

$$\underbrace{(a, b) \otimes (c, d) - (a', b) \otimes (c', d)}_{\text{(i)}} \quad \text{or of the form} \quad \underbrace{(a, b) \otimes (c, d) - (a', d) \otimes (c', b)}_{\text{(ii)}}$$

where $a + c = a' + c'$, the points $(a, b), (a', b)$ resp. $(a, b), (a', d)$ are in Δ, and the points $(c, d), (c', d)$ resp. $(c, d), (c', b)$ are in $(2\Delta)^{(1)}$. As for case (i), these can be decomposed further as a sum (or minus a sum) of expressions of the form $(a, b) \otimes (c, d) - (a + 1, b) \otimes (c - 1, d)$, which can be rewritten as

$$\delta((a, b) \wedge (c - e, d) \otimes (e, 0) - (a + 1, b) \wedge (c - e, d) \otimes (e - 1, 0))$$

and therefore as an element of $\operatorname{im} \delta$, at least if e can be chosen in the interval $[\max(u+1, c - M_d), \min(v, c - m_d)]$. The reader can verify that this is indeed non-empty, from which the claim follows in this case. As for (ii), with e chosen from the non-empty interval $[\max(u, c' - M_b), \min(v, c' - m_b)]$ one verifies that

$$\delta((c' - e, b) \wedge (a', d) \otimes (e, 0)) = (c' - e, b) \otimes (a' + e, d) - (a', d) \otimes (c', b),$$

allowing one to replace (ii) with an expression of type (i), and the claim again follows.

Summing up, we have

$$\begin{aligned} b_{N_\Delta - 4} &= \dim \operatorname{im} f - |(3\Delta)^{(1)} \cap \mathbb{Z}^2| \\ &= \sum_{\{i,j\} \subseteq \{-1,0,1\}} |[m_i + m_j + u, M_i + M_j + v] \cap \mathbb{Z}| - \sum_{i'=-2}^{2} \left| (3\Delta)^{(1)} \cap (\mathbb{Z} \times \{i'\}) \right|. \end{aligned}$$

Each lattice point of $(3\Delta)^{(1)} = 2\Delta + \Delta^{(1)}$ appears in an interval on the left, and conversely. To see this it suffices to note that each lattice point of 2Δ arises as the sum of two lattice points in Δ, which is a well-known property [21]. So all terms with $i + j \neq 0$ cancel out the terms with $i' \neq 0$, and we are left with

$$\begin{aligned} |[m_1 + m_{-1} + u, M_1 + M_{-1} + v] \cap \mathbb{Z}| + |[2m_0 + u, 2M_0 + v] \cap \mathbb{Z}| \\ - \left| (3\Delta)^{(1)} \cap (\mathbb{Z} \times \{0\}) \right|. \end{aligned}$$

Term by term this equals

$$\begin{aligned} \left(|\partial\Delta \cap \mathbb{Z}^2| + N_{\Delta^{(1)}} - 2 - \varepsilon \right) + (2(M_0 - m_0) + N_{\Delta^{(1)}}) \\ - (2(M_0 - m_0) + (2 - \varepsilon) + N_{\Delta^{(1)}}) \end{aligned}$$

where $\varepsilon := (u - m_0) + (M_0 - v) \in \{0, 1, 2\}$ denotes the cardinality of $\partial\Delta \cap (\mathbb{Z} \times \{0\})$. Because the above expression simplifies to $N_\Delta - 4$, this concludes the proof in the $\dim \Delta^{(1)} = 1$ case.

Case $\dim \Delta^{(1)} = 2$ **and** $\Delta \not\cong \Upsilon_2$

In this case our task amounts to proving that $b_{N_\Delta - 4} = 0$, but this follows from Conjecture 1.6 for polygons Δ satisfying $\operatorname{lw}(\Delta) \leq 4$, which was verified in Section 5.

7 Quotienting the Koszul complex

We now start working towards an algorithmic determination of the graded Betti table of the toric surface $X_\Delta \subseteq \mathbb{P}^{N_\Delta - 1}$ associated to a given two-dimensional lattice polygon Δ. Essentially, the method is about reducing the dimensions of the vector spaces involved, in order to make the linear algebra more manageable. This is mainly done by incorporating bigrading and duality. However when dealing with large polygons a further reduction is useful. In this section we show that the Koszul complex always admits certain exact subcomplexes that can be described in a combinatorial way. Quotienting out such a subcomplex does not affect the cohomology, while making the linear algebra easier, at least in theory. For reasons we don't understand our practical implementation shows that the actual gain in runtime is somewhat unpredictable: sometimes it is helpful, but other times the contrary is true. But it is worth the try, and in any case we believe that the material below is also interesting from a theoretical point of view.

We first introduce the subcomplex from an algebraic point of view, then reinterpret things combinatorially, and finally specify our discussion to the case of the Veronese surfaces $X_{d\Sigma}$. In the latter setting the idea of quotienting out such an exact subcomplex is not new: for instance it appears in the recent paper by Ein, Erman and Lazarsfeld [14, p. 2].

7.1 An exact subcomplex

We begin with the following lemma, which should be known to specialists, but we include a proof for the reader's convenience.

Lemma 7.1. *Let M be a graded module over $k[x_1, \dots, x_N]$ and suppose that the multiplication-by-x_N map $M \to M$ is an injection. Then the Koszul complexes*

$$\dots \to \bigwedge^{p+1} V \otimes M \to \bigwedge^{p} V \otimes M \to \bigwedge^{p-1} V \otimes M \to \dots$$

and

$$\dots \to \bigwedge^{p+1} W \otimes M/(x_N M) \to \bigwedge^{p} W \otimes M/(x_N M) \to \bigwedge^{p-1} W \otimes M/(x_N M) \to \dots$$

have the same graded cohomology. Here V and W denote the degree one parts of the polynomial rings $k[x_1, \dots, x_N]$ and $k[x_1, \dots, x_{N-1}]$, respectively.

Proof. Denote by M' the graded module $M/(x_N M)$. For every $p \geq 0$ we have a short exact sequence

$$0 \longrightarrow \left(\bigwedge^{p} W \otimes M\right) \oplus \left(\bigwedge^{p-1} W \otimes M\right) \xrightarrow{\alpha} \bigwedge^{p} V \otimes M \xrightarrow{\beta} \bigwedge^{p} W \otimes M' \longrightarrow 0,$$

by letting

$$\alpha\left(v_1 \wedge \dots \wedge v_p \otimes m \,,\, w_1 \wedge \dots \wedge w_{p-1} \otimes m' \right) \\ = v_1 \wedge \dots \wedge v_p \otimes x_N m \,+\, x_N \wedge w_1 \wedge \dots \wedge w_{p-1} \otimes m'$$

and $\beta(v_1 \wedge \dots \wedge v_p \otimes m) = \pi(v_1) \wedge \dots \wedge \pi(v_p) \otimes \overline{m}$, where $\pi : V \to W$ maps x_i to itself if $i \neq N$ and to zero otherwise, and $\overline{m}$ denotes the residue class of m modulo $x_N M$. As usual if $p = 0$ then it is understood that $\bigwedge^{p-1} W \otimes M = 0$. We leave a verification of the exactness to the reader, but note that the injectivity of the multiplication-by-x_N map is important here.

On the other hand the spaces

$$C_p = \left(\bigwedge^{p} W \otimes M\right) \oplus \left(\bigwedge^{p-1} W \otimes M\right)$$

naturally form a long exact sequence $\ldots \to C_2 \to C_1 \to C_0 \to 0$ along the morphisms

$$d_p : C_p \to C_{p-1} : (a,b) \mapsto (-b + \delta_p(a), -\delta_{p-1}(b))$$

where δ_p and δ_{p-1} are the usual coboundary maps, as described in (3). Exactness holds because if $d_p(a,b) = 0$ then $d_{p+1}(0,-a) = (a,b)$. Overall we end up with a short exact sequence of complexes:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \bigwedge^{p+1} W \otimes M \oplus \bigwedge^p W \otimes M & \to & \bigwedge^{p+1} V \otimes M & \to & \bigwedge^{p+1} W \otimes M' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \bigwedge^{p} W \otimes M \oplus \bigwedge^{p-1} W \otimes M & \to & \bigwedge^{p} V \otimes M & \to & \bigwedge^{p} W \otimes M' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

This gives a long exact sequence in (co)homology, and the result follows from the exactness of the left column. □

Now we explain how to exploit the above lemma for our purposes. We can apply it to the Koszul complex

$$\ldots \to \bigwedge^{p+1} V_\Delta \otimes \bigoplus_{i\geq 0} V_{i\Delta} \to \bigwedge^{p} V_\Delta \otimes \bigoplus_{i\geq 0} V_{i\Delta} \to \bigwedge^{p-1} V_\Delta \otimes \bigoplus_{i\geq 0} V_{i\Delta} \to \ldots$$

as well as to the twisted Koszul complex

$$\ldots \to \bigwedge^{p+1} V_\Delta \otimes \bigoplus_{i\geq 1} V_{(i\Delta)^{(1)}} \to \bigwedge^{p} V_\Delta \otimes \bigoplus_{i\geq 1} V_{(i\Delta)^{(1)}} \to \bigwedge^{p-1} V_\Delta \otimes \bigoplus_{i\geq 1} V_{(i\Delta)^{(1)}} \to \ldots$$

These are complexes of graded modules over the polynomial ring whose variables correspond to the lattice points of Δ. In both cases the variable corresponding to whatever point $P \in \Delta \cap \mathbb{Z}^2$ can be chosen as x_N, because multiplication by x_N will always be injective. Then the lemma yields that we can replace $V_{i\Delta}$ by $V_{(i\Delta)\setminus((i-1)\Delta+P)}$ in the first complex, and that we can replace $V_{(i\Delta)^{(1)}}$ by $V_{(i\Delta)^{(1)}\setminus(((i-1)\Delta)^{(1)}+P)}$ in the second complex. In both cases we must also replace the V_Δ's in the wedge product by $V_{\Delta\setminus\{P\}}$. Splitting these complexes into their graded pieces we conclude that $K_{p,q}(X,L)$ can be computed as the cohomology in the middle of

$$\bigwedge^{p+1} V_{\Delta\setminus\{P\}} \otimes V_{((q-1)\Delta)\setminus((q-2)\Delta+P)} \longrightarrow \bigwedge^{p} V_{\Delta\setminus\{P\}} \otimes V_{(q\Delta)\setminus((q-1)\Delta+P)} \longrightarrow \bigwedge^{p-1} V_{\Delta\setminus\{P\}} \otimes V_{((q+1)\Delta)\setminus(q\Delta+P)},$$

and that the twisted Koszul cohomology spaces $K_{p,q}(X;K,L)$ can be computed as the cohomology in the middle of

$$\bigwedge^{p+1} V_{\Delta\setminus\{P\}} \otimes V_{((q-1)\Delta)^{(1)}\setminus((q-2)\Delta+P)^{(1)}} \longrightarrow \bigwedge^{p} V_{\Delta\setminus\{P\}} \otimes V_{(q\Delta)^{(1)}\setminus((q-1)\Delta+P)^{(1)}}$$

$$\longrightarrow \bigwedge^{p-1} V_{\Delta\setminus\{P\}} \otimes V_{((q+1)\Delta)^{(1)}\setminus(q\Delta+P)^{(1)}}.$$

Here for any $A \subseteq \mathbb{Z}^2$ we let $V_A \subseteq k[x^{\pm 1}, y^{\pm 1}]$ denote the space of Laurent polynomials whose support is contained in A.

Remark 7.2. The coboundary morphisms are still defined as in (3), with the additional rule that $x^i y^j$ is considered zero in V_A as soon as $(i,j) \notin A$.

Remark 7.3. It is important to observe that the above complexes remain naturally bigraded, and that this is compatible with the bigrading described in Section 2.2. In other words, for any $(a,b) \in \mathbb{Z}^2$, also the spaces $K_{p,q}^{(a,b)}(X,L)$ and $K_{p,q}^{(a,b)}(X;K,L)$ can be computed from the above sequences.

7.2 Removing multiple points

In some cases we can remove multiple points from Δ by applying Lemma 7.1 repeatedly. In algebraic terms this works if and only if these points, when viewed as elements of V_Δ, form a regular sequence for the graded module M, where M is either $\bigoplus_{i\geq 0} V_{i\Delta}$ or $\bigoplus_{i\geq 1} V_{(i\Delta)^{(1)}}$. The length of a regular sequence is bounded by the Krull dimension of M, which is equal to 3. So we can never remove more than three points. It is well-known that for graded modules over Noetherian rings any permutation of a regular sequence is again a regular sequence, so the order of removing points does not matter. Concretely, after removing the points $P_1, \ldots, P_m$ we get the complex

$$\ldots \longrightarrow \bigwedge^{p+1} V_{\Delta\setminus\{P_1,\ldots,P_m\}} \otimes \frac{M_{q-1}}{P_1 M_{q-2} + \ldots + P_m M_{q-2}}$$
$$\longrightarrow \bigwedge^{p} V_{\Delta\setminus\{P_1,\ldots,P_m\}} \otimes \frac{M_{q}}{P_1 M_{q-1} + \ldots + P_m M_{q-1}} \longrightarrow \ldots$$

where M_i denotes the degree i part of M. Here, as before, we abuse notation and identify the points $P_i \in \Delta$ with the corresponding monomials in V_Δ. So for $M = \bigoplus_{i\geq 0} V_{i\Delta}$ this gives

$$\ldots \longrightarrow \bigwedge^{p+1} V_{\Delta\setminus\{P_1,\ldots,P_m\}} \otimes V_{(q-1)\Delta\setminus((P_1+(q-2)\Delta)\cup\ldots\cup(P_m+(q-2)\Delta))}$$
$$\longrightarrow \bigwedge^{p} V_{\Delta\setminus\{P_1,\ldots,P_m\}} \otimes V_{q\Delta\setminus((P_1+(q-1)\Delta)\cup\ldots\cup(P_m+(q-1)\Delta))} \longrightarrow \ldots$$

while for $M = \bigoplus_{i\geq 1} V_{(i\Delta)^{(1)}}$ it gives

$$\ldots \longrightarrow \bigwedge^{p+1} V_{\Delta\setminus\{P_1,\ldots,P_m\}} \otimes V_{((q-1)\Delta)^{(1)}\setminus((P_1+((q-2)\Delta)^{(1)})\cup\ldots\cup(P_m+((q-2)\Delta)^{(1)}))}$$
$$\longrightarrow \bigwedge^{p} V_{\Delta\setminus\{P_1,\ldots,P_m\}} \otimes V_{q\Delta\setminus((P_1+((q-1)\Delta)^{(1)})\cup\ldots\cup(P_m+((q-1)\Delta)^{(1)}))} \longrightarrow \ldots$$

The question we study in this section is which sequences of points $P_1, \ldots, P_m \in \Delta \cap \mathbb{Z}^2$ are regular, where necessarily $m \leq 3$.

We first study the problem of which sequences of two points are regular. As for $M = \bigoplus_{i \geq 0} V_{i\Delta}$, if we first remove a point $P \in \Delta \cap \mathbb{Z}^2$ then we end up with M/PM, whose graded components in degree $q \geq 1$ are of the form $V_{q\Delta \setminus (P+(q-1)\Delta)}$, while the degree 0 part is just $V_{0\Delta}$. Multiplication by another point $Q \in \Delta \cap \mathbb{Z}^2$ in M/PM corresponds to

$$V_{q\Delta \setminus (P+(q-1)\Delta)} \xrightarrow{\cdot Q} V_{(q+1)\Delta \setminus (P+q\Delta)}.$$

In order for the sequence P, Q to be regular this map has to be injective for all $q \geq 1$. This means that

$$((q\Delta \setminus (P + (q-1)\Delta)) + Q) \cap (P + q\Delta) \cap \mathbb{Z}^2 = \emptyset.$$

Subtracting $P + Q$ yields

$$(q\Delta - P) \setminus ((q-1)\Delta) \cap (q\Delta - Q) \cap \mathbb{Z}^2 = \emptyset,$$

eventually leading to the criterion

$$P, Q \text{ is regular for } \bigoplus_{i \geq 0} V_{i\Delta} \quad \Leftrightarrow \quad \forall q \geq 1 : (q\Delta - P) \cap (q\Delta - Q) \cap \mathbb{Z}^2 \subseteq (q-1)\Delta. \quad (14)$$

Similarly we find

$$P, Q \text{ is regular for } \bigoplus_{i \geq 1} V_{(i\Delta)^{(1)}} \quad \Leftrightarrow \quad \forall q \geq 1 : (q\Delta - P)^{(1)} \cap (q\Delta - Q)^{(1)} \cap \mathbb{Z}^2 \subseteq ((q-1)\Delta)^{(1)}. \quad (15)$$

These criteria are strongly simplified by the equivalences *1.* $\iff$ *2.* $\iff$ *9.* of the following theorem:

Theorem 7.4. *Let Δ be a two-dimensional lattice polygon. For two distinct lattice points $P, Q \in \Delta$, the following are equivalent:*

1. *P, Q is a regular sequence for $\bigoplus_{i \geq 0} V_{i\Delta}$.*
2. *P, Q is a regular sequence for $\bigoplus_{i \geq 1} V_{(i\Delta)^{(1)}}$.*
3. *$(q\Delta - P) \cap (q\Delta - Q) \subseteq (q-1)\Delta$ for some $q > 1$.*
4. *$(q\Delta - P) \cap (q\Delta - Q) \subseteq (q-1)\Delta$ for all $q \geq 1$.*
5. *$((q\Delta)^\circ - P) \cap ((q\Delta)^\circ - Q) \subseteq ((q-1)\Delta)^\circ$ for all $q \geq 1$, where $^\circ$ denotes the interior for the standard topology on $\mathbb{R}^2$.*
6. *$((q\Delta)^{(1)} - P) \cap ((q\Delta)^{(1)} - Q) \cap \mathbb{Z}^2 \subseteq ((q-1)\Delta)^{(1)} \cap \mathbb{Z}^2$ for all $q \geq 1$.*
7. *$(q\Delta - P) \cap (q\Delta - Q) \cap \mathbb{Z}^2 \subseteq (q-1)\Delta \cap \mathbb{Z}^2$ for all $q \geq 1$.*

8. *Let ℓ be the line through P and Q. For both half-planes H bordered by ℓ, the polygon $H \cap \Delta$ is a triangle with P and Q as two vertices (this may be degenerate, in which case it is the line segment PQ).*

9. *Δ is a quadrangle and P and Q are opposite vertices of this quadrangle (this may be the degenerate case where Δ is a triangle and P, Q are any pair of vertices of Δ).*

Proof. The equivalences 1. $\iff$ 7. and 2. $\iff$ 6. follow from the foregoing discussion. 3. $\implies$ 4.: assume that 3. holds for some $q > 1$. Let $q' \geq 1$, we show that it also holds for q'. Let $W \in (q'\Delta - P) \cap (q'\Delta - Q)$, we need to show that $W \in (q' - 1)\Delta$.

In case $q' > q$, we define $\delta = (q-1)/(q'-1) < 1$. Now consider

$$\begin{aligned} W &\in ((q'-1)\Delta + (\Delta - P)) \cap ((q'-1)\Delta + (\Delta - Q)) \\ \delta W &\in ((q-1)\Delta + \delta(\Delta - P)) \cap ((q-1)\Delta + \delta(\Delta - Q)) \\ &\subseteq ((q-1)\Delta + (\Delta - P)) \cap ((q-1)\Delta + (\Delta - Q)) \\ &= (q\Delta - P) \cap (q\Delta - Q) \subseteq (q-1)\Delta. \end{aligned}$$

We conclude that $W \in (q'-1)\Delta$.

If $q' < q$, we find

$$\begin{aligned} W + (q-q')\Delta &\subseteq \left[(q'\Delta - P) \cap (q'\Delta - Q)\right] + (q-q')\Delta \\ &\subseteq (q'\Delta - P + (q-q')\Delta) \cap (q'\Delta - Q + (q-q')\Delta) \\ &\subseteq (q\Delta - P) \cap (q\Delta - Q) \subseteq (q-1)\Delta. \end{aligned}$$

Since $W + (q-q')\Delta \subseteq (q-1)\Delta$, it follows that $W \in (q'-1)\Delta$.

4. $\implies$ 5.: this holds by taking interiors on both sides and using the fact that $(A \cap B)^\circ = A^\circ \cap B^\circ$.

5. $\implies$ 6.: intersect with $\mathbb{Z}^2$ on both sides and use $\Delta^\circ \cap \mathbb{Z}^2 = \Delta^{(1)} \cap \mathbb{Z}^2$.

6. $\implies$ 7.: let $W \in (q\Delta - P) \cap (q\Delta - Q) \cap \mathbb{Z}^2$.

$$\begin{aligned} W + \left((3\Delta)^{(1)} \cap \mathbb{Z}^2\right) &= \left(W + (3\Delta)^{(1)}\right) \cap \mathbb{Z}^2 \\ &\subseteq \left[q\Delta + (3\Delta)^{(1)} - P\right] \cap \left[q\Delta + (3\Delta)^{(1)} - Q\right] \cap \mathbb{Z}^2 \\ &\subseteq \left[((q+3)\Delta)^{(1)} - P\right] \cap \left[((q+3)\Delta)^{(1)} - Q\right] \cap \mathbb{Z}^2 \\ &\subseteq ((q+2)\Delta)^{(1)} \cap \mathbb{Z}^2. \end{aligned}$$

Since $(3\Delta)^{(1)}$ must contain a lattice point, it follows that $W \in (q-1)\Delta \cap \mathbb{Z}^2$.

7. $\implies$ 8.: we show this by contraposition, so we assume that item 8. is not satisfied for a half-plane H.

Let T a vertex of $H \cap \Delta$ at maximal distance from ℓ, and assume for now that this distance is positive. Let R be a vertex of $H \cap \Delta$, distinct from P, Q and T (the fact that such an R exists follows from the assumption). Without loss of generality, we may assume that R lies in the half-plane bordered by the line PT that does not contain Q. Choose coordinates such that the origin is P.

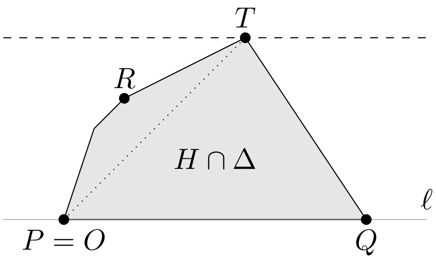


Figure 2: 7. $\Longrightarrow$ 8.

Figure 3: degenerate case (where T may be equal to Q)

Equip R with barycentric coordinates

$$R = \alpha T + \beta Q + \gamma P = \alpha T + \beta Q. \tag{16}$$

Because of the position of R, we know that $0 \leq \alpha \leq 1$ and $\beta < 0$.

Choose an integer $q > \max\{1, -\beta^{-1}\}$. Let $W = qR$. We claim that

$$W \in \left((q\Delta) \cap (q\Delta - Q) \cap \mathbb{Z}^2\right) \setminus (q-1)\Delta, \tag{17}$$

contradicting 7. Since R is a vertex of $H \cap \Delta$, we immediately have $W \in \left((q\Delta) \cap \mathbb{Z}^2\right) \setminus (q-1)\Delta$. It remains to show that $W \in q\Delta - Q$. Using (16), we have

$$\begin{aligned} W + Q = qR + Q &= qR + \beta^{-1}(R - \alpha T) \\ &= (q + \beta^{-1})R + (-\beta^{-1}\alpha)T \end{aligned}$$

This is a convex combination of $qP = O$, qR and qT because

$$q + \beta^{-1} \geq 0, \qquad -\beta^{-1}\alpha \geq 0,$$

and

$$(q + \beta^{-1}) + (-\beta^{-1}\alpha) = q + \beta^{-1}(1 - \alpha) \leq q.$$

It follows that $W + Q \in q\Delta$.

In the degenerate case where $T \in \ell$, without loss of generality one can assume that there is a vertex R such that R and Q lie on opposite sides of $P = O$. One proceeds as above with $\alpha = 0$ and $\beta < 0$.

8. $\Longrightarrow$ *9.*: this follows immediately from the geometry: Δ must be the union of two triangles on the base PQ.

9. $\Longrightarrow$ *3.*: we show this for $q = 2$. By assumption, the lattice polygon Δ is a convex quadrangle $PRQS$ (possibly degenerated into a triangle, i.e. one of R or S may coincide with P or Q). We need to show that

$$(2\Delta - P) \cap (2\Delta - Q) \subseteq \Delta \tag{18}$$

The left hand side is clearly contained in the cones $\widehat{RPS}$ and $\widehat{RQS}$, whose intersection is precisely our quadrangle $PRQS = \Delta$. $\square$

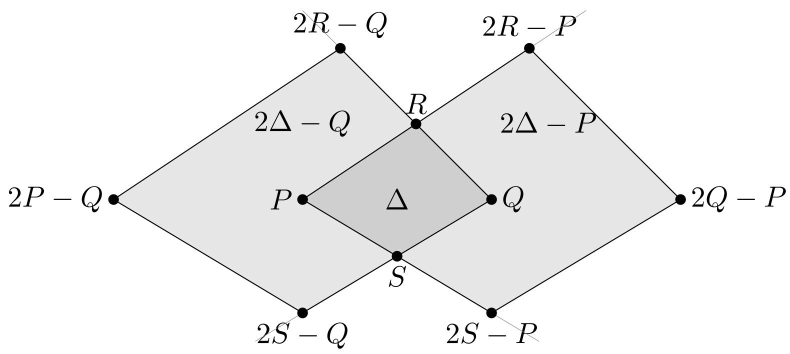


Figure 4: *9.* $\implies$ *3.* with $q = 2$

Now let us switch to regular sequences consisting of three points. We have the following easy fact:

Lemma 7.5. *Let $P, Q, R \in \Delta \cap \mathbb{Z}^2$ be distinct. Then P, Q, R is a regular sequence for $M = \bigoplus_{i\geq 0} V_{i\Delta}$ (resp. $M = \bigoplus_{i\geq 1} V_{(i\Delta)^{(1)}}$) if and only if*

$$P, Q, \qquad Q, R, \qquad P, R$$

are regular sequences.

Proof. It is clearly sufficient to prove the 'if' part of the claim. Assume for simplicity that $M = \bigoplus_{i\geq 0} V_{i\Delta}$, the other case is similar. Since P, Q is regular, all we have to check is that

$$V_{q\Delta\setminus((P+(q-1)\Delta)\cup(Q+(q-1)\Delta))} \xrightarrow{\cdot R} V_{(q+1)\Delta\setminus((P+q\Delta)\cup(Q+q\Delta))}$$

is injective, or equivalently that

$$(q\Delta\setminus((P+(q-1)\Delta)\cup(Q+(q-1)\Delta))+R)\cap((P+q\Delta)\cup(Q+q\Delta))=\emptyset.$$

This condition can be rewritten as

$$q\Delta \cap ((q\Delta + P - R) \cup (q\Delta + Q - R)) \subseteq (P + (q-1)\Delta) \cup (Q + (q-1)\Delta). \tag{19}$$

Since P, R is regular we know that $q\Delta \cap (q\Delta + P - R) \subseteq P + (q-1)\Delta$ by (14). Similarly because Q, R is regular we have $q\Delta \cap (q\Delta + Q - R) \subseteq Q + (q-1)\Delta$. Together these two inclusions imply (19). □

As an immediate corollary, we deduce using Theorem 7.4:

Corollary 7.6. *Let Δ be a two-dimensional lattice polygon. For three distinct lattice points $P, Q, R \in \Delta$, the following statements are equivalent:*

1. *P, Q, R is a regular sequence for $\bigoplus_{i\geq 0} V_{i\Delta}$.*
2. *P, Q, R is a regular sequence for $\bigoplus_{i\geq 1} V_{(i\Delta)^{(1)}}$.*
3. *Δ is a triangle with vertices P, Q and R.*

7.3 Example: the case of Veronese embeddings

Let us apply the foregoing to $\Delta = d\Sigma$ for $d \geq 2$, whose corresponding toric surface is the Veronese surface $\nu_d(\mathbb{P}^2)$ with coordinate ring

$$S_{d\Sigma} \cong k \oplus V_{d\Sigma} \oplus V_{2d\Sigma} \oplus V_{3d\Sigma} \oplus V_{4d\Sigma} \oplus V_{5d\Sigma} \oplus \ldots \tag{20}$$

By the foregoing corollary the sequence of points $(0,d)$, $(d,0)$, $(0,0)$ is regular for $S_{d\Sigma}$. When one removes these points along the above guidelines, the resulting graded module is

$$k \oplus V_{d\Sigma \setminus \{(0,d),(d,0),(0,0)\}} \oplus V_{\mathrm{conv}\{(d-1,d-1),(2,d-1),(d-1,2)\}} \oplus 0 \oplus 0 \oplus 0 \oplus \ldots$$

which can be rewritten as

$$k \oplus V_{d\Sigma \setminus \{(0,d),(d,0),(0,0)\}} \oplus V_{(d,d)-(d\Sigma)^{(1)}} \oplus 0 \oplus 0 \oplus 0 \oplus \ldots \tag{21}$$

We recall from the end of Section 7.1 that multiplication is defined by lattice addition, with the convention that the product is zero whenever the sum falls outside the indicated range. In order to find the graded Betti table of $\nu_d(\mathbb{P}^2)$, it therefore suffices to compute the cohomology of complexes of the following type:

$$\begin{aligned} \bigwedge^{\ell+1} V_{d\Sigma \setminus \{(0,d),(d,0),(0,0)\}} \longrightarrow \bigwedge^{\ell} V_{d\Sigma \setminus \{(0,d),(d,0),(0,0)\}} \otimes V_{d\Sigma \setminus \{(0,d),(d,0),(0,0)\}} \\ \longrightarrow \bigwedge^{\ell-1} V_{d\Sigma \setminus \{(0,d),(d,0),(0,0)\}} \otimes V_{(d,d)-(d\Sigma)^{(1)}} \end{aligned} \tag{22}$$

Indeed, the cohomology in the middle has dimension $\dim K_{\ell,1}(X,L) = b_\ell$ and the cokernel of the second morphism has dimension $\dim K_{\ell-1,2}(X,L) = c_{N_\Delta - 1 - \ell}$.

We can carry out the same procedure in the twisted case. The resulting graded module is

$$k \oplus V_{(d\Sigma)^{(1)}} \oplus V_{(d,d)-d\Sigma \setminus \{(0,d),(d,0),(0,0)\}} \oplus V_{\{(d,d)\}} \oplus 0 \oplus 0 \oplus \ldots$$

For instance, one finds that $K^\vee_{\ell,1}(X,L) \cong K_{N_\Delta - 3 - \ell, 2}(X; K, L)$ is the cohomology in the middle of

$$\begin{aligned} \bigwedge^{N_\Delta - \ell - 2} V_{(d\Sigma) \setminus \{(0,d),(d,0),(0,0)\}} \otimes V_{(d\Sigma)^{(1)}} \longrightarrow \\ \bigwedge^{N_\Delta - \ell - 3} V_{d\Sigma \setminus \{(0,d),(d,0),(0,0)\}} \otimes V_{(d,d)-d\Sigma \setminus \{(0,d),(d,0),(0,0)\}} \\ \longrightarrow \bigwedge^{N_\Delta - \ell - 4} V_{d\Sigma \setminus \{(0,d),(d,0),(0,0)\}} \otimes V_{(d,d)}. \end{aligned}$$

As a side remark, note that this complex is isomorphic to the dual of (22). Thus this gives a combinatorial proof of the duality formula $K^\vee_{\ell,1}(X,L) \cong K_{N_\Delta - 3 - \ell, 2}(X; K, L)$ for Veronese surfaces.

Let us conclude with a visualization of the point removal procedure in the case where $d = 3$ (in the non-twisted setting). Figure 5 shows how the coordinate ring gradually shrinks upon removal of $(0,3)$, then of $(3,0)$, and finally of $(0,0)$. The left column shows the graded parts of the original coordinate ring (20) in degrees $0,1,2,3$, while the right column does the same for the eventual graded module described in (21).

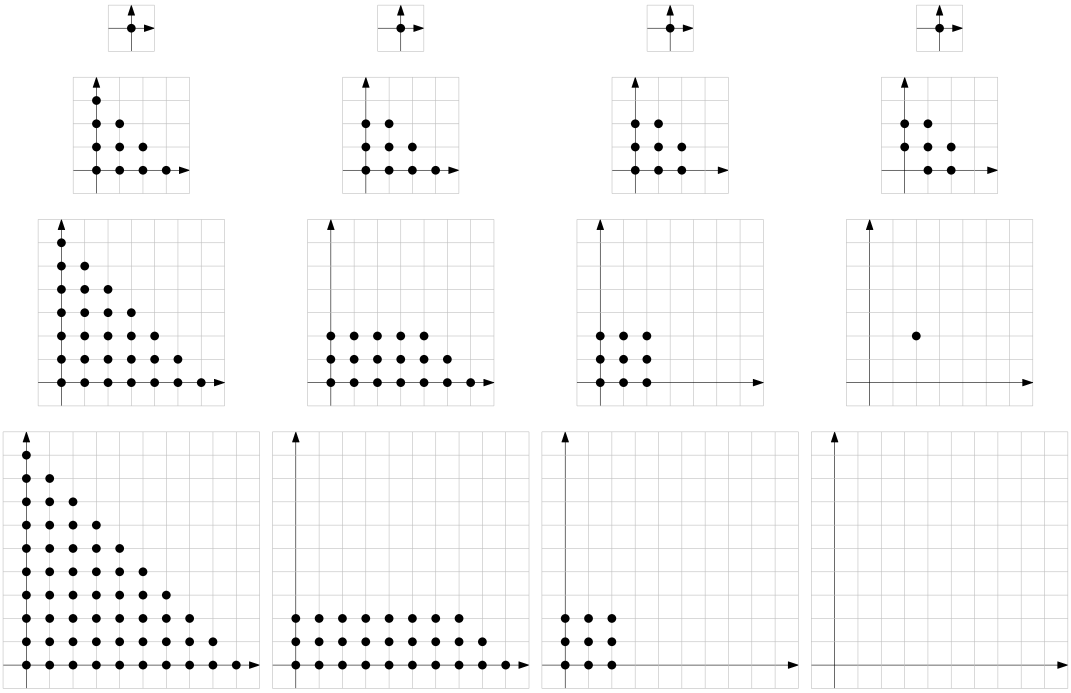


Figure 5: Removing three points for $\Delta = 3\Sigma$

8 Computing graded Betti numbers

8.1 The algorithm

To compute the entries b_ℓ and c_ℓ of the graded Betti table (1) of $X_\Delta \subseteq \mathbb{P}^{N_\Delta - 1}$ we use the formulas (4) and (7). In other words, we determine the b_ℓ's as

$$\dim \ker \left(\bigwedge^{\ell} V_\Delta \otimes V_\Delta \to \bigwedge^{\ell-1} V_\Delta \otimes V_{2\Delta} \right) - \dim \bigwedge^{\ell+1} V_\Delta,$$

while the c_ℓ's are computed as

$$\dim \ker \left(\bigwedge^{\ell-1} V_\Delta \otimes V_{\Delta^{(1)}} \to \bigwedge^{\ell-2} V_\Delta \otimes V_{(2\Delta)^{(1)}} \right).$$

Essentially, this requires writing down a matrix of the respective linear map and computing its rank. As explained in Section 2.2 we can consider these expressions for each bidegree (a, b) independently, and then just sum the contributions $c^\vee_{\ell,(a,b)}$ resp. $b_{\ell,(a,b)}$. This greatly reduces the dimensions of the vector spaces and hence of the matrices that we need to deal with.

Remark 8.1. The subtracted term in the formula for b_ℓ can be made explicit:

$$\dim \bigwedge^{\ell+1} V_\Delta = \binom{N_\Delta}{\ell+1}.$$

However we prefer to compute its contribution in each bidegree separately (which is easily done, see Section 8.2), the reason being that the $b_{\ell,(a,b)}$'s are interesting in their own right; see also Remark 8.2 below.

Speed-ups

Lemma 1.3 allows us to obtain $b_{N_\Delta-1-\ell}$ from c_ℓ and $c_{N_\Delta-1-\ell}$ from b_ℓ, so we only compute one of both. In practice we make an educated guess for what we think will be the easiest option, based on the dimensions of the spaces involved. Moreover, using Hering and Schenck's Theorem 1.4 we find that c_ℓ vanishes as soon as $\ell \geq N_\Delta + 1 - |\partial\Delta \cap \mathbb{Z}^2|$. For this reason the computation of $b_1, \ldots, b_{|\partial\Delta\cap\mathbb{Z}^2|-2}$ can be omitted, which is particularly interesting in the case of the Veronese polygons $d\Sigma$, which have many lattice points on the boundary.

Remark 8.2. From the proof of Lemma 1.3 we can extract the formula

$$b_{\ell,(a,b)} - c_{N_\Delta-1-\ell,(a,b)} = \sum_{j=0}^{\ell+1}(-1)^{j+1}\dim\left(\bigwedge^{\ell+1-j} V_\Delta \otimes V_{j\Delta}\right)_{(a,b)} \tag{23}$$

for each bidegree $(a,b) \in \mathbb{Z}^2$ and each $\ell = 1, \ldots, N_\Delta - 2$. Here the subscript on the right hand side indicates that we consider the subspace of elements having bidegree (a,b). As explained in Section 8.2, we can easily compute the dimensions of the spaces on the right hand side in practice. Together with (8) this allows one to obtain the bigraded parts of the entire Betti table, using essentially the same method. As an illustration, bigraded versions of some of the data gathered in Appendix A have been made available on `http://sage.ugent.be/www/jdemeyer/betti/`.

We use the material from Section 7 to reduce the dimensions further. As soon as we are dealing with an n-gon with $n \geq 5$, then by Theorem 7.4 we can remove one lattice point only. In the case of a quadrilateral we can remove two opposite vertices. In the case of a triangle we can remove its three vertices. For simple computations we just make a random amenable choice. For larger computations it makes sense to spend a little time on optimizing the point(s) to be removed, by computing the dimensions of the resulting quotient spaces.

Remark 8.3. As we have mentioned before, from a practical point of view the effect of removing lattice points is somewhat unpredictable. In certain cases we even observed that, although the resulting matrices are of considerably lower dimension, computing the rank takes more time. We currently have no explanation for this.

Another useful optimization is to take into account symmetries of Δ, which naturally induce symmetries of multiples of Δ and $\Delta^{(1)}$. For example for b_ℓ, consider a symmetry $\psi \in \mathrm{AGL}_2(\mathbb{Z})$ of $(\ell+1)\Delta$ and let (a,b) be a bidegree. Then $b_{\ell,(a,b)} = b_{\ell,\psi(a,b)}$. The analogous remark holds for c_ℓ, using symmetries ψ of $(\ell-1)\Delta + \Delta^{(1)}$.

A final speed-up comes from computing in finite characteristic, thereby avoiding inflation of coefficients when doing rank computations. We believe that this does not affect the outcome, even when computing modulo very small primes such as 2, but we have no proof of this fact. Therefore this speed-up comes at the cost of ending up with conjectural graded Betti tables. However recall from Remark 1.10 that the graded Betti numbers can never decrease, so the zero entries are rigorous (and because of Lemma 1.3 the other entry on the corresponding antidiagonal is rigorous as well).

Writing down the matrices

The maps we need to deal with are of the form

$$\bigwedge^p V_A \otimes V_B \xrightarrow{\delta} \bigwedge^{p-1} V_A \otimes V_C, \tag{24}$$

where A, B and C are finite sets of lattice points and δ is as in (3), subject to the additional rule mentioned in Remark 7.2. For a given bidegree (a, b), as a basis of the left hand side of (24) we make the obvious choice

$$\{\, x^{i_1}y^{j_1} \wedge \ldots \wedge x^{i_p}y^{j_p} \otimes x^{i'}y^{j'} \mid (i', j') = (a, b) - (i_1, j_1) - \ldots - (i_p, j_p) \text{ and } \{(i_1, j_1), \ldots, (i_p, j_p)\} \subseteq A \text{ and } (i', j') \in B\},$$

where $\{(i_1, j_1), \ldots, (i_p, j_p)\}$ runs over all p-element subsets of A. In the implementation, we equip A with a total order $<$ and take subsets such that $(i_1, j_1) < \ldots < (i_p, j_p)$. We do not need to store the part $x^{i'}y^{j'}$ since that is completely determined by the rest (for a fixed bidegree). We use the analogous basis for the right hand side of (24). We then compute the transformation matrix corresponding to the map δ in a given bidegree, and determine its rank.

Note that the resulting matrix is very sparse: it has at most p non-zero entries in every column, while the non-zero entries are 1 or -1. Therefore we use a sparse data structure to store this matrix.

Implementation

We have implemented all this in Python and Cython, using SageMath [33] with LinBox [26] for the linear algebra. In principle the algorithm should work equally fine in characteristic zero (at the cost of some efficiency) but for technical reasons our current implementation does not support this. For the implementation details we refer to the programming code, which is made available at `https://github.com/jdemeyer/toricbetti`.

8.2 Computing the dimensions of the spaces

Given finite subsets $A, B \subseteq \mathbb{Z}^2$, computing the dimension of the space $\bigwedge^p V_A \otimes V_B$ in each bidegree can be done efficiently without explicitly constructing a basis. These dimensions determine the sizes of the matrices involved. Knowing this size allows to estimate the amount of time and memory needed to compute the rank. We use this to decide whether to compute b_ℓ or $c_{N_\Delta - 1 - \ell}$, and which point(s) we remove when applying the material from Section 7.

Namely, consider the generating function (which is actually a polynomial)

$$f_A(X,Y,T) = \prod_{(i,j)\in A} (1 + X^i Y^j T). \tag{25}$$

Then the coefficient of $X^a Y^b T^p$ is the dimension of the component in bidegree (a,b) of $\bigwedge^p V_A$. The generating function for $\bigwedge^p V_A \otimes V_B$ then becomes

$$f_{A,B}(X,Y,T) = \prod_{(i,j)\in A} (1 + X^i Y^j T) \cdot \sum_{(i,j)\in B} X^i Y^j. \tag{26}$$

If we are only interested in a fixed p, we can compute modulo T^{p+1}, throwing away all higher-order terms in T.

8.3 Applications

As a first application we have verified Conjecture 1.6 for all lattice polygons containing at most 32 lattice points with at least one lattice point in the interior (namely we used the list of polygons from [7] and took those polygons for which $N_\Delta \leq 32$). There are 583 095 such polygons; the maximal lattice width that occurs is 8. Apart from the ten exceptional polygons $3\Sigma, \ldots, 6\Sigma, \Upsilon_2, \ldots, \Upsilon_6$ and 2Υ, we verified that the entry $b_{N_\Delta - \mathrm{lw}(\Delta) - 1}$ indeed equals zero. In the exceptional cases, whose graded Betti tables are gathered in Appendix A, we found that $b_{N_\Delta - \mathrm{lw}(\Delta)}$ equals zero. Together with Theorem 4.6 this proves that Conjecture 1.6 is satisfied for each of these lattice polygons. The computation was carried out modulo 40 009 and took 1006 CPU core-days on an Intel Xeon E5-2680 v3.

As a second application we have computed the graded Betti table of the 6-fold Veronese surface $X_{6\Sigma}$, which can be found in Appendix A. Currently the computation was done in finite characteristic only (again 40 009) and therefore some of the non-zero entries are conjectural. The computation took 12 CPU core-days on an IBM POWER8. This new data leads to the guesses stated in Conjecture 1.11, predicting certain entries of the graded Betti table of $X_{d\Sigma} = \nu_d(\mathbb{P}^2)$ for arbitrary $d \geq 2$.

- The first guess states that the last non-zero entry on the row $q = 1$ is given by $d^3(d^2-1)/8$. This is true for $d = 2, 3, 4, 5$ and has been verified in characteristic 40 009 for $d = 6, 7$.

- The second guess is about the first non-zero entry on the row $q = 2$, which we believe to be
$$\binom{N_{(d\Sigma)^{(1)}} + 8}{9}.$$
Here we have less supporting data: it is true for $d = 3, 4, 5$ and has been verified in characteristic 40 009 for $d = 6$. On the other hand our guess naturally fits within the more widely applicable formula
$$\binom{N_{\Delta^{(1)}} - 1 + \left|\{\, v \in \mathbb{Z}^2 \setminus \{(0,0)\} \mid \Delta^{(1)} + v \subseteq \Delta \,\}\right|}{N_{\Delta^{(1)}} - 1},$$

which we have verified for a large number of small polygons. It was discovered and proven to be a lower bound by the fourth author, in the framework of his Ph.D. research; we refer to his upcoming thesis for a proof.

A Some explicit graded Betti tables

This appendix contains the graded Betti tables of $X_\Delta \subseteq \mathbb{P}^{N_\Delta - 1}$ for the instances of Δ that are the most relevant to this paper. The largest of these Betti tables were computed using the algorithm described in Section 8. Because these computations were carried out modulo 40 009 the resulting tables are conjectural, except for the zero entries and the entries on the corresponding antidiagonal. The smaller Betti tables have been verified independently in characteristic zero using the Magma intrinsic [3], along the lines of [9, §2]. For the sake of clarity, we have indicated the conjectural entries by an asterisk. The question marks '???' mean that the corresponding entry has not been computed.

Σ $(N_\Delta = 3)$:

	0
0	1
1	0
2	0

2Σ $(N_\Delta = 6)$:

	0	1	2	3
0	1	0	0	0
1	0	6	8	3
2	0	0	0	0

3Σ $(N_\Delta = 10)$:

	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	27	105	189	189	105	27	0
2	0	0	0	0	0	0	0	1

4Σ $(N_\Delta = 15)$:

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	75	536	1947	4488	7095	7920	6237	3344	1089	120	0	0
2	0	0	0	0	0	0	0	0	0	0	55	24	3

5Σ $(N_\Delta = 21)$:

	0	1	2	3	4	5	6	7	8	···
0	1	0	0	0	0	0	0	0	0	
1	0	165	1830	10710	41616	117300	250920	417690	548080	···
2	0	0	0	0	0	0	0	0	0	

	···	9	10	11	12	13	14	15	16	17	18
0		0	0	0	0	0	0	0	0	0	0
1	···	568854	464100	291720	134640	39780	4858	375	0	0	0
2		0	0	0	0	2002	4200	2160	595	90	6

6Σ $(N_\Delta = 28)$:

	0	1	2	3	4	5	6	7	8	···
0	1	0	0	0	0	0	0	0	0	
1	0	315	4950	41850	240120	1024650	3415500	9164925	20189400	···
2	0	0	0	0	0	0	0	0	0	

	···	9	10	11	12	13	14	15	···
0		0	0	0	0	0	0	0	
1	···	36989865	56831850	73547100	80233200	73547100	56163240	35102025	···
2		0	0	0	0	0	0	0	

	···	16	17	18	19	20	21	22	23	24	25
0		0	0	0	0	0	0	0	0	0	0
1	···	17305200	6177545*	1256310*	160398*	17890*	945*	0	0	0	0
2		48620*	231660*	593028*	473290*	218295*	69300	15525	2376	225	10

$7\Sigma \ (N_\Delta = 36)$:

	···	26	27	28	29	30	31	32	33
0		0	0	0	0	0	0	0	0
1	···	???	53352*	2058*	0	0	0	0	0
2		27821664*	8824410*	2215136	434280	64449	6832	462	15

$\Upsilon = \Upsilon_1 \ (N_\Delta = 4)$:

	0	1
0	1	0
1	0	0
2	0	1

$2\Upsilon \ (N_\Delta = 10)$:

	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	24	84	126	84	20	0	0
2	0	0	0	0	20	36	21	4

$\Upsilon_2 \ (N_\Delta = 7)$:

	0	1	2	3	4
0	1	0	0	0	0
1	0	7	8	3	0
2	0	0	6	8	3

$\Upsilon_3 \ (N_\Delta = 11)$:

	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	30	120	210	189	105	27	0	0
2	0	0	0	21	105	147	105	40	6

$\Upsilon_4 \ (N_\Delta = 16)$:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	81	598	2223	5148	7920	8172	6237	3344	1089	120	0	0	0
2	0	0	0	0	55	450	2376	4488	4950	3630	1859	612	117	10

$\Upsilon_5 \ (N_\Delta = 22)$:

	0	1	2	3	4	5	6	7	8	9	···
0	1	0	0	0	0	0	0	0	0	0	
1	0	175	1995	11970	47481	135660	290820*	476385*	597415*	581724*	···
2	0	0	0	0	0	120*	1575*	9555*	52650*	172172*	

	···	10	11	12	13	14	15	16	17	18	19
0		0	0	0	0	0	0	0	0	0	0
1	···	466102*	291720*	134640*	39780*	4858*	375*	0	0	0	0
2		291720*	338130*	291720*	194782*	102120*	39900	11305	2205	266	15

$\Upsilon_6 \ (N_\Delta = 29)$:

	···	18	19	20	21	22	23	24	25	26
0		0	0	0	0	0	0	0	0	0
1	···	???	160398*	17890*	945*	0	0	0	0	0
2		16095603*	7911490*	3140445*	995280	246675	46176	6150	520	21

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A LOWER BOUND FOR THE GONALITY CONJECTURE

WOUTER CASTRYCK

ABSTRACT. For every integer $k \geq 3$ we construct a k-gonal curve C along with a very ample divisor of degree $2g + k - 1$ (where g is the genus of C) to which the vanishing statement from the Green-Lazarsfeld gonality conjecture does not apply.

The gonality conjecture due to Green and Lazarsfeld [GL86] states that for C a smooth complex projective curve of genus $g \geq 1$ and gonality $k \geq 2$, and L a globally generated divisor on C of sufficiently large degree, one has the following vanishing criterion in Koszul cohomology:

$$K_{i,1}(C, L) \neq 0 \quad \Leftrightarrow \quad 1 \leq i \leq h^0(C, L) - k - 1. \tag{1}$$

This conjecture was proved two years ago by Ein and Lazarsfeld [EL15], who moreover provided the sufficient lower bound $\deg L \geq g^3$. In the meantime this was improved to $\deg L \geq 4g - 3$ by Rathmann [Ra16].

It is likely that this bound can be improved further, although Green and Lazarsfeld already noted [GL86, p. 86] that one at least needs

$$\deg L \geq 2g + k - 1 \tag{2}$$

because of the non-vanishing of $K_{g-1,1}(K_C + D)$, for any divisor D of rank 1 and degree k on C. Very recently Farkas and Kemeny [FK16] showed that if C is sufficiently generic inside the moduli space of k-gonal curves of genus g then the bound (2) is sufficient for the vanishing criterion (1) to hold. Moreover they conjectured that this should be true for *all* curves whose genus is large enough when compared to the gonality, a statement referred to as the effective gonality conjecture. Results due to Green [Gr84, Thm. 3.c.1] resp. Teixidor i Bigas [Te07, Prop. 3.8] imply that this is indeed the case for trigonal resp. tetragonal curves of genus $g > 3$ resp. $g > 6$.

In this note we aim for an improved delimitation of the foregoing considerations, through the following result.

Theorem. *For each $k \geq 3$ there exists a curve C of genus $g = k(k-1)/2$ along with a very ample divisor L of degree $2g + k - 1$ such that*

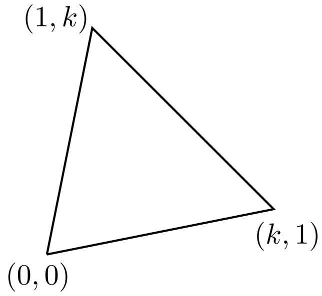
$$K_{h^0(C,L)-k,1}(C, L) \neq 0. \tag{3}$$

In particular, the bound (2) *is not sufficient for the gonality conjecture to apply.*

The main conclusions to be drawn are that on the one hand, at most, one can hope to improve Rathmann's bound to $\deg L \geq 2g + k$, and that on the other hand $g > k(k-1)/2$ is a necessary lower bound in the statement of Farkas and

Kemeny's effective gonality conjecture. In particular one observes that the special cases implied by the works of Green and Teixidor i Bigas are sharp.

The construction is short and explicit: we let C be a smooth plane curve of degree $k+1$ carrying two total inflection points. By positioning these points at the coordinate points $(1:0:0)$ and $(0:1:0)$ at infinity, and moving the intersection point of the corresponding tangent lines to the finite coordinate point $(0:0:1)$, we can assume that inside $\mathbb{A}^2$ our curve is defined by a polynomial $f(x,y)$ whose Newton polygon Γ is as follows:



Conversely, every curve C defined by a sufficiently generic polynomial that is supported on the above polygon will do. We let $L = (k+1)D$ where

$$D + (1:0:0) + (0:1:0)$$

is the effective divisor cut out by the line at infinity. Being a smooth plane curve of degree $k+1$, the genus of C equals $g = k(k-1)/2$, while its gonality is k by [Se87, Prop. 3.13]. Finally the degree of L equals $(k+1)(k-1) = 2g+k-1$, as announced.

Identifying the function field of C with the field of fractions of $\mathbf{C}[x,y]/(f(x,y))$, we have the following lemma:

Lemma. *L is a very ample divisor, and the vector space $H^0(C,L)$ admits the monomial basis*

$$\{\, x^i y^j \,|\, i \geq 1, j \geq 1, i+j \leq k+1 \,\}.$$

In particular $h^0(C,L) = \binom{k+1}{2} = k(k+1)/2$.

Proof. The very-ampleness follows from degree considerations. As for the other statement, it follows from basic principles in toric geometry that $H^0(C,L)$ is generated by all monomials that are supported on Γ; see for example [CDV06, Thm. 2] for a ready-to-use statement[1], where we note that our divisor L is exactly the divisor D_C given there. These monomials are given by

$$\{\, 1 \,\} \;\cup\; \{\, x^i y^j \,|\, i \geq 1, j \geq 1, i+j \leq k+1 \,\},$$

[1]Strictly spoken that theorem only applies if $f(x,y)$ is non-degenerate with respect to its Newton polygon, in our case meaning that C is non-tangent to the line at infinity, but it holds more generally. (On the other hand, for the purpose of proving our main theorem, adding the non-degeneracy assumption is fine.)

which are linearly dependent because $f(x, y) = 0$. Since $f(x, y)$ has a non-zero constant term we can eliminate 1 from the above set, and the remaining monomials are clearly linearly independent functions on C. □

If one uses the above basis of $H^0(C, L)$ to embed C in

$$\mathbb{P}^{\binom{k+1}{2}-1}$$

then it naturally lands inside the $(k-1)$-fold Veronese surface $\nu_{k-1}(\mathbb{P}^2)$. This implies that we have an injection

$$K_{k(k+1)/2-k,1}\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-1)\right) \hookrightarrow K_{k(k+1)/2-k,1}(C, L) = K_{h^0(C,L)-k,1}(C, L).$$

But it is known that the former space is non-trivial; see e.g. [CCDL16, Thm. 1.8]. From this our theorem follows.

Remark. The curve C, when embedded using L, can be viewed as a hyperplane section of the projectively embedded toric surface X_Γ associated to the lattice polygon Γ. Thanks to the Cohen-Macaulay property of toric surfaces, the graded Betti tables of C and X_Γ are the same. In previous joint work with Cools, Demeyer and Lemmens [CCDL16], we had already observed 'exceptional' vanishing behavior among the linear syzygies in the case of X_Γ (there Γ was denoted as Υ_{k-1}). As it turns out, this percolates to the curve level, thereby saying something meaningful about the gonality conjecture.

Remark. We work over **C** because for instance [EL15] does so as well, but the theorem presented above is valid over any algebraically closed field, with the same proof.

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On graded Betti tables of curves in toric surfaces

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Abstract

In a first part of this paper, we prove constancy of the canonical graded Betti table among the smooth curves in linear systems on certain toric surfaces, namely those that are Gorenstein weak Fano. In a second part, we show that Green's canonical syzygy conjecture holds for all smooth curves of genus $g \leq 32$ on arbitrary toric surfaces. Conversely we use known results on Green's conjecture (due to Lelli-Chiesa) to obtain new facts about graded Betti tables of toric surfaces.

Contents

1 Introduction

Let k be an algebraically closed field of characteristic zero, let $\Delta \subseteq \mathbb{R}^2$ be a two-dimensional lattice polygon, and consider an irreducible Laurent polynomial

$$f = \sum_{(i,j) \in \Delta \cap \mathbb{Z}^2} c_{i,j} x^i y^j \in k[x^{\pm 1}, y^{\pm 1}] \tag{1}$$

that is supported on Δ. Let $S_\Delta = k[\, X_{i,j} \,|\, (i,j) \in \Delta \cap \mathbb{Z}^2 \,]$ be the polynomial ring obtained by associating a formal variable to each lattice point in Δ. We think of it as the homogeneous coordinate ring of projective $(N_\Delta - 1)$-space, where $N_\Delta = |\Delta \cap \mathbb{Z}^2|$. Consider the map

$$\varphi_\Delta : (k^*)^2 \hookrightarrow \mathbb{P}^{N_\Delta - 1} : (\alpha, \beta) \mapsto (\alpha^i \beta^j)_{(i,j) \in \Delta \cap \mathbb{Z}^2},$$

the Zariski closure of the image of which is a toric surface that we denote by X_Δ. Let U_f be the curve in $(k^*)^2$ defined by $f = 0$, and assume that the closure C_f of $\varphi_\Delta(U_f)$ inside X_Δ is a smooth hyperplane section, necessarily cut out by

$$\sum_{(i,j) \in \Delta \cap \mathbb{Z}^2} c_{i,j} X_{i,j} = 0.$$

This assumption is generically true, i.e. it holds for a dense open subset of the space of Laurent polynomials that are supported on Δ. For instance a well-known generically satisfied sufficient condition reads that f is Δ-non-degenerate, in the sense of [2].

Remark 1.1. Whenever one is given a non-rational smooth projective curve C inside a toric surface X, equipped with an embedding $\varphi : (k^*)^2 \hookrightarrow X$, then one can show that C arises in the above way. Namely let P_C be the polygon associated to a torus-invariant divisor on X that is linearly equivalent to C; see [10, §4.3] for how this polygon is constructed. Define $\Delta = \operatorname{conv}(P_C \cap \mathbb{Z}^2)$, where we note that if C is Cartier then P_C is a lattice polygon and $\Delta = P_C$. If for f one takes the generator of the ideal of $\varphi^{-1}(C)$ inside $k[x^{\pm 1}, y^{\pm 1}]$ that is supported on Δ, then the above assumption is satisfied and one has $C_f \cong C$. We refer to [6, §4] for more background on these claims.

Under our generic assumption that C_f is a smooth hyperplane section, many of its geometric invariants can be told explicitly from the combinatorics of Δ. The starting result was proven by Khovanskii [17], who obtained that the geometric genus $g(C_f)$ is given by $N_{\Delta^{(1)}} = |\Delta^{(1)} \cap \mathbb{Z}^2|$, where $\Delta^{(1)}$ denotes the convex hull of the lattice points in the interior of Δ. To avoid low genus pathologies, from now on let us assume that $|\Delta^{(1)} \cap \mathbb{Z}^2| \geq 4$. Then recent work of mainly Kawaguchi (a technical assumption was removed by the first two current authors) provides a similar combinatorial interpretation for the Clifford index $\operatorname{ci}(C_f)$; see [9, 11] for some background on this invariant.

Theorem 1.2 (see [6, 16]). *One has* $\operatorname{ci}(C_f) = \operatorname{lw}(\Delta^{(1)})$ *unless* $\Delta^{(1)} \cong \Upsilon$, $\Delta^{(1)} \cong 2\Upsilon$ *or* $\Delta^{(1)} \cong (d-3)\Sigma$ *for some* $d \in \mathbb{Z}_{\geq 5}$, *in which case one has* $\operatorname{ci}(C_f) = \operatorname{lw}(\Delta^{(1)}) - 1$.

Here lw denotes the lattice width [3] and $\Delta \cong \Delta'$ indicates that Δ' can be obtained from Δ using a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2 : (x\ y) \mapsto (x\ y)A + b$, where $A \in \mathrm{GL}_2(\mathbb{Z})$ and $b \in \mathbb{Z}^2$ (unimodular equivalence). The polygons Υ and Σ are given by $\operatorname{conv}\{(-1,-1),(1,0),(0,1)\}$ resp. $\operatorname{conv}\{(0,0),(1,0),(0,1)\}$, and the multiples are in Minkowski's sense. As a corollary to Theorem 1.2, we note that C_f is non-hyperelliptic if and only if $\Delta^{(1)}$ is two-dimensional.

The proof of Theorem 1.2 entails similar interpretations for the gonality and the Clifford dimension. Finer data that is known to be encoded in the combinatorics of Δ includes the scrollar invariants [6] associated to a gonality pencil, which specialize to the Maroni invariants in the trigonal case. Assuming that Δ satisfies a mild condition, it also includes the first scrollar Betti numbers associated to a gonality pencil, which specialize to Schreyer's invariants b_1, b_2 in the case of tetragonal curves (where the mild condition is void); see [5, 7].

An immediate consequence is that all these invariants depend on Δ only. This is an a priori non-trivial fact that can be rephrased as *constancy* (of the Clifford index, the gonality, ...) among the smooth curves in linear systems of curves on toric surfaces. The existing literature contains other results of this type. For instance recent work of Lelli-Chiesa proves constancy of the gonality and the Clifford index for curves in certain linear systems on other types of rational surfaces [19]. A theorem by Green and Lazarsfeld states that constancy of the Clifford index holds in linear systems on K3 surfaces [13], although here constancy of the gonality is not necessarily true.

In view of Theorem 1.2 and Green's canonical syzygy conjecture [12], it is natural to wonder whether similar constancy results hold for the entire graded Betti table

$$\begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & \dots & g-4 & g-3 & g-2 \\ \hline 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & a_1 & a_2 & a_3 & \dots & a_{g-4} & a_{g-3} & 0 \\ 2 & 0 & a_{g-3} & a_{g-4} & a_{g-5} & \dots & a_2 & a_1 & 0 \\ 3 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1, \end{array} \qquad (2)$$

of the canonical image of C_f in $\mathbb{P}^{g-1}$, where $g = g(C_f) = |\Delta^{(1)} \cap \mathbb{Z}^2|$. When writing down the above shape we assume that C_f is non-hyperelliptic, or equivalently that $\Delta^{(1)}$ is two-dimensional, so that the canonical map $\kappa : C_f \to \mathbb{P}^{g-1}$ is an embedding. We will keep making this assumption throughout the rest of the article. An attractive feature of smooth curves in toric surfaces is that κ is well understood. Indeed, a refined version of Khovanskii's theorem provides us with a canonical divisor K_Δ on C_f whose associated Riemann-Roch space $H^0(C_f, K_\Delta)$ admits the basis $\{\, x^i y^j \mid (i,j) \in \Delta^{(1)} \cap \mathbb{Z}^2 \,\}$. Thus for this choice of canonical divisor one has that

$$\kappa \circ \varphi_\Delta|_{U_f} = \varphi_{\Delta^{(1)}}|_{U_f}.$$

As a consequence the canonical model of C_f, which we denote by C, satisfies

$$C \subseteq X_{\Delta^{(1)}} \subseteq \mathbb{P}^{g-1}.$$

We therefore expect some interplay between the graded Betti table of C and that of $X_{\Delta^{(1)}}$, which is known to be of the form

$$\begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & \dots & g-4 & g-3 \\ \hline 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & b_1 & b_2 & b_3 & \dots & b_{g-4} & b_{g-3} \\ 2 & 0 & c_{g-3} & c_{g-4} & c_{g-5} & \dots & c_2 & c_1, \end{array} \tag{3}$$

by [8, Lemma 1.2].

The main result of this article is the following constancy statement, whose proof is given in Section 4:

Theorem 1.3. *If*

- $X_{\Delta^{(1)}}$ *is a Gorenstein weak Fano toric surface, or*
- $|\partial\Delta^{(1)} \cap \mathbb{Z}^2| \geq g/2 + 1$,

then for all $\ell = 1, \dots, g-3$ *we have* $a_\ell = b_\ell + c_\ell$*. In particular, in these cases the graded Betti table of* C *is independent of the coefficients of* f*.*

Here $\partial\Delta^{(1)}$ denotes the boundary of $\Delta^{(1)}$.

We note that the Gorenstein weak Fano condition can be easily rephrased in combinatorial terms, as is done in Section 3 below. This case covers all polygons Δ satisfying $\Delta^{(1)} \cong (d-3)\Sigma$ for some $d \geq 5$, leading to the statement that the canonical graded Betti table of a smooth plane degree d curve depends on d only. On the other hand, the class of polygons Δ for which $|\partial\Delta^{(1)} \cap \mathbb{Z}^2| \geq g/2 + 1$ covers all cases where $\mathrm{lw}(\Delta^{(1)}) \leq 2$ by [5, Lemma 9]. Such polygons correspond to trigonal and certain tetragonal curves, where constancy was known to hold before [5, 24].

We actually believe that the sum formula $a_\ell = b_\ell + c_\ell$ is true for a considerably larger class of polygons than the ones covered by the above theorem. Of course, even when the formula fails, it might still be true that the graded Betti table of C does not depend on f, i.e. the defect might depend on Δ and ℓ only. Examples of such behaviour are given in Section 4. We leave it as an open question whether or not this is true in general.

We end this article in Section 5 with a somewhat disjoint discussion on how Green's canonical syzygy conjecture, concerning graded Betti tables of canonical curves, relates to a conjecture that we have formulated in a previous article [8], concerning graded Betti tables of toric surfaces. In particular, we settle new cases of both conjectures. For instance we find that Green's conjecture holds for all smooth curves on toric surfaces of genus at most 32.

2 An exact sequence involving six terms

Let Δ be a lattice polygon with two-dimensional interior lattice polygon $\Delta^{(1)}$. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be an irreducible Laurent polynomial as in (1) and assume that the corresponding hyperplane section C_f of X_Δ is smooth.

Let $\rho : X \to X_\Delta$ be the *minimal* toric resolution of singularities, i.e. X is the toric surface associated to the smooth subdivision of the inner normal fan to Δ in which no more new rays are introduced than needed. It can be obtained using Hirzebruch-Jung continued fractions as described in [10, §10.2]. Let K be the canonical divisor on X obtained by taking minus the sum of all torus-invariant prime divisors [10, Thm. 8.2.3].

Because C_f does not meet the singular locus of X_Δ, it pulls back to an isomorphic curve C' on X. Define $D_f = C' - \operatorname{div}(f)$, where f is viewed as a function on X by pushing it forward along φ_Δ and then pulling it back along ρ. This is a torus-invariant divisor that is linearly equivalent to C'.

Lemma 2.1. *One has:*

- *The divisor D_f is base-point free, and the polygon P_{D_f} associated to D_f is Δ.*
- *Its adjoint divisor $L := D_f + K$ is also base-point free, and the polygon P_L associated to L is $\Delta^{(1)}$.*

The second statement might be of interest to people studying Fujita type results; see [18, 21]. Here the minimality of our resolution $X \to X_\Delta$ is important, as the reader can tell from the proof below. Also recall that for divisors on a smooth toric surface, the notions of base-point free and nef are synonymous [10, Thms. 6.1.7 and 6.3.12].

Proof. Let Σ_Δ be the fan of X_Δ (i.e., the inner normal fan to Δ) and let Σ be the fan of X. Denote by $U(\Sigma)$ the set of primitive generators of the rays of Σ, and let $U(\Sigma_\Delta) \subseteq U(\Sigma)$ be the subset of vectors that correspond to rays of Σ_Δ. Since the divisor D_f is torus-invariant, it is of the form $\sum_{v \in U(\Sigma)} a_v D_v$, where $D_v \subseteq X$ is the prime divisor corresponding to the ray generated by v. Let $H(v, a_v)$ be the half-plane of points $x \in \mathbb{R}^2$ satisfying $\langle x, v \rangle \geq -a_v$ and let $L(v, a_v)$ be the line defined by $\langle x, v \rangle = -a_v$. As explained in [6, §4], we have that

$$P_{D_f} = \bigcap_{v \in U(\Sigma)} H(v, a_v) = \bigcap_{v \in U(\Sigma_\Delta)} H(v, a_v) = \Delta.$$

Moreover, if $u \in U(\Sigma) \setminus U(\Sigma_\Delta)$ corresponds to a ray that lies in between two consecutive rays of Σ_Δ with primitive generators $v, w \in U(\Sigma_\Delta)$, then $L(u, a_u)$ passes through the vertex $L(v, a_v) \cap L(w, a_w)$ of Δ. In other words, if $v, w \in U(\Sigma)$ correspond to consecutive rays of Σ, then $L(v, a_v) \cap L(w, a_w) \in \Delta$. By [10, Prop. 6.1.1], this just means that D_f is base-point free.

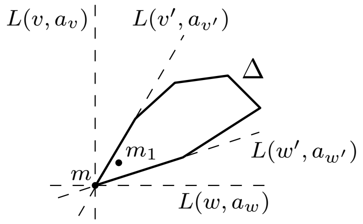
Since $K = -\sum_{v \in U(\Sigma)} D_v$, we have that $L = \sum_{v \in U(\Sigma)} (a_v - 1) D_v$. It follows that the polygon associated to L is

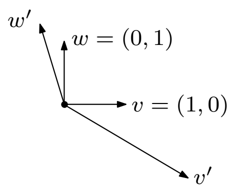
$$P_L = \bigcap_{v \in U(\Sigma)} H(v, a_v - 1).$$

To prove that L is base-point free, again by [10, Prop. 6.1.1] it suffices to show that for all $v, w \in U(\Sigma)$ that correspond to adjacent rays, the lattice point $m_1 = L(v, a_v - 1) \cap L(w, a_w - 1)$ belongs to P_L. Because X is smooth, the vectors v, w form a basis $\mathbb{Z}^2$, hence using a unimodular transformation if needed we may assume that $v = (1, 0)$ and $w = (0, 1)$. Then the point m_1 becomes $(-a_v + 1, -a_w + 1)$. From the base-point-freeness of D_f we know that

$$m = (a_v, a_w) \in \Delta \subseteq H(v, a_v) \cap H(w, a_w).$$

Now consider $v', w' \in U(\Sigma_\Delta)$ such that $m \in L(v', a_{v'}) \cap L(w', a_{w'})$, so v' and w' are the primitive normal vectors of the edges of Δ that are adjacent to the vertex m. We can assume that $L(v', a_{v'})$ is steeper than $L(w', a_{v'})$, and note that it could happen that $v' = v$ and/or $w' = w$. In order to prove that $m_1 \in P_L$, it suffices to show that $L(v', a_{v'})$ passes strictly above m_1 and that $L(w', a_{w'})$ passes strictly below m_1. We only prove the statement for v'; the one for w' follows by symmetry.





Let $v_0 = v, v_1, \ldots, v_n = v'$ be the vectors in $U(\Sigma)$ from v up to v' going clockwise. We claim that all v_i satisfy $x_i > -y_i$, where $v_i = (x_i, y_i)$. For $i = n$, this claim tells us that $L(v', a_{v'})$ passes strictly above m_1. Suppose our claim is false and let i be minimal such that $x_i \leq -y_i$. Note that $i > 0$. It is impossible that $x_i = -y_i$, because in that case $w = (0, 1)$ and $v_i = (1, -1)$ would be a basis of $\mathbb{Z}^2$, so would be able to delete the rays corresponding to $v_j \in U(\Sigma)$ with $j < i$, while the associated toric surface would still be a resolution of singularities of X_Δ, contradicting the minimality assumption. So $x_i < -y_i$. Also $x_{i-1} > -y_{i-1}$, by the minimality of i. Now v_{i-1} and v_i must form a basis of $\mathbb{Z}^2$ and hence the determinant $x_i y_{i-1} - x_{i-1} y_i$ of the matrix formed by v_i and v_{i-1} is ± 1. But $x_{i-1}(-y_i) > x_{i-1} x_i > (-y_{i-1}) x_i$, and since we have two strict inequalities, the difference is at least 2. This contradicts that the determinant is ± 1, proving our claim.

It remains to show that $P_L = \Delta^{(1)}$. Because L is base-point free, again from the criterion [10, Prop. 6.1.1] we see that P_L is a lattice polygon. Now since it is contained in the interior of $P_{D_f} = \Delta$, while on the other hand it clearly contains $\Delta^{(1)}$, the claim follows. □

The above lemma is valuable in investigating how the graded Betti table (2) of the canonical model C of C_f relates to the graded Betti table (3) of $X_{\Delta^{(1)}}$. We assume that the reader is familiar with how the entries a_ℓ, b_ℓ, c_ℓ for $\ell = 1, \dots, g-3$ arise as dimensions of Koszul cohomology spaces. We refer to [1] for more background, and to [8, §2] and [15] for a discussion that is specific to toric surfaces. For what follows, it is convenient to define $a_0 = b_0 = c_0 = a_{g-2} = b_{g-2} = c_{g-2} = 0$.

Our starting point is the standard exact sequence $0 \to \mathcal{O}_X(-C') \to \mathcal{O}_X \to \mathcal{O}_{C'} \to 0$ of sheaves of $\mathcal{O}_X$-modules. It can be rewritten as

$$0 \to \mathcal{O}_X(-D_f) \xrightarrow{\mu_f} \mathcal{O}_X \longrightarrow \mathcal{O}_{C'} \to 0,$$

where μ_f denotes multiplication by the function f. By the adjunction formula $K_{C'} := L|_{C'}$ is a canonical divisor on C'. Tensoring the above exact sequence with $\mathcal{O}_X(qL)$ then gives exact sequences

$$0 \to \mathcal{O}_X((q-1)L + K) \xrightarrow{\mu_f} \mathcal{O}_X(qL) \longrightarrow \mathcal{O}_{C'}(qK_{C'}) \to 0$$

for all $q \geq 0$. We claim that $H^1(X, (q-1)L + K) = 0$, which by Serre duality [10, Thm. 9.2.10] is equivalent with $H^1(X, (1-q)L) = 0$. Indeed for $q = 0$ and $q = 1$ this is true by Demazure vanishing [10, Thm. 9.2.3], while for $q \geq 2$ it follows from Batyrev-Borisov vanishing [10, Thm. 9.2.7(a)]. In both cases we used that L is base-point free, while in the latter case we also used that $P_L = \Delta^{(1)}$ is two-dimensional. Thus by taking cohomology we obtain a short exact sequence

$$0 \to \bigoplus_{q \geq 0} H^0(X, (q-1)L + K) \xrightarrow{\mu_f} \bigoplus_{q \geq 0} H^0(X, qL) \longrightarrow \bigoplus_{q \geq 0} H^0(C', qK_{C'}) \to 0$$

of k-vector spaces. In a natural way, this can be viewed as an exact sequence of graded modules over $S_{\Delta^{(1)}} = S^* V_{\Delta^{(1)}}$, where $V_{\Delta^{(1)}} = H^0(X, L)$ and S^* denotes the symmetric algebra. By [1, Lem. 1.25] we find a long exact sequence in Koszul cohomology:

$$\cdots \to K_{p,q-1}(X; K, L) \xrightarrow{\mu_f} K_{p,q}(X, L) \longrightarrow K_{p,q}(C', K_{C'})$$

$$\longrightarrow K_{p-1,q}(X; K, L) \xrightarrow{\mu_f} K_{p-1,q+1}(X, L) \longrightarrow K_{p-1,q+1}(C', K_{C'}) \to \dots$$

Now note that the image of $X \xrightarrow{|L|} \mathbb{P}^{g-1}$, where $g = h^0(X, L) = |\Delta^{(1)} \cap \mathbb{Z}^2|$, is nothing else but $X_{\Delta^{(1)}}$. Thus

$$b_\ell = \dim K_{\ell,1}(X, L) \qquad \text{and} \qquad c_\ell = \dim K_{g-2-\ell,2}(X, L) = \dim K_{\ell-1,1}(X; K, L)$$

for $\ell = 0, 1, \ldots, g-2$, where the last equality again follows from Serre duality, as explained in more detail in [8, §2.1]. Combining these formulas with $a_\ell = \dim K_{\ell,1}(C', K_{C'})$ we find for each $\ell = 0, 1, \ldots, g-2$ our desired exact sequence, of the form

$$0 \to b_\ell \to a_\ell \to c_\ell \xrightarrow{\mu_f} c_{g-1-\ell} \to a_{g-1-\ell} \to b_{g-1-\ell} \to 0 \tag{4}$$

where we abusingly write the dimensions, rather than the cohomology spaces themselves.

Remark 2.2. It follows that

$$b_\ell + c_\ell - c_{g-1-\ell} - b_{g-1-\ell} = a_\ell - a_{g-1-\ell}.$$

The right hand side is known to be equal to

$$\binom{g-1}{\ell-1} \frac{(g-1-\ell)(g-1-2\ell)}{\ell+1}$$

using the Hilbert polynomial of the canonical curve C. This formula also follows from [8, Lem. 1.3], by using instead the left hand side of the equality.

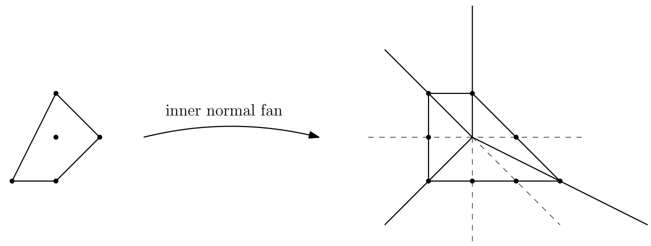
3 Gorenstein weak Fano toric surfaces

As before let Δ be a lattice polygon with two-dimensional interior $\Delta^{(1)}$. Let Σ_Δ denote the inner normal fan to Δ, and as in the proof of Lemma 2.1 let $U(\Sigma_\Delta)$ be the set of primitive generators of its rays. The prime divisor associated to $v \in U(\Sigma_\Delta)$ will again be denoted by D_v. For reasons that will become apparent in the next section, we are interested in situations where the polygon P_{-K_Δ} associated to the anticanonical divisor

$$-K_\Delta = \sum_{v \in U(\Sigma)} D_v$$

on X_Δ is a lattice polygon. Using the criteria in [10, Chapter 6] one sees that this holds if and only if $-K_\Delta$ is base-point free (i.e., nef) and Cartier. Since $-K_\Delta$ is always big, we conclude that we are actually interested in the cases where X_Δ is a so-called *Gorenstein weak Fano* toric surface.

Note that P_{-K_Δ} has one interior lattice point only, namely the origin, therefore as soon as we are in the Gorenstein weak Fano case, it concerns a reflexive polygon. Its dual polygon is the convex hull of the vectors $v \in U(\Sigma_\Delta)$, which is then also reflexive. It is not hard to see that the argument works in both ways, i.e. a toric surface is Gorenstein weak Fano if and only if the convex hull of the primitive generators of the rays of its fan is a reflexive polygon. Up to unimodular equivalence, there are 16 reflexive polygons [22, Prop. 4.1], so a toric surface is Gorenstein weak Fano if and only if its fan is a *coherent crepant refinement* of the inner normal fan to one of these 16 polygons. That is, it is obtained by inserting a number of rays (possibly none) that pass through a lattice point on the boundary of the dual polygon.



A similar criterion was proven to hold in any dimension by Nill [22, Prop. 1.7], to whom's paper we refer for more background.

The aim of the current section is to show that the Gorenstein weak Fano property enjoys a certain robustness.

Lemma 3.1. *If X_Δ is Gorenstein weak Fano and $X \to X_\Delta$ is the minimal toric resolution of singularities, then also X is Gorenstein weak Fano, and moreover $P_{-K} = P_{-K_\Delta}$.*

Here, as in the previous section, K denotes the canonical divisor on X obtained by taking minus the sum of all torus-invariant prime divisors.

Proof. Consider the maximal coherent crepant refinement of Σ_Δ, obtained by inserting a ray for *each* lattice point on the boundary of the reflexive polygon obtained by taking the convex hull of $U(\Sigma)$. This clearly gives a resolution of singularities. Therefore the fan Σ of X must be obtained from Σ_Δ by inserting a number of these rays (possibly none, possibly all). We conclude that Σ is also a coherent crepant refinement of Σ_Δ, and both claims follow. □

For our second robustness statement, we need the following notation. Given a lattice polygon Δ with two-dimensional interior lattice polygon $\Delta^{(1)}$, then $\Delta^{\max}$ is defined as the maximal lattice polygon Γ (with respect to inclusion) satisfying $\Gamma^{(1)} = \Delta^{(1)}$. The polygon $\Delta^{\max}$ can be obtained from $\Delta^{(1)}$ by moving out its edges over an integral distance 1. Therefore each edge of $\Delta^{\max}$ is parallel to an edge of $\Delta^{(1)}$, although the converse may fail, because it could happen that an edge shrinks to length 0. See [6, §2] and the references therein for more background.

Lemma 3.2. *If X_Δ is Gorenstein weak Fano, then also $X_{\Delta^{(1)}}$ is Gorenstein weak Fano. Moreover, the latter property holds if and only if $X_{\Delta^{\max}}$ is Gorenstein weak Fano, and in this case the normal fans to $\Delta^{(1)}$ and $\Delta^{\max}$ are the same.*

Proof. We will rely on the following observation: let X be a Gorenstein weak Fano projective toric surface, and let X' be a toric blow-down of X, i.e. the toric surface obtained by removing a certain number of rays from the fan defining X. Then X' is also Gorenstein weak Fano. Indeed, if the primitive generators of the rays of a fan span a reflexive polygon, then this remains true after dropping some of these rays.

We first prove the last equivalence, namely that $X_{\Delta^{(1)}}$ is Gorenstein weak Fano if and only if the same is true for $X_{\Delta^{\max}}$. As noted above, the inner normal fan to $\Delta^{(1)}$ is a subdivision of the inner normal fan to $\Delta^{\max}$, which by the foregoing observation implies the 'only if' part of the statement. As for the 'if' part, assume that $\Delta^{\max}$ is Gorenstein weak Fano. We will show that the subdivision is in fact trivial, i.e. the normal fans to $\Delta^{(1)}$ and $\Delta^{\max}$ are the same, from which the desired conclusion follows. Indeed, suppose that there is an edge $\tau \subseteq \Delta^{(1)}$ that disappears after moving out the edges, i.e. its length shrinks to 0, and choose it such that there is an adjacent edge τ' that does not disappear. Let v be the vertex common to τ and τ'. Using a unimodular transformation if needed we can assume that τ is supported on the line $y = 0$, that $v = (0, 0)$, and that the next lattice point on τ' is $(-b, a)$ with $a \geq b \geq 1$. The outward shifts of the supporting lines of τ and τ' meet in the point

$$w = \left(\frac{b-1}{a}, -1\right),$$

which is necessarily a lattice point, hence $b = 1$ and $w = (0, -1)$. Now let τ'' be the first non-disappearing edge at the other side of τ; note that it might a priori not be adjacent to it. Denote its primitive inner normal vector by (c, d), so that its supporting line is of the form $cx + dy = e$. Notice that $c \leq -1$ and moreover $e \leq c$ because $(1, 0)$ is contained in the corresponding half-plane. Now the outward shift of this line (defined by $cx + dy = e - 1$) must also pass through w, leading to the identity

$$d = -e + 1 > 1.$$

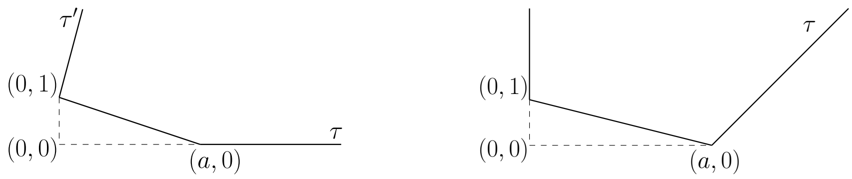
This contradicts the being Gorenstein weak Fano of $\Delta^{\max}$, because the convex hull of (a, b), (c, d) and the other primitive generators of the rays of its normal fan contains $(0, 1)$ as an interior point.

As for the first implication, note that Δ is obtained from $\Delta^{\max}$ by clipping off a number of vertices. We show that these vertices can be glued back on, one by one, while preserving the Gorenstein weak Fano property. Remark that a vertex can only be clipped off if it is 'smooth', meaning that a unimodular transformation takes it to $(0, 0)$ with the adjacent edges lining up with the coordinate axes: otherwise $\Delta^{(1)}$

would be affected. Up to changing the order of the coordinates, the clipping then necessarily happens along the segment connecting $(0,1)$ and $(a,0)$ for some $a \geq 0$. We make a case distinction between three removal types.

- Type 1: none of the adjacent edges was removed completely. This means that glueing back the vertex boils down to dropping a ray from the inner normal fan, which preserves the Gorenstein weak Fano property.

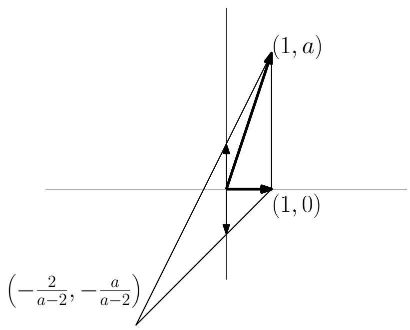
- Type 2: exactly one of the adjacent edges was removed completely. Then the situation is either of the following:



  One of the primitive generators of the rays of the inner normal fan to Δ is given by $(1,a)$.

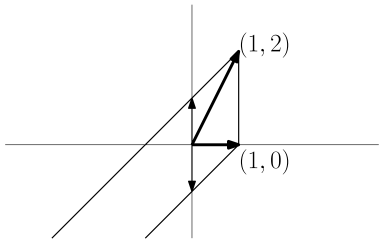
  In the first case, the primitive normal vector to τ' is of the form (b,c) for some $c < 0$ and $b \geq 1$, where the latter inequality holds because τ' cannot be horizontal (otherwise Δ would have an empty interior). This means that $(1,0)$ belongs to the polygon spanned by the primitive generators, and therefore it stays reflexive upon replacement of $(1,a)$ by $(1,0)$, i.e. the Gorenstein weak Fano property is preserved when glueing back our vertex.

  In the second case we find $(1,0)$ among the primitive generators of the rays of the inner normal fan. If $a > 2$ then by the Gorenstein weak Fano property all other primitive generators must belong to the triangle



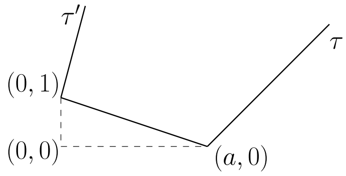
  because otherwise either $(0,1)$ or $(0,-1)$ would belong to the interior of the polygon they span. If $a > 4$ then $2/(a-2) < 1$, so the above region cannot contain the primitive normal vector to τ, which has to have a strictly negative first coordinate: a contradiction. If $a = 4$ then the primitive normal vector to τ is necessarily $(-1, -2)$, which gives a contradiction with the fact that $\Delta^{(1)}$ is two-dimensional. If $a = 3$ one finds $(-1, -1)$, $(-1, -2)$ or $(-2, -3)$, each of which cases again yields a contradiction with the two-dimensionality of $\Delta^{(1)}$.

If $a = 2$ then the region becomes a half-strip



where now the conclusion reads that the primitive normal vector to τ has a negative second coordinate (possibly zero): this means that Δ is contained in a vertical strip of width 2, once again contradicting the fact that $\Delta^{(1)}$ is two-dimensional. Thus we conclude that $a = 1$, and a similar reasoning shows that the primitive normal vector to τ must be of the form $(b, 1)$ for some $b < 0$. But this means that the polygon spanned by the primitive normal vectors contains $(0, 1)$, and therefore the Gorenstein weak Fano property is preserved upon replacement of $(1, a) = (1, 1)$ by $(0, 1)$, i.e. upon glueing back our vertex.

- Type 3: the two adjacent edges are removed completely. Then the situation must be as follows.



This is very similar to before. In the cases where $a \geq 2$ one again obtains a contradiction, either with the Gorenstein weak Fano property or with the two-dimensionality of $\Delta^{(1)}$: the region in which the primitive normal vector to τ should be contained becomes even smaller. If $a = 1$ then we find that the primitive normal vectors to τ and τ' are $(1, b)$ resp. $(b', 1)$ for integers $b, b' < 0$, so we can replace $(1, 1)$ by the pair $(0, 1), (1, 0)$, i.e. we can glue back our vertex.

This concludes the proof. □

One corollary is that, in the statement of Theorem 1.3, the condition that $X_{\Delta^{(1)}}$ is Gorenstein weak Fano can be replaced by X_Δ being Gorenstein weak Fano, although the resulting theorem is strictly weaker.

4 Constancy results

In this section we prove Theorem 1.3. As before let Δ be a lattice polygon with two-dimensional interior $\Delta^{(1)}$, let f be as in (1) and assume that it defines a smooth hyperplane section C_f of X_Δ. Let $X \to X_\Delta$ be the minimal toric resolution of singularities. We use the set-up and notations from Section 2. From (4), we conclude the following:

Lemma 4.1. *For each* $\ell = 0, 1, \ldots, g-2$ *one has that* $a_\ell = b_\ell + c_\ell$ *iff* $a_{g-1-\ell} = b_{g-1-\ell} + c_{g-1-\ell}$ *iff*

$$K_{\ell-1,1}(X; K, L) \xrightarrow{\mu_f} K_{\ell-1,2}(X, L)$$

is the zero map.

Recall that μ_f denotes multiplication by f. Explicitly, this is the map induced by the vertical maps (also denoted by μ_f) of the commutative diagram

$$\begin{array}{ccccc} & & \bigwedge^{\ell-1} V_{\Delta^{(1)}} \otimes V_{\Delta^{(1)(1)}} & \xrightarrow{\delta} & \bigwedge^{\ell-2} V_{\Delta^{(1)}} \otimes V_{(2\Delta^{(1)})^{(1)}} \\ & & \downarrow \mu_f & & \downarrow \mu_f \\ \bigwedge^{\ell} V_{\Delta^{(1)}} \otimes V_{\Delta^{(1)}} & \xrightarrow{\delta} & \bigwedge^{\ell-1} V_{\Delta^{(1)}} \otimes V_{2\Delta^{(1)}} & \xrightarrow{\delta} & \bigwedge^{\ell-2} V_{\Delta^{(1)}} \otimes V_{3\Delta^{(1)}}, \end{array}$$

where as before $V_{\Delta^{(1)}} = H^0(X, L)$ is the space of Laurent polynomials that are supported on $\Delta^{(1)}$. Similarly $V_{\Delta^{(1)(1)}} = H^0(X, L+K)$ denotes the space of Laurent polynomials that are supported on $\Delta^{(1)(1)}$, and so on; see also [8, §2]. The δ's are the usual boundary morphisms

$$v_1 \wedge v_2 \wedge v_3 \wedge v_4 \wedge \cdots \otimes w \mapsto \sum_s (-1)^s v_1 \wedge v_2 \wedge v_3 \wedge v_4 \wedge \cdots \wedge \widehat{v_s} \wedge \cdots \otimes v_s w$$

($\widehat{v_s}$ means that v_s is being omitted) and the μ_f's act like

$$v_1 \wedge v_2 \wedge v_3 \wedge v_4 \wedge \cdots \otimes w \mapsto v_1 \wedge v_2 \wedge v_3 \wedge v_4 \wedge \cdots \otimes fw,$$

where fw indeed ends up in the target space because $\Delta + \Delta^{(1)(1)} \subseteq 2\Delta^{(1)}$ and $\Delta + (2\Delta^{(1)})^{(1)} \subseteq 3\Delta^{(1)}$. From now on, let us denote $\Delta^{(1)(1)}$ by $\Delta^{(2)}$.

Then indeed $K_{\ell-1,1}(X; K, L)$ is the kernel of the top row while $K_{\ell-1,2}(X, L)$ is the cohomology in the middle of the bottom row. In view of Lemma 4.1, our aim is to find conditions under which $\mu_f = 0$ on the level of cohomology. It is convenient to introduce a multiplication map for each monomial x^iy^j that is supported on Δ. That is, for each $(i,j) \in \Delta \cap \mathbb{Z}^2$ we consider the morphism $\mu_{i,j} : K_{\ell-1,1}(X;K,L) \to K_{\ell-1,2}(X,L)$ that is induced by

$$v_1 \wedge v_2 \wedge v_3 \wedge v_4 \wedge \cdots \otimes w \;\mapsto\; v_1 \wedge v_2 \wedge v_3 \wedge v_4 \wedge \cdots \otimes x^iy^jw.$$

Note that

$$\mu_f = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} c_{i,j}\mu_{i,j}.$$

In fact we even have

$$\mu_f = \sum_{(i,j)\in\partial\Delta\cap\mathbb{Z}^2} c_{i,j}\mu_{i,j} \tag{5}$$

thanks to the following observation:

Lemma 4.2. *If $(i,j) \in \Delta^{(1)}$ then $\mu_{i,j} = 0$ on the level of cohomology.*

Proof. This follows from a well-known type of reasoning; see e.g. [1, Lem. 2.19]. Explicitly, if

$$\alpha = \sum_r c_r v_{r,1} \wedge v_{r,2} \wedge \cdots \wedge v_{r,\ell-1} \otimes w_r \;\in\; \bigwedge^{\ell-1} V_{\Delta^{(1)}} \otimes V_{\Delta^{(2)}}$$

is in the kernel of δ, then one verifies that $\mu_{i,j}(\alpha)$ is the coboundary of

$$-\sum_r c_r x^iy^j \wedge v_{r,1} \wedge v_{r,2} \wedge \cdots \wedge v_{r,\ell-1} \otimes w_r \;\in\; \bigwedge^{\ell} V_{\Delta^{(1)}} \otimes V_{\Delta^{(1)}} \tag{6}$$

and therefore vanishes on the level of cohomology. □

The above argument does not work for $(i,j) \in \partial\Delta$ because in that case $x^iy^j \notin V_{\Delta^{(1)}}$ and therefore (6) may not be contained in $\bigwedge^\ell V_{\Delta^{(1)}} \otimes V_{\Delta^{(1)}}$. However, the condition that $(i,j) \in \Delta^{(1)}$ can be relaxed:

Lemma 4.3. *If $(i,j) \in \Delta$ can be written as $(i_1,j_1)+(i_2,j_2)$ such that $(i_1,j_1) \in \Delta^{(1)}$ and $(i_2,j_2) + \Delta^{(2)} \subseteq \Delta^{(1)}$, then $\mu_{i,j} = 0$ on the level of cohomology.*

Proof. In the above proof

$$-\sum_r c_r x^{i_1}y^{j_1} \wedge v_{r,1} \wedge v_{r,2} \wedge \cdots \wedge v_{r,\ell-1} \otimes x^{i_2}y^{j_2}w_r \;\in\; \bigwedge^{\ell} V_{\Delta^{(1)}} \otimes V_{\Delta^{(1)}}$$

serves as a replacement for (6). □

It is natural to try to take $(i_2, j_2) \in P_{-K}$, so that $x^{i_2}y^{j_2} \in H^0(X, -K)$. Indeed recall that $V_{\Delta^{(2)}} = H^0(X, L+K)$ and $V_{\Delta^{(1)}} = H^0(X, L)$, so in this case we indeed have that $(i_2, j_2) + \Delta^{(2)} \subseteq \Delta^{(1)}$. Such an appropriate decomposition of $(i,j) \in \Delta$ can be found only if

$$x^i y^j \in H^0(X, -K) \cdot H^0(X, L). \tag{7}$$

Often $H^0(X, -K)$ consists of the constant functions only, i.e. $P_{-K} \cap \mathbb{Z}^2 = \{(0,0)\}$, in which case (7) is impossible as soon as $(i,j) \in \partial\Delta$. On the other hand, if the right hand side of (7) generates all of $H^0(X, D_f)$, or equivalently, if the map

$$H^0(X, -K) \otimes H^0(X, L) \to H^0(X, D_f) \tag{8}$$

is surjective, then we can conclude that all $\mu_{i,j}$'s are zero on cohomology, and therefore the same is true for μ_f.

We are ready to prove our main result:

Proof of Theorem 1.3. We will assume that $\Delta = \Delta^{\max}$, i.e. Δ is the maximal polygon having $\Delta^{(1)}$ as its interior. This is not a restriction: as soon as $C_f \subseteq X_\Delta$ is a smooth hyperplane section, this is also the case for the Zariski closure of $\varphi_{\Delta^{\max}}(U_f)$ viewed inside $X_{\Delta^{\max}}$, as explained in [6, §4]. Moreover, the statement of Theorem 1.3 only involves $\Delta^{(1)}$, which is left unaffected.

We first deal with the case where $X_{\Delta^{(1)}}$ is Gorenstein weak Fano, which by Lemma 3.2 holds if and only if X_Δ is Gorenstein weak Fano (because of our assumption that Δ is maximal). By Lemma 3.1 then also X is Gorenstein weak Fano, and moreover $P_{-K} = P_{-K_\Delta}$. Now note that

$$P_{-K} + \Delta^{(1)} = \Delta.$$

Indeed, the inclusion $\subseteq$ is obvious, while for the other inclusion it is enough to prove that each vertex m of Δ is in $P_{-K} + \Delta^{(1)}$. Let v, w be consecutive elements of $U(\Sigma)$ such that $m = L(v, a_v) \cap L(w, a_w)$. By the proof of Lemma 2.1 we know that $m_1 = L(v, a_v - 1) \cap L(w, a_w - 1) \in \Delta^{(1)}$, hence $m = m_0 + m_1 \in P_{-K} + \Delta^{(1)}$ with $m_0 = L(v,1) \cap L(w,1)$.

But then also

$$(P_{-K} \cap \mathbb{Z}^2) \ + \ (\Delta^{(1)} \cap \mathbb{Z}^2) \ = \ \Delta \cap \mathbb{Z}^2$$

by [14, Thm 1.1], because the inner normal fan to $P_{-K} = P_{-K_\Delta}$ coarsens that of $\Delta^{(1)}$. Indeed, it obviously coarsens the inner normal fan to Δ, which by Lemma 3.2 is equal to the inner normal fan to $\Delta^{(1)}$. But this precisely means that (8) is surjective, so the maps μ_f are all trivial on the level of cohomology, and the conclusion follows from Lemma 4.1.

As for the other case where $|\partial\Delta^{(1)} \cap \mathbb{Z}^2| \geq g/2 + 1$, the maps μ_f are trivial for a much simpler reason, namely because the dimension c_ℓ of the domain or the dimension $c_{g-1-\ell}$ of the codomain (or both) is zero. This in turn follows from a result due to Hering and Schenck, stating that $\min\{\,\ell \,|\, c_{g-\ell} \neq 0\,\} = |\partial\Delta^{(1)} \cap \mathbb{Z}^2|$; see [15, Thm. IV.20] or [23]. □

We believe that the sum formula $a_\ell = b_\ell + c_\ell$ holds for a considerably larger class of lattice polygons, although there are counterexamples (if there were not, then this would have negative consequences for Green's canonical syzygy conjecture, as explained in Remark 5.6 in the next section). The smallest counterexample we found lives in genus $g = 12$. Namely, consider $f = x^6 + y^2 + x^2y^6$ along with its Newton polygon $\Delta = \mathrm{conv}\{(0,2),(6,0),(2,6)\}$. A computer calculation along the lines of [4] shows that the graded Betti table of the canonical model of C_f is given by

	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	0	45	231	550	693	399	69	0	0	0	0
2	0	0	0	0	69	399	693	550	231	45	0
3	0	0	0	0	0	0	0	0	0	0	1

while that of $X_{\Delta^{(1)}}$ is given by

	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	0	39	186	414	504	295	69	0	0	0
2	0	0	0	0	1	105	189	136	45	6.

Here one sees that the exact sequence (4) for $\ell = 5$ reads:

$$0 \to 295 \to 399 \to 105 \xrightarrow{\mu_f} 1 \to 69 \to 69 \to 0.$$

So μ_f is not trivial in this case, but rather surjective onto its one-dimensional codomain.

Another natural question is whether it is true in general that the graded Betti table of C is independent of the coefficients of f, even if the sum formula does not hold. In general one has for each $\ell = 1, \ldots, g-3$ that

$$a_\ell = b_\ell + c_\ell - \dim \operatorname{im} \mu_f$$

and constancy holds if and only if $\dim \operatorname{im} \mu_f$ depends on Δ and ℓ only. From (5) it follows that, at least, there is no dependence on the coefficients $c_{i,j}$ that are supported on $\Delta^{(1)}$. In other words, only the coefficients that are supported on the boundary might a priori matter. A consequence of this observation is that constancy of the graded Betti table holds for primitive lattice triangles, i.e. lattice triangles without lattice points on the boundary, except for the three vertices. Indeed, using a transformation of the form $f \leftarrow \gamma f(\alpha x, \beta y)$, with $\alpha, \beta, \gamma \in k \setminus \{0\}$, one can always arrange that the three coefficients supported on the vertices $(i_1, j_1), (i_2, j_2), (i_3, j_3)$ are all 1. This means that $a_\ell = b_\ell + c_\ell - \dim \operatorname{im}(\mu_{i_1,j_1} + \mu_{i_2,j_2} + \mu_{i_3,j_3})$, regardless of the coefficients of f.

5 Connections with Green's conjecture

Let C be a smooth projective non-hyperelliptic curve over k of genus $g \geq 4$, and assume that it is canonically embedded in $\mathbb{P}^{g-1}$. Green in [12] showed that the number of leading zeros on the quadratic strand of the graded Betti table of C is at most its Clifford index $\operatorname{ci}(C)$. The so-called canonical syzygy conjecture, or *Green's conjecture*, predicts that equality holds. More precisely, if we denote the graded Betti table of C as in (2), then the conjecture can be phrased as follows:

Conjecture 5.1 (Green)**.**

$$\min\{\ell \,|\, a_{g-\ell} \neq 0\} = \operatorname{ci}(C) + 2.$$

Returning to our smooth curve C_f of genus $g = |\Delta^{(1)} \cap \mathbb{Z}^2|$, where as before we assume that $\Delta^{(1)}$ is two-dimensional and contains at least 4 lattice points, we recall that its canonical model lives inside $X_{\Delta^{(1)}} \subseteq \mathbb{P}^{g-1}$. The aim of this section is to tie Green's conjecture to a conjecture that we stated in a previous paper [8, Conj. 1.6], concerning the length of the linear strand of the graded Betti table of a projectively embedded toric surface. In the particular case of $X_{\Delta^{(1)}}$, it reads:

Conjecture 5.2 (see [8])**.** *Assume that $\Delta^{(1)} \not\cong \Upsilon$. Denoting the graded Betti table of $X_{\Delta^{(1)}} \subseteq \mathbb{P}^{g-1}$ as in (3), we have that*

$$\min\{\ell \,|\, b_{g-\ell} \neq 0\} = \begin{cases} \operatorname{lw}(\Delta^{(1)}) + 1 & \textit{if } \Delta^{(1)} \cong (d-3)\Sigma \textit{ for some } d \geq 5, \\ \operatorname{lw}(\Delta^{(1)}) + 1 & \textit{if } \Delta^{(1)} \cong 2\Upsilon, \\ \operatorname{lw}(\Delta^{(1)}) + 2 & \textit{in all other cases.} \end{cases}$$

In [8] we proved that the right hand side always gives an upper bound, and that the conjecture is true for $g \leq 32$ (through a computer verification).

The connection between both conjectures is as follows.

Lemma 5.3. *If Conjecture 5.1 holds for smooth irreducible hyperplane sections $C_f \subseteq X_\Delta$ then Conjecture 5.2 correctly predicts the length of the linear strand of the graded Betti table of $X_{\Delta^{(1)}}$. Moreover, if $|\partial\Delta^{(1)} \cap \mathbb{Z}^2| \geq \operatorname{lw}(\Delta^{(1)}) + 2$, then also the converse implication holds.*

Remark 5.4. Note that if $\Delta^{(1)} \cong \Upsilon$ then Conjecture 5.2 is tautologically true, while Conjecture 5.1 is known to hold for curves of genus $g = N_\Upsilon = 4$. Therefore we ignore this case in the proofs below.

Proof. Note that the right hand sides of the equalities in Conjecture 5.1 and Conjecture 5.2 agree by Theorem 1.2. Let us denote this common quantity by γ.

First assume that Conjecture 5.1 holds for some smooth irreducible hyperplane section $C_f \subseteq X_\Delta$. To deduce Conjecture 5.2 for $X_{\Delta^{(1)}}$, it suffices to prove that $b_{g-(\gamma-1)} = 0$. This follows from the fact that $a_{g-(\gamma-1)} = 0$, along with the exact sequence (4) for $\ell = g - (\gamma - 1)$.

For the other implication we need to show that $a_{g-(\gamma-1)} = 0$. Since $b_{g-(\gamma-1)} = 0$ by assumption, thanks to (4) it suffices to show that $c_{g-(\gamma-1)} = 0$. But this follows from Hering and Schenck's aforementioned result [15, Thm. IV.20] that $\min\{\ell \,|\, c_{g-\ell} \neq 0\} = |\partial\Delta^{(1)} \cap \mathbb{Z}^2|$. Because of the stated inequality, we have that

$$\gamma - 1 \leq \mathrm{lw}(\Delta^{(1)}) + 1 \leq |\partial\Delta^{(1)} \cap \mathbb{Z}^2| - 1,$$

hence indeed $c_{g-(\gamma-1)} = 0$. □

As a first application, we find:

Corollary 5.5. *Green's conjecture holds for all smooth curves on toric surfaces of genus $g \leq 32$.*

Proof. In view of Remark 1.1 it suffices to prove the conjecture for curves of the form $C_f \subseteq X_\Delta$. Now, as mentioned, our Conjecture 5.2 has been verified computationally for all interior lattice polygons $\Delta^{(1)}$ having at most 32 lattice points. Another computation shows that in this range, up to unimodular equivalence the only interior lattice polygon $\Delta^{(1)}$ that does *not* satisfy the inequality $|\partial\Delta^{(1)} \cap \mathbb{Z}^2| \geq \mathrm{lw}(\Delta^{(1)}) + 2$ is $\Delta^{(1)} = \Upsilon$. Thus the claim follows from Lemma 5.3. □

Remark 5.6. It is not so easy to find interior lattice polygons for which the inequality $|\partial\Delta^{(1)} \cap \mathbb{Z}^2| \geq \mathrm{lw}(\Delta^{(1)}) + 2$ does *not* hold. Here the condition of being interior is crucial: if we omit this assumption, it is trivial to find counterexamples (e.g. the primitive lattice triangles that we encountered at the end of Section 4 can have arbitrarily large lattice width). The smallest interior counterexample that we encountered is $\Delta^{(1)}$ where $\Delta = \mathrm{conv}\{(4,0),(0,10),(10,4)\}$. This concerns a 9-gon without extra points on the boundary, satisfying $g = |\Delta^{(1)} \cap \mathbb{Z}^2| = 36$ and $\mathrm{lw}(\Delta^{(1)}) = 8$, see Figure 1. If we want to check Green's conjecture for this specific polygon Δ, we need to show that $a_{27} = 0$ (since $g - (\gamma - 1) = 27$). Using the algorithm from [8] we checked that $b_{27} = 0$, therefore Conjecture 5.2 holds in this case. On the other hand $c_{27} \neq 0$ by Hering and Schenck's result, so we cannot use (4) to conclude that $a_{27} = 0$. In fact if the sum formula $a_{27} = b_{27} + c_{27}$ from the

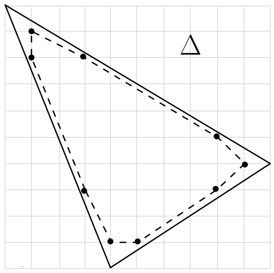


Figure 1: Counterexample to the inequality $|\partial\Delta^{(1)} \cap \mathbb{Z}^2| \geq \mathrm{lw}(\Delta^{(1)}) + 2$

statement of Theorem 1.3 would be true in this case (which we do not believe it is), then from $c_{27} \neq 0$ it would follow that $a_{27} \neq 0$ and hence that Green's conjecture is false!

As a second application, we use known cases of Green's conjecture to deduce new cases of Conjecture 5.2. Here the main input is due to Lelli-Chiesa, who proved Green's conjecture for curves on smooth rational surfaces, modulo certain assumptions, the most restrictive one being the existence of an anticanonical pencil. Let us state her result more precisely, adapting the notation to our specific case of curves of the form $C_f \subseteq X_\Delta$. Because the ambient surface needs to be smooth, as in Section 2 we let $X \to X_\Delta$ be the minimal toric resolution of singularities and write C' for the pull-back of C_f. Again we let $D_f = C' - \mathrm{div}(f)$ and consider the canonical divisor $K = -\sum_v D_v$, where v ranges over the set $U(\Sigma)$ of primitive generators of the rays of the fan Σ of X.

Theorem 5.7 (see [19]). *Assume that the following conditions are satisfied:*

- $L = D_f + K$ *is big and nef,*
- $h^0(X, -K) \geq 2$*,*
- *if* $h^0(X, -K) = 2$*, then the Clifford index of a general curve* $C \in |D_f|$ *is not computed by restricting the anticanonical divisor to* C*.*

Then Green's conjecture is true for C_f*.*

Note that the second condition can be rephrased as $|P_{-K} \cap \mathbb{Z}^2| \geq 2$. The first condition is automatically satisfied for toric surfaces: L is nef because of Lemma 2.1 and big because $\Delta^{(1)}$ is two-dimensional. The next two lemma's show that also the third condition is void in our case.

Lemma 5.8. *Let X be a toric surface with $h^0(X, -K) = 2$. If $\Delta = P_D$ is the polygon of a torus-invariant nef Cartier divisor D on X, then* $\mathrm{lw}(\Delta) < |\partial\Delta \cap \mathbb{Z}^2|$.

Proof. The fact that D is nef and Cartier ensures that Δ is a lattice polygon. Now there is a non-zero lattice point $m_0 \in P_{-K}$ and by using a unimodular transformation if needed, we can assume that $m_0 = (1,0)$. Let y_1 (resp. y_2) be the minimum (resp. maximum) of the second coordinates of the points of Δ. For all $v \in U(\Sigma)$, we have that $\langle m_0, v\rangle \geq -1$, hence the first coordinate of each $v \in U(\Sigma)$ is at least -1. Consider an edge e at the right hand side of Δ, i.e. an edge whose inner normal vector has a strictly negative first coordinate. Then the corresponding $v \in U(\Sigma)$ must have first coordinate equal to -1. Hence, if e intersects a horizontal line at integral height, then this point of intersection is a lattice point. As a consequence $|\partial\Delta \cap \mathbb{Z}^2| > y_2 - y_1 \geq \mathrm{lw}(\Delta)$. □

Lemma 5.9. *Let X be a smooth toric surface with $h^0(X, -K) = 2$. Let D be a torus-invariant nef divisor on X such that $D+K$ is big and nef. Then for a general curve $C \in |D|$ the Clifford index is not computed by $-K|_C$.*

Proof. Note that all divisors are Cartier because of the smoothness assumption. Denote the (lattice) polygon P_D corresponding to D by Δ. The short exact sequence $0 \to \mathcal{O}_X(-D-K) \to \mathcal{O}_X(-K) \to \mathcal{O}_C(-K|_C) \to 0$ yields the long exact sequence

$$0 \to H^0(X, -D-K) \to H^0(X, -K) \to H^0(C, -K|_C) \to H^1(X, -D-K) \to \dots$$

Since $D+K$ is big and nef, the polygon $P_{D+K} = \Delta^{(1)}$ is two-dimensional and we have that $h^0(X, -D-K) = h^1(X, -D-K) = 0$ by Batyrev-Borisov vanishing. It follows that $h^0(C, -K|_C) = h^0(X, -K) = 2$. Hence, the divisor $-K|_C$ gives rise to a linear system on C of rank $r = h^0(C, -K|_C) - 1 = 1$ and degree $\sum_{v\in U(\Sigma)} \deg(D_v|_C) = |\partial\Delta \cap \mathbb{Z}^2|$. Now if the Clifford index of C would be computed by $-K|_C$, then we would have $\mathrm{ci}(C) = |\partial\Delta \cap \mathbb{Z}^2| - 2$. On the other hand, by Theorem 1.2 and Lemma 5.8, we have that $\mathrm{ci}(C) \leq \mathrm{lw}(\Delta^{(1)}) \leq \mathrm{lw}(\Delta) - 2 < |\partial\Delta \cap \mathbb{Z}^2| - 2$, a contradiction. □

We can now conclude:

Corollary 5.10. *If*

$$|P_{-K_{\Delta^{(1)}}} \cap \mathbb{Z}^2| \geq 2,$$

then Conjecture 5.2 correctly predicts the length of the linear strand of the graded Betti table of $X_{\Delta^{(1)}}$.

Proof. Because the statement only involves $\Delta^{(1)}$, we can assume that Δ is maximal. Using a unimodular transformation if needed, we can also assume that $(1,0)$ is contained in the polygon associated to $-K_{\Delta^{(1)}}$, which implies, as in the proof of Lemma 5.8, that all inner normal vectors to $\Delta^{(1)}$ having a strictly negative first coordinate must be of the form $(-1, b)$ for some $b \in \mathbb{Z}$. But then the same must be true for $\Delta = \Delta^{\max}$, which is obtained from $\Delta^{(1)}$ by moving out the edges over an integral distance 1. Then it is not hard to see that the minimal toric resolution of singularities $X \to X_\Delta$ is obtained by inserting rays whose primitive generators are of the form (a, b) with $a \geq -1$. In other words $(1, 0) \in P_{-K}$, and therefore we can apply Lelli-Chiesa's theorem to conclude that Green's conjecture is true for any smooth hyperplane section $C_f \subseteq X_\Delta$. The conclusion now follows from Lemma 5.3. □

As a special case we find that Conjecture 5.2 is true if $X_{\Delta^{(1)}}$ is Gorenstein weak Fano.

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A combinatorial interpretation for Schreyer's tetragonal invariants

Wouter Castryck and Filip Cools

Abstract

Schreyer has proved that the graded Betti numbers of a canonical tetragonal curve are determined by two integers b_1 and b_2, associated to the curve through a certain geometric construction. In this article we prove that in the case of a smooth projective tetragonal curve on a toric surface, these integers have easy interpretations in terms of the Newton polygon of its defining Laurent polynomial. We can use this to prove an intrinsicness result on Newton polygons of small lattice width.

MSC2010: Primary 14H45, Secondary 14M25

1 Introduction

Let k be an algebraically closed field of characteristic 0 and let $\mathbb{T}^2 = (k^*)^2$ be the two-dimensional torus over k. Let $\Delta \subset \mathbb{R}^2$ be a two-dimensional lattice polygon and consider the associated toric surface $\text{Tor}(\Delta)$ over k, i.e. the Zariski closure of the image of

$$\varphi_\Delta : \mathbb{T}^2 \hookrightarrow \mathbb{P}^{\sharp(\Delta\cap\mathbb{Z}^2)-1} : (\alpha, \beta) \mapsto (\alpha^i \beta^j)_{(i,j)\in\Delta\cap\mathbb{Z}^2}.$$

Let

$$f = \sum_{(i,j)\in\mathbb{Z}^2} c_{i,j} x^i y^j \in k[x^{\pm 1}, y^{\pm 1}]$$

be an irreducible Laurent polynomial and consider its Newton polygon

$$\Delta(f) = \text{conv}\left\{ (i,j) \in \mathbb{Z}^2 \,\middle|\, c_{i,j} \neq 0 \right\}.$$

Let $U_f \subset \mathbb{T}^2$ be the curve cut out by f. We say that f is Δ-non-degenerate if $\Delta(f) \subset \Delta$ and for every face $\tau \subset \Delta$ (vertex, edge, or Δ itself) the system

$$f_\tau = \frac{\partial f_\tau}{\partial x} = \frac{\partial f_\tau}{\partial y} = 0$$

has no solutions in $\mathbb{T}^2$. Here

$$f_\tau = \sum_{(i,j) \in \tau \cap \mathbb{Z}^2} c_{i,j} x^i y^j.$$

For a fixed instance of Δ and given that $\Delta(f) \subset \Delta$, the condition of Δ-non-degeneracy is generically satisfied. It implies that the Zariski closure C_f of $\varphi_\Delta(U_f)$ inside $\mathrm{Tor}(\Delta)$ is non-singular. A curve that is isomorphic to C_f for some Δ-non-degenerate Laurent polynomial is in turn called Δ-non-degenerate.

Non-degenerate curves form an attractive class of objects from the point of view of explicit algebraic geometry. On the one hand they vastly generalize well-known families such as elliptic curves, hyperelliptic curves, trigonal curves[1], smooth plane curves, $C_{a,b}$ curves, ... covering a much broader range of geometric situations. On the other hand they remain very tangible, because many important geometric invariants can be told by simply looking at the combinatorics of Δ. Two notable instances are:

- the (geometric) *genus* g, which equals $\sharp(\Delta^{(1)} \cap \mathbb{Z}^2)$, where $\Delta^{(1)}$ is the convex hull of the interior lattice points of Δ; see [10];
- the *gonality* γ, which equals $\mathrm{lw}(\Delta)$, except if $\Delta \cong 2\Upsilon$ or $\Delta \cong d\Sigma$ for some $d \geq 2$, where

  $$\Upsilon = \mathrm{conv}\{(-1,-1),(1,0),(0,1)\} \qquad \text{and} \qquad \Sigma = \mathrm{conv}\{(0,0),(1,0),(0,1)\},$$

  in which case it equals $\mathrm{lw}(\Delta) - 1$; here lw denotes the lattice width, and $\cong$ indicates unimodular equivalence; see [4, Lem. 6.2]. (Shorter characterization: $\gamma = \mathrm{lw}(\Delta^{(1)}) + 2$ except if $\Delta \cong 2\Upsilon$ in which case $\gamma = 3$.)

Similar interpretations exist for the *Clifford index* and the *Clifford dimension* [4, §8], and in some cases for the *minimal degree of a plane model* [6]. The current paper extends the list of combinatorial features of non-degenerate curves, by focusing on tetragonal curves. Namely, we give the following interpretation for the invariants b_1 and b_2, as introduced by Schreyer in [14, (6.2)]. The definition of these invariants will be recalled in Section 2 below.

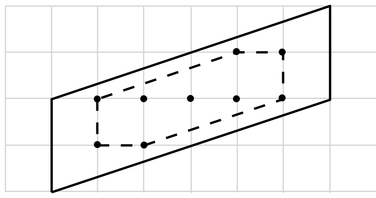
Theorem 1. *Let C be a tetragonal Δ-non-degenerate curve. Then Schreyer's corresponding set of invariants $\{b_1, b_2\}$ is given by*

$$\left\{ \ \sharp(\partial\Delta^{(1)} \cap \mathbb{Z}^2) - 4\,,\ \sharp(\Delta^{(2)} \cap \mathbb{Z}^2) - 1 \ \right\}.$$

[1]Strictly spoken, there do exist trigonal curves that are not non-degenerate; for example see [4, Lem. 4.4]. But all trigonal curves are 'morally' non-degenerate, in the sense that they can always be embedded in a toric surface, which is sufficient for most applications. See also the remark at the end of this section.

Here ∂ denotes the boundary and $\Delta^{(2)} = \Delta^{(1)(1)}$ is the convex hull of the interior lattice points of $\Delta^{(1)}$.

Example 2. The Laurent polynomial $f = 1 + y^2 - x^6y^2 + x^6y^4 \in \mathbb{C}[x, y]$ is Δ-non-degenerate, where Δ is as follows.



The dashed lines indicate $\Delta^{(1)}$. One verifies, purely by looking at the Newton polygon, that C_f is a tetragonal curve of genus 9 with $b_1 = b_2 = 2$. (In view of [4, Cor. 6.3, Thm. 9.1], one can even say that it carries a unique g^1_4, whose scrollar invariants read $1, 1, 4$; see Remark 2 below for more background on this terminology.)

Schreyer's invariants are known to determine the *Betti diagram of the canonical ideal*, and vice versa [14, (6.2)]. In particular, Theorem 1 implies that in the tetragonal case, the Betti diagram is combinatorially determined. We believe that this holds in much greater generality (work in progress).

A second aim of this paper is to initiate a discussion on the *intrinsicness* of Δ. Namely, given the many geometric invariants that are encoded in the Newton polygon, one might wonder to what extent it is possible to reconstruct Δ from the abstract geometry of a given Δ-non-degenerate curve C_f. The best one can hope for is to find back Δ up to unimodular equivalence, because unimodular transformations correspond to automorphisms of $\mathbb{T}^2$. Another relaxation is that (usually) one can only expect to recover $\Delta^{(1)}$, rather than all of Δ. For example, let $f \in k[x, y]$ be $d\Sigma$-non-degenerate for some integer $d \geq 2$ and let $(x_0, y_0) \in U_f$ be sufficiently generic. Then $f' = f(x + x_0, y + y_0)$ is Δ-non-degenerate, where Δ is obtained from $d\Sigma$ by clipping off the point $(0, 0)$. In this case $\Delta \ncong d\Sigma$, while clearly $C_f \cong C_{f'}$. More generally, pruning a vertex off a lattice polygon Δ without affecting its interior boils down to forcing the curve through a certain non-singular point of $\mathrm{Tor}(\Delta)$, which is usually not intrinsic. One is naturally led to the following question.

Question 3 (intrinsicness)**.** *Let Δ, Δ' be two-dimensional lattice polygons for which there exists a curve that is both Δ-non-degenerate and Δ'-non-degenerate. Does it follow that $\Delta^{(1)} \cong \Delta'^{(1)}$?*

Our conjecture is that for 'most' pairs of polygons the answer is yes. E.g., this is known to be true as soon as

(a) $\Delta^{(1)}$ is one-dimensional, because a Δ-non-degenerate curve is hyperelliptic of genus $g \geq 2$ if and only if $\Delta^{(1)} \cap \mathbb{Z}^2$ consists of g collinear points [11, Lem. 3.2.9],

(b) $\Delta^{(1)} = \emptyset$ or $\Delta^{(1)} \cong (d-3)\Sigma$ for some integer $d \geq 3$, because a Δ-non-degenerate curve is abstractly isomorphic to a smooth plane curve if and only if $\Delta^{(1)}$ is a multiple of the standard simplex (up to equivalence) [4, Cor. 8.2].

(c) $\Delta^{(1)} \cong [0,a] \times [0,b]$ for some integers $a \geq b \geq -1$ with $(a+1)(b+1) \neq 4$, because a Δ-non-degenerate curve of genus $g \neq 4$ can be embedded in $\mathbb{P}^1 \times \mathbb{P}^1$ if and only if $\Delta^{(1)}$ is a standard rectangle (up to equivalence); see [5]. The assumption $g \neq 4$ is necessary: see the discussion following (d) below.

Let us indicate why we expect Question 3 to have an affirmative answer for many more instances of Δ, while gathering some material that will be needed in Section 2. Our starting point is a theorem by Khovanskii [10], stating that there exists a canonical divisor K_Δ on C_f such that a basis for the Riemann-Roch space $H^0(C_f, K_\Delta)$ is given by

$$\left\{x^i y^j\right\}_{(i,j)\in\Delta^{(1)}\cap\mathbb{Z}^2} . \tag{1}$$

Here x, y are to be viewed as functions on C_f through φ_Δ. Note that one recovers the statements that $g = \sharp(\Delta^{(1)} \cap \mathbb{Z}^2)$ and that C_f is hyperelliptic if and only if $\Delta^{(1)}$ is one-dimensional; see [7, Lem. 5.1] for more details. If $\Delta^{(1)}$ is two-dimensional, then Khovanskii's theorem implies that the canonical model C_f^{can} of C_f satisfies

$$C_f^{\text{can}} \subset \operatorname{Tor}(\Delta^{(1)}) \subset \mathbb{P}^{g-1}.$$

But surfaces of the form $\operatorname{Tor}(\Delta^{(1)})$ are very special. Most notably, they are of low degree, and they are generated by binomials. The idea is that they are so special that there is room for at most one such surface containing C_f^{can}. This idea is not always true, but the exceptions seem rare. If it *is* true, then the following general and seemingly new statement allows one to recover $\Delta^{(1)}$. A proof will be given in Section 3.

Theorem 4. *Let Δ, Δ' be two-dimensional lattice polygons with*

$$\sharp(\Delta \cap \mathbb{Z}^2) - 1 = \sharp(\Delta' \cap \mathbb{Z}^2) - 1 = N,$$

and suppose that $\operatorname{Tor}(\Delta), \operatorname{Tor}(\Delta') \subset \mathbb{P}^N$ *can be obtained from one another using a projective transformation. Then $\Delta \cong \Delta'$.*

Using this, we can immediately extend the above list to the case where

(d) $\sharp(\Delta^{(1)} \cap \mathbb{Z}^2) \geq 5$ and $\Delta^{(2)} = \emptyset$, which holds if and only if C_f is trigonal of genus $g \geq 5$, or isomorphic to a smooth plane quintic [4, §8]. In this case $\operatorname{Tor}(\Delta^{(1)})$ can be characterized as the unique irreducible surface containing C_f^{can} that is generated by quadrics. Indeed, the fact that it is generated by quadrics follows from [12], while uniqueness follows from Petri's theorem [13].

The above argument breaks down in the genus 4 case where $\Delta \cong 2\Upsilon$, because $\mathrm{Tor}((2\Upsilon)^{(1)}) = \mathrm{Tor}(\Upsilon)$ is not generated by quadrics. And indeed, using this, it is not hard to cook up examples of (2Υ)-non-degenerate curves that are non-degenerate with respect to $[0,3] \times [0,3]$, and also of (2Υ)-non-degenerate curves that are non-degenerate with respect to $\mathrm{conv}\{(0,0),(4,0),(0,2)\}$. (See §5.6 of our unpublished `arXiv` paper `1304.4997` for an extended discussion; see also Example 13 below.)

In Section 2 we will give a similar but more complicated recipe for recovering $\mathrm{Tor}(\Delta^{(1)})$ in most tetragonal cases. More precisely, we extend the list with the situation where

(e) $\mathrm{lw}(\Delta^{(1)}) = 2$ and $\sharp(\partial\Delta^{(1)} \cap \mathbb{Z}^2) \geq \sharp(\Delta^{(2)} \cap \mathbb{Z}^2) + 5$, which holds if and only if C_f is tetragonal and $b_1 \geq b_2 + 2$. In this case $\mathrm{Tor}(\Delta^{(1)})$ can be characterized as the unique surface containing C_f^{can} that is linearly equivalent to $2H - b_1 R$, when viewed as a divisor inside the scroll spanned by a g^1_4.

More explanation will be given in Section 4. Of course, in establishing this, we will make extensive use of Theorem 1 and its proof.

Remark 5. Even though we formulate our results in terms of non-degenerate curves, they remain valid for the slightly more general class of *arbitrary* smooth curves in toric surfaces. Indeed, to a smooth (non-torus-invariant) curve C in a toric surface $\varphi : \mathbb{T}^2 \hookrightarrow X$ one can always associate a 'defining Laurent polynomial' $f \in k[x^{\pm 1}, y^{\pm 1}]$, by which we mean a generator of the ideal of $\varphi^{-1}C$. It is well-defined up to multiplication by $cx^i y^j$ for some $c \in k^*$ and $(i,j) \in \mathbb{Z}^2$. One then just proceeds with f and $\Delta = \Delta(f)$, as if f were Δ-non-degenerate. We refer to [4, §4] for a more extended discussion.

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2 Schreyer's tetragonal invariants

Let C/k be a tetragonal curve of genus $g \geq 5$ and assume it to be canonically embedded in $\mathbb{P}^{g-1}$. Fix a gonality pencil g^1_4 on C and consider

$$S = \bigcup_{D \in g^1_4} \langle D \rangle \subset \mathbb{P}^{g-1},$$

where $\langle D \rangle \subset \mathbb{P}^{g-1}$ denotes the linear span of D. One can show that S is a rational normal threefold scroll whose type we denote by (e_1, e_2, e_3), where we assume $0 \leq$

$e_1 \leq e_2 \leq e_3$. One has $\deg S = e_1 + e_2 + e_3 = g - 3$, and S is non-singular if and only if $e_1 > 0$. If $e_1 = 0$ then the singularities are resolved by the natural map $\mu : \mathbb{P}(\mathcal{E}) \to S$, where $\mathcal{E}$ is the locally free sheaf $\mathcal{O}(e_1) \oplus \mathcal{O}(e_2) \oplus \mathcal{O}(e_3)$ on $\mathbb{P}^1$; if $e_1 > 0$ then μ is an isomorphism. The Picard group of $\mathbb{P}(\mathcal{E})$ is freely generated by the hyperplane class $H = [\mu^*(\mathcal{O}(1))]$ and the ruling class R consisting of the fibers of the projection $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$. The following intersection-theoretic identities hold: $H^3 = g - 3$, $H^2 \cdot R = 1$ and $R^2 = 0$. For more general background and references, see [4, §9] and [14, §2-4].

Remark 6. The numbers e_1, e_2, e_3 are called the *scrollar invariants* of C with respect to our g^1_4.

Now let C' be the strict transform under μ of our canonical curve $C \subset S$. Schreyer proved that C' is the complete intersection of surfaces Y and Z in $\mathbb{P}(\mathcal{E})$, with $Y \sim 2H - b_1 R$, $Z \sim 2H - b_2 R$, $b_1 + b_2 = g - 5$ and $-1 \leq b_2 \leq b_1 \leq g - 4$. He moreover showed that b_1, b_2 are invariants of the curve: they depend neither on the canonical embedding, nor on the choice of the g^1_4, nor on the choice of Y and Z. If $b_1 > b_2$, which is automatic if g is even, then Y is in fact unique, and $\mu(Y) \subset \mathbb{P}^{g-1}$ is independent of the chosen g^1_4. For these particular statements we refer to [14, (6.2)].

The goal of this section is to prove the combinatorial interpretation for Schreyer's invariants b_1, b_2 stated in Theorem 1. Using the abbreviations

$$B = \sharp(\partial\Delta^{(1)} \cap \mathbb{Z}^2) - 4, \qquad B^{(1)} = \sharp(\Delta^{(2)} \cap \mathbb{Z}^2) - 1,$$

we will in fact show:

Theorem 7. *Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$, and suppose that C_f is tetragonal. Then its invariants b_1, b_2 statisfy $\{b_1, b_2\} = \{B, B^{(1)}\}$. If moreover $B > B^{(1)}$ then the surface $\mu(Y)$ associated to the canonical model C_f^{can} from Section 1 equals* $\mathrm{Tor}(\Delta^{(1)})$.

Proof. The assumption that C_f is tetragonal is equivalent to $\mathrm{lw}(\Delta^{(1)}) = 2$ and $\Delta \not\cong 2\Upsilon$. We can also suppose that $\Delta \not\cong 5\Sigma$, because this case can be reduced to

$$\Delta \cong \mathrm{conv}\{(1,0), (5,0), (0,5), (0,1)\}$$

by means of a coordinate transformation, as explained in the discussion preceding Question 3. By [4, Lem. 5.2] we can therefore suppose that

$$\Delta^{(1)} \subset \left\{(X,Y) \in \mathbb{R}^2 \,|\, 0 \leq Y \leq 2\right\} \quad \text{and} \quad \Delta \subset \left\{(X,Y) \in \mathbb{R}^2 \,|\, -1 \leq Y \leq 3\right\}.$$

Then the projection map $U_f \to \mathbb{T}^1 : (x, y) \mapsto x$ has degree 4, i.e. it gives rise to a g^1_4 on C_f. As remarked in Section 1, the canonical model C_f^{can} obtained using the basis (1) of $H^0(C_f, K_\Delta)$ satisfies

$$C_f^{\text{can}} \subset \text{Tor}(\Delta^{(1)}) \subset \mathbb{P}^{g-1}.$$

The scroll S corresponding to our g^1_4 is easily seen to be the Zariski closure of the image of the map

$$\mathbb{T}^3 \hookrightarrow \mathbb{P}^{g-1} : (\alpha, \beta, \gamma) \mapsto \left((\alpha^i)_{(i,0)\in\Delta^{(1)}\cap\mathbb{Z}^2} : (\beta\alpha^i)_{(i,1)\in\Delta^{(1)}\cap\mathbb{Z}^2} : (\gamma\alpha^i)_{(i,2)\in\Delta^{(1)}\cap\mathbb{Z}^2} \right).$$

(Note that the scrollar invariants e_1, e_2, e_3 are precisely the numbers

$$\sharp\{(i', j') \in \Delta^{(1)} \cap \mathbb{Z}^2 \,|\, j' = j\} - 1$$

for $j = 0, 1, 2$, up to order; for a generalization of this observation, see [4, §9].) Moreover, one verifies that S contains $\text{Tor}(\Delta^{(1)})$, i.e. the above chain of inclusions extends to

$$C_f^{\text{can}} \subset \text{Tor}(\Delta^{(1)}) \subset S \subset \mathbb{P}^{g-1}.$$

Now let $\mu : \mathbb{P}(\mathcal{E}) \to S$ be as above and denote by C' the strict transform of C_f^{can} under μ. Similarly, denote by T' the strict transform of $\text{Tor}(\Delta^{(1)})$. Write the divisor class of T' as $aH + bR$ with $a, b \in \mathbb{Z}$. Let F be the fiber of π above $\alpha \in \mathbb{T}^1 \subset \mathbb{P}^1$. Then $\mu(F)$ is a $\mathbb{P}^2$ whose intersection with $\text{Tor}(\Delta^{(1)})$ has $\beta = y$ and $\gamma = y^2$ as parameter equations on $\mathbb{T}^2 \subset \mathbb{P}^2$. In particular this intersection is a conic, so we have that

$$a = (aH + bR) \cdot H \cdot R = T' \cdot H \cdot R = 2.$$

Next, we compute the intersection product $T' \cdot H^2$ in two ways. On the one hand we find the degree of $\text{Tor}(\Delta^{(1)})$, which equals $2\text{Vol}(\Delta^{(1)})$ because the Hilbert polynomial of $\text{Tor}(\Delta^{(1)})$ equals the Ehrhart polynomial of $\Delta^{(1)}$, see [8, Prop. 9.4.3]. On the other hand one has

$$T' \cdot H^2 = (2H + bR) \cdot H^2 = 2(g - 3) + b.$$

We obtain that $b = 2\text{Vol}(\Delta^{(1)}) - 2(g - 3) = -B$, where the latter equality follows from Pick's theorem. In conclusion, $T' \sim 2H - BR$. Now

- if $Y = T'$ then it is immediate that $b_1 = B$ and, consequently, $b_2 = B^{(1)}$,
- if $Y \neq T'$ then if we intersect $Y \sim 2H - b_1 R$ and $T' \sim 2H - BR$ on $\mathbb{P}(\mathcal{E})$, we obtain a (possibly reducible) curve whose image under μ has degree

  $$H \cdot (2H - BR) \cdot (2H - b_1 R) = 4(g-3) - 2b_1 - 2B \leq 4(g-3) - 2(g-5) = 2g - 2.$$

  This follows from $2b_1 \geq b_1 + b_2 = g - 5$ and $2B \geq B + B^{(1)} = g - 5$ if $B \geq B^{(1)}$, and from $2b_1 \geq b_1 + b_2 + 1 = g - 4$ and $2B = g - 6$ if $B < B^{(1)}$; see Lemma 9 below. In both cases, if either one of the inequalities would be strict, then we would run into a contradiction because C' is contained in this intersection (and $\mu(C') = C_f^{\mathrm{can}}$, being a canonical curve, has degree $2g-2$). We conclude that $b_1 = b_2 = B = B^{(1)} = \frac{g-5}{2}$ or $b_1 = B^{(1)} = \frac{g-4}{2}$ and $b_2 = B = \frac{g-6}{2}$.

All conclusions follow. □

Remark 8. Assume that C_f is not isomorphic to a smooth plane quintic, i.e. $\Delta^{(1)} \not\cong 2\Sigma$. Then by Petri's theorem [13] the ideal of C_f^{can} is generated by quadrics. In this case we can construct (instances of) Schreyer's surfaces $Y, Z \subset \mathbb{P}(\mathcal{E})$ in a concrete way, by explicitly giving the defining equations of $\mu(Y), \mu(Z) \subset S$. Indeed, by [3, Thm. 4] the ideal of C_f^{can} is minimally generated by quadrics

$$b_1, \dots, b_r, b'_1 \dots, b'_s, \mathcal{F}_{2,w_1}, \dots, \mathcal{F}_{2,w_t},$$

where

- the $r = \binom{g-3}{2}$ binomials b_i generate $\mathcal{I}(S)$,
- the $s = (4g-6) - \sharp(2\Delta^{(1)} \cap \mathbb{Z}^2)$ binomials b'_i cut $\mathrm{Tor}(\Delta^{(1)})$ out in S,
- $t = \sharp(\Delta^{(2)} \cap \mathbb{Z}^2) = B^{(1)} + 1$ and the quadrics $\mathcal{F}_{2,w_i}$ are constructed in the explicit manner described in [3]. Note that there is some freedom in the way these quadrics arise.

Then if $\mathcal{F}_f \subset \mathbb{P}(\mathcal{E})$ denotes the strict transform under μ of the joint zero locus of the quadrics $\mathcal{F}_{2,w_i}$, one can verify that $\mathcal{F}_f \sim 2H - B^{(1)}R$, so that one can take $Y = T'$ and $Z = \mathcal{F}_f$ if $B \geq B^{(1)}$, and $Y = \mathcal{F}_f$ and $Z = T'$ if $B < B^{(1)}$.

We end this section by explicitly listing the lattice polygons for which $B \leq B^{(1)}$. We will need the following property of two-dimensional lattice polygons of the form $\Delta^{(1)}$. An edge τ of a two-dimensional lattice polygon Γ is always supported on a line $a_\tau X + b_\tau Y = c_\tau$ with $a_\tau, b_\tau, c_\tau \in \mathbb{Z}$ and a_τ, b_τ coprime. When signs are chosen appropriately, we can assume that Γ is contained in the half-plane $a_\tau X + b_\tau Y \leq c_\tau$. Then the line $a_\tau X + b_\tau Y = c_\tau + 1$ is called the *outward shift* of τ. It is denoted by $\tau^{(-1)}$, and the polygon (which may take vertices outside $\mathbb{Z}^2$) that arises as the intersection of the half-planes $a_\tau X + b_\tau Y \leq c_\tau + 1$ is denoted by $\Gamma^{(-1)}$. If $\Gamma = \Delta^{(1)}$ for some lattice polygon Δ, then the outward shifts of two adjacent edges of Γ always intersect in a lattice point, and in fact $\Gamma^{(-1)} = \Delta^{(1)(-1)}$ is a lattice polygon. Moreover, $\Delta \subset \Delta^{(1)(-1)}$, i.e. $\Delta^{(1)(-1)}$ is the maximal lattice polygon with respect to inclusion for which the convex hull of the interior lattice points equals $\Delta^{(1)}$. See [9, §4] or [11, §2.2] for proofs.

Even though the following statement is purely combinatorial, given its geometric interpretation, it is natural to abbreviate $g = \sharp(\Delta^{(1)} \cap \mathbb{Z}^2)$. Similarly, we will write $g^{(1)} = \sharp(\Delta^{(2)} \cap \mathbb{Z}^2)$.

Lemma 9. *Let Δ be a lattice polygon with $\mathrm{lw}(\Delta^{(1)}) = 2$. Then we have:*

- $B < B^{(1)}$ *if and only if*

$$\Delta^{(1)} \cong \Gamma_{4k+4} := \mathrm{conv}\,\{(0,0),(k,0),(2k+2,1),(k+1,2),(1,2)\}$$

for some integer $k \geq 0$. In this case $g = 4k+4$, $B = 2k-1$ and $B^{(1)} = 2k$.

- $B = B^{(1)}$ *if and only if either*

$$\Delta^{(1)} \cong \Gamma^m_{4k+5} := \mathrm{conv}\,\{(0,0),(k,0),(2k+2,1),(k+m,2),(m,2),(0,1))\}$$

for some integers $k \geq 0$ and $0 \leq m \leq k+2$ (in these cases, $g = 4k+5$ and $B = B^{(1)} = 2k$), or

$$\Delta^{(1)} \cong \Gamma_{4k+3} := \mathrm{conv}\,\{(0,0),(k,0),(2k+1,1),(k+1,2),(1,2)\}$$

for some integer $k \geq 1$ (in this case, $g = 4k+3$ and $B = B^{(1)} = 2k-1$), or

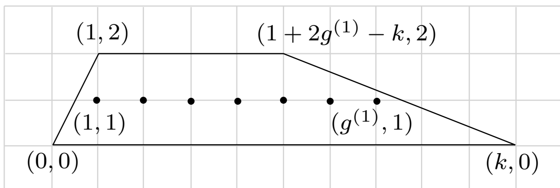
$$\Delta^{(1)} \cong \Gamma_{4k+1} := \mathrm{conv}\,\{(0,0),(k,0),(2k,1),(k,2),(1,2)\}$$

for some integer $k \geq 2$ (in this case, $g = 4k+1$ and $B = B^{(1)} = 2k-2$).

Proof. First we consider the polygons with $g^{(1)}$ equal to 0 and 1 separately. If $g^{(1)} = 0$ then $\Delta^{(1)} \cong 2\Sigma$, hence $B = 2 > B^{(1)} = -1$. If $g^{(1)} = 1$ then $B^{(1)} = 0$, hence $B \leq B^{(1)}$ if and only if $g \leq 5$. It is easy to check that there is one such polygon in genus 4 (namely $\Delta \cong 2\Upsilon$, so $\Delta^{(1)} \cong \Upsilon = \Gamma_4$) and three such polygons in genus 5 (corresponding to $\Delta^{(1)} \cong \Gamma^0_5, \Gamma^1_5, \Gamma^2_5$). Each of these appear in the classification.

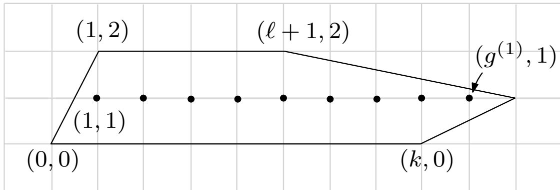
If $g^{(1)} \geq 2$, we can use Koelman's classification [11, Section 4.3] of lattice polygons Γ with lattice width 2. One can assume that $\Gamma = \Delta^{(1)}$ is contained in the strip $\{(X,Y) \in \mathbb{R}^2 \,|\, 0 \leq Y \leq 2\}$. Koelman subdivided these polygons into three types:

- *Type* 0*: there is no boundary lattice point of* Γ *with* $Y = 1$.
  Then up to equivalence $\Gamma = \Delta^{(1)}$ is of the form



with $g^{(1)} \leq k \leq 2g^{(1)}$. One sees that $B = 2g^{(1)} - 2$ and $B^{(1)} = g^{(1)} - 1$, so $B \leq B^{(1)}$ implies that $g^{(1)} \leq 1$: a contradiction.

- *Type* 1*: there is one boundary lattice point of* Γ *with* $Y = 1$.
  Up to equivalence $\Gamma = \Delta^{(1)}$ is of the form

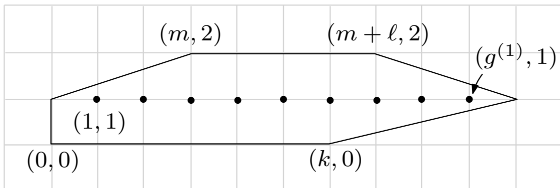


  with $0 \leq k \leq 2g^{(1)} + 1$ and

$$\begin{cases} 0 \leq \ell \leq k & \text{if} \quad 0 \leq k \leq g^{(1)}, \\ 0 \leq \ell \leq 2g^{(1)} - k + 1 & \text{if} \quad g^{(1)} < k \leq 2g^{(1)} + 1. \end{cases}$$

  Since moreover Γ is an interior lattice polygon we have that $\Gamma^{(-1)}$ takes its vertices inside $\mathbb{Z}^2$, leading to the inequalities $k \geq \frac{g^{(1)}-1}{2}$ and $\ell \geq \frac{g^{(1)}-1}{2}$. For this type, $B = k + \ell - 1 \geq g^{(1)} - 2$ and $B^{(1)} = g^{(1)} - 1$. So if $B \leq B^{(1)}$ then either $k = \ell = \frac{g^{(1)}-1}{2}$ (and $g = 4k + 4 \equiv 0 \bmod 4$), or $k = \ell = \frac{g^{(1)}}{2}$ (and $g = 4k + 3 \equiv 3 \bmod 4$), or $k = \frac{g^{(1)}+1}{2}$ and $\ell = \frac{g^{(1)}-1}{2}$ (and $g = 4k + 1 \equiv 0 \bmod 4$). We find back the polygons $\Gamma_{4k+1}, \Gamma_{4k+3}, \Gamma_{4k+4}$ from the statement of the lemma.

- *Type* 2*: there are two boundary lattice points of* Γ *with* $Y = 1$.
  Up to equivalence $\Gamma = \Delta^{(1)}$ is of the form



  with $0 \leq m \leq g^{(1)} + 1$, $0 \leq k \leq 2g^{(1)} + 2 - 2m$ and

$$\begin{cases} 0 \leq \ell \leq k & \text{if} \quad 0 \leq k \leq g^{(1)} + 1 - m, \\ 0 \leq \ell \leq 2g^{(1)} - k - 2m + 2 & \text{if} \quad g^{(1)} + 1 - m < k \leq 2g^{(1)} + 2 - 2m. \end{cases}$$

  Since moreover Γ is an interior lattice polygon, we also get the inequalities $k \geq \frac{g^{(1)}-1}{2}$ and $\ell \geq \frac{g^{(1)}-1}{2}$. If $B \leq B^{(1)}$ then since $B = k + \ell \geq g^{(1)} - 1 = B^{(1)}$, we have that $k = \ell = \frac{g^{(1)}-1}{2}$, $B = B^{(1)} = 2k$ and $g = 4k + 5$. So we get the polygons Γ^m_{4k+5} from the statement.

This concludes the proof. □

Remark 10. For each lattice polygon $\Gamma = \Gamma_g, \Gamma_g^m$ appearing in the statement of the lemma, there is only one polygon Δ for which $\Delta^{(1)} = \Gamma$, namely $\Delta = \Gamma^{(-1)}$. Note that $(\Gamma_4)^{(-1)} \cong 2\Upsilon$ and recall that a (2Υ)-non-degenerate curve is trigonal, rather than tetragonal.

3 From toric surfaces to polygons

This section is devoted to proving Theorem 4. As an a priori remark, note that it is important to impose that $\text{Tor}(\Delta)$ and $\text{Tor}(\Delta')$ are obtained from one another using a transformation of $\mathbb{P}^N$, rather than just isomorphic. For instance, let

$$\Delta = \text{conv}\{(0,0),(3,0),(3,2),(0,2)\} \quad \text{and} \quad \Delta' = \text{conv}\{(0,0),(5,0),(5,1),(0,1)\},$$

then $\text{Tor}(\Delta), \text{Tor}(\Delta') \subset \mathbb{P}^{11}$ are isomorphic (because their normal fans are the same), but not projectively equivalent, as they have different degrees (6 resp. 5). Here clearly $\Delta \not\cong \Delta'$.

Proof. We assume familiarity with the theory of divisors on toric surfaces, along the lines of [4, §3]. Notation-wise, we will write

- Σ_Δ for the (inner) normal fan associated to a given two-dimensional lattice polygon Δ, and
- Δ_D for the polygon (well-defined up to translation) corresponding to a Weil divisor (or a Cartier divisor, or an invertible sheaf) D on a given toric surface.

The proof then works as follows. Let Δ and Δ' be as in the statement of Theorem 4. The projective transformation induces an automorphism $\text{Tor}(\Delta) \to \text{Tor}(\Delta)$ that sends $\mathcal{O}_{\text{Tor}(\Delta)}(1)$ to $\mathcal{O}_{\text{Tor}(\Delta')}(1)$. Because

$$\Delta \cong \Delta_{\mathcal{O}_{\text{Tor}(\Delta)}(1)} \qquad \text{and} \qquad \Delta' \cong \Delta_{\mathcal{O}_{\text{Tor}(\Delta')}(1)}$$

it suffices to prove the following general statement: if

$$\iota : \text{Tor}(\Delta) \xrightarrow{\cong} \text{Tor}(\Delta')$$

is an isomorphism between two toric surfaces, and if D is a Weil divisor on $\text{Tor}(\Delta)$, then

$$\Delta_D \cong \Delta_{\iota(D)}.$$

Now it is known that two isomorphic toric varieties always admit a toric isomorphism between them [1, Thm. 4.1], i.e. an isomorphism that is induced by a $\text{GL}_2(\mathbb{Z})$-transformation taking Σ_Δ to $\Sigma_{\Delta'}$. It is clear that such an isomorphism preserves polygons (up to equivalence). Therefore we may assume that $\Sigma_\Delta = \Sigma_{\Delta'}$ and that ι is an automorphism of $\mathrm{Tor}(\Delta)$. Every such automorphism can be written as the composition of

- a toric automorphism,
- the automorphism induced by the action of an element of $\mathbb{T}^2$,
- a number of automorphisms of the form e_v^λ, where $\lambda \in k$ and $v \in \mathbb{Z}^2$ is a *column vector* of Δ, i.e. a primitive vector v for which there exists an edge $\tau \subset \Delta$ such that $u + v \in \Delta$ for all $u \in (\Delta \setminus \tau) \cap \mathbb{Z}^2$. To describe e_v^λ explicitly, assume that $v = (0, -1)$ and that τ lies horizontally (the general case can be reduced to this case by using an appropriate unimodular transformation). Then $\mathrm{Tor}(\Delta)$ can be viewed as a compactification of $\mathbb{T}^2 \cup (x\text{-axis})$ rather than just $\mathbb{T}^2$. On $\mathbb{T}^2 \cup (x\text{-axis})$, e_v^λ acts as $(x, y) \mapsto (x, y + \lambda)$. The column vector property ensures that this extends nicely to all of $\mathrm{Tor}(\Delta)$.

  Example. Let $\Delta = [0, 1] \times [0, 1]$ and consider the map

  $$\varphi : \mathbb{T}^2 \cup (x\text{-axis}) \hookrightarrow \mathrm{Tor}(\Delta) : (x, y) \mapsto (1, x, y, xy).$$

  The point $(x, y + \lambda)$ is mapped to $(1 : x : y + \lambda : xy + \lambda x)$. So here

  $$e^\lambda_{(0,-1)} : (X_{0,0} : X_{1,0} : X_{0,1} : X_{1,1}) \mapsto (X_{0,0} : X_{1,0} : X_{0,1} + \lambda X_{0,0} : X_{1,1} + \lambda X_{1,0}).$$

See [2, Thm. 3.2] for a proof of this statement, along with a more elaborate discussion. Now the first type of automorphisms preserves polygons up to equivalence, as before. The second type also preserves polygons because it preserves torus-invariant Weil divisors. As for the third type, let D_τ be the torus-invariant prime divisor corresponding to the base edge τ of v. Then by adding a divisor of the form $\mathrm{div}(x^i y^j)$ if needed, one can always find a torus-invariant Weil divisor that is equivalent to D and whose support does not contain D_τ; see [4, §4] for more details. But such a divisor is preserved by e_v^λ, hence the theorem follows. □

4 Intrinsicness for tetragonal curves

We are ready to explain why intrinsicness holds for lattice polygons Δ satisfying

$$\mathrm{lw}(\Delta^{(1)}) = 2 \quad \text{and} \quad B \geq B^{(1)} + 2,$$

that is, for the polygons of type (e) from the introduction. Let C be a Δ-non-degenerate curve. Then it is a tetragonal curve (indeed, $B \geq B^{(1)} + 2$ implies $\Delta \not\cong 2\Upsilon$) whose Schreyer invariants b_1, b_2 satisfy $b_1 \geq b_2 + 2$. By Theorem 7 we find that Schreyer's surface $\mu(Y) \subset \mathbb{P}^{g-1}$ equals $\mathrm{Tor}(\Delta^{(1)})$. Now suppose that C is also Δ'-non-degenerate for some two-dimensional lattice polygon Δ'. By the tetragonality of C we have $\mathrm{lw}(\Delta'^{(1)}) = 2$. In analogy with the previous notation, write

$$B' = \sharp(\partial\Delta'^{(1)} \cap \mathbb{Z}^2) - 4, \qquad B'^{(1)} = \sharp(\Delta'^{(2)} \cap \mathbb{Z}^2) - 1,$$

so that $\{B', B'^{(1)}\} = \{b_1, b_2\}$ by Theorem 7. It follows that either

$$B' \geq B'^{(1)} + 2 \qquad \text{or} \qquad B'^{(1)} \geq B' + 2.$$

But the latter is impossible by Lemma 9, which states that $B'^{(1)}$ is at most $B' + 1$. Therefore $B' > B'^{(1)}$ and, again by Theorem 7, we find that $\mu(Y)$ is given by $\mathrm{Tor}(\Delta'^{(1)})$. We conclude that $\mathrm{Tor}(\Delta^{(1)})$ and $\mathrm{Tor}(\Delta'^{(1)})$ are equal, possibly modulo a projective transformation. Intrinsicness now follows from Theorem 4.

This argument can be refined. For instance, in genus $g \not\equiv 0 \bmod 4$ it suffices that $B \geq B^{(1)} + 1$, because in this case Lemma 9 yields the sharper bound $B'^{(1)} \leq B'$. In genus $g \equiv 2 \bmod 4$ one sees that this is automatically satisfied.

By pushing this type of reasoning, we obtain the following statement.

Theorem 11. *Let Δ, Δ' be two-dimensional lattice polygons and let there be a curve that is both Δ-non-degenerate and Δ'-non-degenerate. Suppose that $\mathrm{lw}(\Delta^{(1)}) = 2$ and define $g = \sharp(\Delta^{(1)} \cap \mathbb{Z}^2) = \sharp(\Delta'^{(1)} \cap \mathbb{Z}^2)$.*

- *Case $g \equiv 0 \bmod 4$. If $\Delta^{(1)}, \Delta'^{(1)} \not\cong \Gamma_g$ then $\Delta^{(1)} \cong \Delta'^{(1)}$. This is automatic if $\sharp(\partial\Delta^{(1)} \cap \mathbb{Z}^2) \geq \sharp(\Delta^{(2)} \cap \mathbb{Z}^2) + 5$.*
- *Case $g \equiv 1 \bmod 4$. If $\Delta^{(1)}, \Delta'^{(1)} \not\cong \Gamma_g^m$ for all $1 \leq m \leq (g+3)/4$ then $\Delta^{(1)} \cong \Delta'^{(1)}$. This is automatic if $\sharp(\partial\Delta^{(1)} \cap \mathbb{Z}^2) \geq \sharp(\Delta^{(2)} \cap \mathbb{Z}^2) + 4$.*
- *Cases $g \equiv 2, 3 \bmod 4$. Here one always has $\Delta^{(1)} \cong \Delta'^{(1)}$.*

Proof. The cases $g \equiv 0, 2 \bmod 4$ follow along the above lines of thought. For the case $g \equiv 3 \bmod 4$ one remarks that Schreyer's invariants coincide if and only if $B = B^{(1)}$, which by Lemma 9 happens if and only if $\Delta^{(1)} \cong \Delta'^{(1)} \cong \Gamma_g$. If not then $B \geq B^{(1)} + 1$, and one proceeds as before.

The most subtle case is when $g \equiv 1 \bmod 4$. If $g = 5$ then Schreyer's invariants coincide if and only if $\Delta^{(1)} \cong \Delta'^{(1)} \cong \Gamma_5^0$ (indeed, the polygons Γ_5^1 and Γ_5^2 appearing in Lemma 9 were excluded in the statement), so this is analogous to the $g \equiv 3 \bmod 4$ case. If $g > 5$ then one draws the weaker conclusion that Schreyer's invariants coincide if and only if $\Delta^{(1)}$ and $\Delta'^{(1)}$ are among Γ_g and Γ_g^0. To distinguish between both cases, one notes that the scrollar invariants e_1, e_2, e_3 are

$$\frac{g-5}{4}, \frac{g-1}{4}, \frac{g-3}{2} \qquad \text{and} \qquad \frac{g-5}{4}, \frac{g-5}{4}, \frac{g-1}{2},$$

respectively. Here we implicitly used that our curve carries a unique g^1_4 by [4, Cor. 6.3], so it does make sense to talk about *the* scrollar invariants. We conclude that $\Delta^{(1)} \cong \Delta'^{(1)} \cong \Gamma^0_g$ if the curve has two coinciding scrollar invariants, and that $\Delta^{(1)} \cong \Delta'^{(1)} \cong \Gamma_g$ if not. □

Remark 12. Note that the theorem remains valid if we replace 'for all $1 \leq m \leq (g+3)/4$' by 'for all $m \in \{0, \ldots, (g+3)/4\} \setminus \{i\}$', for whatever i.

Example 13. Let $g \geq 4$ satisfy $g \equiv 0 \bmod 4$, and denote by Δ_g the (unique) lattice polygon for which $\Delta_g^{(1)} = \Gamma_g$. Then it is possible that a Δ_g-non-degenerate curve is also non-degenerate with respect to a lattice polygon Δ' for which $\Delta'^{(1)} \ncong \Gamma_g$. For instance, consider $f = 1 - x^2y^4 - x^{\frac{g}{2}+2}y^2$ and $f' = (y^4-1)x^{\frac{g}{2}+1} + 4y^2$. Both polynomials are non-degenerate with respect to their respective Newton polygons. Note that $\Delta(f) \cong \Delta_g$ and that $\Delta(f')^{(1)} \ncong \Gamma_g$. Now the rational maps

$$U_f \to U_{f'} : (x,y) \mapsto \left(x, \frac{1-xy^2}{x^{\frac{g}{4}+1}y}\right)$$

$$U_{f'} \to U_f : (x,y) \mapsto \left(x, \frac{2y}{x^{\frac{g}{4}+1}(1+y^2)}\right)$$

are inverses of each other, so C_f and $C_{f'}$ are isomorphic. We conclude that C_f is both Δ_g-non-degenerate and $\Delta(f')$-non-degenerate.

Example 14. We conjecture that for each $g \geq 5$ with $g \equiv 1 \bmod 4$ and each $0 \leq n, m \leq (g+3)/4$, there exists a curve that is both Δ_g^n- and Δ_g^m-non-degenerate. Here Δ_g^n and Δ_g^m are the unique lattice polygons having Γ_g^n and Γ_g^m as their respective interiors.

Loosely speaking, we believe that the following strategy for finding such a curve always works (although we could not prove this). From Sections 1 and 2 we know that the canonical model C_f^{can} of a Δ_g^n-non-degenerate curve C_f satisfies $C_f^{\mathrm{can}} \subset \mathrm{Tor}(\Gamma_g^n) \subset S \subset \mathbb{P}^{g-1}$, where S is a rational normal scroll of type

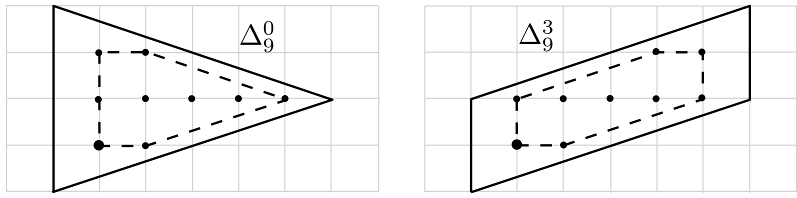
$$\left(\frac{g-5}{4}, \frac{g-5}{4}, \frac{g-1}{2}\right),$$

and that C_f^{can} arises as the intersection of two surfaces Y and Z inside the class

$$2H - \frac{g-5}{2}R$$

(the role of μ, which is only relevant for $g = 5$, is ignored for the sake of exposition). Recall from Remark 8 that one can take $Y = \mathrm{Tor}(\Gamma_g^n)$, and $Z = \mathcal{F}_f$. The idea is to switch the role of Y and Z, in the sense that one chooses f such that $\mathcal{F}_f = \theta(\mathrm{Tor}(\Gamma_g^m))$ for some $\theta \in \mathrm{Aut}(S) \subset \mathrm{Aut}(\mathbb{P}^{g-1})$. Because non-degeneracy is generically satisfied, one expects $\theta^{-1}(Y)$ to be of the form $\mathcal{F}_{f'}$ for some Δ_g^m-non-degenerate Laurent polynomial f'.

Explicit examples in genus $g = 5$ can be found in our unpublished `arXiv` paper `1304.4997`. For $g = 9$ and $\{n, m\} = \{0, 3\}$ we used the above approach to find that



the Δ_9^0-non-degenerate Laurent polynomial

$$f = 8x^5y + 36x^4y + 66x^3y - x^2y^2 + 62x^2y - x^2 + 33xy + 9y - 2x^{-1}y^3 - 2x^{-1}y^2 - 4x^{-1}y - 3x^{-1} - 3x^{-1}y^{-1}$$

and the Δ_9^3-non-degenerate Laurent polynomial

$$f' = 2x^5y^3 + x^5y^2 - x^5y - 6x^4y - 15x^3y + 2x^2y^2 - 14x^2y + x^2 - 15xy - 6y - x^{-1}y + 3x^{-1} + 3x^{-1}y^{-1}$$

define birationally equivalent curves in $\mathbb{T}^2$. To describe the automorphism θ explicitly, we need to pick coordinates of $\mathbb{P}^{g-1}$. When thought of as the ambient space of $\mathrm{Tor}(\Gamma_9^0)$, we will write

$$\mathbb{P}^{g-1} = \mathrm{Proj}\, V \qquad \text{with } V = k[X_{0,0}, X_{1,0}, X_{0,1}, X_{1,1}, X_{2,1}, X_{3,1}, X_{4,1}, X_{0,2}, X_{1,2}],$$

where $X_{i,j}$ is the coordinate corresponding to the lattice point $(i, j) \in \Gamma_9^0$ (the origin is understood to be the bold-marked lattice point). Similarly, when thought of as the ambient space of $\mathrm{Tor}(\Gamma_9^3)$ we write

$$\mathbb{P}^{g-1} = \mathrm{Proj}\, W \qquad \text{with } W = k[X_{0,0}, X_{1,0}, X_{0,1}, X_{1,1}, X_{2,1}, X_{3,1}, X_{4,1}, X_{3,2}, X_{4,2}].$$

Then, on the level of coordinate rings, $\theta : V \to W$ can be defined by

$$\begin{pmatrix} \theta(X_{0,1}) \\ \theta(X_{1,1}) \\ \theta(X_{2,1}) \\ \theta(X_{3,1}) \\ \theta(X_{4,1}) \\ \theta(X_{0,2}) \\ \theta(X_{1,2}) \\ \theta(X_{0,0}) \\ \theta(X_{1,0}) \end{pmatrix} = \begin{pmatrix} 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 9 & 7 & 2 & 0 & 0 & 0 & 0 \\ 1 & 6 & 13 & 12 & 4 & 0 & 0 & 0 & 0 \\ 1 & 7 & 18 & 20 & 8 & 0 & 0 & 0 & 0 \\ 1 & 8 & 24 & 32 & 16 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 & 2 & 4 & 3 & 6 \end{pmatrix} \cdot \begin{pmatrix} X_{0,1} \\ X_{1,1} \\ X_{2,1} \\ X_{3,1} \\ X_{4,1} \\ X_{3,2} \\ X_{4,2} \\ X_{0,0} \\ X_{1,0} \end{pmatrix}.$$

We leave it to the reader to verify that θ maps S to S and sends $\mathrm{Tor}(\Gamma_9^3)$ to $\mathcal{F}_f$ and $\mathcal{F}_{f'}$ to $\mathrm{Tor}(\Gamma_9^0)$ (for an appropriate choice of defining equations for $\mathcal{F}_f$ and $\mathcal{F}_{f'}$).

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Intrinsicness of the Newton polygon for smooth curves on $\mathbb{P}^1 \times \mathbb{P}^1$

Wouter Castryck and Filip Cools

Abstract

Let C be a smooth projective curve in $\mathbb{P}^1 \times \mathbb{P}^1$ of genus $g \neq 4$, and assume that it is birationally equivalent to a curve defined by a Laurent polynomial that is non-degenerate with respect to its Newton polygon Δ. Then we show that the convex hull $\Delta^{(1)}$ of the interior lattice points of Δ is a standard rectangle, up to a unimodular transformation. Our main auxiliary result, which we believe to be interesting in its own right, is that the first scrollar Betti numbers of Δ-non-degenerate curves are encoded in the combinatorics of $\Delta^{(1)}$, if Δ satisfies some mild conditions.

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1 Introduction

Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be an irreducible Laurent polynomial over an algebraically closed field k of characteristic zero and let $U(f)$ be the curve it defines in the two-dimensional torus $\mathbb{T}^2 = (k^*)^2$. The *Newton polygon* $\Delta = \Delta(f)$ of f is the convex hull in $\mathbb{R}^2$ of all the exponent vectors in $\mathbb{Z}^2$ of the monomials that appear in f with a non-zero coefficient. We will always assume that Δ is two-dimensional. We say that f is *non-degenerate* with respect to its Newton polygon Δ (or more briefly, f is Δ-non-degenerate) if and only if for each face $\tau \subset \Delta$ (including $\tau = \Delta$) the system

$$f_\tau = \frac{\partial f_\tau}{\partial x} = \frac{\partial f_\tau}{\partial y} = 0$$

does not have any solutions in $\mathbb{T}^2$. Here, f_τ is obtained from f by only considering the terms that are supported on τ. This condition is generically satisfied. Consider the map

$$\varphi_\Delta : \mathbb{T}^2 \to \mathbb{P}^{\sharp(\Delta \cap \mathbb{Z}^2) - 1} : (x, y) \mapsto (x^i y^j)_{(i,j) \in \Delta \cap \mathbb{Z}^2}.$$

The Zariski closure of its full image $\varphi_\Delta(\mathbb{T}^2)$ is a *toric surface* $\mathrm{Tor}(\Delta)$, while the Zariski closure of $\varphi_\Delta(U(f))$ is a hyperplane section C_f of $\mathrm{Tor}(\Delta)$, which is smooth if f is non-degenerate. We will denote the projective coordinates of $\mathbb{P}^{\sharp(\Delta\cap\mathbb{Z}^2)-1}$ by $X_{i,j}$ where (i,j) runs over $\Delta \cap \mathbb{Z}^2$.

We say that a smooth curve C is Δ-*non-degenerate* if and only if it is birationally equivalent to $U(f)$ for a Δ-non-degenerate Laurent polynomial f. Note that if C is moreover projective, then it is isomorphic to C_f. If C is Δ-non-degenerate, then a lot of its geometric properties are encoded in the combinatorics of the lattice polygon Δ. For instance, its geometric genus $g(C)$ equals the number of interior lattice points of Δ [10]. Similar interpretations were recently provided for the gonality [3, 9], the Clifford index and dimension [3, 9], the scrollar invariants associated to a gonality pencil [3] and Schreyer's tetragonal invariants [5].

Given this long list, the following question (initiated in [5]) naturally arises: to what extent can we recover Δ from the geometry of a Δ-non-degenerate curve? At least, we have to allow two relaxations to this question. First, we can only expect to find back the polygon Δ up to a *unimodular transformation*, i.e. an affine map of the form

$$\chi : \mathbb{R}^2 \to \mathbb{R}^2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix} + B$$

with $A \in \mathrm{GL}_2(\mathbb{Z})$ and $B \in \mathbb{Z}^2$, since these maps correspond to automorphisms of $\mathbb{T}^2$. Secondly, we can (usually) only hope to recover the convex hull of the interior lattice points of Δ, denoted by $\Delta^{(1)}$ (see [5] for an easy example demonstrating the need for this relaxation). In fact, all the aforementioned combinatorial interpretations are in terms of the combinatorics of $\Delta^{(1)}$ rather than Δ (e.g. $g(C) = \sharp(\Delta^{(1)} \cap \mathbb{Z}^2)$).

Given a Δ-non-degenerate curve C, we say that the Newton polygon Δ is *intrinsic* for C if and only if for all Δ'-non-degenerate curves C' that are birationally equivalent to C, we have that $\Delta^{(1)} \cong \Delta'^{(1)}$. Hereby, we use $\cong$ to denote the unimodular equivalence relation. Before stating some intrinsicness results, we give notations for some special lattice polygons:

$$\begin{aligned} \square_{a,b} &= \mathrm{conv}\{(0,0),(a,0),(0,b),(a,b)\} \text{ for } a,b \in \mathbb{Z}_{\geq 0}, \\ \Sigma &= \mathrm{conv}\{(0,0),(1,0),(0,1)\}, \\ \Upsilon &= \mathrm{conv}\{(-1,-1),(1,0),(0,1)\}. \end{aligned}$$

The Newton polygon is intrinsic for all rational ($\Delta^{(1)} = \emptyset$), hyperelliptic ($\Delta^{(1)}$ is one-dimensional, and therefore determined by the genus) and trigonal curves of genus at least 5 ($\Delta^{(1)}$ has lattice width 1, and is determined by the Maroni invariants). However, there are trigonal curves of genus 4 for which Δ is not intrinsic: there exist curves which are non-degenerate with respect to polygons Δ and Δ', with $\Delta^{(1)} = \Upsilon$ and $\Delta'^{(1)} = \square_{1,1}$. Intrinsicness of the Newton polygon for tetragonal curves was studied in [5]: the Newton polygon Δ is intrinsic if $g(C) \bmod 4 \in \{2,3\}$, but it might occasionally be not intrinsic if $g(C) \bmod 4 \in \{0,1\}$. From [3], it follows that non-degenerate smooth plane curves of degree $d \geq 3$ ($\Delta^{(1)} \cong (d-3)\Sigma$) and curves with Clifford dimension 3 ($\Delta^{(1)} \cong 2\Upsilon$) have an intrinsic Newton polygon. Moreover, a partial result was given for non-degenerate curves on Hirzebruch surfaces $\mathcal{H}_n$: the value n is intrinsic.

In this paper, we examine intrinsicness of Δ for curves on $\mathbb{P}^1 \times \mathbb{P}^1$. Namely, we will show that a Δ-non-degenerate curve C of genus $g \neq 4$ can be embedded in $\mathbb{P}^1 \times \mathbb{P}^1$ (if and) only if $\Delta^{(1)} = \emptyset$ or $\Delta^{(1)} \cong \square_{a,b}$ for $a, b \in \mathbb{Z}_{\geq 0}$ satisfying $g = (a+1)(b+1)$; see Theorem 18 in Section 3. In order to prove this result, we give a combinatorial interpretation for the first scrollar Betti numbers of Δ-non-degenerate curves with respect to a gonality pencil, as soon as Δ satisfies some mild conditions (see Section 2).

Notations. Let $\mathbb{P}^N$ be a projective space with coordinates $(X_0 : \ldots : X_N)$. For each projective variety $V \subset \mathbb{P}^N$, we write $\mathcal{I}(V) \subset k[X_0, \ldots, X_N]$ to indicate the homogeneous ideal of V and $\mathcal{I}_d(V) \subset \mathcal{I}(V)$ to indicate its homogeneous degree d piece. If $J \subset k[X_0, \ldots, X_N]$ is a homogeneous ideal, then $\mathcal{Z}(J) \subset \mathbb{P}^N$ is the zero locus of the polynomials in J.

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2 First scrollar Betti numbers

2.1 Definition

We start by recalling the definition and some properties of *rational normal scrolls*.

Let $n \in \mathbb{Z}_{\geq 2}$ and let $\mathcal{E} = \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_n)$ be a locally free sheaf of rank n on $\mathbb{P}^1$. Denote by $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ the corresponding $\mathbb{P}^{n-1}$-bundle. We assume that $0 \leq e_1 \leq e_2 \leq \ldots \leq e_n$ and that $e_1 + e_2 + \cdots + e_n \geq 2$. Set $N = e_1 + e_2 + \ldots + e_n + n - 1$. Then the image $S = S(e_1, \ldots, e_n)$ of the induced morphism

$$\mu : \mathbb{P}(\mathcal{E}) \to \mathbb{P}H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)),$$

when composed with an isomorphism $\mathbb{P}H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \to \mathbb{P}^N$, is called a rational normal scroll of type $(e_1, \ldots, e_n)$. Up to automorphisms of $\mathbb{P}^N$, rational normal scrolls are fully characterized by their type.

They can also be described in a geometric way: consider linearly independent projective subspaces $\mathbb{P}_1^{e_1}, \ldots, \mathbb{P}_n^{e_n} \subset \mathbb{P}^N$ of dimensions $e_1, \ldots, e_n$, so their span is the whole projective space $\mathbb{P}^N$. For each $i \in \{1, \dots, n\}$, fix a rational normal curve in $\mathbb{P}_i$ of degree e_i parametrized by a Veronese map $\nu_i : \mathbb{P}^1 \to \mathbb{P}_i^{e_i}$. Then

$$S = \cup_{P \in \mathbb{P}^1} \langle \nu_1(P), \dots, \nu_n(P) \rangle \subset \mathbb{P}^N$$

is a rational normal scroll of type $(e_1, \dots, e_n)$.

The scroll is smooth if and only if $e_1 > 0$. In this case, $\mu : \mathbb{P}(\mathcal{E}) \to S$ is an isomorphism. If $0 = e_1 = \ldots = e_\ell < e_{\ell+1}$ with $1 \leq \ell < n$, then the scroll is a cone with an $(\ell - 1)$-dimensional vertex. In this case $\mu : \mathbb{P}(\mathcal{E}) \to S$ is a resolution of singularities and

$$\mu_\lambda : \mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(\lambda)) \to S' = S(e_1 + \lambda, \dots, e_n + \lambda)$$

is an isomorphism for all integers $\lambda > 0$.

The Picard group of $\mathbb{P}(\mathcal{E})$ is freely generated by the class H of a hyperplane section (more precisely, the class corresponding to $\mu^*\mathcal{O}_{\mathbb{P}^N}(1)$) and the class R of a fiber of π; i.e.

$$\mathrm{Pic}(\mathbb{P}(\mathcal{E})) = \mathbb{Z}H \oplus \mathbb{Z}R.$$

We have the following intersection products:

$$H^n = e_1 + \ldots + e_n, \ H^{n-1}R = 1 \ \text{ and } \ R^2 = 0$$

(where $R^2 = 0$ means that any appearance of R^2 annihilates the product). If we denote the class which corresponds to $\mu_\lambda^*\mathcal{O}_{\mathbb{P}^{N+n\lambda}}(1)$ by H', we obtain the equality $H' = H + \lambda R$ in $\mathrm{Pic}(\mathbb{P}(\mathcal{E}))$.

Let C/k be a smooth projective curve of genus g and gonality $\gamma \geq 4$. Assume that C is canonically embedded in $\mathbb{P}^{g-1}$ and fix a gonality pencil g^1_γ on C. By [6, Thm. 2],

$$S = \bigcup_{D \in g^1_\gamma} \langle D \rangle \subset \mathbb{P}^{g-1}$$

is a $(\gamma - 1)$-dimensional rational normal scroll containing C. If S is of type $(e_1, \dots, e_{\gamma-1})$, the numbers $e_1, \dots, e_{\gamma-1}$ are called the *scrollar invariants* of C with respect to g^1_γ. Using the Riemann-Roch theorem, one can see that $e_{\gamma-1} \leq \frac{2g-2}{\gamma}$.

The following theorem extends a result from [13] on tetragonal and pentagonal curves to arbitrary curves.

Theorem 1. *Let C be a canonically embedded smooth projective curve of genus g and gonality $\gamma \geq 4$. If g^1_γ is a gonality pencil on C, let $S \subset \mathbb{P}^{g-1}$ be the rational normal scroll swept out by g^1_γ and let C' be the strict transform of C under the resolution $\mu : \mathbb{P}(\mathcal{E}) \to S$. Then there exist effective divisors $D_1, \dots, D_{(\gamma^2-3\gamma)/2}$*

on $\mathbb{P}(\mathcal{E})$ *along with integers* $b_1, \ldots, b_{(\gamma^2-3\gamma)/2}$*, such that* $D_\ell \sim 2H - b_\ell R$ *for all* ℓ *and* C' *is the (scheme-theoretical) intersection of the* D_ℓ*'s. Moreover, the multiset* $\{b_1, \ldots, b_{(\gamma^2-3\gamma)/2}\}$ *does not depend on the choice of the* D_ℓ*'s, and*

$$\sum_{\ell=1}^{(\gamma^2-3\gamma)/2} b_\ell \,=\, (\gamma-3)(g-\gamma-1).$$

Proof. Define $\beta_i = \frac{i(\gamma-2-i)}{\gamma-1}\binom{\gamma}{i+1}$ and note that $\beta_1 = (\gamma^2-3\gamma)/2$. The existence follows from [13, Cor. 4.4] and its proof, where the D_ℓ's come from an exact sequence of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-modules

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-\gamma H + (g-\gamma+1)R) \to \sum_{\ell=1}^{\beta_{\gamma-3}} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-(\gamma-2)H + b_\ell^{(\gamma-3)}R) \to \cdots$$
$$\to \sum_{\ell=1}^{\beta_2} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H + b_\ell^{(2)}R) \to \sum_{\ell=1}^{\beta_1} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H + b_\ell R) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{O}_{C'} \to 0. \quad (1)$$

Tensoring (1) with $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2H + bR)$ for a sufficiently large integer b and computing the Euler characteristics of the terms in the resulting exact sequence, one can show that

$$\sum_{\ell} b_\ell \,=\, (\gamma-3)(g-\gamma-1);$$

see [1, Prop. 2.9].

We are left with showing the independence of the multiset $\{b_1, \ldots, b_{(\gamma^2-3\gamma)/2}\}$. Herefore, consider the exact sequence

$$\sum_{\ell=1}^{\beta_1} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-D_\ell) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{O}_{C'} \to 0. \quad (2)$$

If $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ is the $\mathbb{P}^{\gamma-2}$-bundle and ξ is the generic point of $\mathbb{P}^1$, then

$$\pi^{-1}(\xi) = \mathbb{P}^{\gamma-2}_{k(\xi)} = \operatorname{Proj} \mathcal{S},$$

where $\mathcal{S} = k(\xi)[x_0, \ldots, x_{\gamma-2}]$. Applying $\cdot \otimes_{\mathbb{P}^1} k(\xi)$ to (2) yields an exact sequence of graded $\mathcal{S}$-modules, that can be extended to a minimal free resolution

$$0 \to \mathcal{S}(-\gamma) \to \mathcal{S}(-\gamma+2)^{\oplus \beta_{\gamma-3}} \to \cdots \to \mathcal{S}(-2)^{\oplus \beta_1} \to \mathcal{S} \to \mathcal{S}_{C'} \to 0$$

of the coordinate ring $\mathcal{S}_{C'}$ of C' over $k(\xi)$ (see [13, Lemma 4.2] and [2, Step A of Thm. 2.1]). As explained in [13, proof of Thm. 3.2] and [2, proof of Step B of Thm. 2.1], this resolution can be lifted to a minimal free resolution of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-modules extending (2). This resolution is unique up to isomorphism by [13, Thm. 3.2] or [2, Thm. 1.3], which implies the independence. □

We call the invariants $b_1, \ldots, b_{(\gamma^2-3\gamma)/2}$ the *first scrollar Betti numbers of C with respect to g^1_γ*. The main goal of this section is to give a combinatorial interpretation for these invariants for non-degenerate curves.

In [5], we already treated the case of tetragonal Δ-non-degenerate curves: the first scrollar Betti numbers are given by

$$\sharp(\partial\Delta^{(1)} \cap \mathbb{Z}^2) - 4 \quad \text{and} \quad \sharp(\Delta^{(2)} \cap \mathbb{Z}^2) - 1.$$

These numbers are independent from the choice of the gonality pencil. This will no longer be true for non-degenerate curves of higher gonality.

2.2 Scrollar invariants for non-degenerate curves

Let f be a Δ-non-degenerate Laurent polynomial and consider the corresponding smooth curve $C_f \subset \mathrm{Tor}(\Delta) \subset \mathbb{P}^N$ with $N = \sharp(\Delta \cap \mathbb{Z}^2) - 1$. Assume that the polygon $\Delta^{(1)}$ is two-dimensional.

By [10], C_f is a non-rational and non-hyperelliptic curve and there exists a canonical divisor K_Δ on C_f such that

$$H^0(C_f, K_\Delta) = \langle x^i y^j \rangle_{(i,j)\in\Delta^{(1)}\cap\mathbb{Z}^2}$$

(where x, y are functions on C_f through φ_Δ). In particular, the curve C_f has genus $g = \sharp(\Delta^{(1)} \cap \mathbb{Z}^2) \geq 3$; see [3] for more details. Moreover, the Zariski closure $C = C_f^{can}$ of the image of $U(f)$ under

$$\varphi_{\Delta^{(1)}} : \mathbb{T}^2 \hookrightarrow \mathbb{P}^{g-1} : (x, y) \mapsto (x^i y^j)_{(i,j)\in\Delta^{(1)}\cap\mathbb{Z}^2} \tag{3}$$

is a canonical model for C_f. We end up with the inclusions

$$C \subset T = \mathrm{Tor}(\Delta^{(1)}) = \overline{\varphi_{\Delta^{(1)}}(\mathbb{T}^2)} \subset \mathbb{P}^{g-1},$$

where T is a toric surface since $\Delta^{(1)}$ is two-dimensional.

A *lattice direction* is a primitive integer vector $v = (a, b) \in \mathbb{Z}^2$. The *width* $w(\Delta, v)$ of Δ with respect to a lattice direction v is the smallest integer ℓ such that Δ is contained in the strip $k \leq aY - bX \leq k + \ell$ of $\mathbb{R}^2$ for some $k \in \mathbb{Z}$. The *lattice width* is defined as $\mathrm{lw}(\Delta) = \min_v w(\Delta, v)$. Lattice directions v that attain the minimum are called *lattice-width directions*.

In [3], we gave a combinatorial interpretation for the gonality γ of $C = C_f^{can}$ (or C_f) in terms of the lattice width of Δ:

$$\gamma = \begin{cases} \mathrm{lw}(\Delta) = \mathrm{lw}(\Delta^{(1)}) + 2 & \text{if } \Delta \not\cong 2\Upsilon \text{ and } \Delta \not\cong d\Sigma \text{ for all } d \in \mathbb{Z}_{\geq 4}, \\ \mathrm{lw}(\Delta) - 1 = \mathrm{lw}(\Delta^{(1)}) + 2 & \text{if } \Delta \cong d\Sigma \text{ for some } d \in \mathbb{Z}_{\geq 4}, \\ \mathrm{lw}(\Delta) - 1 = \mathrm{lw}(\Delta^{(1)}) + 1 & \text{if } \Delta \cong 2\Upsilon, \end{cases}$$

where we use our assumption that $\Delta^{(1)}$ is two-dimensional. From now on, we make the stronger assumption that $\gamma = \mathrm{lw}(\Delta) \geq 4$, and that $\Delta^{(1)}$ is not equivalent with $k\Sigma$ for any k or Υ, hence $\Delta \not\cong d\Sigma$ and $\Delta \not\cong 2\Upsilon$. Then each lattice-width direction $v = (a, b)$ gives rise to a rational map

$$C \dashrightarrow \mathbb{P}^1 : (x^i y^j)_{(i,j)\in\Delta^{(1)}\cap\mathbb{Z}^2} \mapsto x^a y^b$$

of degree equal to the gonality γ. We call the corresponding linear pencil g^1_γ of C a *combinatorial gonality pencil.* If Δ is sufficiently big (for a precise statement, see [3, Corollary 6.3]), each gonality pencil on C is combinatorial.

Fix a lattice-width direction v of Δ. After applying a suitable unimodular transformation χ, we may assume that $v = (1, 0)$ and that Δ is contained in the horizontal strip $0 \leq Y \leq \gamma$ in $\mathbb{R}^2$. So, the gonality map $C \dashrightarrow \mathbb{P}^1$ associated to v is the vertical projection to the x-axis. Write

$$i^{(-)}(j) = \min\{i \in \mathbb{Z} \,|\, (i, j) \in \Delta^{(1)}\} \text{ and } i^{(+)}(j) = \max\{i \in \mathbb{Z} \,|\, (i, j) \in \Delta^{(1)}\}$$

for all $j \in \{1, \ldots, \gamma - 1\}$. By [3, Theorem 9.1], the scrollar invariants $e_1, \ldots, e_{\gamma-1}$ of C with respect to g^1_γ are equal to $E_j := i^{(+)}(j) - i^{(-)}(j)$ for $j \in \{1, \ldots, \gamma - 1\}$ (up to order). In fact, a Zariski dense part of the scroll S is parametrized by

$$(a_1, \ldots, a_{\gamma-1}, x) \in \mathbb{T}^\gamma \mapsto (a_j x^{i-i^{(-)}(j)})_{(i,j)\in\Delta^{(1)}\cap\mathbb{Z}^2} = (a_j, \ldots, a_j x^{E_j})_{1\leq j\leq\gamma-1} \in \mathbb{P}^{g-1}.$$

Note that $T = \mathrm{Tor}(\Delta^{(1)}) \subset S$ since the map $\varphi_{\Delta^{(1)}}$ can be obtained from the above parametrization by restricting to $a_j = x^{i^{(-)}(j)} y^j$, so we get the inclusions

$$C \subset T \subset S \subset \mathbb{P}^{g-1}. \tag{4}$$

If S is singular, then $\mu : S' = S(e_1 + \lambda, \ldots, e_{\gamma-1} + \lambda) \cong \mathbb{P}(\mathcal{E}) \to S$ is a resolution of singularities for each integer $\lambda > 0$ (hereby, we slightly abuse notation: μ is the map $\mu \circ \mu_\lambda^{-1}$ using the notations in Section 2.1). Let C' and T' be the strict transforms of respectively C and T under μ. For each lattice polygon $\Gamma \subset \mathbb{R}^2$, write $\Gamma[\lambda]$ to denote the Minkowski sum of Γ and $[(0, 0), (\lambda, 0)] \subset \mathbb{R}^2$. In other words, $\Gamma[\lambda]$ is obtained from Γ by stretching it out in the horizontal direction over a distance λ. Using this notation, one can see that $T' = \mathrm{Tor}(\Delta^{(1)}[\lambda]) = \mathrm{Tor}(\Delta[\lambda]^{(1)})$. We end up with the inclusions

$$C' \subset T' \subset S' \subset \mathbb{P}^{g-1+\lambda(\gamma-1)}. \tag{5}$$

2.3 First scrollar Betti numbers of toric surfaces

Let C be a Δ-non-degenerate curve and fix a combinatorial gonality pencil g^1_γ on C, corresponding to a lattice direction v. We work under the following assumptions:

(i) $\Delta^{(1)}$ is not equivalent with $k\Sigma$ for any k or Υ, and $\gamma = \operatorname{lw}(\Delta) \geq 4$,

(ii) $v = (1, 0)$ and Δ is contained in the horizontal strip $0 \leq Y \leq \gamma$, so that g^1_γ corresponds to the vertical projection,

(iii) the curve C is canonically embedded, so that we obtain the sequence of inclusions $C \subset T \subset S \subset \mathbb{P}^{g-1}$ from (4).

Recall that the scrollar invariants $e_1, \ldots, e_{\gamma-1}$ of C with respect to g^1_γ match with $E_1, \ldots, E_{\gamma-1}$ (up to order). Consider $\mu : \mathbb{P}(\mathcal{E}) \to S$ and let $T' \subset \mathbb{P}(\mathcal{E})$ be the strict transform of $T = \operatorname{Tor}(\Delta^{(1)})$ under μ, as in Section 2.2. If Δ satisfies the condition $\mathcal{P}_1(v)$ defined below (see Definition 4), we will provide effective divisors $D_1, \ldots, D_{\binom{\gamma-2}{2}}$ on $\mathbb{P}(\mathcal{E})$ along with integers $b_1, \ldots, b_{\binom{\gamma-2}{2}}$, such that the following three conditions are satisfied:

- T' is the (scheme-theoretical) intersection of the D_ℓ's,
- $D_\ell \sim 2H - b_\ell R$ for all ℓ (where H is the hyperplane class and R is the class of a fibre in $\operatorname{Pic}(\mathbb{P}(\mathcal{E}))$), and,
- $\sum_{\ell=1}^{\binom{\gamma-2}{2}} b_\ell = (\gamma - 4)g - (\gamma^2 - 3\gamma) + \sharp(\partial\Delta^{(1)} \cap \mathbb{Z}^2)$.

In what follows, we will also assume that $e_1 > 0$, so that $\mathbb{P}(\mathcal{E}) \cong S$. This condition is not essential (see Remark 9), but it allows us to work with the inclusion $T \subset S$ rather than $T' \subset \mathbb{P}(\mathcal{E})$. For convenience, we will use the notation D_{j_1,j_2} for the the divisors, where $j_1, j_2 \in \{1, \ldots, \gamma - 1\}$ such that $j_2 - j_1 \geq 2$, and denote the corresponding invariants by B_{j_1,j_2}. Below, we will first introduce divisors $Y_{j_1,j_2,r}$ of S. Afterwards (see Definition 6), we will define the divisors D_{j_1,j_2} by means of the divisors $Y_{j_1,j_2,r}$.

For each $j_1, j_2 \in \{1, \ldots, \gamma - 1\}$ such that $j_2 - j_1 \geq 2$ and $1 \leq r \leq \frac{j_2 - j_1}{2}$, let $Y_{j_1,j_2,r} \subset S$ be the subvariety defined by the binomials of $\mathcal{I}_2(\operatorname{Tor}(\Delta^{(1)}))$ having the form

$$X_{i_1,j_1} X_{i_2,j_2} - X_{i'_1,j_1+r} X_{i'_2,j_2-r}.$$

One can see that $Y_{j_1,j_2,r}$ is a $(\gamma - 2)$-dimensional toric variety $\operatorname{Tor}(\Omega_{j_1,j_2,r})$, where $\Omega_{j_1,j_2,r} \subset \mathbb{R}^{\gamma-2}$ is a full-dimensional lattice polytope (see Example 3 for a tangible instance). The (Euclidean) volume of this polytope equals

$$\frac{1}{(\gamma - 2)!}(2(E_1 + \ldots + E_{\gamma-1}) - (E_{j_1} + E_{j_2} - \epsilon_{j_1,j_2,r})),$$

where $\epsilon_{j_1,j_2,r}$ is defined as $\epsilon^{(-)}_{j_1,j_2,r} + \epsilon^{(+)}_{j_1,j_2,r}$, with

$$\begin{aligned}\epsilon^{(-)}_{j_1,j_2,r} &= \begin{cases} 0 & \text{if} \quad i^{(-)}(j_1+r) + i^{(-)}(j_2-r) \leq i^{(-)}(j_1) + i^{(-)}(j_2) \\ 1 & \text{if} \quad i^{(-)}(j_1+r) + i^{(-)}(j_2-r) > i^{(-)}(j_1) + i^{(-)}(j_2) \end{cases} \\ &= \max\{0, (i^{(-)}(j_1+r) + i^{(-)}(j_2-r)) - (i^{(-)}(j_1) + i^{(-)}(j_2))\},\end{aligned}$$

and

$$\begin{aligned}\epsilon^{(+)}_{j_1,j_2,r} &= \begin{cases} 0 & \text{if} \quad i^{(+)}(j_1+r) + i^{(+)}(j_2-r) \geq i^{(+)}(j_1) + i^{(+)}(j_2) \\ 1 & \text{if} \quad i^{(+)}(j_1+r) + i^{(+)}(j_2-r) < i^{(+)}(j_1) + i^{(+)}(j_2) \end{cases} \\ &= \max\{0, (i^{(+)}(j_1) + i^{(+)}(j_2)) - (i^{(+)}(j_1+r) + i^{(+)}(j_2-r))\}.\end{aligned}$$

In the above equalities for $\epsilon^{(-)}_{j_1,j_2,r}$ and $\epsilon^{(+)}_{j_1,j_2,r}$, we use the following result.

Lemma 2. *The inequalities*

$$i^{(-)}(j_1+r) + i^{(-)}(j_2-r) \leq i^{(-)}(j_1) + i^{(-)}(j_2) + 1$$

and

$$i^{(+)}(j_1+r) + i^{(+)}(j_2-r) \geq i^{(+)}(j_1) + i^{(+)}(j_2) - 1$$

hold for all $j_1, j_2 \in \{1, \ldots, \gamma - 1\}$ *such that* $j_2 - j_1 \geq 2$ *and* $1 \leq r \leq \frac{j_2 - j_1}{2}$.

Proof. We only show the first inequality; the second one follows by symmetry. Consider the line segment $L = [(i^{(-)}(j_1), j_1), (i^{(-)}(j_2), j_2)]$, and let $(i', j_1 + r)$ and $(i'', j_2 - r)$ be the intersection points of L with the horizontal lines at heights $j_1 + r$ and $j_2 - r$. Note that L is contained in the interior of Δ and that $i' + i'' = i^{(-)}(j_1) + i^{(-)}(j_2)$. If $i^{(-)}(j_1+r) + i^{(-)}(j_2-r) \geq i^{(-)}(j_1) + i^{(-)}(j_2) + 2 = i' + i'' + 2$, then $i' \leq i^{(-)}(j_1+r) - 1$ or $i'' \leq i^{(-)}(j_2 - r) - 1$, so $(i^{(-)}(j_1+r) - 1, j_1 + r)$ or $(i^{(-)}(j_2 - r) - 1, j_2 - r)$ is a lattice point lying in the interior of Δ. This is in contradiction with the definition of $i^{(-)}(\cdot)$. ☐

Example 3. *Assume that* $\Delta = \Delta(f)$ *is as in Figure 1 (here* $\gamma = 5$*).*

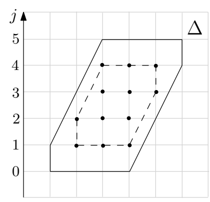


Figure 1: picture of Δ

Appropriate instances of $\Omega_{j_1,j_2,r}$ *can be realized as in Figure 2.*
Here, $\epsilon_{1,3,1} = 1$ *(since* $\epsilon^{(+)}_{1,3,1} = 1$*),* $\epsilon_{1,4,1} = 0$ *and* $\epsilon_{2,4,1} = 1$ *(since* $\epsilon^{(-)}_{2,4,1} = 1$*).*

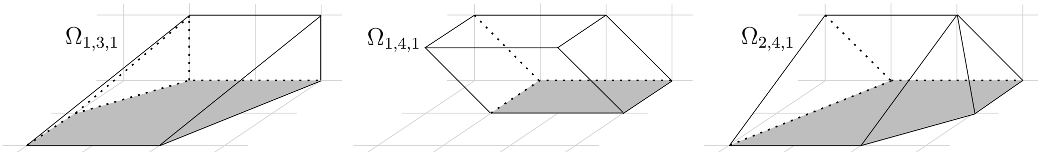


Figure 2: picture of the $\Omega_{j_1,j_2,r}$'s

The intersection of $Y_{j_1,j_2,r}$ with a typical fiber of $S \to \mathbb{P}^1$ is a quadratic hypersurface, hence there is a $B_{j_1,j_2,r} \in \mathbb{Z}$ such that $Y_{j_1,j_2,r} \sim 2H - B_{j_1,j_2,r}R$. Taking the intersection product of the latter equation with $H^{\gamma-2}$, we get

$$\begin{aligned} \deg Y_{j_1,j_2,r} = Y_{j_1,j_2,r} \cdot H^{\gamma-2} &= 2H^{\gamma-1} - B_{j_1,j_2,r}H^{\gamma-2}R \\ &= 2(e_1 + \ldots + e_{\gamma-1}) - B_{j_1,j_2,r} \\ &= 2(E_1 + \ldots + E_{\gamma-1}) - B_{j_1,j_2,r}, \end{aligned}$$

but $\deg Y_{j_1,j_2,r} = (\gamma-2)! \cdot \mathrm{Vol}(\Omega_{j_1,j_2,r})$, so $B_{j_1,j_2,r}$ equals $E_{j_1} + E_{j_2} - \epsilon_{j_1,j_2,r}$.

Write

$$\mathcal{S}^{(-)}_{j_1,j_2} = \left\{ r \in \left\{ 1, \ldots, \left\lfloor \frac{j_2 - j_1}{2} \right\rfloor \right\} \mid \epsilon^{(-)}_{j_1,j_2,r} = 0 \right\}$$

and

$$\mathcal{S}^{(+)}_{j_1,j_2} = \left\{ r \in \left\{ 1, \ldots, \left\lfloor \frac{j_2 - j_1}{2} \right\rfloor \right\} \mid \epsilon^{(+)}_{j_1,j_2,r} = 0 \right\}.$$

Definition 4. *We say that Δ satisfies condition $\mathcal{P}_1(v)$ if and only if there are no integers $j_1, j_2 \in \{1, \ldots, \gamma-1\}$ with $j_2 - j_1 \geq 2$ such that $\mathcal{S}^{(-)}_{j_1,j_2}$ and $\mathcal{S}^{(+)}_{j_1,j_2}$ are non-empty and disjoint.*

In other words, the condition $\mathcal{P}_1(v)$ means that for each pair of integers $j_1, j_2 \in \{1, \ldots, \gamma-1\}$ with $j_2 - j_1 \geq 2$ either at least one of the sets $\mathcal{S}^{(-)}_{j_1,j_2}$, $\mathcal{S}^{(+)}_{j_1,j_2}$ is empty, or there is a common $r \in \left\{1, \ldots, \left\lfloor \frac{j_2-j_1}{2} \right\rfloor\right\}$ for which $\epsilon^{(-)}_{j_1,j_2,r} = \epsilon^{(+)}_{j_1,j_2,r} = 0$. There is a useful criterion to check whether $\mathcal{S}^{(-)}_{j_1,j_2}$ is empty or not (and analogously for $\mathcal{S}^{(+)}_{j_1,j_2}$): $\mathcal{S}^{(-)}_{j_1,j_2} = \emptyset$ if and only if all the lattice points $(i^{(-)}(j), j)$ with $j_1 < j < j_2$ lie strictly right from the line segment $L = \left[(i^{(-)}(j_1), j_1), (i^{(-)}(j_2), j_2)\right]$.

In the above definition, we also allow the lattice direction v to be different from $(1,0)$: in that case, first take a unimodular transformation χ such that $\chi(v) = (1,0)$ and that $\chi(\Delta)$ is contained in the horizontal strip $0 \leq Y \leq \gamma$, and replace Δ by $\chi(\Delta)$ while checking the condition. The definition is independent of the particular choice of the unimodular transformation χ.

In fact, in some of the examples below, the lattice direction v is $(1,0)$, but Δ is contained in a horizontal strip of the form $k \leq Y \leq k + \gamma$ with $k \neq 0$. In that case, we do not really need to apply any unimodular transformation χ first: we can define the sets $\mathcal{S}^{(-)}_{j_1,j_2}$ and $\mathcal{S}^{(+)}_{j_1,j_2}$ for $j_1, j_2 \in \{k+1, \ldots, k+\gamma-1\}$.

Example 5. *Assume that a part of* $\Delta^{(1)}$ *looks as in Figure 3 (for some large enough* n*).*

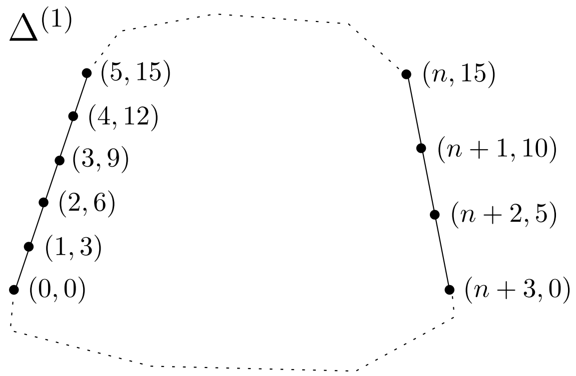


Figure 3: part of $\Delta^{(1)}$

In Table 1, the sets $\mathcal{S}^{(-)}_{j_1,j_2}$ *and* $\mathcal{S}^{(+)}_{j_1,j_2}$ *are given for all couples* (j_1, j_2) *with* $j_1 + j_2 = 15$ *in this part of the polytope* $\Delta^{(1)}$.

(j_1, j_2)	$\mathcal{S}^{(-)}_{j_1,j_2}$	$\mathcal{S}^{(+)}_{j_1,j_2}$
$(0,15)$	$\{3,6\}$	$\{5\}$
$(1,14)$	$\{1,2,3,4,5,6\}$	$\{1,2,3,4,5,6\}$
$(2,13)$	$\{1,2,3,4,5\}$	$\{1,2,3,4,5\}$
$(3,12)$	$\{3\}$	$\{1,2,3,4\}$
$(4,11)$	$\{1,2,3\}$	$\{1,2,3\}$
$(5,10)$	$\{1,2\}$	$\emptyset$
$(6,9)$	$\emptyset$	$\{1\}$

Table 1: table of subsets $\mathcal{S}^{(.)}_{j_1,j_2}$

We conclude that Δ *does not satisfy condition* $\mathcal{P}_1(v)$ *(consider* $j_1 = 0$ *and* $j_2 = 15$*).*

For all polygons Δ that satisfy condition $\mathcal{P}_1(v)$, we give a recipe to construct the divisors D_{j_1,j_2} in terms of the subvarieties $Y_{j_1,j_2,r}$.

Definition 6. *Assume that the lattice polygon* Δ *satisfies condition* $\mathcal{P}_1(v)$.

- *If* $\mathcal{S}^{(-)}_{j_1,j_2} \cap \mathcal{S}^{(+)}_{j_1,j_2} \neq \emptyset$, *we define* D_{j_1,j_2} *as* $Y_{j_1,j_2,r}$ *with* $r \in \mathcal{S}^{(-)}_{j_1,j_2} \cap \mathcal{S}^{(+)}_{j_1,j_2}$ *minimal. Set* $\epsilon_{j_1,j_2} = \epsilon_{j_1,j_2,r} = 0$.
- *If* $\mathcal{S}^{(-)}_{j_1,j_2} = \emptyset$ *and* $\mathcal{S}^{(+)}_{j_1,j_2} \neq \emptyset$ *or vice versa, take* $r \in \mathcal{S}^{(-)}_{j_1,j_2} \cup \mathcal{S}^{(+)}_{j_1,j_2}$ *minimal, define* $D_{j_1,j_2} = Y_{j_1,j_2,r}$ *and set* $\epsilon_{j_1,j_2} = \epsilon_{j_1,j_2,r} = 1$.

- *If* $\mathcal{S}^{(-)}_{j_1,j_2} = \mathcal{S}^{(+)}_{j_1,j_2} = \emptyset$, *define* $D_{j_1,j_2} = Y_{j_1,j_2,1}$ *and set* $\epsilon_{j_1,j_2} = \epsilon_{j_1,j_2,1} = 2$.

Remark 7. *In Definition 6, the divisor* D_{j_1,j_2} *is always of the form* $Y_{j_1,j_2,r}$ *and* r *is chosen such that* $\epsilon_{j_1,j_2,r}$ *is minimal, or equivalently,* $B_{j_1,j_2,r}$ *is maximal. Moreover, if* $D_{j_1,j_2} = Y_{j_1,j_2,r}$ *and if we define* $\epsilon^{(-)}_{j_1,j_2} = \epsilon^{(-)}_{j_1,j_2,r}$ *and* $\epsilon^{(+)}_{j_1,j_2} = \epsilon^{(+)}_{j_1,j_2,r}$, *then*

$$\epsilon^{(-)}_{j_1,j_2} = \min_s \epsilon^{(-)}_{j_1,j_2,s}, \quad \epsilon^{(+)}_{j_1,j_2} = \min_t \epsilon^{(+)}_{j_1,j_2,t} \quad \text{and} \quad \epsilon_{j_1,j_2} = \epsilon^{(-)}_{j_1,j_2} + \epsilon^{(+)}_{j_1,j_2}. \tag{6}$$

Here, it is crucial that Δ *satisfies condition* $\mathcal{P}_1(v)$: *if* $\mathcal{S}^{(-)}_{j_1,j_2}$ *and* $\mathcal{S}^{(+)}_{j_1,j_2}$ *were non-empty and disjoint, then* $\min_r \epsilon_{j_1,j_2,r} = 1$ *(take* $r \in \mathcal{S}^{(-)}_{j_1,j_2} \cup \mathcal{S}^{(+)}_{j_1,j_2}$*), but* $\min_s \epsilon^{(-)}_{j_1,j_2,s} = \min_t \epsilon^{(+)}_{j_1,j_2,t} = 0$.

If we set $B_{j_1,j_2} = E_{j_1} + E_{j_2} - \epsilon_{j_1,j_2}$, *we have that* $D_{j_1,j_2} \sim 2H - B_{j_1,j_2}R$ *and*

$$\begin{aligned} \textstyle\sum_{j_2-j_1\geq 2} B_{j_1,j_2} &= (\gamma-4)(E_1+\ldots+E_{\gamma-1}) + E_1 + E_{\gamma-1} - \textstyle\sum_{j_2-j_1\geq 2} \epsilon_{j_1,j_2} \\ &= (\gamma-4)(g-\gamma+1) + E_1 + E_{\gamma-1} - \textstyle\sum_{j_2-j_1\geq 2} \epsilon_{j_1,j_2}. \end{aligned}$$

Example 8. *If* $\partial\Delta^{(1)}$ *meets each horizontal line of height* $j \in \{2,\ldots,\gamma-2\}$ *in two lattice points, we have* $\epsilon_{j_1,j_2,r} = 0$ *and* $\mathcal{S}^{(-)}_{j_1,j_2} = \mathcal{S}^{(+)}_{j_1,j_2} = \left\{1,\ldots,\left\lfloor\frac{j_2-j_1}{2}\right\rfloor\right\}$ *for all* j_1, j_2, r. *Hence,* Δ *satisfies condition* $\mathcal{P}_1(v)$. *Moreover,* $\epsilon_{j_1,j_2} = 0$ *and* $D_{j_1,j_2} = Y_{j_1,j_2,1}$ *for all* j_1, j_2. *In this case,*

$$\sharp(\partial\Delta^{(1)} \cap \mathbb{Z}^2) = (E_1+1) + (E_{\gamma-1}+1) + 2(\gamma-3)$$

and $\sum \epsilon_{j_1,j_2} = 0$, *so*

$$\sum B_{j_1,j_2} = (\gamma-4)g - (\gamma^2-3\gamma) + \sharp(\partial\Delta^{(1)} \cap \mathbb{Z}^2).$$

Remark 9. *If* S *is singular, let* $\lambda > 0$ *be an integer and consider the inclusions from* (5). *Note that* $\Delta[\lambda]$ *satisfies condition* $\mathcal{P}_1(v)$ *if and only if* Δ *satisfies condition* $\mathcal{P}_1(v)$. *We can define the subvarieties* $Y_{j_1,j_2,r}$ *and* D_{j_1,j_2} *of* S' *in the same way as we did before (using* $\Delta[\lambda]$ *instead of* Δ*). Since* $H' = H + \lambda R$, *we get that*

$$Y_{j_1,j_2,r} \sim 2H' - ((E_{j_1}+\lambda) + (E_{j_2}+\lambda) - \epsilon_{j_1,j_2,r})R = 2H - B_{j_1,j_2,r}R$$

and $D_{j_1,j_2} \sim 2H - B_{j_1,j_2}R$.

We are now able to state and prove the main result of this subsection.

Theorem 10. *If* Δ *satisfies condition* $\mathcal{P}_1(v)$, *there exist* $\binom{\gamma-2}{2}$ *effective divisors* D_{j_1,j_2} *on* $\mathbb{P}(\mathcal{E})$ *(with* $j_1, j_2 \in \{1,\ldots,\gamma-1\}$ *and* $j_2 - j_1 \geq 2$*) such that*

- T' *is the (scheme-theoretical) intersection of the divisors* D_{j_1,j_2},

- $D_{j_1,j_2} \sim 2H - B_{j_1,j_2}R$ *for all* j_1, j_2*, where* $B_{j_1,j_2} = E_{j_1} + E_{j_2} - \epsilon_{j_1,j_2}$*, and,*
- $\sum_{j_2-j_1 \geq 2} B_{j_1,j_2} = (\gamma - 4)g - (\gamma^2 - 3\gamma) + \sharp(\partial\Delta^{(1)} \cap \mathbb{Z}^2)$.

Proof. By Remark 9, we may assume that S is smooth, hence $\mathbb{P}(\mathcal{E}) \cong S$. We need to prove that $\mathcal{I}(\mathrm{Tor}(\Delta^{(1)})) = \mathcal{I}(\bigcap D_{j_1,j_2})$, where the inclusion $\mathcal{I}(\bigcap D_{j_1,j_2}) \subset \mathcal{I}(\mathrm{Tor}(\Delta^{(1)}))$ is trivial. Pick an arbitrary quadratic binomial

$$f = X_{i_1,j_1}X_{i_2,j_2} - X_{i_3,j_3}X_{i_4,j_4} \in \mathcal{I}(\mathrm{Tor}(\Delta^{(1)})).$$

These binomials generate the ideal, so we only need to show that $f \in \mathcal{I}(\bigcap D_{j_1,j_2})$. Note that $j_1 + j_2 = j_3 + j_4$, so we may assume that $j_1 \leq j_3 \leq j_4 \leq j_2$. Moreover, if $j_1 = j_3$ and $j_4 = j_2$, we get that $f \in \mathcal{I}(S) \subset \mathcal{I}(\bigcap D_{j_1,j_2})$. So we may even assume that $j_1 < j_3$.

Take r such that $D_{j_1,j_2} = Y_{j_1,j_2,r}$. We claim that

$$I := i_1 + i_2 = i_3 + i_4 \geq i^{(-)}(j_1 + r) + i^{(-)}(j_2 - r).$$

If $\epsilon^{(-)}_{j_1,j_2} = \epsilon^{(-)}_{j_1,j_2,r} = 0$, this follows from

$$I \geq i^{(-)}(j_1) + i^{(-)}(j_2) \geq i^{(-)}(j_1 + r) + i^{(-)}(j_2 - r).$$

If $\epsilon^{(-)}_{j_1,j_2} = \epsilon^{(-)}_{j_1,j_2,r} = 1$, we have that $\epsilon^{(-)}_{j_1,j_2,j_3-j_1} = 1$ by (6) (since Δ satisfies condition $\mathcal{P}_1(v)$), hence

$$I \geq i^{(-)}(j_3) + i^{(-)}(j_4) = i^{(-)}(j_1 + r) + i^{(-)}(j_2 - r),$$

where we use Lemma 2. Analogously, we can show that $I \leq i^{(+)}(j_1+r)+i^{(+)}(j_2-r)$.

The above claim implies that we can find integers i'_1, i'_2 such that $i'_1 + i'_2 = I$, $i^{(-)}(j_1 + r) \leq i'_1 \leq i^{(+)}(j_1 + r)$, $i^{(-)}(j_2 - r) \leq i'_2 \leq i^{(+)}(j_2 - r)$, hence

$$X_{i_1,j_1}X_{i_2,j_2} - X_{i'_1,j_1+r}X_{i'_2,j_2-r} \in \mathcal{I}(D_{j_1,j_2}) = \mathcal{I}(Y_{j_1,j_2,r}).$$

So we may replace the term $X_{i_1,j_1}X_{i_2,j_2}$ in f by $X_{i'_1,j_1+r}X_{i'_2,j_2-r}$ (and in particular, j_1 by $j_1 + r$ and j_2 by $j_2 - r$). Continuing in this way, we will eventually get that $j_1 = j_3$ and $j_4 = j_2$, hence $f \in \mathcal{I}(S)$. This will happen after a finite number of steps since the maximum of $j_2 - j_1$ and $j_4 - j_3$ decreases after each step.

We are left with proving the formula for the sum of the B_{j_1,j_2}'s. By Remark 7 and the elaboration of Example 8, it suffices to show that the sum of the ϵ_{j_1,j_2} counts the number of times that $\partial\Delta^{(1)}$ intersects the horizontal lines of height $2, \ldots, \gamma - 2$ in a non-lattice point. Let $A^{(-)}$ be the set of couples (j_1, j_2) such that $j_1, j_2 \in \{1, \ldots, \gamma - 1\}$, $j_2 - j_1 \geq 2$ and $\mathcal{S}^{(-)}_{j_1,j_2} = \emptyset$ (or equivalently, the line segment $L = \big[(i^{(-)}(j_1), j_1), (i^{(-)}(j_2), j_2)\big]$ passes left from all the lattice points $(i^{(-)}(j'), j')$ with

$j_1 < j' < j_2$). Let $B^{(-)}$ be the set of integers $j \in \{1, \ldots, \gamma-1\}$ such that $(i^{(-)}(j), j) \notin \partial\Delta^{(1)}$. We claim that the sets $A^{(-)}$ and $B^{(-)}$ have the same cardinality. We will do this by giving a concrete bijection between these sets. Analogously, we can define the sets $A^{(+)}$ and $B^{(+)}$, and prove that they have the same number of elements. The theorem follows directly, since $\sharp(A^{(-)} \cup A^{(+)}) = \sum_{j_1,j_2} \epsilon_{j_1,j_2}$ by (6) and $\sharp(B^{(-)} \cup B^{(+)})$ is the number of non-lattice point intersections.

If $(j_1, j_2) \in A^{(-)}$, then the line segment $L = [(i^{(-)}(j_1), j_1), (i^{(-)}(j_2), j_2)]$ will pass at the left hand side of the lattice points $(i^{(-)}(j), j)$ with $j_1 < j < j_2$. For precisely one of these lattice points, the horizontal distance to L will be equal to the minimal value $\frac{1}{j_2 - j_1}$. Consider the map

$$\alpha^{(-)} : A^{(-)} \to B^{(-)}$$

sending the couple (j_1, j_2) to the value of j of that lattice point; see below for an example. On the other hand, if $j \in B^{(-)}$, thus $(i^{(-)}(j), j) \notin \partial\Delta^{(1)}$, then there should be lattice points $(i^{(-)}(j_1), j_1)$ and $(i^{(-)}(j_2), j_2)$ with $j_1 < j < j_2$ such that $L = [(i^{(-)}(j_1), j_1), (i^{(-)}(j_2), j_2)]$ passes left from $(i^{(-)}(j), j)$. If we take a couple (j_1, j_2) that satisfies this property and has a minimal value for $j_2 - j_1$, then $(j_1, j_2) \in A^{(-)}$. Indeed, if $(i^{(-)}(j'), j')$ with $j_1 < j' < j_2$ lies on or left from L, then either (j_1, j') or (j', j_2) would also satisfy the condition and would have a smaller value for the difference of the heights. Now let's show that the couple (j_1, j_2) is unique. If not, there exists another couple $(j_1', j_2') \in A^{(-)}$ with $j_1' < j < j_2'$ such that $L' = [(i^{(-)}(j_1'), j_1'), (i^{(-)}(j_2'), j_2')]$ passes left from $(i^{(-)}(j), j)$ with $j_2' - j_1' = j_2 - j_1$. We may assume that $j_1' < j_1 < j < j_2' < j_2$. Then L passes left from $(i^{(-)}(j_2'), j_2')$ and L' passes left from $(i^{(-)}(j_1), j_1)$, so $L'' = [(i^{(-)}(j_1'), j_1'), (i^{(-)}(j_2), j_2)]$ passes left from all the lattice points $(i^{(-)}(j'), j')$ with $j_1' < j' < j_2$ (see Figure 4). Let's denote

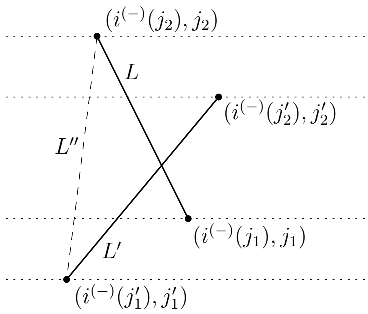


Figure 4: the line segments L, L' and L''

the horizontal distance from the line segment L'' to the lattice point $(i^{(-)}(j'), j')$ by $d(j')$. Using Lemma 2, we obtain that $d(j_1' + r) + d(j_2 - r) = 1$ for all $1 \leq r \leq \frac{j_2 - j_1'}{2}$. For precisely one integer $j_1' < j' < j_2$, the distance $d(j')$ is equal to the minimal value $\frac{1}{j_2 - j_1'}$. On the other hand, except for $j' = j_1$ or $j' = j_2'$, the distance $d(j')$ has to be at least $\frac{1}{j_2 - j_1} = \frac{1}{j_2' - j_1'}$, since $(i^{(-)}(j'), j')$ lies strictly right from L or L'. So we may assume that $j' = j_1$ (the case $j' = j_2'$ is analogous), hence $d(j_1) = \frac{1}{j_2 - j_1'}$ and $d(j_2') = 1 - \frac{1}{j_2 - j_1'}$ (using $r = j_1 - j_1'$ above). It follows that the horizontal distance to L'' from the point on L' on height j_1 is equal to

$$\frac{j_1 - j_1'}{j_2' - j_1'} \cdot d(j_2') = \frac{j_1 - j_1'}{j_2' - j_1'} \cdot \frac{j_2 - j_1' - 1}{j_2 - j_1'} \geq \frac{j_1 - j_1'}{j_2 - j_1'} \geq \frac{1}{j_2 - j_1'} = d(j_1),$$

so L' does not pass left from $(i^{(-)}(j_1), j_1)$, a contradiction. In conclusion we can consider the map

$$\beta^{(-)} : B^{(-)} \to A^{(-)}$$

sending j to the unique such couple (j_1, j_2).

The maps $\alpha^{(-)}$ and $\beta^{(-)}$ are inverse of each other. For instance, to prove that the map $\alpha^{(-)} \circ \beta^{(-)}$ is the identity map on $B^{(-)}$, consider $j \in B^{(-)}$ and write $\beta^{(-)}(j) = (j_1, j_2)$. If $\alpha^{(-)}(j_1, j_2) = j' \neq j$, then the horizontal distance from $(i^{(-)}(j), j)$ to $L = [(i^{(-)}(j_1), j_1), (i^{(-)}(j_2), j_2)]$ is of the form $\frac{d}{j_2 - j_1}$ with $1 < d < j_2 - j_1$. But then either $L' = [(i^{(-)}(j'), j'), (i^{(-)}(j_2), j_2)]$ (the case $j' < j$) or $L'' = [(i^{(-)}(j_1), j_1), (i^{(-)}(j'), j')]$ (the case $j' > j$) passes left from $(i^{(-)}(j), j)$. This is in contradiction with $\beta^{(-)}(j) = (j_1, j_2)$, since $j_2 - j'$ and $j' - j_1$ are both strictly smaller than $j_2 - j_1$. We leave the proof of the equality $\beta^{(-)} \circ \alpha^{(-)} = \mathrm{Id}_{A^{(-)}}$ as an exercise. □

Example 11. *Consider a polygon* Δ *of which a part of the boundary of* $\Delta^{(1)}$ *is as in Figure 5 (the* (i, j)*-coordinates are translated a bit).*

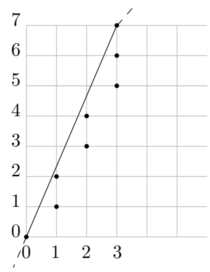


Figure 5: part of $\partial\Delta^{(1)}$

For this horizontal slice of the polygon,

$$A^{(-)} = \{(0,2), (0,7), (2,4), (2,7), (4,6), (4,7)\} \quad \textit{and} \quad B^{(-)} = \{1,2,3,4,5,6\}.$$

The map $\alpha^{(-)}$ *is defined as follows:*

$$(0,2) \mapsto 1 \ , \ (0,7) \mapsto 2 \ , \ (2,4) \mapsto 3 \ , \ (2,7) \mapsto 4 \ , \ (4,6) \mapsto 5 \ , \ (4,7) \mapsto 6.$$

One can show that the D_{j_1,j_2}'s in Theorem 10 can be used to resolve $\mathcal{O}_{T'}$ as an $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-module, following Schreyer [13]. For this one needs that the fibers of $\pi|_{T'} : T' \to \mathbb{P}^1$ have constant Betti numbers and that the corresponding resolutions are pure, but this can be verified. So it is justified to call the B_{j_1,j_2}'s the *first scrollar Betti numbers of the toric surface* $\mathrm{Tor}(\Delta^{(1)})$, even though we will not push this discussion further.

2.4 First scrollar Betti numbers of non-degenerate curves relative to the toric surface

We will use the same set-up and assumptions as in the beginning of Section 2.3. The assumption (i) implies that $\Delta^{(2)} \neq \emptyset$. Moreover, we also use the notations appearing in the inclusions (5), so C' and T' are the strict transforms of the canonically embedded Δ-non-degenerate curve C and the toric surface T under the resolution $\mu : S' \cong \mathbb{P}(\mathcal{E}) \to S$. In this section, we will present divisors on S' that scheme-theoretically cut out C' from T'.

Herefore, we rely on the following construction from [4]. Write

$$f = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} c_{i,j}x^iy^j \in k[x^{\pm1}, y^{\pm1}]$$

and consider $w \in \Delta^{(2)} \cap \mathbb{Z}^2$. For each $(i,j) \in \Delta \cap \mathbb{Z}^2$, there exist $u_{i,j}, v_{i,j} \in \Delta^{(1)} \cap \mathbb{Z}^2$ such that $(i,j) - w = (u_{i,j} - w) + (v_{i,j} - w)$. Hereby, we use that $\Delta + \Delta^{(2)} \subset 2\Delta^{(1)}$ and that the polygon $\Delta^{(1)}$ is normal. Then the quadrics

$$Q_w = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} c_{i,j}X_{u_{i,j}}X_{v_{i,j}} \in k[X_{i,j}]_{(i,j)\in\Delta'^{(1)}\cap\mathbb{Z}^2},$$

where w ranges over $\Delta^{(2)} \cap \mathbb{Z}^2$, scheme-theoretically cut out C from T.

In order to create the divisors D_ℓ, we will need an extra condition on Δ, which garantees that we can choose the lattice points $u_{i,j}, v_{i,j}$ in a particular way.

Definition 12. *We say that Δ satisfies condition $\mathcal{P}_2(v)$ if for each lattice point (i,j) of Δ and each horizontal line L, there exist two (not necessarily distinct) horizontal lines M_1, M_2, such that for all $w \in L \cap \Delta^{(2)} \cap \mathbb{Z}^2$, there exist $u_{i,j} \in M_1 \cap \Delta^{(1)} \cap \mathbb{Z}^2$ and $v_{i,j} \in M_2 \cap \Delta^{(1)} \cap \mathbb{Z}^2$ (dependent on (i,j) and w) such that*

$$(i,j) - w = (u_{i,j} - w) + (v_{i,j} - w).$$

Remark 13. *Write*

$$i^{(--)}(j) = \min\{i \in \mathbb{Z} \,|\, (i,j) \in \Delta^{(2)}\} \text{ and } i^{(++)}(j) = \max\{i \in \mathbb{Z} \,|\, (i,j) \in \Delta^{(2)}\}$$

for all $j \in \{2, \ldots, \gamma-2\}$. *An equivalent definition is as follows:* Δ *satisfies condition* $\mathcal{P}_2(v)$ *if and only if for all* $(i,j) \in \Delta$ *and for all* $j' \in \{2, \ldots, \gamma-2\}$, *there exist* $j_1, j_2 \in \{1, \ldots, \gamma-1\}$ *such that* $j_1 + j_2 = j + j'$ *and*

$$i + [i^{(--)}(j'), i^{(++)}(j')] \subset [i^{(-)}(j_1), i^{(+)}(j_1)] + [i^{(-)}(j_2), i^{(+)}(j_2)]. \qquad (7)$$

This condition is obviously satisfied for $(i,j) \in \Delta^{(1)}$ *(take* $j_1 = j$ *and* $j_2 = j'$*). Moreover, the condition also holds if* (i,j) *lies on the interior of a horizontal edge (i.e. the top or bottom edge) of* Δ. *Indeed, assume for instance that* (i,j) *lies in the interior of the top edge* $[(i^-, j), (i^+, j)]$ *of* Δ. *We have that*

$$i^{(-)}(j'+1) + i^{(-)}(j-1) \leq i^{(--)}(j') + i^- + 1 \leq i^{(--)}(j') + i.$$

Hereby, the first inequality follows by replacing L *in the proof of Lemma 2 by the half-closed line segment* $[(i^{(--)}(j'), j'), (i^-, j)[$. *Analogously, we get that*

$$i^{(+)}(j'+1) + i^{(+)}(j-1) \geq i^{(++)}(j') + i,$$

so (7) *follows for* $j_1 = j' + 1$ *and* $j_2 = j - 1$.

Although at first sight the condition $\mathcal{P}_2(v)$ might seem strong, it is not so easy to cook up instances of lattice polygons Δ for which the condition is not satisfied. The smallest example we have found is a polygon with 46 interior lattice points and lattice width 10.

Example 14. *Let* Δ *be as in Figure 6 (the dashed line indicates* $\Delta^{(1)}$*).*

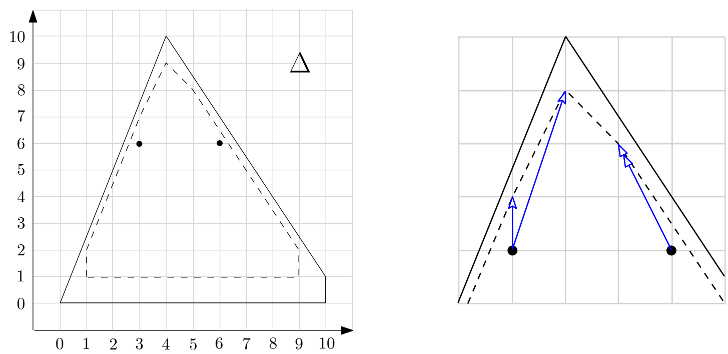


Figure 6: A lattice polygon Δ that does not satisfy condition $\mathcal{P}_2(v)$

We claim that Δ *does not satisfy condition* $\mathcal{P}_2(v)$. *Indeed, take the top vertex* $(i,j) = (4,10)$ *of* Δ *and the horizontal line* L *at height* 6. *For the point* $w \in L \cap \Delta^{(2)} \cap \mathbb{Z}^2$, *consider the bold-marked lattice points* $(3,6)$ *and* $(6,6)$ *on* L. *In both cases, there is a unique decomposition of* $(i,j) - w$:

$$(1,4) = (0,1) + (1,3) \qquad \textit{resp.} \qquad (-2,4) = (-1,2) + (-1,2).$$

So one sees that it is impossible to take the $u_{i,j}$'s and/or the $v_{i,j}$'s on the same line, which proves the claim.

Theorem 15. *If Δ satisfies condition $\mathcal{P}_2(v)$, then there exist $\gamma-3$ effective divisors D_ℓ on $\mathbb{P}(\mathcal{E})$ (with $2 \leq \ell \leq \gamma-2$) such that*

- *C' is the (scheme-theoretical) intersection of T' and the divisors D_ℓ,*
- *$D_\ell \sim 2H - B_\ell R$ for all ℓ, where*

$$B_\ell = i^{(++)}(\ell) - i^{(--)}(\ell) = -1 + \sharp\{(i,j) \in \Delta^{(2)} \cap \mathbb{Z}^2 \,|\, j = \ell\},$$

so

$$\sum_{2\leq \ell \leq \gamma-2} B_\ell = \sharp(\Delta^{(2)} \cap \mathbb{Z}^2) - (\gamma - 3).$$

Proof. The formula for the sum $\sum_\ell B_\ell$ is easily verified, so we focus on the other assertions. Take $\lambda \geq 0$ so that $S' = S(e_1+\lambda, \ldots, e_{\gamma-1}+\lambda)$ is smooth (and isomorphic to $\mathbb{P}(\mathcal{E})$) and define $\Delta' = \Delta[\lambda]$. We are going to use the inclusions

$$C' \subset T' \subset S' \subset \mathbb{P}^{g-1+\lambda(\gamma-1)}$$

from (5), where $T' = \mathrm{Tor}(\Delta'^{(1)})$. Write $(X_{i,j})_{(i,j)\in\Delta'^{(1)}\cap\mathbb{Z}^2}$ for the projective coordinates on $\mathbb{P}^{g-1+\lambda(\gamma-1)}$.

Let $\ell \in \{2, \ldots, \gamma-2\}$ and denote the lattice points of $\Delta[2\lambda]^{(2)}$ of height ℓ by $w_0, \ldots, w_{B_\ell+2\lambda}$. If $w \in \{w_0, \ldots, w_{B_\ell+2\lambda}\}$ and $(i,j) \in \Delta$, then we claim that we can find $u_{i,j}, v_{i,j} \in \Delta'^{(1)}$ such that $(i,j) - w = (u_{i,j} - w) + (v_{i,j} - w)$, in such a way that their second coordinates are independent from w. Indeed, since Δ satisfies condition $\mathcal{P}_2(v)$, there exist $j_1, j_2 \in \{1, \ldots, \gamma-1\}$ such that $j_1 + j_2 = j + \ell$ and

$$i + [i^{(--)}(\ell), i^{(++)}(\ell)] \subset [i^{(-)}(j_1), i^{(+)}(j_1)] + [i^{(-)}(j_2), i^{(+)}(j_2)],$$

hence

$$i + [i^{(--)}(\ell), i^{(++)}(\ell) + 2\lambda] \subset [i^{(-)}(j_1), i^{(+)}(j_1) + \lambda] + [i^{(-)}(j_2), i^{(+)}(j_2) + \lambda].$$

This implies that we can take $u_{i,j}$ and $v_{i,j}$ with second coordinates j_1 and j_2. Define

$$Q_w = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} c_{i,j} X_{u_{i,j}} X_{v_{i,j}} \in k[X_{i,j}]_{(i,j)\in\Delta'^{(1)}\cap\mathbb{Z}^2}.$$

A consequence of the choice of $u_{i,j}, v_{i,j}$ is that

$$X_{w_s} Q_{w_{r+1}} - X_{w_{s+1}} Q_{w_r} \in \mathcal{I}(S')$$

(rather than just $\mathcal{I}(T')$) for all $r \in \{0, \ldots, B_\ell + 2\lambda - 1\}$ and $s \in \{0, \ldots, B_\ell + \lambda - 1\}$. Since

$$\frac{X_{w_1}}{X_{w_0}} = \frac{X_{w_2}}{X_{w_1}} = \ldots = \frac{X_{w_{B_\ell+\lambda}}}{X_{w_{B_\ell+\lambda-1}}}$$

is a local parameter for the $(\gamma - 2)$-plane $R_{(0:1)} = \pi^{-1}(0:1) \subset S'$, it follows that the $R_{(0:1)}$-orders of

$$\mathcal{Z}(Q_{w_1}), \mathcal{Z}(Q_{w_2}), \ldots, \mathcal{Z}(Q_{w_{B_\ell+2\lambda}}) \tag{8}$$

increase by 1 at each step. For a similar reason, with $R_{(1:0)} = \pi^{-1}(1:0) \subset S'$, the $R_{(1:0)}$-orders of (8) decrease by 1 at each step. We conclude that there exists an effective divisor D_ℓ such that for all $i \in \{0, \ldots, B_\ell + 2\lambda\}$ we have

$$\mathcal{Z}(Q_{w_i}) \,=\, i \cdot R_{(0:1)} \,+\, (B_\ell + 2\lambda - i) \cdot R_{(1:0)} \,+\, D_\ell \tag{9}$$

on S'. The divisor D_ℓ is in fact the divisor of S' cut out by the quadrics in (8). Using (9) and Remark 9, we get that

$$D_\ell \sim 2H' - (B_\ell + 2\lambda)R = 2H - B_\ell R,$$

so it is sufficient to show that the quadrics Q_w (where w ranges over $\Delta[2\lambda]^{(2)} \cap \mathbb{Z}^2$) cut out C' from T'. If $\lambda = 0$, this follows from [4, Theorem 3.3].

Before we prove this, we need to introduce one more notion: for each lattice polygon Γ with two-dimensional $\Gamma^{(1)}$, write Γ^{max} to denote the largest lattice polygon with interior lattice polygon equal to $\Gamma^{(1)}$, so $\Gamma^{max} \supset \Gamma$. The polygon Γ^{max} can be constructed as follows. Let $v_1, \ldots, v_r$ be the primitive inward pointing normal vectors of the edges of $\Gamma^{(1)}$ and write $\Gamma^{(1)}$ as an intersection $\cap_{t=1}^r H_t$ of half-planes

$$H_t = \{P \in \mathbb{R}^2 \,|\, \langle P, v_t \rangle \geq a_t\}$$

(where $\langle \cdot, \cdot \rangle$ denotes the standard inner product and $a_t \in \mathbb{Z}$). Then

$$\Gamma^{max} = \cap_{t=1}^r H_t^{(-1)} \quad \text{with} \quad H_t^{(-1)} = \{P \in \mathbb{R}^2 \,|\, \langle P, v_t \rangle \geq a_t - 1\}.$$

We will use the following two properties (see [3, Section 2] for other properties of Γ^{max}):

- If $\Gamma^{(2)} \neq \emptyset$, then $2\Gamma^{(1)} = \Gamma^{(2)} + \Gamma^{max}$, since both lattice polygons are defined by the half-planes $2H_t = \{P \in \mathbb{R}^2 \,|\, \langle P, v_t \rangle \geq -2a_t\}$.

- If Φ_1, Φ_2 are lattice polygons such that $\Phi_1 + \Gamma \subset \Phi_2 + \Gamma^{max}$, then $\Phi_1 \subset \Phi_2$ if Φ_2 satisfies the following condition: it is the intersection of half-planes H_t' with H_t' of the form

$$\{P \in \mathbb{R}^2 \,|\, \langle P, v_t \rangle \geq b_t\}$$

for some $b_t \in \mathbb{Z}$ (hence parallel to H_t).

Indeed, if $\Phi_1 \not\subset \Phi_2$, take a lattice point P in $\Phi_1 \setminus \Phi_2$. Then $\langle P, v_t \rangle < b_t$ for some value of t. Take $Q \in \Gamma$ with $\langle Q, v_t \rangle = a_t - 1$ (this is always possible). We have that $P + Q \in \Phi_1 + \Gamma$ and $\langle P + Q, v_t \rangle < a_t + b_t - 1$, but $\Phi_2 + \Gamma^{max}$ is the intersection of the half-planes $H''_t = \{P \in \mathbb{R}^2 \mid \langle P, v_t \rangle \geq a_t + b_t - 1\}$, so $P + Q \notin \Phi_2 + \Gamma^{max}$, a contradiction.

Now take $F \in \mathcal{I}(C')$ homogeneous of degree d and let

$$\xi : k[X_{i,j}]_{(i,j)\in\Delta'^{(1)}\cap\mathbb{Z}^2} \to k[x^{\pm 1}, y^{\pm 1}]$$

be the ring morphism that maps $X_{i,j}$ to $x^i y^j$. Since $\xi(F)(x, y) = 0$ for all $(x, y) \in \mathbb{T}^2$ with $f(x, y) = 0$ (and f is irreducible), the Laurent polynomial $\xi(F)$ has to be of the form cf for some $c \in k[x^{\pm 1}, y^{\pm 1}]$. The Newton polygon of cf is equal to $\Delta(c) + \Delta$, while the Newton polygon $\Delta(\xi(F))$ is contained in

$$d\Delta'^{(1)} = (d-2)\Delta'^{(1)} + \Delta'^{(2)} + \Delta'^{max} = (d-2)\Delta'^{(1)} + \Delta[2\lambda]^{(2)} + \Delta^{max}$$

(here, we use the first property of maximal polygons with $\Gamma = \Delta'$). So we obtain that

$$\Delta(c) + \Delta \subset (d-2)\Delta'^{(1)} + \Delta[2\lambda]^{(2)} + \Delta^{max}.$$

Now we can use the second property of maximal polygons with $\Phi_1 = \Delta(c)$, $\Phi_2 = (d-2)\Delta'^{(1)} + \Delta[2\lambda]^{(2)}$ and $\Gamma = \Delta$. Note that Φ_2 might have a horizontal (top or bottom) edge while $\Delta^{(1)}$ has not, but this is not an issue (since $\Delta^{(1)} \not\cong k\Sigma$). It follows that

$$\Delta(c) \subset (d-2)\Delta'^{(1)} + \Delta[2\lambda]^{(2)}.$$

So we can write

$$c = \sum_{w=(i,j)\in\Delta[2\lambda]^{(2)}\cap\mathbb{Z}^2} g_{i,j} x^i y^j$$

for polynomials $g_{i,j} \in k[x, y]$ with $\Delta(g_{i,j}) \subset (d-2)\Delta'^{(1)}$. For all lattice points $w = (i, j) \in \Delta[2\lambda]^{(2)} \cap \mathbb{Z}^2$, there is a homogeneous polynomial $G_{i,j} \in k[X_{i,j}]_{(i,j)\in\Delta'^{(1)}\cap\mathbb{Z}^2}$ such that $\xi(G_{i,j}) = g_{i,j}$. On the other hand, $\xi(Q_w) = x^i y^j f$, hence

$$\xi(F) = cf = \sum_{w=(i,j)\in\Delta[2\lambda]^{(2)}\cap\mathbb{Z}^2} \xi(G_{i,j})\xi(Q_w).$$

So $F - \sum_{w=(i,j)} G_{i,j} Q_w$ belongs to the kernel of the map ξ, which implies that it is contained in $\mathcal{I}_d(T')$, which is what we wanted to prove. □

2.5 First scrollar Betti numbers for non-degenerate curves

We are ready to prove the main result of this section, by combining the results from Sections 2.3 and 2.4.

Theorem 16. *Let* Δ *be a lattice polygon with* $lw(\Delta) \geq 4$ *such that* $\Delta^{(1)} \not\cong \Upsilon$ *and* $\Delta^{(1)} \not\cong k\Sigma$ *for any integer* k. *Assume that* Δ *satisfies the conditions* $\mathcal{P}_1(v)$ *and* $\mathcal{P}_2(v)$, *where* v *is a lattice-width direction. Let* C *be a* Δ*-non-degenerate curve and let* g^1_γ *be the combinatorial gonality pencil on* C *corresponding to* v *(with* $\gamma = lw(\Delta)$*). Then the first scrollar Betti numbers of* C *with respect to* g^1_γ *are given by*

$$\{B_\ell\}_{\ell \in \{2,\ldots,\gamma-2\}} \cup \{B_{j_1,j_2}\}_{\substack{j_1,j_2 \in \{1,\ldots,\gamma-1\} \\ j_2 - j_1 \geq 2}}.$$

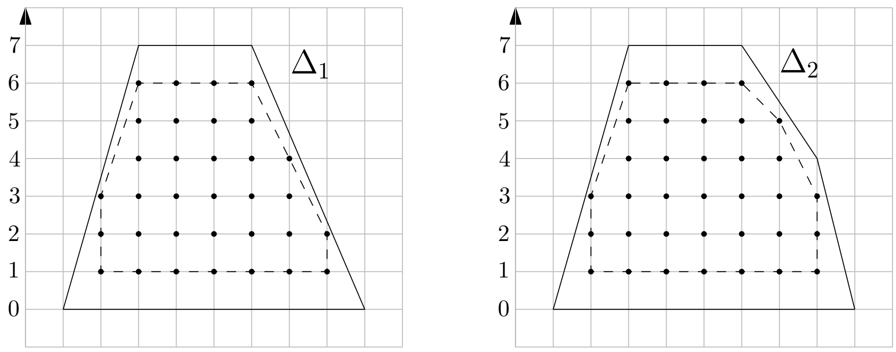
Proof. We use the notations and set-up from Section 2.3. Theorem 10 and Theorem 15 imply that there exist divisors $D_\ell \sim 2H - B_\ell R$ on $\mathbb{P}(\mathcal{E})$, with $\ell \in \{2, \ldots, \gamma - 2\}$, and divisors $D_{j_1,j_2} \sim 2H - B_{j_1,j_2} R$ on $\mathbb{P}(\mathcal{E})$, with $j_1, j_2 \in \{1, \ldots, \gamma - 1\}$ and $j_2 - j_1 \geq 2$, such that C' is the scheme-theoretical intersection of these divisors. So we can use Theorem 1 to conclude the proof. Note that indeed

$$\sum_{\ell \in \{2,\ldots,\gamma-2\}} B_\ell + \sum_{\substack{j_1,j_2 \in \{1,\ldots,\gamma-1\} \\ j_2 - j_1 \geq 2}} B_{j_1,j_2} = (\gamma - 3)(g - \gamma - 1),$$

as announced in the theorem. □

We believe that the above theorem is of independent interest. For instance it is not well-understood which sets of (first) scrollar Betti numbers are possible for canonical curves of a given genus and gonality, and our result can be used to prove certain existence results. It has been conjectured that "most" curves have so-called balanced (first) scrollar Betti numbers, meaning that $\max |b_i - b_j| \leq 1$, see [1] and the references therein. Non-degenerate curves are typically highly non-balanced, since one expects the B_{j_1,j_2}'s to be about twice the B_ℓ's.

Example 17. *Consider the following lattice polygons* Δ_1 *and* Δ_2 *of lattice width 7 (and lattice-width direction* $v = (1, 0)$*), which only differ from each other at the right hand side.*



The polygon Δ_1 does not satisfy all the combinatorial constraints of Theorem 16, since condition $\mathcal{P}_2(v)$ does not hold: $\mathcal{S}_{2,6}^{(-)} = \{1\}$ and $\mathcal{S}_{2,6}^{(+)} = \{2\}$. Although we have not pursued this, we believe that the conditions $\mathcal{P}_1(v)$ and $\mathcal{P}_2(v)$ are always fulfilled if $\gamma < 7$.

On the other hand, the polygon Δ_2 meets all the conditions from the statement, and so we can apply Theorem 16. The first scrollar Betti numbers of a Δ_2-degenerate curve are as follows:

$$\begin{array}{lllll} B_{1,3} = 12 & B_{1,4} = 10 & B_{1,5} = 10 & B_{1,6} = 9 & B_{2,4} = 10 \\ B_{2,5} = 10 & B_{2,6} = 9 & B_{3,5} = 8 & B_{3,6} = 8 & B_{4,6} = 7 \\ B_2 = 4 & B_3 = 4 & B_4 = 4 & B_5 = 3 & \end{array}$$

The sum of these numbers is 108, which agrees with $(\gamma - 3)(g - \gamma - 1)$ for $g = 35$ and $\gamma = 7$.

3 Intrinsicness on $\mathbb{P}^1 \times \mathbb{P}^1$

Theorem 18. *Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its (two-dimensional) Newton polygon $\Delta = \Delta(f)$, and assume that $\Delta \not\cong 2\Upsilon$. Then $U(f)$ is birationally equivalent to a smooth projective genus g curve in $\mathbb{P}^1 \times \mathbb{P}^1$ if and only if $\Delta^{(1)} = \emptyset$ or $\Delta^{(1)} \cong \square_{a,b}$ for some integers $a \geq b \geq 0$, necessarily satisfying $g = (a+1)(b+1)$.*

Proof. We may assume that $U(f)$ is neither rational, nor elliptic or hyperelliptic (and hence that $\Delta^{(1)}$ is two-dimensional) because such curves admit smooth complete models in $\mathbb{P}^1 \times \mathbb{P}^1$. So for the 'if' part we can assume that $b \geq 1$. But then $\operatorname{Tor}(\Delta^{(1)}) \cong \mathbb{P}^1 \times \mathbb{P}^1$, and the statement follows using the canonical embedding (3).

The real deal is the 'only if' part. At least, if a curve C/k is birationally equivalent to a (non-rational, non-elliptic, non-hyperelliptic) smooth projective curve in $\mathbb{P}^1 \times \mathbb{P}^1$, then it is Δ'-non-degenerate with $\Delta' = [-1, a+1] \times [-1, b+1]$ for $a \geq b \geq 1$: this follows from the material in [3, Section 4] (one can use an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ to ensure appropriate behavior with respect to the toric boundary). Note that $\Delta'^{(1)} = \square_{a,b}$. The geometric genus of C equals $g = (a+1)(b+1)$ by [10] and its gonality equals $\gamma = b + 2$ by [3, Cor. 6.2]. We observe that

- g is composite.
- C has Clifford dimension equal to 1 by [3, Theorem 8.1].
- the scrollar invariants of C (with respect to any gonality pencil) are all equal to $g/(\gamma-1)-1$. Indeed, by [3, Theorem 6.1], every gonality pencil is computed by projecting along some lattice-width direction v. If $a > b$, then the only pair

of lattice-width directions is $\pm(1,0)$ and from [3, Theorem 9.1], we find that the corresponding scrollar invariants are $a, a, \ldots, a$. If $a = b$, we also have the pair $\pm(0,1)$, giving rise to the same scrollar invariants.

- if $\gamma \geq 4$, then the first scrollar Betti numbers (with respect to any gonality pencil) take exactly two distinct values: $2g/(\gamma-1)-2$ and $g/(\gamma-1)-3$. Indeed, Δ' satisfies condition $\mathcal{P}_1(v)$ (see Example 8), but also condition $\mathcal{P}_2(v)$: take $(j_1, j_2) = (j, \ell)$ if $j \in \{-1, \ldots, b+1\}$, $(j_1, j_2) = (j+1, \ell-1)$ if $j = -1$ and $(j_1, j_2) = (j-1, \ell+1)$ if $j = b+1$. By Theorem 16 we find that these numbers are $2a, 2a, \ldots, 2a, a-2, a-2, \ldots, a-2$.

A first consequence is that $U(f)$ admits a combinatorial gonality pencil. Indeed, Δ cannot be of the form 2Υ (excluded in the statement of the theorem), nor of the form $d\Sigma$ for some $d \geq 2$: the cases $d = 2$ and $d = 3$ correspond to rational and elliptic curves (excluded at the beginning of this proof), the case $d = 4$ corresponds to curves of genus 3 (not composite), and the cases where $d \geq 5$ correspond to curves of Clifford dimension 2.

Without loss of generality we may then assume that $v = (1,0)$ and $\Delta \subset \{\,(i,j) \in \mathbb{R}^2 \,|\, 0 \leq j \leq \gamma\,\}$, so that our gonality pencil corresponds to the vertical projection. By [3, Theorem 9.1], the numbers $E_\ell = -1 + \sharp\{(i,j) \in \Delta^{(1)} \cap \mathbb{Z}^2 \,|\, j = \ell\}$ (for $\ell = 1, \ldots, \gamma-1$) are the corresponding scrollar invariants. Hence the E_ℓ's must all be equal to $E := g/(\gamma-1) - 1 \geq 1$.

This already puts severe restrictions on the possible shapes of $\Delta^{(1)}$. By horizontally shifting and skewing we may assume that the lattice points at height $j = 1$ are $(0,1), \ldots, (E,1)$ and that the lattice points at height $j = 2$ are $(0,2), \ldots, (E,2)$. If $\gamma = 3$, it follows that $\Delta^{(1)}$ has the desired rectangular shape, so we may suppose that $\gamma \geq 4$. Then by horizontally flipping if needed, we can assume that the lattice points at height $j = 3$ are $(i,3), \ldots, (E+i,3)$ for some $i \geq 0$. Now $i \geq 2$ is impossible, for this would introduce a new lattice point at height $j = 2$; thus $i = 0$ or $i = 1$. Continuing this type of reasoning, we obtain that the lattice points of $\Delta^{(1)}$ can be seen as a pile of n blocks of respectively $m_1, \ldots, m_n$ sheets, where each block is shifted to the right over a distance 1 when compared to its predecessor.

We need to show that $n = 1$, because then $\Delta^{(1)}$ has the desired rectangular shape. We will first prove that Δ statisfies condition $\mathcal{P}_1(v)$. Since $i^{(+)}(j) - i^{(-)}(j) = E$ for each value of j, the inequality

$$\epsilon^{(-)}_{j_1,j_2,r} + \epsilon^{(+)}_{j_1,j_2,r} \leq 1$$

holds (so never $\epsilon^{(-)}_{j_1,j_2,r} = \epsilon^{(+)}_{j_1,j_2,r} = 1$) for all $j_1, j_2 \in \{2, \ldots, \gamma-2\}$ with $j_2 - j_1 \geq 2$ and $r \in \left\{1, \ldots, \left\lfloor \frac{j_2-j_1}{2} \right\rfloor\right\}$. This implies that

$$\mathcal{S}^{(-)}_{j_1,j_2} \cup \mathcal{S}^{(+)}_{j_1,j_2} = \left\{1, \ldots, \left\lfloor \frac{j_2-j_1}{2} \right\rfloor\right\}.$$

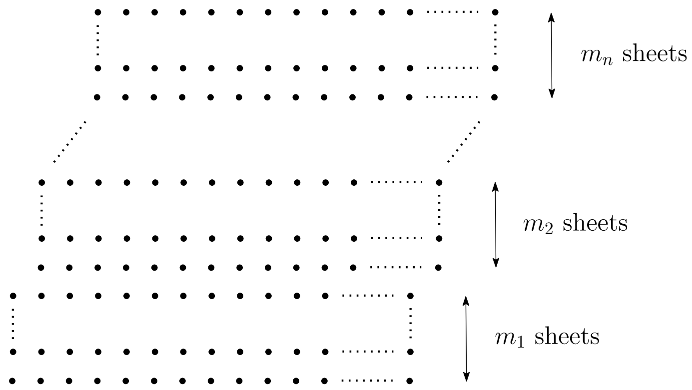


Figure 7: lattice points of $\Delta^{(1)}$ in sheets

Now assume that $\mathcal{S}^{(-)}_{j_1,j_2}$ and $\mathcal{S}^{(+)}_{j_1,j_2}$ are non-empty and disjoint. In this case, we can take $r, s \in \left\{1, \ldots, \left\lfloor \frac{j_2-j_1}{2} \right\rfloor\right\}$ such that

$$\epsilon^{(-)}_{j_1,j_2,r} = \epsilon^{(+)}_{j_1,j_2,s} = 0 \quad \text{and} \quad \epsilon^{(+)}_{j_1,j_2,r} = \epsilon^{(-)}_{j_1,j_2,s} = 1.$$

If $r < s$, we get that

$$i^{(-)}(j_1 + s) + i^{(-)}(j_2 - s) > i^{(-)}(j_1) + i^{(-)}(j_2)$$

and

$$i^{(+)}(j_1) + i^{(+)}(j_2) > i^{(+)}(j_1 + r) + i^{(+)}(j_2 - r).$$

Subtracting E from both sides of the latter equation yields

$$i^{(-)}(j_1) + i^{(-)}(j_2) > i^{(-)}(j_1 + r) + i^{(-)}(j_2 - r),$$

so

$$i^{(-)}((j_1 + r) + (s - r)) + i^{(-)}((j_2 - r) - (s - r)) \geq i^{(-)}(j_1 + r) + i^{(-)}(j_2 - r) + 2,$$

which is in contradiction with Lemma 2. A similar contradiction can be obtained if $r > s$.

Now let's prove that Δ also satisfies property $\mathcal{P}_2(v)$, where we assume that $n \geq 2$. By Remark 13 and a symmetry consideration (rotation over 180°), it suffices to check the condition for lattice points (i, j) that lie on the left side of the boundary of Δ (and even of Δ^{max}). Take $\ell \in \{2, \ldots, \gamma - 2\}$, $w = (i^{(--)}(\ell), \ell)$ and $u_{i,j} = (i_1, j_1), v_{i,j} = (i_2, j_2) \in \Delta^{(1)}$ such that $(i, j) - w = (u_{i,j} - w) + (v_{i,j} - w)$, hence $j_1 + j_2 = j + \ell$. It is sufficient to prove that

$$i + i^{(++)}(\ell) \leq i^{(+)}(j_1) + i^{(+)}(j_2). \tag{10}$$

First assume that $|j - \ell| > |j_2 - j_1|$. If $j \in \{1, \ldots, \gamma - 1\}$, then

$$\begin{aligned} (i+E) + i^{(++)}(\ell) &\leq (i^{(-)}(j) + E) + i^{(+)}(\ell) \\ &= i^{(+)}(j) + i^{(+)}(\ell) \\ &\leq i^{(+)}(j_1) + i^{(+)}(j_2) + 1, \end{aligned}$$

where we use Lemma 2 for the last inequality. Since $E \geq 1$, the desired inequality (10) follows. We still need to check (10) for points $(i, j) \in \partial\Delta$ with $j = 0$ and $j = \gamma$, in particular $i = -1$ resp. $i = n - 1$ because we can assume that (i, j) lies on the left side of the boundary of Δ^{max}.

- If $(i, j) = (-1, 0)$, the line segment L between $(i + E, j) = (E - 1, 0)$ and $w' = (i^{(++)}(\ell), \ell) \in \Delta^{(2)}$ intersects the horizontal lines of heights j_1 and j_2 in points that belong to $\Delta^{(1)}$. Using a similar argument as in the proof of Lemma 2, we obtain that $(i + E) + i^{(++)}(\ell) \leq i^{(+)}(j_1) + i^{(+)}(j_2) + 1$, which gives us (10) using $E \geq 1$.
- Analogously, we can handle the case $(i, j) = (n - 1, \gamma)$: the line segment L between $(i + E, j) = (n + E - 1, \gamma)$ and w' will intersect the horizontal lines of heights j_1 and j_2 in points that are contained in $\Delta^{(1)}$ and (10) follows.

If $|j - \ell| = |j_2 - j_1|$, we may assume that $j_1 = j \in \{1, \ldots, \gamma - 1\}$ and $j_2 = \ell \in \{2, \ldots, \gamma - 2\}$. But then the inequalities $i \leq i^{(-)}(j_1) \leq i^{(+)}(j_1)$ and $i^{(++)}(\ell) \leq i^{(+)}(j_2)$ yield (10).

We still have to consider the case where $|j - \ell| < |j_2 - j_1|$, which implies that $j \in \{1, \ldots, \gamma - 1\}$. By Lemma 2, we have that

$$i^{(-)}(j) + i^{(-)}(\ell) \leq i^{(-)}(j_1) + i^{(-)}(j_2) + 1,$$

hence

$$\begin{aligned} i + i^{(++)}(\ell) &\leq i^{(-)}(j) + i^{(+)}(\ell) \\ &= i^{(-)}(j) + i^{(-)}(\ell) + E \\ &\leq i^{(-)}(j_1) + i^{(-)}(j_2) + E + 1 \\ &\leq i^{(+)}(j_1) + i^{(+)}(j_2). \end{aligned}$$

Since both conditions $\mathcal{P}_1(v)$ and $\mathcal{P}_2(v)$ hold for Δ, we can apply Theorem 16. If $n \geq 2$, then there is at least one first scrollar Betti number having value $E - 1 = g/(\gamma - 1) - 2$, for instance B_2. This is distinct from both $2g/(\gamma - 1) - 2$ and $g/(\gamma - 1) - 3$: contradiction. Therefore $n = 1$, i.e. $\Delta^{(1)}$ has the requested rectangular shape. □

4 Open questions

Here are two interesting open questions related to this paper:

1. In Section 2, we gave a combinatorial interpretation for the first scrollar Betti numbers of Δ-non-degenerate curves C in terms of the combinatorics of Δ, in case Δ satisfies the condition $\mathcal{P}_1(v)$ (see Definition 4) and $\mathcal{P}_2(v)$ (see Definition 12). Can this be generalized to all polygons Δ? There seems to be no geometric reason why this wouldn't be the case, but we did not succeed to get rid of the conditions.

2. In Theorem 18 of Section 3, we showed that non-degenerate curves on $\mathbb{P}^1 \times \mathbb{P}^1$ have an intrinsic Newton polygon (at least, if $g \neq 4$). Can this be generalized to Δ-non-degenerate curves on Hirzebruch surfaces $\mathcal{H}_n$? In this case, we expect $\Delta^{(1)} = \emptyset$ or $\Delta^{(1)} \cong \text{conv}\{(0,0), (a+nb, 0), (a, b), (0, b)\}$ for some integers $a, b, n \geq 0$.

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CURVES IN CHARACTERISTIC 2 WITH NON-TRIVIAL 2-TORSION

WOUTER CASTRYCK, MARCO STRENG, DAMIANO TESTA

Abstract. Cais, Ellenberg and Zureick-Brown recently observed that over finite fields of characteristic two, all sufficiently general smooth plane projective curves of a given odd degree admit a non-trivial rational 2-torsion point on their Jacobian. We extend their observation to curves given by Laurent polynomials with a fixed Newton polygon, provided that the polygon satisfies a certain combinatorial property. We also show that in each of these cases, the sufficiently general condition is implied by being ordinary. Our treatment includes many classical families, such as hyperelliptic curves of odd genus and $C_{a,b}$ curves. In the hyperelliptic case, we provide alternative proofs using an explicit description of the 2-torsion subgroup.

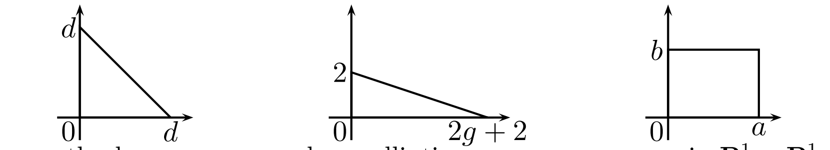
1. Introduction

The starting point of this article is a recent theorem by Cais, Ellenberg and Zureick-Brown [CEZB, Thm. 4.2], asserting that over a finite field k of characteristic 2, almost all smooth plane projective curves of a given odd degree $d \geq 3$ have a non-trivial k-rational 2-torsion point on their Jacobian. Here, 'almost all' means that the corresponding proportion converges to 1 as $\#k$ and/or d tend to infinity. The underlying observation is that such curves admit

- a 'geometric' k-rational half-canonical divisor Θ_{geom}: the canonical class of a smooth plane projective curve of degree d equals $(d-3)H$, where H is the class of hyperplane sections; if d is odd then $\frac{1}{2}(d-3)H$ is half-canonical,
- an 'arithmetic' k-rational half-canonical divisor Θ_{arith} (whose class is sometimes called the *canonical theta characteristic*), related to the fact that over a perfect field of characteristic 2, the derivative of a Laurent series is always a square [Mum, p. 191].

The difference $\Theta_{\text{geom}} - \Theta_{\text{arith}}$ maps to a k-rational 2-torsion point on the Jacobian. The proof of [CEZB, Thm. 4.2] then amounts to showing that, quite remarkably, this point is almost always non-trivial.

There exist many classical families of curves admitting such a geometric half-canonical divisor. Examples include hyperelliptic curves of odd genus g, whose canonical class is given by $(g-1)g^1_2$ (where g^1_2 denotes the hyperelliptic pencil), and smooth projective curves in $\mathbf{P}^1_k \times \mathbf{P}^1_k$ of even bidegree (a, b) (both a and b even, that is), where the canonical class reads $(a-2)R_1 + (b-2)R_2$ (here R_1, R_2 are the two rulings of $\mathbf{P}^1_k \times \mathbf{P}^1_k$). The families mentioned so far are parameterized by sufficiently generic polynomials that are supported on the polygons



smooth plane curves of degree d — hyperelliptic curves of genus g — curves in $\mathbf{P}^1_k \times \mathbf{P}^1_k$ of bidegree (a, b),

respectively. The following lemma, which is an easy consequence of the theory of toric surfaces (see Section 2), gives a purely combinatorial reason for the existence of a half-canonical divisor in these cases.

Lemma 1. *Let k be a perfect field and let Δ be a two-dimensional lattice polygon. For each edge $\tau \subset \Delta$, let $a_\tau X + b_\tau Y = c_\tau$ be its supporting line, where $\gcd(a_\tau, b_\tau) = 1$. Suppose that the system of congruences*

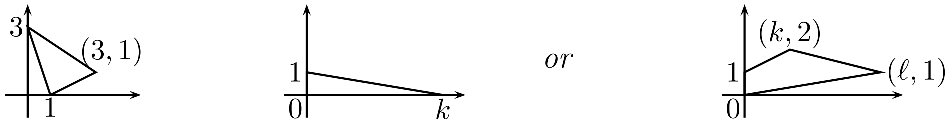
$$\{\ a_\tau X + b_\tau Y \equiv c_\tau + 1 \pmod 2\ \}_{\tau \text{ edge of } \Delta} \tag{1}$$

admits a solution in $\mathbf{Z}^2$. Then any sufficiently general Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$ that is supported on Δ defines a curve carrying a k-rational half-canonical divisor on its non-singular complete model.

In the proof of Lemma 1 below, where we describe this half-canonical divisor explicitly, we will be more precise on the meaning of 'sufficiently general'.

Here again, when specializing to characteristic 2, there is, in addition, an arithmetic k-rational half-canonical divisor. So it is natural to wonder whether the proof of [CEZB, Thm. 4.2] still applies in these cases. We will show that it usually does.

Theorem 2. *Let Δ be a two-dimensional lattice polygon satisfying the conditions of Lemma 1, where in addition we assume that Δ is not unimodularly equivalent to*



for some $k \geq 1$ — *for some $0 \leq k < \ell \geq 3$ with k even and ℓ odd.*

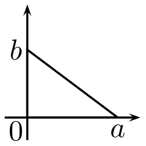
Then there exists a non-empty Zariski open subset $S_\Delta/\mathbf{F}_2$ of the space of Laurent polynomials that are supported on Δ having the following property. For every perfect field k of characteristic 2 and every $f \in S_\Delta(k)$, the Jacobian of the non-singular complete model of the curve defined by f has a non-trivial k-rational 2-torsion point.

(Right before the proof of Theorem 2 we will define the set S_Δ explicitly.) As a consequence, if k is a finite field of characteristic 2, then the proportion of Laurent polynomials that are supported on Δ, which define a curve whose Jacobian has a non-trivial k-rational 2-torsion point, tends to 1 as $\#k \to \infty$. See the end of Section 3, where we also discuss asymptotics for increasing dilations of Δ, i.e. the analogue of $d \to \infty$ in the smooth plane curve case. In Section 4 we give sufficient conditions that have a more arithmetic flavor, involving the rank of the Hasse-Witt matrix.

These observations seem new even for hyperelliptic curves of odd genus[1] (even though this is a well-known fact for the subfamily of hyperelliptic curves having a prescribed

[1] In view of the asymptotic consequences discussed in Section 3, this observation shows that [CFHS, Principle 3] can fail for $g > 2$.

k-rational Weierstrass point; see below). In this case we can give alternative proofs using an explicit description of the 2-torsion subgroup; see Section 5. Another interesting class of examples is given by the polygons



where a and b are not both even. The case $a = b$ corresponds to the smooth plane curves of odd degree considered in [CEZB]. The case $\gcd(a, b) = 1$ corresponds to so-called $C_{a,b}$ curves. The case $b = 2$, $a = 2g + 1$ (a subcase of the latter) corresponds to hyperelliptic curves having a prescribed k-rational Weierstrass point P. Note that in this case $g^1_2 \sim 2P$, so there is indeed always a k-rational half-canonical divisor, regardless of the parity of g.

Remark 3. This explains why Denef and Vercauteren had to allow a factor 2 while generating cryptographic hyperelliptic and $C_{a,b}$ curves in characteristic 2; see Sections 6 of [DV1, DV2].

Finally, the case $b = 3$, $a \geq 4$ corresponds to trigonal curves having maximal Maroni invariant (that is trigonal curves for which the series $(h^0(ng^1_3))_{n\in\mathbf{Z}_{\geq 0}}$ starts increasing by steps of 3 as late as the Riemann-Roch theorem allows it to do); if $a = 6$, these are exactly the genus-4 curves having a unique g^1_3.

We conclude by stressing that the results in this paper are unlikely to generalize to characteristic $p > 2$, by lack of an appropriate analogue of our arithmetic half-canonical divisor Θ_{arith}.

2. Half-canonical divisors from toric geometry

Let k be a perfect field, let $f = \sum_{(i,j)\in\mathbf{Z}^2} c_{i,j} x^i y^j \in k[x^{\pm1}, y^{\pm1}]$ be a Laurent polynomial, and let

$$\Delta(f) = \operatorname{conv}\left\{ (i,j) \in \mathbf{Z}^2 \mid c_{i,j} \neq 0 \right\}$$

be the Newton polygon of f, which we assume to be two-dimensional. We say that f is *non-degenerate with respect to its Newton polygon* if for every face $\tau \subset \Delta(f)$ (vertex, edge, or $\Delta(f)$ itself) the system

$$f_\tau = \frac{\partial f_\tau}{\partial x} = \frac{\partial f_\tau}{\partial y} = 0 \qquad \text{with } f_\tau = \sum_{(i,j)\in\tau\cap\mathbf{Z}^2} c_{i,j} x^i y^j$$

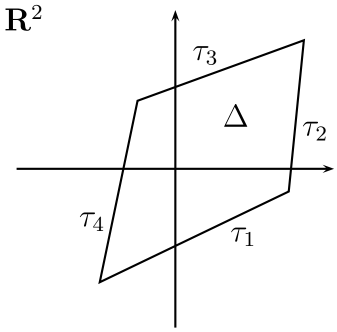
has no solutions[2] over an algebraic closure of k. For a given two-dimensional lattice polygon Δ, we say that f is Δ-*non-degenerate* if $\Delta(f) = \Delta$ and f is non-degenerate with respect to its Newton polygon. The condition of Δ-non-degeneracy is generically satisfied, in the sense that it is characterized by the non-vanishing of

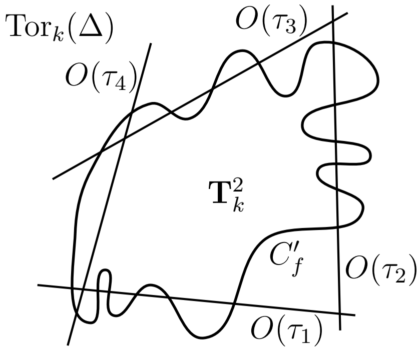
$$\rho_\Delta := \operatorname{Res}_\Delta\left(f, x\frac{\partial f}{\partial x}, y\frac{\partial f}{\partial y}\right) \in \mathbf{Z}[c_{i,j} | (i,j) \in \Delta \cap \mathbf{Z}^2]$$

(where $\operatorname{Res}_\Delta$ is the sparse resultant; ρ_Δ does not vanish identically in any characteristic [CV, §2]). Non-degenerate Laurent polynomials are always (absolutely) irreducible.

[2]Note that this is in fact automatically true if τ is a vertex.

To a two-dimensional lattice polygon Δ one can associate a toric surface $\mathrm{Tor}_k(\Delta)$, which is a compactification of $\mathbf{T}_k^2 = \mathrm{Spec}\, k[x^{\pm 1}, y^{\pm 1}]$ to which the natural self-action of the latter extends algebraically. This extended action decomposes $\mathrm{Tor}_k(\Delta)$ in a finite number of orbits, which naturally correspond (in a dimension-preserving manner) to the faces of Δ; for each face τ, write $O(\tau)$ for the according orbit. Now if $f \in k[x^{\pm 1}, y^{\pm 1}]$ is a Δ-non-degenerate Laurent polynomial, the non-degeneracy condition with respect to Δ itself ensures that it cuts out a non-singular curve C_f in $\mathbf{T}_k^2 = O(\Delta)$. Similarly, one finds that its compactification C'_f in $\mathrm{Tor}_k(\Delta)$ does not contain any of the zero-dimensional $O(\tau)$'s, and that it intersects the one-dimensional $O(\tau)$'s transversally.





In particular, since $\mathrm{Tor}_k(\Delta)$ is normal, the non-degeneracy of f implies that C'_f is a non-singular complete model of C_f. See [CC, §3-4] and [CDV, §2] for more details.

Example 4. Assume that $\Delta = \mathrm{conv}\{(0,0),(d,0),(0,d)\}$. In this case $\mathrm{Tor}_k(\Delta)$ is just the projective plane, and the toric orbits are

- $\mathbf{T}_k^2 = O(\Delta)$,
- the three coordinate points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$, which are the orbits of the form $O(\text{vertex})$,
- the three coordinate axes from which the coordinate points are removed: these are the orbits of the form $O(\text{edge})$.

Thus C'_f is a non-singular projective plane curve that is non-tangent to any of the coordinate axes, and that does not contain any of the coordinate points. This is essentially an if-and-only-if: an absolutely irreducible Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$, for which $\Delta(f) \subset \Delta$, is Δ-non-degenerate if and only if its zero locus in $\mathbf{T}_k^2$ compactifies to a non-singular degree d curve in $\mathbf{P}_k^2$ that is non-tangent to the coordinate axes, and that does not contain the coordinate points.

Example 5. Let $g \geq 2$ be an integer, and consider $f = y^2 + h_1(x)y + h_0(x)$, where $\deg h_1 \leq g{+}1$, $\deg h_0 = 2g{+}2$, and $h_0(0) \neq 0$. Then $\Delta(f) = \mathrm{conv}\{(0,0),(2g{+}2,0),(0,2)\}$, and $\mathrm{Tor}_k(\Delta(f))$ is the weighted projective plane $\mathbf{P}_k(1:g+1:1)$. Here again, if f is non-degenerate with respect to its Newton polygon then C'_f is a non-singular curve that is non-tangent to the coordinate axes and that does not contain any coordinate points. In this case C'_f is a hyperelliptic curve of genus g (cf. Remark 8).

Now for each edge $\tau \subset \Delta$ let $\nu_\tau \in \mathbf{Z}^2$ be the inward pointing primitive normal vector to τ, let p_τ be any element of $\tau \cap \mathbf{Z}^2$, and let D_τ be the k-rational divisor on C'_f cut out by $O(\tau)$. Using the Δ-non-degeneracy of f one can prove

$$\text{div} \frac{dx}{xy\frac{\partial f}{\partial y}} = \sum_{\tau \text{ edge}} (-\langle \nu_\tau, p_\tau \rangle - 1) D_\tau. \tag{2}$$

Here $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbf{R}^2$. See [CDV, Cor. 2.7] for an elementary but elaborate proof of (2). It is possible to give a more conceptual proof using adjunction theory, along the lines of [CLS, Prop. 10.5.8].

Remark 6. From the theory of sparse resultants it follows that $\partial f/\partial y$ does not vanish identically, so that the left-hand side of (2) makes sense. Note also that $0 = df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$, so we could as well have written

$$\text{div} \frac{dy}{xy\frac{\partial f}{\partial x}}. \tag{3}$$

Proof of Lemma 1. Assume that f is Δ-non-degenerate (which, as mentioned above, is a non-empty Zariski open condition). Let $(i_0, j_0) \in \mathbf{Z}^2$ be a solution to the given system of congruences. We claim that the translated polygon $(-i_0, -j_0) + \Delta$ is such that all corresponding $\langle \nu_\tau, p_\tau \rangle$'s are odd. To see this, note that in this case $(0,0)$ is a solution to the according system of congruences (1). This implies that all c_τ's are odd. Together with $\langle \nu_\tau, p_\tau \rangle = \pm c_\tau$ this yields the claim. So by applying the above to $x^{-i_0} y^{-j_0} f$, we find that

$$\Theta_{\text{geom}} = \sum_{\tau \text{ edge}} \frac{-\langle \nu_\tau, p_\tau \rangle - 1}{2} D_\tau$$

is a k-rational half-canonical divisor on $C'_{x^{-i_0} y^{-j_0} f} = C'_f$. □

Example 4 (continued). Assume that d is odd, so that the conditions from Lemma 1 are satisfied. Applying the above proof with $(i_0, j_0) = (1, 1)$ yields

$$\Theta_{\text{geom}} = \frac{d-3}{2} D_\infty$$

where D_∞ is the divisor cut out by the line at infinity. So we recover the divisor class mentioned in the introduction.

Remark 7. Still assume that $\Delta = \text{conv}\{(0,0), (d,0), (0,d)\}$ with d odd. We already noted that the condition of non-degeneracy restricts our attention to smooth plane curves of degree d that do not contain the coordinate points and that intersect the coordinate axes transversally. But of course *any* smooth plane curve of degree d carries a k-rational half-canonical divisor. This shows that the non-degeneracy condition, even though it is generically satisfied, is sometimes a bit stronger than needed.[3] For a general two-dimensional lattice polygon Δ, the according weaker condition reads that f is Δ-*toric*, meaning that $\Delta(f) \subset \Delta$, that $\Delta(f)^{(1)} = \Delta^{(1)}$, and that C_f compactifies to a non-singular

[3]The reader might want to note that there always exists an automorphism of $\mathbf{P}^2_k$ that puts our smooth plane curve in a non-degenerate position (at least if $\#k$ is sufficiently large). But for more general instances of Δ, the automorphism group of $\text{Tor}_k(\Delta)$ may be much smaller (e.g. the only automorphisms may be the ones coming from the $\mathbf{T}^2_k$-action), in which case it might be impossible to resolve tangency to the one-dimensional toric orbits.

curve C'_f in $\mathrm{Tor}_k(\Delta)$. Here $\Delta^{(1)}$ denotes the lattice polygon obtained by taking the convex hull of the $\mathbf{Z}^2$-points that lie in the interior of Δ, and similarly for $\Delta(f)^{(1)}$. See [CC, §4] for more background on this notion. Now we have to revisit Remark 6, however: there do exist instances of Δ-toric Laurent polynomials $f \in k[x^{\pm 1}, y^{\pm 1}]$ for which $\partial f/\partial y$ *does* vanish identically (example: take $f = 1 + x^2y^2 + x^3y^2$ and $\Delta = \Delta(f)$). For these instances the left-hand side of (2) does not make sense. But in that case $\partial f/\partial x$ does not vanish identically (otherwise C_f would have singularities), and one can prove that (2) holds with the left-hand side replaced by (3).

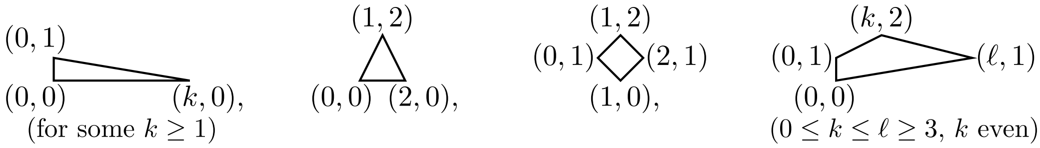
Remark 8. We mention two other well-known features of Δ-non-degenerate (or Δ-toric) Laurent polynomials, that can be seen as consequences to (2); see for instance [CC, CV] and the references therein:

- the genus of C'_f equals $\#\left(\Delta^{(1)} \cap \mathbf{Z}^2\right)$,
- if $\#\left(\Delta^{(1)} \cap \mathbf{Z}^2\right) \geq 2$, then C'_f is hyperelliptic if and only if $\Delta^{(1)} \cap \mathbf{Z}^2$ is contained in a line.

3. Proof of the main result

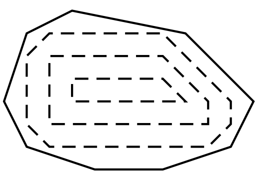
Lemma 9. *Let Δ be a two-dimensional lattice polygon and suppose as in Lemma 1 that* (1) *admits a solution in* $\mathbf{Z}^2$. *If Δ is not among the polygons excluded in the hypothesis of Theorem 2, then there is a solution of* (1) *contained in* $\Delta \cap \mathbf{Z}^2$.

Proof. Let us first classify all two-dimensional lattice polygons Δ for which the reduction-modulo-2 map $\pi_\Delta \colon \Delta \cap \mathbf{Z}^2 \to (\mathbf{Z}/(2))^2$ is not surjective. If the interior lattice points of Δ lie on a line, then surjectivity fails if and only if Δ is among



(a) (b) (c) (d)

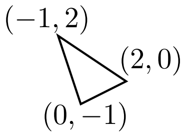
(up to unimodular equivalence). This assertion follows from Koelman's classification; see [Koe, Ch. 4] or [Cas, Thm. 10]. Now any two-dimensional lattice polygon Δ can be peeled into 'onion skins', by subsequently taking the convex hull of the interior lattice points, until one ends up with a lattice polygon whose interior lattice points are contained in a line.



If π_Δ is not surjective, then clearly π_Ω is not surjective for each onion skin Ω. In particular, the last onion skin must necessarily be among (a-d).

But for a lattice polygon to arise as an onion skin of a strictly larger lattice polygon Δ is a stringent condition. Using the criterion from [HS, Lem. 9-11] one sees that the only polygons among (a-d) of this type are the polygons (a) with $k = 1$ or $k = 2$, the polygon

(b) and the polygon (c). The same criterion shows that the only instance of such a larger Δ for which π_Δ is not surjective is



(e)

(up to unimodular equivalence). The latter, again by [HS, Lem. 9-11], is not an onion skin of a strictly bigger lattice polygon itself. This ends the classification: up to unimodular equivalence, the instances of Δ for which π_Δ is not surjective are (a)-(e).

Now let Δ be a two-dimensional lattice polygon and suppose that (1) admits a solution in $\mathbf{Z}^2$. If π_Δ is surjective, then it clearly also admits a solution in $\Delta \cap \mathbf{Z}^2$. So we may assume that Δ is among (a-e). Then the lemma follows by noting that cases (b), (c) and (d) with ℓ even admit the solution $(1,1) \in \Delta \cap \mathbf{Z}^2$, and that cases (a), (e) and (d) with ℓ odd were excluded in the énoncé. □

Remark 10. Because of Remark 8, the excluded polygons correspond to certain classes of smooth plane quartics, rational curves, and hyperelliptic curves, respectively.

We can now define the variety S_Δ mentioned in the statement of Theorem 2. Namely, we will prove the existence of a non-trivial k-rational 2-torsion point under the assumption that

- f is Δ-non-degenerate (i.e. the genericity assumption from Lemma 1), and
- for at least one solution $(i_0, j_0) \in \Delta \cap \mathbf{Z}^2$ to the system of congruences (1), the corresponding coefficient c_{i_0,j_0} is non-zero.

So we can let S_Δ be defined by $c_{i_0,j_0}\rho_\Delta \neq 0$.

Remark 11. Here again, one can weaken the condition of being Δ-non-degenerate to being Δ-toric, as described in Remark 7. When that stronger version is applied to $\Delta = \operatorname{conv}\{(0,0),(d,0),(0,d)\}$ with d odd, one exactly recovers [CEZB, Thm. 4.2].

Proof of Theorem 2. By replacing f with $x^{-i_0}y^{-j_0}f$ if needed, we assume that $(0,0) \in \Delta$ is a solution to the system of congruences (1) and that the constant term of f is non-zero. As explained in [Mum, p. 191], C'_f comes equipped with a k-rational divisor Θ_{arith} such that $2\Theta_{\text{arith}} = \operatorname{div} dx$. (Recall that the derivative of a Laurent series over k is always a square, so the order of dx at a point of C'_f is indeed even.) On the other hand, Lemma 1 and its proof provide us with a k-rational divisor Θ_{geom} such that

$$2\Theta_{\text{geom}} = \operatorname{div} \frac{dx}{xy\frac{\partial f}{\partial y}}.$$

In order to prove that $\Theta_{\text{geom}} \not\sim \Theta_{\text{arith}}$ (and hence that $\operatorname{Jac}(C'_f)$ has a non-trivial k-rational 2-torsion point), we need to show that

$$xy\frac{\partial f}{\partial y}$$

is a non-square when considered as an element of the function field $k(C_f)$. If it were a square, then there would exist Laurent polynomials α, G, H such that

$$H^2 xy \frac{\partial f}{\partial y} + \alpha f = G^2 \qquad \text{in } k[x^{\pm 1}, y^{\pm 1}], \tag{4}$$

where $f \nmid H$. Taking derivatives with respect to y yields

$$(\alpha + H^2 x) \frac{\partial f}{\partial y} = \frac{\partial \alpha}{\partial y} f,$$

which together with (4) results in

$$\left((\alpha + H^2 x)\alpha + H^2 xy \frac{\partial \alpha}{\partial y} \right) f = (\alpha + H^2 x) G^2.$$

Since f is irreducible, it follows that $f \mid (\alpha + H^2 x)$ or $f \mid G^2$. Using (4) and $f \nmid H$, the latter implies that $f \mid \frac{\partial f}{\partial y}$, which is a contradiction (by the theory of sparse resultants, see Remark 6; one can alternatively repeat the argument using (3) if wanted). So we know that $f \mid (\alpha + H^2 x)$. Along with (4) we conclude that there exists a Laurent polynomial $\beta \in k[x^{\pm 1}, y^{\pm 1}]$ such that

$$H^2 x \left(y \frac{\partial f}{\partial y} + f \right) + \beta f^2 = G^2.$$

Taking derivatives with respect to x yields

$$H^2 \left(f + x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + xy \frac{\partial^2 f}{\partial x \partial y} \right) + \frac{\partial \beta}{\partial x} f^2 = 0.$$

Since f has a non-zero constant term, the large factor between brackets is non-zero. On the other hand, since $f \nmid H$, it must be a multiple of f^2. Note that $\Delta(f^2) = 2\Delta(f)$, while $\Delta(f + \cdots + xy\partial^2 f/(\partial x \partial y)) \subset \Delta(f)$. This is a contradiction. □

We end this section by discussing some asymptotic consequences to Theorem 2.

Growing field size. Let Δ be a two-dimensional lattice polygon satisfying the conditions of Theorem 2. Let k be a finite field of characteristic 2. Because non-degeneracy is characterized by the non-vanishing of ρ_Δ, the proportion of Δ-non-degenerate Laurent polynomials $f \in k[x^{\pm 1}, y^{\pm 1}]$ (amongst all Laurent polynomials that are supported on Δ) converges to 1 as $\#k \to \infty$. Then Theorem 2 implies:

$$\lim_{\#k \to \infty} \text{Prob} \left(\text{Jac}(C'_f)(k)[2] \neq 0 \;\middle|\; f \in k[x^{\pm 1}, y^{\pm 1}] \text{ is } \Delta\text{-non-degenerate} \right) = 1.$$

As soon as $\#(\Delta^{(1)} \cap \mathbf{Z}^2) \geq 2$ this is deviating statistical behavior: in view of Katz-Sarnak-Chebotarev-type density theorems [KS, Theorem 9.7.13], for a general smooth proper family of genus g curves, one expects that the probability of having a non-trivial rational 2-torsion point on the Jacobian approaches the chance that a random matrix in $\text{GL}_g(\mathbf{F}_2)$ satisfies $\det(M - \text{Id}) = 0$, which is

$$-\sum_{r=1}^{g} \prod_{j=1}^{r} \frac{1}{1 - 2^j}$$

by [CFHS, Thm. 6]. For $g = 1, 2, 3, 4, \ldots$, these probabilities are $1, \frac{2}{3}, \frac{5}{7}, \frac{32}{45}, \ldots$ (converging to about 0.71121).

In the table below we denote by $\square_i$ the square $[0, i]^2$ (for $i = 2, 3, 4$), by H_g the hyperelliptic polygon $\text{conv}\{(0, 0), (2g+2, 0), (0, 2)\}$ (for $g = 7, 8$), and by E the exceptional polygon $\text{conv}\{(1, 0), (3, 1), (0, 3)\}$ from the statement of Theorem 2. Each entry corresponds to a sample of 10^4 uniformly randomly chosen Laurent polynomials $f \in k[x^{\pm 1}, y^{\pm 1}]$ that are supported on $\square_2, \square_3, \ldots$ The table presents the proportion of f's for which $\text{Jac}(C'_f)$ has a non-trivial k-rational 2-torsion point, among those f's that are non-degenerate with respect to their Newton polygon $\Delta(f) = \square_2, \square_3, \ldots$ The count was carried out using Magma [BCP], either by using the intrinsic function for computing the Hasse-Weil zeta function, or by spelling out the Hasse-Witt matrix [SV, Thm. 1.1] and applying Manin's theorem [Man].

k	$\square_2$ ($g=1$)	$\square_3$ ($g=4$)	$\square_4$ ($g=9$)	H_7 ($g=7$)	H_8 ($g=8$)	E ($g=3$)
$\mathbf{F}_2$	0/0	0.370	0.958	0.995	0.670	0.143
$\mathbf{F}_4$	0.750	0.621	1.000	1.000	0.795	0.449
$\mathbf{F}_8$	0.884	0.654	1.000	1.000	0.852	0.591
$\mathbf{F}_{16}$	0.940	0.697	1.000	1.000	0.872	0.661
$\mathbf{F}_{32}$	0.968	0.704	1.000	1.000	0.877	0.696
$\mathbf{F}_{64}$	0.986	0.716	1.000	1.000	0.880	0.694
$\mathbf{F}_{128}$	0.992	0.703	1.000	1.000	0.889	0.708
$\mathbf{F}_{256}$	0.996	0.709	1.000	1.000	0.888	0.707
asymptotic prediction	1	$\frac{32}{45} \approx 0.711$	1	1	$\frac{8}{9} \approx 0.889$	$\frac{5}{7} \approx 0.714$

Note that the conditions of Theorem 2 are satisfied for $\square_2$, $\square_4$ and H_7. So here we proved that the proportion converges to 1. In the case of H_8, by the material in Section 5 (see Corollary 27) we know that the proportion converges to $\frac{8}{9}$. In the other two cases $\square_4$ and E we have no clue, so our best guess is that these follow the $\text{GL}_g(\mathbf{F}_2)$-model.

Growing polygon. Let k be a finite field of characteristic 2. If Δ is a two-dimensional lattice polygon satisfying the conditions of Lemma 1, then the same holds for each odd Minkowski multiple $(2n + 1)\Delta$. It seems reasonable to assume that the proportion of $(2n + 1)\Delta$-non-degenerate Laurent polynomials $f \in k[x^{\pm 1}, y^{\pm 1}]$, amongst all Laurent polynomials that are supported on $(2n + 1)\Delta$, converges to a certain strictly positive constant.

This is certainly true for the larger proportion of $(2n+1)\Delta$-*toric* Laurent polynomials. Namely, using [Poo2, Thm. 1.2] one can show that this proportion converges to

$$Z_{\text{Tor}_k(\Delta)\setminus S}((\#k)^{-3})^{-1} \cdot Z_S((\#k)^{-1})^{-1}$$

as $n \to \infty$; here S denotes the (finite) set of singular points of $\text{Tor}_k(\Delta)$, and Z stands for the Hasse-Weil Zeta function. It should be possible to prove a similar statement for non-degenerate Laurent polynomials by redoing the closed point sieve in the proof of [Poo2, Thm. 1.2], but we did not work out the details of this.

On the other hand, the number of solutions to (1) inside $(2n+1)\Delta \cap \mathbf{Z}^2$ tends to infinity. So the assumption would allow one to conclude:

$$\lim_{n\to\infty} \text{Prob}\left(\text{Jac}(C'_f)(k)[2] \neq 0 \;\middle|\; f \in k[x^{\pm 1}, y^{\pm 1}] \text{ is } (2n+1)\Delta\text{-non-degenerate} \right) = 1.$$

This is again deviating statistical behavior: in view of Cohen-Lenstra type heuristics, one naively expects a probability of about

$$1 - \prod_{j=1}^{\infty}(1-2^{-j}) \approx 0.71121;$$

see [CEZB] for some additional comments.

When applied to $(2n+1)\Sigma$-toric Laurent polynomials, where Σ is the standard simplex, one recovers the claim made before [CEZB, Thm. 4.2].

4. Connections with the rank of the Hasse-Witt matrix

Let us revisit the proof of Theorem 2 from the previous section. Our sufficient condition that

$$(5) \qquad c_{i_0,j_0} \neq 0 \quad \text{for at least one solution } (i_0,j_0) \in \Delta \cap \mathbf{Z}^2 \text{ to the system (1)}$$

(see right before Remark 11) seems rather equation-specific. However, it is easy to show that automorphisms of $\text{Tor}_k(\Delta)$ cannot alter whether (5) is satisfied or not. For instance, in the case of smooth plane projective curves of odd degree $d \geq 3$, one verifies that if

$$F(X,Y,Z) = \sum_{i+j\leq d} c_{i,j} X^i Y^j Z^{d-i-j} \in k[X,Y,Z]$$

is such that $c_{i,j} = 0$ as soon as both i and j are odd, then applying a linear change of variables does not affect this. This suggests that something more fundamental is going on. In Conjecture 15 below we will formulate a guess for a geometric interpretation of condition (5), involving the rank of the Hasse-Witt matrix (or of the Cartier-Manin operator, if one prefers). We will prove this guess in a number of special cases. Our main references on the Hasse-Witt matrix are [Man, Ser, SV].

Here is a first fact:

Lemma 12. *Let k be a perfect field of characteristic 2, let Δ be a two-dimensional lattice polygon satisfying the conditions of Lemma 1, and let $f = \sum_{(i,j)\in\mathbf{Z}^2} c_{i,j}x^iy^j \in k[x^{\pm 1}, y^{\pm 1}]$ be a Δ-non-degenerate (or Δ-toric) Laurent polynomial. Let*

- *g be the genus of C'_f, i.e. $g = \#(\Delta^{(1)} \cap \mathbf{Z}^2)$, and*
- *ρ be the number of solutions $(i_0,j_0) \in \Delta \cap \mathbf{Z}^2$ to the system of congruences (1).*

If $c_{i_0,j_0} = 0$ for every such solution, then the rank of the Hasse-Witt matrix of C'_f is at most $g - \rho$.

Proof. By [CDV, Cor. 2.6 and 2.7] we find that

$$(6) \qquad \left\{ x^iy^j \frac{dx}{xy\frac{\partial f}{\partial y}} \right\}_{(i,j)\in\Delta^{(1)}\cap\mathbf{Z}^2}$$

is a basis for the space of regular differentials on C'_f. (If in the Δ-toric case the denominator happens to vanish identically, one can replace $dx/(\partial f/\partial y)$ by $dy/(\partial f/\partial x)$ as explained in Remark 7.) Assume that $c_{i_0,j_0} = 0$ for each of the ρ solutions $(i_0, j_0) \in \Delta \cap \mathbf{Z}^2$ to the system (1). Remark that these solutions are all contained in $\Delta^{(1)}$. One then verifies that the ρ corresponding differentials $z_{i_0,j_0}dx$, where

$$z_{i_0,j_0} = \frac{x^{i_0}y^{j_0}}{xy\frac{\partial f}{\partial y}},$$

satisfy $\partial z_{i_0,j_0}/\partial x = 0$. Following the construction from [SV, §1] we conclude that at least ρ rows of the Hasse-Witt matrix with respect to the basis (6) are zero. □

As an interesting corollary we obtain:

Corollary 13. *Let k and Δ be as before and let f be a Δ-non-degenerate (or Δ-toric) Laurent polynomial over k. Assume moreover that Δ is not among the polygons excluded in the statement of Theorem 2. If C'_f is ordinary then it has a non-trivial k-rational 2-torsion point on its Jacobian.*

Proof. In view of Lemma 9, the fact that Δ is not among the excluded polygons ensures that $\rho > 0$. A result by Serre [Ser, Prop. 10] says that C'_f is ordinary if and only if its Hasse-Witt matrix has rank g. So the previous lemma implies that if C'_f is ordinary, then (5) is satisfied. The claim now follows from Theorem 2. □

Remark 14. The following alternative proof of Corollary 13 was suggested to us by Christophe Ritzenthaler. A result by Stöhr and Voloch [SV, Cor. 3.2] states that the Hasse-Witt matrix has rank $g - h^0(C'_f, \Theta_{\text{arith}})$. So if C'_f is ordinary then $h^0(C'_f, \Theta_{\text{arith}}) = 0$, and in particular Θ_{arith} cannot be linearly equivalent to an effective divisor. Now if Δ is not among the excluded polygons, then by Lemma 9 there is at least one solution $(i_0, j_0) \in \Delta \cap \mathbf{Z}^2$ to the system (1). Fix such a solution and consider the corresponding translated polygon $(-i_0, -j_0) + \Delta$, as in the proof of Lemma 1. We again find that all $\langle \nu_\tau, p_\tau \rangle$'s are odd, but now because $(0, 0) \in (-i_0, -j_0) + \Delta$ we also find that they are strictly negative. In other words the resulting half-canonical divisor Θ_{geom} is effective. Hence Θ_{geom} and Θ_{arith} are non-equivalent. Their difference then yields a non-trivial k-rational 2-torsion point on $\text{Jac}(C'_f)$.

Our guess is that Lemma 12 admits the following converse. This would give the desired geometric interpretation of condition (5).

Conjecture 15. *Let k be a perfect field of characteristic 2, let Δ be a two-dimensional lattice polygon satisfying the conditions from Lemma 1, and let f be a Δ-non-degenerate (or Δ-toric) Laurent polynomial. Then the rank of the Hasse-Witt matrix of C'_f is at least $g - \rho$, and the bound is attained if and only if $c_{i_0,j_0} = 0$ for every solution $(i_0, j_0) \in \Delta \cap \mathbf{Z}^2$ to the system of congruences (1).*

We can prove this conjecture in a number of special cases. Because the statements seem interesting in their own right, we will each time reformulate (and sometimes refine) Conjecture 15 accordingly.

Theorem 16 (Conjecture 15 for smooth plane curves of odd degree)**.** *Let k be a perfect field of characteristic 2, let $d \geq 3$ be an odd integer and let $f = \sum_{i+j \leq d} c_{i,j} x^i y^j \in k[x, y]$ define a smooth plane projective curve C/k of degree d and genus $g = (d-1)(d-2)/2$. Then the rank of the Hasse-Witt matrix of C is bounded from below by*

$$g - \frac{d^2 - 1}{8} = \frac{3}{8}(d-1)(d-3)$$

Furthermore equality holds if and only if $c_{i,j} = 0$ as soon as i and j are odd.

Proof. Recall from Remark 14 that Stöhr and Voloch [SV, Cor. 3.2] proved that the rank of the Hasse-Witt matrix is $g - h^0(C, \Theta_{\text{arith}})$. By the Brill-Noether theory of smooth plane curves [Har, Thm. 2.1] we have

$$h^0(C, D) \leq \frac{\frac{d-1}{2}\frac{d+1}{2}}{2} = (d^2-1)/8 \tag{7}$$

for any divisor D on C of degree $g-1$. In particular this also holds for $D = \Theta_{\text{arith}}$, from which the lower bound follows. As for the last statement, by [Har, part 2b of Thm. 2.1] the bound in (7) is attained if and only if D is in the class of $\frac{d-3}{2}H$, i.e. if and only if $D \sim \Theta_{\text{geom}}$. But the proof of Theorem 2 (or of [CEZB, Thm. 4.2]) is precisely about showing that if $c_{i,j} \neq 0$ for some i and j that are both odd, then $\Theta_{\text{arith}} \not\sim \Theta_{\text{geom}}$. This yields the 'only if' part, while the 'if' part follows from Lemma 12. $\square$

Theorem 17 (Conjecture 15 for hyperelliptic curves of odd genus)**.** *Let k be a perfect field of characteristic 2. Let C be a hyperelliptic curve of odd genus $g \geq 3$, given in weighted projective form by*

$$C\colon \quad Y^2 + H_1(X, Z)Y = H_0(X, Z), \tag{8}$$

where H_1 and H_0 in $k[X, Z]$ are homogeneous of degrees $g+1$ and $2g+2$ respectively. Then the rank of the Hasse-Witt matrix of C equals

$$g - \frac{1}{2}\deg\gcd\left(H_1, Z^{-1}\frac{\partial}{\partial X}H_1\right).$$

In particular, it is bounded from below by

$$g - \frac{g+1}{2} = \frac{g-1}{2},$$

where equality holds if and only if $\frac{\partial}{\partial X}H_1 = 0$.

Proof. Write $H_1 = \sum_{i=0}^{g+1} c_i X^i Z^{g+1-i}$ and define

$$P(X, Z) = \sum_{i=0}^{(g+1)/2} c_{2i} X^i Z^{(g+1)/2-i} \qquad \text{and} \qquad Q(X, Z) = \sum_{i=0}^{(g-1)/2} c_{2i+1} X^i Z^{(g-1)/2-i}.$$

Note that $H_1 = P^2 + XZQ^2$ and $\frac{\partial}{\partial X}H_1 = ZQ^2$. Now the polynomial $f = y^2 + H_1(x,1)y + H_0(x, 1)$ is Δ-toric, where $\Delta = \text{conv}\{(0,0), (2g+2, 0), (0,2)\}$; here C'_f is nothing else but C. An explicit computation shows that the Hasse-Witt matrix with respect to the basis

(6) equals, up to a reordering of the rows, the Sylvester matrix of P and Q. It is well-known that the corank of the Sylvester matrix of two polynomials equals the degree of their greatest common divisor, which in our case equals

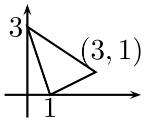
$$\deg \gcd(P, Q) = \frac{1}{2} \deg \gcd(P^2, Q^2) = \frac{1}{2} \deg \gcd \left(H_1, Z^{-1} \frac{\partial}{\partial X} H_1 \right).$$

The remaining claims follow immediately. □

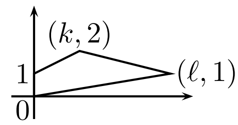
Remark 18. This indeed implies Conjecture 15 for hyperelliptic curves of odd genus because $\frac{\partial}{\partial X} H_1 = 0$ if and only if all terms $c_{i,j} x^i y^j$ in $f = y^2 + H_1(x,1)y + H_0(x,1)$ with i and j odd are 0.

Remark 19. The lower bound $(g-1)/2$ holds for arbitrary curves C of genus g (not necessarily odd) over fields of characteristic 2, and it can be attained by hyperelliptic curves only. This follows from Clifford's theorem, as explained in [SV, Cor. 3.2].

Theorem 20 (Conjecture 15 for the exceptional polygons)**.** *Let k be a perfect field of characteristic 2, let Δ be one of the polygons*



or



for some $0 \leq k < \ell \geq 3$ with k even and ℓ odd

that were excluded in the statement of Theorem 2, and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be Δ-non-degenerate (or Δ-toric). Then the rank of the Hasse-Witt matrix of C'_f is equal to $g = \#(\Delta^{(1)} \cap \mathbf{Z}^2)$. In particular C'_f is ordinary.

Proof. The polygon on the left corresponds to smooth plane quartics of the form

$$c_{1,0} X Z^3 + c_{1,1} XYZ^2 + c_{2,1} X^2 YZ + c_{3,1} X^3 Y + c_{1,2} XY^2 Z + c_{0,3} Y^3 Z.$$

The Hasse-Witt matrices of smooth plane quartics are explicitly described at the end of [SV, §3]. In our case this gives

$$\begin{pmatrix} c_{1,1} & c_{3,1} & 0 \\ 0 & c_{2,1} & c_{0,3} \\ c_{1,0} & 0 & c_{1,2} \end{pmatrix}$$

with determinant $c_{1,1}c_{2,1}c_{1,2} + c_{1,0}c_{3,1}c_{0,3}$. With the aid of a computer algebra package one can verify that this determinant is non-zero (using that the curve is smooth).

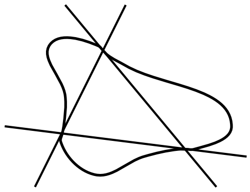
As for the polygons on the right, we have that $f = cx^k y^2 + h_1(x)y + c'$ for non-zero $c, c' \in k$ and a degree $\ell = g + 1$ polynomial $h_1(x) \in k[x]$. Substituting $y \leftarrow yx^{-k}$ and multiplying the equation by $c^{-1}x^k$ puts our curve in the Weierstrass form

$$y^2 + c^{-1} h_1(x) y + c^{-1} c' x^k.$$

Using that k is even one sees that $h_1(x)$ is square-free (otherwise there would be an affine singularity). The result then follows from the previous theorem. □

A fun corollary is the following geometric sufficient condition for ordinarity. Remark that similar conditions have been described before (such as the existence of 7 bitangent lines, which is actually sufficient and necessary; see [SV, §3]).

Corollary 21. *Let C be a smooth plane quartic curve over a field k of characteristic 2 admitting three non-colinear inflection points, such that the corresponding tangent lines are precisely the lines through two of these points.*



Then C is ordinary.

Proof. A projective transformation positions the three inflection points at $(0:0:1)$, $(0:1:0)$ and $(1:0:0)$. One verifies that the dehomogenization of the corresponding defining polynomial is Δ-non-degenerate, where Δ is the left-most polygon in the statement of the previous corollary. □

Remark 22. Theorems 16, 17 and 20 provide several characteristic 2 examples of families of curves whose Hasse-Witt matrices have constant rank. This (partly) addresses Question 2 of [FP, §3.7].

5. Hyperelliptic curves

Let C be a hyperelliptic curve of genus $g \geq 2$ over a perfect field k. Then C has a smooth weighted projective plane model of the form (8). The Newton polygon of (the defining polynomial of) the corresponding affine model $y^2 + H_1(x,1)y - H_0(x,1) = 0$ is contained in a triangle with vertices $(0,0)$, $(2g+2,0)$ and $(0,2)$, and is generically equal to this triangle. In particular, Theorem 2 implies that if the characteristic of k is 2 and C is sufficiently general of odd genus, then its Jacobian has a non-trivial k-rational 2-torsion point. By Corollary 13 we can replace 'sufficiently general' by 'ordinary'.

The purpose of this stand-alone section is to give alternative proofs of these facts (Corollaries 25 and 27), using an explicit description of the 2-torsion subgroup of $\mathrm{Jac}(C)$.

Theorem 23. *Let C/k be a hyperelliptic curve over a perfect field k of characteristic 2 given by a smooth model (8). The Jacobian of C has* no *rational point of order 2 if and only if $H_1(X,Z)$ is a power of an irreducible odd-degree polynomial in $k[X,Z]$.*

Corollary 24. *Let C/k be a hyperelliptic curve of odd 2-rank over a perfect field k of characteristic 2. Then the Jacobian of C has a k-rational point of order 2.*

Corollary 25. *Let C/k be an ordinary hyperelliptic curve of odd genus over a perfect field k of characteristic 2. Then the Jacobian of C has a k-rational point of order 2.*

Corollary 26. *Let C/k be a hyperelliptic curve of genus $2^m - 1$ over a perfect field k of characteristic 2, for some integer $m \geq 2$. If the Jacobian of C has no k-rational point of order 2, then it has 2-rank zero, but it is not supersingular.*

Finally, for integers $g, r \geq 1$, let $c_{g,r}$ be the proportion of equations (8) over $\mathbf{F}_{2^r}$ that define a curve of genus g whose Jacobian has at least one rational point of order 2.

Corollary 27. *The limit* $\lim_{r\to\infty} c_{g,r}$ *exists and we have*

$$\lim_{r\to\infty} c_{g,r} = \begin{cases} 1 & \text{if } g \text{ is odd,} \\ g/(g+1) & \text{if } g \text{ is even.} \end{cases}$$

Proof of Theorem 23. All we need to do is describe the two-torsion of the Jacobian $\mathrm{Jac}(C)$ of C. Since we were not able to find a ready-to-use statement in the literature, we give a stand-alone treatment, even though what follows is undoubtedly known to several experts in the field; for instance, it is implicitly contained in [EP, PZ]. Let $\overline{k}$ be an algebraic closure of k. Note that C has a unique point $Q_{(a:b)} = (a : \sqrt{F(a,b)} : b) \in C(\overline{k})$ for every root $(a : b) \in \mathbf{P}^1_{\overline{k}}$ of $H_1 = H_1(X,Z)$. This gives n points, where $n \in \{1,\ldots,g+1\}$ is the number of distinct roots of H_1. Let D be the divisor of zeroes of a vertical line, so D is effective of degree 2. All such divisors D are linearly equivalent, and are linearly equivalent to $2Q_{(a:b)}$ for each $(a : b)$. In particular, if we let

$$A = \ker\left(\bigoplus_{(a:b)} (\mathbf{Z}/2\mathbf{Z}) \xrightarrow{\Sigma} (\mathbf{Z}/2\mathbf{Z})\right),$$

then we have a homomorphism

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Jac}(C)(\overline{k})[2] \\ (c_{(a:b)} \bmod 2)_{(a:b)} & \longmapsto & (\sum_{(a:b)} c_{(a:b)} Q_{(a:b)}) - (\frac{1}{2}\sum_{(a:b)} c_{(a:b)})D. \end{array}$$

In fact, this map is an isomorphism. Indeed, it is injective because if the divisor of a function is invariant under the hyperelliptic involution, then so is the function itself, i.e. it is contained in $\overline{k}(x)$. But at the points $Q_{(a:b)}$ such functions can only admit poles or zeroes having an even order. Surjectivity follows from the fact that $\mathrm{Jac}(C)(\overline{k})[2]$ is generated by divisors that are supported on the Weierstrass locus of C. This can be seen using Cantor's algorithm [Kob, Appendix.§6-7], for the application of which one needs to transform the curve to a so-called imaginary model; this is always possible over $\overline{k}$. Alternatively, surjectivity follows from the injectivity and the fact that $\#\mathrm{Jac}(C)(\overline{k})[2] = 2^{n-1}$ by [EP, Thm. 1.3].

Then in particular, the rational 2-torsion subgroup $\mathrm{Jac}(C)(k)[2]$ is isomorphic to the subgroup of elements of A that are invariant under $\mathrm{Gal}(\overline{k}/k)$, that is, to

$$A_k = \ker\left(\bigoplus_{P|H_1} (\mathbf{Z}/2\mathbf{Z}) \to (\mathbf{Z}/2\mathbf{Z}) : (c_P)_P \mapsto \sum_P c_P \deg(P)\right)$$

where the sum is taken over the irreducible factors P of H_1.

The only way for A_k to be trivial is for H_1 to be the power of an irreducible factor P of odd degree. □

Proof of Corollary 24. Let n be the degree of the radical R of H_1. The 2-rank of C equals $n-1$ (as in the proof of Theorem 23; see e.g. [EP, Thm. 1.3]). So if the 2-rank is odd, then R has even degree, which implies that H_1 is not a power of an irreducible odd-degree polynomial. In particular, Theorem 23 implies that C has a non-trivial k-rational 2-torsion point. □

Proof of Corollary 25. This is a special case of Corollary 24 since in characteristic 2, the 2-rank of an ordinary abelian variety equals its dimension. □

Proof of Corollary 26. If there is no rational point of order 2, then H_1 is a power of a polynomial of odd degree dividing $\deg H_1 = g + 1 = 2^m$. In other words, it is a power of a linear polynomial and hence the 2 rank of C is zero. There are no supersingular hyperelliptic curves of genus $2^m - 1$ in characteristic 2 by [SZ, Thm. 1.2]. □

Proof of Corollary 27. As $r \to \infty$, the proportion of equations (8) for which H_1 is not separable becomes negligible. By Theorem 23 it therefore suffices to prove the corresponding limit for the proportion of degree $g + 1$ polynomials that are *not* irreducible of odd degree. If g is odd then this proportion is clearly 1. If g is even then this is the same as the proportion of reducible polynomials of degree $g + 1$, which converges to $1 - (g + 1)^{-1}$. □

Remark 28. In Corollary 27, instead of working with the proportion of equations (8), we can work with the corresponding proportion of $\mathbf{F}_{2^r}$-isomorphism classes of hyperelliptic curves of genus g. This is because the subset of equations (8) that define a hyperelliptic curve of genus g whose only non-trivial geometric automorphism is the hyperelliptic involution (inside the affine space of all equations of this form) is non-empty [Poo1], open, and defined over $\mathbf{F}_2$ (being invariant under the $\mathrm{Gal}(\overline{\mathbf{F}}_2, \mathbf{F}_2)$-action). See also [Zhu].

We finish by identifying the 2-torsion point from the proof of Theorem 2 in the hyperelliptic case with one of the 2-torsion points from the proof of Theorem 23. The former proof provides Θ_{arith} and Θ_{geom} with $2\Theta_{\mathrm{arith}} \sim 2\Theta_{\mathrm{geom}}$, hence the class of $T = \Theta_{\mathrm{arith}} - \Theta_{\mathrm{geom}}$ is two-torsion. We have $2\Theta_{\mathrm{arith}} = \mathrm{div}\, dx$. To compute $2\Theta_{\mathrm{geom}}$, we need to take an appropriate model as in the proof of Lemma 1. The bivariate polynomial $y^2 + H_1(x, 1)y + H_0(x, 1)$ gives an affine model of our hyperelliptic curve C, and if g is odd, then the system from Lemma 1 admits the solution $(1, 1)$. By the proof of that lemma, we should then look at the toric model C'_f where

$$f = x^{-1}(y + H_1(x, 1) + y^{-1}H_0(x, 1)).$$

Then Θ_{geom} is given by $2\Theta_{\mathrm{geom}} = \mathrm{div}\frac{1}{xy\frac{\partial f}{\partial y}}dx$, so we compute

$$\frac{\partial f}{\partial y} = x^{-1}(1 + y^{-2}H_0(x, 1)) = x^{-1}y^{-1}\ H_1(x, 1).$$

We find

$$T = \Theta_{\mathrm{arith}} - \Theta_{\mathrm{geom}} = \frac{1}{2}\ \mathrm{div}\, xy\frac{\partial f}{\partial y} = \frac{1}{2}\mathrm{div}\, H_1(x, 1),$$

where $\mathrm{div}\, H_1(x, 1)$ is twice the sum of all points $P_{(a:b)}$ as $(a : b)$ ranges over the roots of $H_1(X, Z)$ in $\mathbf{P}^1_{\overline{k}}$ (with multiplicity), minus $(g + 1)$ times the divisor D of degree 2 at infinity. This is the 2-torsion point from the proof of Theorem 23 corresponding to the element $(1, 1, \ldots, 1) \in A_k$.

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