USTL
FLANDRES ARTOIS



LABORATOIRE D'ANALYSE NUMERIQUE
ET D'OPTIMISATION





n° d'ordre : 790

# THESE

présentée à

l'Université des Sciences et Techniques de Lille Flandres-Artois

pour obtenir le titre de

DOCTEUR D'ETAT ES SCIENCES MATHEMATIQUES

par

Marc PREVOST

# APPROXIMANTS DE TYPE-PADE ET MOMENTS MODIFIES



soutenue le 5 avril 1991 devant la commission d'examen

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*Monsieur Maroni m'a fait l'amitié d'accepter la présidence de ce jury. Je l'en remercie vivement.*

*Les professeurs Varga et Golub m'ont fait l'honneur de se pencher sur mes travaux, qu'ils en trouvent ici l'expression de ma gratitude.*

*Monsieur Bessis m'a fait découvrir les utilisations possibles de certains de mes résultats. Je lui en suis vivement reconnaissant.*

*Je remercie également Jeannette Van Iseghem et Bernard Germain-Bonne d'avoir accepté de participer à ce jury après avoir, durant plusieurs années, répondu avec gentillesse et disponibilité à mes questions.*

*Enfin, la totale liberté de recherche, la confiance et l'indulgence dont C. Brezinski m'a fait bénéficier ont été prépondérantes dans l'élaboration de ce travail. En outre, sa connaissance exceptionnelle de tous les travaux de recherche liés de près ou de loin à l'approximation de Padé m'a fait gagner un temps précieux.*

*Et je remercie particulièrement les chercheurs français et étrangers que j'ai eu l'occasion de solliciter lors de diverses rencontres, en la personne de A. Magnus, A. Ronveaux, S. Paskowski, P. Marcellan, A. Matos, J. Antolin, ...*

*Que toute l'équipe du Laboratoire A.N.O. de Lille et plus particulièrement H. Sadok, F. Cordellier pour leurs précieux conseils, F. Tailly pour la dactylographie et H. Glanc pour la mise en forme définitive, recoivent ici toute mon amitié.*

*à Catherine et Juliette*
*pour leur patience*

## Introduction

This report contains the synthesis of my works published or to appear and which are listed in the bibliography.

These different researches deal with the problem of the approximation of a function $f$ of the complex variable $t$ whose coefficients $(c_i)_{i\in\mathbb{N}}$ of the Taylor expansion around $t = 0$ are given. This problem is often dealt with and has many applications in Physics.

Generally, one try to approximate $f$ by a rational fraction in $t$ whose coefficients depend on those of $f$. If we know, about $f$, only the value of a finite number of coefficients $(c_i)_{i=0,\ldots,N}$, a useful method is the Padé rational approximation : it consists to built a rational fraction whose Taylor expansion around $t = 0$ agrees with $f$ as far as possible that is, at least, up to the term whose degree equals the sum of the degrees of the numerator and the denominator of the rational fraction.

The importance of this theory can be explained by many convergence properties for some classes of functions (Stieltjes, Polya, meromorphic). Generally, these results proceed from the knowledge of the location of poles of the Padé approximant which are actually the inverses of the zeros of orthogonal polynomials. For the above classes, the poles of Padé approximants converge to those of $f$.

However, Padé approximation does not permit to use some additional informations about $f$, as for example, the location of some poles or zeros, or the domain of holomorphy. It is the justification of the Padé-Type Approximants (P.T.A.) (whose poles are fixed), Partial Padé Approximants (with zeros and poles preassigned) or Cauchy-Type Approximants (whose denominator is a function), all three introduced by C. BREZINSKI ([Brez 1], [Brez 3], [Brez 4]).

The section 1 contains, mainly :

- some approximation results for P.T.A. of Markov-Stieltjes functions ([Prev 1], [Prev 0]).

-a determinantal expression for Partial Padé approximants.

- a convergence theorem for the Padé approximation to the product of two Markov-Stieltjes functions whose cuts are respectively included in $\mathbb{R}^-$ and $\mathbb{R}^+$. ([Prev 7]).

For function which can be written as following :

$$f(t) = \int_I \frac{1}{1 - xt} d\alpha(t)$$

where $I$ is a compact interval of $\mathbb{R}$ and $\alpha$ a bounded variation function, another way to approach $f$ consists, from the knowledge of a finite number of coefficients $(c_i)_{i=0,\dots,N}$, in approximating the distribution $d\alpha$ on $I$, this approximation allowing to continue the sequence of Padé Approximants ( P.A.) by what we have called *"Approached Padé Approximants"*. ([Prev 5]).

This method is effective if $d\alpha$ is absolutely continuous with respect to the Lebesgue measure on $I$, and is directly related to P.T.A. studied in [Prev 0], [Prev 1].

The section 2 contains some results about Dirac masses detection and acceleration of convergence of some sequences :

If the function $\alpha$ has some jumps, that is if $d\alpha$ contains Dirac masses $A_i$ at the points $a_i$, we have built a method using modified moments in order to detect and calculate the jump of $\alpha$ ([Prev 6], [Prev 8]).

In [Prev 10], we have applied these results to accelerate the logarithmic sequences of moments $(c_n)_{n \in \mathbb{N}}$ (i.e : $c_n = \int_0^1 x^n d\alpha(x)$) because the limit of $(c_n)$ is the jump of $\alpha$ at the point 1.

At last, the experience gained about the moment sequences has been useful for proving an acceleration theorem for the columns of $E$-algorithm regarding a set of sequences with an asymptotic expansion on a scale of comparison ([MaPr 11]).

This report contains two sections :

* Padé-type approximants

* Modified moments

in which we have especially stressed the applications, the theorems being stated without proof.

**Section 1.**



**Section 2.**

## Présentation

L'objectif de ce mémoire est de faire la synthèse de mes travaux parus ou à paraître et dont la liste figure dans la bibliographie.

Le fil conducteur de ces différentes recherches est le problème de l'approximation d'une fonction $f$ de la variable complexe $t$ dont on connaît les coefficients $(c_i)_{i\in\mathbb{N}}$ du développement en série de Taylor au voisinage de l'origine. Ce problème est une question souvent abordée et dont l'importance en physique n'est plus à démontrer. En général on cherchera à approximer $f$ par une fraction rationnelle de la variable $t$ dont les coefficients dépendent de ceux de $f$. Si on ne dispose sur $f$ d'aucun autre renseignement que la valeur d'un nombre fini de coefficients $(c_i)_{i=0,\ldots,N}$, alors la méthode généralement utilisée est l'approximation rationnelle dont le développement en série de Taylor au voisinage de l'origine coïncide avec celui de la fonction $f$ jusque l'ordre maximum.

L'importance prise par cette théorie des approximants de Padé est justifiée par de nombreux théorèmes de convergence pour certaines classes de fonctions (fonctions de Stieltjes, de Polya, méromorphes). En général, ces résultats de convergence passent par la connaissance du comportement des pôles des approximants de Padé, qui sont en fait les inverses des racines de polynômes orthogonaux.

Pour les classes de fonctions indiquées ci-dessus, les pôles des approximants se rapprochent de ceux de la fonction $f$.

L'approximation rationnelle de Padé ne permet pas, cependant, d'utiliser des informations supplémentaires sur $f$, comme par exemple la connaissance de certains pôles ou zéros, ou le domaine d'holomorphie. C'est la raison d'être des approximants rationnels de type Padé (dont les pôles sont fixés), de Padé partiels (avec des pôles et des zéros imposés) ou de type Cauchy (dont le dénominateur est en fait une fonction) tous trois introduits par C. BREZINSKI [Brez 1, Brez 3, Brez 4].

La section 1 contient principalement :

- pour l'approximation de type Padé des résultats d'approximation pour des fonctions de Markov-Stieltjes [Prev 1, Prev 0].

- l'expression sous forme d'un rapport de deux déterminants pour les approximants de Padé-Partiels [Prev 4].

- pour l'approximation de Padé un théorème de convergence vers le produit de deux fonctions de Stieltjes dont les coupures sont respectivement incluses dans $\mathbb{R}^-$ et $\mathbb{R}^+$ [Prev 7] ainsi que l'utilisation et l'accélération d'algorithmes déjà existants pour le calcul des pôles des fonctions méromorphes [Prev 3].

Pour les fonctions qui admettent une représentation intégrale sur un intervalle compact de $\mathbb{R}$

$$f(t) = \int_I \frac{1}{1-xt} d\alpha(t)$$

où $\alpha$ est une fonction à variation bornée, une autre façon d'aborder le problème de l'approximation de $f$ consiste, toujours en supposant connus un certain nombre de coefficients $(c_i)_{i=0,N}$, à approcher la distribution $d\alpha$ sur l'intervalle $I$, cette approximation de $d\alpha$ permettant de prolonger la suite des approximants par ce que l'on a appelé *"approximants de Padé approchés"* ([Prev 5]). Cette méthode est efficace si $d\alpha$ est absolument continue par rapport à la mesure de Lebesgue sur $I$, et est directement liée aux approximants de type Padé étudiés dans [Prev 0, Prev 1] (section 2, paragraphe 2).

La section 2 traite aussi de la détection de distributions de Dirac dans une densité ainsi que l' accélération de la convergence de suites de moments.

Dans le cas où la fonction $\alpha$ comporte des sauts, c'est à dire quand $d\alpha$ contient des masses de Dirac $A_i$ en des points $a_i$, nous avons établi une technique utilisant les moments modifiés et permettant de détecter ([Prev 6]) et de calculer ([ Prev 8]) les sauts de $d\alpha$

Dans [Prev 10] on a appliqué les résultats précédents pour accélérer des suites logarithmiques de moments $(c_n)_{n\in\mathbb{N}}$, c'est à dire vérifiant :

$$c_n = \int_0^1 x^n da(x)$$

la limite de $(c_n)$ étant le saut de la fonction $\alpha$ au point 1.

Enfin l' expérience acquise à propos des suites de moments a permis de démontrer un théorème d'accération pour les colonnes du $E$-algorithme concernant un ensemble de suites admettant un développement asymptotique selon une échelle de fonctions [MaPr 11].

Cette synthèse se divise en deux sections :

- approximants de type Padé

- moments modifiés

dans lesquelles on a surtout insisté sur les applications, les théorèmes étant donnés sans démonstration.

La section 1 : *"Approximants de type Padé"* se divise en 6 paragraphes :

1 : Introduction

2 : Fonctions avec représentation intégrale

3 : Fonctions méromorphes

4 : Fonctions de Markov-Stieltjes

5 : Approximants de Padé Partiels

6 : Polynôme de type-Stieltjes et -Geronimus.

La section 2 *"Moments modifiés"* contient 4 parties

1 : Introduction

2 : $c$ admet une représentation intégrale

3 : $c$ est la somme d'une autre fonctionnelle et de distributions de Dirac .

3.1 : Cas général.

3.2 : Détection et calcul des masses de Dirac dans une densité sur [-1, +1].

3.3 : Extension à un intervalle infini.

3.4 : Applications au calcul des singularités d'une fonction à partir de ses moments.

3.5 : Accélération des suites de moments.

4 : Propriété d'accélération pour le E-algorithme.

**Section 1**

# Approximants de type Padé

## 1. Introduction

Rappelons en premier lieu le formalisme introduit et utilisé par C. BREZINSKI dans son ouvrage "Padé-Type Approximants and General Orthogonal Polynomials" [Brez 1].

Soit $f$ une série formelle de puissances de la variable $t$

$$f(t) = \sum_{i=0}^{\infty} c_i t^i, c_i \in \mathbb{C}.$$

On définit une fonctionnelle linéaire $c$ agissant sur l'ensemble des polyômes $\mathcal{P}$ c'est à dire un élément de $\mathcal{P}'$, dual algébrique de $\mathcal{P}$ (on dira aussi distribution sur $\mathcal{P}$)

$$\begin{aligned} c :& \mathcal{P} \to \mathbb{C} \\ & x^i \to c(x^i) :< c, x^i >:= c_i \end{aligned} \tag{1}$$

Les $(c_i)$ sont appelés les moments de la fonctionnelle $c$. En appliquant formellement la fonctionnelle linéaire $c$ aux deux membres de l'égalité ci-dessous :

$$\frac{1}{1 - xt} = 1 + xt + x^2t^2 + \ldots \tag{2}$$

on obtient :

$$f(t) =< c, \frac{1}{1 - xt} > \tag{3}$$

où $c$ agit sur la variable $x$.

Bien entendu, l'égalité (3) n'est que formelle car $c$ étant un élément de $\mathcal{P}'$, la quantité $< c, \frac{1}{1 - xt} >$ n'a pas de signification.

Donc sans autre hypothèse sur $c$, on ne pourra calculer que l'image par $c$ d'un polynôme en $x$. Selon le polynôme choisi, on obtiendra un approximant de Padé ou de type Padé, les théorèmes de convergence des approximants ne s'appliquant que si certaines informations supplémentaires sur $f$ (donc sur $c$) sont connues.

Soit donc $v$ un polynôme arbitraire de degré $k$ :

$$v(x) = b_0 + b_1 x + \ldots + b_k x^k$$

et soit

$$w(t) = < c, \frac{v(x) - v(t)}{x - t} > \tag{4}$$

où $c$ agit sur $x$, $t$ étant un paramètre. $w$ est un polynôme de degré $k-1$ en $t$.

Si l'on pose :

$$\tilde{w}(t) = t^{k-1} w(t^{-1})$$
$$\tilde{v}(t) = t^k v(t^{-1})$$

on obtient

$$\frac{\tilde{w}(t)}{\tilde{v}(t)} - f(t) = 0(t^k) \qquad t \to 0$$

Un tel approximant rationnel est appelé *"Approximant de Type Padé"* (A.T.P.) et est noté :

$$\tilde{w}(t)/\tilde{v}(t) = (k-1/k)_f(t) \tag{5}$$

C. BREZINSKI a montré que si $P$ est le polynôme d'interpolation d'Hermite de la fonction $x \to (1-xt)^{-1}$ en $k$ points arbitraires $t_i \in \mathbb{C}$ alors

$$< c, P > = (k-1/k)_f(t).$$

où $v(x) = \prod_{i=1}^{k}(x - t_i)$

Dans ce cas, $P$ s'exprime par :

$$P(x) = \frac{1 - v(x)/v(t^{-1})}{1 - xt}$$

et donc

$$< c, P > = \frac{1}{v(t^{-1})} c \left( \frac{v(t^{-1}) - v(x)}{1 - xt} \right) = \frac{\tilde{w}(t)}{\tilde{v}(t)}$$

Dans l'expression formelle (3) la fonction $x \to (1-xt)^{-1}$ a été remplacée par le polynôme $P$.

Les A.T.P. sont donc une généralisation des Approximants de Padé (A.P.) car dans ces derniers le dénominateur n'est pas arbitraire mais doit vérifier :

$$< c, x^m v(x) > = 0 \qquad m = 0, \ldots, k-1.$$

En contrepartie l'ordre d'approximation est plus élevé :

$$\frac{\tilde{w}(t)}{\tilde{v}(t)} - f(t) = 0(t^{2k})$$

et $\tilde{w}(t)/\tilde{v}(t)$ est noté $[k-1/k]_f(t)$.

## 2. Fonctions avec représentation intégrale

Pour obtenir des résultats de convergence des A.T.P. il faut donner des informations sur $f$.

Dans [Prev. 1] on suppose que $f$ est une fonction admettant une représentation intégrale sur un intervalle compact de $\mathbb{R}$ qui peut sans perte de généralité être supposé $[-1,1]$.

$$f(t) = \int_{-1}^{1} \frac{d\alpha(x)}{1-xt}, \qquad \alpha \text{ fonction à variation bornée} \tag{6}$$

Pour espérer approcher $f$ avec des A.T.P., on constate qu'il est préférable d'interpoler $x \to (1-xt)^{-1}$ en des points de l'intervalle $[-1,1]$. Cette condition sera automatiquement satisfaite si ces points sont les racines de polynômes $R_n$ orthogonaux par rapport à un poids quelconque $d\mu$ positif sur $[-1,1]$. De plus le comportement connu des polynômes orthogonaux par rapport à un poids positif, à l'intérieur et à l'extérieur de $[-1,1]$ a permis de montrer le théorème suivant :

**Theorème 1** *Soit $f$ définie par (6), $t^{-1} \in \mathbb{C}-[-1,1]$. Soit $(R_n)_{n\in\mathbb{N}}$ une suite de polynômes orthogonaux par rapport à une distribution $d\mu$ dont le support est $[-1,1]$ et telle que $\mu' > 0$ presque partout dans $[-1,1]$ (cette dernière condition peut être remplacée par $\overline{\lim}_{n\to\infty}|R_n(x)|^{1/n} \leq 1 \quad \forall x \in [-1,1]$)*

*Soit $P_n(x) = \frac{1-R_n(x)/R_n(t^{-1})}{1-xt}$.*

*Alors la suite d'A.T.P. $(< c, P_n >)_n = (c(P_n))_n$ converge uniformément vers $f$ sur tout compact inclus dans $\mathbb{C}-]-\infty,-1] \cup [1,+\infty[$ et de plus*

$$\overline{\lim}_{n\to\infty}|c(P_n) - f(t)|^{1/n} \leq \rho < 1 \tag{7}$$

*avec $\rho = |t^{-1} \pm \sqrt{t^{-2}-1}|$ (On choisit la détermination de la racine carrée qui est positive si l'argument est positif)* ■

Dans le cas particulier où $d\mu = d\alpha$ (si $\alpha$ est non décroissante), $c(P_n)$ devient l'A.P. $[n-1/n]$ pour la fonction $f$ et on retrouve ainsi la formule d'erreur des A.P. :

$$\overline{\lim}_{n\to\infty} \left|[n-1/n]_f - f(t)\right|^{1/n} \leq \rho^2 < 1 \tag{8}$$

qui a été obtenue par G. BAKER ([Bak] p. 220.) mais d'une autre façon.

A l'aide du théorème de Perron-Knopp, le théorème 1 se généralise immédiatement de la manière suivante :

**Theorème 2.** *La conclusion du théorème 1 est maintenue si $f$ est seulement holomorphe dans* $\mathbb{C} - ]-\infty, -1] \cup [1, +\infty[$.

**Remarque.** Le théorème 2 a été étendu à un domaine d'holomorphie quelconque par M. EIERMANN ([Eir]).

**Exemples** : les hypothèses du théorème 1 concernant $R_n$ (donc $P_n$) sont notamment satisfaites dans les cas suivants

a) $P_n$ meilleur approximant dans $L^p([-1,1])$ de la fonction $x \to (1-xt)^{-1}$ pour $p = 1, 2, \infty$. Dans ces 3 cas S. PASKOWSKI a donné explicitement la distribution $d\mu$ par rapport à laquelle les polynômes $R_n$ sont orthogonaux. (voir [Pas 1])

b) $P_n$ = développement tronqué à l'ordre $n$ de la fonction $x \to (1-xt)^{-1}$ en série de polynômes de Chebychev de première ou seconde espèce (ce dernier cas redonne la transformation $E_{\alpha,\beta}$ de W. NIETHAMMER [Niet ] qui est elle même une généralisation du procédé de sommation d'Euler).

Conclusion : Dans les théorèmes 1 et 2 les A.T.P. ont permis de tenir compte du domaine d'holomorphie de $f$ ce que ne font pas les A.P. sauf si $f$ a la représentation intégrale (6) avec une fonction $\alpha$ bornée et non décroissante car dans ce cas les pôles (et les zéros) des A.P. se placent automatiquement sur la coupure de $f$.

### 3. Fonctions méromorphes.

La classe des fonction méromorphes vérifie une propriété similaire précisée par le

**Theorème 3.** (Montessus de Ballore [MdB], [Saff]) *Si $f$ est une fonction méromorphe dans $D(0,R)$ $0 < R \leq +\infty$ dont les pôles $\pi_1, \ldots, \pi_n$ sont ordonnés par module croissant, chaque pôle étant compté avec sa multiplicité.*

*Alors la suite d'approximants de Padé $([m/n]_f(t))_{m \in \mathbb{N}}$ converge vers $f$ uniformément sur tout compact de $D(0,R) - \cup_{i=1}^{m}\{\pi_i\}$* ∎

Ce théorème a permis à Henrici et Rutishauser ([Hen], [Rut]) d'obtenir une approximation des pôles d'une fonction méromorphe de la façon suivante :

Soit

$$H_k^{(n)} = \begin{vmatrix} c_n & c_{n+1} & \cdots & c_{n+k-1} \\ \vdots & & \vdots & \\ c_{n+k-1} & c_{n+k} & \cdots & c_{n+2k-2} \end{vmatrix} \qquad \begin{matrix} n \geq 0 \\ k \geq 1 \end{matrix}$$

les déterminants de Hankel reliés à la série $f(t) = \sum_{i=0}^{\infty} c_i t^i$. Si le k$^{ieme}$ pôle a un module strictement inférieur à celui du suivant alors :

$$\lim_{n \to \infty} H_k^{(n)} / H_k^{(n+1)} = \pi_1 \pi_2 \ldots \pi_k$$

et donc si $|\pi_{k-1}| < |\pi_k| < |\pi_{k+1}|$ on a :

$$\lim_n \frac{H_k^{(n)}}{H_k^{(n+1)}} \cdot \frac{H_{k-1}^{(n+1)}}{H_{k-1}^{(n)}} = \pi_k$$

Le k$^{ieme}$ pôle d'une fonction méromorphe peut donc être approximé par une suite de rapports de déterminants de Hankel. Cette suite peut être calculée récursivement par l'algorithme Q-D de Rutihauser.

Dans [Prev 3] j'ai montré que deux autres algorithmes déjà existants $R-S$ [PyAt] et $\varepsilon$ [Wynn] permettent également de calculer les pôles $\pi_k$.

Si de plus les pôles vérifient :

$$|\pi_{k-2}| < |\pi_{k-1}| < |\pi_k| < |\pi_{k+1}| < |\pi_{k+2}|$$

et

$$|\pi_k / \pi_{k+1}| \neq |\pi_{k-1} / \pi_k|$$

utilisant un récent résultat démontré simultanément et indépendamment par J.P. DELAHAYE [Del] et J. WIMP [Wimp 2] et concernant le rapport des différences successives des termes d'une suite à convergence linéaire, j'ai pu montrer que la convergence des $Q-D$, $R-S$ et $\varepsilon$ algorithmes se fait vers $\pi_k$ de façon linéaire ce qui permet de les accélérer à l'aide par exemple du $\Delta^2$-d'Aitken.

## 4. Fonctions de Markov-Stieltjes

Pour certaines classes de fonctions, des théorèmes de convergence pour la suite des A.P. existent. (Pour une mise à jour récente sur ce sujet on pourra consulter [Br Va]). Ces théorèmes concernent la classe :

a) des fonctions de Markov-Stieltjes, c'est à dire qui s'écrivent

$$f(t) = \int_I \frac{1}{1-xt} d\alpha(x)$$

$I$ intervalle compact de $\mathbb{R}$, $\alpha$ fonction bornée non décroissante sur $I$ avec le théorème de A.A. MARKOV ([Sze] théorème 3.5.4 ).

b) des fonctions méromorphes avec le théorème de Montessus de Ballore.

c) des fonctions qui s'écrivent comme le produit ou la somme d'une fonction de Markov-Stieltjes avec une fraction rationnelle sans pôle sur la coupure de celle-ci [Gon].

Utilisant le produit de fonctionnelles linéaires défini par P. MARONI de la façon suivante (voir [Mar]) :

$c$, $d \in \mathcal{P}'$, le produit $cd$ est défini par

$$< cd, x^n >:= \sum_{i=0}^{n} c_i d_{n-i} = \sum_{i=0}^{n} d_i c_{n-i}$$

et en l'appliquant à des fonctionnelles définies positives c'est à dire vérifiant $< c,p >= \int_{\mathbb{R}} p(x) d\alpha(x)$ $\alpha$ bornée, non décroissante sur $\mathbb{R}$, j'ai pu prouver un théorème de convergence de la suite des A.P. vers le produit de deux fonctions de Markov-Stieltjes dont les coupures sont respectivement incluses dans $\mathbb{R}^+$ et $\mathbb{R}^-$. [Prev 7].

**Theorème 4** *Soit* $f(t) = \int_0^a \frac{1}{1-xt} d\alpha(x)$ $\quad a \in \mathbb{R}^+$, *a fini et* $\alpha$ *non décroissante* $g(t) = \int_b^0 \frac{1}{1-xt} d\beta(x)$ $\quad b \in \mathbb{R}^-$, *b fini et* $\beta$ *non décroissante.*

*Alors la suite paradiagonale des A.P. :*

$$([M+J/M]_{f.g})_{M \in \mathbb{N}} \qquad J \geq -1$$

*converge, quand $M$ tend vers l'infini, vers $f(t)g(t)$ uniformément sur tout compact de* $\mathbb{C}-]-\infty, b^{-1}] \cup [a^{-1}, +\infty[$.

La preuve de ce théorème passe par la détermination explicite de la fonction non décroissante $\gamma$ vérifiant

$$f(t)g(t) = \int_b^a \frac{1}{1-xt} d\gamma(x)$$

. ∎

## 5. Approximants de Padé partiels

L'approximation de type Padé permet de choisir les pôles des approximants rationnels (notés $N(t)/D(t)$) de la fonction f. On peut généraliser de la façon suivante en imposant des valeurs à $N(t)/D(t)$ en des points donnés.

Soit donc les conditions :

$N$ polynôme de degré $m+p$

$D$ polynôme de degré $n+p$

et vérifiant

$$\begin{cases} N(t) - D(t)f(t) = 0(t^{m+n+p+1}) & \\ \dfrac{N(\alpha_i)}{D(\alpha_i)} = \beta_i \quad i = 1, \ldots, p & \alpha_i \in \mathbb{C} - \{0\}, \text{ distincts} \\ & \beta_i \in \mathbb{C} \quad (\beta_i \text{ peut valoir l'infini}) \end{cases}$$

$N(t)/D(t)$ s'exprime sous forme d'un rapport de deux déterminants [Prev 4] de taille $p+1$ et dont les éléments dépendent des approximants de Padé

$$[m/n]_f, [m+1/n+1], \cdots [m+p/n+p]_f$$

et de leurs valeurs en $\alpha_1, \cdots, \alpha_p$.

Noter que la taille $p+1$ des déterminants peut compromettre tout calcul numérique mais leur structure particulière permet d'utiliser le $E$-algorithme pour leur calcul.

Si $\beta_i \in \{0, \infty\}$ pour $i = 1$ à $p$, cela revient à imposer pôles et zéros de l'approximant $N(t)/D(t)$ qui porte alors le nom de *Approximant de Padé Partiel* (Voir [Brez 4]).

## 6. Polynômes de type-Stieltjes et -Geronimus

Dans l'approximation de Padé, une écriture plus précise du reste est :

$$f(t) - [n-1/n]_f(t) = \frac{t^n}{\tilde{P}_n(t)} c\left(\frac{P_n(x)}{1-xt}\right) \tag{9}$$

où $P_n$ est le n$^{ieme}$ polynôme orthogonal par rapport à $c$ :

$$< c, P_n.x^k >= 0 \qquad 0 \leq k \leq n-1.$$

Posons

$$q_n(t) :=< c, \frac{P_n(x)}{1-xt} >=< P_n c, \frac{1}{1-xt} >$$

. Les premiers moments de la fonctionnelle transformée $P_n c$ sont nuls car :

$$< P_n c, x^k >=< c, P_n(x).x^k >= 0 \qquad k = 0, \cdots, n-1$$

On considère donc le polyôme $E_{n+1}$ vérifiant :

$$< P_n c, E_{n+1}(x) x^k >= 0, \qquad k = 0, \cdots, n. \tag{10}$$

*$E_{n+1}$ apparaît alors comme étant le polynôme de degré $(n+1)$ orthogonal par rapport à la fonctionnelle $P_n c$.*

Dans le cas particulier où $c$ est une fonctionnelle définie positive sur $[-1, 1]$, ces polynômes ont été étudiés par G. MONEGATO [Mon] et portent le nom de polynômes de type-Stieltjes.

Le système (10) est triangulaire et de diagonale $c(P_n^2)$, l'existence et l'unicité de $E_{n+1}$ sont donc résolues par le théorème suivant [Prev 2].

**Théorème 5.** *$E_{n+1}$ est uniquement déterminé par (10) si et seulement si la fonctionnelle $c$ est définie (c'est à dire si $c(P_n^2) \neq 0 \quad \forall n \in \mathbb{N}$)* ■

Le polynôme associé à $E_{n+1}$ (supposé unitaire) pour la fonctionnelle $P_n c$ est :

$$< P_n c, \frac{E_{n+1}(x) - E_{n+1}(t)}{x-t} >=< P_n c, P_n >= c(P_n^2) := h_n$$

Par suite

$$[n/n+1]_{q_n}(t) = \frac{h_n t^n}{\tilde{E}_{n+1}(t)}$$

où $\tilde{E}_{n+1}(t) = t^{n+1} E_{n+1}(t^{-1})$ et le reste de l'approximation de Padé pour $q_n$ est :

$$q_n(t) - [n/n+1]_{q_n}(t) = 0(t^{2n}) \qquad t \to 0.$$

On en déduit :

$$f(t) - [n-1/n]_f(t) = \frac{h_n t^{2n}}{\tilde{P}_n(t)\tilde{E}_{n+1}(t)} - [n-1/n]_f(t) + 0(t^{3n+2}) \tag{11}$$

L'A.T.P. de dénominateur $\tilde{P}_n \tilde{E}_{n+1}$ est égal à :

$$(2n/2n+1)_f(t) = \frac{h_n t^{2n}}{\tilde{P}_n(t)\tilde{E}_{n+1}(t)} - [n-1/n]_f(t).$$

La relation (11) contient donc une estimation du reste de l'approximation de Padé (Voir [Brez 2]).

Le polynôme $E_{n+1}$ peut servir à construire un A.T.P. d'une autre fonction de la façon suivante :

Soit

1) $g(t) = \sum_{i=0}^{\infty} \bar{c}_i d^i = < \bar{c}, \dfrac{1}{1-xt} >$ où $\bar{c}_i :=< \bar{c}, x^i > i \in \mathbb{N}$.

2) $E_{n+1}$ polynôme de degré $n+1$ défini par :

$$< P_n c, E_{n+1}(x).x^k >= 0 \qquad k = 0, \cdots, n$$

3) $S_n(t) =< \bar{c}, \dfrac{E_{n+1}(x) - E_{n+1}(t)}{x-t} >$ le polynôme de degré $n$ associé à $E_{n+1}$ pour la fonctionnelle $\bar{c}$.

Le polynôme $S_n$ possède la propriété fondamentale suivante :

**Propriété :** *Soit $\bar{c}$ une fonctionnelle définie ($\Longleftrightarrow H_0(\bar{c}_n) \neq 0 \quad \forall n \in \mathbb{N}$) et $(G_k)_{k\in\mathbb{N}}$ la famille de polynômes orthogonaux (unitaires) par rapport à $\bar{c}$.*

*Si la fonctionnelle $c$ est définie alors :*

*a)*

$$< P_n c, S_n G_k >= 0 \qquad k = 1, \cdots, n \tag{12}$$

*où $P_n$ est la suite de polynômes orthogonaux par rapport à $c$*

*b)*

$$\frac{\tilde{S}_n(t)}{\tilde{E}_{n+1}(t)} = (n/n+1)_g(t). \tag{13}$$

■

Le cas particulier $g(t) = (1 - t^2)^{-1/2}$ a été traité par GERONIMUS [Ger] et étudié par G. MONEGATO [Mon], aussi on a appelé $S_n$, dans le cas général, polynôme de type-Geronimus.

Comme remarqué par P. MARONI, on peut définir le polynôme $S_n$ en prenant pour $G_k$ non plus une suite de polynômes orthogonaux par rapport à une fonctionnelle $\bar{c}$ mais une suite quelconque de polynômes de degré $k$.

Cette extension reste à faire.

# Section 2

## Moments modifiés.

### 1. Introduction

Cette section traite de l'utilisation de la suite des moments modifié dans le but d'approcher la fonctionnelle $c$ pour certaines classes.

**Définition :** *Soit $c$ un élément de $\mathcal{P}'$ dual algébrique de $\mathcal{P}$ espace vectoriel des polynômes.*

*Soit $(\pi_n)_{n\in\mathbb{N}}$ une suite arbitraire de polynômes de degré $n$.*

*La suite $< c, \pi_n >$ notée aussi $c(\pi_n)$ est appelée suite des moments modifiés.*

**Remarque 1 :** Dans le cas où $(\pi_n)$ est une suite de polynômes orthogonaux par rapport à une autre fonctionnelle, la suite des moments modifiés permet de stabiliser le calcul des coefficients de la relation de récurrence à 3 termes des polynômes orthogonaux par rapport à $c$. Voir [Gau 1], [WPB], [Whe].

**Remarque 2 :** Si $c$ n'est constituée que d'une combinaison linéaire finie de distributions de Dirac et de ses dérivées, alors à partir d'un nombre fini de moments de $c$, l'approximation de Padé permet de trouver $c$ de la façon suivante :
Soit

$$c = \sum_{i=1}^{m} \sum_{j=0}^{r_i - 1} (-1)^j A_{ij} D^j \delta_{a_i} \tag{14}$$

où

$$A_{ij} \in \mathbb{R}, i = 1, ..., m; A_{i,r_i-1} \neq 0$$

$$a_i \quad \text{complexes distincts}; j = 0, ...r_i - 1.$$

$$q := \sum_{i=1}^{m} r_i$$

La distribution de Dirac et sa dérivée d'ordre $m$ étant définies par :

$$< \delta_{x_0}, \phi > := \phi(x_0) \qquad < D^m \delta_{x_0}, \phi > := \phi^{(m)}(x_0)(-1)^m$$

En termes de fonctions génératrices la relation (14) peut être réécrite de la façon suivante :

$$f(t) = \frac{P(t)}{Q(t)}$$

où

$$f(t) =< c, \frac{1}{1-xt} >:= \sum_{i=0}^{\infty} c_i t^i$$

et $P(t)/Q(t)$ est une fraction rationnelle de degré $(M, q)$ où $M = q - Inf\{r_i, i = 1, \ldots, m\}$ et dont les pôles sont $a_i^{-1}$, $i = 1, \ldots, m$. Par suite $f$ est une fraction rationnelle de degré $(M, q)$ et donc l'approximant de Padé $[M'/q]_f(t)$ est identique à $f$ pour tout M' supérieur ou égal à M.

**2. $c$ admet une représentation intégrale**

On suppose dans ce paragraphe que $c$ est définie par :

$$< c, p >= \int_I p(x) w(x) d\mu(x) \tag{15}$$

où $I$ est un intervalle compact de $\mathbb{R}$ (on peut supposer $I = [-1, 1]$), $w$ est une fonction de $L^2(d\mu)$ et $\mu(x)$ une fonction non décroissante sur $[-1, 1]$. Soit $\pi_k(d\mu, x)$ la suite de polynômes orthonormés par rapport à $d\mu$ i.e :

$$\begin{aligned} \int_{-1}^{1} \pi_k(d\mu, x).\pi_l(d\mu, x) d\mu(x) &= 0 \qquad k \neq l \\ &= 1 \qquad k = l \end{aligned}$$

Une méthode classique pour approcher $c$ donc $w$ consiste à développer $w$ sur la base $\pi_0, \pi_1, \ldots$

$$w(.) = \sum_{k=0}^{\infty} c(\pi_k(d\mu, x)).\pi_k(d\mu, .) \tag{16}$$

puis à tronquer cette série à l'ordre $N$

$$w_N(.) = \sum_{k=0}^{N} c(\pi_k(d\mu, x)).\pi_k(d\mu, .) \tag{17}$$

$w_N$ se trouve donc être la projection dans $L^2(d\mu)$ de la fonction $w$ sur l'espace vectoriel engendré par $\pi_0, \pi_1, \ldots, \pi_N$ (voir [WPB]).

Il existe un lien entre cette méthode de projection dans $L^2(d\mu)$ et les A.T.P. obtenus en remplaçant la fonction $x \rightarrow (1-xt)^{-1}$ par son polynôme d'interpolation aux racines de polynômes orthogonaux par rapport à $d\mu$ (voir Section 1, paragraphe 2).

En effet si $c$ vérifie (15) alors la fonction $f(t) = < c, \frac{1}{1-xt} >$ s'écrit :

$$f(t) = \int_{-1}^{1} \frac{1}{1-xt} w(x) d\mu(x). \tag{18}$$

$f$ est holomorphe sur $\mathbb{C}-]-\infty,-1] \cup [1,+\infty[$ et change de détermination quand la variable complexe $t$ traverse la coupure $]-\infty,-1] \cup [1,+\infty[$.

Par la formule d'inversion de Stieltjes ([Chi] p. 90), $wd\mu$ est interprété comme le changement de détermination de $F(t) = t^{-1}f(t^{-1})$ quand $t$ traverse $[-1,1]$.

Ainsi une bonne approximation de $f$ donc de $F$ dans le plan coupé $\mathbb{C}-]-\infty,-1] \cup [1,+\infty[$ et particulièrement près de la coupure fournira une bonne approximation de $w$. Plus précisément, soit

$$\frac{1}{1-xt} = \sum_{k=0}^{\infty} \beta_k(t)\pi_k(d\mu, x) \tag{19}$$

le développement de $(1-xt)^{-1}$ en série de polynômes orthonormés $\pi_0, \pi_1, \ldots$

On a :

$$\beta_k(t) = \int_{-1}^{1} \frac{\pi_k(d\mu, x)}{1-xt} d\mu(x) \tag{20}.$$

La troncature de la série de polynômes orthogonaux (19) à l'ordre $N$ donne l'approximant suivant de $f$ :

$$< c, \sum_{k=0}^{N} \beta_k(t)\pi_k(d\mu, x) > = \sum_{k=0}^{N} \beta_k(t)c(\pi_k(d\mu, x))$$

dont le changement de détermination pour $t$ traversant la coupure est précisément $w_N.d\mu$ (voir [Prev 5]).

**3.** $c$ **est la somme d'une autre fonctionnelle et de distributions de Dirac** (Voir [Prev 6] et [Prev 8])

### 3.1 Cas général

Supposons que $c$ est définie par :

$$c = \omega + \sum_{i=1}^{m} \sum_{j=0}^{r_i - 1} (-1)^j A_{ij} D^j \delta_{a_i} \tag{21}$$

et posons

$$q := \sum_{i=1}^{m} r_i$$

En termes de fonction génératrice la relation (21) devient :

$$f(t) = g(t) + \frac{P(t)}{Q(t)} \tag{22}$$

où $f(t) =< c, \frac{1}{1-xt} >= \sum_0^\infty c_i t^i, g(t) =< \omega, \frac{1}{1-xt} >= \sum_0^\infty \omega_i t^i$ et $P(t)/Q(t)$ est une fraction rationnelle de degré $(M, q)$ avec $M = q - Inf\{r_i, i = 1, \cdots, m\}$ et dont les pôles sont en $a_i^{-1} \quad i = 1, \cdots, m$.

Par conséquent le théorème de Montessus de Ballore peut être réécrit de la manière suivante :

**Théorème** (Montessus de Ballore) *Si $c$ et $\omega$ vérifient (21) et si*

$$\lim_{n \to \infty} \omega_n / a_i^n = 0 \qquad i = 1, \cdots, m \tag{23}$$

*alors la suite des A.P. $([p/q]_f)_{p \in \mathbb{N}}$ converge vers $f$ uniformément sur tout compact inclus dans $D\ (0, Max_{1 \le i \le m} |a_i^{-1}|) - \cup_{i=1}^m \{a_i^{-1}\}$ quand $p$ tend vers l'infini.* ■

Les points $a_i, i = 1, \cdots, m$, peuvent alors être approximés par les pôles de $[p/q]$ que l'on peut calculer avec le $Q - D$, le $R - S$ ou l'$\varepsilon$-algorithme (Voir Section 1, paragraphe 3). L'hypothèse (23) est équivalente à :

$$g \text{ holomorphe dans } D(0, Max_{1 \le i \le m} |a_i^{-1}|)$$

ou encore à :

$$a_i^{-1} \in \text{domaine d'holomorphie de } g, \text{ pour } i = 1, \cdots, m.$$

Le théorème de Montessus de Ballore et le $Q-D$ algorithme ne sont donc pas utilisables si $g$ n'est pas holomorphe en $a_i^{-1}$, $i=1,\cdots,m$. C'est le cas notamment si $g$ est holomorphe dans $\mathbb{C}$ sauf sur une coupure, et que les points $a_i^{-1}$ sont précisément sur cette coupure.

J'ai donc été amené à transformer la relation (23) en une autre relation asymptotique concernant $\omega$ et n'incluant pas les points $a_1,\cdots,a_m$ (*).

On suppose que $\omega$ satisfait :

$$\lim_{k\to\infty} <\omega,\pi_i\pi_k>=0 \qquad i=0,\cdots,q \tag{24}$$

où $(\pi_k)_{k\in\mathbb{N}}$ est une suite arbitraire de polynômes unitaires de degré exact $k$.

La multiplication des 2 membres de (21) par le polynôme

$$P(x)=\prod_{i=1}^{m}(x-a_i)^{r_i}$$

donne :

$$P.c=P.\omega \tag{25}$$

L'égalité (25) est équivalente à

$$<P.c,\pi_k>=<P.\omega,\pi_k> \qquad \forall k\in\mathbb{N}$$

ou à

$$c(P.\pi_k)=\omega(P.\pi_k) \qquad \forall k\in\mathbb{N}.$$

Pour trouver les points $a_1,\cdots,a_m$, on développe le polynôme $P$ sur la base $\pi_0,\cdots,\pi_q$ :

$$P(x)=\sum_{i=0}^{q} b_i\pi_i(x) \text{ avec } b_q=1 \tag{26}$$

et on obtient l'équivalence :

$$P.c=P.\omega \iff \sum_{i=0}^{q} b_i c(\pi_i\pi_k)=\sum_{i=0}^{q} b_i\omega(\pi_i\pi_k), \qquad \forall k\in\mathbb{N}. \tag{27}$$

Les $(b_i)_{i=0,q}$ sont donc les solutions d'un système dont le second membre tend vers 0 grâce à (24).

---

(*) Je remercie P. MARONI de m'avoir indiqué cet aspect des choses.

Sous des hypothèses concernant les polynômes $(\pi_i)_{i\in\mathbb{N}}$ on a montré dans [Prev 6] que si la solution $(\beta_0^{(N)},\cdots,\beta_q^{(N)})$ du système :

$$\sum_{i=0}^{q}\beta_i^{(N)}c(\pi_i\pi_k)=0,\beta_q^{(N)}=1 \tag{28}$$

converge, c'est vers la solution $(b_0,b_1,\ldots,b_q)$ de l'identité (27).

De plus, si $\pi_0,\pi_1,\ldots$ vérifient des conditions supplémentaires, alors on peut montrer que la suite $(\beta_0^{(N)},\ldots,\beta_q^{(N)})$ admet $(b_0,b_1,\ldots,b_q)$ comme point d'accumulation.

## 3.2 Détection et calcul des masses de Dirac dans une densité sur $[-1,1]$

La question qui se pose alors est de déterminer les fonctionnelles $\omega$ et les polynômes $(\pi_k)_{k\in\mathbb{N}}$ qui vérifient (24). La réponse est apportée en prenant pour $\omega$ une distribution absolument continue par rapport à la mesure de Lebesgue sur un intervalle compact (par exemple $[-1,1]$) et $\pi_k=T_k$ polynômes de Chebyshev de première espèce.

Soit donc $\omega$ définie par :

$$\forall p\in\mathcal{P},<\omega,p>=\int_{-1}^{1}p(x)w(x)dx,w\in L^1 \tag{29}$$

et

$$\pi_k(x)=\frac{1}{2^{k-1}}T_k(x)=\frac{1}{2^{k-1}}\cos(kArccosx).$$

Le lemme de Riemann-Lebesgue ([Rud] p. 99) et la règle très simple de multiplication des polynômes $T_k$ assurent la validité de l'hypothèse (24) et donc le théorème suivant :

**Théorème 6 :** ([Prev 6])*Soit c et $\omega$ reliés par (21) et $\omega$ absolument continue par rapport à dx dans* $[-1,1]$. *Soit le système* $(\mathcal{T})$

$$(\mathcal{T})\begin{cases}\sum_{i=0}^{q}\beta_i^{(N)}(c(T_{i+k})+c(T_{|i-k|})=0,k=N,\ldots,N+q-1\\ \beta_q^{(N)}=1\end{cases}$$

*Alors*

**1)** *la suite* $(\beta_0^{(N)},\ldots,\beta_q^{(N)})$ *admet* $(b_0,\ldots,b_q)$ *comme point d'accumulation.*

**2)** *sauf pour une position particulière des points $a_1, \ldots, a_q$, la limite de la solution $(\beta_0^{(N)}, \ldots, \beta_q^{(N)})$ si elle existe est égale à $(b_0, b_1, \ldots, b_q)$ où les coefficients $b_0, \ldots, b_q$ sont les coefficients du développement $\prod_{i=1}^{m}(x-a_i)^{r_i}$ sur la base $T_0, \ldots, T_q$.* ■

**Remarque 1 :** En termes de fonction génératrice les relations (21) et (29) s'écrivent :

$$f(t) = \int_{-1}^{1} \frac{1}{1-xt} w(x)dx + \frac{P(t)}{Q(t)} \tag{30}$$

$P(t)/Q(t)$ est une fraction rationnelle dont les pôles sont $a_1^{-1}, \ldots, a_q^{-1}$. Si ces pôles sont à l'intérieur du disque unité, le théorème de Montessus de Ballore et le $Q-D$ algorithme permettent de les calculer.Par contre, si ces pôles sont à l'extérieur de ce disque, ou pire sur la coupure de $f$ qui est $]-\infty,-1] \cup [1,+\infty[$, le théorème ci-dessus permet de les approcher. Voir les applications numériques dans [Prev 6].

**Remarque 2 :** La solution du système $(\mathcal{T})$ ci-dessus est liée à l'approximation de Padé-Chebychev (appelée aussi approximation de Maehly) de la fonction $w$ dans (16). Le théorème 6 dans le cas particulier où $q=0$ fournit un critère de détection d'une distribution de Dirac dans une densité.

**Critère de détection :** Soit $c$ et $\omega$ reliés par (21). Soit $\omega$ une distribution de densité $w(x)$ dans $[-1,1], w \in L^1$ alors :

$$\lim_{k\to\infty} c(T_k) = 0 \iff q = 0 \qquad \text{(pas de masse de Dirac)}$$

■

Une fois la position des points $(a_i)_{i=1,\ldots,m}$ déterminée, il reste à calculer la valeur des coefficients $A_{ij}$ dans (21). Ce fut l'objet de l'article [Prev 8] dans le cas particulier où $c$ ne comporte pas de dérivée de distribution de Dirac.

**Thérème 7 :** *Soit $\omega$ définie par*

$$< \omega, p >= \int_{-1}^{1} p(x)w(x)dx$$

*Soit*

$$c = \omega + \sum_{i=1}^{q} A_i \delta a_i, \qquad a_i \in [-1,1] \text{ distincts} \qquad A_i \in \mathbf{C} - \{0\} \tag{31}$$

*Alors :*

$$A_i = \lim_{n\to\infty} c(l_n(x, a_i)) \qquad i = 1, \ldots, q \tag{32}$$

*où* $l_n(x,a) = \dfrac{k_n(x,a)}{k_n(a,a)}$.

$k_n$ *désignant le noyau reproduisant des polynômes de Chebyshev de première espèce* $T_n$ *:*

$$k_n(x,a) = \frac{1}{2} + \sum_{k=1}^{n} T_k(x)T_k(a)$$

**Preuve :** Les polynômes $l_n(x,a)$ pour $a \in [-1,1]$ ont la propriété remarquable suivante :

$$\lim_{n\to\infty} l_n(x,y) = 0 \qquad x \neq y$$
$$l_n(x,x) = 1$$

De plus les propriétés de $T_n$ permettent d'appliquer le théorème de Lebesgue de la convergence dominée.

La relation (31) donne :

$$< c, l_n(x,a_k) > = < \omega, l_n(x,a_k) > + \sum_{i=1}^{q} A_i l_n(a_k,a_i) \tag{32}$$

et en faisant tendre $n$ vers l'infini, on trouve :

$$A_k = \lim_{n\to\infty} c(l_n(x,a_k)).$$

Le polynôme $l_n(x,a_i)$ agit comme un filtre ne laissant passer que la distribution $\delta a_i$, les autres masses de Dirac ainsi que la densité devenant négligeables dans la suite des moments modifiés $(< c, l_n(x,a_i) >)_n$

■

**Remarque 1 :** Il est possible, en utilisant une primitive de $l_n(x,a)$ de calculer les coefficients $A_{ij}$ dans le cas où $c$ contient des dérivées de distributions de Dirac.

**Remarque 2 :** Un autre critère de détection de distribution de Dirac dans une distribution de la forme (21) est le suivant : on représente le graphe de la fonction $a \to c(l_n(x,a))$ pour $a$ parcourant $[-1,1]$. Si $c$ a une masse de Dirac en $a_0 \in [-1,1]$ alors $\lim_{n\to\infty} c(l_n(x,a_0)) \neq 0$ (0 sinon) et cette courbe présentera alors un *"pic"* au point $a_0$ (voir exemples dans [Prev 8]).

### 3.3 Extension à un intervalle infini

On suppose maintenant que $\omega$ est définie positive sur $\mathbb{R}$,

$$< \omega, p >= \int_{\mathbb{R}} p(x) d\alpha(x) \qquad \alpha \text{ bornée, non décroissante} \tag{33}$$

et que la donnée de $\int_{\mathbb{R}} x^i d\alpha(x)$ pour $i \in \mathbb{N}$ détermine $\alpha$ de manière unique.

Si $c$ ne diffère de $\omega$ que par l'adjonction d'une masse de Dirac $A$ en $a \in \mathbb{R}$, alors cette masse $A$ peut être calculée en utilisant le

**Théorème 8 :** *Si $\omega$ vérifie (33) et $c = \omega + A\delta_a$*

*Si $\alpha(x)$ continue en $a$ alors*

$$\lim_{n \to \infty} \frac{1}{\sum_{k=0}^{n} P_k^2(a)} = A \tag{34}$$

*où $(P_k)_{k \in \mathbb{N}}$ est la suite de polynômes orthonormés par rapport à $c$ ($c(P_k P_l) = \delta_{kl}$).*

La preuve repose dans une large mesure sur la théorie des moments ([Fr. ] chap.2)

∎

Le calcul des quantités $\sum_{k=0}^{n} P_k^2(a)$ dans (34) est facilité par l'utilisation de l'$\varepsilon$ - algorithme : En effet, si l'on pose :

$$\tilde{f}(t) = \sum_{i=0}^{\infty} \tilde{c}_i t^i$$

avec

$$\tilde{c}_i = c_i - a c_{i-1} \qquad i = 1, 2, 3, \ldots$$

$$\tilde{c}_0 := c_0$$

on obtient :

- pour $a \neq 0$ $[n/n]_{\tilde{f}}(a^{-1}) = \varepsilon_{2n}^{(0)} = \dfrac{1}{\sum_{k=0}^{n} P_k^2(a)}$ où $\varepsilon_{2n}^{(0)}$ est obtenu en appliquant l'$\varepsilon$-algorithme à la suite des sommes partielles de $\tilde{f}$ en $a^{-1}$ c'est-à-dire à $S_n = c_n a^{-n}$.

- pour $a = 0$, on peut calculer $\sum_{k=0}^{n} P_k^2(0)$ avec le $w$-algorithm de Wynn qui permet de calculer le valeur d'un A.P. à l'infini ou avec l'$\varepsilon$-algorithme appliqué à la suite $(S_n)_n$ définie par :

$$S_n = \sum_{k=0}^{n} \binom{n}{k} c_k$$

Des exemples numériques sont proposés dans [Prev 8].

## 3.4 Applications au calcul des singularités de fonctions à partir de ses moments

Une application (*) possible de ce qui précède concerne la détection de singularités d'une fonction à partir de ses moments.

Soit donc une fonction $w(x)$ réelle de la variable réelle dont les moments

$$c_n = \int_{\mathbb{R}} x^n w(x)dx$$

sont supposés connus. Supposons que $w$ ait un nombre fini $q$ de discontinuités en $a_i$ d'amplitude $A_i$, alors dérivant $w$ au sens des distributions, ces sauts vont faire apparaître des masses de Dirac $A_i$ aux points $a_i$

$$w' = \{w\}' + \sum_{i=1}^{q} A_i \delta a_i \tag{33}$$

où $\{w\}'$ représente la dérivée usuelle si elle existe.

Les moments $c'_n$ de $w'$ peuvent être calculés à partir des moments $c_n$ de $w$ par :

$$c'_n = \int x^n w'(x)dx = -n \int x^{n-1} w(x)dx = -nc_{n-1}$$

Les points $a_i$ et les masses $A_i$ peuvent donc être détectés puis calculés avec les méthodes précédentes. Voir des exemples dans [Prev 6], [Prev 8].

---

(*) Je remercie D. BESSIS de m'avoir indiqué cette application possible.

### 3.5 Accélération des suites de moments.

Puisque l'on dispose désormais de méthodes permettant de calculer les sauts d'une fonction à partir de ses moments, on peut envisager d'accélérer certaines suites logarithmiques de moments à l'aide de la suite des moments modifiés.

En effet, si une suite $(c_n)_{n\in\mathbb{N}}$ de réels peut s'écrire

$$c_n = \int_0^1 x^n d\alpha(x) \quad \text{avec} \quad \alpha \quad \text{à variation bornée} \tag{34}$$

on sait que la limite $l$ de la suite $(c_n)$ est la valeur du saut de $\alpha$ au point 1

$$l = \alpha(1) - \alpha(1^-)$$

En choisissant une suite convenable de polynômes $\pi_n$ on peut calculer plus rapidement le saut de $\alpha$ en 1 donc la valeur de $l$.

Soit donc $(c_n)$ définie par :

$$c_n = \int_0^1 x^n w(x)dx + l \tag{35}$$

ou bien

$$c = \omega + l\delta_1.$$

La suite des moments modifiés s'écrit alors :

$$c(\pi_n) = \omega(\pi_n) + l$$

(on suppose $\pi_n$ normalisé par $\pi_n(1) = 1$).

On sait que la vitesse de convergence de la suite $(c_n)$ vers $l$ dépend de l'allure de la fonction au voisinage de $1^-$. Il faudra donc en tenir compte pour le choix de la suite $\pi_n$ de la façon suivante :

**Théorème 9 :** *Soit $c$ définie par (35) avec $w(x) \sim K(1-x)^\rho \qquad x \to 1$ et $w \in L^2[0,1]$ alors la suite des moments modifiés :*

$$u_n := c(l_n^{(\rho)}(x))$$

*converge plus vite que $(c_n)$ vers $l$, où $l_n^{(\rho)}(x)$ est le polynôme de Jacobi normalisé et ramené à $[0,1]$ :*

$$l_n^{(\rho)}(x) = P_n^{(\rho,0)}(2x-1)/P_n^{(\rho,0)}(1)$$

∎

On dispose de critères simples permettant de vérifier les hypothèses de ce théorème :

* si $w$ vérifie $w(x) \sim K(1-x)^{\rho} \quad x \to 1^-$, $K \in \mathbb{R}$ alors $c_n \sim l + Cn^{-\rho-1}, C \in \mathbb{R}$ (Voir [Wid 1] chap V). (La réciproque est vraie si $w > 0$)

* de plus on dispose également d'une condition nécessaire et suffisante portant sur les moments $c_n$ pour que $w \in L^2[0,1]$ (Voir [Wid 1] p. 110).

Dans le cas particulier où $w$ admet une représentation intégrale du type :

$$w(x) = \int_0^1 \frac{1}{1-(ax+b)t} d\mu(t) \qquad b \le 1, a+b \le 1 \tag{37}$$

avec $\mu$ fonction à variation bornée dans $[-1,1]$, (c'est le cas si par exemple $(c_n)$ est la $n^{\text{ieme}}$ somme partielle de certaines séries) alors la suite des moments modifiés

$$\left(c(l_n^{(0)}(x))\right)_{n \in \mathbb{N}}$$

d'abord converge plus vite que la suite $(c_n)$ vers $l$ mais de plus est elle même une suite de moments avec une représentation intégrale sur $[-1,1]$ qui se réduit à $[-1,0]$ si $a = -1$ et $b = 1$, ce qui permet, en utilisant les méthodes de [Prev 1] d'accélérer cette nouvelle suite en une suite à convergence au moins géométrique et de raison $3 - 2\sqrt{2} \sim 0,17$. Pour les détails voir [Prev 10].

### 3.6 Propriété d'accélération pour le $E$-algorithme

Comme on l'a vu dans le théorème 9, l'accélération de la suite $(c_n)$ par la suite des moments modifiés passe par l'utilisation d'une suite auxiliaire de polynômes dépendant eux mêmes de l'allure du poids $w(x)$ au voisinage de $1^-$.

Par exemple, si $w$ admet un développement asymptotique

$$w(x) = \sum_{i=1}^{\infty} a_i \varphi_i(x)$$

sur une échelle de fonction $(\varphi_i)_{i \in \mathbb{N}}$ vérifiant

$$\frac{\varphi_{i+1}(x)}{\varphi_i(x)} \to 0 \qquad x \to 1^-,$$

alors le remplacement formel de $w(x)$ dans (35) fournit :

$$c_n = l + \sum_{i=1}^{\infty} a_i g_i(n) \tag{38}$$

où

$$g_i(n) = \int_0^1 x^n \varphi_i(x)dx$$

C'est précisément ce développement asymptotique (38) qui a donné naissance au $E-$algorithme simultanément découvert en 1980 par C. BREZINSKI et HÅVIE (voir [Brez 5] ) et que l'on rappelle ci-dessous :

Soit $(c_n)$ une suite de réels vérifiant :

$$c_n = l + a_1 g_1(n) + a_2 g_2(n) + \dots \tag{39}$$

où les $a_i$ sont inconnus, $g_i(n)$ étant des fonctions de $n$ arbitraires.

Le $E$-algorithme est un procédé récursif permettant de calculer la valeur de $l$ si le développement (39) ne contient qu'un nombre fini de termes.

On pose

$$E_k^{(n)} = \frac{|c_n, g_1(n), g_2(n), \cdots, g_k(n)|}{|1, g_1(n), g_2(n), \cdots, g_k(n)|}$$

où $|c_n, g_1(n), g_2(n), \cdots, g_k(n)|$ désigne le déterminant

$$\begin{vmatrix} c_n & g_1(n) & g_2(n) & \cdots & g_k(n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n+k} & g_1(n+k) & g_2(n+k) & \cdots & g_k(n+k) \end{vmatrix}$$

Si l'on pose

$$\pi_k^{(n)}(x) = \begin{vmatrix} 1 & g_1(n) & \cdots & g_k(n) \\ x & g_1(n+1) & \vdots & g_k(n+1) \\ \vdots & \vdots & \vdots & \vdots \\ x^k & g_1(n+k) & \cdots & g_k(n+k) \end{vmatrix}$$

et

$$l_k^{(n)}(x) = \pi_k^{(n)}(x)/\pi_k^{(n)}(1).$$

$l_k^{(n)}(x)$ est un polynôme de degré $k$ en $x$ donc la suite $(E_k^{(n)})_{k\in\mathbb{N}}$ est une suite de moments modifiés pour la fonctionnelle $c^{(n)}$ :

$$E_k^{(n)} = c^{(n)}(l_k^{(n)}(x))$$

où $c^{(n)}$ est la fonctionnelle définie par :

$$< c^{(n)}, p(x) > := < c, x^n p(x) >$$

On place les quantités $E_k^{(n)}$ dans un tableau à double entrée :

$$\begin{array}{lll} E_0^{(0)} = c_0 & & \\ E_0^{(1)} = c_1 & E_1^{(0)} & \\ E_0^{(2)} = c_2 & E_1^{(1)} & E_2^{(0)} \\ \vdots & \vdots & \ddots \end{array}$$

C. BREZINSKI a montré dans [Brez 5] que chaque colonne du $E$-algorithme converge plus vite que la précédente, dans le cas où les suites auxiliaires $g_i(n)$ sont à convergence linéaire :

$$g_i(n+1)/g_i(n) \to b_i \neq 1 \quad , \quad b_i \neq b_j \text{ pour } i \neq j$$

et si elles forment une échelle de comparaison c'est-à-dire si

$$g_{i+1}(n) = o(g_i(n)) \quad n \to \infty \quad \forall i \in \mathbb{N}^*.$$

Ce résultat d'accélération n'existe plus si les suites $g_i(n)$ sont toutes à convergence logarithmique car dans ce cas tous les $b_i$ valent 1.

Cependant avec des hypothèses supplémentaires concernant les suites $g_i(n)$ dans leur ensemble on a pu montrer le théorème suivant :

**Théorème 10 :** *Soit $(c_n)$ une suite de réels vérifiant :*

$$c_n = l + a_1 g_1(n) + a_2 g_2(n) + \dots$$

*Si* $g_{i+1}(n) =_{n\to\infty} o(g_i(n)) \qquad \forall i = 1, 2, \dots$

*et si* $\forall i \in \mathbb{N}, \forall p \in \mathbb{N}, \exists N \in \mathbb{N}, \forall n > N,$

$$\begin{vmatrix} g_{i+p}(n) & \dots & g_{i+1}(n) & g_i(n) \\ \vdots & \vdots & \vdots & \vdots \\ g_{i+p}(n+p) & \dots & g_{i+1}(n+p) & g_i(n+p) \end{vmatrix} \geq 0 \qquad (H)$$

*avec* $g_0(n) := 1,$

***alors chaque colonne du E-algorithme converge plus vite que la précédente i.e. :***

$$\lim_{n\to\infty} \frac{E_{k+1}^{(n)} - l}{E_k^{(n)} - l} = 0, \qquad \forall k = 1, 2, \cdots$$

■

L'hypothèse $(H)$ est satisfaite notamment dans les cas suivants :

*a)* $g_i(n)$ suite à convergence linéaire vérifiant :

$$g_0(n) > 0 \qquad , \lim_n g_i(n+1)/g_i(n) = b_i \in ]0,1[, b_i \neq b_j, \text{pour } i \neq j$$

*b)* $g_1(n) = g(n)$ totalement monotone et $g_i(n) = (-1)^i \Delta^i g(n)$

*c)* $g_i(n) = x_n^{\alpha_i} \qquad 1 \geq x_1 > x_2 \ldots > 0$ et $0 \leq \alpha_1 < \alpha_2 < \alpha_3 \ldots < \infty$

*d)* $g_i(n) = \rho_i^n \qquad 1 \geq \rho_1 > \rho_2 > \ldots > 0$

et plus généralement

$$g_i(n) = K(x_n, \alpha_i)$$

avec $(x_n)_{n \in \mathbb{N}}$ suite réelle décroissante et $(\alpha_i)_{i \in \mathbb{N}}$ croissante et $K$ fonction totalement positive (voir [Kar])

L'hypothèse $g_{i+1}(n) = o(g_i(n))$ est réalisée dans les cas a), c) et d) et si $g(n)$ est logarithmique dans le cas b).

## Conclusion

Après avoir permis d'approximer certaines fonctions holomorphes à partir de leur développement en série de Taylor, l'Approximation de Type Padé a servi également à l'examen des singularités des fonctions connaissant leurs moments, ainsi qu'à l'accélération de suites de moments à convergence logarithmique.On peut supposer qu'au vu des résultats déjà obtenus, ce dernier sujet va donner lieu à des extensions intéressantes.

◇◇◇◇◇◇◇◇◇

La bibliographie ci-dessous reprend toutes les références citées dans les articles [Prev 0] à [Prev 11].

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Journal of Computational and Applied Mathematics 9 (1983) 333-346
North-Holland


# Padé-type approximants with orthogonal generating polynomials

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Received 3 December 1982
Revised 18 March 1983

*Abstract:* The interpolation of the function $x \to 1/(1-xt)$ generating the series $f(t)=\sum_{i=0}^{\infty} c_i t^i$ at the zeros of an orthogonal polynomial with respect to a distribution $d\alpha$ satisfying some conditions will give us a process for accelerating the convergence of $f_n(t)=\sum_{i=0}^{n} c_i t^i$. Then, we shall see that the polynomial of best approximation of $x \to 1/(1-xt)$ over some interval or its development in Chebyshev polynomials $T_n$ or $U_n$ are only particular cases of the main theorem.

At last, we shall show that all these processes accelerate linear combinations with positive coefficients of totally monotonic and oscillating sequences.

*Keywords:* Padé-type approximants, Stieltjes-series, totally monotonic sequence.

*AMS (MOS) Subject Classification:* 41A20, 65B10.

## 1. Introduction

Let us first recall the definition and main properties of Padé-type approximants [3, p. 9-11].

Let $f$ be a formal power series in one variable

$$f(t)=\sum_{i=0}^{\infty} c_i t^i, \quad c_i \in \mathbb{R}, \ \forall i.$$

Let us define the linear functional $c$ acting on the space of real polynomials by:

$$c(x^i)=c_i \quad \text{for } i=0,1,\dots$$

The number $c_i$ is called the moment of order $i$ of the functional $c$. We get immediately

$$f(t)=c\left(\frac{1}{1-xt}\right).$$

The function $x \to 1/(1-xt)$ is called the generating function of the series $f$.

Let $v$ be an arbitrary polynomial of degree $k$

$$v(x)=b_0+b_1x+\cdots+b_kx^k$$

and let us define $w$ by:

$$w(t)=c\left(\frac{v(x)-v(t)}{x-t}\right)$$

where $c$ acts on $x$, $t$ being a parameter. $w$ is a polynomial of degree $k-1$.

Let us now define $\tilde{w}$ and $\tilde{v}$ by $\tilde{w}(t)=t^{k-1}w(t^{-1})$ and $\tilde{v}(t)=t^k v(t^{-1})$. We have:

$$\tilde{w}(t)/\tilde{v}(t)-f(t)=O(t^k), \quad t \to 0.$$

Such a rational approximant to $f$ is called a Padé-type approximant and it is denoted

$$(k-1/k)_f(t).$$

$v$ is called the generating polynomial of the approximant. Brezinski [3, p. 21] proved the following results:
- If $P$ is the Lagrange interpolation of the generating function of $f$ at $k$ arbitrary points $x_i$ in the complex plane, then

$$c(P)=(k-1/k)_f(t)$$

where the generating polynomial $v$ of the approximant is

$$v(x)=(x-x_1)(x-x_2)\cdots(x-x_k).$$

If some of the interpolation points $x_i$ coïncide, we still have:

$$c(P)=(k-1/k)_f(t).$$

$P$ being the general Hermite interpolation polynomial of the generating function of $f$. This property will be systematically used in the sequel.
- Pointwise convergence. Let $t \in D$, domain of convergence of $f$, if for all $k$, the zeros $x_i$ of $v$ are such that $tx_i \in [-a, 0]$, with $a > 0$, then the sequence $(k-1/k)_f(t)$ converges to $f(t)$ when $k \to \infty$.

In this paper, we shall only consider the particular case

$$f(t)=\int_a^b \frac{\mathrm{d}\alpha(x)}{1-xt}$$

where $\alpha$ is bounded and non decreasing in $[a, b]$.

Let us now begin with the main theorem.

## 2. Approximation of series

We consider a real-valued non decreasing bounded function $\mu$ which takes infinitely many different values. By a weight function $w(x)=\mu'(x)$ will be meant the derivative of the absolutely continuous distribution $\mu$. The support of $\mathrm{d}\mu$, i.e. the set of the points of increase of $\mu(x)$ is noted $\mathrm{Br}(\mathrm{d}\mu)$

**Lemma 2.1.** [9, p. 117]. *If* $\mathrm{Br}(\mathrm{d}\mu) \subseteq [-1, 1]$, *then*

$$\varliminf_{n\to\infty} \left| \frac{\sqrt[n]{|R_n(z)|}}{z+\sqrt{z^2-1}} \right| \geqslant 1 \quad \forall z \in \mathbb{C}-[-1,1]$$

*where $R_n$ is the orthogonal polynomial of degree $n$ with respect to the distribution $\mathrm{d}\mu$ and where $\sqrt{z^2-1}$ is the branch which takes real positive values for $z>1$.*

**Lemma 2.2.** [9, p. 123–124]. *If* $\mathrm{Br}(\mathrm{d}\mu)=[-1, 1]$ *and* $\mu'(x)>0$ *for almost all* $x \in [-1, 1]$, *then*

$$\varlimsup_{n\to\infty} \sqrt[n]{|R_n(x)|} \leqslant 1 \quad \forall x \in [-1, 1].$$

Let $\tau$ be defined on $\mathbb{C}$ by:

$$\tau(y)=\frac{y-\frac{1}{2}(a+b)}{\frac{1}{2}(b-a)}.$$

Set $X=\tau(x)$, $A=\tau(t^{-1})$ for $x \in \mathbb{C}$, $t \in \mathbb{C}^*$. $\tau$ maps the real interval $[a, b]$ one-to-one onto $[-1, 1]$

**Main Theorem**: *Let* $f(t)=\int_a^b \mathrm{d}\alpha(x)/(1-xt)$, $t^{-1}\in\mathbb{C}-[a,b]$, $\alpha$ *bounded and nondecreasing. Let* $R_n^*(x)=R_n(\tau(x))=R_n(X)$ *for* $x\in[a,b]$, *where* $R_n$ *is the orthogonal polynomial of degree* $n$ *with respect to some distribution* $\mathrm{d}\mu$ *such that* $\mathrm{Br}(\mathrm{d}\mu)=[-1,1]$ *and* $\mu'(X)>0$ *for almost all* $X\in[-1,1]$ (*this last condition may be replaced by*

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|R_n(X)|}\leqslant 1,\quad \forall X\in[-1,1]).$$

*Let*

$$P_n(x)=\frac{1-R_n^*(x)/R_n^*(t^{-1})}{1-xt}$$

*be the polynomial interpolating* $x\to 1/(1-xt)$ *at the the zeros of* $R_n^*$. *Under these conditions, we have*

$$\overline{\lim_{n\to+\infty}}\left[|c(P_n)-f(t)|\right]^{1/n}\leqslant|B|<1$$

*with* $B=A-\sqrt{A^2-1}$. (*In the cut plane, we have* $|B|<1$, $\forall A\in\mathbb{C}-[-1,1]$.)

**Proof.** Let $e_n=c(P_n)-f(t)$ be the error

$$e_n=\frac{1}{R_n^*(t^{-1})}\int_a^b\frac{R_n^*(x)}{1-xt}\,\mathrm{d}\alpha(x)=\frac{2}{b-a}\,\frac{1}{tR_n(A)}\int_{-1}^{1}\frac{R_n(X)}{A-X}\,\mathrm{d}\beta(X)$$

with $\mathrm{d}\beta(X)=\mathrm{d}\alpha(\tau^{-1}(X))$. It follows that

$$|e_n|\leqslant\frac{1}{|t|\cdot|R_n(A)|}\cdot\sup_{-1\leqslant X\leqslant 1}|R_n(X)|\cdot\sup_{-1\leqslant X\leqslant 1}\left|\frac{1}{A-X}\right|\cdot\int_{-1}^{1}|\mathrm{d}\beta(X)|$$

$$(t^{-1}\notin[a,b]\ \Rightarrow\ A\notin[-1,1]).$$

Using Lemma 2.2, we get

$$\overline{\lim_{n\to\infty}}|e_n|^{1/n}\leqslant\frac{1}{\underline{\lim}_n|R_n(A)|^{1/n}}\quad\Rightarrow\quad\overline{\lim_{n\to\infty}}|e_n|^{1/n}\leqslant|B|<1\quad\text{(Lemma 2.1)}.$$

$B=A-\sqrt{A^2-1}$ such that $|B|<1$. □

As we shall see now, this theorem has many applications. Indeed, many summation processes satisfy the conditions of the main theorem (Sections 2.1 to 2.6).

### 2.1. Padé approximants

**Theorem 2.3.** *The Padé-approximant* $[n-1/n]_f(t)$ *to the series* $f(t)=\sum_{n=0}^{\infty}c_nt^n$ *with*

$$c_n=\int_a^b x^n\mathrm{d}\alpha(x),\quad t^{-1}\notin[a,b],$$

$\alpha$ *bounded and nondecreasing in* $[a,b]$ *satisfies*

$$\overline{\lim_{n\to\infty}}\left|[n-1/n]_f(t)-f(t)\right|^{1/n}\leqslant|B|^2$$

*with* $B$ *defines as above.*

**Proof.** We know that the generating polynomial of the Padé-approximant $[n-1/n]_f(t)$ is orthogonal with respect to the distribution defining $(c_n)_{n\in\mathbb{N}}$. We have

$$f(t)=\int_a^b \frac{\mathrm{d}\alpha(x)}{1-xt}=\frac{2}{b-a}\int_{-1}^1 \frac{\mathrm{d}\beta(X)}{A-X}$$

where $\mathrm{d}\beta(X)=\mathrm{d}\alpha(\tau^{-1}(X))=\mathrm{d}\alpha(x)$.

If we consider the sequence of polynomials $R_n$ which are orthogonal with respect to the distribution $\mathrm{d}\beta$, we obtain:

$$e_n=\frac{1}{tR_n(A)}\int_{-1}^1 \frac{R_n(X)}{A-X}\mathrm{d}\beta(X).$$

We have $\underline{\lim}_{n\to\infty}|R_n(A)|^{1/n}\geqslant |A+\sqrt{A^2-1}\,|>1$ (Lemma 2.1).

Using the development of the function $X\to 1/(A-X)$ in $T_n$, Chebyshev polynomials of the first kind [14, p. 153; 16]:

$$\frac{1}{A-X}=\frac{4B}{1-|B|^2}\sum_{k=0}^{\infty}{}' B^kT_k(X). \qquad A=\tfrac{1}{2}\left(B+\frac{1}{B}\right),\quad |B|<1$$

(where $\sum'_{k=0}u_k=\frac{1}{2}u_0+u_1+u_2+u_3+\cdots$) we get

$$\int_{-1}^1 \frac{R_n(X)}{A-X}\mathrm{d}\beta(X)=\frac{4B}{1-|B|^2}\sum_{k=n}^{\infty}B^k\int_{-1}^1 T_k(X)R_n(X)\mathrm{d}\beta(X)$$

since

$$\int_{-1}^1 T_k(X)R_n(X)\mathrm{d}\beta(X)=0,\quad k=0,1,\ldots,n-1.$$

Now, for all $X\in[-1,1]$ $|T_n(X)|\leqslant 1$ by construction of $T_n$. Thus

$$\left|\int_{-1}^1 T_k(X)R_n(X)\mathrm{d}\beta(X)\right|\leqslant\int_{-1}^1|R_n(X)|\mathrm{d}\beta(X)$$
$$\leqslant\left[\int_{-1}^1\mathrm{d}\beta(X)\cdot\int_{-1}^1|R_n(X)|^2\mathrm{d}\beta(X)\right]^{1/2}\leqslant\left[\int_{-1}^1\mathrm{d}\beta(X)\right]^{1/2}=M$$

and

$$\left|\int_{-1}^1\frac{R_n(X)}{A-X}\mathrm{d}\beta(X)\right|\leqslant\frac{4|B|}{1-|B|^2}\cdot\frac{|B|^n}{1-|B|}\cdot M.$$

We obtain

$$\overline{\lim_{n\to\infty}}\,|e_n|^{1/n}\leqslant\frac{|B|}{|A+\sqrt{A^2-1}\,|}=|B|^2<1. \qquad \square$$

This result has been obtained by Baker [2, p. 220], but in a different way.

### 2.2. Best approximation of the generating function

The polynomial of best approximation of $X\to 1/(X-A)$ for $X\in[-1,1]$ is given by [12, p. 33–34]:

$$P_n(X)=\frac{1}{X-A}-\frac{B^n}{A^2-1}\cos(n\phi+\delta)$$

where $B=A-\operatorname{sign}(A)\cdot\sqrt{A^2-1}$, $\cos\phi=X$ and $(AX-1)/(X-A)=\cos\delta$; $A\in R$, $|A|>1$.

**Theorem 2.4.** $\overline{\lim}_{n\to\infty}|c(P_n)-f(t)|^{1/n}\leqslant|B|<1$ *where $P_n$ is defined as above.*

**Proof.** $P_n(X)$ interpolates $X \to 1/(X-A)$ at the zeros of $\cos(n\phi+\delta)$: let us set:

$$R_{n+1}(X) = (X-A)\cos(n\phi+\delta).$$

$R_{n+1}$ is a polynomial of degree $(n+1)$. We have:

$$R_{n+1}(X) = -T_n(X) + \frac{1}{2B}T_{n+1}(X) + \frac{B}{2}T_{n-1}(X),$$

$$R_1(X) = -1 + \frac{X}{2B} + \frac{B}{2}X = -1 + AX.$$

Using the recurrence relation between the Chebyshev polynomials $T_n$:

$$T_{n+1}(X) = 2XT_n(X) - T_{n-1}(X)$$

We get

(1) $R_{n+2}(X) = 2XR_{n+1}(X) - R_n(X)$, $\forall\, n \geqslant 1$;

(2) $P_{n+1}(X) = 2BXP_n(X) - B^2P_{n-1}(X) - 2B$, $\forall\, n \geqslant 1$.

Since the coefficient of $R_n$ in the relation (1) is positive, it follows, from a result due to Favard [7], that $(R_n)_{n\geqslant 1}$ is orthogonal with respect to some distribution $d\mu$. From the alternance theorem of Chebyshev, we know that $R_n$ has all its zeros in $[-1, 1]$.

The sequence $(R_n)_{n\in\mathbb{N}}$ satisfies the conditions of the main theorem and thus:

$$\overline{\lim_{n\to\infty}} |e_n|^{1/n} = \overline{\lim_{n\to\infty}} |f(t) - c(P_n)|^{1/n} \leqslant |B| < 1. \qquad \square$$

Since $R_n$ is explicitly known, we have a more precise expression of $|e_n|$

$$|e_n| \leqslant \frac{|B|^n}{|t||A^2-1|}\left[\int_{-1}^{1} |d\beta(X)|\right] \cdot \frac{2}{(b-a)}.$$

**Remark 2.5.** If $t \in \mathbb{R}$, $P_n(X)$ is no more the polynomial of best approximation but it is still the interpolation polynomial of $X \to 1/(X-A)$ at some $x_i \in \mathbb{C}$.

**Theorem 2.6.** *When $n$ tends to infinity $c(P_n)$ tends to $f(t)$ uniformly in $t$ on every compact $K$ subset of* $\mathbb{C} - [a^{-1}, b^{-1}]$.

**Proof.**

$$\sup_{t\in K} |f(t) - c(P_n)| \leqslant \frac{2|B|^n}{|t||A^2-1|}\,\frac{M}{b-a},\; M = \int_{-1}^{1} |d\beta(X)|$$

$$\leqslant \frac{2M}{b-a} \sup_{t\in K} \frac{|B|^n}{|t||A^2-1|}, \quad t \neq 0.$$

$1/(|t||A^2-1|)$ is bounded on $K$ and $\sup_{t\in K}|B|^n \to 0$ when $n \to \infty$. $\square$

Let us give an example

$$f(t) = \frac{\operatorname{Log}(1-t)}{t} = \sum_{i=0}^{n} c_i t^i \quad \text{with } c_i = \frac{1}{i+1} = \int_0^1 x^i \mathrm{d}x.$$

For $\operatorname{Log} 2 = f(-1)$ we have $t = -1$, $[a, b] = [0, 1]$

$$A = \frac{t^{-1} - \frac{1}{2}(a+b)}{\frac{1}{2}(b-a)} = -3 \quad \text{and} \quad B = A \pm \sqrt{A^2-1}, \quad |B| < 1, \quad B = -3 + \sqrt{8},$$

$$M = \int_0^1 \mathrm{d}x = 1.$$

We get

$$|f(-1)-c(P_n)|=|e_n|\leqslant \tfrac{1}{4}|-3+\sqrt{8}|^n \simeq \tfrac{1}{4}(0.172)^n.$$

This implies $|e_{20}|\leqslant 10^{-16}$; in practice we have $|e_{16}|=10^{-16}$.

*2.3. Development of the generating function in a series of Chebyshev polynomials of the first kind*

Instead of replacing $1/(X-A)$ by its polynomial of best approximation, we shall replace it by its development in Chebyshev polynomials of the first kind:

$$\frac{1}{X-A}=\frac{-4B}{1-B^2}\sum_{k=0}^{\infty}{}' B^k T_k(X) \quad \text{for } |A|>1,\ A\in\mathbb{C}$$

where $B$ is defined as in Section 2.2 and $\sum_{k=0}^{\prime\infty} u_k=\frac{1}{2}u_0+u_1+u_2+\cdots$. See [14, p. 153, 16].

Let us set

$$P_n(X)=\frac{-4B}{1-B^2}\sum_{k=0}^{n}{}' B^k T_k(X)$$

where the prime has the same meaning as above.

Remark 2.7. $P_n$ used in this paragraph is different from that in theorem 2.4. The same for $R_n$.

Theorem 2.8. $\overline{\lim}_{n\to\infty}|f(t)-c(P_n)|^{1/n}\leqslant |B|<1$ *with* $P_n$ *defined above.*

Proof. The relation

$$P_n(X)=\frac{1-R_{n+1}(X)/R_{n+1}(A)}{X-A}$$

gives

$$\frac{1}{X-A}\cdot\frac{R_{n+1}(X)}{R_{n+1}(A)}=\frac{1}{X-A}-P_n(X)=-\frac{4B}{1-B^2}\sum_{k=n+1}^{\infty} B^k T_k(X),$$

$$\frac{R_{n+1}(X)}{R_{n+1}(A)}=\frac{-4B^{n+2}}{1-B^2}\left[\sum_{k=0}^{\infty} B^k T_{k+n+1}(X)\right](X-A).$$

We set:

$$R_{n+1}(X)=(X-A)\sum_{k=0}^{\infty} B^k T_{n+k+1}(X);\ R_{n+1}(A)=\frac{B^2-1}{4B^{n+2}},$$

so that $\lim|R_n(A)|^{1/n}=1/|B|$;

$$R_{n+1}(X)=-A\sum_{k=0}^{\infty} B^k T_{n+k+1}(X)+\sum_{k=0}^{\infty} B^k\cdot X\cdot T_{n+1+k}(X).$$

From the relation between $(T_n)_{n\in\mathbb{N}}$, we get:

$$R_{n+1}(X)=-\frac{1}{2B}T_{n+1}(X)+\tfrac{1}{2}T_n(X).$$

As in Section 2.2, the sequence $(R_{n+1})_{n\geqslant 0}$ verifies:

$$R_{n+1}(X)=2XR_n(X)-R_{n-1}(X),\qquad R_1(X)=\frac{-X+B}{2B},\ R_2(X)=\frac{-2X^2+BX+1}{2B}$$

and

$$\overline{\lim_{n\to\infty}} |R_{n+1}(X)|^{1/(n+1)} \leqslant \overline{\lim_{n\to\infty}} \left(\tfrac{1}{2} + \frac{1}{2|B|}\right)^{1/(n+1)} = 1.$$

If $A \in \mathbb{R}$, the polynomials $(R_n)_{n\geqslant 1}$ are orthogonal with respect to some distribution $d\mu$.

Let us prove that $\mathrm{Br}(d\mu) \subseteq [-1, 1]$ or that $R_{n+1}$ has all its zeros in $[-1, 1]$. Because of the relation $R_{n+1}(X) = 2XR_n(X) - R_{n-1}(X)$, $R_{n+1}$ has all its zeros in some interval of $\mathbb{R}$. Let $X$ be a zero of $R_{n+1}$:

$$R_{n+1}(X) = 0 \quad \Leftrightarrow \quad |T_{n+1}(X)| = |B| \cdot |T_n(X)|$$
$$\Leftrightarrow |T_{n+1}(X)| < |T_n(X)|$$

which is impossible if $X \in [-1, 1]$. So $(R_{n+1})_{n\geqslant 0}$ satisfies the assumptions of the main theorem and we obtain:

$$\overline{\lim_{n\to\infty}} |f(t) - c(P_n)|^{1/n} = \overline{\lim_{n\to\infty}} |e_n|^{1/n} \leqslant |B| < 1. \qquad \square$$

Since $R_{n+1}(X) = \frac{1}{2}T_n(X) - (1/2B)T_{n+1}(X)$, we obtain a more precise error bound

$$|e_n| \leqslant \frac{4|B|^{n+2}M}{|B^2 - 1|(1 - |B|)|t|} \cdot \frac{2}{b - a} \quad (t \neq 0).$$

For Log 2, $t = -1$, $A = -3$, $M = 1$, $B \simeq -0.172$ and we have

$$|e_n| \leqslant \frac{(0.172)^n}{3.9}.$$

This error bound is slightly greater than the one using the polynomial of best approximation.

### *2.4. Development of the generating function in a series of Chebyshev polynomials of the second kind*

In

$$f(t) = \int_a^b \frac{d\alpha(x)}{1 - xt} = \frac{2}{(b-a)} \int_{-1}^1 \frac{d\beta(X)}{A - X},$$

we replace $1/(X - A)$ by its expansion in a series of Chebyschev polynomials of the second kind:

$$\frac{1}{X - A} = -2B \sum_{k=0}^{\infty} B^k U_k(X).$$

The sequence $(U_n)_{n\geqslant 0}$ verifies [14, p. 49, 16]

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \qquad U_0 = 1, \quad U_1(x) = 2x$$
$$\sup_{x\in[-1,1]} |U_n(x)| = U_n(1) = (n + 1).$$

Let us set $P_n(X) = -2B\Sigma_{k=0}^n B^k U_k(X)$.

**Theorem 2.9.** $\overline{\lim}_{n\to\infty} |f(t) - c(P_n)|^{1/n} \leqslant |B| < 1$ *with $P_n$ defined above.*

**Proof.** $P_n$ interpolates $X \to 1/(X - A)$ at the zeros of $R_{n+1}$ with

$$\frac{1}{X - A}\frac{R_{n+1}(X)}{R_{n+1}(A)} = -2B \sum_{k=n+1}^{\infty} B^k U_k(X).$$

As in Section 2.3 we get:

$$R_{n+1}(X)=\tfrac{1}{2}U_n(X)-\frac{1}{2B}U_{n-1}(X), \qquad R_{n+1}(A)=-\frac{1}{2B^{n+2}}.$$

$$|R_{n+1}(X)|\leqslant\frac{n+2}{2|B|}+\frac{n+1}{2} \quad\Rightarrow\quad \varlimsup_{n\to\infty}|R_{n+1}(X)|^{1/n+1}\leqslant 1.$$

The sequence $(R_n)_{n\geqslant 1}$ satisfies

$$R_{n+1}(X)=2XR_n(X)-R_{n-1}(X).$$

$$R_1(X)=\tfrac{1}{2}-\frac{X}{2B}, \qquad R_2(X)=\frac{-2X^2}{B}+X+\frac{1}{2B}$$

and so, $(R_{n+1})_{n\geqslant 0}$ are orthogonal with respect to some distribution $d\mu$ over some interval of $\mathbb{R}$; as in Section 2.3, this interval is $[-1, 1]$. $(R_n)_{n\geqslant 0}$ satisfies the assumptions of the main theorem and we have:

$$\varlimsup_{n\to\infty}|c(P_n)-f(t)|^{1/n}\leqslant|B|<1. \qquad \square$$

This summation process is exactly the same which has been studied by Niethammer [13]; this will be proved in [15].

More precisely:

$$|e_n|\leqslant\frac{4B^{n-1}(n+1-n|B|)}{|t|(b-a)(1-B^2)}\cdot M.$$

For Log 2 we get $|e_n|\leqslant(0.172)^n\cdot 0.7\cdot(0.828n+1)$. This error bound is greater than that of Sections 2.2 and 2.3.

*2.5. $L^1$ approximation of the generating function*

After having approximated $1/(X-A)$ on $[-1, 1]$ by its best approximation polynomial, viz. approximation in $L^\infty$,

$$L^p=\left\{\Psi:\mathbb{R}\to\mathbb{R}\,\middle|\,\left[\int_{-1}^{1}|\Psi(X)|^p\,dX\right]^{1/p}<+\infty\right\},$$

$$L^\infty=\left\{\Psi:\mathbb{R}\to\mathbb{R}\,\middle|\,\sup_{-1\leqslant X\leqslant +1}|\Psi(X)|<+\infty\right\},$$

we shall use the approximation of $1/(X-A)$ in $L^1$ and $L^2$ norms (see Section 2.6). We shall see that the points of interpolation of $1/(X-A)$ corresponding to the two different approximations are the zeros of orthogonal polynomials which satisfy the assumptions of the main theorem.

Akhiezer [1] has proved that the polynomial $P_n^*(X)$ which minimizes:

$$\int_{-1}^{1}\left|P_n^*(X)-\frac{1}{X-A}\right|dX$$

is the interpolation polynomial of $X\to 1/(X-A)$ at the zeros of $U_{n+1}(X)$. We are precisely in the hypothesis of main theorem with:

$$R_{n+1}(X)=U_{n+1}(X) \quad\text{and}\quad P_n^*(X)=P_n^*(\tau^{-1}(x))=P_n(x).$$

Thus

**Theorem 2.10.**

$$\varlimsup_{n\to\infty}|e_n|^{1/n}=\varlimsup_{n\to\infty}|f(t)-c(P_n)|^{1/n}\leqslant|B|<1,$$

*where $P_n$ is the best $L^1$-appromation of the function $X\to 1/(X-A)$ on $[a, b]$.*

### 2.6. $L^2$-approximation of the generating function

In [11] Jurkatt and Shawyer give the best $L^2$-approximation polynomial of $1/(X-A)$ with $A>1$ over the interval $[-1, 1]$

$$P_n^*(X) = -\sum_{k=0}^{n} (2k+1) q_k(A) p_k(X)$$
$$= \frac{1}{X-A} - \frac{n+1}{X-A}\{p_{n+1}(X)q_n(A) - p_n(X)q_{n-1}(A)\}$$

where $p_n(X)$ is the Legrendre polynomial of degree $n$ and $q_n(X)$ is the associated Legendre function.

$$q_n(X) = \tfrac{1}{2}\int_{-1}^{1} \frac{p_n(t)}{X-t}\,\mathrm{d}t.$$

where the integral is a Cauchy principal value, when $X \in [-1, 1]$. $P_n^*(X)$ interpolates $1/(X-A)$ at the zeros of $p_{n-1}(X)q_n(A) - p_n(X)q_{n-1}(A)$. Let us set

$$R_{n-1}(X) = p_{n-1}(X) - p_n(X)\frac{q_{n-1}(A)}{q_n(A)}.$$

we have $R_{n-1}(A) = 1/(n+1)q_n(A)$

**Theorem 2.11.** $\overline{\lim}_{n\to\infty} |f(t) - c(P_n)|^{1/n} \leqslant |B| < 1$ *where* $P_n$ *is the best* $L^2$*-approximation of* $1/(X-A)$ *over* $[a, b]$.

**Proof.** Let us show that the polynomial $R_{n+1}(X)$ satisfies the assumptions of the main theorem, viz. $R_{n+1}(X)$ is orthogonal with respect to $\mathrm{d}\mu$. $\mathrm{Br}(\mathrm{d}\mu) \subseteq [-1, 1]$ and $\overline{\lim}_{n\to\infty}|R_n(X)|^{1/n} \leqslant 1$.

Jurkatt and Shawyer proved [11]:

$$\frac{q_{n+1}(A)}{q_n(A)} \sim B^{-1} \quad (n \to \infty).$$

B is defined as above: $A = \frac{1}{2}(B + 1/B)$, $B = A - \sqrt{A^2-1}$ such that $|B| < 1$. Thus $\overline{\lim}_{n\to\infty}|R_{n-1}(X)|^{1/n+1} \leqslant 1$ since $\lim_{n\to\infty}|p_n(X)|^{1/n} = 1$, $\forall X \in [-1, 1]$, [17, p. 195].

To show that $R_{n+1}(X)$ is orthogonal with respect to some distribution $\mathrm{d}\mu$ over $[-1, 1]$, we must find its three terms recurrence relationship. The recurrence relation between $p_{n-1}$, $p_n$ and $p_{n+1}$ is:

$$p_{n+1}(x) = \frac{2n+1}{n+1}xp_n(x) - \frac{n}{n+1}p_{n-1}(x).$$

Replacing in the definition of $R_{n+2}$ and eliminating $p_{n-1}$, we find:

$$R_{n+2}(X) = \left[A_{n+2}X - \frac{q_{n+2}(A)}{q_{n+1}(A)} + \frac{q_{n+1}(A)}{q_n(A)}\cdot\frac{A_{n+2}}{A_{n+1}}\right]R_{n+1}(X)$$
$$-q_{n-1}(A)\frac{q_{n+1}(A)}{q_n^2(A)}C_{n+1}\cdot\frac{A_{n+2}}{A_{n+1}}R_n(X)$$

with $A_{n+1} = (2n+1)/(n+1)$ and $C_{n+1} = n/(n+1)$.

$q_{n+1}(A)$ has the same sign as $A^{n+1}$, thus

$$\frac{q_{n-1}(A)q_{n+1}(A)}{q_n^2(A)}C_{n+1}\frac{A_{n+2}}{A_{n+1}} > 0$$

and the $R_{n+1}$'s are orthogonal with respect to some distribution $\mathrm{d}\mu$.

Let us show now that $\mathrm{Br}(d\mu) \subseteq [-1, 1]$ viz. $R_{n+1}(X) \neq 0$, $\forall X \notin [-1, 1]$. We know that

$$q_{n+1}(A) = \frac{2n+1}{n+1} A q_n(A) - \frac{n}{n+1} q_n(A)$$

$$\Rightarrow \quad \frac{q_n(A)}{q_{n+1}(A)} = \frac{1}{\dfrac{2n+1}{n+1} A - \dfrac{n}{n+1} \dfrac{q_{n-1}(A)}{q_{n(A)}}}$$

while the same holds for $p_n$, $p_{n-1}$, $p_{n+1}$.

We shall show by induction that

$$\frac{q_{n-1}(A)}{q_n(A)} > \frac{p_{n-1}(X)}{p_n(X)} \quad \forall X > 1, \ \forall A > 1.$$

This is true for $n = 1$

$$\frac{q_0(A)}{q_1(A)} = \frac{1}{A - \dfrac{2}{\mathrm{Log}\left|\dfrac{A+1}{A-1}\right|}} \quad \text{and} \quad p_0(X) = \frac{1}{X}.$$

A simple calculation shows that:

$$\frac{q_0(A)}{q_1(A)} > \frac{p_0(X)}{p_1(X)} \quad \forall A > 1, \ \forall X > 1.$$

Now, if

$$\frac{q_{n-1}(A)}{q_n(A)} > \frac{p_{n-1}(X)}{p_n(X)},$$

we have

$$\frac{2n+1}{n+1} A - \frac{n}{n+1} \frac{q_{n-1}(A)}{q_n(A)} < \frac{2n+1}{n+1} - \frac{n}{n+1} \frac{p_{n-1}(X)}{p_n(X)}$$

$$\Rightarrow \quad \frac{q_{n+1}(A)}{q_n(A)} < \frac{p_{n+1}(X)}{p_n(X)}.$$

$q_{n+1}(A)/q_n(A)$ has the same sign as $A$, thus it is strictly positive and we deduce:

$$\frac{q_n(A)}{q_{n+1}(A)} > \frac{p_n(X)}{p_{n+1}(X)} \quad \forall A > 1, \ \forall X > 1, \ \forall n \geqslant 0.$$

It follows that $p_{n+1}(X) > p_n(X) \cdot q_{n+1}(A)/q_n(A)$ since $p_{n+1}(X) > 0$ for $X > 1$. Thus $R_{n+1}(X) > 0$, $\forall X > 1$.

If now $X < -1$, we have:

$$\frac{p_n(X)}{p_{n+1}(X)} < 0 \quad \forall X < -1 \quad \text{and} \quad \frac{q_{n+1}(A)}{q_n(A)} > 0$$

$$\Rightarrow \quad R_{n+1}(X) = p_{n+1}(X) - p_n(X) \frac{q_{n+1}(A)}{q_n(A)} = p_n(X) \left[ \frac{p_{n+1}(X)}{p_n(X)} - \frac{q_{n+1}(A)}{q_n(A)} \right].$$

Thus $R_{n+1}(X)$ and $p_n(X)$ have different signs and

$$R_{n+1}(X) \neq 0 \quad \text{for } X < -1.$$

$(R_n)_{n \in \mathbb{N}}$ obey a three term recurrence with a positive coefficient for $R_{n-1}$, and are nonzero for $X > 1$, $X < -1$. Therefore, $(R_{n+1})_{n \geqslant 0}$ are orthogonal with respect to $d\mu$ such that $\mathrm{Br}(d\mu) \subseteq [-1, 1]$ [7] and $\overline{\lim}_{n \to \infty} |R_{n+1}(X)|^{1/n} \leqslant 1$.

Table 1

| $n$ | (1) | (2) | (3) |
|---|---|---|---|
| 0 | $6\cdot10^{-2}$ | 1 | $3\cdot10^{-2}$ |
| 6 | $2\cdot10^{-8}$ | $2.6\cdot10^{-5}$ | $7\cdot10^{-7}$ |
| 10 | $3\cdot10^{-12}$ | $2\cdot10^{-8}$ | $2\cdot10^{-8}$ |
| 14 | $10^{-15}$ | $2\cdot10^{-11}$ | $6\cdot10^{-13}$ |
| 16 | $10^{-16}$ | $5.6\cdot10^{-13}$ | $2.3\cdot10^{-14}$ |

The main theorem gives:

$$\overline{\lim_{n\to\infty}} |e_n|^{1/n} \leqslant |B| < 1. \qquad \square$$

An open problem whether the best approximation of $X \to 1/(X-A)$ over $[-1, 1]$ in the $L^p$ norm is a polynomial which interpolates $1/(X-A)$ at the zeros of some orthogonal polynomial over $[-1, 1]$ or not: the result is true for $p = 1, 2$ and $\infty$.

Remark 2.12. From the recurrence relation between $R_n(X)$ and from $P_n(X) = (1 - R_n(X)/R_n(A))/(X - A)$, we can deduce a relation giving $c(P_{n-1})$ in terms of $c(P_n)$, $c(P_{n-1})$ and $c(xP_n)$. Thus the calculation of $c(P_{n-1})$ is very simple:

$$c(P_{n+1}) = A_n c(xP_n) + B_n c(P_n) - C_n c(P_{n-1}).$$

To end this section, let us give a numerical example.

For Log 2 we get the following results (see Table 1):

(1) errors with the best approximation in $L^\infty$;

(2) $(0.172)^n$;

(3) errors with Padé-approximants.

## 3. Convergence acceleration of sequences

In this section, we shall study the sequences accelerated by the various processes described in Section 2.

Let $(S_n)_{n\geqslant0}$ be a real sequence which converges to $S$. We may change $S_n$ into the partial sums of series by setting.

$$c_0 = S_0, \qquad c_i = \frac{\Delta S_{i-1}}{t^i}, \quad i \geqslant 1, \quad t \in \mathbb{R} - \{0\}$$

$$\Rightarrow \quad S_n = \sum_{i=0}^{n} c_i t^i \quad \text{and} \quad S = \sum_{i=0}^{\infty} c_i t^i \text{ if it exists.}$$

Let $f(t) = \sum_{i=0}^{\infty} c_i t^i$; we have $f(t) = S$.

Let us choose an arbitrary sequence of polynomials $R_n$ satisfying the conditions of the main theorem. We obtain a process denoted $T \cdot T: (S_n)_{n\in\mathbb{N}} \to (c(P_n))_{n\in\mathbb{N}}$ and we set $T(S_n) = c(P_n)$.

The sequence $(S_n)$ is said to be *totally monotonic* if:

$$(-1)^k \Delta^k S_n \geqslant 0 \quad \text{for } n, k = 0, 1, 2, \ldots$$

where $\Delta$ is defined as $\Delta^0 S_n = S_n$ and $\Delta^{k+1} S_n = \Delta(\Delta^k S_n) = \Delta^k S_{n+1} - \Delta^k S_n$ for $n, k = 0, 1, \ldots$ We shall write $(S_n) \in \mathrm{TM}$.

We shall say that $(S_n)$ is *totally oscillating* and we shall write $(S_n) \in \mathrm{TO}$ if $((-1)^n S_n) \in \mathrm{TM}$. A convergent TO sequence converges to zero.

Brezinski [4] proved that

$$\lim_{n\to\infty}\frac{S_{n+1}-S}{S_n-S}=\rho\in\,]0,1]\quad\text{for } (S_n)\in \mathrm{TM}$$

and

$$\lim_{n\to\infty}\frac{S_{n+1}-S}{S_n-S}=\rho\in[-1,0[\quad\text{for } (S_n)\in \mathrm{TO}.$$

**Theorem 3.1.** *Let $(S_n)$ be a real sequence converging to $S$. If there exists a real $t\in\,]-\infty,1[$ such that*

$$\left(\frac{\Delta S_{n-1}}{t^n}\right)\in \mathrm{TM},$$

*then $T(S_n)$ converge to $S$ and we have*

$$\overline{\lim_{n\to\infty}}\ |T(S_n)-S|^{1/n}\leqslant|B|<1\quad (B \textit{ follows via } A=2R/t-1).$$

**Proof.** If $(\Delta S_{n-1}/t^n)\in\mathrm{TM}$, then $(c_n)_{n\geqslant1}\in\mathrm{TM}$ and $f(t)$ is a Stieltjes series

$$f(t)=\int_0^{1/R}\frac{\mathrm{d}\alpha(x)}{1-xt},\quad t<1.$$

We have $1/t\in[0,1]\supseteq[0,1/R]$ because $R\geqslant1$ [4, p. 177]

$$R=\left[\overline{\lim_{n\to\infty}}\left|\frac{c_{n-1}}{c_n}\right|\right]^{-1}\neq0.$$

We apply the main theorem with $[a,b]=[0,1/R]$ and we get Theorem 3.1. □

**Theorem 3.2.** *Some of the processes $T$ accelerate all convergent $(S_n)\in\mathrm{TO}$ which satisfy* $\lim_{n\to\infty}S_n/S_{n-1}=-1$.

**Proof.** Because $(S_n)$ is convergent and $\lim_{n\to\infty}S_n/S_{n-1}=-1$, we have $\lim_{n\to\infty}\Delta S_n/\Delta S_{n-1}=-1$ which leads to $R=1/\overline{\lim}|\Delta S_n/\Delta S_{n-1}|=1$. Therefore application of Theorem 3.1 with $\tau=-1$, $R=1$ (thus $A=-3$, $|B|=3-\sqrt{8}$) leads to:

$$\overline{\lim_{n\to\infty}}\ |T(S_n)-S_n|^{1/n}\leqslant3-\sqrt{8},$$

$$\lim_{n\to\infty}|S_n-S|^{1/n}=\lim_{n\to\infty}|S_n|^{1/n}=1\quad(\text{since } S_n\to0,\ n\to\infty).$$

Thus

$$\lim_{n\to\infty}\left|\frac{T(S_n)-S}{S_n-S}\right|=\lim_{n\to\infty}\left|\frac{T(S_n)}{S_n}\right|=0.\quad\square$$

Before proving that $T$ accelerates all the convergent TO sequences, we need:

**Lemma 3.3.** *If $(S_n)\in\mathrm{TM}$, then $(-\Delta S_{n-1}/\rho^n)\in\mathrm{TM}$ where*

$$\rho=\lim_{n\to\infty}\frac{S_{n+1}-S}{S_n-S},\quad\rho\in\,]0,1].$$

**Proof.** It is based on the remark that the product of an MS sequence and a TM sequence is an MS sequence. (MS = the set of sequences of Stieltjes moments) and on the fact that a converging MS sequence is a TM sequence [6]. □

**Theorem 3.4.** *Some of the processes T accelerate all the convergent* TO *sequences*

**Proof.**

$$(S_n)_{n\geqslant 0}\in \mathrm{TO} \;\Rightarrow\; (-\Delta S_{n-1})_{n\geqslant 1}\in \mathrm{TO}$$
$$\Rightarrow\; \left((-1)^{n+1}\Delta S_{n-1}\right)_{n\geqslant 1}\in \mathrm{TM}.$$

If

$$\rho=\lim_{n\to\infty}\frac{S_{n+1}-S}{S_n-S}=\lim_{n\to\infty}\frac{S_{n+1}}{S_n},$$

we know that $\rho\in[-1,0[$ and we have

$$\rho=\lim_{n\to\infty}\frac{\Delta S_{n+1}}{\Delta S_n}\quad\Rightarrow\quad\left(\frac{-\Delta S_{n-1}}{\rho^n}\right)\in \mathrm{TM}\quad\text{(Lemma 3.3).}$$

Theorem 3.1 with $t=\rho$ gives

$$\overline{\lim_{n\to\infty}}\,|T(S_n)-S|^{1/n}=\overline{\lim_{n\to\infty}}\,|T(S_n)|^{1/n}\leqslant|B|$$

where $B=A+\sqrt{A^2-1}$ and $A=2/\rho-1<0$. (The $R$ in Theorem 3.1 is now $|t|/|\rho|=1$.) This, compared with $\lim_{n\to\infty}|S_n-S|^{1/n}=\lim_{n\to\infty}|S_n|^{1/n}=|\rho|$ shows that:

$$\overline{\lim_{n\to\infty}}\,\frac{|T(S_n)-S|}{S_n-S}=0$$

because

$$\forall\,\rho\in[-1,0[,\ |B|=-(2/\rho-1)-\sqrt{(2/\rho-1)^2-1}<|\rho|.\qquad\square$$

We obtain easily:

**Corollary 3.5.** *If $(S_n)_{n\geqslant 0}$ converges to $S$ and if two constants $a\neq 0$ and $b$ exist such that $(aS_n+b)\in$ TO, then $T(S_n)$ converges to $S$ faster than $(S_n)$.*

We also have, for TM sequences:

**Definition 3.6.** The sequence $(S_n)$ is said to be logarithmic if:

$$\rho=\lim_{n\to\infty}\frac{S_{n+1}-S}{S_n-S}=1.$$

**Theorem 3.7.** *If $(S_n)\in$ TM and $\rho=\lim_{n\to\infty}(S_{n+1}-S)/(S_n-S)\neq 1$, then $T$ accelerates $(S_n)$.*

**Proof.** We have $(-\Delta S_{n-1}/\rho^n)\in$ TM (Lemma 3.3).
Theorem 3.1 gives, with $t=\rho$

$$\overline{\lim_{n\to\infty}}\,|T(S_n)-S|^{1/n}\leqslant|B|\quad\text{where}\quad B=A-\sqrt{A^2-1},\ A=\frac{2}{\rho}-1>0.$$

This, compared with $\lim_{n\to\infty}|S_n-S|^{1/n}=\rho$ shows that $T$ accelerates $S_n$ (because

$$\frac{2}{\rho}-1-\sqrt{\left(\frac{2}{\rho}-1\right)^2-1}<\rho,\quad\forall\,\rho\in]0,1[).\qquad\square$$

**Example 3.8.** If $S_n = (0.9)^n/(n+1)$, $(S_n) \in \mathrm{TM}$ and $\rho = 0.9$

$$B = \frac{2}{\rho} - 1 - \sqrt{\left(\frac{2}{\rho} - 1\right)^2 - 1} = 0.52 < 0.9.$$

**Corollary 3.9.** *If* $(S_n)$ *converges to* $S$ *and if there exist two constants* $a \neq 0$ *and* $b$ *such that* $(aS_n + b) \in \mathrm{TM}$ *and* $\rho = \lim_{n\to\infty}(S_{n+1} - S)/(S_n - S) \neq 1$, *then* $T$ *accelerates* $(S_n)$.

And we have, at least:

**Theorem 3.10.** *If* $(S_n) = (au_n + bv_n)$ *where*

(i) $(u_n)$ *satisfies the conditions of Theorem* 3.7;

(ii) $(v_n) \in \mathrm{TO}$ *converges*;

(iii) $a, b \in \mathbb{R}^+$,

*then* $T$ *accelerates* $(S_n)$.

**Proof.** We only need to apply the main theorem of the second section where $[a, b] = [-\rho_2/\rho_1, 1]$ ($\rho_1$ and $\rho_2$ are, respectively, $\lim \sqrt[n]{u_n}$ and $\lim \sqrt[n]{v_n}$) and to observe that a linear combination with positive coefficients of non decreasing bounded functions is still a non decreasing bounded function. □

Theorem 3.10 can easily be generalized to finite linear combinations with positive coefficients of non logarithmic TM sequences or convergent TO sequences.

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Journal of Computational and Applied Mathematics 21 (1988) 133–144
North-Holland 

# Stieltjes- and Geronimus-type polynomials

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Received 17 December, 1984
Revised 15 December 1986 and 1 March 1987

*Abstract:* This work is an extension to the formal case of a previous work by Monegato [5].

A Stieltjes-type polynomial is a polynomial of degree $(n+1)$, $E_{n+1}$, orthogonal with respect to the functional $\{\tilde{c}_i = c(P_n(x)x^i),\ i = 0, 1, 2, \ldots\}$, where $c$ is a functional defined by the series $f(t) = \sum_{i=0}^{\infty} c_i t^i$. $E_{n+1}$ is of important use in the estimation of the error in Padé approximation. An iteration of the construction of $E_{n+1}$ is attempted. (Sections 1, 2, 3, 4).

In the last section, we study the properties of the polynomial $S_n$ associated with $E_{n+1}$ with respect to another functional $\tilde{c}$. $S_n$ is called a Geronimus-type polynomial. It is shown that $S_n$ verifies the system $c(P_n S_n G_k) = 0$ for $k = 1, \ldots, n$, where the $G_k$'s are orthogonal with respect to $\tilde{c}$.

## 1. Introduction

In [5], Monegato defines the Stieltjes polynomials $E_{n+1}(x)$ associated with a nonnegative weight function $p(x)$ on an interval $(a, b)$,

$$\int_a^b p(x)P_n(x)E_{n+1}(x)x^k \,\mathrm{d}x, \quad k = 0, 1, \ldots, n,$$

where $P_n(x)$ is the $n$th-degree polynomial orthogonal on $(a, b)$ with respect to $p(x)$. This class of polynomials plays an important role in the estimation of the error of Gaussian rules (see [4,5]). In the same paper, Monegato considers another special class of polynomials $\{S_n(X)\}$, introduced by Geronimus [3],

$$\int_a^b p(x)P_n(x)S_n(x)\cos(k \cdot \arccos(x))\,\mathrm{d}x = 0, \quad k = 1, 2, \ldots, n,$$

where $P_n(x)$ is the $n$th-degree polynomial orthogonal on $(a, b)$ with respect to $p(x)$.

In this paper we consider the case of orthogonal polynomials defined by a linear functional $c$ (see [1]) and define the corresponding Stieltjes and Geronimus polynomials. In particular we present properties of these latter polynomials and introduce some generalizations. The generalization of Stieltjes polynomials we consider, has been recently introduced by Brezinski in [2] to estimate the error in Padé approximation.

In this preliminary section, we recall some definitions and properties about Padé approximants and Padé-type approximants which can be found in [1,Ch. 1].

Let $f$ be a formal power series in one variable

$$f(t) = \sum_{i=0}^{\infty} c_i t^i, \quad c_i \in \mathbb{R}.$$

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Let $c$ be a linear functional acting on the space of real polynomials, defined by

$$c(x^i) = c_i, \quad i = 0, 1, 2, \ldots;$$

the number $c_i$ is called the moment of order $i$ of the functional $c$. We formally have [1],

$$f(t) = c\left((1-xt)^{-1}\right).$$

If $v$ is some polynomial, then

$$P(x) := \left(1 - v(x)/v(t^{-1})\right)/(1-xt)$$

is the Hermite interpolation polynomial of the function $x \to 1/(1-xt)$ at the zeros of $v$ and we get

$$f(t) - c(P(x)) = \frac{1}{v(t^{-1})} c\left(\frac{v(x)}{1-xt}\right) = \frac{t^n}{\tilde{v}(t)} c\left(\frac{v(x)}{1-xt}\right) = \mathrm{O}(t^n), \tag{1}$$

where $n$ = degree of $v$ and $\tilde{v}(t) = t^n v(t^{-1})$.

In the following, the $\tilde{}$ on top of a polynomial $p(t)$ of degree $d$ will have the meaning $\tilde{p}(t) = t^d p(t^{-1})$. Notice that $c(P)$ is a rational function of degree $(n-1/n)$ which coincides with the series $\sum c_i t^i$ up to the term $c_{n-1}t^{n-1}$. Indeed we have

$$c(P) = \tilde{w}(t)/\tilde{v}(t),$$

where

$$w(t) = c\left(\frac{v(x)-v(t)}{x-t}\right), \quad \deg w = n-1,$$

is the associated polynomial to $v$ with respect to $c$. We denote $c(\mathrm{P}) := (n-1/n)_f(t)$. $(n-1/n)_f$ is called a Padé-type approximant (PTA) with $v$ as generating polynomial.

To obtain an approximant with a higher order of approximation, we take $v$ such that

$$c\left(v(x)x^i\right) = 0 \quad \text{for } i = 0, \ldots, n-1.$$

In this case, if the Hankel determinant

$$H_n(c_0) = \begin{vmatrix} c_0 & \cdots & c_{n-1} \\ \vdots & & \vdots \\ c_{n-1} & \cdots & c_{n-2} \end{vmatrix}$$

is different from 0, then $v$ is the orthogonal polynomial of degree $n$ with respect to the functional $c$ and is denoted by $P_n$.

The relation (1) becomes:

$$\begin{aligned} f(t) - c\left(\frac{1 - P_n(x)/P_n(t^{-1})}{1-xt}\right) &= \frac{t^n}{\tilde{P}_n(t)} c\left(\frac{P_n(x)}{1-xt}\right) \\ &= \frac{t^{2n}}{P_{n(t)}} c\left(\frac{P_n(x)x^n}{1-xt}\right) = \frac{t^{2n}}{\tilde{P}_n^2(t)} c\left(\frac{P_n^2(x)}{1-xt}\right), \end{aligned}$$

$$c\left(\frac{1 - P_n(x)/P_n(t^{-1})}{1-xt}\right) = \frac{1}{P_n(t^{-1})} c\left(\frac{P_n(t^{-1}) - P_n(x)}{1-xt}\right) = \frac{\tilde{Q}_n(t)}{\tilde{P}_n(t)},$$

where

$$Q_n(t) = c\left(\frac{P_n(x) - P_n(t)}{x - t}\right)$$

of degree $(n-1)$ is the associated polynomial to $P_n$. The rational function $\tilde{Q}_n/\tilde{P}_n$ is denoted by $n-1/n_f(t)$ called a Padé approximant to $f$.

## 2. Stieltjes-type polynomials

In this section we will consider known result about Stieltjes polynomials defined by nonnegative weight functions, and extend them to the case of orthogonal polynomials defined by the linear functional $c$ (acting on the space of real polynomials) we have introduced in Section 1.

**Definition** The polynomial $E_{n+1}$ of degree $(n+1)$, orthogonal with respect to the functional $\tilde{c}$ defined by $\tilde{c}_i = c(P_n(x)x^i)$, $i \in \mathbb{N}$, is called Stieltjes-type polynomial. (See Monegato [5] and Stieltjes [6]).

If we set $\tilde{c}_i = c(P_n(x)x^i) i = 0, 1, 2, \ldots,$ $E_{n+1}$ satisfies

$$\tilde{c}\left(E_{n+1}(x)x^k\right) = 0, \quad k = 0, 1, \ldots, n \iff c\left(E_{n+1}(x)P_n(x)x^k\right) = 0, \qquad k = 0, 1, 2, \ldots, n. \tag{2}$$

*Existence and unicity of $E_{n+1}$*

**Theorem 1.** *$E_{n+1}$ is uniquely determined by* (2) *if and only if $c(P_n^2) \neq 0$.*

**Proof.** $E_{n+1}$ is assumed to be monic, i.e., $E_{n+1} = x^{n+1} + e_n^{(n+1)}x^n + \cdots + e_0^{(n+1)}$. The system (2) is triangular and the diagonal term is $c(x^n P_n(x)) = c(P_n^2(x))$. $\square$

The associated polynomial to $E_{n+1}$ with respect to

$$\tilde{c}_k = c\left(P_n(x)x^k\right)$$

is

$$\tilde{c}\left(\frac{E_{n+1}(x) - E_{n+1}(t)}{x - t}\right) = c\left(P_n(x)\frac{E_{n+1}(x) - E_{n+1}(t)}{x - t}\right) = c\left(P_n(x)x^n\right) = c\left(P_n^2\right) := h_n.$$

So, $h_n t^n/\tilde{E}_{n+1}(t)$ is the Padé Approximant of $q_n(t) := c(P_n(x)/(1-xt))$

$$[n/n+1]_{q_n}(t) = h_n t^n/\tilde{E}_{n+1}(t).$$

**Property 1.** *The relation*

$$\left[c(P_n(x)/(t-x))\right]^{-1} = \frac{1}{h_n}E_{n+1}(t) + \frac{a_1^{(n)}}{t} + \frac{a_2^{(n)}}{t^2} + \cdots \tag{3}$$

*holds. Thus $(1/h_n)E_{n+1}(t)$ is the entire part of the function $[c(P_n(x)/(t-x))]^{-1}$*

**Proof.** Since $h_n t^n/\tilde{E}_{n+1}(t) = [n/n+1]_{q_n}(t)$, we have $h_n t^n/\tilde{E}_{n+1}(t) - q_n(t) = O(t^{2n-2})$ and $h_n/E_{n+1}(t) - t^{-1}q_n(t^{-1}) = O(t^{-2n-3})$. So,

$$1/t^{-1}q_n(t^{-1}) - (1/h_n)E_{n+1}(t) = O(t^{-1})$$

which is (3), since $t^{-1}\,q_n(t^{-1}) = c(P_n(x)/(t-x))$. □

*Calculation of $E_{n+1}$*

Unfortunately, the sequence $(E_{n+1})_{n\in\mathbb{N}}$ does not satisfy a three-term recurrence relation [5]. So we must calculate $E_{n+1}$ by solving the triangular system (2)

$$c(P_n(x)E_{n+1}(x)x^k) = 0, \quad k = 0,\dots,n.$$

If we set

$$E_{n+1}(x) = x^{n+1} + e_n^{(n+1)}x^n + e_{n-1}^{(n+1)}x^{n-1} + \cdots + e_0^{(n+1)},$$

then (2) is equivalent to

$$\begin{cases} e_n^{(n+1)}\alpha_n^{(n)} & = -\alpha_{n+1}^{(n)}, \\ e_{n-1}^{(n+1)}\alpha_n^{(n)} + e_n^{(n+1)}\alpha_{n+1}^{(n)} & = -\alpha_{n+2}^{(n)}, \\ \vdots & \vdots \\ e_0^{(n+1)}\alpha_n^{(n)} + e_1^{(n+1)}\alpha_{n+1}^{(n)} + \cdots + e_n^{(n+1)}\alpha_{2n}^{(n)} & = -\alpha_{2n+1}^{(n+1)}; \end{cases}$$

where $\alpha_k^{(n)} = c(x^k P_n(x))$, $\alpha^{(n)}{}_k = 0$ if $k < n$.

The quantities $\alpha_k^{(n)}$ may be calculated recursively (see [1,2].)

*Integral representation of $E_{n+1}$*

**Theorem 2.** *If $x/q_n(x^{-1})$ is holomorphic in the exterior of a path $C$, except at infinity, and $t$ in the interior of $C$, then*

$$E_{n+1}(t) = \frac{1}{2\mathrm{i}\pi}h_n\int_C \frac{x}{q_n(x^{-1})(x-t)}\,\mathrm{d}x.$$

**Proof.** Using

$$1/t^{-1}q_n(t^{-1}) = (1/h_n)E_{n+1}(t) + O(t^{-1}),$$

we get

$$\int_C \frac{x}{q_n(x^{-1})(x-t)}\,\mathrm{d}x = \frac{2\mathrm{i}\pi}{h_n}E_{n+1}(t).$$

*Determinantal expression of $E_{n+1}$*

(1) Since $E_{n+1}$ is an orthogonal polynomial of degree $(n+1)$ with respect to $\tilde{c}$, we have

$$E_{n+1}(x) = \begin{vmatrix} 0 & & \cdots & 0 & c(P_n x^n) & c(P_n x^{n+1}) \\ \vdots & & & & \vdots & \vdots \\ 0 & & & & & \\ c(P_n x^n) & & & & c(P_n x^{2n}) & c(P_n x^{2n+1}) \\ 1 & x & x^2 & \cdots & x^n & x^{n+1} \end{vmatrix} \times \frac{1}{(h_n)^n}$$

(2) We have also

$$h_n^n E_{n+1}(x)=\begin{vmatrix} c(P_n^2) & c(x^nP_1P_n) & \cdots & c(x^nP_nP_k) & \cdots & c(x^nP_nP_{n-1}) & c(x^nP_nP_{n+1}) \\ 0 & c(P_n^2) & & c(x^{n-1}P_nP_k) & & & \\ 0 & 0 & & \vdots & & & \\ & 0 & & c(x^{n-k+1}P_nP_k) & & & \\ & & & c(P_n^2) & & & \\ & & & 0 & & & \\ & & & 0 & & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots & c(x^2P_nP_{n+1}) \\ 0 & 0 & & 0 & & c(P_n^2) & c(xP_nP_{n+1}) \\ P_0 & P_1 & & P_k & \cdots & P_{n-1} & P_{n+1} \end{vmatrix}$$

Proof. Multiplying $E_{n+1}(x)$ by $x^kP_n(x)$, and applying the functional $c$ shows that two rows are identical for $k=0,1,\ldots,n$.

## 3. Padé-type approximant $(2n/2n+1)_f$ with $P_nE_{n+1}$ as generating polynomial

Recently, Brezinski [2] used a method due to Kronrod [4] to estimate the error $f(t)-[n-1/n]_f(t)$ in Padé approximation. More details about this method can be found in [2]. Here we recall the main idea.

If we take

$$v(x)=P_n(x)E_{n+1}(x)$$

in relation (1), then

$$c\big(v(x)x^k\big)=0 \quad \text{for } k=0,\ldots,n$$

and

$$f(t)-(2n/2n+1)_f(t)=\frac{t^{2n+1}}{\tilde{v}(t)}c\left(\frac{v(x)}{1-xt}\right)=\frac{t^{3n+2}}{\tilde{v}(t)}c\left(\frac{v(x)x^{n+1}}{1-xt}\right).$$

The associated polynomial to $v$, of degree $2n$, is

$$w(t)=c\left(\frac{v(x)-v(t)}{x-t}\right)=c\left(\frac{P_n(x)E_{n+1}(x)-P_n(t)E_{n+1}(t)}{x-t}\right),$$

which can be put in the following form:

$$w(t)=c\big(P_n^2(x)\big)+E_{n+1}(t)Q_n(t).$$

Thus we have

$$\begin{aligned}(2n/2n+1)_f(t)&=\frac{h_nt^{2n}+\tilde{E}_{n+1}(t)\tilde{Q}_n(t)}{\tilde{P}_n(t)\tilde{E}_{n+1}(t)}\\ &=\frac{h_nt^{2n}}{\tilde{P}_n(t)\tilde{E}_{n+1}(t)}+\frac{\tilde{Q}_n(t)}{\tilde{P}_n(t)}=\frac{h_nt^{2n}}{\tilde{P}_n(t)\tilde{E}_{n+1}(t)}+[n-1/n]_f(t).\end{aligned}$$

From this expression we obtain

$$(2n/2n+1)_f(t) - [n-1/n]_f(t)$$
$$= f(t) - [n-1/n]_f(t) - \frac{t^{3n+2}}{\tilde{P}_n(t)\tilde{E}_{n+1}(t)} c\left(\frac{P_n(x)E_{n+1}(x)x^{n+1}}{1-xt}\right)$$

and

$$(2n/2n+1)_f(t) - [n-1/n]_f(t) = f(t) - [n-1/n]_f(t) + O(t^{3n+2})$$

Thus $(2n/2n+1)_f(t)$ can be used to estimate the error $f(t)-[n-1/n]_f(t)$ in Padé approximation.

## 4. Iteration

In this part, we try to iterate the method of Section 3.

Let us write the error in the Padé Approximation (PA)

$$f(t) - [n-1/n]_f(t) = \frac{t^n}{\tilde{P}_n(t)} c\left(\frac{P_n(x)}{1-xt}\right)$$

and replace $c(P_n(x)/(1-xt)) := q_n(t)$ by its PA

$$[n/n+1]_{q_n}(t) = h_n t^n/\tilde{E}_{n+1}(t).$$

We obtain

$$f(t) - [n-1/n]_f(t) = \frac{t^n}{\tilde{P}_n(t)}\left([n/n+1]_{q_n}(t) + \frac{t^{n+1}}{\tilde{E}_{n+1}(t)} c\left(\frac{E_{n+1}(x)P_n(x)}{1-xt}\right)\right)$$

and

$$f(t) - [n-1/n]_f(t) - \frac{h_n t^{2n}}{\tilde{P}_n(t)\tilde{E}_{n+1}(t)} = \frac{t^{2n+1}}{\tilde{P}_n(t)\tilde{E}_{n+1}(t)} c\left(\frac{E_{n+1}(x)P_n(x)}{1-xt}\right).$$

The idea for iterating the process of construction of $E_{n+1}$ consists in replacing

$$c(E_{n+1}(x)P_n(x)/1-xt) := q_n^{(1)}(t)$$

by its PA $[(n+1)/(n+2)]_{q_n^{(1)}}(t)$ if it exists.

We are lead to find some polynomial $E_{n+2}^{(1)}$ such that

$$c\left(E_{n+1}(x)P_n(x)E_{n+2}^{(1)}(x)x^k\right) = 0 \quad \text{for } k = 0, \ldots, n+1. \tag{4}$$

*Existence and unicity of $E_{n+2}^{(1)}$*

Let us write $E_{n+2}^{(1)} = x^{n+2} + f_{n+1}x^{n+1} + \cdots + f_0$. (Each $f_i$, $i = 0, \ldots, n+1$, depends upon $n$.)
System (4) is triangular with diagonal terms

$$c\left(E_{n+1}(x)P_n(x)x^{n+1}\right).$$

Hence we can state the following theorem.

**Theorem 2.** *The system*

$$c\left(E_{n+1}(x)P_n(x)E_{n+2}^{(1)}(x)x^k\right)=0,\quad k=0,\dots,n+1$$

*has a unique solution* $E_{n+2}^{(1)}$ *of degree* $(n+2)$ *if and only if*

$$c\left(E_{n+1}(x)P_n(x)x^{n+1}\right)\neq 0.$$

*Calculation of* $E_{n+2}^{(1)}$

The $f_i$'s, $i=0,\dots,(n+1)$, satisfy the triangular system

$$\begin{aligned}
&\beta_0^{(n)}f_{n+1}+\beta_1^{(n)} &&=0,\\
&\beta_1^{(n)}f_{n+1}+\beta_0^{(n)}f_n+\beta_2^{(n)} &&=0,\\
&\vdots &&\vdots\\
&\beta_{n+1}^{(n)}f_{n+1}+\beta_n^{(n)}f_n+\cdots+\beta_0^{(n)}f_0+\beta_{n+2}^{(n)} &&=0,
\end{aligned}$$

where

$$\beta_i^{(n)}=c\left(E_{n+1}(x)P_n(x)x^{n+1+i}\right)=\alpha_{2n+2+i}^{(n)}+\sum_{k=0}^{n}e_k^{(n)}\alpha_{n+1+i+k}^{(n)}.$$

If the quantities $a_i^{(n)}$ are known in (3), then the above formulas allow us to calculate the $\beta_i^{(n)}$'s. From

$$\frac{1}{t^{-1}q_n(t^{-1})}=\frac{1}{h_n}E_{n+1}(t)+\frac{a_1^{(n)}}{t}+\frac{a_2^{(n)}}{t^2}+\cdots+\frac{a_i^{(n)}}{t^i}+\cdots,$$

where

$$t^{-1}q_n(t^{-1})=c\left(\frac{P_n(x)}{t-x}\right)=\sum_{k=0}^{\infty}\alpha_{n+k}^{(n)}t^{-k-n-1},$$

we deduce

$$\left\{\begin{aligned}
&a_1^{(n)}\alpha_n^{(n)}+h_n^{-1}\left(\alpha_{n+1}^{(n)}e_0^{(n)}+\alpha_{n+2}^{(n)}e_1^{(n)}+\cdots+e_n^{(n)}\alpha_{2n+1}^{(n)}+\alpha_{2n+2}^{(n)}\right)=0,\\
&a_2^{(n)}\alpha_n^{(n)}+a_1^{(n)}\alpha_{n+1}^{(n)}+h_n^{-1}\left(\alpha_{n+2}^{(n)}e_0^{(n)}+\cdots+e_n^{(n)}\alpha_{2n+2}^{(n)}+\alpha_{2n+3}^{(n)}\right)=0,\\
&\vdots\\
&a_i^{(n)}\alpha_n^{(n)}+a_{i-1}^{(n)}\alpha_{n+1}^{(n)}+\cdots+a_1^{(n)}\alpha_{n+i-1}\\
&\quad+h_n^{-1}\left(e_0^{(n)}\alpha_{n+i}^{(n)}+\cdots+e_n^{(n)}\alpha_{2n+i}^{(n)}+\alpha_{2n+i+1}^{(n)}\right)=0\\
&\vdots
\end{aligned}\right.$$

$$\Leftrightarrow\left\{\begin{aligned}
&a_1^{(n)}\alpha_n^{(n)}+h_n^{-1}\beta_0^{(n)}=0,\\
&a_2^{(n)}\alpha_n^{(n)}+a_1^{(n)}\alpha_{n+1}^{(n)}+h_n^{-1}\beta_1^{(n)}=0,\\
&\vdots\\
&a_{n+3}^{(n)}\alpha_n^{(n)}+a_{n+2}^{(n)}\alpha_{n+1}^{(n)}+\cdots+a_1^{(n)}\alpha_{2n+2}+h_n^{-1}\beta_{n+2}^{(n)}=0,
\end{aligned}\right.$$

hence

$$\beta_0^{(n)} = -h_n \alpha_n^{(n)} a_1^{(n)},$$
$$\beta_1^{(n)} = -h_n\left(\alpha_n^{(n)} a_2^{(n)} + \alpha_{n+1}^{(n)} a_1^{(n)}\right),$$
$$\vdots$$
$$\beta_{n+2} = -h_n\left(a_{n+3}^{(n)}\alpha_n^{(n)} + a_{n+2}^{(n)}\alpha_{n+1}^{(n)} + \cdots + a_1^{(n)}\alpha_{2n+2}^{(n)}\right).$$

*Padé-type approximant* $(3n+2)/(3n+3)$ *with* $P_n E_{n+1} E_{n+2}^{(1)}$ *as generating polynomial*

If we take, in (1)

$$v(x) = P_n(x) E_{n+1}(x) E_{n+2}^{(1)}(x), \quad \text{degree } v = 3n+3,$$

then $c(v(x)x^k) = 0$ for $k = 0,\ n+1$ and

$$f(t) - \left(\frac{3n+2}{3n+3}\right)_f(t) = \frac{t^{3n+3}}{\tilde{v}(t)} c\left(\frac{v(x)}{1-xt}\right) = \frac{t^{4n+5}}{\tilde{v}(t)} c\left(\frac{v(x)x^{n+2}}{1-xt}\right).$$

The associated polynomial to $v$ is

$$w(t) = c\left(\frac{P_n(x)E_{n+1}(x)E_{n+2}^{(1)}(x) - P_n(t)E_{n+1}(t)E_{n+2}^{(1)}(t)}{x-t}\right)$$
$$= c\left(P_n(x)E_{n+1}(x)\frac{E_{n+2}^{(1)}(x) - E_{n+2}^{(1)}(t)}{x-t} + P_n(x)E_{n+2}^{(1)}(t)\frac{E_{n+1}(x) - E_{n+1}(t)}{x-t}\right.$$
$$\left. + E_{n+1}(t)E_{n+2}^{(1)}(t)\frac{P_n(x) - P_n(t)}{x-t}\right.$$
$$= c\left(P_n(x)E_{n+1}(x)x^{n+1}\right) + E_{n+2}^{(1)}(t)c\left(P_n^2(x)\right) + E_{n+1}(t)E_{n+2}^{(1)}(t)Q_n(t),$$

i.e,

$$w(t) = \beta_0^{(n)} + h_n E_{n+2}^{(1)}(t) + E_{n+1}(t)E_{n+2}^{(1)}(t)Q_n(t) \in P_{3n+2}.$$

We deduce

$$\left(\frac{3n+2}{3n+3}\right)_f(t) = \frac{\tilde{w}(t)}{\tilde{v}(t)} = \frac{\beta_0^{(n)}t^{3n+2} + h_n\tilde{E}_{n+2}^{(1)}(t)t^{2n} + \tilde{E}_{n+1}(t)\tilde{E}_{n+2}^{(1)}(t)\tilde{Q}_n(t)}{\tilde{P}_n(t)\tilde{E}_{n+1}(t)\tilde{E}_{n+2}^{(1)}(t)},$$

$$\left(\frac{3n+2}{3n+3}\right)_f(t) = \beta_0^{(n)}\frac{t^{3n+2}}{\tilde{P}_n(t)\tilde{E}_{n+1}(t)\tilde{E}_{n+2}^{(1)}(t)} + \frac{h_n t^{2n}}{\tilde{P}_n(t)\tilde{E}_{n+1}(t)} + \left[\frac{n-1}{n}\right]_f(t),$$

and the error in the Padé approximation becomes

$$f(t) - \left[\frac{n-1}{n}\right]_f(t) = \left(\frac{3n+2}{3n+3}\right)_f(t) - \left[\frac{n-1}{n}\right]_f(t) + \frac{t^{4n+5}}{\tilde{v}(t)} c\left(\frac{v(x)x^{n+2}}{1-xt}\right).$$

## 5. Geronimus-type polynomials

Consider the Stieltjes polynomial $E_{n+1}$ defined, for example, by

$$c\left(P_n(x)E_{n+1}(x)x^k\right) = 0, \quad k = 0, 1, \ldots, n.$$

Let $S_n$ be the polynomial of degree $n$ associated with $E_{n+1}$ with respect to a functional $\bar{c}$ with

$$g(t)=\bar{c}\left((1-xt)^{-1}\right)=\sum_{i=0}^{\infty}\bar{c}_i t^i$$

and defined by the relation

$$S_n(t)=\bar{c}\left(\frac{E_{n+1}(x)-E_{n+1}(t)}{x-t}\right);$$

we have

$$\tilde{S}_n(t)/\tilde{E}_{n+1}(t)=(n/(n+1))_g(t).$$

Since the case $g(t)=(1-t^2)^{-1/2}$ was treated by Geronimus [3], following Monegato we call $S_n$ a Geronimus-type polynomial.

*Fundamental properties of $S_n$*

**Property 1.** *If the functional $\bar{c}$ is definite (i.e. $H_0(\bar{c}_n)\neq 0$, $\forall n$) and $\{G_k\}_{k\in\mathbb{N}}$ the family of monic orthogonal polynomials with respect to $\bar{c}$, then*

$$c(P_n(x)S_n(x)G_k(x))=0 \quad \textit{for } k=1 \textit{ to } n.$$

**Proof.** Since $E_{n+1}\in P_{n+1}$ and $G_k\in P_k$, we can write

$$E_{n+1}=G_{n+1}+\gamma_nG_n+\gamma_{n-1}G_n+\cdots+\gamma_1G_1+\gamma_0G_0 \quad \text{for some } \gamma_i.$$

So, $S_n=H_{n+1}+\gamma_nH_n+\cdots+\gamma_1H_1$ where

$$H_k(t)=\bar{c}\left(\frac{G_k(x)-G_k(t)}{x-t}\right), \quad \bar{c} \text{ acts on } x;$$

$$\deg H_k=k-1, \quad H_0=0.$$

To prove the property it is sufficient to evaluate the quantity

$$\begin{aligned} &c(P_nS_nG_k)-c(P_nE_{n+1}H_k)\\ &\quad=c\left(P_n(G_kH_{n+1}+\cdots+G_k\gamma_1H_1)-P_n(H_kG_{n+1}+\cdots+\gamma_1G_1H_k+\gamma_0H_k)\right)\\ &\quad=c(P_n(G_kH_{n+1}-H_kG_{n+1}))+\gamma_nc(P_n(G_kH_n-H_kG_n))\\ &\qquad+\cdots+\gamma_1c(P_n(G_kH_1-H_kG_1))-\gamma_0c(P_nG_0H_k)=0. \end{aligned}$$

Indeed, since $\deg(G_kh_i-H_kG_i)=|i-k-1|\leqslant(n-k)<n$ for $k=1,\ n$, we have

$$c(P_nS_nG_k)=c(P_nE_{n+1}H_k)=0 \quad \text{for } k=1,\ n. \qquad \square$$

**Property 2.** *We have*

(i) $$\frac{g(t^{-1})}{q_n(t^{-1})}=h_n^{-1}S_n(t)+\frac{b_1^{(n)}}{t}+\frac{b_2^{(n)}}{t^2}+\cdots$$

*where*

$$q_n(t)=c(P_n(x)/(1-xt)), \qquad g(t)=\bar{c}\left((1-xt)^{-1}\right).$$

(ii) *If $g(x^{-1})/q_n(x^{-1})$ is holomorphic in the exterior of a path $C$, except at infinity, with $t$ in the interior of $C$. Then*

$$S_n(t)=\frac{h_n}{2i\pi}\int_C\frac{g(x^{-1})}{q_n(x^{-1})(x-t)}\,dx.$$

**Proof.** $S_n$ is the associated polynomial to $E_{n+1}$ with respect to $\tilde{c}$, so we have

$$\tilde{S}_n(t) - g(t)\tilde{E}_{n+1}(t) = O(t^n).$$

By property 1 (3)

$$E_{n+1}(t) = h_n\left[\frac{t}{q_n(t^{-1})} - \frac{a_1^{(n)}}{t} - \frac{a_2^{(n)}}{t^2}\cdots\right],$$

$$\frac{1}{h_n}G(t)E_{n+1}(t) = \frac{tG(t)}{q_n(t^{-1})} + O(t^{-2}),$$

where $G(t) = t^{-1}g(t^{-1})$. We obtain

$$\frac{1}{h_n}S_n(t) = \frac{G(t)}{t^{-1}q_n(t^{-1})} + O(t^{-1}) = \frac{g(t^{-1})}{q_n(t^{-1})} + O(t^{-1}). \qquad \square$$

By computing a residue, we get (ii).

*Determinantal expression of $S_n$*

$$h_n^n S_n(x) = \begin{vmatrix} c(P_n^2) & c(P_nP_1G_n) & & c(P_nP_kG_n) & & c(P_nP_{n-1}G_n) & c(P_n^2G_n) \\ 0 & c(P_n^2) & & \vdots & & & \\ 0 & 0 & \ddots & c(P_nP_kG_{n-k+1}) & & & \\ & & & c(P_n^2) & & & \\ & & & 0 & \ddots & & \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & & 0 & \cdots\ 0 & c(P_n^2) & c(P_n^2G_1) \\ P_0 & P_1 & & P_k & & P_{n-1} & P_n \end{vmatrix}$$

**Proof.** Multiplying the last row by $P_nG_k$, $k = 1, \ldots, n$, and applying the functional $c$, we get

$$c(P_n(x)S_n(x)G_k(x)) = 0 \quad \text{for } k = 1, \ldots, n$$

since two rows of the determinant are identical. $\square$

*Nonexistence of a three-term recurrence relation for $\{S_n\}_{n\in\mathbb{N}}$*

**Theorem.** *Let $(P_n)_{n\in\mathbb{N}}$ be an orthogonal polynomial system* (OPS) *with respect to c,*

$$\begin{cases} P_{n+1}(x) = (x + B_{n+1})P_n(x) - C_{n+1}P_{n-1}(x), \\ P_0 = 1, \qquad P_{-1} = 0. \end{cases}$$

*Let $(S_n)$ be a family of polynomials satisfying some three-term recurrence relation*

$$\begin{cases} S_{n+1}(x) = (x + \mu_{n+1})S_n - \delta_{n+1}S_{n+1}, \quad n \in \mathbb{N}, \\ S_0 = 1, \quad S_{-1} = 0. \end{cases}$$

*Then, if $G_k$ is some polynomial of degree k,*

$$c(P_nS_nG_k) = 0 \quad \textit{for } k = 1, 2, 3, 4,$$

*implies*

$$\mu_n = B_{n+1}, \quad n \geqslant 2; \qquad B_n = B_{n+2}, \quad n \geqslant 3;$$
$$\delta_n = C_{n+2}, \quad n \geqslant 3; \qquad C_{n+1} = C_{n+3}, \quad n \geqslant 5.$$

**Proof.** By taking
$k = 1$, we get $\mu_n = B_{n+1}$, $n \geqslant 2$;
$k = 2$, $c_{n+2} = \delta_n$, $n \geqslant 3$;
$k = 3$, $B_n = B_{n+2}$ $n \geqslant 3$; and for
$k = 4$, $C_{n+1} = C_{n+3}$, $n \geqslant 5$.

So, in the general case, the $\{S_n\}_{n\in\mathbb{N}}$ do not satisfy a three-term recurrence relation. Let us now examine some particular cases.

*First particular case*

**Theorem.** *Let $(P_n)_{n\in\mathbb{N}}$ an OPS with respect to $c$, assumed to be definite.*
*Let $(S_n)_{n\in\mathbb{N}}$ be the unique family of monic polynomials such that $c(P_nS_nG_k) = 0$, $k = 1,\ldots,n$, where the $G_k$'s are the orthogonal polynomials with respect to $\bar{c}$ definite, then*

$$S_n \equiv P_n \quad \forall n \in \mathbb{N} \iff \begin{cases} G_{n+1}(x) = (x+\beta)G_n(x) - \gamma G_{n-1}(x) & \forall n \geqslant 1, \\ G_0 = 2, \qquad G_1 = x + \beta \quad \textit{and} \\ P_{n+1}(x) = (x+\beta)P_n(x) - \gamma P_{n-1}(x) & \forall n \geqslant 1, \\ P_0 = C/\gamma, \qquad P_1 = x + B, \\ B \textit{ and } C \textit{ being arbitrary constants of } \mathbb{R}. \end{cases}$$

**Proof.** We have $P_{n+1}(x) = (x + B_{n+1})P_n(x) - C_{n+1}P_{n-1}(x)$,

$$C_{n+1} \neq 0 \quad \forall n \geqslant 1, \qquad P_{-1} = 0, \quad P_0 = 1$$

and

$$G_{n+1}(x) = (x + \beta_{n+1})G_n(x) - \gamma_{n+1}G_{n-1}(x), \quad n \geqslant 0$$
$$\gamma_{n+1} \neq 0 \quad \forall n \geqslant 1, \qquad G_0 = 1, \quad G_{-1} = 0.$$
$$S_n \equiv P_n \quad \forall n \in \mathbb{N} \iff c\left(P_n^2G_k\right) = 0 \quad \forall k = 1,\ldots,n.$$

By induction, we prove that

$$B_{n+1} = \beta_{n+1} = \beta_1 := \beta \quad \forall n \geqslant 1$$

and

$$C_{n+1} = \gamma_{n+1} = \gamma_3 = \tfrac{1}{2}\gamma_2 := \gamma \quad \forall n \geqslant 2,$$

$C_2$ and $B_1$ taking arbitrary values. □

*Second particular case: $c = \bar{c}$*

If the two functionals $c$ and $\bar{c}$ are identical, then by using the determinantal expression of

$E_{n+1}$ and $S_n$ we get the identity

$$\begin{vmatrix} c(P_n^2) & c(x^n P_1 P_n) & & c(x^n P_n P_{n-1}) & c(x^n P_n P_{n+1}) \\ & c(P_n^2) & & & \\ & 0 & \ddots & & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots\ 0 & c(P_n^2) & c(xP_n P_{n+1}) \\ Q_0 & Q_1 & & Q_{n-1} & Q_{n+1} \end{vmatrix}$$

$$= \begin{vmatrix} c(P_n^2) & c(P_n^2 P_1) & & & c(P_n^3) \\ 0 & c(P_n^2) & & & c(P_n^2 P_{n-1}) \\ & 0 & \ddots & & \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \dots\ 0 & c(P_n^2) & c(P_n^2 P_1) \\ P_0 & P_1 & & & P_n \end{vmatrix} \times c(P_0^2).$$

where

$$Q_i(t) = c\left(\frac{P_i(t) - P_i(x)}{t - x}\right)$$

is the associated polynomial of $P_i$ (notice that $Q_0 = 0$).

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Journal of Computational and Applied Mathematics 19 (1987) 89–98
North-Holland


# Calculation of poles of meromorphic functions with q-d, r-s and $\epsilon$-algorithms. Acceleration of these processes

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Received 9 December 1985

*Abstract:* We study (Sections 1 and 2) the speed of convergence of the q-d algorithm to poles of a meromorphic function in order to accelerate it.

In Section 3 and 4, we show that the quotient of two successive vertical terms of two well known algorithms (r-s and $\epsilon$) approaches also these poles. At last, a theorem of acceleration is given.

*Keywords:* q-d, r-s and $\epsilon$-algorithm, meromorphic function, acceleration.

## Introduction

The calculation of poles of a meromorphic function is based on the fact that the quotient of two successive Hankel determinants of order $k$ converges to the product of the first $k$ poles of this function.

With Lemma 1, we will show that this convergence is linear (see Definition 1) and so, Aitken's $\delta^2$ will accelerate this sequence.

**Lemma 1** [4, p. 218]. *Let* $(u_n)_{n\in\mathbb{N}}$ *a sequence of complex numbers such that*:

$$u_{n+1} \neq u_n \quad \forall n > N.$$

*Let* $\lambda \in \mathbb{C}$, $|\lambda| \neq 1$, *then*:

$$\left.\begin{array}{l} \exists u \in \mathbb{C} \text{ such that} \\ \lim\limits_{n\to\infty} \dfrac{u_{n+1}-u}{u_n-u} = \lambda \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \lim\limits_{n\to\infty} \dfrac{\Delta u_{n+1}}{\Delta u_n} = \lambda \\ \text{where } \Delta u_k = u_{k+1} - u_k,\ k = 0, 1, \ldots \end{array}\right.$$

**Definition 1.** If $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty}(u_{n+1}-u)/(u_n-u) = \lambda$, $\lambda \neq 0$ and $\lambda \neq 1$ then the sequence $(u_n)_{n\in\mathbb{N}}$ is said to be linearly convergent and $\lambda$ is the speed of convergence of $(u_n)_{n\in\mathbb{N}}$.

**Lemma 2** (Product of two linear sequences). *If* $(u_n)_{n\in\mathbb{N}}$ *and* $(v_n)_{n\in\mathbb{N}}$ *are two linear sequences and*

$$\lim_{n\to\infty}(u_{n+1}-u)(u_n-u)^{-1}=\lambda,\qquad \lim_{n\to\infty}(v_{n+1}-v)(v_n-v)^{-1}=\mu,$$

*then*

$$|\lambda|>|\mu| \Rightarrow \lim_{n\to\infty}(u_{n+1}v_{n+1}-uv)(u_nv_n-uv)^{-1}=\lambda.$$

**Lemma 3** (Inverse of a linear sequence). *If* $(u_n)_{n\in\mathbb{N}}$ *is a linear sequence convergent to* $u\neq 0$ *then* $(u_n^{-1})_{n\in\mathbb{N}}$ *is a linear sequence. Moreover,*

$$\lim_{n\to\infty}(u_{n+1}^{-1}-u^{-1})(u_n^{-1}-u^{-1})^{-1}=\lim_{n\to\infty}(u_{n+1}-u)(u_n-u)^{-1}.$$

**Corollary 1.** *If* $(u_n)_{n\in\mathbb{N}}$ *and* $(v_n)_{n\in\mathbb{N}}$ *are two linear sequences such that* $\lim_{n\to\infty}(v_n)\neq 0$ *and* $|\lambda|\neq|\mu|$, *then the sequence* $(u_n/v_n)_{n\in\mathbb{N}}$ *is linear.*

## 1. The speed of convergence of the q-d algorithm

Theorem 1 is an improvement of Gragg's result on the speed of convergence of the quotient of two consecutive Hankel determinants, to the poles of a meromorphic function.

**Theorem 1.** *Let f be a meromorphic function for* $|z|<R$, $0<R\leqslant+\infty$. *For sufficiently small z*

$$f(z)=c_0+c_1z+c_2z^2+\cdots.$$

*Let f have poles* $\pi_n$ *ordered*:

$$0<|\pi_1|\leqslant|\pi_2|\leqslant\cdots<R,$$

*each pole being counted with proper multiplicity. Let N the number of poles* $0\leqslant N\leqslant+\infty$. *If* $k<N-1$ *and* $|\pi_k|<|\pi_{k+1}|$ (*where* $\pi_0:=0$) *then*

$$\lim_{n\to\infty}\frac{H_k^{(n)}}{H_k^{(n+1)}}=\pi_1\pi_2\cdots\pi_k \quad (\textit{see } [5,6,8]), \tag{1}$$

*where* $H_k^{(n)}$ *is the classical Hankel determinant.*

*Moreover, if*

$$1\leqslant k\leqslant N-2 \quad \textit{and} \quad |\pi_{k-1}|<|\pi_k|<|\pi_{k+1}|<|\pi_{k+2}|$$

($\pi_0:=0$), *then the convergence of the sequence*

$$\left(H_k^{(n)}/H_k^{(n+1)}\right)_{n\in\mathbb{N}} \quad \textit{to } \pi_1\pi_2\cdots\pi_k$$

*is linear. I.e.,*

$$\lim\frac{H_k^{(n)}/H_k^{(n+1)}-\pi_1\pi_2\cdots\pi_k}{H_k^{(n-1)}/H_k^{(n)}-\pi_1\pi_2\cdots\pi_k}=\frac{\pi_k}{\pi_{k+1}}. \tag{2}$$

**Remark.** Since $f$ is meromorphic, the three sequences $(H_{k-1}^{(n)})_{n\in\mathbb{N}}$, $(H_k^{(n)})_{n\in\mathbb{N}}$, $(H_{k+1}^{(n)})_{n\in\mathbb{N}}$ are not zero from a certain rank [6, p. 605].

**Proof.** By using Lemma 1, we only need to prove

$$\lim_{n\to\infty} \Delta\left(H_k^{(n)}/H_k^{(n+1)}\right)\left[\Delta\left(H_k^{(n-1)}/H_k^{(n)}\right)\right]^{-1} = \rho, \quad |\rho| \neq 1,$$

where $\Delta$ acts on the index $n$.

With Jacobi's identity between Hankel determinants, we obtain

$$\Delta\left(H_k^n/H_k^{n+1}\right) = -\left(H_{k-1}^{(n+2)}\cdot H_{k+1}^{(n)}\right)/\left(H_k^{(n+1)}\cdot H_k^{(n+2)}\right),$$

and

$$\begin{aligned}\Delta\left(H_k^n/H_k^{n+1}\right)&\cdot\left[\Delta\left(H_k^{n-1}/H_k^n\right)\right]^{-1}\\ &= \left(H_{k-1}^{n+2}/H_{k-1}^{n+1}\right)\left(H_k^n/H_k^{n+1}\right)\cdot\left(H_k^{n+1}/H_k^{n+2}\right)\left(H_{k+1}^n/H_{k+1}^{n-1}\right).\end{aligned}$$

By applying (1) twice, we get:

$$\begin{aligned}\lim_{n\to\infty}\Delta\left(H_k^n/H_k^{n+1}\right)\cdot\left[\Delta\left(H_k^{n-1}/H_k^n\right)\right]^{-1} &= (\pi_1\cdots\pi_{k-1})^{-1}(\pi_1\cdots\pi_{k+1})^{-1}(\pi_1\cdots\pi_k)^2\\ &= \pi_k/\pi_{k+1} \quad\text{and}\quad |\pi_k/\pi_{k+1}| \neq 1. \qquad \square\end{aligned}$$

**Corollary 2.** *With hypothesis of Theorem* 1, *the convergence of the sequence* $(H_k^n/H_k^{n+1})_{n\in\mathbb{N}}$ *to* $\pi_1\pi_2\cdots\pi_k$ *can be accelerated by Aitken's* $\delta^2$.

**Remark.** Calculation of the sequence

$$\alpha_k^{(n)} := H_k^{(n)}/H_k^{(n+1)} \quad \text{with } H_k^{(n)} \neq 0, \qquad k = 1, 2, \ldots, \quad n = 0, 1, \ldots .$$

Using Jacobi's identity, we get

$$\alpha_{k+1}^{(n-1)}\left[1/\alpha_k^{(n+1)} - 1/\alpha_k^{(n)}\right] = \left[\alpha_k^{(n-1)} - \alpha_k^{(n)}\right]/\alpha_{k-1}^{(n+1)}$$

with initial conditions

$$\alpha_0^{(n)} = 1, \qquad \alpha_1^{(n)} = c_n/c_{n+1}, \quad n = 0, 1, \ldots .$$

## 2. Acceleration of the q-d algorithm

Since the sequence $(\alpha_k^{(n)})_{n\in\mathbb{N}}$ converges to $\pi_1\pi_2\cdots\pi_k$, the pole $\pi_k$ can be obtained as the limit of $(\alpha_k^{(n)}/\alpha_{k-1}^{(n)})_{n\in\mathbb{N}}$. This last sequence can be calculated recursively with the q-d algorithm [6,8].

We set

$$q_k^{(n)} := \left(H_k^{n+1}\cdot H_{k-1}^n\right)/\left(H_k^n\cdot H_{k-1}^{n+1}\right).$$

We know that, if $|\pi_{k-1}| < |\pi_k| < |\pi_{k+1}|$ and $k < N-1$ [6],

$$\lim_{n\to\infty} q_k^{(n)} = \pi_k^{-1}.$$

If, in addition, $k < N-2$ and $|\pi_{k-2}| < |\pi_{k-1}| < |\pi_k| < |\pi_{k+1}| < |\pi_{k+2}|$, then the sequence $(H_k^{(n)}/H_k^{(n+1)})_{n\in\mathbb{N}}$ and $(H_{k-1}^{(n)}/H_{k-1}^{(n+1)})_{n\in\mathbb{N}}$ are linear. □

So, by Lemma 2, we obtain the following theorem.

**Theorem 2.** *Let $f$ satisfy the same hypothesis as in Theorem 1. If $k < N-2$,*

$$|\pi_{k-2}| < |\pi_{k-1}| < |\pi_k| < |\pi_{k+1}| < |\pi_{k+2}| \quad \text{and} \quad |\pi_k/\pi_{k+1}| \neq |\pi_{k-1}/\pi_k|. \tag{3}$$

*Then, the convergence of $(q_k^{(n)})_{n\in\mathbb{N}}$ to $1/\pi_k$ is linear, i.e.,*

$$\lim_{n\to\infty} \left(q_k^{(n+1)} - \pi_k^{-1}\right)/\left(q_k^{(n)} - \pi_k^{-1}\right) = \theta_k$$

$$\text{where } \begin{cases} \theta_k = \pi_k/\pi_{k+1} & \text{if } |\pi_k/\pi_{k+1}| > |\pi_{k-1}/\pi_k|, \\ \theta_k = \pi_{k-1}/\pi_k & \text{otherwise.} \end{cases}$$

*Particular case: if $k=1$, then (3) is replaced by*

$$|\pi_0| < |\pi_1| < |\pi_2| < |\pi_3| \quad \text{where } \pi_0 := 0. \tag{4}$$

**Proof.** Condition (3) and Theorem 1 lead to the following result: $(\alpha_k^{(n)})_{n\in\mathbb{N}}$ and $(\alpha_{k-1}^{(n)})_{n\in\mathbb{N}}$ are linear sequences, with respective speeds, $\pi_k/\pi_{k+1}$ and $\pi_{k-1}/\pi_k$.

So, by Lemmas 1 and 2, we obtain Theorem 2 since:

$$q_k^{(n)} = \alpha_{k-1}^{(n)}/\alpha_k^{(n)}.$$

**Corollary 3.** *With hypothesis of Theorem 2, the convergence of the sequence $(q_k^{(n)})_{n\in\mathbb{N}}$ to $\pi_k^{-1}$ can be accelerated by Aitken's $\delta^2$.*

## 3. r-s algorithm and acceleration

Let $(c_n)_{n\in\mathbb{N}}$ a sequence of complex numbers. The r-s algorithm is a recursive method of calculation of the two quantities

$$r_k^{(n)} = H_k^{(n)}/\overline{H}_{k-1}^{(n)} \quad \text{and} \quad s_k^{(n)} = \overline{H}_k^{(n)}/H_k^{(n)}$$

where $\overline{H}_k^{(n)} := H_k(\Delta c_n)$.

The rules are [7]:

$$\begin{cases} s_k^{(n)}/s_{k-1}^{(n+1)} + 1 = r_k^{(n+1)}/r_k^{(n)}, \\ s_k^{(n+1)}/s_k^{(n)} = r_{k+1}^{(n)}/r_k^{(n+1)} + 1. \end{cases}$$

With initial conditions

$$s_0^{(n)} = 1, \qquad r_1^{(n)} = c_n, \quad n = 0, 1, 2, \ldots .$$

**Theorem 3.** *Let $f$ satisfy the same hypothesis as in Theorem 1. If $k < N-1$, $|\pi_{k-1}| < |\pi_k| < |\pi_{k+1}|$, and $\pi_i \neq 1$ $\forall i = 0, 1, \ldots, k-1$. Then*

$$\lim_{n\to\infty} r_k^{(n)}/r_k^{(n+1)} = \pi_k.$$

*If, in addition*

$$|\pi_{k-2}| < |\pi_{k-1}| < |\pi_k| < |\pi_{k+1}| < |\pi_{k+2}|, \quad k < N-2$$

*and*

$$|\pi_k/\pi_{k+1}| \neq |\pi_{k-1}/\pi_k|$$

*then the convergence of the sequence* $(r_k^{(n)}/r_k^{(n+1)})_{n\in\mathbb{N}}$ *is linear. With*

$$\lim_{n\to\infty} \left[r_k^{(n+1)}/r_k^{(n+2)}) - \pi_k\right] / \left[r_k^{(n)}/r_k^{(n+1)} - \pi_k\right] = \theta_k$$

*where* $\theta_k$ *is defined as in Theorem* 2.

**Proof.** If 1 is not a pole of $f$, the function

$$g(t) = \sum_{i=0}^{\infty} (\Delta c_i) t^i$$

has the same poles as $f$, in $|z| < |\pi_k|$. Indeed, $g(t) = t^{-1}[(1-t)f(t) - c_0]$, since these two last functions are holomorphic in a neighbourhood of zero and have the same Taylor expansion.

So, $\lim_{n\to\infty}(\overline{H}_{k-1}^{(n)}/\overline{H}_{k-1}^{(n+1)}) = \pi_1 \cdots \pi_{k-1}$ (see Theorem 1). We deduce

$$\lim_{n\to\infty} \left(r_k^{(n)}/r_k^{(n+1)}\right) = \lim_{n\to\infty} \frac{H_k^{(n)} \times \overline{H}_{k-1}^{(n+1)}}{\overline{H}_{k-1}^{(n)} \times H_k^{(n+1)}} = \frac{\pi_1 \cdots \pi_k}{\pi_1 \cdots \pi_{k-1}} = \pi_k.$$

The second part of Theorem 3 results from Lemmas 2 and 3 since the two sequences

$$\left(H_k^{(n)}/H_k^{(n+1)}\right)_{n\in\mathbb{N}} \quad \text{and} \quad \left(\overline{H}_{k-1}^{(n)}/\overline{H}_{k-1}^{(n+1)}\right)_{n\in\mathbb{N}}$$

are linear with respectively speeds

$$\pi_k/\pi_{k+1} \quad \text{and} \quad \pi_{k-1}/\pi_k. \qquad \square$$

**Corollary 4.** *With the hypothesis of Theorem* 3, *the sequence* $(r_k^{(n)}/r_k^{(n+1)})_{n\in\mathbb{N}}$ *can be accelerated by Aitken's* $\delta^2$.

## 4. Use of $\epsilon$-algorithm to evaluate poles of a meromorphic function. Acceleration

Let $(c_n)_{n\in\mathbb{N}}$ a sequence of complex numbers. $\epsilon$-algorithm is a recursive scheme for the calculation of the quantities [9]:

$$e_k(c_n) = H_{k+1}(c_n)/\tilde{H}_k(c_n)$$

where $\tilde{H}_k(c_n) = H_k(\Delta^2 c_n)$ by the rules

$$\epsilon_{k+1}^{(n)} = \epsilon_{k-1}^{(n+1)} + \left[\epsilon_k^{(n+1)} - \epsilon_k^{(n)}\right]^{-1}, \quad k,n = 0, 1, \ldots$$

$$\epsilon_{-1}^{(n)} = 0, \qquad \epsilon_0^{(n)} = c_n,$$

where $\epsilon_{2k}^{(n)} = e_k(c_n)$ and $\epsilon_{2k+1}^{(n)} = e_k(\Delta c_n)$.

By Theorem 4, we show how to calculate the poles of a meromorphic function by applying $\epsilon$-algorithm to the sequence of the coefficients of the Taylor expansion of $f$.

**Theorem 4.** *Let $f$ be a meromorphic function in $D(0, R)$, $0 < R \leqslant +\infty$. Let $f(z) = \sum_{i=0}^{\infty} c_i z^i$ have poles $\pi_n$ ordered $0 < |\pi_1| \leqslant |\pi_2| \leqslant \cdots < R$, each pole being countered with proper multiplicities. Let $N$ the number of poles $0 \leqslant N \leqslant \infty$. If $k < N-1$ and $|\pi_{k-1}| < |\pi_k| < |\pi_{k+1}|$ and $\pi_i \neq 1$, $\forall i = 0, 1, \ldots, k-1$.*

*Then*

*(a) $\lim_{n\to\infty} \epsilon_{2k-2}^{(n)} / \epsilon_{2k-2}^{(n+1)} = \pi_k$, and*

*(b) $\lim_{n\to\infty} \epsilon_{2k-1}^{(n)} / \epsilon_{2k-1}^{(n+1)} = 1/\pi_k$.*

*Moreover, if $k < N-2$ and $|\pi_{k-2}| < |\pi_{k-1}| < |\pi_k| < |\pi_{k+1}| < |\pi_{k+2}|$ and $|\pi_{k-1}/\pi_k| \neq |\pi_k/\pi_{k+1}|$ then, the convergence of the two sequences $(\epsilon_{2k-2}^{(n)}/\epsilon_{2k-2}^{(n+1)})_{n\in\mathbb{N}}$ and $(\epsilon_{2k-1}^{(n)}/\epsilon_{2k-1}^{(n+1)})_{n\in\mathbb{N}}$ are linear, with speed $\theta_k$ defined as in Theorem 2.*

**Proof.** The function

$$h(t) = \sum_{i=0}^{\infty} (\Delta^2 c_n) \cdot t^n$$

has the same poles as the function $f$, since

$$h(t) = t^{-2}\left[f(t)(1-t)^2 - c_0 - c_1 t + 2c_0 t\right].$$

We deduce

$$\lim_{n\to\infty} \frac{H_{k-1}(\Delta^2 c_n)}{H_{k-1}(\Delta^2 c_{n+1})} := \lim_{n\to\infty} \frac{\tilde{H}_{k-1}^{(n)}}{\tilde{H}_{k-1}^{(n+1)}} = \pi_1 \cdots \pi_{k-1}.$$

This leads to

$$\lim_{n\to\infty} \left(\epsilon_{2k-2}^{(n)}/\epsilon_{2k-2}^{(n+1)}\right) = \lim_{n\to\infty} \frac{H_k^{(n)}}{\tilde{H}_{k-1}^{(n)}} \times \frac{\tilde{H}_{k-1}^{(n+1)}}{H_k^{(n+1)}} = \pi_k,$$

which is (a) of the theorem. The proof for (b) is analogous. □

For the second part of Theorem 4, we must notice that $(\epsilon_{2k-2}^{(n)}/\epsilon_{2k-2}^{(n+1)})_{n\in\mathbb{N}}$ is the quotient of two linear sequences $(H_k^{(n)}/H_k^{(n+1)})_{n\in\mathbb{N}}$ and $(\tilde{H}_{k-1}^{(n)}/\tilde{H}_{k-1}^{(n+1)})_{n\in\mathbb{N}}$ with respective speeds

$$\pi_k/\pi_{k+1} \quad \text{and} \quad \pi_{k-1}/\pi_k.$$

So, Lemmas 2 and 3 permit to conclude.

**Remark.** If 1 is a simple pole of $f$, then only (b) holds, for $\pi_k \neq 1$.

**Corollary 5.** *If $f$ satisfies the conditions of Theorem 4, then the two sequences*

$$\left(\epsilon_{2k-2}^{(n)}/\epsilon_{2k-2}^{(n+1)}\right)_{n\in\mathbb{N}} \quad \text{and} \quad \left(\epsilon_{2k-1}^{(n)}/\epsilon_{2k-1}^{(n+1)}\right)_{n\in\mathbb{N}}$$

*can be accelerated by Aitken's $\delta^2$ process.*

### *4.1. Calculation of zeros and poles of a meromorphic function. Acceleration*

Let $g(t) = \sum_{i=0}^{\infty} d_i t^i$ a meromorphic function, $d_0 \neq 0$, and

$$f(t) = 1/g(t) = \sum_{i=0}^{\infty} c_i t^i.$$

The poles $\pi_n$ of $f$ are the zeros $\zeta_n$ of $g$. To calculate zeros of $g$, we must use the progressive form of $\epsilon$-algorithm. See [2].

We display the $\epsilon_k^{(n)}$'s in an array. The lower index denotes a column and the upper index denotes a diagonal. The index $k$ is an element of $\{-2, -1, 0, 1, 2, \dots\}$ and $n \in \mathbb{Z}$, $n \geqslant -[\frac{1}{2}k] - 1$. We progress from top to bottom with the rhombus rule:

$$\epsilon_k^{(n+1)} = \epsilon_k^{(n)} + \left[\epsilon_{k+1}^{(n)} - \epsilon_{k-1}^{(n+1)}\right]^{-1}.$$

With initial conditions

$$\epsilon_{2k}^{(-k-1)} = 0, \quad k \geqslant 0$$

for the first line.

$$\epsilon_{2k-1}^{(-k)} = \sum_{i=0}^{k-1} V_i, \quad k \geqslant 1 \quad \text{with } V_i := \sum_{j=0}^{i} d_j$$

for the second line, and

$$\epsilon_{-1}^{(n)} = 0, \quad n \geqslant 0$$

for the first column. The quantities $(\epsilon_{2k-1}^{(-k)})_{k \geqslant 1}$ may be calculated by mean of the recursive computation

$$\begin{cases} \epsilon_{2k+1}^{(-k-1)} = d_k + 2\epsilon_{2k-1}^{(-k)} - \epsilon_{2k-3}^{(-k+1)}, & k \geqslant 1, \\ \epsilon_{-1}^{(0)} = 0, \qquad \epsilon_1^{(-1)} = d_0. \end{cases}$$

We know that the second column is

$$\epsilon_0^{(n)} = \sum_{i=0}^{n} c_i, \quad n = 0, 1, 2, \dots .$$

All these above quantities are related with Padé approximants by the relation

$$\epsilon_{2k}^{(n)} = [n + k/k]_f(1) = [k/n + k]_g(1)$$

with

$$f(t) = \sum_{i=0}^{\infty} c_i t^i, \qquad g(t) = \sum_{i=0}^{\infty} d_i t^i.$$

The result of Theorem 4 becomes:

$$\lim_{n \to \infty} \left(\epsilon_{2k-1}^{(n)} / \epsilon_{2k-1}^{(n+1)}\right) = 1/\zeta_k, \qquad |\zeta_{k-1}| < |\zeta_k| < |\zeta_{k+1}|.$$

The convergence is linear, with the speed $\theta_k$ if:

$$|\zeta_{k-2}| < |\zeta_{k-1}| < |\zeta_k| < |\zeta_{k+1}| < |\zeta_{k+2}|$$

and

$$|\zeta_{k-1}/\zeta_k| \neq |\zeta_k/\zeta_{k+1}|,$$

where

$$\begin{cases} \theta_k = \zeta_{k-1}/\zeta_k & \text{if } |\zeta_{k-1}/\zeta_k| > |\zeta_k/\zeta_{k+1}|, \\ \theta_k = \zeta_k/\zeta_{k+1} & \text{otherwise.} \end{cases}$$

So, the convergence of $(\epsilon_{2k-1}^{(n)}/\epsilon_{2k-1}^{(n+1)})$ to the inverse of the $k$th zero of $g$ is linear and thus can be accelerated by Aitken's $\delta^2$.

### 4.2. *Calculation of poles of g*

The function

$$G(t)=\sum_{i=0}^{k} T_k t^k \quad \text{where } T_k=\sum_{i=0}^{k} V_i \quad \text{and} \quad V_k=\sum_{j=0}^{k} d_j$$

has the same poles of $g$ and the pole 1.

So, using the same table as above we can write: If the poles of $G$ are $W_i$, and $|W_{k-1}|<|W_k|<|W_{k+1}|$, then

$$\lim_{n\to\infty} \epsilon_{2n+1}^{(-n-k-2)}/\epsilon_{2n-1}^{(-n-k-1)}=W_k^{-1} \quad \text{if } |W_k|<1$$

(quotient of two successive terms of an odd line) and

$$\lim_{n\to\infty} \epsilon_{2n}^{(-n-k-1)}/\epsilon_{2n-2}^{(-n-k)}=W_k.$$

Moreover, if $|W_{k-2}|<|W_{k-1}|<|W_k|<|W_{k+1}|<|W_{k+2}|$ and $|W_{k-1}/W_k| \neq |W_k/W_{k-1}|$, then the two convergences above are linear and can be accelerated by Aitken's $\delta^2$.

So, we obtain a similar result to the one obtained with q-d algorithm for the simultaneous calculation of poles and zeros of a meromorphic function with a unique table.

### 4.3. *Particular case*

Let $P_k(z)=\sum_{i=0}^{k} d_i z^i$, $d_0 \neq 0$, $d_k \neq 0$, be a polynomial of degree $k$. Its zeros can be found as above, with $\epsilon$-algorithm. The quotient of two consecutive vertical terms of each odd column converges to a zero of $P_k$. I.e.

$$\lim \epsilon_{2i-1}^{(n)}/\epsilon_{2i-1}^{(n+1)}=1/\zeta_i \quad \text{if } |\zeta_{i-1}|<|\zeta_i|<|\zeta_{i+1}|$$
$$\text{for } i=1,2,\ldots,k \quad (\zeta_0:=0 \text{ and } \zeta_{k+1}:=\infty).$$

We remark that this particular case is similar to the one obtained by Brezinski in [3] since the numbers $(u_i^{(k)})_{i=l,l-k}:=\epsilon_{2k-1}^{(-k+i)}$ are linked by a linear combination, precised under, by relation (5).

Let $d_i=0$ for $i=k+1,\ k+2,\ldots$ . Let us write

$$u_i^{(k)}:=\epsilon_{2k-1}^{(-k+i)}=[k-1/i]W(t)(1), \quad i=0,1,2,\ldots$$

where $W(t)=\sum_{i=0}^{\infty} V_i t^i$. *Then*

$$d_0u_l^{(k)}+d_1u_{l+1}^{(k)}+d_2u_{l+2}^{(k)}+\cdots+d_ku_{l+k}^{(k)}=0 \quad \forall l=0,1,2,\ldots . \tag{5}$$

**Proof of relation (5).**

$$u_i^{(k)}=[k-1/i]W(t)(1)=\epsilon_{2i}^{-i+k-1}(T_n)$$

where

$$T_0:=V_0,$$
$$T_n=\sum_{i=0}^{n} V_i \quad n=0,1,\ldots, \qquad T_n=0 \quad \text{if } n<0.$$

We have

$$u_i^{(k)} = \begin{vmatrix} \sum_{j=0}^{k-1-i} V_j & \cdots & \sum_{j=0}^{k-1} V_j \\ V_{k-i} & \cdots & V_k \\ \vdots & & \vdots \\ V_{k-1} & \cdots & V_{k+i-1} \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 \\ V_{k-i} & & \cdots & V_k \\ \vdots & & & \vdots \\ V_{k-1} & & \cdots & V_{k+i-1} \end{vmatrix}^{-1}.$$

By using

$$V_i = d_0 + d_1 \cdots + d_i, \quad i \geqslant 0, \qquad V_i = V_k \quad \forall i \geqslant k.$$
$$V_i := 0, \quad i < 0.$$

We obtain

$$u_i^{(k)} = \begin{vmatrix} T_{k-1-i} & \cdots & T_{k-1} \\ V_{k-i} & \cdots & V_k \\ \vdots & & \vdots \\ V_{k-1} & \cdots & V_{k+i-1} \end{vmatrix} \cdot \begin{vmatrix} d_{k-i-1} & \cdots & & d_k \\ & & d_k & 0 \\ \vdots & & & \vdots \\ d_k & 0 & \cdots & 0 \end{vmatrix}^{-1} = \frac{(-1)^i D_{k,i,1}}{d_k^i}.$$

where

$$D_{k,i,j} = \begin{vmatrix} V_{k-j} & V_{k-j+1} & \cdots & V_{k-i} & T_{k-i-1} \\ V_k & V_{k-1} & \cdots & V_{k-i+j} & T_{k-i+j-1} \\ \vdots & \vdots & & \vdots & \vdots \\ V_k & V_k & \cdots & V_k & T_{k-1} \end{vmatrix}.$$

The determinants $D_{k,i,j}$ satisfy the relation:

$$D_{k,i,j} = d_{k-j} D_{k,i-j,1} - d_k D_{k,i,j+1}, \quad j = 1, \ldots, k. \tag{6}$$

Let us multiply $D_{k,i,j}$ by $(-d_k)^{j-1}$ and sum the $k$ equations obtained.

$$D_{k,i,1} = d_{k-1} D_{k,i-1,1} - d_k D_{k,i,2},$$
$$-d_k D_{k,i,2} = -d_k d_{k-2} D_{k,i-2,1} + d_k^2 D_{k,i,3},$$
$$\vdots$$
$$(-d_k)^{k-1} D_{k,i,k-1} = (-d_k)^{k-2} d_1 D_{k,i-k+1,1} + (-d_k)^{k-1} D_{k,i,k},$$
$$(-d_k)^{k-1} D_{k,i,k} = d_0 D_{k,i-k,1} (-d_k)^{k-1}$$

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$$D_{k,i,1} = d_{k-1} D_{k,i-1,1} - d_k d_{k-2} D_{k,i-2,1} + \cdots + (-d_k)^{k-2} d_1 D_{k,i-k+i,1} + (-d_k)^{k-1} d_0 D_{k,i-k,1}$$
$$\Leftrightarrow (-d_k)^i u_i^{(k)} = d_{k-1}(-d_k)^{i-1} u_{i-1}^{(k)} + (-d_k)^{i-1} u_{i-2}^{(k)} + \cdots + (-d_k)^{i-1} d_1 u_{i-k+1}^{(k)}$$
$$+ (-d_k)^{i-1} d_o u_{i-k}^{(k)} \quad \forall i \geqslant k.$$

Which is the relation (5).

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Applied Numerical Mathematics 6 (1989/90) 221–224
North-Holland 

# DETERMINANTAL EXPRESSION FOR PARTIAL PADÉ APPROXIMANTS

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Recently, Brezinski [1] introduced the notion of partial Padé approximants (PPA) which consists in finding a rational function with preassigned zeros and poles and which coincides with the expansion of a function $f$ up the maximal degree. Here, we obtain a rational function with preassigned values (even $\infty$) which coincides with the expansion of this function $f$ as far as possible and this leads us, in the particular case when these values are 0 or $\infty$, to a determinantal expression of the PPA.

## 1. Introduction

The purpose of this work is to find a rational function $R(t) = P(t)/Q(t)$ with degree $P = m + p$ and degree $Q = n + p$ such that

$$\begin{aligned} &P(t) - Q(t)f(t) = \mathrm{O}(t^{m+n+p+1}), \\ &R(\alpha_i) = \beta_i, \qquad \alpha_i \in \mathbb{C}^*, \quad \alpha_i \text{ distincts } (\beta_i \text{ possibly } \infty), \quad i = 1, \ldots, p. \end{aligned} \tag{1}$$

In Section 2 we give the expression of $R(t)$ as the quotient of two determinants. Section 3 is devoted to the study of the particular case

$$\beta_i = \begin{cases} 0 & \text{for } i = 1, \ldots, k, \\ \infty & \text{for } i = k+1, \ldots, p, \end{cases}$$

(partial Padé approximants are recovered). Finally, we show how to calculate $R(t)$ with the $E$-algorithm.

## 2. Expression of $R$

**Notation 2.1.** Let $\tilde{N}_m(t)/\tilde{D}_n(t)$ be the Padé approximant of a function $f$. $\tilde{N}_m(t)/\tilde{D}_n(t) = [m/n]_f(t)$ is supposed to exist. Then, we have

$$\tilde{D}_n(t)f(t) - \tilde{N}_m(t) = \mathrm{O}(t^{m+n+1}).$$

Theorem 2.2. *If system* (1) *has a unique solution, then*

$$R(t)=\frac{\begin{vmatrix} \tilde{N}_m(t)t^p & e_{n,m}(\alpha_1) & \cdots & e_{n,m}(\alpha_p) \\ \tilde{N}_{m+1}(t)t^{p-1} & e_{n+1,m+1}(\alpha_1) & & e_{n+1,m+1}(\alpha_p) \\ \vdots & \vdots & & \vdots \\ \tilde{N}_{m+p}(t) & e_{n+p,m+p}(\alpha_1) & \cdots & e_{n+p,m+p}(\alpha_p) \end{vmatrix}}{\begin{vmatrix} \tilde{D}_n(t)t^p & e_{n,m}(\alpha_1) & \cdots & e_{n,m}(\alpha_p) \\ \tilde{D}_{n+1}(t)t^{p-1} & e_{n+1,m+1}(\alpha_1) & \cdots & e_{n+1,m+1}(\alpha_p) \\ \vdots & \vdots & & \vdots \\ \tilde{D}_{n+p}(t) & e_{n+p,m+p}(\alpha_1) & \cdots & e_{n+p,m+p}(\alpha_p) \end{vmatrix}} =: \frac{P(t)}{Q(t)},$$

*where*

$$e_{n+k,m+k}(\alpha_i)=\begin{cases} \alpha_i^{p-k}\left[\tilde{D}_{n+k}(\alpha_i)\cdot\beta_i-\tilde{N}_{m+k}(\alpha_i)\right], & \beta_i\neq\infty, \\ \alpha_i^{p-k}\tilde{D}_{n+k}(\alpha_i), & \beta_i=\infty, \end{cases}$$

$$i=1,\ldots,p.$$

**Proof.** $R(\alpha_i)=\beta_i$ is obtained by subtracting the first column from the $i$th in the numerator and in the denominator. Moreover

$$Q(t)f(t)-P(t)=\begin{vmatrix} [f(t)\tilde{D}_n(t)-\tilde{N}_m(t)]t^p & e_{n,m}(\alpha_1) & \cdots & e_{n,m}(\alpha_p) \\ \vdots & \vdots & & \vdots \\ [f(t)\tilde{D}_{n+p}(t)-\tilde{N}_{m+p}(t)]t & e_{n+p,m+p}(\alpha_1) & \cdots & e_{n+p,m+p}(\alpha_p) \end{vmatrix}.$$

So $P(t)-Q(t)f(t)=O(t^{n+m+p+1})$. □

**Remark 2.3.** If $\beta_i=f(\alpha_i)$ for $i=1,\ldots,p$; then $R(t)$ is the multipoint Padé approximant at zero (order $m+n+p$) and at $\alpha_i$ $(i=1,\ldots,p)$. Let us set

$$\boldsymbol{D}(t)=\begin{pmatrix} \tilde{D}_n(t)t^p \\ \vdots \\ \tilde{D}_{n+p}(t) \end{pmatrix},\qquad \boldsymbol{N}(t)=\begin{pmatrix} N_m(t)t^p \\ \vdots \\ \tilde{N}_{m+p}(t) \end{pmatrix},\qquad \boldsymbol{e}_{n,m}^{(p)}(t)=f(t)\boldsymbol{D}(t)-\boldsymbol{N}(t),$$

three vectors in $\mathbb{R}^{p+1}$. From the expression of $R(t)$ we obtain:

$$\det\left(R(t)\boldsymbol{D}(t)-\boldsymbol{N}(t),\ f(\alpha_1)\boldsymbol{D}(\alpha_1)-\boldsymbol{N}(\alpha_1),\ldots,f(\alpha_p)\boldsymbol{D}(\alpha_p)-\boldsymbol{N}(\alpha_p)\right)=0.$$

### *2.1. Geometrical interpretation*

The vectors in $\mathbb{R}^{p+1}$ depending on $t$, $f(t)\boldsymbol{D}(t)-\boldsymbol{N}(t)$, are known for $t=\alpha_1,\ldots,\alpha_p$. We want to approximate the value $f(t)$. So, $R(t)$ is such that the error $R(t)\boldsymbol{D}(t)-\boldsymbol{N}(t)$ is in the hyperplane determined by the errors committed at $\alpha_1,\ldots,\alpha_p$.

## 3. Particular case: Partial Padé approximants

Let us now examine the case:

$$\beta_1 = \beta_2 = \cdots = \beta_k = 0,$$
$$\beta_{k+1} = \beta_{k+2} = \cdots = \beta_p = \infty.$$

We have

$$R(t) = \frac{\begin{vmatrix} \tilde N_m(t)t^p & \alpha_1^p \tilde N_m(\alpha_1) & \dots & \alpha_k^p \tilde N_m(\alpha_k) & \alpha_{k+1}^p \tilde D_n(\alpha_{k-1}) & \dots & \alpha_p^p \tilde D_n(\alpha_p) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \tilde N_{m-p}(t) & \tilde N_{m-p}(\alpha_1) & \dots & \tilde N_{m-p}(\alpha_k) & \tilde D_{n-p}(\alpha_{k-1}) & \dots & \tilde D_{n-p}(\alpha_p) \end{vmatrix}}{\begin{vmatrix} \tilde D_n(t)t^p & \alpha_1^p \tilde N_m(\alpha_1) & \dots & \alpha_k^p \tilde N_m(\alpha_k) & \alpha_{k-1}^p \tilde D_n(\alpha_{k-1}) & \dots & \alpha_p^p \tilde D_n(\alpha_p) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \tilde D_{n-p}(t) & \tilde N_{m-p}(\alpha_1) & \dots & \tilde N_{m-p}(\alpha_k) & \tilde D_{n-p}(\alpha_{k-1}) & \dots & \tilde D_{n-p}(\alpha_p) \end{vmatrix}}.$$

Let $r := p - k$. We set

$$\tilde v_k(t) = \prod_{i=1}^{k} (t - \alpha_i) \qquad \text{(degree } k\text{)},$$
$$\tilde w_r(t) = \prod_{i=k+1}^{p} (t - \alpha_i) \quad \text{(degree } r\text{)}.$$

Then $R(t)$ can be written as

$$R(t) = \frac{\tilde v_k(t) \times \tilde p_\mu(t)}{\tilde w_r(t) \times \tilde q_\nu(t)},$$

where $\tilde p_\mu$ and $\tilde q_\nu$ are some polynomials with deg $\tilde p_\mu := \mu = m + p - k = m + r$ and deg $\tilde q_\nu := \nu = n + p - r = n + k$. We have

$$\tilde v_k(t)\tilde p_\mu(t) - f(t)\tilde w_r(t)\tilde q_\nu(t) = \mathrm{O}(t^{\mu+\nu+1}),$$

or

$$\tilde p_\mu(t) - \tilde q_\nu(t) f(t) \frac{\tilde w_r(t)}{\tilde v_k(t)} = \mathrm{O}(t^{\mu+\nu+1}).$$

Thus $\tilde p_\mu(t)/\tilde q_\nu(t)$ is the Padé approximant $[\mu/\nu]$ of $\tilde w_r(t)f(t)/\tilde v_k(t)$ for $\mu \geqslant r$ and $\mu \geqslant k$. Thus $R$ is a partial Padé approximant as defined in [1].

**Remark 3.1.** The expression for $n \geqslant k$ of the $n$th orthogonal polynomial with respect to the linear functional (supposed to be definite) is defined by the coefficients of the series $\tilde w_r f/\tilde v_k$:

$$q_\nu(t) = t^\nu \tilde q_\nu(t^{-1}),$$

$$q_\nu(t) = \frac{1}{w_r(t)} \begin{vmatrix} D_n(t) & \alpha_1^p \tilde N_m(\alpha_1) & \dots & \alpha_k^p \tilde N_m(\alpha_k) & \alpha_{k+1}^p \tilde D_n(\alpha_{k+1}) & \dots & \alpha_p^p \tilde D_n(\alpha_p) \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ D_{n+p}(t) & \tilde N_{m+p}(\alpha_1) & \dots & \tilde N_{m+p}(\alpha_k) & \tilde D_{n+p}(\alpha_{k+1}) & \dots & \tilde D_{n+p}(\alpha_p) \end{vmatrix}.$$

where $n = \nu - k$, $m = \mu - r$, and $D_n, \ldots, D_{n+p}$ are the orthogonal polynomials with respect to $c^{(m-n+1)}$, where $c^{(s)}$ is the functional defined by $c^{(s)}(x_i) = c_{i+s}$.

## 4. Calculation of $R$ with $E$-algorithm

By dividing the $j$th row of the numerator and of the denominator by $\tilde{D}_{n+j-1}(t)t^{p+j-1}$, we obtain $R(t)$ as a ratio of determinants, the first column of the denominator being 1.

We set

$$S_j = \tilde{N}_{j+m-n}(t)/\tilde{D}_j(t), \quad j = 0, 1, \ldots$$

and

$$g_i(j) = \frac{\beta_i \tilde{D}_j(\alpha_i) - \tilde{N}_{j-r}(\alpha_i)}{\tilde{D}_j(t)} \times \left(\frac{\alpha_i}{t}\right)^{n+p-j}, \quad i = 1, \ldots, p.$$

So

$$R(t) = \frac{\begin{vmatrix} S_j(t) & g_1(j) & \cdots & g_p(j) \\ \vdots & \vdots & & \vdots \\ S_{j+p}(t) & g_1(j+p) & \cdots & g_p(j+p) \end{vmatrix}}{\begin{vmatrix} 1 & g_1(j) & \cdots & g_p(j) \\ \vdots & \vdots & & \vdots \\ 1 & g_1(j+p) & \cdots & g_p(j+p) \end{vmatrix}}.$$

This shows that $R$ can be computed using the $E$-algorithm with the initializations:

$$E_0^{(j)} = \tilde{N}_{j-r}(t)/\tilde{D}_j(t) = S_j, \quad j = 0, 1, \ldots,$$

$$g_{0,i}^{(j)} = g_i(j), \qquad i = 1, 2, \ldots, \quad j = 0, 1, \ldots.$$

By applying $E$-algorithm of Brezinski [2], we obtain:

$$R(t) = E_p^{(j)}.$$

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Journal of Computational and Applied Mathematics 32 (1990) 237–252
North-Holland


# Approximation of weight function and approached Padé approximants

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Received 24 September 1989
Revised 25 January 1990

*Abstract:* We discuss the orthogonal expansion of a distribution $d\psi$ whose support $[-1, 1]$ and first $N+1$ moments $c_0, c_1, \ldots, c_N$ are given. The truncated orthogonal expansion $d\bar{\psi}_N$ gives rise to approached orthogonal polynomials and so to approached Padé approximants of the Stieltjes function with the distribution $d\psi$.

*Keywords:* Padé approximants, orthogonal expansion, weight function.

## 1. Introduction

Here we recall the formalism introduced by Brezinski [2]. Let $f$ be a formal power series in one variable:

$$f(t) = \sum_{i=0}^{\infty} c_i t^i, \quad c_i \in \mathbb{R}.$$

Let us define a linear functional acting on the space of real polynomials by

$$c(x^i) = c_i \quad \text{for } i = 0, 1, \ldots .$$

$c_i$ is called the moment of order $i$ of the functional $c$. By expanding the function $x \to (1-xt)^{-1}$ in a power series we formally get

$$f(t) \sim c\left((1-xt)^{-1}\right).$$

If we replace the function $x \to (1-xt)^{-1}$ by an appropriate function $g_t(x)$, we may hope that $c(g_t(x))$, if it exists, will be an approximation of $f(t)$.

We suppose that the first $N+1$ moments are known. In order to calculate $c(g_t(x))$, $g_t(x)$ needs to be a polynomial in the variable $x$, and the degree of $g_t(x)$ must be less than or equal to $N$:

$$g_t(x) = \sum_{k=0}^{N} \alpha_k(t) x^k.$$

(Note that the functions $\alpha_k$ may depend on $c_0, \ldots, c_N$.)

Many cases of $g_t(x)$ as a polynomial approximant of the function $x \to (1-xt)^{-1}$ have been studied in [13] (see [14] for a summary).

In this paper, we study the approximation $\mathrm{d}\bar{\psi}_N$ of the distribution $\mathrm{d}\psi$, provided by $g_t(x)$ in the following cases:

- $g_t$ as orthogonal polynomial with respect to the linear functional $c$ (Section 2),
- $g_t$ as interpolation polynomial of the generating function $x \to (1-xt)^{-1}$ or $g_t$ as the truncated orthogonal expansion of this generating function (Section 3).

The convergence of $\mathrm{d}\bar{\psi}_N$ to $\mathrm{d}\psi$ is studied in Section 4 and the orthogonal polynomial with respect to $\mathrm{d}\bar{\psi}_N$ will be called approached orthogonal polynomial and studied in Section 5. These approached orthogonal polynomials lead us to the construction of the approached Padé approximant (Section 6). Section 7 deals with the construction of the Padé–Chebyshev approximant of the weight function $w(x)$ when $\mathrm{d}\psi = w(x)\,\mathrm{d}x$.

## 2. Padé approximants

If only $N+1$ moments of the function $f$ are given, we can use a Padé approximant of $f$, $[p/q]_f$ with $p+q+1 \leqslant N$. $[p/q]_f$ is the rational fraction whose Taylor series coincides with that of $f$ up to the maximal order.

In the generic case, we have

$$f(t) - [p/q]_f(t) = \mathrm{O}(t^{p+q+1}) \quad \text{and} \quad g_t(x) = \frac{1 - P_q(x)/P_q(t^{-1})}{1-xt},$$

where $P_q$ is the polynomial of degree $q$ belonging to the family of orthogonal polynomials with respect to the functional $c^{(p-q+1)}$ defined by

$$c^{(p-q+1)}(x^i) = c_{i+p-q+1}.$$

We now suppose that

$$f(t) = \int_{-1}^{1} \frac{1}{1-xt}\,\mathrm{d}\psi(x), \tag{A}$$

where $\psi$ is a real function of bounded variation on $[-1, 1]$. If $\psi$ is nondecreasing, then $f$ is called a Stieltjes function.

The Gaussian quadrature method applied to the integral corresponds exactly to this Padé approximation of $f$,

$$f(t) = \int_{-1}^{1} \frac{1}{1-xt}\,\mathrm{d}\psi(x) \sim \sum_{i=1}^{M} \frac{A_i^{(M)}}{1-x_i^{(M)}t} = [M-1/M]_f(t), \quad 2M \leqslant N,$$

and provides an approximation of $\mathrm{d}\psi$ by

$$\mathrm{d}\psi \sim \mathrm{d}\bar{\psi}_N := \sum_{i=1}^{M} A_i^{(M)}\,\mathrm{d}\delta x_i^{(M)}$$

($\mathrm{d}\delta a$ is the Dirac distribution on the point $a$, i.e., $\int h(x)\,\mathrm{d}\delta a = h(a)$ and $\bar{\psi}_N$ is a step function with jumps of magnitude $A_i^{(M)}$ at points $x_i^{(M)}$, $i = 1,\ldots, M$, (zeros of $P_M$).

If $\psi$ is a nondecreasing function, it is well known that the points $x_i^{(M)}$, the zeros of the orthogonal polynomial $P_n$ with respect to $c$, belong to $[-1, 1]$ and that the weights $A_i^{(M)}$ are positive.

**Remark 1.** If $\psi$ is a nondecreasing function, with infinitely many points of increase, Van Assche has used Turan determinants to approximate $\mathrm{d}\psi$ [16].

## 3. Orthogonal expansion

Our aim is still to replace in (A) the integrand $x \to (1-xt)^{-1}$ by a polynomial $g_t(x)$ of degree $N$ in the variable $x$.

Let $\mathrm{d}\mu$ be a known positive distribution and $\{P_k(\mathrm{d}\mu, x), k \in \mathbb{N}\}$ the sequence of orthonormal polynomials with respect to $\mathrm{d}\mu$. We suppose that the support of $\mathrm{d}\mu$ is $[-1, 1]$. It is well known that $P_k(\mathrm{d}\mu, x)$ has degree $k$, has all its zeros in $[-1, +1]$ and satisfies a three-term recurrence relationship.

The expansion of $g_t$ in the basis $P_0(\mathrm{d}\mu, x), \ldots, P_N(\mathrm{d}\mu, x)$ is:

$$g_t(x) = \sum_{k=0}^{N} \beta_k(t) P_k(\mathrm{d}\mu, x), \tag{B}$$

where the $\beta_k$'s satisfy

$$\beta_k(t) = \int_{-1}^{1} g_t(x) P_k(\mathrm{d}\mu, x)\, \mathrm{d}\mu(x),$$

because of the relation $\int_{-1}^{1} P_k(\mathrm{d}\mu, x) P_l(\mathrm{d}\mu, x)\, \mathrm{d}\mu(x) = \delta_{kl}$. So, an approximation of $f$ is:

$$f(t) \sim \sum_{k=0}^{N} \beta_k(t) c(P_k(\mathrm{d}\mu, x)). \tag{1}$$

The terms

$$c(P_k(\mathrm{d}\mu, x)) := \int_{-1}^{1} P_k(\mathrm{d}\mu, x)\, \mathrm{d}\psi(x)$$

are the modified moments of the distribution $\mathrm{d}\psi$ (see [8]).

The Stieltjes inversion formula [4, p.90] allows us to express $\mathrm{d}\psi$ in terms of $F(z) = z^{-1} f(z^{-1})$:

$$\psi(t) - \psi(s) = -\frac{1}{\pi} \lim_{y \to 0^+} \int_s^t \operatorname{Im}\{F(x + \mathrm{i}y)\}\, \mathrm{d}x. \tag{2}$$

$\psi$ is supposed to be normalized by setting, if necessary, $\psi(x) = \frac{1}{2}[\psi(x+0) - \psi(x-0)]$ at the points of discontinuity. By (2) $\psi$ appears as the change of determination of the analytic function $f$ through its cut $]-\infty, -1[\cup]1, +\infty[$.

If we replace in (2) the function $f$ by the approximation given by (1), we obtain an approximation $\bar{\psi}_N$ of $\psi$ as the change of determination of the $\beta_k$'s through $\mathbb{R}$. This method fails because it needs two approximations, the first given by (1) and the second given by "$\lim_{y \to 0^+}$" in (2).

If, moreover, we suppose that for each $k$ a function $\alpha_k$ exists such that

$$\beta_k(t) = \int_{-1}^{1} \frac{1}{1-xt} \, \mathrm{d}\alpha_k(x),$$

then we obtain an approximation $\mathrm{d}\bar{\psi}_N$ of $\mathrm{d}\psi$ by

$$\mathrm{d}\psi \sim \mathrm{d}\bar{\psi}_N := \sum_{k=0}^{N} \mathrm{d}\alpha_k(x) c(P_k(\mathrm{d}\mu, x)). \tag{3}$$

**Example 2.**

$$g_t(x) = \frac{1 - v(x)/v(t^{-1})}{1-xt},$$

where $v$ is some polynomial of degree $N+1$. Then, $g_t$ is the polynomial interpolant of $1/(1-xt)$ at the zeros of $v$. Now the $\beta_k(t)$'s are rational fractions with poles at the inverse of the zeros $x_i$, $i = 0, \dots, N$, of $v$. So $\beta_k$ has no change of determination, except at the points $x_i^{-1}$, $i = 0, \dots, N$, and $\bar{\psi}_N$ is a step function with jumps at $x_i^{-1}$.

**Example 3.** Let $g_t$ be given by (B) with

$$\beta_k(t) = \int_{-1}^{1} (1-xt)^{-1} P_k(\mathrm{d}\mu, x) \, \mathrm{d}\mu(x),$$

i.e., let $g_t$ be the truncation of the orthogonal expansion of the function $x \to (1-xt)^{-1}$ in $P_k(\mathrm{d}\mu, x)$ [18]. Then the function $\alpha_k$ in (3) is $\mathrm{d}\alpha_k(x) = P_k(\mathrm{d}\mu, x) \, \mathrm{d}\mu(x)$ and

$$\mathrm{d}\bar{\psi}_N(t) = \mathrm{d}\mu(t) \sum_{k=0}^{N} P_k(\mathrm{d}\mu, t) c(P_k(\mathrm{d}\mu, x)). \tag{4}$$

**Remark 4.** *If* $\mathrm{d}\psi(x) = w(x) \, \mathrm{d}\mu(x)$ *with* $w \in L^2(\mathrm{d}\mu)$, *then*

$$\lim_{k \to \infty} c(P_k(\mathrm{d}\mu, x)) = 0.$$

**Proof.** Just recall that the quantities $c(P_k(\mathrm{d}\mu, x))$ are just the coefficients in the orthogonal expansion of $w$ in terms of $\{P_k(\mathrm{d}\mu, x)\}$:

$$w(x) = \sum_{k=0}^{\infty} \left[ \int_{-1}^{1} P_k(\mathrm{d}\mu, x) w(x) \, \mathrm{d}\mu(x) \right] P_k(\mathrm{d}\mu, x).$$

If $w \in L^2(\mathrm{d}\mu)$, then

$$\left( \int_{-1}^{1} P_k(\mathrm{d}\mu, x) w(x) \, \mathrm{d}\mu(x) \right)_{k \in \mathbb{N}} = c(P_k(\mathrm{d}\mu, x))_{k \in \mathbb{N}} \in L^2$$

and $\lim_{k \to \infty} c(P_k(\mathrm{d}\mu, x)) = 0$. □

The result of Remark 4 no longer holds if $\mathrm{d}\psi$ is not absolutely continuous with respect to $\mathrm{d}\mu(x)$.

**Example 5.** $d\psi = d\delta_0$ (Dirac function), $\int_{-1}^{1} P_k(d\mu, x)\, d\delta_0 = P_k(d\mu, 0)$ and $\lim_{k\to\infty} P_k(d\mu, 0)$, in general, does not exist.

This remark will provide us a mean to detect a Dirac mass (and its derivatives) in a distribution (see [15]).

**Example 6.** Let $g_t$ be the polynomial in $x$ of best approximation of the function $x \to (1-xt)^{-1}$ on $[-1, 1]$. Different expressions of $g_t$ exist [11, p.33, 34]. Hornecker [9] has obtained an expression of $g_t$ in terms of Chebyshev polynomials of the first kind $T_n$, orthogonal with respect to $dx/\sqrt{1-x^2}$:

$$g_t(x) = \frac{2(1+\rho^2)}{1-\rho^2}\left[\sum_0^{N-1}{}' \rho^k T_k(x) + \frac{\rho^N}{1-\rho^2} T_N(x)\right], \tag{5}$$

where

$$\sum_0^N{}' a_i = \tfrac{1}{2}a_0 + a_1 + \cdots + a_N \quad \text{(Paszkowski's notation [12])}$$

and $\rho = t^{-1} \pm \sqrt{t^2-1}$, the sign before the square root being chosen such that $|\rho| < 1$. Note that the orthogonal expansion of $(1-xt)^{-1}$ over the basis $T_0, \ldots, T_N$ is

$$\frac{2(1+\rho^2)}{1-\rho^2}\sum_{k=0}^{N}{}' \rho^k T_k(x).$$

After having calculated the change of determination of $c(g_t(x))$, we obtain

$$d\bar{\psi}_N(t) = \frac{2dt}{\pi\sqrt{1-t^2}}\left[\sum_{k=0}^{N-1}{}' T_k(t)c(T_k) - \tfrac{1}{2}U_{N-2}(t)c(T_N)\right],$$

where $c(T_k) = \int_{-1}^{1} T_k(x)\, d\psi(x)$.

## 4. Convergence

In this section, we are interested by the convergence of $d\bar{\psi}_N$ to $d\psi$.

### 4.1. Gauss quadrature

Let $\psi$ be nondecreasing with bounded variation on $[-1, 1]$. We have

$$d\bar{\psi}_N = \sum_{i=1}^{N} A_i^{(N)}\, d\delta_{x_i}^{(N)}$$

(see Section 2); $d\bar{\psi}_N$ satisfies

$$\int_{-1}^{1} x^k\, d\bar{\psi}_N(x) = \int_{-1}^{1} x^k\, d\psi(x), \quad k = 0, \ldots, 2N-1.$$

Let $l_N : C([-1, 1], \mathbb{R}) \to \mathbb{R}$:

$$h \to l_N(h) = \int_{-1}^{1} h(x)\, \mathrm{d}\bar{\psi}_N(x),$$

$$\| l_N \| = \sum_{i=1}^{N} |A_i^{(N)}| = \sum_{i=1}^{N} A_i^{(N)} = c_0, \quad A_i^{(N)} > 0 \ \forall i.$$

So, $l_N \to c$ on $\mathscr{P}$, the space of polynomials and $\{l_N\}_{N \in \mathbb{N}}$ is uniformly bounded. Then, by applying the Banach–Steinhaus theorem,

$$\mathrm{d}\bar{\psi}_N \underset{\mathrm{w}}{\to} \mathrm{d}\psi \quad \text{(weak convergence)}, \quad N \to \infty.$$

### 4.2. Orthogonal expansion

Let $\psi$ be of bounded variation on $[-1, 1]$, not necessarily nondecreasing. The approximation $\mathrm{d}\bar{\psi}_N$ of $\mathrm{d}\psi$ is (see (4))

$$\mathrm{d}\bar{\psi}_N(t) = \mathrm{d}\mu(t) \sum_{k=0}^{N} P_k(\mathrm{d}\mu, t) c(P_k(\mathrm{d}\mu, x)),$$

with

$$c(P_k(\mathrm{d}\mu, x)) = \int_{-1}^{1} P_k(\mathrm{d}\mu, x)\, \mathrm{d}\psi(x),$$

where $P_k(\mathrm{d}\mu, x)$ is the orthonormal polynomial with respect to $\mathrm{d}\mu$. $\mathrm{d}\bar{\psi}_N$ satisfies:

$$\int_{-1}^{1} P_k(\mathrm{d}\mu, x)\, \mathrm{d}\bar{\psi}_N(x) = c(P_k(\mathrm{d}\mu, x)) = \int_{-1}^{1} P_k(\mathrm{d}\mu, x)\, \mathrm{d}\psi(x)$$

for $k = 0, \ldots, N$, and then

$$\int_{-1}^{1} x^k\, \mathrm{d}\bar{\psi}_N(x) = \int_{-1}^{1} x^k\, \mathrm{d}\psi(x) = c_k, \quad k = 0, \ldots, N.$$

because $\deg P_k(\mathrm{d}\mu, x) = k$.

Let $l_N : C([-1, 1], \mathbb{R}) \to \mathbb{R}$:

$$h \to l_N(h) = \int_{-1}^{1} h(x)\, \mathrm{d}\bar{\psi}_N(x), \quad l_N(P) = P \ \forall P \in \mathscr{P}_N,$$

$$\| l_N \| = \sup_{\|h\|_\infty \leqslant 1} | l_N(h) | \leqslant \| h \|_\infty \operatorname{var}(\bar{\psi}_N, [-1, 1]) \quad \text{(total variation)}.$$

So

$$\| l_N \| = \int_{-1}^{1} \left| \sum_{k=0}^{N} c(P_k(\mathrm{d}\mu, x)) P_k(\mathrm{d}\mu, t) \right| \mathrm{d}\mu(t).$$

By the Banach–Steinhaus theorem we obtain the next theorem.

**Theorem 7.** *Let* $(P_k(\mathrm{d}\mu, x))$ *be the sequence of orthonormal polynomials with respect to* $\mathrm{d}\mu$, *where* $\mu$ *is a nondecreasing function of bounded variation, whose support is* $[-1, 1]$, *with infinitely many* *points of increase. Let $\psi$ be a function of bounded variation on $[-1, 1]$. Suppose that $c_i := \int_{-1}^{1} x^i \,\mathrm{d}\psi(x)$ are known for $i = 0, 1, 2, \ldots$ . Let*

$$c(P_k(\mathrm{d}\mu, x)) := \int_{-1}^{1} P_k(\mathrm{d}\mu, x)\,\mathrm{d}\psi(x), \quad k \in \mathbb{N},$$

*and*

$$\mathrm{d}\bar{\psi}_N(t) = \mathrm{d}\mu(t) \sum_{k=0}^{N} c(P_k(\mathrm{d}\mu, x)) P_k(\mathrm{d}\mu, t).$$

*Suppose that*

$$\int_{-1}^{1} \left| \sum_{k=0}^{N} c(P_k(\mathrm{d}\mu, x)) P_k(\mathrm{d}\mu, t) \right| \mathrm{d}\mu(t) < M,$$

*where $M$ is some constant independent of $N$. Then*

$$\mathrm{d}\bar{\psi}_N \underset{\mathrm{w}}{\rightarrow} \mathrm{d}\psi \quad \text{(weak convergence)}, \quad N \rightarrow \infty.$$

Let us now examine the particular case where $\mathrm{d}\psi$ is absolutely continuous with respect to $\mathrm{d}\mu$ (given distribution). Let us set

$$\mathrm{d}\psi(t) = w(t)\,\mathrm{d}\mu(t)$$

and

$$\bar{w}_N(t) := \sum_{k=0}^{N} c(P_k(\mathrm{d}\mu, x)) P_k(\mathrm{d}\mu, t). \tag{6}$$

In (6), $\bar{w}_N$ appears as the best $L^2(\mathrm{d}\mu)$-polynomial approximation of $w$. We can, then, apply the classical theorems.

**Theorem 8** (Cheney [3, p.110]). *If $\mathrm{d}\psi = w\,\mathrm{d}\mu$, with $w$ continuous on $[-1, 1]$, then $w - \bar{w}_N$ has at least $N + 1$ zeros in $[-1, 1]$.*

**Theorem 9** (Freud [7]). *If $w \in L^2(\mathrm{d}\mu)$, then $\bar{w}_N \rightarrow_{N \rightarrow \infty} w$ in $L^2(\mathrm{d}\mu)$.*

**Theorem 10** (Freud [7]). *If $\mathrm{d}\mu$ is absolutely continuous with respect to $\mathrm{d}x$ and $\mu'(x) \leqslant M(1 - x^2)^{-1/2}$, $-1 < x < 1$, if $w$ is continuous on $[-1, 1]$ and of bounded variation, then $\bar{w}_N$ converges uniformly on every set of points $x_0$ for which*

$$\sum_{k=0}^{n-1} P_k^2(\mathrm{d}\mu, x_0) \leqslant Ln$$

*is satisfied with a fixed $L$.*

**Corollary 11.** *If $\mathrm{d}\mu = \mathrm{d}x/\sqrt{1 - x^2}$ on $]-1, 1[$, then*

$$P_k(\mathrm{d}\mu, x) = a_k T_k(x), \quad a_0 = 1/\sqrt{\pi}, \quad a_k = \sqrt{2/\pi}, \quad k \geqslant 1,$$

*and so*

$$\frac{2}{\pi}\sum_{k=0}^{N}{}' T_k(t)c(T_k)$$

*converges uniformly on* $[-1, 1]$ *to* $\sqrt{1-x^2}\psi'(x)$ *if* $\psi'$ *is continuous and of bounded variation on* $[-1, 1]$.

This result holds if we use the best approximation for $g_t$ as in Example 6. See the numerical examples in the Appendix.

## 5. Approached orthogonal polynomials

Let $f$ be the function

$$f(t)=\int_{-1}^{1}\frac{1}{1-xt}\,\mathrm{d}\psi(x),$$

with $\psi$ a function of bounded variation (not necessarily nondecreasing) and where $c_i = \int_{-1}^{1}x^i\,\mathrm{d}\psi(x)$, $i=0,\ldots,N$, are supposed to be given.

We use an approximation in $L^2(\mathrm{d}\mu)$:

$$\mathrm{d}\bar{\psi}_N(t)=\mathrm{d}\mu(t)\sum_{k=0}^{N}c\big(P_k(\mathrm{d}\mu,\,x)\big)P_k(\mathrm{d}\mu,\,t)$$

and calculate the orthogonal polynomials with respect to $\mathrm{d}\bar{\psi}_N$, denoted by $P_k(\mathrm{d}\bar{\psi}_N,\,x)$. Because of

$$\int_{-1}^{1}x^i\,\mathrm{d}\bar{\psi}_N(x)=\int_{-1}^{1}x^i\,\mathrm{d}\psi(x),\quad i=0,\ldots,N,$$

we obtain

$$P_k\big(\mathrm{d}\bar{\psi}_N,\,x\big)=P_k(\mathrm{d}\psi,\,x),\quad k=0,\ldots,\big[\tfrac{1}{2}N\big].$$

The moments of $\mathrm{d}\bar{\psi}_N$ will be denoted by

$$\bar{c}_i:=\int_{-1}^{1}x^i\,\mathrm{d}\bar{\psi}_N(x). \tag{7}$$

So $\bar{c}_i=c_i$ for $i=0,\ldots,N$.

In order to calculate the other coefficients $\bar{c}_i$, $i\geqslant N+1$, we can remark that

$$\bar{c}\big(P_j(\mathrm{d}\mu,\,x)\big)=0,\quad j\geqslant N+1.$$

We have

$$\begin{aligned}\bar{c}\big(P_j(\mathrm{d}\mu,\,x)\big)\\&=\int P_j(\mathrm{d}\mu,\,x)\,\mathrm{d}\bar{\psi}_N(x)=\int P_j(\mathrm{d}\mu,\,x)\sum_{k=0}^{N}c\big(P_k(\mathrm{d}\mu,\,x)\big)P_k(\mathrm{d}\mu,\,x)\,\mathrm{d}\mu(x)\\&=\sum_{k=0}^{N}c\big(P_k(\mathrm{d}\mu,\,x)\big)\int_{-1}^{1}P_j(\mathrm{d}\mu,\,x)P_k(\mathrm{d}\mu,\,x)\,\mathrm{d}\mu(x)=0\quad\text{for } j\geqslant N+1.\end{aligned}$$

So, from the expression of $P_j(\mathrm{d}\mu, x)$ in the basis $1, x, \ldots, x^j$, $\bar{c}(P_j(\mathrm{d}\mu, x)) = 0$ gives $\bar{c}_j$ in terms of $c_0, \ldots, c_N$ and $\bar{c}_{N+1}, \ldots, \bar{c}_{j-1}$. The difference

$$c\big(P_j(\mathrm{d}\mu, x)\big) - \bar{c}\big(P_j(\mathrm{d}\mu, x)\big) \text{ is equal to } c\big(P_j(\mathrm{d}\mu, x)\big) \quad \text{for } j \geqslant N+1,$$

and then tends to zero when $j$ tends to infinity, if $\mathrm{d}\psi = \psi'(x)\,\mathrm{d}x$ with $\psi' \in L^2(\mathrm{d}\mu)$ (see Remark 4).

### 5.1. *Definition*

The polynomials $P_j(\mathrm{d}\bar{\psi}_N, x)$ orthogonal with respect to $\mathrm{d}\bar{\psi}_N$ will be called *approached orthogonal polynomials* with respect to $\mathrm{d}\psi$ at order $N$.

### 5.2. *Expression of $P_j(\mathrm{d}\bar{\psi}_N, x)$ in terms of $P_j(\mathrm{d}\mu, x)$*

$$P_j\big(\mathrm{d}\bar{\psi}_N, x\big) = D_j \begin{vmatrix} \bar{c}(P_0(\mathrm{d}\mu)P_0(\mathrm{d}\mu)) & \bar{c}(P_0(\mathrm{d}\mu)P_1(\mathrm{d}\mu)) & \cdots & \bar{c}\big(P_j(\mathrm{d}\mu)P_0(\mathrm{d}\mu)\big) \\ \bar{c}(P_0(\mathrm{d}\mu)P_1(\mathrm{d}\mu)) & \bar{c}(P_1(\mathrm{d}\mu)P_1(\mathrm{d}\mu)) & \cdots & \bar{c}\big(P_j(\mathrm{d}\mu)P_1(\mathrm{d}\mu)\big) \\ \vdots & \vdots & & \vdots \\ \bar{c}\big(P_0(\mathrm{d}\mu)P_{j-1}(\mathrm{d}\mu)\big) & \bar{c}\big(P_1(\mathrm{d}\mu)P_{j-1}(\mathrm{d}\mu)\big) & \cdots & \bar{c}\big(P_j(\mathrm{d}\mu)P_{j-1}(\mathrm{d}\mu)\big) \\ P_0(\mathrm{d}\mu, x) & P_1(\mathrm{d}\mu, x) & \cdots & P_j(\mathrm{d}\mu, x) \end{vmatrix}, \tag{8}$$

where $\bar{c}(P) = \int_{-1}^{1} P(x)\,\mathrm{d}\bar{\psi}_N(x)$ and $D_j$ a coefficient of normalization.

Formula (8) allows us to calculate the orthogonal polynomial $P_j(\mathrm{d}\bar{\psi}_N, x)$, which gives a continuation of the sequence $P_j(\mathrm{d}\psi, x)$ for $j \geqslant [\frac{1}{2}N] + 1$, if only the $N+1$ first moments of $\mathrm{d}\psi$ are given. (Similar ideas have been used: in [17], it consisted in calculating $P_k(\mathrm{d}\psi, x)$ with the three-term recurrence relationship up to $k = [\frac{1}{2}N]$, and for a greater index $k$, in calculating all the sequences of $P_k$ by taking the coefficients in the recurrence relation constant and equal to their last computed value. In [10], the method involves interpolation between computed and terminator coefficients in the continued fraction.)

**Remark 12.** We can obtain another expression of $P_j(\mathrm{d}\bar{\psi}_N, x)$ because $\mathrm{d}\bar{\psi}_N$ is the product of $\mathrm{d}\mu$ with a polynomial. See [12].

### 5.3. *Particular case* $\mathrm{d}\mu = \mathrm{d}x/\sqrt{1-x^2}$

In this case

$$P_j(\mathrm{d}\mu, x) = T_j(x)a_j, \quad a_j = \sqrt{\frac{2}{\pi}}, \quad j \geqslant 1, \quad a_0 = \frac{1}{\sqrt{\pi}}.$$

$T_j$ is the Chebyshev polynomial of the first kind. Note that

$$T_jT_l = \tfrac{1}{2}\big(T_{j+l} + T_{|j-l|}\big).$$

So, the expressions

$$\mathrm{d}\bar{\psi}_N = \frac{2}{\pi} \frac{\mathrm{d}x}{\sqrt{1-x^2}} \sum_{i=0}^{N}{}' c(T_j) T_j(x) \tag{9}$$

and

$$P_j(\mathrm{d}\bar{\psi}_N,\, x) = D_j \begin{vmatrix} \bar{c}(T_0) & \bar{c}(T_1) & \cdots & \bar{c}(T_j) \\ \bar{c}(T_1) & \bar{c}(\frac{1}{2}(T_2+T_0)) & \cdots & \bar{c}(\frac{1}{2}(T_{j+1}+T_{j-1})) \\ \vdots & \vdots & & \vdots \\ \bar{c}(T_{j-1}) & \bar{c}(\frac{1}{2}(T_j+T_{j-2})) & \cdots & \bar{c}(\frac{1}{2}(T_{2j-1}+T_1)) \\ T_0(x) & T_1(x) & \cdots & T_j(x) \end{vmatrix} \tag{10}$$

hold. The above determinant can be simplified since

$$\begin{cases} \bar{c}(T_j) = c(T_j), & j \leqslant N, \\ \bar{c}(T_j) = 0, & j \geqslant N+1. \end{cases}$$

**Property 13.** *If* $\mathrm{d}\mu = \mathrm{d}x/\sqrt{1-x^2}$ *and* $\psi' \in L^2(\mathrm{d}\mu)$, *then*

$$|c_i - \bar{c}_i| \leqslant A \frac{2^{1-i}}{\pi} \sum_{k=N+1}^{i} \binom{i}{\frac{1}{2}(i-k)}, \tag{11}$$

$c_i - \bar{c}_i = 0$ *if* $i \leqslant N$ *(*$A$ *is a constant:* $A = \int_{-1}^{1} |\mathrm{d}\psi(x)|$*).*

**Proof.** For $i \geqslant N+1$,

$$c_i - \bar{c}_i = \frac{2}{\pi} \sum_{k=N+1}^{i} c(T_k) \int_{-1}^{1} x^i T_k(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}}.$$

But

$$\int_{-1}^{1} x^i T_k(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int_{-1}^{1} 2^{-i} \sum_{j=0}^{i} \binom{i}{j} T_{k-i+2j}(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}}$$

$$= 2^{-i} \binom{i}{\frac{1}{2}(i-k)} \quad \left(0 \text{ if } \tfrac{1}{2}(i-k) \notin \mathbb{N}\right).$$

So

$$c_i - \bar{c}_i = \frac{2}{\pi} \sum_{k=N+1}^{i} c(T_k) 2^{-i} \binom{i}{\frac{1}{2}(i-k)}.$$

An upper bound for $c(T_k)$ is

$$|c(T_k)| = \left| \int_{-1}^{1} T_K(x)\, \mathrm{d}\psi(x) \right| \leqslant \int_{-1}^{1} |\mathrm{d}\psi(x)| = A.$$

*Numerical example*

(a) $\quad d\psi = dx, \qquad N=5, \qquad c_i = \dfrac{1+(-1)^i}{i+1},$

$$c_0 = 2, \qquad c_1 = 0, \qquad c_2 = \tfrac{2}{3}, \qquad c_3 = 0, \qquad c_4 = \tfrac{2}{5}, \qquad c_5 = 0.$$

Calculation of $\bar{c}_6$:

$$\bar{c}(T_6) = 0 \;\Rightarrow\; 32\bar{c}_6 = 48c_4 - 18c_2 + c_0 \;\Rightarrow\; \bar{c}_6 = \tfrac{23}{80} = 0.2875.$$

Formula (11) gives $|c_6 - \bar{c}_6| \leqslant (\pi \cdot 2^4)^{-1} \simeq 0.02$ (note that $c_6 = 0.2857$); $\bar{c}(T_7) = \bar{c}(T_8) = 0$ gives $\bar{c}_7 = 0$, $\bar{c}_8 = 0.2260\ldots$ (compare with $c_8 = 0.222\ldots$), (11) gives $|c_8 - \bar{c}_8| \leqslant 0.045$.

(b) $\quad d\psi = x\,dx \;$ on $[-1, 1], \qquad c_i = \dfrac{1+(-1)^{i+1}}{i+2}, \qquad A = \displaystyle\int_{-1}^{1} |x|\,dx = 1.$

Let us take $N = 5$. From $\bar{c}(T_6) = 0$, $\bar{c}_6 = 0$. From $\bar{c}(T_7) = 0$, $\bar{c}_7 = 0.2229$ (compare with $c_7 = 0.2222$) and

$$|c_7 - \bar{c}_7| \leqslant \frac{A}{\pi}\frac{1}{2^6} \simeq 5 \cdot 10^{-3}.$$

## 6. Approached Padé approximants

### *6.1. Introduction*

We only study approached Padé approximants for the function

$$f(t) = \int_{-1}^{1} \frac{1}{1-xt}\,d\psi(x),$$

where $\psi$ is a function of bounded variation on $[-1, 1]$. We suppose that $c_0, \ldots, c_N$ are known:

$$c_i := \int_{-1}^{1} x^i\,d\psi(x).$$

Let us approximate $d\psi$ by

$$d\bar{\psi}_N(t) = d\mu(t) \sum_{k=0}^{N} c(P_k(d\mu, x))P_k(d\mu, t),$$

and search the Padé approximant for

$$\bar{f}_N(t) := \int_{-1}^{1} \frac{1}{1-xt}\,d\bar{\psi}_N(x).$$

**Theorem 14.**

$$\bar{f}_N(t) - f(t) = O(t^{N+1}).$$

**Proof.**

$$\bar{f}_N(t) = \int_{-1}^{1} \frac{1}{1-xt} \sum_{k=0}^{N} c(P_k(\mathrm{d}\mu,\, x)) P_k(\mathrm{d}\mu,\, x)\, \mathrm{d}\mu(x)$$

$$= \sum_{k=0}^{N} c(P_k(\mathrm{d}\mu,\, x)) q_k(\mathrm{d}\mu,\, t),$$

where

$$q_k(\mathrm{d}\mu,\, t) = \int_{-1}^{1} \frac{1}{1-ut} P_k(\mathrm{d}\mu,\, u)\, \mathrm{d}\mu(u).$$

$q_k(\mathrm{d}\mu,\, t)$ is holomorphic in $\mathbb{C} \setminus ]-\infty,\, -1[ \cup ]1,\, +\infty[$ (associated function), and $q_k(\mathrm{d}\mu,\, t) = O(t^k)$ since

$$q_k(\mathrm{d}\mu,\, t) = \int_{-1}^{1} \frac{t^k u^k}{1-ut} P_k(\mathrm{d}\mu,\, u)\, \mathrm{d}\mu(u)$$

by the orthogonality of $P_k(\mathrm{d}\mu,\, u)$. This proof uses, in fact, the property that the first $N+1$ moments of $\mathrm{d}\bar{\psi}_N$ are $c_0, \ldots, c_N$. (See Section 4.2.) □

**Definition 15.** Let $c_0, \ldots, c_N$ be given and $\bar{c}_{N+1}$, $\bar{c}_{N+2}, \ldots$ be estimated as explained in Section 5; we can construct a Padé approximant for the function $\bar{f}_N(t)$. We shall call it an *approached Padé approximant of* $f$.

*6.2. Convergence of approached Padé approximants*

In [13], we proved that, under some conditions about the distribution $\mathrm{d}\mu$, (a) $\bar{f}_N$ converges to $f$ uniformly on every compact set of the complex plane cut along $]-\infty,\, -1] \cup [1,\, +\infty[$; (b) the factor of convergence is such that

$$\overline{\lim_{N \to \infty}} \left| f(t) - \bar{f}_N(t) \right|^{1/N} \leqslant \left| t^{-1} \pm \sqrt{t^{-2} - 1} \right| < 1.$$

Note that assertions (a) and (b) remain true if $f$ is only analytic on $\mathbb{C} \setminus ]-\infty,\, 1] \cup [1,\, +\infty[$.

For the other terms of Table 1, the problem is more complicated since the Padé approximant of a function $f$ does not generally converge to $f$. Nevertheless, if $\psi$ is nondecreasing on $[-1,\, 1]$,

Table 1

| | $c_0$ | $c_1$ | $c_2$ | $c_3$ | $\cdots$ | $c_N$ | $\bar{c}_{N+1}$ | $\bar{c}_{N+2}$ |
|---|---|---|---|---|---|---|---|---|
| $c_0$ | $[0/0]_f$ | | | | | $[N/0]_f$ \| $[N+1/0]_{\bar{f}_N}$ | | |
| $c_1$ | | | | | | | | |
| $c_2$ | | Exact Padé | | | | | | |
| $c_3$ | | approximants of $f$ | | | | | | |
| $\vdots$ | | | | | | Approached Padé | | |
| | | | | | | approximants of $f$ | | |
| $c_N$ | $[0/N]_f$ | | | | | | | |
| $\bar{c}_{N+1}$ | $[0/N+1]_{\bar{f}_N}$ | | | | | | | |
| $\bar{c}_{N+2}$ | | | | | | | | |

$\bar{c}(P_k(\mathrm{d}\mu,\, x)) = 0$, $k \geqslant N+1$, gives $\bar{c}_{N+1}$, $\bar{c}_{N+2}, \ldots$ .

it is well known that the Padé approximants $[N/N]$ converge to $f$. So, if the function $\bar{\psi}_N$ is nondecreasing [1], the same is true for the Padé approximant of $\bar{f}_N$ and then for the approached Padé approximant of $f$.

## 7. Padé–Chebyshev approximant of the weight function

In this section, we restrict ourselves to the particular case

$$\begin{cases} \mathrm{d}\mu = \dfrac{\mathrm{d}x}{\sqrt{1-x^2}}, \\ \mathrm{d}\psi(x) = w(x)\,\mathrm{d}x, \end{cases} \qquad w \geqslant 0 \text{ on } [-1, 1].$$

Formula (9) gives an approximation of the weight function $w$:

$$w(x) \sim \bar{w}_N(x) = \frac{2}{\pi} \frac{1}{\sqrt{1-x^2}} \sum_{j=0}^{N}{}' c(T_j) T_j(x),$$

where

$$c(T_j) = \int_{-1}^{1} T_j(x)\,\mathrm{d}\psi(x).$$

Instead of using the approximation $\bar{w}_N$, we can try to construct a Padé–Chebyshev approximant of $w$ in the following way: find $a_0, a_1, \ldots, a_p, b_0, \ldots, b_q$ such that

$$\left[\tfrac{1}{2}a_0T_0 + a_1T_1 + \cdots + a_pT_p\right] - \left[\sum_0^N{}' c(T_j)T_j\right]\left[\tfrac{1}{2}b_0T_0 + b_1T_1 + \cdots + b_qT_q\right] = \mathrm{O}(T_{N+1}). \tag{12}$$

The product on the left-hand side of (12) gives terms from $T_0$ up to $T_{N+q}$ with the very simple rule of multiplication

$$T_iT_j = \tfrac{1}{2}\left[T_{i+j} + T_{|i-j|}\right].$$

Equation (12) gives rise to the system (see [1])

$$\tfrac{1}{2}\sum_{j=0}^{q}{}' b_j\left(c(T_{i+j}) + c(T_{|i-j|})\right) = 0, \qquad i = p+1, \ldots, p+q,$$

$$\tfrac{1}{2}\sum_{j=0}^{q}{}' b_j\left(c(T_{i+j}) + c(T_{|i-j|})\right) = a_i, \qquad i = 0, 1, \ldots, p.$$

From the definition of Padé–Chebyshev approximants, we immediately obtain the following theorem.

[1] If $\mathrm{d}\psi(x) = w(x)\,\mathrm{d}x$ with $w \geqslant 0$ on $[-1, 1]$, it is possible to use an approximation of $w$ using Legendre polynomials which is positive on $[-1, 1]$. These approximants were introduced by Durmeyer [6] and studied by Deriennic [5].

**Theorem 16.** *If the weight function* $w$ *is the product of* $1/\sqrt{1-x^2}$ *by a rational fraction of degree* $p/q$, *then*

$$[p/q]\frac{2}{\pi\sqrt{1-x^2}} = w(x) \quad \forall x \in ]-1, 1[,$$

*where* $[p/q]$ *denotes the Padé–Chebyshev approximant of the series* $\Sigma_0^{\prime\infty} c(T_i)T_i(x)$.

Note that the construction of $[p/q]$ needs the coefficients $c(T_0)$, $c(T_1), \ldots, c(T_{p+2q+1})$.

**Remark 17.** This kind of rational approximant is called a Maehly approximant. See [1, II, p.58].

**Remark 18.** It can be proved that the poles of the Padé–Chebyshev approximant approach the points where the distribution $d\psi$ has Dirac masses [15].

**Acknowledgements**

I thank C. Brezinski for the idea used in Section 7 and A. Magnus for having pointed out to me the references [10,18].

**Appendix**

Figure 1 shows approximations of $d\psi(x) = w(x)\,dx$ for different values of $w$. $w$ is in dotted line, its approximation $\bar{w}_N$ in solid line.

$$d\bar{\psi}_N = \bar{w}_N(x)\,dx, \quad \bar{w}_N = \frac{2}{\pi}\frac{1}{\sqrt{1-x^2}}\sum_0^{N}{}' c(T_i)T_i$$

for Chebyshev polynomials of the first kind:

$$T_{i+1}(x) = 2xT_i(x) - T_{i-1}(x), \quad T_0 = 1, \ T_1(x) = x.$$

$$\bar{w}_N = \frac{2}{\pi}\sqrt{1-x^2}\sum_0^N c(U_i)U_i$$

for Chebyshev polynomials of the second kind:

$$U_{i+1}(x) = 2xU_i(x) - U_{i-1}(x), \quad U_0 = 1, \ U_1(x) = 2x.$$

$$\bar{w}_N = \sum_0^N (i+\tfrac{1}{2})c(P_i)P_i,$$

$P_i$ a Legendre orthogonal polynomial:

$$P_{i+1}(x) = \frac{2i+1}{i+1}xP_i(x) - \frac{i}{i+1}P_{i-1}(x), \quad P_0 = 1, \ P_1(x) = x.$$

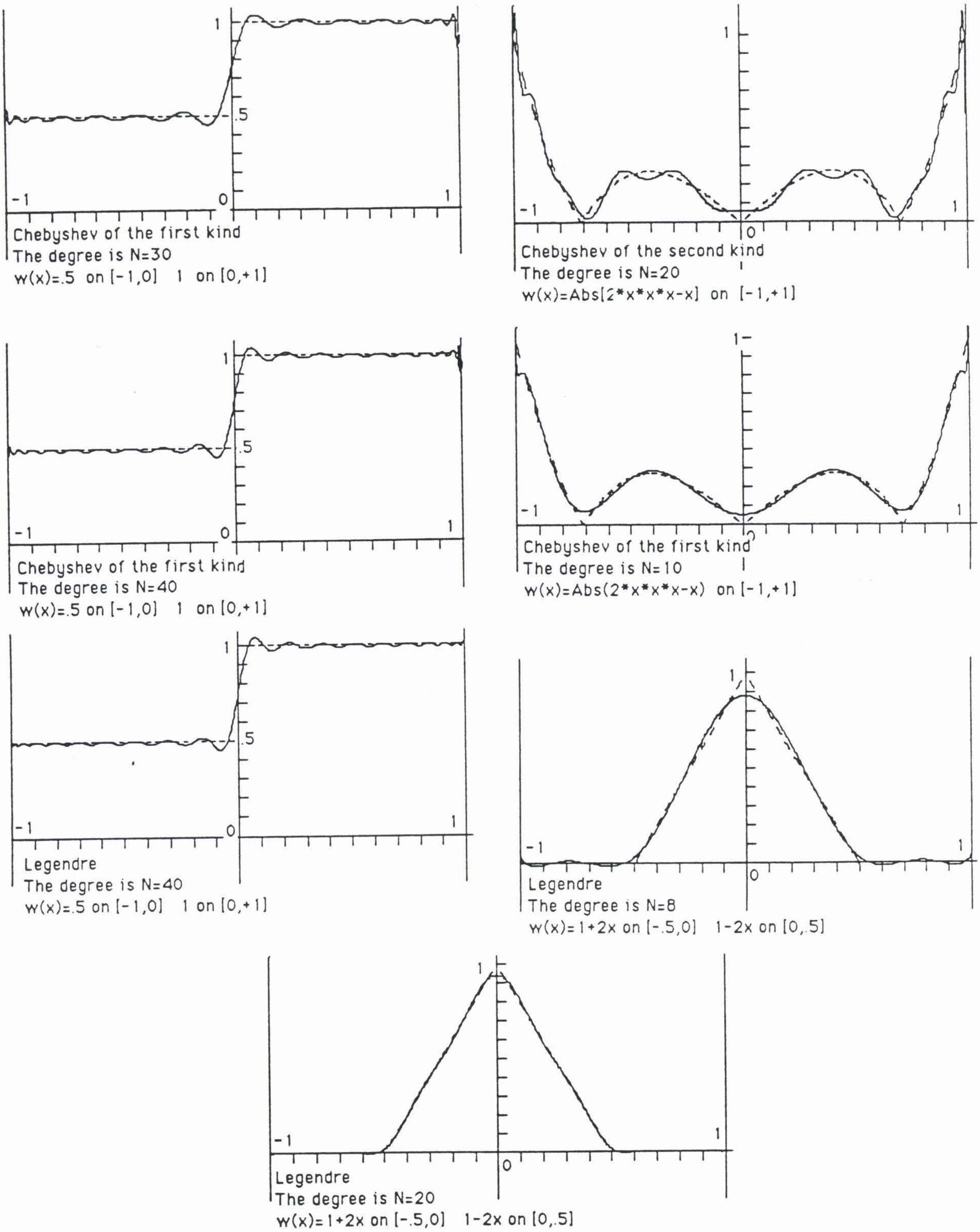
Chebyshev of the first kind
The degree is N=30
w(x)=.5 on [-1,0] 1 on [0,+1]
Chebyshev of the second kind
The degree is N=20
w(x)=Abs[2*x*x*x-x] on [-1,+1]
Chebyshev of the first kind
The degree is N=40
w(x)=.5 on [-1,0] 1 on [0,+1]
Chebyshev of the first kind
The degree is N=10
w(x)=Abs(2*x*x*x-x) on [-1,+1]
Legendre
The degree is N=40
w(x)=.5 on [-1,0] 1 on [0,+1]
Legendre
The degree is N=8
w(x)=1+2x on [-.5,0] 1-2x on [0,.5]
Legendre
The degree is N=20
w(x)=1+2x on [-.5,0] 1-2x on [0,.5]

Fig. 1.

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# Dirac Masses detection in a density on [-1,1] from its moments. Applications to singularities of a function.

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**ABSTRACT.**

Let us consider a distribution $c$ which is the sum of another one (with an asymptotic property on a given sequence of orthogonal polynomials) and a linear combination of Dirac distributions (and their derivatives) at the points $a_1, \ldots, a_q$.

In the section 3 we present a method to compute the points $a_1, \ldots, a_q$ from the moments of $c$. In order to detect the singularities of a function $f$, we differentiate $f$ in the sense of the distributions. The jumps of $f$ appears as Dirac masses in the derivatives. The jumps of $f'$ will appear as Dirac masses in $f''$ and so on. The singularities of $f$ can therefore be computed by mean of a method explained in section 4.

## 1 Introduction and notations.

The space of linear functional (distributions) defined on $\mathcal{P}$, space of polynomials, will be denoted by $\mathcal{P}'$.

The Dirac distribution $\delta_{x_0}$ and its derivative of order $m$, $D^m\delta_{x_0}$ are defined as follows :

$$\delta_{x_0}(\varphi) := <\delta_{x_0}, \varphi> := \varphi(x_0)$$

$$<D^m\delta_{x_0}, \varphi> = \varphi^{(m)}(x_0).(-1)^m$$

.

Let $c$ and $\omega$ be two elements of $\mathcal{P}'$. We assume that the moments $c_n := <c, x^n>$ for $n = 0, 1, \ldots$ are given and that $c$ and $\omega$ are linked by the following relation :

$$(1) \qquad c = \omega + \sum_{i=1}^{m}\left[\sum_{j=0}^{r_i-1}(-1)^j A_{ij} D^j \delta_{a_i}\right]$$

$A_{i,j} \in \mathbb{R}, i = 1, m$
$a_i$ distinct, $j = 0, r_i - 1$
$A_{i,r_i-1} \neq 0, q := \sum_{i=1}^{m} r_i$

Relation (1) shows that $c$ is equal to $\omega$ apart from a linear combination of Dirac distributions and their derivatives at the points $a_1, \cdots, a_m$.

If the distribution $\omega$ is such that $\omega_n := <\omega, x^n> \to 0$ when $n \to \infty$ faster than $\max_{i=1,m}|a_i|^n$, then the $a_i$'s can be computed by the Q-D algorithm (Section 2).

The aim of this paper is to compute the $a_i$'s when the distribution $\omega$ satisfies

$$\lim_{k\to\infty} <\omega, P_k P_i> = 0, i = 0, \cdots, q$$

for some sequence of polynomials $P_j$.

## 2 Padé Approximation - QD algorithm.

**Theorem 1** *([4])*
*If $c = \omega + \sum_{i=1}^{m}\sum_{j=0}^{r_i-1}(-1)^j A_{ij} D^j \delta_{a_i}$, $a_i$ distinct points of* $\mathbb{C}$ *and*
$<\omega, x^n> = 0$, $\forall n \in \mathbb{N}$ *then*

$$\sum_{n=0}^{\infty} c_n t^n = \sum_{i=1}^{m}\sum_{j=0}^{r_i-1}\left(\frac{A_{ij}}{1 - a_i t}\right)^{(j)}$$

*(rational fraction of degree $(M, q)$ )*
*where $c_n := c(x^n) := <c, x^n>$, $q = \sum_{i=1}^{m} r_i$,*
*$M = q - Min\{r_i, i = 1, \cdots, m\}$, and the Padé approximant of $\sum_{n=0}^{\infty} c_n t^n$ is*

$$[p/q] = \sum_{n=0}^{\infty} c_n t^n \qquad \text{for } p \geq M$$

$\triangle$

Note that the poles of the Padé approximant $[p/q]$, $p \geq M$ are the points $a_i^{-1}, i = 1, \ldots, m$ with order $r_i$ ; the latter one can be computed with the QD algorithm of Rutishauser [8].

In some particular case, the convergence of the P.A. to the function $f$ can be established:

**Theorem 2** *(Montessus de Ballore)*
*If $c$ and $\omega$ satisfy (1) and $\lim_{n\to\infty} \frac{\omega_n}{a_i^n} = 0$, $i = 1, \cdots, m$ then the Padé approximant $[p/q]$ tends to*

$$f(t) = \sum_{n=0}^{\infty} c_n t^n$$

*uniformly on every compact included in*

$$D\left(0, \max_{1\leq i\leq m} |a_i^{-1}|\right) - \bigcup_{i=1}^{m}\{a_i^{-1}\}$$

*when the degree p of the numerator tends to infinity. The poles of $[p/q]$ tend to $a_i^{-1}, i = 1, \cdots, m$ with the order $r_i$ when p tends to infinity.*

**Proof.** See [3] for $r_i = 1$, $i = 1, \ldots, m$ and also the elegant proof of Saff [9] for the general case. The $a_i$'s can be approximated by the columns of the QD algorithm. In general the convergence of these columns to $a_i^{-1}$ is linear [5].The $a_i$'s can also be approximated by the $\varepsilon$ and $r-s$ algorithms, the convergence being also linear [5] and so can be accelerated.

## 3 Extraction of Dirac distribution.

In this part, we assume another kind of asymptotic condition on $\omega$. Let $(P_k)_{k\in\mathbb{N}}$ be a sequence of monic polynomials of exact degree $k$. We make the following assumption on $\omega$ :

(2) $$\lim_{k\to\infty} \omega(P_i P_k) = 0 \quad \text{for} \quad i = 0, \cdots, q$$

After multiplication of the two handsides of equality (1) by the polynomial $P(x) = \prod_{i=1}^{m}(x - a_i)^{r_i}$ of degree $q$, we get :

(3) $$P.c = P.\omega$$

The equality (3) is equivalent to

$$c(P(x)x^k) = \omega(P(x).x^k) \quad \forall k \in \mathbb{N}$$

$$\text{or} \quad c(P(x)P_k(x)) = \omega(P(x)P_k(x)) \quad \forall k \in \mathbb{N}$$

In order to find the points $a_i$'s, let us expand the polynomial $P(x)$ in terms of $P_0, P_1, \cdots, P_q$

$$P(x) = \prod_{i=1}^{m}(x - a_i)^{r_i} = \sum_{i=0}^{q} b_i P_i(x) \quad , \quad b_q = 1.$$

$$(3) \Leftrightarrow \sum_{i=0}^{q} b_i c(P_i P_k) = \sum_{i=0}^{q} b_i \omega(P_i P_k) \qquad (4)$$

The coefficients $b_i$ will be computed by solving the following linear system :

$$(S_N^q)\begin{cases} \sum_{i=0}^{q} b_i^{(N)} c(P_i P_k) = 0, \text{for } \; k = N, \ldots, N+q-1 \\ b_q^{(N)} = 1 \end{cases}$$

and applying the following theorem 3.

**Theorem 3** *Let c and $\omega$ be defined as in (1). We assume that the sequence $(P_k)_{k\in\mathbb{N}}$ has the properties (i) and (ii)*

$$(i) \lim_{k\to\infty} \sum_{i=1}^{m}\sum_{j=0}^{r_i-1} \alpha_{ij} P_k^{(j)}(a_i) = 0 \iff \alpha_{ij} = 0, \begin{cases} i = 1, \cdots, m \\ j < r_i \end{cases}$$

*(ii)*

$$\begin{vmatrix}
P_0(a_1) & 0 & 0 & \cdots & 0 & \cdots & P_0(a_m) & 0 & 0 & \cdots & 0 \\
P_1(a_1) & P_1'(a_1) & 0 & \cdots & \cdots & \cdots & P_1(a_m) & P_1'(a_m) & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
P_{r_1-1}(a_1) & P'_{r_1-1}(a_1) & \cdots & \cdots & P_{r_1-1}^{(r_1-1)}(a_1) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & P_{r_q-1}(a_m) & P'_{r_q-1}(a_m) & \vdots & \vdots & P_{r_m-1}^{(r_m-1)}(a_m) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
P_{q-1}(a_1) & P'_{q-1}(a_1) & \cdots & \cdots & P_{q-1}^{(r_1-1)}(a_1) & \cdots & \cdots & P_{q-1}(a_m) & \cdots & \cdots & P_{q-1}^{(r_m-1)}(a_1)
\end{vmatrix}$$

*different from 0.*

*If the solution $(b_0^{(N)}, \cdots, b_q^{(N)})$ of the system $(S_N^q)$ converges to $(l_0, \cdots, l_q)$ when N tends to infinity, then, this limit satisfies $l_i = b_i \quad i = 0, \cdots, q$.*

**Proof.** Let us set $\Delta = \sum_{i=1}^{m}\sum_{j=0}^{r_i-1} A_{ij}(-1)^j D^j \delta_{a_i}$, $q = \sum_{i=1}^{m} r_i$ and

$$A_N = \begin{pmatrix} c(P_0 P_N) & \cdots & c(P_q P_N) \\ \vdots & & \vdots \\ c(P_0 P_{N+q}) & \cdots & c(P_q P_{N+q}) \end{pmatrix}$$

$$\Omega_N = \begin{pmatrix} \omega(P_0 P_N) & \cdots & \omega(P_q P_N) \\ \vdots & & \vdots \\ \omega(P_0 P_{N+q}) & \cdots & \omega(P_q P_{N+q}) \end{pmatrix}$$

$(S_N^q)$ becomes $A_N \bar{b}^{(N)} = 0$ where $\bar{b}^{(N)} = (b_0^{(N)}, \cdots, b_q^{(N)})$, $b_q^{(N)} = 1$. The $b_i$'s satisfy the system $A_N \bar{b} = \Omega_N \bar{b}$ where $\bar{b} = (b_0, \cdots, b_q)$, $b_q = 1$.
From $\lim_{N\to\infty} \bar{b}^{(N)} = \bar{l} = (l_0, \cdots, l_q)$ it arises $\lim_{N\to\infty}(A_N - \Omega_N)(\bar{b} - \bar{l}) = 0$ and $\lim_{k\to\infty} \sum_{s=o}^{q}(b_s - l_s)\,\Delta(P_s P_k) = 0$.
By applying Leibniz formula for the derivation of the product $P_s P_k$, we get :

$$\lim_{k\to\infty} \sum_{i=1}^{m}\sum_{l=0}^{r_i-1} \alpha_{il} P_k^{(l)}(a_i) = 0$$

with

$$\alpha_{il} = \sum_{n=l}^{r_i-1} A_{i,n} \binom{n}{l} \sum_{j=0}^{q-1}(b_j - l_j)P_j^{(n)}(a_i) \quad (b_q = l_q = 1)$$

The property (i) of the sequence $(P_k)$ implies

$(S)$ $\quad \alpha_{il} = 0$ for $i = 1, \cdots, m$ and $l = 0, \cdots, r_i - 1$

$(b_0 - l_0, \cdots, b_{q-1} - l_{q-1})$ satisfies the above system (S) whose determinant is non zero (property (ii)) and so $b_i = l_i$, $i = 0, \cdots, q$. $\triangle$

**Remark 1.** Properties (i) and (ii) seem to be very constrained. In the appendix, we show that a large class of orthogonal polynomials satisfies (i). The property (ii) is a condition on the location of the points $a_1, \cdots, a_m$.

**Remark 2.** With the same proof as in theorem 3, it is possible to prove that the solution of $(S_N^{q'})$ cannot have a limit for $q' \neq q$.
The above theorem 3 admits a reciprocal precised by

**Theorem 4** *Let $c$ and $\omega$ be defined as in (1). We assume that the sequence $(P_k)_{k\in\mathbb{N}}$ has the properties (iii) (and (ii))*
*(iii) $D_n =$*

$$\begin{vmatrix} P_n(a_1) & \cdots & P_n^{(r_1-1)}(a_1) & \cdots & P_n(a_m) & \cdots & P_n^{(r_m-1)}(a_m) \\ \vdots & & \vdots & & \vdots & & \vdots \\ P_{n+q-1}(a_1) & \cdots & P_{n+q-1}^{(r_1-1)}(a_1) & \cdots & P_{n+q-1}(a_m) & \cdots & P_{n+q-1}^{(r_m-1)}(a_m) \end{vmatrix}$$

*doesn't converge to 0.*
*Then $(b_0, \cdots, b_q)$ is a cluster point of the sequence : $((b_0^{(N)}, \cdots, b_q^{(N)}))_{N\in\mathbb{N}}$ of solutions of the system $(S_N^q)$.*

**Proof.**
We have :

$$A_N\vec{b} = \Omega_N\vec{b}$$
$$A_N\vec{b}^{(N)} = 0$$

and so :

$$\begin{pmatrix} \Delta(P_0P_N) & \cdots & \Delta(P_qP_N) \\ \vdots & & \vdots \\ \Delta(P_0P_{N+q}) & \cdots & \Delta(P_qP_{N+q}) \end{pmatrix} \begin{pmatrix} b_0^{(N)} \\ \vdots \\ b_q^{(N)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$A_N - \Omega_N = (c-\omega)(P_iP_{N+j}), i = 0,q, j = 0,q) =$$
$$(\Delta(P_iP_{N+j}), i = 0,q, j = 0,q)$$

$$\lim_{N\to\infty}(A_N - \Omega_N)(\vec{b}^{(N)} - \vec{b}) = 0 \Rightarrow$$
$$\lim_{N\to\infty}\sum_{s=0}^{q}(b_s^{(N)} - b_s)\Delta(P_sP_N) = 0$$

and

$$\sum_{i=1}^{m}\sum_{l=0}^{r_i-1} P_N^{(l)}(a_i)\left(\sum_{n=l}^{r_i-1} A_{i,n}\begin{pmatrix} n \\ l \end{pmatrix}\sum_{j=0}^{q-1}(b_j^{(N)} - b_j)P_j^{(n)}(a_i)\right)$$

converges to 0 when $N$ converge to $\infty$.
A sequence $(N_k)_{k\in\mathbb{N}}$ can be choosen such that the determinant $D_{N_k}$ involved in (iii) has an absolute value greater than a given strictly positive real number independant of $k$ (or equivalently the matrix of (iii) is uniformly inversible).
So, the sequence $\sum_{n=l}^{r_i-1} A_{i,n}\binom{n}{l}\sum_{j=0}^{q-1}(b_j^{(N_k)} - b_j)P_j^{(n)}(a_i)$ converges to 0 when $k$ goes to infinity, for each $i = 1, \cdots, m$.
From property (ii), we can conclude that $\lim_{k\to\infty} b_j^{(N_k)} = b_j$ for each $j = 0, \cdots, q-1$.

$\triangle$

# 4 Applications

In this section, we assume that the support of the distribution $\omega$ is included in $[-1,1]$ and is absolutely continuous with respect to the Lebesgue measure on $[-1,1]$.

$$(5) \qquad \forall p \in \mathcal{P}, \omega(p) = \int_{-1}^{1} p(x)w(x)dx, w \in L^1$$

$c$ and $\omega$ are assumed to be linked with relation (1).
Assumption (2) will be satisfied by the Chebyshev polynomials on $[-1,1]$ as follows : the Riemann-Lebesgue lemma ([7] page 99), and the very simple multiplication rule

$$T_iT_k = \frac{1}{2}(T_{i+k} + T_{|i-k|})$$

insure that

$$(6) \qquad lim_{n\to\infty}\omega(T_n) = lim_{n\to\infty}\int_{-1}^{1} T_n(x)w(x)dx = 0$$

and $lim_{k\to\infty}\omega(T_iT_k) = 0$ for each fixed $i \in \mathbb{N}$.
The hypothesis (i) of theorem 3 will proceed from the following lemma

**Lemma 1** *Let $\{\alpha_{ij}, i = 1, \cdots, m; j = 0, \cdots, r_i - 1\}$ be some set of complex numbers and let $a_1, \cdots, a_m$ be $m$ distinct points of $\mathbb{C}$, then*

$$\lim_{k\to\infty}\sum_{i=1}^{m}\sum_{j=0}^{r_i-1}\alpha_{ij}T_k^{(j)}(a_i) = 0 \Rightarrow \alpha_{ij} = 0, \quad \begin{matrix} i = 1, \cdots, m, \\ j = 0, \cdots, r_i - 1 \end{matrix}$$

**Proof.** If $\alpha_{l,r_l-1}$ is non zero, let us consider the polynomial of degree $q-1$

$$\tilde{P}(x) = \frac{P(x)}{(x-a_l)} = \prod_{\substack{k=1 \\ k\neq l}}^{m}(x-a_k)^{r_k}(x-a_l)^{r_l-1} = \sum_{i=0}^{q-1}{}'\beta_iT_i(x)$$

where $q = \sum_{i=1}^{m} r_i$ and $\sum_{i=0}^{m}{}'u_i := \frac{1}{2}u_0 + \sum_{i=1}^{m} u_i$.
By expanding $\tilde{P}.T_k$ in terms of $T_{k-q+1}, T_{k-q+2}, \cdots, T_{k+q-1}$

$$\tilde{P}T_k = \sum_{i=0}^{q-1}{}'\beta_iT_iT_k = \frac{1}{2}\sum_{i=0}^{q-1}{}'\beta_i[T_{i+k} + T_{|i-k|}] =$$

$$\frac{1}{2}(\beta_0T_k + \beta_1(T_{k-1} + T_{k+1}) + \cdots + \beta_{q-1}(T_{k+q-1} + T_{k-q+1}))$$

We get :

$$lim_{k\to\infty}\sum_{i=1}^{m}\sum_{j=0}^{r_i-1}\alpha_{ij}(\tilde{P}T_k)^{(j)}(a_i) =$$
$$lim_{k\to\infty}\alpha_{l,r_l-1}T_k(a_l)\tilde{P}^{(r_l-1)}(a_l) = 0$$

$\tilde{P}^{(r_l-1)}(a_l) \neq 0$ and $lim_{k\to\infty}T_k(a_l) \neq 0$ leads to a contradiction. $\triangle$

With Chebychev polynomials, the system $(S_N^q)$ and theorem 3 can be rewritten in the following way :

**Theorem 5** *Let $c$ and $\omega$ be defined as in (1). We assume that $\omega$ satisfies (5). If the points $a_1, \cdots, a_m$ are such that the determinant*

$$\Delta_q = \begin{vmatrix}
T_0(a_1) & 0 & 0 & \cdots & 0 & \cdots & T_0(a_m) & 0 & 0 & \cdots & 0 \\
T_1(a_1) & T_1'(a_1) & 0 & \cdots & \cdots & \cdots & T_1(a_m) & T_1'(a_m) & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
T_{r_1-1}(a_1) & T'_{r_1-1}(a_1) & \cdots & \cdots & T^{(r_1-1)}_{r_1-1}(a_1) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & T_{r_q-1}(a_m) & T'_{r_q-1}(a_m) & \vdots & \vdots & T^{(r_m-1)}_{r_m-1}(a_m) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
T_{q-1}(a_1) & T'_{q-1}(a_1) & \cdots & \cdots & T^{(r_1-1)}_{q-1}(a_1) & \cdots & \cdots & T_{q-1}(a_m) & \cdots & \cdots & T^{(r_m-1)}_{q-1}(a_1)
\end{vmatrix}$$

*is different from 0 and if the solution $(b_0^{(N)}, \cdots, b_q^{(N)})$ of the system $(T_N^q)$*

$$\begin{cases} \sum_{i=0}^q b_i^{(N)}(c(T_{i+k}) + c(T_{|i-k|})) = 0, k = N, \cdots, N+q-1 \\ b_q^{(N)} = 1 \end{cases}$$

*converges to a limit $(l_0, l_1, \cdots, l_q)$ when $N$ goes to infinity, then*

$$l_i = b_i \quad i = 0, \cdots, q$$

*where the $b_i$'s are defined by*

$$\prod_{i=1}^m (x - a_i)^{r_i} = \sum_{i=0}^q b_i T_i(x)$$

The theorem 5, in the particular case $q = 0$, provides a criterion to detect Dirac masses in a density as follows.

**Theorem 6** *Let $c$ and $\omega$ be defined as in (1). We assume that $\omega$ satisfies (5) then*

$$\lim_{n\to\infty} c(T_n) = 0 \Leftrightarrow q = 0 \; (no \;\; Dirac \;\; mass)$$

**Proof.** Let us assume $q = 0$ then $c$ is equal to $\omega$ and

$$c(T_n) = \omega(T_n) \to 0 \quad n \to \infty$$

Conversely, we assume that $\lim_{n\to\infty} c(T_n) = 0$ and $q \neq 0$. Then, $m$ points $a_1, \cdots, a_m$, $m$ integers $r_1, ..., r_m$, and $q$ coefficients $A_{ij}$, $i = 1, \cdots, m$ , $j = 0, \cdots, r_i - 1$, exist such that

$$c - \omega = \sum_{i=1}^m \sum_{j=0}^{r_i-1} A_{ij}(-1)^j D^j \delta_{a_i}, \quad A_{i,r_i-1} \neq 0 \quad i = 1, \cdots, m$$

From $\lim_{n\to\infty} \omega(T_n) = 0$, (6) and $\lim_{n\to\infty} c(T_n)$, we deduce

$$\lim_{n\to\infty} \sum_{i=1}^m \sum_{j=0}^{r_i-1} A_{ij} T_n^{(j)}(a_i) = 0,$$

which is impossible unless $A_{ij} = 0 \quad i = 1, \cdots, m$ , $j = 0, \cdots, r_i - 1$ by lemma 1. $\triangle$

With Chebychev polynomials, theorem 4 becomes:

**Theorem 7** *Let $c$ and $\omega$ be defined as in (1). If the points $a_1, \cdots, a_m$ are such that the determinant $\Delta_q$ of theorem 5 is non zero, then $(b_0, \cdots, b_q)$ is a cluster point of the sequence $(b_0^{(N)}, \cdots, b_q^{(N)})_{N\in\mathbb{N}}$ of solutions of the system $(T_N^q)$*

The proof is only based on the fact that

$$\begin{vmatrix} T_n(a_1) & \cdots & T_n^{(r_1-1)}(a_1) & \cdots & T_n(a_m) & \cdots & T_n^{(r_m-1)}(a_m) \\ \vdots & & \vdots & & \vdots & & \vdots \\ T_{n+q-1}(a_1) & \cdots & T^{(r_1-1)}_{n+q-1}(a_1) & \cdots & T_{n+q-1}(a_m) & \cdots & T^{(r_m-1)}_{n+q-1}(a_m) \end{vmatrix}$$

can't converge to 0, which will be an easy consequence of lemma 3 (See Appendix).

## 5 Appendix.

We shall show that a large class of orthogonal polynomials $(P_k)_{k\in\mathbb{N}}$ satisfy the property (i) :

$$\lim_{k\to\infty} \sum_{i=1}^m \sum_{j=0}^{r_i-1} \alpha_{ij} P_k^{(j)}(a_i) = 0 \Leftrightarrow \alpha_{ij} = 0, \quad \begin{matrix} i = 1, \cdots, m, \\ j = 0, \cdots, r_i - 1 \end{matrix}$$

for $a_1, \cdots, a_m$ distinct points of $\mathbb{C}$. It is well known that, if $(P_k)_{k\in\mathbb{N}}$ is a sequence of orthogonal polynomials with respect to a linear functional, then the product $P_i P_k$ can be expanded in terms of $P_{|i-k|}, P_{|i-k|+1}, \cdots, P_{i+k}$

$$P_i(x)P_k(x) = \sum_{j=|i-k|}^{i+k} \alpha_{ikj} P_j(x)$$

So, if it exits some real $M$ such that

$$|\alpha_{i,k,j}| < M, \; \forall k \in \mathbb{N} \quad \forall j = |k-i|, \cdots, k+i,$$

and if $(P_k(a))_{k\in\mathbb{N}}$ doesn't converge to 0 whatever $a \in \mathbb{C}$ (that is true for Jacobi polynomials) then the sequence $(P_k)_{k\in\mathbb{N}}$ satisfy (i). (Same proof as in lemma 1).

Let us now show that under some conditions, the sequence $(P_k)_{k\in\mathbb{N}}$ can also satisfy (iii).

**Lemma 2** *Let $d$ and $e$ be two linear functionals belonging to $\mathcal{P}'$ linked by the relation :*

$$\pi.e = d$$

*where $\pi$ is some polynomial of degree $q$, $\pi(x) = \prod_{i=1}^{q}(x-a_i)$ then*

$$(7) \quad \tilde{Q}_n(t)\tilde{P}_n(t) - \tilde{Q}_{n-1}(t)\tilde{P}_{n+1}(t) =$$

$$(-1)^q A_{n+1}\cdots A_{n+q+1}.h_n. \begin{vmatrix} P_{n+1}(a_1) & \dots & P_{n+1}(a_q) \\ \vdots & & \vdots \\ P_{n+q}(a_1) & \dots & P_{n+q}(a_q) \end{vmatrix}^2$$

*$(P_n)_{n\in\mathrm{N}}$ is orthogonal sequence w.r.t $e$*
*$(\tilde{P}_n)_{n\in\mathrm{N}}$ is orthogonal sequence w.r.t $d$*
*$(\tilde{Q}_n)_{n\in\mathrm{N}}$ associated with $\tilde{P}_n$*
*$(Q_n)_{n\in\mathrm{N}}$ associated with $P_n$*
*$A_k$ coefficients of the recurrence relation of the $P_k$ 's and $h_n = e(P_n^2)$.*
*In the case of a zero $a_k$, of multiplicity $r_k > 1$, we replace the corresponding columns of (7) by the derivatives of order $0, 1, \cdots, r_k - 1$ of the polynomials $P_{n+1}(x)$, $P_{n+2}(x), \cdots, P_{n+q}(x)$ at $x = a_k$.*

**Proof.**
For seak of simplicity, we only write the proof with $r_i = 1$, $i = 1, \cdots, m$. The sequence $(P_n)_{n\in\mathrm{N}}$ satisfies:

$$e(P_n P_m) = 0 \text{ if } n \neq m.$$

We have ([11, p. 30])

$$(8) \quad \tilde{P}_n(x) = \frac{1}{\pi(x)} \begin{vmatrix} P_n(x) & P_n(a_1) & \cdots & P_n(a_q) \\ \vdots & \vdots & & \vdots \\ P_{n+q}(x) & P_{n+q}(a_1) & \cdots & P_{n+q}(a_q) \end{vmatrix}$$

The associated polynomial $\tilde{Q}_n$ is defined by

$$\tilde{Q}_n(t) = d\left(\frac{\tilde{P}_{n+1}(x) - \tilde{P}_{n+1}(t)}{x-t}\right)$$

($d$ acts on the variable $x$) and with (8) is equal to :

$$\tilde{Q}_n(t) = \begin{vmatrix} Q_n(t) & P_{n+1}(a_1) & \cdots & P_{n+1}(a_q) \\ \vdots & \vdots & & \vdots \\ Q_{n+q}(t) & P_{n+q+1}(a_1) & \cdots & P_{n+q+1}(a_q) \end{vmatrix}$$

$$-\tilde{P}_{n+1}(t)e\left(\frac{\pi(x)-\pi(t)}{x-t}\right)$$

The left handside of equality (7) is

$$\tilde{Q}_n(t)\tilde{P}_n(t) - \tilde{Q}_{n-1}(t)\tilde{P}_{n+1}(t) =$$

$$\begin{vmatrix} Q_n(t) & P_{n+1}(a_1) & \cdots & P_{n+1}(a_q) \\ \vdots & \vdots & & \vdots \\ Q_{n+q}(t) & P_{n+q+1}(a_1) & \cdots & P_{n+q+1}(a_q) \end{vmatrix} \times \tilde{P}_n(t)$$

$$-\begin{vmatrix} Q_{n-1}(t) & P_n(a_1) & \cdots & P_n(a_q) \\ \vdots & \vdots & & \vdots \\ Q_{n+q-1}(t) & P_{n+q}(a_1) & \cdots & P_{n+q}(a_q) \end{vmatrix} \times \tilde{P}_{n+1}(t)$$

$$= \begin{vmatrix} Q_{n-1}(t) & P_n(a_1) & \cdots & P_n(a_q) & P_n(t) \\ \vdots & \vdots & & \vdots & \vdots \\ Q_{n+q}(t) & P_{n+q+1}(a_1) & \cdots & P_{n+q+1}(a_q) & P_{n+q+1}(t) \end{vmatrix} \times$$

$$\frac{(-1)^q}{\pi(t)} \begin{vmatrix} P_{n+1}(a_1) & \cdots & P_{n+1}(a_q) \\ \vdots & & \vdots \\ P_{n+q}(a_1) & \cdots & P_{n+q}(a_q) \end{vmatrix}$$

(by Sylvester identity).
From the recurrence relationships

$$\begin{aligned} P_{n+1}(x) &= (A_{n+1}x + B_{n+1})P_n(x) - C_{n+1}.P_{n-1}(x) \\ Q_n(x) &= (A_{n+1}x + B_{n+1})Q_{n-1}(x) - C_{n+1}.Q_{n-2}(x) \end{aligned}$$

it arises

$$\tilde{Q}_n(t)\tilde{P}_n(t) - \tilde{Q}_{n-1}(t)\tilde{P}_{n+1}(t) =$$

$$(-1)^q A_{n+2}\cdots A_{n+q+1} \begin{vmatrix} P_{n+1}(a_1) & \cdots & P_{n+1}(a_q) \\ \vdots & & \vdots \\ P_{n+q}(a_1) & \cdots & P_{n+q}(a_q) \end{vmatrix}^2$$

$$\times(Q_{n-1}(t)P_{n+1}(t) - Q_n(t)P_n(t)$$

which is relation (7) since

$$Q_{n-1}(t)P_{n+1}(t) - Q_n(t)P_n(t) = A_{n+1}e(P_n^2).$$

△

Formula (7) will allow us to write some conditions about the given linear functional $e$ in order to obtain a sequence of polynomials $(P_k)$ satisfying the relation (iii).

**Lemma 3** *The notations are the same as in lemma 2. Let us assume that $e$ is definite*

$$\text{i.e.} \quad e(P_n^2) \neq 0. \quad \forall n \in \mathrm{N}$$

*and $(P_n)_{n\in\mathrm{N}}$ satisfy $e(P_nP_m) = \delta_{nm}$. If $(P_n^{(j)}(a_i))_{n\in\mathrm{N}}$ doesn't converge to 0 for $i = 1, \cdots, m, j = 0, \cdots, r_i - 1$, if $(Q_n(a_i))_{n\in\mathrm{N}}$ and $(P_n(a_i))_{n\in\mathrm{N}}$ are bounded for $i = 1, \cdots, m$ and if $(A_n)_{n\in\mathrm{N}}$ doesn't converge to 0 then the sequence $(P_n)_{n\in\mathrm{N}}$ satisfies (iii)*

**Proof.** As in lemma 2, we display the proof for $r_i = 1$, $i = 1, \cdots, m$. Let us rewrite (7) for $\pi(x) = (x - a_1)$

$$\tilde{Q}_n(t)\tilde{P}_n(t) - \tilde{Q}_{n-1}(t)\tilde{P}_{n+1}(t) = -A_{n+1}A_{n+2}[P_{n+1}(a_1)]^2$$

with

$$\tilde{P}_n(t) = \frac{1}{t-a_1} \begin{vmatrix} P_n(t) & P_n(a_1) \\ P_{n+1}(t) & P_{n+1}(a_1) \end{vmatrix}$$

$$\tilde{Q}_n(t) = \begin{vmatrix} Q_n(t) & P_{n+1}(a_1) \\ Q_{n+1}(t) & P_{n+2}(a_1) \end{vmatrix} - \tilde{P}_{n+1}(t)e(1)$$

From hypothesis, we deduce that $\tilde{Q}_n(a_2)$ and $\tilde{Q}_{n-1}(a_2)$ are bounded and

$$\tilde{P}_n(a_2) = \frac{1}{a_2 - a_1} \begin{vmatrix} P_n(a_2) & P_n(a_1) \\ P_{n+1}(a_2) & P_{n+1}(a_1) \end{vmatrix}$$

can't converge to 0. The complete proof follows by induction. △

**Remark.** The assumption $(Q_n(a_i))_{n\in\mathbb{N}}$ bounded for $i = 1,\cdots,m$ can be replaced by $(h(a_i)Q_n(a_i))_{n\in\mathbb{N}}$ bounded with $h(a_i) \neq 0$ for $i = 1,\cdots,m$.

**Example.** If $c$ is the linear functional defined by

$$c(p(x)) = \int_{-1}^{1} p(x)\frac{dx}{\sqrt{1-x^2}}$$

the corresponding orthonormal polynomials are the Chebychev polynomials $T_n$, which are bounded on $[-1,1]$ and are such that the sequence $(T_n(a))_{n\in\mathbb{N}}$ doesn't converge to 0 for each $a \in [-1,1]$. The associated polynomials are $U_n$ and satisfy $|\sqrt{1-x^2}U_n(x)| \leq 1 \quad \forall x \in [-1,1]$.
So, by lemma 3, the sequence

$$\begin{vmatrix} T_{n+1}(a_1) & \cdots & T_{n+1}(a_q) \\ \vdots & & \vdots \\ T_{n+q}(a_1) & \ldots & T_{n+q}(a_q) \end{vmatrix}$$

can't converge to 0 when $n$ goes to infinity for $a_i \in [-1,1]$ $i = 1,\cdots,q$.(But it can be proved that a subsequence can converge to 0).

**Numerical example.** Let us now explain the method of detection of Dirac masses in a density and the approximation of the location of these masses. We suppose that a distribution $c$ is the sum of a density over a compact interval (which without lost of generality can be supposed to be $[-1,1]$) and a finite sum of Dirac distributions as follows :

$$c = \omega + \sum_{i=1}^{m}\sum_{j=0}^{r_i-1} A_{ij}(-1)^j D^j \delta_{a_i}$$

$a_i$ distinct points of $\mathbf{C}$, $q = \sum_{i=1}^{m} r_i$ and

$$<\omega, p(x)> = \omega(p(x)) = \int_{-1}^{1} p(x)w(x)dx$$

with $w \in L^1$ and $p \in \mathcal{P}$.

Only the moments $c_n := <c, x^n>$ and the compact containing the support of $\omega$ are supposed to be given. The difficult point is when the points $a_i$ belong to the support of $\omega$ because in this case the function $f$ defined by the formal expansion $c(\frac{1}{1-xt}) = \sum_{i=0}^{\infty} c_i t^i$ is the sum of a Stieltjes function analytic in $\mathbf{C}-]-\infty,-1] \cup [1,+\infty[$ and a rational fraction whose poles $a_i^{-1} \quad i = 1,\cdots,m$ are on the cut $]-\infty,-1] \cup [1,+\infty[$.

**First example.**

$c = \omega + 3\delta_{0.5}$ where $\omega_n = \omega(x^n) = \int_{-1}^{1} x^n dx$. The moments $c_n = \frac{1-(-1)^{n+1}}{n+1} + 3(0,5)^n, \quad n \in \mathbb{N}$ are supposed to be given. The difficulty is to extract the geometric sequence $(0,5)^n$ from the sequence $(c_n)$ because the two sequences $\omega_n$ and $(0,5)^n$ both converge to 0.

Theorems 5 and 6 provide us a tool to detect Dirac masses and their places. From the values of $c(T_n)$, (see table 1, column 2) where $T_n$ is the $n^{th}$ Chebyshev polynomial of the first kind, we can conclude that $c(T_n)$ doesn't converge to 0 and then there are Dirac masses in the distribution $c$ (theorem 6).The system $(T_N^q)$ with $q = 1$ is

$$2b_0^{(n)}c(T_n) + b_1^{(n)}(c(T_{n+1} + c(T_{n-1})) = 0$$

with $b_1^{(n)} = 1$. If the sequence $(b_0^{(n)})_{n\in\mathbb{N}}$ converges to $b_0$ then by theorem 5 the limit $b_0$ satisfies

$$(x - a_1) = b_0 T_0(x) + 1 \times T_1(x) = -0.5 + x$$

and so $a_1 = 0.5$. The coefficient $A_{10} = 3$ will be computed by another method explained in a further paper [6].

**Second example.**

Let the distribution $c$ be defined as follows :

$$c = \omega \quad \text{with} \qquad \omega(x^n) = \int_{-1}^{1} x^n w(x)dx$$

$$\text{where} \quad w(x) = x + 0.5 \qquad \text{on} \quad [-0.5, 0.5]$$

The coefficients

$$c_n = 0.5^{n+1}/(n+2) \qquad n \qquad \text{even}$$

$$c_n = 0.5^{n+1}/(n+1) \qquad n \qquad \text{odd}$$

are supposed to be given. The goal is, here, to detect the jump of the function $w$. In view of table 2, column 2, the sequence $(c(T_n))_{n\in\mathbb{N}}$ converges to 0 and so, by theorem 6, there is no Dirac mass.

The jumps of $w$ will appear in the derivative (in the sense of distribution [10]) of $w$. If $w$ has a finite number of jumps of finite magnitude $A_i$ at points $a_1,\cdots,a_q$ then the derivative of $w$ satisfies :

$$w' = \{w\}' + \sum_{i=1}^{q} A_i \delta_{a_i}$$

where $\{w\}'$ is the derivative of $w$ in the usual sense if it exists.

The moments of $w'$ can be computed as follows :

$$\forall p \in P, \int_{-1}^{1} p(x)w'(x)dx = -\int_{-1}^{1} p'(x)w(x)dx$$

with an integration by part. (Or $<c', p> = -<c, p'>$ with the notation of Schwartz [10]) and so

$$\begin{aligned} c_n' &: = <c', x^n> = -nc_{n-1} \quad n \geq 1 \\ c_0' &= 0 \end{aligned}$$

The sequence $c'(T_n)$ (see column (4), table 2) doesn't converge to 0. So, we can conclude that there are some Dirac masses in the distribution $c'$ and therefore some jumps in the function $w$. The system $(T_N^1)$ for $c'$ is:

$$2b_0^{(n)}c'(T_n) + b_1^{(n)}(c'(T_{n+1}) + c'(T_{n-1})) = 0 \quad \text{with} \quad b_1^{(n)} = 1.$$

The sequence $b_0^{(n)}$, (column 5) seems to converge to -.5. By theorem 5, the limit satisfies

$$x - a_1 = -0,5T_0(x) + 1 \times T_1(x) = -0.5 + x$$

and so the function $w$ has surrely one jump at the point $a_1 = 0.5$.

**Third example.**

Let the distribution be defined as follows

$$c = \omega \text{ with } \quad \omega(x^n) = \int_{-1}^{1} x^n w(x)dx$$

where $w(x) = (x+1)^2$ on $[-1,0], -2x+1$ on $[0,0.5]$
The moments $c_n$ are

$$c_n = (2 + 0.5^{n+1}.(n+3))/((n+1)(n+2)(n+3)) \ n \text{ even}$$

$$c_n = (-2 + 0.5^{n+1}.(n+3))/((n+1)(n+2)(n+3)) \ n \text{ odd}$$

The sequence $c(T_n)$ seems to converge to 0, then there is no Dirac mass. (Table 3, Column 2). The sequence $c'(T_n) = -c(T_n{}') = -nc(U_{n-1})$ converges to 0 and therefore there is no jump in the function $w$.
Let us examine the sequence $c''(T_n) = -c'(T_n') = +c(T_n'')$. This sequence doesn't converge to 0, thus we are led to solve the system $(T_N^q)$ with $q = 1,2,\dots(T_N^1)$ provides a solution $(b_0^{(N)})$ which doesn't converge (See table 3,column 7). The system $(T_n^2)$ is :

$$\begin{cases} 2b_0^{(n)}u_n + b_1^{(n)}(u_{n+1}+u_{n-1}) + b_2^{(n)}(u_{n+2}+u_{n-2}) &= 0 \\ 2b_0^{(n)}u_{n+1} + b_1^{(n)}(u_{n+2}+u_n) + b_2^{(n)}(u_{n+3}+u_{n-1}) &= 0 \end{cases}$$

with $b_2^{(n)} = 1$ and $u_k := c''(T_k)$. The vector solution $(b_0^{(n)}, b_1^{(n)})$(columns (8) and (9),table 3) seems to converge to $(b_0, b_1) = (1, -1)$. The points $a_1 = 0$, $a_2 = \frac{1}{2}$ singularities of $w$ will be obtained by solving

$$\begin{aligned} (x-a_1)(x-a_2) &= b_0T_0(x) + b_1T_1(x) + b_2T_2(x) \\ &= 1 - x + (2x^2 - 1) \\ &= 2x^2 - x = 0 \end{aligned}$$

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Table 1:

| $n$ | (1) $c_n$ | (2) $c(T_n)$ | (3) $b_0^{(n)}$ |
|---|---|---|---|
| 0 | 5.000000 | 2.50000 | |
| 1 | 1.500000 | 1.50000 | -0.111111 |
| 2 | 1.416667 | -2.16667 | -0.346154 |
| 3 | 0.375000 | -3.00000 | -0.633333 |
| 4 | 0.587500 | -1.63333 | -0.459184 |
| 5 | 0.093750 | 1.50000 | -0.436508 |
| 6 | 0.332589 | 2.94286 | -0.509709 |
| 7 | 0.023438 | 1.50000 | -0.470370 |
| 8 | 0.233941 | -1.53175 | -0.489637 |
| 9 | 0.005859 | -3.00000 | -0.508658 |
| 10 | 0.184748 | -1.52020 | -0.493355 |
| 20 | 0.095241 | -1.50501 | -0.498335 |
| 30 | 0.064516 | 2.99778 | -0.500371 |
| 40 | 0.048780 | -1.50080 | -0.499733 |

Table 2:

| $n$ | (1) $c_n$ | (2) $c(T_n)$ | (3) $c'_n$ | (4) $c'(T_n)$ | (5) $b_0^{(n)}$ |
|---|---|---|---|---|---|
| 0 | 0.50000000 | 0.25000 | 0.00000000 | 0.00000 | |
| 1 | 0.08333333 | 0.08333 | -0.5000000 | -0.50000 | -0.333333 |
| 2 | 0.04166667 | -0.41667 | -0.16666667 | -0.33333 | 0.750000 |
| 3 | 0.01250000 | -0.20000 | -0.12500000 | 1.00000 | -0.300000 |
| 4 | 0.00625000 | 0.21667 | -0.05000000 | 0.93333 | -0.267857 |
| 5 | 0.00223214 | 0.20238 | -0.03123000 | -0.50000 | -0.095238 |
| 6 | 0.00111607 | -0.01429 | -0.01339286 | -1.02857 | -0.486111 |
| 7 | 0.00043403 | -0.10556 | -0.00781250 | -0.50000 | -0.711111 |
| 8 | 0.00021701 | -0.09127 | -0.00347222 | 0.31746 | -0.787500 |
| 9 | 0.00008878 | -0.01299 | -0.00195313 | 1.00000 | -0.487013 |
| 10 | 0.00004439 | 0.07828 | -0.00088778 | 0.65657 | -0.380769 |
| 20 | 0.00000002 | -0.03697 | -0.00000091 | 0.42607 | -0.586765 |
| 30 | 0 | -0.00056 | -0.00000000 | -1.00111 | -0.499444 |
| 40 | 0 | 0.01892 | 0 | 0.53784 | |

Table 3:

| $n$ | (1) $c_n$ | (2) $c(T_n)$ | (3) $c'_n$ | (4) $c'(T_n)$ | (5) $c''_n$ | (6) $c''(T_n)$ | (7) $b_0^{(n)}$ | (8) $(b_0^{(n)}$, | (9) $b_1^{(n)})$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.583333 | 0.2916 | 0.0000 | 0.0000 | 0.000 | 0.00000 | | | |
| 1 | -0.041666 | -0.0416 | -0.5833 | -0.5833 | 0.000 | 0.00000 | | | |
| 2 | 0.043750 | -0.4958 | 0.0833 | 0.1666 | 1.166 | 2.33333 | 0.2142 | 0.99823 | -0.47493 |
| 3 | -0.013541 | 0.0708 | -0.1312 | 1.2250 | -0.250 | -1.00000 | -1.4000 | 0.81735 | -0.34572 |
| 4 | 0.010565 | 0.3178 | 0.0541 | -0.2333 | 0.525 | -5.13333 | -0.0324 | -0.86579 | -1.83777 |
| 5 | -0.005580 | -0.0267 | -0.0528 | -1.1369 | -0.270 | 0.66667 | -0.6071 | 1.27899 | -2.51824 |
| 6 | 0.004107 | -0.1715 | 0.0334 | -0.0285 | 0.316 | 5.94286 | -0.1682 | 1.12718 | -1.61614 |
| 7 | -0.002723 | -0.0159 | -0.0287 | 0.8097 | -0.234 | 1.33333 | -0.3416 | 0.99166 | -1.21949 |
| 8 | 0.002041 | 0.0835 | 0.0217 | 0.2174 | 0.230 | -5.03175 | -0.0861 | 0.95705 | -0.81760 |
| 9 | -0.001506 | 0.0224 | -0.0183 | -0.4634 | -0.196 | -2.20000 | -0.4663 | 1.02177 | -0.95639 |
| 10 | 0.001169 | -0.0363 | 0.0150 | -0.1767 | 0.182 | 2.97980 | 0.1677 | 0.99900 | -1.09208 |
| 20 | 0.000188 | 0.0122 | 0.0043 | 0.0790 | 0.095 | -5.00501 | -0.0978 | 0.99312 | -0.97106 |
| 30 | 0.000061 | -0.0061 | 0.0020 | -0.0011 | 0.064 | 5.99778 | -0.1667 | 1.00429 | -1.02127 |
| 40 | 0.000027 | 0.0029 | 0.0011 | -0.0365 | 0.048 | -5.00104 | -0.0995 | 1.00029 | -1.01097 |

# Product of forms with applications to Padé approximants.

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A formal power series $f$ defines a form $c$ whose moments are the coefficients of $f$. In this paper, we apply the product of two forms as defined by Maroni in [6].The particular case of positive distributions is studied. Applications are given for the convergence of Padé approximants to a product of Stieltjes functions whose supports are respectively included in the positive and negative real half lines and for the acceleration of convergence of some particular sequences.

## I - PRODUCT OF FORMS ([6]).

Let $P$ be the space of polynomials with coefficients in $\mathbf{C}$. The set of the linear forms on $P$ (i.e. : $L(P, \mathbf{C})$) will be denoted by $P'$.

$$c \in P' c : P \to \mathbf{C}$$
$$x^n \to < c, x^n > := c_n$$

An element of $P'$ is determined by the sequence $(c_n)_{n \in \mathbf{N}}$ which are called the moments of the form $c$.

Let us now recall the definition of the tensor product of two forms.

**Definition 1.** *Let $c$ and $d$ be two elements of $P'$. $c \otimes d$ is the form whose moments are given by :*

$$< c \otimes d, x^n y^m > = c_n d_m \quad n, m \in \mathbf{N} \tag{1}$$

*or equivalently :*

$$< c \otimes d, p(x) q(y) > = < c, p(x) > . < c, q(y) > \qquad p, q \in P.$$

We can now, define the product of two forms.

**Definition 2.** *Let $c$ and $d$ be two elements of $P'$. The product of $c$ and $d$ is the form $e$ defined by :*

$$< e, z^n > := < c \otimes d, \frac{x^{n+1} - y^{n+1}}{x - y} > \tag{2}$$

*$e$ will be denoted by $cd$.*

**Remark 1** : the moments of $cd$ are

$$(cd)_n = \sum_{k=0}^{n} c_{n-k} d_k$$

and thus are the coefficients of the Cauchy product of the corresponding two formal series :

$$\sum_{i=0}^{\infty} c_i t^i \text{ and } \sum_{i=0}^{\infty} d_i t^i$$

**Remark 2** : The definition 2 is equivalent to :

$$< cd, p(z) >=< c \otimes d, \frac{xp(x) - yp(y)}{x - y} > \quad \forall p \in P \qquad (3).$$

## II - GENERAL PROPERTIES OF THE PRODUCT OF FORMS.

**Proposition 1 :**

The following assertions $(i)$ to (vi) hold, for $c$, $d$, $e \in P'$

(i) $cd = dc$

(ii) $c(d+e) = cd + ce$

(iii) $c(de) = (cd)e$

(iv) $\delta_a \delta_b = \frac{1}{a-b}(a\delta_a - b\delta_b)$ for $a \neq b$
where $\delta_a$ is defined by : $< \delta_a, x^n >= a^n \quad n \in \mathbf{N}, a \in \mathbf{C}$.
and $\lambda.c$, $\lambda \in \mathbf{C}$, $c \in P'$ by : $< \lambda.c, x^n >:= \lambda. < c, x^n >$.

(v) $\delta_o c = c$

(vi) $\delta_a \delta_a = -z\delta_a^{'}$
where $\delta_a^{'}$ and $z.c$ are defined as following :

$$< \delta_a^{'}, p(x) >= - < \delta_a, p'(x) >= -p'(a) \quad \forall p \in P$$
$$< zc, p(z) >=< c, zp(z) > \text{ or } < zc, z^n >:= c_{n+1} \quad n \in \mathbf{N}$$

**Proof.**

(i)
$$\begin{aligned}(cd)_n &= \textstyle\sum_{k=0}^{n} c_{n-k} d_k = c_n d_o + c_{n-1} d_1 + \ldots + c_1 d_{n-1} + c_o d_n \\ &= \textstyle\sum_{k=0}^{n} c_k d_{n-k} = (dc)_n \quad \forall n \in \mathbf{N}\end{aligned}$$

(ii)

$$< c(d+e), z^n >=< c \otimes (d+e), \frac{x^{n+1} - y^{n+1}}{x - y} >=$$
$$< c \otimes d, \frac{x^{n+1} - y^{n+1}}{x - y} > + < c \otimes e, \frac{x^{n+1} - y^{n+1}}{x - y} > \text{(property of tensor product)}$$
$$= (cd)_n + (ce)_n.$$

(iii)

$$< c(de), z^n >= \sum_{k=0}^{n} c_{n-k}(de)_k =$$

$$\sum_{k=0}^{n} c_{n-k}(\sum_{j=0}^{k} d_{k-j}e_j) = \sum_{k=0}^{n} e_k(\sum_{j=0}^{n-k} c_{n-k-j}d_j) =< (cd)e, z^n >$$

(iv) $$(\delta_a\delta_b)_n = \sum_{k=0}^{n} a^{n-k}b^k = \frac{a^{n+1}-b^{n+1}}{a-b} = \frac{a}{a-b}(\delta_a)_n - \frac{b}{a-b}(\delta_b)_n.$$

(v) $$(\delta_o c)_n = \sum_{k=0}^{n}(\delta_0)_k c_{n-k} = c_n \quad \forall n \in \mathbf{N}$$

(vi)

$$(\delta_a\delta_a)_n = \sum_{k=0}^{n} a^{n-k}a^k = \sum_{k=0}^{n} a^n = (n+1)a^n$$

and $(-z\delta_a^{'})_n =< -z\delta_a^{'}, z^n >= - < \delta_a^{'}, z^{n+1} >=< \delta_a, (n+1)z^n >= (n+1)a^n$.

■

The relation between the distributions $c$, $d$ and $cd$ is precised by the following theorem 1.

**Definition 3 :** *If $c \in P'$, $\varphi : \mathbf{C} \to \mathbf{C}$. The product $\varphi(z)c$ is defined by : $p \in P$, $\varphi \in P$, $< \varphi(z)c, p(z) >:=< c, \varphi(z)p(z) >$ if it exists.*

**Theorem 1 :** *Let $f(t) = \sum_{i=0}^{\infty} c_i t^i$ and $g(t) = \sum_{i=0}^{\infty} d_i t^i$.*

*If $f(z^{-1})d$* and *$g(z^{-1})c$ exist, then :*

$$cd = f(z^{-1})d + g(z^{-1})c \tag{4}$$

**Proof.**

The function $f(z^{-1})$ is formally defined by :

$$f(z^{-1}) = \Sigma c_i z^{-i} = c(\frac{z}{z-x})$$

where $c$ acts on $x$, and the distribution $f(z^{-1})d$ defined by :

$$< f(z^{-1})d, p(z) >=< d, f(z^{-1})p(z) >, \forall p \in P,$$

may exist, if for instance, $d$ has an integral representation and $f(z^{-1})$ is an integrable function for all $z$ belonging to the support of $d$.

$$< cd, p(z) >=< c_x \otimes d_y, \frac{xp(x) - yp(y)}{x - y} >$$

$$f(z^{-1})d_z + g(z^{-1})c_z = c_x(\frac{z}{z-x})d_z + d_y(\frac{z}{z-y})c_z \quad \begin{array}{l} c_x \text{ acts on } x \\ d_y \text{ acts on } y \end{array}$$

and

$$< f(z^{-1})d_z + g(z^{-1})c_z, p(z) >=< c_x(\frac{z}{z-x})d_z, p(z) > + < d_y(\frac{z}{z-y})c_z, p(z) >$$

$$=< d_z, c_x(\frac{zp(z)}{z-x}) > + < c_z, d_y(\frac{zp(z)}{z-y}) >$$

$$=< c_x \otimes d_z, \frac{zp(z)}{z-x} > + < c_z \otimes d_y, \frac{zp(z)}{z-y} >$$

$$=< c_x \otimes d_y, \frac{yp(y)}{y-x} > + < c_x \otimes d_y, \frac{xp(x)}{x-y} >=< c_x \otimes d_y, \frac{xp(x) - yp(y)}{x-y} >$$

**Remark 3 :** The product of distributions can be defined in the same way as above by :

$$< cd, \varphi >=< c_x \otimes d_y, \frac{x\varphi(x) - y\varphi(y)}{x - y} >$$

in the space $D$ of all functions $\varphi$ indefinetely differentiable with compact support (See [8])

$$c, d \in D' \text{ and } \begin{cases} \psi(x,y) = \frac{x\varphi(x) - y\varphi(y)}{x-y} \\ \psi(x_o, x_o) = (x\varphi(x))'_{x=x_o} \end{cases}, \psi \in D$$

From $supp(c \otimes d) = supp(c) \times supp(d)$, we deduce :

$$supp(cd) = supp(c) \cup supp(d)$$

.

**Examples of product of two forms.**

**Example 1.**

$$c = \chi_{[0,1]} \quad d = \chi_{[-1,0]}$$

$$< c, p > = \int_0^1 p(x)dx, \quad p \in P \quad < d, p > = \int_{-1}^0 p(x)dx, \quad p \in P.$$

Relation (4) gives

$$cd = |z| ln(1 + \frac{1}{|z|}) \chi_{[-1,1]}$$

and so the following relation

$$-\frac{1}{t^2} ln(1-t) ln(1+t) = \int_{-1}^1 \frac{1}{1-zt} |z| ln(1 + \frac{1}{|z|}) dz$$

holds.

In fact

$$f(z^{-1}) = c_x(\frac{z}{z-x}) = \int_0^1 \frac{z}{z-x} dx = z ln(\frac{z}{z-1})$$

and $f(z^{-1})\chi_{[-1,0]}$ is integrable on $[-1, 0]$. The same result holds for $g(z^{-1})\chi_{[0,1]}$ because

$$g(z^{-1}) = z ln(\frac{z+1}{z})$$

.

So,

$$\begin{aligned} cd &= z ln(\tfrac{z}{z-1})\chi_{[-1,0]} + z ln(\tfrac{z+1}{z})\chi_{[0,1]} \\ &= |z| ln(1 + \tfrac{1}{|z|})\chi_{[-1,1]} \end{aligned}$$

This example is in fact the starting point of the paper : during a talk, M. Huttner who uses Padé approximation in order to prove the irrationality of certain real numbers (for example $ln2.ln3$) wondered whether the sequence of Padé approximants to the product $t^{-1}ln(1-t)$ and $t^{-1}ln(1+bt)(b > 0)$ converges to this function. The above example 1 gives a positive answer to this question and moreover provides the explicit weight function (which is positive on $[-1, 1]$) (see theorem 5).

**Example 2** : $c = \chi_{[0,1]}$

$cc = 2z ln(\frac{z}{1-z})\chi_{[0,1]}$ which is integrable on $[0, 1]$ but non positive

$$2z\,ln(\tfrac{z}{1-z}) \quad \begin{array}{l} < 0 \text{ for } z \in [0, 0.5] \\ > 0 \text{ for } z \in [0.5, 1] \end{array}$$

In terms of generating function :

$$\frac{1}{t^2} ln^2(1-t) = 2\int_0^1 \frac{1}{1-zt} z\,ln(\frac{z}{1-z})dz.$$

**Example 3 :** Generalization of examples 1 and 2.

$$\begin{aligned}
0 < b < a \quad \chi_{[0,a]}\chi_{[0,b]} &= z\,ln(\frac{z^2}{(z-a)(z-b)})\chi_{[0,b]} - z\,ln(\frac{z-b}{z})\chi_{[b,a]} \\
a = b > 0 \quad \chi_{[0,a]}\chi_{[0,a]} &= 2z\,ln(\frac{z}{a-z})\chi_{[0,a]} \\
b < 0 < a \quad \chi_{[0,a]}\chi_{[b,0]} &= -z\,ln(\frac{z-a}{z})\chi_{[b,0]} + z\,ln(\frac{z-b}{z})\chi_{[0,a]}
\end{aligned}$$

(it is a positive weight on $[b, a]$)

$$a = -b > 0 \quad \chi_{[0,a]}\chi_{[-a,0]} = |z|ln(1 + \frac{a}{|z|})\chi_{[-a,a]}(z) \geq 0$$

**Example 4.**

If $\frac{z}{a-z}c$ is defined in $P'$ and $f(a^{-1})$ exists then

$$\delta_a c = \frac{z}{a-z}c + f(a^{-1})\delta_a.$$

where $f(a^{-1}) = c_x(\frac{a}{a-x})$.

So, the product of a form with a Dirac form appears as a sum of another form and a Dirac form.

If $c = \chi_{[0,1]}$, then we get :

$$\delta_a\chi_{[0,1]} = \frac{z}{a-z}\chi_{[0,1]} + a\,ln(\frac{a}{a-1})\delta_a$$

for $a \in \mathbf{C} - [0, 1]$ and

$$\frac{1}{1-at}\frac{ln(1-t)}{-t} = \int_0^1 \frac{z}{z-a}\frac{dz}{1-zt} + \frac{a}{1-at} ln(\frac{a}{a-1})$$

(which can be easily checked by another way).

**Example 5.**

In the same way, the relation

$$\frac{1}{\sqrt{z}}\chi_{[0,1]}\frac{1}{\sqrt{|z|}}\chi_{[-1,0]} = 2\;Arctan(\frac{1}{\sqrt{|z|}})\chi_{[-1,1]} \quad \text{holds}$$

**Example 6.**

$$\delta'_a\chi_{[0,1]} = \frac{-z}{(z-a)^2}\chi_{[0,1]} - z ln(1-\frac{1}{z})\delta'_a \qquad a \notin [0,1]$$

$$= \frac{-z}{(z-a)^2}\chi_{[0,1]} + (ln(1-\frac{1}{a}) + \frac{1}{a-1})\delta_a \;- a ln(1-\frac{1}{a})\delta'_a$$

## III - APPLICATIONS TO PADE APPROXIMANTS.

The main result of this section is the convergence of the paradiagonal sequence Padé approximants to the product of two Stieltjes functions whose supports are respectively included in $\mathbf{R}^+$ and $\mathbf{R}^-$. (The same result also holds for the product of a Stieltjes function with a rational fraction whose poles are on $\mathbf{R}^-$). At first, let us recall the definition of a positive distribution.

**Definition ([4] or [2] page 115)**

a) The distribution $c \in P'$ is said to be positive if

$$c(p) \geq 0 \quad \forall p \in P, p(x) \geq 0 \quad \forall x \in \mathbf{R}$$

b) $c$ is said to be positive definite if

$$c(p) > 0 \quad \forall p \in P, p(x) \geq 0 \text{ and } p(x) \neq 0, \forall x \in \mathbf{R}$$

The following results are well known.

**Theorem 2.**

A necessary and sufficient condition that there should exist a non decreasing function $\alpha$ such that :

$$c_n := < c, x^n > := \int_{-\infty}^{+\infty} x^n d\alpha(x) \quad n = 0, 1, \ldots$$

is that the distribution $c$ should be positive.

In this case the sequence $(c_n)_n$ is called a Hamburger moment sequence.

**Theorem 3 :**

A necessary and sufficient condition that there should exist a nondecreasing function $\alpha$ such that :

$$c_n = < c, x^n > = \int_0^\infty x^n d\alpha(x) \quad n = 0, 1, \ldots$$

is that the distribution $c$ and $z.c$ should be positive.

In this case the sequence $(c_n)_n$ is called a Stieltjes moment sequence.

Let us now rewrite the theorem 1 in the particular case where $c$ and $d$ are positive with support respectively included in $\mathbf{R}^+$ and $\mathbf{R}^-$.

**Theorem 4.**

If $c$ and $d$ have the representation :

$$\begin{array}{ll} < c, p > = \int_0^a p(x) d\alpha(x) & a \in \mathbf{R}^+ \text{ or } a = +\infty \\ & \alpha \text{ nondecreasing with no jump at } 0 \\ < d, p > = \int_b^0 p(x) d\beta(x) & b \in \mathbf{R}^- \text{ or } b = -\infty \\ & \beta \text{ nondecreasing with no jump at } 0 \end{array}$$

then $cd$ is a positive distribution with support $[b, a]$ and thus has the integral representation

$$< cd, p > = \int_b^a p(x) d\gamma(x)$$

where $\gamma$ is a nondecreasing function :

$$d\gamma(z) = d\beta(z) \int_0^a \frac{z}{z - x} d\alpha(x) + d\alpha(z) \int_b^0 \frac{z}{z - y} d\beta(y)$$

**Proof.**

At first, we prove that the distributions $f(z^{-1})d$ and $g(z^{-1})c$ are well defined.

$$f(z^{-1}) = c_x\left(\frac{z}{z - x}\right) = \int_0^a \frac{z}{z - x} d\alpha(x)$$

$$g(z^{-1}) = d_y\left(\frac{z}{z - y}\right) = \int_b^0 \frac{z}{z - y} d\beta(y)$$

So, $f(z^{-1})d$ is well defined since :

$$< f(z^{-1})d, p(z) > := < d, f(z^{-1})p(z) > = \int_{-\infty}^0 f(z^{-1}) p(z) d\beta(z)$$

and $f(z^{-1})d\beta(z)$ is integrable on $]-\infty,0]$ because $\frac{z}{z-x}d\alpha(x)d\beta(z)$ is integrable on $[0,+\infty[\times]-\infty,0]$. We can now write :

$$cd = f(z^{-1})d + g(z^{-1})c \quad (\text{see}(4))$$

Moreover $f(z^{-1})d$ is a positive distribution because

$$\begin{aligned}< f(z^{-1})d, p(z) > &= < d, f(z^{-1})p(z) > \\ &= \int_{-\infty}^{0} f(z^{-1})p(z)d\beta(z) \quad \forall p \geq 0\end{aligned}$$

and

$$f(z^{-1}) = \int_0^\infty \frac{z}{z-x}d\alpha(x) \geq 0 \quad \forall z \in ]-\infty,0[.$$

The same result holds for $< g(z^{-1})c, p(z) >= \int_0^\infty g(z^{-1})p(z)d\alpha(z)$. Thus the product $cd$ is positive and the support of $cd$ is $supp(c) \cup supp(d)$. The relation (4) gives, in terms of weight functions :

$$d\gamma(z) = f(z^{-1})d\beta(z) + g(z^{-1})d\alpha(z).$$

∎

We are now able to prove a result of convergence for a paradiagonal sequence of Padé approximants to a product of two Markov-Stieltjes functions. A general result on the convergence of paradiagonal Padé approximants was given by A.A. Markov (see Szegö [9], theorem 3.5.4) who proved that if $f(t) = \int_{\mathbf{R}} \frac{d\alpha(x)}{1-xt}$, where $\alpha$ is a bounded nondecreasing function, with compact support, then the sequence of Padé approximants $[M+J/M], J \geq -1$ to the function $f$ converges uniformely on every compact set of $\mathbf{C} \setminus \Delta^{-1}$ when $M$ tends to infinity, ($\Delta^{-1}$ is the inverse of $\Delta$, compact support of $\alpha$ ). From Markov's theorem and theorem 4 we deduce :

**Theorem 5 :**

If

$f(t) = \int_0^a \frac{1}{1-xt}d\alpha(x)$ $\quad$ $a \in \mathbf{R}^+, a$ finite, $\alpha$ *nondecreasing function with no jump at* 0

$g(t) = \int_b^0 \frac{1}{1-xt}d\beta(x)$ $\quad$ $b \in \mathbf{R}^-, b$ finite, $\beta$ *nondecreasing function with no jump at* 0 then the sequence of Padé approximants

$$([M+J/M]_{fg})_{M\in\mathbb{N}} \quad J\geq -1,$$

converges uniformely on every compact set of $\mathbf{C}-]-\infty,b^{-1}[\cup]a^{-1},+\infty[$.

**Remark 4 :** If the supports of $d\alpha$ and $d\beta$ do not satisfy $supp(d\alpha)\subset\mathbf{R}^+$ and $supp(d\beta)\subset\mathbf{R}^-$, for example, in the case where $supp(d\alpha)$ intersect $supp(d\beta)$, then the *"weight function"* $d\gamma$ underlying $f(t)g(t)$ may change its sign in $supp(d\alpha)\cap supp(d\beta)$ and therefore Markov's theorem cannot be used.

**Remark 5 :** In [5], Goncar proved a similar result for the product of a Markov-Stieltjes function with a rational fraction whose poles ar not on the cut.

**Remark 6 :** Theorem 5 can be a first step in order to prove the convergence of Cauchy type approximants defined in [3].

## IV - APPLICATIONS TO CONVERGENCE ACCELERATION.

In this section, we prove that the Cauchy product of a totally monotonic sequence ($TM$ sequence) by a convergent totally oscillating ($TO$) sequence is accelerated by $\varepsilon$-algorithm (or some related algorithm).

At first, we establish a result about $\varepsilon$-algorithm when applied to the sequences $(e_n)_n$ such that

$$e_n=\int_b^a x^n d\gamma(x)+l$$

where $\gamma$ is a nondecreasing function and $[b,a]\subset[-1,1[$ and $\gamma$ has no jump in $-1$ .

**Theorem 6 :** *Let $e_n=\int_b^a x^n d\gamma(x)+l$ defined as above. The $\varepsilon$ -algorithm accelerates the convergence of $(e_n)_n$ to $l$. More precisely :*

$$\lim_{k\to\infty}\frac{\varepsilon_{2M_k}^{(J_k)}-l}{e_{2M_k+J_k}-l}=0 \qquad \begin{array}{c}\text{for } J_k\geq -1\\ \lim_{k\to\infty}M_k=+\infty\\ \lim_{k\to\infty}M_k/J_k=+\infty\end{array}$$

**Proof.**

$\lim_{n\to\infty}e_n=l$ because $\gamma$ has no jump at the point $-1$ and $a<1$. It is well known that the $\varepsilon$-algorithm is linked with Padé approximants by the following relation [2, p. 159]

$$\varepsilon_{2M_k}^{(J_k)}(e_n)=[M_k+J_k/M_k]_f(1)$$

where $f(t) = \sum_{n=0}^{\infty} \Delta e_{n-1} t^n = \int_b^a \frac{1}{1-xt} d\alpha(x), e_{-1} := 0$

Remark that $f(1) = \lim_{n\to\infty} e_n = l$.

At first, we have to evaluate the error of Padé approximation ([1], page 191).

$$|[M_k + J_k/M_k]_f(1) - f(1)| \leq |\frac{a-b}{1-b}|^{J_k} . |\frac{1-\sqrt{\frac{1-a}{1-b}}}{1+\sqrt{\frac{1-a}{1-b}}}|^{2M_k} \times cste$$

Next, the factor of convergence of the sequence $(e_n)_n$ is :

$$\lim_{k\to\infty} |e_{2M_k+J_k} - l|^{\frac{1}{2M_k+J_k}} = \sup(|a|,|b|).$$

The quotient $r_k = \frac{[M_k+J_k/M_k]_f(1)-f(1)}{e_{2M_k+J_k}-l}$ satisfies :

$$\overline{\lim_{k\to\infty}} |r_k|^{\frac{1}{2M_k+J_k}} \leq \overline{\lim_{k\to\infty}} |\frac{a-b}{1-b}|^{\frac{J_k}{2M_k+J_k}} . |\frac{1-\sqrt{\frac{1-a}{1-b}}}{1+\sqrt{\frac{1-a}{1-b}}}|^{\frac{2M_k}{2M_k+J_k}} \div \sup(|a|,|b|)$$

$$\leq |\frac{1-\sqrt{\frac{1-a}{1-b}}}{1+\sqrt{\frac{1-a}{1-b}}}| \div \sup(|a|,|b|) = \rho < 1$$

Therefore, the convergence of $\varepsilon_{2M_k}^{(J_k)}(e_n)$ is faster than that of $(e_n)_n$ to $l$, the acceleration factor being $\rho$. ■

**Definition.** *a) A sequence $(c_n)_n$ is said to be totally monotonic ($(c_n) \in TM$) if*

$(-1)^k \Delta^k c_n \geq 0$ *for* $n, k = 0, 1, \ldots$

*or equivalently there exists a nondecreasing function $\alpha$ such that*

$c_n = \int_0^1 x^n d\alpha(x) \quad n = 0, 1, \ldots.$

*b) A sequence $(d_n)_n$ is said to be totally oscillating*

$$((d_n) \in T0) \text{ if } ((-1)^n d_n)_n \in TM$$

.

Theorem 4 and 6 applied to this particular case immediately give :

**Theorem 7 :**

If $(c_n) \in TM$, $d_n \in T0$ and converging, then the sequence $e_n = \sum_{k=0}^{n} c_{n-k} d_k$ has the following integral representation (where $\gamma$ is a nondecreasing function) :

$$e_n = \int_{-1}^{1} x^n d\gamma(x).$$

Moreover, if $(c_n)$ is non logarithmic ( $\Longleftrightarrow \lim_{n\to\infty} \frac{c_{n+1}-l}{c_n-l} = a < 1$) then the convergence of $(e_n)_n$ can be accelerated by $\varepsilon$-algorithm (in the sense of theorem 6).

**Proof :** By theorem 4, we have :

$$e_n = \int_{-1}^{1} x^n d\gamma(x)$$

with

$$d\gamma(x) = d\alpha(x) \int_{-1}^{0} \frac{x}{x-y} d\beta(y) + d\beta(x) \int_{0}^{1} \frac{x}{x-y} d\alpha(y) \qquad \geq 0$$

Moreover, if $(c_n)$ is non logarithmic, then $(c_n)_n$ has the representation

$$c_n = \int_{0}^{a} x^n d\alpha(x) \qquad 0 \leq a < 1$$

and therefore the sequence

$$e_n = \int_{-1}^{a} x^n d\gamma(x)$$

can be accelerated by $\varepsilon$-algorithm (by theorem 6) ∎

**Remark 6 :** A similar result has been established for linear combination of $TM$ and $TO$ sequences with positive coefficients and for a general process based on orthogonal polynomials. [7]

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**Dirac masses determination with orthogonal polynomials and $\varepsilon$-algorithm Application to totally monotonic sequences.**

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Let us consider a distribution $c$ with is the sum of another distribution, $\omega$, and a linear combination of Dirac distribution with masses $A_1, \ldots, A_q$ at the points $a_1, \ldots, a_q$. In [4], we have proposed a method to compute the points $a_1, \ldots, a_q$ when $\omega$ has an asymptotic property when applied to a given sequence of orthogonal polynomials. This paper is devoted to the computation of the masses $A_1, \ldots, A_q$ with the aid of Chebyshev polynomials and $\varepsilon$-algorithm according to the distribution $\omega$. An application to finding the limit of totally monotonic sequence is also given.

## I - INTRODUCTION AND NOTATIONS.

The space of linear functionals (distributions) defined on $P$, space of polynomials, will be denoted by $P'$. The Dirac distribution $\delta_{x_0}$ is defined as follows :

$$< \delta_{x_0}, p >= p(x_o),\ p \in P, x_0 \in \mathbf{C}$$

Let $c$ and $\omega$ be two elements of $P'$. We assume that the moments

$$c_n := c(x^n) :=< c, x^n > \ n = 0, 1, \ldots$$

are given and that $c$ and $\omega$ are related by the following relation :

$$c = \omega + \sum_{i=1}^{q} A_i \delta_{a_i} \quad a_i \in [-1,1], a_i \text{ distincts},\ A_i \in \mathbf{C} - \{0\} \tag{1}$$

If $\omega$ has an integral representation whose support is a compact interval of $\mathbf{R}$, then a general convergence theorem of Goncar [3] for the Padé approximants to the sum of a Markov-Stieltjes function $h(t)$ and a rational fraction $r(t)$ permits to compute the points $a_1, \ldots, a_q$ (which are the inverse of the poles of $r(t)$)and the residues $A_1, \ldots, A_q$ in the case where $a_1, \ldots, a_q$ do not lie on the cut of $h(t)$.

Here, we present a method for computing the residues $A_1, \ldots, A_q$ when points $a_1, \ldots, a_q$ are known (or computed by the method described in [4]) when $\omega$ is absolutely continuous ($\omega$ non necessarily positive) with respect to the Lebesgue measure on a compact interval of $\mathbf{R}$ (section II) or when $\omega$ is a positive distribution on $\mathbf{R}$ (section III).

## II - CHEBYSHEV POLYNOMIALS.

Let $c$ and $\omega$ be two distribution defined on $P$ and which satisfy relation (1).

Moreover, we suppose that $\omega$ has an integral representation on a compact interval of $\mathbf{R}$ which can be assumed to be $[-1,1]$.

$$< \omega, p >:= \int_{-1}^{1} p(x) w(x) dx \quad w \in L^1 \tag{2}$$

In order to get the mass $A_l$ at the point $a_l$, the idea is to apply both sides of equality (1) to the charactersitic function $\chi_{\{a_l\}}$. This formally gives

$$< c, \chi_{\{a_l\}} > = < \omega, \chi_{\{a_l\}} > + A_l = A_l$$

since $\omega$ is absolutely continuous with respect do $dx$. Thus, $A_l$ appears as the moment of $\chi_{\{a_l\}}$ by the distribution $c$. Since only the polynomial moments $c_n$ are known, it is realistic to approximate the characteristic function by a sequence of polynomials $l_n \in P_n$ in a way such that the equality

$$A_l = \lim_{n \to \infty} < c, l_n >$$

holds.

A convenient mean to construct such polynomials $l_n$ is the kernel polynomial for the weight $dx/\sqrt{1-x^2}$. Let us firt recall some properties of Chebyshev polynomials of the first kind $T_n$ :

$$T_n(x) = \cos(nArccosx) \quad n = 0, 1, \ldots$$

Three-terms recurrence relationship :

$$\left| \begin{array}{lll} T_{n+1}(x) & = 2xT_n(x) - T_{n-1}(x) & n = 1, 2, \ldots \\ T_o = 1 & \qquad\qquad T_1 = x & \end{array} \right.$$

Orthogonality relation :

$$k \in \mathbf{N}, l \in \mathbf{N}^*, \frac{2}{\pi} \int_{-1}^{+1} T_k(x) T_l(x) \frac{dx}{\sqrt{1-x^2}} = \delta_{kl} \text{ (Kronecker symbol)}$$

The sequence $\frac{1}{\sqrt{\pi}} T_0, \sqrt{\frac{2}{\pi}} T_1(x), \cdots, \sqrt{\frac{2}{\pi}} T_n(x)$ is the system of orthogonal polynomials with respect to the weight function $\frac{1}{\sqrt{1-x^2}}$ $-1 < x < 1$.

Let us consider, now, the reproducing kernel polynomial :

(which is orthogonal $w.r.t.(x-a)\frac{dx}{\sqrt{1-x^2}}$)

(3) $$k_n(x,a) = [T_{n+1}(x)T_n(a) - T_n(x)T_{n+1}(a)]/(x-a)$$

which, due to Christoffel identity can be rewritten as:

$$k_n(x,a) = \sum_{k=0}^{n}{}' T_k(x)T_k(a)$$

where $\sum_{i=0}^{\prime n} u_i := \frac{1}{2}u_0 + u_1 + \ldots + u_n$.

From the definition of $T_n$, it follows immediately that

$$|(x-a)k_n(x,a)| \leq 2 \quad -1 \leq x \leq +1$$

The value of $k_n(x,a)$ at $x = a$ is :

$$k_n(a,a) = \sum_{k=0}^{n}{}' T_k^2(a) = T'_{n+1}(a)T_n(a) - T'_n(a)T_{n+1}(a).$$

An identity between $T_n$ and $U_n$, Chebyshev polynomial of second kind gives also :

$$k_n(a,a) = \frac{1}{4}(U_{2n}(a) + 2n + 1)$$

The polynomial

$$l_n(x,a) = k_n(x,a)/k_n(a,a)$$

satisfies the following properties : ([2] p. 102)

(i) $l_n(a,a) = 1$

(ii) $|l_n(x,a)| \leq \frac{4}{n|x-a|} \quad n \geq 3, x, a \in [-1,1]$

(iii) $|l_n(x,a)| \leq 4 \quad n \geq 3,\ x, a \in [-1,1]$.

See Appendix for graphical example (Figure 1).

The Lebesgue theorem insures that

$$\lim_{n\to\infty}\int_{-1}^{1} l_n(x,a)w(x)dx = 0$$

and so

$$\lim_{n\to\infty} c(l_n(x,a)) = \lim_{n\to\infty}\int_{-1}^{1} l_n(x,a)w(x)dx + \sum_{i=1}^{q} A_i l_n(a_i,a)$$
$$= \lim_{n\to\infty}\sum_{i=1}^{q} A_i l_n(a_i,a)$$

If $a = a_l$ and $a_i$ distincts $\in [-1,1]$ then

$$\lim_{n\to\infty} l_n(a_i,a_l) = 0 \quad i \neq l$$
$$l_n(a_l,a_l) = 1$$

and thus we get :

$$A_l = \lim_{n\to\infty} c[l_n(x,a_l)]$$

and the

**Theorem 1.**

Let $c$ and $\omega$ be two distributions satisfying (1). If $\omega$ satisfy

$$< \omega, p >= \int_{-1}^{1} p(x)w(x)dx \quad w \in L^1$$

then

$$1 \leq l \leq q \quad A_l = \lim_{n\to\infty}\frac{\sum_{k=0}^{\prime n} c(T_k)T_k(a_l)}{\sum_{k=0}^{\prime n} T_k^2(a_l)} = \lim_{n\to\infty}\frac{\sum_{k=0}^{\prime n} c(T_k)T_k(a_l)}{\frac{1}{4}(U_{2n}(a_l)+2n+1)}$$

See section IV for numerical examples.

**Remark 1 :** If two of the $a_i$ 's are equal then relation (1) becomes

$$c = \omega + \sum_{i=1}^{q-2} A_i \delta_{a_i} + A.\delta'_a$$

and the mass $A$ at the point $a \in [-1, 1]$ can be obtained by

$$A = \lim_{n \to \infty} -c(L_{n+1}(x, a))$$

$$L_{n+1}(x, a) = \int_{-1}^{x} l_n(t, a) dt$$

**Remark 2.** If we do not know that the distribution has Dirac masses, a convenient way is to plot the function

$$c(l_n(x, a)) \text{ for } a \in [-1, 1].$$

If $c$ has a Dirac mass at the point $a$ then $\lim_{n \to \infty} c(l_n(x, a)) \neq 0$ (0 otherwise). See Appendix for a graphical example (Figure 2).

## III - PADE APPROXIMANTS AND THE $\varepsilon$-ALGORITHM.

We assume now that $c$ and $\omega$ are related by (5)

(5) $$c = \omega + A\delta_a \quad a \in \mathbf{R}, A \in \mathbf{C}$$

and that $\omega$ satisfies

(6) $$< \omega, p > = \int_{\mathbf{R}} p(x) d\alpha(x) \quad \alpha \text{ bounded, nondecreasing function.}$$

Relation (5) is equivalent to :

$$(x - a)c = (x - a)\omega$$

Let us set

$$\tilde{c} = (x-a)c \text{ and } \tilde{w} = (x-a)\omega$$

the moments $\tilde{c}_n$ and $\tilde{\omega}_n$ satisfy :

$$\tilde{c}_n = c_{n+1} - ac_n = \tilde{\omega}_n = \tilde{\omega}_n = \omega_{n+1} - a\omega_n \quad n = 0, 1, \ldots$$

The orthogonal polynomials with respect to $\tilde{c} = \tilde{\omega}$ will be denoted by $P_k(\tilde{c}, x)$ or $P_k(\tilde{\omega}, x)$. From the Christoffel Darboux identity, it arises :

$$P_n(\tilde{c}, x) = \sum_{k=0}^{n} P_k(c, x) P_k(c, a) = P_k(\tilde{\omega}, x) = \sum_{k=0}^{n} P_k(\omega, x) P_k(\omega, a)$$

where $P_k(d, x)$ are orthonormal with respect to the functional $d$ :

$$d(P_k(d, x) P_j(d, x)) = \delta_{kj} \quad k, j \in \mathbb{N}$$

Let us now consider the sequence of polynomials :

$$\begin{aligned} l_n(x, a) &:= P_n(\tilde{c}, x) / P_n(\tilde{c}, a) && (7) \\ &= \sum_{k=o}^{n} P_k(\omega, x) P_k(\omega, a) / \sum_{k=0}^{n} P_k^2(\omega, a) && (8) \\ &= \sum_{k=0}^{n} P_k(c, x) P_k(c, a) / \sum_{k=0}^{n} P_k^2(c, a) && (9) \end{aligned}$$

the moment $c(l_n(x, a))$ satisfy

$$c(l_n(x, a)) = c\left[\frac{\sum_{k=0}^{n} P_k(c, x) P_k(c, a)}{\sum_{k=0}^{n} P_k^2(c, a)}\right] = \frac{1}{\sum_{k=0}^{n} P_k^2(c, a)}$$

From relation (5), the moments $c(l_n(x, a))$ can also be expressed as :

$$\begin{aligned} c(l_n(x, a)) &= \omega(l_n(x, a)) + A l_n(a, a) \\ &= \frac{1}{\sum_{k=0}^{n} P_k^2(\omega, a)} + A \quad \text{(From (8))}. \end{aligned}$$

A condition which insures that

$$\lim_{n\to\infty} \sum_{k=0}^{n} P_k^2(\omega, a) = 0$$

is that $\alpha$ be continuous at the point $a$ and the distribution $d\alpha$ belongs to the set $E$ of distributions uniquely determined by their moments [2, p. 62].

**Theorem 2.**

If $c$ and $\omega$ satisfy the relation (5). If $\omega$ satisfies (6) with $d\alpha \in E$ and $\alpha$ continuous at the point $a \in \mathbf{R}$.

Then

$$A = \lim_{n\to\infty} c[l_n(x,a)] = \lim_{n\to\infty} \frac{1}{\sum_{k=0}^{n} P_k^2(c,a)} \tag{10}$$

**Remark.**

If $\alpha(x)$ has some jumps at the points $a_1, \ldots, a_q$ of magnitude $A_1, \ldots, A_q \in \mathbf{R}^+$ then the relation (5) becomes

$$c = \bar{\omega} + \sum_{i=1}^{q} A_i \delta a_i + A\delta_a$$

which is equivalent to the relation (1).

We are now interested by the computation of the quantities involved in (10). Let $f(t)$ be the power series

$$f(t) = \sum_{i=0}^{\infty} \tilde{c}_i t^i \text{ with } \tilde{c}_i = c_i - ac_{i-1} \; i = 0, 1, \ldots \quad c_{-1} := 0$$

whose $n^{th}$ partial sum is :

$$f_n(t) = \sum_{i=0}^{n} \tilde{c}_i t^i.$$

It is well known that Padé approximants and the $\varepsilon$ algorithm are related by : [1, p. 159]

$$(11) \qquad [n/n]_f(t) = \varepsilon_{2n}^{(0)}$$

where the $\varepsilon$-algorithm is applied to the sequence of partial sum $(f_n(t))$. On another hand, Padé approximants are also related to orthogonal polynomials by :[1, p. 128]

$$(12) \qquad [n/n]_f(t) = \tilde{c}_o + \frac{t\tilde{Q}_{n+1}(t)}{\tilde{P}_n(\tilde{c},t)}$$

$$(13) \qquad \text{where} \qquad \tilde{P}_n(\tilde{c},t) = t^n P_n(\tilde{c},t^{-1})$$

$$(14) \qquad \text{and} \qquad Q_{n-1}(t) = \tilde{c}\left(\tfrac{Pn(\tilde{c},x)-P_n(\tilde{c},t)}{x-t}\right) \qquad \tilde{c} \text{ acts on } x$$

$$(15) \qquad \tilde{Q}_{n-1}(t) = t^{n-1}Q_{n-1}(t^{-1})$$

If we take $t = a^{-1}$ in (12) :

$$\begin{gathered}
[n/n]_f(a^{-1}) = c_o + a^{-1}\tfrac{\tilde{Q}_{n-1}(a^{-1})}{\tilde{P}_n(\tilde{c}^{(1)},a^{-1})} = c_o + \tfrac{Q_{n-1}(a)}{P_n(\tilde{c}^{(1)},a)} \\
\tilde{c}_n^{(1)} = \tilde{c}_{n+1} = c_{n+1} - ac_n \quad n \geq 0 \Rightarrow \tilde{c}^{(1)} = (x-a)c \\
Q_{n-1}(a) = \tilde{c}^{(1)}\left(\tfrac{P_n(\tilde{c}^{(1)},x)-P_n(\tilde{c}^{(1)},a)}{x-a}\right) = c(P_n(\tilde{c}^{(1)},x) - P_n(\tilde{c}^{(1)},a)) \\
Q_{n-1}(a) = c(\textstyle\sum_{k=0}^n P_k(c,x)P_k(c,a)) - c_o \sum_{k=0}^n P_k^2(c,a) = 1 - c_o - \sum_{k=0}^n P_k^2(c,a)
\end{gathered}$$

therefore

$$[n/n]_f(a^{-1}) = c_o + \frac{1 - c_o\sum_{k=0}^n P_k^2(c,a)}{\sum_{k=0}^n P_k^2(c,a)} = \frac{1}{\sum_{k=0}^n P_k^2(c,a)}$$

For $a \neq 0$ we get :

$$(16) \qquad [n/n]_f(a^{-1}) = \varepsilon_{2n}^{(0)} = \frac{1}{\sum_{k=0}^n P_k^2(c,a)}$$

where $\varepsilon_{2n}^{(0)}$ is obtained by the $\varepsilon$-algorithm when applied to the sequence of partial sums

$$f_n(a^{-1}) = \sum_{i=0}^{n} \tilde{c}_i a^{-i} = \sum_{i=0}^{n} (c_i - ac_{i-1}) a^{-i} = c_n a^{-n}.$$

If $a = 0$ then the $\varepsilon$ -algorithm can no more be used but it is possible to compute the value of $[n/n]_f(\infty)$ by using the quotient of the coefficients of the highest degree term as follows : [1, p. 36]

$$(17) \qquad [n/n]_f(t) = \begin{vmatrix} \tilde{c}_o t^n & \tilde{c}_o t^{n-1} + \tilde{c}_1 t^n & \cdots & \sum_{i=0}^{n} \tilde{c}_i t^i \\ \tilde{c}_1 & \tilde{c}_2 & \cdots & \tilde{c}_{n+1} \\ \vdots & & \vdots & \vdots \\ \tilde{c}_n & \tilde{c}_{n+1} & \cdots & \tilde{c}_{2n} \end{vmatrix} \div \begin{vmatrix} t^n & t^{t-1} & \cdots & 1 \\ \tilde{c}_1 & \tilde{c}_2 & \cdots & \tilde{c}_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{c}_n & \tilde{c}_{n+1} & \cdots & \tilde{c}_{2n} \end{vmatrix}$$

and the value of $[n/n]_f(t)$ at $t = \infty$ is :

$$(18) \qquad [n/n]_f(\infty) = \begin{vmatrix} \tilde{c}_0 & \cdots & \tilde{c}_n \\ \vdots & \vdots & \vdots \\ \tilde{c}_n & \cdots & \tilde{c}_{2n} \end{vmatrix} \div \begin{vmatrix} \tilde{c}_2 & \cdots & \tilde{c}_{n+1} \\ & \vdots & \vdots \\ \tilde{c}_{n+1} & \cdots & \tilde{c}_{2n} \end{vmatrix}$$

$$= H_{n+1}(\tilde{c}_0)/H_n(\tilde{c}_2) \quad \text{(Hankel determinants)}$$

Since $\tilde{c}_i = c_i - ac_{i-1} \quad i = 0, 1, \ldots$ and $a = 0$ we get

$$[n/n]_f(\infty) = H_{n+1}(c_0)/H_n(c_2).$$

The above quantity can be computed recursively with the $w$-algorithm of Wynn [1, p. 168] or by the $\varepsilon$-algorithm as follows:

After manipulations on the rows of the determinants,

$$H_{n+1}(c_0)/H_n(c_2) = H_{n+1}(S_0)/H_n(\Delta^2 S_0)$$

with $S_n = (I + E)^n c_0 = \sum_{k=0}^{n} C_n^k c_k$ and the quotient $H_{n+1}(S_0)/H_n(\Delta^2 S_0)$ can be computed by $\varepsilon$-algorithm.

So, if $a = 0$ the quantity involved in (10) can be computed by applying the $\varepsilon$-algorithm to the sequence $S_n = (I + E)^n c_0$.

## IV - Numerical examples.

**Example 1.** $c = dx + \delta_{0.5}$ on $[-1, 1]$. Only the moments and the support of $c$ are assumed to be given. Since the distribution $dx$ is positive, $\varepsilon$-algorithm can be used as well as the Chebyshev polynomials. The mass at the point $a = 0.5$ is $A = 1$.

$$c_n = \frac{1 - (-1)^n}{n+1} + 0.5^n \qquad \text{column}(2)$$

$$c[l_n(x, a)] = \sum_{k=0}^{n}{}' c(T_k)T_k(a) / \sum_{k=0}^{n}{}' T_k^2(a) \qquad \text{column}(4)$$

$$\varepsilon_{2n}^{(0)}(c_n a^{-n}) \qquad \text{column}(5)$$

($c(T_n)$ (column (3)) does not converge to 0 which confirms the presence of Dirac masses ; see [4]).

We can see that both $c(l_n(x, a))$ and $\varepsilon_{2n}^{(0)}(c_n a^{-n})$ converge to $A = 1$ and that the convergence of the first one is faster than that of the second one (See table 1).

**Example 2.** $c = \omega + 4\delta_{7.5}$ where $< \omega, p >= \int_0^\infty p(x)e^{-x}dx$. Here, the method of II cannot be used, since the interval is infinite. By applying the $\varepsilon$-algorithm to the sequence $c_n\ 7.5^{-n}$, the limit of $\varepsilon_{2n}^{(0)}$ is $A = 4$.

$$c_n = n! + 4 * 7.5^n \Longrightarrow c_n 7.5^{-n} = n! 7.5^{-n} + 4$$

(See table 2).

**Example 3.** Let the distribution be defined as follows :

$$c = \omega \text{ with } \omega(x^n) = \int_{-1}^{1} x^n w(x)dx$$

where

$$\begin{aligned} w(x) &= x + 1 \quad \text{on} \quad [-1, 0.5[ \\ w(x) &= 1 - x \quad \text{on} \quad ]0.5, 1] \end{aligned}$$

The coefficients

$$c_n = \frac{0.5^{n+1}}{n+2} + \frac{2}{(n+1)(n+2)} \qquad n \text{ even}$$
$$c_n = 0.5^{n+1}/(n+2) \qquad n \text{ odd}$$

are supposed to be given. The goal is, here, to compute the jump of $w$ at the point $a = 0.5$ (which can be approximated by the method explained in [4]).

The jumps of $w$ will appear in the derivative (in the sense of distributions) of $w$. If $w$ has a finite number of jumps of finite magnitude $A_i$ at points $a_1, \cdots, a_q$ then the derivative of $w$ satisfies :

$$w' = \{w\}' + \sum_{i=1}^{q} A_i \delta_{a_i}$$

where $\{w\}'$ is the derivative of $w$ in the usual sense, if it exists.

The moments of $w'$ can be computed as follows :

$$\forall p \in P, \int_{-1}^{1} p(x) w'(x) dx = -\int_{-1}^{1} p'(x) w(x) dx$$

by integration by part, and so :

$$c'_n := < c', x^n > = -n c_{n-1} \quad n \geq 1$$
$$c'_0 = 0.$$

$c'$ is not a positive distribution, so the $\varepsilon$-algorithm is uneffective. We can only use the Chebyshev kernel (See table 3). The sequence $c'(T_n)$ does not converge to 0, which indicates that the distribution $c'$ has Dirac masses. The point $a = 0.5$ can be calculated with

$$a = \lim_{n \to \infty} [c'(T_{n+1}) + c'(T_{n-1})]/2c'(T_n) \text{ (see[4])}$$

and the mass $A = -1$ satisfies

$$A = \lim_{n\to\infty} c'(l_n(x,a))$$

## V - Limit of totally monotonic sequences.

An important application of the previous sections is the calculation of the limit of totally monotonic sequences.

**Definition.** *A sequence $(c_n)_{n\in\mathbb{N}}$ is said to be totally monotonic ($c_n \in TM$) if*

$$(-1)^n \Delta^k c_n \geq 0 \text{ for } n,k = 0,1,\cdots$$

*or equivalently if there exists a nondecreasing function $\alpha$ such that :*

$$c_n = \int_0^1 x^n d\alpha(x) \quad n = 0,1,\cdots$$

∎

It is well known that a $TM$ sequence is always convergent and that the limit $l$ satisfies

$$l = \alpha(1) - \alpha(1^-) \quad [1, p.116-120]$$

Since $\alpha$ is nondecreasing, we can apply theorem 2 and thus the limit of the sequence $(c_n)_n$ can be found by the $\varepsilon$-algorithm [1, p. 165]. It is possible to generalize the result of the section two for Jacobi polynomials on $[0,1]$.

In the particular case where $a = 1$, it is possible to extend the properties of the Chebyshev reproducing kernel $l_n(x,a)$ of section II to Jacobi polynomials on $[0,1]$ (Shifted Jacobi polynomials).

**Lemma** *Let $P_n^{*(\alpha,\beta)}(x)$ be the shifted Jacobi polynomial with $\beta > -1$, $\alpha > -1$*

*Let $l_n(x) = P_n^{*(\alpha,\beta)}(x)/P_n^{*(\alpha,\beta)}(1)$ then*

*(i)* $l_n(1) = 1$

*(ii)* $\beta < \alpha, \alpha > -\frac{1}{2} \Longrightarrow \lim_{n\to\infty} l_n(x) = 0, \forall x \in [0,1[$

*(iii)* $|l_n(x)| \leq 1 \quad \forall x \in [0,1]$

See appendix for graphical examples (Figure 3).

**Proof.**

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ satisfy [5, p. 58]

$$\int_{-1}^{1} P_n^{(\alpha,\beta)} P_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx = 0 \text{ if } n \neq m$$

The normalization of $P_n^{(\alpha,\beta)}$ is taken such that :

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$$

The shifted Jacobi polynomials are defined by :

$$P_n^{*(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1)$$

$P_n^{*(\alpha,\beta)}$ satisfies :

$$\int_0^1 P_n^{*(\alpha,\beta)}(x) P_n^{*(\alpha,\beta)}(x)(1-x)^\alpha x^\beta dx = 0 \text{ if } n \neq m.$$

$$P_n^{*(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$$

$$P_n^{*(\alpha,\beta)}(0) = P_n^{(\alpha,\beta)}(-1) = (-1)^n \binom{n+\beta}{n}$$

Moreover, if $\beta < \alpha$, $\alpha > -\frac{1}{2}$

$$\max_{0 \leq x \leq 1} |P_n^{*(\alpha,\beta)}(x)| = \binom{n+\alpha}{n} \approx n^\alpha, \text{ if } \alpha > -\frac{1}{2},$$

and

$$|P_n^{*(\alpha,\beta)}(x)| = 0(n^{-\frac{1}{2}}), x \in [0,1[ \qquad ([5, p.168])$$

Thus

(i) $l_n(1) = P_n^{*(\alpha,\beta)}(1)/P_n^{*(\alpha,\beta)}(1) = 1$

(ii) $|l_n(x)| = \dfrac{P_n^{*(\alpha,\beta)}(x)}{\binom{n+\alpha}{n}} \leq 1 \quad \forall x \in [0,1]$

(iii) $|l_n(x)| = \dfrac{0(n^{-\frac{1}{2}})}{\binom{n+\alpha}{n}} \longrightarrow 0$ *when* $n \to \infty$ for $\alpha > -\frac{1}{2}, \forall x \in [0,1[$.

■

The Lebesgue theorem insures that :

$$\lim_{n\to\infty} c(l_n(x)) = \lim_{n\to\infty} \int_0^1 l_n(x)d\alpha(x) = l$$

**Theorem 3.** *If the sequence* $(c_n)_{n\in\mathbb{N}} \in TM$ *and* $\alpha > -\frac{1}{2}$, $\alpha > \beta$ *then :*

$$\lim_{n\to\infty} \frac{c(P_n^{*(\alpha,\beta)}(x))}{P_n^{*(\alpha,\beta)}(1)} = \lim_{n\to\infty} c_n$$

■

The computation of $c(P_n^{*(\alpha,\beta)}(x))$ is very easy with the following expression of $P_n^{*(\alpha,\beta)}(x)$ [5, p. 68]

$$P_n^{*(\alpha,\beta)}(x) = \sum_{\nu=0}^{n} \binom{n+\alpha}{n-\nu}\binom{n+\beta}{\nu}(x-1)^\nu x^{n-\nu}$$

and

$$\begin{aligned} c(P_n^{*(\alpha,\beta)}(x)) &= \sum_{\nu=0}^{n} \binom{n+\alpha}{n-\nu}\binom{n+\beta}{\nu} c((x-1)^\nu x^{n-\nu}) \\ &= \sum_{\nu=0}^{n} \binom{n+\alpha}{n-\nu}\binom{n+\beta}{\nu} \Delta^\nu c_{n-\nu} \end{aligned}$$

**Numerical examples:** $c_n = \sum_{k=1}^{n}[1-(1-\frac{1}{k^3})^k] \quad n \geq 1.$

The sequence $(c_n)_n$ is totally monotonic and the convergence is logarithmic.

Column (3) contains the diagonal of the $\varepsilon$-array, for the sequence $(c_n)_n$. Column (4) contains $c(l_n(x))$ for $\alpha = 1$, $\beta = 0$. Using exact arithmetic,we saw that $c(l_n)$ is totally oscillating up to 50 that is $((-1)^n c(l_n(x))) \in TM$, and applying the $\varepsilon$-algorithm to it gives column (5). The limit seems to be : 0.62231122649. (See table 4).

## APPENDIX

The kernel polynomial $l_n(x,a) = \frac{\sum'^{n}_{k=0} T_k(x)T_k(a)}{\sum'^{n}_{k=0} T_k^2(a)}$ is plotted in figure 1 with $a = 0.5$ and $n = 20$

Figure 1

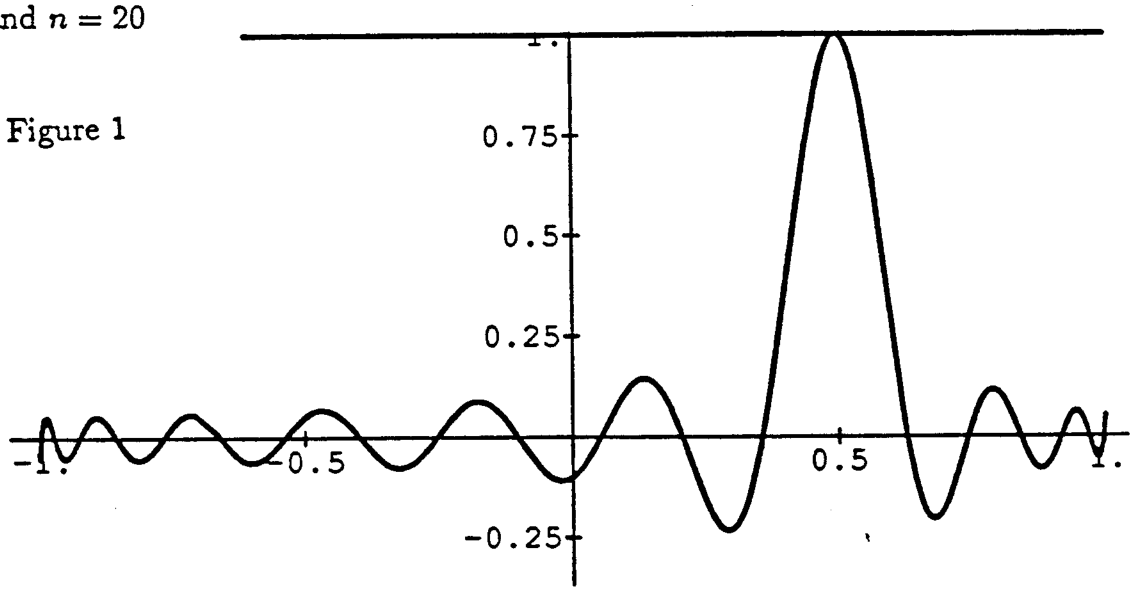

Figure 2 represents $c(l_n(x,a))$ ($c$ acts on $x$), for $a \in [-1,1], c = \chi_{[-1,1]}dx + \delta_{0.5}$ and $n = 20$.

Figure 2

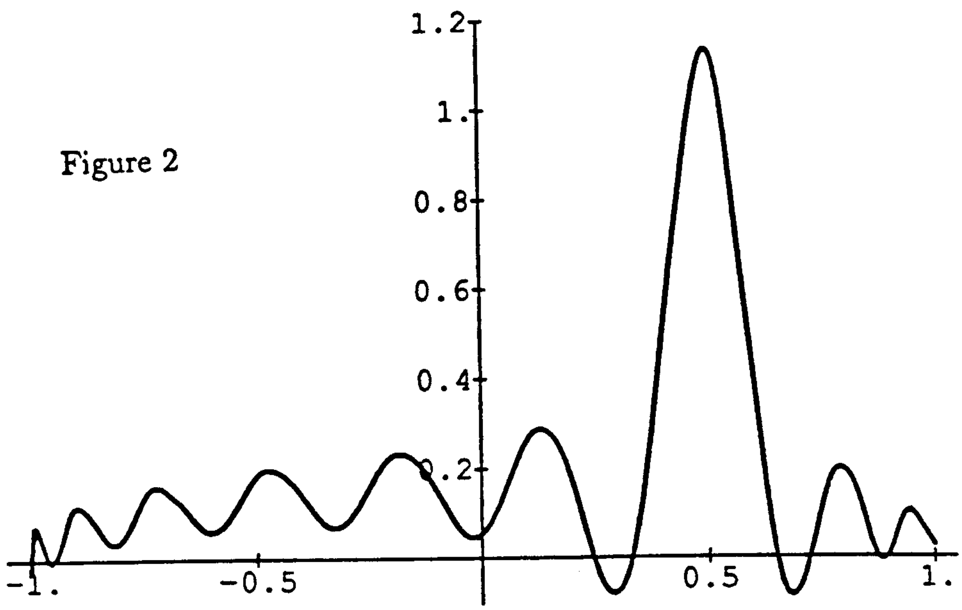


Figure 3 represents $l_n(x)$ (see V) for $\alpha = 1 \quad \beta = 0 \quad n = 15$

Figure 3

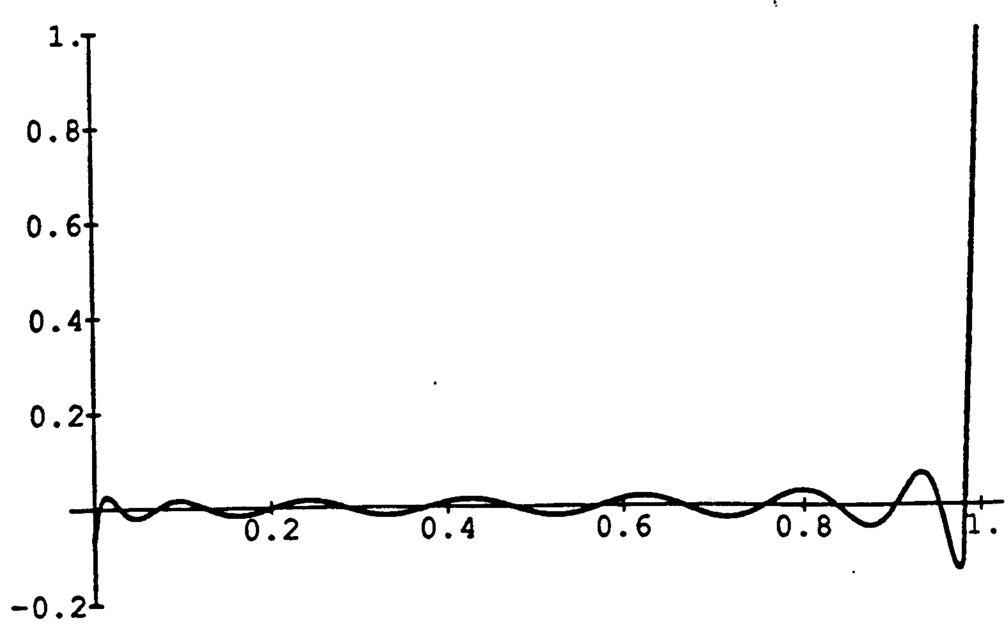

Table 1

| (1) n | (2) $c_n$ | (3) $c(T_n)$ | (4) $c(l_n)$ | (5) $\varepsilon_n^{(o)}$ |
|---|---|---|---|---|
| 0 | 3.0000 | 1.5000 | 3.0000 | 3.0000 |
| 1 | 0.5000 | 0.5000 | 2.3330 | −0.5000 |
| 2 | 0.9167 | −1.1667 | 2.3330 | 2.1429 |
| 3 | 0.1250 | −1.0000 | 1.6670 | 5.6250 |
| 4 | 0.4625 | −0.6333 | 1.6220 | 2.0940 |
| 5 | 0.0313 | 0.5000 | 1.5600 | 0.5324 |
| 6 | 0.3013 | 0.9429 | 1.3840 | 1.6313 |
| 7 | 0.0078 | 0.5000 | 1.3580 | −4.4275 |
| 8 | 0.2261 | −0.5317 | 1.3400 | 1.5102 |
| 9 | 0.0020 | −1.0000 | 1.2720 | 25.1566 |
| 10 | 0.1828 | −0.5202 | 1.2610 | 1.4989 |
| 11 | 0.0005 | 0.5000 | 1.2490 | 2.1000 |
| 12 | 0.1541 | 0.9860 | 1.2080 | 1.3726 |
| 13 | 0.0001 | 0.5000 | 1.2010 | −12.1628 |
| 14 | 0.1334 | −0.5103 | 1.1940 | 1.3271 |
| 15 | 0.0000 | −1.0000 | 1.1700 | 58.9267 |
| 16 | 0.1177 | −0.5078 | 1.1650 | 1.3223 |
| 17 | 0.0000 | 0.5000 | 1.1600 | 4.6932 |
| 18 | 0.1053 | 0.9938 | 1.1430 | 1.2642 |
| 19 | 0.0000 | 0.5000 | 1.1390 | −23.7156 |
| 20 | 0.0952 | −0.5050 | 1.1360 | 1.2406 |
| 21 | 0.0000 | −1.0000 | 1.1240 | 106.9488 |
| 22 | 0.0870 | −0.5041 | 1.1210 | 1.2379 |
| 23 | 0.0000 | 0.50000 | 1.1180 | 8.3106 |
| 24 | 0.0800 | 0.9965 | 1.1090 | 1.2046 |
| 25 | 0.0000 | 0.50000 | 1.1070 | −39.0873 |
| 26 | 0.0741 | −0.5030 | 1.1050 | 1.1902 |
| 27 | 0.0000 | −1.0000 | 1.0970 | 169.2253 |
| 28 | 0.0690 | −0.5026 | 1.0960 | 1.1885 |
| 29 | 0.0000 | 0.50000 | 1.0940 | 12.9518 |
| 30 | 0.0645 | 0.9978 | 1.0880 | 1.1670 |
| 31 | 0.0000 | 0.50000 | 1.0860 | −58.2783 |
| 32 | 0.0606 | −0.5020 | 1.0850 | 1.1572 |
| 33 | 0.0000 | −1.000 | 1.0800 | 245.7564 |
| 34 | 0.0571 | −0.5017 | 1.0790 | 1.1561 |
| 35 | 0.0000 | 0.50000 | 1.0780 | 18.6168 |
| 36 | 0.0541 | 0.9985 | 1.0730 | 1.1410 |
| 37 | 0.0000 | 0.5000 | 1.0730 | −81.2902 |
| 38 | 0.0513 | −0.5014 | 1.0720 | 1.1340 |
| 39 | 0.0000 | −1.0000 | 1.0680 | 336.5967 |
| 40 | 0.0488 | −0.5008 | 1.0670 | 1.1332 |
| 41 | 0.0000 | 0.50000 | 1.0660 | 25.2922 |
| 42 | 0.0465 | 0.9989 | 1.0630 | 1.1220 |

Table 2

| n | $\varepsilon_n^{(0)}$ | n | $\varepsilon_n^{(0)}$ | n | $\varepsilon_n^{(0)}$ | n | $\varepsilon_n^{(0)}$ |
|---|---|---|---|---|---|---|---|
| 0 | 5.000000 | 12 | 4.002010 | 24 | 4.001458 | 36 | 4.001115 |
| 2 | 4.023121 | 14 | 4.001910 | 26 | 4.001338 | 38 | 4.001087 |
| 4 | 4.004119 | 16 | 4.001858 | 28 | 4.001284 | 40 | 4.001086 |
| 6 | 4.003355 | 18 | 4.001635 | 30 | 4.001283 | 42 | 4.001041 |
| 8 | 4.002554 | 20 | 4.001532 | 32 | 4.001254 | 44 | 4.001099 |
| 10 | 4.002495 | 22 | 4.001532 | 34 | 4.001181 | 46 | 4.001074 |

Table 3

| (1) n | (2) c'$_n$ | (3) c'(T$_n$) | (4) c'(l$_n(x,a)$) |
|---|---|---|---|
| 0 | 0.000 | 0.000 | 0.00000 |
| 1 | −1.250 | −1.250 | −0.83333 |
| 2 | −0.167 | −0.333 | −0.45833 |
| 3 | −0.594 | 1.375 | −0.91667 |
| 4 | −0.050 | 0.933 | −1.02222 |
| 5 | −0.359 | −0.125 | −0.94500 |
| 6 | −0.013 | −1.029 | −0.968888 |
| 7 | −0.257 | −0.688 | −0.99595 |
| 8 | −0.003 | 0.317 | −0.97339 |
| 9 | −0.202 | 1.037 | −0.98621 |
| 10 | −0.001 | 0.657 | −1.00178 |
| 11 | −0.167 | −0.350 | −0.98806 |
| 12 | −0.000 | −1.007 | −0.99097 |
| 13 | −0.143 | −0.607 | −0.99924 |
| 14 | −0.000 | 0.395 | −0.99176 |
| 15 | −0.125 | 1.013 | −0.99447 |
| 16 | −0.000 | 0.596 | −1.00046 |
| 17 | −0.111 | −0.406 | −0.99493 |
| 18 | −0.000 | −1.003 | −0.99579 |
| 19 | −0.100 | −0.575 | −0.99974 |
| 20 | −0.000 | 0.426 | −0.99605 |
| 21 | −0.091 | 1.007 | −0.99703 |
| 22 | −0.000 | 0.569 | −1.00018 |

| (1) n | (2) c$_n$ | (3) c'(T$_n$) | (4) c'(l$_n(x,a)$) |
|---|---|---|---|
| 23 | −0.083 | −0.432 | −0.99721 |
| 24 | −0.000 | −1.002 | −0.99757 |
| 25 | −0.077 | −0558 | −0.99988 |
| 26 | −0.000 | 0.443 | −0.99769 |
| 27 | −0.071 | 1.004 | −0.99815 |
| 28 | −0.000 | 0.554 | −1.00009 |
| 29 | −0.067 | −0.446 | −0.99824 |
| 30 | −0.000 | −1.001 | −0.99842 |
| 31 | −0.063 | −0.547 | −0.99994 |
| 32 | −0.000 | 0.454 | −0.99849 |
| 33 | −0.059 | 1.003 | −0.99874 |
| 34 | −0.000 | 0.545 | −1.00005 |
| 35 | −0.056 | −0.456 | −0.99879 |
| 36 | −0.000 | −1.001 | −0.99890 |
| 37 | −0.053 | −0.540 | −0.99996 |
| 38 | −0.000 | 0.461 | −0.99893 |
| 39 | −0.050 | 1.002 | −0.99909 |
| 40 | −0.000 | 0.538 | −1.00004 |
| 41 | −0.048 | −0.463 | −0.99915 |
| 42 | −0.000 | −1.001 | −0.99921 |
| 43 | −0.045 | −0.527 | −0.99984 |

| n | $c_n$ | $\varepsilon_n^{(o)}(c_n)$ | $c(l_n)$ | $\varepsilon_n^{(o)}(c(l_n))$ |
|---|---|---|---|---|
| 0 | 1.000000 | 0.000000000000000 | 1.00000000000000 | |
| 1 | 0.234375 | 0.000000000000000 | 0.46875000000000 | .4687500000000000 |
| 2 | 0.341422 | 4.266666666666667 | 0.64228014022253 | 5.762687673263252 |
| 3 | 0.402472 | .4314173386971078 | 0.61929020831748 | .6219796891591877 |
| 4 | 0.441837 | 28.54070766854518 | 0.62296588902627 | 2041.872461560933 |
| 5 | 0.469295 | .5302231719587736 | 0.62207959957785 | .6223074894681753 |
| 6 | 0.489526 | 100.4135643550456 | 0.62244458151444 | 86788.54353453308 |
| 7 | 0.505045 | .568169785337829 | 0.62221173725566 | .622313355279943 |
| 8 | 0.517323 | 259.4309287987356 | 0.62239217549190 | -580807.4148428634 |
| 9 | 0.527278 | .5866926812734348 | 0.62224376258451 | .6223112246271211 |
| 10 | 0.535511 | 557.1543794684833 | 0.62236786185632 | 87277012.35547296 |
| 11 | 0.542434 | .5971079453425248 | 0.62226349373451 | .6223112283730024 |
| 12 | 0.548335 | 1057.17072493674 | 0.62235161654952 | -415711646.7747317 |
| 13 | 0.553425 | .6035417954722929 | 0.62227689289343 | .6223112264950859 |
| 14 | 0.557860 | 1835.083149896018 | 0.62234055731544 | 73208176641.16921 |
| 15 | 0.561759 | .607792874630161 | 0.62228609146925 | .6223112264948491 |
| 16 | 0.565214 | 2978.500734319092 | 0.62233234119527 | 11137522677.15121 |

Table 4

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# Aitken's $\Delta^2$ and Lubkin's $w$ processes as Padé-type approximants. Generalization.

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**Abstract** :In this paper we show that Aitken's $\Delta^2$ and Lubkin's $w$ processes can be interpreted as Padé-type approximants with generating polynomials depending on the sequence to be accelerated. This remark will allow us to generalize the first algorithm.

**I - Introduction.** See [1].

Let $(S_n)_{n\in N}$ be a sequence of complex numbers. We transform it into the partial sums of a series by setting :

$$c_0 := S_0, \quad c_i = \frac{\Delta S_{i-1}}{t_0^i} \quad i = 1, 2, \ldots$$

where $\Delta S_{i-1} = S_i - S_{i-1}$ and $t_0 \in \mathbf{C}, t_0 \neq 0$.
We get :

$$S_n = \sum_{i=0}^{n} c_i t_0^i \text{ and } S = \sum_{i=0}^{\infty} c_i t_0^i = \lim_{i\to\infty} S_n \text{ if it exists}$$

Let us set :

$$f(t) = \sum_{i=0}^{\infty} c_i t^i$$

and $P_n(x) = \frac{1-R_{n+1}(x)/R_{n+1}(t^{-1})}{1-xt}$ where $R_{n+1}$ is an arbitrary polynomial of degree $(n+1)$. $P_n$ is the interpolation polynomial of the generating functional : $x \to (1-xt)^{-1}$ at the zeros of $R_{n+1}$.

If we define the linear functional $c$ by :

$$c(x^i) = c_i, \; i = 0, 1, \ldots$$

it formally arises :

$$f(t) = c((1-xt)^{-1})$$

and

$$c(P_n(x)) = \frac{1}{R_{n+1}(t^{-1})} c\left[\frac{R_{n+1}(t^{-1}) - R_{n+1}(x)}{1-xt}\right] .(c \text{ acts on } x).$$

$c(P_n(x))$ is a rational fraction, in the variable $t$, of type $(n/n+1)$. It is called a Padé-type approximant to $f$ and satisfies:

$$c(P_n(x)) - f(t) = 0(t^n)$$

If we write :

$$R_{n+1}(x) = \sum_{i=0}^{n+1} a_i^{(n+1)} x^i \ (a_i^{(n)} \text{ can depend on } t_0 \text{ and } S_n).$$

It arises

$$R_{n+1}(t_0^{-1}) - R_{n+1}(x) = \sum_{i=1}^{n+1} a_i^{(n+1)}(1 - x^i t_0^i) t_0^{-i}$$

The i$^{th}$ term of the sequence $(S_n)_{n\in \mathbb{N}}$ can be written as :

$$S_i = c(1 + xt_0 + \cdots + x^i t_0^i) = c\left(\frac{1 - x^{i+1}t_0^{i+1}}{1 - xt_0}\right)$$

and $c(P_n(x))$ becomes :

$$c(P_n(x)) = \frac{1}{R_{n+1}(t_0^{-1})} \sum_{i=1}^{n+1} a_i^{(n+1)} S_{i-1} t_0^{-i}$$

which can be expressed by :

$$(1) \qquad c(P_n(x)) = \sum_{i=0}^{n} \mu_i^{(n)} S_i \quad \text{with } \mu_i^{(n)} = a_{i+1}^{(n+1)} t_0^{-i-1} / R_{n+1}(t_0^{-1})$$

We set :

$$c(P_n(x)) = T(S_n)$$

A general theorem of convergence can be established [3].

**Theorem 1 :** *If the sequence $(S_n)_{n\in \mathbb{N}}$ is such that the zeros of $R_{n+1}$ belong to the interval $[-\delta, 0], \delta > 0$ and if the sequence $(S_n)$ is convergent then*

$$\lim_{n\to\infty} T(S_n) = \lim_{n\to\infty} S_n$$

## II - Aitken's $\Delta^2$ process as Padé-type approximant.

Aitken's $\Delta^2$ process is an algorithm which accelerates the convergence of linear sequences :

$$\delta_2(S_n) = S_{n+1} \cdot \frac{\Delta S_{n+1}}{\Delta S_{n+1} - \Delta S_n} - S_{n+2} \cdot \frac{\Delta S_n}{\Delta S_{n+1} - \Delta S_n}$$

or

$$\delta_2(S_n) = S_{n+1} \cdot \frac{c_{n+2}}{c_{n+2} - c_{n+1}} - S_{n+2} \frac{c_{n+1}}{c_{n+2} - c_{n+1}}$$

$\delta_2(S_n)$ appears as a linear combination of $S_{n+1}$ and $S_{n+2}$ with coefficients depending on $S_n$, $S_{n+1}$, $S_{n+2}$. If we set :

$$R_{n+3}(x) = \frac{c_{n+2}}{c_{n+2}-c_{n+1}} x^{n+2} - \frac{c_{n+1}}{c_{n+2}-c_{n+1}} x^{n+3}$$

$$t_0 = 1, R_{n+3}(1) = 1 \quad .$$

$$P_{n+2}(x) = \frac{1 - R_{n+3}(x)}{1-x}$$

Then $P_{n+2}$ interpolates the generating function $x \to (1-x)^{-1}$ at the zeros of $R_{n+3}$, that is at 0 with multiplicity $(n+2)$ and at $c_{n+2}/c_{n+1}$ with multiplicity 1.

It is a well known result that Aitken's $\Delta^2$ process is the first row of Padé table :

$$\delta_2(S_n) = [n+2/1]_f(1)$$

**Other interpretation.**

We can try to interpolate generating function at 0 with multiplicity $(n+1)$ and at $c_{n+2}/c_{n+1}$ with multiplicity 2.

$$R_{n+3}(x) = x^{n+1}(x - c_{n+2}/c_{n+1})^2/(1 - c_{n+2}/c_{n+1})^2 \quad R_{n+3}(1) = 1$$

with the notations of the introduction :

$$c(P_{n+2}(x)) = \frac{(c_{n+2}/c_{n+1})^2 S_n - 2(c_{n+2}/c_{n+1}).S_{n+1} + S_{n+2}}{(1 - c_{n+2}/c_{n+1})^2}$$

By using $c_{n+2} = \Delta S_{n+1}$, $c_{n+1} = \Delta S_n$, $c_n = \Delta S_{n-1}$ and after simplification, we get another form of $c(P_{n+2}(x))$

$$c(P_{n+2}(x)) = \delta_2(S_n)$$

So we have

**Theorem 2 :**

*Aitken's $\Delta^2$ process is equal to $c(P_{n+2}(x))$ where $P_{n+2}$ is the interpolation polynomial of the function $x \to (1-x)^{-1}$ at 0 with multiplicity $(n+2)$ and at $(c_{n+2}/c_{n+1})$ with multiplicity 1 or at 0 with multiplicity $(n+1)$ and at $(c_{n+2}/c_{n+1})$ with multiplicity 2.*

**III - Generalization of Aitken's process**

We can try to generalize the precedent results as following. Let

$$R_{n+3}(x) = x^{n+3-k}\left(x - \frac{c_{n+2}}{c_{n+1}}\right)^k /(1 - c_{n+2}/c_{n+1})^k$$

0 is a zero of $R_{n+3}$ with multiplicity $n+3-k, c_{n+2}/c_{n+1}$ is a zero of $R_{n+3}$ with multiplicity $k$.

We obtain a process denoted by $\Delta_k^{(n)}$ :

$$\Delta_k^{(n)} := c(P_{n+3}(x)) = \frac{\sum_{p=0}^{k} \binom{k}{p} \left(-\frac{c_{n+2}}{c_{n+1}}\right)^p S_{n+2-p}}{\left(1 - \frac{c_{n+2}}{c_{n+1}}\right)^k}$$

which can be simply written (symbolic notation)

$$\Delta_k^{(n)} = (1 - c_{n+2}/c_{n+1})^{-k}.(I - \frac{c_{n+2}}{c_{n+1}} E)^k \circ S_{n+2-k}$$

where $E(S_i) = S_{i+1}$ and $I(S_i) = S_i$.

**particular case :**

$k = 1$ or $k = 2$ gives $\Delta_2^{(n)} = \delta_2(S_n)$. See II

$k = 3$, gives, after simplification : $\Delta_3^{(n)} = S_{n+2} - (\Delta S_{n+1})^3 \frac{\Delta^3 S_{n-1}}{(\Delta^2 S_n)^3}$.

or $\Delta_3^{(n)} = \Delta_2^{(n)} + \frac{(\Delta S_{n+1})^2}{(\Delta^2 S_n)^3}((\Delta S_n)^2 - \Delta S_{n-1}.\Delta S_{n+1})$

**Numerical application.**

**1)** $\Delta_2^{(n)}$, $\Delta_3^{(n)}$, $\Delta_{n+2}^{(n)}$ applied to $S_n = \sum_{k=0}^{n} \frac{(-1)^k}{k+1}; \lim_{n\to\infty} S_n = \log 2$.

| $n$ | $S_n$ | $\Delta_2^{(n)}$ | $\Delta_3^{(n)}$ | $\Delta_{n+2}^{(n)}$ |
|---|---|---|---|---|
| 0 | 1.000 | | | |
| 1 | 0.500 | | | |
| 2 | 0.833 | 0.7 | | |
| 3 | 0.583 | 0.6904761904761904 | 0.6948493683187560 | 0.6948493683187560 |
| 4 | 0.783 | 0.6944444444444444 | 0.6926154549611340 | 0.6938347812833409 |
| 5 | 0.617 | 0.6924242424242424 | 0.6933633859253694 | 0.6934151293689576 |
| 6 | 0.760 | 0.6935897435897436 | 0.6930435442269762 | 0.6932546153507766 |
| 7 | 0.635 | 0.6928571428571428 | 0.6932028218694885 | 0.6931911353582859 |
| 8 | 0.746 | 0.6933473389355742 | 0.6931147199364174 | 0.6931654824266621 |
| 9 | 0.646 | 0.6930033416875522 | 0.6931673597659893 | 0.6931549158427455 |
| 10 | 0.737 | 0.6932539682539682 | 0.6931339908097277 | 0.6931504920196000 |
| 11 | 0.653 | 0.6930657506744463 | 0.6931561591564057 | 0.6931486139521721 |
| 12 | 0.730 | 0.6932106782106782 | 0.6931408600288600 | 0.6931478070153731 |
| 13 | 0.659 | 0.6930967180967180 | 0.6931517571320955 | 0.6931474566694211 |
| 14 | 0.725 | 0.6931879423258733 | 0.6931437862277585 | 0.6931473031658790 |
| 15 | 0.663 | 0.6931137858557213 | 0.6931497506964930 | 0.6931472353652639 |
| 16 | 0.722 | 0.6931748806748806 | 0.6931451991024993 | 0.6931472052040062 |
| 17 | 0.666 | 0.6931239512121865 | 0.6931487325532943 | 0.6931471917008954 |
| 18 | 0.719 | 0.6931668512550865 | 0.6931459477838329 | 0.6931471856208846 |
| 19 | 0.669 | 0.6931303775344022 | 0.6931481720952606 | 0.6931471828690705 |
| 20 | 0.716 | 0.6931616470778669 | 0.6931463740612400 | 0.6931471816177483 |
| 21 | 0.671 | 0.6931346368409871 | 0.6931478432253306 | 0.6931471810463044 |
| 22 | 0.714 | 0.6931581275621523 | 0.6931466310568286 | 0.6931471807843213 |
| 23 | 0.673 | 0.6931375704648144 | 0.6931476400500263 | 0.6931471806637814 |
| 24 | 0.713 | 0.6931556628081347 | 0.6931467933892627 | 0.6931471806081372 |
| 25 | 0.674 | 0.6931396564055737 | 0.6931475090916949 | 0.6931471805823719 |
| 26 | 0.711 | 0.6931538856095920 | 0.6931468999771572 | 0.6931471805704077 |
| 27 | 0.676 | 0.6931411799365090 | 0.6931474216287011 | 0.6931471805648375 |
| 28 | 0.710 | 0.6931525720531642 | 0.6931469722894743 | 0.6931471805622379 |
| 29 | 0.677 | 0.6931423184823582 | 0.6931473614238747 | 0.6931471805610218 |
| 30 | 0.709 | 0.6931515803143486 | 0.6931470227401084 | 0.6931471805604518 |
| 31 | 0.678 | 0.6931431863345109 | 0.6931473188909237 | 0.6931471805601840 |
| 32 | 0.708 | 0.6931508175921422 | 0.6931470588008285 | 0.6931471805600580 |
| 33 | 0.679 | 0.6931438593762287 | 0.6931472881556996 | 0.6931471805599984 |
| 34 | 0.707 | 0.6931502214278631 | 0.6931470851270912 | 0.6931471805599706 |
| 35 | 0.679 | 0.6931443893338498 | 0.6931472655007312 | 0.6931471805599572 |
| 36 | 0.706 | 0.6931497487353118 | 0.6931471047083707 | 0.6931471805599510 |
| 37 | 0.680 | 0.6931448122920192 | 0.6931472485060110 | 0.6931471805599480 |
| 38 | 0.706 | 0.6931493691386813 | 0.6931471195159224 | 0.6931471805599465 |
| 39 | 0.681 | 0.6931451539445927 | 0.6931472355563709 | 0.6931471805599460 |
| 40 | 0.705 | 0.6931490608050397 | 0.6931471308805031 | 0.6931471805599455 |
| 41 | 0.681 | 0.6931454329240487 | 0.6931472255496243 | 0.6931471805599455 |
| 42 | 0.705 | 0.6931488077679622 | 0.6931471397195292 | 0.6931471805599452 |
| 43 | 0.682 | 0.6931456629655344 | 0.6931472177185481 | 0.6931471805599455 |
| 44 | 0.704 | 0.6931485981680046 | 0.6931471466774938 | 0.6931471805599453 |
| 45 | 0.682 | 0.6931458543450361 | 0.6931472115194400 | 0.6931471805599450 |
| 46 | 0.704 | 0.6931484230713075 | 0.6931471522149313 | 0.6931471805599452 |
| 47 | 0.683 | 0.6931460148544717 | 0.6931472065607034 | 0.6931471805599455 |
| 48 | 0.703 | 0.6931482756611275 | 0.6931471566660485 | 0.6931471805599452 |
| Ln(2) | | 0.6931471805599453 | 0.6931471805599453 | 0.6931471805599453 |

**2)** $\Delta_2^{(n)}$ and $\Delta_3^{(n)}$ applied to $S_n = 1/n$, $S = \lim_{n\to\infty} S_n = 0$. We have

$$\Delta_2^{(n)} = 1/(2(n+1))$$
$$\Delta_3^{(n)} = 1/\left[4(n+2)(1+\frac{3}{n^2-4})\right]$$

$|\Delta_3^{(n)} - S| < \frac{1}{2}|\Delta_2^{(n)} - S|$ but $\Delta_3^{(n)}$ doesn't accelerate $\Delta_2^{(n)}$.

**3)** $S_n = \lambda^n/(n+1)$ $\lambda \in [-1,1[$

Algebrically we can prove that $(\Delta_3^{(n)})_n$ converges faster than $(S_n)_n$ but not faster than $(\Delta_2^{(n)} = \delta_2(S_n))_n$

**Properties of $\Delta_k$**

**Property 1 :** *Let $\Delta_k$ the sequence transformation*

$$\Delta_k : S_n \to \Delta_k^{(n)} \quad n \in \mathbf{N}$$

*then $\Delta_k$ accelerates the set of linear converging sequences $(S_n)_n$ ($\Longleftrightarrow \lim_{n\to\infty} \frac{S_{n+1}-S}{S_n-S} = \lambda \in [-1,1[$.)*

**Proof.**

$$\frac{\Delta_k^{(n)} - S}{S_{n+k} - S} = \sum_{p=0}^{k} \binom{k}{p} \frac{S_{n+p} - S}{S_{n+k} - S} . \frac{(\Delta S_{n+k-2})^p (\Delta S_{n+k-1})^{k-p}}{(\Delta^2 S_{n+k-2})^k}$$

We have $\lim_{n\to\infty} \frac{S_{n+p}-S}{S_{n+k}-S} = \lambda^{p-k} = 0$ and

$\lim_{n\to\infty} \frac{\Delta_k^{(n)}-S}{S_{n+k}-S} = \frac{1}{(1-\lambda)^k} \sum_{p=0}^{k} \binom{k}{p} (-1)^{k-p} = 0.$

We can also prove the

**Property 2 :** *If $S_n = S + \alpha\lambda^n$ then $\Delta_k^{(n)} = S$. $\forall n = 1, \ldots, k = 0, 1, 2, \ldots$*

**IV - Interprétation of Lubkin' $W$ process.**

The $W$-process is the sequence transformation given by :

$$T(S_n) = \frac{S_{n+1}(\Delta S_{n+1} \Delta^2 S_{n-1}) - S_{n+2}(\Delta S_{n-1} \Delta^2 S_n)}{\Delta S_{n+1} \Delta^2 S_{n-1} - \Delta S_{n-1} . \Delta^2 S_n}$$

with notations of $I$, we have

$$T(S_n) = c(P_{n+2}(x))$$

where $P_{n+2}(x) = \frac{1-R_{n+3}(x)}{1-x}$ and $R_{n+3}(x) = \frac{x^{n+2}(\Delta S_{n+1}\Delta^2 S_{n-1}) - x^{n+3}(\Delta S_{n-1}\Delta^2 S_n)}{\Delta S_{n+1}\Delta^2 S_{n-1} - \Delta S_{n-1}.\Delta^2 S_n}$

Thus $P_{n+2}$ interpolates $x \to (1-x)^{-1}$ at 0 with multiplicity $(n+2)$ and at $\xi_n = \frac{\Delta S_{n+1}\Delta^2 S_{n-1}}{\Delta S_{n-1}\Delta^2 S_n}$ with multiplicity 1.

**Application to totally monotonic sequences.**

**Definition :** *A sequence $(c_n)_n$ is said to be totally monotonic ($(c_n) \in TM$) if $(-1)^k \Delta^k c_n \geq 0$ for $n, k = 0, 1, \ldots$ or equivalently there exists a nondecreasing function $\alpha$ such that $c_n = \int_0^1 x^n d\alpha(x)$ $n = 0, 1, \ldots$ If $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \frac{1}{\rho}$ then $c_n = \int_0^{1/\rho} x^n d\alpha(x)$ and $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=0}^{n} c_i = \int_0^{\frac{1}{\rho}} \frac{d\alpha(x)}{1-x}$.*

Let $R_{n+3}$ be arbitrary polynomial of degree $n+3$ and $P_{n+2}$ defined as in I.

$$S - T(S_n) = S - c(P_{n+2}(x)) = \int_0^{1/\rho} \frac{R_{n+3}(x)}{1-x} d\alpha(x)$$

$*$ $R_{n+3}(x) = x^{n+3}$ gives $T(S_n) = S_{n+2}$

$\left.\begin{array}{l} * \; R_{n+3}(x) = x^{n+2}\frac{(c_{n+2}-c_{n+1}x)}{(c_{n+2}-c_{n+1})} \\ * \; R_{n+3}(x) = x^{n+1}\frac{(c_{n+2}-c_{n+1}x)^2}{(c_{n+2}-c_{n+1})^2} \end{array}\right\}$ gives $T(S_n) = \Delta_2^{(n)} = \delta_2(S_n)$Aitken's process

$*$ $R_{n+3(x)} = x^{n+3-k}\frac{(c_{n+2}-c_{n+1}x)^k}{(c_{n+2}-c_{(n+1)})^k}$ gives $T(S_n) = \Delta_k^{(n)}$, $n \geq k-3$

$*$ $R_{n+3}(x) = x^{n+2}\left(\frac{\Delta S_{n+1}\Delta^2 S_{n-1} - x\Delta S_{n-1}\Delta^2 S_n}{\Delta S_{n+1}\Delta^2 S_{n-1} - \Delta S_{n-1}\Delta^2 S_n}\right)$ gives $T(S_n) =$ Lubkin's process.

We can also take $R_{n+3}$ as the best approximant of $1/(1-x)$ on $[0, 1/\rho]$ with respect to some norm (see [2]).

For Lubkin's process, if $(S_n)$ is a totally monotonic $(TM)$ sequence, then :

$$\lim_{n\to\infty} \frac{\Delta S_{n+1}}{\Delta S_n} = \frac{1}{\rho} \in ]0,1]$$
$$\text{and } \lim_{n\to\infty} \xi_n < \frac{1}{\rho} \text{ with } \xi_n < \frac{1}{\rho}$$

because $\frac{c_{n+2}}{c_{n+3}} < \frac{c_{n+1}}{c_{n+2}}$.*Remark* $\frac{c_{n+2}}{c_{n+1}} < \xi_n < \frac{1}{\rho}$. So $\xi_n$ is a better approximation of $\frac{1}{\rho}$ than $c_{n+2}/c_{n+1}$ (for $\Delta_2^{(n)}$). That fact explains that $W$ gives better results that $\Delta^2$ on $TM$ sequences.

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# Acceleration of Some Logarithmic Sequences of Moments.

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**Abstract**

In [5], the authors have proved that the set of logarithmic sequences (LOG) is not accelerable .Nevertheless,some subsets of LOG can be accelerated [6,7,11,12].Most of algorithms are based on the asymptotic expansion of the sequence to be accelerated.In this paper the point of view is quite different,since an analytic property of the whole sequence is used.If $(c_n)_{n\in\mathbb{N}}$ is a sequence of moments with an integral representation in the interval [0,1],then we will prove that the (Jacobi) modified moments accelerate $(c_n)_{n\in\mathbb{N}}$ (Section 2).In a particular case (Section 3) the modified sequence has an integral representation in [-1,0] and so can be also accelerated.

## 1 Introduction

Let $(c_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with limit $l$. $(c_n)_{n\in\mathbb{N}}$ is said to be logarithmic ($(c_n)_{n\in\mathbb{N}} \in$ LOG) if :

$$\lim_{n\to\infty} \frac{c_{n+1}-l}{c_n-l} = 1 \tag{1}$$

Moreover we assume that $(c_n)_{n\in\mathbb{N}}$ has an integral representation in [0,1]:

$$c_n = \int_0^1 x^n d\alpha(x) \tag{2}$$

where $\alpha$ is a function with bounded variation. Since $(c_n)_{n\in\mathbb{N}}$ is assumed to be logarithmic,1 is the upper bound of the support of $\alpha$ and it can be proved that the limit of $(c_n)_{n\in\mathbb{N}}$ is the jump of $\alpha$ at the point 1 ([1]p.119,[14]p.107):

$$l = \alpha(1) - \alpha(1^-)$$

If $(c_n)_{n\in\mathbb{N}}$ satisfies (2),then the number

$$c(P_n) := \int_0^1 P_n(x) d\alpha(x)$$

where $P_n$ is a polynomial of degree $n$,is called *modified moment* [3].
In [8] the modified moments,where $P_n$ is some suitable kernel polynomial,are used to compute the jump of $\alpha$ at some point $a \in [0,1]$.
In the particular case where $a = 1$, that is to approximate the limit of $(c_n)_{n\in\mathbb{N}}$ ,we proved the following result:([8])

**Theorem 1** *Let $P_n^{*(\alpha,\beta)}$ be the shifted Jacobi polynomial with $\beta > -1$ and $\alpha > -1$ then: for $\alpha > \frac{-1}{2}, \alpha > \beta$,*

$$\lim_{n\to\infty} \frac{c(P_n^{*(\alpha,\beta)})}{P_n^{*(\alpha,\beta)}(1)} = \lim_{n\to\infty} c_n = l$$

In this paper,we suppose that the distribution $d\alpha$ is the sum of an absolutely continuous distribution with respect to the Lebesgue measure and the Dirac distribution of magnitude $l$ at the point 1.

$$d\alpha(x) = w(x)dx + l d\delta_1 \tag{3}$$

So the sequence $(c_n)_{n \in \mathbb{N}}$ satisfies the following relation:

$$c_n = \int_0^1 x^n w(x)dx + l \tag{4}$$

In Section 2,we assume that there exist a real constant $\rho > -1$ such that :

$$w(x) \sim K(1-x)^\rho \quad \text{when } x \to 1, K \in \mathbb{R}. \tag{5}$$

We shall see that the sequence of modified moments accelerates the convergence of $(c_n)_{n \in \mathbb{N}}$ to $l$.
In Section 3 ,$w$ is assumed to have an integral representation.We shall prove that,in this case ,the sequence of modified moments accelerates $(c_n)_{n \in \mathbb{N}}$ and moreover can be accelerated in a sequence with a geometric convergence factor $(3-2\sqrt{2}) \sim 0.17...$

## 2 $w(x) \sim K(1-x)^\rho, x \to 1, \rho > -1, K \in \mathbb{R}$ .

In this case ,it can be proved that the sequence $e_n := c_n - l$ converges to 0 with order $\rho + 1$:

$$e_n \sim K n^{-\rho-1}/\Gamma(\rho+2)$$

(The converse is true if w is positive in [0,1] ,[14],chap.V) So,if the constant $\rho$ is not known,it can be deduced from the sequence $(e_n)$ or from $(\Delta c_n) := c_{n+1} - c_n$ since $\Delta c_n = \int_0^1 x^n(1-x)w(x)dx$.

We obtain the following result:

**Theorem 2** *If the sequence $(c_n)_{n \in \mathbb{N}}$ of real numbers satisfies:*

$$c_n = \int_0^1 x^n w(x)dx + l \tag{6}$$

*where $w(x) \sim K(1-x)^\rho, x \to 1^-$ and $w \in L^2[0,1]$ then the sequence:*

$$u_n = c(P_n^{*(\rho,0)})/P_n^{*(\rho,0)}(1)$$

*converges faster than $(c_n)_{n \in \mathbb{N}}$ to $l$ ,i.e.:*

$$\lim_{n\to\infty} \frac{u_n - l}{c_n - l} = 0$$

*Proof:* The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ satisfy ([13] p.58):

$$\int_{-1}^1 P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx = 0 \text{ if } n \neq m$$

The normalization of $P_n^{(\alpha,\beta)}$ is taken such that:

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$$

The shifted Jacobi polynomials are defined by:

$$P_n^{*(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1).$$

So ,the $P_n^{*(\alpha,\beta)}$ 's satisfy :

$$\int_{-1}^{1} P_n^{*(\alpha,\beta)}(x) P_m^{*(\alpha,\beta)}(x)(1-x)^\alpha x^\beta dx = 0 \text{ if } n \neq m$$

and $P_n^{*(\alpha,\beta)}(1) = \begin{pmatrix} n+\alpha \\ n \end{pmatrix}$, $P_n^{*(\alpha,\beta)}(0) = (-1)^n \begin{pmatrix} n+\beta \\ n \end{pmatrix}$.

Let us set

$$l_n^{*(\alpha,\beta)}(x) = P_n^{*(\alpha,\beta)}(x)/P_n^{*(\alpha,\beta)}(1) \tag{7}$$

and

$$u_n := c(l_n^{*(\rho,0)}).$$

Since $w \in L^2$ and $w(x) \sim K(1-x)^\rho$ when $x \to 1^-$ then $w \in L^2((1-x)^\rho)$ and the following expansion

$$w(x)/(1-x)^\rho = \sum_{n=0}^{\infty} \alpha_n P_n^{*(\rho,0)}(x) \tag{8}$$

holds.

The coefficients $\alpha_n$ of (8) are:

$$\alpha_n = \int_0^1 w(x) P_n^{*(\rho,0)}(x) dx / (h_n^{*(\rho,0)})$$

where

$$h_n^{*(\rho,0)} := \int_0^1 \left(P_n^{*(\rho,0)}(x)\right)^2 (1-x)^\rho dx \tag{9}$$

$$= \frac{2^{\rho+1}}{2n+\rho+1} \quad \text{([13]p.68)}. \tag{10}$$

$w(x)(1-x)^{-\rho} \in L^2((1-x)^\rho) \Longrightarrow \sum_{n=0}^{\infty} \alpha_n^2 < \infty$ and so

$$\lim_{n \to \infty} \alpha_n = 0$$

Therefore

$$\begin{aligned} u_n &= c(l_n^{*(\rho,0)}(x)) \\ &= \int_0^1 w(x) P_n^{*(\rho,0)}(x) dx / P_n^{*(\rho,0)}(1) + l \\ &= \frac{\alpha_n . 2^{\rho+1}}{(2n+\rho+1) . \begin{pmatrix} n+\rho \\ n \end{pmatrix}} + l \end{aligned}$$

.

The quotient: $\dfrac{u_n - l}{c_n - l} = \dfrac{\alpha_n 2^{\rho+1}}{(2n+\rho+1)\begin{pmatrix} n+\rho \\ n \end{pmatrix}} . O(n^{\rho+1}) = \alpha_n . O(1)$

Table 1: m-m-s for $c_{n+1} = 1 + \log(c_n)$

| $n$ | $c_n - l$ | $c(l_n^{*(0,0)}) - l$ | $n$ | $c_n - l$ | $c(l_n^{*(0,0)}) - l$ |
|---|---|---|---|---|---|
| 0 | 0.5 | 0.5 | 13 | 0.12 | -2.1 D-5 |
| 1 | 0 .40 | 0.31 | 14 | 0.11 | -1.5 D-5 |
| 2 | 0.34 | 1.1 D-1 | 15 | 0.11 | -1.2 D-5 |
| 3 | 0.30 | 1.3 D-2 | 16 | 0.10 | -0.9 D-6 |
| 4 | 0.26 | -3.8 D-3 | 17 | 0.09 | -7.3 D-6 |
| 5 | 0.23 | -1.6 D-3 | 18 | 0.09 | -5.8 D-6 |
| 6 | 0.20 | -4.8 D-4 | 19 | 0.09 | -4.7 D-6 |
| 7 | 0.19 | -2.6 D-4 | 20 | 0.08 | -3.8 D-6 |
| 8 | 0.17 | -1.5 D-4 | 21 | 0.08 | -3.1 D-6 |
| 9 | 0.16 | -0.9 D-4 | 22 | 0.08 | -2.6 D-6 |
| 10 | 0.15 | -6.0 D-5 | 23 | 0.07 | -2.2 D-6 |
| 11 | 0.14 | -4.1 D-5 | 24 | 0.07 | -1.9 D-6 |
| 12 | 0.13 | -2.9 D-5 | 25 | 0.07 | -1.6 D-6 |

since $\binom{n+\rho}{n} \sim n^\rho$ and so :

$$\lim_{n\to\infty} \frac{u_n - l}{c_n - l} = 0 \quad \text{since} \lim_{n\to\infty} \alpha_n = 0.$$

*Remark:* If $w$ does not belong to $L^2[0,1]$ but only $x^q w(x) \in L^2[0,1]$ for some $q \in \mathbb{N}$,then by the same way it can be proved that the sequence $u_n^{(q)} := c^{(q)}(l_n^{*(\rho,0)}(x)) := c(x^q l_n^{(\rho,0)}(x))$ accelerates the convergence of $(c_{n+q})_{n\in\mathbb{N}}$ and so accelerates $(c_n)_{n\in\mathbb{N}}$ because $(c_n)_{n\in\mathbb{N}}$ is a logarithmic sequence.

## Numerical examples

In the following numerical examples,the quantities $c(l_n^{*(\alpha,\beta)}(x))$ can be computed by mean of the expression of Jacobi polynomials ([13]p.68):

$$P_n^{*(\alpha,\beta)}(x) = \sum_{\nu=0}^{n} \binom{n+\alpha}{n-\nu} \binom{n+\beta}{\nu} (x-1)^\nu x^{n-\nu}$$

which leads to:

$$c(l_n^{*(\alpha,\beta)}(x)) = \sum_{\nu=0}^{n} \binom{n+\alpha}{n-\nu} \binom{n+\beta}{\nu} \Delta^\nu c_{n-\nu} / \binom{n+\alpha}{n}$$

where $\Delta^{k+1} c_i := \Delta^k c_{i+1} - \Delta^k c_i, \Delta^0 c_i := c_i$.

a) Let us consider the sequence :

$$c_{n+1} = 1 + \log(c_n), \quad c_0 = 1.5$$

which is a logarithmic sequence converging to $l = 1$.I don't know if $(c_n)_{n\in\mathbb{N}}$ is a moment sequence but it can be proved that $c_n = 1 + O(1/n)$([11]).So we can use the modified moment sequence (m-m-s) $c(l_n^{*(\rho,0)}(x))$ with $\rho = 0$.See Table 1.

Table 2: m-m-s for $c_n = \log(1+(n+1)^{1/2})$

| $n$ | $c_n$ | $c(l_n^{*(-1/2,0)})$ |
|---|---|---|
| 0 | 0.69 | 0.69 D-1 |
| 1 | 0.53 | 0.22 D-1 |
| 2 | 0.45 | 0.35 D-1 |
| 3 | 0.40 | 0.30 D-1 |
| 4 | 0.36 | 0.10 D-1 |
| 5 | 0.34 | 0.11 D-1 |
| 6 | 0.32 | 0.49 D-2 |
| 7 | 0.30 | 0.57 D-2 |
| 8 | 0.28 | 0.28 D-2 |
| 9 | 0.27 | 0.35 D-2 |
| 10 | 0.26 | 0.18 D-2 |
| 11 | 0.25 | 0.24 D-2 |
| 12 | 0.24 | 0.13 D-2 |

| $n$ | $c_n$ | $c(l_n^{*(-1/2,0)})$ |
|---|---|---|
| 13 | 0.23 | 0.17 D-2 |
| 14 | 0.22 | 0.10 D-2 |
| 15 | 0.22 | 0.13 D-2 |
| 16 | 0.21 | 0.74 D-3 |
| 17 | 0.21 | 0.10 D-2 |
| 18 | 0.20 | 0.59 D-3 |
| 19 | 0.20 | 0.81 D-3 |
| 20 | 0.19 | 0.48 D-3 |
| 21 | 0.19 | 0.67 D-3 |
| 22 | 0.19 | 0.39 D-3 |
| 23 | 0.19 | 0.56 D-3 |
| 24 | 0.18 | 0.33 D-3 |
| 25 | 0.18 | 0.47 D-3 |

b) The sequence :

$$c_n = \log(1+1/\sqrt{n+1})$$

is totally monotonic and logarithmic ([1],p 121).It converges to 0 as $1/\sqrt{n}$.So we have to compute the modified moment $c(l_n^{*(\rho,0)})$ with $\rho = -1/2$.See Table 2.

# 3 Case where $w$ is a Markov-Stieltjes function

In this section,we assume that the sequence $(c_n)_{n\in\mathbb{N}}$ satisfies (4) with a weight w(x) having an integral representation as following:

$$w(x) = \int_0^1 \frac{1}{1-(ax+b)t}d\mu(t) \tag{11}$$

where $\mu$ is a function with bounded variation ( $b \le 1$ and $a+b \le 1$).

If $a+b=1$ or $b=1$ then we assume that $\mu(1)-\mu(1^-)=0$ (this last assumption makes $w$ to be integrable on [0,1]).

We will show that the modified moment $c(P_n^*)$ where $P_n^*$ is the Legendre polynomial shifted on [0,1] has an integral representation on an interval depending on $a$ and $b$.

**Calculation of $c(P_n^*)$**

The Rodrigues formula for the shifted Legendre polynomial is:

$$P_n^*(x) = \frac{(-1)^n}{n!}\frac{d^n}{dx^n}(x^n(1-x)^n)$$

So $c(P_n^*) = \displaystyle\int_0^1 \frac{(-1)^n}{n!}\frac{d^n}{dx^n}(x^n(1-x)^n)w(x)dx + lP_n^*(1)$.

By $n$ integrations by parts,we obtain:

$$c(P_n^*) - lP_n^*(1) = \int_0^1\int_0^1 x^n(1-x)^n \frac{a^n t^n}{[1-(ax+b)t]^{n+1}}d\mu(t)dx.$$

After the change of variable: $t \longrightarrow v = \dfrac{x(1-x)t}{1-(ax+b)t}$
we obtain:

$$c(P_n^*) - lP_n^*(1) = \int_0^1 \int_0^1 a^n v^n \frac{1}{[1-(ax+b)t]} d\mu\left(\frac{v}{x(1-x)+v(ax+b)}\right) dx$$

and

$$c(P_n^*) - lP_n^*(1) = a^n \int_{v=0}^{v=1} v^n \int_{x=x_1(v)}^{x=x_2(v)} \frac{x(1-x)+v(ax+b)}{x(1-x)} d\mu\left(\frac{v}{x(1-x)+v(ax+b)}\right) dx$$

where $x_1(v)$ and $x_2(v)$ are the roots of the quadratic equation $\frac{x(1-x)}{1-ax-b} = v$ belonging to [0,1].
If $a = 1$ and $b = 0$ then $x_1(v) = v$ and $x_2(v) = 1$.
If $a = -1$ and $b = 1$ then $x_1(v) = 0$ and $x_2(v) = 1 - v$.
If we set :

$$l_n^*(x) = P_n^*(x)/P_n^*(1)$$

then the sequence $(c(l_n^*))_n$ satisfies:

$$c(l_n^*(x)) = l + a^n \int_0^1 v^n d\gamma(v) \tag{12}$$

where

$$d\gamma(v) = \int_{x=x_1(v)}^{x=x_2(v)} \frac{x(1-x)+v(ax+b)}{x(1-x)} d\mu\left(\frac{v}{x(1-x)+v(ax+b)}\right) dx. \tag{13}$$

**1)** If $a = 1$ and $b = 0$ then

$$d\gamma(v) = \int_v^1 \frac{1-x+v}{1-x} d\mu\left(\frac{v}{x(1-x+v)}\right) dx$$

In this case the sequence $c(l_n^*)$ is still logarithmic but the result of section 2 shows that the sequence $(c(l_n^*))_{n\in\mathbb{N}}$ accelerates the convergence of $(c_n)_{n\in\mathbb{N}}$ to $l$.

**2)** If $a = -1$ and $b = 1$ then :

$$d\gamma(v) = \int_0^{1-v} \frac{x+v}{x} d\mu\left(\frac{v}{(1-x)(x+v)}\right) dx$$

and

$$c(l_n^*) = l + (-1)^n \int_0^1 v^n d\gamma(v) = l + \int_{-1}^0 v^n d\overline{\gamma}(v). \tag{14}$$

Thus the sequence $(c(l_n^*))_{n\in\mathbb{N}}$ can be accelerated by a general summation process ([10]): If we set $u_n := c(l_n^*)$, the sequence $u(\overline{l_n^*}) = l + \int_{-1}^0 \overline{l_n^*}(x) d\overline{\gamma}(v)$ where $\overline{l_n^*}(x) = P_n(2x+1)/P_n(3)$ satisfies:

$$u(\overline{l_n^*}) = l + \int_{-1}^0 \frac{P_n(2x+1)}{P_n(3)} d\overline{\gamma}(x) = l + \frac{1}{2P_n(3)} \int_{-1}^1 P_n(x) d\overline{\gamma}\left(\frac{u-1}{2}\right)$$

and:

$$\limsup_{n\to\infty} \mid u(\overline{l_n^*}) - l \mid^{1/n} = \limsup_{n\to\infty} \mid P_n(3) \mid^{-1/n} = 3 - \sqrt{8} \quad ([13], p.194)$$

since $\mid P_n(x) \mid \leq 1 \ \ \forall x \in [-1,1]$.

*Remark 1:* If the function $\mu$ in (11) is non decreasing,then the function $\gamma$ defined by (13) is non decreasing too and thus the sequence $(c(l_n^*))_{n\in\mathbb{N}}$ of (12) is totally monotonic if $a \geq 0$ and totally oscilating around $l$ if $a \leq 0$.In the latter case ,the sequence $c(l_n^*)$ can be accelerated by the $\epsilon$-algorithm (with the same factor of convergence if $b = 1$ and $a = -1$ as before.For details see [9]).

*Remark 2:* The result of this section remains valid if $x^{-\beta}(1-x)^{-\alpha}w(x)$ satisfies (11).In that case,we have to compute the m-m-s $u_n = c(l_n^{*(\alpha,\beta)})$ and after the sequence $u(\overline{l_n^*})$.

### Numerical examples:

a)Let us consider the series:

$$c_n = \sum_{k=0}^{n-1} \frac{1}{(k+1)^2} \qquad c_0 = 0$$

The sequence $(c_n)_{n\in\mathbb{N}}$ converges to $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(k+1)^2} = \pi^2/6$ and the Riemann Formula gives:([14] p.232)

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)^2} = \int_0^1 \frac{-\log(x)}{1-x} dx.$$

So $c_n$ has the following integral representation:

$$\begin{aligned} c_n &= \int_0^1 x^n \frac{\log(x)}{1-x} dx + \int_0^1 \frac{-\log(x)}{1-x} dx \\ &= \int_0^1 x^n \frac{\log(x)}{1-x} dx + \frac{\pi^2}{6}\delta_1 \\ &= \frac{\pi^2}{6} + O(1/n) \end{aligned} \tag{15}$$

The weight $\dfrac{\log(x)}{1-x}$ has an integral representation :

$$\frac{\log(x)}{1-x} = \int_0^1 \frac{-1}{1-(1-x)t} dx.$$

So,it satisfies the conditions of (11) of section 3 and thus the modified moments $c(P_n^*)$ have an integral representation :

$$c(P_n^*) = \int_{-1}^{0} v^n \frac{-2\log \mid v \mid}{1-v} dv + \frac{\pi^2}{6}.$$

It can be proved,from the value of $\dfrac{-2\log \mid v \mid}{1-v}$ around $v = 1$ that

$$c(P_n^*) - \frac{\pi^2}{6} \sim \frac{(-1)^n}{n^2}.$$

Table 3: error sequence for example a)

| $n$ | $c_n - \pi^2/6$ | $c(P_n^*) - \pi^2/6$ | $u(\overline{l_n^*}) - \pi^2/6$ |
|---|---|---|---|
| 0 | -1.64 | -1.64 D 0 | -1.64 D 00 |
| 1 | -0.64 | 0.35 D 0 | -0.31 D 00 |
| 2 | -0.39 | -0.14 D 0 | -0.29 D-01 |
| 3 | -0.28 | 0.77 D-1 | -0.29 D-02 |
| 4 | 0.22 | -0.47 D-1 | -0.33 D-03 |
| 5 | -0.18 | 0.32 D-1 | -0.41 D-04 |
| 6 | -0.15 | -0.23 D-1 | -0.54 D-05 |
| 7 | -0.13 | 0.17 D-1 | -0.74 D-06 |
| 8 | -0.11 | -0.13 D-1 | -0.10 D-06 |
| 9 | -0.10 | 0.10 D-1 | -0.15 D-07 |
| 10 | -0.10 | -0.90 D-2 | -0.22 D-08 |
| 11 | -0.08 | 0.75 D-2 | -0.34 D-09 |
| 12 | -0.08 | -0.63 D-2 | -0.51 D-10 |
| 13 | -0.07 | 0.54 D-2 | -0.78 D-11 |
| 14 | -0.07 | -0.47 D-2 | -0.12 D-11 |
| 15 | -0.06 | 0.41 D-2 | -0.18 D-12 |
| 16 | -0.06 | -0.36 D-2 | -0.29 D-13 |
| 17 | -0.06 | 0.32 D-2 | -0.46 D-14 |
| 18 | -0.05 | -0.29 D-2 | -0.72 D-15 |
| 19 | -0.05 | 0.26 D-2 | -0.11 D-15 |
| 20 | -0.05 | -0.23 D-2 | -0.18 D-16 |
| 21 | -0.04 | 0.21 D-2 | -0.29 D-17 |
| 22 | -0.04 | -0.19 D-2 | -0.47 D-18 |
| 23 | -0.04 | 0.18 D-2 | -0.75 D-19 |
| 24 | -0.04 | -0.16 D-2 | -0.12 D-19 |
| 25 | -0.04 | 0.15 D-2 | -0.19 D-20 |

So the sequence $c(P_n^*)$ converges faster than $(c_n)_{n\in\mathbb{N}}$ . Since $\dfrac{-2\log\mid v\mid}{1-v} \geq 0, \quad \forall x \in [-1,0]$,the sequence $c(P_n^*)$ is totally oscilating around its limit $\pi^2/6$ and so can be accelerated,for example by the $\epsilon$-algorithm or by the following modified moment sequence:([10])

$$u(\overline{l_n^*}) := u\left(\frac{P_n(2x+1)}{P_n(3)}\right) \qquad \text{where } u_n = c(P_n^*).$$

The error sequence $e_n = u(\overline{l_n^*}) - \pi^2/6$ satisfies (See Table 3)

$$\limsup_{n\to\infty} \mid e_n \mid^{1/n} = 3 - \sqrt{8}$$

b)The sequence

$$c_n = \sum_{k=0}^{n-1} \frac{1}{\sqrt{k+1}(k+2)} \qquad c_0 = 0$$

has the following integral representation:[4]

$$c_n = \int_0^1 x^n w(x)dx + l$$

where l is the limit of the series $c_n$ and

$$w(x) = \frac{2x}{(1-x)\sqrt{\pi}} \int_0^{\sqrt{-\log x}} e^{t^2} dt.$$

From the definition of the series $c_n$,the relation

$$c_n - l = O\left(\frac{1}{\sqrt{n}}\right)$$

holds. So,we have to compute the m-m-s $u_n := c(l_n^{*(\rho,\beta)})$ with $\rho = -1/2$. I don't know if $(1-x)^{1/2}x^{-\beta}w(x)$ satisfies (11) with $\beta = 1/2$ but with exact arithmetics,it can be remarked that $(u_n)_n$ is a totally oscilating sequence around its limit(computed and accelerated by mean of $\varepsilon$-algorithm with exact arithmetics)

$$l = 1.86002507922119030718069591571733669 3...$$

So,we can apply the $\epsilon$-algorithm to $u_n := c(l_n^{*(-1/2,1/2)})$.(See Table 4)

Table 4: sequence of example b)

| $n$ | $c_n - l$ | $u_n - l$ | $\epsilon_n^{(0)}(u_n) - l$ | $n$ | $c_n - l$ | $u_n - l$ | $\epsilon_n^{(0)}(u_n) - l$ |
|---|---|---|---|---|---|---|---|
| 0 | -1.86 | -1.86 D 0 | -1.86 | 15 | -0.49 | 0.13 D-1 | |
| 1 | -1.36 | 0.13 D 0 | | 16 | -0.48 | -0.12 D-1 | -0.29 D-12 |
| 2 | -1.12 | -0.88 D-1 | -0.65 D-01 | 17 | -0.47 | 0.11 D-1 | |
| 3 | -0.97 | 0.63 D-1 | | 18 | -0.45 | -0.10 D-1 | -0.84 D-14 |
| 4 | -0.88 | -0.49 D-1 | -0.11 D-02 | 19 | -0.44 | 0.10 D-1 | |
| 5 | -0.80 | 0.4 D-1 | | 20 | -0.43 | -0.95 D-2 | -0.24 D-15 |
| 6 | -0.74 | -0.33 D-1 | -0.23 D-04 | 21 | -0.42 | 0.90 D-2 | |
| 7 | -0.70 | 0.28 D-1 | | 22 | -0.41 | -0.86 D-2 | -0.68 D-17 |
| 8 | -0.66 | -0.25 D-1 | -0.55 D-06 | 23 | -0.40 | -0.82 D-2 | |
| 9 | -0.62 | 0.22 D-1 | | 24 | -0.39 | -0.78 D-2 | -0.19 D-18 |
| 10 | -0.59 | -0.19 D-1 | -0.14 D-07 | 25 | -0.39 | 0.75 D-2 | |
| 11 | -0.57 | 0.18 D-1 | | 26 | -0.38 | -0.72 D-2 | -0.57 D-20 |
| 12 | -0.55 | -0.16 D-1 | -0.38 D-09 | 27 | -0.37 | 0.69 D-2 | |
| 13 | -0.53 | 0.15 D-1 | | 28 | -0.37 | -0.66 D-2 | -0.16 D-21 |
| 14 | -0.51 | -0.14 D-1 | -0.10 D-10 | 29 | -0.36 | 0.64 D-2 | |

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# Acceleration property of the E-algorithm

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May 14, 1991

**Abstract**

A convergence result for the E-algorithm is proved for sequences such that the error has an asymptotic expansion on a scale of comparison for which a determinantal relation holds.

# 1 E-algorithm (see [1])

## 1.1 Introduction

Let $(S_n)_{n\in N}$ be a sequence of complex numbers and suppose that $(S_n)_{n\in N}$ is extrapolated as following:

$$S_n = S + a_1 g_1(n) + a_2 g_2(n) + \cdots + a_k g_k(n) \tag{1}$$

where the sequences $g_i$ are given .

The problem of extrapolation is to compute the value of $S$ (which is the limit of $(S_n)$ if it converges or its antilimit if it diverges).To compute the value of $S$ let us write the relation (1) for the index $n$ to $n+k$

$$S_{n+i} = S + a_1 g_1(n+i) + a_2 g_2(n+i) + \cdots + a_k g_k(n+i) \quad i = 0, \cdots, k \tag{2}$$

If the above system (2) is nonsingular then the solution $S$ can be computed by the relation

$$S = \frac{\begin{vmatrix} S_n & g_1(n) & \cdots & g_k(n) \\ S_{n+1} & g_1(n+1) & \cdots & g_k(n+1) \\ \vdots & \vdots & \cdots & \vdots \\ S_{n+k} & g_1(n+k) & \cdots & g_k(n+k) \end{vmatrix}}{\begin{vmatrix} 1 & g_1(n) & \cdots & g_k(n) \\ 1 & g_1(n+1) & \cdots & g_k(n+1) \\ \vdots & \vdots & \cdots & \vdots \\ 1 & g_1(n+k) & \cdots & g_k(n+k) \end{vmatrix}} \tag{3}$$

which can be written as:

$$S = \frac{\begin{vmatrix} S_n & g_1(n) & \cdots & g_k(n) \end{vmatrix}}{\begin{vmatrix} 1 & g_1(n) & \cdots & g_k(n) \end{vmatrix}}$$

if the determinants are only denoted by their first row. If the sequence $(S_n)$ has not the exact form (1) then the value of $S$ given by (3) depends on the indexes $n$ and $k$.It is denoted by $E_k(S_n)$:

$$E_k(S_n) := \frac{\begin{vmatrix} S_n & g_1(n) & \cdots & g_k(n) \end{vmatrix}}{\begin{vmatrix} 1 & g_1(n) & \cdots & g_k(n) \end{vmatrix}} \tag{4}$$

The E-algorithm is a recursive algorithm which permits to compute the numbers $E_k(S_n)$ for all $k$ and $n$ without computing the determinants occurring in (4).

It is as following:

$$\begin{aligned} E_0^{(n)} &= S_n \quad n = 0, 1, \cdots \\ g_{0,i}^{(n)} &= g_i(n) \quad i = 1, 2, \cdots, \text{and} \qquad n = 0, 1, \cdots \end{aligned}$$

for $k = 1, 2, \cdots$ , $n = 0, 1, \cdots$ and $i = k+1, k+2, \cdots$:

$$\begin{aligned} E_k^{(n)} &= \frac{E_{k-1}^{(n)} g_{k-1,k}^{(n+1)} - E_{k-1}^{(n+1)} g_{k-1,k}^{(n)}}{g_{k-1,k}^{(n+1)} - g_{k-1,k}^{(n)}} \\ g_{k,i}^{(n)} &= \frac{g_{k-1,i}^{(n)} g_{k-1,k}^{(n+1)} - g_{k-1,i}^{(n+1)} g_{k-1,k}^{(n)}}{g_{k-1,k}^{(n+1)} - g_{k-1,k}^{(n)}} \end{aligned}$$

BREZINSKI proved that:

**Theorem 1** *[1]*

$$E_k^{(n)} = E_k(S_n) \qquad k, n = 0, 1, \cdots$$

The quantities $E_k^{(n)}$ are placed in a double entry array:

$$\begin{array}{lllll}
E_0(S_0) = S_0 & & & & \\
E_0(S_1) = S_1 & E_1(S_0) & & & \\
E_0(S_2) = S_2 & E_1(S_1) & E_2(S_0) & & \\
E_0(S_3) = S_3 & E_1(S_2) & E_2(S_1) & E_3(S_0) & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

## 1.2 Convergence result

If the sequences $g_i$ are linear and form a scale of comparison then BREZINSKI showed that each row of the $E$-array converges faster than the preceding one:

**Theorem 2** *[1] Suppose that* $S_n = S + a_1 g_1(n) + a_2 g_2(n) + \cdots$.
*If*

$$\lim_{n\to\infty} \frac{g_i(n+1)}{g_i(n)} = b_i \neq 1, \forall i = 1, 2, \cdots \text{with } b_i \neq b_j \ i \neq j$$

*then*

$$\lim_{n\to\infty} E_k^{(n)} = S \quad k = 0, 1, \cdots$$

*Moreover if*

$$g_{i+1}(n) = o_{n\to\infty}(g_i(n)) \ \forall i = 1, 2, \cdots$$

*then*

$$E_k(S_n) - S = o_{n\to\infty}(E_{k-1}(S_n) - S) \ \ \forall k = 0, 1, \cdots$$

# 2 Acceleration property of the $E$-algorithm

The above theorem 2 cannot be applied if two of the $b_i's$ are equal or if one of the $b_i's$ equals 1,by example for sequences that have an asymptotic expansion on the following scale of functions:$1/n, 1/n^2, 1/n^3, \cdots$ because in this case all the $b_i's$ equal 1. In this section ,we have obtained the same acceleration property by replacing the condition of theorem 2 by a determinantal property on the $g_i's$.

Let us first prove a result which can be useful in a general situation:

**Theorem 3** *Let* $\{(g_i(n))_n, i = 0, 1, \cdots\}$ *be a set of sequences satisfying*
*(i)* $\lim_{n\to\infty} \dfrac{g_{i+1}(n)}{g_i(n)} = 0 \ i = 0, 1, \cdots$
*(ii)* $\forall p \in \mathbb{N}, \forall i \in \mathbb{N}, \exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow$

$$\begin{vmatrix} g_{i+p}(n) & \cdots & g_i(n) \\ \vdots & \vdots & \vdots \\ g_{i+p}(n+p) & \cdots & g_i(n+p) \end{vmatrix} \geq 0$$

*then*

$$\forall k \in \mathbb{N}, \forall l \geq k+1, \forall i \in \mathbb{N} \lim_{n\to\infty} \frac{\left| g_{i+l+1}(n) \quad g_{i+k}(n) \quad \cdots \quad g_i(n) \right|}{\left| g_{i+l}(n) \quad g_{i+k}(n) \quad \cdots \quad g_i(n) \right|} = 0$$

For the proof, we need the following lemma:

**Lemma 1** *[2] Let $(u_n)$ and $(v_n)$ be two convergent sequences of real or complex numbers. If the sequence $(v_n)$is monotonic for $n > N$ then:*

$$\lim_{n\to\infty} \frac{\Delta u_n}{\Delta v_n} = \rho \in \mathbb{R} \quad \Rightarrow \lim_{n\to\infty} \frac{u_n - u}{v_n - v} \text{ exists and } \lim_{n\to\infty} \frac{u_n - u}{v_n - v} = \rho$$

*where $u = \lim_{n\to\infty} u_n, v = \lim_{n\to\infty} v_n$.*

*Proof of theorem* 3

From a theorem of KARLIN (see [3],p.58),the property (ii) implies that all minors in (ii) are positive.

So the following determinant :

$$\left| g_{i+l+1}(n) \quad g_{i+l}(n) \quad g_{i+k}(n) \quad \cdots \quad g_i(n) \right| = (-1)^{k+1} \left| g_{i+l+1}(n) \quad g_{i+k}(n) \quad \cdots \quad g_i(n) \quad g_{i+l}(n) \right|$$

is positive.

Sylvester's identity applied to this determinant gives:

$$(-1)^{k+1} \left| g_{i+l+1}(n) \quad g_{i+k}(n) \quad \cdots \quad g_i(n) \right| . \left| g_{i+k}(n+1) \quad \cdots \quad g_i(n+1) \quad g_{i+l}(n+1) \right| \geq$$

$$(-1)^{k+1} \left| g_{i+l+1}(n+1) \quad g_{i+k}(n+1) \quad \cdots \quad g_i(n+1) \right| . \left| g_{i+k}(n) \quad \cdots \quad g_i(n) \quad g_{i+l}(n) \right|$$

since the determinant $\left| g_{i+k}(n+1) \quad \cdots \quad g_i(n+1) \right|$ is positive (by (ii)).

By moving the row $g_{i+l}$ or $g_{i+l+1}$ and after dividing each member of the above inequality by the product :

$$\left| g_{i+l}(n) \quad g_{i+k}(n) \quad \cdots \quad g_i(n) \right| . \left| g_{i+l}(n+1) \quad g_{i+k}(n+1) \quad \cdots \quad g_i(n+1) \right|$$

which is positive (because of (ii))

we obtain :

$$\frac{\left| g_{i+l+1}(n) \quad g_{i+k}(n) \quad \cdots \quad g_i(n) \right|}{\left| g_{i+l}(n) \quad g_{i+k}(n) \quad \cdots \quad g_i(n) \right|} \geq \frac{\left| g_{i+l+1}(n+1) \quad g_{i+k}(n+1) \quad \cdots \quad g_i(n+1) \right|}{\left| g_{i+l}(n+1) \quad g_{i+k}(n+1) \quad \cdots \quad g_i(n+1) \right|} \geq 0$$

and thus the sequence :

$$Q_n := \frac{\left| g_{i+l+1}(n) \quad g_{i+k}(n) \quad \cdots \quad g_i(n) \right|}{\left| g_{i+l}(n) \quad g_{i+k}(n) \quad \cdots \quad g_i(n) \right|}$$

which is a positive and decreasing sequence (for $n > N$) converges to some limit $\sigma \geq 0$.

Let us show now that the limit $\sigma$ equals 0. For that,let us write $Q_n$ as the quotient of the differences of two sequences as in lemma 1.It is possible by applying Sylvester's identity to the numerator and denominator of $Q_n$ after having moved the row $g_{i+k}$

$$Q_n := \frac{\begin{vmatrix} g_{i+l+1}(n) & g_{i+k-1}(n) & \cdots & g_i(n) & g_{i+k}(n) \end{vmatrix}}{\begin{vmatrix} g_{i+l}(n) & g_{i+k-1}(n) & \cdots & g_i(n) & g_{i+k}(n) \end{vmatrix}}$$

$$= \frac{\begin{aligned} &\begin{vmatrix} g_{i+l+1}(n) & g_{i+k-1}(n) & \cdots & g_i(n) \end{vmatrix} \times \begin{vmatrix} g_{i+k-1}(n+1) & \cdots & g_i(n+1) & g_{i+k}(n+1) \end{vmatrix} \\ -&\begin{vmatrix} g_{i+l+1}(n+1) & g_{i+k-1}(n+1) & \cdots & g_i(n+1) \end{vmatrix} \times \begin{vmatrix} g_{i+k-1}(n) & \cdots & g_i(n) & g_{i+k}(n) \end{vmatrix} \end{aligned}}{\begin{aligned} &\begin{vmatrix} g_{i+l}(n) & g_{i+k-1}(n) & \cdots & g_i(n) \end{vmatrix} \times \begin{vmatrix} g_{i+k-1}(n+1) & \cdots & g_i(n+1) & g_{i+k}(n+1) \end{vmatrix} \\ -&\begin{vmatrix} g_{i+l}(n+1) & g_{i+k-1}(n+1) & \cdots & g_i(n+1) \end{vmatrix} \times \begin{vmatrix} g_{i+k-1}(n) & \cdots & g_i(n) & g_{i+k}(n) \end{vmatrix} \end{aligned}}$$

and so the quotient

$$Q_n = \frac{\Delta\left(\frac{\begin{vmatrix} g_{i+l+1}(n) & g_{i+k-1}(n) & \cdots & g_i(n) \end{vmatrix}}{\begin{vmatrix} g_{i+k}(n) & g_{i+k-1}(n) \cdots & g_i(n) \end{vmatrix}}\right)}{\Delta\left(\frac{\begin{vmatrix} g_{i+l}(n) & g_{i+k-1}(n) & \cdots & g_i(n) \end{vmatrix}}{\begin{vmatrix} g_{i+k}(n) & g_{i+k-1}(n) \cdots & g_i(n) \end{vmatrix}}\right)} \tag{5}$$

where $\Delta$ acts on the index $n$,converges to $\sigma$ when $n$ tends to infinity.

Let us first prove that $\sigma = 0$ for $k = 0$ and $l \geq 1$. If $k = 0$ the quotient $Q_n$ reduces to :

$$Q_n := \frac{\begin{vmatrix} g_{i+l+1}(n) & g_i(n) \end{vmatrix}}{\begin{vmatrix} g_{i+l}(n) & g_i(n) \end{vmatrix}} = \frac{\Delta\left(\frac{g_{i+l+1}(n)}{g_i(n)}\right)}{\Delta\left(\frac{g_{i+l}(n)}{g_i(n)}\right)}$$

converges to $\sigma$ when $n$ tends to infinity,for $k = 0$. Since $\lim\limits_{n\to\infty} \dfrac{g_{i+l+1}(n)}{g_i(n)} = 0$ and $\lim_{n\to\infty} \frac{g_{i+l}(n)}{g_i(n)} = 0 \ \forall l \geq 1$, lemma 1 and condition (i) imply that

$$\sigma = \lim_{n\to\infty} \frac{g_{i+l+1}(n)/g_i(n)}{g_{i+l}(n)/g_i(n)} = \lim_{n\to\infty} \frac{g_{i+l+1}(n)}{g_{i+l}(n)} = 0.$$

The complete proof for all $k$ follows by induction by using the relation (5).

□

We can now prove an acceleration property for the $E-$ algorithm applied to a sequence $S_n$ which has an asymptotic expansion on some scale of comparison.

**Theorem 4** *Let $(S_n)_n$ be a sequence converging to $S$ such that:*

$$S_n = S + a_1 g_1(n) + a_2 g_2(n) + \cdots$$

*If*

*(i)* $\lim_{n\to\infty} \frac{g_{i+1}(n)}{g_i(n)} = 0$ *for* $i = 0, 1, 2, \cdots$, *where* $g_0(n) := 1$

*(ii)* $\forall p \in \mathbb{N}, \forall i \in \mathbb{N}, \exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow$

$$\begin{vmatrix} g_{i+p}(n) & \cdots & g_i(n) \\ \vdots & \vdots & \vdots \\ g_{i+p}(n+p) & \cdots & g_i(n+p) \end{vmatrix} \geq 0$$

*then*

$$\lim_{n\to\infty} \frac{E_{k+1}^{(n)} - S}{E_k^{(n)} - S} = 0 \quad k = 0, 1, \cdots.$$

*Proof:*

With the expansion of $E_k^{(n)}$ given by 4,

$$\frac{E_{k+1}^{(n)} - S}{E_k^{(n)} - S} = \frac{\begin{vmatrix} S_n - S & g_1(n) & \cdots & g_{k+1}(n) \end{vmatrix}}{\begin{vmatrix} 1 & g_1(n) & \cdots & g_{k+1}(n) \end{vmatrix}} \times \frac{\begin{vmatrix} 1 & g_1(n) & \cdots & g_k(n) \end{vmatrix}}{\begin{vmatrix} S_n - S & g_1(n) & \cdots & g_k(n) \end{vmatrix}} =$$

$$\frac{\begin{vmatrix} 1 & g_1(n) & \cdots & g_k(n) \end{vmatrix}}{\begin{vmatrix} 1 & g_1(n) & \cdots & g_{k+1}(n) \end{vmatrix}} \cdot \frac{a_{k+2} \begin{vmatrix} g_{k+2}(n) & g_1(n) & \cdots & g_{k+1}(n) \end{vmatrix} + a_{k+3} \begin{vmatrix} g_{k+3}(n) & g_1(n) & \cdots & g_{k+1}(n) \end{vmatrix} + \cdots}{a_{k+1} \begin{vmatrix} g_{k+1}(n) & g_1(n) & \cdots & g_k(n) \end{vmatrix} + a_{k+2} \begin{vmatrix} g_{k+2}(n) & g_1(n) & \cdots & g_k(n) \end{vmatrix} + \cdots}$$

With theorem 3 , the second quotient is equivalent to the quotient of its first terms and :

$$\lim_{n\to\infty} \frac{E_{k+1}^{(n)} - S}{E_k^{(n)} - S} = \frac{a_{k+2}}{a_{k+1}} \lim_{n\to\infty} \frac{\begin{vmatrix} 1 & g_1(n) & \cdots & g_k(n) \end{vmatrix}}{\begin{vmatrix} 1 & g_1(n) & \cdots & g_{k+1}(n) \end{vmatrix}} \times \frac{\begin{vmatrix} g_{k+2}(n) & g_1(n) & \cdots & g_{k+1}(n) \end{vmatrix}}{\begin{vmatrix} g_{k+1}(n) & g_1(n) & \cdots & g_k(n) \end{vmatrix}}$$

By substracting each row from the preceding one:

$$\lim_{n\to\infty} \frac{E_{k+1}^{(n)} - S}{E_k^{(n)} - S} =$$

$$\frac{a_{k+2}}{a_{k+1}} \lim_{n\to\infty} \frac{\begin{vmatrix} \Delta g_1(n) & \cdots & \Delta g_k(n) \end{vmatrix}}{\begin{vmatrix} \Delta g_1(n) & \cdots & \Delta g_{k+1}(n) \end{vmatrix}} \cdot \frac{\begin{vmatrix} g_{k+2}(n) & g_1(n) & \cdots & g_{k+1}(n) \\ \Delta g_{k+2}(n) & \Delta g_1(n) & \cdots & \Delta g_{k+1}(n) \\ \Delta g_{k+2}(n+1) & \Delta g_1(n+1) & \cdots & \Delta g_{k+1}(n+1) \\ \vdots & \vdots & \cdots & \vdots \\ \Delta g_{k+2}(n+k) & \Delta g_1(n+k) & \cdots & \Delta g_{k+1}(n+k) \end{vmatrix}}{\begin{vmatrix} g_{k+1}(n) & g_1(n) & \cdots & g_k(n) \\ \Delta g_{k+1}(n) & \Delta g_1(n) & \cdots & \Delta g_k(n) \\ \Delta g_{k+1}(n+1) & \Delta g_1(n+1) & \cdots & \Delta g_k(n+1) \\ \vdots & \vdots & \cdots & \vdots \\ \Delta g_{k+1}(n+k-1) & \Delta g_1(n+k-1) & \cdots & \Delta g_k(n+k-1) \end{vmatrix}} \quad (6)$$

With Sylvester's identity applied to its numerator,(6) becomes:

$$\lim_{n\to\infty} \frac{E_{k+1}^{(n)} - S}{E_k^{(n)} - S} = \frac{a_{k+2}}{a_{k+1}} \times$$

$$\lim_{n\to\infty} \frac{\begin{vmatrix} g_1(n) & \cdots & g_{k+1}(n) \end{vmatrix} \times \begin{vmatrix} \Delta g_{k+2}(n) & \Delta g_1(n) & \cdots & \Delta g_k(n) \end{vmatrix} - \begin{vmatrix} g_{k+2}(n) & g_1(n) & \cdots & g_k(n) \end{vmatrix} \times \begin{vmatrix} \Delta g_1(n) & \cdots & \Delta g_{k+1}(n) \end{vmatrix}}{(-1)^{k+1} \begin{vmatrix} g_1(n) & \cdots & g_{k+1}(n) \end{vmatrix} \times \begin{vmatrix} \Delta g_1(n) & \cdots & \Delta g_{k+1}(n) \end{vmatrix}} =$$

$$\frac{a_{k+2}}{a_{k+1}} \lim_{n\to\infty} \left( \frac{\begin{vmatrix} \Delta g_1(n) & \cdots & \Delta g_k(n) & \Delta g_{k+2}(n) \end{vmatrix}}{\begin{vmatrix} \Delta g_1(n) & \cdots & \Delta g_k(n) & \Delta g_{k+1}(n) \end{vmatrix}} - \frac{\begin{vmatrix} g_1(n) & \cdots & g_k(n) & g_{k+2}(n) \end{vmatrix}}{\begin{vmatrix} g_1(n) & \cdots & g_k(n) & g_{k+1}(n) \end{vmatrix}} \right).$$

But $\lim_{n\to\infty} \frac{\begin{vmatrix} \Delta g_1(n) & \cdots & \Delta g_k(n) & \Delta g_{k+2}(n) \end{vmatrix}}{\begin{vmatrix} \Delta g_1(n) & \cdots & \Delta g_k(n) & \Delta g_{k+1}(n) \end{vmatrix}} = \lim_{n\to\infty} \frac{\begin{vmatrix} 1 & g_1(n) & \cdots & g_k(n) & g_{k+2}(n) \end{vmatrix}}{\begin{vmatrix} 1 & g_1(n) & \cdots & g_k(n) & g_{k+1}(n) \end{vmatrix}} =$ 0 by theorem 3

and

$$\lim_{n\to\infty} \frac{\begin{vmatrix} g_1(n) & \cdots & g_k(n) & g_{k+2}(n) \end{vmatrix}}{\begin{vmatrix} g_1(n) & \cdots & g_k(n) & g_{k+1}(n) \end{vmatrix}} = 0 \text{ by theorem 3}$$

Thus

$$\lim_{n\to\infty} \frac{E_{k+1}^{(n)} - S}{E_k^{(n)} - S} = 0, \quad k = 0, 1, \cdots.$$

□

**Remark**

In the E-algorithm the sequence $g_{k,i}(n)$ can be expressed by the quotient of two determinants as following (see ([1])):

$$g_{k,i}(n) = \frac{\begin{vmatrix} g_i(n) & g_1(n) & \cdots & g_k(n) \end{vmatrix}}{\begin{vmatrix} 1 & g_1(n) & \cdots & g_k(n) \end{vmatrix}}$$

which can be rewritten as:

$$g_{k,i}(n) = (-1)^k \frac{\begin{vmatrix} g_i(n) & g_k(n) & \cdots & g_1(n) \end{vmatrix}}{\begin{vmatrix} g_k(n) & \cdots & g_1(n) & 1 \end{vmatrix}}$$

Thus the condition (ii) of the theorem 4 implies that

$$(-1)^k g_{k,i}(n) \geq 0 \ \forall i \in \mathbf{N}, \ \forall \mathrm{k} \in \mathbf{N} \ \forall \mathrm{n} \geq \mathrm{N}$$

and this last condition is easily checked during the computation of the E-algorithm.

# 3 Applications

Let us now give some examples for which the sequences $g_i(n)$ satisfy (ii).

**Example 1**
All the $g_i$'s,$i \geq 1$ are linear and are such that $g_i(n)$ positive sequence and

$$\lim_{n\to\infty} \frac{g_i(n+1)}{g_i(n)} = b_i, \ , b_i < b_j \text{ for } i > j$$

In fact, the determinant of (ii) is asymptotically the following Vandermonde determinant:

$$\begin{vmatrix} 1 & \cdots & 1 \\ b_{i+p} & \cdots & b_i \\ b_{i+p}^2 & \cdots & b_i^2 \\ \cdots & \cdots & \cdots \\ b_{i+p}^p & \cdots & b_i^p \end{vmatrix}$$

which is positive.
This first example is a consequence of the theorem 2 of BREZINSKI.
**Example 2**
The first sequence $g_1(n) := c_n$ is a Stieltjes moment sequence ($c_n := \int_0^1 x^n d\alpha(x), support(\alpha) = [0,1]$) and $g_i(n) = (-1)^i\Delta^i c_n$. In this case the set $\{1, c_n, -\Delta c_n, \Delta^2 c_n, -\Delta^3 c_n, \cdots\}$ satisfies (ii) because :

$$\begin{vmatrix} (-1)^l\Delta^l c_n & \cdots & \Delta c_n & c_n & 1 \end{vmatrix} = \begin{vmatrix} 1 & c_n & \Delta c_n & \cdots & \Delta^l c_n \end{vmatrix} =$$
$$\begin{vmatrix} c_{n+1} & \Delta c_{n+1} & \Delta^2 c_{n+1} & \cdots & \Delta^l c_{n+1} \end{vmatrix} = \begin{vmatrix} c_{n+1} & c_{n+2} & c_{n+3} & \cdots & c_{n+l+1} \end{vmatrix} \geq 0.$$

If $(c_n)$ is a logarithmic convergent sequence then the set $\{1, c_n, -\Delta c_n, \Delta^2 c_n, -\Delta^3 c_n, \cdots\}$ satisfies also (i) of theorem 4 because the sequence $((-1)^i\Delta^i c_{n+1})_n$ also has an integral representation on the same support [0,1] than the sequence $(c_n)_n$ and so is logarithmic.

**Example 3**
Let us take

$$g_i(n) := K(\alpha_n, \beta_i)$$

where $\alpha_n$ and $\beta_i$ are decreasing sequences of real numbers:

$$\alpha_1 < \alpha_2 < \alpha_3 < \cdots, \beta_1 > \beta_2 > \beta_3 > \cdots$$

and where $K$ is a totally positive function defined as following (see [3]chap.1).
$K$ satisfies :

$\forall r \in \mathbb{N}$ for all $x_1 < x_2 < x_3 < \cdots < x_m, y_1 < y_2 < y_3 < \cdots < y_m, x_i, y_i \in \mathbb{R}$ ; $1 \leq m \leq r$

$$K\begin{pmatrix} x_1 & x_2 & \cdots & x_m \\ y_1 & y_2 & \cdots & y_m \end{pmatrix} = \begin{vmatrix} K(x_1,y_1) & K(x_1,y_2) & \cdots & K(x_1,y_m) \\ K(x_2,y_1) & K(x_2,y_2) & \cdots & K(x_2,y_m) \\ \vdots & \vdots & & \vdots \\ K(x_m,y_1) & K(x_m,y_2) & \cdots & K(x_m,y_m) \end{vmatrix} \geq 0$$

and so the set $\{1, g_1(n), g_2(n), \cdots\}$ satisfies (ii).

For example if $K(x,y) = x^y$ then

$$g_i(n) = \alpha_n^{\beta_i}$$

satisfy (ii) for $(\alpha_n)_n$ positive increasing sequence of real numbers and $(\beta_i)_i$ decreasing sequence. Moreover if

$$\lim_{n\to\infty} \alpha_n = 0 \text{ and } \beta_i > \beta_{i+1}, \forall i$$

then the set $\{1, g_1(n), g_2(n), \cdots\}$ satisfies also (i) of theorem 4. This includes the particular case :

$$g_i(n) = n^{\beta_i}, 0 > \beta_1 > \beta_2 > \cdots$$

which does not satisfy the conditions of the theorem (2)of BREZINSKI as remarked at the beginning of section 2

**Example 4**

Let us consider the set of functions:

$$g_i(n) := n^{\alpha_i} \log^{\beta_i} n$$

with $\alpha_1 > \alpha_2 > \alpha_3 > \cdots, \beta_i \in \mathbb{R}$

or if some of the $\alpha$'s are equal:$\alpha_{i_0} = \alpha_{i_0+1}$ for some $i_0$ ,$\beta_{i_0} > \beta_{i_0+1}$.

By computing the Wronskien of the functions

$$k_i(x) := x^{\alpha_i} \log^{\beta_i} x$$

it is possible,using a theorem of KARLIN (see [3] p.52),to prove that the set $\{1, g_1(n), g_2(n), \cdots\}$ satisfies (i) and (ii) of theorem 4.

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## Abstract

The advantage of the rational Padé-type approximants (P.T.A.) in comparison with the Padé-approximants is the freee choice of the poles. This work first consists to take advantage of this peculiarity to improve the approximation to some analytic functions in incorporating in the P.T.A. the known informations about these functions. For the class of Markov-Stieltjes functions, an other way consists, from the knowledge of a finite number of the Taylor coefficients, in approximating the weight function ; some results of convergence are then proved. The second part deals with the use of modified moments in order to detect Dirac masses in a density from its classical moments. These methods are also used to compute the jumps of a distribution and to accelerate the convergence of some logarithmic sequences.

## Keywords

Padé approximants

Padé-type approximants

Orthogonal polynomials

Weight function

Convergence acceleration

E-algorithm

Dirac Mass

Modified moments

## Résumé

L'avantage de l'approximation rationnelle de type-Padé (A.T.P.) par rapport à l'approximation rationnelle de Padé est le libre choix des pôles des approximants.Le travail a d'abord consisté à profiter de cette particularité pour améliorer l'approximation de certaines fonctions analytiques en incorporant dans les A.T.P. les renseignements connus sur ces fonctions .Pour les fonctions de Markov-Stieltjes, une autre façon d' aborder le problème est d'approximer d'abord leur fonction poids pour ensuite obtenir une approximation de la fonction elle-même.Des théoremes de convergence sont alors démontrés.La seconde partie traite de l'utilisation des moments modifiés pour la détection des masses de Dirac dans une densité à partir de ses moments classiques.Les méthodes obtenues ont également permis la détection et le calcul des sauts d'une distribution ainsi que l'accélération de la convergence de certaines suites logarithmiques.



## Mots-clés

Approximants de Padé

Approximants de type-Padé

Polynomes orthogonaux

Fonction poids

Accélération de la convergence

E-algorithme

Masse de Dirac

Moments modifiés