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Marc Jules MORONVAL

CONTRIBUTION A LA FORMULATION VARIATIONNELLE DU PROBLEME DE L'OPTIMISATION DE STRUCTURES DU TYPE ARC

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MEMBRES DU JURY : MM. GONTIER G., *Président*
PARSY F., *Rapporteur*
FAURE R., *Examinateur*
KOITER W., *Invité*

A mes parents et beaux-parents,

Cette thèse est l'aboutissement d'un cycle d'études dans le service de Mécanique de l'Université des Sciences et Techniques de LILLE. La résolution des problèmes présentés a été effectuée lors d'un séjour dans le département d'Aéronautique du California Institute of Technology. La formulation des problèmes et les méthodes de résolution utilisées sont décrites en Français et en Anglais. Les problèmes traités sont présentés sous leur forme originale en Anglais. La partie française de cette thèse est numérotée en chiffres romains, la partie anglaise en chiffres arabes.

Je tiens à exprimer à Monsieur le Professeur Parsy ma reconnaissance et mon amitié pour les encouragements et conseils qu'il m'a toujours donnés.

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Je tiens à exprimer ma gratitude envers Monsieur le Professeur W.T. Koiter non seulement pour avoir accepté une invitation pour la soutenance de cette thèse, mais aussi pour l'excellence de son enseignement.

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RESUME

On étudie le problème de l'optimisation de structures du type arc isostatique avec comme inconnues la forme de la ligne moyenne, la distribution de l'épaisseur et la pente du support. Les structures supportent une pression uniforme. La condition de comportement imposée est soit la raideur, soit l'état des contraintes.

On utilise la théorie de la commande optimale pour obtenir les conditions d'optimalité. Les solutions des systèmes d'équations différentielles aux conditions limites sont obtenues numériquement à l'aide de la méthode du "Parallel shooting".

L'inclusion de la contribution de l'effort tranchant dans la définition des conditions de comportement a été suffisante pour assurer l'existence des solutions.

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I - INTRODUCTION.

L'application des méthodes d'optimisation à la conception des structures a pour but de déterminer la "meilleure" solution possible. Dans des sociétés à orientation économique, on voudrait pouvoir minimiser le prix de revient global des constructions. Si l'on tient compte à la fois du prix des matériaux et aussi des coûts indirects résultant de leur poids, par exemple les dépenses en carburant pour le transport des structures aérospatiales, la minimisation du volume des matériaux utilisés est souvent choisie comme objectif primordial. Lorsque le projeteur utilise la méthode traditionnelle d'études, basée sur une procédure d'essais successifs, son expérience lui permet de réduire le nombre de cycles de conception et de vérification infructueux. Dans des domaines où cette expérience n'a pu être acquise, les résultats d'études d'optimisation peuvent guider l'ingénieur pendant la phase de conception et lui fournir les moyens d'évaluer l'efficacité de la solution retenue.

De rapides développements sont intervenus dans le domaine de l'optimisation des structures depuis le début des années 60. Une analyse des publications, réalisée en 1963 par Wasiuntynski et Brandt [1] présente l'historique de ce domaine dont Galilée fut un précurseur. Barnett [2] en 1966 a exposé les techniques et les principes importants de conception ainsi qu'un grand nombre de résultats. L'analyse effectuée par Sheir et Prager [3] relate les progrès réalisés entre 1963 et 1968. Une liste des orientations actuelles de la recherche est contenue dans l'article par Niordson et Pedersen [4].

On peut expliquer, en partie, l'ampleur des travaux effectués durant les quinze dernières années dans le domaine de l'optimisation des structures à l'aide des deux faits suivants. Les programmes spatiaux ont posé aux ingénieurs des problèmes nouveaux dans lesquels le poids des éléments de structure jouait un rôle beaucoup plus important que celui de leur prix de revient. Le développement de calculateurs à grande capacité a permis l'automatisation des méthodes de conception. On pensait que les techniques d'optimisation permettraient d'obtenir la meilleure structure possible dans des domaines où l'expérience était limitée. Mais le jour où des "boîtes noires" vont concevoir automatiquement des structures est encore à venir.

Un problème d'optimisation consiste à trouver la meilleure solution possible dans un ensemble de structures admissibles. Une ou plusieurs conditions imposées sur le comportement caractérisent en partie cet ensemble. Prager et Taylor [5] ont présenté une méthode générale de formulation des problèmes d'optimisation de structures ayant des sections du type "sandwich", soumises à une condition de raideur, de fréquence propre minimale ou de charge de flambement. Huang [6] a traité le problème d'une plaque circulaire supportant une pression uniforme et soumise à une condition de raideur. Niordson [7] a analysé la conception optimale de poutres de fréquence propre minimale donnée. Des problèmes dans lesquels la charge de flambement est imposée ont été résolus respectivement par Frauenthal [9] pour des plaques circulaires et par Taylor [10] pour des colonnes à section du type "sandwich". Une formulation générale pour minimiser la masse de structures soumises à une condition sur l'état des contraintes a été effectuée par Giraudbit [11]. Stroud [12] a présenté une analyse de l'état d'avancement des recherches pour l'optimisation de structures soumises à des conditions du type aéroélastique.

Après la présentation des différents types de conditions pouvant être utilisées pour caractériser le comportement admissible d'une structure, les méthodes de résolution sont exposées. Quand les inconnues du problème sont des fonctions, les conditions d'optimalité sont obtenues à partir du calcul des variations. A cause du caractère hautement non linéaire des systèmes d'équations différentielles aux conditions limites obtenues, seuls des éléments simples d'une structure peuvent être ainsi analysés. Pour des réalisations plus complexes, un ensemble de paramètres est choisi a priori. Ces inconnues peuvent être alors déterminées directement à l'aide des techniques de la programmation mathématique (Réf. [13]) ou indirectement en utilisant des critères d'optimalité (Réf. [14]). Une amélioration de l'efficacité des méthodes de la programmation mathématique peut être obtenue en utilisant des techniques d'approximation (Réf. [15]) ou le concept de dualité de la programmation géométrique (Réf. [16]). Une combinaison des techniques de programmation mathématique et des critères d'optimalité a été utilisée pour le dimensionnement de structures complexes dans la Réf. [17]. Une différence fondamentale existe entre la formulation analytique et la formulation discrète des problèmes. Les méthodes variationnelles ont pour but de déterminer la meilleure structure, alors que les techniques de programmation mathématique améliorent des solutions admissibles. Quand la solution optimale n'existe pas, une méthode de résolution utilisant la deuxième approche pourra améliorer une approximation initiale de la solution. On doit par conséquent utiliser la formulation variationnelle en vue de déterminer si un problème donné est ou n'est pas résoluble.

Dans cette thèse, on considère des problèmes avec deux fonctions inconnues. Un de nos objectifs est de définir une formulation assurant l'existence de solutions pour les cas considérés. On impose soit une condition sur l'état de contrainte, soit une condition sur la raideur de la structure. Une définition générale de ces deux types de problèmes est exposée dans le chapitre II. La théorie de la commande optimale et les méthodes numériques, décrites dans le chapitre III, sont utilisées respectivement pour obtenir les équations d'optimalité et pour résoudre les systèmes d'équations différentielles aux conditions limites obtenues. L'étude de structures isostatiques du type arc ayant comme inconnues la forme de la ligne moyenne, la distribution de l'épaisseur et la pente du support est exposée au chapitre IV.

En considérant les résultats obtenus, il est nécessaire de se souvenir du jugement exprimé par Niordson et Pedersen sur l'approche variationnelle : "Puisque les équations différentielles obtenues sont hautement non linéaires et n'admettent en général aucune solution régulière (des singularités apparaissent très souvent aux bornes de l'intervalle) ces problèmes sont en général très difficiles à résoudre et le domaine semble plutôt obscur".

II - DEFINITION DES PROBLEMES.

On peut définir le problème de la détermination de structures ayant un volume de matériaux minimal de la manière suivante :

"Dans un ensemble donné de structures admissibles, trouver celle ayant un volume de matériaux minimal".

Nous ne saurions insister suffisamment sur le fait que seule la définition de l'ensemble de structures admissibles détermine la solution optimale. Les techniques de minimisation ne sont que les moyens de les trouver.

La définition de l'ensemble des structures admissibles doit inclure :

a) le type de structure et la théorie approchée utilisée pour obtenir ses équations de comportement,

b) les conditions géométriques,

c) les charges supportées par la structure,

d) les conditions aux limites imposées,

e) les conditions imposées sur le comportement de la structure qui limitent les variables caractérisant l'ouvrage.

Les inconnues devant être déterminées par les méthodes de minimisation constituent un ensemble de paramètres et de fonctions.

On ne considère ici que deux types de conditions de comportement :

- *la condition de contrainte*, si on impose à l'état des contraintes dans la structure d'être admissible.

- *la condition de raideur*, si le travail effectué, pendant la déformation par les forces appliquées à la structure doit prendre une valeur imposée.

Une présentation générale de ces deux types de problèmes est incluse dans ce chapitre dans le but de déterminer les outils de minimisation nécessaire à leur résolution.

2.1. - *Définitions*.

On considère des structures pouvant être décrites à l'aide d'une coordonnée x seulement. Soit $\underset{\sim}{s}(x)$ le vecteur des inconnues à déterminer. On représente respectivement par $\underset{\sim}{u}(x)$ et $\underset{\sim}{\sigma}(x)$ le champ des déplacements et le champ des contraintes dans la structure. Soient respectivement $\{\underset{\sim}{F}(x)\}$ et $\underset{\sim}{p}(x)$ l'ensemble des forces concentrées et la charge linéique appliquées sur la structure.

Les équations de comportement de la structure fournissent les deux opérateurs suivants :

a) les équations d'équilibre :

$$L_1(\underset{\sim}{s},\underset{\sim}{u},\{\underset{\sim}{F}\},\underset{\sim}{p}) = 0 \qquad (2.1.)$$

b) les relations entre les contraintes et les déplacements :

$$L_2(\underset{\sim}{s},\underset{\sim}{u},\underset{\sim}{\sigma}) = 0. \qquad (2.2.)$$

On inclut les conditions aux limites dans l'opérateur L_1. Quand on emploie la théorie linéaire de l'élasticité, ou ses approximations, les opérateurs L_1 et L_2 sont linéaires par rapport à $\underset{\sim}{u},\underset{\sim}{\sigma}$, $\{\underset{\sim}{F}\}$ et $\underset{\sim}{p}$ mais sont en général non linéaires par rapport à $\underset{\sim}{s}$.

Le volume des matériaux de la structure est :

$$V = \int v(\underset{\sim}{s})dx \qquad (2.3.)$$

où l'intégrale doit être évaluée sur la structure complète.

2.2. - *Condition de contrainte.*

On suppose l'existence d'une fonction $f(\underset{\sim}{\sigma})$ qui décrit l'état des contraintes à chaque point de la structure. Un état de contrainte est par définition admissible si :

$$f(\underset{\sim}{\sigma}) \leqslant 0 \qquad (2.4.)$$

Le problème de la détermination de structures ayant un volume de matériaux minimal peut être formulé de la manière suivante :

$$\min_{\underset{\sim}{s}} V = \int v(\underset{\sim}{s})dx \qquad (2.5.)$$

avec les conditions :

$$L_1(\underset{\sim}{s},\underset{\sim}{u},\{\underset{\sim}{F}\},\underset{\sim}{p}) = 0 \qquad (2.6.)$$

$$L_2(\underset{\sim}{s},\underset{\sim}{u},\underset{\sim}{\sigma}) = 0 \qquad (2.7.)$$

$$f(\underset{\sim}{\sigma}) \leqslant 0 \qquad (2.8.)$$

Pour résoudre un tel problème, des équations d'optimalité prenant en compte des conditions du type inégalité sont nécessaires.

2.3. - *Condition de raideur.*

Au lieu d'exiger que l'état des contraintes soit admissible, on peut imposer une borne supérieure sur la valeur des déplacements que subit la structure durant sa déformation. Ceci peut être réalisé en imposant la valeur du travail effectué par les charges appliquées à la structure pendant sa déformation. Dans cette deuxième formulation, on impose une valeur moyenne des déplacements, en utilisant les charges comme fonction de pondération. Soit $U(\underset{\sim}{s},\{\underset{\sim}{F}\},\underset{\sim}{p})$ le travail effectué par les forces appliquées. Sa valeur est donnée par :

$$U(\underset{\sim}{s},\{\underset{\sim}{F}\},\underset{\sim}{p}) = \frac{1}{2}\{\Sigma\ \underset{\sim}{F}.\underset{\sim}{u} + \int \underset{\sim}{p}.\underset{\sim}{u}\ dx\} \qquad (2.9.)$$

où la sommation doit être effectuée pour toutes les forces concentrées et l'intégrale est évaluée sur la structure complète. Le champ des déplacements $\underset{\sim}{u}$ est obtenu à partir de l'opérateur d'équilibre L_1.

Comme il a été démontré par Wasiuntisky [18], le problème de la minimisation du volume de matière utilisé pour une condition de raideur est équivalent à celui de la minimisation du travail des forces extérieures avec un volume de matériaux imposé. Dans ces deux problèmes, les équations d'équilibre doivent être satisfaites.

Par conséquent, le problème de la minimisation du volume des matériaux, pour une condition de raideur, peut être formulé de la manière alternative suivante :

$$\underset{\underset{\sim}{s}}{\mathrm{Min}}\ U = \frac{1}{2}\{\Sigma\ \underset{\sim}{F}.\underset{\sim}{u} + \int \underset{\sim}{p}.\underset{\sim}{u}\ dx\} \qquad (2.10.)$$

avec les conditions

$$L_1(\underset{\sim}{s},\underset{\sim}{u},\{\underset{\sim}{F}\},\underset{\sim}{p}) = 0 \qquad (2.11.)$$

$$\int v(\underset{\sim}{s})dx = V_o. \qquad (2.12.)$$

Pour résoudre ce problème, des conditions d'optimalité prenant en compte des conditions intégrales sont requises.

On remarque que dans le contexte de la théorie linéaire de l'élasticité, ou de ses approximations, la valeur du travail effectué par les charges externes U est égale à celle de l'énergie de déformation de la structure W calculée dans la configuration d'équilibre. Ceci fournit une méthode de formulation particulièrement attractive pour des structures isostatiques. Dans ce cas particulier, le calcul de l'énergie de déformation qui nécessite la détermination du champ des contraintes $\underset{\sim}{\sigma}$ peut être effectué sans évaluer le champ des déplacements $\underset{\sim}{u}$. Les équations d'équilibre d'un système isostatique peuvent s'écrire :

$$L_3(\underset{\sim}{s},\underset{\sim}{\sigma},\{\underset{\sim}{F}\},\underset{\sim}{p}) = 0. \qquad (2.13.)$$

Le problème de la minimisation du volume des matériaux utilisés peut être formulé dans ce cas particulier de la manière suivante :

$$\underset{\underset{\sim}{s}}{\text{Min}}\ J = \int W(\underset{\sim}{\sigma})dx \qquad (2.14.)$$

avec les conditions

$$L_3(\underset{\sim}{s},\underset{\sim}{\sigma},\{\underset{\sim}{F}\},\underset{\sim}{p}) = 0 \qquad (2.15.)$$

$$\int v(\underset{\sim}{s})dx = V_o.$$

2.4. - *Résumé.*

On a montré que le problème de la minimisation du volume des matériaux d'une structure fait intervenir des conditions du type inégalité, pour une condition de contrainte, et du type intégral pour une condition de raideur. Les méthodes d'obtention des équations d'optimalité et les méthodes numériques utilisées pour les résoudre sont décrites dans le chapitre suivant.

III - CONDITIONS D'OPTIMALITE ET METHODES NUMERIQUES.

Quand les inconnues devant être déterminées sont des fonctions, et non des paramètres, on obtient les équations d'optimalité en utilisant le calcul des variations. A cause du caractère hautement non linéaire des systèmes d'équations différentielles obtenues, une résolution numérique est en général nécessaire.

La théorie de la commande optimale est utilisée pour obtenir les conditions d'optimalité du premier ordre. On a préféré ce formalisme à celui du calcul des variations classique car il fournit directement un système d'équations différentielles du premier ordre, bien adapté pour une résolution numérique. Les systèmes d'équations différentielles aux conditions limites ainsi obtenus sont résolus en utilisant la méthode du "Parallel shooting". Un programme très général a été développé pour la résolution de familles de problèmes dépendant d'un paramètre et dans lesquels des conditions sont imposées en n points de l'intervalle d'intégration.

On rappelle d'abord les conditions d'optimalité du premier ordre. Une présentation détaillée de la méthode de la commande optimale est contenue dans la Réf. [19]. La méthode du "Parallel shooting" et les possibilités du programme réalisé sont décrites dans la deuxième partie de ce chapitre. Une analyse des différentes méthodes de résolution numérique des systèmes d'équations différentielles aux conditions limites est contenue dans la Réf. [20].

3.1. - *Conditions d'optimalité.*

Le problème de minimisation, sans condition imposée, est d'abord défini et les conditions d'optimalité du premier ordre sont obtenues. On établit ensuite les conditions d'optimalité relatives à des variantes du premier problème.

3.1.1. - *Minimisation sans condition.*

Les équations de comportement d'un système physique peuvent s'écrire :

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x,\underset{\sim}{y}(x),\underset{\sim}{u}(x)) \qquad (3.1.)$$

où $\underset{\sim}{y}(x)$ est le vecteur des variables d'état avec m composantes.

$\underset{\sim}{u}(x)$ est le vecteur des variables de contrôle avec n composantes.

Les équations (3.1.) sont appelées équations d'état du système.

On considère le problème suivant :

$$\underset{\underset{\sim}{u}(x)}{\text{Min}}\ J = \phi(\underset{\sim}{y}(x_0),\underset{\sim}{y}(x_1)) + \int_{x_0}^{x_1} L(x,\underset{\sim}{y}(x),\underset{\sim}{u}(x))dx \qquad (3.2.)$$

avec les conditions

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x,\underset{\sim}{y}(x),\underset{\sim}{u}(x)) \qquad (3.3.)$$

$$\left.\begin{array}{lll} \underset{\sim}{y}(x_0) & \text{donné} & (a) \\ \underset{\sim}{y}(x_1) & \text{arbitraire} & (b) \end{array}\right\} \quad (3.4.)$$

L'objectif du problème est J, son Lagrangien est L.

On obtient les conditions nécessaires d'optimalité du premier ordre en imposant que J soit stationnaire par rapport à des variations arbitraires des variables de contrôle $\underset{\sim}{u}(x)$. On définit

$$\tilde{J} = J + \int_{x_0}^{x_1} \underset{\sim}{\lambda}^T \cdot \left[\underset{\sim}{f}(x,\underset{\sim}{y},\underset{\sim}{u}) - \frac{d\underset{\sim}{y}}{dx}\right] dx \qquad (3.5.)$$

$$H = L + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f} \qquad (3.6.)$$

où :

a) $\underset{\sim}{\lambda}^T$ est le transposé d'un vecteur de m multiplicateurs de Lagrange qui seront choisis de manière telle que le coefficient des variations $\delta \tilde{y}(x)$ dans $\delta\tilde{J}$ soit nul.

b) H est l'Hamiltonien du système.

Les variations $\delta\tilde{J}$ de $\tilde{J}$ sont :

$$\delta\tilde{J} = \left[\frac{\partial\psi}{\partial y(x_0)} + \underset{\sim}{\lambda}^T\right]_{x=x_0} \partial y(x_0) + \left[\frac{\partial\psi}{\partial y(x_1)} - \lambda^T\right]_{x=x_1} \delta\underset{\sim}{y}(x_1)$$

$$+ \int_{x_0}^{x_1}\left[\{L_{\underset{\sim}{u}} + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f}_{\underset{\sim}{u}}\}\, \delta\underset{\sim}{u}(x) + \{L_{\underset{\sim}{y}} + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f}_{\underset{\sim}{y}} + \frac{d\underset{\sim}{\lambda}^T}{dx}\}\, \delta\underset{\sim}{y}(x) dx\right]$$

(3.7.)

où $L_{\underset{\sim}{u}} \equiv \dfrac{\partial L(x,y,u)}{\partial \underset{\sim}{u}}$.

En choisissant les multiplicateurs de Lagrange de la manière suivante :

$$\frac{d\underset{\sim}{\lambda}^T}{dx} = - \left[L_{\underset{\sim}{y}} + \underset{\sim}{\lambda}.\underset{\sim}{f}_{\underset{\sim}{y}}\right] , \qquad (3.8.)$$

les variations $\delta\tilde{J}$ de $\tilde{J}$ s'annulent si ;

$$L_{\underset{\sim}{u}} + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f}_{\underset{\sim}{u}} = 0 \qquad (3.9.)$$

$$\underset{\sim}{\lambda}^T(x_1) = \frac{\partial\psi}{\partial \underset{\sim}{y}(x_1)} \qquad (3.10.)$$

Aucune condition n'est imposée sur la valeur des multiplicateurs de Lagrange à $x = x_0$ puisque $\underset{\sim}{y}(x_0)$ est prescrit. On doit remarquer que ces conditions ne sont que suffisantes pour que $\tilde{J}$ soit stationnaire, puisque l'on a supposé l'existence des multiplicateurs de Lagrange $\underset{\sim}{\lambda}$.

On résume les résultats précédents :

a) Problème

$$\underset{\underset{\sim}{u}(x)}{\text{Min}}\ J = \psi(\underset{\sim}{y}(x_0), \underset{\sim}{y}(x_1)) + \int_{x_0}^{x_1} L(x, \underset{\sim}{y}(x), \underset{\sim}{u}(x))dx \qquad (3.11.)$$

avec les conditions

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}(x), \underset{\sim}{u}(x)) \qquad (3.12.)$$

$$\underset{\sim}{y}(x_0) \text{ donné} \qquad (3.13.)$$

b) Conditions d'optimalité du premier ordre :

$$H \equiv L + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f} \qquad (3.14.)$$

$$H_{\underset{\sim}{u}} = 0 \qquad (3.15.)$$

$$\frac{d\underset{\sim}{\lambda}^T}{dx} = -H_{\underset{\sim}{y}} \qquad (3.16.)$$

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x,\underset{\sim}{y},\underset{\sim}{u}) \tag{3.17.}$$

$$\left.\begin{array}{ll} \underset{\sim}{y}(x_0) \quad \text{donné} & (a) \\ \underset{\sim}{\lambda}^T(x_1) = \dfrac{\partial\psi}{\partial \underset{\sim}{y}(x_1)} & (b) \end{array}\right\} \quad (3.18.)$$

Les variables de contrôle $\underset{\sim}{u}$ sont calculées à partir du système de n équations algébriques (3.15.). Les équations différentielles (3.16.) et (3.17.) forment un système dont les conditions limites sont données par (3.18.).

A cause de leur caractère hautement non linéaire, il n'est en général pas possible de résoudre directement les équations (3.15.). Cependant, elles peuvent être utilisées pour obtenir un système d'équations différentielles pour les variables de contrôle $\underset{\sim}{u}$, Puisque (3.15.) doit être satisfait pour tout x, $x \in [x_0,x_1]$ alors

$$\frac{dH_{\underset{\sim}{u}}}{dx} = 0 \tag{3.19.}$$

$$H_{\underset{\sim}{u}}\Big|_{x=x^*} = 0, \quad \text{où} \quad x^* \in [x_0,x_1]. \tag{3.20.}$$

Les équations (3.19.) se réduisent à :

$$H_{\underset{\sim}{u}\underset{\sim}{u}} \frac{d\underset{\sim}{u}}{dx} = - \left[H_{\underset{\sim}{u}x} + H_{\underset{\sim}{u}\underset{\sim}{y}}\underset{\sim}{f} - H_{\underset{\sim}{u}\underset{\sim}{\lambda}} H_{\underset{\sim}{y}}\right] \tag{3.21.}$$

Les équations (3.21.) déterminent $\dfrac{d\underset{\sim}{u}}{dx}$ d'une manière unique quand :

$H_{\underset{\sim}{u}\underset{\sim}{u}}$ n'est pas singulier, c'est-à-dire quand

$$\text{dét}\left[H_{\underset{\sim}{u}\underset{\sim}{u}}\right] \neq 0 \tag{3.22.}$$

On peut écrire, quand la condition (3.22.) est satisfaite, l'ensemble II des conditions d'optimalité de la manière suivante :

$$\left.\begin{aligned} \frac{d\underset{\sim}{u}}{dx} &= - H_{\underset{\sim}{u}\underset{\sim}{u}}^{-1}\left[H_{\underset{\sim}{u}x} + H_{\underset{\sim}{u}\underset{\sim}{y}}\,\underset{\sim}{f} - H_{\underset{\sim}{u}\underset{\sim}{\lambda}}\,H_{\underset{\sim}{y}}\right] && (a)\\ \frac{d\underset{\sim}{\lambda}^T}{dx} &= - H_{\underset{\sim}{y}} && (b)\\ \frac{d\underset{\sim}{y}}{dx} &= \underset{\sim}{f}(x,\underset{\sim}{y},\underset{\sim}{u}) && (c) \end{aligned}\right\} \quad (3.23.)$$

$$\left.\begin{aligned} & H_{\underset{\sim}{u}}\big|_{x=x^*} = 0 && (a)\\ & \underset{\sim}{y}(x_0) \quad \text{prescrit} && (b)\\ & \underset{\sim}{\lambda}^T(x_1) = \frac{\partial\psi}{\partial\underset{\sim}{y}(x_1)} && (c) \end{aligned}\right\} \quad (3.24.)$$

Les équations (3.23.) définissent un système d'équations différentielles dont les conditions limites, spécifiées à des valeurs différentes de x, sont données par (3.24.).

3.1.2. - *Problème dépendant d'un paramètre inconnu.*

Problème :

$$\min_{\underset{\sim}{u}(x),\alpha} \quad J = \int_{x_0}^{x_1} L(x,\underset{\sim}{y}(x),\underset{\sim}{u}(x),\alpha)dx \qquad (3.25)$$

avec les conditions

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x,\underset{\sim}{y},\underset{\sim}{u},\alpha) \qquad (3.26.)$$

On peut appliquer les résultats précédents en considérant le nouveau vecteur des variables d'état ;

$$\bar{\underset{\sim}{y}} = \begin{bmatrix} \underset{\sim}{y} \\ \alpha \end{bmatrix} \qquad (3.27.)$$

L'équation d'état associée avec y_{m+1} est :

$$\frac{dy_{m+1}}{dx} = 0 \qquad (3.28.)$$

Puisque la valeur de $y_{m+1}(x_0)$ n'est pas imposée, les variations admissibles $\delta y_{m+1}(x_0)$ ne sont pas nulles. Les conditions d'optimalité sont données par (3.23.) et (3.24.) où l'Hamiltonien est :

$$H = L + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f} + \lambda_{m+1} \times (0) \qquad (3.29.)$$

Les conditions limites pour λ_{m+1} sont :

$$\lambda_{m+1}(x_0) = \lambda_{m+1}(x_1) = 0 \qquad (3.30.)$$

3.1.3. - *Cas où certaines variables d'état sont prescrites à $x = x_1$.*

Problème :

$$\underset{\underset{\sim}{u}(x)}{\text{Min}} \; J = \phi(\underset{\sim}{y}(x_0), \underset{\sim}{y}(x_1)) + \int_{x_0}^{x_1} L(x, \underset{\sim}{y}, \underset{\sim}{u})\,dx \qquad (3.31.)$$

avec les conditions

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}, \underset{\sim}{u})$$

$y_i(x_0)$ donné, $i \in I$

$y_j(x_1)$ donné, $j \in J$.

Les conditions d'optimalité (3.15.) à (3.17.) restent applicables à condition que le système soit contrôlable. Les conditions limites sont dans ce cas :

$$y_i(x_0) \text{ donné, } i \in I \tag{3.32.}$$

$$\lambda_k(x_0) = - \frac{\partial \phi}{\partial y_k(x_0)} \quad k \notin I \tag{3.33.}$$

$$y_j(x_1) \text{ donné, } j \in J \tag{3.34.}$$

$$\lambda_\ell(x_1) = \frac{\partial \phi}{\partial y_\ell(x_1)} \quad \ell \notin J \tag{3.35.}$$

<u>*Remarque*</u> : Si les variables d'état doivent satisfaire une condition du type $\Psi(\underset{\sim}{y}(x_0),\underset{\sim}{y}(x_1)) = 0$, les conditions aux limites sont obtenues en ajoutant Ψ à ϕ avec le multiplicateur de Lagrange ν :

$$\tilde{\phi} = \phi + \nu \Psi \tag{3.36.}$$

Les conditions aux limites sont :

$$y_i(x_0) \text{ prescrit, } i \in I \tag{3.37.}$$

$$\lambda_k(x_0) = - \left[\frac{\partial \phi}{\partial y_k(x_0)} + \nu \frac{\partial \Psi}{\partial y_k(x_o)} \right] \quad k \notin I \tag{3.38.}$$

$$y_j(x_1) \text{ prescrit, } j \in I \tag{3.39.}$$

$$\lambda_\ell(x_1) = \left[\frac{\partial \phi}{\partial y_\ell(x_1)} + \nu \frac{\partial \Psi}{\partial y_\ell(x_0)} \right] \quad \ell \notin J \tag{3.40.}$$

$$\Psi(\underset{\sim}{y}(x_0), \underset{\sim}{y}(x_1)) = 0 \tag{3.41.}$$

3.1.4. - *Problème avec une condition du type intégral.*

Problème :

$$\underset{\underset{\sim}{u}(x)}{\text{Min}} \; J = \phi(\underset{\sim}{y}(x_0), \underset{\sim}{y}(x_1)) + \int_{x_0}^{x_1} L(x, \underset{\sim}{y}, \underset{\sim}{u})\,dx \tag{3.42.}$$

avec les conditions :

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}, \underset{\sim}{u}) \tag{3.43.}$$

$$\int_{x_0}^{x_1} g(x, \underset{\sim}{y}, \underset{\sim}{u})\,dx = C \tag{3.44.}$$

Les résultats précédents peuvent être appliqués en prenant comme vecteur d'état :

$$\bar{\underset{\sim}{y}} = \begin{bmatrix} \underset{\sim}{y} \\ y_{m+1} \end{bmatrix} \tag{3.45.}$$

L'équation d'état associée à y_{m+1} est :

$$\frac{dy_{m+1}}{dx} = g(x, \underset{\sim}{y}, \underset{\sim}{u}) \tag{3.46.}$$

Les conditions aux limites sur y_{m+1} sont :

$$y_{m+1}(x_0) = 0, \quad y_{m+1}(x_1) = C \; . \tag{3.47.}$$

L'Hamiltonien du système devient :

$$H = L + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f} + \lambda_{m+1}\, g(x,\underset{\sim}{y},\underset{\sim}{u}) \; . \tag{3.48.}$$

Puisque le Lagrangien L et les équations d'état (3.43.) ne sont pas fonction de y_{m+1} l'équation différentielle pour λ_{m+1} est :

$$\frac{d\lambda_{m+1}}{dx} = 0 \tag{3.49.}$$

Les conditions aux limites sur λ_{m+1} sont inconnues puisque les valeurs de y_{m+1} sont données à x_0 et x_1. Le multiplicateur de Lagrange λ_{m+1} est une constante inconnue dont la valeur est déterminée par la satisfaction de la condition (3.44.).

3.1.5. - *Problème avec une condition du type inégalité.*

Problème :

$$\underset{\underset{\sim}{u}(x)}{\text{Min}} \; J = \int_{x_0}^{x_1} L(x,\underset{\sim}{y},\underset{\sim}{u})dx \tag{3.50.}$$

avec les conditions

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x,\underset{\sim}{y},\underset{\sim}{u}) \tag{3.51.}$$

$$C(x,\underset{\sim}{y},\underset{\sim}{u}) \leq 0 \tag{3.52.}$$

Soit H^* défini par :

$$H^* = L + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f} \tag{3.53.}$$

Les variations de $\tilde{J}$ données par (3.7.) sont :

$$\delta\tilde{J} = \int_{x_0}^{x_1} H^*_{\underset{\sim}{u}} \cdot \delta\underset{\sim}{u}(x)dx \qquad (3.54.)$$

La condition (3.52.) est dite "effective" sur un arc de solution si

$$C(x,\underset{\sim}{y},\underset{\sim}{u}) = 0 \qquad (3.55.)$$

Quand l'inégalité stricte est satisfaite, la condition est dite ineffective.

Pour que $\tilde{J}$ soit minimum, il faut que $\delta\tilde{J}$ soit positif ou nul pour toute variation admissible des variables de contrôle,

Quand la condition est effective, les variations admissibles de $\underset{\sim}{u}$ doivent être telles que

$$C_{\underset{\sim}{u}} \cdot \delta\underset{\sim}{u} \leqslant 0, \quad \int_{x_0}^{x_1} H^*_{\underset{\sim}{u}} \cdot \delta\underset{\sim}{u}\, dx \geqslant 0 \qquad (3.56.)$$

Une condition suffisante est obtenue en introduisant un multiplicateur de Lagrange non négatif tel que ;

$$H^*_{\underset{\sim}{u}} + \eta\, C_{\underset{\sim}{u}} = 0 \qquad (3.57.)$$

Quand la condition n'est pas effective, les variations admissibles $\delta\underset{\sim}{u}$ ne sont pas restreintes par la condition (3.52.). Dans ce cas, les conditions d'optimalité (3.15.) sont applicables.

Un ensemble de conditions d'optimalité semblable à (3.15.) et (3.16.) peut être obtenu en transformant la condition du type inégalité par une condition du type égalité à l'aide d'une variable de contrôle additionnelle μ telle que :

$$C(x,\underset{\sim}{y},\underset{\sim}{u}) + \mu^2 = 0 \qquad (3.58.)$$

L'Hamiltonien H est obtenu en ajoutant à (3.53.) la condition (3.58.) à l'aide d'un multiplicateur de Lagrange non négatif η

$$H = L + \underset{\sim}{\lambda}^T \underset{\sim}{f} + \eta\left[C(x,\underset{\sim}{y},\underset{\sim}{u}) + \mu^2\right] \qquad (3.59.)$$

3.1.6. - *Résumé.*

Les conditions d'optimalité du premier ordre exposées dans cette section ne sont que des conditions suffisantes puisqu'on a supposé l'existence des divers multiplicateurs de Lagrange. Cependant les conditions du premier ordre ne sont que nécessaires pour qu'une solution soit un optimum local. Pour déterminer si une solution particulière correspond, ou non, à un minimum local, on utilisera une condition suffisante du second ordre : nous appliquerons la condition de convexité, De plus des perturbations des solutions seront effectuées.

3.2. - *Méthodes Numériques.*

La technique du "shooting" est une méthode générale de résolution de systèmes d'équations différentielles du premier ordre aux conditions limites. On décrit d'abord la méthode pour le cas d'un système d'équations différentielles dont les conditions limites sont spécifiées en deux points. Les possibilités du programme général réalisé sont ensuite exposées.

3.2.1. - *Méthode.*

Problème 1 :

Trouver les solutions de :

$$\left.\begin{array}{ll}
\dfrac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x,\underset{\sim}{y}(x)) \quad \text{n équations} & \text{(a)} \\
\text{avec les conditions limites} & \\
y_i(x_0) = a_i, \quad i \in I \quad \text{(q conditions)} & \text{(b)} \\
y_j(x_1) = b_j, \quad j \in J \quad \text{(n - q conditions)} & \text{(c)}
\end{array}\right\} \qquad (3.60.)$$

Ce n'est pas un problème classique aux valeurs initiales car les valeurs de certaines composantes de $\underset{\sim}{y}$ ne sont pas spécifiées à $x = x_0$.

En effet :

$$y_k(x_0) \text{ est inconnu, } k \notin I \qquad (3.61.)$$

Cependant la solution du problème 1 peut être obtenue en étudiant le problème 2 aux valeurs initiales défini ci-dessous.

Problème 2 :

Trouver la solution de :

$$\frac{d\underset{\sim}{y}}{dx} = f(x,\underset{\sim}{y}(x)) \qquad (3.62.)$$

avec les conditions initiales

$$\left.\begin{array}{ll}
y_i(x_0) = a_i & i \in I \\
y_k(x_0) = s_k & j \notin I
\end{array}\right\} \qquad (3.63.)$$

Soit $u(x,s_k)$ la solution du problème 2. Elle sera une solution du problème 1 si :

$$\Phi_j(s_k) \equiv u_j(x_1, s_k) - b_j = 0 \qquad j \in J, \quad k \notin I \qquad (3.64.)$$

On appelle fonction d'erreur la fonction Φ des valeurs initiales s_k ci-dessus définie. Le problème 1 a été transformé en un problème aux conditions initiales où les valeurs initiales s_k doivent être déterminées pour satisfaire aux $n - p$ équations algébriques (3.64.).

Ces équations algébriques sont résolues à l'aide de la méthode de Newton [20], ce qui nécessite la résolution d'une séquence de problèmes aux valeurs initiales du type 2 où

$$y_k(x_0) = t_k^{\nu}, \qquad k \notin I \qquad (3.65.)$$

L'approximation $t_k^{\nu+1}$ de la solution s_k est obtenue à partir de la précédente, t_k^{ν}, à l'aide de :

$$\frac{\partial \Phi_j}{\partial s_k}\Big|_{t_k^{\nu}} (t_k^{\nu+1} - t_k) = - \Phi(t_k^{\nu}) \qquad j \in J, \quad k \notin I \qquad (3.66.)$$

où $$\frac{\partial \Phi_j}{\partial s_k} = \frac{\partial y_k(x_1)}{\partial y_k(x_0)} \qquad j \in J, \quad k \notin I \qquad (3.67.)$$

On obtient le gradient des valeurs finales $y_j(x_1)$ par rapport aux conditions initiales inconnues $y_k(x_0)$ par intégration du système variationnel de (3.62.) :

$$\frac{d}{dx} \frac{\partial y_j}{\partial y_k(x_0)} = \frac{\partial f_j}{\partial y_\ell} \frac{\partial y_\ell}{\partial y_k(x_0)} \qquad \text{(a)}$$

avec les valeurs initiales

$$\left.\frac{\partial y_\ell}{\partial y_k(x_0)}\right|_{x_0} = \begin{cases} 0 & \text{si} \quad k \neq \ell \\ 1 & \text{si} \quad k = \ell \end{cases} \qquad \text{(b)} \quad (3.68.)$$

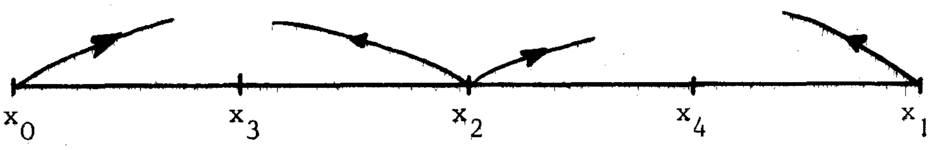
La répétition de l'indice muet ℓ dans (3.38.) implique une sommation de $\ell = 1$ à $\ell = n$. Par conséquent, l'évalution de $\frac{\partial \Phi_j}{\partial s_k}$ nécessite l'intégration de $(n-p) \times n$ équations différentielles linéaires.

On obtient la solution du problème 1 aux conditions limites en résolvant une séquence de problèmes 2 aux valeurs initiales. Le nombre total d'équations différentielles devant être intégrées à chaque itération est $n + n \times (n-p)$. La méthode itérative, décrite ci-dessus, nécessite une approximation initiale t_k^o des valeurs inconnues.

Remarques :

a) Le choix de l'approximation initiale doit être effectué judicieusement surtout quand la solution du problème 1 n'est pas unique.

b) On divise en général l'intervalle d'intégration $[x_0, x_1]$ en plusieurs sous intervalles. Ceci a le double avantage de réduire la propagation des erreurs numériques d'intégration et d'éviter les difficultés pratiques causées par la croissance exponentielle des solutions du système variationnel. Des conditions initiales sont données à l'origine de chaque sous intervalle. La continuité des solutions est obtenue à l'extrémité de chaque intervalle. Cette modification de la méthode exposée ci-dessus est appelée méthode du "parallel shooting". La figure suivante illustre la méthode.



Les points :

- x_0, x_2, x_1 sont les origines des sous intervalles.
- x_3, x_4 sont les extrémités des sous intervalles.

Les sous intervalles sont :

$$[x_0,x_3],\ [x_2,x_3],\ [x_2,x_4] \text{ et } [x_4,x_1] .$$

3.2.2. - *Programme général.*

Quand on utilise la méthode du "parallel shooting", la plus grande partie du temps de programmation est utilisée pour construire la matrice variationnelle (3.64.). Pour pallier à cet inconvénient d'ordre pratique, un programme général a été développé. Il utilise la méthode du "parallel shooting" pour résoudre le problème :

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x,\underset{\sim}{y},\underset{\sim}{\alpha},\gamma) \qquad n \text{ équations} \qquad (a)$$

$$\frac{d\underset{\sim}{u}}{dx} = g(x,\underset{\sim}{y},\underset{\sim}{u},\underset{\sim}{\alpha},\gamma) \qquad m \text{ équations} \qquad (b)$$

$$\int_{x_0}^{x_1} \underset{\sim}{h}(x,\underset{\sim}{y},\underset{\sim}{\alpha},\gamma) = a \qquad q \text{ intégrales} \qquad (c)$$

avec les conditions limites :

$$\left.\begin{array}{lll} y_{m_k}(x_k) & \text{imposé} & (d) \\ y_{j_k}(x_k) & \text{inconnu} & (e) \\ y_{n_\ell}(z_\ell) & \text{imposé} & (f) \\ y_{i_\ell}(z_\ell) & \text{inconnu} & (g) \\ N(y_{j_k}(x_k),\ y_{i_\ell}(z_\ell)) = 0 & & (h) \end{array}\right\} \quad (3.69.)$$

où :

x_k, $k = 1, \ldots, K$ sont les origines des sous intervalles

z_ℓ, $\ell = 1, \ldots, L$ sont les extrémités des sous intervalles

$\underset{\sim}{\alpha}$ est un vecteur de coefficients inconnus

γ est un paramètre par rapport auquel une famille de problèmes est définie.

$\underset{\sim}{u}$ est un vecteur devant être calculé sur la solution.

$N(y_{j_k}(x_k), y_{i_\ell}(z_\ell)) = 0$ est un ensemble de conditions non linéaires devant être satisfaites.

Le système d'équations différentielles (3.69. a), (3.69 b) et les intégrales sont définies par l'utilisateur dans un sous-programme. Les conditions non linéaires (3.69.c) sont aussi définies dans un sous-programme. Les conditions aux limites et les caractéristiques des origines et des extrémités des sous intervalles sont définies par cartes de données. On a trouvé le programme très utile car le nombre de sous intervalles peut être modifié sans que cela entraine un travail excessif.

3.3. - *Résumé*.

On distingue trois phases dans les problèmes d'optimisation présentés dans le chapitre suivant :

- définition du problème
- obtention d'un système d'équations différentielles aux conditions limites.
- résolution du système obtenu en utilisant, d'une manière générale, la méthode du "shooting".

Nous insistons de nouveau sur le fait que seule la première phase, c'est-à-dire la définition du problème détermine la solution.

V - CONCLUSION.

On a étudié le problème de l'optimisation de structures du type arc isostatique soumis à une condition de raideur ou à une condition de contrainte. Les équations d'optimalité ont été obtenues à l'aide de la théorie de la commande optimale. Un programme général a été réalisé pour résoudre des systèmes d'équations différentielles aux conditions limites.

L'inclusion de la contribution de l'effort tranchant dans la définition de l'énergie de déformation et de l'état des contraintes a été suffisante pour assurer l'existence de solutions.

Lorsqu'on impose une condition de raideur, une structure en état de membrane, c'est-à-dire un arc sur lequel le moment fléchissant est nul, ne satisfait pas les équations d'optimalité. Cependant, la meilleure solution de membrane est une très bonne approximation de la solution pour des applications pratiques

Lorsqu'on impose une condition de contrainte définie à l'aide du critère du cisaillement maximal, la meilleure solution de membrane est la solution optimale pour des valeurs courantes du coefficient de charge. Cependant, le nombre et le caractère des solutions des équations d'optimalité du premier ordre sont fonctions de ce paramètre.

Nous avons donc mis en évidence que certaines hypothèses simplificatrices, justifiées pour l'analyse des structures, ne pouvaient être adoptées pour des études d'optimisation.

I. INTRODUCTION

The purpose of structural optimization is to provide systematic ways of obtaining "better" structures. In economically oriented societies, one would like to minimize the cost likely to be associated with the structure during its lifetime. Considering the price of the material and the indirect cost associated with the weight of a structure, as for example the fuel needed for the transportation of aerospace structures, the minimization of the material volume can often be chosen as the primary objective. Furthermore the traditional design procedure of trial and error relies heavily on previous experience in order to maintain the number of design and analysis cycles as small as possible. In areas for which previous knowledge is not readily available the structural optimization techniques provide the engineer with guidelines for the conception of the structures and with a reference to evaluate the merits of the final design.

Since the early 60's the field of structural optimization underwent a rapid development. A review article published in 1963 by Wasiuntynski [1] describes the history of the field, starting with Galileo. Barnett [2] in 1966 presented a survey of the important design techniques and principles together with a number of significant results. The review by Sheu and Prager [3] covers the progress made between 1963 and 1968. An extensive reference list of the recent areas of research is contained in the article by Niordson and Pedersen [4].

Two reasons can be found for the extensive work for the past 15 years in the field of structural optimization. The space programs provided the designers with new challenges in which the structural weight was much more critical than the construction cost. Also the development of large computers made it feasible to automate the design process. It was felt that the optimization procedures would provide means of finding the best possible structures in those domains for which previous experience was limited. But the day where "black boxes" will design structural systems is still to come.

A structural optimization problem consists of finding the "best" possible element in a given class of admissible designs. One or several behavioral constraints characterize, in part, the feasibility of a given design. Prager and Taylor [5] presented a uniform method of treating problems of optimal design of sandwich structures subjected to constraint on the stiffness, on the fundamental frequency or on the buckling load. Huang [6] treated the problem of a solid circular plate supporting a uniform pressure with a stiffness constraint. Niordson [7] analyzed the optimal elastic design of solid beams with a prescribed natural frequency. Cases where the buckling load is imposed have been treated by Budiansky and Frauenthal [8] for arches, by Frauenthal [9] for circular plates, and by Taylor and Liu [10] for sandwich columns. A general formulation of the mass minimization of structures subjected to stress constraints has been given by Giraudbit [11]. A review and an assessment of the state of the art in optimal aeroelastic design has been presented by Stroud [12].

Having discussed some of the behavioral constraints entering in the formulation of a structural optimization problem, let us review the methods of solution. An analytical formulation, using the Calculus of Variations to obtain the optimality conditions, can be made when unknown design functions are chosen. Optimization of simple structural elements only can be treated in this fashion due to the highly non-linear character of the resulting differential equations. For large structural systems, an a priori discretization is performed. The unknown design variables are reduced into a set of parameters. They can be determined directly using the mathematical programming techniques of Ref. [13], or indirectly by means of optimality criteria, cf. Ref. [14]. An increase of the efficiency of the mathematical programming techniques can be achieved by using approximative concepts, cf. Ref. [15], or by utilizing the geometric programming methods, cf. Ref. [16]. To size complex structures a combination of the mathematical programming and the optimality criteria approach is used in Ref. [17]. An important difference exists between the analytical and the "discretized" formulations. The variational approach is concerned with finding the optimal solution as the mathematical programming techniques improve feasible designs. When the optimal solution does not exist, a solution technique based on the second approach may yield a "better" design even when there is no "best" design. Investigations using the analytical method are therefore necessary to determine which classes of problems are, or are not, well posed. The aim of this thesis is to investigate optimization problems in which two design functions are unknown. One of our objectives is to determine formulations of the investigated cases, leading to well-posed problems. The behavioral constraint is either the state of stress or the stiffness of the structure. A definition of the problems is given in Chapter II. The Optimal Control Theory and the Shooting Techniques, described in Chapter III, are used respectively to derive the optimality conditions and to solve the resulting two points boundary value problems. Statically determinant arches of unknown middle line shape, unknown thickness distribution and unknown slope of the support are investigated in Chapter IV. In evaluating the following work, one has to remember the assessment made on the analytical approach by Niordson and Pedersen [4]:

"Due to the fact that the differential equations are often highly non-linear and have no regular solution (singularities very often appear at the boundaries) such problems may be rather cumbersome and tricky to solve and the whole field seems rather unclarified."

II. DEFINITION OF THE PROBLEMS

The problem of determining structures of minimum material volume can be defined as:

"In a given class of admissible structures, find the one of minimum material volume."

It is to be understood that only the definition of the class of admissible structures will determine the optimal design. The minimization techniques are only means of finding the solution.

The definition of the class of admissible structures should specify:

a) the type of structures and the approximative theory used to derive their governing equations

b) the geometrical constraints

c) the loading case

d) the imposed boundary conditions

e) the design requirements which constrain the design variables.

The design variables constitute a set of parameters and of functions with respect to which the minimization procedure is to be performed.

In the present analysis, two types of design requirements are taken into consideration. When the state of stress in the structure is constrained to be admissible, the problem will be referred to as "stress case." The work done by the prescribed external forces on the structure is constrained for the "stiffness case."

A general discussion of those two problems is included in this chapter, for the purpose of defining the minimization tools needed to find their solutions.

2-1. Definitions

Let us consider only structures which can be described using one coordinate x only. Let $\underset{\sim}{s}(x)$ be the set of design parameters. Let $\underset{\sim}{u}(x)$ and $\underset{\sim}{\sigma}(x)$ represent respectively the field of displacement and the field of stress in the structure. Let $\underset{\sim}{F}(x)$ and $\underset{\sim}{p}(x)$ be respectively the set of concentrated forces and the pressure applied on the structure.

The governing equations of the structure provide the following two operators:

a) the equilibrium equations

$$L_1(\underset{\sim}{s}, \underset{\sim}{u}, \underset{\sim}{F}, \underset{\sim}{p}) = 0 \tag{2.1}$$

b) the stress displacement relations

$$L_2(\underset{\sim}{s}, \underset{\sim}{u}, \underset{\sim}{\sigma}) = 0 \tag{2.2}$$

The boundary conditions are included in the operator L_1. Let us remark that, using approximations of the linear theory of elasticity, the operators L_1 and L_2 are linear with respect to $\underset{\sim}{u}$, $\underset{\sim}{\sigma}$, $\underset{\sim}{F}$ and $\underset{\sim}{p}$, but in general they are non-linear with respect to $\underset{\sim}{s}$.

The material volume of the structure can be written as:

$$V = \int v(\underset{\sim}{s})\, dx \tag{2.3}$$

where the integral is to be evaluated on the entire structure.

2-2. Stress Case

Let us assume the existence of a function $f(\underset{\sim}{\sigma})$ which describes the state of stress at each point of the structure. An admissible state of stress is defined as being such that:

$$f(\underset{\sim}{\sigma}) \leqslant 0 \tag{2.4}$$

The problem of minimizing the material volume of a structure can be formulated as:

$$\min_{\underset{\sim}{s}} \ V = \int v(\underset{\sim}{s})\, dx \tag{2.5}$$

subjected to:

$$L_1(\underset{\sim}{s}, \underset{\sim}{u}, \underset{\sim}{F}, \underset{\sim}{p}) = 0 \tag{2.6}$$

$$L_2(\underset{\sim}{s}, \underset{\sim}{u}, \underset{\sim}{\sigma}) = 0 \tag{2.7}$$

$$f(\underset{\sim}{\sigma}) \leqslant 0 \tag{2.8}$$

To solve the present problem, optimality conditions which include inequality constraints are needed.

2-3. Stiffness Case

Rather than requiring the state of stress in the structure to be admissible, a limit on the displacements occurring during the deformation of the structure can be imposed. This can be achieved by requiring the work done by the external forces, during the deformation process, to be equal to a given amount. In the previous formulation, the average displacement of the structure, using the applied loads as weighting functions, is prescribed. Let $U(\underset{\sim}{s}, \underset{\sim}{F}, \underset{\sim}{p})$

be the work done by the external forces. Its value is given by

$$U(\underset{\sim}{s}, \underset{\sim}{F}, \underset{\sim}{p}) = \frac{1}{2} \{\Sigma \underset{\sim}{F} \cdot \underset{\sim}{u} + \int \underset{\sim}{p} \cdot \underset{\sim}{u}\, dx\} \tag{2.9}$$

where the summation is to be made for all the concentrated external forces, and the integral is to be evaluated over the entire structure. The field of displacement $\underset{\sim}{u}$ is obtained from the equilibrium operator L_1.

As it has been shown by Wasiutynski [18], the problem of minimizing the material volume of the structure subjected to a stiffness constraint is equivalent to the problem of minimizing the work done by the external forces in which the material volume is imposed as a constraint. For both problems the equilibrium equations have to be enforced.

The problem of minimizing the material volume of a structure can therefore be formulated in an alternate way as:

$$\underset{\underset{\sim}{s}}{\text{Min}}\ U = \frac{1}{2}\{\Sigma \underset{\sim}{F} \cdot \underset{\sim}{u} + \int \underset{\sim}{p} \cdot \underset{\sim}{u}\, dx\} \tag{2.10}$$

subjected to:

$$L_1(\underset{\sim}{s}, \underset{\sim}{u}, \underset{\sim}{F}, \underset{\sim}{p}) = 0 \tag{2.11}$$

$$\int v(\underset{\sim}{s})\, dx = V_0 \tag{2.12}$$

To solve the present problem, optimality conditions which include integral constraints are needed.

It is to be noted that, in the context of linear elasticity theory, or of its approximations, the work done by the external forces is equal to the strain energy of the structure in its equilibrium configuration. This provides an alternate formulation found to be convenient for statically determinant structures. In this particular case the computation of the strain energy density which requires the evaluation of the stress field $\underset{\sim}{\sigma}$ can be made without considering the field of displacement $\underset{\sim}{u}$. The equilibrium equations for statically determinant structures can be expressed as:

$$L_3(\underset{\sim}{s}, \underset{\sim}{\sigma}, \underset{\sim}{F}, \underset{\sim}{p}) = 0 \tag{2.13}$$

The problem of minimizing the material volume of a statically determinant structure can therefore be formulated in an alternate way as:

$$\text{Min } J = \int W(\underset{\sim}{\sigma})\, dx \tag{2.14}$$

subjected to:

$$L_3(\underset{\sim}{s}, \underset{\sim}{\sigma}, \underset{\sim}{F}, \underset{\sim}{p}) = 0 \tag{2.15}$$

$$\int v(\underset{\sim}{s})\, dx = V_0 \tag{2.16}$$

2-4. Summary

We have shown that the problem of minimizing the material volume of a structure will involve inequality constraints for the stress case and integral constraints for the stiffness case. The method of deriving the optimality conditions and the numerical techniques used to find their solutions are described in the following chapter.

III. OPTIMALITY CONDITIONS AND METHOD OF SOLUTION

When the design variables consist of unknown functions, rather than a set of parameters, the optimality conditions are obtained by means of the Calculus of Variations. Due to the highly non-linear character of the resulting system of differential equations, the solution is generally obtained by numerical integration.

The Optimal Control Theory is used to derive the necessary first order optimality conditions. This formalism was preferred to the classical Calculus of Variations since it provides directly a system of first order differential equations, well suited for numerical integration. The resulting Two Point Boundary Value Problems are solved using the parallel shooting technique. A general purpose computer code which can solve a one parameter family of n-point boundary value problems was developed.

In order to define the terminology used in the sequel, the first order optimality conditions are recalled. A detailed derivation of them can be found in Ref. [19]. The parallel shooting technique and the computer code capabilities are described in the second part of this chapter. A general analysis of the numerical methods available for solving two points boundary value problems is given in Ref. [20].

3-1. Optimality Conditions

The unconstrained minimization problem is first defined and the corresponding set of first order conditions is derived. Optimality conditions for different variations of the simplest problem are then obtained.

3-1-1. Unconstrained Minimization

Let the governing equations of a physical system be:

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}(x), \underset{\sim}{u}(x)) \qquad (3.1)$$

where:

$\underset{\sim}{y}$ is the state variables vector with m components

$\underset{\sim}{u}$ is the control variables vector with n components.

The equations (3. 1) are called the state equations of the system.

Let us consider the following problem:

$$\underset{\underset{\sim}{u}(x)}{\text{Min}} \quad J = \varphi(\underset{\sim}{y}(x_0), \underset{\sim}{y}(x_1)) + \int_{x_0}^{x_1} L(x, \underset{\sim}{y}(x), \underset{\sim}{u}(x))\, dx \qquad (3.2)$$

subjected to

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}(x), \underset{\sim}{u}(x)) \qquad (3.3)$$

$$\left.\begin{array}{ll} \underset{\sim}{y}(x_0) \text{ prescribed} & (a) \\ \underset{\sim}{y}(x_1) \text{ not prescribed} & (b) \end{array}\right\} \qquad (3.4)$$

The objective function is J and L is the Lagrangian of the problem.

The first order necessary conditions for J to be a minimum are obtained by requiring J to be stationary with respect to arbitrary variations of the control variables $\underset{\sim}{u}(x)$. Let us define

$$\tilde{J} = J + \int_{x_0}^{x_1} \underset{\sim}{\lambda}^T \cdot [\underset{\sim}{f}(x, \underset{\sim}{y}, \underset{\sim}{u}) - \frac{d\underset{\sim}{y}}{dx}] \, dx \tag{3.5}$$

$$H = L + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f} \tag{3.6}$$

where: a) $\underset{\sim}{\lambda}^T$ is a vector of m Lagrange multipliers which will be chosen such that the coefficient of $\delta \underset{\sim}{y}(x)$ in $\delta \tilde{J}$ vanishes.

b) H is the Hamiltonian of the system.

The variations $\delta\tilde{J}$ of $\tilde{J}$ are:

$$\delta\tilde{J} = \left[\frac{\partial\varphi}{\partial \underset{\sim}{y}(x_0)} + \underset{\sim}{\lambda}^T\right]_{x=x_0} \delta\underset{\sim}{y}(x_0) + \left[\frac{\partial\varphi}{\partial \underset{\sim}{y}(x_1)} - \underset{\sim}{\lambda}^T\right]_{x=x_1} \delta\underset{\sim}{y}(x_1)$$

$$+ \int_{x_0}^{x_1} \left[\left\{L_{\underset{\sim}{u}} + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f}_{\underset{\sim}{u}}\right\} \delta\underset{\sim}{u}(x) + \left\{L_{\underset{\sim}{y}} + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f}_{\underset{\sim}{y}} + \frac{d\underset{\sim}{\lambda}^T}{dx}\right\} \delta y(x) \, dx\right] \tag{3.7}$$

where $L_{\underset{\sim}{u}} \equiv \dfrac{\partial L(x, \underset{\sim}{y}, \underset{\sim}{u})}{\partial \underset{\sim}{u}}$

Choosing the Lagrange's multipliers $\underset{\sim}{\lambda}^T$ such that

$$\frac{d\underset{\sim}{\lambda}^T}{dx} = - [L_{\underset{\sim}{y}} + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f}_{\underset{\sim}{y}}] \tag{3.8}$$

the variation $\delta\tilde{J}$ will vanish if:

$$L_{\underset{\sim}{u}} + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f}_{\underset{\sim}{u}} = 0 \tag{3.9}$$

$$\underset{\sim}{\lambda}^T(x_1) = \frac{\partial\varphi}{\partial \underset{\sim}{y}(x_1)} \tag{3.10}$$

The Lagrange's multipliers are not specified at $x = x_0$ since $\underset{\sim}{y}(x_0)$ is specified. It is to be noted that the previous conditions are only sufficient conditions for $\tilde{J}$ to be stationary because we assumed that the defined Lagrange's multipliers do exist.

Let us summarize the previous results:

a) Problem

$$\min_{\underset{\sim}{u}(x)} J = \varphi(y(x_0), y(x_1)) + \int_{x_0}^{x_1} L(x, \underset{\sim}{y}(x), \underset{\sim}{u}(x))dx \tag{3.11}$$

subjected to

$$\frac{d\underset{\sim}{y}}{dx} = f(x, \underset{\sim}{y}, \underset{\sim}{u}) \tag{3.12}$$

$$\underset{\sim}{y}(x_0) \text{ specified} \tag{3.13}$$

b) First order optimality conditions are

$$H \equiv L + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f} \tag{3.14}$$

$$H_{\underset{\sim}{u}} = 0 \tag{3.15}$$

$$\frac{d\underset{\sim}{\lambda}^T}{dx} = - H_{\underset{\sim}{y}} \tag{3.16}$$

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}, \underset{\sim}{u}) \tag{3.17}$$

$$\underset{\sim}{y}(x_0) \text{ prescribed} \tag{3.18a}$$

$$\underset{\sim}{\lambda}^T(x_1) = \frac{\partial\varphi}{\partial \underset{\sim}{y}(x_1)} \tag{3.18b}$$

The equations (3.15) define a set of n algebraic equations to compute the control variables $\underset{\sim}{u}$. The equations (3.16), (3.17) along with the boundary conditions (3.18) define a two point boundary value problem.

Due to their highly non-linear character, it is often not possible to solve directly the equations (3.15). However, they can be used to generate a system of differential equations for the control variables $\underset{\sim}{u}$. Since (3.15) is satisfied for all x in $[x_0, x_1]$ then

$$\frac{dH_{\underset{\sim}{u}}}{dx} = 0 \qquad (3.19)$$

$$H_{\underset{\sim}{u}}\Big|_{x=x^*} = 0, \quad x^* \in [x_0, x_1] \qquad (3.20)$$

The equations (3.19) reduce to:

$$H_{\underset{\sim}{u}\underset{\sim}{u}} \frac{d\underset{\sim}{u}}{dx} = - [H_{\underset{\sim}{u}x} + H_{\underset{\sim}{u}\underset{\sim}{y}} \underset{\sim}{f} - H_{\underset{\sim}{u}\underset{\sim}{\lambda}} H_{\underset{\sim}{y}}] \qquad (3.21)$$

The equations (3.21) determine $d\underset{\sim}{u}/dx$ uniquely when:

$H_{\underset{\sim}{u}\underset{\sim}{u}}$ is not singular, i.e.:

$$\det [H_{\underset{\sim}{u}\underset{\sim}{u}}] \neq 0 \qquad (3.22)$$

The set II of optimality conditions for the problem under consideration, provided (3.22) holds, can be written as:

$$\frac{d\underset{\sim}{u}}{dx} = - H^{-1}_{\underset{\sim}{u}\underset{\sim}{u}} [H_{\underset{\sim}{u}x} + H_{\underset{\sim}{u}\underset{\sim}{y}} \underset{\sim}{f} - H_{\underset{\sim}{u}\underset{\sim}{\lambda}} H_{\underset{\sim}{y}}] \qquad \text{(a)}$$

$$\frac{d\underset{\sim}{\lambda}^T}{dx} = - H_{\underset{\sim}{y}} \qquad \text{(b)} \qquad (3.23)$$

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}, \underset{\sim}{u}) \qquad \text{(c)}$$

$$H_{\underset{\sim}{u}}\Big|_{x=x^*} = 0 \qquad \text{(a)}$$

$$\underset{\sim}{y}(x_0) \text{ specified} \qquad \text{(b)} \qquad (3.24)$$

$$\underset{\sim}{\lambda}^T(x_1) = \frac{\partial \varphi}{\partial \underset{\sim}{y}(x_1)} \qquad \text{(c)}$$

The equations (3. 23) define a system of differential equations for which the boundary conditions (3, 24) are specified at different values of x.

3-1-2. Problem Depending on an Unknown Coefficient

Problem:

$$\min_{\underset{\sim}{u}(x),\, a} \quad J = \int_{x_0}^{x_1} L(x, \underset{\sim}{y}(x),\ \underset{\sim}{u}(x),\ a)\, dx \tag{3. 25}$$

$$\text{subjected to:} \quad \frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}, \underset{\sim}{u}, a) \tag{3. 26}$$

The previous results can be applied by considering the extended state vector $\bar{\underset{\sim}{y}}$:

$$\bar{\underset{\sim}{y}} = \begin{bmatrix} \underset{\sim}{y} \\ \\ a \end{bmatrix} \tag{3. 27}$$

The state equation associated with y_{m+1} is:

$$\frac{dy_{m+1}}{dx} = 0 \tag{3. 28}$$

Since the value of $y_{m+1}(x_0)$ is not specified, the admissible $\delta y_{m+1}(x_0)$ is not zero. The optimality conditions are given by (3. 23) and (3. 24) where the Hamiltonian H is given by:

$$H = L + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f} + \lambda_{m+1} \times (0) \tag{3. 29}$$

The boundary conditions on λ_{m+1} are:

$$\lambda_{m+1}(x_0) = \lambda_{m+1}(x_1) = 0 \tag{3. 30}$$

3-1-3. Some State Variables Prescribed at $x = x_1$

Problem:

$$\underset{u(x)}{\text{Min}} \quad J = \varphi(\underset{\sim}{y}(x_0),\ \underset{\sim}{y}(x_1)) + \int_{x_0}^{x_1} L(x, \underset{\sim}{y}, \underset{\sim}{u})\, dx \tag{3.31}$$

subjected to:

$$\frac{d\underset{\sim}{y}}{dx} = f(x, \underset{\sim}{y}, \underset{\sim}{u})$$

$$y_i(x_0) \text{ prescribed}, \quad i \in I$$

$$y_j(x_1) \text{ prescribed}, \quad j \in J$$

The optimality conditions (3.15) to (3.17) hold, provided the system is controllable. The boundary conditions are in this case:

$$y_i\,(x_0) \text{ prescribed} \qquad i \in I \tag{3.32}$$

$$\lambda_k(x_0) = -\frac{\partial\varphi}{\partial y_k(x_0)} \qquad k \notin I \tag{3.33}$$

$$y_j(x_1) \text{ prescribed} \qquad j \in J \tag{3.34}$$

$$\lambda_\ell(x_1) = \frac{\partial\varphi}{\partial y_\ell\,(x_1)} \qquad \ell \notin J \tag{3.35}$$

Note: If the state variables must also satisfy a constraint $\psi(\underset{\sim}{y}(x_0),\ \underset{\sim}{y}(x_1)) = 0$, the boundary conditions can be obtained by adjoining ψ to φ with a Lagrange's multiplier ν

$$\tilde{\varphi} = \varphi + \nu\psi \tag{3.36}$$

The boundary conditions become

$$y_i(x_0) \text{ prescribed} \qquad i \in I \tag{3.37}$$

$$\lambda_k(x_0) = -\left[\frac{\partial \varphi}{\partial y_k(x_0)} + \nu \frac{\partial \psi}{\partial y_k(x_0)}\right] \quad k \notin I \tag{3.38}$$

$$y_j(x_1) \text{ prescribed} \qquad j \in J \tag{3.39}$$

$$\lambda_\ell(x_1) = \left[\frac{\partial \varphi}{\partial y_\ell(x_1)} + \nu \frac{\partial \psi}{\partial y_\ell(x_1)}\right] \quad \ell \notin J \tag{3.40}$$

$$\psi(\underset{\sim}{y}(x_0), \underset{\sim}{y}(x_1)) = 0 \tag{3.41}$$

3-1-4. Problem with an Integral Constraint

Problem:

$$\underset{\underset{\sim}{u}(x)}{\text{Min}} \quad J = \varphi(\underset{\sim}{y}(x_0), \underset{\sim}{y}(x_1)) + \int_{x_0}^{x_1} L(x, \underset{\sim}{y}, \underset{\sim}{u})\, dx \tag{3.42}$$

subjected to:

$$\frac{d\underset{\sim}{y}}{dx} = f(x, \underset{\sim}{y}, \underset{\sim}{u}) \tag{3.43}$$

$$\int_{x_0}^{x_1} g(x, \underset{\sim}{y}, \underset{\sim}{u})\, dx = c \tag{3.44}$$

The previous result can be applied by considering the extended state vector

$$\bar{\underset{\sim}{y}} = \begin{bmatrix} \underset{\sim}{y} \\ \\ y_{m+1} \end{bmatrix} \tag{3.45}$$

The state equation associated with y_{m+1} is:

$$\frac{dy_{m+1}}{dx} = g(x, \underset{\sim}{y}, \underset{\sim}{u}) \tag{3.46}$$

The boundary conditions on y_{m+1} are:

$$y_{m+1}(x_0) = 0, \quad y_{m+1}(x_1) = c \tag{3.47}$$

The Hamiltonian of the problem becomes

$$H = L + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f} + \lambda_{m+1}\, g(x, \underset{\sim}{y}, \underset{\sim}{u}) \tag{3.48}$$

Since the state variable y_{m+1} does not appear in the Lagrangian L and the state equations (3.43), the governing equation for λ_{m+1} is:

$$\frac{d\lambda_{m+1}}{dx} = 0 \tag{3.49}$$

The boundary conditions on λ_{m+1} are unspecified since the values of y_{m+1} are given. The Lagrange's multiplier λ_{m+1} appears as an unknown constant. Its value is to be determined from the satisfaction of the integral constraint (3.44).

3-1-5. Problem with Inequality Constraint

Problem

$$\underset{\underset{\sim}{u}(x)}{\text{Min}} \quad J = \int_{x_0}^{x_1} L(x, \underset{\sim}{y}, \underset{\sim}{u})\, dx \tag{3.50}$$

subjected to:

$$\frac{d\underset{\sim}{y}}{dx} = f(x, \underset{\sim}{y}, \underset{\sim}{u}) \tag{3.51}$$

$$C(x, \underset{\sim}{y}, \underset{\sim}{u}) \leq 0 \tag{3.52}$$

Let H^* be defined as:

$$H^* = L + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f} \tag{3.53}$$

The variations of $\tilde{J}$ given by (3.7) are:

$$\delta \tilde{J} = \int_{x_0}^{x_1} H^*_{\underset{\sim}{u}} \cdot \delta \underset{\sim}{u}(x)\, dx \tag{3.54}$$

The constraint (3.52) is said to be "effective" on an arc of the solution if:

$$C(x, \underset{\sim}{y}, \underset{\sim}{u}) = 0 \tag{3.55}$$

When the strict inequality is satisfied the constraint is said to be "not effective."

In order for $\tilde{J}$ to be a minimum, we must have $\delta \tilde{J} \geq 0$ for all admissible variations of the control variable.

When the constraint is effective, the admissible $\delta \underset{\sim}{u}$ are such that:

$$C_{\underset{\sim}{u}} \cdot \delta \underset{\sim}{u} \leq 0 \; , \qquad \int_{x_0}^{x_1} H^*_{\underset{\sim}{u}} \cdot \delta \underset{\sim}{u}\, dx \geq 0 \tag{3.56}$$

A sufficient condition of optimality is obtained by introducing a non-negative Lagrange multiplier η such that

$$H^*_{\underset{\sim}{u}} + \eta C_{\underset{\sim}{u}} = 0 \tag{3.57}$$

When the constraint is not effective the class of admissible $\delta \underset{\sim}{u}$ is not restricted by the constraint. The optimality conditions (3.15) hold.

A set of optimality conditions analogous to the equations (3. 15) and (3. 16) can be obtained by a transformation of the inequality constraint (3. 52) into an equality constraint using an additional control variable μ such that:

$$C(x, \underset{\sim}{y}, \underset{\sim}{u}) + \mu^2 = 0 \tag{3.58}$$

The Hamiltonian H is obtained by adjoining to (3. 53) the equality constraint (3. 57) using a non-negative Lagrange's multiplier η

$$H = L + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f} + \eta \left[C(x, \underset{\sim}{y}, \underset{\sim}{u}) + \mu^2\right] \tag{3.59}$$

3-1-6. Summary

The first order optimality conditions recalled in this section are only sufficient conditions. However, the first order conditions are nothing but necessary conditions for a solution to be optimal. To determine whether or not a solution corresponds to a local minimum a second order sufficiency condition, the convexity condition, will be used. In addition a restricted perturbation of the solutions will be performed in some cases.

3-2. Shooting Technique

The shooting technique is a general method to solve systems of first order ordinary differential equations for which the boundary conditions are specified at several points. The technique is first illustrated for the simplest case of a two points boundary value problem. The capabilities of the general shooting program are then described.

3-2-1. Method

Problem 1:

Find the solution of

$$\left.\begin{aligned} &\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}(x)) && \text{(n equations)} && \text{(a)} \\ &\text{with the boundary conditions} && && \\ &y_i(x_0) = a_i && i \in I \quad \text{(p conditions)} && \text{(b)} \\ &y_j(x_1) = b_j && j \in J \quad \text{(n-p conditions)} && \text{(c)} \end{aligned}\right\} \quad (3.60)$$

The present problem is not an initial value problem since some of the initial conditions are not specified, i.e.,

$$y_k(x_0) \text{ unspecified } \quad k \notin I \quad \text{(n-p values)} \qquad (3.61)$$

However the solution of problem 1 can be obtained from the following initial value problem.

Problem 2

Find the solution of

$$\left.\begin{aligned} &\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}(x)) && \text{(a)} \\ &\text{with the initial values} && \\ &y_i(x_0) = a_i \qquad i \in I && \text{(b)} \\ &y_k(x_0) = s_k \qquad k \notin I && \text{(c)} \end{aligned}\right\} \quad (3.62)$$

where the initial values s_k are such that

$$y_j(x_1) = b_j \qquad j \in J \tag{3.63}$$

Let $\underset{\sim}{u}(x, s_k)$ be the solution of problem 2. It will be a solution of problem 1 iff

$$\Phi_j(s_k) \equiv u_j(x_1, s_k) - b_j = 0 \qquad j \in J, \quad k \notin I \tag{3.64}$$

The function Φ of the unknown initial values s_k is called the ''mis-match'' function. Problem 1 is reduced into an initial value problem, where the initial values s_k must be determined from the solution of the n-p algebraic equations (3.64).

The algebraic equations are solved with the Newton-Ralphson's method, which requires the solution of a sequence of initial value problems 2 where:

$$y_k(x_0) = t_k^{\nu} \qquad k \notin I \tag{3.65}$$

The approximation $t_k^{\nu+1}$ of the solution s_k is obtained from the previous iterate t_k^{ν} using:

$$\frac{\partial \Phi_j}{\partial s_k} (t_k^{\nu+1} - t_k^{\nu}) = -\Phi(t_k^{\nu}) \qquad j \in J, \; k \notin I \tag{3.66}$$

where:

$$\frac{\partial \Phi_j}{\partial s_k} = \frac{\partial y_j(x_1)}{\partial y_k(x_0)} \qquad j \in J, \quad k \notin I \tag{3.67}$$

The gradient of the final values $y_j(x_1)$ with respect to the unknown initial values $y_k(x_0)$ is obtained by integration of the variational system of (3.62a):

$$\left.\begin{array}{ll} \dfrac{d}{dx}\, \dfrac{\partial y_j}{\partial y_k(x_0)} = \dfrac{\partial f_j}{\partial y_\ell}\, \dfrac{\partial y_\ell}{\partial y_k(x_0)} & \text{(a)} \\ \text{with the initial values:} & \\ \dfrac{\partial y_\ell}{\partial y_k(x_0)} = \begin{cases} 0 & \text{if } k \neq \ell \\ 1 & \text{if } k = \ell \end{cases} & \text{(b)} \end{array}\right\} \quad (3.68)$$

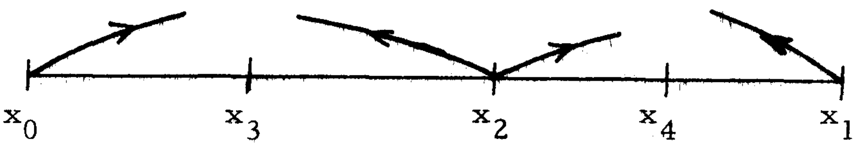
Therefore the evaluation of $\frac{\partial \Phi_j}{\partial s_k}$ requires the integration of $(n-p) \times n$ linear differential equations.

The solution of the two points boundary value problem 1 is obtained by solving a sequence of initial value problems 2. The total number of differential equations to be solved for each iteration is $n + n\,(n-p)$. The iterative process requires an initial guess t_k^o of the unknown initial values.

Remarks:

a) The choice of the initial guess of the unknown initial values should be made carefully, especially when the solution of problem 1 is not unique.

b) Due to the exponential character of the solutions of the variational system (3.68), and to reduce the numerical errors, the integration interval $[x_0, x_1]$ is divided into subintervals called shooting intervals. Initial values are guessed at each starting point of the shooting intervals. The solutions are matched at each final point of the shooting intervals. This variation of the previous method is called parallel shooting technique. The following diagram illustrates the method.



where: x_0, x_2, x_1 are the shooting points

x_3, x_4 are the matching points

$[x_0, x_3]$, $[x_2, x_3]$, $[x_2, x_4]$ and $[x_4, x_1]$ are the shooting intervals.

3-2-2. General Shooting Program

When using the shooting method, most of the programming effort is spent to construct the variational matrix (3.64) for each particular problem. To alleviate this practical problem, a general computer program was realized. It utilizes the parallel shooting technique to solve the following problem:

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}, \underset{\sim}{a}, \gamma) \qquad \text{n equations} \qquad \text{(a)}$$

$$\frac{d\underset{\sim}{u}}{dx} = \underset{\sim}{g}(x, \underset{\sim}{y}, \underset{\sim}{u}, \underset{\sim}{a}, \gamma) \qquad \text{m equations} \qquad \text{(b)}$$

$$\int_{x_0}^{x_1} \underset{\sim}{h}(x, \underset{\sim}{y}, \underset{\sim}{a}, \gamma) = \underset{\sim}{a} \qquad \text{q integrals} \qquad \text{(c)}$$

with the boundary conditions:

$$y_{m_k}(x_k) \text{ prescribed} \qquad \text{(d)}$$

$$\left.\begin{array}{ll} y_{j_k}(x_k) \text{ not prescribed} & \text{(e)} \\ y_{n_\ell}(z_\ell) \text{ prescribed} & \text{(f)} \\ y_{i_\ell}(z_\ell) \text{ not prescribed} & \text{(g)} \\ N(y_{j_k}(x_k),\ y_{i_\ell}(z_\ell)) = 0 & \text{(h)} \end{array}\right\} \quad (3.69)$$

where:

x_k, $k = 1, \ldots K$ are the shooting points

z_ℓ, $\ell = 1, \ldots L$ are the matching points

$\underset{\sim}{a}$ is a vector of unknown coefficients

γ is a parameter with respect to which a one parameter family of problems is defined

$\underset{\sim}{u}$ a vector to be evaluated on the solution

$N(y_{j_k}(x_k),\ y_{i_\ell}(z_\ell)) = 0$ is a set of non-linear equations to be satisfied by the solutions.

The set of differential equations (3.69a), (3.69b), the integrals (3.69c), as well as their variational system, are specified by a user's written subroutine. The non-linear conditions (3.69c) are also defined by a subroutine. The boundary conditions and the characteristics of the different shooting and matching points are specified by data cards. The program was found to be very convenient since the number of shooting and matching points could be modified as needed with a minimum amount of work.

3-3. Summary

The optimization problems of the following sections will consist of these three steps:

a) definition of the problem

b) derivation of a set of differential equations with boundary values prescribed at several points

c) resolution of the resulting n-points boundary value problems using, in general, the shooting technique. It is to be noted again that only the first step, i.e., the definition of the problem, will determine the solutions.

IV. ARCH STRUCTURES

The problem of determining the optimal thickness distribution only of beams or arches of known geometry has been studied extensively in the past. Huang and Sheu [21] treated the case of circular sandwich beams for the stiffness case. For sandwich sections, the bending and the extensional rigidities of the cross section are linear functions of the face sheet thickness. Giraudbit [11] investigated the case of a clamped circular arch with a solid cross section subjected to a stress constraint.

In this chapter the problem of determining the thickness distribution, the shape of the middle line and the slope of the support of statically determinant arches of minimum material volume, satisfying either a stress or a stiffness constraint, is treated. The applied load is a uniform pressure normal to the middle line of the structures. One of the objectives is to determine whether or not the best geometry is such that the bending moment vanishes everywhere for the optimal structure.

A formal problem definition is first given. The optimality problem for a stiffness constraint is then investigated. The shear force contribution is included in the strain energy definition. The same problem is then treated for the stress constraint case. The shear force contribution is also taken into account in the failure criterion.

4-1. Problem Definition

Let S be the set of arches such that:

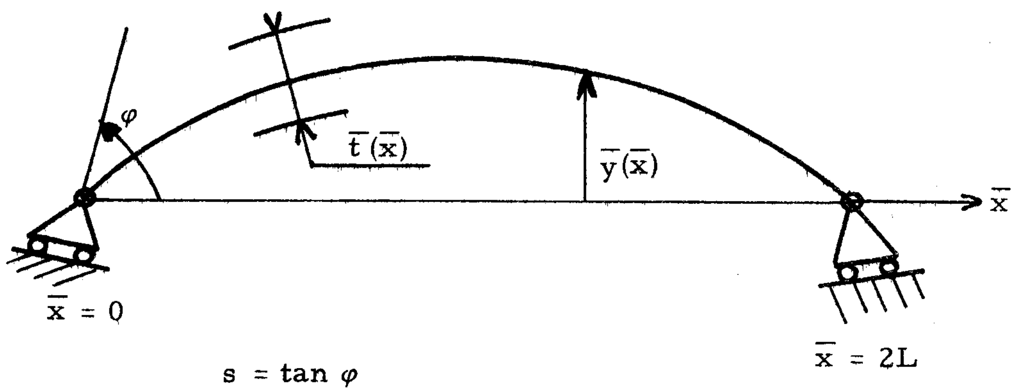
a) their governing equations include the effects of the shear stress.

b) they are subjected to a uniform pressure normal to their middle line and have simple support type of boundary conditions.

c) they satisfy either a given stress or a given stiffness constraint.

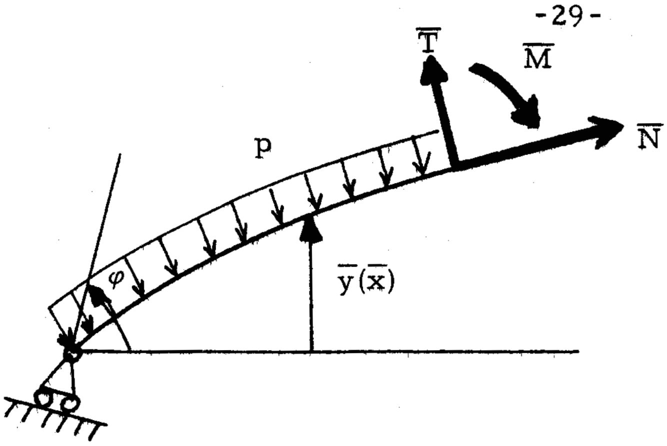
d) their length is 2L, their shape is such that $\bar{y}(o) = \bar{y}(2L) = 0$ and they have a constant unit width.

We seek the middle line shape $\bar{y}(\bar{x})$, the thickness distribution $\bar{t}(\bar{x})$ and the slope of the support s of the element in S of minimum material volume.



4-2. Equilibrium Equations

Since the arches are simply supported, the normal force $\bar{N}$, the bending moment $\bar{M}$ and the shear force $\bar{T}$ can be computed directly from the three equilibrium equations of a plane structure.



$s = \tan\varphi$

For symmetric arches with respect to $\overline{x} = L$:

$$\left.\begin{aligned}
\overline{N} &= \frac{pL}{\sqrt{1+\overline{y},_{\overline{x}}^{2}}}\left[\overline{y},_{\overline{x}}\left(\frac{\overline{x}}{L}-1\right)-\left(\frac{1}{s}+\frac{\overline{y}}{L}\right)\right] && \text{(a)}\\
\overline{T} &= \frac{pL}{\sqrt{1+\overline{y},_{\overline{x}}^{2}}}\left[\left(\frac{\overline{x}}{L}-1\right)+\overline{y},_{\overline{x}}\left(\frac{1}{s}+\frac{\overline{y}}{L}\right)\right] && \text{(b)}\\
\overline{M} &= pL^2\left[\tfrac{1}{2}\left(\frac{\overline{y}}{L}\right)^2+\frac{\overline{y}}{Ls}+\tfrac{1}{2}\left(\frac{\overline{x}}{L}\right)^2-\frac{\overline{x}}{L}\right] && \text{(c)}
\end{aligned}\right\} \qquad (4.1)$$

Note that the solution (4. 1) satisfies the simply supported boundary conditions since

$$\overline{M}\Big|_{\overline{x}=0} = \overline{M}\Big|_{\overline{x}=2L} = 0 \qquad (4.2)$$

Upon introduction of the following dimensionless variables:

$$\left.\begin{aligned}
&x = \frac{\overline{x}}{L},\quad z_1 = \frac{\overline{y}}{L},\quad z_2 = \overline{y},_{\overline{x}},\quad z_3 = s\\
&N = \frac{\overline{N}}{pL},\quad T = \frac{\overline{T}}{pL},\quad M = \frac{\overline{M}}{pL^2}
\end{aligned}\right\} \qquad (4.3)$$

the equilibrium equations (4. 1) reduce to:

$$\left.\begin{aligned} N &= \frac{1}{\sqrt{1+z_2^2}}\left[z_2(x-1) - (\frac{1}{z_3} + z_1)\right] && \text{(a)} \\ T &= \frac{1}{\sqrt{1+z_2^2}}\left[(x-1) + z_2(\frac{1}{z_3} + z_1)\right] && \text{(b)} \\ M &= \tfrac{1}{2} z_1^2 + \frac{z_1}{z_3} + \tfrac{1}{2} x^2 - x && \text{(c)} \end{aligned}\right\} \quad (4.4)$$

A shape on which M = 0 everywhere, hereafter referred to as "membrane design," is a circular arch defined by:

$$(z_1)_M = -\frac{1}{z_3} + \sqrt{\frac{1}{z_3^2} + 2(x - \frac{x^2}{2})} \quad (4.5a)$$

when $\frac{1}{z_3} > 0$.

On such a design, the shear force T vanishes and the normal traction N is constant

$$N = -\sqrt{1 + (\frac{1}{z_3})^2} \quad (4.5b)$$

4-3. Optimal Arches with a Stiffness Constraint

Since the structure is statically determinant, the strain energy of the structure, in its equilibrium configuration, is used as the objective function $\bar{J}$.

$$\bar{J} = \tfrac{1}{2}\int_0^{2L}\left(\frac{\bar{N}^2+\beta^2\bar{T}^2}{EA} + \frac{\bar{M}^2}{EI}\right)\sqrt{1+\bar{y}_{,\bar{x}}^2}\, d\bar{x} \quad (4.6)$$

where:

E ..... material Young's modulus

A = t..... cross section area

$I = \frac{\bar{t}^3}{12}$ ..... cross section moment of inertia

$\beta^2 = \frac{E}{G}k$ ... where G is the material shear modulus and k is a coefficient to take into account the real distribution of the shear stress in the cross section.

Note that the width of the structure does not enter in the previous expressions since it is assumed to be constant and unity.

The material volume V_0 is used as a constraint

$$V_0 = \int_0^{2L} A\sqrt{1 + \bar{y},_{\bar{x}}^2} \; d\bar{x} \tag{4.7}$$

Upon introduction of the previously defined dimensionless variables (4.3) and of:

$$t = \bar{t}\,(\frac{2L}{V_0}) \tag{4.8}$$

the dimensionless objective function J and the volume constraint reduce respectively to:

$$J = \tfrac{1}{2} \int_0^2 \left(\frac{N^2 + \beta^2 T^2}{t} + \frac{12\alpha^2 M^2}{t^3} \right) \sqrt{1 + z_2^2} \; dx \tag{4.9}$$

$$\int_0^2 t \sqrt{1 + z_2^2} \; dx = 2 \tag{4.10}$$

where: $\alpha = \frac{2L^2}{V_0}$ characterizes the length to the thickness ratio of a straight beam of uniform thickness satisfying the volume constraint:

$$\bar{J} = \frac{p^2 L^2}{E} \alpha J \tag{4.11}$$

4-3-1. Formulation

Because of the symmetry in the imposed boundary conditions the problem can be formulated as:

$$\text{Min } J = \int_0^1 \left(\frac{N^2+\beta^2 T^2}{t} + \frac{12M^2\alpha^2}{t^3} \right) \sqrt{1+z_2^2}\, dx \qquad (4.12)$$

subjected to:

i) the material volume constraint

$$\int_0^1 t \sqrt{1+z_2^2}\, dx = 1 \qquad (4.13)$$

ii) the state equations

$$\left.\begin{array}{ll} \dfrac{dz_1}{dx} = z_2 & \text{(a)} \\[2ex] \dfrac{dz_3}{dx} = 0 & \text{(b)} \end{array}\right\} \qquad (4.14)$$

iii) the equilibrium equations (4.4)

iv) the boundary conditions:

$$\left.\begin{array}{ll} z_1(0) = 0 & z_1(1) \text{ unknown} \\ z_3(0) \text{ unknown} & z_3(1) \text{ unknown} \end{array}\right\} \qquad (4.15)$$

Note: a) the unknown parameter z_3 is treated as a state variable bounded to be constant by (4.14b). b) the symmetry condition $z_2(1)=0$ cannot be inforced directly since z_2 is not a state variable. It will be a consequence of the boundary conditions since the symmetry information is contained in the equilibrium equations (4.4).

The Hamiltonian H of the system is obtained by adjoining to its Lagrangian the material volume constraint (4.13), the state equations (4. 14) by means of the Lagrange's multipliers λ_4, λ_1, and λ_3. The state variables are z_1 and z_3 and the control variables are z_2 and t.

$$H = \left(\frac{N^2+\beta^2T^2}{t} + \frac{12M^2\alpha^2}{t^3} + \lambda_4 t\right)\sqrt{1+z_2^2} + \lambda_1 z_2 + \lambda_3 \times(0) \qquad (4.16)$$

4-3-2. Optimality Conditions

The first order optimality conditions are:

$$\left.\begin{aligned} H_t &= \left(\lambda_4 - \frac{N^2+\beta^2T^2}{t^2} - \frac{36M^2\alpha^2}{t^4}\right)\sqrt{1+z_2^2} = 0 && \text{(a)} \\ H_{z_2} &= \left(\frac{12M^2\alpha^2}{t^3} + \frac{N^2+\beta^2T^2}{t} + \lambda_4 t\right)\frac{z_2}{\sqrt{1+z_2^2}} \\ &\quad + \lambda_1 + \frac{2}{t}\sqrt{1+z_2^2}\left(N\frac{\partial N}{\partial z_2} + \beta^2 T\frac{\partial T}{\partial z_2}\right) = 0 && \text{(b)} \end{aligned}\right\} \qquad (4.17)$$

$$\left.\begin{aligned} \frac{d\lambda_1}{dx} &= -H_{z_1} = -\frac{24M\alpha^2}{t^3}\left(z_1 + \frac{1}{z_3}\right)\sqrt{1+z_2^2} \\ &\quad + 2\frac{N}{t} - 2\beta^2 z_2\frac{T}{t} && \text{(a)} \\ \frac{d\lambda_3}{dx} &= -H_{z_3} = \frac{1}{z_3^2}\left[24\frac{M^2\alpha^2}{t^3} z_1\sqrt{1+z_2^2}\right. \\ &\quad \left. - \frac{2N}{t} + 2\beta^2 z_2\frac{T}{t}\right] && \text{(b)} \end{aligned}\right\} \qquad (4.18)$$

The material volume constraint (4. 13) and the state equation (4. 14) are also part of the optimality conditions. The previous conditions define the set I of optimality conditions.

The boundary conditions are:

at $x = 0$

$$\left.\begin{array}{lll} z_1(0) = 0 & \lambda_1(0) \text{ unknown} & (a) \\ z_3(0) \text{ unknown} & \lambda_3(0) = 0 & (b) \\ \text{at } x = 1 & & \\ z_1(1) \text{ unknown} & \lambda_1(1) = 0 & (c) \\ z_3(1) \text{ unknown} & \lambda_3(1) = 0 & (d) \end{array}\right\} \quad (4.19)$$

Note that the condition (4. 19c) is implied by (4. 17b) when $z_2(1) = 0$.

To avoid the difficulty of solving (4. 17a) and (4. 17b) for the control variables z_2 and t the following set of equations is used to generate a differential equation for z_2.

$$\left.\begin{array}{ll} \dfrac{dH_t}{dx} = 0 \quad , & H_t = 0 \quad x^*, \; x^* \epsilon \; [0, 1] \\ \dfrac{dH_{z_2}}{dx} = 0 \quad , & H_{z_2} = 0 \quad \text{at } x = 1 \end{array}\right\} \quad (4.20)$$

Solving (4. 20) for $\frac{dz_2}{dx}$ and t, keeping only the positive root for the thickness, one obtains:

$$\left.\begin{array}{ll} t = \sqrt{\dfrac{B_1 + \sqrt{B_1^2 + 144M^2\alpha^2\lambda_4}}{2\lambda_4}} & (a) \\ \dfrac{dz_2}{dx} = -(1+z_2^2)^{3/2} \, N \, \dfrac{T^2 B_2 + B_3}{C} & (b) \end{array}\right\} \quad (4.21)$$

$$\left.\begin{aligned} \text{where } B_1 &= N^2+\beta^2 T^2 && \text{(a)} \\ B_2 &= \frac{12M-a^2}{t^2}(4\beta^2-3)+\beta^2 && \text{(b)} \\ B_3 &= \left(\frac{12Ma^2}{t^2}+2-\beta^2\right)\left(\frac{72M^2a^2}{t^2}+N^2\right) && \text{(c)} \end{aligned}\right\} \quad (4.22)$$

$$\left.\begin{aligned} C &= M^2 C_1 + \beta^2 C_2 && \text{(a)} \\ C_1 &= \frac{24a^2}{t^2}\left[\frac{72M^2a^2}{t^2}+N^2(1+3\beta^2)+T^2(3+\beta^2)\right] && \text{(b)} \\ C_2 &= (N^2+T^2)^2 && \text{(c)} \end{aligned}\right\} \quad (4.23)$$

The equivalence between the systems of equations (4. 17) and (4. 22) hold only when the Jacobian of (4. 17), which is proportional to C, does not vanish, i. e.,

$$C \sim H_{tt} H_{z_2 z_2} - (H_{tz_2})^2 \neq 0 \quad (4.24)$$

Theorem 4-1

If the effects of the shear forces are not taken into account in the strain energy density, i. e., $\beta^2 = 0$, then $C = 0$ whenever $M = 0$.

Proof

(4. 23) and $\beta^2 = 0 \Longrightarrow C = \frac{24a^2}{t^2} M^2 \left[\frac{72M^2a^2}{t^2} + N^2+3T^2\right]$ q. e. d.

This case is undesirable since the Hamiltonian H becomes locally linear in the control variable z_2. Under these conditions it can be shown that in order for $\frac{dz_2}{dx}$ to remain finite (i. e., the radius of curvature of the middle line to be non-zero) the traction N has to vanish. A physical interpretation could be found from the fact that when $\beta = 0$, a penalty on the system is set for only N and M but not for the shear force T. When $M = 0$, the condition $N = 0$ implies that all the force is transmitted to the structure as a shear force. This difficulty was alleviated here by including the contribution of the shear force into the strain energy density.

Since λ_1 does not appear in (4. 22) and because the boundary condition (4. 19c) is satisfied setting $z_2(1) = 0$, the optimality conditions reduce to the equations (4. 21), (4.18b), along with the material volume constraint (4. 13). In the boundary conditions (4. 19), (4. 19c) is replaced by $z_2(1) = 0$. This defines the set II of optimality conditions.

4-3-3. Straight Design

Theorem 4-2

The straight design defined by:

$$z_1(x) = 0 \ , \qquad \frac{1}{z_3} = 0$$

satisfies all the optimality conditions.

Proof

Since $z_1(x) = 0 \quad \forall\, x \in [0, 1]$, $z_2(x) = 0$

By virtue of (4. 4), $N = 0$.

Furthermore (4. 18b), (4. 19b) imply that $\lambda_3 = 0$ which satisfies (4. 19c). The thickness distribution is obtained from (4. 22a) and λ_4 is computed from the material volume constraint (4. 13) which reduces to:

$$\int_0^1 \sqrt{D_1 + \sqrt{D_1^2 + \lambda_4 \alpha^2 D_2}}\ dx = \sqrt{2\lambda_4} \tag{4. 25}$$

where $D_1 = \beta^2 (x-1)^2$, $\qquad D_2 = 144\left(\frac{x^2}{2} - x\right)^2$

The integral equation (4. 25) can be solved easily by taking an arbitrary value of $\lambda_4 a^2$ and then deducing, from the evaluation of the integral, the values of λ_4 and a^2.

4-3-4. Membrane Design

Theorem 4-3

The membrane design defined by (4. 5) does not satisfy the optimality conditions.

Proof

It is most convenient to use, for the present proof, the original set of optimality conditions. If z_1 as defined by (4. 5) were a solution then

$$(4.17) \implies \quad \lambda_1 + 2\lambda_4 t \frac{z_2}{\sqrt{1+z_2^2}} = 0 \qquad (4.26)$$

$$(4.18a) \implies \frac{d\lambda_1}{dx} = \frac{2N}{t} \qquad (4.27)$$

Since $z_2(x) \geq 0 \quad \forall x,\ x \in [0, 1]$, (4. 26) implies that $\lambda_1(x) \leq 0$ $\forall x,\ x \in [0, 1]$. But (4. 5b) implies that $N < 0 \ \forall x,\ x \in [0, 1]$ therefore according to (4. 27) $\frac{d\lambda_1}{dx} < 0$. This is a contradiction since $\lambda_1(1) = 0$. q. e. d.

It is to be noted that the derivatives of λ_1 given respectively by (4. 27), and the derivative of (4. 26) differ only by their sign.

The membrane designs are a one parameter family of designs with respect to the slope of their support. Let us seek in that subclass of admissible designs the "best" element according to our stiffness criterion. We consider t and z_3 to be respectively an unknown function and an unknown parameter. The shape z_1 and its derivative z_2 are two known functions of z_3 defined by (4.5). The optimality condition (4.17a) remains valid, and a direct minimization of the strain energy with respect to z_3 is performed.

$$(4.17a) \implies \frac{N^2}{t^2} = \lambda_4 \tag{4.28}$$

The volume constraint and the objective function become respectively

$$\int_0^1 |N| \sqrt{1 + z_2^2}\, dx = \sqrt{\lambda_4} \tag{4.29}$$

$$J = \int_0^1 \frac{N^2}{t} \sqrt{1+z_2^2}\, dx = \lambda_4 \tag{4.30}$$

Therefore λ_4 has to be minimized with respect to z_3. The evaluation of the integral (4.29), after substitution of the value of N defined in (4.5b) yields:

$$\sqrt{\lambda_4} = a \sin^{-1} \left(\frac{1}{\sqrt{a}}\right) \tag{4.31}$$

where $a = 1 + \left(\frac{1}{z_3}\right)^2$

Noting that: $$\sin^{-1}\left(\frac{1}{\sqrt{a}}\right) = \tan^{-1} (z_3) \tag{4.32}$$

one obtains:

$$\frac{d\sqrt{\lambda_4}}{dz_3} = \frac{1}{z_3^2} \left[- \frac{2 \tan^{-1} z_3}{z_3} + 1 \right] \tag{4.33}$$

A stationary value, here a minimum, of $\sqrt{\lambda_4}$ is obtained for the positive root of:

$$z_3 = 2 \tan^{-1} z_3, \quad z_3 \neq 0 \tag{4.34}$$

The minimum was found to be:

$$\sqrt{\lambda_4} = 1.3800\ , \qquad \lambda_4 = 1.9044$$

$$\text{at } \frac{1}{z_3} = 0.4290 \tag{4.35}$$

4-3-5. Results

Since no closed form solutions beside the straight design were found, a numerical solution of the two point boundary value problem defined by the set II of optimality conditions was performed. The integration was performed up to $\alpha^2 = 8$, the value at which numerical difficulties occurred due to the sensitivity of the solution with respect to the unknown initial slope $z_2(0)$. The results for the middle line shape, the thickness distribution, and the slope of the support as well as the initial slope of the structures are plotted respectively in figure 1 through 3 for different values of α and $\beta^2 = 2.5$. The value of the objective function corresponding to the straight designs, the optimal designs and the "best" membrane design are plotted on figure 4.

4-3-6. Conclusion

For the shape and the thickness optimization of simply supported arches, subjected to a uniform pressure, satisfying a stiffness constraint, it has been found necessary to include the effects of the shear force contribution in the strain energy definition. Two families of local optimal solutions have been found. The straight design which satisfies all the necessary optimality conditions was not found to be a global minimum for the achieved numerical solutions. A one parameter family of optimal designs, with respect to an average thickness to length ratio $(\frac{1}{\alpha})$ was generated. Although the "best" membrane design does not satisfy the optimality conditions, it represents for practical values of $\alpha(\alpha > 5)$ a very good approximation to the "best" design.

4-4. Optimal Arches with a Stress Constraint

The approximative two-dimensional state of stress used for the present analysis is defined as follows:

a) On an element normal to the middle line of the arch the stress vector is composed of a normal stress $\bar{\sigma}_t$, due to the normal force $\overline{N}$ and the bending moment $\overline{M}$, and of a shear stress $\bar{\tau}$ due to the shear force $\overline{T}$.

b) On an element parallel to the middle line, the normal stress is zero and the shear stress is $-\bar{\tau}$.

The normal stress $\bar{\sigma}_t$ is a linear function of the distance $\bar{v}$ from the middle line

$$\bar{\sigma}_t = \frac{\overline{M}\,\bar{v}}{I} + \frac{\overline{N}}{A}, \quad -\frac{t}{2} \leqslant \bar{v} \leqslant \frac{t}{2} \tag{4.36}$$

where $I = \frac{\bar{t}^3}{12}$: cross section moment of inertia

$A = \bar{t}$: cross section area

Note: the width of the section does not appear in those formulas since it was assumed to be a unit constant.

Although theoretically this is not true, the shear stress $\bar{\tau}$ is assumed to be a constant on the cross section.

$$\bar{\tau} = \tfrac{1}{2}\beta\,\frac{\overline{T}}{A}$$

where $\frac{1}{2}\beta$ is a coefficient which can be used to study the influence of the shear stress on the solution.

The maximum shear failure criterion defines the admissible state of stress. At each point of the cross section, there exists a direction for which the shear stress is maximum. The state of stress is said to be admissible if the magnitude of the maximum shear is less than or equal to a given limit value $\bar{\tau}_{adm}$. Given our approximative state of stress:

$$T_{max}(\bar{v}) = \frac{\sqrt{\sigma_t^2(\bar{v}) + 4\tau^2}}{2} \qquad (4.38)$$

Since the magnitude of the tensile stress, in a given cross section, is maximum at the top or bottom fiber, the stress criterion can be formulated as:

$$\left.\begin{aligned} &\frac{1}{\bar{t}^2}\left(\frac{6\overline{M}}{\bar{t}} + \overline{N}\right)^2 + \beta^2\frac{\overline{T}^2}{t^2} \leqslant 4\,\bar{\tau}^2_{adm} && \text{(a)} \\ &\frac{1}{t^2}\left(\frac{6\overline{M}}{\bar{t}} - \overline{N}\right)^2 + \beta^2\frac{T^2}{t^2} \leqslant 4\,\bar{\tau}^2_{adm} && \text{(b)} \end{aligned}\right\} \qquad (4.39)$$

Our assumption on the shear stress distribution $\bar{\tau}$ in the cross section was made in order to avoid the search of the location of the fiber in a cross section for which the shear has its largest value.

The material volume is the objective function $\bar{J}$.

$$\bar{J} = \int_0^{2L} \bar{t}\,\sqrt{1 + \bar{y},_{\bar{x}}^2}\;d\bar{x} \qquad (4.40)$$

Upon introduction of the dimensionless variables defined by (4.3) and of

$$t = \frac{\bar{t}}{L} \quad \frac{2\,\bar{\tau}_{adm}}{p} \qquad (4.41)$$

the objective function (4.40) and the stress constraint (4.39) reduce respectively to

$$J = \frac{1}{2}\int_0^2 t\,\sqrt{1+z_2^2}\;dx \qquad (4.42)$$

$$\left.\begin{aligned} &\frac{1}{t^2}\left(\frac{6M\gamma}{t} + N\right)^2 + \frac{\beta^2 T^2}{t^2} - 1 \leq 0 \\ &\frac{1}{t^2}\left(\frac{6M\gamma}{t} - N\right)^2 + \frac{\beta^2 T^2}{t^2} - 1 \leq 0 \end{aligned}\right. \qquad (4.43)$$

where

$$\left.\begin{aligned} \bar{J} &= \frac{pL^2}{\tau_{adm}}\,J && \text{(a)} \\ \gamma &= \frac{2\tau_{adm}}{p} && \text{(b)} \end{aligned}\right\} \qquad (4.44)$$

For large values of γ, referred to as the load coefficient, the applied pressure is small with respect to the admissible maximum shear.

Since the result cannot depend on the sign of the pressure, it will be taken as positive. Therefore we will limit our investigations to the case $\gamma > 0$, and $\frac{1}{z_3} \geq 0$.

4-4-1. Formulation

Because of the symmetry in the imposed boundary conditions, the problem can be formulated as:

$$\text{Min } J = \int_0^1 t\,\sqrt{1+z_2^2}\;dx \qquad (4.45)$$

subjected to:

i) the state equations

$$\left.\begin{aligned} \frac{dz_1}{dx} &= z_2 \qquad &(a) \\ \frac{dz_3}{dx} &= 0 \qquad &(b) \end{aligned}\right\} \qquad (4.46)$$

ii) the stress constraints:

$$\left.\begin{aligned} \frac{1}{t^2}\left(\frac{6M\gamma}{t} + N\right)^2 + \beta^2 \frac{T^2}{t^2} - 1 + \mu_1^2 &= 0 \qquad &(a) \\ \frac{1}{t^2}\left(\frac{6M\gamma}{t} - N\right)^2 + \beta^2 \frac{T^2}{t^2} - 1 + \mu_2^2 &= 0 \qquad &(b) \end{aligned}\right\} \qquad (4.47)$$

iii) the equilibrium equations (4. 4)

iv) the boundary conditions

$$\left.\begin{aligned} &z_1(0) = 0 \qquad && z_1(1) \text{ unknown} \\ &z_3(0) \text{ unknown} \qquad && z_3(1) \text{ unknown} \end{aligned}\right\} \qquad (4.48)$$

Note. a) the unknown slope of the support is treated as a state variable bounded to be a constant by (4. 46b).

b) the inequality constraints (4. 43) have been transformed into the equality constraints (4. 47) by introducing two additional control variables μ_1 and μ_2. A stress constraint (4. 46) will be said to be "effective" when its corresponding μ_i (i = 1, 2) is zero.

The Hamiltonian H of the system is obtained by adjoining to its Lagrangian the state equations (4. 46) and the stress constraint (4. 47) by means of the Lagrange's multipliers λ_1, λ_3, η_1 and η_3. The state variables are z_1 and z_2. The control variables are t,

z_2, μ_1 and μ_2.

$$H = t\sqrt{1+z_2^2} + \lambda_1 z_2 + \lambda_3 x(0) + \eta_1\left[\frac{1}{t^2}\left(\frac{6M\gamma}{t} + N\right)^2 + \beta^2\frac{T^2}{t^2} - 1 + \mu_1^2\right] + \eta_2\left[\frac{1}{t^2}\left(\frac{6M\gamma}{t} - N\right)^2 + \beta^2\frac{T^2}{t^2} - 1 + \mu_2^2\right] \tag{4.49}$$

4-4-2. Optimality Conditions

The first order optimality conditions are:

$$\left.\begin{aligned}
&H_{\mu_i} = 2\mu_i\eta_i = 0 \qquad i = 1, 2 && \text{(a)}\\
&H_t = \sqrt{1+z_2^2} - \frac{2\eta_1}{t}\left[1 + \frac{6M\gamma}{t^3}\left(\frac{6M\gamma}{t} + N\right) - \mu_1^2\right] + \frac{2\eta_2}{t}\left[1 + \frac{6M\gamma}{t^3}\left(\frac{6M\gamma}{t} - N\right) - \mu_2^2\right] && \text{(b)}\\
&H_{z_2} = \frac{t z_2}{\sqrt{1+z_2^2}} + \frac{2\eta_1}{t^2}\left[\left(\frac{6M\gamma}{t} + N\right)\frac{\partial N}{\partial z_2} + \beta^2 T\frac{\partial T}{\partial z_2}\right] + \frac{2\eta_2}{t^2}\left[-\left(\frac{6M\gamma}{t} - N\right)\frac{\partial N}{\partial z_2} + \beta^2 T\frac{\partial T}{\partial z_2}\right] + \lambda_1 = 0 && \text{(c)}
\end{aligned}\right\} \tag{4.50}$$

$$\frac{d\lambda_i}{dx} = -H_{z_i} = -\frac{2\eta_1}{t^2}\left[\left(\frac{6M\gamma}{t} + N\right)\left(\frac{6\gamma}{t}\frac{\partial M}{\partial z_i} + \frac{\partial N}{\partial z_i}\right) + \beta^2 T\frac{\partial T}{\partial z_i}\right] - \frac{2\eta_2}{t^2}\left[\left(\frac{6M\gamma}{t} - N\right)\left(\frac{6\gamma}{t}\frac{\partial M}{\partial z_i} - \frac{\partial N}{\partial z_i}\right) + \beta^2 T\frac{\partial T}{\partial z_i}\right]$$

$$i = 1 \text{ or } 3 \tag{4.51}$$

The stress constraints (4.47) are also part of the optimality conditions. Since they result from the transformation of the inequality constraints (4.43), their associated Lagrange's multipliers must be non-negative on an optimal solution. The boundary conditions are:

at $x = 0$

$$\left.\begin{array}{lll} z_1(0) = 0 & \lambda_1(0) \text{ unknown} & \text{(a)} \\ z_3(0) \text{ unknown} & \lambda_3(0) = 0 & \text{(b)} \\ \text{at } x = 1 & & \\ z_1(1) \text{ unknown} & \lambda_1(1) = 0 & \text{(c)} \\ z_3(0) \text{ unknown} & \lambda_3(1) = 0 & \text{(d)} \end{array}\right\} \quad (4.52)$$

As for the stiffness case, the condition (4.52c) is implied by the symmetry condition:

$$z_2(1) = 0 \qquad (4.53)$$

Two constraints enter into the problem, therefore three cases should be considered:

a) case where no constraint is effective

b) case where one constraint is effective

c) case where two constraints are effective

Theorem 4-4

At every point of the optimal structure, at least one stress constraint has to be effective.

Proof

Let us suppose that no stress constraint is effective, i.e., that $\mu_1 \neq 0$ and $\mu_2 \neq 0$. Then (4.50a) implies that $\eta_1 = \eta_2$, which contradicts (4.50b).

The previous theorem rules out the first possibility, as expected from physical considerations. Since the structure is statically determinant, a change of thickness at a point does not affect the values of M, N and T at the other locations. Therefore if the value of the thickness at a point was higher than the one required to satisfy the stress constraints, it could be reduced until one constraint becomes effective.

<u>Theorem 4-5</u>

A feasible state of stress for which the stress constraint (4.47a) is effective is such that

$$NM \geqslant 0$$

<u>Proof</u>

Let us suppose $NM < 0$. Then the constraint (4.47b) is violated. q.e.d.

<u>Theorem 4-6</u>

When two constraints are effective, then either $N = 0$ or $M = 0$.

<u>Proof</u>

When the two constraints are effective, then $\mu_1 = \mu_2 = 0$. The two constraints (4.49) imply that $NM = 0$. A segment of arch on which $N = 0$ everywhere is, from (4.4a), a straight line. Using (4.4b), M is found to be non-zero on such an arc, except at $x = 0$.

q.e.d.

When two constraints are effective, the shape and the thickness distribution are fully determined by the stress conditions. However the optimality conditions will determine whether or not such structures are optimal.

4-4-3. Membrane Design

Theorem 4-7

The "best" membrane design, i.e., the membrane design with $\frac{1}{z_3} = 0.429$ satisfies all the optimality conditions provided $\gamma \geqslant 1/3$.

Proof

We will first show that any membrane design satisfies all the optimality conditions, except the boundary condition on λ_3. For a membrane design $M = 0 \quad \forall x, x \in [0,1]$. Therefore the two stress constraints are effective and $T = 0$.

Let us compute the two Lagrange's multipliers η_1 and η_2 from the optimality conditions.

$$(4.47) \Longrightarrow \quad \frac{N^2}{t^2} = 1 \tag{4.54}$$

$$(4.5b) \Longrightarrow \quad t = \sqrt{1 + (\frac{1}{z_3})^2} \tag{4.55}$$

$$(4.50b) \Longrightarrow \quad \frac{2\eta_1}{t} + \frac{2\eta_2}{t} = \sqrt{1+z_2^2} \tag{4.56}$$

$$(4.50c) \text{ and } (4.5a) \Longrightarrow \lambda_1 = \frac{t\, z_2}{\sqrt{1+z_2^2}} = (x-1) \tag{4.57}$$

$$(4.57) \Longrightarrow \frac{d\lambda_1}{dx} = 1 \tag{4.58}$$

Using (4. 51) with i = 1, (4. 56) and (4. 58) one obtains:

$$\frac{2\eta_1}{t} - \frac{2\eta_2}{t} = \frac{1}{3\gamma}\sqrt{1+z_2^2} \qquad (4.59)$$

(4. 59) and (4. 56) define a system of two linear equations for η_1 and η_2 which has the following solution:

$$\left.\begin{aligned} \eta_1 &= \frac{t\sqrt{1+z_2^2}}{4}\left(1+\frac{1}{3\gamma}\right) && \text{(a)} \\ \eta_2 &= \frac{t\sqrt{1+z_2^2}}{4}\left(1-\frac{1}{3\gamma}\right) && \text{(b)} \end{aligned}\right\} \qquad (4.60)$$

when $0 < \gamma < \frac{1}{3}$ then $\eta_2 < 0$ which indicates that the membrane design is not optimal. When $\gamma > \frac{1}{3}$ then $\eta_1 > 0$ and $\eta_2 > 0$. In this case the membrane design of minimum material volume, i. e., satisfying also the boundary conditions on λ_3, is a local optimum. This solution corresponds to the "best" membrane defined in section 4-3-4. The slope of the support is given by $\frac{1}{z_3} = 0.429$ and the material volume is 1. 3800.

4-4-4. Straight Design

<u>Theorem 4-8</u>

The straight design $z_1(x) = 0$, $\frac{1}{z_3} = 0$ satisfies all the optimality conditions for all values of γ.

<u>Proof</u>

For the straight design N = 0, the two constraints (4. 47) are effective.

$$(4.47) \Longrightarrow \quad t = \sqrt{\frac{\beta^2 T^2 + \sqrt{\beta^4 T^4 + 144\, M^2 \gamma^2}}{2}} \qquad (4.61)$$

where T and M are given by the equilibrium equation (4.4):

$$T = x - 1 , \quad M = \frac{x^2}{2} - x \tag{4.62}$$

$$(4.50b) \Longrightarrow \left(\frac{2\eta_1}{t} + \frac{2\eta_2}{t}\right)\left(1 + \frac{36M^2\gamma^2}{t^4}\right) = 1 \tag{4.63}$$

Using (4.51) with i = 1 and the boundary condition $\lambda_2(1) = 0$ one obtains:

$$\frac{d\lambda_1}{dx} = 0 \quad \Longrightarrow \quad \lambda_1(x) = 0 \tag{4.64}$$

$$(4.50c) \Longrightarrow 2\{\eta_1 - \eta_2\}\left[\frac{6M\gamma}{t^3} T\right] = 0 \tag{4.65}$$

Since M and T are not identically zero for all x in the interval $[0, 1]$:

$$(4.63) \text{ and } (4.65) \Longrightarrow \eta_1 = \eta_2 = \frac{t}{4\left(1 + \frac{36M^2\gamma^2}{t^4}\right)} \tag{4.66}$$

A direct computation of λ_3 from (4.51) with i = 3 shows that $\lambda_3(x) = 0$. Therefore the straight design satisfies all the optimality conditions.

The material volume of the straight design V_s can be computed by quadratures using (4.61) and (4.62). A lower bound of the material volume is obtained when $\beta = 0$.

$$[V_s]_{\beta=0} = \sqrt{3\gamma} \; \frac{\pi}{4} \tag{4.67}$$

The straight design corresponds to a local optimum, but it cannot be a global optimum for all values of γ, since the material volume associated with the "best" membrane design is independent of γ.

4-4-5. Designs with One Effective Constraint

The membrane design ceased to be a local optimum for $\gamma < \frac{1}{3}$ because η_2, the Lagrange's multiplier associated with the stress constraint (4.47b), became negative. This gives an indication that better designs, on which the constraint (4.47b) is not effective, might be found. Let us give a physical interpretation of the previous argument. On a membrane design:

$$z_1(x) = [z_1(x)]_M, \quad M = 0, \; N < 0$$

A direct substitution in (4.4c) for the case $\frac{1}{z_3} > 0$ shows that:

a) if $z_1(x) > [z_1(x)]_M$ then $M > 0$

b) if $z_1(x) < [z_1(x)]_M$ then $M < 0$

Therefore the length of the middle line of an arch, on which $M > 0$ everywhere, cannot be shorter than the one of the membrane design having the same initial slope. A study of the one parameter family of designs:

$$z_1(x) = [z_1(x)]_M + \varepsilon\, x^2(-2x+3)$$

showed that the "best" membrane design was improved for $\varepsilon < 0$ and $\gamma \leqslant \gamma_\ell < \frac{1}{3}$. The decrease of the material volume was obtained even though there was an increase of the average thickness of the arch, due to the bending moment. However the decrease in the middle line length more than compensated for the increase of the average thickness. For the investigated family of designs, $M \times N$

is positive when $\varepsilon < 0$. These were the determining factors for investigating the case where only the constraint (4.47a) is effective.

When the constraint (4.47a) only is effective, the optimality conditions are obtained from the ones defined in the section 4-4-2 setting

$$\mu_1 = \eta_2 = 0 \qquad (4.68)$$

$$(4.50b) \Rightarrow \frac{2\eta_1}{t} = \frac{\sqrt{1+z_2^2}}{1+\frac{6M\gamma}{t^3}\left(\frac{6M\gamma}{t} + N\right)} \qquad (4.69)$$

When the state of stress constrained by (4.47a) only is feasible, theorem 4-5 and (4.69) imply that $\frac{2\eta_1}{t} > 0$.

To avoid the difficulty of solving the constraint (4.47a) and the equation (4.50c) for z_2 and t, total derivatives of those equations with respect to x are used to generate a system of two linear equations for $\frac{dt}{dx}$ and $\frac{dz_2}{dx}$. The total derivative of the stress constraint (4.47a) with respect to x can be written as:

$$B_1 \frac{dt}{dx} + B_2 \frac{dz_2}{dx} = B_3 \qquad (4.70)$$

where

$$\left.\begin{aligned} B_1 &= 4t^3 - 2t(\beta^2T^2+N^2) - 12\,MN\gamma && (a)\\ B_2 &= -\frac{2t^2}{1+z_2^2}\,T\left[N(1-\beta^2) + \frac{6M\gamma}{t}\right] && (b)\\ B_3 &= 2T\sqrt{1+z_2^2}\,\left[\beta^2t^2 + 6\gamma tN + 36\gamma^2M\right] && (c)\end{aligned}\right\} \quad (4.71)$$

The conditions (4.50) can be expressed as:

$$H_{z_2} = \frac{tz_2}{\sqrt{1+z_2^2}} + DC + \lambda_1 = 0 \qquad \text{(a)}$$

$$\text{where } D \equiv \frac{2\eta_1}{t} \qquad \text{(b)} \qquad (4.72)$$

$$C \equiv \frac{T}{t(1+z_2^2)} \left[\frac{6M\gamma}{t} + N(1-\beta^2)\right] \qquad \text{(c)}$$

The total derivatives with respect to x of D and C are respectively:

$$\frac{dD}{dx} = D_1 \frac{dt}{dx} + D_2 \frac{dz_2}{dx} + D_3 \qquad \text{(a)}$$

$$\text{where: } D_1 = \frac{D^2}{\sqrt{1+z_2^2}} \frac{18M\gamma}{t^4} \left(\frac{8M\gamma}{t} + N\right) \qquad \text{(b)}$$

$$D_2 = \frac{D}{\sqrt{1+z_2^2}} \left(\frac{z_2}{\sqrt{1+z_2^2}} - \frac{6M\gamma}{t^3} D \frac{\partial N}{\partial z_2}\right) \qquad \text{(c)} \qquad (4.73)$$

$$D_3 = -D^2 T \frac{6\gamma}{t^3} \left(\frac{12M\gamma}{t} + N\right) \qquad \text{(d)}$$

$$\frac{dC}{dx} = C_1 \frac{dt}{dx} + C_2 \frac{dz_2}{dx} + C_3 \qquad \text{(a)}$$

$$\text{where } C_1 = -\frac{T}{t^2(1+z_2^2)} \left[\frac{12M\gamma}{t} + N(1-\beta^2)\right] \qquad \text{(b)}$$

$$C_2 = \frac{1}{t^2(1+z_2^2)} \left[\left(-\frac{6M\gamma}{t}(N+2Tz_2) + (1-\beta^2)(T^2-N^2-2TNz_2)\right]\right. \qquad \text{(c)} \qquad (4.74)$$

$$C_3 = \frac{1}{t\sqrt{1+z_2^2}} \left[\frac{6}{t}(T^2+M) + N(1-\beta^2)\right] \qquad \text{(d)}$$

The total derivative with respect to x of the condition (4.50c) can be written as

$$\frac{dH_{z_2}}{dx} = F_1 \frac{dt}{dx} + F_2 \frac{dz_2}{dx} - F_3 = 0 \qquad (4.75)$$

where

$$\left.\begin{aligned} F_1 &= \frac{z_2}{\sqrt{1+z_2^2}} + DC_1 + D_1C && \text{(a)} \\ F_2 &= \frac{t}{(1+z_2^2)^{3/2}} + DC_2 + D_2C && \text{(b)} \\ F_3 &= -\frac{d\lambda_1}{dx} - DC_3 - D_3C && \text{(c)} \end{aligned}\right\} \qquad (4.76)$$

$$\frac{d\lambda_1}{dx} = -\frac{D}{t}\left[\left(\frac{6M\gamma}{t} + N\right)\left(\frac{6\gamma}{t}\left\{z_1 + \frac{1}{z_3}\right\} - \frac{1}{\sqrt{1+z_2^2}}\right) + \beta^2 T \frac{z_2}{\sqrt{1+z_2^2}}\right] \qquad (4.77)$$

The relation (4.77) is derived from the condition (4.51) with $i = 1$. The conditions (4.70) and (4.75) define a linear system of equations for $\frac{dz_2}{dx}$ and $\frac{dt}{dx}$ which has a unique solution when

$$G \equiv B_1F_2 - B_2F_1 \neq 0 \qquad (4.78)$$

The solution is given by:

$$\left.\begin{aligned} \frac{dz_2}{dx} &= \frac{B_1F_3 - B_3F_1}{G} && \text{(a)} \\ \frac{dt}{dx} &= \frac{B_3F_2 - B_2F_3}{G} && \text{(b)} \end{aligned}\right\} \qquad (4.79)$$

Theorem 4-9

If $\beta = 0$ then $G = 0$ whenever $M = 0$.

Proof

The thickness can be computed directly from (4. 47a)

$$t^2 = N^2 \tag{4.80}$$

A direct evaluation of the variables defined by (4. 71) through (4. 77) gives:

$$B_1 = 2N^2 t \, , \qquad B_2 = -\frac{2N^2}{1+z_2^2} NT$$

$$D = \sqrt{1+z_2^2} \, , \qquad D_1 = 0 \, , \quad D_2 = \frac{z_2}{\sqrt{1+z_2^2}}$$

$$C = \frac{NT}{t(1+z_2^2)} \, , \qquad C_1 = -\frac{NT}{t^2(1+z_2^2)}$$

$$C_2 = \frac{1}{t(1+z_2^2)^2} \{T^2 - 2TNz_2 - N^2\}$$

$$F_1 = \frac{z_2}{\sqrt{1+z_2^2}} - \frac{NT}{t^2\sqrt{1+z_2^2}}$$

$$F_2 = \frac{t}{1+z_2^2)^{3/2}} [1 + \frac{1}{t^2} (T^2 - TNz_2 - N^2)]$$

Using (4. 78) to evaluate G, one finds:

$$G = \frac{2N^2}{(1+z_2^2)^{3/2}} [t^2 - N^2 + T^2 (1 - \frac{N^2}{t^2})]$$

Therefore by virtue of (4. 80) : $G = 0$ q. e. d.

The shear force contribution was introduced into the stress criterion to insure that $\frac{dt}{dx}$ and $\frac{dz_2}{dx}$ can be uniquely determined from the equations (4.70) and (4.75).

The system of equations (4.79), along with equation (4.51) with $i = 3$, define the set II of optimality field equations. The boundary conditions are given by (4.52), where (4.52c) is replaced by (4.53). The initial value of the thickness is given by the stress constraint (4.47a) evaluated at $x = 0$. This defines the set II of optimality conditions.

Numerical solution of the set II of optimality conditions was first performed for $\gamma = .3$ using a direct shooting technique as described in Chapter III. The unknown initial values were $z_2(0)$ and $\frac{1}{z_3}$. The matching conditions were $z_2(1) = \lambda_3(1) = 0$. Due to the peculiar changes in the unknown initial conditions from one iteration to the next one, this search method was abandoned. A direct computation of $z_2(1)$ and $\lambda_3(1)$ as a function of $z_2(0)$ and $\frac{1}{z_3}$ was performed for different values of γ and $\beta = 2$. Figure 6 shows the loci of the points in the $(z_2(0), \frac{1}{z_3})$ plane for which either $z_2(1) = 0$ or $\lambda_3(1) = 0$. A solution of our problem, which must satisfy both conditions, is an intersection point of the two loci. For $\gamma = .25$ the figure 6a shows that no intersection point exists. When $\gamma = .3$ there exist two intersection points A and B as shown on figure 6b. The point A corresponds to a local minimum of the material volume, but the point B is neither a minimum nor a maximum. As γ was increased up to the value 1/3, the point A moved toward point C which represents the best membrane design. For $\gamma = 0.35$ only one intersection point B of the two loci was found as shown on figure 6c. The problems encountered during the iterative search can be explained by the fact that the solutions correspond to the intersection points of two curves which are almost tangent when $\gamma = .3$.

When one constraint only is effective, the number and the character of the solutions to the optimality conditions depend on γ in the following manner:

a) $\gamma < \gamma^*$: no solution. It was found that $\gamma^* = 0.27$ for $\beta = 2$.

b) $\gamma^* < \gamma < \frac{1}{3}$: two solutions, only one of which corresponds to a local minimum of the material volume.

c) $\gamma > \frac{1}{3}$: one solution which is not a local minimum.

To verify these surprising results, the problem of determining the "best" parabolic arch satisfying the stress constraints (4.47) was investigated. The thickness t(x) is considered as an unknown function. The initial slope of the structure $z_2(0)$ and the slope of the support are two unknown parameters. The dependence of the solutions on the parameter γ, when one stress constraint only is effective, was found to be similar to the one of the general problem.

Remark: In the previous analysis we considered the cases where the same set of constraints was effective on the entire structure. Theoretically, solutions on which several sets of constraints are effective on different arcs of the solution should be investigated. Two types of switching points could exist:

a) switching point between an arc on which the two constraints are effective and an arc on which only one constraint is effective,

b) switching point between an arc on which one of the two stress constraints is effective and an arc on which the other stress constraint is effective.

A detailed analysis of those cases was not performed, since they can be ruled out on physical considerations. When one constraint is effective, its associated Lagrange's multiplier is positive. It indicates that no branching to another type of arc will be locally improving as long as the other constraint is of course not violated. This was found to be verified during our numerical computations.

4-4-6. Results

Figure 5 shows the value of the material volume as a function of γ for the different types of solutions. The numerical computations were made for the case $\beta = 2.0$.

The straight design is a local optimum for any value of γ, and corresponds to the global minimum for $\gamma < \gamma^+$. The "best" membrane design is a local optimum for $\gamma > \frac{1}{3}$ and corresponds to the global minimum for $\gamma > \gamma^+$. When one stress constraint only is effective, solutions to the optimality conditions exist when $\gamma > \gamma^*$. The local minimum found for $\gamma^* < \gamma < \frac{1}{3}$ did not appear to be a global minimum. The values of γ^+ and γ^* obtained in our computations are respectively $\gamma^* = 0.27$ and $\gamma^+ = 0.44$. These values are functions of the coefficient β.

4-4-7. Conclusion

The influence of the shear force was introduced in the definition of the admissible state of stress. The maximum shear was used as the failure criterion, introducing two stress constraints.

For the considered cases the global optimum was obtained when the two stress constraints were effective. It was either the "best" membrane design, when the load coefficient γ is larger than γ^+, or the straight design when $\gamma < \gamma^+$. Furthermore the "best" membrane design ceased to be a local optimum for $\gamma < \frac{1}{3}$.

When only one stress constraint was effective, two solutions of the optimality conditions were shown to exist for $\gamma^* < \gamma < \frac{1}{3}$. One of these only was a local minimum. When $\gamma > \gamma^*$ one solution of this type did exist, but it was neither a maximum nor a minimum.

Although for practical applications the "best" membrane design corresponds to the minimal material volume design, it has been shown that other solutions to the optimality conditions do exist.

V. CONCLUSION

The material volume minimization with respect to the shape and the thickness has been investigated for a type of structures subjected to either a stress or a stiffness constraint. The optimality conditions have been derived using the Optimal Control Theory, and a general purpose computer program solving n-points boundary value problems with the parallel shooting techniques has been developed.

The inclusion of the contribution of the shear force in the strain energy density or in the failure criterion was found sufficient to obtain well-posed problems when dealing with statically determinant arches. For the stiffness problem, a membrane design, i.e., a structure on which the bending moment is identically zero, does not correspond to the optimal structure. However it was found to be a very good approximation of the "best" design for practical cases. When the maximum shear failure criterion is imposed, the "best" membrane design corresponds to the true optimum for practical values of the load coefficient. However, the number of solutions of the first order optimality conditions was found to depend on the value of the load coefficient.

Evidence has been shown that a modification of the governing equations of the structures to include some of the effects judged insignificant in structural analysis may transform an ill-posed optimization problem into a well-posed problem.

REFERENCES

1. Wasiuntynski, Z., Brandt, A., "The Present Stage of Knowledge in the Field of Optimum Design of Structures," Applied Mechanics Review, 16 (1963), 341-350.

2. Barnett, R., "Survey of Optimal Structural Design," Experimental Mechanics, 19A-26A, Dec. 1966.

3. Sheu, C. Y., Prager, W., "Recent Development in Optimal Structural Design," Applied Mechanics Review, 21 (1968), 985-992.

4. Niordson, F. Y., Pedersen, P., "A Review of Optimal Structural Design," Proceedings of 13th International Congress of Theoretical and Applied Mechanics, Springer Verlag, 1973, 264-278.

5. Prager, W., Taylor, J. E.,"Problems of Optimal Structural Design," Transactions of ASME, March 1966, 102-106.

6. Huang, N. C., "Optimal Design of Elastic Structures for Maximum Stiffness," Int. J. Solids Structures, Vol. 4, 689-700, Pergamon Press.

7. Niordson, F. I., "On the Optimal Design of a Vibrating Beam," Q. Appl. Math., 23, 47-53 (1965).

8. Budiansky, B., Frauenthal, J. C. and Hutchinson, J., "On Optimal Arches," Report SM-30, Harvard University (1969).

9. Frauenthal, J. C., "Constrained Optimal Design of Circular Plates Against Buckling," Report SM 50, Harvard University (1971).

10. Taylor, J. E. and Liu, C. Y., "Optimal Design of Columns,"

REFERENCES (Cont'd)

AIAA Journal, Vol. 6, No. 8, Aug. 1968, 1497-1502.

11. Giraudbit, J. N., "Optimal Simple Structures with Bending and Membrane Stress." Thesis in Partial Fulfillment of the Requirements for the Degree of Aeronautical Engineer, California Institute of Technology (1 72).

12. Stroud, W. J., "Automated Structural Design with Aeroelastic Constraints: A Review and Assessment of the State of the Art." Presented at the ASME Symposium on Structural Optimization, 1974 Winter Annual Meeting.

13. Pope, G. G., Schmit, L. A., eds., "Structural Design Applications of Mathematical Programming Techniques," AGARDograph No. 149, Feb. 1971.

14. Berke, L., Khot, N. S., "Use of Optimality Criteria Methods for Large Scale Systems." AGARD Lecture Series No. 70 on Structural Optimization.

15. Schmit, L. A., Farshi, B., "Some Approximation Concepts for Structural Synthesis." AIAA Journal, Vol. 12, No. 5, May 1974, 692-699.

16. Templeman, A. B., "The Use of Geometric Programming Methods for Structural Optimization." AGARD Lecture Series No. 70 on Structural Optimization.

17. Sobieszczanski, J., "Sizing of Complex Structures by the Integration of Several Different Optimal Design Algorithms." AGARD Lecture Series No. 70 on Structural Optimization.

REFERENCES (Cont'd)

18. Wasiutynski, Z., "On the Equivalence of Design Principles: Minimum Potential-Constant Volume and Minimum Volume-Constant Potential." Bulletin de l'Academie Polonaise des Sciences, Serie des Sciences Techniques, Vol. XIV, No. 9, 1966.

19. Bryson, A. E. and Ho, Y. C., "Applied Optimal Control Theory." Waltham, Mass., Blaisdell Publ. Co., 1969.

20. Keller, H. B., "Numerical Methods for Two Point Boundary-Value Problems." Blaisdell Publishing Company. A Division of Ginn and Company, Waltham, Massachusetts. Toronto-London, 1968.

21. Huang, N. C., Sheu, C. Y., "Optimal Design of Elastic Circular Sandwich Beams for Minimum Compliance," Journal of Applied Mechanics, September 1970, 569-577.

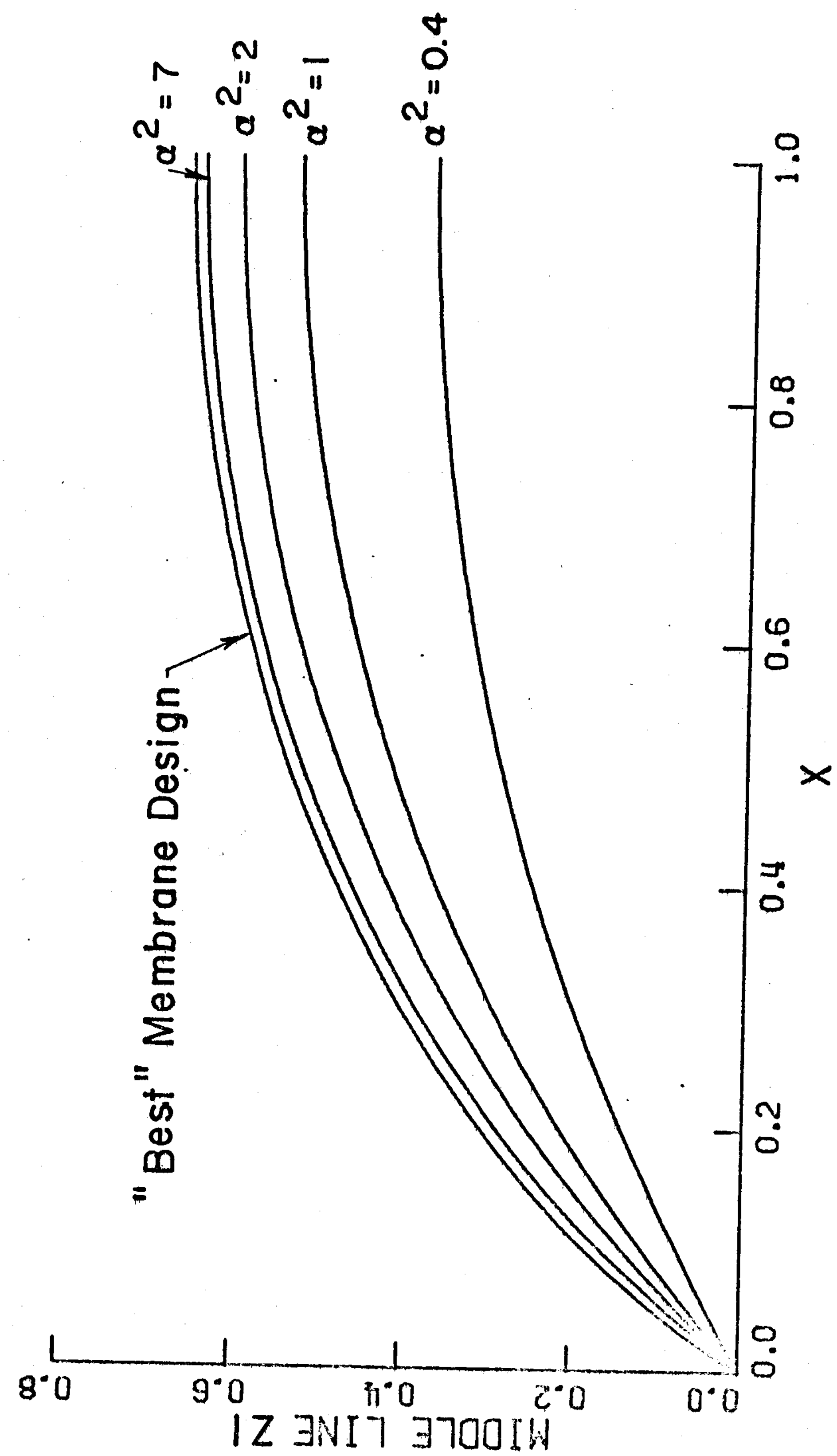


FIG. 1 OPTIMAL ARCH
STIFFNESS CONSTRAINT
MIDDLE LINE SHAPE Z1 $\beta^2 = 2.5$

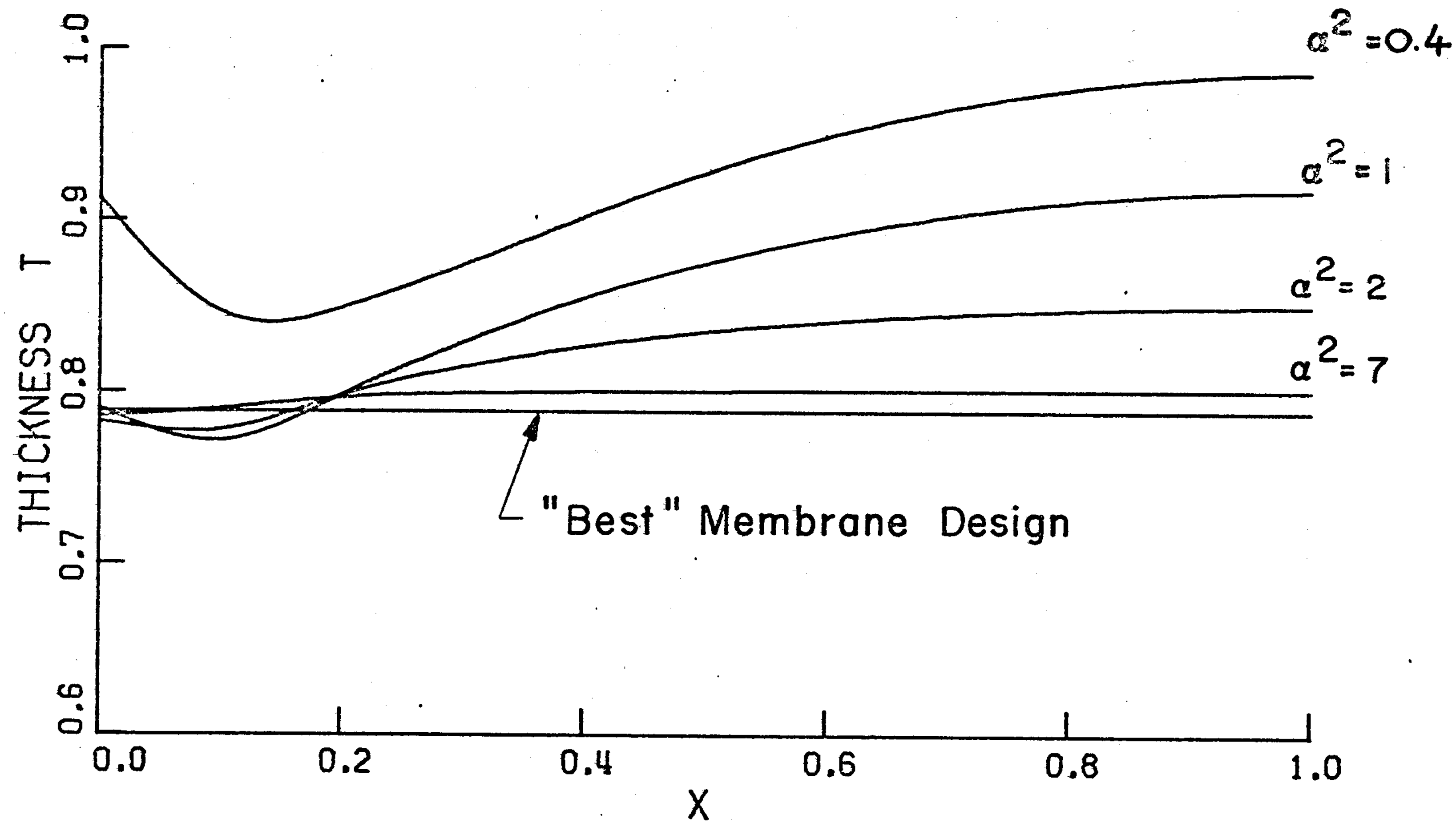
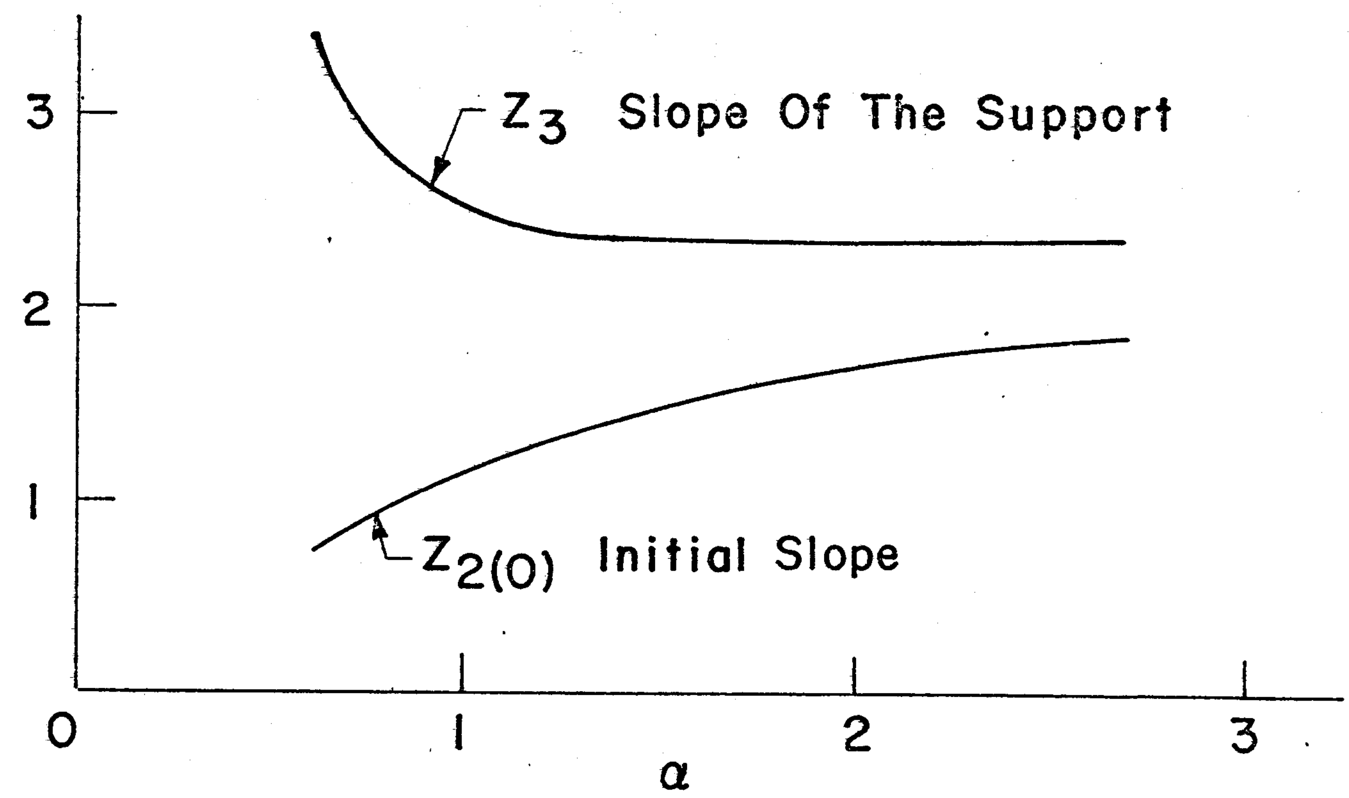
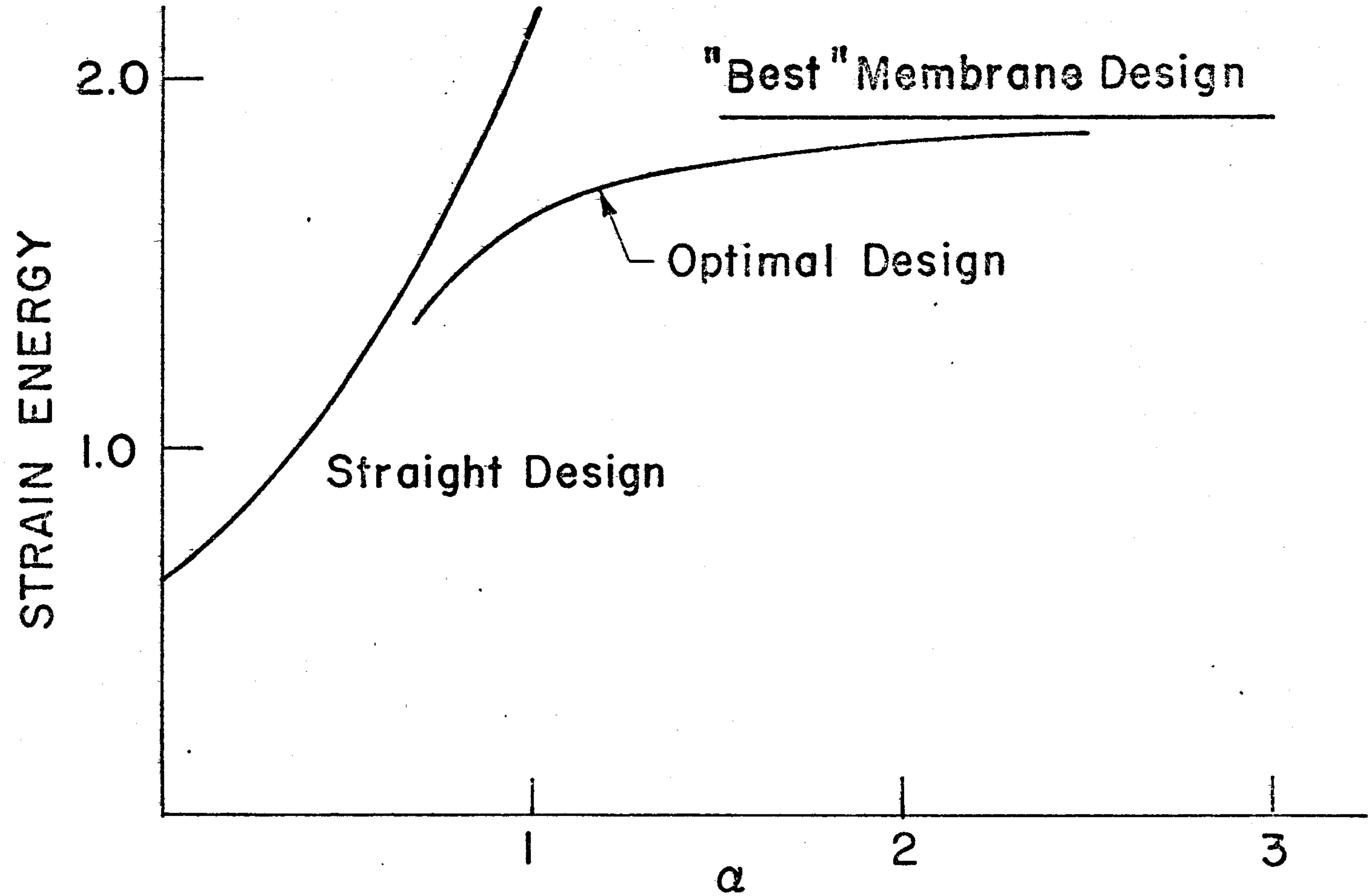


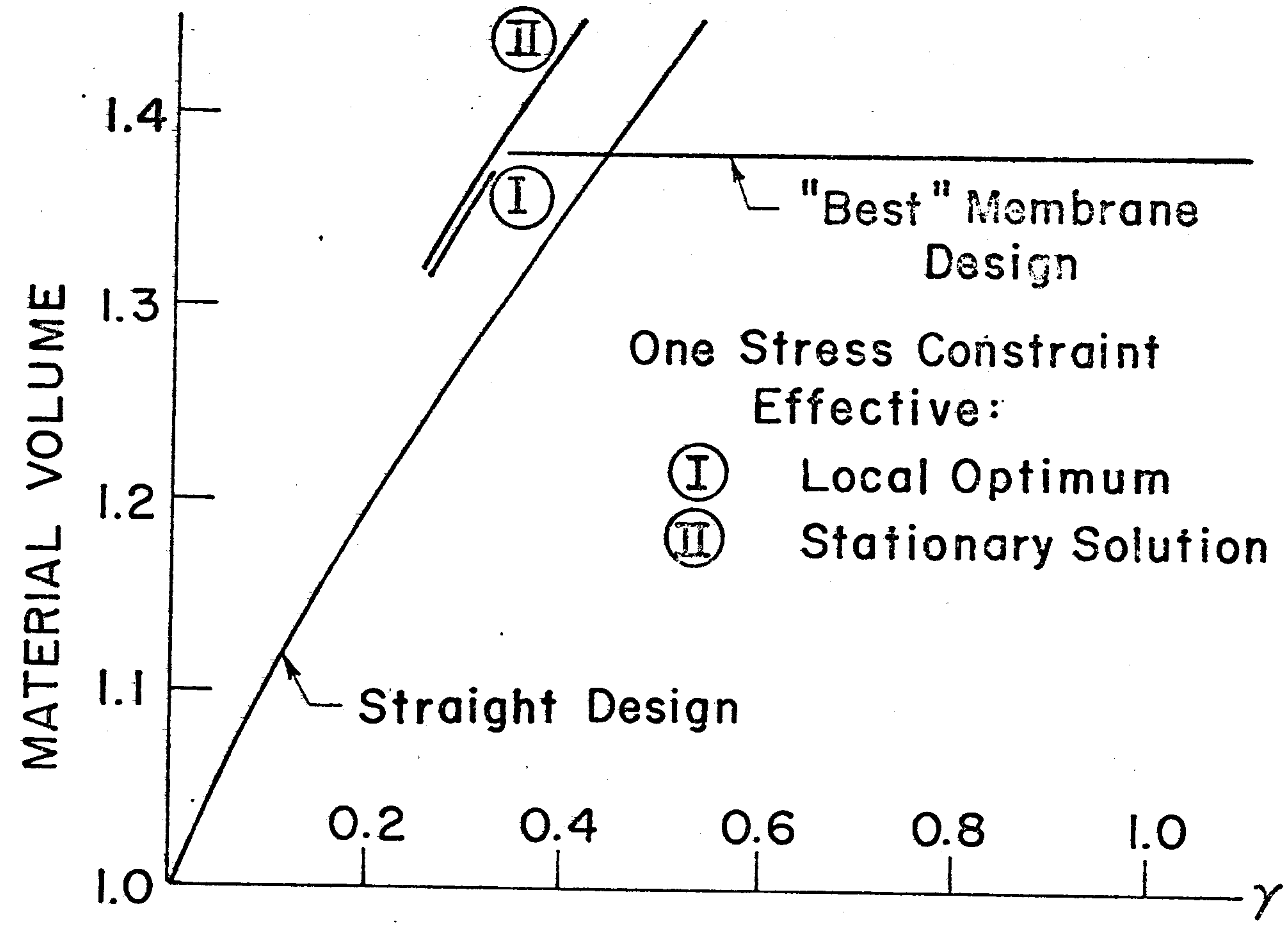
FIG.2 ØPTIMAL ARCH
STIFFNESS CØNSTRAINT
THICKNESS T $\beta^2 = 2.5$



OPTIMAL ARCH
STIFFNESS CONSTRAINT $\beta^2 = 2.5$
FIG. 3 SLOPE OF THE SUPPORT AND INITIAL SLOPE



OPTIMAL ARCH
STIFFNESS CONSTRAINT $\beta^2 = 2.5$
FIG. 4 STRAIN ENERGY



OPTIMAL ARCH STRESS CONSTRAINT
FIG. 5 MATERIAL VOLUME

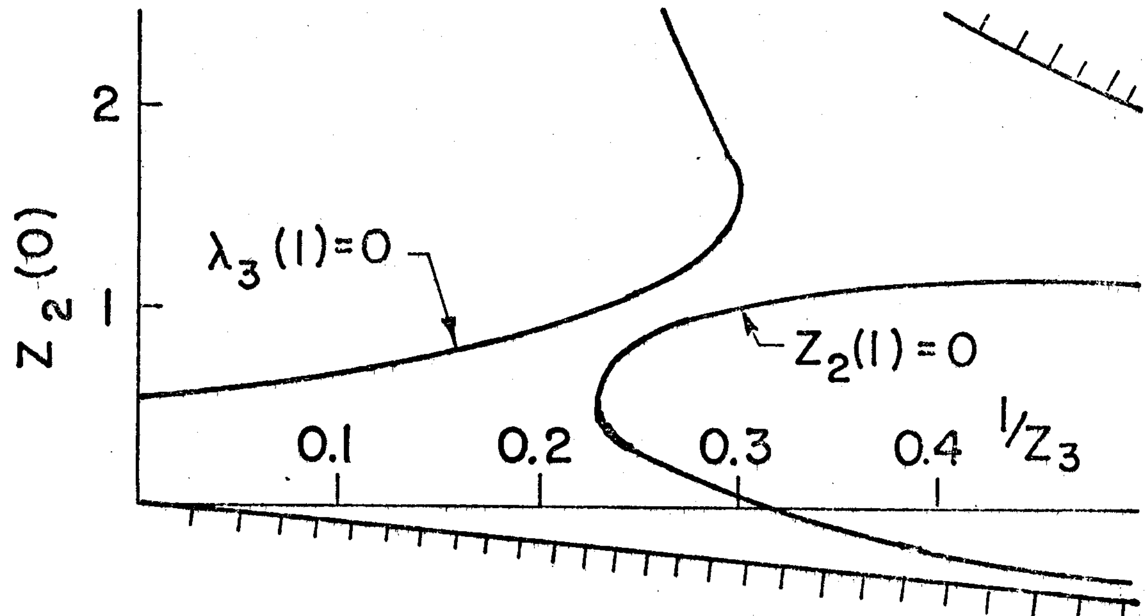


Fig. 6a $\gamma = 0.25$

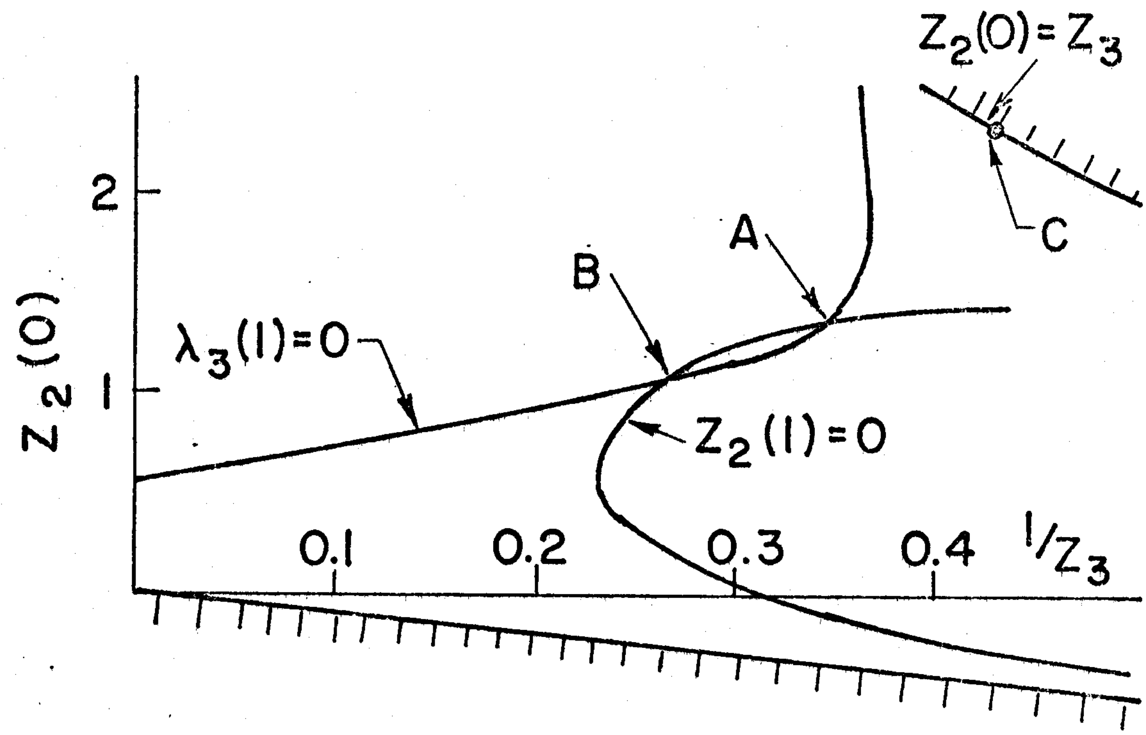


Fig. 6b $\gamma = 0.3$

OPTIMAL ARCH STRESS CONSTRAINT

FIG. 6 LOCI OF $Z_2(1)=0, \lambda_3(1)=0$

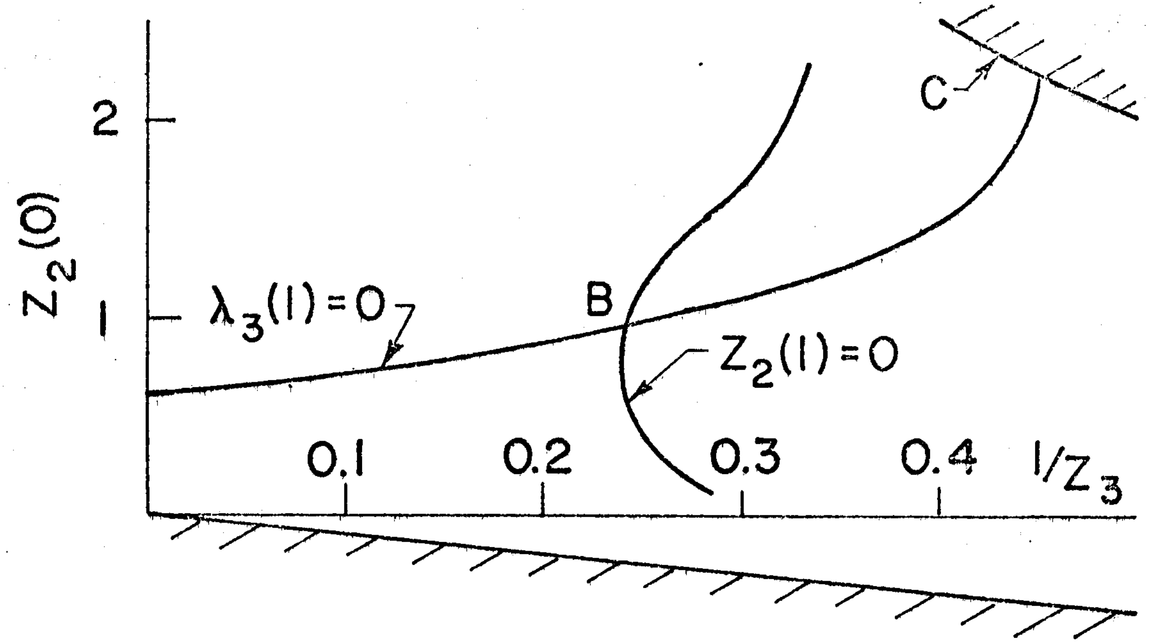


Fig. 6c $\gamma = 0.35$

OPTIMAL ARCH STRESS CONSTRAINT

FIG. 6 LOCI OF $Z_2(1)=0, \lambda_3(1)=0$