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Modèles mathématiques de la théorie du transfert radiatif

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Modèles mathématiques de la théorie du transfert radiatif

Résumé On s'intéresse dans ce travail à différents modèles de transfert radiatif, décrivant les interactions entre la matière et les photons. Les radiations sont décrites en termes d'énergie et flux d'énergie, dans le cas macroscopique, le fluide environnant est quant à lui décrit par les équations d'Euler (modèle d'hydrodynamique radiative). Dans le cas microscopique, le champ radiatif est vu comme une collection des photons interagissant avec la matière par des mécanismes d'absorption-émission. Ces mécanismes dépendent des états d'excitation interne et d'ionisation de la matière.

On commence par monter l'existence locale de solutions régulières pour un système couplant les équations d'Euler et l'équation du transfert radiatif. Ce système est obtenu à partir du bilan d'énergie et d'impulsion totale. Puis on fait une discussion asymptotique pour ce modèle dans le régime hors équilibre et on obtient un système simple couplant les équations d'Euler et une équation elliptique. On montre l'existence des profils de choc (réguliers) pour ce système, et la régularité de ces profils en fonction de l'amplitude du choc. Puis on étudie la stabilité asymptotique de ces profils. Enfin, on présente une étude d'un système décrivant le champ radiatif et les états internes de la matière. On montre l'existence de solutions pour ce système et on établit rigoureusement la convergence vers l'équilibre statistique. Les résultats théoriques sont illustrés par des simulations numériques.

Mots-clés : transfert radiatif, hydrodynamique, équations d'Euler, existence locale, analyse asymptotique, profils de choc, stabilité asymptotique, états internes, équilibre statistique.

Mathematical models of the theory of the radiative transfer

Abstract We are interested in various different models arising in radiative transfer, which describe the interactions between the medium and the photons. The radiation is described in terms of energy and energy flux in the macroscopic view, the material being described by the Euler equations (radiative hydrodynamic model). In another way, the radiation can be seen as a collection of photons, in the microscopic view point ; the photons can be absorbed or emitted by the material. The absorption and the emission of photons depend on the internal excitation and ionization state of the material.

We begin with the local existence (in time) of smooth solutions to a system coupling the Euler equations and the transfer equation. This system describes the exchange of energy and moment between the radiation and the material. Next, we give an asymptotic discussion for this model in the NON-LTE regime and get a simple system : coupling the Euler equations with an elliptic equation. We show the existence of (smooth) shock profiles to this system and the regularity of the shock profile as a function of the strength of the shock. Then we study the asymptotic stability of the shock profile. Finally, we study a system describing the radiation and the internal state of the material, in the microscopic view point. We prove the existence of the solution to this system and study the convergence towards the statistical equilibrium. The theoretical results are illustrated by numerical simulations.

Keywords : radiative transfer, hydrodynamic, Euler equations, local existence, asymptotic analysis, shock profile, asymptotic stability, internal state, statistical equilibrium.

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Introduction

1.1 Présentation générale

Dans ce travail on s'intéresse à différents modèles de transfert radiatif, décrivant les interactions entre la matière et les photons. Plus précisément, on abordera ces questions soit d'un point de vue macroscopique soit d'un point de vue microscopique.

Dans le cas macroscopique, les radiations sont décrites en termes d'énergie et flux d'énergie. Le fluide environnant est quant à lui décrit par les équations d'Euler et on obtient un système couplé par des termes d'échange d'énergie et d'impulsion provenant du bilan d'énergie totale et d'impulsion totale. Nous aborderons certains points de l'analyse mathématique de ces modèles, notamment l'existence de solutions particulières de type onde progressive, définissant ainsi des profils de choc, et l'étude de la stabilité de ces solutions.

Dans le cas microscopique, le champ radiatif est vu comme une collection des photons interagissant avec la matière par des mécanismes d'absorption-émission. Ces mécanismes dépendent des états d'excitation interne et d'ionisation de la matière, les effets radiatifs agissant sur la population des états atomiques. Ainsi, le champ radiatif et les états internes de la matière doivent être déterminés simultanément. On étudiera un système d'équations décrivant de telles interactions, en justifiant l'existence de solutions et en établissant rigoureusement la convergence vers l'équilibre statistique.

1.2 Modèle d'hydrodynamique radiative

1.2.1 Transfert radiatif

Le champ radiatif est composé par des photons, particules ponctuelles sans masse. Un photon avec fréquence ν représente l'énergie $h\nu$, où h est la constante de Planck, et aussi représente la quantité du mouvement (de module) $p = \frac{h\nu}{c}$, où c est la vitesse de lumière, vitesse à laquelle les photons se déplacent. On définit la densité des photons $\psi(t, x, v, \nu)$ telle que

$$\psi(t, x, v, \nu)\mathrm{d}x\mathrm{d}\nu\mathrm{d}v$$

soit le nombre des photons en temps $t \in \mathbb{R}^+$, et à la position $x \in \mathbb{R}^N$ dans un élément de volume $\mathrm{d}x$, avec les fréquences ν dans un intervalle $(\nu, \nu + \mathrm{d}\nu)$, et déplacement sur l'angle solide $\mathrm{d}v$ autour de la direction $v \in \mathbb{S}^{N-1}$. Notons que $\mathrm{d}v$ est la mesure de Lebesgue normalisée sur la sphère $\mathbb{S}^{N-1}$.

Le champ radiatif est plutôt décrit par son intensité spécifique $f(t, x, v, \nu)$, une fonction du temps $t \in \mathbb{R}^+$, de la position $x \in \mathbb{R}^N$, de la direction de déplacement des photons $v \in \mathbb{S}^{N-1}$ et de leur fréquence $\nu \in \mathbb{R}^+$, qui est définie par

$$f(t, x, v, \nu) = ch\nu\psi(t, x, v, \nu). \tag{1.1}$$

L'avantage de l'intensité spécifique $f(t, x, v, \nu)$ est de permettre d'écrire l'énergie trans-

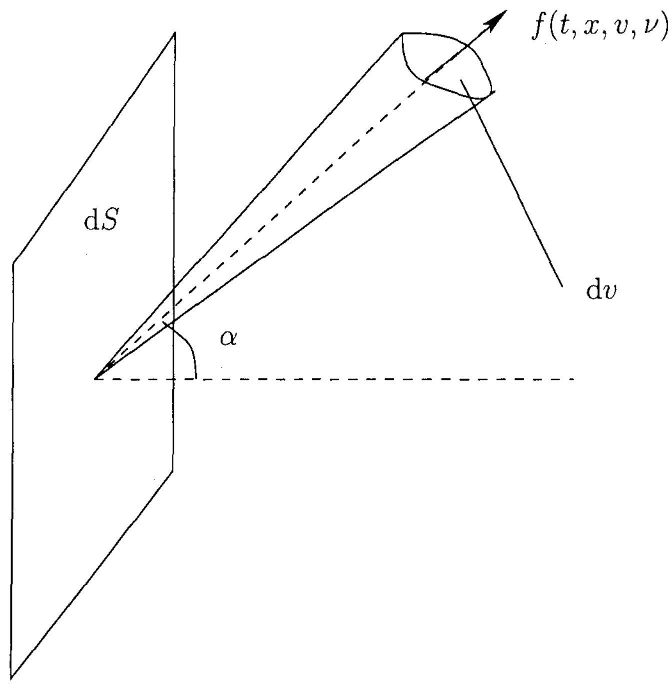

FIG. 1.1 – Intensité spécifique

portée traversant une surface dS entre un angle α mesurant la direction du rayonnement, pour les fréquences dans l'intervalle $(\nu, \nu + \mathrm{d}\nu)$, à la position $x + \mathrm{d}x$, dans un intervalle du temps $[t, t + \mathrm{d}t]$, (voir Fig 1.1). Cette énergie est donnée par

$$\mathrm{d}E = f(t, x, v, \nu) \cos \alpha \, \mathrm{d}S \mathrm{d}\nu \mathrm{d}v \mathrm{d}t.$$

Dans le cas où l'intensité spécifique est indépendante de la direction v, c'est-à dire $f \equiv f(t, x, \nu)$, le champ radiatif est dit "isotrope". De plus si l'intensité spécifique est indépendante de la position x, le champ radiatif est dit "homogène" et "isotrope". L'exemple le plus important de champ radiatif homogène et isotrope est l'équilibre thermodynamique de la matière, qui est décrit par une distribution particulière, la distribution planckienne $B(\nu, \theta)$ paramétrée par la température θ,

$$B(\nu, \theta) = \frac{2h\nu^3}{c^2} \left(\mathrm{e}^{h\nu/(k\theta)} - 1 \right)^{-1}, \tag{1.2}$$

où k est la constante de Boltzmann.

L'équation de transfert radiatif s'écrit

$$\frac{1}{c} \partial_t f + v \cdot \nabla_x f = Q, \tag{1.3}$$

Le terme à gauche dans (1.3) caractérise le transport des photons avec la vitesse cv suivant la ligne droite $x + cvt$. Le terme source $Q = Q^s + Q^t$, représente les interactions avec le milieu où Q^s décrit les phénomèmes de dispersion ("scattering" en anglais), qui se traduisent par des changements de la direction des trajectoires des photons, alors que Q^t décrit les interactions sous forme d'échanges avec la matière consistant en des mécanismes d'absorption et d'émission de photons.

On définit trois quantités particulières qui sont intéressantes en physique : la densité d'énergie radiative, le vecteur de flux radiatif et le tenseur de pression radiative. Elles correspondent aux moments sur l'espace des fréquences et des directions, et sont définies respectivement par

$$\begin{aligned} E_R &= \frac{1}{c}\int_{\mathbb{R}^+}\int_{\mathbb{S}^{N-1}} f(t,x,v,\nu)\,\mathrm{d}v\mathrm{d}\nu, \\ F_R &= \int_{\mathbb{R}^+}\int_{\mathbb{S}^{N-1}} vf(t,x,v,\nu)\,\mathrm{d}v\mathrm{d}\nu, \\ P_R &= \frac{1}{c}\int_{\mathbb{R}^+}\int_{\mathbb{S}^{N-1}} v\otimes vf(t,x,v,\nu)\,\mathrm{d}v\mathrm{d}\nu. \end{aligned} \tag{1.4}$$

Le sens physique du vecteur flux radiatif F_R et du tenseur de pression radiative s'explique de la manière suivante : on considère une surface $\mathrm{d}S$ de normale n, le flux radiatif est défini tel que l'énergie radiative traversant cette surface soit $F_R\cdot n\,\mathrm{d}S$. Par définition de la densité des photons, le nombre total des photons de fréquence ν et dans toutes les directions est

$$\left(\int_{\mathbb{S}^{N-1}} \psi(t,x,v,\nu)cv\mathrm{d}v\right)\cdot n\mathrm{d}S.$$

Comme un photon de fréquence ν représente l'énergie $h\nu$, l'énergie totale traversant cette surface est

$$\left(\int_{\mathbb{R}^+}\int_{\mathbb{S}^{N-1}} h\nu\psi(t,x,v,\nu)cv\mathrm{d}v\mathrm{d}\nu\right)\cdot n\mathrm{d}S.$$

D'après la relation entre l'intensité spécifique et la densité des photons (voir (1.1)), on obtient ainsi le flux radiatif. Dans la théorie cinétique des gaz, la pression est définie comme le flux du moment qui traverse la surface. On définit ainsi le tenseur de pression radiative P_R, dont l'élément P_R^{ij} est le flux du moment à la i ème composante traversant une surface orthogonale au j ème axe :

$$P_R^{ij} = \int_{\mathbb{R}^+}\int_{\mathbb{S}^{N-1}} f(t,x,v,\nu)v_iv_j\mathrm{d}v\mathrm{d}\nu.$$

On peut l'obtenir par le même traitement que dans la dérivation de flux radiatif.

Notons que l'énergie radiative E_R est le moment d'ordre zéro, le vecteur de flux radiatif F_R est d'ordre un, et le tenseur de pression radiative P_R est d'ordre deux. On peut définir les moments d'ordre plus élevé, mais ils ont peu de sens physique. En multipliant (1.3) par 1 et v/c, et en intégrant sur l'espace des fréquences et des directions, on obtient le système aux moments suivant

$$\left\{\begin{array}{rcl} \partial_t E_R + \nabla_x\cdot F_R &=& Q_E := \displaystyle\int_{\mathbb{R}^+}\int_{\mathbb{S}^{N-1}} Q\mathrm{d}v\mathrm{d}\nu, \\ \dfrac{1}{c^2}\partial_t F_R + \nabla_x\cdot P_R &=& Q_F := \dfrac{1}{c}\displaystyle\int_{\mathbb{R}^+}\int_{\mathbb{S}^{N-1}} vQ\mathrm{d}v\mathrm{d}\nu, \end{array}\right. \tag{1.5}$$

où Q_E représente l'énergie gagnée ou perdue par les radiations et Q_F le moment gagné ou perdu par le champ radiatif.

A l'équilibre thermodynamique, l'intensité spécifique est donnée par la fonction planckienne (voir (1.2)). Lorsque les radiations sont à l'état d'équilibre thermodynamique, on peut calculer l'énergie radiative, le vecteur de flux radiatif, et le tenseur de pression radiative. Par exemple, en dimension trois, on calcule l'énergie radiative comme

$$\begin{aligned}
E_R^* &= \frac{1}{c}\int_{\mathbb{R}+}\int_{\mathbb{S}^2} B(\theta,\nu)\mathrm{d}v\mathrm{d}\nu \\
&= \frac{2k^4\theta^4}{h^3c^3}\int_{\mathbb{R}+}\frac{x^3}{\mathrm{e}^x-1}\mathrm{d}x = \frac{2k^4\theta^4}{h^3c^3}\int_{\mathbb{R}+} x^3\sum_{n=1}^{\infty}\mathrm{e}^{-nx}\mathrm{d}x \\
&= \frac{12k^4\theta^4}{h^3c^3}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{2k^4\pi^4}{15h^3c^3}\theta^4 = \frac{\sigma}{c\pi}\theta^4,
\end{aligned}$$

où l'indice supérieur $*$ indique l'équilibre thermodynamique, et $\sigma = \frac{2k^4\pi^5}{15h^3c^2}$ est la constante de Stefan-Boltzmann. Rappelons que $\mathrm{d}v$ est la mesure de Lebesgue normalisée sur la sphère $\mathbb{S}^2$. Comme le champ radiatif est isotrope et homogène, on a $F_R^* = 0$, car à l'équilibre thermodynamique, il n'y a pas de flux d'énergie. Pour le tenseur de pression radiative, on a

$$P_R^* = \frac{\sigma\theta^4}{3\pi}\mathbb{I}_3 = \frac{\bar{p}^*}{3}\mathbb{I}_3,$$

où $\bar{p}$ est la pression moyenne définie par

$$\bar{p} = \frac{1}{3}\mathrm{tr}(P_R) = \frac{1}{3}\left(P_R^{11} + P_R^{22} + P_R^{33}\right).$$

Rappelons qu'à l'équilibre thermodynamique, l'énergie radiative, le vecteur de flux radiatif et le tenseur de pression radiative sont définis par

$$E_R^* = \frac{\sigma}{c\pi}\theta^4, \qquad F_R^* = 0, \qquad P_R^* = \frac{\sigma\theta^4}{3c\pi}\mathbb{I}_3. \tag{1.6}$$

1.2.2 Effets Doppler, effets relativistes

On a introduit l'intensité spécifique radiative et les quantités associées avec la description du champ radiatif. Maintenant on va étudier les interactions entre la matière et la radiation, plus particulièrement trois phénomènes : l'absorption des photons, l'émission des photons et la dispersion des photons.

Jusqu'ici, nous discutions dans un repère fixe (le repère terrestre). Mais, dans beaucoup d'applications la vitesse du milieu change d'un endroit à un autre et on doit considérer un repère attaché en mouvement (le repère mobile) et il y a donc des différences suivant qu'une quantité est mesurée dans le repère en mouvement ou le repère fixe. Lorsqu'on tient compte des effets relativistes et de l'effet Doppler, des termes supplémentaires liés à ces formules de changement de repères interviennent dans les équations. Mais les deux choix de repères ont chacun des avantages et des inconvénients. Comme indiqué dans le travail de Mihalas et Klein [48], l'équation (1.3) écrite dans le repère fixe concerne les dérivées en temps t et en espace x. Quant au repère mobile, il nous donne une équation plus compliquée, prenant en compte les dérivées non seulement en temps t et en espace x, mais aussi en direction v et en fréquence ν. Par contre dans le repère en mouvement, on peut facilement spécifier les propriétés thermodynamiques, particulièrement quand les coefficients d'absorption et d'émission du photon par les atomes sont isotropes, alors qu'elles deviennent anisotropes dans le repère fixe. On va désormais introduire le lien entre les quantités mesurées dans chaque repère.

Soit $u(t,x)$ la vitesse du milieu, on définit le facteur de Lorentz comme

$$\Upsilon(t,x) = \frac{1}{\sqrt{1-\frac{|u(t,x)|^2}{c^2}}}. \tag{1.7}$$

Alors les formules suivantes lient la fréquence ν^0 et la direction v^0 des photons mesurées dans le repère en mouvement, à la fréquence ν et la direction v mesurées dans le repère fixe,

$$\nu^0 = \nu\Upsilon(1-\frac{v\cdot u}{c}), \quad v^0 = \frac{\nu}{\nu^0}\Big(v - \frac{\Upsilon}{c}u(1-\frac{u\cdot v}{c}\frac{\Upsilon}{\Upsilon+1})\Big), \tag{1.8}$$

où on a omis la dépendance en (t,x) de la vitesse u du milieu. A partir de maintenant, l'indice supérieur 0 indique les quantités mesurées dans le repère en mouvement.

Il est naturel d'écrire les équations dans le repère fixe. Mais comme indiqué dans le travail de Mihalas et Klein [48], les interactions matière/radiation sont quant à elles mieux décrites dans le repère en mouvement, où on a

$$\begin{aligned} Q^{s,0} &= \frac{1}{l_s}\Big(\int_{\mathbb{S}^{N-1}} \sigma_s^0(\nu^0,v^0,v^{0\prime})f^0(v^{0\prime},\nu^0)\mathrm{d}v^{0\prime} \\ &\qquad -f^0(v^0,\nu^0)\int_{\mathbb{S}^{N-1}} \sigma_s^0(\nu^0,v^{0\prime},v^0)\mathrm{d}v^{0\prime}\Big), \\ Q^{t,0} &= \frac{1}{l_a}\sigma_a^0(\nu^0,v^0)\left(B(\nu^0,\theta)-f^0(v^0,\nu^0)\right), \end{aligned} \tag{1.9}$$

avec l_s le libre parcours moyen ("mean free path" en anglais) de la dispersion, l_a le libre parcours moyen de l'absorption-émission, $\sigma_s(\nu,\ v,\ v')$ le coefficient caractérisant le changement d'un photon de fréquence ν, de la direction de v' à v, $\sigma_a(\nu,\ v)$ le coefficient caractérisant les processus thermodynamiques, et $B(\nu,\theta)$ la distribution planckienne. Ici on fait l'hypothèse que les photons sont émis à l'équilibre thermique avec un corps noir. Un photon est censé voyager en ligne droite avec la vitesse c jusqu'à collision. Alors le mouvement du photon peut être considéré comme une suite des chemins en ligne droite qui se terminent par des collisions, desquelles le photon émerge sur un nouveau chemin. Chacun de ces vols entre les collisions s'appelle un chemin libre. Le libre parcours moyen est défini comme la longueur moyenne des chemins libres. Alors on peut définir ainsi le libre parcours moyen de la dispersion l_s et le libre parcours moyen de l'absorption-émission l_a. Notons que le terme $\int\sigma_s(\nu,\ v,\ v')f(v')/l_s\mathrm{d}v'$ caractérise, en fréquence ν, le nombre de photon lorsque l'on passe de direction v à la direction v'; et le terme $\int\sigma_s(\nu,\ v',\ v)f(v)/l_s\mathrm{d}v'$ caractérise, en fréquence ν, le nombre de photon changé lorsque l'on passe de direction v' à la direction v. Dans le repère mobile, la dispersion est conservée, c'est-à-dire $\iint Q^{s,0}\mathrm{d}v^0\mathrm{d}\nu^0 = 0$, qui est non valide dans repère fixe.

En utilisant les relations de conversion obtenues par (1.8), on obtient le terme source écrit dans le repère fixe

$$
\begin{aligned}
Q &= Q^s + Q^t, \\
Q^s &= \frac{1}{l_s}\Big\{ \frac{1}{\Lambda(v)^2} \int_{\mathbb{S}^2} \sigma_s(\nu, v, v') f(v', \nu') \Lambda(v') \mathrm{d}v' \\
&\qquad - \Lambda(v) f(v, \nu) \int_{\mathbb{S}^2} \frac{\sigma_s(\nu, v', v)}{\Lambda(v')^2} \mathrm{d}v' \Big\}, \\
Q^t &= \frac{\sigma_a(\nu, v)}{l_a} \Big\{ \frac{B(\nu^0, \theta)}{\Lambda(v)^2} - \Lambda(v) f(v, \nu) \Big\},
\end{aligned}
\tag{1.10}
$$

où $\Lambda(v)$ est défini par

$$
\Lambda(t, x, v) = \frac{\nu^0}{\nu} = \frac{1 - \frac{v \cdot u(t,x)}{c}}{\sqrt{1 - \frac{|u(t,x)|^2}{c^2}}}. \tag{1.11}
$$

Notons que l'on a omis la dépendance en (t, x) dans (1.10). Finalement, en tenant compte de ces corrections dans le repère fixe, l'équation de transfert radiatif (1.3) devient

$$
\begin{aligned}
\frac{1}{c}\partial_t f + v \cdot \nabla_x f \;=\; & \frac{1}{l_s}\Big\{ \frac{1}{\Lambda(v)^2} \int_{\mathbb{S}^{N-1}} \sigma_s(\nu, v, v') f(v', \nu') \Lambda(v') \mathrm{d}v' \\
& \qquad - \Lambda(v) f(v, \nu) \int_{\mathbb{S}^{N-1}} \frac{\sigma_s(\nu, v', v)}{\Lambda(v')^2} \mathrm{d}v' \Big\} \\
& + \frac{\sigma_a(\nu, v)}{l_a} \Big\{ \frac{B(\nu^0, \theta)}{\Lambda(v)^2} - \Lambda(v) f(v, \nu) \Big\}.
\end{aligned}
\tag{1.12}
$$

1.2.3 Couplage avec l'hydrodynamique

On a introduit le champ radiatif et les interactions entre la matière et la radiation. En ce qui concerne le problème de l'hydrodynamique radiative, on s'intéresse aussi à l'écoulement de la matière. Pour l'hydrodynamique radiative, la matière est généralement dans l'état gazeux. On considère que le gaz est modélisé comme un continuum, ce qui signifie en particulier que le libre parcours moyen moléculaire est beaucoup plus court que la longueur d'écoulement et le libre parcours moyen des photons. Dans les conditions physiques auxquelles on s'intéresse, les effets visqueux sont négligeables. Les équations hydrodynamiques résultent des conservations de la masse, du moment et de l'énergie, à un volume différentiel de l'espace. On commence par les équations d'Euler relativistes qui s'écrivent, en négligeant pour le moment les échanges radiatifs,

$$
\begin{cases}
\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\
\partial_t (\Upsilon H \rho u) + \nabla_x \cdot (\Upsilon H \rho u \otimes u) + \nabla_x p = 0, \\
\partial_t (\Upsilon H \rho - \frac{p}{c^2}) + \nabla_x \cdot (\Upsilon H \rho u) = 0,
\end{cases}
\tag{1.13}
$$

où ρ est la densité du gaz mesurée dans le repère fixe, et on note ρ^0 la densité mesurée dans le repère mobile, tel que $\rho = \Upsilon \rho^0$, u est la vitesse du fluide, alors que Υ est le facteur de Lorentz, défini par (1.7), H est l'enthalpie, définie par

$$
H = 1 + \frac{e + \Upsilon p / \rho}{c^2},
$$

e étant l'énergie interne spécifique, et p la pression. On suppose que le gaz satisfait la loi des gaz parfaits $p = \frac{\Gamma e \rho}{\Upsilon}$, de sorte que $H = 1 + \frac{(1+\Gamma)e}{c^2}$, avec la constante adiabatique Γ.

Notons que (1.13) représente les équations relativistes de l'hydrodynamique sous forme conservative pour un volume fixé dans l'espace. En d'autres termes, l'intégration de ces équations sur un volume arbitraire fixé dans l'espace donne des résultats qui peuvent être interprétés en tant que conservations de la densité relative, du moment et de l'énergie dans ce volume. Les équations hydrodynamiques relativistes en l'absence d'un champ radiatif sont données dans beaucoup de littérature sur la relativité et la mécanique de fluide, par exemple dans [49, 17, 43, 9, 61, 51].

En supposant $|u| \ll c$, on néglige les termes d'ordre 2 en $\frac{|u|}{c}$ et on obtient le système d'Euler classique

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x p = 0, \\ \partial_t(\rho E) + \nabla_x \cdot (\rho E u + p u) = 0, \end{cases} \tag{1.14}$$

où $E = e + \frac{|u|^2}{2}$ est l'énergie totale spécifique. Notons que la loi des gaz parfaits devient $p = \Gamma \rho e$ en négligeant les termes d'ordre 2 en $\frac{|u|}{c}$ dans Υ, et que (1.14) est aussi la limite non-relativiste du système hydrodynamique, voir [49, 39, 40, 43].

Maintenant on considère les effets radiatifs dans les équations hydrodynamiques (1.14), les conservations de la densité, du moment et de l'énergie. Si une radiation significative est présente, on doit inclure le moment radiatif et l'énergie radiative dans ces conservations. La prise en compte des échanges d'énergie et du moment avec le champ radiatif conduit au système couplé suivant

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x p = -Q_F, \\ \partial_t(\rho E) + \nabla_x \cdot (\rho E u + p u) = -Q_E, \\ \dfrac{1}{c}\partial_t f + v \cdot \nabla_x f = Q, \end{cases} \tag{1.15}$$

où on rappelle que Q est défini par (1.10), et Q_E, Q_F sont définis par (1.5). Rappellons qu'en dérivant (1.15), on néglige les termes d'ordre 2 en $\frac{|u|}{c}$ dans les équations hydrodynamiques, mais pas pour l'équation du transfert radiatif (1.12). On mentionne que Mihalas et Klein ont étudié l'approximation de l'équation radiative à l'ordre de $O(\frac{u}{c})$ dans [48].

1.2.4 Modèle gris

L'équation de transfert radiatif (1.12) peut se simplifier en supposant que les coefficients σ_s et σ_a sont indépendants de la variable de fréquence ν : c'est l'hypothèse dite du transfert radiatif "gris". Cette hypothèse peut être raisonnable pour le coefficient de dispersion σ_s, par exemple en considérant la dispersion de Thomson (voir [6, 51]), mais elle est plus discutable pour le coefficient d'absorption σ_a. Cette hypothèse permet toutefois d'intégrer (1.12) suivant la variable ν sur l'ensemble $\mathbb{R}^+$. Tout d'abord on introduit les notations suivantes

$$\overline{f}(t,x,v) = \int_{\mathbb{R}^+} f(t,x,v,\nu)\mathrm{d}\nu,$$

$$\begin{aligned} \int_{\mathbb{R}^+} B(\nu^0,\theta)\mathrm{d}\nu &= \int_{\mathbb{R}^+} \frac{2h\left(\nu^0\right)^3}{c^2} \frac{\mathrm{d}\nu}{e^{h\nu^0/(k\theta)} - 1} \\ &= \frac{1}{\Lambda}\mathbb{B}(\theta), \end{aligned}$$

où $\mathbb{B}(\theta) = \sigma\theta^4/\pi$ est la fonction intégrée de la distribution planckienne, on rappelle que $\sigma = \frac{2\pi^5 k^4}{15h^3c^2}$ est la constante de Stefan-Boltzmann. Avec ces notations, on intègre (1.12) par rapport à la variable ν sur l'ensemble $\mathbb{R}^+$, et on obtient l'équation intégrée suivante

$$\frac{1}{c}\partial_t\overline{f} + v\cdot\nabla_x\overline{f} = \overline{Q}, \tag{1.16}$$

avec

$$\overline{Q} = \overline{Q}^s + \overline{Q}^t, \tag{1.17}$$

$$\overline{Q}^s = \int_{\mathbb{R}^+} Q^s \mathrm{d}\nu = \frac{1}{l_s}\left(\frac{\langle\sigma_s\Lambda^2\overline{f}\rangle}{\Lambda^3} - \left\langle\frac{\sigma_s}{\Lambda^2}\right\rangle\Lambda\overline{f}\right), \tag{1.18}$$

$$\overline{Q}^t = \int_{\mathbb{R}^+} Q^t \mathrm{d}\nu = \frac{\sigma_a}{l_a}\left(\frac{\mathbb{B}(\theta)}{\Lambda^3} - \Lambda\overline{f}\right), \tag{1.19}$$

où $< \cdot >$ désigne l'intégration par rapport à la variable de direction v' sur la sphère $\mathbb{S}^{N-1}$.

Dans ce cas "gris", les termes de bilan d'énergie Q_E et d'impulsion Q_F peuvent être écrits comme :

$$Q_E = \int_{\mathbb{S}^{N-1}} Q\mathrm{d}v = \langle Q\rangle,$$
$$Q_F = \int_{\mathbb{S}^{N-1}} vQ\mathrm{d}v = \frac{1}{c}\langle vQ\rangle,$$

où on a omis la notation surlignée. Le modèle (1.15) se simplifie alors sous la forme suivante

$$\begin{cases} \partial_t\rho + \nabla_x\cdot(\rho u) = 0, \\ \partial_t(\rho u) + \nabla_x\cdot(\rho u\otimes u) + \nabla_x p = -\frac{1}{c}\langle vQ\rangle, \\ \partial_t(\rho E) + \nabla_x\cdot(\rho E u + pu) = -\langle Q\rangle, \\ \frac{1}{c}\partial_t f + v\cdot\nabla_x f = Q, \end{cases} \tag{1.20}$$

où Q est désormais défini par (1.17). Notons que l'énergie totale spécifique E est la somme de l'énergie interne spécifique e et l'énergie cinétique $u^2/2$, et que la pression p satisfait la loi des gaz parfaits $p = (\gamma-1)\rho e$. C'est le premier modèle que l'on étudiera : dans le chapitre 2 on établit l'existence locale en temps et l'unicité de solutions régulières de (1.20), en suivant les approches classiques pour les systèmes hyperboliques.

1.2.5 Existence de solutions en temps petit

Le premier résultat de ce travail est consacré à l'étude mathématique du système (1.20) qui couple une équation cinétique et les équations d'Euler. On s'inspire de la démarche introduite par Kato dans [24] pour obtenir un théorème d'existence locale général pour le problème de Cauchy associé à un système hyperbolique. La théorie des solutions locales en temps pour un système de lois de conservation de la forme

$$\partial_t u + \sum_{i=1}^{N}\partial_{x_j} f_j(u) = 0, \tag{1.21}$$

où $u = u(t,x)\in\mathbb{R}^m$, et $x = (x_1, x_2, \cdots, x_N)\in\mathbb{R}^N$, est bien établie dans [35, 1, 34, 45, 55]. En particulier, cette approche s'adapte pour le système d'Euler, sans couplage avec l'équation cinétique, pourvu que les données initiales soient suffisamment régulières : il existe où $u = u(t, x) \in \mathbb{R}^m$, et $x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N$, est bien établie dans [35, 1, 34, 45, 55]. En particulier, cette approche s'adapte pour le système d'Euler, sans couplage avec l'équation cinétique, pourvu que les données initiales soient suffisamment régulières : il existe un intervalle de temps maximal sur lequel il existe une unique solution régulière. Ce résultat est général aux systèmes de lois de conservation ayant des propriétés particulières : hyperbolique et symétrisable. Par exemple, dans [45], Majda utilise la notion de système symétrisable au sens de Friedrichs, c'est-à-dire qu'il existe une matrice $A_0 = A_0(u)$ symétrique et définie positive, telle que $u \mapsto A_0(u)$ soit bornée uniformément sur tout compact G, et $A_j(u) = A_0(u)\partial_u f_j$ est symétrique pour $j = 1, 2 \cdots, N$. Alors on montre l'existence locale de solutions de l'équation quasi-linéaire

$$A_0(u)\partial_t u + \sum_{i=1}^{N} A_j(u)\partial_{x_j} u = 0,$$

par la stratégie introduite par Kato dans [24]. On obtient ainsi l'existence locale de solutions du système de lois de conservation.

Nous utilisons ici la même technique pour montrer l'existence en temps petit pour le système couplant les équations d'Euler et l'équation cinétique. En pratique, il est plus agréable d'écrire les équations hydrodynamiques en prenant pour variables indépendantes la pression p, la vitesse u et l'entropie S. Les équations d'Euler en les variables p, u, S sont symétrisables au sens de Friedrichs. Pour l'équation cinétique, on intègre sur les caractéristiques en supposant données les inconnues "fluides" et on obtient ainsi les estimations clefs qui permettent de mettre en place une méthode de point fixe.

Un tel système couplé Euler-cinétique a été étudié dans [4] et [3], motivé par la modélisation d'écoulements fluide/particules. Toutefois le couplage traité dans [4] et [3] a une nature différente puisqu'il provient de termes de force de frottement, au lieu de termes source collisionnels . Nous mentionnons également les travaux récents [62] exécutés indépendamment du nôtre, qui sont consacrés à l'analyse d'un problème très proche de (1.20). Le modèle étudié dans [62] est non-relativiste et incorpore la dépendance par rapport à la variable de fréquence.

Sans surprise notre résultat d'existence local est le théorème suivant :

Théorème 1.1. *Soit s un nombre entier tel que $s > N/2 + 1$. Soit G_1 un ouvert relativement compact dans $\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^+$. On suppose que les données initiales (ρ_0, u_0, e_0, f_0) satisfont*

$$\begin{pmatrix} \rho_0(x) \\ u_0(x) \\ e_0(x) \end{pmatrix} \in G_1, \qquad \begin{pmatrix} \rho_0 - \overline{\rho}_0 \\ u_0 \\ e_0 - \overline{e}_0 \end{pmatrix} \in H^s(\mathbb{R}^N),$$

et

$$f_0 \geq 0, \ f_0 - \overline{f}_0 \in L^2(\mathbb{S}^{N-1}; H^s(\mathbb{R}^N)) \bigcap L^\infty(\mathbb{R}^N \times \mathbb{S}^{N-1}),$$

où $\overline{\rho}_0$, $\overline{e}_0$ et $\overline{f}_0$ sont des constantes strictement positives telles que $(\overline{\rho}_0,\ 0,\ \overline{e}_0,\ \overline{f}_0)$ soit une solution constante du système (1.20). Alors, il existe un temps $T > 0$, tel qu'il y a une unique solution régulière (ρ, u, e, f) du système (1.20), qui vérifie

$$(\rho - \bar{\rho}_0, u, e - \bar{e}_0) \in C([0, T]; H^s(\mathbb{R}^n)),$$

et

$$f - \bar{f}_0 \in C([0, T]; L^2(\mathbb{S}^{N-1}; H^s(\mathbb{R}^N))).$$

1.3 Un modèle simple de gaz radiatif

1.3.1 Discussion asymptotique : le régime Non-TEL

Maintenant on présente une discussion asymptotique du système qui couple l'évolution des radiations et la dynamique des gaz. Ici, on ne tient pas compte de l'effet Doppler et on suppose que les coefficients σ_t, σ_s sont des constantes positives. Alors l'équation cinétique peut s'écrire simplement

$$\frac{1}{c}\partial_t f + v \cdot \nabla_x f = \frac{\sigma_s}{l_s}\Big(\langle f\rangle - f\Big) + \frac{\sigma_a}{l_a}\Big(\frac{\sigma}{\pi}\theta^4 - f\Big).$$

Notons que le premier terme à droite caractérise la dispersion et le deuxième caractérise l'absorption-émission. Avec ces simplifications, Q_E et Q_F définis dans (1.5) peuvent s'écrire

$$\begin{aligned} Q_E &= \frac{\sigma_a}{l_a}\left(\frac{\sigma}{\pi}\theta^4 - \langle f\rangle\right), \\ Q_F &= -\frac{1}{c}\left(\frac{\sigma_s}{l_s} + \frac{\sigma_a}{l_a}\right)\langle vf\rangle . \end{aligned}$$

On obtient le système couplé

$$\left\{ \begin{aligned} &\partial_t\rho + \nabla_x \cdot (\rho u) = 0, \\ &\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x p = \frac{1}{c}\left(\frac{\sigma_s}{l_s} + \frac{\sigma_a}{l_a}\right)\langle vf\rangle , \\ &\partial_t(\rho E) + \nabla_x \cdot (\rho E u + pu) = -\frac{\sigma_a}{l_a}\left(\frac{\sigma}{\pi}\theta^4 - \langle f\rangle\right), \\ &\frac{1}{c}\partial_t f + v \cdot \nabla_x f = \frac{\sigma_s}{l_s}\Big(\langle f\rangle - f\Big) + \frac{\sigma_a}{l_a}\Big(\frac{\sigma}{\pi}\theta^4 - f\Big). \end{aligned} \right. \tag{1.22}$$

Notons que l'énergie totale

$$\frac{1}{c}\int_{\mathbb{R}^N}\int_{\mathbb{S}^{N-1}} f \mathrm{d}v\mathrm{d}x + \int_{\mathbb{R}^N} \rho E \mathrm{d}x$$

est conservée formellement.

Afin de juger plus facilement l'importance relative des divers phénomènes dans un flux, et la nature quantitative de flux, il est utile d'étudier les équations sous forme adimensionnelle. D'abord on introduit quelques échelles hydrodynamiques. Soit L l'unité de longueur d'observation, u_∞ la vitesse du son typique du gaz qui est l'unité de vitesse du fluide, ρ_∞ l'unité de densité, et θ_∞ l'unité de température. Alors on pose

$$x = \mathrm{L}\hat{x}, \quad t = \frac{\mathrm{L}}{u_\infty}\hat{t}, \quad \rho = \rho_\infty\hat{\rho}, \quad \theta = \theta_\infty\hat{\theta}. \tag{1.23}$$

Notons que le chapeau désigne une quantité adimensionnelle. Et on définit les unités de la pression p et d'énergie spécifique E comme

$$p = \rho_\infty u_\infty^2 \hat{p}, \quad E = u_\infty^2 \hat{E}, \quad f = \frac{\sigma}{\pi}\theta_\infty^4 \hat{f}, \tag{1.24}$$

notons que l'unité de l'intensité spécifique f n'est que $\mathbb{B}(\theta_\infty)$, la fonction intégrée de la distribution planckienne à la température θ_∞, voir (1.6). Pour écrire (1.22) sous forme adimensionnelle, quatre paramètres sans dimension interviennent :

- $\mathcal{C} = \dfrac{c}{u_\infty}$, le rapport de la vitesse de la lumière et de la vitesse du son typique du gaz,
- $\mathcal{L}_s = \dfrac{l_s}{\mathrm{L}}$, le nombre de Knudsen associé à la dispersion,
- $\mathcal{L}_a = \dfrac{l_a}{\mathrm{L}}$, le nombre de Knudsen associé à l'absorption-émission,
- $\mathcal{P} = \dfrac{\frac{\sigma}{c\pi}\theta_\infty^4}{\rho_\infty u_\infty^2}$, facteur qui compare l'énergie radiative et l'énergie du gaz.

Le système (1.22) devient ainsi

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, u) = 0\,, \\ \partial_t(\rho\, u) + \nabla_x \cdot (\rho\, u \otimes u) + \nabla_x p = \mathcal{P}\left(\dfrac{\sigma_s}{\mathcal{L}_s} + \dfrac{\sigma_a}{\mathcal{L}_a}\right)\langle v f\rangle\,, \\ \partial_t(\rho\, E) + \nabla_x \cdot (\rho\, E\, u + p\, u) = -\dfrac{\mathcal{P}\mathcal{C}}{\mathcal{L}_a}\sigma_a\left(\theta^4 - \langle f\rangle\right), \\ \dfrac{1}{\mathcal{C}}\,\partial_t f + v \cdot \nabla_x f = \dfrac{\sigma_s}{\mathcal{L}_s}\left(\langle f\rangle - f\right) + \dfrac{\sigma_a}{\mathcal{L}_a}\left(\theta^4 - f\right). \end{cases} \tag{1.25}$$

Notons que l'on a omis le chapeau dans toutes les quantités. D'abord on suppose que $\mathcal{C} \gg 1$ et $\mathcal{P}$ d'ordre 1. On s'intéresse au régime hors-équilibre, où la dispersion est dominante dans les interactions, donc $\mathcal{L}_s$ et $\mathcal{L}_a$ vérifient dans ce cas

$$\mathcal{L}_s = \frac{1}{\mathcal{C}},\ \mathcal{L}_a = \mathcal{C}.$$

On introduit le développement de Hilbert

$$f = f^{(0)} + \frac{1}{\mathcal{C}} f^{(1)} + \frac{1}{\mathcal{C}^2} f^{(2)} + \cdots\,,$$

puis on l'insère dans l'équation cinétique, c'est-à-dire la quatrième équation du système (1.25). En identifiant les termes de même d'ordre en puissances de $\mathcal{C}$, il vient

- termes en $\mathcal{C}$: $< f^{(0)} > - f^{(0)} = 0$, alors $f^{(0)}$ ne dépend pas de la variable v, et on note $f^{(0)}(t,x,v) = n(t,x)$,
- termes en $\mathcal{C}^0$: $v \cdot \nabla_x f^{(0)} = \sigma_s\left(< f^{(1)} > - f^{(1)}\right)$, comme $< f^{(1)} >$ ne dépend pas de la variable v, donc $f^{(1)}(t,x,v) = -\frac{1}{\sigma_s} v \cdot \nabla_x n(t,x)$,
- termes en $\mathcal{C}^{-1}$: $\partial_t f^{(0)} + v \cdot \nabla_x f^{(1)} = \sigma_s\left(< f^{(2)} > - f^{(2)}\right) + \left(\theta^4 - f^{(0)}\right)$, on l'intégre en v sur l'ensemble $\mathbb{S}^{N-1}$, notons que $f^{(0)}(t,x,v) = n(t,x)$, et $f^{(1)}(t,x,v) = -\frac{1}{\sigma_s} v \cdot \nabla_x n(t,x)$, et on obtient

$$\partial_t n - \frac{1}{N\,\sigma_s}\,\Delta_x n = \sigma_a\left(\theta^4 - n\right).$$

Notons que pour le terme source de la deuxième équation dans (1.25), on a

$$\left(\frac{\sigma_s}{\mathcal{L}_s} + \frac{\sigma_a}{\mathcal{L}_a}\right)\langle v f\rangle \simeq \sigma_s < v f^{(1)} > = -\frac{\sigma_s}{N}\nabla_x n.$$

Alors, on obtient le système limite suivant

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, u) = 0\,, \\ \partial_t(\rho\, u) + \nabla_x \cdot (\rho\, u \otimes u) + \nabla_x p = -\dfrac{\mathcal{P}\sigma_s}{N}\,\nabla_x n\,, \\ \partial_t(\rho\, E) + \nabla_x \cdot (\rho\, E\, u + p\, u) = -\mathcal{P}\,\sigma_a\left(\theta^4 - n\right), \\ \partial_t n - \dfrac{1}{N\,\sigma_s}\,\Delta_x n = \sigma_a\left(\theta^4 - n\right). \end{cases}$$

La hiérarchie de modèles peut être complétée en supposant maintenant que $\mathcal{P}$, σ_s et σ_a vérifient les relations suivantes

$$\mathcal{P} \ll 1\,, \qquad \mathcal{P}\,\sigma_a = 1\,, \qquad N\,\sigma_s = 1/\sigma_a\,,$$

et on obtient le système couplé de type hyperbolique-elliptique

$$\begin{cases} \partial_t\rho + \nabla_x \cdot (\rho\, u) = 0\,, \\ \partial_t(\rho\, u) + \nabla_x \cdot (\rho\, u \otimes u) + \nabla_x p = 0\,, \\ \partial_t(\rho\, E) + \nabla_x \cdot (\rho\, E\, u + p\, u) = n - \theta^4\,, \\ -\Delta_x n = \theta^4 - n\,. \end{cases} \tag{1.26}$$

Ce modèle décrit encore des échanges d'énergie entre radiations et matière, ici dans un régime diffusif hors équilibre. C'est le deuxième modèle qu'on analyse dans ce manuscrit.

1.3.2 Existence de profils de choc

Le deuxième résultat dans ce travail est consacré à l'étude mathématique du système (1.26) qui couple une équation elliptique et un système hyperbolique, via l'échange d'énergie, dans le cas monodimensionnel $x \in \mathbb{R}$. Rappellons que les trois premières équations dans (1.26) ne sont que les équations de dynamique des gaz influencé par l'énergie radiative n. Dans le travail de Sideris [57], il est montré que, pour une certaine classe de données initiales, les solutions régulières des équations de la dynamique des gaz, explosent en temps fini. Cette explosion en temps fini des solutions de la dynamique des gaz est liée à la formation d'ondes de choc. Si on étudie les solutions faibles, il y a toujours des problèmes d'unicité. On introduit le critère entropique qui nous permet de trouver la solution physique parmi les solutions faibles et définit l'onde de choc des équations de la dynamique des gaz.

On écrit le systèmes de la dynamique des gaz, (1.14), sous forme vectorielle, comme

$$\partial_t \begin{pmatrix} \rho \\ \rho u \\ \rho\left(e + u^2/2\right) \end{pmatrix} + \partial_x \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \left(\rho u^2/2 + \rho e + p\right) u \end{pmatrix} = 0, \tag{1.27}$$

où on a écrit l'énergie totale E comme la somme de l'énergie interne et l'énergie cinétique : $E = e + u^2/2$. Rappelons que la pression p satisfait la loi des gaz parfaits $p = (\gamma - 1)\rho e$. Alors le système (1.27) peut s'écrire sous forme de quasi-linéaire suivante

$$\partial_t \begin{pmatrix} \rho \\ u \\ e \end{pmatrix} + \begin{pmatrix} u & \rho & 0 \\ \frac{(\gamma-1)e}{\rho} & u & \gamma - 1 \\ 0 & (\gamma-1)e & u \end{pmatrix} \partial_x \begin{pmatrix} \rho \\ u \\ e \end{pmatrix} = 0, \tag{1.28}$$

où la matrice admet comme valeurs propres

$$\lambda_1 = u - \sqrt{\gamma(\gamma-1)e}, \quad \lambda_2 = u, \quad \lambda_3 = u + \sqrt{\gamma(\gamma-1)e},$$

avec les vecteurs propres suivants

$$r_1 = \begin{pmatrix} -\rho \\ \sqrt{\gamma(\gamma-1)e} \\ -(\gamma-1)e \end{pmatrix}, \quad r_2 = \begin{pmatrix} (\gamma-1)\rho \\ 0 \\ -(\gamma-1)e \end{pmatrix}, \quad r_3 = \begin{pmatrix} \rho \\ \sqrt{\gamma(\gamma-1)e} \\ (\gamma-1)e \end{pmatrix}.$$

Notons que $d\lambda_j \cdot r_j = (1+\gamma)\sqrt{\gamma(\gamma-1)e}/2$, for $j = 1, 3$, et que $d\lambda_2 \cdot r_2 = 0$. Alors, le premier et le troisième champs caractéristiques sont vraiment non-linéaires, et le deuxième est linéairement dégénéré, voir [35, 58, 55].

Une onde de choc est une fonction de la forme

$$(\rho, u, e)(t,x) = \begin{cases} (\rho_-, u_-, e_-) & \text{si } x < \sigma t, \\ (\rho_+, u_+, e_+) & \text{si } x > \sigma t, \end{cases} \tag{1.29}$$

qui satisfait la condition de Rankine-Hugoniot :

$$-\sigma \begin{bmatrix} \rho \\ \rho u \\ \rho\left(e + u^2/2\right) \end{bmatrix} + \begin{bmatrix} \rho u \\ \rho u^2 + \gamma\rho e, \\ \left(\rho u^2/2 + \rho e + p\right) u \end{bmatrix} = 0, \tag{1.30}$$

où $[\cdot]$ désigne le saut a travers le choc, défini par,

$$[v] = v_+ - v_-\,, \quad v = \rho,\ \rho u, \ \text{ ou } \rho(e + u^2/2),$$

et la condition entropique

$$\lambda_1^+ < \sigma < \min\{\lambda_1^-, \lambda_2^+\} \quad \text{ou bien } \max\{\lambda_2^-, \lambda_3^+\} < \sigma < \lambda_3^-, \tag{1.31}$$

où λ_1^- désigne la valeur de λ_1 évaluée à l'état (ρ_-, u_-, e_-), σ s'appelle la vitesse du choc, et (ρ_-, u_-, e_-) (resp. (ρ_+, u_+, e_+)) l'état de gauche (resp. l'état de droite). Plus précisément, pour $1-$ choc on a

$$\sigma < u_+ < u_-, \tag{1.32}$$

(pour cette inequalité, on peut consulter dans [58, 55].) Donc par (1.30) on obtient

$$[\rho] > 0, \tag{1.33}$$

alors 1-choc est dit compressif. Et 3-choc est dit extensif au sens suivant

$$\sigma > u_+ > u_-, \quad [\rho] < 0. \tag{1.34}$$

Maintenant nous revenons à notre système (1.26) qui couple une équation elliptique et un système hyperbolique, via l'échange d'énergie. Avec l'influence de l'énergie radiative, que devient la structure d'onde du choc ? On fixe quelques notations et on donne le résultat principal.

D'abord on écrit l'équation elliptique sous la forme

$$n(t,x) - \partial_{xx} n(t,x) = \theta(t,x)^4,$$

où on rappelle que n représente l'énergie radiative, et que cette équation décrit les radiations dans un régime stationnaire de diffusion. On introduit l'opérateur K, l'inverse de l'opérateur $1 - \partial_{xx}$. On a

$$n(t,x) = (K\theta^4)(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\,\theta(t,y)^4\,dy\,. \tag{1.35}$$

Définissons une nouvelle quantité q par

$$q(t,x) := -\partial_x n\,(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\,\mathrm{sgn}(x-y)\,\theta(t,y)^4\,dy\,, \tag{1.36}$$

qui est interprétée comme le flux de chaleur radiative. On obtient $-\partial_x q = n - \theta^4$. Alors en utilisant q, les équations fluides se simplifient en

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, u) = 0\,, \\ \partial_t(\rho\, u) + \partial_x(\rho\, u^2 + p) = 0\,, \\ \partial_t(\rho\, E) + \partial_x(\rho\, E\, u + p\, u + q) = 0\,, \end{cases} \tag{1.37}$$

avec l'énergie totale $E = e + u^2/2$ où l'énergie interne e et la pression p satisfont $p = (\gamma - 1)\rho e$. Notons que la différence entre (1.37) et les équations d'Euler usuelles, voir (1.27), est la présence du flux de chaleur radiative q. Rappelons que l'on s'intéresse à l'influence des échanges d'énergie sur la structure des ondes de chocs. Plus précisément, on suppose que $(\rho, u, e)(t, x)$ définie par (1.29) est une onde de choc, solution des équations d'Euler usuelles ($q = 0$ dans (1.37)), c'est-à-dire que $(\rho, u, e)(t, x)$ définie par (1.29) satisfait la condition de Rankine-Hugoniot (1.30) et la condition entropique (1.31). On voudrait savoir s'il existe un profil de choc régulier $(\rho, u, e)(x - \sigma t)$, solution du système (1.37), avec q donnée par (1.36), qui lisse (1.29), solution discontinue des équations d'Euler. En d'autres termes, on va chercher une solution régulière au système (1.37), du type

$$(\rho, u, e)(t, x) = (\rho, u, e)(x - \sigma t),$$

et vérifiant la condition asymptotique

$$\lim_{\xi \to \pm\infty} (\rho, u, e)(\xi) = (\rho_\pm, u_\pm, e_\pm)\,. \tag{1.38}$$

Ce problème présente des analogies avec la régularisation visqueuse des équations de la dynamique des gaz compressibles, telle qu'elle est traitée par D. Gilbarg dans [16]. Concernant le transfert radiatif, une analyse formelle est présentée dans [22] appuyée par des simulations numériques. Un modèle plus simple a été introduit et étudié dans [27, 29, 54]. Ce modèle simple consiste en une équation de type de Burgers, couplée avec une équation linéaire de diffusion,

$$\begin{cases} \partial_t u + \partial_x \Big(\dfrac{u^2}{2}\Big) = -\partial_x q\,, \\ -\partial_{xx} q + q = -\partial_x u\,. \end{cases} \tag{1.39}$$

Notons que ces deux équations peuvent se simplifier en une équation scalaire

$$\partial_t u + \partial_x \Big(\frac{u^2}{2}\Big) = Ku - u\,, \tag{1.40}$$

où K est un opérateur intégral, voir (1.35). Cette équation est analysée dans [50, 33, 41, 56]. On mentionne particulièrement le travail de Kawashima et Nishibata [27], qui ont montré l'existence et la régularité de profils de choc, et analysé leur stabilité asymptotique. Nos travaux sont une première tentative de prolonger les résultats connus pour (1.40) au modèle plus physique (1.37). Nos résultats s'énoncent ainsi :

Théorème 1.2. *Supposons que la constante adiabatique γ satisfait*

$$1 < \gamma < \frac{\sqrt{7} + 1}{\sqrt{7} - 1} \simeq 2.215\,.$$

Soit l'état de gauche (ρ_-, u_-, e_-) fixé. Alors, il existe une constante positive δ (dépendant de ρ_-, u_-, e_- et γ), telle que pour tout état de droite (ρ_+, u_+, e_+), qui satisfait

- $\|(\rho_+, u_+, e_+) - (\rho_-, u_-, e_-)\| \le \delta$,
- *la fonction* (1.29) *est une onde de choc, avec la vitesse* σ, *pour les équations classiques d'Euler,*

il existe une onde progressive ("traveling wave" en anglais), $(\rho, u, e)(x - \sigma t)$, *de classe* C^2, *solution du système* (1.37), *avec* q *donnée par* (1.36) *et la condition asymptotique* (1.38). *De plus, l'onde progressive satisfait*

$$\left|\frac{\mathrm{d}}{\mathrm{d}\xi}(\rho, u, e)(\xi)\right| \le C|u_+ - u_-|^2,$$

pour une certaine constante $C > 0$.

Théorème 1.3. *Supposons que la constante adiabatique* γ *satisfait*

$$1 < \gamma < \frac{\sqrt{7}+1}{\sqrt{7}-1} \simeq 2.215\,.$$

Soit l'état de gauche (ρ_-, u_-, e_-) *fixé. Alors, il existe une suite décroissante de constantes positives* $(\delta_n)_n$ *(dépendent de* ρ_-, u_-, e_- *et* γ*), telles que pour tout état de droite* (ρ_+, u_+, e_+), *qui satisfait*

- $\|(\rho_+, u_+, e_+) - (\rho_-, u_-, e_-)\| \le \delta_n$,
- *la fonction* (1.29) *est une onde de choc, avec la vitesse* σ, *pour les équations classiques d'Euler,*

il existe une onde progressive $(\rho, u, e)(x - \sigma t)$, *de classe* C^{n+2}, *solution du système* (1.37), *avec* q *donnée par* (1.36) *et la condition asymptotique* (1.38). *De plus, l'onde progressive satisfait*

$$\left|\frac{\mathrm{d}^l}{\mathrm{d}\xi^l}(\rho, u, e)(\xi)\right| \le C|u_+ - u_-|^{l+1}.$$

où $l = 1, 2, \cdots, n+1$, *et* C *est une certaine constante.*

Remarque 1.4. *La restriction sur* γ *est peut-être superflue, mais elle simplifie la démonstration et elle couvre les cas physiquement intéressants* $1 < \gamma \le 2$.

Remarque 1.5. *Les petits paramètres* δ_n *sont explicites pour le modèle scalaire* (1.39), *voir [27]. Mais ici, pour le système* (1.37), *ils dépendent de l'état de gauche* ρ_-, e_-, u_-, *et de la constant* γ.

En effet l'existence du profil de choc est construit en "collant" deux orbites hétéroclines d'un système auxiliaire. Les régularités sont obtenues en étudiant au point où on colle les deux orbites hétéroclines.

Plus précisément, on écrit le système (1.37) en une seule variable $\xi = x - \sigma t$. Tout d'abord on écrit le flux de chaleur radiative q en cette variable par

$$q(x - \sigma t) = \frac{1}{2}\int_{\mathbb{R}} \mathrm{e}^{-|x-\sigma t-y|}\,\mathrm{sgn}(x - \sigma t - y)\,\theta(y)^4\,dy\,. \tag{1.41}$$

Par la condition asymptotique (1.38), $\theta = \dfrac{\gamma-1}{R}e$ et le théorème de Lebesgue, on déduit que q tend vers zéro quand ξ tend vers $\pm\infty$. Puis on introduit une nouvelle variable $v = u - \sigma$, qui peut être interprétée comme la vitesse dans le repère du choc. Alors le système (1.37) se réduit au système

$$\begin{cases} (\rho\, v)' = 0\,, \\ (\rho\, v^2 + (\gamma - 1)\, \rho\, e)' = 0\,, \\ (\rho\, v\, (e + \frac{v^2}{2}) + (\gamma - 1)\, \rho\, v\, e + q)' = 0\,. \end{cases} \tag{1.42}$$

Notons que on écrit l'énergie specifique E sous la somme $e + u^2/2$, et la pression $p = (\gamma - 1)\rho e$. Puis on intègre (1.42) sur l'ensemble $(-\infty, \xi]$ et on obtient

$$\begin{cases} (\rho\, v)(\xi) = j\,, \\ (\rho\, v^2 + (\gamma - 1)\, \rho\, e)(\xi) = j\, C_1\,, \\ (\rho\, v\, (e + \frac{v^2}{2}) + (\gamma - 1)\, \rho\, v\, e + q)(\xi) = j\, C_2\,, \end{cases} \tag{1.43}$$

où j, C_1, C_2 sont constantes d'intégration définies par l'état de gauche comme

$$\begin{cases} j = \rho_-\, v_-\,, \\ j\, C_1 = \rho_-\, v_-^2 + (\gamma - 1)\, \rho_-\, e_-\,, \\ j\, C_2 = \rho_-\, v_-\, (e_- + \frac{v_-^2}{2}) + (\gamma - 1)\, \rho_-\, v_-\, e_-\,, \end{cases}$$

En utilisant ces constantes d'intégration, on obtient les relations suivantes :

$$\rho(\xi) = \frac{j}{v(\xi)}\,, \quad e(\xi) = \frac{(C_1 - v(\xi))\, v(\xi)}{\gamma - 1}\,. \tag{1.44}$$

On insère ces relations dans la troisième équation de (1.43) et on obtient l'équation suivante :

$$v(\xi)^2 - \frac{2\,\gamma\, C_1}{\gamma + 1}\, v(\xi) + \frac{2\,(\gamma - 1)\, C_2}{\gamma + 1} = \frac{2\,(\gamma - 1)}{j\,(\gamma + 1)}\, q(\xi)\,. \tag{1.45}$$

Notons que $q(\xi)$ est donnée par (1.41) en terme de la température θ, qui peut s'écrire par v comme :

$$\theta(\xi) = \frac{(\gamma - 1)\, e(\xi)}{R} = \frac{(C_1 - v(\xi))\, v(\xi)}{R}\,. \tag{1.46}$$

Alors (1.45) se réduit en une fonction d'intégration par

$$\begin{aligned} &v(\xi)^2 - \frac{2\,\gamma\, C_1}{\gamma + 1}\, v(\xi) + \frac{2\,(\gamma - 1)\, C_2}{\gamma + 1} \\ &\quad = \frac{(\gamma - 1)}{j\,(\gamma + 1)\, R^4} \int_{\mathbb{R}} \mathrm{e}^{-|\xi - y|}\, \mathrm{sgn}(\xi - y)\, v(y)^4 (C_1 - v(y))^4\, dy\,. \end{aligned} \tag{1.47}$$

Alors nous voulons chercher une solution v, de classe C^2 de (1.47), telle que $v(\xi)$ tends vers $v_\pm$ quand ξ tends vers $\pm\infty$. En revanche s'il existe une telle fonction v de classe C^2, on peut obtenir le profil de choc $\rho(\xi)$ et $e(\xi)$ (ou $\theta(\xi)$) par les relations (1.44) (1.46).

Notons que (1.47) est équivalente à l'équation suivante

$$(v(\xi) - v_-)(v(\xi) - v_+) = \frac{(\gamma - 1)}{j\,(\gamma + 1)\, R^4} \int_{\mathbb{R}} \mathrm{e}^{-|\xi - y|}\, \mathrm{sgn}(\xi - y)\, v(y)^4\, (C_1 - v(y))^4\, dy\,,$$

car le flux de chaleur radiative s'annule en $\pm\infty$.

Comme nous voulons chercher une solution v de classe C^2, on différencie (1.47) deux fois, on obtient l'équation suivante

$$\begin{aligned}&(v-\frac{\gamma C_1}{\gamma+1})\,v''+(v')^2-\frac{4\,(\gamma-1)}{j\,(\gamma+1)\,R^4}\,(C_1-v)^3\,v^3\,(C_1-2v)\,v'\\&-\frac{1}{2}\,(v-v_-)\,(v-v_+)=0\,.\end{aligned}\tag{1.48}$$

Notons que (1.48) est une équation différentielle d'ordre 2, qui est plus facile à étudier que (1.47), une équation intégrale. De plus on peut écrire (1.48) sous la forme d'un système différentiel d'ordre 1. Pour cela, une nouvelle variable intervient $\hat{v} := v - \frac{v_- + v_+}{2}$. On a le système différentiel d'ordre 1 comme :

$$\left\{\begin{array}{lcl}\hat{v}' & = & w\,,\\ \hat{v}\,w' & = & -w^2-f(\hat{v})\,w+\dfrac{\hat{v}^2-a^2}{2}\,,\end{array}\right.\tag{1.49}$$

où $f(\hat{v})$ est un polynôme d'ordre 7, et $a = \frac{|v_- - v_+|}{2}$ caractérise la taille du choc.

La condition asymptotique du système (1.49) est donnée par

$$\lim_{\xi\to\pm\infty}(\hat{v},w)=(\mp a,0),\ \text{1-choc},\ \lim_{\xi\to\pm\infty}(\hat{v},w)=(\pm a,0),\ \text{3-choc},$$

où on a utilisé les propriétés du 1-choc et 3-choc, voir (1.32)-(1.34). Notons que $(\pm a, 0)$, sont des points stationnaires du système (1.49). Alors l'existence du profil de choc est équivalente à l'existence d'orbite hétérocline qui lie ces points stationnaires. S'il existe une telle orbite, alors $\hat{v}$ doit s'annuler au moins en un point. Mais le système (1.49) est singulier en $\hat{v} = 0$.

Donc il est pratique de travailler avec un système auxiliaire sans singularité. Le système auxiliaire est donné par

$$\left\{\begin{array}{lcl}V' & = & V\,W\,,\\ W' & = & -W^2-f(V)\,W+\dfrac{(V^2-a^2)}{2}\,.\end{array}\right.\tag{1.50}$$

Comme nous venons de le voir, le profil de choc est construit par deux orbites hétéhoclines du système (1.50). Dans le chapitre 3, on prouvera que (1.50) a un point stationnaire $(0, w_0)$ sur l'axe $V = 0$, et qu'il existe deux orbites hétéhoclines reliant $(a, 0)$ et $(0, w_0)$, $(-a, 0)$ et $(0, w_0)$, respectivement.

Ces sont les idées des preuves des notre résultat. Les détails sont donnés dans le chapitre 3. On remarque que pour le modèle simple (1.39), Kawashima et Nishibata résolvent le système

$$\left\{\begin{array}{lcl}\hat{v}' & = & w\,,\\ \hat{v}\,w' & = & -w^2-\ w+\dfrac{\hat{v}^2-a^2}{2}\,,\end{array}\right.$$

qui est plus simple (mais aussi très proche) que (1.49). En particulier, ici, $f(V)$ n'est pas une constante, mais un polynôme d'ordre 7, de plus, $f(0)$ dépendent de choc par les constantes d'intégrations j, C_1, C_2. C'est la première restriction nonexplicite du choc de déduire l'existence du profil de choc.

1.3.3 Stabilité asymptotique des profils de choc

Le troisième résultat dans ce travail est la discussion de la stabilité du profil de choc du système (1.26). Dans la section 1.3.2, on a étudié l'existence et la régularité du profil de choc, voir Théorème 1.2 et Théorème 1.3, en coordonnées euleriennes. Ici on traite la stabilité du profil de choc en coordonnées lagrangiennes. Les variables conservées sont la densité ρ, l'impulsion ρu et l'énergie totale $\rho\ E$, avec $E = e + u^2/2$, en coordonnées euleriennes, voir (1.26). Mais en coordonnées lagrangiennes, on a une forme plus simple des variables conservées : le volume spécifique $1/\rho$, la vitesse u et l'énergie totale spécifique $E = e + u^2/2$. Alors, on écrit le système (1.26) en coordonnées lagrangiennes. Soit $s \in \mathbb{R}^+$ et $y \in \mathbb{R}$ les variables du temps et de l'espace en coordonnées euleriennes. On écrit le système (1.26) avec q la variable de flux de chaleur radiative, comme

$$\begin{cases} \partial_s\rho + \partial_y(\rho\, u) = 0\,, \\ \partial_s(\rho\, u) + \partial_y(\rho\, u^2 + p) = 0\,, \\ \partial_s(\rho\, E) + \partial_y(\rho\, E\, u + p\, u + q) = 0\,, \\ -\partial_{yy}q + q + \left(\theta^4\right)_y = 0.\,, \end{cases} \tag{1.51}$$

De l'équation de la conservation de la masse

$$\partial_s\rho + \partial_y(\rho u) = 0,$$

on déduit que $\rho\ \mathrm{d}y\ -\ \rho u\ \mathrm{d}s$ est fermée, donc exacte. On introduit une variable x définie par

$$\mathrm{d}x \ = \ \rho\ \mathrm{d}y\ -\ \rho u\ \mathrm{d}s,$$

qui s'appelle coordonnée lagrangienne de masse. On note par $v = 1/\rho$ le volume spécifique et par t la variable du temps en coordonnées lagrangiennes. On obtient la relation suivante entre les coordonnées lagrangiennes et les coordonnées euleriennes :

$$\begin{aligned} \mathrm{d}y &= v\ \mathrm{d}x\ +\ u\ \mathrm{d}t, \\ \mathrm{d}s &= \mathrm{d}t. \end{aligned}$$

On considère des équations en coordonnées euleriennes de la forme

$$\partial_s\varphi\ +\ \partial_y\psi\ =\ 0, \tag{1.52}$$

qui est équivalente à la forme suivante

$$\mathrm{d}\,(\varphi\ \mathrm{d}y\ -\ \psi\ \mathrm{d}s)\ =\ 0.$$

En utilisant la relation entre les coordonnées lagrangiennes et les coordonnées euleriennes, on obtient

$$\mathrm{d}\,(\varphi v\ \mathrm{d}x\ +\ (\varphi u\ -\ \psi)\,\mathrm{d}t)\ =\ 0.$$

Donc, on obtient cette équation écrite en coordonnées lagrangiennes

$$\partial_t\,(\varphi v) - \partial_x\,(\varphi u - \psi) = 0.$$

Pour le système (1.51), on choisit des fonctions particulières φ et ψ, et on obtient les trois premieres équations du système (1.51) en coordonnées lagrangiennes,

$$\begin{cases} \partial_t v - \partial_x u = 0\,, \\ \partial_t u + \partial_x p = 0\,, \\ \partial_t E + \partial_x\,(p+q) = 0\,, \end{cases} \tag{1.53}$$

couplées avec l'équation elliptique via l'échange de l'énergie

$$-\partial_x \left(\frac{\partial_x q}{v} \right) + vq + \partial_x(\theta^4) = 0. \tag{1.54}$$

Notons que l'énergie radiative n et le flux de chaleur radiative q sont reliés par

$$q = -\frac{\partial_x n}{v},$$

ce n'est qu'une dérivation en coordonnées lagrangiennes. Rappellons que la pression p satisfait $p = \frac{R\theta}{v}$, $e = \frac{R\theta}{\gamma - 1} = C_v\theta$, par la loi des gaz parfaits, et que $E = e + u^2/2 = C_v\theta + u^2/2$. On ferme ainsi le système couplé (1.53) et (1.54).

Maintenant on va préciser le problème mathématique auquel on s'intéresse. On suppose que $(V, U, \Theta, Q)(x - \sigma t)$ est une onde progressive régulière, disons au moins C^4, du système (1.53) et (1.54), qui satisfait la condition asymptotique

$$\lim_{\xi \to \pm\infty} (V, U, \Theta, Q)(\xi) = (v_\pm, u_\pm, \theta_\pm, 0), \tag{1.55}$$

où $(v_\pm, u_\pm, \theta_\pm)$ définissent une onde de choc, avec la vitesse σ, solution des équations d'Euler ((1.53) avec $q \equiv 0$). L'existence d'une telle solution onde progressive est assurée par le Théorème 1.3. On se pose la question suivante : est-ce qu'il existe des solutions globales du problème de Cauchy pour les équations (1.53) et (1.54), avec les données initiales

$$(v, u, \theta)(0, x) = (v_0, u_0, \theta_0)(x), \tag{1.56}$$

dans un voisinage du profil de choc $(V, U, \Theta, Q)(x - \sigma t)$?

Encore une fois, de tels problèmes de stabilité asymptotique des profils de chocs ont été bien étudiés pour des systèmes hyperboliques avec régularisation visqueuse : voir par exemple, [23, 25] pour le cas scalaire, [21] pour des systèmes généraux et [25, 47] pour les équations de Navier-Stokes compressibles. Ici on s'inspire particulièrement des résultats de [25, 47].

Matsumura et Nishihara, dans [47], ont montré la stabilité asymptotique des profils de choc du p−système avec la régularisation visqueuse. Ils ont supposé que

- les perturbations initiales du volume spécifique v et la vitesse u sont de masse nulle,
- la taille du choc, $|v_+ - v_-|$, est suffisamment petite,
- les primitives des perturbations initiales sont suffisamment petites dans l'espace $H^2(\mathbb{R})$.

Sous ces hypothèses, une méthode d'énergie est appliquée au système intégré, complété par les données initiales qui sont les primitives des perturbations initiales. L'existence globale de solutions du système intégré est établie dans [47].

Dans [25], Kawashima et Matsumura ont généralisé les résultats de [47] au système complet de la dynamique des gaz en traitant deux difficultés. La première difficulté, par rapport au système général traité dans [21], est que le système n'est pas parabolique, puisqu'il s'agit d'un système entre une équation hyperbolique et deux équations paraboliques. Cette difficulté est traitée par la technique développée dans [28, 46]. La deuxième difficulté provient du fait que la variable conservée dans l'équation de la conservation de l'énergie est $C_v\theta + u^2/2$, alors qu'on a une régularisation visqueuse sur la température θ.

Ici nous adaptons la méthode de Kawashima et Matsumura pour traiter la stabilité asymptotique du profil de choc du système (1.53) et (1.54), qui couple un système hyperbolique et une équation elliptique. En utilisant les idées de [25], on introduit une nouvelle variable pour la perturbation de la variable conservée de l'énergie, par l'approximation de la perturbation de la température. On développe cette méthode pour la perturbation de la quantité q. La stratégie de [25] ne peut être suivie directement, la linéarisation des équations intégrées autour du profil ne permettant pas un contrôle des termes d'erreur par de simples estimations d'énergie. Néanmoins, une symétrisation classique des équations quasi-linéaires permet de traiter la difficulté dans le cas considéré ici. Par ailleurs, la dissipation introduite par le terme d'échange d'énergie est plus faible que celle introduite par la viscosité. Ceci conduit à considérer des perturbations dont les intégrales sont dans l'espace $H^3(\mathbb{R})$, au lieu de $H^2(\mathbb{R})$ dans [25, 21]. Enfin, le contrôle de la variable q, qui satisfait l'équation elliptique est une difficulté supplémentaire. Pour cette difficulté, on utilise la méthode introduite dans [26], qui conduit à des inégalités entre la variable q et la température θ, conséquence de l'équation elliptique. On établit ainsi des estimations $L_t^\infty(W_x^{1,\infty})$ de la solution, au lieu de $L_t^2(W_x^{1,\infty})$ dans [25, 21].

Notre résultat est le théorème suivant

Théorème 1.6. *Soit (v_-, u_-, θ_-) fixé, et (v_+, u_+, θ_+) définissant une onde de choc avec (v_-, u_-, θ_-), et la vitesse de choc σ. On note $(V, U, \Theta, Q)(\xi)$, $\xi = x - \sigma t$, une onde progressive, solution du système* (1.53)-(1.54), *qui satisfait la condition asymptotique* (1.55). *Supposons que les données initiales v_0, u_0, et θ_0 satisfont*

- $(v_0 - V, u_0 - U, \theta_0 - \Theta) \in H^2(\mathbb{R})$;
- *il existe des fonctions $\Phi_0, \Psi_0, \tilde{W}_0 \in H^3(\mathbb{R})$, telles que*

$$\Phi_0' = v_0 - V,\ \Psi_0' = u_0 - U,\ (\tilde{W}_0)' = -\left(C_v\theta_0 + \frac{u_0^2}{2}\right) - \left(C_v\Theta + \frac{U^2}{2}\right),$$

et $\|(\Phi_0, \Psi_0, \tilde{W}_0)\|_{H^3(\mathbb{R})}$ est suffisamment petit.

On suppose de plus que le taille d'onde de choc, $|v_+ - v_-|$, est suffisamment petite. Alors, il existe une unique solution globale $(v, u, \theta, q)(t, x)$ telle que

$$\begin{aligned} (v - V, u - U, \theta - \Theta) &\in C^0(0, \infty; H^2(\mathbb{R})), \\ q - Q &\in C^0(0, \infty; H^3(\mathbb{R})), \end{aligned}$$

et cette solution converge vers le profil de choc au sens suivant

$$\sup_{x \in \mathbb{R}} |(v, u, \theta, q)(t, x) - (V, U, \Theta, Q)(x - \sigma t)| \to 0 \quad \textit{lorsque} \quad t \to +\infty.$$

Plus précisément, pour étudier la stabilité asymptotique du profil de choc, il est naturel de définir les variables de perturbations, $(\phi,\ \psi,\ w,\ z)(t, \xi)$, des solutions $(v,\ u,\ \theta,\ q)(t, x)$ autour des profils de choc $(V,\ U,\ \Theta,\ Q)(\xi)$, où $\xi = x - st$,

$$(v, u, \theta, q)(t, x) = (V, U, \Theta, Q)(\xi) + (\phi, \psi, w, z)(t, \xi). \tag{1.57}$$

En introduisant $(\phi,\ \psi,\ \theta,\ z)(t, \xi)$ dans système (1.53)-(1.54) avec la condition initiale (1.56), on obtient le système satisfait par les variables de perturbation

$$\begin{cases} \phi_t - s\phi_\xi - \psi_\xi = 0, \\ \psi_t - s\psi_\xi + \left(R\dfrac{\Theta + w}{V+\phi} - R\dfrac{\Theta}{V}\right)_\xi = 0, \\ \left(C_v w + U\psi + \dfrac{\psi^2}{2}\right)_t - s\left(C_v w + U\psi + \dfrac{\psi^2}{2}\right)_\xi \\ \qquad + R\left(\dfrac{\Theta + w}{V+\phi}(U+\psi) - \dfrac{\Theta}{V}U\right)_\xi + z_\xi = 0, \\ \dfrac{1}{V}\left(\dfrac{Q'}{V} - \Theta^4\right)' - \dfrac{1}{V+\phi}\left(\dfrac{Q'+z_\xi}{V+\phi}\right)_\xi + z + \dfrac{((\Theta+w)^4)_\xi}{V+\phi} = 0, \end{cases} \tag{1.58}$$

avec la donnée initiale

$$(\phi, \psi, w)(0,\xi) = (\phi_0, \psi_0, w_0)(\xi) \equiv (v_0 - V, u_0 - U, \theta_0 - \Theta)(\xi). \tag{1.59}$$

Notons que le prime désigne la dérivation en ξ des fonctions d'une seule variable ξ. Alors la stabilité du profil de choc est équivalent à

- l'existence globale en temps d'une solution $(\phi,\, \psi,\, w,\, z)(t,\xi)$,
- la solution $(\phi,\, \psi,\, w,\, z)(t,\xi)$ satisfait

$$\sup_{\xi\in\mathbb{R}} |(\phi,\, \psi,\, w,\, z)(t,\xi)| \to 0 \quad \text{lorsque} \quad t \to +\infty.$$

Notons que (1.58) est un système couplé un système hyperbolique (les trois premières équations) et une équation elliptique (la quatrième équation). Pour montrer l'existence globale en temps de solution du système (1.58), il faut travailler sur le système intégré, parce que la monotonie du profil de choc donne une propriété de dissipation pour le système intégré. Cet avantage nous permet d'obtenir les estimations a priori. Donc nous introduisons les variables intégrées Ψ, Ψ, W, Z par

$$\begin{cases} \Phi_\xi = \phi, \\ \Psi_\xi = \psi, \\ W_\xi = w - \dfrac{1}{C_v}\left(U'\Psi - \dfrac{\psi^2}{2}\right), \\ Z_\xi = z(V+\phi) + \left(\dfrac{Q'}{V} - \Theta^4\right)' \dfrac{\phi}{V}. \end{cases} \tag{1.60}$$

Les définitions de Φ et Ψ sont naturelles. Mais la définition de W tient compte du fait que la troisième variable conservée dans l'équation de la conservation de l'énergie n'est pas $C_v\theta$, mais $C_v\theta + u^2/2$. De plus on a considéré la non linéarité de l'équation elliptique dans la définition de Z. En utilisant ces variables intégrées et en intégrant système (1.58) sur l'ensemble $(-\infty, \xi]$, on obtient le système intégré suivant

$$\begin{cases} \Phi_t - s\Phi_\xi - \Psi_\xi = 0, \\ \Psi_t - s\Psi_\xi + R\left(\dfrac{\Theta + w}{V+\phi} - \dfrac{\Theta}{V}\right) = 0, \\ (C_v W + U\Psi)_t - s\left(C_v W + U\Psi\right)_\xi + R\dfrac{\Theta + w}{V+\phi}(U+\psi) \\ \qquad -R\dfrac{\Theta}{V}U + \dfrac{1}{V+\phi}\left(Z_\xi - \left(\dfrac{Q'}{V} - \Theta^4\right)'\dfrac{\phi}{V}\right) = 0, \\ \dfrac{Q'\phi}{V(V+\phi)} - \dfrac{1}{V+\phi}\left(\dfrac{1}{V+\phi}\left(Z_\xi - \left(\dfrac{Q'}{V} - \Theta^4\right)'\dfrac{\phi}{V}\right)\right)_\xi \\ \qquad +Z + (\Theta + w)^4 - \Theta^4 = 0, \end{cases} \tag{1.61}$$

avec la donnée initiale

$$(\Phi, \Psi, W)(0, \xi) = (\Phi_0, \Psi_0, W_0)(\xi)\,, \tag{1.62}$$

où Φ_0, Ψ_0, W_0 sont définies par ϕ_0, ψ_0, w_0 et le profil de choc V, U, W, Q. Alors le problème est équivalent à montrer l'existence globale en temps du système (1.61) avec la condition initiale (1.62). La preuve de l'existence globale consiste en les trois étapes suivantes :

- On utilise la symétrisation classique des équations quasi-linéaire et obtient les estimations $L^\infty(L^2)$ de Φ (ou ϕ), Ψ (ou ψ) et W (ou w) et leurs dérivées, et les estimations $L^2(L^2)$ de Z (ou w) et leurs dérivées. Notons que la linéarisations des équations autour du profil de choc, utilisée dans [25], n'est plus valable ici.
- On linéarise les équations autour du profil de choc, comme dans [25], et déduit les estimations $L^2(L^2)$ de ϕ, ψ et leurs dérivées.
- On déduit les estimations elliptiques par l'équation elliptique. Ceci nous permet de contrôler la perturbation z par la perturbation w, et réciproquement.

Les détails, par exemple, la convergence de ϕ, ψ, w, z, sont donnés dans le chapitre 4.

1.4 Modèle NON-LTE *line radiative transfer*

1.4.1 Equation de transport en transfert radiatif

Le dernier modèle abordé dans ce travail vise à décrire des interactions entre les états internes atomiques de la matière et les radiations. Ce travail a été réalisé en collaboration avec L. Desvillettes. Les radiations sont décrites par son intensité specifique $f(t, x, v, \nu)$, fonction du temps $t \in \mathbb{R}^+$, de la position $x \in \mathbb{R}^3$, de la direction de déplacement des photons $v \in \mathbb{S}^2$, et de la fréquence $\nu \in \mathbb{R}^+$. Cette quantité satisfait l'équation de transport suivante

$$\frac{1}{c}\partial_t f + v \cdot \nabla_x f = \eta - \chi f, \tag{1.63}$$

où c est la vitesse de la lumière. Le photon se déplace à la vitesse de la lumière dans la direction v, η est le coefficient d'émissivité des photons par la matière et χ est le coefficient d'extinction, qui caractérise l'absorption des photons par la matière. Si η et χ sont donnés, alors l'équation (1.63) est linéaire en f et on obtient l'expression de f en intégrant sur les caractéristiques. Mais en réalité, ces coefficients dépendent des états internes de la matière. Pour trouver cette dépendance, on étudie les interactions entre les radiations et la matière. On suppose dans la suite que la matière est statique, la vitesse du milieu est nulle. Les discussions sont menées dans un seul repère, le repère fixe.

Notons que η et χ sont le coefficient (total) d'émissivité et le coefficient (total) d'absorption des photons, qui comprennent les vrais processus thermiques et aussi les processus de la dispersion (scattering en anglais). Ils peuvent être écrits comme la somme d'une partie thermique et d'une partie de dispersion, voir [49] §72. Mais en réalité, il est difficile de distinguer un processus particulier, la plupart des processus est un mélange parce que les processus radiatifs et de collision fonctionnent simultanément. Comme indiqué dans [49], p.328, la véritable physique de la situation émerge seulement quand l'équation de transfert est couplée directement aux équations de l'équilibre statistique, qui décrivent explicitement comment des niveaux atomiques sont peuplés et dépeuplés.

1.4.2 Coefficients d'Einstein, transitions lié-lié

Pour étudier les interactions entre *line radiative transfer* et la matière, on se donne deux états d'énergie stationnaires : un état fondamental d'indice 1 avec l'énergie E_1, le poids statistique g_1, et un état excité d'indice 2 avec l'énergie E_2 ($E_2 > E_1$), le poids statistique g_2 . La transition énergétique ne pourra s'effectuer que si la quantité d'énergie fournie est au minimum égale à $\Delta E = E_2 - E_1$, la différence d'énergie entre les deux états. Il y a trois processus qui peuvent se produire lorsqu'un système de deux états est soumis à un rayonnement de fréquence ν, correspondant à la différence d'énergie ΔE, c'est-à-dire que $h\nu = \Delta E$. Ces trois processus sont :

- Absorption (induite) : On définit B_{12} le coefficient d'Einstein d'absorption induite tel que la probabilité d'excitation du niveau E_1 vers le niveau E_2 par seconde soit $B_{12}\rho$, et que l'énergie absorbée par chaque unité de volume soit

  $$\dot{E}_{absorp} = n_1 B_{12} f \phi(\nu) h\nu \mathrm{d}v \mathrm{d}\nu\,,$$

  où n_1 est la fonction de densité d'atomes au niveau E_1, $\rho(t,x)$ est l'intensité moyenne intégrée sur le profil de ligne

  $$\rho(t,x) = \int_{\mathbb{R}^+}\int_{\mathbb{S}^2} f(t,x,v,\nu)\phi(\nu)\mathrm{d}v\mathrm{d}\nu\,,$$

  et la fonction de profil $\phi(\nu)$ est décrite par exemple par le profil de Voigt, la convolution du profil de Lorentz et du profil Gaussien.
- Émission induite : On définit B_{21} le coefficient d'Einstein d'émission induite tel que la probabilité de désexcitations du niveau E_2 vers le niveau E_1 par seconde soit $B_{12}\rho$, et que l'énergie absorbée par chaque unité de volume soit

  $$\dot{E}_{induite} = n_2 B_{21} f \phi(\nu) h\nu \mathrm{d}v \mathrm{d}\nu.$$

- Émission spontanée : On définit A_{21} le coefficient d'Einstein d'émission spontanée tel que la probabilité de désexcitations du niveau E_2 vers le niveau E_1 par seconde soit A_{12}, et que l'énergie absorbée par chaque unité de volume soit

  $$\dot{E}_{spont} = n_2 A_{21} \phi(\nu) h\nu \mathrm{d}v \mathrm{d}\nu.$$

A ces trois transitions radiatives, s'ajoutent des transitions lié-lié :

- Excitation de collision : On définit C_{12} le coefficient d'Einstein d'excitation de collision qui est la probabilité de transition de collision du niveau E_1 vers le niveau E_2,

- Désexcitation de collision : On définit C_{21} le coefficient d'Einstein de la désexcitation de collision qui est la probabilité de transition de collision du niveau E_2 vers le niveau E_1.

Notons que le coefficient de désexcitation de collision, C_{21}, est souvent défini par $C_{21} = N_{col}K_{21}$, où N_{col} est la densité des particules associés de collision, et K_{21} est le taux de collision, voir [12, 60].

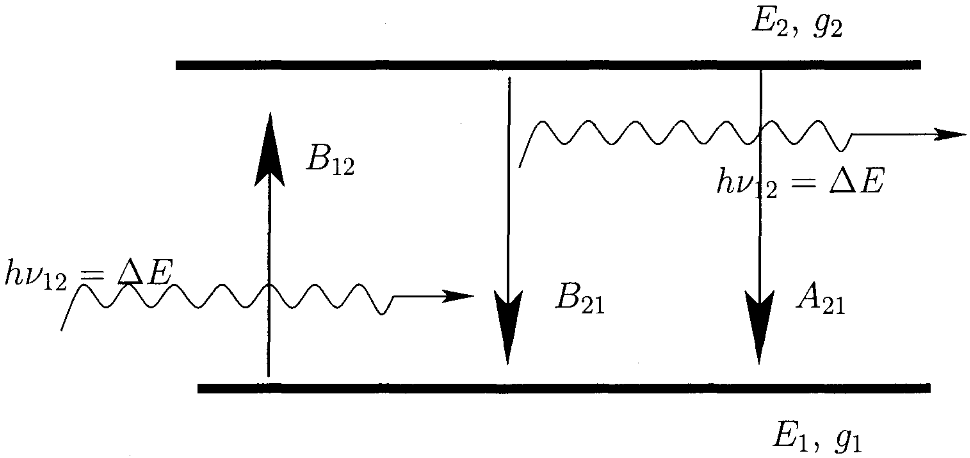

FIG. 1.2 – Trois processus radiatifs

A partir du bilan radiatif et de collisions et de la loi de Plank d'équilibre thermodynamique, on obtient la relation suivante entre les coefficients d'Einstein :

Lemme 1.7. *A_{21}, B_{12}, B_{21}, C_{12}, et C_{21} satisfont*

$$\frac{B_{12}}{B_{21}} = \frac{g_2}{g_1}, \quad \frac{A_{21}}{B_{21}} = \frac{2h\nu_{12}^3}{c^2},$$
$$\frac{C_{21}}{C_{12}} = \frac{g_1}{g_2} e^{(h\nu_{12})/(k\theta)},$$

où $h\nu_{12}$ est l'énergie qui sépare les niveaux E_1 et E_2, k est la constante de Boltzmann, g_1 et g_2 sont poids statistiques, et θ est la température cinétique.

1.4.3 Couplage entre les radiations et les états atomiques

En utilisant les coefficients d'Einstein et les densités d'atomes au niveau E_1 et E_2, on a les expressions du coefficient d'absorption η et du coefficient d'extinction χ comme

$$\begin{aligned} \eta &= n_2 A_{21} h\nu\phi(\nu), \\ \chi &= (n_1 B_{12} - n_2 B_{21})\, h\nu\phi(\nu), \end{aligned}$$

et les équations d'évolution des densités d'atomes

$$\begin{cases} \partial_t n_1 &= n_2 A_{21} - (n_1 B_{12} - n_2 B_{21})\rho + (n_2 C_{21} - n_1 B_{12}), \\ \partial_t n_2 &= -n_2 A_{21} + (n_1 B_{12} - n_2 B_{21})\rho - (n_2 C_{21} - n_1 B_{12}). \end{cases}$$

Ainsi on obtient un système couplant les radiations avec les densités d'atomes sur les niveaux d'énergies

$$\begin{cases} \dfrac{1}{c}\partial_t f + v \cdot \nabla_x f = (n_2 A_{21} - (n_1 B_{12} - n_2 B_{21})\, f)\, h\nu\phi(\nu), \\ \partial_t n_1 = n_2 A_{21} - (n_1 B_{12} - n_2 B_{21})\rho + (n_2 C_{21} - n_1 B_{12}), \\ \partial_t n_2 = -n_2 A_{21} + (n_1 B_{12} - n_2 B_{21})\rho - (n_2 C_{21} - n_1 B_{12}). \end{cases} \tag{1.64}$$

Le système (1.64) converge rapidement vers l'équilibre statistique, c'est-à-dire qu'après un intervalle de temps très court, les transitions radiatives et les transitions de collision se compensent. A l'équilibre, on est conduit au système couplé

$$\begin{cases} \dfrac{1}{c}\partial_t f + v \cdot \nabla_x f = (n_2 A_{21} - (n_1 B_{12} - n_2 B_{21})\, f)\, h\nu\phi(\nu), \\ \partial_t n_1 \;=\; 0 \;=\; n_2 A_{21} - (n_1 B_{12} - n_2 B_{21})\rho + (n_2 C_{21} - n_1 B_{12}), \\ \partial_t n_2 \;=\; 0 \;=-n_2 A_{21} + (n_1 B_{12} - n_2 B_{21})\rho - (n_2 C_{21} - n_1 B_{12}). \end{cases} \tag{1.65}$$

On définit un paramètre ϵ qui va caractériser ce régime asymptotique et dans la suite on étudie le système

$$\begin{cases} \dfrac{1}{c}\partial_t f^\epsilon + v \cdot \nabla_x f^\epsilon = (n_2^\epsilon A_{21} - (n_1^\epsilon B_{12} - n_2^\epsilon B_{21})\, f^\epsilon)\, h\nu\phi(\nu), \\ \epsilon\partial_t n_1^\epsilon \;=\; n_2^\epsilon A_{21} - (n_1^\epsilon B_{12} - n_2^\epsilon B_{21})\rho^\epsilon + (n_2^\epsilon C_{21} - n_1^\epsilon B_{12}), \\ \epsilon\partial_t n_2^\epsilon \;=-n_2^\epsilon A_{21} + (n_1^\epsilon B_{12} - n_2^\epsilon B_{21})\rho^\epsilon - (n_2^\epsilon C_{21} - n_1^\epsilon B_{12}). \end{cases} \tag{1.66}$$

On considère ce système dans le domaine $x \in X$, borné dans $\mathbb{R}^3$, alors l'équation cinétique est équipée de la condition initiale

$$f^\epsilon(0, x, v, \nu) = f_0(x, v, \nu) \geq 0, \tag{1.67}$$

et la condition au bord

$$f^\epsilon(t, x, v, \nu)|_{\mathbb{R}^+ \times (\partial X \times \mathbb{S}^2)_- \times \mathbb{R}^+} = g(t, x, v, \nu) \geq 0, \tag{1.68}$$

où $(\partial X \times \mathbb{S}^2)_-$ est défini par

$$(\partial X \times \mathbb{S}^2)_- = \{(x, v) \in \partial X \times \mathbb{S}^2 : v \cdot \Gamma_x < 0\},$$

avec Γ_x la normale extérieure au point $x \in \partial X$. Les densités initiales sont

$$n_1^\epsilon(0, x) = n_{10}(x) \geq 0, \;\; n_2^\epsilon(0, x) = n_{20}(x) \geq 0. \tag{1.69}$$

On note $n(x) = n_{10}(x) + n_{20}(x)$, car on a la relation $\partial_t(n_1^\epsilon + n_2^\epsilon) = 0$, et on obtient la conservation de la densité atomique,

$$n_1^\epsilon(t, x) + n_2^\epsilon(t, x) = n_{10}(x) + n_{20}(x) \equiv n(x). \tag{1.70}$$

Avec cette condition on ferme (1.65), le système limite.

Dans ce travail, en premier lieu, on étudiera l'existence et l'unicité de solution sur $[0, T]$ avec T fixé, du système (1.66) à ϵ fixé. Puis on montera l'existence et l'unicité de solution sur $[0, T]$ avec T fixé, du système (1.65). Alors on montrera la convergence du système (1.66) vers (1.65) quand $\epsilon \to 0$, c'est-à-dire qu'on justifie l'approximation quasi-stationaire. L'existence de solution sur $[0, T]$ avec T fixé est obtenu par un schéma itératif en construisant une suite de solutions convergente. Et l'unicité est assurée par le lemme de Gronwall. On a le résultat suivant :

Théorème 1.8. *On suppose que*

- *les données initiales et la donnée au bord satisfont*

$$0 \leq f_0 \in L^\infty(X \times \mathbb{S}^2 \times \mathbb{R}^+), \qquad 0 \leq g \in L^\infty(\mathbb{R}^+ \times (\partial X \times \mathbb{S}^2)_- \times \mathbb{R}^+),$$

$$n_{10} \geq 0, \;\; n_{20} \geq 0, \quad n_{10} + n_{20} = n \in L^\infty(X),$$

- *les coefficients d'Einstein satisfont*

$$A_{21}, \;\; B_{21}, \;\; B_{12} \text{ sont constantes,}$$

$$\delta_* \leq C_{12}(x), \;\; C_{21}(x) \leq \delta^*,$$

avec les constantes numériques δ_*, δ^*.

- *la fonction de profil* $\phi(\nu)$ *satisfait*

$$0 \leq \phi(\nu) h\nu \leq \delta, \;\; \forall \nu \in \mathbb{R}^+.$$

Alors, pour tout $T \in \mathbb{R}^+$ *fixe, il existe une unique soltuion* f^ϵ, n_1^ϵ *et* n_2^ϵ *dans* $L^\infty([0,T] \times X \times \mathbb{S}^2 \times \mathbb{R}^+)$ *et* $L^\infty([0,T] \times X)$ *respectivement. De plus, cette solution* f^ϵ, n_1^ϵ *et* n_2^ϵ *converge vers les fonctions* f, n_1 *et* n_2, *dans* $L^\infty([0,T] \times X \times \mathbb{S}^2 \times \mathbb{R}^+)$ *faible-$*$ et dans* $L^\infty([0,T] \times X)$ *faible-$*$ respectivement. Ces fonctions définissent l'unique solution du système limite* (1.65).

Ce théorème est démontré dans le chapitre 5 où de plus le comportement quand $\epsilon \to 0$ est confirmé par des tests numériques.

Remarque 1.9. *L'équilibre dynamique dont nous venons de parler ne constitue par un équilibre thermique, au sens usuel du terme, comme expliqué dans [10] pour deux raisons valables dans les conditions expérimentales courantes :*

- *les atomes n'échangent leur énergie d'excitation qu'avec la radiation et non pas avec le thermostat que constitue le milieu environnant;*
- *la radiation n'a aucun rapport avec la radiation d'équilibre décrit par la loi planckienne.*

1.5 Conclusions et perspectives

1.5.1 Conclusions

Quand les effets visqueux et la conduction thermique sont négligeables, le transport d'énergie radiative peut fortement influencer la structure du choc faible. Quand la taille de l'onde de choc est suffisamment petite, nous montrons l'existence des profils de choc faible et le stabilité aysmptotique des profils de choc.

Nous étudions un modèle décrivant l'influence des radiations sur la répartition énergétique des atomes de la matière. Nous montrons l'existence des solutions et la convergence des solutions vers un régime d'équilibre statistique. Cette étude est complétée par un travail de validation numérique.

1.5.2 Perspectives

Les principales perspectives de recherche qui apparaissent à l'issu de cette thèse concernent l'influence du transport d'énergie radiative sur la structure des chocs de grande amplitude. Dans un fluide thermoconductible, non visqueux, pour un choc de grande amplitude, seulement le profil de la température devrait changer continûment, et les autres inconnues éprouvent un saut de manière discontinue (Cf [49, 61] et leurs références). Aussi pour le système plus simple couplant une équation de Burgers et une équation elliptique, Kawashima et Nishibata ont montré que les profils de choc avaient des discontinuités pour les chocs de grande amplitude. L'existence de ce résultat au cas du système de la dynamique des gaz est un enjeu, tant du point de vue de la modélisation physique que des développements mathématiques.

La deuxième perspective concerne l'explosion des solutions régulières définies localement en temps. Dans le Chapitre 2, nous montrons l'existence locale en temps des solutions régulières. En dynamique des gaz, Sideris a construit des solutions qui explosent en temps fini, pour une classe de donnée initiales, [57]. Pour un système couplé de transfert radiatif proche de celui étudié au Chapitre 2, Zhong et Jiang ont construit des solutions régulières explosant en temps fini par un raisonnement "à la Sideris", [62]. Toutefois, ils exhibent des solutions qui explosent en temps fini au prix de nombreuses simplifications du modèle : absence de dispersion ($\sigma_s = 0$) et avec une loi d'émission un peu éloignée des modèles physiques. Il serait intéressant de chercher à étendre un tel résultat d'explosion pour un modèle plus complet sur le plan physique, ou au contraire de déterminer si ces termes peuvent limiter la croissance des solutions.

La dernière perspective concerne le modèle décrivant l'influence des radiations sur la répartition énergique du plasma. Si l'on considère les transitions libre-lié, les équations de densités ne sont plus linéaires en fonctions de la densité, parce que les coefficients des transitions libre-lié dépendent de la densité d'électrons libres. Il est donc intéressant d'étendre l'analyse du Chapitre 5 à ce cas, de comparer les transitions et d'étudier le comportement du terme source (le facteur qui compare le coefficient d'émissivité et le coefficient d'extinction).

Chapitre 2

Radiative hydrodynamics with Doppler corrections : Local existence of smooth solutions

2.1 Introduction and main result

This chapter is devoted to the local-in-time existence of smooth solutions of the following PDEs system

$$\left\{\begin{array}{rcl}\partial_t\rho+\nabla_x\cdot(\rho u) &=& 0,\\ \partial_t(\rho u)+\mathrm{Div}_x(\rho u\otimes u+pI) &=& -\dfrac{1}{c}\displaystyle\int_{\mathbb{S}^{N-1}} vQ\mathrm{d}v,\\ \partial_t(\rho(e+\frac{u^2}{2}))+\nabla_x\cdot(\rho(e+\frac{u^2}{2})u+pu) &=& -\displaystyle\int_{\mathbb{S}^{N-1}} Q\mathrm{d}v,\\ \frac{1}{c}\partial_t f+v\cdot\nabla_x f &=& Q.\end{array}\right. \tag{2.1}$$

The system arises in radiative transfer theory : it is intended to describe interaction between a fluid, described by its density $\rho(t,x)$, its velocity $u(t,x)$, and its specific energy $e(t,x)$, and a radiation field, described by its specific intensity $f(t,x,v)$. Throughout this chapter, the pressure p is defined by means of ρ and the specific energy e by the perfect gas constitutive law. Denote the total energy by $E=e+|u|^2/2$. Here and below, $t\geq 0$, and $x\in\mathbb{R}^N$ stand for time and space variable respectively. The intensity of radiation f depends also on a direction variable $v\in\mathbb{S}^{N-1}$. Throughout this chapter dv denotes the normalized Lebesgue measure on $\mathbb{S}^{N-1}$. The system is completed by imposing initial data

$$\left\{\begin{array}{rcl}\begin{pmatrix}\rho\\ u\\ e\end{pmatrix}(0,x) &=& \begin{pmatrix}\rho_0\\ u_0\\ e_0\end{pmatrix}(x),\\ f(0,x,v) &=& f_0(x,v).\end{array}\right. \tag{2.2}$$

We recall below a few facts about the physical background, only saying here that we restrict to a grey model where dependence with respect to the frequency of the radiation has been neglected. The interaction term Q depends non-linearly on the unknowns ρ, u, e and f. The coupling is due to energy and impulsion exchanges between the fluid and radiation. The precise definition of Q will be given in the next section.

Well posedness theory for (2.1) naturally appeals to classical fixed point strategies for hyperbolic systems. We refer to [15] for linear situation and [45], [24] for the non linear framework. Such a coupled system involving the Euler system and a kinetic equation has been investigated in [4] and [3], motivated by modeling of fluid/particles flows. However, the coupling dealt with in [4] and [3], has a different nature since it arises through friction force terms, instead of being related to "collision-like source term" as in (2.1). We also mention the recent work [62] performed independently to ours, which is devoted to the analysis of a problem very close to (2.1), we only mention that the non-relativistic model studied in [62] incorporates the dependence of the frequency variable.

Let us denote by G the state space $\mathbb{R}^+ \times \{u \in \mathbb{R}^N | |u| < c\} \times \mathbb{R}^+$, where we assume that the fluid velocity is always less than the light speed. Unsurprisingly our main result states as the following :

Theorem 2.1. *Let G_1 be a relatively compact set of G and s be an integer such that $s > N/2 + 1$. Assume that*

$$\begin{pmatrix} \rho_0(x) \\ u_0(x) \\ e_0(x) \end{pmatrix} \in G_1, \quad \text{and} \quad \begin{pmatrix} \rho_0 - \bar{\rho}_0 \\ u_0 \\ e_0 - \bar{e}_0 \end{pmatrix} \in H^s(\mathbb{R}^N),$$

and

$$f_0 \geq 0, \;\; f_0 - \bar{f}_0 \in L^2(\mathbb{S}^{N-1}; H^s(\mathbb{R}^N)) \bigcap L^\infty(\mathbb{R}^N \times \mathbb{S}^{N-1}),$$

where $\overline{\rho}_0$, $\overline{e}_0$ and $\bar{f}_0$ are positive constants such that $(\bar{\rho}_0, 0, \bar{e}_0; \bar{f}_0)$ is a constant solution to the system (2.1) and (2.2). Then, there exists a $T > 0$, such that the problem (2.1) and (2.2) has a unique smooth solution (ρ, u, e, f) on $[0, T]$, verifying

$$(\rho - \bar{\rho}_0, u, e - \bar{e}_0) \in C([0, T]; H^s(\mathbb{R}^n)),$$

and

$$f - \bar{f}_0 \in C([0, T]; L^2(\mathbb{S}^{N-1}; H^s(\mathbb{R}^N))).$$

This chapter is organized as follows. First we discuss modeling issues concerning the system (2.1) and we setup a few notations. Section 3 presents the iterative procedure that leads to the existence-uniqueness result. The crucial estimates are detailed in Section 4 and 5. The former shows uniform estimates with a "high norm" involving many derivatives of the unknowns, while the latter justifies that the scheme is contractive in a "low norm" that is nothing but the L^2−norm.

Throughout this chapter we shall denote the usual Sobolev spaces by $H^s(\mathbb{R}^N)$, with the norm $\| \cdot \|_{H^s(\mathbb{R}^N)}$, $s \in \mathbb{N}$, defined by :

$$\|f\|_{H^s(\mathbb{R}^N)} = \left(\sum_{|\alpha| \leq s} \int_{\mathbb{R}^N} |\partial_x^\alpha f(x)|^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

The space $L^2(\mathbb{S}_v^{N-1}, H^s(\mathbb{R}_x^N))$ stands for the space of functions $f(x, v)$, verifying for a.e $v \in \mathbb{S}^{N-1}$, $f(\cdot, v) \in H^s(\mathbb{R}_x^N)$ and

$$\int_{\mathbb{S}^{N-1}} \|f(\cdot, v)\|^2_{H^s(\mathbb{R}_x^N)} \mathrm{d}v < \infty.$$

We denote the norm of this space by :

$$N_s(f) = \left(\int_{\mathbb{S}^{N-1}} \|f(\cdot, v)\|^2_{H^s(\mathbb{R}_x^N)} \mathrm{d}v \right)^{\frac{1}{2}}. \tag{2.3}$$

2.2 Modelling issue

In this section we present more precisely the physical motivation of (2.1). We refer for further details to [49, 43, 9, 17]. We also introduce some notations that will be used in the sequel.

2.2.1 Radiative transfer equation

For the sake of simplicity, let us restrict for the presentation to the 3-dimensional framework. We go back temporarily to a more complete model : radiation is seen as a set of photons characterized by their position $x \in \mathbb{R}^3$, their frequency $\nu \geq 0$, and the direction of their flight $v \in \mathbb{S}^2$. All photons fly with a velocity equal to the speed of light c. Let us denote by h the Planck constant, so that a photon with frequency ν has energy $h\nu$. Hence we are interested in the specific intensity of radiation $f(t, x, v, \nu)$ so that

$$\frac{1}{c} f(t, x, v, \nu) \mathrm{d}v \mathrm{d}\nu \mathrm{d}x$$

gives the radiation energy at time t, in the volume $\mathrm{d}x$ centered x, corresponding to photons with frequencies in $(\nu, \nu + \mathrm{d}\nu)$, flying in the solid angle $\mathrm{d}v$ around the direction $v \in \mathbb{S}^2$.

Remark that the specific intensity f provides a complete macroscopic description of the radiation field. It can be related to the photon distribution function $f_R(t, x, v, p)$, and the photon number density $\psi(t, x, v, \nu)$. We refer to [49] and [17].

The evolution of the photons is influenced by various interaction processes : scattering produces change in the direction of flight, and photons are also absorbed or emitted by the surrounding fluid. Therefore we get a kinetic evolution equation

$$\frac{1}{c} \partial_t f + v \cdot \nabla_x f = Q \tag{2.4}$$

which tells that the simple motion on straight lines with velocity cv is perturbed by the interaction mechanisms embodied into the source term Q, which splits as

$$Q = \eta - \chi f,$$

where η is the total emissivity, being split itself into a thermal part, the true (thermal) emissivity η^t, and a scattering part η^s; χ is the total absorption coefficient (sometimes called the total extinction coefficient), also being split into a thermal part, the true (thermal) absorption χ^t, and a scattering part χ^s. Hence, we have

$$\begin{aligned} Q &= (\eta^t + \eta^s) - (\chi^t + \chi^s) f \\ &= (\eta^t - \chi^t f) + (\eta^s - \chi^s f) \\ &= Q^t + Q^s, \end{aligned}$$

where Q^t and Q^s denote the thermal part and the scattering part respectively. Remark that some physicists usually use κ, σ to denote the true thermal extinction coefficient and the scattering extinction coefficient respectively.

An important feature for many applications consists in taking into account relativistic effects and Doppler corrections. The latter defines how quantities measured in the comobile frame (in which the velocity of a fluid particle always equals to zero) are related to quantities evaluated in the reference frame (a fixed laboratory frame). The correction terms involve the Lorentz factor

$$\Upsilon(t,x) = \frac{1}{\sqrt{1-\frac{|u(t,x)|^2}{c^2}}},$$

that depends on the velocity of the fluid $u(t,x)$. The following formulae link the frequency ν^0, and the direction v^0 of the photons measured in the comobile frame to the ones denoted by ν and v measured in the fixed reference frame,

$$\nu^0 = \nu\Upsilon(1-\frac{v\cdot u}{c}), \quad v^0 = \frac{\nu}{\nu^0}\Big(v - \frac{\Upsilon}{c}u(1-\frac{u\cdot v}{c}\frac{\Upsilon}{\Upsilon+1})\Big). \tag{2.5}$$

Here and below, the superscript '0' denotes quantities measured in the co-moving frame.

In the co-moving frame, we can write the emissivity and the extinction coefficient as :

$$\eta^{s,0} = \frac{1}{l_s}\int_{\mathbb{S}^2}\sigma_s^0(\nu^0,v^0,v^{0\prime})f^0(v^{0\prime},\nu^0)\mathrm{d}v^{0\prime}, \tag{2.6}$$

$$\chi^{s,0} = \frac{1}{l_s}\int_{\mathbb{S}^2}\sigma_s^0(\nu^0,v^{0\prime},v^0)\mathrm{d}v^{0\prime}, \tag{2.7}$$

$$\eta^{t,0} = \frac{1}{l_a}\sigma_a^0(\nu^0,v^0)B(\nu^0,\theta), \tag{2.8}$$

$$\chi^{t,0} = \frac{1}{l_a}\sigma_a^0(\nu^0,v^0), \tag{2.9}$$

where l_s(resp. l_a) denotes the scattering mean free path (resp. the thermal absorption mean free path), the coefficient $\sigma_s(\nu,v,v')$ characterizes the change of photons direction from v' to v, $\sigma_a(\nu,v)$ is the non-negative coefficient characterizing the thermal process, the function $B(\nu,\theta)$ characterize the emission law, depending on the temperature θ of the fluid. We assume that the photons are emitted in a thermal equilibrium with a black body system, so B is given by the Planck function,

$$B(\nu,\theta) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/(k\theta)}-1}, \tag{2.10}$$

where k is the Boltzmann constant. From (2.6) to (2.9) we get the source term Q^0 measured in the co-moving frame.

Remark 2.2. *From (2.6) to (2.9) we get that the scattering term $Q^{s,0}$ is conservative in the co-moving frame, i.e*

$$\int_{\mathbb{S}^2}\int_{\mathbb{R}+}(\eta^{s,0} - \chi^{s,0}f^0)\mathrm{d}\nu^0\mathrm{d}v^0 = 0.$$

Because in the co-moving frame, scattering interactions only produce a change in the direction of photon's flight, thus it has no distribution on the total energy. It is not true in the lab-fixed frame because of Doppler shift and aberration of the light (2.5), as we will see latter on.

Now we go back to the lab-fixed regime using the formulae (2.5). In order to write the source term Q in the lab-fixed frame, let us set

$$\Lambda(t,x,v) = \frac{\nu^0}{\nu} = \frac{1-\frac{v\cdot u(t,x)}{c}}{\sqrt{1-\frac{|u(t,x)|^2}{c^2}}}. \tag{2.11}$$

If $v' \in \mathbb{S}^2$ we denote for convenience,

$$\nu' = \frac{\nu(1 - \frac{u \cdot v}{c})}{1 - \frac{u \cdot v'}{c}}.$$

Note that we omit the time and space variables when no confusion can arise. Now we give the conversion relations (see [49] or Appendix 2.6.1 in a simple case) :

$$\begin{aligned}
\nu^0 \mathrm{d}v^0 \mathrm{d}\nu^0 &= \nu \mathrm{d}v \mathrm{d}\nu, && (2.12)\\
\frac{1}{\nu^3} f(\nu, v) &= \frac{1}{(\nu^0)^3} f^0(\nu^0, v^0), && (2.13)\\
\frac{1}{\nu^2} \eta(\nu, v) &= \frac{1}{(\nu^0)^2} \eta^0(\nu^0, v^0), && (2.14)\\
\nu \chi(\nu, v) &= \nu^0 \chi^0(\nu^0, v^0). && (2.15)
\end{aligned}$$

Furthermore we have the following equality (see [17] for the proof, or Appendix 2.6.1 in a simple case) :

$$\mathrm{d}v^0 = \frac{1}{\Lambda(v)^2} \mathrm{d}v. \tag{2.16}$$

Using the above conversion relations (2.12)-(2.16) in (2.6)-(2.9) we can write the emissivity and the extinction coefficient as :

$$\begin{aligned}
\Lambda(v)^2 \eta^s &= \frac{1}{l_s} \int_{\mathbb{S}^2} \sigma_s(\nu, v, v') f(v', \nu') \Lambda(v') \mathrm{d}v', && (2.17)\\
\frac{1}{\Lambda(v)} \chi^s &= \frac{1}{l_s} \int_{\mathbb{S}^2} \sigma_s(\nu, v', v) \frac{1}{\Lambda(v')^2} \mathrm{d}v', && (2.18)\\
\Lambda(v)^2 \eta^t &= \frac{1}{l_a} \sigma_a(\nu, v) B(\nu^0, \theta), && (2.19)\\
\frac{1}{\Lambda(v)} \chi^a &= \frac{1}{l_a} \sigma_a(\nu, v). && (2.20)
\end{aligned}$$

Thus we get the source term Q measured in the lab-fixed regime as

$$\begin{aligned}
Q &= Q^s + Q^t,\\
Q^s &= \frac{1}{l_s} \left\{ \frac{1}{\Lambda(v)^2} \int_{\mathbb{S}^2} \sigma_s(\nu, v, v') f(v', \nu') \Lambda(v') \mathrm{d}v' - \Lambda(v) f(v, \nu) \int_{\mathbb{S}^2} \frac{\sigma_s(\nu, v', v)}{\Lambda(v')^2} \mathrm{d}v' \right\},\\
Q^t &= \frac{\sigma_a(\nu, v)}{l_a} \left\{ \frac{B(\nu^0, \theta)}{\Lambda(v)^2} - \Lambda(v) f(v, \nu) \right\}.
\end{aligned}$$

Equation (2.4) is coupled to the Euler equation by energy and impulse exchanges, thus let us define

$$Q_E = \int_0^{+\infty} \int_{\mathbb{S}^2} Q \mathrm{d}v \mathrm{d}\nu, \quad Q_F = \frac{1}{c} \int_0^{+\infty} \int_{\mathbb{S}^2} v Q \mathrm{d}v \mathrm{d}\nu.$$

Then the evolution of ρ, u, E is driven by

$$\left\{ \begin{aligned}
\partial_t \rho + \nabla_x \cdot (\rho u) &= 0,\\
\partial_t (\rho u) + \mathrm{Div}_x (\rho u \otimes u) + \nabla p &= -Q_F,\\
\partial_t (\rho (e + \frac{u^2}{2})) + \nabla_x \cdot (\rho (e + \frac{u^2}{2}) u + p u) &= -Q_E,\\
\frac{1}{c} \partial_t f + v \cdot \nabla_x f &= Q.
\end{aligned} \right. \tag{2.21}$$

As said above, we assume that the gas obeys the perfect gas pressure law,

$$e = \frac{p}{(\gamma - 1)\rho}, \quad p = R\rho\theta, \tag{2.22}$$

where R is the perfect gas constant and $\gamma > 1$ is the ratio of specific heats at constant pressure and volume.

Remark that in this subsection we restrict our discussion to the 3-dimensional framework. In what follows we go back to the general case, that is $x \in \mathbb{R}^N$ and $v \in \mathbb{S}^{N-1}$.

2.2.2 Grey assumption

The problem is simplified by considering coefficients σ_s and σ_a that do not depend on the frequency variable ν. That is the so-called "grey assumption". We can integrate the transport equation with respect to ν. Let us introduce the following notations,

$$\overline{f}(t, x, v) = \int_0^{+\infty} f(t, x, v, \nu) \mathrm{d}\nu,$$

$$\begin{aligned} \int_0^{+\infty} B(\nu^0, \theta) \mathrm{d}\nu &= \int_0^{+\infty} \frac{2h\left(\nu^0\right)^3}{c^2} \frac{\mathrm{d}\nu}{e^{h\nu^0/(k\theta)} - 1} \\ &= \frac{1}{\Lambda} \mathbb{B}(\theta), \end{aligned}$$

where $\mathbb{B}(\theta) = \sigma\theta^4/\pi$ is the integrated Planck function and $\sigma = \frac{2\pi^5 k^4}{15h^3c^2}$ is the Stefan-Boltzmann constant. Observe that the Stefan-Boltzmann constant σ is different from the $\sigma_s(v, v')$ and $\sigma_a(v)$.

Furthermore integrating the scattering emissivity leads to

$$\begin{aligned} &\frac{1}{\Lambda(v)^2} \int_0^{+\infty} \int_{\mathbb{S}^2} \sigma_s(v, v') \Lambda(v') f(v', \nu') \mathrm{d}v' \mathrm{d}\nu \\ &= \frac{1}{\Lambda(v)^2} \int_{\mathbb{S}^2} \sigma_s(v, v') \Lambda(v') \mathrm{d}v' \left(\int_0^{+\infty} f(v', \nu') \frac{\Lambda(v')}{\Lambda(v)} \mathrm{d}\nu' \right) \\ &= \frac{1}{\Lambda(v)^3} \int_{\mathbb{S}^2} \sigma_s(v, v') \Lambda(v')^2 \overline{f}(v') \mathrm{d}v'. \end{aligned}$$

Thus integrating the transport equation (2.4) with respect ν yields

$$\frac{1}{c} \partial_t \overline{f} + v \cdot \nabla_x \overline{f} = \overline{Q}, \tag{2.23}$$

with

$$\overline{Q} = \overline{Q}^s + \overline{Q}^t, \tag{2.24}$$

$$\overline{Q}^s = \int_0^{+\infty} Q^s \mathrm{d}\nu = \frac{1}{l_s} \left(\frac{\langle \sigma_s \Lambda^2 \overline{f} \rangle}{\Lambda^3} - \left\langle \frac{\sigma_s}{\Lambda^2} \right\rangle \Lambda \overline{f} \right), \tag{2.25}$$

$$\overline{Q}^t = \int_0^{+\infty} Q^t \mathrm{d}\nu = \frac{\sigma_a}{l_a} \left(\frac{\mathbb{B}(\theta)}{\Lambda^3} - \Lambda \overline{f} \right), \tag{2.26}$$

where $<\cdot>$ denotes the integration with respect to v', precisely we have

$$\langle \sigma_s \Lambda^2 \overline{f} \rangle = \int_{\mathbb{S}^{N-1}} \sigma_s(v,v')\Lambda^2(v')\overline{f}(t,x,v')\mathrm{d}v', \tag{2.27}$$

and

$$\left\langle \frac{\sigma_s}{\Lambda^2} \right\rangle = \int_{\mathbb{S}^{N-1}} \frac{\sigma_s(v',v)}{\Lambda^2(v')}\mathrm{d}v'. \tag{2.28}$$

Finally the system we are dealing with reads

$$\left\{\begin{array}{rcl} \partial_t \rho + \nabla_x \cdot (\rho u) &=& 0, \\ \partial_t(\rho u) + \mathrm{Div}_x(\rho u \otimes u) + \nabla p &=& -\frac{1}{c}\int_{\mathbb{S}^{N-1}} vQ\mathrm{d}v, \\ \partial_t(\rho(e+\frac{u^2}{2})) + \nabla_x \cdot (\rho(e+\frac{u^2}{2})u + pu) &=& -\int_{\mathbb{S}^{N-1}} Q\mathrm{d}v, \\ \frac{1}{c}\partial_t f + v\cdot\nabla_x f &=& Q, \end{array}\right. \tag{2.29}$$

where we have omitted the over-line to simplify the notation, where Q depends on f, ρ, u, θ, by (2.24)-(2.26) and the pressure p satisfies (2.22). As a matter of fact it is worth remarking that the total energy of this system is formally conserved

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\mathbb{R}^N} \rho(e+\frac{u^2}{2})\mathrm{d}x + \frac{1}{c}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} f\mathrm{d}v\mathrm{d}x\right\} = 0.$$

2.2.3 Change of unknowns

It is convenient to introduce the entropy S, that is the function of ρ and p verifying

$$\mathrm{d}S = \frac{1}{\theta}\left\{\mathrm{d}e - \frac{p}{\rho^2}\mathrm{d}\rho\right\}. \tag{2.30}$$

By (2.22) we get $\exp S = p\,\rho^{-\gamma}$. Notice that from (2.30) we get the so-called Gibbs equation

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{1}{\theta}\left\{\frac{\mathrm{d}e}{\mathrm{d}t} - \frac{p}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t}\right\},$$

where $\frac{\mathrm{d}}{\mathrm{d}t}$ stands for the derivative along fluid particle path, namely

$$\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + u\cdot\nabla_x.$$

For smooth solutions the Euler system (2.29) can be written in the new variables (p,u,S) as

$$\left\{\begin{array}{rcl} \frac{\mathrm{d}p}{\mathrm{d}t} + \gamma\, p\, \nabla_x\cdot u &=& R\int_{\mathbb{S}^{N-1}}(\frac{u\cdot v}{c}-1)Q\mathrm{d}v, \\ \frac{\mathrm{d}u}{\mathrm{d}t} + \frac{1}{\rho}\,\nabla_x p &=& -\frac{1}{c\,\rho}\int_{\mathbb{S}^{N-1}} v\,Q\mathrm{d}v, \\ \frac{\mathrm{d}S}{\mathrm{d}t} &=& -\frac{R}{p}\int_{\mathbb{S}^{N-1}}(\frac{u\cdot v}{c}-1)Q\mathrm{d}v, \\ \frac{1}{c}\partial_t f + v\cdot\nabla_x f &=& Q. \end{array}\right. \tag{2.31}$$

We set $U = (p, u, S)^t$ and (2.31) reads

$$\begin{cases} \partial_t U + \sum_{j=1}^{N} A_j(U)\partial_{x_j} U &= b(U, f), \\ \frac{1}{c}\partial_t f + v \cdot \nabla_x f &= Q, \end{cases} \tag{2.32}$$

with

$$A_j(U) = \begin{pmatrix} u_j & 0 & 0 & \cdots & \gamma p & \cdots & 0 & 0 \\ 0 & u_j & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 0 & u_j & \vdots & 0 & \vdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\rho} & 0 & 0 & \cdots & u_j & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots & u_j & 0 \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & u_j \end{pmatrix} \in \mathfrak{M}_{N+2} \tag{2.33}$$

and

$$b(U, f) = \begin{pmatrix} R\left\langle (\frac{u \cdot v}{c} - 1)\ Q \right\rangle \\ -\frac{1}{c\ \rho} \langle v\ Q \rangle \\ -\frac{R}{p}\left\langle (\frac{u \cdot v}{c} - 1)\ Q \right\rangle \end{pmatrix}. \tag{2.34}$$

From (2.2) we define the initial data for (2.32) as

$$\begin{cases} U(0, x) = U_0(x), \\ f(0, x, v) = f_0(x, v). \end{cases} \tag{2.35}$$

As we discuss in the unknown variable $U = (p, u, S)^t$, the state space is also noted as $G = \mathbb{R}^+ \times \{u \in \mathbb{R}^N \| u| < c\} \times \mathbb{R}^+$. Thus Theorem 2.1 is equivalent to prove the following :

Theorem 2.3. *Let G_1 be a relatively compact set of G and s be an integer such that $s > N/2 + 1$. We assume that*

1. *$U_0(x) \in G_1$ for all $x \in \mathbb{R}^N$, and $U_0 - \bar{U}_0 \in H^s(\mathbb{R}^N)$ where $\bar{U}_0 = (\bar{p}_0, 0, \bar{S}_0)$ is a constant state;*
2. *$f_0 \geq 0$, $f_0 - \bar{f}_0 \in L^2(\mathbb{S}_v^{N-1}, H^s(\mathbb{R}_x^N))$, where $\bar{f}_0$ is the constant state for radiation, such that $(\bar{U}_0, \bar{f}_0)$ is a constant solution to the system (2.32).*

Then, there is a time interval $[0, T]$ with $T > 0$ so that there exists a solution (U, f), such that

$$U - \bar{U}_0 \in C([0, T]; H^s(\mathbb{R}^N)),$$

and

$$f - \bar{f}_0 \in C([0, T]; L^2(\mathbb{S}_v^{N-1}; H^s(\mathbb{R}_x^N))).$$

Remark 2.4. *Note that from (2.11) we have $\Lambda = 1$ at the constant state $\bar{U}_0$ for which the velocity of the fluid is $\bar{u}_0 = 0$. Thus by (2.25) it is clear that $Q^s = 0$ at the constant state $\bar{U}_0$. And if we take $\bar{f}_0 = \mathbb{B}(\bar{\theta}_0)$, the integrated Planck function in the state of temperature $\bar{\theta}_0$, defined by the fluid constant state $\bar{U}_0$, thus by (2.26) it is clear that $Q^t = 0$. Thus we get the expression of the constant state for radiation.*

2.3 Local existence of smooth solutions, proof of Theorem 2.3

In this section we prove Theorem 2.3. First we recall a common structure of symmetrisability in the sense of Friedrichs, and study the stability estimates. Secondly we smooth the initial data to avoid technical difficulties and we fix once and for all two constants ε_0, R and a subset G_2 in which the fluid solution will take its values. Finally we construct a sequence of solutions to linearized problems and complete the proof of Theorem 2.3 under two technical lemmas which will be proved later on. Recall that the state space is denoted by $G = \mathbb{R}^+ \times \{u \in \mathbb{R}^N ||u| < c\} \times \mathbb{R}^+$.

2.3.1 The common structure and the Friedrichs' energy estimate

At first we recall a common structure by the following basic definition.

Definition 2.5. *The system*

$$\partial_t U + \sum_{j=1}^{N} A_j(U)\partial_{x_j} U = b$$

is symmetrisable in the sense of Friedrichs, if for all $U \in G$, there exists a positive definite symmetric matrix $\tilde{A}_0(U)$, smoothly varying with U, so that there hold

1. *For any $\mathcal{O}$, open set with $\bar{\mathcal{O}} \subset\subset G$, for any $U \in \mathcal{O}$,*

$$CI \leq \tilde{A}_0(U) \leq C^{-1}I \tag{2.36}$$

with a constant $C = C(\mathcal{O}) < 1$,

2. *$\tilde{A}_j(U) = \tilde{A}_0(U)A_j(U)$ is symmetric for any $j = 1, 2, \cdots N$.*

Remark 2.6. *According to [45] (p.11), if a system is symmetrisable in the sense of Definition 2.5, then the following linearized problem*

$$\begin{cases} \tilde{A}_0(V)\partial_t W + \sum_{j=1}^{N} \tilde{A}_j(V)\partial_{x_j} W - \tilde{B}(V,t,x)W &= F, \\ W(0,x) &= W_0(x), \end{cases}$$

is well-posed over $[0,T] \times \mathbb{R}^N$ for any $T > 0$, provided $W_0 \in L^2(\mathbb{R}^N)$, $V(t,x)$ in C^1 with $V(t,x) \in \bar{\mathcal{O}} \subset\subset G$, $\tilde{B}$ being a smoothly varying $N \times N$ matrix function of its arguments. We define the energy

$$E(t) = \int_{\mathbb{R}^N} \tilde{A}_0(V(t,x))W(t,x) \cdot W(t,x)\mathrm{d}x.$$

Then we can compute the basic energy identity of Friedrichs :

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = \int_{\mathbb{R}^N} (\partial_t \tilde{A}_0(V) + \sum_{j=1}^{N} \partial_{x_j} \tilde{A}_j(V) + \tilde{B} + \tilde{B}^T)W \cdot W\mathrm{d}x + 2\int_{\mathbb{R}^N} F \cdot W\mathrm{d}x. \tag{2.37}$$

Using (2.36) yields the following inequality

$$
\begin{aligned}
\|W(t)\|_{L^2(\mathbb{R}^N)} \leq \frac{1}{C}\left(\|W_0\|_{L^2(\mathbb{R}^N)} + \int_0^t \|F(\tau)\|_{L^2(\mathbb{R}^N)}\mathrm{d}\tau\right) \\
+\frac{1}{2C}\left\|\partial_t\tilde{A}_0(V) + \sum_{j=1}^N \partial_{x_j}\tilde{A}_j(V) + \tilde{B} + \tilde{B}^T\right\|_{L^\infty([0,t]\times\mathbb{R}^N)} \int_0^t \|W(\tau)\|_{L^2(\mathbb{R}^N)}\mathrm{d}\tau,
\end{aligned}
\tag{2.38}
$$

where the constant C is defined by (2.36)as V varies in $\mathcal{O}$. The derivation is given in the Appendix 2.6.2. Then by using the Gronwall Lemma, we also deduce the following stability estimate :

$$
\begin{aligned}
\sup_{0\leq t\leq T} \|W(t)\|_{L^2(\mathbb{R}^N)} \leq \frac{1}{C}\left(\|W_0\|_{L^2(\mathbb{R}^N)} + \int_0^T \|F(t)\|_{L^2(\mathbb{R}^N)}\mathrm{d}t\right) \\
\exp\left\{\frac{1}{2C}\|\partial_t(\tilde{A}_0(V)) + \sum_{j=1}^N \partial_{x_j}(\tilde{A}_j(V)) + \tilde{B} + \tilde{B}^T\|_{L^\infty([0,T]\times\mathbb{R}^N)}T\right\}.
\end{aligned}
$$

For the Euler system (2.31) in the variables (p, u, S), the matrix $\tilde{A}_0(U)$ is given by

$$
\tilde{A}_0(U) = \begin{pmatrix} \dfrac{1}{\gamma\rho\, p} & 0 \\ 0 & \mathrm{Id}_{N+1} \end{pmatrix} \in \mathfrak{M}_{N+2}. \tag{2.39}
$$

2.3.2 Smoothing the initial data

At first we need to smooth the initial data U_0, f_0 to avoid some technical difficulties. We choose a smooth function $j(x)$ such that

$$
j \in C_0^\infty(\mathbb{R}^N),\ j \geq 0,\ \mathrm{supp} j \subseteq \left\{x \in \mathbb{R}^N \middle| |x| \leq 1\right\},\ \text{and } \int_{\mathbb{R}^N} j(x)\mathrm{d}x = 1.
$$

Set $j_\varepsilon(x) = \varepsilon^{-N} j(\frac{x}{\varepsilon})$. It is easy to verify that

$$
j_\varepsilon \in C_0^\infty(\mathbb{R}^N),\ \mathrm{supp} j_\varepsilon \subseteq \left\{x \in \mathbb{R}^N \middle| |x| \leq \varepsilon\right\},\ \text{and } \int_{\mathbb{R}^N} j_\varepsilon(x)\mathrm{d}x = 1.
$$

And we define a smoothing operator J_ε by

$$
\begin{aligned}
J_\varepsilon:\ H^s(\mathbb{R}^N) &\rightarrow \bigcap_{t\geq 0} H^t(\mathbb{R}^N), \\
V &\mapsto j_\varepsilon * V.
\end{aligned}
$$

Immediately we have the following :

Lemma 2.7. *1. If $V \in H^s(\mathbb{R}^N)$, then there holds*

$$
\|J_\varepsilon V - V\|_{H^s(\mathbb{R}^N)} \rightarrow 0 \text{ as } \varepsilon \text{ goes to } 0; \tag{2.40}
$$

2. If $V \in H^1(\mathbb{R}^N)$, and $\varepsilon \leq 1$, there holds

$$
\|J_\varepsilon V - V\|_{L^2(\mathbb{R}^N)} \leq C\varepsilon\|V\|_{H^1(\mathbb{R}^N)}. \tag{2.41}
$$

Now we introduce the regularization of the initial data U_0 and f_0. Let $\varepsilon_0 > 0$ that will be precised later and set $\varepsilon_k = 2^{-k}\varepsilon_0$, for $k = 1, 2, \cdots$. We define

$$\begin{aligned} U_0^k &= \bar{U}_0 + J_{\varepsilon_k}(U_0 - \bar{U}_0), \\ f_0^k(x,v) &= \bar{f}_0 + J_{\varepsilon_k}(x) * (f_0(x,v) - \bar{f}_0) \\ &= \bar{f}_0 + \int_{\mathbb{R}^N} j_{\varepsilon_k}(x-y)(f_0(y,v) - \bar{f}_0)\mathrm{d}y. \end{aligned} \tag{2.42}$$

for $k = 0, 1, \cdots$. By the properties of the operator J_ε, immediately we have

$$\begin{aligned} U_0^k - \bar{U}_0 &\in H^\infty(\mathbb{R}^N), \\ f_0^k - \bar{f}_0 &\in L^2\left(\mathbb{S}_v^{N-1}; H^\infty(\mathbb{R}_x^N)\right). \end{aligned}$$

The following claim determines once and for all two constants ε_0 and R that will be used to preserve stability properties of the iterative scheme. Recall that G_1 is a relatively compact subset of $G = \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^+$, such that U_0 takes its values in G_1, and $\varepsilon_k = \varepsilon_0 2^{-k}$.

Lemma 2.8. *Let G_2 be a relatively compact open subset of G such that $\bar{G}_1 \subset G_2$. Then there exist three constants R, $\hat{C}$ and ε_0 such that $\hat{C} < 1$,*

1. *$\|U - U_0^0\|_{H^s(\mathbb{R}^N)} \le R$ implies $U(x) \in \bar{G}_2$, for all x.*
2. *For all $k \in \mathbb{N}$,*
$$\|U_0 - U_0^k\|_{H^s(\mathbb{R}^N)} \le \hat{C}\frac{R}{4}. \tag{2.43}$$
3. *For all $k \in \mathbb{N}$,*
$$N_s(f_0 - f_0^k) \le \frac{1}{4} N_s(f_0 - \bar{f}_0), \tag{2.44}$$
recall that $N_s(\cdot)$ is defined in (2.3) in §2.1.

Proof. Recall that $s > N/2 + 1$. By using (2.40) and the Sobolev embedding theorem there exists a constant η_0 such that if $0 < \varepsilon \le \eta_0$ we have U_0^0 takes values in G_2. Thus by the Sobolev embedding theorem we have :

$$\|U - U_0^0\|_{L^\infty(\mathbb{R}^N)} \le C\|U - U_0^0\|_{H^s(\mathbb{R}^N)}.$$

Since U_0^0 takes its values in G_2, there exists a constant R such that if $\|U - U_0^0\|_{H^s(\mathbb{R}^N)} \le R$ we have $U \in \bar{G}_2$.

From (2.40) we deduce

$$\|J_\varepsilon U_0 - U_0\|_{H^s(\mathbb{R}^N)} \to 0,$$

as $\varepsilon \to 0$. Thus there exists $\eta_1 > 0$, such that if $0 < \varepsilon \le \eta_1$, we have

$$\|J_\varepsilon U_0 - U_0\|_{H^s(\mathbb{R}^N)} \le \frac{R}{4}\hat{C},$$

where $\hat{C} = C(G_2)$ is defined by (2.36) with G_2.

Similarly, if $f_0 \ne \bar{f}_0$, then we have $N_s(f_0 - \bar{f}_0) > 0$. By (2.40) there exists $\eta_2 > 0$, such that if $0 < \varepsilon \le \eta_2$, we have

$$N_s(J_\varepsilon f_0 - f_0) \le \frac{1}{4} N_s(f_0 - \bar{f}_0).$$

In the case that $f_0 \equiv \bar{f}_0$, we have $f_0^k \equiv f_0$ for all $k \in \mathbb{N}$. Thus the above inequality is also valid for all $\varepsilon > 0$.

Hence ε_0 is fixed such that $0 < \varepsilon_0 \le \min(\eta_0,\ \eta_1,\ \eta_2,\ 1)$, we have (2.43) and (2.44). $\square$

Remark 2.9. *We use the notation $\hat{C}$ to distinguish from the general numerical constants which are noted by C. Furthermore $\hat{C}$ is a fixed constant.*

2.3.3 Construction of the solution through an iterative scheme and the proof of Theorem 2.3

Before giving the iterative scheme to construct the solution of (2.32) with the initial data U_0 and f_0, we fix a few notations for the transport equation :

$$\frac{1}{c}\partial_t f + v \cdot \nabla_x f = Q.$$

We split the source term Q into a gain and a loss term :

$$Q = Q^+ - Q^-,$$

where Q^+ and Q^- are written as :

$$\begin{aligned} Q^+(f,U) &= F_1(f,u,v) + F_3(u,\theta,v) \\ &= \frac{1}{l_s}\frac{\langle \sigma_s \Lambda^2 f\rangle}{\Lambda^3} + \frac{\sigma_a}{l_a}\frac{\mathbb{B}(\theta)}{\Lambda^3}, \\ Q^-(f,U) &= F_2(u,v)f \\ &= \left(\frac{1}{l_s}\left\langle\frac{\sigma_s}{\Lambda^2}\right\rangle + \frac{\sigma_a}{l_a}\right)\Lambda f. \end{aligned}$$

Recall that $\sigma_s(\cdot,\cdot)$ (resp. $\sigma_a(\cdot)$) is a function over $\mathbb{S}^{N-1}\times\mathbb{S}^{N-1}$ (resp. $\mathbb{S}^{N-1}$) under the grey assumption, and that $\mathbb{B}(\theta) = \sigma\theta^4/\pi$ is the integrated Planck function, σ is Stefan-Boltzmann constant. For the expressions of $< \sigma_s\Lambda^2 f >$ and $< \frac{\sigma_s}{\Lambda^2} >$, see (2.27) and (2.28). Let us denote by M the maximum of

$$\frac{1}{l_s}\frac{\sigma_s(v',v)\Lambda(v')^2}{\Lambda^3} \text{ and } \frac{\sigma_a}{l_a}\frac{\mathbb{B}(\theta)}{\Lambda^3} \tag{2.45}$$

for v, v' varying in $\mathbb{S}^{N-1}$ and U varying in G_2. Thus if U take values in G_2, Q^+ will be bounded as

$$Q^+ \leq M\left(\int_{\mathbb{S}^{N-1}} f(t,x,v')\mathrm{d}v' + 1\right) \leq M\left(\|f(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^n_x\times\mathbb{S}^{N-1}_v)} + 1\right), \tag{2.46}$$

recall that $\mathrm{d}v$ denotes the normalized Lebesgue measure on $\mathbb{S}^{N-1}$.

Hence the transport equation can be written as

$$\frac{1}{c}\partial_t f + v\cdot\nabla_x f + F_2(u,v)f = Q^+(f,U).$$

Now we construct a sequence of solutions through the following scheme,

1. For $t \geq 0$, we set $(U^0, f^0) = (U_0^0, f_0^0)$;
2. For $k = 0,1,2,\cdots$, define (U^{k+1}, f^{k+1}) inductively as solution of the following linearized equation,

$$\left\{\begin{aligned} \tilde{A}_0(U^k)\partial_t U^{k+1} + \sum_{j=1}^N \tilde{A}_j(U^k)\partial_{x_j}U^{k+1} &= \tilde{A}_0(U^k)b(U^k,f^k), \\ U^{k+1}(0,x) &= U_0^{k+1}(x), \end{aligned}\right. \tag{2.47}$$

and

$$\begin{cases} \frac{1}{c}\partial_t f^{k+1} + v\cdot\nabla f^{k+1} + F_2(u^k,v)f^{k+1} &= Q^+(f^k,U^k), \\ f^{k+1}(0,x,v) &= f_0^{k+1}(x,v). \end{cases} \tag{2.48}$$

It follows immediately that

$$U^{k+1}\in C^\infty([0,T_k]\times\mathbb{R}^N), \quad f^{k+1}\in L^2(\mathbb{S}_v^{N-1};C^\infty([0,T_k]\times\mathbb{R}_x^N)),$$

where T_k is the largest time such that the following estimate

$$\sup_{0\le t\le T_k}\|U^k(t)-U_0^0\|_{H^s(\mathbb{R})}\le R, \tag{2.49}$$

is valid.

Note that the solution f^{k+1} can be obtained by integrating along characteristics. And we can write it as

$$\begin{aligned} f^{k+1}(t,x,v) &= f_0^{k+1}(x-cvt,v)\exp\left\{-\int_0^t cF_2(v,u^k(\tau,x+cv(\tau-t)))\mathrm{d}\tau\right\} \\ &+c\int_0^t \exp\left\{-c\int_\tau^t F_2(v,u^k(\tilde\tau,x+cv(\tilde\tau-t)))\mathrm{d}\tilde\tau\right\} Q^+(f^k,u^k)\Big|_{(\tau,x+cv(\tau-t),v)}\mathrm{d}\tau, \end{aligned}$$

and it is easy to see that $f^{k+1}\ge 0$, for all $k\in\mathbb{N}$.

The following lemma is crucial since it shows that we can bound the times T_k from below by some $T_*>0$.

Lemma 2.10. *There exist two constants $L>0$ and $T_*>0$ such that U^k and f^k defined by (2.47) and (2.48), for $k=0,1,\cdots$, satisfy*

$$\sup_{0\le t\le T_*}\|U^k(t)-U_0^0\|_{H^s(\mathbb{R}^N)}\le R, \tag{2.50}$$

$$\sup_{0\le t\le T_*}\|\partial_t U^k(t)\|_{H^{s-1}(\mathbb{R}^N)}\le L, \tag{2.51}$$

$$\sup_{0\le t\le T_*} N_s(f^k(t)-f_0^0)\le 2N_s(f_0-\bar f_0)+2MT_*, \tag{2.52}$$

$$\|f^k\|_{L^\infty([0,T_*]\times\mathbb{R}_x^N\times\mathbb{S}_v^{N-1})}\le 2\|f_0\|_{L^\infty(\mathbb{R}_x^N\times\mathbb{S}_v^{N-1})}+2MT_*, \tag{2.53}$$

where M is defined by (2.45).

We assume temporarily that Lemme 2.10 holds and continue the argument. We aim at proving that U^{k+1} and f^{k+1} are Cauchy sequences in some norms. According to the idea of Kato [24] or see [45], it is relevant to use the L^2 norm for that purpose. In particular, we have the following contraction property in this low norm.

Lemma 2.11. *There exist $T_{**}\in]0,T_*]$, $a<1$ and a nonnegative sequence $(\beta_k)_k$ with $\sum_k\beta_k<+\infty$, such that U^k and f^k defined by (2.47) and (2.48), for $k=1,2,\cdots$, satisfy*

$$\begin{aligned} &\sup_{0\le t\le T_{**}}\left(\|(U^{k+1}-U^k)(t)\|_{L^2(\mathbb{R}^N)}+\|(f^{k+1}-f^k)(t)\|_{L^2(\mathbb{R}_x^N\times\mathbb{S}_v^{N-1})}\right) \\ &\le a\sup_{0\le t\le T_{**}}\left(\|(U^k-U^{k-1})(t)\|_{L^2(\mathbb{R}^N)}+\|(f^k-f^{k-1})(t)\|_{L^2(\mathbb{R}_x^N\times\mathbb{S}_v^{N-1})}\right)+\beta_k. \end{aligned} \tag{2.54}$$

We also assume Lemma 2.11 for the time being and use it to complete the proof of Theorem 2.3. Immediately from Lemma 2.11 we have

$$\sum_{k=0}^{+\infty} \sup_{0\le t\le T_{**}} \|(U^{k+1}-U^k)(t)\|_{L^2(\mathbb{R}^N)} < +\infty,$$
$$\sum_{k=0}^{+\infty} \sup_{0\le t\le T_{**}} \|(f^{k+1}-f^k)(t)\|_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)} < +\infty.$$

Thus there exist two functions U and f such that

$$\begin{aligned} &\sup_{0\le t\le T_{**}} \|(U^k-U)(t)\|_{L^2(\mathbb{R}^N)} \to 0,\\ &\sup_{0\le t\le T_{**}} \|(f^k-f)(t)\|_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)} \to 0, \end{aligned} \tag{2.55}$$

as k goes to $+\infty$. Furthermore (2.55) implies that $U-\bar{U}_0\in C([0,T_{**}];L^2(\mathbb{R}^N))$, $f-\bar{f}_0\in C([0,T_{**}];L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v))$, since the uniform limit of continuous functions is continuous.

For any $w\in H^s(\mathbb{R}^N)$, the Sobolev interpolation inequality reads

$$\|w\|_{H^{s'}(\mathbb{R}^N)} \le C_s\|w\|^{1-s'/s}_{L^2(\mathbb{R}^N)}\|w\|^{s'/s}_{H^s(\mathbb{R}^N)}, \text{ for any } 0<s'<s$$

we refer to [45]. Thus using (2.50) in Lemma 2.10 we have for any s', $0<s'<s$, for any $0\le t\le T_{**}$

$$\|(U^{k+1}-U^k)(t)\|_{H^{s'}(\mathbb{R}^N)} \le C\|(U^{k+1}-U^k)(t)\|^{1-s'/s}_{L^2(\mathbb{R}^N)}.$$

Together with (2.55), we conclude that,

$$\sup_{0\le t\le T_{**}} \|(U^k-U)(t)\|_{H^{s'}(\mathbb{R}^N)} \to 0, \tag{2.56}$$

as $k\to+\infty$, for any $0<s'<s$, thus $U-\bar{U}_0\in C([0,T_{**}];H^{s'}(\mathbb{R}^N))$. If we choose s' such that $N/2+1<s'<s$, use the Sobolev inequality and get

$$U^k-U\to 0 \text{ in } C([0,T_{**}],C^1(\mathbb{R}^N)). \tag{2.57}$$

We reproduce a similar analysis to the sequence f^k. We get

$$\sup_{0\le t\le T_{**}} \int_{\mathbb{S}^{N-1}} \|(f^k-f)(t,\cdot,v)\|^2_{H^{s'}(\mathbb{R}^N_x)}\mathrm{d}v \to 0$$

as $k\to+\infty$, for any $0<s'<s$, thus $f-\bar{f}_0\in C([0,T_{**}];L^2(\mathbb{S}^{N-1};H^{s'}(\mathbb{R}^N)))$. So that if we still choose s' such that $N/2+1<s'<s$, and the Sobolev embedding theorem and the Cauchy-Schwarz inequality yield

$$\begin{aligned} &\sup_{0\le t\le T_{**}} \int_{\mathbb{S}^{N-1}} \|(f^k-f)(t,\cdot,v)\|_{L^\infty(\mathbb{R}^N_x)}\mathrm{d}v \to 0,\\ &\sup_{0\le t\le T_{**}} \int_{\mathbb{S}^{N-1}} \|\partial_{x_j}(f^k-f)(t,\cdot,v)\|_{L^\infty(\mathbb{R}^N_x)}\mathrm{d}v \to 0,\ j=1,\cdots,N, \end{aligned} \tag{2.58}$$

as $k\to+\infty$. We are going to show that (U,f) is a solution to (2.32) with the initial data (2.35).

Lemma 2.12. *There holds*

$$b(U^k, f^k) \to b(U, f) \text{ in } L^\infty([0, T_{**}] \times \mathbb{R}^N). \tag{2.59}$$

Proof. We observe that the components of $b(U, f)$ are either in the form of

$$\int_{\mathbb{S}^{N-1}} g_1(v, U) f(t, x, v) \mathrm{d}v, \tag{2.60}$$

or in the form of

$$\int_{\mathbb{S}^{N-1}} g_2(v, U) \mathrm{d}v, \tag{2.61}$$

where g_1 and g_2 are smooth functions of their arguments.

We have for any $0 \le t \le T_{**}$,

$$\begin{aligned}
&\left| \int_{\mathbb{S}^{N-1}} g_1(v, U^k) f^k(t, x, v) \mathrm{d}v - \int_{\mathbb{S}^{N-1}} g_1(v, U) f(t, x, v) \mathrm{d}v \right| \\
&\le \int_{\mathbb{S}^{N-1}} |g_1(v, U^k)| |f^k(t, x, v) - f(t, x, v)| \mathrm{d}v \\
&\quad + \int_{\mathbb{S}^{N-1}} |g_1(v, U^k) - g_1(v, U)| |f(t, x, v)| \mathrm{d}v \\
&\le C \int_{\mathbb{S}^{N-1}} |f^k(t, x, v) - f(t, x, v)| \mathrm{d}v + C|U^k - U| \int_{\mathbb{S}^{N-1}} |f(t, x, v)| \mathrm{d}v.
\end{aligned}$$

Then (2.57) and (2.58) lead

$$\left| \int_{\mathbb{S}^{N-1}} g_1(v, U^k) f^k(t, x, v) \mathrm{d}v - \int_{\mathbb{S}^{N-1}} g_1(v, U) f(t, x, v) \mathrm{d}v \right| \to 0,$$

as $k \to +\infty$ in $L^\infty([0, T_{**}] \times \mathbb{R}^N)$. With the same argument we can get the convergence of the term $g_2(v, U^k)$. We have thus proved (2.59). □

From (2.57), we derive that

$$\sum_{j=1}^N A_j(U^k) \partial_{x_j} U^{k+1} \to \sum_{j=1}^N A_j(U) \partial_{x_j} U \text{ in } L^\infty([0, T_{**}] \times \mathbb{R}^N).$$

Together with (2.59) there holds

$$\partial_t U^{k+1} \to -\sum_{j=1}^N A_j(U) \partial_{x_j} U + b(U, f) \text{ in } L^\infty([0, T_{**}] \times \mathbb{R}^N).$$

Since we also have that $\partial_t U^{k+1} \to \partial_t U$ in the sense of distributions, thus we get :

$$\partial_t U = -\sum_{j=1}^N A_j(U) \partial_{x_j} U + b(U, f),$$

and

$$U^{k+1} - U \to 0 \text{ in } C^1([0, T_{**}] \times \mathbb{R}^N). \tag{2.62}$$

For the initial data, from (2.40) we have $U_0^k - U_0 \to 0$, in $H^s(\mathbb{R}^N)$, and also in $C^1(\mathbb{R}^N)$, since $s > N/2 + 1$.

Finally the limit U of U^k belongs to $C^1([0,T_{**}] \times \mathbb{R}^N)$ and it satisfies

$$\begin{cases} \partial_t U + \sum_{j=1}^N A_j(U)\partial_{x_j} U &= b(U,f), \\ U(0,x) &= U_0(x). \end{cases}$$

By using (2.58) and reasoning as in Lemma 2.12 we can also pass to the limit in (2.48) which yields

$$\begin{cases} \frac{1}{c}\partial_t f + v \cdot \nabla_x f + F_2(v,u)f &= F_1(f,u,v) - F_3(u,\theta,v), \\ f(0,x,v) &= f_0(x,v). \end{cases}$$

In what follows we prove that

$$\begin{aligned} U - \bar{U}_0 &\in C([0,T_{**}]; H^s(\mathbb{R}^N)), \\ f - \bar{f}_0 &\in C([0,T_{**}]; L^2(\mathbb{S}^{N-1}; H^s(\mathbb{R}^N))), \end{aligned} \tag{2.63}$$

with which we complete the proof of Theorem 2.3. Firstly we prove the following :

Lemma 2.13. *The solution that we obtained through the iteration scheme satisfies*

$$U - \bar{U}_0 \in C_w([0,T_{**}]; H^s(\mathbb{R}^N)), \qquad f - \bar{f}_0 \in C_w([0,T_{**}]; L^2(\mathbb{S}_v^{N-1}; H^s(\mathbb{R}_x^N))),$$

here C_w denotes the continuity of $U - \bar{U}_0$ (respectively $f - \bar{f}_0$) on the time interval $[0,T_{**}]$ with values in the weak topology of $H^s(\mathbb{R}^N)$ (respectively $L^2(\mathbb{S}_v^{N-1}; H^s(\mathbb{R}_x^N))$), that is, for any $\phi \in H^{-s}(\mathbb{R}^N)$ (respectively $\phi \in L^2(\mathbb{S}_v^{N-1}; H^{-s}(\mathbb{R}_x^N))$), $(\phi, U - \bar{U}_0)_s$ (respectively $\int_{\mathbb{S}_v^{N-1}} (\phi, f - \bar{f}_0)_s \mathrm{d}v$) is a continuous function on $[0,T_{**}]$. Here $(\cdot,\cdot)_s$ denotes the inner product of the Hilbert space $H^s(\mathbb{R}^N)$.

Proof. We recall two results obtained in the above discussions and use them to get the weak continuity of the fluid variable. On the one hand from (2.56) we have

$$U^k - U \to 0,$$

in $H^{s'}(\mathbb{R}^N)$ with $s' < s$ as $k \to +\infty$, uniformly on $[0,T_{**}]$. On the other hand from (2.50) in Lemma 2.10 we have also

$$\|U^k - \bar{U}_0\|_{H^s(\mathbb{R}^N)} \le R + \|U_0^0 - \bar{U}_0\|_{H^s(\mathbb{R}^N)},$$

uniformly over $[0,T_{**}]$.

Since $s' < s$, $H^{-s'}(\mathbb{R}^N)$ is dense in $H^{-s}(\mathbb{R}^N)$. Thus by a $3-\varepsilon$ argument using the above results yields, for any $\phi \in H^{-s}(\mathbb{R}^N)$,

$$(\phi, U^k - \bar{U}_0)_s \to (\phi, U - \bar{U}_0)_s, \tag{2.64}$$

as $k \to +\infty$, uniformly on $[0,T_{**}]$. That implies that

$$U - \bar{U}_0 \in C_w([0,T_{**}], H^s(\mathbb{R}^N)),$$

since the uniform limit of continuous functions is also continuous.

By the same discussion we can get the weak continuity of the kinetic function f, we omit the details. $\square$

As (U, f) is a local smooth solution to the system (2.32) with the initial condition (2.35), from (2.50) and Lemma 2.8, we have that $U(t,x) \in \bar{G}_2$ for any $(t,x) \in [0, T_{**}] \times \mathbb{R}^N$. Furthermore from the definition of $\tilde{A}_0(U)$, we have

$$CI \leq \tilde{A}_0(U) \leq C^{-1}I,$$

where $C = C(G_2) < 1$. Thus we can define an equivalent norm $\|\cdot\|_{s,\tilde{A}_0(t)}$ in $H^s(\mathbb{R}^N)$, as

$$\|V\|^2_{s,\tilde{A}_0(t)} = \sum_{|\alpha|\leq s} \int_{\mathbb{R}^N} \tilde{A}_0(t)\partial^\alpha V \cdot \partial^\alpha V \mathrm{d}x,$$

for any $0 \leq t \leq T_{**}$, where we have used the abbreviation $\tilde{A}_0(t) = \tilde{A}_0(U(t,x))$. We denote $(\cdot,\cdot)_{s,\tilde{A}_0(t)}$ the inner product in the Hilbert space $H^s(\mathbb{R}^N)$ associated with the norm $\|\cdot\|_{s,\tilde{A}_0(t)}$. According to [45], p.44, we have

$$\overline{\lim_{t\to 0^+}} \|V(t)\|_{s,\tilde{A}_0(t)} = \overline{\lim_{t\to 0^+}} \|V(t)\|_{s,\tilde{A}_0(0)}, \tag{2.65}$$

for any $V \in C_w([0, T_{**}], H^s(\mathbb{R}^N))$.

Lemma 2.14. *We have the strong continuity of $U - \bar{U}_0$ (respectively $f - \bar{f}_0$) at 0^+ in $H^s(\mathbb{R}^N)$ (respectively in $L^2(\mathbb{S}_v^{N-1}; H^s(\mathbb{R}_x^N))$) that is*

$$\|U(t,\cdot) - U_0\|_{s,\tilde{A}_0(0)} \to 0, \text{ (respectively } N_s(f(t) - f_0) \to 0),$$

as $t \to 0^+$.

Proof. Recall that if $(w_n)_n$ converges weakly to w in a Hilbert space, then $(w_n)_n$ converges strongly to w if and only if

$$\|w\| \geq \overline{\lim_{n\to+\infty}} \|w_n\|.$$

Thus to prove the Lemma, it remains to prove the following inequalities

$$\overline{\lim_{t\to 0^+}} \|U(t) - \bar{U}_0\|_{s,\tilde{A}_0(t)} \leq \|U_0 - \bar{U}_0\|_{s,\tilde{A}_0(0)},$$

and

$$\overline{\lim_{t\to 0^+}} N_s(f(t) - \bar{f}_0) \leq N_s(f_0 - \bar{f}_0).$$

We first show the continuity for the fluid variable, and we use the iteration scheme to derive this bound. We write the system (2.47) as

$$\begin{cases} \tilde{A}_0(U^k)\partial_t(U^{k+1} - \bar{U}_0) + \displaystyle\sum_{j=1}^N \tilde{A}_j(U^k)\partial_{x_j}(U^{k+1} - \bar{U}_0) &= \tilde{A}_0(U^k)b(U^k, f^k), \\ U^{k+1}(0,x) - \bar{U}_0 &= U_0^{k+1}(x) - \bar{U}_0. \end{cases}$$

Applying ∂_x^α to the above equation and taking the inner product of the resulting equation with $\partial_x^\alpha(U^{k+1} - \bar{U}_0)$, we integrate the resulting equation over $[0,t] \times \mathbb{R}^N$, with $t \leq T_{**}$, and finally sum over all α, $|\alpha| \leq s$:

$$\frac{1}{2}\sum_{|\alpha|\leq s} \int_{\mathbb{R}^N} \tilde{A}_0(U^k)\partial_x^\alpha(U^{k+1} - \bar{U}_0)\cdot\partial_x^\alpha(U^{k+1} - \bar{U}_0)\mathrm{d}x\Big|_0^t = \int_0^t \int_{\mathbb{R}^N} \sum_{|\alpha|\leq s} F_\alpha(\tau,x)\mathrm{d}x\mathrm{d}\tau, \tag{2.66}$$

where

$$F_\alpha(\tau,x)=\frac{1}{2}\left(\partial_t\left(\tilde{A}_0(U^k)\right)+\sum_{j=1}^N\partial_{x_j}\left(\tilde{A}_j(U^k)\right)\right)\partial_x^\alpha(U^{k+1}-\bar{U}_0)\cdot\partial_x^\alpha(U^{k+1}-\bar{U}_0)$$
$$+\partial_x^\alpha(U^{k+1}-\bar{U}_0)\cdot\partial_x^\alpha\left(\tilde{A}_0(U^k)b(U^k,f^k)\right)$$
$$+\sum_{j=1}^N\left(\left[\tilde{A}_j(U^k)\partial_{x_j},\partial_x^\alpha\right](U^{k+1}-\bar{U}_0)\right)\cdot\partial_x^\alpha(U^{k+1}-\bar{U}_0).$$

By (2.50), (2.51), (2.52), (2.53) in Lemma 2.10, $\sum_{|\alpha|\le s}\int_{\mathbb{R}^N}F_\alpha(\tau,x)\mathrm{d}x$ can be bounded by an integrable function $\tilde{F}(\tau)\ge 0$, independent of k, that is

$$\sum_{|\alpha|\le s}\int_{\mathbb{R}^N}F_\alpha(\tau,x)\mathrm{d}x\le\tilde{F}(\tau).$$

In fact $\tilde{F}$ is bounded uniformly on $[0,T_{**}]$. On the one hand by (2.62) we have , as $k\to+\infty$,

$$\sup_{0\le t\le T_{**}}\|\tilde{A}_0(U^k(t))-\tilde{A}_0\left(U(t)\right)\|_{L^\infty(\mathbb{R}^N)}\to 0. \tag{2.67}$$

On the other hand from the weak convergence (2.64) we have

$$\overline{\lim_{k\to+\infty}}(U^{k+1}-\bar{U}_0,U^{k+1}-\bar{U}_0)_{s,\tilde{A}_0(t)}\ge(U-\bar{U}_0,U-\bar{U}_0)_{s,\tilde{A}_0(t)}.$$

Thus we conclude with

$$\overline{\lim_{k\to+\infty}}\sum_{|\alpha|\le s}\int_{\mathbb{R}^N}\tilde{A}_0(U^k)\partial_x^\alpha(U^{k+1}-\bar{U}_0)\cdot\partial_x^\alpha(U^{k+1}-\bar{U}_0)\ge\|U(t)-\bar{U}_0\|_{s,\tilde{A}_0(0)}, \tag{2.68}$$

for any fixed $t\in[0,T_{**}]$. As U_0^{k+1} is defined by (2.42) we have

$$\sum_{|\alpha|\le s}\int_{\mathbb{R}^N}\tilde{A}_0(U_0^k)\partial_x^\alpha(U_0^{k+1}-\bar{U}_0)\cdot\partial_x^\alpha(U_0^{k+1}-\bar{U}_0)\to\|U_0-\bar{U}_0\|_{s,\tilde{A}_0(0)}, \tag{2.69}$$

as $k\to+\infty$. Together (2.68) and (2.69), we take the sup limit of (2.66) as $k\to+\infty$, and get

$$\|U-\bar{U}_0\|_{s,\tilde{A}_0(t)}\le\|U_0-\bar{U}_0\|_{s,\tilde{A}_0(0)}+\int_0^t\tilde{F}(\tau)\mathrm{d}\tau. \tag{2.70}$$

Thus we get

$$\overline{\lim_{t\to 0^+}}\|U(t)-\bar{U}_0\|_{s,\tilde{A}_0(t)}\le\|U_0-\bar{U}_0\|_{s,\tilde{A}_0(0)},$$

which yields that $U-\bar{U}_0$ converges strongly to $U_0-\bar{U}_0$ in $H^s(\mathbb{R}^N)$ as $t\to 0^+$.

On the other hand with the same discussion we can get the result for the kinetic variable. First we write (2.48) as

$$\begin{cases}\dfrac{1}{c}\partial_t(f^{k+1}-\bar{f}_0)+v\cdot\nabla_x(f^{k+1}-\bar{f}_0)+F_2(v,u^k)(f^{k+1}-\bar{f}_0))\\ \qquad=F_1(f^k,u^k,v)+F_3(u^k,\theta^k,v)-F_2(u^k,v)\bar{f}_0,\\ (f^{k+1}-\bar{f}_0)(0,x,v)=f_0^{k+1}(x,v)-\bar{f}_0.\end{cases}$$

Applying ∂_x^α to the above equation and taking the inner product of the resulting equation with $\partial_x^\alpha(f^{k+1}-\bar{f}_0)$, we integrate the resulting equation over $[0,t]\times\mathbb{R}_x^N\times\mathbb{S}_v^{N-1}$, with $t\le T_{**}$, and finally sum over all α, $|\alpha|\le s$:

$$\frac{1}{2c}\sum_{|\alpha|\le s}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|\partial_x^\alpha(f^{k+1}-\bar{f}_0)|^2\mathrm{d}x\mathrm{d}v\Big|_0^t=\sum_{|\alpha|\le s}\int_0^t\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\tilde{G}_\alpha(\tau,x,v)\mathrm{d}x\mathrm{d}v\mathrm{d}\tau,$$

where

$$\begin{aligned}\tilde{G}_\alpha &= \partial_x^\alpha\left(F_1(f^k,u^k,v)+F_3(u^k,\theta^k,v)-F_2(u^k,v)\bar{f}_0\right)\partial_x^\alpha(f^{k+1}-\bar{f}_0)\\ &\quad+\partial_x^\alpha(f^{k+1}-\bar{f}_0)[F_2(v,u^k),\partial_x^\alpha](f^{k+1}-\bar{f}_0)\end{aligned}$$

By (2.50), (2.51), (2.52),(2.53) we obtain that $\sum_{|\alpha|\le s}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\tilde{G}_\alpha(\tau,x,v)\mathrm{d}x\mathrm{d}v$ can be bounded by an integrable function $\tilde{G}$ independent of k, that is

$$\sum_{|\alpha|\le s}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\tilde{G}_\alpha(\tau,x,v)\mathrm{d}x\mathrm{d}v\le\tilde{G}(\tau).$$

Then we take the sup limit to the integrated equation, and get

$$\overline{\lim_{k\to\infty}}\frac{1}{2c}N_s(f^{k+1}(t)-\bar{f}_0)\le\frac{1}{2c}N_s(f_0-\bar{f}_0)+\int_0^t\tilde{G}(\tau)\mathrm{d}\tau. \tag{2.71}$$

While using the weak convergence we have

$$\overline{\lim_{k\to\infty}}N_s(f^{k+1}(t)-\bar{f}_0)\ge N_s(f(t)-\bar{f}_0). \tag{2.72}$$

Apply the (2.71) and (2.72) and get :

$$\frac{1}{2c}N_s(f(t)-\bar{f}_0)\le\frac{1}{2c}N_s(f_0-\bar{f}_0)+\int_0^t\tilde{G}(\tau)\mathrm{d}\tau,$$

from which immediately we have

$$\overline{\lim_{t\to 0^+}}N_s(f(t)-\bar{f}_0)\le N_s(f_0-\bar{f}_0).$$

Thus we get the continuity of $N_s(f(t)-\bar{f}_0)$ at 0^+. □

By the above argument we can prove the strong right continuity of $U-\bar{U}_0$, $f-\bar{f}_0$ at any other time $\hat{T}$, $0\le\hat{T}<T_{**}$. Furthermore in the system (2.32) by introducing a new variable $\tilde{t}=T-t$, and using the same argument we can get also the strong left continuity of $U-\bar{U}_0$, $f-\bar{f}_0$ on $(0,T_{**}]$. Therefore, $U-\bar{U}_0\in C([0,T_{**}];H^s(\mathbb{R}^N))$ and $f-\bar{f}_0\in C([0,T_{**}];L^2(\mathbb{S}_v^{N-1};H^s(\mathbb{R}_x^N)))$. And it follows directly from (2.32) that

$$U-\bar{U}_0\in C^1([0,T_{**}];H^{s-1}(\mathbb{R}^N)),$$

and

$$f-\bar{f}_0\in C^1([0,T_{**}];L^2(\mathbb{S}^{N-1};H^{s-1}(\mathbb{R}^N))).$$

This ends the proof of Theorem 2.3, up to the justification of Lemma 2.10 and Lemma 2.11.

2.4 Estimates in high norm, proof of Lemma 2.10

In this section we prove Lemma 2.10. At first we show that the fluid unknown defined by (2.47) fulfills a uniform $L^\infty(H^s)$ estimate; then we deduce a bound on $\partial_t U$ in $L^\infty(H^{s-1})$. Eventually, we discuss the kinetic unknown.

At first we assume that (2.50)-(2.53) are satisfied for U^k, f^k, and we prove that the property extends iteratively.

2.4.1 $L^\infty(H^s)$ Estimate of the fluid unknown

We define $W^{k+1} = U^{k+1} - U_0^0$, and compute that W^{k+1} satisfies

$$\left\{\begin{array}{rcl} \tilde{A}_0(U^k)\partial_t W^{k+1} + \sum_{j=1}^N \tilde{A}_j(U^k)\partial_{x_j} W^{k+1} & = & \tilde{A}_0(U^k)b(U^k, f^k) - \sum_{j=1}^N \tilde{A}_j(U^k)\partial_{x_j} U_0^0, \\ W^{k+1}(0,x) & = & W_0^{k+1} = U_0^{k+1} - U_0^0. \end{array}\right.$$

Multiplying by $\tilde{A}_0^{-1}(U^k)$ to the above equation, applying ∂_x^α to the resulting equation, for any multi-index $\alpha \in \mathbb{N}^N$, $0 \le |\alpha| \le s$, multiplying then by $\tilde{A}_0(U^k)$ to the resulting equation, we denote $W_\alpha^{k+1} = \partial_x^\alpha (W^{k+1})$, and get

$$\left\{\begin{array}{rcl} \tilde{A}_0(U^k)\partial_t W_\alpha^{k+1} + \sum_{j=1}^N \tilde{A}_j(U^k)\partial_{x_j} W_\alpha^{k+1} & = & I_1 + I_2 + I_3, \\ W_\alpha^{k+1}(0,x) & = & \partial_x^\alpha W_0^{k+1} = \partial_x^\alpha (U_0^{k+1} - U_0^0), \end{array}\right. \tag{2.73}$$

with

$$\begin{aligned} I_1 &= \tilde{A}_0(U^k)\partial_x^\alpha b(U^k, f^k), \\ I_2 &= -\sum_{j=1}^N \tilde{A}_0(U^k)\partial_x^\alpha \left(A_j(U^k)\partial_{x_j} U_0^0\right), \\ I_3 &= \sum_{j=1}^N \tilde{A}_0(U^k)\left[A_j(U^k), \partial_x^\alpha\right]\partial_{x_j} W^{k+1}, \end{aligned}$$

where $[\cdot,\cdot]$ stands for the commutator. Observe that $I_3 \equiv 0$ when $|\alpha| = 0$.

Applying the energy inequality (2.38) to (2.73) yields

$$\begin{aligned} \|W_\alpha^{k+1}(t)\|_{L^2(\mathbb{R}^N)} &\le \frac{1}{\hat{C}}\left(\|W_\alpha^{k+1}(0)\|_{L^2(\mathbb{R}^N)} + \int_0^t \|(I_1 + I_2 + I_3)(\tau)\|_{L^2(\mathbb{R}^N)} \mathrm{d}\tau\right) \\ &+ \frac{1}{2\hat{C}}\|\partial_t(\tilde{A}_0(U^k)) + \sum_{j=1}^N \partial_{x_j}(\tilde{A}_j(U^k))\|_{L^\infty([0,t]\times\mathbb{R}^N)} \int_0^t \|W_\alpha^{k+1}(\tau)\|_{L^2(\mathbb{R}^N)} \mathrm{d}\tau, \end{aligned} \tag{2.74}$$

we recall that $\hat{C} = C(G_2)$ is defined by (2.36) as V varies in G_2.

By the induction hypothesis and Lemma 2.8, we have

$$U^k(t,x) \in G_2, \text{ for } (t,x) \in [0, T_*] \times \mathbb{R}^N,$$

where the restriction on T_* will be clarified later.

First we have the following inequalities

$$\sup_{V\in G_2} |D_V(\tilde{A}_j(V))| \le C, \text{ for } j = 0, 1, \cdots, N.$$

Thus as $0 \le \tau \le t$, we have

$$\begin{aligned} &\|\partial_t(\tilde{A}_0(U^k(\tau))) + \sum_{j=1}^{N} \partial_{x_j}(\tilde{A}_j(U^k(\tau)))\|_{L^\infty(\mathbb{R}^N)} \\ &\le C\left(\|\partial_t U^k(\tau))\|_{L^\infty(\mathbb{R}^N)} + \|\nabla_x U^k(\tau)\|_{L^\infty(\mathbb{R}^N)}\right) \\ &\le C\left(\|\partial_t U^k(\tau))\|_{H^{s-1}(\mathbb{R}^N)} + \|\nabla_x U^k(\tau)\|_{H^{s-1}(\mathbb{R}^N)}\right), \end{aligned} \tag{2.75}$$

where we have used the Sobolev embedding theorem. Then the triangle inequality gives

$$\begin{aligned} \|\nabla_x U^k(\tau)\|_{H^{s-1}(\mathbb{R}^N)} &\le \|\nabla_x(U^k(\tau) - U_0^0)\|_{H^{s-1}(\mathbb{R}^N)} + \|\nabla_x U_0^0\|_{H^{s-1}(\mathbb{R}^N)} \\ &\le C\|U^k(\tau) - U_0^0\|_{H^s(\mathbb{R}^N)} + C(U_0) \end{aligned}$$

Using this inequality in (2.75) yields

$$\|\partial_t(\tilde{A}_0(U^k)) + \sum_{j=1}^{N} \partial_{x_j}(\tilde{A}_j(U^k))\|_{L^\infty([0,t]\times\mathbb{R}^N)} \le C(L + R + 1) \le C. \tag{2.76}$$

Next we estimate I_3. Since $I_3 \equiv 0$ if $|\alpha| = 0$, we limit our discussion to the case $0 < |\alpha| \le s$. To this end we shall combine the classical tame estimate for commutators [45] p.43, and the Sobolev embedding theorem. For $0 \le \tau \le t$, we have

$$\begin{aligned} \|I_3(\tau)\|_{L^2(\mathbb{R}^N)} &\le \sum_{j=1}^{N} \sup_{V\in G_2} |\tilde{A}_0(V)| \left\| \left[A_j(U^k(\tau)) - A_j(\bar{U}_0), \partial_x^\alpha\right] \partial_{x_j} W^{k+1}(\tau) \right\|_{L^2(\mathbb{R}^N)} \\ &\le \sum_{j=1}^{N} C\Big(\|\nabla_x(A_j(U^k(\tau)) - A_j(\bar{U}_0))\|_{L^\infty(\mathbb{R}^N)} \|\partial_{x_j} W^{k+1}(\tau)\|_{H^{s-1}(\mathbb{R}^N)} \\ &\qquad + \|\partial_{x_j} W^{k+1}(\tau)\|_{L^\infty(\mathbb{R}^N)} \|A_j(U^k(\tau)) - A_j(\bar{U}_0)\|_{H^s(\mathbb{R}^N)} \Big) \\ &\le \sum_{j=1}^{N} C\|A_j(U^k(\tau)) - A_j(\bar{U}_0)\|_{H^s(\mathbb{R}^N)} \|W^{k+1}(\tau)\|_{H^s(\mathbb{R}^N)}. \end{aligned}$$

By using the classical tame estimate for a composed function, [[45] p.43], the triangle inequality, and the induction hypothesis (2.50) for U^k, we have

$$\begin{aligned} \|A_j(U^k(\tau)) - A_j(\bar{U}_0)\|_{H^s(\mathbb{R}^N)} &\le C(\|U^k - U_0^0\|_{H^s(\mathbb{R}^N)} + \|U_0^0 - \bar{U}_0\|_{H^s(\mathbb{R}^N)}) \\ &\le C(R + C(U_0)) \le C. \end{aligned} \tag{2.77}$$

Thus we are led to

$$\|I_3(\tau)\|_{L^2(\mathbb{R}^N)} \le C\|W^{k+1}(\tau)\|_{H^s(\mathbb{R}^N)}.$$

We estimate I_2 as follows

$$\|I_2(\tau)\|_{L^2(\mathbb{R}^N)} \leq \sum_{j=1}^{N} \sup_{V\in G_2} |\tilde{A}_0(V)| \left\|\partial_x^\alpha \left(A_j(U^k(\tau))\partial_{x_j} U_0^0\right)\right\|_{L^2(\mathbb{R}^N)}$$
$$\leq C\sum_{j=1}^{N}\left(\left\|\partial_x^\alpha \left((A_j(U^k) - A_j(\bar{U}_0))(\tau)\partial_{x_j} U_0^0\right)\right\|_{L^2(\mathbb{R}^N)} + \|A_j(\bar{U}_0)\partial_x^\alpha \partial_{x_j} U_0^0\|_{L^2(\mathbb{R}^N)}\right).$$

Obviously the second term only depends on the initial value U_0^0 and $\bar{U}_0$. For the first term we use the classical tame estimate for a product of functions, see [45] p.43, which yields

$$\begin{aligned} &\|\partial_x^\alpha \left((A_j(U^k) - A_j(\bar{U}_0))(\tau)\partial_{x_j} U_0^0\right)\|_{L^2(\mathbb{R}^N)} \\ &\quad \leq C\|(A_j(U^k) - A_j(\bar{U}_0))(\tau)\|_{H^s(\mathbb{R}^N)}\|\partial_{x_j} U_0^0\|_{H^s(\mathbb{R}^N)} \\ &\quad \leq C(R + C(U_0)) \leq C, \end{aligned} \tag{2.78}$$

where we have used (2.77). Note that (2.78) is valid in both $|\alpha| = 0$ and $0 < |\alpha| \leq s$. We can conclude that, for all $0 \leq \tau \leq t$,

$$\|I_2(\tau)\|_{L^2(\mathbb{R}^N)} \leq C, \text{ for all } 0 \leq |\alpha| \leq s.$$

Similarly to estimate I_1, for all $0 \leq \tau \leq t$, we begin with

$$\|I_1(\tau)\|_{L^2(\mathbb{R}^N)} \leq \sup_{V\in G_2} |\tilde{A}_0(V)| \|\partial_x^\alpha b(U^k, f^k)(\tau)\|_{L^2(\mathbb{R}^N)}.$$

So we are lead to estimate the term $\|\partial_x^\alpha b(U^k, f^k)(\tau)\|_{L^2(\mathbb{R}^N)}$. Firstly we study the case $|\alpha| \neq 0$. Next we study the case that $|\alpha| = 0$. Recall that to prove (2.59), we have used the fact that the components of $b(U, f)$ can be recast either the form (2.60) or (2.61). Obviously we need to estimate the following terms

$$\left\|\partial_x^\alpha \int_{\mathbb{S}^{N-1}} g_1(v, U^k)(\tau) f^k(\tau, x, v)\mathrm{d}v\right\|_{L^2(\mathbb{R}_x^N)}$$

and

$$\left\|\partial_x^\alpha \int_{\mathbb{S}^{N-1}} g_2(v, U^k)(\tau)\mathrm{d}v\right\|_{L^2(\mathbb{R}_x^N)}.$$

First we have

$$\begin{aligned} &\left\|\partial_x^\alpha \left(\int_{\mathbb{S}^{N-1}} g_1(v, U^k)(\tau) f^k(\tau, x, v)\mathrm{d}v\right)\right\|_{L^2(\mathbb{R}_x^N)} \\ &\quad \leq \int_{\mathbb{S}^{N-1}} \left\|\partial_x^\alpha \left((g_1(v, U^k) - g_1(v, U_0^0))(\tau)(f^k(\tau, x, v) - f_0^0(x, v))\right)\right\|_{L^2(\mathbb{R}_x^N)} \mathrm{d}v \\ &\qquad + \int_{\mathbb{S}^{N-1}} \left\|\partial_x^\alpha \left(g_1(v, U_0^0)(f^k(\tau, x, v) - f_0^0(x, v))\right)\right\|_{L^2(\mathbb{R}_x^N)} \mathrm{d}v \\ &\qquad + \int_{\mathbb{S}^{N-1}} \left\|\partial_x^\alpha \left(g_1(v, U^k)(\tau) f_0^0(x, v)\right)\right\|_{L^2(\mathbb{R}_x^N)} \mathrm{d}v \end{aligned}$$

Observe that in the case $|\alpha| \neq 0$ the third term in the right hand side can be written as

$$\|\partial_x^\alpha \left(g_1(v, U^k)(\tau) f_0^0(x, v)\right)\|_{L^2(\mathbb{R}_x^N)} = \|\partial_x^\alpha \left(g_1(v, U^k)(\tau) f_0^0(x, v) - g_1(v, 0)\bar{f}_0\right)\|_{L^2(\mathbb{R}_x^N)}.$$

Using the induction hypothesis (2.50), (2.52), (2.53) and the tame estimate for product yields

$$\int_{\mathbb{S}^{N-1}} \left\|\partial_x^\alpha(g_1(v,U^k)f^k(\tau,x,v))\right\|_{L^2(\mathbb{R}_x^N)} \mathrm{d}v \le C(N_s(f_0-\bar f_0)+M\tau);$$

While for g_2, the tame estimate for composed functions yields

$$\left\|\partial_x^\alpha \int_{\mathbb{S}^{N-1}} g_2(v,U^k)(\tau)\mathrm{d}v\right\|_{L^2(\mathbb{R}_x^N)} \le C(N_s(f_0-\bar f_0)+M\tau).$$

Thus we are led to

$$\|\partial_x^\alpha b(U^k,f^k)(\tau)\|_{L^2(\mathbb{R}^N)} \le C(N_s(f_0-\bar f_0)+M\tau), \tag{2.79}$$

for $0<|\alpha|\le s$, and $0\le\tau\le t$.

Then in the case of $|\alpha|=0$ we need to estimate $\|b(U^k,f^k)(\tau)\|_{L^2(\mathbb{R}^N)}$. In what follows we study the estimate of $\|<|Q(U^k,f^k)|>\|_{L^2(\mathbb{R}^N)}$ which gives the estimate $\|b(U^k,f^k)(\tau)\|_{L^2(\mathbb{R}^N)}$ immediately. From the expression of Q we have

$$\begin{aligned}\int_{\mathbb{S}^{N-1}} |Q(u^k,f^k)|\mathrm{d}v \;\le\; & \int_{\mathbb{S}^{N-1}} \left|\int_{\mathbb{S}^{N-1}} \Big(\tilde g_1(v,v',u^k)f(v') - \tilde g_2(v,v',u^k)f(v)\Big)\mathrm{d}v'\right| \mathrm{d}v\\ &+\int_{\mathbb{S}^{N-1}} \left|\tilde g_3(v,u^k)\mathbb{B}(\theta^k)-\tilde g_4(v,u^k)f(v)\right|\mathrm{d}v,\end{aligned}$$

where we use the abbreviation $\theta^k=\theta(U^k)$. Note that $\tilde g_1|_{u=0}=\frac{\sigma_s(v,v')}{l_s}$, $\tilde g_2|_{u=0}=\frac{\sigma_s(v',v)}{l_s}$, $\tilde g_3|_{u=0}=\tilde g_4|_{u=0}=\frac{\sigma_a(v)}{l_a}$. Immediately using the triangle inequality we have

$$\begin{aligned}\int_{\mathbb{S}^{N-1}} |Q(u^k,f^k)|\mathrm{d}v \;\le\; & C\left(\int_{\mathbb{S}^{N-1}} |u^k f^k(v)|\mathrm{d}v + \int_{\mathbb{S}^{N-1}} |f^k(v)-\bar f_0|\mathrm{d}v\right)\\ &+C\left(\int_{\mathbb{S}^{N-1}} |u^k\mathbb{B}(\theta^k)|\mathrm{d}v + \int_{\mathbb{S}^{N-1}} |\mathbb{B}(\theta^k)-\bar f_0|\mathrm{d}v\right).\end{aligned}$$

Together with the fact that $\bar f_0=\mathbb{B}(\bar\theta_0)$, the velocity of the fluid $\bar u=0$, we can write the above inequality as

$$\int_{\mathbb{S}^{N-1}} |Q(u^k,f^k)|\mathrm{d}v \;\le\; C\sup_{x,v}|f^k(\tau,x,v)||u^k| + C\|f^k(\tau,x)-\bar f_0\|_{L^2(\mathbb{S}_v^{N-1})} + C|U^k-\bar U_0|.$$

Thus we have

$$\begin{aligned}\left\|\int_{\mathbb{S}^{N-1}} |Q(u^k,f^k)|\mathrm{d}v\right\|_{L^2(\mathbb{R}_x^N)} &\le C\left\|u^k\right\|_{L^2(\mathbb{R}^N)}\left\|f^k\right\|_{L^\infty([0,t]\times\mathbb{R}_x^N\times\mathbb{S}_v^{N-1})}\\ +C\left\|U^k-\bar U_0\right\|_{L^2(\mathbb{R}^N)} &+ C\left\|f^k(\tau)-\bar f_0\right\|_{L^2(\mathbb{R}_x^N\times\mathbb{S}_v^{N-1})}.\end{aligned}$$

Now we have the $L^2(\mathbb{R}^N)$ norm of $b(U^k,f^k)$:

$$\begin{aligned}\|b(U^k,f^k)\|_{L^2(\mathbb{R}^N)} \;&\le\; C\left\|\int_{\mathbb{S}^{N-1}} |Q(u^k,f^k)|\mathrm{d}v\right\|_{L^2(\mathbb{R}^N)}\\ &\le\; C\left\|u^k\right\|_{L^2(\mathbb{R}^N)}\left\|f^k\right\|_{L^\infty([0,t]\times\mathbb{R}_x^N\times\mathbb{S}_v^{N-1})} + C\left\|U^k-\bar U_0\right\|_{L^2(\mathbb{R}^N)}\\ &\quad +C\left\|f^k(\tau)-\bar f_0\right\|_{L^2(\mathbb{R}_x^N\times\mathbb{S}_v^{N-1})}.\end{aligned}$$

Using the induction hypothesis (2.50), (2.52) and (2.53) for U^k and f^k yields

$$\|b(U^k, f^k)\|_{L^2(\mathbb{R}^N)} \leq C(N_s(f_0 - \bar{f}_0) + M\tau + 1).$$

Together with the inequality (2.79), we have

$$\|\partial_x^\alpha b(U^k, f^k)(\tau)\|_{L^2(\mathbb{R}^N)} \leq C(N_s(f_0 - \bar{f}_0) + M\tau + 1), \quad \text{for } 0 \leq |\alpha| \leq s. \tag{2.80}$$

Now as a conclusion that we get

$$\|I_1(\tau)\|_{L^2(\mathbb{R}^N)} \leq C(N_s(f_0 - \bar{f}_0) + M\tau + 1) \leq C(1 + M\tau).$$

We come back to (2.74) by combining together (2.76) and the estimates to I_1, I_2, , I_3, and we end up with

$$\begin{aligned}
&\|W_\alpha^{k+1}(t)\|_{L^2(\mathbb{R}^N)} \\
&\leq \frac{1}{\hat{C}} \left(\|W_\alpha^{k+1}(0)\|_{L^2(\mathbb{R}^N)} + \int_0^t C\left((1 + M\tau) + \|W^{k+1}(\tau)\|_{H^s(\mathbb{R}^N)}\right) \mathrm{d}\tau \right) \\
&\qquad + C\int_0^t \|W_\alpha^{k+1}(\tau)\|_{L^2(\mathbb{R}^N)} \mathrm{d}\tau.
\end{aligned}$$

Summing over all $\alpha \in \mathbb{N}^N$, $0 \leq |\alpha| \leq s$, we obtain

$$\begin{aligned}
&\|W^{k+1}(t)\|_{H^s(\mathbb{R}^N)} \\
&\leq \frac{1}{\hat{C}} \left(\|W^{k+1}(0)\|_{H^s(\mathbb{R}^N)} + C(1 + Mt)t \right) + C\int_0^t \|W^{k+1}(\tau)\|_{H^s(\mathbb{R}^N)} \mathrm{d}\tau.
\end{aligned}$$

The Gronwall Lemma yields

$$\sup_{0 \leq t \leq T} \|W^{k+1}\|_{H^s(\mathbb{R}^N)} \leq \exp(CT)\frac{1}{\hat{C}} \left(\|W^{k+1}(0)\|_{H^s(\mathbb{R}^N)} + C(1 + MT)T \right),$$

for any $0 \leq T \leq T_{k+1}$. Using (2.43) twice, we get

$$\sup_{0 \leq t \leq T} \|W^{k+1}\|_{H^s(\mathbb{R}^N)} \leq \exp(CT)\frac{1}{\hat{C}} \left(R\frac{\hat{C}}{2} + C(1 + MT)T \right).$$

Thus there exists a $T_* > 0$ and $T_* < 1$, (the restriction $T_* < 1$ will be used to determine the constant L), such that there holds

$$\sup_{0 \leq t \leq T_*} \|W^{k+1}\|_{H^s(\mathbb{R}^N)} \leq R. \tag{2.81}$$

2.4.2 $L^\infty(H^{s-1})$ Estimate of the time derivative

Next we prove (2.51). By definition of U^{k+1} we have

$$\partial_t U^{k+1} = -\sum_{j=1}^N A_j(U^k)\partial_{x_j} U^{k+1} + b(U^k, f^k).$$

Hence for any multi-index $\alpha \in \mathbb{N}^N$, $0 \leq |\alpha| \leq s-1$, applying ∂_x^α to the above equation yields :

$$\begin{aligned}
&\|\partial_x^\alpha(\partial_t U^{k+1})(t)\|_{L^2(\mathbb{R}^N)} \\
&\leq \|\partial_x^\alpha b(U^k, f^k)(t)\|_{L^2(\mathbb{R}^N)} + \sum_{j=1}^N \left\|\partial_x^\alpha\left(A_j(U^k)\partial_{x_j}U^{k+1}\right)\right\|_{L^2(\mathbb{R}^N)} \\
&\leq \|\partial_x^\alpha b(U^k, f^k)(t)\|_{L^2(\mathbb{R}^N)} + \sum_{j=1}^N \left\|\partial_x^\alpha\left(A_j(\bar{U}_0)\partial_{x_j}U^{k+1}\right)(t)\right\|_{L^2(\mathbb{R}^N)} \\
&\qquad + \sum_{j=1}^N \left\|\partial_x^\alpha\left((A_j(U^k) - A_j(\bar{U}_0))\partial_{x_j}U^{k+1}\right)(t)\right\|_{L^2(\mathbb{R}^N)}
\end{aligned} \tag{2.82}$$

The term $\|\partial_x^\alpha b(U^k, f^k)(t)\|_{L^2(\mathbb{R}^N)}$ has been studied in Section 2.4.1, see (2.80), we have :

$$\|\partial_x^\alpha b(U^k, f^k)(t)\|_{L^2(\mathbb{R}^N)} \leq C(1 + MT_*) \text{ for } 0 \leq |\alpha| \leq s.$$

The second term in (2.82) can be estimated by using (2.81) as :

$$\sum_{j=1}^N \left\|\partial_x^\alpha\left(A_j(\bar{U}_0)\partial_{x_j}U^{k+1}\right)(t)\right\|_{L^2(\mathbb{R}^N)} \leq C(\|W^{k+1}(t)\|_{H^s(\mathbb{R}^N)} + \|U_0 - \bar{U}_0\|_{H^s(\mathbb{R}^N)}) \leq C.$$

Finally estimate of the third term again relies on the tame estimate for product and the Sobolev inequality. We obtain

$$\begin{aligned}
&\left\|\partial_x^\alpha\left((A_j(U^k) - A_j(\bar{U}_0))\partial_{x_j}U^{k+1}\right)(t)\right\|_{L^2(\mathbb{R}^N)} \\
&\leq \left\|\partial_x^\alpha\left((A_j(U^k) - A_j(\bar{U}_0))\partial_{x_j}(U^{k+1} - U_0^0)\right)(t)\right\|_{L^2(\mathbb{R}^N)} \\
&\quad + \left\|\partial_x^\alpha\left((A_j(U^k) - A_j(\bar{U}_0))\partial_{x_j}(U_0^0 - \bar{U}_0)\right)(t)\right\|_{L^2(\mathbb{R}^N)} \\
&\leq C,
\end{aligned}$$

where we have used (2.77) and (2.81). Hence we deduce that

$$\|\partial_x^\alpha(\partial_t U^{k+1})(t)\|_{L^2(\mathbb{R}^N)} \leq C(1 + MT_*), \text{ for all } 0 \leq t \leq T_*,$$

holds. Note that the constant C does depend on R, not on L. Summing over the multi-index $\alpha \in \mathbb{N}^N$, $0 \leq |\alpha| \leq s-1$, yields

$$\sup_{0 \leq t \leq T_*} \|\partial_t U^{k+1}\|_{H^{s-1}(\mathbb{R}^N)} \leq C(1 + MT_*) \leq C(1 + M) := L.$$

2.4.3 Estimates of the kinetic unknown

We recall that f^{k+1} is defined by (2.48). By integration (2.48) along the characteristics we obtain

$$\begin{aligned}
f^{k+1}(t,x,v) &= f_0(x - cvt, v)\exp\left\{-c\int_0^t F_2(v, u^k(\tau, x + cv(\tau - t)))\mathrm{d}\tau\right\} \\
&\quad + c\int_0^t \exp\left\{-c\int_\tau^t F_2(v, u^k(\tilde{\tau}, x + cv(\tilde{\tau} - t)))\mathrm{d}\tilde{\tau}\right\} Q^+(f^k, U^k)\Big|_{(\tau, x+cv(\tau-t), v)} \mathrm{d}\tau.
\end{aligned}$$

Recall that

$$\begin{aligned} Q^+(f,U) &= F_1(f,u,v) + F_3(u,\theta,v), \\ Q^-(f,U) &= F_2(u,v)f(t,x,v), \end{aligned}$$

are the gain term and the loss one in the source function Q, $F_2 \geq 0$, and in fact Q^+ can be written as :

$$Q^+ = \frac{1}{l_s}\frac{\langle \sigma_s \Lambda^2 f\rangle}{\Lambda^3} + \frac{\sigma_a}{l_a}\frac{\mathbb{B}(\theta)}{\Lambda^3}.$$

In 2.3.3 we have defined the constant M by (2.45) and the Q^+ could be bounded as in (2.46). With these discussions we have, for any $0 \leq t \leq T_*$,

$$\|f^{k+1}(t)\|_{L^\infty(\mathbb{R}_x^N\times\mathbb{S}_v^{N-1})} \leq \|f_0\|_{L^\infty(\mathbb{R}_x^N\times\mathbb{S}_v^{N-1})} + Mt\left(\sup_{0\leq t\leq T_*}\|f^k\|_{L^\infty(\mathbb{R}_x^N\times\mathbb{S}_v^{N-1})} + 1\right).$$

Using the induction hypothesis (2.53) for f_k yields

$$\begin{aligned} \sup_{0\leq t\leq T_*}\|f^{k+1}\|_{L^\infty(\mathbb{R}_x^N\times\mathbb{S}_v^{N-1})} &\leq (1+2MT_*)\|f_0\|_{L^\infty(\mathbb{R}_x^N\times\mathbb{S}_v^{N-1})} + MT_*(1+2MT_*) \\ &\leq 2\|f_0\|_{L^\infty(\mathbb{R}_x^N\times\mathbb{S}_v^{N-1})} + 2MT_*. \end{aligned}$$

if $MT_* \leq 1/2$, thus we get (2.53).

Next we seek an estimate on the space derivative of f^{k+1}. To this purpose, write (2.48) in $f^{k+1} - f_0^0$ and apply ∂_x^α to the resulting system, for any multi-index α, $0 \leq |\alpha| \leq s$, and get the following :

$$\left\{ \begin{array}{l} \frac{1}{c}\partial_t\partial_x^\alpha(f^{k+1} - f_0^0) + v\cdot\nabla_x\partial_x^\alpha(f^{k+1} - f_0^0) + F_2(v,u^k)\partial_x^\alpha(f^{k+1} - f_0^0)) \\ \quad = \partial_x^\alpha(Q^+(f^k,U^k) - F_2(v,u^k)f_0^0) + \partial_x^\alpha\left(v\cdot\nabla_x f_0^0\right) \\ \quad +[F_2(v,u^k) - F_2(v,0), \partial_x^\alpha](f^{k+1} - f_0^0) \\ \partial_x^\alpha(f^{k+1} - f_0^0)(0,x,v) = \partial_x^\alpha(f_0^{k+1}(x,v) - f_0^0). \end{array} \right. \tag{2.83}$$

Also integrating along the characteristics, we can write the solution as

$$\partial_x^\alpha(f^{k+1} - f_0^0) = \overline{I}_1 + \overline{I}_2 + \overline{I}_3,$$

with

$$\begin{aligned} \overline{I}_1 &= \left(\partial_x^\alpha(f_0^{k+1} - f_0^0)\right)(x - cvt, v)\exp\left\{-c\int_0^t F_2(v,u^k(\tau, x + cv(\tau - t)))\mathrm{d}\tau\right\}, \\ \overline{I}_2 &= c\int_0^t \exp\left\{-c\int_\tau^t F_2(v,u^k(\tau', x + cv(\tau' - t)))\mathrm{d}\tau'\right\} \\ &\quad \times\partial_x^\alpha\left(Q^+(f^k,U^k) - F_2(v,u_k)f_0^0 - v\cdot\nabla_x f_0^0\right)\Big|_{(\tau, x+cv(t-\tau), v)}\mathrm{d}\tau, \\ \overline{I}_3 &= -c\int_0^t \exp\left\{-c\int_\tau^t F_2(v,u^k(\tau', x + cv(\tau' - t)))\mathrm{d}\tau'\right\} \\ &\quad \times[F_2(v,u^k) - F_2(v,0), \partial_x^\alpha](f^{k+1} - f_0^0)\Big|_{(\tau, x+cv(t-\tau), v)}\mathrm{d}\tau. \end{aligned}$$

At first we study $\bar{I}_1$, since $F_2 \geq 0$, we have for any t, $0 \leq t \leq T_*$, $0 \leq |\alpha| \leq s$,

$$\|\overline{I}_1(t)\|_{L^2(\mathbb{R}_x^N\times\mathbb{S}_v^{N-1})} \leq \|\partial_x^\alpha(f_0^{k+1} - f_0^0)\|_{L^2(\mathbb{R}_x^N\times\mathbb{S}_v^{N-1})}.$$

Then to estimate $\overline{I}_2$ at first we use the Cauchy-Schwarz inequality and triangle inequality and get :

$$
\begin{aligned}
&\|\overline{I}_2(t)\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)}\\
&\le ct\int_0^t\int_{\mathbb{S}^2}\|\partial_x^\alpha\left(Q^+(f^k,U^k)-F_2(v,u^k)f_0^0-v\cdot\nabla_x f_0^0\right)(\tau,v)\|^2_{L^2(\mathbb{R}^N_x)}\mathrm{d}v\mathrm{d}\tau\\
&\le ct\int_0^t\int_{\mathbb{S}^2}\|\partial_x^\alpha\left(Q^+(f^k,U^k)-F_2(v,u^k)f_0^0\right)(\tau,v)\|^2_{L^2(\mathbb{R}^N_x)}\mathrm{d}v\mathrm{d}\tau\\
&\quad+ct\int_0^t\int_{\mathbb{S}^2}\|v\cdot\nabla_x f_0^0(\tau,v)\|^2_{L^2(\mathbb{R}^N_x)}\mathrm{d}v\mathrm{d}\tau\\
&\le Ct\int_0^t\int_{\mathbb{S}^2}\|\partial_x^\alpha\left(Q^+(f^k,U^k)-F_2(v,u^k)f_0^0\right)(\tau,v)\|^2_{L^2(\mathbb{R}^N_x)}\mathrm{d}v\mathrm{d}\tau+Ct^2N_s(f_0^0-\bar{f}_0)^2.
\end{aligned}
$$

To estimate the first term in the right hand side, firstly recall that

$$
F_2(v,u)=\left\langle\frac{\sigma_s}{l_s\Lambda^2}\right\rangle+\frac{\sigma_a}{l_a}\Lambda,
$$

$$
Q^+(f,U)=\frac{1}{l_s}\frac{\langle\sigma_s\Lambda^2 f\rangle}{\Lambda^3}+\frac{\sigma_a}{l_a}\frac{\mathbb{B}(\theta)}{\Lambda^3},
$$

then $Q^+(f,U)-F_2(v,u^k)f_0^0$ can be written as

$$
\begin{aligned}
Q^+(f^k,U^k)-F_2(v,u^k)f_0^0 \;=\;& \int_{\mathbb{S}^{N-1}}\left(\tilde{g}_1(v,v',u^k)f^k(t,x,v')-\tilde{g}_2(v,v',u^k)f_0^0(x,v)\right)\mathrm{d}v'\\
&+\left(\tilde{g}_3(v,u^k)\mathbb{B}(\theta^k)-\tilde{g}_4(v,u^k)f_0^0\right).
\end{aligned}
$$

Note that there hold $\tilde{g}_1|_{u=0}=\frac{\sigma_s(v,v')}{l_s}$, $\tilde{g}_2|_{u=0}=\frac{\sigma_s(v',v)}{l_s}$, $\tilde{g}_3|_{u=0}=g_4|_{u=0}=\frac{\sigma_a(v)}{l_a}$. With these notations we can write $Q^+(f^k,U^k)-F_2(v,u^k)f_0^0$ as

$$
\begin{aligned}
&Q^+(f^k,U^k)-F_2(v,u^k)f_0^0\\
&=\int_{\mathbb{S}^{N-1}}\tilde{g}_1(v,v',u^k)(f^k(t,x,v')-f_0^0(x,v'))\mathrm{d}v'+\int_{\mathbb{S}^{N-1}}\tilde{g}_1(v,v',u^k)(f_0^0(x,v')-\bar{f}_0)\mathrm{d}v'\\
&\quad-\int_{\mathbb{S}^{N-1}}\tilde{g}_2(v,v',u^k)(f_0^0(t,x,v)-\bar{f}_0)\mathrm{d}v'+\int_{\mathbb{S}^{N-1}}(\tilde{g}_1(v,v',u^k)-\tilde{g}_2(v,v',u^k))\bar{f}_0\mathrm{d}v'\\
&\quad+\tilde{g}_3(v,u^k)(\mathbb{B}(\theta^k)-\bar{f}_0)-\tilde{g}_4(v,u^k)(f_0^0-\bar{f}_0)+(\tilde{g}_3(v,u^k)-\tilde{g}_4(v,u^k))\bar{f}_0.
\end{aligned}
$$

Recall that $\bar{f}_0=\mathbb{B}(\bar{\theta}_0)$. By the tame estimate for a product we can get

$$
\begin{aligned}
&\|\partial_x^\alpha(Q^+(f^k,U^k)-F_2(v,u^k)f_0^0)\|_{L^2(\mathbb{R}^N)}\\
&\quad\le C\|U^k-\bar{U}_0\|_{H^s(\mathbb{R}^N)}+C\|f^k-f_0^0\|_{H^s(\mathbb{R}^N_x)}+C\|f_0^0-\bar{f}_0\|_{H^s(\mathbb{R}^N_x)},
\end{aligned}
$$

for all $0\le|\alpha|\le s$. Immediately we have that

$$
\|\overline{I}_2(t)\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)}\le Ct^2\left(\|U^k-\bar{U}_0\|^2_{H^s(\mathbb{R}^N)}+N_s(f^k-f_0^0)^2+N_s(f_0^0-\bar{f}_0)^2\right)
$$

Using the induction hypothesis (2.50) and (2.52) yields

$$
\|\overline{I}_2(t)\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)}\le Ct^2\left((2N_s(f_0^0-\bar{f}_0)+Mt)^2+N_s(f_0^0-\bar{f}_0)^2+1\right).
$$

It remains to discuss $\overline{I}_3$. Observe that $I_3 = 0$ if $|\alpha| = 0$. Thus we limit our discussion to the case $|\alpha| \neq 0$. The Cauchy-Schwarz inequality leads to

$$\|\overline{I}_3(t)\|^2_{L^2(\mathbb{R}^N_x \times \mathbb{S}^{N-1}_v)} \leq ct \int_0^t \int_{\mathbb{S}^2} \|[F_2(v,u^k) - F_2(v,0), \partial^\alpha_x](f^{k+1} - f^0_0)(\tau, v)\|^2_{L^2(\mathbb{R}^N_x)} \mathrm{d}\tau \mathrm{d}v.$$

Using the tame estimate for a commutator and the Sobolev inequality yields for any $0 \leq \tau \leq t$, and $v \in \mathbb{S}^{N-1}$,

$$\begin{aligned}
&\|[F_2(v,u^k) - F_2(v,0), \partial^\alpha_x](f^{k+1} - f^0_0)(\tau, v)\|_{L^2(\mathbb{R}^N_x)} \\
&\leq C\|F_2(v,u^k) - F_2(v,0)\|_{H^s(\mathbb{R}^N)} \|f^{k+1} - f^0_0\|_{H^s(\mathbb{R}^N_x)} \\
&\leq C\|f^{k+1}(\tau, v) - f^0_0\|_{H^s(\mathbb{R}^N_x)},
\end{aligned}$$

Thus we get

$$\|\overline{I}_3(t)\|^2_{L^2(\mathbb{R}^N_x \times \mathbb{S}^{N-1}_v)} \leq ct \int_0^t N_s(f^{k+1}(\tau) - f^0_0)^2 \mathrm{d}\tau.$$

Combining the estimates of $\overline{I}_1$, $\overline{I}_2$ and $\overline{I}_3$ gives

$$\begin{aligned}
&\|\partial^\alpha_x(f^{k+1}(t) - f^0_0)\|^2_{L^2(\mathbb{R}^N \times \mathbb{S}^{N-1})} \\
&\leq 3\left(\|\overline{I}_1(t)\|^2_{L^2(\mathbb{R}^N_x \times \mathbb{S}^{N-1}_v)} + \|\overline{I}_2(t)\|^2_{L^2(\mathbb{R}^N_x \times \mathbb{S}^{N-1}_v)} + \|\overline{I}_3(t)\|^2_{L^2(\mathbb{R}^N_x \times \mathbb{S}^{N-1}_v)}\right) \\
&\leq 3\left\|\partial^\alpha_x(f^{k+1}_0 - f^0_0)\right\|^2_{L^2(\mathbb{R}^N \times \mathbb{S}^{N-1})} + Ct^2\left(1 + N_s(f^0_0 - \bar{f}_0)^2 + \left(2N_s(f^0_0 - \bar{f}_0) + Mt\right)^2\right) \\
&+Ct \int_0^t N_s(f^{k+1}(\tau) - f^0_0)^2 \mathrm{d}\tau.
\end{aligned}$$

The above inequality holds for all $\alpha \in \mathbb{N}^N$, $0 \leq |\alpha| \leq s$, sum over all such α, and get,

$$\begin{aligned}
N_s(f^{k+1} - f^0_0)^2 \ \leq\ & 3N_s(f^{k+1}_0 - f^0_0)^2 + Ct \int_0^t N_s(f^{k+1}(\tau) - f^0_0)^2 \mathrm{d}\tau \\
&+CT^2_*\left(\left(2N_s(f^0_0 - \bar{f}_0) + MT_*\right)^2 + N_s(f^0_0 - \bar{f}_0)^2 + 1\right).
\end{aligned}$$

Finally Using the Gronwall Lemma, the triangle inequality and (2.44) in Lemma 2.8 gives

$$\sup_{0 \leq t \leq T_*} N_s(f^{k+1} - f^0_0)^2 \ \leq\ (2N_s(f^0 - \bar{f}_0) + MT_*)^2,$$

thus we get (2.52) for T_* small enough.

2.5 Contraction in L^2 norm : proof of Lemma 2.11

In this section we prove Lemma 2.11. Again we split the proof, discussing first $U^{k+1} - U^k$ to get the $L^\infty(L^2)$ estimate, then dealing with $f^{k+1} - f^k$ to get its estimate. Finally we get the inequality (2.54) by concluding the two estimates.

2.5.1 Estimate of the fluid unknown

In this subsection we use the energy inequality (2.38) and the Gronwall Lemma to get the estimate of $U^{k+1} - U^k$. Firstly by definition we compute that $U^{k+1} - U^k$ satisfies :

$$\left\{\begin{aligned}
\tilde{A}_0(U^k)\partial_t(U^{k+1} - U^k) + \sum_{j=1}^N \tilde{A}_j(U^k)\partial_{x_j}(U^{k+1} - U^k) &= H_1 + H_2, \\
(U^{k+1} - U^k)(0,x) &= U^{k+1}_0 - U^k_0,
\end{aligned}\right.$$

with H_1 and H_2 defined as

$$H_1 = \tilde{A}_0(U^k)\left(b(U^k, f^k) - b(U^{k-1}, f^{k-1})\right),$$
$$H_2 = \sum_{j=1}^{N} \tilde{A}_0(U^k)\left(A_j(U^k) - A_j(U^{k-1})\right)\partial_{x_j} U^k.$$

Using the energy inequality (2.38) for any $0 \le t \le T_*$ yields

$$\begin{aligned}
&\|(U^{k+1} - U^k)(t)\|_{L^2(\mathbb{R}^N)} \\
&\quad \le \frac{1}{\hat{C}}\left(\|U_0^{k+1} - U_0^k\|_{L^2(\mathbb{R}^N)} + \int_0^t \|(H_1 + H_2)(\tau)\|_{L^2(\mathbb{R}^N)} \mathrm{d}\tau\right) \\
&\quad + \frac{1}{2\hat{C}}\Big\|\partial_t(\tilde{A}_0(U^k)) + \sum_{j=1}^{N} \partial_{x_j}(\tilde{A}_j(U^k))\Big\|_{L^\infty([0,t]\times\mathbb{R}^N)} \\
&\qquad \int_0^t \|(U^{k+1} - U^k)(\tau)\|_{L^2(\mathbb{R}^N)} \mathrm{d}\tau.
\end{aligned} \tag{2.84}$$

At first we study H_1 in L^2 norm, since $U^k \in G_2$ we have

$$\|H_1(\tau)\|_{L^2(\mathbb{R}^N)} \le C\|(b(U^k, f^k) - b(U^{k-1}, f^{k-1}))(\tau)\|_{L^2(\mathbb{R}^N)}.$$

Recall that to get the estimate of I_1 in the Subsection 2.4.1, we have studied the components of $b(U, f)$ by involving two functions g_1 and g_2, see (2.60) and (2.61). Following this idea, we thus need to study the following quantities :

$$\begin{aligned}
&\left\|\int_{\mathbb{S}^{N-1}} \left(g_1(v, U^k) f^k - g_1(v, U^{k-1}) f^{k-1}\right)(\tau)\mathrm{d}v\right\|^2_{L^2(\mathbb{R}^N_x)} \\
&\le \int_{\mathbb{S}^{N-1}} \left\|\left(g_1(v, U^k) f^k - g_1(v, U^{k-1}) f^{k-1}\right)(\tau, v)\right\|^2_{L^2(\mathbb{R}^N_x)} \mathrm{d}v
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\int_{\mathbb{S}^{N-1}} \left(g_2(v, U^k) - g_2(v, U^{k_1})\right)(\tau)\mathrm{d}v\right\|^2_{L^2(\mathbb{R}^N_x)} \\
&\le \int_{\mathbb{S}^{N-1}} \left\|\left(g_2(v, U^k) - g_2(v, U^{k_1})\right)(\tau, v)\right\|^2_{L^2(\mathbb{R}^N_x)} \mathrm{d}v.
\end{aligned}$$

On the one hand using the triangle inequality and the induction hypothesis, we have the following estimate for g_1 :

$$\begin{aligned}
&\left\|\left(g_1(v, U^k(\tau)) f^k(\tau, v) - g_1(v, U^{k-1}(\tau)) f^{k-1}(\tau, v)\right)\right\|^2_{L^2(\mathbb{R}^N_x)} \\
&\le 2\left\|\left(g_1(v, U^k(\tau)) - g_1(v, U^{k-1}(\tau))\right) f^k(\tau, v)\right\|^2_{L^2(\mathbb{R}^N_x)} \\
&\quad + 2\left\|g_1(v, U^{k-1}(\tau))(f^k(\tau, v) - f^{k-1}(\tau, v))(\tau, v)\right\|^2_{L^2(\mathbb{R}^N_x)} \\
&\le C\left(\|(U^k - U^{k-1})\|^2_{L^2(\mathbb{R}^N)} + \|(f^k - f^{k-1})(\tau, v)\|^2_{L^2(\mathbb{R}^N_x)}\right).
\end{aligned}$$

On the other hand, for the term g_2, there holds

$$\|(g_2(v, U^k) - g_2(v, U^{k-1}))(\tau)\|^2_{L^2(\mathbb{R}^N)} \le C\|(U^k - U^{k-1})(\tau)\|^2_{L^2(\mathbb{R}^N)}.$$

Combining the above estimates for g_1, g_2 we have the following estimate for H_1 :

$$\|H_1(\tau)\|^2_{L^2(\mathbb{R}^N)} \le C\left(\|(U^k - U^{k-1})(\tau)\|^2_{L^2(\mathbb{R}^N)} + \|(f^k - f^{k-1})(\tau)\|^2_{L^2(\mathbb{R}^N_x \times \mathbb{S}^{N-1}_v)}\right).$$

Next using the Sobolev embedding theorem, the triangle inequality and the induction hypothesis (2.50) we estimate $\|H_2(\tau)\|^2_{L^2(\mathbb{R}^N)}$ as follows

$$\begin{aligned}\|H_2(\tau)\|^2_{L^2(\mathbb{R}^N)} &\le \sum_{j=1}^N C\|(A_j(U^k) - A_j(U^{k-1}))(\tau)\|^2_{L^2(\mathbb{R}^N)}\|\partial_{x_j}U^k(\tau)\|^2_{L^\infty(\mathbb{R}^N)}\\ &\le C\|(U^k - U^{k-1})(\tau)\|^2_{L^2(\mathbb{R}^N)}.\end{aligned}$$

Finally to estimate $\|U_0^{k+1} - U_0^k\|_{L^2(\mathbb{R}^N)}$, we first use the triangle inequality, and then use (2.41), we have

$$\begin{aligned}\|U_0^{k+1} - U_0^k\|_{L^2(\mathbb{R}^N)} &\le \|U_0^{k+1} - U_0\|_{L^2(\mathbb{R}^N)} + \|U_0^k - U_0\|_{L^2(\mathbb{R}^N)}\\ &\le C\varepsilon_0 2^{-k}.\end{aligned} \tag{2.85}$$

Using (2.76), (2.85) and the estimates of H_1, H_2 in (2.84) yields

$$\begin{aligned}&\|(U^{k+1} - U^k)(t)\|_{L^2(\mathbb{R}^N)} \le C\varepsilon_0 2^{-k} + C\int_0^t \|(U^{k+1} - U^k)(\tau)\|_{L^2(\mathbb{R}^N)}\mathrm{d}\tau\\ &\quad + CT\sup_{0\le t\le T}\left(\|(U^k - U^{k-1})(t)\|_{L^2(\mathbb{R}^N)} + \|(f^k - f^{k-1})(t)\|_{L^2(\mathbb{R}^N_x \times \mathbb{S}^{N-1}_v)}\right),\end{aligned}$$

for any T, $0 < T \le T_*$. Then using the Gronwall Lemma to the above inequality yields

$$\begin{aligned}&\sup_{0\le t\le T}\|(U^{k+1} - U^k)(t)\|_{L^2(\mathbb{R}^N)}\\ &\quad \le CT\exp(CT)\sup_{0\le t\le T}\left(\|U^k - U^{k-1}\|_{L^2(\mathbb{R}^N)} + \|f^k - f^{k-1}\|_{L^2(\mathbb{R}^N)}\right)\\ &\quad + C\exp(CT)\varepsilon_0 2^{-k},\end{aligned} \tag{2.86}$$

for any T, $0 < T \le T_*$.

2.5.2 Estimate of the kinetic unknown

In this subsection we write $f^{k+1} - f^k$ explicitly and give its L^2 estimate. Firstly by definition we compute that $f^{k+1} - f^k$ satisfies

$$\begin{cases} \frac{1}{c}\partial_t(f^{k+1} - f^k) + v\cdot\nabla_x(f^{k+1} - f^k) + F_2(v, u^k)(f^{k+1} - f^k) = K_1 + K_2,\\ (f^{k+1} - f^k)(0, x, v) = (f_0^{k+1} - f_0^k)(x, v),\end{cases}$$

with

$$\begin{aligned}K_1 &= Q^+(f^k, U^k) - Q^+(f^{k-1}, U^{k-1}),\\ K_2 &= (F_2(v, u^k) - F_2(v, u^{k-1}))f^k.\end{aligned}$$

Thus integrating along the characteristics, we can write $f^{k+1} - f^k$ explicitly as :

$$\begin{aligned}&(f^{k+1} - f^k)(t, x, v)\\ &= (f_0^{k+1} - f_0^k)(x - cvt, v)\exp\left\{-c\int_0^t F_2(v, u^k(\tau, x + cv(\tau - t)))\mathrm{d}\tau\right\}\\ &\quad + c\int_0^t \exp\left\{-c\int_\tau^t F_2(v, u^k(\tau', x + cv(\tau' - t)))\mathrm{d}\tau'\right\}(K_1 + K_2)\Big|_{(\tau, x+cv(t-\tau), v)}\mathrm{d}\tau.\end{aligned}$$

For convenience let us split $f^{k+1}-f^k$ into 3 parts $\tilde{I}_1$, $\tilde{I}_2$ and $\tilde{I}_3$ as

$$\tilde{I}_1=(f_0^{k+1}-f_0^k)(x-cvt,v)\exp\left\{-c\int_0^t F_2(v,u^k(\tau,x+cv(\tau-t)))\mathrm{d}\tau\right\},$$
$$\tilde{I}_2=c\int_0^t \exp\left\{-c\int_\tau^t F_2(v,u^k(\tau',x+cv(\tau'-t)))\mathrm{d}\tau'\right\}K_1\Big|_{(\tau,x+cv(t-\tau),v)}\mathrm{d}\tau,$$
$$\tilde{I}_3=-c\int_0^t \exp\left\{-c\int_\tau^t F_2(v,u^k(\tau',x+cv(\tau'-t)))\mathrm{d}\tau'\right\}K_2\Big|_{(\tau,x+cv(t-\tau),v)}\mathrm{d}\tau.$$

In what follows we give the estimates for $\tilde{I}_1$, $\tilde{I}_2$ and $\tilde{I}_3$, which immediately give the estimate for $f^{k+1}-f^k$.

Note that $F_2\geq 0$, then using the triangle inequality and (2.41) we estimate $\tilde{I}_1$ as, for any $0\leq t\leq T_*$,

$$\begin{aligned}\|\tilde{I}_1(t)\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)} &\leq \|f_0^{k+1}-f_0^k\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)}\\ &\leq C\|f_0^{k+1}-f_0\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)}+C\|f_0^{k+1}-f_0\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)}\\ &\leq C\varepsilon_0^2 2^{-2k}.\end{aligned}$$

Next to estimate $\tilde{I}_2$, we use the Cauchy-Schwarz inequality, and obtain

$$\|\tilde{I}_2(t)\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)}\leq t\int_0^t\left\|\left(Q^+(f^k,U^k)-Q^+(f^{k-1},U^{k-1})\right)(\tau)\right\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)}\mathrm{d}\tau.$$

Recall that

$$Q^+(f,U)=\frac{1}{l_s}\frac{\langle\sigma_s\Lambda^2 f\rangle}{\Lambda^3}+\frac{\sigma_a}{l_a}\frac{\mathbb{B}(\theta)}{\Lambda^3},$$

thus we can decompose $Q^+(f,U)$ into two parts as :

$$\int_{\mathbb{S}^{N-1}}\overline{g}_1(v,v',u(\tau,x))f(\tau,x,v')\mathrm{d}v',\qquad \overline{g}_2(v,U(\tau,x)).$$

Following the strategy used in Subsection 2.5.1 to estimate H_1, we get the following estimate :

$$\begin{aligned}&\left\|\left(Q^+(f^k,U^k)-Q^+(f^{k-1},U^{k-1})\right)(\tau)\right\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)}\\ &\leq\left(\left\|(U^k-U^{k-1})(\tau)\right\|^2_{L^2(\mathbb{R}^N)}+\left\|(f^k-f^{k-1})(\tau)\right\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)}\right)\end{aligned}$$

Thus we deduce the estimate for I_2, i.e.

$$\|\tilde{I}_2\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)}\leq Ct\int_0^t\left(\left\|(U^k-U^{k-1})(\tau)\right\|^2_{L^2(\mathbb{R}^N)}+\left\|(f^k-f^{k-1})(\tau)\right\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)}\right)\mathrm{d}\tau.$$

Similarly for $\tilde{I}_3$ we use the Cauchy-Schwarz inequality, the induction hypothesis and get

$$\begin{aligned}\left\|\tilde{I}_3(t)\right\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)} &\leq C\,t\int_0^t\left\|(F_2(v,u^k)-F_2(v,u^{k-1}))(\tau)f^k(\tau,x,v)\right\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)}\mathrm{d}\tau\\ &\leq C\,t\int_0^t\left\|(U^k-U^{k-1})(\tau)\right\|^2_{L^2(\mathbb{R}^N)}\mathrm{d}\tau.\end{aligned}$$

By combining the estimates of I_1, I_2 and I_3, we get

$$\begin{aligned}\|(f^{k+1}-f^k)(t)\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)} \;\le\; & \frac{C\varepsilon_0^2}{2^{2k}}+Ct\int_0^t\|(U^k-U^{k-1})(\tau)\|^2_{L^2(\mathbb{R}^N)}\mathrm{d}\tau \\ & +Ct\int_0^t\|(f^k-f^{k-1})(\tau)\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)}\mathrm{d}\tau,\end{aligned}$$

which implies

$$\begin{aligned}&\sup_{0\le t\le T}\|(f^{k+1}-f^k)(t)\|_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)} \\ &\le CT\sup_{0\le t\le T}\left(\|U^k-U^{k-1}\|_{L^2(\mathbb{R}^N)}+\|f^k-f^{k-1}\|_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)}\right)+\frac{C\varepsilon_0}{2^k},\end{aligned} \tag{2.87}$$

for any $0<T\le T_*$.

2.5.3 Conclusion, end of the proof of Lemma 2.11

By combining (2.86) and (2.87), we conclude with

$$\begin{aligned}&\sup_{0\le t\le T}\left(\|(U^{k+1}-U^k)(t)\|_{L^2(\mathbb{R}^N)}+\|(f^{k+1}-f^k)(t)\|_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)}\right) \\ &\le \frac{C\varepsilon_0}{2^k}+CT\exp(CT)\sup_{0\le t\le T}\left(\|U^k-U^{k-1}\|^2_{L^2(\mathbb{R}^N)}+\|f^k-f^{k-1}\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{S}^{N-1}_v)}\right),\end{aligned}$$

for any $0<T\le T_*$.

Then there exists a $T_{**}\in]0,T_*]$, such that

$$a=C\exp(CT_{**})T_{**}<1.$$

We end the proof of Lemma 2.11 by setting

$$\beta_k=\frac{C\varepsilon_0}{2^k}.$$

2.6 Appendix

2.6.1 One dimensional radiation and the conversion relations between the co-moving frame and the lab-fixed frame

Recall that the specific intensity of radiation $f(t,x,v,\nu)$ is defined to be such that the radiation energy at time t corresponding to the radiation of frequencies $(\nu,\nu+\mathrm{d}\nu)$ in a time interval $\mathrm{d}t$, in a position $x+\mathrm{d}x$, into the solid angle $\mathrm{d}v$ around the direction $v\in\mathbb{S}^2$, is

$$\frac{1}{c}f(t,x,v,\nu)\mathrm{d}v\mathrm{d}\nu\mathrm{d}x.$$

where $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ and $t\in\mathbb{R}^+$.

Below we pay attention to a simple case : assuming a one-dimensional structure, and we verify the conversion relations between the co-moving frame and the lab-fixed frame. In this simple case we use a special Lorentz transformation. And the same results hold by applying the general Lorentz transformation.

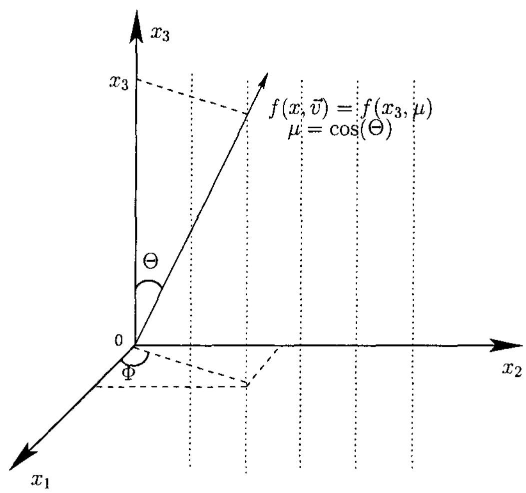

FIG. 2.1 – The planar geometry

We assume that the material is homogeneous in the horizontal directions (in $x_1 - x_2$ plan), with properties varying only as a function of x_3 and time t. Thus the intensity has azimuthal symmetry about the x_3-axis. Its angular distribution can be described completely in terms of the polar angle Θ, or $\mu = \cos\Theta$. Note by Φ the azimuthal angle. Recall that

$$\mathrm{d}v = \frac{\sin\Theta}{4\pi}\mathrm{d}\Theta\mathrm{d}\Phi = \frac{1}{4\pi}\mathrm{d}\mu\mathrm{d}\Phi,$$

and we can write the intensity $f(t, x_3, \mu, \nu)$ as function of x_3, μ and ν, see Fig2.1.

Taking into account the Doppler shift and aberration of light between frames in relative motion, we have the formulae linking the frequency ν^0, v^0 measured in the co-moving frame and the ones ν, v, measured in the lab-fixed frame,

$$\nu^0 = \nu\Upsilon(1 - \frac{v \cdot u}{c}), \quad v^0 = \frac{\nu}{\nu^0}\left(v - \frac{\Upsilon}{c}u(1 - \frac{u \cdot v}{c}\frac{\Upsilon}{\Upsilon + 1})\right),$$

where Υ is the Lorentz factor : $\Upsilon = 1/\sqrt{1 - (|u|/c)^2}$. Since we restrict our discussion to the planar symmetry, we introduce the fluid velocity u as $u = (0, 0, \beta c)^t$. Thus there holds $u \cdot v = \beta\mu c$. Recall that $v_3 = \mu$ and $\Upsilon = 1/\sqrt{1 - \beta^2}$. The above formulae then yield the following special Lorentz transformation :

$$\begin{aligned}
\nu^0 &= \Upsilon\nu\left(1 - \frac{u \cdot v}{c}\right) = \Upsilon\nu(1 - \beta\mu), \\
\nu^0 v_1^0 &= \nu v_{x_1}, \\
\nu^0 v_2^0 &= \nu v_{x_2}, \\
\nu^0 v_3^0 &= \Upsilon\nu(\mu - \beta).
\end{aligned}$$

Note that $\nu^0 v^0_{x_3} = \nu^0 \mu^0$. Then we can easily compute that

$$\begin{aligned} \nu^0 &= \Upsilon\nu(1-\beta\mu), \\ \mu^0 &= \frac{\Upsilon\nu}{\nu^0}(\mu-\beta) = \frac{\mu-\beta}{1-\beta\mu} \\ \left(1-\mu^{0^2}\right)^{\frac{1}{2}} &= \frac{(1-\mu^2)^{\frac{1}{2}}}{\Upsilon(1-\beta\mu)} = (1-\mu^2)^{\frac{1}{2}}\frac{\nu}{\nu^0}, \\ \Phi^0 &= \Phi. \end{aligned}$$

Immediately we get

$$\nu^0 \mathrm{d}\nu^0 \mathrm{d}\mu^0 \mathrm{d}\Phi^0 = \nu \mathrm{d}\nu \mathrm{d}\mu \mathrm{d}\Phi, \tag{2.88}$$

which means that $\nu \mathrm{d}\nu \mathrm{d}v$ is a Lorentz invariant, i.e (2.12) in this simple case. Furthermore we have

$$\mathrm{d}\mu^0 \mathrm{d}\Phi^0 = \frac{1}{\Lambda(v)^2}\mathrm{d}\mu \mathrm{d}\Phi, \tag{2.89}$$

where $\Lambda(v) = \frac{1-\beta\mu}{\sqrt{1-\beta^2}}$. And (2.89) gives (2.16). Note that

$$\mu^0 + \frac{u}{c} = \frac{\mu}{\Upsilon}\frac{1}{1-\beta\mu} = \frac{\mu}{\Upsilon}\frac{\nu}{\nu^0}, \tag{2.90}$$

which will be used for the transformation properties of the specific intensity f.

Next to determine the transformation properties of the specific intensity f we fellow the ideas of L.H Thomas (introduced in [49] p.413). The number of photons, N, in a frequency interval $\mathrm{d}\nu$, passing through an element of area $\mathrm{d}S$ oriented perpendicular to the x_3-axis, into a solid angle $\mathrm{d}v$ along an angle $\Theta = \cos^{-1}\mu$ to the x_3-axis, in a time interval $\mathrm{d}t$, is

$$N = \frac{f(\mu,\nu)}{h\nu}\mathrm{d}v\mathrm{d}\nu\mathrm{d}S\cos\Theta\mathrm{d}t,$$

by letting $\mathrm{d}S$ be stationary in the lab frame.

Then to an observer in a frame moving with velocity u along the x_3-axis, $\mathrm{d}S$ appears to be moving with a velocity u in the negative x_3 direction, then the number of photons that can be counted by the observer is

$$N^0 = \frac{f^0(\mu^0,\nu^0)}{h\nu^0}\mathrm{d}\nu^0\mathrm{d}v^0\left(\mathrm{d}S\cos\Theta^0\mathrm{d}t^0 + \frac{u}{c}\mathrm{d}S\mathrm{d}t^0\right).$$

The first term gives the number of photons that would have been counted if $\mathrm{d}S$ had been stationary, while the second is the photon number density $f^0/(h\nu^0)$ times the volume $u\ \mathrm{d}S\ \mathrm{d}t^0$. Using (2.90) we have

$$N^0 = \frac{f^0(\mu^0,\nu^0)}{h\nu^0}\mathrm{d}\nu^0\mathrm{d}v^0\mathrm{d}S\mathrm{d}t^0\frac{\mu}{\Upsilon}\frac{\nu}{\nu^0}.$$

Both observers must count the same number of photons passing through $\mathrm{d}S$, hence, $N = N^0$, using the fact $\mathrm{d}t^0 = \Upsilon\mathrm{d}t$, and using (2.90) we thus get

$$f(\mu,\nu) = \left(\frac{\nu}{\nu^0}\right)^3 f^0(\mu^0,\nu^0). \tag{2.91}$$

Now we determine the transformation properties of the emissivity η. Observers in all frames will count the same number of photons emitted from a definite volume element into a particular solid angle and frequency interval in a time interval, hence

$$\frac{\eta(\mu,\nu)\mathrm{d}v\mathrm{d}\nu\mathrm{d}x\mathrm{d}t}{h\nu}=\frac{\eta^0(\nu^0)\mathrm{d}v^0\mathrm{d}\nu^0\mathrm{d}x^0\mathrm{d}t^0}{h\nu^0}.$$

Recall that $\mathrm{d}x\mathrm{d}t$ is an invariant, thus we have

$$\eta(\mu,\nu)=\left(\frac{\nu}{\nu^0}\right)^2\eta^0(\nu^0), \tag{2.92}$$

where we note that η^0 is isotropic in the co-moving frame, i.e η^0 is independent of direction variable.

Similarly, observers in all frames will count the same number of photons absorbed by a defined volume element into a particular solid angle and frequency interval in a time interval, hence

$$\frac{\chi(\mu,\nu)f(\mu,\nu)\mathrm{d}v\mathrm{d}\nu\mathrm{d}x\mathrm{d}t}{h\nu}=\frac{\chi^0(\nu^0)f^0(\mu^0,\nu^0)\mathrm{d}v^0\mathrm{d}\nu^0\mathrm{d}x^0\mathrm{d}t^0}{h\nu^0}.$$

Hence

$$\chi(\mu,\nu)=\frac{\nu^0}{\nu}\chi^0(\nu^0). \tag{2.93}$$

Thus we verify (2.12)-(2.16) in the planar radiation. As pointed in [49], p.414, the same results apply for arbitrary relative motion of the two frames provided that μ is replaced by $v\in\mathbb{S}^2$.

2.6.2 Derivation of (2.38)

The energy identity (2.37) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\tilde{A}_0(V)W\cdot W\mathrm{d}x=\int_{\mathbb{R}^N}(\partial_t\tilde{A}_0(V)+\sum_{j=1}^N\partial_{x_j}\tilde{A}_j(V)+\tilde{B}+\tilde{B}^T)W\cdot W\mathrm{d}x+2\int_{\mathbb{R}^N}F\cdot W\mathrm{d}x.$$

By (2.36) we can write the left-hand side as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\tilde{A}_0(V)W\cdot W\mathrm{d}x=2\left(\int_{\mathbb{R}^N}\tilde{A}_0(V)W\cdot W\mathrm{d}x\right)^{\frac12}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^N}\tilde{A}_0(V)W\cdot W\mathrm{d}x\right)^{\frac12}.$$

And the terms in the right-hand side can also be written as

$$\begin{aligned}
&\int_{\mathbb{R}^N}(\partial_t\tilde{A}_0(V)+\sum_{j=1}^N\partial_{x_j}\tilde{A}_j(V)+\tilde{B}+\tilde{B}^T)W\cdot W\mathrm{d}x+\int_{\mathbb{R}^N}F\cdot W\mathrm{d}x\\
&\le\left\|\partial_t\tilde{A}_0(V)+\sum_{j=1}^N\partial_{x_j}\tilde{A}_j(V)+\tilde{B}+\tilde{B}^T\right\|_{L^\infty([0,T]\times\mathbb{R}^N)}\int_{\mathbb{R}^N}W^2\mathrm{d}x\\
&\quad+2\left(\int_{\mathbb{R}^N}F^2\mathrm{d}x\right)^{\frac12}\left(\int_{\mathbb{R}^N}W^2\mathrm{d}x\right)^{\frac12}\\
&=\|\mathrm{DIV}_\tilde{A}+\tilde{B}+\tilde{B}^T\|_{L^\infty([0,T]\times\mathbb{R}^N)}\int_{\mathbb{R}^N}W^2\mathrm{d}x+2\left(\int_{\mathbb{R}^N}F^2\mathrm{d}x\right)^{\frac12}\left(\int_{\mathbb{R}^N}W^2\mathrm{d}x\right)^{\frac12},
\end{aligned}$$

where $\mathrm{DIV\_}\tilde{A} = \partial_t \tilde{A}_0(V) + \sum_{j=1}^N \partial_{x_j} \tilde{A}_j(V)$. Thus the energy identity can be written as

$$\begin{aligned}
&2\left(\int_{\mathbb{R}^N} \tilde{A}_0(V)W \cdot W \mathrm{d}x\right)^{\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^N} \tilde{A}_0(V)W \cdot W \mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \|\mathrm{DIV_}\tilde{A} + \tilde{B} + \tilde{B}^T\|_{L^\infty([0,T]\times\mathbb{R}^N)} \int_{\mathbb{R}^N} W^2 \mathrm{d}x \\
&+2\left(\int_{\mathbb{R}^N} F^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} W^2 \mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.94}$$

Using (2.36) in (2.94) we get

$$\begin{aligned}
&2\sqrt{C}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^N} \tilde{A}_0(V)W \cdot W \mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \left\|\mathrm{DIV_}\tilde{A} + \tilde{B} + \tilde{B}^T\right\|_{L^\infty([0,T]\times\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} W^2 \mathrm{d}x\right)^{\frac{1}{2}} + 2\left(\int_{\mathbb{R}^N} F^2 \mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned}$$

Integrating the above inequality with respect to t yields

$$\begin{aligned}
&2\sqrt{C}\left(\int_{\mathbb{R}^N} \tilde{A}_0(V)W \cdot W \mathrm{d}x\right)^{\frac{1}{2}} \leq 2\sqrt{C}\left(\int_{\mathbb{R}^N} \tilde{A}_0(V(0))W_0 \cdot W_0 \mathrm{d}x\right)^{\frac{1}{2}} \\
&+\|\mathrm{DIV_}\tilde{A} + \tilde{B} + \tilde{B}^T\|_{L^\infty([0,T]\times\mathbb{R}^N)} \int_0^t \|W^2(t)\|_{L^2(\mathbb{R}^N)} \mathrm{d}t + 2\int_0^t \|F^2(t)\|_{L^2(\mathbb{R}^N)} \mathrm{d}t.
\end{aligned}$$

Using (2.36) once again, we get

$$\begin{aligned}
2C\|W(t)\|^2_{L^2(\mathbb{R}^N)} &\leq 2\|W_0\|^2_{L^2(\mathbb{R}^N)} + 2\int_0^t \|F^2(t)\|_{L^2(\mathbb{R}^N)} \mathrm{d}t \\
&+\|\mathrm{DIV_}\tilde{A} + \tilde{B} + \tilde{B}^T\|_{L^\infty([0,T]\times\mathbb{R}^N)} \int_0^t \|W^2(t)\|_{L^2(\mathbb{R}^N)} \mathrm{d}t.
\end{aligned}$$

Thus we end with (2.38).

Chapitre 3

Shock Profiles for Non Equilibrium Radiating Gases

Ce chapitre fait l'objet d'une publication dans la revenue Physcia D.

3.1 Introduction and main results

We are interested in a system of PDEs describing astrophysical flows, where a gas interacts with radiation through energy exchanges. Similar questions arise in the modeling of reentry problems, or high temperature combustion phenomena. The gas is described by its density $\rho > 0$, its bulk velocity $u \in \mathbb{R}$, and its specific total energy $E = e + u^2/2$, where e stands for the specific internal energy. (Our analysis is restricted to a one-dimensional framework, but this is not a loss of generality, as shown below.) We consider a situation where the gas is not in thermodynamical equilibrium with the radiations, which are thus described by their own energy n. The evolution of the gas flow is governed by the system :

$$\begin{cases} \partial_t \rho + \partial_x (\rho\, u) = 0\,, \\ \partial_t (\rho\, u) + \partial_x (\rho\, u^2 + P) = 0\,, \\ \partial_t (\rho\, E) + \partial_x (\rho\, E\, u + P\, u) = n - \theta^4\,, \end{cases} \tag{3.1}$$

where the right-hand side in the last equation accounts for energy exchanges with the radiations, P being the pressure of the gas, and θ its temperature. Throughout the paper, we always assume that the gas obeys the perfect gas pressure law :

$$P = R\, \rho\, \theta = (\gamma - 1)\, \rho\, e\,, \tag{3.2}$$

where R is the perfect gas constant, and $\gamma > 1$ is the ratio of the specific heats at constant pressure, and volume. This assumption yields many algebraic simplifications, but we believe that our results still hold for a general pressure law satisfying the usual requirements of thermodynamics. System (3.1) is completed by considering that radiations are described by a stationary diffusion regime that reads :

$$-\partial_{xx} n = \theta^4 - n\,. \tag{3.3}$$

We detail in Appendix 3.4 how the system (3.1), (3.3) can be formally derived by asymptotics arguments, starting from a more complete system involving a kinetic equation for the specific intensity of radiation.

As a matter of fact, the operator $(1 - \partial_{xx})$ can be explicitly inverted, and (3.3) can be recast as a convolution :

$$n(t,x) = \frac{1}{2} \int_{\mathbb{R}} \mathrm{e}^{-|x-y|}\, \theta(t,y)^4 \, dy \,. \tag{3.4}$$

Let us introduce the quantity :

$$q(t,x) := -\partial_x n\,(t,x) = \frac{1}{2} \int_{\mathbb{R}} \mathrm{e}^{-|x-y|} \operatorname{sgn}(x-y)\, \theta(t,y)^4 \, dy \,, \tag{3.5}$$

where sgn is the sign function :

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

The quantity q can be interpreted as the radiative heat flux. Then, we can rewrite (3.1), (3.3) as follows :

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, u) = 0 \,, \\ \partial_t(\rho\, u) + \partial_x(\rho\, u^2 + P) = 0 \,, \\ \partial_t(\rho\, E) + \partial_x(\rho\, E\, u + P\, u + q) = 0 \,, \end{cases} \tag{3.6}$$

with q given by (3.5). Recall that $E = e + u^2/2$, and P is given by (3.2).

In this paper, we address the question of the influence of the energy exchanges on the structure of shock waves. More precisely, let us consider given states at infinity $(\rho_\pm, u_\pm, e_\pm)$, and let us asume that :

$$(\rho, u, e)(t,x) = \begin{cases} (\rho_-, u_-, e_-) & \text{if } x < \sigma\, t, \\ (\rho_+, u_+, e_+) & \text{if } x > \sigma\, t, \end{cases} \tag{3.7}$$

is a shock wave, with speed σ, solution to the standard Euler equations (that is, system (3.6) with $q \equiv 0$). We refer to [35, 55, 58] for a detailed study of shock waves for the Euler equations. The question we ask is the following : does there exist a traveling wave $(\rho, u, e)(x - \sigma t)$ solution to (3.6), with q given by (3.5), that satisfies the asymptotic conditions :

$$\lim_{\xi \to \pm\infty} (\rho, u, e)(\xi) = (\rho_\pm, u_\pm, e_\pm) \,. \tag{3.8}$$

In other words, we are concerned with the existence of a shock profile, and a natural expectation (at least for shocks of small amplitude) is that the step shock (3.7) is smoothed into a continuous profile, due to the dissipation introduced by (3.3). The analogous problem for the compressible Navier-Stokes system has been treated a long time ago, see [16], without any smallness assumption on the shock wave. Concerning radiative transfer, a formal analysis of shock profiles has been performed in [22], together with rough numerical simulations. (We refer also to [61, 49] for the physical background.) The main purpose of this work is to make the analysis of [22] rigorous. Since we are only concerned in this paper with the existence of shock profiles, and not with their stability, the problem is purely one-dimensional (due to the Galilean invariance of the Euler equations). This is why we have directly restricted to the one-dimensional case. However, the formal derivation of Appendix 3.4 is made in several space dimensions.

Before stating our main results, let us mention that a simplified version of (3.6), (3.5) has been introduced, and studied in [27, 29]. This 'baby-model' consists in a Burgers type equation :

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = -\partial_x q \,,$$

coupled to the diffusion equation :

$$-\partial_{xx} q + q = -\partial_x u \,.$$

These two equations can be seen as a scalar version of (3.6), (3.5) since they can be recast as :

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = Ku - u \,, \tag{3.9}$$

where K is the integral operator already arising in (3.5) :

$$Ku(t,x) = \frac{1}{2} \int_{\mathbb{R}} \mathrm{e}^{-|x-y|}\, u(t,y)\, dy \,.$$

The thorough study of (3.9) has motivated a lot of works ; we mention in particular [50, 33, 41, 56]. Clearly (3.9) can be seen as a prototype for discussing (3.6), (3.5) ; nevertheless, replacing (3.6), and (3.5) by (3.9) has two important consequences : the equation becomes scalar, and the 'diffusion' $K-1$ applies to the unique unknown (while in (3.6), the 'diffusion' appears through the radiative heat flux q only in the third equation). Our work is a first attempt to extend the known results for (3.9) to the more physical model (3.6), (3.5).

Let us now state our main results. The first result deals with the existence of smooth shock profiles when the strength of the shock is small :

Theorem 3.1. *Let γ satisfy*

$$1 < \gamma < \frac{\sqrt{7}+1}{\sqrt{7}-1} \simeq 2.215 \,,$$

and let (ρ_-, u_-, e_-) be fixed. Then there exists a positive constant δ (that depends on (ρ_-, u_-, e_-), and γ) such that, for all state (ρ_+, u_+, e_+) verifying :

– $\|(\rho_+, u_+, e_+) - (\rho_-, u_-, e_-)\| \le \delta$,

– the function (3.7) is a shock wave, with speed σ, for the (standard) Euler equations, then there exists a C^2 traveling wave $(\rho, u, e)(x - \sigma t)$ solution to (3.6), (3.5), (3.8).

As in the study of the 'baby-model' (3.9), the existence of a smooth shock profile is linked to a smallness assumption on the shock strength, see [27]. Here the smallness parameter δ may depend on the state (ρ_-, u_-, e_-), while for (3.9), the smallness parameter is uniform (and even explicit !).

The restriction on the adiabatic constant γ might be unnecessary, but it simplifies the proof, and it covers the main physical cases $1 < \gamma \le 2$.

Our second result is also in the spirit of [27], and deals with the smoothness of the shock profile constructed in the previous Theorem :

Theorem 3.2. *Let γ satisfy*

$$1 < \gamma < \frac{\sqrt{7}+1}{\sqrt{7}-1} \simeq 2.215 \,,$$

and let (ρ_-, u_-, e_-) be fixed. Then there exists a decreasing sequence of positive numbers $(\delta_n)_{n\in\mathbb{N}}$ (the sequence depends on (ρ_-, u_-, e_-), and γ) such that, for all $n \in \mathbb{N}$, and for all state (ρ_+, u_+, e_+) verifying :

- $\|(\rho_+, u_+, e_+) - (\rho_-, u_-, e_-)\| \leq \delta_n$,
- *the function* (3.7) *is a shock wave, with speed σ, for the (standard) Euler equations, then there exists a C^{n+2} traveling wave $(\rho, u, e)(x - \sigma t)$ solution to* (3.6), (3.5), (3.8).

To a large extent, our analysis follows the arguments of [22], and [27]. The proof of Theorem 3.1 is presented in Section 3.2, while Section 3.3 is devoted to the proof of Theorem 3.2. The investigation of strong shocks, as well as stability issues will be addressed in a forthcoming work.

3.2 Existence of smooth shock profiles

In this section, we prove Theorem 3.1. We first recall some basic facts on shock waves for the Euler equations. Then, we make some transformations on the traveling wave equation. Eventually, we prove Theorem 3.1 by using an auxiliary system of Ordinary Differential Equations, that is introduced and studied in the last paragraph of this section.

3.2.1 Shock wave solutions to the Euler equations

In this paragraph, we recall some basic facts about the (entropic) shock wave solutions to the Euler equations :

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, u) = 0\,, \\ \partial_t(\rho\, u) + \partial_x(\rho\, u^2 + P) = 0\,, \\ \partial_t(\rho\, E) + \partial_x(\rho\, E\, u + P\, u) = 0\,, \end{cases}$$

where P, and E are given as in the introduction. We refer to [35, 55, 58] for all the details, and omit the calculations. In all what follows, we only consider shock waves that satisfy Lax shock inequalities. We shall thus speak of 1-shock waves, or 3-shock waves.

We consider a fixed 'left' state (ρ_-, u_-, e_-). Then the 'right' states (ρ_+, u_+, e_+) such that $(\rho_\pm, u_\pm, e_\pm)$ define a 1-shock wave, with speed σ, is a half-curve initiating at (ρ_-, u_-, e_-). Introducing the notation $v_\pm = u_\pm - \sigma$, the Rankine-Hugoniot jump conditions can be re-written as :

$$\begin{cases} \rho_+\, v_+ = \rho_-\, v_- =: j\,, \\ \rho_+\, v_+^2 + P_+ = \rho_-\, v_-^2 + P_- =: j\, C_1\,, \\ \rho_+\, v_+(e_+ + \dfrac{v_+^2}{2}) + P_+\, v_+ = \rho_-\, v_-(e_- + \dfrac{v_-^2}{2}) + P_-\, v_- =: j\, C_2\,. \end{cases} \tag{3.10}$$

Observe that v_- does not only depend on the 'left' state (ρ_-, u_-, e_-), but also on (ρ_+, u_+, e_+), because v_- is defined with the help of the shock speed σ. Consequently, the constants j, C_1, and C_2 depend on both (ρ_-, u_-, e_-), and (ρ_+, u_+, e_+).

For 1-shocks, that is when the inequalities :

$$u_+ - c_+ < \sigma < u_+\,, \quad \sigma < u_- - c_-\,,$$

are satisfied (c denotes the sound speed), all quantities j, C_1, and C_2 are positive. Moreover, when the strength of the shock tends to zero, that is when (ρ_+, u_+, e_+) tends to (ρ_-, u_-, e_-), one has

$$\begin{pmatrix} \sigma \\ j \\ C_1 \\ C_2 \end{pmatrix} \longrightarrow \begin{pmatrix} u_- - c_- \\ \rho_- c_- \\ c_- + (\gamma - 1) e_- / c_- \\ \gamma e_- + c_-^2/2 \end{pmatrix} . \tag{3.11}$$

Consequently, when the strength of the shock is small, all quantities j, C_1, C_2 are bounded away from zero. Recall also that 1-shocks are compressive, in the sense that $\rho_+ > \rho_-$. This inequality immediately implies that $0 < v_+ < v_-$. Eventually, the strength of the shock tends to zero if, and only if u_+ tends to u_- (in that case, we also have $\rho_+ \longrightarrow \rho_-$, and $e_+ \longrightarrow e_-$ because of the Rankine-Hugoniot jump conditions).

In all what follows, we limit our discussion to the case of 1-shocks for simplicity, but the extension to 3-shocks is immediate.

3.2.2 Reduction of the traveling wave equation

In this paragraph, we derive, and transform the equation satisfied by traveling wave solutions to (3.6), (3.5). A traveling wave solution to (3.6), (3.5) with speed σ is a solution $(\rho, u, e)(x - \sigma t)$. For such solutions, the radiative heat flux q also depends on the sole variable $x - \sigma t$:

$$q(x - \sigma t) = \frac{1}{2} \int_{\mathbb{R}} \mathrm{e}^{-|x - \sigma t - y|} \operatorname{sgn}(x - \sigma t - y)\, \theta(y)^4 \, dy \,,$$

and (3.6) reads :

$$\begin{cases} (\rho\,(u - \sigma))' = 0 \,, \\ (\rho\, u\,(u - \sigma) + (\gamma - 1)\,\rho\, e)' = 0 \,, \\ (\rho\,(e + \frac{u^2}{2})\,(u - \sigma) + (\gamma - 1)\,\rho\, e\, u + q)' = 0 \,, \end{cases}$$

where $'$ denotes differentiation with respect to the variable $\xi = x - \sigma t$. Introducing the new unknown $v = u - \sigma$, the above system is easily seen to be equivalent to :

$$\begin{cases} (\rho\, v)' = 0 \,, \\ (\rho\, v^2 + (\gamma - 1)\,\rho\, e)' = 0 \,, \\ (\rho\, v\,(e + \frac{v^2}{2}) + (\gamma - 1)\,\rho\, v\, e + q)' = 0 \,. \end{cases} \tag{3.12}$$

Since we are looking for a shock profile, the traveling wave solution should also satisfy :

$$\lim_{\xi \to \pm\infty} (\rho, v, e) = (\rho_\pm, v_\pm, e_\pm) \,, \tag{3.13}$$

where $v_\pm = u_\pm - \sigma$, and $(\rho_\pm, u_\pm, e_\pm)$ defines a 1-shock wave with speed σ for the Euler equations. Notice that the quantity $a = |u_+ - u_-|/2$, that measures the strength of the shock, is invariant with respect to our change of velocity, that is $a = |u_+ - u_-|/2 = |v_+ - v_-|/2$. Recall also that for 1-shocks, there holds $v_- > v_+ > 0$.

Observing that we have :

$$q(\xi) = \frac{1}{2} \int_0^{+\infty} \mathrm{e}^{-y} \left(\theta(\xi + y)^4 - \theta(\xi - y)^4 \right) dy \,,$$

we conclude that q tends to zero at $\pm\infty$ by Lebesgue's Theorem (because θ is necessarily bounded since it has finite limits at $\pm\infty$). We can thus integrate the system (3.12)-(3.13) once, and (3.12) reads equivalently :

$$\begin{cases} (\rho\, v)(\xi) = j\,, \\ (\rho\, v^2 + (\gamma-1)\,\rho\, e)(\xi) = j\, C_1\,, \\ (\rho\, v\,(e+\frac{v^2}{2}) + (\gamma-1)\,\rho\, v\, e + q)(\xi) = j\, C_2\,, \end{cases} \tag{3.14}$$

where the constants j, C_1 and C_2 are given by the Rankine-Hugoniot conditions (3.10). For small shocks, the positive constants j, C_1, C_2 have the asymptotic behavior (3.11).

From the two first equations of (3.14), we derive the relations :

$$\rho(\xi) = \frac{j}{v(\xi)}\,, \quad e(\xi) = \frac{(C_1 - v(\xi))\, v(\xi)}{\gamma - 1}\,.$$

The third equation of (3.14) thus reduces to :

$$v(\xi)^2 - \frac{2\,\gamma\, C_1}{\gamma+1}\, v(\xi) + \frac{2\,(\gamma-1)\, C_2}{\gamma+1} = \frac{2\,(\gamma-1)}{j\,(\gamma+1)}\, q(\xi)\,. \tag{3.15}$$

Using the equation of state (3.2), as well as the second equation of (3.14), we get :

$$\theta(\xi) = \frac{(\gamma-1)\, e(\xi)}{R} = \frac{(C_1 - v(\xi))\, v(\xi)}{R}\,.$$

Consequently, (3.15) can be recast as an integral equation with a single unknown function v :

$$v(\xi)^2 - \frac{2\,\gamma\, C_1}{\gamma+1}\, v(\xi) + \frac{2\,(\gamma-1)\, C_2}{\gamma+1} = \frac{(\gamma-1)}{j\,(\gamma+1)\, R^4} \int_{\mathbb{R}} \mathrm{e}^{-|\xi-y|}\, \mathrm{sgn}(\xi-y)\, v(y)^4 (C_1 - v(y))^4\, dy\,. \tag{3.16}$$

We are searching for a solution v to (3.16), that satisfies the asymptotic conditions $v(\xi) \to v_\pm$, as $\xi \to \pm\infty$.

Remark 3.3. *If we find a C^2 solution v to (3.16) that does not vanish, and that satisfies $v(\xi) \to v_\pm$ as $\xi \to \pm\infty$, then we obtain a C^2 shock profile (ρ, u, e) by simply setting :*

$$\rho(\xi) = \frac{j}{v(\xi)}\,, \quad u(\xi) = v(\xi) + \sigma\,, \quad e(\xi) = \frac{(C_1 - v(\xi))\, v(\xi)}{\gamma-1}\,.$$

In particular, if $v(\xi) \in [v_+, v_-]$ for all ξ, then v does not vanish.

Remark 3.4. *Since the heat flux q vanishes at $\pm\infty$, (3.16) can be also rewritten as :*

$$(v(\xi) - v_-)(v(\xi) - v_+) = \frac{(\gamma-1)}{j\,(\gamma+1)\, R^4} \int_{\mathbb{R}} \mathrm{e}^{-|\xi-y|}\, \mathrm{sgn}(\xi-y)\, v(y)^4\, (C_1 - v(y))^4\, dy\,.$$

We are going to rewrite (3.16) as a second order differential equation, that will be easier to study than the integral equation (3.16). Indeed, assuming that v is a C^2 function of ξ, and differentiating twice (3.16) with respect to ξ, we get (see [22] for the details of the computations) :

$$(v - \frac{\gamma\, C_1}{\gamma+1})\, v'' + (v')^2 - \frac{4\,(\gamma-1)}{j\,(\gamma+1)\, R^4}\, (C_1 - v)^3\, v^3\, (C_1 - 2v)\, v' - \frac{1}{2}\, (v - v_-)\, (v - v_+) = 0\,. \tag{3.17}$$

Conversely, if v is a C^2 solution to (3.17) that satisfies $v(\xi) \longrightarrow v_\pm$ as $\xi \longrightarrow \pm\infty$, then v is also a solution to (3.16). If in addition v takes its values in the interval $[v_+, v_-]$, then we can construct a C^2 shock profile, and thus prove Theorem 3.1.

The differential equation (3.17) can be simplified by introducing the new unknown function $\hat{v} = v - (v_- + v_+)/2$, and by rewriting the second order differential equation as a first order system :

$$\left\{ \begin{array}{rcl} \hat{v}' & = & w \, , \\ \hat{v}\, w' & = & -w^2 - f(\hat{v})\, w + \dfrac{\hat{v}^2 - a^2}{2} \, , \end{array} \right. \tag{3.18}$$

where f is the following polynomial function :

$$f(\hat{v}) = \frac{4\,(\gamma-1)}{j\,R^4\,(\gamma+1)} \left(\frac{C_1}{\gamma+1} - \hat{v} \right)^3 \left(\hat{v} + \frac{\gamma\, C_1}{\gamma+1} \right)^3 \left(2\hat{v} + \frac{(\gamma-1)\, C_1}{\gamma+1} \right) . \tag{3.19}$$

We recall that $a = |v_- - v_+|/2$, and that a measures the strength of the shock.

Remark 3.5. *The asymptotic behavior* (3.11) *of* j, C_1, *and* C_2 *shows that when the strength of the shock tends to zero* ($a \longrightarrow 0^+$)*, the limit of* $f(0)$ *is given by :*

$$f(0) \longrightarrow \frac{4\,\gamma^3\,(\gamma-1)^2}{R^4\,(\gamma+1)^8} \, \frac{(c_- + (\gamma-1)\,\frac{e_-}{c_-})^7}{\rho_-\, c_-} > 0 \, .$$

Since $v_+ < v_-$ for a 1-shock, we are searching for a solution to (3.18) that is defined on all $\mathbb{R}$, and that satisfies :

$$\lim_{\xi \to -\infty} (\hat{v}, w)(\xi) = (a, 0) \, , \quad \lim_{\xi \to +\infty} (\hat{v}, w)(\xi) = (-a, 0) \, . \tag{3.20}$$

To prove Theorem 3.1, we are thus reduced to showing the existence of a heteroclinic orbit for (3.18) that connects the stationary solutions $(\pm a, 0)$. Due to the previous transformation $\hat{v} = v - (v_- + v_+)/2$, if $\hat{v}$ takes its values in $[-a, a]$, then $v = \hat{v} + (v_- + v_+)/2$ will take its values in the interval $[v_+, v_-]$, and therefore will not vanish.

Remark 3.6. *The system* (3.18) *is 'singular' at* $\hat{v} = 0$. *Nevertheless, we are searching for a smooth solution connecting* $(\pm a, 0)$*, so that* $\hat{v}$ *vanishes in at least one point. Because* $w' = \hat{v}''$ *should also have a limit at this point, a* C^2 *shock profile can exist only if the equation :*

$$w^2 + f(0)\, w + \frac{a^2}{2} = 0 \, ,$$

has real roots. The corresponding discriminant condition turns out to be much less simple than the one found in [27] for the 'baby model' (3.9). *(In particular,* $f(0)$ *depends on the shock through the constants* j, *and* C_1*). This is a first 'nonexplicit' restriction on the shock strength to derive the existence of a smooth shock profile.*

Due to the singular nature of the system (3.18) at $\hat{v} = 0$, it is more convenient to work on an auxiliary system of ODEs, where the singularity has been eliminated (at least formally) thanks to a change of variables. This procedure was already used in [27]. In the next paragraph, we shall introduce this auxiliary system, and complete the proof of Theorem 3.1.

3.2.3 Existence of a heteroclinic orbit

We begin with a result on an auxiliary system of ODEs, where the singularity at $\hat{v} = 0$ has been eliminated :

Proposition 3.7. *Assume that γ satisfies $1 < \gamma < (\sqrt{7}+1)/(\sqrt{7}-1)$, and consider the following system of ODEs :*

$$\begin{cases} V' &= V\,W\,, \\ W' &= -W^2 - f(V)\,W + \dfrac{(V^2-a^2)}{2}\,. \end{cases} \tag{3.21}$$

There exists a positive constant a_0, that depends only on (ρ_-, u_-, e_-), and γ such that if the shock strength a satisfies $a \in (0, a_0]$, the following properties hold :

- *$f(0)^2 - 2a^2 > 0$, and we define $w_0 := \big(-f(0) + \sqrt{f(0)^2 - 2a^2}\big)/2 < 0$.*
- *There exists a solution $(V_\flat, W_\flat)$ to (3.21) that is defined on all $\mathbb{R}$, and that satisfies*

$$\lim_{\eta\to-\infty} (V_\flat, W_\flat)(\eta) = (a, 0)\,, \quad \lim_{\eta\to+\infty} (V_\flat, W_\flat)(\eta) = (0, w_0)\,.$$

  Furthermore, $V_\flat$ is decreasing, and the convergence of $V_\flat$ to 0 as $\eta \to +\infty$ is exponential.
- *There exists a solution $(V_\sharp, W_\sharp)$ to (3.21) that is defined on all $\mathbb{R}$, and that satisfies*

$$\lim_{\eta\to-\infty} (V_\sharp, W_\sharp)(\eta) = (-a, 0)\,, \quad \lim_{\eta\to+\infty} (V_\sharp, W_\sharp)(\eta) = (0, w_0)\,.$$

  Furthermore, $V_\sharp$ is increasing, and the convergence of $V_\sharp$ to 0 as $\eta \to +\infty$ is exponential.

Assuming that the result of Proposition 3.7 holds, the existence of a heteroclinic orbit for (3.18) connecting $(\pm a, 0)$ can be derived by following the analysis of [27]. We briefly recall the method. Using the solution $(V_\flat, W_\flat)$, we introduce the change of variable :

$$\Xi_\flat(\eta) = -\int_\eta^{+\infty} V_\flat(\zeta)\, d\zeta\,.$$

Since $V_\flat$ tends to 0 exponentially as η tends to $+\infty$, $\Xi_\flat$ is well-defined, and it is an increasing C^∞ diffeomorphism from $\mathbb{R}$ to $(-\infty, 0)$. Then $(\hat{v}_\flat, w_\flat) := (V_\flat, W_\flat) \circ \Xi_\flat^{-1}$ is a C^∞ solution to (3.18) on the interval $(-\infty, 0)$, that satisfies :

$$\lim_{\xi\to-\infty} (\hat{v}_\flat, w_\flat)(\xi) = (a, 0)\,, \quad \lim_{\xi\to0^-} (\hat{v}_\flat, w_\flat)(\xi) = (0, w_0)\,.$$

Similarly, with the help of the solution $(V_\sharp, W_\sharp)$ we can construct a C^∞, decreasing diffeomorphism $\Xi_\sharp$ from $\mathbb{R}$ to $(0, +\infty)$, and a C^∞ solution $(\hat{v}_\sharp, w_\sharp)$ to (3.18) on the interval $(0, +\infty)$. This solution $(\hat{v}_\sharp, w_\sharp)$ connects $(0, w_0)$ and $(-a, 0)$, as ξ varies from 0^+ to $+\infty$. Let us now 'glue' the solutions $(\hat{v}_\flat, w_\flat)$, and $(\hat{v}_\sharp, w_\sharp)$, by defining :

$$(\hat{v}, w)(\xi) := \begin{cases} (\hat{v}_\flat, w_\flat)(\xi) & \text{if } \xi < 0, \\ (\hat{v}_\sharp, w_\sharp)(\xi) & \text{if } \xi > 0, \end{cases} \tag{3.22}$$

and extend the functions $\hat{v}$, and w at 0 by setting $(\hat{v}, w)(0) = (0, w_0)$. In this way, $\hat{v}$, and w are continuous on $\mathbb{R}$, and C^∞ on $\mathbb{R} \setminus \{0\}$. It remains to show that $\hat{v} \in C^2(\mathbb{R})$, that $(\hat{v}, w)$ solves (3.18) on $\mathbb{R}$, and that $\hat{v}$ takes its values in $(-a, a)$.

Observe first of all that $\hat{v}$ is a decreasing function, because of the monotonicity properties of $V_\flat, V_\sharp, \Xi_\flat, \Xi_\sharp$. Using the asymptotic behavior of $V_\flat$, $V_\sharp$ at $-\infty$, we get that $\hat{v}(\xi) \in (-a, a)$ for all $\xi \in \mathbb{R}$.

Let us now note that the above construction of $(\hat{v}, w)$ shows that $(\hat{v}, w)$ is a solution to (3.18) on $\mathbb{R} \setminus \{0\}$. In particular, $\hat{v}'(\xi) = w(\xi)$ if $\xi \neq 0$. Moreover, w is continuous on $\mathbb{R}$, so $\hat{v} \in C^1(\mathbb{R})$, and $\hat{v}'(0) = w(0) = w_0$. To prove that $\hat{v} \in C^2(\mathbb{R})$, it is sufficient to show that $w \in C^1(\mathbb{R})$, which is equivalent to showing that w' has a limit at 0 (because we already know that w is C^∞ on $\mathbb{R} \setminus \{0\}$). To prove that w' has a limit at 0, we are going to study the asymptotic behavior of $(V_\flat, W_\flat)$, and $(V_\sharp, W_\sharp)$ at $+\infty$. More precisely, let us denote $U(V, W)$ the vector field associated to the ODE (3.21) :

$$U(V,W) = \begin{pmatrix} V\,W \\ -W^2 - f(V)\,W + \dfrac{(V^2 - a^2)}{2} \end{pmatrix}, \tag{3.23}$$

where f is given by (3.19). The Jacobian matrix of U at $(0, w_0)$ is :

$$\begin{pmatrix} w_0 & 0 \\ -f'(0)\,w_0 & -2\,w_0 - f(0) \end{pmatrix} = \begin{pmatrix} \lambda_1^{(0)} & 0 \\ b_0 & \lambda_2^{(0)} \end{pmatrix}.$$

For a sufficiently small, one checks that $\lambda_2^{(0)} < \lambda_1^{(0)} < 0$ (see Proposition 3.7 for the definition of w_0). The eigenvectors corresponding to the eigenvalues $\lambda_1^{(0)}$ and $\lambda_2^{(0)}$ are :

$$e_1^{(0)} = \begin{pmatrix} f(0) + 3\,w_0 \\ b_0 \end{pmatrix}, \quad e_2^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The standard theory of autonomous ODEs, see e.g. [52], shows that there are exactly two solutions to (3.21) that tend to $(0, w_0)$ as η tends to $+\infty$, and that are tangent to the straight line $(0, w_0) + \mathbb{R}\, e_2^{(0)}$. Moreover, all the other solutions to (3.21) that tend to $(0, w_0)$ as η tends to $+\infty$ are tangent to the straight line $(0, w_0) + \mathbb{R}\, e_1^{(0)}$. Now, it is rather simple to see that the two solutions to (3.21) that tend to $(0, w_0)$ as η tends to $+\infty$, and that are tangent to the straight line $(0, w_0) + \mathbb{R}\, e_2^{(0)}$, satisfy $V \equiv 0$, and :

$$W' = -W^2 - f(0)\,W - \frac{a^2}{2}.$$

Because the solutions $(V_\flat, W_\flat)$, and $(V_\sharp, W_\sharp)$ given by Proposition 3.7 cannot satisfy $V_\flat \equiv 0$, and $V_\sharp \equiv 0$, we can conclude that the solutions $(V_\flat, W_\flat)$, and $(V_\sharp, W_\sharp)$ are tangent to $(0, w_0) + \mathbb{R}\, e_1^{(0)}$ as η tends to $+\infty$. In particular, this yields :

$$\lim_{\eta \to +\infty} \frac{W_\flat'(\eta)}{V_\flat'(\eta)} = \lim_{\eta \to +\infty} \frac{W_\sharp'(\eta)}{V_\sharp'(\eta)} = \frac{-f'(0)\,w_0}{f(0) + 3\,w_0}. \tag{3.24}$$

(A quick verification shows that $f(0) + 3\,w_0 > 0$ for small enough a.) From the construction of the solutions $(\hat{v}_\flat, w_\flat)$, and $(\hat{v}_\sharp, w_\sharp)$, we get :

$$\lim_{\xi \to 0^-} w_\flat'(\xi) = \lim_{\xi \to 0^+} w_\sharp'(\xi) = \frac{-f'(0)\,w_0^2}{f(0) + 3\,w_0}.$$

As a consequence, when a is small enough, $w \in C^1(\mathbb{R})$, and therefore $\hat{v} \in C^2(\mathbb{R})$. Moreover, $(\hat{v}, w)$ solves (3.18) on $\mathbb{R} \setminus \{0\}$, so by continuity, it solves (3.18) on $\mathbb{R}$. This completes the proof of Theorem 3.1, provided that the result of Proposition 3.7 holds.

3.2.4 Proof of Proposition 3.7

In this paragraph, we prove Proposition 3.7, which will complete the proof of Theorem 3.1. At first, we define the set :

$$P = \left\{(V,W)|V \in [-a,a], W^2 + f(V)\,W - \frac{V^2-a^2}{2} = 0\right\},$$

so that the points $(\pm a, 0)$ belong to P. The following Lemma gives a description of P for $a > 0$ small enough. We refer to figure 3.1 for a schematic picture.

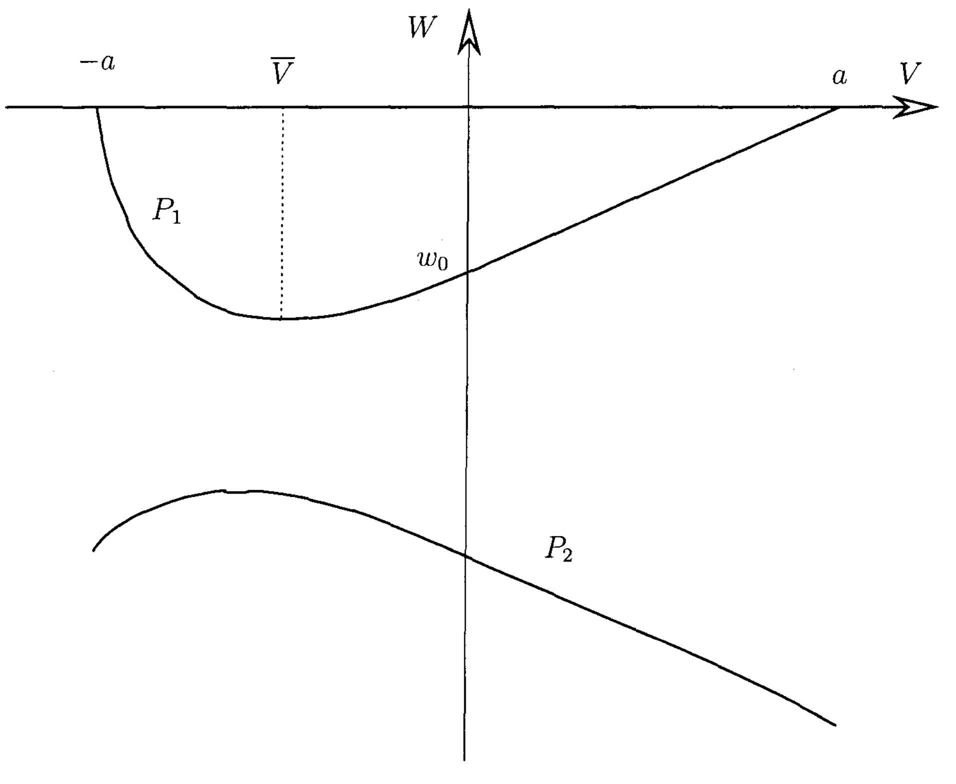

FIG. 3.1 – The set $P = P_1 \cup P_2$.

Lemma 3.8. *Assume that $1 < \gamma < (\sqrt{7}+1)/(\sqrt{7}-1)$. Then there exists a constant $a_0 > 0$, that only depends on (ρ_-, u_-, e_-) and γ such that if the shock strength a satisfies $a \in (0, a_0]$, we have the following results :*

- *For all $V \in [-a,a]$, $f(V)^2 + 2\,(V^2-a^2) > 0$. We can thus define*

$$\forall\, V \in [-a,a]\,, \quad \mathbb{W}_1(V) = \frac{-f(V) + \sqrt{f(V)^2 + 2\,(V^2-a^2)}}{2}\,,$$
$$\mathbb{W}_2(V) = \frac{-f(V) - \sqrt{f(V)^2 + 2\,(V^2-a^2)}}{2}\,.$$

- *$P = P_1 \cup P_2$, where P_1 and P_2 are two curves defined by*

$$P_1 = \left\{(V, \mathbb{W}_1(V))|V \in [-a,a]\right\}, \quad P_2 = \left\{(V, \mathbb{W}_2(V))|V \in [-a,a]\right\},$$

so that the points $(\pm a, 0)$, and $(0, w_0)$ belong to P_1.

- *There exists a unique point* $\overline{V} \in (-a, 0)$ *such that* $\mathbb{W}_1$ *is increasing on the interval* $[\overline{V}, a]$, *and* $\mathbb{W}_1$ *is decreasing on the interval* $[-a, \overline{V}]$.
- *For all* $V \in [\overline{V}, 0]$, *one has* $\mathbb{W}_2(V) < \mathbb{W}_1(\overline{V})$.

Proof. Let us first define a function Δ by setting :

$$\Delta(V) := f(V)^2 + 2\,(V^2 - a^2)\,.$$

Using (3.11), for a small enough, we have :

$$\frac{C_1}{\gamma+1} - a \geq \kappa > 0\,, \quad \frac{\gamma\, C_1}{\gamma+1} - a \geq \kappa > 0\,, \quad \frac{(\gamma-1)\, C_1}{\gamma+1} - 2a \geq \kappa > 0\,,$$

where κ is a positive constant that only depends on (ρ_-, u_-, e_-) and γ. Moreover, (3.11) also shows that $j \geq \kappa$ for $a \in (0, a_0]$, up to restricting κ. Consequently, there exist $a_0 > 0$, and $\kappa > 0$ such that for $a \in (0, a_0]$, we have $f(V) \geq \kappa$, and $\Delta(V) \geq \kappa$ for all $V \in [-a, a]$. This directly shows that the set P is the union of the two curves P_1, and P_2. It is rather clear from the definition of P_1 that $(\pm a, 0)$, and $(0, w_0)$ belong to P_1 (recall that w_0 is defined in Proposition 3.7). Observe also that $\mathbb{W}_2(V) < \mathbb{W}_1(V) \leq 0$ for $V \in [-a, a]$, and $\mathbb{W}_1(V) < 0$ if $V \in (-a, a)$.

The functions $\mathbb{W}_1$, and $\mathbb{W}_2$ are C^∞ on $[-a, a]$. Moreover, we compute the relation :

$$\forall\, V \in [-a, a]\,, \quad \sqrt{\Delta(V)}\, \mathbb{W}_1'(V) = V - \mathbb{W}_1(V)\, f'(V)\,, \tag{3.25}$$

and from (3.19), we also compute

$$f'(V) = \frac{14\,(\gamma-1)}{j\, R^4\,(\gamma+1)}$$
$$\left(\frac{C_1}{\gamma+1} - \hat{v}\right)^2 \left(\hat{v} + \frac{\gamma\, C_1}{\gamma+1}\right)^2 \left(2\hat{v} + \frac{(\gamma-1)\, C_1}{\gamma+1} + \frac{C_1}{\sqrt{7}}\right) \left(-2\hat{v} - \frac{(\gamma-1)\, C_1}{\gamma+1} + \frac{C_1}{\sqrt{7}}\right). \tag{3.26}$$

As we have done for f, and Δ, a careful analysis shows that for $1 < \gamma < (\sqrt{7}+1)/(\sqrt{7}-1)$, and for a small enough, one has $f'(V) \geq \kappa > 0$ for all $V \in [-a, a]$, because each term in the product (3.26) is positive. Using this information in (3.25), we can already conclude that $\mathbb{W}_1$ is increasing on the interval $[0, a]$ (see figure 3.1). Moreover, the relation (3.25) also shows that $\mathbb{W}_1'(0) > 0$, and $\mathbb{W}_1'(-a) < 0$. Consequently, there exists some $\overline{V} \in (-a, 0)$ such that $\mathbb{W}_1'(\overline{V}) = 0$. Let us prove that $\overline{V}$ is the only zero of $\mathbb{W}_1'$. We claim that it is sufficient to show the following property :

$$\mathbb{W}_1'(V) = 0 \Longrightarrow \mathbb{W}_1''(V) > 0\,. \tag{3.27}$$

Indeed, if the property (3.27) holds true, then any point where $\mathbb{W}_1'$ vanishes is a local strict minimum. If there existed two such local strict minima $-a < \overline{V}_1 < \overline{V}_2 < a$, then $\mathbb{W}_1$ would admit a local maximum $\overline{V}_3 \in (\overline{V}_1, \overline{V}_2)$, which is obviously impossible. Therefore let us prove that the property (3.27) holds true.

Differentiating (3.25) with respect to V, we obtain that if $\mathbb{W}_1'(\overline{V}) = 0$, then

$$\sqrt{\Delta(\overline{V})}\, \mathbb{W}_1''(\overline{V}) = 1 - f''(\overline{V})\, \mathbb{W}_1(\overline{V})\,.$$

Observing that

$$|f''(\overline{V})\,\mathbb{W}_1(\overline{V})| \le C\,|\mathbb{W}_1(\overline{V})| \le C\,\frac{a^2-\overline{V}^2}{f(\overline{V})} \le \frac{C\,a^2}{\kappa}\,,$$

for suitable positive constants C, and κ (that are independent of $a \in (0,a_0]$), we can conclude that $\mathbb{W}_1''(\overline{V}) > 0$, provided that a is small enough. This completes the proof that $\mathbb{W}_1'$ has a unique zero $\overline{V} \in (-a,0)$, and therefore $\mathbb{W}_1$ is decreasing on $[-a,\overline{V}]$, and is increasing on $[\overline{V},a]$.

For the last point of the lemma, we use the relation :

$$\mathbb{W}_1'(V) + \mathbb{W}_2'(V) = -f'(V) < 0\,.$$

Because $\mathbb{W}_1'(V) \ge 0$ for $V \in [\overline{V},0]$, we get $\mathbb{W}_2'(V) < 0$ for $V \in [\overline{V},0]$. Thus for $V \in [\overline{V},0]$, we have $\mathbb{W}_2(V) \le \mathbb{W}_2(\overline{V}) < \mathbb{W}_1(\overline{V})$, and the proof of the Lemma is complete. □

Using Lemma 3.8, we are going to prove Proposition 3.7. The analysis follows [16].

As we have already seen in the preceeding paragraph, the point $(w_0,0)$ is a stable node of (3.21). We now study the nature of the equilibrium points $(\pm a,0)$. Recall that the vector field associated to (3.21) is denoted U, see (3.23). The Jacobian matrix of U at $(a,0)$ is :

$$\begin{pmatrix} 0 & a \\ a & f(a) \end{pmatrix}\,,$$

so it has exactly one negative eigenvalue μ_1, and one positive eigenvalue μ_2 (the equilibrium point $(a,0)$ is a saddle point) :

$$\mu_1 = \frac{-f(a) - \sqrt{f(a)^2+4\,a^2}}{2}\,,\quad \mu_2 = \frac{-f(a) + \sqrt{f(a)^2+4\,a^2}}{2}\,.$$

An eigenvector associated to μ_2, and is $r_2 = (a,\mu_2)$. Moreover, using the relation (3.25), we can check that for a small enough, the following inequality holds :

$$0 < \frac{\mu_2}{a} < \frac{a}{f(a)} = \mathbb{W}_1'(a)\,, \tag{3.28}$$

where the function $\mathbb{W}_1$ is defined in Lemma 3.8. Let us now define a compact set K_1 by :

$$K_1 := \Big\{(V,W) \in [0,a]\times\mathbb{R} \,|\, \mathbb{W}_1(V) \le W \le 0\Big\}\,,$$

Then the inequalities (3.28) show that for $s<0$ small enough, the point $(a,0)+s\,r_2$ belongs to the interior of K_1. We refer to figure 3.2 for a detailed picture of the situation.

From the standard theory of autonomous ODEs, see e.g. [52], we know that there exists a maximal solution $(V_\flat,W_\flat)$ to (3.21) that tends to the saddle point $(a,0)$ as η tends to $-\infty$, and that is tangent to the half-straight line $(a,0)+\mathbb{R}^-\,r_2$. This solution is defined on an open interval $(-\infty,\eta_*)$ (with possibly $\eta_* = +\infty$). For large negative η, the preceeding analysis shows that $(V_\flat,W_\flat)(\eta)$ belongs to the interior of K_1. Moreover, $(V_\flat,W_\flat)$ cannot reach the boundary of K_1. Indeed $V_\flat$ cannot identically vanish so $(V_\flat,W_\flat)(\eta) \notin \partial K_1 \cap \{V=0\}$. Similarly, we have $(V_\flat,W_\flat)(\eta) \ne (a,0)$. Eventually, on the set :

$$\Big\{(V,0)\,|\,V\in(0,a)\Big\} \cup \Big\{(V,\mathbb{W}_1(V))\,|\,V\in(0,a)\Big\}\,,$$

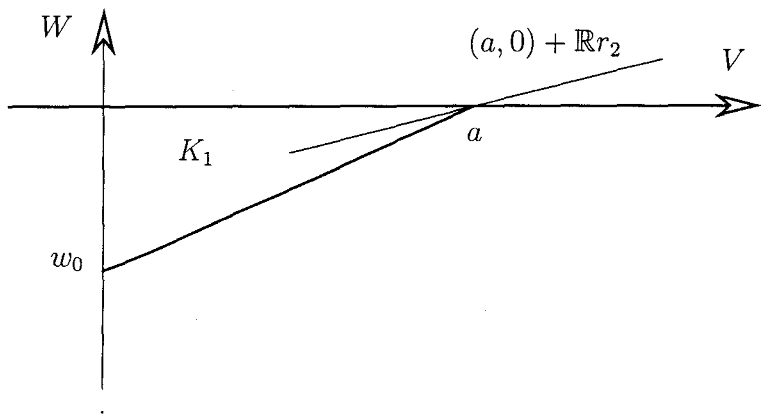

FIG. 3.2 – The compact set K_1.

the vector field U is not zero, and is directed towards the interior of K_1. Therefore the solution $(V_\flat, W_\flat)$ cannot reach ∂K_1, so it takes its values in the compact set K_1. The maximal solution $(V_\flat, W_\flat)$ is thus defined on $\mathbb{R}$. It cannot reach the boundary of K_1, so $W_\flat$ takes negative values, which means that $V_\flat$ is decreasing (because $V_\flat' = V_\flat W_\flat$). Because $(V_\flat, W_\flat)$ takes values in the interior of K_1, the function $W_\flat$ is also decreasing. This shows that $(V_\flat, W_\flat)(\eta)$ has a limit as η tends to $+\infty$, and this limit is necessarily be a stationary solution of (3.21). The only possibility is that $(V_\flat, W_\flat)(\eta)$ tends to $(0, w_0)$ as η tends to $+\infty$. The convergence is necessarily exponential, because the Jacobian matrix of U at $(0, w_0)$ has two negative eigenvalues, see e.g. [52].

To construct the other solution $(V_\sharp, W_\sharp)$, we argue similarly by defining a compact set K_2 :

$$K_2 := \Big\{(V,W) \in [-a, \overline{V}] \times \mathbb{R} | \mathbb{W}_1(V) \leq W \leq 0\Big\} \cup \Big\{(V,W) \in [\overline{V}, 0] \times \mathbb{R} | \mathbb{W}_1(\overline{V}) \leq W \leq 0\Big\},$$

see figure 3.3. The Jacobian matrix of U at $(-a, 0)$ has one negative eigenvalue ν_1, and one positive eigenvalue ν_2, with :

$$\nu_2 = \frac{-f(-a) + \sqrt{f(-a)^2 + 4\,a^2}}{2}\,.$$

An eigenvector associated to the eigenvalue ν_2 is $R_2 = (-a, \nu_2)$. As was done earlier, we check that the inequalities :

$$\mathbb{W}_1'(-a) = \frac{-a}{f(-a)} < \frac{\nu_2}{-a} < 0\,,$$

hold true. Therefore, one can reproduce the above analysis, and show that there exists a solution $(V_\sharp, W_\sharp)$ to (3.21) that takes its values in K_2 (and is thus defined on $\mathbb{R}$), and that tends to $(-a, 0)$ at $-\infty$. Moreover, $(V_\sharp, W_\sharp)$ can not reach the boundary of K_2, so $V_\sharp$ is increasing. It only remains to study the monotonicity of $W_\sharp$. This is slightly more complicated than for $W_\flat$. Observe that K_2 is the union of the sets :

$$K_2^1 := \Big\{(V,W) \in [-a, 0] \times \mathbb{R} \,|\, \mathbb{W}_1(V) \leq W \leq 0\Big\}\,,$$
$$K_2^2 := \Big\{(V,W) \in [\overline{V}, 0] \times \mathbb{R} \,|\, \mathbb{W}_1(\overline{V}) \leq W \leq \mathbb{W}_1(V)\Big\}\,.$$

When $(V_\sharp, W_\sharp)$ takes its values in the interior of K_2^1, the function $W_\sharp$ is decreasing (this is the case for large negative η). At the opposite, when $(V_\sharp, W_\sharp)$ takes its values in the interior of K_2^2, the function $W_\sharp$ is increasing, because thanks to Lemma 3.8, we have :

$$\begin{aligned} W_\sharp'(\eta) &= -W_\sharp(\eta)^2 - f(V_\sharp(\eta))\,W_\sharp(\eta) + \frac{V_\sharp(\eta)^2 - a^2}{2} \\ &= \big(\mathbb{W}_1(V_\sharp(\eta)) - W_\sharp(\eta)\big)\,\big(W_\sharp(\eta) - \mathbb{W}_2(V_\sharp(\eta))\big) \\ &\geq \big(\mathbb{W}_1(V_\sharp(\eta)) - W_\sharp(\eta)\big)\,\big(\mathbb{W}_1(\overline{V}) - \mathbb{W}_2(V_\sharp(\eta))\big) > 0\,. \end{aligned}$$

Moreover, if $(V_\sharp, W_\sharp)(\eta_0)$ belongs to the interior of K_2^2 for some $\eta_0 \in \mathbb{R}$, then $(V_\sharp, W_\sharp)(\eta)$ belongs to the interior of K_2^2 for all $\eta \geq \eta_0$ (because it cannot reach the boundary of K_2^2 for $\eta \geq \eta_0$). Summing up, either $(V_\sharp, W_\sharp)(\eta)$ belongs to K_2^1 for all η, and $W_\sharp$ is monotonic on $\mathbb{R}$, either $(V_\sharp, W_\sharp)(\eta)$ belongs to K_2^2 for all η greater than some η_0, and $W_\sharp$ is monotonic on $[\eta_0, +\infty)$. In any case, the function $W_\sharp$ is monotonic on a neighborhood of $+\infty$, and thus has a limit at $+\infty$. This shows that $(V_\sharp, W_\sharp)(\eta)$ tends to $(0, w_0)$ as η tends to $+\infty$, and the convergence is exponential. As a matter of fact, we have seen in the preceeding paragraph that $(V_\sharp, W_\sharp)$ is tangent to the straight line $(0, w_0) + \mathbb{R}\, e_1^{(0)}$ as η tends to $+\infty$, so one can check that $(V_\sharp, W_\sharp)(\eta)$ belongs to the interior of K_2^2 for large positive η. This means that $W_\sharp$ is decreasing on some interval $(-\infty, \eta_0)$, and increasing on $[\eta_0, +\infty)$. The proof of Proposition 3.7 is now complete.

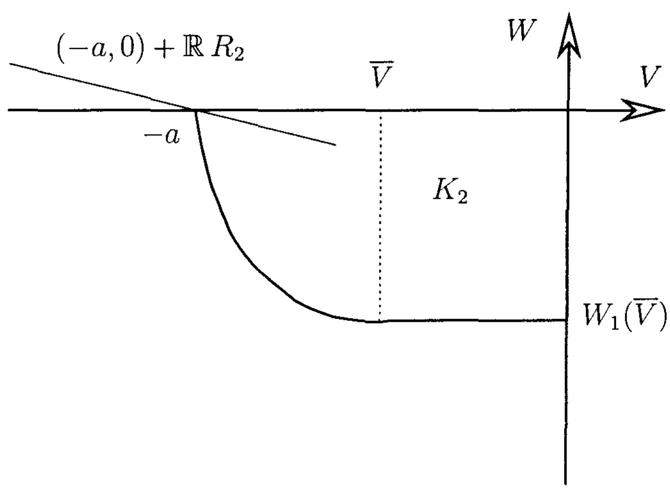

FIG. 3.3 – The compact set K_2.

3.2.5 Estimate for the shock profile

In this paragraph we give the estimate for C^2 shock profile obtained §3.7. We have the following

Lemma 3.9. *The C^2 shock profile obtained in §3.7 verifies*

$$\|\rho',\, u',\, e'\|_{L^\infty(\mathbb{R})} \leq C a^2,$$

where the constant C is independent on the shock strength a, but may dependent on (ρ_-, u_-, e_-) and γ.

Proof. We prove the estimate for the shock profile of the velocity u, with which we can easily get the other estimates. As $v = u - \sigma$, $\hat{v} = v - \dfrac{v_- + v_+}{2}$, and $\hat{v}, w$ satisfy (3.18), we have

$$|u'| = |v'| = |\hat{v}'| = |w|.$$

As we have seen that the shock profile was constructed by two heteroclinic orbits of the system (3.21), thus we get

$$|u'| \leq \sup_{V\in[-a,a]} |\mathbb{W}_1(V)| = \sup_{V\in[-a,a]} \left| \frac{-f(V) + \sqrt{f(V)^2 + 2\,(V^2 - a^2)}}{2} \right|$$

Recall that in Remark 3.5 we have shown that $\lim_{a\to 0} f(0) > 0$. Thus for sufficient small a we have $\varsigma_1 < f(V) < \varsigma_2$, for any $V \in [-a, a]$, where $\varsigma_1 = \frac{1}{2}\lim_{a\to 0} f(0)$ and $\varsigma_1 = \frac{3}{2}\lim_{a\to 0} f(0)$. Note that $\lim_{a\to 0} f(0)$ depends only on (ρ_-, u_-, e_-) and γ, consequently ς_1 and ς_2 are independent on a. Thus the estimate of $|u'|$ can be written as

$$\begin{aligned} |u'| &\leq \sup_{V\in[-a,a]} \frac{f(V) - \sqrt{f(V)^2 + 2\,(V^2 - a^2)}}{2} \\ &\leq \sup_{V\in[-a,a]} \frac{f(V)}{2}\frac{2(a^2 - V^2)}{f(V)^2} \\ &\leq Ca^2, \end{aligned}$$

where the constant C is independent on a, but on (ρ_-, u_-, e_-) and γ. Thus completes the proof. $\square$

3.3 Additional regularity of shock profiles

As should be clear from the preceeding section, the key point in the construction of a shock profile is Proposition 3.7 that gives the existence of two heteroclinic orbits for the system (3.21). To prove Theorem 3.2, we are going to study the behavior of the derivatives of $(V_\flat, W_\flat)$, and $(V_\sharp, W_\sharp)$ near $+\infty$. The proof of Theorem 3.2 follows from an induction argument. To make the arguments clear, we deal with the first case separately. In all what follows, $(V_\flat, W_\flat)$, and $(V_\sharp, W_\sharp)$ are the solutions to (3.21) that are defined in Proposition 3.7, and $(\hat{v}, w)$ denotes the solution to (3.18) that is defined by (3.22). We have the following :

Proposition 3.10. *Under the assumptions of Proposition 3.7, there exists a positive constant $a_1 \leq a_0$ (that depends on (ρ_-, u_-, e_-), and γ), such that for all $a \in (0, a_1]$, one has $w \in C^2(\mathbb{R})$, $\hat{v} \in C^3(\mathbb{R})$, and :*

$$w(\xi) = w_0 + w_1\,\hat{v}(\xi) + w_2\,\hat{v}(\xi)^2 + o(\hat{v}(\xi)^2)\,, \quad \text{as } \xi \to 0,$$

for some suitable constants w_1, w_2 (w_0 has already been defined in Proposition 3.7).

Proof. Recall that V_b, and $V_\sharp$ do not vanish on $\mathbb{R}$, so we can introduce some C^∞ functions $W_{\flat,1}$, and $W_{\sharp,1}$ that are defined by :

$$W_\flat = w_0 + V_\flat\, W_{\flat,1}\,, \quad W_\sharp = w_0 + V_\sharp\, W_{\sharp,1}\,.$$

Substituting in (3.21) shows that $(V_\flat, W_{\flat,1})$, and $(V_\sharp, W_{\sharp,1})$ are solutions to the system :

$$\begin{cases} V' = V(w_0 + V\,W_1)\,, \\ W_1' = -w_0\,\dfrac{f(V)-f(0)}{V} - 2\,V\,W_1^2 - 3\,w_0\,W_1 - f(V)\,W_1 + \dfrac{V}{2}\,. \end{cases} \tag{3.29}$$

Moreover, we already know from Proposition 3.7, and (3.24) that :

$$\lim_{\eta\to+\infty}(V_\flat, W_{\flat,1})(\eta) = \lim_{\eta\to+\infty}(V_\sharp, W_{\sharp,1})(\eta) = \Big(0, \frac{-f'(0)\,w_0}{f(0)+3\,w_0}\Big)\,.$$

We denote $U_1(V, W_1)$ the vector field associated with (3.29) :

$$U_1(V, W_1) := \begin{pmatrix} V(w_0 + V\,W_1) \\ -w_0\,\dfrac{f(V)-f(0)}{V} - 2\,V\,W_1^2 - 3\,w_0\,W_1 - f(V)\,W_1 + \dfrac{V}{2} \end{pmatrix}\,.$$

Recall that f is a polynomial function of degree 7, see (3.19), thus $F(V) := (f(V)-f(0))/V$ is a polynomial function of degree 6, and we have $F(0) = f'(0)$, $F'(0) = f''(0)/2$. Obviously the system of ODEs (3.29) admits the equilibrium point $(0, w_1)$, where :

$$w_1 := \frac{-F(0)\,w_0}{f(0)+3\,w_0} = \frac{-f'(0)\,w_0}{f(0)+3\,w_0}\,.$$

We are now going to study the nature of the equilibrium point $(0, w_1)$, and show that for a small enough, this equilibrium point is a stable node for (3.29). Then we shall show that $w \in C^2(\mathbb{R})$, and $\hat{v} \in C^3(\mathbb{R})$. In the end, we shall derive the asymptotic expansion near $\xi = 0$.

Step 1 : the Jacobian matrix of U_1 at $(0, w_1)$ is :

$$\begin{pmatrix} w_0 & 0 \\ \dfrac{1}{2} - 2\,w_1^2 - f'(0)\,w_1 - \dfrac{f''(0)}{2}\,w_0 & -f(0) - 3\,w_0 \end{pmatrix} = \begin{pmatrix} \lambda_1^{(1)} & 0 \\ b_1 & \lambda_2^{(1)} \end{pmatrix}\,.$$

Using Remak 3.5, we can conclude that for sufficiently small a, that is $a \in (0, a_1]$ for some positive number a_1 less than a_0, one has $\lambda_2^{(1)} < \lambda_1^{(1)} < 0$, that is, $f(0) + 4\,w_0 > 0$. Moreover, the eigenvectors corresponding to the eigenvalues $\lambda_1^{(1)}$, and $\lambda_2^{(1)}$ are :

$$e_1^{(1)} = \begin{pmatrix} f(0) + 4\,w_0 \\ b_1 \end{pmatrix}\,, \quad e_2^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,.$$

Consequently, $(0, w_1)$ is a stable node of (3.29), and there are exactly two solutions to (3.29) that tend to $(0, w_1)$ as η tends to $+\infty$, and that are tangent to the straight line $(0, w_1) + \mathbb{R}\,e_2^{(1)}$. All the other solutions to (3.29) that tend to $(0, w_1)$ as η tends to $+\infty$ are tangent to the straight line $(0, w_1) + \mathbb{R}\,e_1^{(1)}$. As in the preceeding section, we can thus conclude that :

$$\lim_{\eta\to+\infty} \frac{W_{\flat,1}'(\eta)}{V_\flat'(\eta)} = \lim_{\eta\to+\infty} \frac{W_{\sharp,1}'(\eta)}{V_\sharp'(\eta)} = \frac{b_1}{f(0)+4\,w_0}\,. \tag{3.30}$$

Step 2 : if we let g_1 denote the second coordinate of the vector field U_1, we have $W_{\flat,1}' = g_1(V_\flat, W_{\flat,1})$, and $W_{\sharp,1}' = g_1(V_\sharp, W_{\sharp,1})$. Differentiating once with respect to η, and using (3.30), we end up with :

$$\lim_{\eta\to+\infty} \frac{W_{\flat,1}''(\eta)}{V_\flat'(\eta)} = \lim_{\eta\to+\infty} \frac{W_{\sharp,1}''(\eta)}{V_\sharp'(\eta)} = \ell_1\,, \tag{3.31}$$

where the real number ℓ_1 can be explicitely computed (but its exact expression is of no use). Following the analysis of the preceeding section, we define some functions $w_{\flat,1} := W_{\flat,1} \circ \Xi_\flat^{-1}$, and $w_{\sharp,1} := W_{\sharp,1} \circ \Xi_\sharp^{-1}$. First of all, (3.30) yields :

$$\lim_{\xi \to 0^-} w'_{\flat,1}(\xi) = \lim_{\xi \to 0^+} w'_{\sharp,1}(\xi) = \frac{b_1\, w_0}{f(0) + 4\, w_0} . \tag{3.32}$$

Observe now that we have the relations :

$$\hat{v}_\flat\, w'_{\flat,1} = W'_{\flat,1} \circ \Xi_\flat^{-1} , \quad \hat{v}_\sharp\, w'_{\sharp,1} = W'_{\sharp,1} \circ \Xi_\sharp^{-1} ,$$

and combining with (3.31), we get :

$$\begin{aligned} \lim_{\xi \to 0^-} (\hat{v}_\flat\, w'_{\flat,1})'(\xi) &= \lim_{\eta \to +\infty} \frac{W''_{\flat,1}(\eta)}{V_\flat(\eta)} = \lim_{\eta \to +\infty} \frac{W''_{\flat,1}(\eta)\, W_\flat(\eta)}{V'_\flat(\eta)} = \ell_1\, w_0 , \\ \lim_{\xi \to 0^+} (\hat{v}_\sharp\, w'_{\sharp,1})'(\xi) &= \lim_{\eta \to +\infty} \frac{W''_{\sharp,1}(\eta)}{V_\sharp(\eta)} = \lim_{\eta \to +\infty} \frac{W''_{\sharp,1}(\eta)\, W_\sharp(\eta)}{V'_\sharp(\eta)} = \ell_1\, w_0 . \end{aligned} \tag{3.33}$$

Differentiating twice the relations $w_\flat = w_0 + \hat{v}_\flat\, w_{\flat,1}$, and $w_\sharp = w_0 + \hat{v}_\sharp\, w_{\sharp,1}$, we obtain :

$$\begin{aligned} w''_\flat = \hat{v}''_\flat\, w_{\flat,1} + \hat{v}'_\flat\, w'_{\flat,1} + (\hat{v}_\flat\, w'_{\flat,1})' &= w'_\flat\, w_{\flat,1} + w_\flat\, w'_{\flat,1} + (\hat{v}_\flat\, w'_{\flat,1})' , \\ w''_\sharp &= w'_\sharp\, w_{\sharp,1} + w_\sharp\, w'_{\sharp,1} + (\hat{v}_\sharp\, w'_{\sharp,1})' . \end{aligned}$$

Using (3.32), and (3.33), we get $w''_\flat(0^-) = w''_\sharp(0^+)$. Using the definition (3.22), this shows that $w \in C^2(\mathbb{R})$, and using $\hat{v}' = w$, we obtain $\hat{v} \in C^3(\mathbb{R})$.

Step 3 : note that we have the following expansions near $\xi = 0$:

$$\begin{aligned} w(\xi) &= w(0) + w'(0)\, \xi + \frac{w''(0)}{2}\, \xi^2 + o(\xi^2) , \\ \hat{v}(\xi) &= w(0)\, \xi + \frac{w'(0)}{2}\, \xi^2 + o(\xi^2) , \end{aligned}$$

with $w(0) = w_0 < 0$. We can thus combine these expansions, and derive :

$$w(\xi) = w_0 + \alpha\, \hat{v}(\xi) + \beta\, \hat{v}(\xi)^2 + o(\hat{v}(\xi)^2) ,$$

for some appropriate real numbers α, and β, that we are going to determine. From the relation $w_\flat(\xi) = w_0 + \hat{v}_\flat(\xi)\, w_{\flat,1}(\xi)$, and using that $w_{\flat,1}(\xi)$ tends to w_1 as ξ tends to 0^-, we first get $\alpha = w_1$. Then from (3.32), and from the relation $\hat{v}'_\flat(0^-) = w_0$, we can obtain :

$$w_{\flat,1}(\xi) = w_1 + \frac{b_1}{f(0) + 4\, w_0}\, \hat{v}_\flat(\xi) + o(\hat{v}_\flat(\xi)) , \quad \text{as } \xi \to 0^- .$$

We thus obtain $\beta = b_1/(f(0) + 4\, w_0)$, which yields :

$$w(\xi) = w_0 + w_1\, \hat{v}(\xi) + w_2\, \hat{v}(\xi)^2 + o(v(\xi)^2) ,$$

where $w_2 := b_1/(f(0) + 4\, w_0)$. This latter expansion will be generalized to any order in what follows. □

We now turn to the proof of Theorem 3.2. More precisely, we are going to prove the following result, that is a refined version of Theorem 3.2 :

Theorem 3.11. *Let the assumptions of Proposition 3.7 be satisfied. Then there exists a nonincreasing sequence of positive numbers $(a_n)_{n\in\mathbb{N}}$ such that, for all integer n, if $a \in (0, a_n]$, then $w \in C^{n+1}(\mathbb{R})$, and $\hat{v} \in C^{n+2}(\mathbb{R})$. Moreover, w admits the following asymptotic expansion near $\xi = 0$:*

$$w(\xi) = w_0 + w_1\,\hat{v}(\xi) + \cdots + w_{n+1}\,\hat{v}(\xi)^{n+1} + o(\hat{v}(\xi)^{n+1})\,, \tag{3.34}$$

where the real numbers $w_0, \ldots, w_{n+1}$ are defined by :

$$\begin{cases} w_0 = \dfrac{-f(0) + \sqrt{f(0)^2 - 2\,a^2}}{2}\,, & \\ w_k = \dfrac{b_{k-1}}{f(0) + (k+2)\,w_0}\,, & \text{for } k = 1, \ldots, n+1, \end{cases}$$

and the real numbers $b_0, \ldots, b_n$ are given by :

$$\begin{cases} b_0 = -f'(0)\,w_0\,, \\ b_1 = \dfrac{1}{2} - 2\,w_1^2 - f'(0)\,w_1 - \dfrac{f''(0)}{2}\,w_0\,, \\ b_k = -\displaystyle\sum_{i=1}^{k+1} \frac{f^{(i)}(0)}{i!}\,w_{k+1-i} - \sum_{i=1}^{k}(i+1)\,w_i\,w_{k+1-i}\,, \quad \text{for } k = 2, \ldots, n. \end{cases}$$

Proof. The case $n = 0$ has been proved in the preceeding section, while the case $n = 1$ is proved in Proposition 3.10. (The reader can check that the definition of w_0, w_1, w_2, b_0, and b_1 coincide with our previous notations.) We prove the general case by using an induction with respect to n, and we thus assume that the result of Theorem 3.11 holds up to the order $n \geq 1$. We are going to construct a_{n+1} so that the conclusion of Theorem 3.11 holds for $a \in (0, a_{n+1}]$. In particular, the real numbers $w_0, \ldots, w_{n+1}$, and $b_0, \ldots, b_n$ are given as in Theorem 3.11, and we can already define the real number b_{n+1} by the formula :

$$b_{n+1} := -\sum_{i=1}^{n+2} \frac{f^{(i)}(0)}{i!}\,w_{n+2-i} - \sum_{i=1}^{n+1}(i+1)\,w_i\,w_{n+2-i}\,.$$

(Observe indeed that this definition only involves $w_0, \ldots, w_{n+1}$, and not w_{n+2}.)

Step 1 : because $V_\flat$, and $V_\sharp$ do not vanish, we can introduce some functions $W_{\flat,n+1}$, and $W_{\sharp,n+1}$ by the relations :

$$W_\flat = w_0 + w_1\,V_\flat + \cdots + w_n\,V_\flat^n + W_{\flat,n+1}\,V_\flat^{n+1}\,, \quad W_\sharp = w_0 + w_1\,V_\sharp + \cdots + w_n\,V_\sharp^n + W_{\sharp,n+1}\,V_\sharp^{n+1}\,.$$

Thanks to Taylor's formula, we can write the polynomial function f as :

$$f(V) = f(0) + f'(0)\,V + \frac{f''(0)}{2}\,V^2 + \cdots + \frac{f^{(n)}(0)}{n!}\,V^n + V^{n+1}\,F_{n+1}(V)\,,$$

where F_{n+1} is a polynomial function such that :

$$F_{n+1}(0) = \frac{f^{(n+1)}(0)}{(n+1)!}\,, \quad F'_{n+1}(0) = \frac{f^{(n+2)}(0)}{(n+2)!}\,.$$

Substituting the expression of $W_\flat$, and $W_\sharp$ in (3.21) shows (after a tedious computation!) that $(V_\flat, W_{\flat,n+1})$, and $(V_\sharp, W_{\sharp,n+1})$ are solutions to the following system of ODEs :

$$\begin{cases} V' = V\,(w_0 + w_1\,V + \cdots + w_n\,V^n + W_{n+1}\,V^{n+1})\,, \\ W'_{n+1} = g_{n+1}(V, W_{n+1})\,, \end{cases} \tag{3.35}$$

where the function g_{n+1} is given by :

$$\begin{aligned} g_{n+1}(V, W_{n+1}) :=& -(n+2)\,W_{n+1}\left(\sum_{k=0}^{n} w_k\,V^k + V^{n+1}\,W_{n+1}\right) - W_{n+1}\sum_{k=0}^{n}(k+1)\,w_k\,V^k \\ & - W_{n+1}\,f(V) - F_{n+1}(V)\sum_{k=0}^{n} w_k\,V^k + b_n + \frac{f^{(n+1)}(0)}{(n+1)!} + V\,Q_{n+1}(V)\,, \end{aligned} \tag{3.36}$$

and Q_{n+1} is a polynomial function that satisfies :

$$Q_{n+1}(0) = b_{n+1} + (n+4)\,w_1\,w_{n+1} + f'(0)\,w_{n+1} + \frac{f^{(n+1)}(0)}{(n+1)!}\,w_1 + \frac{f^{(n+2)}(0)}{(n+2)!}\,w_0\,.$$

When $n = 1$, one has $Q_2 \equiv Q_2(0) = 0$ (see the above definition for b_2). Using the expansion (3.34), which is part of the induction assumption, we also know that :

$$\lim_{\eta\to+\infty}(V_\flat, W_{\flat,n+1})(\eta) = \lim_{\eta\to+\infty}(V_\sharp, W_{\sharp,n+1})(\eta) = (0, w_{n+1}) = \left(0, \frac{b_n}{f(0) + (n+2)\,w_0}\right).$$

With the above definitions for g_{n+1}, and Q_{n+1}, we can check that $(0, w_{n+1})$ is a stationary solution to (3.35). (Recall that w_{n+1} is defined as in Theorem 3.11 by the induction assumption.) We can also evaluate the Jacobian matrix of the vector field associated with the system of ODEs (3.35) :

$$\begin{pmatrix} w_0 & 0 \\ b_{n+1} & -f(0) - (n+3)\,w_0 \end{pmatrix} = \begin{pmatrix} \lambda_1^{(n+1)} & 0 \\ b_{n+1} & \lambda_2^{(n+2)} \end{pmatrix}.$$

There exists a positive number $a_{n+1} \le a_n$ such that for all $a \in (0, a_{n+1}]$, one has $\lambda_2^{(n+2)} < \lambda_1^{(n+2)} < 0$, or equivalently $f(0) + (n+4)\,w_0 > 0$. In that case, the eigenvectors corresponding to the eigenvalues $\lambda_1^{(n+1)}$ and $\lambda_2^{(n+1)}$ are :

$$e_1^{(n+1)} = \begin{pmatrix} f(0) + (n+4)\,w_0 \\ b_{n+1} \end{pmatrix}, \quad e_2^{(n+1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Using the same argument as in the proof of Proposition 3.10, we can conclude that the solutions $(V_\flat, W_{\flat,n+1})$, and $(V_\sharp, W_{\sharp,n+1})$ of (3.35) are tangent to the straight line $(0, w_{n+1}) + \mathbb{R}\,e_1^{(n+1)}$ as η tends to $+\infty$. In particular, this yields :

$$\lim_{\eta\to+\infty}\frac{W'_{\flat,n+1}(\eta)}{V'_\flat(\eta)} = \lim_{\eta\to+\infty}\frac{W'_{\sharp,n+1}(\eta)}{V'_\sharp(\eta)} = \frac{b_{n+1}}{f(0) + (n+4)\,w_0} =: w_{n+2}\,. \tag{3.37}$$

Step 2 : let us define the function $\widetilde{w}_{n+1}$ by the formula :

$$\widetilde{w}_{n+1}(\xi) := \begin{cases} W_{\flat,n+1} \circ \Xi_\flat^{-1}(\xi) & \text{if } \xi < 0, \\ w_{n+1} & \text{if } \xi = 0, \\ W_{\sharp,n+1} \circ \Xi_\sharp^{-1}(\xi) & \text{if } \xi > 0. \end{cases}$$

With this definition, $\widetilde{w}_{n+1}$ is continuous, and we have the relation :

$$w = w_0 + w_1\,\hat{v} + \cdots + w_n\,\hat{v}^n + \widetilde{w}_{n+1}\,\hat{v}^{n+1}. \tag{3.38}$$

Moreover, using (3.37), we obtain :

$$\lim_{\xi\to 0^-} \frac{\widetilde{w}'_{n+1}(\xi)}{\hat{v}'(\xi)} = \lim_{\xi\to 0^+} \frac{\widetilde{w}'_{n+1}(\xi)}{\hat{v}'(\xi)} = w_{n+2}\,, \tag{3.39}$$

which yields $\widetilde{w}'_{n+1}(0^+) = \widetilde{w}'_{n+1}(0^-)$. Therefore, we have $\widetilde{w}_{n+1} \in C^1(\mathbb{R})$. Moreover, using (3.35), we can compute :

$$\widetilde{w}'_{n+1}\,\hat{v} = g_{n+1}(\hat{v}, \widetilde{w}_{n+1})\,, \tag{3.40}$$

so we get $\widetilde{w}'_{n+1}\,\hat{v} \in C^1(\mathbb{R})$.

Step 3 : we use an induction argument to show that $w \in C^{n+2}(\mathbb{R})$ (which will imply immediately $\hat{v} \in C^{n+3}(\mathbb{R})$). More precisely, we assume that for some $k \in \{0, \ldots, n\}$, we have :

$$\widetilde{w}_{n+1}\,\hat{v}^k \in C^{k+1}(\mathbb{R})\,, \quad w \in C^{k+1}(\mathbb{R})\,, \quad \widetilde{w}'_{n+1}\,\hat{v}^{k+1} \in C^{k+1}(\mathbb{R})\,. \tag{3.41}$$

We are going to show that this property implies the same property with k replaced by $k+1$. (Observe that step 2 above shows that the property (3.41) holds for $k = 0$.)

We note that $\hat{v} \in C^{k+2}(\mathbb{R})$, because $\hat{v}' = w \in C^{k+1}(\mathbb{R})$. Moreover, we have $\widetilde{w}_{n+1}\,\hat{v}^{k+1} = (\widetilde{w}_{n+1}\,\hat{v}^k)\,\hat{v} \in C^{k+1}(\mathbb{R})$, and we also have :

$$(\widetilde{w}_{n+1}\,\hat{v}^{k+1})' = \widetilde{w}'_{n+1}\,\hat{v}^{k+1} + (k+1)\,(\widetilde{w}_{n+1}\,\hat{v}^k)\,w \in C^{k+1}(\mathbb{R})\,.$$

Therefore, we get $\widetilde{w}_{n+1}\,\hat{v}^{k+1} \in C^{k+2}(\mathbb{R})$.

Using the relation (3.38), we immediately obtain $w \in C^{k+2}(\mathbb{R})$.

We have $\widetilde{w}'_{n+1}\,\hat{v}^{k+2} = (\widetilde{w}'_{n+1}\,\hat{v}^{k+1})\,\hat{v} \in C^{k+1}(\mathbb{R})$, and using (3.40), we derive :

$$\begin{aligned} &(\widetilde{w}'_{n+1}\,\hat{v}^{k+2})' = \big(g_{n+1}(\hat{v}, \widetilde{w}_{n+1})\,\hat{v}^{k+1}\big)' \\ &= (\partial_1 g_{n+1})(\hat{v}, \widetilde{w}_{n+1})\,\hat{v}^{k+1}\,w + (\partial_2 g_{n+1})(\hat{v}, \widetilde{w}_{n+1})\,\widetilde{w}'_{n+1}\,\hat{v}^{k+1} + (k+1)\,g_{n+1}(\hat{v}, \widetilde{w}_{n+1})\,\hat{v}^k\,w\,, \end{aligned} \tag{3.42}$$

where $\partial_1 g_{n+1}$ (resp. $\partial_2 g_{n+1}$) denotes the partial derivative of g_{n+1} with respect to its first (resp. second) variable. From the definition (3.36), we see that $g_{n+1}(\hat{v}, \widetilde{w}_{n+1})$ can be decomposed as follows :

$$g_{n+1}(\hat{v}, \widetilde{w}_{n+1}) = -(n+2)\,\widetilde{w}_{n+1}^2\,\hat{v}^{n+1} + \widetilde{w}_{n+1}\,P_1(\hat{v}) + P_0(\hat{v})\,,$$

where P_0, and P_1 are polynomial functions. Using this decomposition, and the induction assumption (3.41), we can show that each term of the sum in the right-hand side of (3.42) belongs to $C^{k+1}(\mathbb{R})$. Consequently $\widetilde{w}'_{n+1}\,\hat{v}^{k+2}$ belongs to $C^{k+2}(\mathbb{R})$, and (3.41) holds with k replaced by $k+1$. Because (3.41) holds for $k = 0$, we get that (3.41) holds for $k = n+1$, so we have proved $w \in C^{n+2}(\mathbb{R})$, and $\hat{v} \in C^{n+3}(\mathbb{R})$.

Step 4 : it remains to show that w satisfies the asymptotic expansion (3.34) at the order $n+1$. Using (3.39), and $\widetilde{w}_{n+1} \in C^1(\mathbb{R})$, we obtain :

$$\widetilde{w}_{n+1}(\xi) - w_{n+1} = w_{n+2}\,\hat{v}(\xi) + o(\hat{v}(\xi)), \quad \text{as } \xi \to 0.$$

Plugging this expansion in (3.38), we obtain (3.34) at the order $n+1$, so the proof of the induction is complete. □

Once we know that the function $\hat{v}$ belongs to $C^{n+2}(\mathbb{R})$, for $a \in (0, a_n]$, then $v = \hat{v} + (v_- + v_+)/2$ also belongs to $C^{n+2}(\mathbb{R})$, and we have already seen in the previous section that v does not vanish because $v(\xi) > v_+ > 0$ for all ξ. Moreover, the components (ρ, u, e) of the shock profile are given by :

$$\rho(\xi) = \frac{j}{v(\xi)}, \quad u(\xi) = v(\xi) + \sigma, \quad e(\xi) = \frac{(C_1 - v(\xi))\,v(\xi)}{\gamma - 1},$$

so one has $(\rho, u, e) \in C^{n+2}(\mathbb{R})$, and the proof of Theorem 3.2 is complete. (Recall that the strength of the shock tends to zero if, and only if $a = |u_+ - u_-|/2$ tends to zero.)

Finally we give the estimates for the shocks profile. We have the following

Lemma 3.12. *The C^{n+2} shock profile obtained in Theorem 3.11 has the following estimates*

$$\left\| \frac{d^l}{d\xi^l}(\rho, u, e) \right\|_{L^\infty(\mathbb{R})} \le C a^{l+1}, \ \textit{for } l = 1, 2, \cdots, n+1,$$

where the constant C is independent on the shock strength a, but may dependent on (ρ_-, u_-, e_-) and γ.

Proof. The case $n = 0$ was proved in Lemma 3.9. Now we prove in the case $n = 1$, the shock profile is C^3. As a consequence of Lemma 3.9, we only need to show the estimate $\|u''\|_{L^\infty(\mathbb{R})} \le Ca^3$, which is equivalent to show $\|w'\|_{L^\infty(\mathbb{R})} \le Ca^3$. We fellow [27]. By the construction of shock profile, we know that $\hat{v}''$ decays exponentially as $\xi \to \pm\infty$, and $\hat{v}''$ attains its maximum at a point ξ_0 where $\hat{v}^{(3)} = w'' = 0$. Then differentiate the second equation in (3.18) at ξ_0 and get

$$(f(\hat{v}) + 3w)w' = (f'(\hat{v})w^2 + \hat{v}w).$$

Using the fact that $|w| \le Ca^2$ and $0 < \varsigma_1 \le f(\hat{v}) \le \varsigma_2$, which is derived in the proof of Lemma 3.9, we get

$$|w'| \le Ca^3.$$

where the constant C is independent on the shock strength a, but may dependent on (ρ_-, u_-, e_-) and γ. Thus we prove for the case $n = 1$.

The general case $n+1$ can be obtained by the induction with respect to n. We differentiate $n+1$ times the second equation in (3.18) at the point where $w^{(n+1)}$ attains its maximum, i.e $w^{(n+2)} = 0$, thus we get

$$\begin{aligned}(f(\hat{v}) + (n+3)w)\,w^{(n+1)} &= -\sum_{j=2}^{n+1}\binom{n+2}{j} w^{(j-1)} w^{(n+2-j)} - \sum_{j=1}^{n+1}\binom{n+1}{j} f(\hat{v})^{(j)} w^{(n+1-j)} \\ &\quad + \frac{1}{2}\sum_{j=1}^{n+1}\binom{n+1}{j} w^{(j-1)} w^{(n-j)}.\end{aligned}$$

Note that by the induced condition :

$$|w^{(l)}| \leq Ca^{l+2}, \text{ for } l = 0, 1, \cdots n,$$

then we deduce the following estimate

$$|f(\hat{v})^{(l)}| \leq Ca^{l+1}, \text{ for } l = 1, 2, \cdots, n+1.$$

Similarly we use the fact $0 < \varsigma_1 < f(\hat{v}) < \varsigma_2$, we derive that

$$|w^{(n+1)}| \leq Ca^{n+3},$$

where the constant C is independent on the shock strength a, but may dependent on (ρ_-, u_-, e_-) and γ. Thus we complete the proof of Lemma 3.12. $\square$

3.4 Appendix : Formal derivation of the model

It is worth describing how the model (3.1), (3.3) can be obtained from a more complete physical system. The derivation we propose below remains formal – a rigorous proof being certainly delicate and beyond the scope of this work – and we refer to [4, 17, 43, 49] for further details. Let us introduce the specific intensity of radiation $f(t, x, v)$, that depends on a time variable $t \geq 0$, a space variable $x \in \mathbb{R}^N$, and a direction $v \in \mathbb{S}^{N-1}$. We make the 'grey assumption', which means that the frequency dependence is ignored (all photons have the same frequency). Photons are subject to two main interaction phenomena :

- scattering produces changes in the direction of the photons,
- absorption/emission where photons are lost/produced through a transfer mecanism with the surrounding gas.

The scattering phenomenon is described by the operator :

$$Q_s(f)(t, x, v) = \sigma_s \left(\int_{\mathbb{S}^{N-1}} f(t, x, v')\, dv' - f(t, x, v) \right),$$

(with dv the normalized Lebesgue measure on $\mathbb{S}^{N-1}$), and the absorption/emission phenomenon is described by the operator :

$$Q_a(f)(t, x, v) = \sigma_a \left(\frac{\sigma}{\pi}\, \theta(t, x)^4 - f(t, x, v) \right),$$

where we used the Stefan-Boltzmann emission law, θ being the temperature of the gas, and σ the Stefan-Boltzmann constant. In these definitions, the coefficients $\sigma_{s,a}$ are given positive quantities. These phenomena are both characterized by a typical mean free path, denoted ℓ_s, ℓ_a respectively. Therefore, the evolution of the specific intensity is driven by :

$$\frac{1}{c}\, \partial_t f + v \cdot \nabla_x f = \frac{1}{\ell_s}\, Q_s(f) + \frac{1}{\ell_a}\, Q_a(f) = Q(f), \tag{3.43}$$

where c stands for the speed of light. The equation (3.43) is coupled to the Euler system describing the evolution of the fluid :

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, u) = 0, \\ \partial_t (\rho\, u) + \nabla_x \cdot (\rho\, u \otimes u) + \nabla_x P = -\dfrac{1}{c} \displaystyle\int_{\mathbb{S}^{N-1}} v\, Q(f)\, dv, \\ \partial_t (\rho\, E) + \nabla_x \cdot (\rho\, E\, u + P\, u) = -\displaystyle\int_{\mathbb{S}^{N-1}} Q(f)\, dv. \end{cases} \tag{3.44}$$

The equations (3.43), (3.44) are thus coupled by the exchanges of both momentum and energy, and by the Stefan-Boltzmann emission law. Observe that only the emission/absorption operator enters into the energy equation since the scattering operator is conservative (this would be different if Doppler corrections were taken into account). Note also that the total energy :

$$\frac{1}{c}\int_{\mathbb{R}^N}\int_{\mathbb{S}^{N-1}} f\,dv\,dx + \int_{\mathbb{R}^N} \rho\, E\,dx\,,$$

is (formally) conserved. Writing the system (3.44), and the kinetic equation (3.43) in the dimensionless form, we can make four dimensionless parameters appear :

- $\mathcal{C}$, the ratio of the speed of light over the typical sound speed of the gas,
- $\mathcal{L}_s$, the Knudsen number associated to the scattering,
- $\mathcal{L}_a$, the Knudsen number associated to the absorption/emission,
- $\mathcal{P}$, which compares the typical energy of radiation and the typical energy of the gas.

We thus obtain the rescaled equations :

$$\begin{cases} \dfrac{1}{\mathcal{C}}\,\partial_t f + v\cdot\nabla_x f = \dfrac{1}{\mathcal{L}_s}\,Q_s(f) + \dfrac{1}{\mathcal{L}_a}\,Q_a(f)\,, \\ \partial_t\rho + \nabla_x\cdot(\rho\, u) = 0\,, \\ \partial_t(\rho\, u) + \nabla_x\cdot(\rho\, u\otimes u) + \nabla_x P = \mathcal{P}\left(\dfrac{\sigma_s}{\mathcal{L}_s} + \dfrac{\sigma_a}{\mathcal{L}_a}\right)\displaystyle\int_{\mathbb{S}^{N-1}} v\, f(v)\,dv\,, \\ \partial_t(\rho\, E) + \nabla_x\cdot(\rho\, E\, u + P\, u) = -\dfrac{\mathcal{P}\,\mathcal{C}}{\mathcal{L}_a}\,\sigma_a\left(\theta^4 - \displaystyle\int_{\mathbb{S}^{N-1}} f(v)\,dv\right). \end{cases} \tag{3.45}$$

System (3.1), (3.3) is then obtained in two steps. First of all, we assume $\mathcal{C}\gg 1$. Next, we keep $\mathcal{P}$ of order 1, and we are concerned here with a regime where scattering is the leading phenomenon : the mean free paths are rescaled according to :

$$\mathcal{L}_s \simeq \frac{1}{\mathcal{C}}\,,\qquad \mathcal{L}_a \simeq \mathcal{C}\,.$$

The asymptotics can be readily understood by means of the Hilbert expansion :

$$f = f^{(0)} + \frac{1}{\mathcal{C}}\,f^{(1)} + \frac{1}{\mathcal{C}^2}\,f^{(2)} + \dots$$

Identifying the terms arising with the same power of $1/\mathcal{C}$, we get :

- at the leading order, $f^{(0)}$ belongs to the kernel of the scattering operator, so that is does not depend on the microscopic variable v : $f^{(0)}(t,x,v) = n(t,x)$,
- the relation $Q_s(f^{(1)}) = v\cdot\nabla_x f^{(0)}$ then leads to : $f^{(1)}(t,x,v) = -\frac{1}{\sigma_s}\,v\cdot\nabla_x n(t,x)$,
- integrating the equation for $f^{(2)}$ over the sphere yields :

$$\partial_t n - \frac{1}{N\,\sigma_s}\,\Delta_x n = \sigma_a\,(\theta^4 - n)\,.$$

Note also that in the momentum equation, we have :

$$\left(\frac{\sigma_s}{\mathcal{L}_s} + \frac{\sigma_a}{\mathcal{L}_a}\right)\int_{\mathbb{S}^{N-1}} v\,f(v)\,dv \simeq \sigma_s\int_{\mathbb{S}^{N-1}} v\,f^{(1)}(v)\,dv = -\frac{\sigma_s}{N}\,\nabla_x n\,.$$

Finally, we obtain the limit system :

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, u) = 0\,, \\ \partial_t(\rho\, u) + \nabla_x \cdot (\rho\, u \otimes u) + \nabla_x P = -\dfrac{\mathcal{P}}{N}\, \nabla_x n\,, \\ \partial_t(\rho\, E) + \nabla_x \cdot (\rho\, E\, u + P\, u) = -\mathcal{P}\, \sigma_a\, (\theta^4 - n)\,, \\ \partial_t n - \dfrac{1}{N\, \sigma_s}\, \Delta_x n = \sigma_a\, (\theta^4 - n)\,. \end{cases} \tag{3.46}$$

The system (3.46) describes a nonequilibrium regime, where the material and the radiations have different temperatures ($\theta \neq n^{1/4}$) ; the equilibrium regime would correspond to assuming that the emission/absorption is the leading contribution.

After this first asymptotics, we perform a second asymptotics where we set :

$$\mathcal{P} \ll 1\,, \qquad \mathcal{P}\, \sigma_a = 1\,, \qquad N\, \sigma_s = 1/\sigma_a\,.$$

This leads to (3.1), (3.3). Of course, one might wonder how this second approximation modifies the shock profiles compared to (3.46), in particular when we get rid of the radiative pressure in the momentum equation. We refer to [49, page 579] for some aspects of this problem.

Chapitre 4

Asymptotic Stability of Shock Profiles for Non Equilibrium Radiating Gases

Ce chapitre fait l'objet d'un projet de Note aux Comptes Rendus de l'Académie des Science.

4.1 Introduction

We are interested in a system of PDEs describing astrophysical flows, where the gas interacts with radiation through energy exchanges. The evolution of the gas is governed by the following system of equations written in Lagrangian coordinates :

$$\left\{\begin{array}{rcl} v_t - u_x & = & 0, \\ u_t + p_x & = & 0, \\ (e + \frac{u^2}{2})_t + (pu)_x + q_x & = & 0, \end{array}\right. \tag{4.1}$$

where $v > 0$, u and $e > 0$ represent the specific volume, the velocity and the specific internal energy of the gas, respectively, p is the pressure and q denotes the radiative heat flux, satisfying the following elliptic equation still written in Lagrangian coordinates :

$$-\frac{1}{v}\left(\frac{q_x}{v}\right)_x + q + \frac{(\theta^4)_x}{v} = 0, \tag{4.2}$$

where $\theta > 0$ is the absolute temperature of the gas. The advantage of writing (4.1) in Lagrangian coordinates is to have a simple expression for the conserved variables v, u and $e + u^2/2$. We choose v, u, θ and q as the unknowns in (4.1)-(4.2). The pressure p and the specific internal energy e are related to v and θ by the following perfect gas law :

$$p = R\frac{\theta}{v}, \quad e = \frac{R}{\gamma - 1}\theta := C_v\theta, \tag{4.3}$$

where $R > 0$ is the gas constant, and $\gamma > 1$ is the adiabatic exponent, and $C_v = R/(\gamma - 1)$ is the specific heat at constant volume. In what follows, the adiabatic exponent will be restricted to take values in an interval $(1, \gamma_*)$, with $\gamma_* \simeq 2.215$ (see [38] for the origin of this restriction). However, the main physical cases are covered. We mention that the system (4.1)-(4.2) can be formally derived by asymptotic arguments, starting from a more complete system involving a kinetic equation for the specific intensity of radiation, see the Appendix of [38]. We also refer to [49, 61] for the physical background.

In [38] we have proved that the system (4.1)-(4.2) admits smooth traveling wave solutions of the form

$$\begin{array}{rcll} (v,u,\theta,q)(t,x) & = & (V,U,\Theta,Q)(\xi), & \xi = x - s\,t, \\ (V,U,\theta,Q)(\xi) & \longrightarrow & (v_\pm,u_\pm,\theta_\pm,0), & \text{as } \xi \to \pm\infty, \end{array} \tag{4.4}$$

where $(v_\pm, u_\pm, \theta_\pm, s)$ is a shock wave solution to the standard Euler equation (that is, system (4.1) with $q \equiv 0$). In other words, the constants $(v_\pm, u_\pm, \theta_\pm, s)$ satisfy the following Rankine-Hugoniot conditions :

$$\left\{ \begin{array}{rcl} -s(v_+ - v_-) - (u_+ - u_-) & = & 0 \\ -s(u_+ - u_-) + (p_+ - p_-) & = & 0 \\ -s\left(\left(e_+ + \dfrac{u_+^2}{2}\right) - \left(e_- + \dfrac{u_-^2}{2}\right)\right) + (p_+u_+ - p_-u_-) & = & 0, \end{array} \right. \tag{4.5}$$

and the Lax entropy condition :

$$\text{either } \lambda_1^+ < s < \lambda_1^-, \quad \text{or } \lambda_3^+ < s < \lambda_3^-, \tag{4.6}$$

where $\lambda_1 = -\sqrt{\gamma\, R\theta}/v$ and $\lambda_3 = \sqrt{\gamma\, R\theta}/v$ are the first and the third characteristic speeds of the standard Euler system. The traveling wave (4.4) is called a shock profile solution to (4.1)-(4.2). We refer to [55, 58, 35] for a detailed study of shock waves for the Euler system. Note that (4.6) is equivalent to

$$u_+ < u_- \ (\text{or } s(v_+ - v_-) > 0), \tag{4.7}$$

which will be proved in Appendix 4.4.1.

The shock profile $(V, U, \Theta, Q)(\xi)$ is determined by the ODE system

$$\left\{ \begin{array}{rcl} -sV' - U' & = & 0, \\ -sU' + P' & = & 0, \\ -s\left(\dfrac{R}{\gamma-1}\Theta + \dfrac{U^2}{2}\right)' + (PU)' + Q' & = & 0, \\ -\dfrac{1}{V}\left(\dfrac{Q'}{V}\right)' + Q + \dfrac{(\Theta^4)'}{V} & = & 0, \end{array} \right. \tag{4.8}$$

where $'$ denotes differentiation with respect to the variable $\xi = x - s\,t$, and it also satisfies the asymptotic conditions (4.4). Integrating the first three equations with respect to ξ over $(-\infty, \xi]$ yields

$$\left\{ \begin{array}{rcl} -sV - U & = & a_1, \\ -sU + P & = & a_2, \\ -s\left(\dfrac{R}{\gamma-1}\Theta + \dfrac{U^2}{2}\right) + (PU) + Q & = & a_3, \end{array} \right.$$

where the integrated constants a_1, a_2 and a_3 are defined as

$$\begin{array}{rcl} a_1 & = & -s\, v_\pm - u_\pm, \\ a_2 & = & -su_\pm + p_\pm, \\ a_3 & = & -s\left(C_v\,\theta_\pm + \dfrac{u_\pm^2}{2}\right) + p_\pm u_\pm, \end{array} \tag{4.9}$$

and $P = R\,\Theta/V$ satisfies

$$P = sU + a_2 = a_2 - s\;a_1 - s^2V := b_1 - s^2V := \frac{1}{K(V)}, \tag{4.10}$$

with $b_1 = a_2 - sa_1$. We refer [38] for the detailed study of the existence when the strength of the shock is small. Existence of shock profiles is still open when the strength of shock is arbitrary. Precisely, the following statement is the starting point of the present work.

Proposition 4.1. *[38] Let γ satisfy*

$$1 < \gamma < \frac{\sqrt{7}+1}{\sqrt{7}-1} \simeq 2.215$$

and let (v_-, u_-, θ_-) be fixed. Then there exists a positive number δ (that may depend on v_-, u_-, θ_- and γ), such that if (v_+, u_+, θ_+) satisfies

- *$|v_+ - v_-| \leq \delta$,*
- *$(v_\pm, u_\pm, \theta_\pm)$ is a shock wave, with speed s, for the standard Euler equations,*

then there exists a C^4 shock profile $(V, U, \Theta, Q)(x - st)$ satisfying

$$U' < 0 \text{ (or equivalently } sV' > 0). \tag{4.11}$$

Furthermore we have the following estimate

$$\left\| \frac{\mathrm{d}^l}{\mathrm{d}\xi^l}(V, U, \Theta) \right\|_{L^\infty(\mathbb{R})} \leq C|v_+ - v_-|^{l+1}, \quad \text{for } l = 1,\ 2,\ 3, \tag{4.12}$$

$$\left\| \frac{\mathrm{d}^l}{\mathrm{d}\xi^l} Q \right\|_{L^\infty(\mathbb{R})} \leq C|v_+ - v_-|^{l+1}, \quad \text{for } l = 1,\ 2,\ 3,\ 4, \tag{4.13}$$

for some constant C that does not depend on (v_+, u_+, θ_+) but may depend on (v_-, u_-, θ_-) and γ.

The question we address concerns the asymptotic stability of the shock profiles : does the initial value problem (4.1)-(4.2) with initial data (v_0, u_0, θ_0) given in a neighborhood of a shock profile (V, U, Θ), have a unique global solution $(v, u, \theta)(t, x)$ and $q(t, x)$, such that the solution converges to the shock profile in some norm as $t \to +\infty$?

Such a problem has been well studied for viscous regularization of hyperbolic systems. For example, the scalar case is treated in [23, 25], the general case is addressed in [21], the compressible Navier-Stokes equations are treated in [25, 47]. A typical result assumes a suitably small strength of the shock, restricts to perturbations with vanishing integrals (zero mass perturbations) and establishes the convergence in sup norm of the solution to the shock profile as $t \to +\infty$. Proofs rely on the energy method applied to the integrated system. More precisely, Matsumura and Nishihara in [47] showed the asymptotic stability of shock profiles to the viscous $p-$system under the assumption that the initial perturbations for the specific volume v and the velocity u are derivatives of sufficiently small functions in $H^2(\mathbb{R})$, and that the strength of the travelling wave is suitably small. In §2 of [25], Kawashima and Matsumura generalised this result to the full Navier-Stokes system. The same energy method as in [47] was used for the integrated system. Dealing with the integrated system has two advantages, that are explained in the introduction of [21]. As pointed out also by Kawashima and Matsumura in [25], the monotonicity of the shock profile gives a dissipative mechanism for the integrated system. And this dissipation combined with the viscosity is sufficient to prove the stability of viscous shock profiles, see [25]. Here we will use this dissipation due to the monotonicity of the shock profile combined with the radiative heat conductivity to treat the stability of the shock profiles for the system (4.1)-(4.2).

For the hyperbolic-elliptic system (4.1)-(4.2), the existence of shock profiles was proved in [38] in Eulerian coordinates. Existence of shock profiles was also showed in [32] for general hyperbolic system coupled with a linear elliptic equation. Kawashima, Nikkuni and Nishibata in [26] showed the global in time existence of solutions in the neighborhood of a constant state for the coupled system (4.1)-(4.2) in Eulerian coordinates. This corresponds to a shock of zero strength. Here we will derive and use the elliptic estimates as in [26] to show the global existence of solutions in a neighborhood of a shock profile. Combining the energy estimate for the integrated system and the elliptic estimates derived for the elliptic equation, we will prove the global existence of solutions to the integrated system.

In [27], Kawashima and Nishibata showed the existence of shock profiles and the stability of small shock profiles, for a simplified model of radiating gas dynamics. This simple model can be seen as a prototype for discussing the coupled system (4.1)-(4.2). As we will see later, under the same restrictions on the initial data, shock profiles being of class C^4 (the regularity C^4 gives the estimates of shock profile in norm $W^{3,\infty}(\mathbb{R})$), the initial perturbations being in $H^2(\mathbb{R})$, and the integrals of the initial perturbations over $(-\infty, x]$ being small in $H^3(\mathbb{R})$, we get a similar result, namely the global existence of solutions in a neighborhood of a shock profile, and the convergence to the shock profile as time becomes large.

Now we setup some notations for the initial data, and give the statement of our main result. Consider the initial problem for (4.1)-(4.2) in the neighborhood of a shock profile, with the initial value :

$$(v, u, \theta)(0, x) = (v_0, u_0, \theta_0)(x). \tag{4.14}$$

Let (V, U, Θ, Q) be a given C^4 shock profile as in proposition 4.1. We assume that

$$(v_0 - V, u_0 - U, \theta_0 - \Theta) \in H^2(\mathbb{R}), \tag{4.15}$$

and that there exist some functions Φ_0, Ψ_0, $\tilde{W}_0 \in H^3(\mathbb{R})$ such that

$$\begin{aligned} v_0 - V &= \Phi_0', &(4.16)\\ u_0 - U &= \Psi_0', &(4.17)\\ \left(C_v\theta_0 + \frac{u_0^2}{2}\right) - \left(C_v\Theta + \frac{U^2}{2}\right) &= \tilde{W}_0'. &(4.18)\end{aligned}$$

Here $'$ denotes the differentiation with respect to the real variable x. We define

$$W_0 = \frac{1}{C_v}(\tilde{W}_0 - U\Psi_0) \in H^3(\mathbb{R}).$$

Let $q_0 \in Q + H^3(\mathbb{R})$ be the solution to the equation

$$-\frac{1}{v_0}\left(\frac{q_{0x}}{v_0}\right)_x + q_0 + \frac{(\theta_0^4)_x}{v_0} = 0. \tag{4.19}$$

We refer to Appendix 4.4.2 for the existence and uniqueness of q_0. Then there exists $\tilde{Z}_0 \in H^2(\mathbb{R})$, such that

$$\tilde{Z}_0' = v_0(q_0 - Q) - \left(\frac{Q'}{V} - \Theta^4\right)\left(\frac{v_0 - V}{V}\right)'. \tag{4.20}$$

and we define

$$Z_0 = \tilde{Z}_0 + \frac{v_0 - V}{V}(\frac{Q'}{V} - \Theta^4) \in H^2(\mathbb{R}).$$

Our main result states as follows :

Theorem 4.2. *Let (v_-, u_-, θ_-) be fixed, and let (v_+, u_+, θ_+) define a shock wave with speed s for the standard Euler equations, i.e $(v_\pm, u_\pm, \theta_\pm, s)$ satisfy the Rankine-Hugoniot condition (4.5), and the Lax entropy condition (4.6). There exists $\delta > 0$ such that if $|v_+ - v_-| \leq \delta$, if $(V, U, \Theta, Q)(\xi)$, $\xi = x - s\,t$, is the shock profile given by proposition 4.1, if the initial data (v_0, u_0, θ_0) satisfies (4.15)- (4.18), and if $\|\Phi_0, \Psi_0, \tilde{W}_0\|_{H^3(\mathbb{R})} \leq \delta$, then the initial value problem (4.1), (4.2) and (4.14) has a unique global solution $(v, u, \theta, q)(t, x)$ with*

$$\begin{aligned} (v - V, u - U, \theta - \Theta) &\in C^0(0, \infty; H^2(\mathbb{R})), \\ q - Q &\in C^0(0, \infty; H^3(\mathbb{R})). \end{aligned} \tag{4.21}$$

Furthermore, the solution (v, u, θ, q) tends to the shock profile in the following sense :

$$\sup_{x \in \mathbb{R}} |(v, u, \theta, q)(t, x) - (V, U, \Theta, Q)(x - st)| \to 0 \quad \textit{as} \quad t \to +\infty. \tag{4.22}$$

4.2 Reformulation of the problem, the integrated system

Let us set

$$(v, u, \theta, q)(t, x) = (V, U, \Theta, Q)(\xi) + (\phi, \psi, w, z)(t, \xi), \tag{4.23}$$

where $\xi := x - st$ and (ϕ, ψ, w, z) represent the perturbation of the shock profile. Then we rewrite the system (4.1)-(4.2) as

$$\left\{ \begin{aligned} &\phi_t - s\phi_\xi - \psi_\xi = 0, \\ &\psi_t - s\psi_\xi + \left(R\frac{\Theta + w}{V + \phi} - R\frac{\Theta}{V} \right)_\xi = 0, \\ &\left(C_v w + U\psi + \frac{\psi^2}{2} \right)_t - s\left(C_v w + U\psi + \frac{\psi^2}{2} \right)_\xi \\ &\qquad + R\left(\frac{\Theta + w}{V + \phi}(U + \psi) - \frac{\Theta}{V}U \right)_\xi + z_\xi = 0, \\ &\frac{1}{V}\left(\frac{Q'}{V} - \Theta^4 \right)' - \frac{1}{V + \phi}\left(\frac{Q' + z_\xi}{V + \phi} \right)_\xi + z + \frac{((\Theta + w)^4)_\xi}{V + \phi} = 0, \end{aligned} \right. \tag{4.24}$$

with the initial data

$$(\phi, \psi, w)(0, \xi) = (\phi_0, \psi_0, w_0)(\xi) \equiv (v_0 - V, u_0 - U, \theta_0 - \Theta)(\xi). \tag{4.25}$$

In (4.24), $'$ denotes the differentiation with respect to ξ of functions that depend only on the real variable ξ, while the subscript ξ denotes the partial differentiation with respect to ξ. Note that the fourth equation in (4.24) can be written equivalently as

$$\begin{aligned} &(V + \phi)z + ((\Theta + w)^4 - \Theta^4)_\xi + \left(\frac{Q'\phi}{V(V + \phi)} \right)_\xi - \left(\frac{z_\xi}{V + \phi} \right)_\xi \\ &\qquad + \left(\frac{\phi}{V}\left(\frac{Q'}{V} - \Theta^4 \right) \right)_\xi - \left(\frac{\phi}{V} \right)_\xi \left(\frac{Q'}{V} - \Theta^4 \right) = 0. \end{aligned} \tag{4.26}$$

Now we extend the idea used in [25] to deal with the Navier-Stokes equations to the hyperbolic-elliptic coupled system (4.24). For the initial data, let us define functions w_0, $\tilde{w}_0$, z_0, $\tilde{z}_0$ such that

$$\begin{aligned}\tilde{w}_0 = \tilde{W}_0' &= C_v(\theta_0 + \frac{u_0^2}{2}) - C_v(\Theta + \frac{U^2}{2}) \\ &= C_v w_0 + \frac{1}{2}\psi_0^2 + U\psi_0,\end{aligned}$$

and

$$\tilde{z}_0 = \tilde{Z}_0' = (q_0 - Q)(V + \phi_0) - \left(\frac{\phi_0}{V}\right)'\left(\frac{Q'}{V} - \Theta^4\right) = z_0(V + \phi_0) - \left(\frac{\phi_0}{V}\right)'\left(\frac{Q'}{V} - \Theta^4\right).$$

Then from (4.16)-(4.18) and (4.20) we have

$$(\phi_0, \psi_0, \tilde{w}_0, \tilde{z}_0) = (\Phi_0, \Psi_0, \tilde{W}_0, \tilde{Z}_0)'.$$

Taking these information into account, we seek the solution of problem (4.24) under the form

$$(\phi, \psi, \tilde{w}, \tilde{z}) = (\Phi_\xi, \Psi_\xi, \tilde{W}_\xi, \tilde{Z}_\xi), \tag{4.27}$$

where

$$\tilde{w} = C_v w + \frac{1}{2}\psi^2 + U\psi,$$

and

$$\tilde{z} = z(V + \phi) - \left(\frac{Q'}{V} - \Theta^4\right)\left(\frac{\phi}{V}\right)_\xi.$$

Let us introduce :

$$W = \frac{1}{C_v}(\tilde{W} - U\Psi), \quad Z = \tilde{Z} + \frac{\phi}{V}\left(\frac{Q'}{V} - \Theta^4\right). \tag{4.28}$$

Thus from (4.27), (4.28) we derive the following relations between w, z and W, Z :

$$w = W_\xi + \frac{1}{C_v}\left(U'\Psi - \frac{\psi^2}{2}\right), \quad z = \frac{1}{V + \phi}\left(Z_\xi - \left(\frac{Q'}{V} - \Theta^4\right)'\frac{\phi}{V}\right). \tag{4.29}$$

Then substituting (4.29) and $(\phi, \psi) = (\Phi_\xi, \Psi_\xi)$ into (4.24) and integrating once with respect to ξ, we get the following integrated system :

$$\left\{\begin{array}{l} \Phi_t - s\Phi_\xi - \Psi_\xi = 0, \\ \Psi_t - s\Psi_\xi + R\left(\dfrac{\Theta + w}{V + \phi} - \dfrac{\Theta}{V}\right) = 0, \\ (C_v W + U\Psi)_t - s\left(C_v W + U\Psi\right)_\xi + R\dfrac{\Theta + w}{V + \phi}(U + \psi) \\ \qquad -R\dfrac{\Theta}{V}U + \dfrac{1}{V + \phi}\left(Z_\xi - \left(\dfrac{Q'}{V} - \Theta^4\right)'\dfrac{\phi}{V}\right) = 0, \\ \dfrac{Q'\phi}{V(V + \phi)} - \dfrac{1}{V + \phi}\left(\dfrac{1}{V + \phi}\left(Z_\xi - \left(\dfrac{Q'}{V} - \Theta^4\right)'\dfrac{\phi}{V}\right)\right)_\xi \\ \qquad +Z + (\Theta + w)^4 - \Theta^4 = 0, \end{array}\right. \tag{4.30}$$

with the initial data

$$(\Phi, \Psi, W)(0, \xi) = (\Phi_0, \Psi_0, W_0)(\xi), \quad W_0 = \frac{1}{C_v}\left(\tilde{W}_0 - U\Psi_0\right). \tag{4.31}$$

Thus we are going to prove the following result :

Theorem 4.3. *Suppose* $(\Phi_0, \Psi_0, \tilde{W}_0) \in H^3(\mathbb{R})$, *then there exist positive constants* δ, ϵ *and* C *such that if* $|v_+ - v_-| \leq \delta$ *and*

$$\|(\Phi_0, \Psi_0, \tilde{W}_0)\|_{H^3(\mathbb{R})} \leq \epsilon,$$

then the problem (4.30), (4.31) *has a unique global solution* $(\Phi, \Psi, W) \in C(0, +\infty; H^3(\mathbb{R}))$, *and* $Z \in C(0, +\infty; H^4(\mathbb{R}))$, *satisfying the estimate :*

$$\begin{aligned}
&\|(\Phi, \Psi, W)(t)\|^2_{L^2(\mathbb{R})} + \|(\phi, \psi, w)(t)\|^2_{H^2(\mathbb{R})} + \|z(t)\|^2_{H^3(\mathbb{R})} \\
&+ \int_0^t \||V'|^{1/2}(\Psi, W)(\tau)\|^2_{L^2(\mathbb{R})} + \|Z(\tau)\|^2_{H^1(\mathbb{R})} + \|z(\tau)\|^2_{H^3(\mathbb{R})} + \|(\phi, \psi, w)(\tau)\|^2_{H^2(\mathbb{R})} \mathrm{d}\tau \\
&\leq C\|(\Phi_0, \Psi_0, \tilde{W}_0)\|^2_{H^3(\mathbb{R})}
\end{aligned} \tag{4.32}$$

for all $t \geq 0$.

We can go back to the functions (ϕ, ψ, w, z) from the solutions obtained in Theorem 4.3 and define in this way a global solution of the original system (4.24) (4.25). Thus we obtain the desired solution belonging to the functional space specified in Theorem 4.2. Furthermore the solution of the system (4.24) (4.25) is unique in that space. Thus with Theorem 4.3, to complete the proof of Theorem 4.2, it remains to prove the asymptotic convergence. For example, for the specific volume v, we have the following inequality :

$$\begin{aligned}
\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |v(t,x) - V(x-st)| &= \lim_{t\to\infty} \sup_{x\in\mathbb{R}} |\phi(t, x-st)| \\
&= \lim_{t\to\infty} \sup_{\xi\in\mathbb{R}} |\phi(t,\xi)| \\
&\leq C \lim_{t\to\infty} \|\phi(t)\|_{H^1(\mathbb{R})},
\end{aligned}$$

and we also show the $L^\infty-$ decay of the derivatives of ϕ, we fellow the strategy used in [26]. Form the first equation in (4.24), we have

$$\int_0^t \|\phi_t(\tau)\|^2_{H^1(\mathbb{R})} \mathrm{d}\tau \leq C \int_0^t \|(\phi_\xi, \psi_\xi)(\tau)\|^2_{H^1(\mathbb{R})} \mathrm{d}\tau \leq C\|(\Phi_0, \Psi_0, \tilde{W}_0)\|^2_{H^3(\mathbb{R})}, \tag{4.33}$$

for any $t \geq 0$. Note that for the last inequality we used the a priori estimate (4.32). And there holds that the map $\Sigma : t \mapsto \|\phi_\xi(t)\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \phi_\xi(t,\xi)^2 \mathrm{d}\xi$ is integrable over $\mathbb{R}^+$. Furthermore, using (4.32) and (4.33), we have that Σ' is also integrable over $\mathbb{R}^+$. It then follows that $\Sigma(t) \to 0$ as $t \to +\infty$, that is that $\|\phi_x(t)\|^2_{L^2(\mathbb{R})} \to 0$ as $t \to +\infty$. Using the Sobolev inequality

$$\|f\|_{L^\infty(\mathbb{R})} \leq \sqrt{2}\|f\|^{1/2}_{L^2(\mathbb{R})}\|f_x\|^{1/2}_{L^2(\mathbb{R})},$$

to the functions ϕ and ϕ_ξ, we have

$$\begin{cases}
\|\phi(t)\|_{L^\infty(\mathbb{R})} \leq \sqrt{2}\|\phi\|^{1/2}_{L^2(\mathbb{R})}\|\phi_\xi\|^{1/2}_{L^2(\mathbb{R})} \to 0, \\
\|\phi_\xi(t)\|_{L^\infty(\mathbb{R})} \leq \sqrt{2}\|\phi_\xi\|^{1/2}_{L^2(\mathbb{R})}\|\phi_{\xi\xi}\|^{1/2}_{L^2(\mathbb{R})} \to 0,
\end{cases}$$

as $t \to +\infty$ since there hold $\|\phi_x(t)\|_{L^2(\mathbb{R})} \to 0$ and $\|\phi(t)\|_{H^2(\mathbb{R})}$ is bounded in $\mathbb{R}^+$. Thus we get the convergence of $\phi(t)$. By a similar analysis, we obtain the convergence of ψ and w. While for the convergence of z, we claim that there holds

$$\|z(t)\|^2_{H^1(\mathbb{R})} \leq C\left(\|w(t)\|^2_{L^2(\mathbb{R})} + \|\phi(t)\|^2_{L^2(\mathbb{R})}\right). \tag{4.34}$$

Thus we can get the convergence of z provided (4.34). In deeded (4.34) can be obtained form (4.26). We multiply (4.26) by z and integrated it with respect to $\xi \in \mathbb{R}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}+} (V+\phi)z^2 + \frac{1}{V+\phi} z_\xi^2 \, \mathrm{d}\xi \\
&= \int_{\mathbb{R}+} \left(4\Theta^3 w + 6\Theta^2 w^2 + 4\Theta w^3 + w^4\right) z_\xi \mathrm{d}\xi + \int_{\mathbb{R}+} \frac{Q'\phi z_\xi}{V(V+\phi)} - \left(\frac{Q'}{V} - \Theta^4\right)' \frac{\phi}{V} z \, \mathrm{d}\xi.
\end{aligned}$$

Using the Cauchy-Schwarz inequality and the a priori estimate (4.32), we get (4.34). Thus we obtain the convergence (4.22) and the proof of Theorem 4.2 is complete. It only remains to prove Theorem 4.3. The local existence of solutions of such a hyperbolic-elliptic system was shown in [26]. In order to prove Theorem 4.3 it suffices to show the following a priori estimate, which will be proved in §4.3.

Proposition 4.4. *Let* $(\Phi, \Psi, W) \in C(0,T; H^3(\mathbb{R}))$ *and* $Z \in C(0,T; H^4(\mathbb{R}))$ *be a local solution to the system* (4.30) (4.31). *Set*

$$\begin{aligned}
N(t)^2 \;=\; & \sup_{0 \le \tau \le t} \left(\|(\Phi, \Psi, W)(\tau)\|^2_{L^2(\mathbb{R})} + \|(\phi, \psi, w)(\tau)\|^2_{H^2(\mathbb{R})} + \|z(\tau)\|^2_{H^3(\mathbb{R})} \right) \\
& + \int_0^t \left(\| |V'|^{1/2} (\Psi, W)(\tau)\|^2_{L^2(\mathbb{R})} + \|Z(\tau)\|^2_{H^1(\mathbb{R})} \right) \mathrm{d}\tau \\
& + \int_0^t \left(\|(\phi, \psi, w)(\tau)\|^2_{H^2(\mathbb{R})} + \|z(\tau)\|^2_{H^3(\mathbb{R})} \right) \mathrm{d}\tau.
\end{aligned} \tag{4.35}$$

Then there exist positive constants δ, ϵ and C which are independent of T such that if $|v_+ - v_-| \le \delta$ and $N(T) \le \epsilon$, then the following estimate :

$$N(t)^2 \le C(N(0)^2 + N(t)^3) \tag{4.36}$$

holds for any $t \in [0, T]$.

4.3 A priori estimate

Let $(\Phi, \Psi, W) \in C(0,T; H^3(\mathbb{R}))$ and $Z \in C(0,T; H^4(\mathbb{R}))$ for some $T > 0$, be a local solution to the system (4.30) (4.31). We choose sufficiently small constants δ, and ϵ, such that the assumptions

$$|v_+ - v_-| \le \delta, \quad N(T) \le \epsilon,$$

imply

$$v_+ \ge \frac{1}{2} v_-, \quad \theta_+ \ge \frac{1}{2}\theta_-$$

$$\|\phi, \psi, w\|_{L^\infty([0,T]\times\mathbb{R})} < \frac{1}{4}\min\{v_-, \theta_-\}.$$

Let (ϕ, ψ, w, z) be functions defined by (Φ, Ψ, W, Z) through (4.27) (4.29). In this section we will give an energy estimate for these functions. In what follows the constants in the inequalities may depend on v_-, u_-, θ_-, but not on T.

4.3.1 $L^\infty(L^2)$ Estimates

We will show the $L^\infty(0,T;L^2(\mathbb{R}))$ estimates for (Φ,Ψ,W) and $\partial_\xi^l(\phi,\psi,w)$ with $l=0$, 1 and 2. And we also get the $L^2(0,T;L^2(\mathbb{R}))$ estimate for $|V'|^{1/2}(\Psi,W)$, Z and $\partial_\xi^l z$ with $l=0$, 1 and 2.

System (4.30) can be simplified as

$$\begin{cases} \Phi_t - s\Phi_\xi - \Psi_\xi = 0, \\ \Psi_t - s\Psi_\xi - \dfrac{1}{VK(V)}\Phi_\xi + \dfrac{R}{V}W_\xi + \dfrac{\gamma-1}{V}U'\Psi = F_1, \\ C_v(W_t - sW_\xi) + \dfrac{1}{K(V)}\Psi_\xi - sU'\Psi + \dfrac{1}{V}Z_\xi - \left(\dfrac{Q'}{V} - \Theta^4\right)' \dfrac{\Phi_\xi}{V(V+\phi)} = F_2, \\ Z - \dfrac{1}{V+\phi}\left(\dfrac{Z_\xi}{V+\phi}\right)_\xi + 4\Theta^3 W_\xi + \dfrac{Q'\Phi_\xi}{V(V+\phi)} + \dfrac{1}{V+\phi}\left(\left(\dfrac{Q'}{V} - \Theta^4\right)'\dfrac{\Phi_\xi}{V}\right)_\xi \\ \qquad + \dfrac{4(\gamma-1)}{R}\Theta^3 U'\Psi = F_3, \end{cases} \tag{4.37}$$

where $K(V)$ is defined by (4.10), and the source terms F_1, F_2 and F_3 are defined as follows :

$$\begin{cases} F_1 = \dfrac{\gamma-1}{2V}\psi^2 + \dfrac{R}{V(V+\phi)}w\phi - \dfrac{R\Theta\phi^2}{V^2(V+\phi)}, \\ F_2 = \dfrac{Z_\xi}{V(V+\phi)}\phi + \dfrac{R\Theta}{V(V+\phi)}\phi\psi - \dfrac{R}{V+\phi}w\psi, \\ F_3 = \dfrac{2}{C_v}\Theta^3\psi^2 - 6\Theta^2 w^2 - 4\Theta w^3 - w^4. \end{cases} \tag{4.38}$$

The decomposition (4.37) follows the analysis of [25], and corresponds to linearizing the principal part of the equations around the shock profile and treating the linearization errors as source terms F_1, F_2, F_3.

Then we are going to prove the following Proposition :

Proposition 4.5. *There exist positive constants $\tilde\delta$, $\tilde\epsilon$ and $\tilde C$ such that if $|v_+ - v_-| \le \tilde\delta$, and $N(T) \le \tilde\epsilon$, then*

$$\begin{aligned} &\|(\Phi,\Psi,W)(t)\|^2_{L^2(\mathbb{R})} + \|(\phi,\psi,w)(t)\|^2_{H^2(\mathbb{R})} \\ &+ \int_0^t \left(\||V'|^{1/2}(\Psi,W)(\tau)\|^2_{L^2(\mathbb{R})} + \|Z(\tau)\|^2_{H^1(\mathbb{R})} + \|z(\tau)\|^2_{H^3(\mathbb{R})}\right)\mathrm{d}\tau \\ &\le \tilde C\left(N(0)^2 + |v_+ - v_-|^2\int_0^t \|(\phi,\psi,w)(\tau)\|^2_{H^2(\mathbb{R})}\mathrm{d}\tau + N(t)^3\right) \end{aligned} \tag{4.39}$$

holds for any $t\in[0,T]$.

We split the proof of Proposition 4.5 into some lemmata. Firstly we prove the following

Lemma 4.6. *There exist positive constants δ_1, ϵ_1 and C_1 such that if $|v_+ - v_-| \le \delta_1$ and $N(T) \le \epsilon_1$, then*

$$\begin{aligned} &\|(\Phi,\Psi,W)(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \left(\||V'|^{1/2}(\Psi,W)(\tau)\|^2_{L^2(\mathbb{R})} + \|Z(\tau)\|^2_{H^1(\mathbb{R})}\right)\mathrm{d}\tau \\ &\le C_1\left(N(0)^2 + |v_+ - v_-|^2\int_0^t \|\phi(\tau)\|^2_{H^1(\mathbb{R})}\mathrm{d}\tau + N(t)^3\right) \end{aligned} \tag{4.40}$$

holds for any $t\in[0,T]$.

Proof. We follow the method developped in [25]. We multiply the equations of Φ, Ψ, W and Z in (4.37) by Φ, $K(V)V\Psi$, $RK(V)^2W$ and $\frac{RK(V)^2}{4\Theta^3 V}Z$ respectively, then calculate their sums and get (see [25] for a similar computation) :

$$\begin{aligned} &E_1(\Phi,\Psi,W)_t + E_2(\Psi,W,Z_\xi) + E_3(\phi,\Psi,W,Z) + \{\cdots\}_\xi \\ &\quad = F_1K(V)V\Psi + F_2RK(V)^2W + F_3\frac{RK(V)^2}{4\Theta^3V}Z, \end{aligned} \tag{4.41}$$

where

$$\begin{aligned} E_1(\Phi,\Psi,W) &= \frac{1}{2}\left(\Phi^2 + K(V)V\Psi^2 + C_vRK(V)^2W^2\right), \\ E_2(\Psi,W,Z_\xi) &= A\Psi^2 + sC_vRK(V)K(V)'W^2 + \frac{RK(V)^2}{4\Theta^3V}Z^2 + \frac{RK(V)^2}{4\Theta^3V(V+\phi)^2}Z_\xi^2, \\ E_3(\phi,\Psi,W,Z) &= -R\left(\frac{K(V)^2}{V}\right)'ZW + \left(\frac{Q'}{V} - \Theta^4\right)'\frac{RK(V)^2\phi W}{V(V+\phi)} \\ &\quad + \frac{ZZ_\xi}{V+\phi}\left(\frac{RK(V)^2}{4\Theta^3V(V+\phi)}\right)_\xi + \frac{Q'\phi}{V(V+\phi)}\frac{RK(V)^2}{4\Theta^3V}Z \\ &\quad + \left(\frac{1}{(V+\phi)}\left(\left(\frac{Q'}{V} - \Theta^4\right)'\frac{\phi}{V}\right)_\xi + \frac{4\Theta^3}{C_v}U'\Psi\right)\frac{RK(V)^2}{4\Theta^3V}Z \end{aligned}$$

with $A = \frac{s}{2}(VK(V))' + (\gamma-1)K(V)U'$ and $\{\cdots\}_\xi$ denotes the terms which disappear after integration with respect to $\xi \in \mathbb{R}$. From the last Lemma in Appendix 4.4.1 we have that $A \geq C|V'|$ for some constant C depending on v_-, u_-, θ_-.

Next we use the properties of the shock profile functions $(V,U,W,Q)(\xi)$ to estimate the terms E_1, E_2, E_3 and the terms in the right-hand side of (4.41). Note that $K(V)' = s^2K(V)^2V'$ and $sV' > 0$. Thus we have

$$C^{-1}(\Phi^2 + \Psi^2 + W^2) \leq \quad E_1 \quad \leq C\left(\Phi^2 + \Psi^2 + W^2\right), \tag{4.42}$$

$$E_2 \quad \geq C^{-1}\left(||V'|^{1/2}(\Psi,W)|^2 + Z^2 + Z_\xi^2\right), \tag{4.43}$$

for some positive constant C. Then for some $\alpha > 0$ there are positive constants C_α and C such that there holds :

$$\begin{aligned} |E_3| \;\leq\; & \alpha|V'|W^2 + C_\alpha|v_+ - v_-|^2\left(Z^2 + \phi^2\right) + C|\phi_\xi ZZ_\xi| \\ & + C|v_+ - v_-|^2\left(\phi^2 + Z^2 + Z_\xi^2 + |V'|\Psi^2 + |\phi\phi_\xi Z|\right). \end{aligned} \tag{4.44}$$

Finally with the explicit formulae for F_1, F_2 and F_3, we have the following estimate for the right-hand side of (4.41) :

$$\begin{aligned} &\left|F_1K(V)V\Psi + F_2RK(V)^2W + F_3\frac{RK(V)^2}{4\Theta^3V}Z\right| \\ &\quad \leq C\Big(\left(\psi^2 + |w\phi| + \phi^2\right)|\Psi| + \left(|Z_\xi\phi| + |\phi\psi| + |w\psi|\right)|W| \\ &\qquad + (\psi^2 + w^2 + |w|^3 + w^4)|Z|\Big), \end{aligned} \tag{4.45}$$

for some constant C.

Choosing α and δ_1 small enough, we then integrate (4.41) over $[0,t]\times\mathbb{R}$ and use the estimates (4.42)-(4.45) to obtain the desired estimate (4.40). Thus ends the proof of Lemma 4.6. $\square$

Next we are going to derive the $L^\infty(0,T;L^2(\mathbb{R}))$ estimate for $\partial_\xi^l(\phi,\psi,w)$ with $l=0$, 1 and 2. Linearizing the equations around the shock profile, as in the proof of Lemma 4.6, is not useful any longer to show the $L^\infty(0,T;L^2(\mathbb{R}))$ estimate for $\partial_{\xi\xi}^2(\phi,\psi,w)$, because the dissipation introduced by the elliptic equation is not strong enough to control the linearization errors. But the standard symmetrization of the quasi-linear form of (4.24) works as we shall see below. So we rewrite the system (4.24) as :

$$\begin{cases} \phi_t - s\phi_\xi - \psi_\xi = 0, \\ \psi_t - s\psi_\xi - R\dfrac{\Theta+w}{(V+\phi)^2}\phi_\xi + R\dfrac{w_\xi}{V+\phi} = f_1, \\ C_v(w_t - sw_\xi) + R\dfrac{\Theta+w}{V+\phi}\psi_\xi + z_\xi = f_2, \\ -\left(\dfrac{z_\xi}{V+\phi}\right)_\xi + 4\Theta^3 w_\xi + (V+\phi)z + \left(\dfrac{Q'\phi}{V(V+\phi)}\right)_\xi = f_3, \end{cases} \tag{4.46}$$

where the source terms f_1, f_2 and f_3 are defined as

$$\begin{cases} f_1 = \dfrac{R\phi\Theta'}{V(V+\phi)} - \dfrac{R(\phi^2+2V\phi)}{V^2(V+\phi)^2}\Theta V' + \dfrac{RwV'}{(V+\phi)^2}, \\ f_2 = U'\left(\dfrac{R\phi\Theta}{V(V+\phi)} - \dfrac{Rw}{V+\phi}\right), \\ f_3 = -\left((4\Theta^3)'w + (6\Theta^2w^2 + 4\Theta w^3 + w^4)_\xi + \dfrac{\phi}{V}\left(\dfrac{Q'}{V} - \Theta^4\right)'\right). \end{cases}$$

And we are going prove the following lemma :

Lemma 4.7. *Let $l=0$, 1 and 2. There exist positive constants δ_2, ϵ_2 and $C_2>0$ such that if $|v_+-v_-|\le\delta_2$ and $N(T)\le\epsilon_2$, then*

$$\begin{aligned} &\|\partial_\xi^l(\phi,\psi,w)(t)\|_{L^2(\mathbb{R})}^2 + \int_0^t \|\partial_\xi^l z(\tau)\|_{H^1(\mathbb{R})}^2 \\ &\quad \le C_2\left(N(0)^2 + N(t)^3 + |v_+-v_-|^2\int_0^t \|(\phi,\psi,w)(\tau)\|_{H^l(\mathbb{R})}^2 + \|z(\tau)\|_{H^{l+1}(\mathbb{R})}^2 \mathrm{d}\tau\right) \end{aligned} \tag{4.47}$$

holds for any $t\in[0,T]$.

Proof. Let $l=0$, 1 and 2. We apply ∂_ξ^l to the system (4.46), and we obtain the following system

$$
\begin{cases}
\partial_\xi^l \phi_t - s\partial_\xi^l \phi_\xi - \partial_\xi^l \psi_\xi = 0, \\
\partial_\xi^l \psi_t - s\partial_\xi^l \psi_\xi - R\dfrac{\Theta+w}{(V+\phi)^2}\partial_\xi^l \phi_\xi - \left[\partial_\xi^l, R\dfrac{\Theta+w}{(V+\phi)^2}\right]\phi_\xi \\
\qquad + \dfrac{R}{V+\phi}\partial_\xi^l w_\xi + \left[\partial_\xi^l, \dfrac{R}{V+\phi}\right] w_\xi = \partial_\xi^l f_1, \\
C_v\left(\partial_\xi^l w_t - s\partial_\xi^l w_s\right) + R\dfrac{\Theta+w}{V+\phi}\partial_\xi^l \psi_\xi + \left[\partial_\xi^l, R\dfrac{\Theta+w}{V+\phi}\right]\psi_\xi + \partial_\xi^l z_\xi = \partial_\xi^l f_2, \\
(V+\phi)\,\partial_\xi^l z + \left[\partial_\xi^l, V+\phi\right] z - \left(\dfrac{\partial_\xi^l z_\xi}{V+\phi}\right)_\xi - \left(\left[\partial_\xi^l, \dfrac{1}{V+\phi}\right] z_\xi\right)_\xi \\
\qquad + 4\Theta^3 \partial_\xi^l w_\xi + 4\left[\partial_\xi^l, \Theta^3\right] w_\xi + \partial_\xi^{l+1}\left(\dfrac{Q'\phi}{V(V+\phi)}\right) = \partial_\xi^l f_3,
\end{cases}
$$

where $[\cdot,\cdot]$ denotes the commutator, which vanishes when $l=0$.

Next we multiply the above equations by :

$$
\partial_\xi^l \phi,\ \frac{1}{R}\frac{(V+\phi)^2}{\Theta+w}\partial_\xi^l \psi,\ \frac{1}{R}\frac{(V+\phi)^2}{(\Theta+w)^2}\partial_\xi^l w, \text{ and } \frac{1}{4R\Theta^3}\frac{(V+\phi)^2}{(\Theta+w)^2}\partial_\xi^l z,
$$

respectively. We calculate their sums and get :

$$
\begin{aligned}
&H_1(\phi,\psi,w)_t + H_2(z) + H_3(\phi,\psi,w,z) + H_4(\phi,\psi,w,z) + H_5(\phi,z) + \{\cdots\}_\xi \\
&\quad = \partial_\xi^l f_1 \frac{1}{R}\frac{(V+\phi)^2}{\Theta+w}\partial_\xi^l \psi + \partial_\xi^l f_2 \frac{1}{R}\frac{(V+\phi)^2}{(\Theta+w)^2}\partial_\xi^l w + \partial_\xi^l f_3 \frac{1}{4R\Theta^3}\frac{(V+\phi)^2}{(\Theta+w)^2}\partial_\xi^l z,
\end{aligned}
\tag{4.48}
$$

where

$$
\begin{aligned}
H_1(\phi,\psi,w) &= \frac{1}{2}\left(\left(\partial_\xi^l \phi\right)^2 + \frac{1}{R}\frac{(V+\phi)^2}{\Theta+w}\left(\partial_\xi^l \psi\right)^2 + \frac{C_v}{R}\frac{(V+\phi)^2}{(\Theta+w)^2}\left(\partial_\xi^l w\right)^2\right), \\
H_2(z) &= \frac{1}{4R\Theta^3}\frac{(V+\phi)^2}{(\Theta+w)^2}\left((V+\phi)\left(\partial_\xi^l z\right)^2 + \frac{1}{V+\phi}\left(\partial_\xi^{l+1} z\right)^2\right), \\
H_3(\phi,\psi,w,z) &= -\frac{1}{2R}\left(\frac{(V+\phi)^2}{\Theta+w}\right)_t \left(\partial_\xi^l \psi\right)^2 - \frac{C_v}{2R}\left(\frac{(V+\phi)^2}{(\Theta+w)^2}\right)_t \left(\partial_\xi^l w\right)^2 \\
&\quad + \frac{s}{2R}\left(\frac{(V+\phi)^2}{\Theta+w}\right)_\xi \left(\partial_\xi^l \psi\right)^2 + s\frac{C_v}{2R}\left(\frac{(V+\phi)^2}{(\Theta+w)^2}\right)_\xi \left(\partial_\xi^l w\right)^2 \\
&\quad - \left(\frac{V+\phi}{\Theta+w}\right)_\xi \partial_\xi^l \psi \partial_\xi^l w - \frac{1}{R}\left(\frac{(V+\phi)^2}{(\Theta+w)^2}\right)_\xi \partial_\xi^l w \partial_\xi^l z, \\
H_4(\phi,\psi,w,z) &= -\frac{1}{R}\frac{(V+\phi)^2}{\Theta+w}\partial_\xi^l \psi\left\{\left[\partial_\xi^l, \frac{R}{V+\phi}\right] w_\xi - \left[\partial_\xi^l, R\frac{\Theta+w}{(V+\phi)^2}\right]\phi_\xi\right\} \\
&\quad + \frac{1}{R}\frac{(V+\phi)^2}{(\Theta+w)^2}\partial_\xi^l w\left[\partial_\xi^l, R\frac{\Theta+w}{V+\phi}\right]\psi_\xi + \left(\frac{1}{4R\Theta^3}\frac{(V+\phi)^2}{(\Theta+w)^2}\partial_\xi^l z\right)_\xi \left[\partial_\xi^l, \frac{1}{V+\phi}\right] z_\xi \\
&\quad + \frac{1}{4R\Theta^3}\frac{(V+\phi)^2}{(\Theta+w)^2}\partial_\xi^l z\left\{\left[\partial_\xi^l, V+\phi\right] z + 4\left[\partial_\xi^l, \Theta^3\right] w_\xi\right\}, \\
H_5(\phi,z) &= -\partial_\xi^l\left(\frac{Q'\phi}{V(V+\phi)}\right)\left(\frac{1}{4R\Theta^3}\frac{(V+\phi)^2}{(\Theta+w)^2}\partial_\xi^l z\right)_\xi,
\end{aligned}
$$

and $\{\cdots\}$ denotes the terms that disappear after integration with respect to ξ. Let us integrate the resulting equation (4.48) over the strip $[0,t]\times\mathbb{R}$, we have

$$\int_{\mathbb{R}} H_1(\phi,\psi,w)\mathrm{d}\xi\Big|_0^t + \int_0^t\int_{\mathbb{R}} H_2(z) + H_3(\phi,\psi,w,z) + H_4(\phi,\psi,w,z) + H_5(\phi,z)\ \mathrm{d}\xi\mathrm{d}\tau$$
$$= \int_0^t\int_{\mathbb{R}} \partial_\xi^l f_1 \frac{1}{R}\frac{(V+\phi)^2}{\Theta+w}\partial_\xi^l\psi + \partial_\xi^l f_2 \frac{1}{R}\left(\frac{V+\phi}{\Theta+w}\right)^2 \partial_\xi^l w + \partial_\xi^l f_3 \frac{1}{4R\Theta^3}\frac{(V+\phi)^2}{(\Theta+w)^2}\partial_\xi^l z\mathrm{d}\xi\mathrm{d}\tau. \tag{4.49}$$

Immediately we deduce the following estimates for H_1 and H_2

$$\frac{1}{C}\left(\left(\partial_\xi^l\psi\right)^2 + \left(\partial_\xi^l\psi\right)^2 + \left(\partial_\xi^l w\right)^2\right) \le H_1(\phi,\psi,w) \ \le\ C\left(\left(\partial_\xi^l\psi\right)^2 + \left(\partial_\xi^l\psi\right)^2 + \left(\partial_\xi^l w\right)^2\right),$$
$$H_2(z) \ \ge\ \frac{1}{C}\left(\left(\partial_\xi^l z\right)^2 + \left(\partial_\xi^{l+1} z\right)^2\right),$$

for some constant C. Next for the term H_3, using (4.46) to substitute the term involving ϕ_t and w_t in H_3, we have

$$\begin{aligned}\int_0^t\int_{\mathbb{R}} |H_3(\phi,\psi,w,z)|\,\mathrm{d}\xi\mathrm{d}\tau &\le C\int_0^t\int_{\mathbb{R}} (|\phi_t| + |w_t| + |\phi_\xi| + |w_\xi|)\left(|\partial_\xi^l\psi|^2 + |\partial_\xi^l w|^2\right)\mathrm{d}\xi\mathrm{d}\tau \\ &\le C\left(|v_+ - v_-|^2\int_0^t \|(\partial_\xi^l\psi,\partial_\xi^l w)(\tau)\|^2_{L^2(\mathbb{R})}\mathrm{d}\tau + N(T)^3\right).\end{aligned}$$

Then H_4 collects terms involving a commutation with ∂_ξ, and $H_4 = 0$ when $l=0$, so that it remains to study the case $l=1$ and 2. After some tedious calculations we obtain the following estimate :

$$\int_0^t\int_{\mathbb{R}} |H_4(\phi,\psi,w,z)|\,\mathrm{d}\xi\mathrm{d}\tau$$
$$\le C\Big(|v_+ - v_-|^2\int_0^t \|(\phi_\xi,\psi_\xi,w_\xi)(\tau)\|^2_{H^{l-1}(\mathbb{R})} + \|z(\tau)\|^2_{H^{l+1}(\mathbb{R})}\ \mathrm{d}\tau + N(t)^3\Big).$$

Next for the term H_5, we have

$$\int_0^t\int_{\mathbb{R}} |H_5(\phi,z)|\,\mathrm{d}\xi\mathrm{d}\tau \le C\left(|v_+ - v_-|^2\int_0^t \|\phi(\tau)\|^2_{H^l(\mathbb{R})} + \|\partial_\xi^l z(\tau)\|^2_{H^1(\mathbb{R})}\mathrm{d}\tau + N(t)^3\right).$$

Finally we estimate the right-hand side of (4.49), and we have

$$\int_0^t\int_{\mathbb{R}} \left|\partial_\xi^l f_1 \frac{1}{R}\frac{(V+\phi)^2}{\Theta+w}\partial_\xi^l\psi + \partial_\xi^l f_2 \frac{1}{R}\left(\frac{V+\phi}{\Theta+w}\right)^2 \partial_\xi^l w + \partial_\xi^l f_3 \frac{1}{4R\Theta^3}\frac{(V+\phi)^2}{(\Theta+w)^2}\partial_\xi^l z\right|\mathrm{d}\xi\mathrm{d}\tau$$
$$\le C\left(|v_+ - v_-|^2\int_0^t \|(\phi,w)(\tau)\|^2_{H^l(\mathbb{R})} + \|\partial_\xi^l(\psi,z)(\tau)\|^2_{L^2(\mathbb{R})}\mathrm{d}\tau + N(t)^3\right).$$

Combining the above estimates, with a sufficiently small constant δ_2, we get the desired estimate (4.47), and we complete the proof of Lemma 4.7. □

From the relation (4.28) we can derive the $L^\infty(L^2)$ estimate for z as

$$\int_0^t \|z(\tau)\|^2_{L^2(\mathbb{R})}\mathrm{d}\tau \le C\left(\int_0^t \|Z_\xi(\tau)\|^2_{L^2(\mathbb{R})}\mathrm{d}\tau + |v_+ - v_-|^2\int_0^t \|\phi(\tau)\|^2_{L^2(\mathbb{R})}\mathrm{d}\tau\right). \tag{4.50}$$

Thus using (4.40), (4.47) and (4.50) we obtain the desired estimate (4.39) and we finish the proof of Proposition 4.5.

4.3.2 $L^2(L^2)$ Estimates

In this subsection we show the $L^2(0,T;L^2(\mathbb{R}))$ estimates for $\partial_\xi^l(\phi,\psi)$ for $l=0$, 1 and 2. The main result in this paragraph is stated as follows :

Proposition 4.8. *There exist positive constants δ_3, ϵ_3 and C_3, such that if $|v_+-v_-|\le\delta_3$ and $N(T)\le\epsilon_3$, then*

$$\begin{aligned}\int_0^t &\|(\phi,\psi)(\tau)\|_{H^2(\mathbb{R})}^2\mathrm{d}\tau\\ &\le C_3\Big(N(0)^2+N(t)^3+\|(\phi,\psi)(t)\|_{H^2(\mathbb{R})}^2+\|(\Psi,W)(t)\|_{L^2(\mathbb{R})}^2+\|w(t)\|_{H^1(\mathbb{R})}^2\\ &\quad+\int_0^t\Big(\|w(\tau)\|_{H^2(\mathbb{R})}^2+\|z(\tau)\|_{H^2(\mathbb{R})}^2+\||V'|^{1/2}(\Psi,W)(\tau)\|_{L^2(\mathbb{R})}^2\Big)\mathrm{d}\tau\Big)\end{aligned}\tag{4.51}$$

holds for any $t\in[0,T]$.

We also split the proof of Property 4.8 into some lemmata. We start by showing

Lemma 4.9. *There exist positive constants δ_4, ϵ_4 and $C_4>0$ such that if $|v_+-v_-|\le\delta_4$ and $N(T)\le\epsilon_4$ then*

$$\begin{aligned}\int_0^t\|(\phi,\psi)(\tau)\|_{L^2(\mathbb{R})}^2\mathrm{d}\tau\le C_4\Big(&N(0)^2+N(t)^3+\|(\phi,\psi)(t)\|_{L^2(\mathbb{R})}^2+\|(\Psi,W)(t)\|_{L^2(\mathbb{R})}^2\\ &+\int_0^t\Big(\|Z_\xi(\tau)\|_{L^2(\mathbb{R})}^2+\||V'|^{1/2}(\Psi,W)(\tau)\|_{L^2(\mathbb{R})}^2+\|W_\xi(\tau)\|_{L^2(\mathbb{R})}^2\Big)\mathrm{d}\tau\Big)\end{aligned}\tag{4.52}$$

holds for any $t\in[0,T]$.

Proof. We also follow the method developped in [25]. Multiply the equations of Φ and Ψ in (4.37) by $-(V\Psi)_\xi$ and $-V\Phi_\xi$, respectively, and calculate their sums. Thus we arrive at the following equality :

$$\begin{aligned}&\frac{1}{K(V)}\phi^2-(V\Phi_\xi\Psi)_t+\{\cdots\}_\xi-V\psi^2\\ &\quad-V'(s\Phi_\xi+\Psi_\xi)\Psi-RW_\xi\Phi_\xi-(\gamma-1)U'\Psi\Phi_\xi=-F_1V\Phi_\xi.\end{aligned}\tag{4.53}$$

Then we multiply the equations of Φ and W in (4.37) by $\dfrac{2R}{\gamma-1}(VK(V)W)_\xi$ and $2VK(V)\Psi_\xi$, respectively, and calculate their sums. Thus we arrive the following equality :

$$\begin{aligned}&2V\psi^2+2C_v\left(VK(V)\Psi_\xi W\right)_t+\{\cdots\}_\xi+2C_vRW_\xi^2\Phi_\xi+2K(V)Z_\xi\Psi_\xi\\ &\quad-2C_vK(V)W_\xi^2+2C_v\left(VK(V)\right)'W\left(s\Psi_\xi+\frac{1}{VK(V)}\Phi_\xi-\frac{R}{V}W_\xi+\frac{\gamma-1}{V}U'\Psi\right)\\ &\quad-2RK(V)U'\Psi W_\xi-2sVK(V)U'\Psi\Psi_\xi-\left(\frac{Q'}{V}-\Theta^4\right)'\frac{2K(v)}{V+\phi}\Psi_\xi\Phi_\xi\\ &=F_22VK(V)\Psi_\xi-F_12C_v\left(VK(V)W\right)_\xi.\end{aligned}\tag{4.54}$$

Thus we calculate the sum of (4.53) and (4.54) and simplify it as

$$\begin{aligned}&\tilde{E}_1(\phi,\psi)+\{\cdots\}_\xi+\tilde{E}_2(\phi,\psi,\Psi,W)_t+\tilde{E}_3(W_\xi,\phi,Z_\xi)+\tilde{E}_4(\phi,\psi,\Psi,W_\xi)\\ &\quad=F_22VK(V)\Psi_\xi-F_12C_v\Big(\left(VK(V)W\right)_\xi+V\phi\Big),\end{aligned}\tag{4.55}$$

where

$$\begin{aligned}
\tilde{E}_1(\phi,\psi) &= \frac{1}{K(V)}\phi^2 + V\psi^2, \\
\tilde{E}_2(\phi,\psi,\Psi,W) &= -V\phi\psi + 2C_v VK(V)\psi W, \\
\tilde{E}_3(W_\xi,\phi,Z_\xi) &= -RW_\xi\phi + 2C_v W_\xi\phi + 2K(V)Z_\xi\psi, \\
\tilde{E}_4(\phi,\psi,\Psi,W_\xi) &= -2C_v\, RK(V)W_\xi^2 - 2RK(V)U'\Psi W_\xi - V'(s\phi+\psi)\Psi - (\gamma-1)U'\Psi\phi \\
&\quad +2C_v\,(VK(V))'\,W\left(s\psi + \frac{1}{VK(V)}\phi - \frac{R}{V}W_\xi + \frac{\gamma-1}{V}U'\Psi\right) \\
&\quad -2sVK(V)U'\Psi\psi + 2K(V)\left(\frac{Q'}{V} - \Theta^4\right)'\frac{\phi\psi}{V+\phi},
\end{aligned}$$

and $\{\cdots\}_\xi$ denotes the terms which will disappear after integration with respect to ξ.

Next we use the properties of the shock profile to estimate the functions $\tilde{E}_1$, $\tilde{E}_2$, $\tilde{E}_3$, $\tilde{E}_4$ and the right-hand term of (4.55). Thus we have

$$\frac{1}{C}(\phi^2+\psi^2) \le \quad \tilde{E}_1 \quad \le C(\phi^2+\psi^2), \tag{4.56}$$

$$|\tilde{E}_2| \quad \le C(\phi^2+\psi^2+W^2), \tag{4.57}$$

$$|\tilde{E}_4| \quad \le C\left(\left||V'|^{1/2}(\Psi,W)\right|^2 + |v_+ - v_-|(\phi^2+\psi^2) + W_\xi^2\right), \tag{4.58}$$

for some constant C. Furthermore for some $\alpha > 0$, there exist constant C_α such that

$$|\tilde{E}_3| \le \alpha\left(\phi^2+\psi^2\right) + C_\alpha\left(W_\xi^2 + Z_\xi^2\right). \tag{4.59}$$

Finally we have the following estimate for the term in the right-hand side of (4.55)

$$\begin{aligned}
&\left|F_2 2VK(V)\Psi_\xi - F_1\frac{2R}{\gamma-1}\left((VK(V)W)_\xi + V\phi\right)\right| \\
&\quad \le |\psi|\left(|Z_\xi\phi| + |\phi\psi| + |w\psi|\right) + \left(|W_\xi| + |V'W| + |\phi|\right)\left(\psi^2 + |w\phi| + \phi^2\right).
\end{aligned} \tag{4.60}$$

Now we choose α and δ_4 small enough. We integrate (4.55) over $[0,t]\times\mathbb{R}$, use the estimate (4.56)-(4.60), and we obtain the desired estimate (4.52). □

Next we estimate $\partial_\xi^l(\phi_\xi,\psi_\xi)$, with $l = 0$ and 1, in the $L^2(0,T;L^2(\mathbb{R}))$ norm, by a similar analysis :

Lemma 4.10. *Let $l = 0,\ 1$. There exist positive constants δ_5, ϵ_5 and C_5 such that if $|v_+ - v_-| \le \delta_5$ and $N(T) \le \epsilon_5$, then*

$$\begin{aligned}
&\int_0^t \|\partial_\xi^{l+1}(\phi,\psi)(\tau)\|_{L^2(\mathbb{R})}^2 \mathrm{d}\tau \\
&\le C_5\Bigg(N(0)^2 + N(t)^3 + \|\partial_\xi^{l+1}(\phi,\psi)(t)\|_{L^2(\mathbb{R})}^2 + \|\partial_\xi^l(\psi,W)(t)\|_{L^2(\mathbb{R})}^2 \\
&\quad + \int_0^t \|\partial_\xi^l z_\xi(\tau)\|_{L^2(\mathbb{R})}^2 + |v_+ - v_-|\left(\|(\phi,\psi)(\tau)\|_{H^l(\mathbb{R})}^2 + \|w(\tau)\|_{H^{l+1}(\mathbb{R})}^2\right)\mathrm{d}\tau\Bigg)
\end{aligned} \tag{4.61}$$

holds for any $t \in [0,T]$.

Proof. Using the same strategy as in the proof of Lemma 4.9, we rewrite the first three equations in (4.46) as :

$$\begin{cases} \phi_t - s\phi_\xi - \psi_\xi = 0 \\ \psi_t - s\psi_\xi - R\dfrac{\Theta}{V^2}\phi_\xi + R\dfrac{w_\xi}{V} = \tilde{f}_1, \\ C_v(w_t - sw_\xi) + R\dfrac{\Theta}{V}\psi_\xi + z_\xi = \tilde{f}_2, \end{cases} \tag{4.62}$$

where

$$\begin{cases} \tilde{f}_1 = \dfrac{Rw_\xi}{V(V+\phi)}\phi + f_1, \\ \tilde{f}_1 = \dfrac{R\Theta\phi}{V(V+\phi)}\psi_\xi - \dfrac{Rw\psi_\xi}{V+\phi} + f_2. \end{cases}$$

Then we apply ∂_ξ^l to (4.62) with $l = 0$ and 1, and multiply the equations of $\partial_\xi^l\psi$ and $\partial_\xi^l w$ by $-V\partial_\xi^l\phi_\xi$ and $2K(V)V\partial_\xi^l\psi_\xi$ respectively, then multiply the equations of $\partial_\xi^l\phi$ and $\partial_\xi^l\psi$ by $-(V\psi)_\xi$ and $(VK(V)w)_\xi$ respectively. Finally we calculate their sums and use the properties of the shock profile (V, U, Θ, Q) to obtain the estimate (4.61). We omit the details here. $\square$

To complete the proof of Proposition 4.8, we need the following estimate :

$$\int_0^t \|W_\xi(\tau)\|_{L^2(\mathbb{R})}^2 \mathrm{d}\tau \leq C\left(\int_0^t \|w(\tau)\|_{L^2(\mathbb{R})}^2 \mathrm{d}\tau + \int_0^t \||V'|^{1/2}\Psi(\tau)\|_{L^2(\mathbb{R})}^2 \mathrm{d}\tau + N(t)^4\right), \tag{4.63}$$

which can be obtained immediately from the definition (4.28). Using the estimates (4.52), (4.61), (4.63) we get the desired estimate (4.51), and thus we complete the proof of Proposition 4.8.

4.3.3 Elliptic estimates

In this subsection we derive the $L^2(0,T;H^2(\mathbb{R}))$ estimate for w and the $L^\infty(0,T;H^3(\mathbb{R}))$ estimate for z from the elliptic equation. With these estimates we complete the proof of Proposition 4.4.

Lemma 4.11. *There exist positive constants δ_6, ϵ_6 and C_6 such that if $|v_+ - v_-| \leq \delta_6$ and $N(T) \leq \epsilon_6$, then*

$$\begin{aligned} \int_0^t \|w(\tau)\|_{H^2(\mathbb{R})}^2 \mathrm{d}\tau \leq\ & C_6\bigg(\int_0^t \left(\|Z(\tau)\|_{L^2(\mathbb{R})}^2 + \|z(\tau)\|_{H^2(\mathbb{R})}^2\right) \mathrm{d}\tau \\ & + |v_+ - v_-|^2 \int_0^t \|\phi(\tau)\|_{H^2(\mathbb{R})}^2 \mathrm{d}\tau + N(t)^3\bigg) \end{aligned} \tag{4.64}$$

holds for any $t \in [0,T]$.

Proof. Recall that from the fourth equation in (4.30) we have the following relation between Z and w

$$4\Theta^3 w = -Z + \frac{z_\xi}{V+\phi} - \frac{Q'\phi}{V(V+\phi)} - \left(6\Theta^2 w^2 + 4\Theta w^3 + w^4\right). \tag{4.65}$$

To show the $L^2(0,t;L^2(\mathbb{R}))$ estimate of w, let us multiply (4.65) by w and integrate the resulting equation over $[0,t]\times\mathbb{R}$, we arrive at

$$\int_0^t\int_{\mathbb{R}} 4\Theta^3 w^2 \mathrm{d}\xi\mathrm{d}\tau \leq \int_0^t\int_{\mathbb{R}} |Zw| + \left|\frac{z_\xi w}{V+\phi}\right| + \left|\frac{Q'\phi w}{V(V+\phi)}\right| + \left|\left(6\Theta^2 w^2 + 4\Theta w^3 + w^4\right) w\right| \mathrm{d}\xi\mathrm{d}\tau$$

By the properties of the shock profile, we have

$$\begin{aligned}\int_0^t\int_{\mathbb{R}} 4\Theta^3 w^2 \mathrm{d}\xi\mathrm{d}\tau &\geq C^{-1}\int_0^t \|w(\tau)\|^2_{L^2(\mathbb{R})}\mathrm{d}\tau,\\ \int_0^t\int_{\mathbb{R}} \left|\frac{Q'\phi w}{V(V+\phi)}\right| \mathrm{d}\xi\mathrm{d}\tau &\leq C|v_+-v_-|^2\int_0^t \|(\phi,w)(\tau)\|^2_{L^2}\mathrm{d}\tau,\end{aligned}$$

for some constant C. The Cauchy-Schwarz inequality used for the products Zw and $z_\xi w$ yields

$$\int_0^t\int_{\mathbb{R}} |Zw| + \left|\frac{z_\xi w}{V+\phi}\right| \mathrm{d}\xi\mathrm{d}\tau \leq \alpha\int_0^t \|w(\tau)\|^2_{L^2(\mathbb{R})}\mathrm{d}\tau + C_\alpha\int_0^t \|(Z,z_\xi)(\tau)\|^2_{L^2(\mathbb{R})}\mathrm{d}\tau.$$

Using the definition of $N(t)$ and the Sobolev embedding theorem, we get the following $L^2(0,t;L^2(\mathbb{R}))$ estimate for w,

$$\begin{aligned}C^{-1}\int_0^t \|w(\tau)\|^2_{L^2(\mathbb{R})}\mathrm{d}\tau &\leq \alpha\int_0^t \|w(\tau)\|^2_{L^2(\mathbb{R})}\mathrm{d}\tau + C_\alpha\int_0^t \left(\|Z(\tau)\|^2_{L^2(\mathbb{R})} + \|z_\xi(\tau)\|^2_{L^2(\mathbb{R})}\right)\mathrm{d}\tau\\ &\quad + C\left(|v_+-v_-|\int_0^t \left(\|\phi(\tau)\|^2_{L^2(\mathbb{R})} + \|w(\tau)\|^2_{L^2(\mathbb{R})}\right)\mathrm{d}\tau + N(t)^3\right)\end{aligned} \tag{4.66}$$

for some constants α, C and C_α.

Next let $l=1$ and 2. We apply ∂_ξ^l to (4.65), then multiply the resulting equation by $\partial_\xi^l w$. Then integrate it over $[0,t]\times\mathbb{R}$, and get

$$\begin{aligned}C^{-1}\int_0^t \|\partial_\xi^l w(\tau)\|^2_{L^2(\mathbb{R})}\mathrm{d}\tau &\leq \alpha\int_0^t \|\partial_\xi^l w(\tau)\|^2_{L^2(\mathbb{R})}\mathrm{d}\tau + C_\alpha\int_0^t \left(\|\partial_\xi^l Z(\tau)\|^2_{L^2(\mathbb{R})} + \|z_\xi(\tau)\|^2_{H^l(\mathbb{R})}\right)\mathrm{d}\tau\\ &\quad + C\left(|v_+-v_-|^2\int_0^t \left(\|\phi(\tau)\|^2_{H^l(\mathbb{R})} + \|w(\tau)\|^2_{H^l(\mathbb{R})}\right)\mathrm{d}\tau + N(t)^3\right),\end{aligned} \tag{4.67}$$

Thus using the estimates (4.66) and (4.67), we arrive at

$$\begin{aligned}C^{-1}\int_0^t \|w(\tau)\|^2_{H^2(\mathbb{R})}\mathrm{d}\tau &\leq \alpha\int_0^t \|w(\tau)\|^2_{H^2(\mathbb{R})}\mathrm{d}\tau + C_\alpha\int_0^t \left(\|Z(\tau)\|^2_{H^2(\mathbb{R})} + \|z_\xi(\tau)\|^2_{H^2(\mathbb{R})}\right)\mathrm{d}\tau\\ &\quad + C\left(|v_+-v_-|^2\int_0^t \left(\|\phi(\tau)\|^2_{H^2(\mathbb{R})} + \|w(\tau)\|^2_{H^2(\mathbb{R})}\right)\mathrm{d}\tau + N(t)^3\right),\end{aligned} \tag{4.68}$$

Choosing α and δ_6 small enough, we have

$$\begin{aligned}\int_0^t \|w(\tau)\|^2_{H^2(\mathbb{R})}\mathrm{d}\tau &\leq C\left(\int_0^t \left(\|Z(\tau)\|^2_{H^2(\mathbb{R})} + \|z_\xi(\tau)\|^2_{H^2(\mathbb{R})}\right)\mathrm{d}\tau\right.\\ &\quad \left. + |v_+-v_-|^2\int_0^t \|\phi(\tau)\|^2_{H^2(\mathbb{R})}\mathrm{d}\tau + N(t)^3\right),\end{aligned} \tag{4.69}$$

Note that from (4.29) we have the following relations

$$\begin{aligned}&\int_0^t \|Z_\xi(\tau)\|_{H^1(\mathbb{R})}^2 \mathrm{d}\tau \\ &\le C\left(\int_0^t \|z(\tau)\|_{H^1(\mathbb{R})}^2 \mathrm{d}\tau + |v_+ - v_-|^2 \int_0^t \|\phi(\tau)\|_{H^1(\mathbb{R})}^2 \mathrm{d}\tau + N(t)^3\right).\end{aligned} \tag{4.70}$$

Substituting (4.70) into the right-hand side of (4.69) we get the desired estimate (4.64) and complete the proof. □

Lemma 4.12. *There exist a positive constant C such that*

$$\|z(t)\|_{H^3(\mathbb{R})}^2 \le C\left(\|w(t)\|_{H^2(\mathbb{R})}^2 + \|\phi(t)\|_{H^2(\mathbb{R})}^2 + N(t)^3\right), \tag{4.71}$$

holds for any $t \in [0, T]$.

Proof. We begin with the equation (4.26), and rewrite it as

$$\begin{aligned}&\left(\frac{Q'\phi}{V(V+\phi)}\right)_\xi - \left(\frac{z_\xi}{V+\phi}\right)_\xi + (v+\phi)z + \left(4\Theta^3 w\right)_\xi \\ &\quad + \left(\frac{Q'}{V} - \Theta^4\right)' \frac{\phi}{V} + \left(6\Theta^2 w^2 + 4\Theta w^3 + w^4\right)_\xi = 0.\end{aligned} \tag{4.72}$$

We multiply (4.72) by z and get

$$\begin{aligned}&\frac{z_\xi^2}{V+\phi} + (V+\phi)z^2 + \{\cdots\}_\xi \\ &= 4\Theta^3 w z_\xi + \frac{Q'\phi}{V(V+\phi)} z_\xi - \left(\frac{Q'}{V} - \Theta^4\right)' \frac{\phi z}{V} + \left(6\Theta^2 w^2 + 4\Theta w^3 + w^4\right) z_\xi.\end{aligned} \tag{4.73}$$

Integrating (4.73) with respect to ξ, noting that $\{\cdots\}_\xi$ denotes the terms that disappear after integration, and using the Cauchy-Schwarz inequality, we have

$$\|z(t)\|_{H^1(\mathbb{R})}^2 \le C\Big(\|w(t)\|_{L^2(\mathbb{R})}^2 + |v_+ - v_-|^2 \|\phi(t)\|_{L^2(\mathbb{R})}^2 + N(t)^3\Big). \tag{4.74}$$

Next, we apply ∂_ξ to (4.72) and get

$$\begin{aligned}&\left(\frac{Q'\phi}{V(V+\phi)}\right)_{\xi\xi} - \left(\frac{z_\xi}{V+\phi}\right)_{\xi\xi} + (v+\phi)z_\xi + (V'+\phi_\xi)z + \left(4\Theta^3 w\right)_{\xi\xi} \\ &\quad + \left(\left(\frac{Q'}{V} - \Theta^4\right)' \frac{\phi}{V}\right)_\xi + \left(6\Theta^2 w^2 + 4\Theta w^3 + w^4\right)_{\xi\xi} = 0.\end{aligned} \tag{4.75}$$

On the one hand in order to derive the $L^\infty(0, T; H^1(\mathbb{R}))$ estimate to z_ξ, we multiply (4.75) by z_ξ, then integrate it with respect to ξ, and use the Cauchy-Schwarz inequality, we have

$$\|z_\xi(t)\|_{H^1(\mathbb{R})}^2 \le C\Big(\|w(t)\|_{H^1(\mathbb{R})}^2 + |v_+ - v_-|^2 \left(\|\phi(t)\|_{H^1(\mathbb{R})}^2 + \|z(t)\|_{L^2(\mathbb{R})}^2\right) + N(t)^3\Big). \tag{4.76}$$

While on the other hand in order to treat the $L^\infty(0,T;H^1(\mathbb{R}))$ estimate to $z_{\xi\xi}$,, we multiply (4.75) by $z_{\xi\xi\xi}$, thus we get

$$\begin{aligned}
&-\frac{z_{\xi\xi\xi}^2}{V+\phi}+2\frac{z_{\xi\xi\xi}z_{\xi\xi}}{(V+\phi)^2}\left(V'+\phi_\xi\right)-z_{\xi\xi\xi}z_\xi\left(\frac{1}{V+\phi}\right)_{\xi\xi}\\
&\quad-(V+\phi)z_{\xi\xi}^2+\left(V'+\phi_\xi\right)\left(zz_{\xi\xi\xi}-z_\xi z_{\xi\xi}\right)+\{\cdots\}_\xi\\
&\quad+\left(4\Theta^3 w\right)_{\xi\xi}z_{\xi\xi\xi}+\left(\frac{Q'\phi}{V(V+\phi)}\right)_{\xi\xi}z_{\xi\xi\xi}+\left(\left(\frac{Q'}{V}-\Theta^4\right)'\frac{\phi}{V}\right)_\xi z_{\xi\xi\xi}\\
&\quad+\left(6\Theta^2w^2+4\Theta w^3+w^4\right)_\xi z_{\xi\xi\xi}=0,
\end{aligned}\tag{4.77}$$

where $\{\cdots\}_\xi$ denotes the terms that disappear after integration with respect to ξ. Then integrating with respect to ξ and using the Cauchy-Schwarz inequality, we have

$$\|z_{\xi\xi}(t)\|^2_{H^1(\mathbb{R})}\le C\Big(\|w(t)\|^2_{H^2(\mathbb{R})}+|v_+-v_-|^2\left(\|\phi(t)\|^2_{H^2(\mathbb{R})}+\|z(t)\|^2_{H^1(\mathbb{R})}\right)+N(t)^3\Big).\tag{4.78}$$

Then with (4.74), (4.76) and (4.78) we can get the desired inequality (4.71) for $|v_+-v_-|$ small enough. Thus we complete the proof. $\square$

Using the Proposition 4.5, Proposition 4.8 and the elliptic estimates in Lemma 4.11, Lemma 4.12, we get the desired a priori estimate (4.32). Thus we complete the proof to Proposition 4.4, and obtain our main result.

4.4 Appendix

4.4.1 Some properties of the shock wave solutions to the (standard) Euler equations

In this subsection we recall a few facts about the entropic shock wave solutions to (standard) Euler equations :

$$\begin{cases}
v_t-u_x &= 0,\\
u_t+p_x &= 0,\\
\left(e+\dfrac{u^2}{2}\right)_t+(pu)_x &= 0.
\end{cases}\tag{4.79}$$

Using the perfect gas law (4.3), system (4.79) can be written in the quasi-linear form, and in the variables v, u and θ as

$$\begin{pmatrix} v\\ u\\ \theta\end{pmatrix}_t+\begin{pmatrix}0 & -1 & 0\\ -\frac{R\theta}{v^2} & 0 & \frac{R}{v}\\ 0 & (\gamma-1)\frac{\theta}{v} & 0\end{pmatrix}\begin{pmatrix} v\\ u\\ \theta\end{pmatrix}_x=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Under this quasi-linear form, we calculate the distinct eigenvalues as

$$\lambda_1=-\frac{\sqrt{\gamma R\theta}}{v},\ \lambda_2=0,\ \lambda_3=\frac{\sqrt{\gamma R\theta}}{v},$$

It is obvious to see that the first and the thirth characteristic fields are genuinely non linear.

Definition 4.13. *The constants* $(v_\pm, u_\pm, \theta_\pm)$ *define a shock wave solution with speed* s*, to the system* (4.79) *if the following conditions are satisfied :*

– *the Rankine-Hugoniot jump condition :*

$$\left\{ \begin{array}{rcl} -s[v] - [u] & = & 0 \\ -s[u] + [p] & = & 0 \\ -s\left[e + \dfrac{u^2}{2}\right] + [pu] & = & 0, \end{array} \right. \tag{4.80}$$

where $[\cdot]$ *denotes the jump value from left to right.*

– *the Lax entropy condition :*

$$\text{either 1-shock } \lambda_1^+ < s < \lambda_1^-, \text{ or 3-shock } \lambda_3^+ < s < \lambda_3^-. \tag{4.81}$$

Remark 4.14. *Remark that using the formulae for* λ_1*,* λ_3 *and the perfect gas law, we can rewrite the Lax entropy condition as*

$$\text{either 1-shock } \left\{ \begin{array}{l} s < 0, \\ \dfrac{\gamma R\theta_-}{v_-^2} < s^2 < \dfrac{\gamma R\theta_+}{v_+^2}, \end{array} \right. \quad \text{or 3-shock } \left\{ \begin{array}{l} s > 0, \\ \dfrac{\gamma R\theta_+}{v_+^2} < s^2 < \dfrac{\gamma R\theta_-}{v_-^2}. \end{array} \right. \tag{4.82}$$

Next we study the the Rankine-Hugoniot jump condition (4.80) and derive some properties of the shock wave.

Lemma 4.15. *From the Rankine-Hugoniot jump condition* (4.80) *we have the following equalities :*

- $[e] + \dfrac{p_+ + p_-}{2}[v] = 0;$
- $\dfrac{\gamma+1}{\gamma-1}[\theta] - v_+v_-\left[\dfrac{\theta}{v^2}\right] = 0;$
- $\left(\dfrac{p_-}{v_+} + s^2\right)[v] + \dfrac{R}{v_+}[\theta] = 0.$

Proof. First from (4.80) we can write the jump of u, p and e in term of $[v]$ as

$$[u] = -s[v],\ [p] = s[u] = -s^2[v],\ [e] + \frac{p_+ + p_-}{2}[v] = 0. \tag{4.83}$$

Then using (4.3) we can write the last equation in (4.83) as

$$\frac{1}{\gamma - 1}[\theta] + \frac{1}{2}\left(\frac{\theta_+}{v_+} - \frac{\theta_-}{v_-}\right)[v] = 0. \tag{4.84}$$

Note that

$$\left(\frac{\theta_+}{v_+} - \frac{\theta_-}{v_-}\right)[v] = \theta_+ + \theta_-\frac{v_+}{v_-} - \theta_+\frac{v_-}{v_+} - \theta_- = [\theta] - v_+v_-\left[\frac{\theta}{v^2}\right].$$

Thus (4.84) is equivalent to the following equality

$$\frac{\gamma+1}{\gamma-1}[\theta] - v_+v_-\left[\frac{\theta}{v^2}\right] = 0. \tag{4.85}$$

Thus it remains to prove the last equality in Lemma 4.15. From (4.3) we write the jump value of p as

$$[p] = R\left[\frac{\theta}{v}\right] = R\left(\frac{1}{v_+}[\theta] - \theta_-\left[\frac{1}{v}\right]\right) = R\left(\frac{1}{v_+}[\theta] + \frac{\theta_-}{v_+v_-}[v]\right).$$

Together with the second equality in (4.83) we complete the proof of Lemma 4.15. □

With the help of Lemma 4.15 we derive the following equivalent form of the Lax entropy condition,

Lemma 4.16. *The Lax entropy condition* (4.81) *is equivalent to*

$$[u] < 0 \ (\ or\ s[v] > 0). \tag{4.86}$$

Proof. First we suppose that (4.81) holds. Thus (4.82) holds. For example, in the case of a 1-shock, we have

$$s < 0, \ \left[\frac{p}{v}\right] = \left[\frac{R\theta}{v^2}\right] > 0.$$

Using Lemma 4.15, we derive $[v] < 0$, then there hold $s[v] > 0$ and $[u] < 0$. Similarly we can get the same result in the case of a 3-shock.

Next let us suppose that (4.86) holds. We want to check the inequality (4.81). For example for a 1-shock we have $s < 0$, $[v] < 0$, $[u] < 0$. From the Rankine-Hugoniot condition (4.80), we have

$$[p] > 0, \ \left[\frac{1}{v}\right] > 0, \tag{4.87}$$

and

$$[e] + \frac{p_- + p_+}{2}[v] = 0. \tag{4.88}$$

While the jump of $e = pv/(\gamma - 1)$ can be written as

$$[e] = \frac{1}{\gamma - 1}\left([p]v_+ + p_-[v]\right) = \frac{1}{\gamma - 1}\left(-s^2 v_+[v] + p_-[v]\right),$$

the equality between v and e (4.88) reads

$$\left(-s^2\frac{v_+}{\gamma - 1} + \frac{\gamma + 1}{2(\gamma - 1)}p_+ + \frac{1}{2}p_-\right)[v] = 0.$$

As $[v] \neq 0$, we have

$$s^2 = \frac{\gamma + 1}{2}\frac{p_+}{v_+} + \frac{\gamma - 1}{2}\frac{p_-}{v_+}.$$

Using the inequalities in (4.87) we get (4.82) in the case $s < 0$. Similarly we can get (4.82) in the case $s > 0$. Thus we prove the equivalence and complete the proof. □

In §4.3 we have used the following estimate for the shock profile solution to (4.8) :

Lemma 4.17. *Let U, V, Θ, and Q be a shock profile satisfying (4.8), and let $A = \frac{s}{2}(VK(V))' + (\gamma-1)K(V)U'$. If the strength of the shock is small enough, then there holds :*

$$A \geq C|V'|,$$

for some constant C that may depend on the v_-, u_-, θ_-, but that does not depend on v_+, u_+, θ_+.

Proof. Recall that $K(V)$ is defined by (4.10). And A can be rewritten as

$$A = \frac{sV'}{2} K(V)^2 (b_1 - 2(\gamma-1)P).$$

From (4.11), the property of shock profile, we get

$$A = \frac{|s||V'|}{2} K(V)^2 \tilde{A},$$

where $\tilde{A} = (b_1 - 2(\gamma-1)P)$, with $b_1 = p_\pm + s^2 v_\pm$.

Let us focus on the case of a 1-shock, i.e $s < 0$, so we have

$$s < 0,\ V' < 0,\ U' < 0,\ P' > 0,\ \frac{\gamma p_-}{v_-} < s^2 < \frac{\gamma p_+}{v_+}.$$

Note that the inequality satisfied by s^2 yields

$$|s| \geq \sqrt{\frac{\gamma p_-}{v_-}}.$$

On the one hand there holds $p_+^2 \leq K(V)^2 \leq p_-^2$, that is

$$K(V)^2 \geq p_+^2 = p_-^2 + [p^2] \geq p_-^2 - C|v_+ - v_-|,$$

where the constant C may depend on the left values v_-, u_-, θ_- and γ. On the other hand, $\tilde{A}$ can be estimated as follows

$$\begin{aligned} \tilde{A} = b_1 - 2(\gamma-1)P = p_+ + s^2 v_+ - 2(\gamma-1)P &\geq p_+ + \frac{\gamma p_-}{v_-} v_+ - 2(\gamma-1)p_+ \\ &\geq (3-\gamma)p_- - |(3-2\gamma)[p]| \\ &\geq (3-\gamma)p_- - C|v_+ - v_-|, \end{aligned}$$

where the constant C may depend on the left values v_-, u_-, θ_- and γ. Thus for sufficiently small $|v_+ - v_-|$, we have

$$A \geq |V'| \frac{1}{4} \sqrt{\frac{\gamma p_-}{v_-}} p_-^2 (3-\gamma) p_-.$$

Thus we get the desired estimate in the case $s < 0$. For the other case, a similar analysis can be applied. Thus ends the proof. □

4.4.2 Regularity of the perturbation of the radiative heat flux

Here we show the existence, uniqueness and the regularity of the perturbation of the radiative heat flux variable q_0 :

$$q_0 \in Q + H^3(\mathbb{R}),$$

under the assumption (4.15) for the fluid initial data (v_0, u_0, θ_0). Recall that q_0 satisfies (4.19). Let us denote (ϕ_0, ψ_0, w_0) by $\phi_0 = v_0 - V$, $\psi_0 = u_0 - U$, $w_0 = \theta_0 - \Theta$ and $z_0 = q_0 - Q$. With these notations, (4.19) can be written as

$$\begin{aligned} &\left(\frac{Q'\phi_0}{V(V+\phi_0)}\right)_x + \frac{\phi_0}{V}\left(\frac{Q'}{V} - \Theta^4\right)' + \left((\Theta + w_0)^4 - \Theta^4\right)_x \\ &\qquad + (V+\phi_0)z_0 - \left(\frac{z_{0x}}{V+\phi_0}\right)_x = 0. \end{aligned} \tag{4.89}$$

We multiply (4.89) by z_0 and get

$$\overline{E}_1(z_0) + \{\cdots\}_x + \overline{E}_2(\phi_0, z_0) + \overline{E}_3(w_0, z_0) = 0, \tag{4.90}$$

where

$$\begin{aligned} \overline{E}_1(z_0) &= (V+\phi_0)z_0^2 + \frac{z_0^2}{V+\phi_0}, \\ \overline{E}_2(\phi_0, z_0) &= \left(\frac{Q'}{V} - \Theta^4\right)' \frac{\phi_0 z_0}{V} - \frac{Q'\phi_0 z_{0x}}{V(V+\phi_0)}, \\ \overline{E}_3(w_0, z_0) &= -\left(4\Theta^3 w + 6\Theta^2 w^2 + 4\Theta w^3 + w^4\right) z_{0x}, \end{aligned}$$

and $\{\cdots\}_x$ denotes the terms that disappear after integration with respect to x. The assumption (4.15) gives $(\phi_0, w_0) \in H^2(\mathbb{R})$. Then the Sobolev embedding theorem yields that $(\phi_0, w_0) \in W^{1,\infty}(\mathbb{R})$. Together with the Cauchy-Schwarz inequality, we have the following estimates :

$$\frac{1}{C}\left(z_0^2 + z_{0x}^2\right) \le \quad \overline{E}_1(z_0) \quad \le C\left(z_0^2 + z_{0x}^2\right), \tag{4.91}$$

$$|\overline{E}_2(\phi_0, z_0)| \le C_\alpha \phi_0^2 + \alpha(z_0^2 + z_{0x}^2), \tag{4.92}$$

$$|\overline{E}_3(w_0, z_0)| \le C_\alpha w_0^2 + \alpha z_{0x}^2. \tag{4.93}$$

Let us choose $\alpha \le 1/(4C)$. Integrate (4.90) with respect to x , and use the above estimates, we have

$$\|z_0\|_{H^1(\mathbb{R})}^2 \le C\left(\|\phi_0\|_{L^2(\mathbb{R})}^2 + \|w_0\|_{L^2(\mathbb{R})}^2\right). \tag{4.94}$$

Next we apply ∂_x to (4.89) and get

$$\begin{aligned} &\left(\frac{Q'\phi_0}{V(V+\phi_0)}\right)_{xx} + \left(\frac{\phi_0}{V}\left(\frac{Q'}{V} - \Theta^4\right)'\right)_x + \left((\Theta + w_0)^4 - \Theta^4\right)_{xx} + (V+\phi_0)z_{0x} \\ &\qquad + (V' + \phi_{0x})z_0 - \left(\frac{z_{0xx}}{V+\phi_0}\right)_x + \left(\frac{z_{0x}}{(V+\phi_0)^2}(V' + \phi_{0x})\right)_x = 0. \end{aligned} \tag{4.95}$$

On the one hand in order to derive the $H^1(\mathbb{R})$ estimate to z_{0x}, we multiply (4.95) by z_{0x}, then we integrate it with respect to x, with a similar discussion we get

$$\|z_{0x}\|_{H^1(\mathbb{R})}^2 \le C\left(\|\phi_0\|_{H^1(\mathbb{R})}^2 + \|w_0\|_{H^1(\mathbb{R})}^2\right). \tag{4.96}$$

Thus together with (4.94) we have $z_0 \in H^2(\mathbb{R})$, consequently $z_0 \in W^{1,\infty}(\mathbb{R})$ by the Sobolev embedding theorem. On the other hand in order to get the $H^1(\mathbb{R})$ estimate to z_{0xx}, we multiply (4.95) by z_{0xxx}, then we integrate it with respect to x, and use the fact that $(\phi_0, w_0, z_0) \in W^{1,\infty}(\mathbb{R})$, we end up with

$$\|z_{0xx}\|^2_{H^1(\mathbb{R})} \leq C\left(\|\phi_0\|^2_{H^2(\mathbb{R})} + \|w_0\|^2_{H^2(\mathbb{R})} + \|z_0\|^2_{H^1(\mathbb{R})}\right), \tag{4.97}$$

thus we get $z_0 \in H^3(\mathbb{R})$. Furthermore from (4.89), we obtain that

$$\tilde{Z}_0 = \frac{z_{0x}}{V+\phi_0} - \frac{\phi_0}{V}\left(\frac{Q'}{V} - \Theta^4\right) - \left((\Theta + w_0)^4 - \Theta^4\right) - \left(\frac{Q'\phi_0}{V(V+\phi_0)}\right) \in H^2(\mathbb{R}).$$

Chapitre 5

NON-LTE line radiative transfer, quasi-stationary approximation

5.1 Introduction

We consider a coupling between a radiating field and a plasma. The radiation is described by the following transport equation

$$\frac{1}{c}\partial_t f + v \cdot \nabla_x f = \eta - \chi f, \tag{5.1}$$

where the constant c is the speed of light, $v \in \mathbb{S}^2$ is the direction of propagation of photons, η is the emission coefficient (or emissivity) of matter, χ is the absorption coefficient (or extinction coefficient) of matter, and the unknown is the specific intensity f which is a function of time $t \in \mathbb{R}^+$, space position $x \in X \subset \mathbb{R}^3$, direction $v \in \mathbb{S}^2$, and frequency $\nu \in \mathbb{R}^+$.

If we consider the coefficients η and χ as given, the transfer equation (5.1) is linear and its solution can be written explicitly by integrating along the characteristics. These coefficients depend however in reality upon the internal excitation and ionization states of the plasma. These states are fixed in part by radiative processes that populate and depopulate atomic levels. For the line radiative transfer (bound-bound transitions without photon-ionization), they depend on the Einstein coefficients, the spontaneous emission probability A_{ji} (with $i, j \in \{1, .., K\}$, $i < j$), the absorption probability B_{ij} and the induced (stimulated) emission probability B_{ji}, and can be written as

$$\eta = \sum_i \sum_{j>i} n_j A_{ji} h\nu \phi_{ij}(\nu), \tag{5.2}$$

$$\chi = \sum_i \sum_{j>i} \left(n_i B_{ij} - n_j B_{ji}\right) h\nu \phi_{ij}(\nu), \tag{5.3}$$

where n_i (respectively n_j) denotes the population density at the atomic level i (respectively j), and $\phi_{ij}(\nu)$ represents the line profile for these transitions, which are centered at the frequencies ν_{ij} such that $h\nu_{ij}$ characterizes the energy difference between the upper level j and the lower level i. It is properly described as a Voigt profile, that is a convolution of a Lorentz profile and a Gaussian profile. Note that the extinction coefficient (5.3) is corrected by the stimulated emission.

The population density n_i at level i satisfies the following rate equation, in a static medium,

$$\frac{\partial n_i}{\partial t} = \sum_{j \neq i} n_j P_{ji} - n_i \sum_{j \neq i} P_{ij}, \tag{5.4}$$

where P_{ij} denotes the total (radiative plus collisional) transition rate from level i to level j. We refer to [49] §85 for the rate equation in the general case of a moving medium. Let us denote by n the total molecular number :

$$\sum_i n_i = n.$$

It is clear that it is conserved along time :

$$\frac{\partial n}{\partial t} = 0.$$

Bound-bound transitions (line transitions) between the lower energy level i and the upper energy level j may occur as radiative excitation, spontaneous radiative de-excitation, induced radiative de-excitation, collisional excitation and collision de-excitation. Let us denote C_{ij} (respectively C_{ji}) the rate of collisional excitation rate (respectively the rate of collisional de-excitation rate), such that in (5.4), the total excitation rate P_{ij} and the total de-excitation rate P_{ji} can be written as

$$P_{ij} = B_{ij}\rho_{ij} + C_{ij}, \qquad P_{ji} = A_{ji} + B_{ji}\rho_{ij} + C_{ji}, \tag{5.5}$$

where ρ_{ij} is the integrated mean intensity over the line profile $\phi_{ij}(\nu)$:

$$\rho_{ij}(t,x) = \int_{\mathbb{R}^+}\int_{\mathbb{S}^2} f(t,x,v,\nu)\,\phi_{ij}(\nu)\,\mathrm{d}v\mathrm{d}\nu, \tag{5.6}$$

with $\mathrm{d}v$ denoting the normalized Lebesgue measure on $\mathbb{S}^2$. For the physical background, we refer to the next section, or to [49] §85, [53] §2.6.

In general, the radiation field and the internal state of the matter must be determined simultaneously and self-consistently. In many situtions, the characteristic time of the excitation and de-excitation processes of the matter is much smaller than the characteristic time of the evolution of the radiative field. After rescaling in time the equations, it is therefore possible to introduce a parameter ϵ such that our coupled system becomes

$$\begin{cases} \dfrac{1}{c}\partial_t f^\epsilon + v \cdot \nabla_x f^\epsilon \\ \qquad = \displaystyle\sum_i \sum_{j>i} n_j^\epsilon A_{ji} h\nu \phi_{ij}(\nu) - \sum_i \sum_{j>i} \left(n_i^\epsilon B_{ij} - n_j^\epsilon B_{ji}\right) h\nu\phi_{ij}(\nu) f^\epsilon, \\ \epsilon\dfrac{\partial n_i^\epsilon}{\partial t} = \displaystyle\sum_{j\neq i} n_j^\epsilon P_{ji} - \sum_{j \neq i} n_i^\epsilon P_{ij}. \end{cases} \tag{5.7}$$

Remark 5.1. *For the line radiative transfer, the emission coefficient η and the absorption coefficient χ can also be written as*

$$\begin{aligned} \eta &= \sum_i \sum_{j>i} n_j A_{ji} h\nu_{ij}\phi_{ij}(\nu), \\ \chi &= \sum_i \sum_{j>i} \left(n_i B_{ij} - n_j B_{ji}\right) h\nu_{ij}\phi_{ij}(\nu), \end{aligned}$$

see [49] §73, [51] §7.3 or [61] §5.10, instead of (5.2)-(5.3) *used in [53, 12]. In fact over the line profile* $\phi_{ij}(\nu)$, *centered at* ν_{ij}, *using the effect of profile we have the following approximation*

$$h\nu_{ij}\phi_{ij}(\nu) \approx h\nu\phi_{ij}(\nu).$$

In this case the factor $h\nu$, *in the right-hand side of the transport equation, is no longer a function of frequency but it becomes a constant given by the difference of energy between two energy levels* i *and* j. *The result we obtain can be extended easily to this framework.*

We consider the system (5.7) in the case when the position variable x varies in a bounded (regular, open) domain $X \subset \mathbb{R}^3$. We add therefore the initial condition

$$f^\epsilon(0, x, v, \nu) = f_0(x, v, \nu), \tag{5.8}$$

and the incoming boundary condition

$$f^\epsilon|_{\mathbb{R}^+ \times (\partial X \times \mathbb{S}^2)_- \times \mathbb{R}^+} = g(t, x, v, \nu), \tag{5.9}$$

where $(\partial X \times \mathbb{S}^2)_-$ is defined as

$$(\partial X \times \mathbb{S}^2)_- = \left\{(x, v) \in \partial X \times \mathbb{S}^2 : \Gamma_x \cdot v < 0\right\},$$

with Γ_x denoting the outward normal to X at the point $x \in \partial X$. Finally, the initial population densities $n_i^\epsilon(0, x)$ are given by

$$n_0^\epsilon(0, x) = n_{i0}(x) \geq 0, \ \forall i, \tag{5.10}$$

satisfying

$$\sum_i n_{i0}(x) = n(x). \tag{5.11}$$

We are interested in the existence of solutions to (5.7) – (5.10) (when $\epsilon > 0$ is fixed), and in the behavior of the solutions f^ϵ, n_i^ϵ, as $\epsilon \longrightarrow 0$ (quasi-stationary approximation).

In the sequel, we shall consider the following assumption on the data :

Assumption A : The initial condition f_0 and the boundary condition g satisfy

$$0 \leq f_0 \in L^\infty(X \times \mathbb{S}^2 \times \mathbb{R}^+), \qquad 0 \leq g \in L^\infty(\mathbb{R}^+ \times (\partial X \times \mathbb{S}^2)_- \times \mathbb{R}^+). \tag{5.12}$$

and the initial occupation numbers are such that (5.10) and (5.11) hold, with $n \in L^\infty(\mathbb{R}^+ \times X)$.

The Einstein Coefficients A_{ji}, B_{ij} and B_{ji} are (strictly positive) constants, and the collisional coefficients C_{ij} and C_{ji} are (nonnegative) functions of the position $x \in X$ verifying

$$\delta_* \leq C_{ij}(x), \ C_{ji}(x) \leq \delta^*, \tag{5.13}$$

for some δ_*, $\delta^* > 0$.

Finally, the line profile ϕ is integrable on $\mathbb{R}^+$ and satisfying

$$0 \leq \phi(\nu)h\nu \leq \delta, \ \forall \nu \in \mathbb{R}^+ \tag{5.14}$$

for some $\delta > 0$.

Our main result is stated as

Theorem 5.2. *Let the assumption A on the data be satisfied. Then for any given $T > 0$, there exists a unique nonnegative solution f^ϵ, $(n_i^\epsilon)_{i=1,..,K}$, to (5.5) – (5.11), which belongs to $L^\infty([0,T] \times X \times \mathbb{S}^2 \times \mathbb{R}^+) \times (L^\infty([0,T] \times X))^K$. Furthermore, as $\epsilon \to 0$, this solution converges in $L^\infty([0,T] \times X \times \mathbb{S}^2 \times \mathbb{R}^+) \times (L^\infty([0,T] \times X))^K$ weak * to f, $(n_i)_{i=1,..,K}$, unique nonnegative solution in $L^\infty([0,T] \times X \times \mathbb{S}^2 \times \mathbb{R}^+) \times (L^\infty([0,T] \times X))^K$ to the system*

$$\left\{ \begin{array}{l} \frac{1}{c}\partial_t f + v \cdot \nabla_x f \\ \qquad = \displaystyle\sum_i \sum_{j>i} n_j A_{ji} h\nu \phi_{ij}(\nu) - \sum_i \sum_{j>i} \left(n_i B_{ij} - n_j B_{ji}\right) h\nu \phi_{ij}(\nu) f, \\ 0 = \displaystyle\sum_{j \neq i} n_j P_{ji} - \sum_{j \neq i} n_i P_{ij}, \\ f(0,x,v,\nu) = f_0(x,v,\nu), \qquad f|_{\mathbb{R}^+ \times (\partial X \times \mathbb{S}^2)_- \times \mathbb{R}^+} = g(t,x,v,\nu), \end{array} \right. \tag{5.15}$$

with (5.5), (5.6).

Most of the rest of the paper is devoted to the proof of Theorem 5.2. Existence and uniqueness of a solution to (5.7) – (5.11), for a given ϵ, is proven in §5.2. At the end of this section, we show that a result of existence and uniqueness also holds for the limiting system (5.15) (together with the initial condition (5.8) and the boundary condition (5.9)).

Then, in §5.3, we prove the validity of the quasi-stationary approximation, that is the convergence of solutions of (5.7) – (5.11) when $\epsilon \to 0$ toward solutions of (5.15) (together with the initial condition (5.8) and the boundary condition (5.9)).

In §5.4, we present a numerical test in order to illustrate this convergence.

Finally, a small appendix (§5.5) is devoted to the physical background underlying the transfer equation (5.1) and the rate equation (5.4).

In all the sequel, we shall restrict ourselves in the proof, for the sake of simplicity, to a two-level molecular model. The proof in the general case is identical. This means that there are only two unknowns for the occupation numbers : n_1^ϵ and n_2^ϵ, and that the system (5.7) becomes

$$\left\{ \begin{array}{l} \frac{1}{c}\partial_t f^\epsilon + v \cdot \nabla_x f^\epsilon = \left(n_2^\epsilon A_{21} - (n_1^\epsilon B_{12} - n_2^\epsilon B_{21}) f^\epsilon\right) \phi(\nu) h\nu, \\ f^\epsilon(0,x,v,\nu) = f_0(x,v,\nu), \\ f^\epsilon|_{\mathbb{R}^+ \times (\partial X \times \mathbb{S}^2)_- \times \mathbb{R}^+} = g(t,x,v,\nu), \\ \varepsilon \partial_t n_1^\epsilon = n_2^\epsilon A_{21} + (n_2^\epsilon B_{21} - n_1^\epsilon B_{12})\rho^\epsilon + (n_2^\epsilon C_{21} - n_1^\epsilon C_{12}), \\ \varepsilon \partial_t n_2^\epsilon = -n_2^\epsilon A_{21} - (n_2^\epsilon B_{21} - n_1^\epsilon B_{12})\rho^\epsilon - (n_2^\epsilon C_{21} - n_1^\epsilon C_{12}), \\ n_1^\epsilon(0,x) = n_{10}(x), n_2^\epsilon(0,x) = n_{20}(x), \end{array} \right. \tag{5.16}$$

together with

$$\rho(t,x) = \int_{\mathbb{R}^+} \int_{\mathbb{S}^2} f(t,x,v,\nu)\, \phi(\nu)\, \mathrm{d}v \mathrm{d}\nu, \tag{5.17}$$

where we have omitted the subscript in the density ρ_{12} and the profile ϕ_{12}.

In this paper we limit our discussion to the bound-bound transitions, we refer for details on the bound-free transitions or the free-free transitions to [49, 53], or the papers [14, 7, 11], etc.

We refer to [5] for the existence theory of the radiative transfer equation for a 'grey' model, by using the compactness result introduced in [19, 20], that is, the averaging lemma.

In [60, 12], the authors studied some numerical methods for the line radiative transfer, and the comparison was given between a number of independent computer programs for radiative transfer in molecular rotational lines. Our numerical tests are inspired from the model introduced in [60, 12].

5.2 Proof of existence and uniqueness to system (5.7) for a given ϵ

We begin with a classical explicit resolution of the linear kinetic equation.

5.2.1 Duhamel's formula

We study here the linear transport equation (5.53), together with the initial condition and the boundary condition, for $(t, x, v, \nu) \in \mathbb{R}^+ \times (\partial X \times \mathbb{S}^2)_- \times \mathbb{R}^+$, where X is a bounded set in $\mathbb{R}^3$.

We write the solution explicitly by integrating along the characteristic lines and obtain an L^∞ estimate summarized in the following proposition :

Proposition 5.3. *Let X be a bounded open set in $\mathbb{R}^3$. We consider the following system :*

$$\left\{\begin{array}{rcl} \frac{1}{c}\partial_t f + v\cdot\nabla_x f &=& \eta - \chi f, \\ f(0,x,v,\nu) &=& f_0(x,v,\nu), \\ f|_{\mathbb{R}^+\times(\partial X\times\mathbb{S}^2)_-\times\mathbb{R}^+} &=& g(t,x,v,\nu), \end{array}\right. \tag{5.18}$$

where the initial data f_0 and the boundary data g satisfy

$$0 \le f_0 \in L^\infty(X\times\mathbb{S}^2\times\mathbb{R}^+), \qquad 0 \le g \in L^\infty(\mathbb{R}^+\times(\partial X\times\mathbb{S}^2)_-\times\mathbb{R}^+),$$

and the coefficients η, χ satisfy

$$\eta,\ \chi \in L^\infty(\mathbb{R}^+\times X\times\mathbb{S}^2\times\mathbb{R}^+), \qquad \eta \ge 0.$$

Then, for any given $T > 0$, there exists a constant $\delta(T) > 0$ (depending only on T and the L^∞ norms of η, χ, f_0 and g) such that

$$0 \le f(t,x,v,\nu) \le \delta(T),$$

for all $(t,x,v,\nu) \in [0,T]\times X\times\mathbb{S}^2\times\mathbb{R}^+$.

Proof. Let us denote $Q = \{(t,x)|t\in\mathbb{R}^+, x\in X\}$, and denote by Σ the boundary of Q, see figure 5.1. The boundary Σ has thus two parts :

$$\Sigma = \Sigma_1\bigcup\Sigma_2 = \{(0,x)|x\in X\}\bigcup\{(t,x)|t\in\mathbb{R}^+, x\in\partial X\}.$$

Let us fix a point $M^* = (t^*, x^*)$ in Q, and introduce a characteristic line through M^* as

$$t \longmapsto x(t) = x^* - c\,v(t^*-t).$$

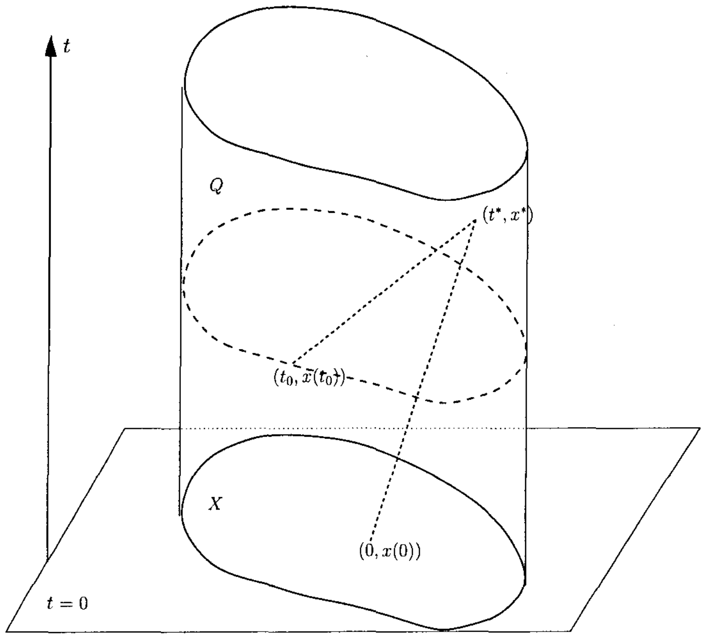

FIG. 5.1 – The characteristic line and the set Q

We look for the intersection of this characteristic line with Σ, the boundary of Q. There are two cases : either the line remains in Q and intersects Σ_1, (that is, the plane $t = 0$) at the point $x(0) = x_0 = x^* - c\ v\ t^*$, or the line intersects $\Sigma_2 = \{(t,x)|x \in \partial X, t > 0\}$ at some point $(t_0, x(t_0))$ with $0 \le t_0 < t^*$, see fig. 5.1.

We denote by $t_0 = t_0(M^*)$ the time when the characteristic line intersects $\Sigma = \partial Q$, i.e

$$t_0 = \inf\{t \ge 0 | (t, x(t)) \in \bar{Q}\}, \tag{5.19}$$

and by x_0 the coordinates of the intersection point.

In order to integrate along this characteristic line, we discuss our two cases :

1. Case $t_0 = 0$: the characteristic line belongs to $\bar{Q}$ when $t \in [0, t^*]$. Then, integrating over the characteristic line from 0 to t^* gives

$$\begin{aligned} & f(t^*, x^*, v, \nu) \\ & = f_0(x^* - cvt^*, v, \nu) \exp\left(-c \int_0^{t^*} \chi(s, x^* + cv^*(s - t^*), v, \nu) \mathrm{d}s \right) \\ & \quad + c \int_0^{t^*} \exp\left(-c \int_s^{t^*} \chi(s', x^* + cv(s' - t^*), v, \nu) \mathrm{d}s' \right) \\ & \qquad \eta(s, x^* + cv(s - t^*), v, \nu) \mathrm{d}s. \end{aligned} \tag{5.20}$$

2. Case $t_0 > 0$: the characteristic line belongs to $\bar{Q}$ when $t \in [t_0, t^*]$. Then integrating

over the characteristic line from t_0 to t^* gives

$$\begin{aligned} & f(t^*,x^*,v,\nu) \\ & = g(t_0,x^*-cv(t^*-t_0),v,\nu)\exp\left(-c\int_{t_0}^{t^*}\chi(s,x^*+cv^*(s-t^*),v,\nu)\mathrm{d}s\right) \\ & \quad +c\int_{t_0}^{t^*}\exp\left(-c\int_{s}^{t^*}\chi(s',x^*+cv(s'-t^*),v,\nu)\mathrm{d}s'\right) \\ & \qquad \eta(s,x^*+cv(s-t^*),v,\nu)\mathrm{d}s. \end{aligned} \tag{5.21}$$

We first obtain the nonnegativity of the solution f at (t^*,x^*,v,ν). Then, thanks to the boundedness of coefficients η and χ, we obtain the estimate for the solution. First, in (5.20),

$$\begin{aligned} f(t^*,x^*,v,\nu) & \leq \|f_0\|_{L^\infty_{x,v,\nu}}\exp\left(c\|\chi\|_{L^\infty_{t,x,v,\nu}}t^*\right)+c\int_0^{t^*}\exp\left(c\|\chi\|_{L^\infty_{t,x,v,\nu}}t^*\right)\|\eta\|_{L^\infty_{t,x,v,\nu}}\mathrm{d}s \\ & = \exp\left(c\|\chi\|_{L^\infty_{t,x,v,\nu}}t^*\right)\left(\|f_0\|_{L^\infty_{x,v,\nu}}+c\|\eta\|_{L^\infty_{t,x,v,\nu}}t^*\right)\equiv\delta_1(t^*). \end{aligned}$$

Similarly, in (5.21),

$$f(t^*,x^*,v,\nu) \leq \exp\left(c\|\chi\|_{L^\infty_{t,x,v,\nu}}t^*\right)\left(\|g\|_{L^\infty_{t,x,v,\nu}}+c\|\eta\|_{L^\infty_{t,x,v,\nu}}t^*\right)\equiv\delta_2(t^*).$$

We complete the proof of the Proposition 5.3, by taking $\delta(T)=\max\{\delta_1(T),\delta_2(T)\}$. □

5.2.2 Existence through an interation procedure

In this subsection, we prove the existence of solutions to system (5.16) (when ϵ is fixed) thanks to an iteration procedure. More precisely we construct iteratively sequences f^k, u_i^k, then we show that they are Cauchy sequences, and prove that their limits satisfy our system. In order to simplify the notations, we do not write explicitly the dependence of f and n_i with respect to ϵ. We write the coupled system as follows

$$\left\{\begin{array}{l} \frac{1}{c}\partial_t f+v\cdot\nabla_x f=\left(n_2A_{21}-\left(n_1B_{12}-n_2B_{21}\right)f\right)\phi(\nu)h\nu, \\ f(0,x,v,\nu)=f_0(x,v,\nu), \\ f|_{\mathbb{R}^+\times(\partial X\times\mathbb{S}^2)_-\times\mathbb{R}^+}=g(t,x,v,\nu), \end{array}\right. \tag{5.22}$$

and

$$\left\{\begin{array}{l} \epsilon\partial_t n_1=n_2A_{21}+(n_2B_{21}-n_1B_{12})\rho+(n_2C_{21}-n_1C_{12}), \\ \epsilon\partial_t n_2=-n_2A_{21}-(n_2B_{21}-n_1B_{12})\rho-(n_2C_{21}-n_1C_{12}), \\ n_1(t,x)+n_2(t,x)=n(x), \\ n_1(0,x)=n_{10}(x),n_2(0,x)=n_{20}(x). \end{array}\right. \tag{5.23}$$

We define the following iterative procedure :
– For $t\geq 0$, we set

$$f^0(t,x,v,\nu)=f_0(x,v,\nu),\quad n_i^0(t,x)=n_{i0}(x),\ i=1,2;$$

– For $k = 0, 1, 2, \ldots$, we assume that (f^k, n_1^k, n_2^k) are defined. We defined $f^{k+1}, n_1^{k+1}, n_2^{k+1}$ by

$$\left\{\begin{array}{l} \frac{1}{c}\partial_t f^{k+1} + v\cdot\nabla_x f^{k+1} = \Big(n_2^k A_{21} - (n_1^k B_{12} - n_2^k B_{21}) f^{k+1}\Big)\,\phi(\nu)h\nu, \\ f^{k+1}(0,x,v,\nu) = f_0(x,v,\nu), \\ f^{k+1}|_{\mathbb{R}^+\times(\partial X\times\mathbb{S}^2)_-\times\mathbb{R}^+} = g(t,x,v,\nu), \end{array}\right. \tag{5.24}$$

and

$$\left\{\begin{array}{l} \epsilon\partial_t n_1^{k+1} = n_2^{k+1}A_{21} + (n_2^{k+1}B_{21} - n_1^{k+1}B_{12})\rho^{k+1} + (n_2^{k+1}C_{21} - n_1^{k+1}C_{12}), \\ \epsilon\partial_t n_2^{k+1} = -\big[n_2^{k+1}A_{21} + (n_2^{k+1}B_{21} - n_1^{k+1}B_{12})\rho^{k+1} + (n_2^{k+1}C_{21} - n_1^{k+1}C_{12})\big], \\ n_i^{k+1}|_{t=0} = n_{i0},\ i = 1,2. \end{array}\right. \tag{5.25}$$

Note that (5.25) can be written in the following equivalent form :

$$\left\{\begin{array}{l} \epsilon\partial_t n_1^{k+1} = n(A_{21} + B_{21}\rho^{k+1} + C_{21}) - n_1^{k+1}(A_{21} + (B_{21}+B_{12})\rho^{k+1} + C_{21} + C_{12}), \\ \epsilon\partial_t n_2^{k+1} = n(B_{12}\rho^{k+1} + C_{12}) - n_2^{k+1}(A_{21} + (B_{21}+B_{12})\rho^{k+1} + C_{21} + C_{12}), \\ n_i^{k+1}|_{t=0} = n_{i0},\ i = 1,2. \end{array}\right.$$

Thus the solution n_1^{k+1} and n_2^{k+1} can be written explicitly as

$$\begin{array}{l} n_1^{k+1}(t,x) = n_{10}(x)\exp\left\{-\frac{1}{\epsilon}\int_0^t \Big(A_{21} + (B_{21}+B_{12})\,\rho^{k+1}(s,x) + C_{21}(x) + C_{12}(x)\Big)\,\mathrm{d}s\right\} \\ \quad +\frac{1}{\epsilon}\int_0^t \exp\left\{-\frac{1}{\epsilon}\int_s^t (A_{21} + (B_{21}+B_{12})\rho^{k+1}(s',x) + C_{21}(x) + C_{12}(x))\mathrm{d}s'\right\} \\ \qquad \times n(x)(A_{21} + B_{21}\rho^{k+1}(s,x) + C_{21}(x))\mathrm{d}s, \\ n_2^{k+1}(t,x) = n_{20}(x)\exp\left\{-\frac{1}{\epsilon}\int_0^t (A_{21} + (B_{21}+B_{12})\rho^{k+1}(s,x) + C_{21}(x) + C_{12}(x))\mathrm{d}s\right\} \\ \quad +\frac{1}{\epsilon}\int_0^t \exp\left\{-\frac{1}{\epsilon}\int_s^t (A_{21} + (B_{21}+B_{12})\rho^{k+1}(s',x) + C_{21}(x) + C_{12}(x))\mathrm{d}s'\right\} \\ \qquad \times n(x)(B_{12}\rho^{k+1}(s,x) + C_{12}(x))\mathrm{d}s. \end{array}$$

Using the nonnegativity of the initial population densities n_{i0} and of f_0, we see by induction that

$$n_i^k \geq 0, \quad n_1^k + n_2^k = n_{10} + n_{20} = n, \ \forall k \in \mathbb{N}.$$

This estimate, still used with proposition 5.3, ensures that f^k (is well defined and) satisfies $0 \leq f^{k+1}(t,x,v,\nu) \leq \delta(T)$, for all $(t,x,v,\nu) \in [0,T]\times X\times\mathbb{S}^2\times\mathbb{R}^+$.

We now prove that the sequences $(f^k)_k$ and $(n_i^k)_k$ defined by (5.24), (5.25) are Cauchy sequences in $L^\infty([0,T]\times X\times\mathbb{S}^2\times\mathbb{R}^+)$ and in $L^\infty([0,T]\times X)$ respectively.

We observe that $f^{k+1} - f^k$ satisfies the equation :

$$\left\{\begin{array}{l} \frac{1}{c}\partial_t(f^{k+1} - f^k) + v\cdot\nabla(f^{k+1} - f^k) + (n_1^k B_{12} - n_2^k B_{21})\phi(\nu)h\nu(f^{k+1} - f^k) \\ \quad = (n_2^k - n_2^{k-1})A_{21}\phi(\nu)h\nu - f^k\big[(n_1^k - n_1^{k-1})B_{12} - (n_2^k - n_2^{k-1})B_{21}\big]\phi(\nu)h\nu, \\ (f^{k+1} - f^k)|_{t=0} = 0, \\ (f^{k+1} - f^k)|_{\mathbb{R}^+\times(\partial X\times\mathbb{S}^2)_-\times\mathbb{R}^+} = 0. \end{array}\right.$$

Then integrating along the characteristic lines, as in the proof of Proposition 5.3, we write $f^{k+1} - f^k$ explicitly as

$$(f^{k+1} - f^k)(t, x, v, \nu) = c \int_{t_0}^t \exp\left\{-c \int_s^t (n_1^k B_{12} - n_2^k B_{21})|_{(s', x+cv(s'-t))}\phi(\nu)h\nu \mathrm{d}s'\right\}$$
$$\times \left[(n_2^k - n_2^{k-1})A_{21} - \left((n_1^k - n_1^{k-1})B_{12} - (n_2^k - n_2^{k-1})B_{21}\right)f^k\right]\Big|_{(s, x+v(s-t))} \phi(\nu)h\nu \mathrm{d}s,$$

where $t_0 \in [0, t]$ is defined by (5.19). Thus, we obtain the following $L^\infty_{x,v,\nu}(X \times \mathbb{S}^2 \times \mathbb{R}^+)$ estimate :

$$\begin{aligned}
&\left\|\left(f^{k+1} - f^k\right)(t)\right\|_{L^\infty_{x,v,\nu}} \\
&\leq \int_0^t \exp\left\{c\|n\|_{L^\infty_x}(B_{12} + B_{21}) \sup_\nu (\phi(\nu)h\nu)\, t\right\} \times \left\|\left(n_1^k - n_1^{k-1}\right)(s)\right\|_{L^\infty_x} \\
&\quad \times c\left(A_{21} + (B_{21} + B_{12})\|f^k\|_{L^\infty_{t,x,v,\nu}}\right) \sup_\nu (\phi(\nu)h\nu)\, \mathrm{d}s \\
&\leq \mathrm{e}^{C_1 T} c\left(A_{21} + (B_{21} + B_{12})\delta(T)\right) \sup_\nu (\phi(\nu)h\nu) \int_0^t \left\|\left(n_1^k - n_1^{k-1}\right)(s)\right\|_{L^\infty_x} \mathrm{d}s \\
&= \xi_1(T) \int_0^t \left\|\left(n_1^k - n_1^{k-1}\right)(s)\right\|_{L^\infty_x} \mathrm{d}s,
\end{aligned}$$

where $C_1 = c\,\|n\|_{L^\infty_x}(B_{12}+B_{21}) \sup_\nu (\phi(\nu)h\nu)$, $\xi_1(T) = \mathrm{e}^{C_1 T} c\left(A_{21}+(B_{21}+B_{12})\delta(T)\right) \sup_\nu (\phi(\nu)h\nu)$, and where we have used the formula $n_2^k - n_2^{k-1} = n_1^k - n_1^{k-1}$.

Then, we observe that $n_1^{k+1} - n_1^k$ is solution to the following equation :

$$\begin{cases}
\epsilon\partial_t(n_1^{k+1} - n_1^k) = -(n_1^{k+1} - n_1^k)(A_{21} + (B_{12} + B_{12})\rho^{k+1} + C_{12} + C_{12}) \\
\qquad -(\rho^{k+1} - \rho^k)n_1^k(B_{12} + B_{21}) \\
(n_1^{k+1} - n_1^k)|_{t=0} = 0.
\end{cases}$$

We recall that the Einstein coefficients A_{ji}, B_{ij} and B_{ji} are constants, and that C_{ij} and C_{ji} are bounded functions of $x \in X$. We write the solution $n_1^{k+1} - n_1^k$ of this equation as

$$\begin{aligned}
&(n_1^{k+1} - n_1^k)(t, x) \\
&= -\frac{1}{\epsilon} \int_0^t \exp\left\{-\frac{1}{\epsilon} \int_s^t (A_{21} + (B_{12} + B_{12})\rho^{k+1}(s', x) + C_{12}(x) + C_{12}(x))\mathrm{d}s'\right\} \\
&\quad \times (\rho^{k+1} - \rho^k)|_{(s,x)} n_1^k(s, x)(B_{12} + B_{21})\mathrm{d}s,
\end{aligned}$$

which yields the following estimate :

$$\begin{aligned}
\|n_1^{k+1} - n_1^k\|_{L^\infty_x}(t) &\leq \frac{1}{\epsilon} \int_0^t \left\|\left(\rho^{k+1} - \rho^k\right)(s)\right\|_{L^\infty_x} \times \|n\|_{L^\infty_x}(B_{12} + B_{12})\mathrm{d}s \\
&\leq \xi_2(T) \int_0^t \left\|\left(f^{k+1} - f^k\right)(s)\right\|_{L^\infty_{x,v,\nu}} \mathrm{d}s \\
&\leq \xi_2(T) T \sup_{s\in[0,t]} \left\|\left(f^{k+1} - f^k\right)(s)\right\|_{L^\infty_{x,v,\nu}},
\end{aligned}$$

with $\xi_2(T) = \int_{\mathbb{R}^+} \phi(\nu)\, \mathrm{d}\nu\, (B_{12} + B_{12})\|n\|_{L^\infty_x}/\epsilon$, which is independent of k.

Let us denote

$$\xi(T) = T\xi_1(T)\xi_2(T).$$

Note that $\xi(T)$ is independent of k. Substituting the estimate of $f^{k+1} - f^k$ into the estimate of $n_1^{k+1} - n_1^k$, we have

$$\begin{aligned}
&\left\| \left(n_1^{k+1} - n_1^k\right)(t) \right\|_{L_x^\infty} \\
&\le \xi(T) \int_0^t \left\| \left(n_1^k - n_1^{k-1}\right)(s) \right\|_{L_x^\infty} \mathrm{d}s \\
&\le \xi(T)^k \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \left\| \left(n_1^1 - n_1^0\right)(s) \right\|_{L_x^\infty} \mathrm{d}s\mathrm{d}t_{k-1} \cdots \mathrm{d}t_1 \\
&= C \times \xi(T)^k \frac{t^k}{k!},
\end{aligned}$$

which holds for any $t \in [0, T]$. Then, for all $k \in \mathbb{N},\ p \in \mathbb{N}^*$,

$$\|n_1^{k+p} - n_1^k\|_{L^\infty([0,T]\times X)} \le C \sum_{l=k}^{k+p-1} \frac{(\xi(T)T)^l}{l!}.$$

Thus we deduce that $(n_1^k)_k$ is a Cauchy sequence in $L^\infty([0,T] \times X)$. The same holds of course for $(n_2^k)_k$. Then using the estimate obtained on $f^{k+1} - f^k$, we obtain that $(f^k)_k$ is a Cauchy sequence in $L^\infty([0,T] \times X \times \mathbb{S}^2 \times \mathbb{R}^+)$. We therefore pass to the limit in (the integral formulations) of equations (5.24) and (5.25), and obtain a bounded solution f, n_1, n_2 to the coupled system (5.22),(5.23).

5.2.3 Uniqueness

Let (f, n_1, n_2) and $(\overline{f}, \overline{n}_1, \overline{n}_2)$ be two solutions to the coupled system (5.22),(5.23) with the same initial and boundary conditions. As in the proof of existence, we introduce $f - \overline{f}$, which is solution to the following system,

$$\begin{cases}
\dfrac{1}{c}\partial_t(f - \overline{f}) + v \cdot \nabla_x(f - \overline{f}) + (n_1 B_{12} - n_2 B_{21})\phi(\nu)h\nu(f - \overline{f}) \\
\qquad = (n_2 - \overline{n}_2)A_{21}\phi(\nu)h\nu - f\left[(n_1 - \overline{n}_1)B_{12} - (n_2 - \overline{n}_2)B_{21}\right]\phi(\nu)\, h\nu, \\
(f - \overline{f})|_{t=0} = 0, \\
(f - \overline{f})|_{\mathbb{R}^+ \times (\partial X \times \mathbb{S}^2)_- \times \mathbb{R}^+} = 0.
\end{cases}$$

Then we integrate the above equation along the characteristic line, and write the solution $f - \overline{f}$ explicitly. We thus study the $L^\infty_{x,v,\nu}(X \times \mathbb{S}^2 \times \mathbb{R}^+)$ estimate for $f - \overline{f}$ and get (as in the proof of existence)

$$\left\|\left(f - \overline{f}\right)(t)\right\|_{L^\infty_{x,v,\nu}} \le \xi_1(T) \int_0^t \|(n_1 - \overline{n}_1)(s)\|_{L_x^\infty} \mathrm{d}s.$$

With a similar analysis, we get the $L_x^\infty(X)$ estimate for $n_1 - \overline{n}_1$ as

$$\|(n_1 - \overline{n}_1)(t)\|_{L_x^\infty} \le \xi_2(T) \int_0^t \left\|\left(f - \overline{f}\right)(s)\right\|_{L^\infty_{x,v,\nu}} \mathrm{d}s.$$

Then summing the above two estimates for $f-\overline{f}$ and $n_1-\overline{n}_1$, we end up with the following estimate :

$$\begin{aligned}&\left\|\left(f-\overline{f}\right)(t)\right\|_{L^\infty_{x,v,\nu}}+\left\|\left(n_1-\overline{n}_1\right)(t)\right\|_{L^\infty_x}\\&\quad\leq\left(\xi_1(T)+\xi_2(T)\right)\int_0^t\left(\left\|\left(f-\overline{f}\right)(s)\right\|_{L^\infty_{x,v,\nu}}+\left\|\left(n_1-\overline{n}_1\right)(s)\right\|_{L^\infty_x}\right)\mathrm{d}s.\end{aligned}$$

Then, using the Gronwall Lemma, the above inequality reads

$$\|f-\overline{f}\|_{L^\infty_{t,x,v,\nu}}+\|n_1-\overline{n}_1\|_{L^\infty_{t,x}}=0.$$

so that we get the uniqueness : $f\equiv\overline{f}$, and $n_i\equiv\overline{n}_i$, in $[0,T]\times X\times\mathbb{S}^2\times R^+$ and $[0,T]\times X$, respectively.

5.2.4 Existence of solution to the limit system ($\epsilon = 0$)

In this subsection, we study the existence and uniqueness to the limiting system (5.15).

We introduce the following iterative procedure :

– We set

$$\begin{aligned}f^0(t,x,v,\nu)&=f_0(x,v,\nu),\\ n_1^0(t,x)&=\frac{A_{21}+B_{21}\rho^0(t,x)+C_{21}(x)}{A_{21}+(B_{21}+B_{12})\rho^0(t,x)+C_{12}(x)+C_{21}(x)}n(x),\\ n_2^0(t,x)&=\frac{B_{12}\rho^0+C_{12}(x)}{A_{21}+(B_{21}+B_{12})\rho^0(t,x)+C_{12}(x)+C_{21}(x)}n(x).\end{aligned}$$

– For $k=0,1,2,...$, we assume that (f^k,n_1^k,n_2^k) have been defined. We define $f^{k+1},n_1^{k+1},n_2^{k+1}$ as solutions to

$$\begin{cases}\frac{1}{c}\partial_t f^{k+1}+v\cdot\nabla_x f^{k+1}=\left(n_2^kA_{21}-(n_1^kB_{12}-n_2^kB_{21})f^{k+1}\right)\phi(\nu)h\nu,\\ f^{k+1}(0,x,v,\nu)=f_0(t,x,v,\nu),\\ f^{k+1}|_{\mathbb{R}^+\times(\partial X\times\mathbb{S}^2)_-\times\mathbb{R}^+}=g(t,x,v,\nu),\end{cases}$$
$$\begin{cases}n_1^{k+1}=\dfrac{A_{21}+B_{21}\rho^{k+1}+C_{21}}{A_{21}+(B_{21}+B_{12})\rho^{k+1}+C_{12}+C_{21}}n,\\ n_2^{k+1}=\dfrac{B_{12}\rho^{k+1}+C_{12}}{A_{21}+(B_{21}+B_{12})\rho^{k+1}+C_{12}+C_{21}}n.\end{cases}\tag{5.26}$$

We recall that ρ^{k+1} is the integrated mean intensity of f^{k+1} over the line profile.

For all $k\in\mathbb{N}$, it is clear by induction that $0\leq n_i^k\leq n(x)$, and (thanks to Proposition 5.3) that f^{k+1} (is well defined and) satisfies $0\leq f^{k+1}(t,x,v,\nu)\leq\delta(T)$, for any $(t,x,v,\nu)\in[0,T]\times X\times\mathbb{S}^2\times\mathbb{R}^+$.

We now prove that for any $T>0$ fixed, $(f^k)_k$ and $(n_i^k)_k$ are Cauchy sequences in $L^\infty([0,T]\times X\times\mathbb{S}^2\times\mathbb{R}^+)$ and in $L^\infty([0,T]\times X)$ respectively.

By proceeding as in the proof of existence to the solution of system (5.16) for a given $\epsilon>0$, we obtain the estimate

$$\left\|\left(f^{k+1}-f^k\right)(t)\right\|_{L^\infty_{x,v,\nu}}(t)\leq\xi_1(T)\int_0^t\left\|\left(n_1^k-n_1^{k-1}\right)(s)\right\|_{L^\infty_x}\mathrm{d}s,$$

where $\xi_1(T) = \mathrm{e}^{C_1 T}\big(A_{21} + (B_{21} + B_{12})\delta(T)\big) \sup_\nu (\phi(\nu) h\nu)$.

Noticing that

$$|n_1^k - n_1^{k-1}| = \frac{n(x)|B_{12}A_{21} + B_{12}C_{21} - B_{21}C_{12}||\rho^k - \rho^{k-1}|}{\big(A_{21} + (B_{21} + B_{12})\rho^k + C_{21} + C_{21}\big)\big(A_{21} + (B_{21} + B_{12})\rho^{k-1} + C_{21} + C_{21}\big)},$$

and that

$$\|(\rho^k - \rho^{k-1})(t)\|_{L_x^\infty} \leq \int_{\mathbb{R}^+} \phi(\nu)\,\mathrm{d}\nu\, \|(f^k - f^{k-1})(t)\|_{L^\infty_{x,v,\nu}},$$

we see that

$$\|n_1^k - n_1^{k-1}\|_{L_x^\infty}(t) \leq \tilde{\xi}_2(T)\|f^k - f^{k-1}\|_{L^\infty_{x,v,\nu}}(t),$$

where $\tilde{\xi}_2(T) = \int_{\mathbb{R}^+} \phi(\nu)\,\mathrm{d}\nu\, B_{12}A_{21}^{-1}\, ||n||_{L_x^\infty}$.

We consider $\xi(T) = \xi_1(T)\tilde{\xi}_2(T)$, which is independent of k. Using the estimates for $f^{k+1} - f^k$ and $n_1^k - n_1^{k-1}$, we get

$$\begin{aligned}
\left\|\left(f^{k+1} - f^k\right)(t)\right\|_{L^\infty_{x,v,\nu}} &\leq \xi(T)\int_0^t \left\|\left(f^k - f^{k-1}\right)(s)\right\|_{L^\infty_{x,v,\nu}} \mathrm{d}s \\
&\leq \xi(T)^k \int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}} \left\|\left(f^1 - f^0\right)(s)\right\|_{L^\infty_{x,v,\nu}}(s)\mathrm{d}s \\
&\leq C\xi(T)^k \frac{t^k}{k!}.
\end{aligned}$$

Finally, for all $k \in \mathbb{N},\ p \in \mathbb{N}^*$, we have

$$\|f^{k+p} - f^k\|_{L^\infty([0,T]\times X\times\mathbb{S}^2\times\mathbb{R}^+)} \leq C \sum_{l=k}^{k+p-1} \frac{(\xi(T)T)^l}{l!}.$$

Thus, $(f^k)_k$ is a Cauchy sequence in $L^\infty([0,T] \times X \times \mathbb{S}^2 \times \mathbb{R}^+)$, and $(n_1^k)_k$ is also a Cauchy sequence in $L^\infty([0,T] \times X)$, together with $(n_2^k)_k$. Passing to the limit in (the integral formulation of) eq. (5.26), we get a solution of our limiting system.

We get then the uniqueness thanks to an analysis similar to that used in the proof of uniqueness for system (5.7) for a given $\epsilon > 0$.

5.3 Quasi-stationary approximation, convergence

We have studied the existence of the coupled system (5.16) when $\epsilon \neq 0$ is given, and for the limiting system (that is, when $\epsilon = 0$). In this subsection, we prove the convergence of the solution f^ϵ, n_i^ϵ toward the solution f, n_i of the limiting system.

Let f^ϵ be the unique solution to system (5.16). We already know that

$$0 \leq n_i^\epsilon(t,x) \leq ||n||_{L^\infty}.$$

As a consequence of Proposition 5.3, we also know that $(f^\epsilon)_\epsilon$ is bounded in $L^\infty([0,T] \times X \times \mathbb{S}^2 \times R^+)$, so that

$$\int_{\mathbb{R}^+} f^\epsilon(t,x,v,\nu)\,\phi(\nu)\,\mathrm{d}\nu$$

is bounded in $L^\infty([0,T]\times X\times\mathbb{S}^2)$. Furthermore, this quantity solves the following system :

$$\begin{cases} \frac{1}{c}\partial_t \int_{\mathbb{R}+} f^\epsilon \phi(\nu)\mathrm{d}\nu + v\cdot\nabla_x \int_{\mathbb{R}+} f^\epsilon\phi(\nu)\mathrm{d}\nu \\ \qquad = n_2^\epsilon A_{21}\int_{\mathbb{R}+}\phi^2(\nu)h\nu\mathrm{d}\nu - (n_1^\epsilon B_{12} - n_2^\epsilon B_{21})\int_{\mathbb{R}+} f^\epsilon\phi^2(\nu)h\nu\mathrm{d}\nu, \\ \left(\int_{\mathbb{R}+} f^\epsilon\phi(\nu)\mathrm{d}\nu\right)(0,x,v) = \int_{\mathbb{R}+} f_0(x,v,\nu)\phi(\nu)\mathrm{d}\nu, \\ \left(\int_{\mathbb{R}+} f^\epsilon\phi(\nu)\mathrm{d}\nu\right)\Big|_{\mathbb{R}+\times(\partial X\times\mathbb{S}^2)_-} = \int_{\mathbb{R}+} g(t,x,v,\nu)\phi(\nu)\mathrm{d}\nu. \end{cases}$$

Using the L^∞ bound of n_i^ϵ, we see that

$$(\frac{1}{c}\partial_t + v\cdot\nabla_x)\int_{\mathbb{R}+} f^\epsilon\phi(\nu)\mathrm{d}\nu$$

is bounded in $L^\infty([0,T]\times X\times\mathbb{S}^2)$. Thanks to the averaging lemma ([5, 19, 20]), we deduce that the sequence $\int_{\mathbb{R}+}\int_{\mathbb{S}^2} f^\epsilon\phi(\nu)\mathrm{d}v\mathrm{d}\nu = \rho^\epsilon(t,x)$ is strongly compact in $L^2([0,T]\times X)$. This ensures that ρ^ϵ converges (up to a subsequence) a.e.

Thus (still up to a subsequence), we can assume that

$$\begin{array}{l} n_i^\epsilon \rightharpoonup n_i \text{ weakly* in } L^\infty([0,T]\times X), i=1,2; \\ f^\epsilon \rightharpoonup f \text{ weakly* in } L^\infty([0,T]\times X\times\mathbb{S}^2\times\mathbb{R}^+); \\ \rho^\epsilon \rightarrow \rho \text{ strongly in } L^1([0,T]\times X), \end{array}$$

where

$$\rho = \int_{\mathbb{R}+}\int_{\mathbb{S}^2} f\phi(\nu)\mathrm{d}v\mathrm{d}\nu.$$

The sequence $n_i^\epsilon\rho^\epsilon$ converges therefore to $n_i\rho$ weakly in $L^1([0,T]\times X)$.

It remains also to pass to the limit in the quantity $n_i^\epsilon\, f^\epsilon\,\phi(\nu)\, h\nu$. This is done by observing that for any test function $\psi_1(v)\,\psi_2(\nu)$ (with $\psi_1,\psi_2\in\mathcal{D}$), the quantity

$$\int_{\mathbb{R}+}\int_{\mathbb{S}^2} f^\epsilon(t,x,v,\nu)\,\phi(\nu)\,h\nu\,\psi_1(v)\,\psi_2(\nu)\,\mathrm{d}v\mathrm{d}\nu$$

converges a.e. This is due to the fact that $\int_{\mathbb{R}+} f^\epsilon(t,x,v,\nu)\,\phi(\nu)\,h\nu\,\psi_2(\nu)\,\mathrm{d}\nu$ satisfies a kinetic equation (like $\int_{\mathbb{R}+} f^\epsilon(t,x,v,\nu)\,\phi(\nu)\,\mathrm{d}\nu$).

Finally, when ϵ tend to 0, the solution to (5.16) converges to the solution (in the weak sense) of :

$$\begin{cases} \frac{1}{c}\partial_t f + v\cdot\nabla_x f = (n_2A_{21} - (n_1B_{12} - n_2B_{21})f)\,\phi(\nu)\,h\nu, \\ f(0,x,v,\nu) = f_0(x,v,\nu), \\ f|_{\mathbb{R}+\times(\partial X\times\mathbb{S}^2)_-\times\mathbb{R}+} = g(t,x,v,\nu), \\ 0 = n_2A_{21} + (n_2B_{21} - n_1B_{12})\rho + (n_2C_{21} - n_1C_{12}), \\ 0 = -n_2A_{21} - (n_2B_{21} - n_1B_{12})\rho - (n_2C_{21} - n_1C_{12}), \\ n_1(t,x) + n_2(t,x) = n(x). \end{cases}$$

Thanks to the result of uniqueness for the limit problem, the convergence is in fact not restricted to a subsequence, which ends the proof of Theorem 5.2.

5.4 Numerical simulation

5.4.1 Reformulation of the problem, definition of the parameters

Since the light speed c is a huge number in the SI system, we introduce a new variable of time, t' to get rid of this difficulty, such that $t' = c\, t$, and therefore

$$\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t'}.$$

We also introduce a new parameter ϵ' that characterizes the convergence of the rate equation to the statistical equilibrium, defined by $\epsilon' = \epsilon\ c$, so that there also holds :

$$\epsilon\frac{\mathrm{d}}{\mathrm{d}t} = \epsilon c\frac{\mathrm{d}}{\mathrm{d}t'} = \epsilon'\frac{\mathrm{d}}{\mathrm{d}t'}.$$

Taking into account these notations, the system (5.16) can thus be rewritten as

$$\begin{cases} \partial_t f^\epsilon + v\cdot\nabla_x f^\epsilon = (n_2^\epsilon A_{21} - (n_1^\epsilon B_{12} - n_2^\epsilon B_{21})f^\epsilon)\,\phi(\nu)h\nu, \\ f^\epsilon(0,x,v,\nu) = f_0(x,v,\nu), \\ f^\epsilon|_{\mathbb{R}^+\times(\partial X\times\mathbb{S}^2)_-\times\mathbb{R}^+} = g(t,x,v,\nu), \\ \varepsilon\partial_t n_1^\epsilon = n_2^\epsilon A_{21} + (n_2^\epsilon B_{21} - n_1^\epsilon B_{12})\rho^\epsilon + (n_2^\epsilon C_{21} - n_1^\epsilon C_{12}), \\ \varepsilon\partial_t n_2^\epsilon = -n_2^\epsilon A_{21} - (n_2^\epsilon B_{21} - n_1^\epsilon B_{12})\rho^\epsilon - (n_2^\epsilon C_{21} - n_1^\epsilon C_{12}), \\ n_1^\epsilon(0,x) = n_{10}(x), n_2^\epsilon(0,x) = n_{20}(x), \end{cases} \tag{5.27}$$

where we have omitted the sign $'$, in order to simplify the notations.

The numerical test is inspired from the problem that was introduced in [60, 12]. We fix the parameters as follows : The domain X is defined as

$$X = \left\{x\in\mathbb{R}^3 : r_{in} < |x| < r_{out}\right\},$$

where $r_{in} = 1$ and $r_{out} = 3$ with units m, and the density is assumed to be zero out of X. The initial radiation f_0 is defined as

$$f_0(x,v,\nu) = \frac{|x| - r_{in}}{r_{out} - r_{in}}B(\nu, T_0),$$

where $B(\nu, T_0)$ is the intensity defined by the Planck function at the temperature $T_0 = 2.728$ K. The incoming radiation at the outer boundary is given by

$$g(t,x,v,\nu) = B(\nu, T_0),$$

and we choose the incoming radiation at the inner boundary to be zero. Then as in [60, 12], we choose a fictive two-level molecule that is specified as follows : The energy difference between these two states is given by

$$\Delta E = E_2 - E_1 = 1.191838281\cdot 10^{-22}\ \ \mathrm{J},$$

thus the frequency $\nu_0 = \nu_{12}$ corresponding to the energy difference ΔE between these two states, is calculated as

$$\nu_0 = \frac{\Delta E}{h} = 1.798710857\cdot 10^{11}\ \ \mathrm{Hz},$$

The ratio of the statistical weights of these two states is given by

$$\frac{g_2}{g_1} = 3.$$

The Einstein coefficient A_{21} of spontaneous emission is given by

$$A_{21} = 1.0 \cdot 10^{-4} \text{ s}^{-1}.$$

Using the Einstein relations (Cf. eq. (5.48) in the appendix), we can calculate the Einstein coefficients of stimulated emission, B_{12}, and of the absorption, B_{21}, as

$$B_{12} = 3.495914724 \cdot 10^{12} \text{ m}^2 \text{ (js)}^{-1}, \qquad B_{21} = 1.165304908 \cdot 10^{12} \text{ m}^2 \text{ (js)}^{-1}.$$

Also, as in [60, 12], the downward collision rate C_{21} is defined by $C_{21} = N_{col} K_{21}$, where N_{col} is the density of the collision partners in units m^{-3}, and K_{21} is the collisional rate in units m^3s^{-1}, they are given by

$$K_{21} = 2.0 \cdot 10^{-16} \text{ m}^3\text{s}^{-1}, \qquad N_{col}(x) = 2.0 \cdot 10^{13} \frac{r_1^2}{|x|^2} \text{ m}^{-3}.$$

Thus we get the value of the downward collision rate, C_{21}, using the Einstein relation (Cf. eq. (5.49) in the appendix, with the constant kinetic temperature $T_{\text{kin}} = 20\text{K}$), we get the upward collision rate C_{12} as

$$C_{21}(x) = 4.0 \cdot 10^{-3} \frac{r_1^2}{|x|^2} \text{ s}^{-1}, \qquad C_{12} = 7.793375810 \cdot 10^{-3} \frac{r_1^2}{|x|^2} \text{ s}^{-1}.$$

The line profile function $\phi(\nu)$, as in [60, 12], is assumed to be Gaussian :

$$\phi(\nu) = \frac{c}{a\nu_0\sqrt{\pi}} \exp\left(-\frac{c^2(\nu - \nu_0)^2}{a^2\nu_0^2}\right),$$

where c is the speed of light, ν_0 is the frequency at the line centre corresponding to the energy difference between E_1 and E_2, and a is the total line width, given as $a = 150 \text{ ms}^{-1}$. Note that the profile has unit s. With these parameters, we define the initial atomic density functions n_{10} and n_{20} by

$$\begin{cases} n_{10}(x) = \dfrac{A_{21} + B_{21}\rho^0 + C_{21}}{A_{21} + (B_{21} + B_{12})\rho^0 + C_{12} + C_{21}} n(x), \\ n_{20}(x) = \dfrac{B_{12}\rho^0 + C_{12}}{A_{21} + (B_{21} + B_{12})\rho^0 + C_{12} + C_{21}} n(x), \end{cases} \tag{5.28}$$

where $n(x)$ is defined by

$$n(x) = 2.0 \cdot 10^8 \frac{r_1^2}{|x|^2} \text{ m}^{-3}.$$

Note that the initial condition of the rate equations (5.28) is consistent with the initial condition of the kinetic equation.

5.4.2 Description of the numerical procedure

Now we describe the numerical method for solving system (5.27), for t in $[0,T]$, x varying in X, v in $\mathbb{S}^2$, and ν in $\mathbb{R}^+$. The method used to solve the rate equation is the classical Euler Method, while for solving the kinetic equation, we use a particle method.

We introduce briefly the particle method which we use to solve the kinetic equation in the form

$$\partial_t f + v\cdot\nabla_x f + \chi f = \eta,$$

where the coefficients χ and η are the functions of $(t,x,v,\nu)\in\mathbb{R}^+\times X\times\mathbb{S}^2\times\mathbb{R}^+$. In the phase space, $(x,v,\nu)\in X\times\mathbb{S}^2\times\mathbb{R}^+$, the unknown function f is approximated at a discrete time t^k, by a discrete measure, namely the numerical particles :

$$f\simeq\sum_{i=1}^{N} a_i(t^k)\delta_{x_i(t^k),v_i(t^k),\nu_i(t^k)},$$

where N is the total number of particles, and each particle i is characterized by its weight $a_i=a_i(t^k)$, its position $x_i=x_i(t^k)$, its direction $v_i=v_i(t^k)$, and its frequency $\nu_i=\nu_i(t^k)$.

The parameters of the particle at next time step are calculated thanks to the following procedure :

$$\begin{cases} a_i(t^{k+1}) = a_i(t^k)\exp\left(-\Delta t\chi(t^k,x_i(t^k),v_i(t^k),\nu_i(t^k))\right), \\ x_i(t^{k+1}) = x_i(t^k)+v_i(t^k)\Delta t, \\ v_i(t^{k+1}) = v_i(t^k), \\ \nu_i(t^{k+1}) = \nu_i(t^k). \end{cases}$$

Then, taking into account the boundary condition and the source term η, we create new particles (labeled from $N+1$ to N').

The unknown function is thus approximated at the discrete time t^{k+1} by

$$f\simeq\sum_{i=1}^{N'} a_i(t^{k+1})\delta_{x_j(t^{k+1}),v_j(t^{k+1}),\nu_j(t^{k+1})},$$

and the integrated mean intensity ρ at time t^{k+1}, can be computed thanks to

$$\rho^{k+1}\simeq\sum_{i=1}^{N'} a_i(t^{k+1})\delta_{x_j(t^{k+1})}\phi(\nu_j(t^{k+1})). \tag{5.29}$$

For more details of the particle method, we refer to [31, 44].

Now we give the numerical method to solve the rate equations. We observe that these equations involve only time derivative, and the position variable $x\in X$ can be seen as a parameter. We use the classical explicit Euler scheme to solve the rate equations in (5.27) as

$$
\begin{aligned}
n_1(t^{k+1},x) = \quad & n_1(t^k,x) + \frac{\Delta t}{\epsilon}\Big(n_2(t^k,x)A_{21} + \Big(n_2(t^k,x)B_{21} - n_1(t^k,x)B_{12}\Big)\,\rho(t^k,x) \\
& \qquad + n_2(t^k,x)C_{21}(x) - n_1(t^k,x)C_{12}(x)\Big), \\
n_2(t^{k+1},x) = \quad & n_2(t^k,x) + \frac{\Delta t}{\epsilon}\Big(- n_2(t^k,x)A_{21} + \Big(n_1(t^k,x)B_{12} - n_2(t^k,x)B_{21}\Big)\,\rho(t^k,x) \\
& \qquad + n_1(t^k,x)C_{12}(x) - n_2(t^k,x)C_{21}(x)\Big).
\end{aligned}
\tag{5.30}
$$

Notice that there holds $n_1(t^{k+1},x) + n_2(t^{k+1},x) = n(x)$.

Now we derive the stability condition for the classical explicit Euler scheme (5.30). First substituting the equality $n_1(t^k,x) = n(x) - n_2(t^{k+1},x)n(x)$ in (5.30) yields

$$
\begin{aligned}
n_1^{k+1} \quad = \quad & n(x)\frac{\Delta t}{\epsilon}\left(A_{21} + B_{21}\rho^k + C_{21}\right) \\
& + n_1^k\left(1 - \frac{\Delta t}{\epsilon}\left(A_{21} + \left(B_{21} + B_{12}\right)\rho^k + C_{21} + C_{12}\right)\right).
\end{aligned}
$$

Note that we omit the dependence of the space variable x, and use the notations $\rho^k = \rho(t^k,x)$ and $n_1^k = n_1^k(t^k,x)$. Then by the induction with respect to k, we arrive at

$$
n_1^{k+1} \quad = \quad \sum_{j=0}^{k} n\frac{\Delta t}{\epsilon}\left(A_{21} + B_{21}\rho^{k-j} + C_{21}\right) \tag{5.31}
$$

$$
\times\prod_{l=0}^{j-1}\left(1 - \frac{\Delta t}{\epsilon}\left(A_{21} + \left(B_{21} + B_{12}\right)\rho^{k-l} + C_{21} + C_{12}\right)\right) \tag{5.32}
$$

$$
+ n_1^0\prod_{j=0}^{k}\left(1 - \frac{\Delta t}{\epsilon}\left(A_{21} + \left(B_{21} + B_{12}\right)\rho^{j} + C_{21} + C_{12}\right)\right) \tag{5.33}
$$

Recall that from the definition of the initial atomic density function n_{10} we can write the total density function n in term of n_{10}, that is n_1^0, as

$$
n = \frac{A_{21} + (B_{21} + B_{12})\rho^0 + C_{12} + C_{21}}{A_{21} + B_{21}\rho^0 + C_{21}} n_1^0.
$$

Substituting this equality in (5.31) yields

$$
\begin{aligned}
n_1^{k+1} \quad = \quad n_1^0\Bigg[& \sum_{j=0}^{k}\frac{\Delta t}{\epsilon}\left(A_{21} + B_{21}\rho^{k-j} + C_{21}\right)\frac{A_{21} + (B_{21} + B_{12})\rho^0 + C_{12} + C_{21}}{A_{21} + B_{21}\rho^0 + C_{21}} \\
& \times\prod_{l=0}^{j-1}\left(1 - \frac{\Delta t}{\epsilon}\left(A_{21} + \left(B_{21} + B_{12}\right)\rho^{k-l} + C_{21} + C_{12}\right)\right) \\
& + \prod_{j=0}^{k}\left(1 - \frac{\Delta t}{\epsilon}\left(A_{21} + \left(B_{21} + B_{12}\right)\rho^{j} + C_{21} + C_{12}\right)\right)\Bigg].
\end{aligned}
$$

Sufficient condition for the stability : the following condition

$$1 - \frac{\Delta t}{\epsilon}\left(A_{21} + (B_{21}+B_{12})\sup_j \|\rho^j\|_{L^\infty(X)} + \|C_{21}+C_{12}\|_{L^\infty(X)}\right) > 0, \tag{5.34}$$

is the sufficient condition for the stability of the scheme (5.30).

For the parameters given in §5.4.1, we compare the orders of A_{21}, $(B_{21}+B_{12})\sup_\nu B(\nu, T_0)$, $\sup_x C_{21}(x)$ and $\sup_x C_{12}(x)$, and we have that (5.34) holds provided that there holds the following inequality

$$\epsilon > 1.2 \cdot 10^{-2}\Delta t.$$

In the numerical test, we take $\Delta t = 0.02$, and $\epsilon = 0.8, 0.08, 0.008, 0.0008$, thus ϵ and Δt verify this inequality and the numerical scheme (5.30) is stable.

Next we describe the discrete setting in time and position variables. Let T be the final time, $(t^k)_{k=0,1,\cdots,N_1}$ be the discrete time in the time interval $[0,T]$, defined as

$$t^k = k * \frac{T}{N_1}, \quad k = 0, 1\cdots, N_1.$$

Notice that the length of the small interval $[t^k, t^{k-1}]$ is $\Delta t = \frac{T}{N_1}$.

Let $x = (x_1, x_2, x_3)$ be a point in domain X in Cartesian coordinates. We write x in spherical coordinates (r, α, β), where r is the distance (radius) from the point to the origin, with $r_{in} < r < r_{out}$, α is the polar angle from the x_3-axis, with $0 \le \alpha \le \pi$, and β is the azimuthal angle in the x_1x_2-plane from the x_1-axis with $0 \le \beta < 2\pi$, see figure 5.2, defined as

$$\begin{cases} x_1 &= r\sin\alpha\cos\beta, \\ x_2 &= r\sin\alpha\sin\beta, \\ x_3 &= r\cos\alpha. \end{cases}$$

Let $(r_i)_{i=0,1,\cdots,n_2}$ be the discrete points in $[r_{in}, r_{out}]$, uniformly in r^3, which are defined as

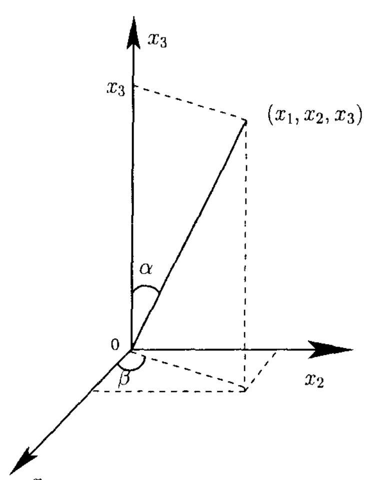

FIG. 5.2 – The spherical coordinates

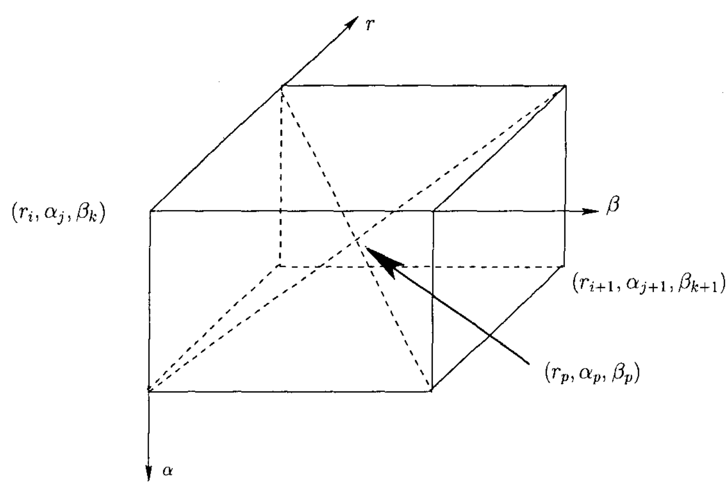

FIG. 5.3 – The small cuboid X_p with the center point

$$r_i = \left(r_{in}^3 + i\frac{r_{out}^3 - r_{in}^3}{N_2}\right)^{\frac{1}{3}}, \quad i = 0, 1, \cdots, N_2,$$

$(\alpha_j)_{j=0,1,\cdots,N_3}$ be the discrete points in $[0,\pi]$, defined as

$$\alpha_j = j\frac{\pi}{N_3}, \quad i = 0, 1, \cdots, N_3,$$

and $(\beta_k)_{k=0,1,\cdots,N_4}$ be the discrete points in $[0,2\pi]$, defined as

$$\beta_k = k\frac{2\pi}{N_4}, \quad i = 0, 1, \cdots, N_4.$$

Thus X, the cuboid in the spherical coordinates,

$$X = \{(r,\alpha,\beta)|\ r_{in} < r < r_{out}, 0 \le \alpha \le \pi, 0 \le \beta < 2\pi\},$$

is the union of $N_2\ N_3\ N_4$ small cuboids, see figure 5.3, as

$$X = \bigcup_{p=1}^{N_2\ N_3\ N_4} X_p = \bigcup_{p=1}^{N_2\ N_3\ N_4} \{(r,\alpha,\beta)|\ r_{i-1} \le r < r_i, \alpha_{j-1} \le \alpha < \alpha_j, \beta_{k-1} \le \beta \le \beta_k\},$$

where we have the relation between p and $i,\ j,\ k$ as $p = i\ N_2\ N_3 + j\ N_3 + k$. In the cuboid X_p we define the center point as

$$\begin{aligned} r_p &= \frac{r_i + r_{i-1}}{2}, \\ \alpha_p &= \frac{\alpha_j + \alpha_{j-1}}{2}, \\ \beta_p &= \frac{\beta_k + \beta_{k-1}}{2}, \end{aligned}$$

which is denoted by x_p in the Cartesian coordinates, see figure 5.3.

Let us now state the numerical procedure for solving the coupled system (5.27) as :

- *Step 1* Give the discrete points over the time interval $[0,T]$ and the domain X, as above discussed. We take

$$T = 1.5,\ N_1 = 75,\ N_2 = 20,\ N_3 = 5 \text{ and } N_4 = 10.$$

- *Step 2* Let $t = 0,\ k = 0$.
- *Step 3* Create the particles according to the initial condition $f_0(x,v,\nu)$. Then compute ρ^0 using formula (5.29).
- *Step 4* Compute $n_1^0,\ n_2^0$ thanks to formula (5.28).
- *Step 5* We compute $n_1^n,\ n_2^n$ by the explicit Euler scheme, (5.30), using the values of $n_1^{n-1},\ n_2^{n-1}$ and ρ^{n-1}.
- *Step 6* Compute ρ^n using the values of $n_1^n,\ n_2^n$ and the boundary condition, thanks to the particle method.
- *Step 7* Let $t = t + \Delta t,\ k = k + 1$.
- *Step 8* If $t \le T$, go to *Step 5.*
- *Step 9* Save the results of $\rho,\ n_1,\ n_2$.

Note that in the above procedure, n_i^n denotes the value of n_i at time $t^n = n\Delta t$.

5.4.3 Numerical results

We take different values of ϵ, such that $\epsilon \to 0$. We compute the solutions f^ϵ (and ρ^ϵ) and the density n_1^ϵ and n_2^ϵ. To verify the convergence of solutions, we compute the (relative) differences between ρ^ϵ, n_i^ϵ and ρ, n_i, (solution of the limiting system) i.e

$$\frac{|\rho^\epsilon(t,x)-\rho(t,x)|}{\rho(t,x)}, \quad \frac{|n_i^\epsilon(t,x)-n_i(t,x)|}{n_i(t,x)}, \; i=1,2. \tag{5.35}$$

At first, we use the same initial data of the specific intensity of radiation, independent of ϵ, thus we have $\rho^\epsilon(0,x) \equiv \rho(0,x) = \int f_0\,\phi(\nu)\mathrm{d}\nu$. Recall that we take the initial condition and the boundary condition as

$$f_0 = \frac{|x| - r_{in}}{r_{out} - r_{in}} B(\nu, T_0), \quad g = B(\nu, T_0),$$

where $B(\nu, T_0)$ is the intensity defined by the Planck function at the temperature $T_0 = 2.728$K. Note that there holds $n_i^\epsilon(0,x) = n_i(0,x)$, because we compute $n_i(0,x)$ through the initial condition of the kinetic equation, $f_0(x,v,\nu)$, see (5.28). This is consistent with the initial condition of the rate equations.

Then in what follows, we plot the relative errors at the different points over the line with the zenith angle $\alpha = \frac{\alpha_1-\alpha_0}{2} = \frac{\pi}{2N_3}$, and the azimuth angle $\beta = \frac{\alpha_1-\alpha_0}{2} = \frac{\pi}{N_4}$.

In figure 5.4-5.5 we plot the relative errors on the mean intensity ρ and the density n_1, respectively, see (5.35), at time $t=0.3$, for different values of ϵ. Figure 5.6-5.13 represent the evolution of these quantities looking at $t=0.6$, $t=0.9$, $t=1.2$ and $t=1.5$, respectively.

The first remark is that the relative error on the mean intensity does not vary too much as $\epsilon \to 0$, nor as time growth, the maximal error still remains of order 10^{-6}. Note that the relative error on ρ is a decreasing function of the function of the radius, which adopts a "linear-like" behavior for large time. In contrast, the relative error on the density n_1 is more sensible to the variations of ϵ ant t. It seems that the error on n_1 is less important for small radii. For ϵ very small, the errors do not change too much as time grows, but the influence of time becomes sensible for moderate values of ϵ, the maximum relative error changing for a factor 0.5 between $t=0.3$ and $t=1.5$ for $\epsilon=0.8$, $\epsilon=0.08$, and a factor 1 for $\epsilon=0.008$, $\epsilon=0.0008$). This is consistent to the fact that we proved a convergence statement on bounded time intervals but we are not able to establish estimates uniform with respect to time.

5.5 Appendix : Einstein coefficients, transfer equation and non-LTE rate equation

In this section, we recall some physical background and derive the transfer Equation (5.1) and the non-LTE rate equation (5.4) for the problem of molecular line transfer. For more details we refer to [49]-[53]. For the problem of molecular line transfer, the bound-bound transitions between the lower level i (with the statistical weight g_i and the energy E_i) and the upper level j (with the statistical weight g_j and the energy E_j) may occur as radiative excitation, spontaneous radiative de-excitation, induced (stimulated) de-excitation, collisional excitation and collisional de-excitation. In what follows, we study these transitions quantitatively by defining the Einstein coefficients that characterize these

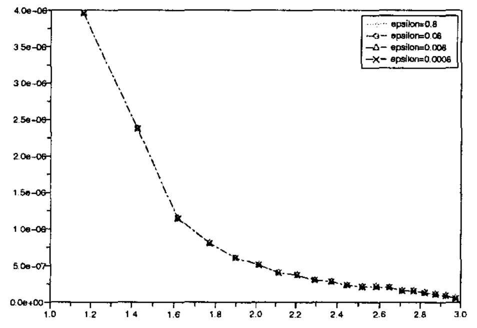

FIG. 5.4 – abs$(\rho^\epsilon - \rho)/\rho$ at time $t = 0.3$

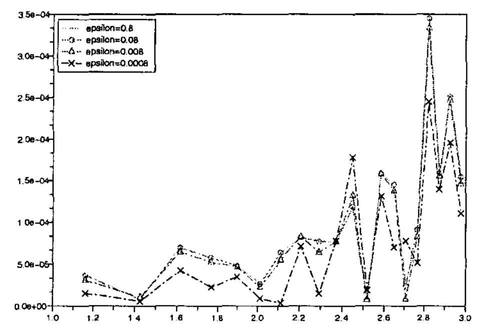

FIG. 5.5 – abs$(n_1^\epsilon - n_1)/n_1$ at time $t = 0.3$

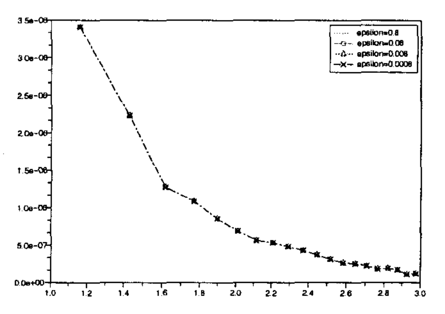

FIG. 5.6 – abs$(\rho^\epsilon - \rho)/\rho$ at time $t = 0.6$

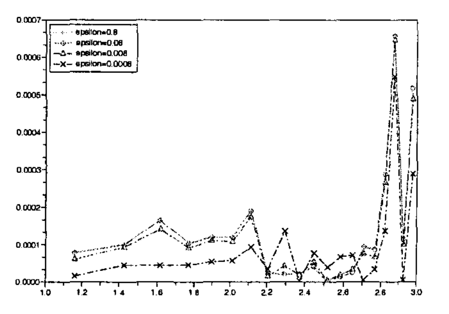

FIG. 5.7 – abs$(n_1^\epsilon - n_1)/n_1$ at time $t = 0.6$

transitions. Let us denote by $h\nu_{ij}$ the energy that separates the lower energy level i and the upper level j, that is $h\nu_{ij} = E_j - E_i$, where E_j and E_i are measured related to the ground state.

5.5.1 Specific intensity

We define the specific intensity $f(t, x, v, \nu)$ of radiation at position $x \in \mathbb{R}^3$, at time $t \in \mathbb{R}^+$, traveling in direction $v \in \mathbb{S}^2$ with frequency $\nu \in \mathbb{R}^+$,. This intensity is such that the amount of energy transported by radiation of frequencies $(\nu, \nu + \mathrm{d}\nu)$ across a surface element $\mathrm{d}S$, in a time $\mathrm{d}t$, into a solid angle $\mathrm{d}v$ around the direction v, is

$$\mathrm{d}E = f(t, x, v, \nu)\mathrm{d}S(l \cdot v)\mathrm{d}v\mathrm{d}\nu\mathrm{d}t = f(t, x, v, \nu)\mathrm{d}S \cos\alpha \, \mathrm{d}v\mathrm{d}\nu\mathrm{d}t, \tag{5.36}$$

where l is the normal to $\mathrm{d}S$ and α in the angle between l and v. The specific intensity provides a complete macroscopic description of the radiation field. From the microscopic point of view, the radiation field is composed of photons, the relation between the specific intensity and the photon number density is introduced in §63 in [49]. In 5.5.3 we shall derive the transfer equation for the specific intensity of radiation.

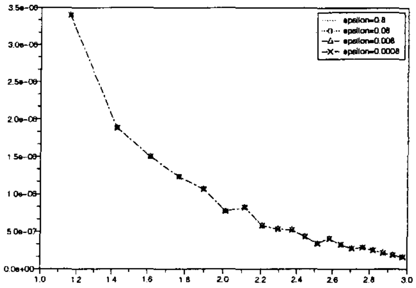

FIG. 5.8 – $\mathrm{abs}(\rho^\epsilon - \rho)/\rho$ at time $t = 0.9$

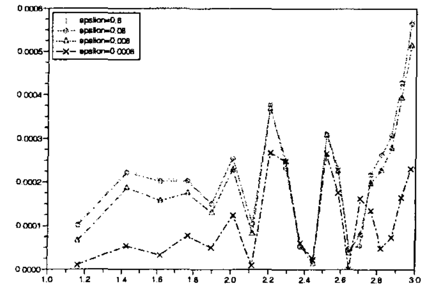

FIG. 5.9 – $\mathrm{abs}(n_1^\epsilon - n_1)/n_1$ at time $t = 0.9$

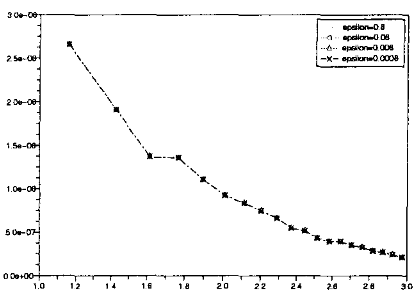

FIG. 5.10 – $\mathrm{abs}(\rho^\epsilon - \rho)/\rho$ at time $t = 1.2$

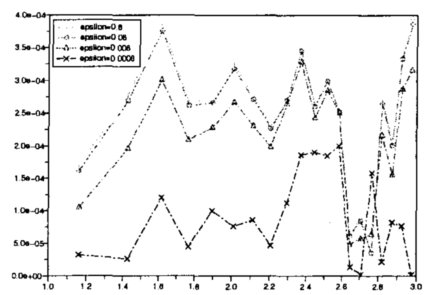

FIG. 5.11 – $\mathrm{abs}(n_1^\epsilon - n_1)/n_1$ at time $t = 1.2$

5.5.2 Einstein coefficients

The Einstein coefficient for radiative excitation, B_{ij}, characterizing the absorption probability, is defined in such a way that the number of photons absorbed in the line per unit volume and per unit time is

$$r_{ij} = n_i B_{ij} f \phi_{ij}(\nu) \mathrm{d}v \mathrm{d}\nu, \tag{5.37}$$

where $\phi_{ij}(\nu)$ is the line profile. Thus, the energy absorbed per unit volume and per unit time is

$$\dot{E}_{absorped} = n_i B_{ij} f \phi_{ij}(\nu) h\nu \mathrm{d}v \mathrm{d}\nu. \tag{5.38}$$

The number of radiative excitations from level i to level j per unit time is

$$N_{ij}^{(1)} = n_i B_{ij} \rho_{ij}, \tag{5.39}$$

where ρ_{ij} is the integrated mean intensity over the line profile ϕ_{ij}, defined by (5.6).

Next we consider a molecule in the upper level j, which can either decay spontaneously to the lower level i, or be stimulated to decay by radiation in the line. The spontaneously emission probability A_{ji} is the transition probability for spontaneous de-excitation from level j to level i per unit time and per particle in level j. Then the number of photons emitted in the line per unit volume and per unit time is

$$r_{ji}^{(1)} = n_j A_{ji} \phi_{ij}(\nu) \mathrm{d}v \mathrm{d}\nu. \tag{5.40}$$

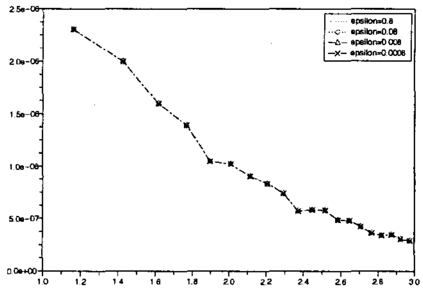

FIG. 5.12 – $\mathrm{abs}(\rho^\epsilon - \rho)/\rho$ at time $t = 1.5$

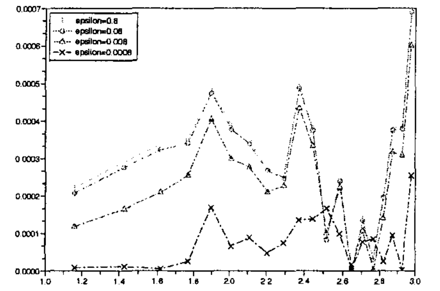

FIG. 5.13 – $\mathrm{abs}(n_1^\epsilon - n_1)/n_1$ at time $t = 1.5$

The energy emitted per unit volume and per unit time is

$$\dot{E}_{spontaneous} = n_j A_{ji} \phi_{ij}(\nu) h\nu \mathrm{d}v \mathrm{d}\nu, \tag{5.41}$$

and the number of spontaneous de-excitation from level j to level i per unit time is

$$N_{ji}^{(1)} = n_j A_{ji}. \tag{5.42}$$

The induced (stimulated) emission probability B_{ji} is defined in such a way that the number of photons emitted in the line per unit volume and per unit time is

$$r_{ji}^{(2)} = n_j B_{ji} f \phi_{ij}(\nu) \mathrm{d}v \mathrm{d}\nu, \tag{5.43}$$

and the energy absorbed per unit volume and per unit time is

$$\dot{E}_{induced} = n_j B_{ji} f \phi_{ij}(\nu) h\nu \mathrm{d}v \mathrm{d}\nu. \tag{5.44}$$

Finally, the number of radiative excitations from level i to level j per unit time is

$$N_{ji}^{(2)} = n_j B_{ji} \rho_{ij}. \tag{5.45}$$

The Einstein coefficient for collisional excitation C_{ij} is defined as the number of collisional excitation from the lower level i to the upper level j per unit time and per particle in level i. Then, the number of collisional excitation from the lower level i to the upper level j per unit time is

$$N_{ij}^{(2)} = n_i C_{ij}. \tag{5.46}$$

Similarly we define the collisional de-excitation C_{ji}, the number of collision de-excitation from the upper level j to the lower level i. Then the number of collisional excitation from the upper level j to the lower level i per unit time is

$$N_{ji}^{(3)} = n_j C_{ji}. \tag{5.47}$$

The collision coefficients, C_{ij} and C_{ji}, are proportional to the density of the collision partner N_{col}. That is, $C_{ji} = N_{col} K_{ij}$, where the collisional rate K_{ij} depend on the temperature and on the type of collisional partner.

The collisional excitation and the collisional de-excitation have no link with the radiation field. For the general expression of the collisional rates, we refer to [49] §85, or [53] §2.6.

From the detailed balance relations, we derive the Einstein relations (Cf. [49] in §73 and §85) :

$$\frac{B_{ij}}{B_{ji}} = \frac{g_j}{g_i}, \quad \frac{A_{ji}}{B_{ji}} = \frac{2h\nu_{ij}^3}{c^2}, \tag{5.48}$$

$$\frac{C_{ji}}{C_{ij}} = \frac{g_i}{g_j} \mathrm{e}^{(h\nu_{ij})/(kT)}, \tag{5.49}$$

where $h\nu_{ij}$ is the energy that separates the energy levels i and j, k is the Boltzmann constant, T is the kinetic temperature, and g_i, g_j are the statistical weights of the energy levels i and j.

5.5.3 Transfer equation and rate equation

Form the above discussion, we write the line absorption coefficient, corrected by the stimulated emission, in the static medium, as

$$\chi = \sum_i \sum_{j>i} (n_i B_{ij} - n_j B_{ji})\, h\nu \phi_{ij}(\nu), \tag{5.50}$$

and the line emission coefficient

$$\eta = \sum_i \sum_{j>i} n_j A_{ji} h\nu \phi_{ij}(\nu). \tag{5.51}$$

When writing the transfer equation, it is often convenient to use the ratio of emissivity to opacity (extinction), that is the so-called the source function, denoted by S. For a line radiative transfer, the source function is

$$S = \frac{n_j A_{ji}}{n_i B_{ij} - n_j B_{ji}}.$$

Using the Einstein relations (5.48), it can be rewritten as

$$S = \frac{2h\nu_{ij}^3}{c^2} \frac{1}{\frac{g_j n_i}{g_i n_j} - 1}. \tag{5.52}$$

Since the frequency variation of order 3 is weak compared to the variation of the profile ϕ_{ij}, (5.52) is often called the frequency-independent line source function, see [49] §73.

Let us consider an element of material of length $\mathrm{d}s$, cross section $\mathrm{d}S$, fixed in the laboratory frame. We consider a radiation field contained in a frequency interval $(\nu, \nu + \mathrm{d}\nu)$, traveling into a solid angle $\mathrm{d}v$ around a direction $v \in \mathbb{S}^2$, normal to $\mathrm{d}S$. When the radiation passes through the element of material, the difference between the amount of energy that emerges at time $t + \mathrm{d}t$, at position $x + \mathrm{d}x$ and the amount of energy that incidents at time t, at position x, must be equal to the difference between the amount of energy created by emission and the energy absorbed by the material, using (5.36) – (5.44), in the volume $\mathrm{d}s\ \mathrm{d}S$, and during the time $\mathrm{d}t$, that is

$$(f(t + \mathrm{d}t, x + \mathrm{d}x, v, \nu) - f(t, x, v, \nu))\, \mathrm{d}S \mathrm{d}v \mathrm{d}\nu \mathrm{d}t = (\eta - \chi f)\, \mathrm{d}s \mathrm{d}S \mathrm{d}v \mathrm{d}\nu \mathrm{d}t.$$

Let s be the path along the ray, we have $\mathrm{d}t = \mathrm{d}s/c$, and

$$\begin{aligned} f(t+\mathrm{d}t, x+\mathrm{d}x, v\nu) - f(t,x,v,\nu) &= \left(\frac{1}{c}\partial_t f + \frac{\partial f}{\partial s}\right)\mathrm{d}s \\ &= \left(\frac{1}{c}\partial_t f + v\cdot\nabla_x f\right)\mathrm{d}s. \end{aligned}$$

We end up with the following transfer equation :

$$\frac{1}{c}\partial_t f + v\cdot\nabla_x f = \eta - \chi f. \tag{5.53}$$

From the radiative excitation number formula (5.39) and the collisional excitation formula (5.46), we derive the total excitation rate P_{ij} as

$$P_{ij} = B_{ij}\rho_{ij} + C_{ij}. \tag{5.54}$$

Similarly using the spontaneous de-excitation formula (5.42), the induced de-excitation number (5.45), and the collisional de-excitation (5.47), we get the total de-excitation rate P_{ji} as

$$P_{ji} = A_{ji} + B_{ji}\rho_{ij} + C_{ji}, \tag{5.55}$$

where ρ_{ij} is defined by (5.6). Thus we get the rate equation (5.4).

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